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Approximate and Renormgroup Symmetries

With 7 figures
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Preface

This is an introduction to a new field in applied group analysis. Namely, the book deals with the so-called renormalization group (briefly renormgroup) symmetries considered in the framework of approximate transformation groups. The notion of the renormalization group and the renormalization group method were introduced in theoretical physics by N. N. Bogoliubov and D. V. Shirkov in 1950s. Renormgroup symmetries provide a basis for the renormgroup algorithm for improving solutions to boundary value problems by converting “less applicable solutions” into “more applicable solutions”. The algorithm is particularly useful for improving approximate solutions given by the perturbation theory.

We present in a concise form the essence of the mathematical apparatus for computing approximate and renormgroup symmetries using the infinitesimal techniques of the modern group analysis. In order to make the book self-contained, we provide in Chapter 1 an outline of basic notions from the classical Lie group analysis of differential equations. Chapters 2 and 3 reflect new trends in the modern group analysis. Chapter 2 contains a brief discussion of approximate transformation groups. In Chapter 3 we discuss methods for calculating symmetries of integro-differential equations. Renormgroup symmetries are introduced and illustrated by several examples in Chapter 4. The renormgroup algorithm is applied to various nonlinear problems in mathematical physics in Chapter 5.

The authors wish to express their gratitude to Professor Dmitry V. Shirkov, a world leader in the study of renormalization groups in quantum field theory. Our collaboration with him over many years plays a decisive role in preparing the “physical part” (Chapters 3, 4 and 5) of the monograph. We also would like to say a word of genuine appreciation in memory of late Dr. Veniamin V. Pustovalov who made our collaboration possible and who inspired many ideas that form a ground of this book.

Nail H. Ibragimov and Vladimir F. Kovalev
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Chapter 1
Lie Group Analysis in Outline

The mathematical discipline known today as the Lie group analysis, was originated in 1870s by an outstanding mathematician of the 19th century, Sophus Lie (1842–1899).

One of the most remarkable achievements of Lie was the discovery that the majority of known methods of integration of ordinary differential equations, which until then had seemed artificial and not intrinsically related to one another, could be derived in a unified manner using his theory. Moreover, Lie provided a classification of all ordinary differential equations in terms of their symmetry groups, and thus described the whole set of equations integrable by group-theoretical methods. These results are presented, e.g. in his textbook [10].

This chapter is aimed at discussing basic concepts from the Lie group analysis: continuous transformation groups and their generators, definition and calculation of symmetry groups of differential equations, simplest methods of integration of non-linear equations using their symmetries. It contains also an introduction to the theory of Lie-Backlund transformation groups and approximate groups. The reader interested in studying more about the Lie group methods of integration of differential equations is referred to [7] and to the recent textbook [8].

1.1 Continuous point transformation groups

1.1.1 One-parameter groups

We will consider here only one-parameter groups. Let $T_a$ be an invertible transformation depending on a real parameter $a$ and acting in the $(x,y)$-plane:

$$
\bar{x} = f(x,y,a), \quad \bar{y} = g(x,y,a),
$$

where the functions $f$ and $g$ satisfy the conditions
\[ f \big|_{a=0} = x, \quad g \big|_{a=0} = y. \tag{1.2} \]

The invertibility is guaranteed if one requires that the Jacobian of \( f, g \) with respect to \( x, y \) is not zero in a neighborhood of \( a = 0 \). Further, it is assumed that the functions \( f \) and \( g \) as well as their derivatives that appear in the subsequent discussion are continuous in \( x, y, a \).

**Definition 1.1.1.** A set \( G \) of transformations (1.1) is a one-parameter transformation group if it contains the identical transformation \( I = T_0 \) and includes the inverse \( T_a^{-1} \) as well as the composition \( T_b T_a \) of all its elements \( T_a, T_b \in G \). By a suitable choice of the group parameter \( a \), the main group property \( T_b T_a \in G \) can be written

\[ T_b T_a = T_{a+b}, \]

that is

\[ f \left( f(x,y,a), g(x,y,a), b \right) = f(x,y,a+b), \]
\[ g \left( f(x,y,a), g(x,y,a), b \right) = g(x,y,a+b). \tag{1.3} \]

In practical applications, the conditions (1.3) hold only for sufficiently small values of \( a \) and \( b \). Then one arrives at what is called a local one-parameter group \( G \). For brevity, local groups are also termed groups.

### 1.1.2 Infinitesimal transformations

The expansion of the functions \( f, g \) into the Taylor series in \( a \) near \( a = 0 \), taking into account the initial condition (1.2), yields the infinitesimal transformation of the group \( G \) (1.1):

\[ \bar{x} \approx x + \xi(x,y)a, \quad \bar{y} \approx y + \eta(x,y)a, \tag{1.4} \]

where

\[ \xi(x,y) = \left. \frac{\partial f(x,y,a)}{\partial a} \right|_{a=0}, \quad \eta(x,y) = \left. \frac{\partial g(x,y,a)}{\partial a} \right|_{a=0}. \tag{1.5} \]

The vector \((\xi, \eta)\) with components (1.5) is the tangent vector (at the point \((x,y))\) to the curve described by the transformed points \((\bar{x}, \bar{y})\), and is therefore called the tangent vector field of the group \( G \).

**Example 1.1.1.** The group of rotations

\[ \bar{x} = x \cos a + y \sin a, \quad \bar{y} = y \cos a - x \sin a \]

has the following infinitesimal transformation:

\[ \bar{x} \approx x + ya, \quad \bar{y} \approx y - xa. \]

The tangent vector field (1.5) is sometimes also written as a first-order differential operator
which behaves as a scalar under an arbitrary change of variables, unlike the vector \((\xi, \eta)\). Lie called the operator \((1.6)\) the symbol of the infinitesimal transformation \((1.4)\) or of the corresponding group \(G\). In the current literature, the operator \(X\) \((1.6)\) is called the generator of the group \(G\) of transformations \((1.1)\).

**Example 1.1.2.** The generator of the group of rotations from Example 1.1.1 has the form

\[
X = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}.
\]  

### 1.1.3 Lie equations

Given an infinitesimal transformation \((1.4)\), or the generator \((1.6)\), the transformations \((1.1)\) of the corresponding one-parameter group \(G\) are defined by solving the following equations known as the Lie equations:

\[
\frac{df}{da} = \xi(f, g), \quad f\big|_{a=0} = x,
\]

\[
\frac{dg}{da} = \eta(f, g), \quad g\big|_{a=0} = y.
\]  

We will write Eq. \((1.8)\) also in the following equivalent form:

\[
\frac{d\bar{x}}{da} = \xi(\bar{x}, \bar{y}), \quad \bar{x}\big|_{a=0} = x,
\]

\[
\frac{d\bar{y}}{da} = \eta(\bar{x}, \bar{y}), \quad \bar{y}\big|_{a=0} = y.
\]  

**Example 1.1.3.** Consider the infinitesimal transformation

\[
\bar{x} \approx x + ax^2, \quad \bar{y} \approx y + axy.
\]

The corresponding generator has the form

\[
X = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}.
\]  

The Lie equations \((1.9)\) are written as follows:

\[
\frac{d\bar{x}}{da} = \bar{x}^2, \quad \bar{x}\big|_{a=0} = x,
\]

\[
\frac{d\bar{y}}{da} = \bar{x}\bar{y}, \quad \bar{y}\big|_{a=0} = y.
\]
The differential equations of this system are easily solved and yield
\[ \ddot{x} = \frac{1}{a + C_1}, \quad \ddot{y} = \frac{C_2}{a + C_1}. \]
The initial conditions imply that \( C_1 = -1/x \), \( C_2 = -y/x \). Consequently we arrive at the following one-parameter group of projective transformations:
\[ \ddot{x} = \frac{x}{1 - ax}, \quad \ddot{y} = \frac{y}{1 - ax}. \] (1.11)

### 1.1.4 Exponential map

One can represent the solution to the Lie equations (1.9) by means of infinite power series (Taylor series). Then the group transformation (1.1) for a generator \( X \) (1.6) is given by the so-called exponential map:
\[ \ddot{x} = e^{aX}(x), \quad \ddot{y} = e^{aX}(y), \] (1.12)
where
\[ e^{aX} = 1 + \frac{a}{1!}X + \frac{a^2}{2!}X^2 + \cdots + \frac{a^s}{s!}X^s + \cdots. \] (1.13)

**Example 1.1.4.** Consider again the generator (1.10) discussed in Example 1.1.3:
\[ X = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}. \]

According to (1.12)–(1.13), one has to find \( X^s(x) \) and \( X^s(y) \) for all \( s = 1, 2, \ldots \).
We calculate several terms, e.g.
\[ X(x) = x^2, \quad X^2(x) = X(X(x)) = X(x^2) = 2!x^3, \quad X^3(x) = X(2!x^3) = 3!x^4, \]
and then make a guess:
\[ X^s(x) = s!x^{s+1}. \]
The proof of the latter equation is given by induction:
\[ X^{s+1}(x) = X(s!x^{s+1}) = (s + 1)!x^{s+2}. \]
Furthermore, one obtains
\[ X(y) = xy, \quad X^2(y) = X(xy) = yX(x) + xX(y) = xy^2 + xxy = 2lyx^2, \]
\[ X^3(y) = 2![yX(x^2) + x^2X(y)] = 2![y(2x^3) + x^2xy] = 3!yx^3, \]
then makes a guess
\[ X^s(y) = s!yx^s. \]
and proves it by induction:

\[ X^{s+1}(y) = s!X(\gamma^s) = s![s\gamma x^{s+1} + x'(xy)] = (s + 1)!\gamma x^{s+1}. \]

Substitution of the above expressions in the exponential map yields

\[ e^{aX}(x) = x + ax^2 + \cdots + a^s x^{s+1} + \cdots. \]

One can rewrite the right-hand side as \( x(1 + ax + \cdots + a^s x^s + \cdots) \). The series in brackets is manifestly the Taylor expansion of the function \( 1/(1 - ax) \) provided that \( |ax| < 1 \). Consequently,

\[ \bar{x} = e^{aX}(x) = \frac{x}{1 - ax}. \]

Likewise, one obtains

\[ e^{aX}(y) = y + ayx + a^2 yx^2 + \cdots + a^s yx^s + \cdots = y(1 + ax + \cdots + a^s x^s + \cdots). \]

Hence,

\[ \bar{y} = e^{aX}(y) = \frac{y}{1 - ax}. \]

Thus, we have arrived at the transformations (1.11):

\[ \bar{x} = \frac{x}{1 - ax}, \quad \bar{y} = \frac{y}{1 - ax}. \]

### 1.1.5 Canonical variables

**Theorem 1.1.1.** Every one-parameter group of transformations (1.1) reduces to the group of translations \( \bar{t} = t + a, \bar{u} = u \) with the generator

\[ X = \frac{\partial}{\partial t} \]

by a suitable change of variables

\[ t = t(x,y), \quad u = u(x,y). \]

The variables \( t, u \) are called canonical variables.

**Proof.** Under a change of variables the differential operator (1.6) transforms according to the formula

\[ X = X(t) \frac{\partial}{\partial t} + X(u) \frac{\partial}{\partial u}. \]  

(1.14)
Therefore, canonical variables are found from the linear partial differential equations of the first order:

\[
X(t) \equiv \xi(x,y) \frac{\partial t(x,y)}{\partial x} + \eta(x,y) \frac{\partial t(x,y)}{\partial y} = 1, \\
X(u) \equiv \xi(x,y) \frac{\partial u(x,y)}{\partial x} + \eta(x,y) \frac{\partial u(x,y)}{\partial y} = 0.
\]  

(1.15)

1.1.6 Invariants and invariant equations

Definition 1.1.2. A function \( F(x,y) \) is an invariant of the group \( G \) of transformations (1.1) if \( F(\bar{x},\bar{y}) = F(x,y) \), i.e.,

\[
F\left( f(x,y,a), g(x,y,a) \right) = F(x,y)
\]  

(1.16)

identically in the variables \( x,y \) and the group parameter \( a \).

Theorem 1.1.2. A function \( F(x,y) \) is an invariant of the group \( G \) if and only if it solves the following first-order linear partial differential equation:

\[
XF \equiv \xi(x,y) \frac{\partial F}{\partial x} + \eta(x,y) \frac{\partial F}{\partial y} = 0.
\]  

(1.17)

Proof. Let \( F(x,y) \) be an invariant. Let us take the Taylor expansion of \( F\left( f(x,y,a), g(x,y,a) \right) \) with respect to \( a \):

\[
F\left( f(x,y,a), g(x,y,a) \right) \approx F(x+a\xi,y+a\eta) \approx F(x,y) + a\left( \xi \frac{\partial F}{\partial x} + \eta \frac{\partial F}{\partial y} \right),
\]

or

\[
F(\bar{x},\bar{y}) = F(x,y) + aX(F) + o(a),
\]

and substitute it in to Eq. (1.16):

\[
F(x,y) + aX(F) + o(a) = F(x,y).
\]

It follows that \( aX(F) + o(a) = 0 \), whence \( X(F) = 0 \), i.e., Eq. (1.17).

Conversely, let \( F(x,y) \) be a solution of Eq. (1.17). Assuming that the function \( F(x,y) \) is analytic and using its Taylor expansion, one can extend the exponential map (1.12) to the function \( F(x,y) \) as follows:

\[
F(\bar{x},\bar{y}) = e^{aX}F(x,y) \overset{\text{def}}{=} \left( 1 + \frac{a}{1!}X + \frac{a^2}{2!}X^2 + \cdots + \frac{a^s}{s!}X^s + \cdots \right) F(x,y).
\]

Since \( XF(x,y) = 0 \), one has \( X^2F = X(XF) = 0, \ldots, X^sF = 0 \). We conclude that \( F(\bar{x},\bar{y}) = F(x,y) \), i.e., Eq. (1.16) thus proving the theorem.
It follows from Theorem 1.1.2 that every one-parameter group of transformations in the plane has one independent invariant, which can be taken to be the left-hand side of any first integral $\psi(x,y) = C$ of the characteristic equation for (1.17):

$$\frac{dx}{\xi(x,y)} = \frac{dy}{\eta(x,y)}. \quad (1.18)$$

Any other invariant $F$ is then a function of $\psi$, i.e., $F(x,y) = \Phi(\psi(x,y))$.

**Example 1.1.5.** Consider the group with the generator (see Exercise 1.1)

$$X = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}. \quad (1.18)$$

The characteristic equation (1.18) is written as

$$\frac{dx}{x} = \frac{dy}{2y}$$

and yields the first integral $\psi = y/x^2$. Hence, the general invariant is given by $F(x,y) = \Phi(y/x^2)$ with an arbitrary function $\Phi$ of one variable.

The concepts introduced above can be generalized in an obvious way to the multi-dimensional case by considering groups of transformations

$$\bar{x}^i = f^i(x,a), \quad i = 1, \ldots, n, \quad (1.19)$$

in the $n$-dimensional space of points $x = (x^1, \ldots, x^n)$ instead of transformations (1.1) in the $(x,y)$-plane. The generator of the group of transformations (1.19) is written as

$$X = \xi^i(x)\frac{\partial}{\partial x^i}, \quad (1.20)$$

where

$$\xi^i(x) = \frac{\partial f^i(x,a)}{\partial a} \bigg|_{a=0}. \quad (1.21)$$

The Lie equations (1.9) become

$$\frac{d\bar{x}^i}{da} = \xi^i(\bar{x}), \quad \bar{x}^i|_{a=0} = x^i. \quad (1.22)$$

The exponential map (1.12) is written:

$$\bar{x}^i = e^{aX}(x^i), \quad i = 1, \ldots, n, \quad (1.23)$$

where

$$e^{aX} = 1 + \frac{a}{1!}X + \frac{a^2}{2!}X^2 + \cdots + \frac{a^s}{s!}X^s + \cdots. \quad (1.23)$$
Definition 1.1.2 of invariant functions of several variables remains the same, namely an invariant is defined by the equation \( F(\vec{x}) = F(x) \). The invariant test given by Theorem 1.1.2 has the same formulation with the evident replacement of Eq. (1.17) by its \( n \)-dimensional version:

\[
\sum_{i=1}^{n} \xi^i(x) \frac{\partial F}{\partial x^i} = 0. \tag{1.24}
\]

Then \( n - 1 \) functionally independent first integrals \( \psi_1(x), \ldots, \psi_{n-1}(x) \) of the characteristic system for Eq. (1.24)

\[
\frac{dx^1}{\xi^1(x)} = \frac{dx^2}{\xi^2(x)} = \cdots = \frac{dx^n}{\xi^n(x)} \tag{1.25}
\]

provides a basis of invariants. Namely, any invariant \( F(x) \) is given by

\[
F(x) = \Phi(\psi_1(x), \ldots, \psi_{n-1}(x)). \tag{1.26}
\]

Let us dwell on this higher-dimensional case and consider a system of equations

\[
F_1(x) = 0, \ldots, F_s(x) = 0, \quad s < n. \tag{1.27}
\]

We shall assume that the rank of the matrix \( \| \frac{\partial F_k}{\partial x^i} \| \) is equal to \( s \) at all points \( x \) satisfying the system of Eqs. (1.27). The system of equations (1.27) then defines an \((n-s)\)-dimensional surface \( M \).

**Definition 1.1.3.** The system of Eqs. (1.27) is said to be invariant with respect to the group \( G \) of transformations (1.19) if each point \( x \) on the surface \( M \) is moved by \( G \) along the surface \( M \), i.e., \( x \in M \) implies \( \bar{x} \in M \).

**Theorem 1.1.3.** The system of Eqs. (1.27) is invariant with respect to the group \( G \) of transformations (1.19) with the generator \( X \) (1.20) if and only if

\[
XF_k \bigg|_M = 0, \quad k = 1, \ldots, s. \tag{1.28}
\]

### 1.2 Symmetries of ordinary differential equations

#### 1.2.1 Frame of differential equations

Any differential equation has two components, namely, the **frame** and the **class of solutions** (see [7]). For example, the frame of a first-order ordinary differential equation

\[
F(x,y,y') = 0
\]
is the surface $F(x,y,p) = 0$ in the space of three independent variables $x,y,p$. It is obtained by replacing the first derivative $y'$ in the differential equation $F(x,y,y') = 0$ by the variable $p$.

The class of solutions is defined in accordance with certain “natural” mathematical assumptions or from a physical significance of the differential equations under discussion.

The crucial step in integrating differential equations is a “simplification” of the frame by a suitable change of the variables $x,y$. The Lie group analysis suggests methods for simplification of the frame by using symmetry groups (or admissible groups) of differential equations.

Consider, as an example, the following Riccati equation:

$$y' + y^2 - \frac{2}{x^2} = 0. \quad (1.29)$$

Its frame is defined by the algebraic equation

$$p + y^2 - \frac{2}{x^2} = 0 \quad (1.30)$$

and is a “hyperbolic paraboloid”. For the Riccati equation (1.29), a one-parameter symmetry group is provided by the following scaling transformations (non-homogeneous dilations) obtained in Sect. 1.3.1:

$$\bar{x} = xe^a, \quad \bar{y} = ye^{-a}. \quad (1.31)$$

Indeed, transformations (1.31) after the extension to the first derivative $y'$ and the substitution $y' = p$ are written as

$$\bar{x} = xe^a, \quad \bar{y} = ye^{-a},$$

$$\bar{p} = pe^{-2a}. \quad (1.32)$$

One can readily verify that the frame of Eq. (1.30) is invariant with respect to the transformations (1.32). Let us check the infinitesimal invariance condition (1.28). The generator (1.20) of the group of transformations (1.32) has the form

$$X = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} - 2p \frac{\partial}{\partial p}.$$ 

One can readily verify that the invariance condition is satisfied. Indeed,

$$X \left( p + y^2 - \frac{2}{x^2} \right) = -2p - 2y^2 + \frac{4}{x^2} = -2 \left( p + y^2 - \frac{2}{x^2} \right),$$

and hence $X \left( p + y^2 - \frac{2}{x^2} \right) \bigg|_{(1.30)} = 0$. For the transformations (1.31), the canonical variables are (Exercise 1.3)

$$t = \ln x, \quad u = xy. \quad (1.33)$$
In the canonical variables (1.33), the Riccati equation (1.29) becomes

\[ u' + u^2 - u - 2 = 0 \quad (u' = du/dt). \tag{1.34} \]

Its frame is obtained by substituting \( u' = q \) in (1.34) and is given by the following algebraic equation:

\[ q + u^2 - u - 2 = 0. \tag{1.35} \]

The left-hand side of Eq. (1.35) does not involve the variable \( t \). Thus the curved frame (1.30) has been reduced to a cylindrical surface protracted along the \( t \)-axis. Namely it is a “parabolic cylinder”. We see that, in integrating differential equations, the decisive step is that of simplifying the frame by converting it into a cylinder. For such purpose, it is sufficient to simplify the symmetry group by introducing canonical variables. In consequence, e.g. the Riccati equation (1.29) takes the integrable form (1.34).

### 1.2.2 Extension of group actions to derivatives

The transformation of derivatives \( y', y'', \ldots \) under the action of the point transformations (1.1), regarded as a change of variables, is well-known from Calculus. It is convenient to write these transformation formulae by using the operator of total differentiation:

\[ D = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + \cdots. \]

Then the transformation formulae, e.g. for the first and second derivatives are written as

\[ \frac{dy'}{dx} = \frac{Dg}{df} = \frac{g_x + y'g_y}{f_x + y'f_y} \equiv P(x,y,y',a), \tag{1.36} \]

\[ \frac{dy''}{dx} = \frac{DP}{df} = \frac{P_x + y'P_y + y''P_y'}{f_x + y'f_y}. \tag{1.37} \]

Starting from the group \( G \) of point transformations (1.1) and then adding the transformation (1.36), one obtains the group \( G_{(1)} \), which acts in the space of the three variables \( (x,y,y') \). Further, by adding the transformation (1.37) one obtains the group \( G_{(2)} \) acting in the space \( (x,y,y',y'') \).

**Definition 1.2.1.** The groups \( G_{(1)} \) and \( G_{(2)} \) are termed the first and second prolongations of \( G \), respectively. The higher prolongations are determined similarly.

### 1.2.3 Generators of prolonged groups

Substituting into (1.36), (1.37) the infinitesimal transformation (1.4),
1.2 Symmetries of ordinary differential equations

\[ \ddot{x} \approx x + a\xi, \quad \ddot{y} \approx y + a\eta, \]

and neglecting all terms of higher order in \( a \), one obtains the following infinitesimal transformations of derivatives:

\[
\ddot{y}' = \frac{y' + aD(\eta)}{1 + aD(\xi)} \approx [y' + aD(\eta)][1 - aD(\xi)] \\
\approx y' + [D(\eta) - y'D(\xi)]a \equiv y' + a\zeta_1,
\]

\[
\ddot{y}'' = \frac{y'' + aD(\zeta_1)}{1 + aD(\xi)} \approx [y'' + aD(\zeta_1)][1 - aD(\xi)] \\
\approx y'' + [D(\zeta_1) - y''D(\xi)]a \equiv y'' + a\zeta_2,
\]

Therefore the generators of the prolonged groups \( G_1 \), \( G_2 \) are

\[
X_1 = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \zeta_1 \frac{\partial}{\partial y'}, \quad \zeta_1 = D(\eta) - y'D(\xi), \quad (1.38)
\]

\[
X_2 = X_1 + \zeta_2 \frac{\partial}{\partial y''}, \quad \zeta_2 = D(\zeta_1) - y''D(\xi). \quad (1.39)
\]

These are called the first and second prolongations of the infinitesimal operator (1.9). The term prolongation formulae is frequently used to denote the expressions for the additional coordinates:

\[
\zeta_1 = D(\eta) - y'D(\xi) = \eta_x + (\eta_y - \xi_x)y' - y^2 \xi_y, \quad (1.40)
\]

\[
\zeta_2 = D(\zeta_1) - y''D(\xi) \\
= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y^2 - y^3 \xi_{yy} \\
+ (\eta_y - 2\xi_x - 3y'\xi_y)y''. \quad (1.41)
\]

1.2.4 Definition of a symmetry group

Let \( G \) be a group of point transformations and let \( G_1, G_2 \) be its first and second prolongations, defined in the previous section.

**Definition 1.2.2.** We say that a group \( G \) of point transformations (1.1) is a symmetry group of a first-order ordinary differential equation

\[
F(x,y,y') = 0, \quad (1.42)
\]

or that Eq. (1.42) admits the group \( G \) if Eq. (1.42) is form invariant under the transformations (1.1), or, in other words, if the frame of Eq. (1.42) is invariant (in the sense of Definition 1.1.3) with respect to the first prolongation \( G_1 \) of the group \( G \).
Likewise, an \( n \)th order differential equation

\[
F(x, y, y', \ldots, y^{(n)}) = 0
\]  

(1.43)

admits a group \( G \) if the frame (the surface in the space \( x, y, y', \ldots, y^{(n)} \)) is invariant with respect to the \( n \)th prolongation \( G(n) \) of \( G \).

### 1.2.5 Main property of symmetry groups

Consider Eq. (1.43) written in the form solved with respect to the \( y^{(n)} \):

\[
y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})
\]  

(1.44)

with a smooth function \( f \). The main property of a symmetry group first proved by S. Lie (the proof for first-order equations is given, e.g. in Lie 1891, Chap. 16, Sect. 1, Theorem 1) is the following.

**Theorem 1.2.1.** A group \( G \) is a symmetry group for Eq. (1.44) if and only if \( G \) converts any classical solution (i.e., \( n \)-times continuously differentiable) of Eq. (1.44) into a classical solution of the same equation.

### 1.2.6 Calculation of infinitesimal symmetries

According to Sect. 1.1.3, it is sufficient to find infinitesimal symmetries, i.e., generators (1.6) of symmetry groups.

Here, the algorithm of construction of infinitesimal symmetries is discussed for second-order equations

\[
F(x, y, y', y'') = 0.
\]  

(1.45)

The infinitesimal invariance criterion has the form

\[
X(2)F\big|_{F=0} = (\xi F_x + \eta F_y + \zeta_1 F_y' + \zeta_2 F_y'')|_{F=0} = 0,
\]  

(1.46)

where \( \xi, \eta, \zeta_1, \zeta_2 \) are computed from the prolongation formulae (1.40) and (1.41). Eq. (1.46) is called the determining equation for the group admitted by the ordinary differential equation (1.45).

If the differential equation is written in the explicit form

\[
y'' = f(x, y, y'),
\]  

(1.47)

the determining equation (1.46), after substituting the values of \( \xi, \eta, \zeta_1, \zeta_2 \) from (1.40), (1.41) with \( y'' \) given by the right-hand side of (1.47), assumes the form

\[
\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2.
\]
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\[ -y'^3 \xi_{yy} + (\eta_y - 2\xi_x - 3y'\xi_y)f \]
\[ -[\eta_x + (\eta_y - \xi_x)y' - y'^2\xi_y]f_y - \xi f_x - \eta f_y = 0. \]  \hspace{1cm} (1.48)

Here \( f(x,y,y') \) is a known function (we are dealing with a given differential equation (1.47) while the coordinates \( \xi \) and \( \eta \) of the generator (1.6)),

\[ X = \xi(x,y) \frac{\partial}{\partial x} + \eta(x,y) \frac{\partial}{\partial y}, \]

are unknown functions of \( x, y \). Since the left-hand side of (1.48) contains the quantity \( y' \) considered as an independent variable along with \( x, y \), the determining equation splits into several independent equations, thus becoming an overdetermined system of differential equations for \( \xi(x,y), \eta(x,y) \). Solving this system, we find all the infinitesimal symmetries of Eq. (1.47).

1.2.7 An example

Let us find the operators

\[ X = \xi(x,y) \frac{\partial}{\partial x} + \eta(x,y) \frac{\partial}{\partial y} \]

admitted by the second-order equation

\[ y'' + \frac{1}{x} y' - e^y = 0. \]  \hspace{1cm} (1.49)

Here \( f = e^y - \frac{1}{x} y' \) and the determining equation (1.48) has the form

\[ \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 \]
\[ -y'^3 \xi_{yy} + (\eta_y - 2\xi_x - 3y'\xi_y) \left( e^y - \frac{y'}{x} \right) \]
\[ + \frac{1}{x} [\eta_x + (\eta_y - \xi_x)y' - y'^2\xi_y] - \xi \frac{y'}{x^2} - \eta e^y = 0. \]

The left-hand side of this equation is a third-degree polynomial in the variable \( y' \). Therefore, the determining equation decomposes into the following four equations, obtained by setting the coefficients of the various powers of \( y' \) equal to zero:

\[ (y')^3 : \quad \xi_{yy} = 0, \]  \hspace{1cm} (1.50)
\[ (y')^2 : \quad \eta_{yy} - 2\xi_{xy} + \frac{2}{x} \xi_y = 0, \]  \hspace{1cm} (1.51)
I Lie Group Analysis in Outline

\[ y' : \quad 2\eta_{xy} - \xi_{xx} + \left( \frac{\xi}{x} \right)_x - 3\xi_y e^y = 0, \quad (1.52) \]

\[ (y')^0 : \quad \eta_{xx} + \frac{1}{x} \eta_x + (\eta_y - 2\xi_x - \eta)e^y = 0. \quad (1.53) \]

Integration of Eqs. (1.50) and (1.51) with respect to \( y \) yields

\[ \xi = p(x)y + a(x), \quad \eta = \left( p' - \frac{p}{x} \right) y^2 + q(x)y + b(x). \]

Let us substitute these expressions for \( \xi, \eta \) into (1.52), (1.53). As the dependence of \( \xi \) and \( \eta \) on \( y \) is polynomial, while the left-hand sides of Eqs. (1.52), (1.53) contain \( e^y \), we must have

\[ \xi_y = 0, \quad \eta_y - 2\xi_e - \eta = 0. \]

The first of these gives us \( p = 0 \), that is, the equality \( \xi = a(x) \); taking this into account, the second condition can be written in the form

\[ q(x) - 2a'(x) - b(x) - q(x)y = 0. \]

Hence \( q = 0, 2a' + b = 0 \). Therefore,

\[ \xi = a(x), \quad \eta = -2a'(x). \]

Substituting these expressions into (1.52), we have

\[ \left( a' - \frac{a}{x} \right)' = 0, \]

from which \( a = C_1 x \ln x + C_2 x \); here Eq. (1.53) is satisfied identically.

As a result, we have obtained the general solution of the determining equations (1.50)–(1.53) in the form

\[ \xi = C_1 x \ln x + C_2 x, \quad \eta = -2[C_1(1 + \ln x) + C_2 x], \]

with constant coefficients \( C_1, C_2 \). In view of the linearity of the determining equations, the general solution can be represented as a linear combination of two independent solutions

\[ \xi_1 = x \ln x, \quad \eta_1 = -2(1 + \ln x); \quad \xi_2 = x, \quad \eta_2 = -2. \]

This means that (1.49) admits two linearly independent operators

\[ X_1 = x \ln x \frac{\partial}{\partial x} - 2(1 + \ln x) \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial x} - 2 \frac{\partial}{\partial y}, \quad (1.54) \]

and that the set of all admissible operators is a two-dimensional vector space with basis (1.54).
1.2 Symmetries of ordinary differential equations

1.2.8 Lie algebras

Definition 1.2.3. Let $X$ and $X'$ be first-order linear differential operators of the form (1.6):

$$X = \xi(x,y) \frac{\partial}{\partial x} + \eta(x,y) \frac{\partial}{\partial y}, \quad X' = \xi'(x,y) \frac{\partial}{\partial x} + \eta'(x,y) \frac{\partial}{\partial y}.$$  

Their commutator $[X, X']$ is defined by $[X, X'] = XX' - X'X$. It is a first-order linear differential operator and has the form

$$[X, X'] = (X(\xi) - X'(\xi)) \frac{\partial}{\partial x} + (X(\eta) - X'(\eta)) \frac{\partial}{\partial y}.$$  

Definition 1.2.4. A vector space $L$ of operators (1.6) is called a Lie algebra if it is closed under the commutator, i.e., if $[X, X'] \in L$ for any $X, X' \in L$. The Lie algebra is denoted by the same letter $L$, and its dimension is the dimension of the vector space $L$.

If a Lie algebra $L$ has the dimension $r < \infty$ it is denoted by $L_r$. If the vector space $L_r$ is spanned by linearly independent operators $X_1, \ldots, X_r$, then the operators $X_1, \ldots, X_r$ provide a basis of the Lie algebra $L_r$. The condition that $[X, X'] \in L$ for any $X, X' \in L$ is equivalent to the following:

$$[X_i, X_j] = c_{ij}^k X_k, \quad c_{ij} = \text{const.}, \quad i, j, k = 1, \ldots, r.$$  

Definition 1.2.5. Let $L_r$ be a Lie algebra spanned by $X_1, \ldots, X_r$. A subspace $K_s$ ($s < r$) of the vector space $L_r$ spanned by linearly independent operators $Y_1, \ldots, Y_s \in L_r$ is called a subalgebra of $L_r$ if

$$[Y, Y'] \in K_s \quad \text{for any} \quad Y, Y' \in K_s.$$  

This condition is equivalent to the following:

$$[Y_i, Y_j] \in K_s, \quad i, j = 1, \ldots, s.$$  

Let us return to general properties of determining equations. As can be seen from (1.48), a determining equation is a linear partial differential equation with the unknown functions $\xi$ and $\eta$ of the variables $x$ and $y$. Therefore, the set of its solutions forms a vector space, which was already noted in the previous example. However, a specific property of determining equations is given by the following statement due to S. Lie.

Theorem 1.2.2. The set of all solutions of any determining equation forms a Lie algebra.
Investigation of the determining equations for symmetries of second-order ordinary differential equations leads Lie to the following significant result [10] (see also [7]).

**Theorem 1.2.3.** For a second-order equation (1.47), the symmetry Lie algebra $L$ has the dimension $r \leq 8$. The maximal dimension $r = 8$ is attained if and only if Eq. (1.47) either is linear or can be linearized by a change of variables.

### 1.3 Integration of first-order equations

Let us discuss the methods of integration of the first-order ordinary differential equations with a known infinitesimal symmetry.

#### 1.3.1 Lie’s integrating factor

We begin with the method of the Lie’s integrating factor. Consider a first-order ordinary differential equation written in the form

$$Q(x,y)dx + P(x,y)dy = 0. \quad (1.58)$$

Lie [10] showed that if Eq. (1.58) admits a one-parameter group with the operator

$$X = \xi(x,y) \frac{\partial}{\partial x} + \eta(x,y) \frac{\partial}{\partial y} \quad (1.6)$$

and if $\xi Q + \eta P \neq 0$, then the function

$$\mu = \frac{1}{\xi Q + \eta P} \quad (1.59)$$

is an integrating factor for Eq. (1.58).

**Example 1.3.1.** Consider the Riccati equation (1.29):

$$y' + y^2 - \frac{2}{x^2} = 0. \quad (1.60)$$

Its symmetry group can be readily found by considering dilations $\bar{x} = ax$, $\bar{y} = by$. Substitution in (1.60) yields

$$\bar{y}' + \bar{y}^2 - \frac{2}{\bar{x}^2} = \frac{b}{a}y' + b^2y^2 - \frac{2}{a^2x^2}. \quad (1.61)$$

The invariance of Eq. (1.60) requires $b/a = b^2 = 1/a^2$. Hence $b = 1/a$. Therefore, the equation admits a one-parameter group of dilations (which can be written in the
1.3 Integration of first-order equations

form $\tilde{x} = xe^a$, $\tilde{y} = ye^{-a}$) with the generator

$$X = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}. \quad (1.61)$$

Writing (1.60) in the form (1.58),

$$dy + (y^2 - 2/x^2)dx = 0 \quad (1.62)$$

and applying formula (1.59), one obtains the integrating factor

$$\mu = \frac{x}{x^2y^2 - xy - 2}.$$ 

After multiplication by this factor, (1.62) is brought to the following form:

$$\frac{xdy + (xy^2 - 2/x)dx}{x^2y^2 - xy - 2} = \frac{xdy + ydx}{x^2y^2 - xy - 2} + \frac{dx}{x} = d\left(\ln x + \frac{1}{3} \ln \frac{xy - 2}{xy + 1}\right) = 0,$$

whence

$$\frac{xy - 2}{xy + 1} = C \quad \text{or} \quad y = \frac{2x^3 + C}{x(x^3 - C)}.$$ 

1.3.2 Method of canonical variables

Given a one-parameter symmetry group, one can use the canonical variables introduced in Sect. 1.1.5 for integrating the first-order equations. Since the property of invariance of an equation with respect to a group is independent of the choice of variables, introduction of canonical variables reduces the equation in question to an equation which does not depend on one of the variables, and hence can be integrated by quadrature. Consider examples.

**Example 1.3.2.** Let us solve the Riccati equation (1.60),

$$y' + y^2 - \frac{2}{x^2} = 0,$$

by the method of canonical variables using symmetry (1.61):

$$X = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}.$$ 

The partial differential equations

$$X(t) = x \frac{\partial t}{\partial x} - y \frac{\partial t}{\partial y} = 1, \quad X(u) = x \frac{\partial u}{\partial x} - y \frac{\partial u}{\partial y} = 0$$

yield the following canonical variables:
Let us rewrite Eq. (1.60) in the canonical variables. We have
\[
\frac{dy}{dx} = \frac{d}{dx} \left( \frac{u}{x} \right) = -\frac{u}{x^2} + \frac{1}{x} \frac{du}{dx} = -\frac{u}{x^2} + \frac{1}{x} \frac{du}{dt} \frac{dt}{dx} = -\frac{u}{x^2} + u'.
\]
Therefore, the left-hand side of the equation in question is written as follows:
\[
\frac{dy}{dx} + y^2 - \frac{2}{x^2} = \frac{u'}{x^2} - \frac{u}{x^2} + \frac{u^2}{x^2} - \frac{2}{x^2} = \frac{1}{x^2} (u' + u^2 - u - 2) = 0.
\]
Thus, the Riccati equation is rewritten in the canonical variables in the following integrable form:
\[
\frac{du}{dt} + u^2 - u - 2 = 0.
\]
It is integrated by separation of variables:
\[
\frac{du}{u^2 - u - 2} = -dt.
\]
Decomposing the integrand into elementary fractions:
\[
\frac{1}{u^2 - u - 2} = \frac{1}{3} \left( \frac{1}{u - 2} - \frac{1}{u + 1} \right),
\]
we evaluate the integral in elementary functions and obtain
\[
\ln \left( \frac{u - 2}{u + 1} \right) = -3t + \ln C.
\]
Now we solve this equation with respect to \( u \),
\[
u = \frac{C + 2e^{3t}}{e^{3t} - C},
\]
substitute
\[
t = \ln |x|, \quad u = xy
\]
and arrive at the solution of the Riccati equation (cf. Example 1.3.1):
\[
y = \frac{2x^3 + C}{x(x^3 - C)}.
\]

**Example 1.3.3.** The equation
\[
y' = \frac{y}{x} + \frac{y^2}{x^2}
\]
is homogeneous, i.e., it admits the group of simultaneous dilation of both variables (scaling transformation)
1.3 Integration of first-order equations

\[ \ddot{x} = xe^a, \quad \ddot{y} = ye^a, \]

with the generator

\[ X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}. \] (1.64)

Canonical variables for the operator (1.64) are

\[ t = \ln |x|, \quad u = \frac{y}{x}. \] (1.65)

In these variables, Eq. (1.63) is written as

\[ \frac{du}{dt} = u^2. \]

Whence, upon integration:

\[ \frac{1}{u} = C - t. \]

Substituting here

\[ t = \ln |x|, \quad y = xu, \]

we obtain the solution of the original equation:

\[ y = \frac{x}{C - \ln |x|}. \]

**Example 1.3.4.** The equation

\[ y' = \frac{y}{x} + \frac{y^3}{x^4} \] (1.66)

admits the group of projective transformations

\[ \ddot{x} = \frac{x}{1 - ax}, \quad \ddot{y} = \frac{y}{1 - ax}, \]

with the generator

\[ X = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}. \] (1.67)

Introducing the canonical variables

\[ t = -\frac{1}{x}, \quad u = \frac{y}{x^3}, \] (1.68)

we rewrite Eq. (1.66) in the form

\[ \frac{du}{dt} = u^3. \]

Integration yields

\[ u = \pm \frac{1}{\sqrt{C - 2t}}, \]
whence, substituting the expressions for \( t \) and \( u \), we obtain the following general solution to our equation:

\[
y = \pm x \sqrt{\frac{x}{2 + Cx}}.
\]

### 1.4 Integration of second-order equations

The Lie’s method of integration of the second-order ordinary differential equations employs the canonical variables in two-dimensional Lie algebras. Introduction of the canonical variables reduces any second-order differential equation admitting a two-dimensional Lie algebra \( L_2 \) into an integrable form.

#### 1.4.1 Canonical variables in Lie algebras \( L_2 \)

The canonical variables reduce a basis of every two-dimensional Lie algebra \( L_2 \) to the simplest form and provide four standard forms of the second-order equations with two symmetries. The basic statements are as follows.

**Theorem 1.4.1.** Any two-dimensional Lie algebra can be transformed, by a proper choice of its basis and suitable variables \( t, u \), called canonical variables, to one of the four non-similar standard forms presented in Table 1.1.

<table>
<thead>
<tr>
<th>Type</th>
<th>Structure of ( L_2 )</th>
<th>Standard form of ( L_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>([X_1, X_2] = 0, \xi_1\eta_2 - \eta_1\xi_2 \neq 0)</td>
<td>(X_1 = \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial u})</td>
</tr>
<tr>
<td>II</td>
<td>([X_1, X_2] = 0, \xi_1\eta_2 - \eta_1\xi_2 = 0)</td>
<td>(X_1 = \frac{\partial}{\partial u}, X_2 = t \frac{\partial}{\partial u})</td>
</tr>
<tr>
<td>III</td>
<td>([X_1, X_2] = X_1, \xi_1\eta_2 - \eta_1\xi_2 \neq 0)</td>
<td>(X_1 = \frac{\partial}{\partial u}, X_2 = t \frac{\partial}{\partial t} + u \frac{\partial}{\partial u})</td>
</tr>
<tr>
<td>IV</td>
<td>([X_1, X_2] = X_1, \xi_1\eta_2 - \eta_1\xi_2 = 0)</td>
<td>(X_1 = \frac{\partial}{\partial u}, X_2 = u \frac{\partial}{\partial u})</td>
</tr>
</tbody>
</table>

**Remark 1.4.1.** In types III and IV, the condition \([X_1, X_2] = X_1\) can be satisfied by a proper change of the basis in \( L_2 \) provided that \([X_1, X_2] \neq 0\).

Let a second-order equation
admit two or more symmetries. Let us single out from these symmetries a two-dimensional Lie algebra $L_2$, determine its type according to Table 1.1, find canonical variables $t, u$ for $L_2$, and rewrite Eq. (1.69) in the variables $t, u$:

$$u'' = g(t,u,u').$$

(1.70)

Theorem 1.4.1 guarantees that Eq. (1.70) belongs to one of four integrable equations given in the following Table 1.2.

Table 1.2 Four types of second-order equations admitting $L_2$

<table>
<thead>
<tr>
<th>Type</th>
<th>Standard form of $L_2$</th>
<th>Canonical form of the Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$X_1 = \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial u}$</td>
<td>$u'' = f(u')$</td>
</tr>
<tr>
<td>II</td>
<td>$X_1 = \frac{\partial}{\partial u}, X_2 = t \frac{\partial}{\partial u}$</td>
<td>$u'' = f(t)$</td>
</tr>
<tr>
<td>III</td>
<td>$X_1 = \frac{\partial}{\partial u}, X_2 = t \frac{\partial}{\partial t} + u \frac{\partial}{\partial u}$</td>
<td>$u'' = \frac{1}{t} f(u')$</td>
</tr>
<tr>
<td>IV</td>
<td>$X_1 = \frac{\partial}{\partial u}, X_2 = u \frac{\partial}{\partial u}$</td>
<td>$u'' = f(t)u'$</td>
</tr>
</tbody>
</table>

1.4.2 Integration method

The method of integration of the second-order non-linear differential equations (1.69) requires the following calculations. First of all, one needs to find the symmetries of the equation in question. Let the equation have two or more symmetries. We single out from these symmetries a two-dimensional Lie algebra $L_2$ and determine its type according to the Structure column of Table 1.1. Then we find the canonical variables by solving the following equations in accordance with the type:

**Type I:** $X_1(t) = 1, X_2(t) = 0; \quad X_1(u) = 0, X_2(u) = 1.$

**Type II:** $X_1(t) = 0, X_2(t) = 0; \quad X_1(u) = 1, X_2(u) = t.$

**Type III:** $X_1(t) = 0, X_2(t) = t; \quad X_1(u) = 1, X_2(u) = u.$

**Type IV:** $X_1(t) = 0, X_2(t) = 0; \quad X_1(u) = 1, X_2(u) = u.$

(1.71)

Now we rewrite the differential equation in the canonical variables choosing $t$ as a new independent variable and $u$ as a dependent one. It will have one of the integrable
forms given in Table 1.2. It remains to integrate the resulting equation and rewrite the solution in the original variables \( x, y \). This completes the integration procedure.

**Example 1.4.1.** Let us apply the integration method to the following non-linear second-order equation:

\[
y'' + e^{3y}y^4 + y'^2 = 0. \tag{1.72}
\]

First, we have to find the symmetries of Eq. (1.72). Here

\[
f = -(e^{3y}y^4 + y'^2)
\]

and the determining equation (1.48) is written as follows:

\[
\begin{align*}
\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - y'^3\xi_{yy} + 3e^{3y}y^4\eta - (\eta_y - 2\xi_x - 3y'\xi_y)(e^{3y}y^4 + y'^2) + & [\eta_x + (\eta_y - \xi_x)y' - y'^2\xi_y](4e^{3y}y^3 + 2y') = 0.
\end{align*}
\]

The left-hand side of this equation is a polynomial of fifth degree in \( y' \). Since it should vanish identically in \( y' \), we equate to zero the coefficients of \( y'^5, y'^4, \ldots \) and obtain the following four independent equations:

\[
\begin{align*}
(y')^5 & : \xi_y = 0, \\
(y')^4 & : 3(\eta_y + \eta) - 2\xi_x = 0, \\
(y')^3 & : \eta_x = 0, \\
(y')^1 & : \xi_{xx} = 0.
\end{align*}
\]

The coefficients for \( (y')^2 \) and \( (y')^0 \) vanish together with the coefficients of \( (y')^4 \) and \( (y')^1 \), respectively. The above four differential equations for two unknown functions \( \xi(x, y) \) and \( \eta(x, y) \) are readily solved and yield:

\[
\xi = C_1 + 3C_3x, \quad \eta = 2C_3 + C_2e^{-y}, \quad C_1, C_2, C_3 = \text{const}.
\]

Hence, the general form of the operator

\[
X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}
\]

admitted by Eq. (1.72) is

\[
X = C_1X_1 + C_2X_2 + C_3X_3,
\]

where

\[
X_1 = \frac{\partial}{\partial x}, \quad X_2 = e^{-y}\frac{\partial}{\partial y}, \quad X_3 = 3x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y}. \tag{1.73}
\]
In other words, Eq. (1.72) admits the three-dimensional Lie algebra \( L_3 \) spanned by the operators (1.73).

The operators \( X_1 \) and \( X_2 \) span a two-dimensional subalgebra \( L_2 \subset L_3 \) and has the type I. Canonical variables \( t \) and \( u \) are obtained by solving Eqs. (1.71) for type I, i.e., the following equations:

\[
\frac{\partial t}{\partial x} = 1, \quad e^{-y} \frac{\partial t}{\partial y} = 0; \quad \frac{\partial u}{\partial x} = 0, \quad e^{-y} \frac{\partial u}{\partial y} = 1.
\]

We take the following solutions to this system:

\[ t = x, \quad u = e^y. \]

Thus, we set \( u = u(t) \) and rewrite the equation in question in the new variables to obtain

\[ u'' + u'^4 = 0. \]

The standard substitution \( u' = v \) reduces it to the first-order equation \( v' + v^4 = 0 \), whence

\[ v = \frac{1}{\sqrt[4]{3x+C_1}}. \]

Now we integrate the equation

\[ \frac{du}{dx} = \frac{1}{\sqrt[4]{3x+C_1}} \]

and obtain:

\[ u = \frac{1}{2} \left[ \sqrt[4]{(3x+C_1)^2 + C_2} \right]. \]

Substitution of the expressions for \( t, u \) yields the solution to Eq. (1.72):

\[ y = \ln \left| \sqrt[4]{(3x+C_1)^2 + C_2} \right| - \ln 2. \]

**Example 1.4.2.** Integrate the non-linear equation

\[ y'' + 2 \left( y' - \frac{y}{x} \right)^3 = 0 \]

which admits the algebra \( L_2 \) of type II spanned by

\[
X_1 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}, \quad X_2 = xy \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}.
\]

**Solution.** The equations \( X_1(t) = 0, X_1(u) = 1; X_2(t) = 0, X_2(u) = t \) provide the canonical variables

\[ t = \frac{y}{x}, \quad u = -\frac{1}{x}. \]
Since the variable $t$ involves the dependent variable $y$, $t$ can be a new independent variable only if one excludes the singular solutions of Eq. (1.74) along which $t$ is identically constant. These singular solutions are the straight lines:

$$y = Kx, \quad K = \text{const}.$$ 

In variables (1.76), Eq. (1.74) becomes

$$u'' = 2$$

and yields $u = t^2 + C_1 t + C_2$. Substituting the expressions for $t$ and $u$, we obtain

$$y^2 + C_1 xy + C_2 x^2 + x = 0.$$ 

Solving this equation with respect to $y$ and introducing the new constants $A = -C_1/2$, $B = A^2 - C_2$, we obtain the solution to Eq. (1.74):

$$y = Kx, \quad y = Ax \pm \sqrt{Bx^2 - x}. \quad (1.77)$$

### 1.5 Symmetries of partial differential equations

#### 1.5.1 Main concepts illustrated by evolution equations

Consider the evolutionary partial differential equations of the second order with one spatial variable $x$:

$$u_t = F(t, x, u, u_x, u_{xx}), \quad \partial F / \partial u_{xx} \neq 0. \quad (1.78)$$

**Definition 1.5.1.** A one-parameter group $G$ of transformations (1.19) of the variables $t, x, u$:

$$\tilde{t} = f(t, x, u, a), \quad \tilde{x} = g(t, x, u, a), \quad \tilde{u} = h(t, x, u, a) \quad (1.79)$$

is called a group admitted by Eq. (1.78), or a symmetry group of Eq. (1.78), if Eq. (1.78) has the same form in the new variables $\tilde{t}, \tilde{x}, \tilde{u}$:

$$\tilde{u}_t = F(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{u}_\tilde{x}, \tilde{u}_{\tilde{x}\tilde{x}}). \quad (1.80)$$

The function $F$ has the same form in both Eqs. (1.78) and (1.80).

According to this definition, the transformations (1.79) of the group $G$ map every solution $u = u(t, x)$ of Eq. (1.78) into a solution $\tilde{u} = \tilde{u}(\tilde{t}, \tilde{x})$ of Eq. (1.80). Since Eq. (1.80) is identical with Eq. (1.78), the definition of an admitted group can be formulated as follows.
Definition 1.5.2. A one-parameter group $G$ of transformations (1.79) is called a group admitted by Eq. (1.78) if the transformations (1.79) map any solution of Eq. (1.78) into a solution of the same equation.

The infinitesimal transformations of the group $G$ of transformations (1.79) are written

$$
\tilde{t} \approx t + a\tau(t,x,u), \quad \tilde{x} \approx x + a\xi(t,x,u), \quad \tilde{u} \approx u + a\eta(t,x,u)
$$

and provide the following generator of the group $G$:

$$
X = \tau(t,x,u) \frac{\partial}{\partial t} + \xi(t,x,u) \frac{\partial}{\partial x} + \eta(t,x,u) \frac{\partial}{\partial u}
$$

acting on any differentiable function $J(t,x,u)$ as follows:

$$
X(J) = \tau(t,x,u) \frac{\partial J}{\partial t} + \xi(t,x,u) \frac{\partial J}{\partial x} + \eta(t,x,u) \frac{\partial J}{\partial u}.
$$

The generator (1.82) of a group $G$ admitted by Eq. (1.78) is known as an infinitesimal symmetry of Eq. (1.78).

The transformations (1.79) of the group with the generator (1.82) are found by solving the Lie equations

$$
\frac{d\tilde{t}}{da} = \tau(\tilde{t},\tilde{x},\tilde{u}), \quad \frac{d\tilde{x}}{da} = \xi(\tilde{t},\tilde{x},\tilde{u}), \quad \frac{d\tilde{u}}{da} = \eta(\tilde{t},\tilde{x},\tilde{u}),
$$

with the initial conditions:

$$
\tilde{t}|_{a=0} = t, \quad \tilde{x}|_{a=0} = x, \quad \tilde{u}|_{a=0} = u.
$$

Let us turn now to Eq. (1.80). The quantities $\tilde{u}_t$, $\tilde{u}_x$ and $\tilde{u}_{xx}$ involved in (1.80) are obtained via the usual rule of change of derivatives by treating Eqs. (1.79) as a change of variables. Then, expanding the resulting expressions for $\tilde{u}_t$, $\tilde{u}_x$, $\tilde{u}_{xx}$ into Taylor series with respect to the parameter $a$ and keeping only the terms linear in $a$, one obtains the infinitesimal form of these expressions:

$$
\tilde{u}_t \approx u_t + a\zeta_0(t,x,u,u_t,u_x),
$$

$$
\tilde{u}_x \approx u_x + a\zeta_1(t,x,u,u_t,u_x),
$$

$$
\tilde{u}_{xx} \approx u_{xx} + a\zeta_2(t,x,u,u_t,u_x,u_{tx},u_{xx}),
$$

where $\zeta_0, \zeta_1, \zeta_2$ are given by the following prolongation formulae:

$$
\zeta_0 = D_t(\eta) - u_tD_t(\tau) - u_xD_t(\xi),
$$

$$
\zeta_1 = D_x(\eta) - u_tD_x(\tau) - u_xD_x(\xi),
$$

$$
\zeta_2 = D_{xx}(\zeta_1) - u_{tx}D_x(\tau) - u_{xx}D_x(\xi).
$$
Here $D_t$ and $D_x$ denote the total differentiations with respect to $t$ and $x$:

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x};$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x}.$$

Substitution of (1.81) and (1.85) in equation (1.80) yields

$$\bar{u}_t - F(\bar{t}, \bar{x}, \bar{u}, \bar{u}_x, \bar{u}_{xx}) \approx u_t - F(t, x, u, u_x, u_{xx})$$

$$+ a \left( \zeta_0 - \frac{\partial F}{\partial u_{xx}} \zeta_2 - \frac{\partial F}{\partial u_x} \zeta_1 - \frac{\partial F}{\partial u} \eta - \frac{\partial F}{\partial x} \xi - \frac{\partial F}{\partial t} \tau \right).$$

Therefore, by virtue of Eq. (1.78), Eq. (1.80) yields

$$\zeta_0 - \frac{\partial F}{\partial u_{xx}} \zeta_2 - \frac{\partial F}{\partial u_x} \zeta_1 - \frac{\partial F}{\partial u} \eta - \frac{\partial F}{\partial x} \xi - \frac{\partial F}{\partial t} \tau = 0,$$

(1.87)

where $u_t$ is replaced by $F(t, x, u, u_x, u_{xx})$ in $\zeta_0, \zeta_1, \zeta_2$.

Eq. (1.87) determines all infinitesimal symmetries of Eq. (1.78) and therefore it is called the determining equation. Conventionally, it is written in the compact form

$$X (u_t - F(t, x, u, u_x, u_{xx})) \big|_{u_t = F} = 0,$$

(1.88)

where the prolongation of the operator $X$ (1.82) to the first and second order derivatives is understood:

$$X = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \zeta_0 \frac{\partial}{\partial u_t} + \zeta_1 \frac{\partial}{\partial u_x} + \zeta_2 \frac{\partial}{\partial u_{xx}},$$

and the symbol $|_{u_t = F}$ means that $u_t$ is replaced by $F(t, x, u, u_x, u_{xx})$.

The determining equation (1.87) (or its equivalent (1.88)) is a linear homogeneous partial differential equation of the second order for unknown functions $\tau(t, x, u), \xi(t, x, u), \eta(t, x, u)$. In consequence, the set of all solutions to the determining equation is a vector space $L$. Furthermore, the determining equation possesses the following significant and less evident property. The vector space $L$ is a Lie algebra, i.e., it is closed with respect to the commutator. In other words, $L$ contains, together with any operators $X_1, X_2$, their commutator $[X_1, X_2]$ defined by

$$[X_1, X_2] = X_1 X_2 - X_2 X_1.$$

In particular, if $L = L_r$ is finite-dimensional and has a basis $X_1, \ldots, X_r$, then the Lie algebra condition is written in the form

$$[X_\alpha, X_\beta] = c_{\alpha\beta}^\gamma X_\gamma,$$

with constant coefficients $c_{\alpha\beta}^\gamma$ known as the structure constants of $L_r$. 
Note that Eq. (1.87) should be satisfied identically with respect to all the variables involved, the variables \( t, x, u, u_x, u_{xx}, u_{tx} \) are treated as five independent variables. Consequently, the determining equation decomposes into a system of several equations. As a rule, this is an over-determined system since it contains more equations than three unknown functions \( \tau, \xi \) and \( \eta \). Therefore, in practical applications, the determining equation can be readily solved. The following statement due to Lie [9] simplifies the calculation of the symmetries of evolution equations\(^1\).

**Lemma 1.5.1.** The symmetry transformations (1.79) of Eqs. (1.78) have the form
\[
\bar{t} = f(t, a), \quad \bar{x} = g(t, x, u, a), \quad \bar{u} = h(t, x, u, a).
\] (1.89)

It means that one can search the infinitesimal symmetries in the form
\[
X = \tau(t) \frac{\partial}{\partial t} + \xi(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u}.
\] (1.90)

For the operators (1.90), the prolongation formulae (1.86) are written as follows:
\[
\zeta_0 = D_t(\eta) - u_x D_t(\xi) - \tau'(t) u_t, \quad \zeta_1 = D_x(\eta) - u_x D_x(\xi),
\]
\[
\zeta_2 = D_x(\zeta_1) - u_{xx} D_x(\xi) = D_x^2(\eta) - u_x D_x^2(\xi) - 2u_{xx} D_x(\xi).
\] (1.91)

**Example 1.5.1.** Let us find the symmetries of the Burgers equation
\[
U_t = U_{xx} + uU_x.
\] (1.91)

According to Lemma 1.5.1, the infinitesimal symmetries have the form (1.90). For the Burgers equation, the determining equation (1.87) has the form
\[
\zeta_0 - \zeta_2 - u\zeta_1 - \eta u_x = 0,
\] (1.93)

where \( \zeta_0, \zeta_1 \) and \( \zeta_2 \) are given by (1.91). Let us single out and annul the terms with \( u_{xx} \). Bearing in mind that \( u_t \) has to be replaced by \( u_{xx} + uu_x \) and substituting in \( \zeta_2 \) the expressions
\[
D_x^2(\xi) = D_x(\xi_x + \xi_u u_x) = \xi_u u_{xx} + \xi_{uu} u_x^2 + 2\xi_{xu} u_x + \xi_{xx},
\]
\[
D_x^2(\eta) = D_x(\eta_x + \eta_u u_x) = \eta_u u_{xx} + \eta_{uu} u_x^2 + 2\eta_{xu} u_x + \eta_{xx},
\] (1.94)

we arrive at the following equation:
\[
2\xi_{uu} u_x + 2\xi_u + \tau'(t) = 0.
\]

It splits into two equations, namely \( \xi_u = 0 \) and \( 2\xi_u - \tau'(t) = 0 \). The first equation shows that \( \xi \) depends only on \( t, x \), and integration of the second equation yields

\(^{1}\) In [9], Sect. III, Lie proves a more general statement about contact transformations of parabolic equations.
\[ \xi = \frac{1}{2} \tau'(t)x + p(t). \] \hspace{1cm} (1.95)

It follows from (1.95) that \( D_\xi^2(\xi) = 0 \). Now the determining equation (1.93) reduces to the form

\[ u_x^2 \eta_{uu} + \left[ \frac{1}{2} \tau'(t)u + \frac{1}{2} \tau''(t)x + p'(t) + 2 \eta_{xx} + \eta \right] u_x + u \eta_x + \eta_{xx} - \eta_t = 0 \]

and splits into three equations:

\[ \eta_{uu} = 0, \]
\[ \frac{1}{2} (\tau'(t)u + \tau''(t)x) + p'(t) + 2 \eta_{xx} + \eta = 0, \hspace{1cm} (1.96) \]
\[ u \eta_x + \eta_{xx} - \eta_t = 0. \]

The first equation of (1.96) yields \( \eta = \sigma(t,x)u + \mu(t,x) \), and the second equation of (1.96) becomes

\[ \left( \frac{1}{2} \tau'(t) + \sigma \right) u + \frac{1}{2} \tau''(t)x + p'(t) + 2 \sigma_x + \mu = 0, \]

whence

\[ \sigma = -\frac{1}{2} \tau'(t), \quad \mu = -\frac{1}{2} \tau''(t)x - p'(t). \]

Thus, we have

\[ \eta = -\frac{1}{2} \tau'(t)u - \frac{1}{2} \tau''(t)x - p'(t). \] \hspace{1cm} (1.97)

Finally, substitution of (1.97) into the third equation of (1.96) yields

\[ \frac{1}{2} \tau'''(t)x + p''(t) = 0, \]

whence \( \tau'''(t) = 0, \ \ p''(t) = 0, \) and hence

\[ \tau(t) = C_1 t^2 + 2C_2 t + C_3, \quad p(t) = C_4 t + C_5. \]

Invoking (1.95) and (1.97), we ultimately arrive at the following general solution of the determining equation (1.93):

\[ \tau(t) = C_1 t^2 + 2C_2 t + C_3, \]
\[ \xi = C_1 tx + C_2 x + C_4 t + C_5, \]
\[ \eta = -(C_1 t + C_2)u - C_1 x - C_4. \]

It contains five arbitrary constants \( C_i \). Hence, the infinitesimal symmetries of the Burgers equation (1.92) form the five-dimensional Lie algebra \( L_5 \) spanned by the following linearly independent operators:
1.5 Symmetries of partial differential equations

A convenient way to expose the structure of a Lie algebra \( L_r \), e.g. subalgebra and other properties, is to dispose the commutators (1.57) of a basis of \( L_r \) in a commutator table whose entry at the intersection of the \( X_i \) row with the \( X_j \) column is the value of their commutator \([X_i, X_j]\). Since the commutator (1.56) is antisymmetric, the commutator table will be antisymmetric as well, with zeros on the main diagonal.

Let us consider as an illustration the five-dimensional Lie algebra \( L_5 \) with the basis (1.98) (Table 1.3).

Table 1.3 Commutator table

<table>
<thead>
<tr>
<th></th>
<th>( X_1 )</th>
<th>( X_2 )</th>
<th>( X_3 )</th>
<th>( X_4 )</th>
<th>( X_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_1 )</td>
<td>0</td>
<td>0</td>
<td>( 2X_1 )</td>
<td>( X_4 )</td>
<td></td>
</tr>
<tr>
<td>( X_2 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( X_2 )</td>
<td>( X_3 )</td>
</tr>
<tr>
<td>( X_3 )</td>
<td>( -X_2 )</td>
<td>0</td>
<td>0</td>
<td>( -X_3 )</td>
<td>0</td>
</tr>
<tr>
<td>( X_4 )</td>
<td>( -2X_1 )</td>
<td>( -X_2 )</td>
<td>( X_3 )</td>
<td>0</td>
<td>( 2X_5 )</td>
</tr>
<tr>
<td>( X_5 )</td>
<td>( -X_4 )</td>
<td>( -X_3 )</td>
<td>0</td>
<td>( -2X_3 )</td>
<td>0</td>
</tr>
</tbody>
</table>

Let \( G \) be a group admitted by Eq. (1.78). Then every transformation (1.79) belonging to the group \( G \) carries over any solution of the differential equation (1.78) into a solution of the same equation. It means that the solutions of a partial differential equation are permuted among themselves under the action of a symmetry group. The solutions may also be individually unaltered, then they are called invariant or more specifically group invariant solutions. Accordingly, group analysis provides two basic ways for construction of exact solutions: group transformations of known solutions and construction of invariant solutions.

1.5.2 Invariant solutions

As said before, if a group transformation maps a solution into itself, we arrive at what is called a group invariant solutions known also as a selfsimilar solution given an infinitesimal symmetry (1.82) of Eq. (1.78), the invariant solutions under the one-parameter group generated by \( X \) are obtained as follows. One calculates two independent invariants \( J_1 = \lambda(t,x) \) and \( J_2 = \mu(t,x,u) \) by solving the first-order linear partial differential equation

\[
X(J) \equiv \tau(t,x,u) \frac{\partial J}{\partial t} + \xi(t,x,u) \frac{\partial J}{\partial x} + \eta(t,x,u) \frac{\partial J}{\partial u} = 0,
\]

or its characteristic system:
Then one designates one of the invariants as a function of the other, e.g.

$$\mu = \phi(\lambda),$$

and solves Eq. (1.100) with respect to $u$. Finally, one substitutes the expression for $u$ in Eq. (1.78) and obtains an ordinary differential equation for the unknown function $\phi(\lambda)$ of one variable. This procedure reduces the number of independent variables by one.

**Example 1.5.2.** Let us find the solutions of the Burgers equation that are invariant under the time translations generated by the operator $X_1$ from (1.98). The invariance condition leads to the stationary solutions

$$u = \Phi(x)$$

for which the Burgers equation is written as

$$\Phi'' + \Phi \Phi' = 0.$$  

(1.101)

Integrating once, we obtain

$$\Phi' + \frac{\Phi^2}{2} = C_1.$$

We integrate now this first-order equation by setting $C_1 = 0$, $C_1 = V^2 > 0$, and $C_1 = -\omega^2 < 0$ and obtain

$$\Phi(x) = \frac{2}{x+C},$$

$$\Phi(x) = V \tanh \left( C + \frac{V}{2} x \right),$$

$$\Phi(x) = \omega \tan \left( C - \frac{\omega}{2} x \right).$$

(1.102)

**1.5.3 Group transformations of solutions**

Let (1.79) be a symmetry transformation group of Eq. (1.78), and let a function

$$u = \Phi(t,x)$$

solve Eq. (1.78). Since (1.79) is a symmetry transformation, the above solution can be also written in the new variables:

$$\tilde{u} = \Phi(\tilde{t},\tilde{x}).$$
Replacing here $\bar{u}, \bar{t}, \bar{x}$ from (1.79), we get
\[ h(t,x,u,a) = \Phi \left( f(t,x,u,a), g(t,x,u,a) \right). \] (1.103)

Solving Eq. (1.103) with respect to $u$ one obtains a one-parameter family (with the parameter $a$) of new solutions to Eq. (1.78).

**Example 1.5.3.** Consider the Burgers equation (1.92),
\[ u_t = u_{xx} + uu_x, \]
and apply the above procedure to the admitted one-parameter group generated by the operator $X_5$ from (1.98):
\[ X_5 = t^2 \frac{\partial}{\partial t} + tx \frac{\partial}{\partial x} - (x + tu) \frac{\partial}{\partial u}. \]
The one-parameter group generated by $X_5$ has the form
\[ \bar{t} = \frac{t}{1-\alpha t}, \quad \bar{x} = \frac{C_2}{1-\alpha t}, \quad \bar{u} = (1-\alpha t)u - ax. \] (1.104)

Using the transformations (1.104) and applying Eq. (1.103) to any known solution $u = \Phi(t,x)$ of the Burgers equation, one obtains the following one-parameter set of new solutions:
\[ u = \frac{ax}{1-\alpha t} + \frac{1}{1-\alpha t} \Phi \left( \frac{t}{1-\alpha t}, \frac{x}{1-\alpha t} \right). \] (1.105)

Let us apply the transformation (1.105), e.g. to the first stationary solution (1.102):
\[ \Phi(x) = \frac{2}{x+C}, \quad C = \text{const.}, \]
on one obtains the new non-stationary solutions
\[ u = \frac{ax}{1-\alpha t} + \frac{2}{x+C(1-\alpha t)} \]
depending on the parameter $\alpha$.

**1.6 Three definitions of symmetry groups**

**1.6.1 Frame and extended frame**

We will use the following notation. Consider the algebraically independent variables
\[ x = \{x^i\}, \quad u = \{u^\alpha\}, \quad u(1) = \{u^\alpha_i\}, \quad u(2) = \{u^\alpha_{ij}\}, \ldots, \] (1.106)
where $\alpha = 1, \ldots, m$, and $i, j = 1, \ldots, n$. The variables $u_{ij}^\alpha, \ldots$ are assumed to be symmetric in subscripts, i.e., $u_{ij}^\alpha = u_{ji}^\alpha$. The operator

$$D_i = \frac{\partial}{\partial x^i} + u_{ij}^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u^\alpha} + \ldots \quad (i = 1, \ldots, n)$$

is called the total differentiation with respect to $x^i$. The operator $D_i$ is a formal sum of an infinite number of terms. However, it truncates when acting on any function of a finite number of the variables $x, u, u^{(1)}, \ldots$. In consequence, the total differentiations $D_i$ are well defined on the set of all functions depending on a finite number of $x, u, u^{(1)}, \ldots$.

Though the variables (1.106) are assumed to be algebraically independent, they are connected by the following differential relations:

$$u_{ij}^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j(u^\alpha) = D_jD_i(u^\alpha).$$

The variables $x^i$ are called independent variables, and the variables $u^\alpha$ are known as differential (or dependent) variables with the successive derivatives $u^{(1)}, u^{(2)}, \ldots$ etc. The universal space of modern group analysis is the space $\mathcal{A}$ of differential functions introduced by Ibragimov [2] (see also [3], Sect. 19) as a generalization of differential polynomials considered by J.F. Ritt in the 1950s.

**Definition 1.6.1.** A locally analytic function (i.e., locally expandable in a Taylor series with respect to all arguments) of a finite number of variables (1.106) is called a differential function. The highest order of derivatives appearing in the differential function is called the order of this function. The set of all differential functions of all finite orders is denoted by $\mathcal{A}$. This set is a vector space with respect to the usual addition of functions and becomes an associative algebra if multiplication is defined by the usual multiplication of functions. A significant property of the space $\mathcal{A}$ is that it is closed under the action of total derivatives (1.107).

**Definition 1.6.2.** A group $G$ of transformations of the form

$$\begin{align*}
\xi^i &= f^i(x, u, a), \quad f^i|_{a=0} = x^i, \\
\eta^\alpha &= \varphi^\alpha(x, u, a), \quad \varphi^\alpha|_{a=0} = u^\alpha,
\end{align*}$$

is called a group of point transformations in the space of dependent and independent variables. The generator of the group $G$ is

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha},$$

where

$$\begin{align*}
\xi^i &= \frac{\partial f^i}{\partial a}|_{a=0}, \\
\eta^\alpha &= \frac{\partial \varphi^\alpha}{\partial a}|_{a=0}.
\end{align*}$$

Let $F_k \in \mathcal{A}$ be any differential functions and let $p$ be the maximum of orders of the differential functions $F_k$, $k = 1, \ldots, s$. Consider the system of equations
1.6 Three definitions of symmetry groups

If one treats the variables $u^\alpha$ as functions of $x$ so that

$$u^\alpha = u^\alpha(x), \quad u_i^\alpha = \frac{\partial u^\alpha}{\partial x^i}, \ldots,$$

then one arrives at the usual concept of a system of differential equations (1.113) of order $p$.

Recall the definitions of the frame and extended frame of differential equations given in [4] (see also [5], Chap. 1).

**Definition 1.6.3.** Let us treat $x, u, u_{(1)}, \ldots$ as functionally independent variables connected only by the differential relations (1.108). Then Equations (1.113) determine a surface in the space of the independent variables $x, u, u_{(1)}, \ldots, u_{(p)}$. This surface is called the frame (or skeleton) of the system of differential equations (1.113).

**Definition 1.6.4.** Consider the frame equation (1.113) together with its differential consequences,

$$F_k(x, u, u_{(1)}, \ldots) = 0, \quad D_i F_k = 0, \quad D_i D_j F_k = 0, \ldots$$

The totality of points $(x, u, u_{(1)}, \ldots)$ satisfying Eqs. (1.114) is called the extended frame of the system of differential equations (1.113) and is denoted by $[F]$.

We will assume that the following equations holds on the frame of the differential equations under consideration:

$$\text{rank} \left| \frac{\partial F_k}{\partial x^i}, \frac{\partial F_k}{\partial u^\alpha}, \frac{\partial F_k}{\partial u_i^\alpha}, \ldots \right| = s.$$

### 1.6.2 First definition of symmetry group

The first definition of a symmetry group of an arbitrary system of differential equations coincides with Definition 1.5.2 for a single evolution equation.

**Definition 1.6.5.** The system of differential equations (1.113) is said to be invariant under the group $G$ of transformations (1.109), (1.110) if the transformations (1.109), (1.110) convert every solution of the system (1.113) into a solution of the same system. Here the solutions of differential equations are considered as classical ones, i.e., are assumed to be smooth functions $u^\alpha = u^\alpha(x)$. If the system of equations (1.113) is invariant under the group $G$, then $G$ is also known as a symmetry group for the system (1.113) or a group admitted by this system.
1.6.3 Second definition

Though the first definition is conceptually simple, it depends upon knowledge of solutions. Therefore, in a practical calculation of symmetries the following geometric definition is more efficient.

**Definition 1.6.6.** The system of differential equations (1.113) is said to be an invariant under the group $G$ if the frame of the system is an invariant surface with respect to the prolongation of the transformations (1.109), (1.110) of the group $G$ to the derivatives $u_{(1)}, \ldots, u_{(p)}$.

According to this definition and the invariance test of equations given by Theorem 1.1.3, one obtains the following infinitesimal test for obtaining symmetries of differential equations.

**Theorem 1.6.1.** The group $G$ with the generator $X$ is admitted by the system of differential equations (1.113) if and only if

$$X_{(p)}F_k\big|_{(1.113)} = 0, \quad k = 1, \ldots, s,$$

where $X_{(p)}$ is the $p$-th prolongation of $X$ and $\big|_{(1.113)}$ means evaluated on the frame of the system of differential equations (1.113). Eq. (1.115) are known as the determining equations.

Let $z_0 = (x_0, u_0, \ldots, u_{0(p)})$ be a point on the frame of system (1.113), i.e., $F_k(x_0, u_0, \ldots, u_{0(p)}) = 0$ ($k = 1, \ldots, s$). The system of differential equations (1.113) is said to be locally solvable at $z_0$ if there is a solution passing through this point, i.e., there exist a solution $u = h(x)$ of differential equations (1.113) defined in a neighborhood of the point $x_0$ such that $u_0 = h(x_0), \ldots, u_{0(p)} = \partial p_{h}/\partial x_{p}(x_0)$ The system (1.113) is said to be locally solvable if it has this property at every generic point of the frame.

It can be shown that for locally solvable systems the first and the second definitions of the symmetry group are equivalent, i.e., Definition 1.6.5 and Definition 1.6.6 provide exactly the same symmetry group.

1.6.4 Third definition

If the system (1.113) is not locally solvable, e.g. if the system (1.113) is over-determined, it may happen (see further Example 1.6.1) that Definition 1.6.6 provides only a subgroup of the symmetry group given by Definition 1.6.5. Therefore, it has been proposed in [3], Sect. 17.1 (see also [5], Chap. 1) the following third definition and proved the appropriate infinitesimal test for the invariance of over-determined systems of differential equations.
Definition 1.6.7. The system of differential equations (1.113) is said to be invariant under the group $G$ if the extended frame $[F]$ is invariant with respect to the infinite-order prolongation of $G$.

The infinitesimal test for this invariance is written as follows (see [3], Theorem 17.1).

Theorem 1.6.2. Let $X$ be the generator of a group $G$. The system of differential equations (1.113) are invariant under the group $G$ in the sense of Definition 1.6.7 if and only if the following equations are satisfied:

$$X_{(p)}F_k |_{[F]} = 0, \quad k = 1, \ldots, s. \quad (1.116)$$

Eqs. (1.116) are also called determining equations.

Remark 1.6.1. According to Theorem 1.6.2, the invariance test does not involve all the differential consequences (1.114) of the differential equations (1.113). In fact, it can be easily shown that it suffices to consider only a finite number of the differential consequences (1.114) such that they form a system in involution. It is also worth noting that we do not need to take into account the additional equations such as $X_{(p)}(D,F_k) = 0$ since they are satisfied identically due to Eqs. (1.116).

For locally solvable systems, all three definitions of symmetry groups are equivalent. For over-determined systems, the first and third definitions are equivalent, whereas the second definition provides, in general, only a subgroup of the symmetry group given by the third definition.

Example 1.6.1. Consider the over-determined system ([5], Sect. 1.3.10)

$$u_t = (u_x)^{-4/3}u_{xx}, \quad v_t = -3(u_x)^{-1/3}, \quad v_x = u. \quad (1.117)$$

This is a system of three equations for two dependent variables $u$ and $v$. The maximal order of equations involved in the system is $p = 2$. Let us first solve the determining equations (1.115). The left-hand side of Eqs. (1.115) depends upon the variables $x, t, u, v, u_x, u_{xx}, u_{xt}$, and $v_{xx}$ in accordance with the prolongation formulae. The solution of the determining equations yields the 6-dimensional Lie algebra spanned by

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = \frac{\partial}{\partial v}, \quad X_4 = \frac{\partial}{\partial u} + x\frac{\partial}{\partial v}, \quad X_5 = 4t\frac{\partial}{\partial t} + 3u\frac{\partial}{\partial u} + 3v\frac{\partial}{\partial v}, \quad X_6 = 2x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}. \quad (1.118)$$

This is the Lie algebra of the maximal symmetry group for Eqs. (1.117) obtained by the second definition (Definition 1.6.6).

Consider now the determining equations (1.116). Differentiation of the third equation of (1.117) yields $v_{xx} = u_x$. Therefore, we replace $v_{xx}$ in the determining equation by $u_x$. Then the left-hand side of Eqs. (1.116) involves only the variables $x, t, u, v, u_x, u_{xx}$ and $u_{xt}$. Solving the determining equations (1.116), one obtains the 7-dimensional Lie algebra spanned by operators (1.118) and by
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Thus, the third definition (Definition 1.6.7) provides a more general symmetry group than the second definition.

1.7 Lie-Backlund transformation groups

This section provides an introduction to the theory of the Lie-Backlund transformation groups and contains the basic definitions, theorems and algorithms used for computation of the Lie-Backlund symmetries of differential equations. The space \( \mathcal{A} \) of the differential functions introduced in Sect. 1.6.1 play a central role in this theory.

1.7.1 Lie-Backlund operators

Geometrically, the Lie-Backlund transformations appear in attempting to find a higher-order generalization of the classical contact (the first-order tangent) transformations (see Bäcklund’s paper [1]) and are identified with the infinite-order tangent transformations. A historical survey of the development of this branch of group analysis and a detailed discussion of the modern theory with many applications are to be found in [3] (see also [6], Chap. 1). We will use here a shortcut to the theory of the Lie-Backlund transformation groups by using a generalization of the infinitesimal generators of point and contact transformation groups. The generalization is known as a Lie-Backlund operator and is defined as follows.

**Definition 1.7.1.** Let \( \xi^i, \eta^\alpha \in \mathcal{A} \) be differential functions depending on any finite number of variables \( x, u, u_1, u_2, \ldots \). A differential operator

\[
X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \xi^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_{i_1i_2} \frac{\partial}{\partial u^\alpha} + \cdots, \tag{1.120}
\]

where

\[
\xi^\alpha = D_i(\eta^\alpha - \xi^i u^\alpha) + \xi^i u^\alpha, \\
\zeta_{i_1i_2} = D_{i_1} D_{i_2}(\eta^\alpha - \xi^i u^\alpha) + \xi^i u^\alpha, \tag{1.121}
\]

is called a Lie-Backlund operator. The Lie-Backlund operator (1.120) is often written in the abbreviated form

\[
X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \cdots, \tag{1.122}
\]
where the prolongation given by (1.120)–(1.121) is understood.

The operator (1.120) is formally an infinite sum. However, it truncates when acting on any differential function. Hence, the action of the Lie-Bäcklund operators is well-defined on the space $\mathcal{A}$.

Consider two Lie-Bäcklund operators

$$X_\nu = \xi^i_\nu \frac{\partial}{\partial x^i} + \eta^\alpha_\nu \frac{\partial}{\partial u^\alpha} + \cdots, \quad \nu = 1, 2,$$

and define their commutator by the usual formula:

$$[X_1, X_2] = X_1 X_2 - X_2 X_1.$$

**Theorem 1.7.1.** The commutator $[X_1, X_2]$ is identical with the Lie-Bäcklund operator given by

$$[X_1, X_2] = (X_1(\xi^i_1) - X_2(\xi^i_2)) \frac{\partial}{\partial x^i} + (X_1(\eta^\alpha_1) - X_2(\eta^\alpha_2)) \frac{\partial}{\partial u^\alpha} + \cdots, \quad (1.123)$$

where the terms denoted by dots are obtained by prolonging the coefficients of $\partial/\partial x^i$ and $\partial/\partial u^\alpha$ in accordance with Eqs. (1.120) and (1.121).

According to Theorem 1.7.1, the set of all Lie-Bäcklund operators is an infinite dimensional Lie algebra with respect to commutator (1.123). It is called the Lie-Bäcklund algebra and denoted by $L_B$. The Lie-Bäcklund algebra is endowed with the following properties (see [3]).

**I.** $D_i \in L_B$. In other words, the total differentiation (1.107) is a Lie-Bäcklund operator. Furthermore,

$$X_* = \xi^i_\nu D_i \in L_B \quad (1.124)$$

for any $\xi^i_\nu \in \mathcal{A}$.

**II.** Let $L_*$ be the set of all Lie-Bäcklund operators of form (1.124). Then $L_*$ is an ideal of $L_B$, i.e., $[X, X_*] \in L_*$ for any $X \in L_B$. Indeed,

$$[X, X_*] = (X(\xi^i_\nu) - X_* (\xi^i_\nu)) D_i \in L_*.$$

**III.** In accordance with property II, two operators $X_1, X_2 \in L_B$ are said to be equivalent (i.e., $X_1 \sim X_2$) if $X_1 - X_2 \in L_*$. In particular, every operator $X \in L_B$ is equivalent to an operator (1.120) with $\xi^i = 0, i = 1, \ldots, n$. Namely, $X \sim \tilde{X}$ where

$$\tilde{X} = X - \xi^i D_i = (\eta^\alpha - \xi^i u^\alpha) \frac{\partial}{\partial u^\alpha} + \cdots. \quad (1.125)$$

**Definition 1.7.2.** The operators of the form

$$X = \eta^\alpha \frac{\partial}{\partial u^\alpha} + \cdots, \quad \eta^\alpha \in \mathcal{A}, \quad (1.126)$$

are called canonical Lie-Bäcklund operators.
Using this definition, we can formulate property III as follows.

**Theorem 1.7.2.** Any operator \( X \in L_B \) is equivalent to a canonical Lie-Bäcklund operator.

**Example 1.7.1.** Let us take \( n = m = 1 \) and denote \( u_1 = u_x \). The generator of the group of translations along the \( x \)-axis and its canonical Lie-Bäcklund form (1.125) are written as follows:

\[
X = \frac{\partial}{\partial x} \sim \tilde{X} = u_x \frac{\partial}{\partial u} + \cdots.
\]

**Example 1.7.2.** Let \( x, y \) be the independent variables, and \( k, c = \text{const.} \) The generator of non-homogeneous dilations and its canonical Lie-Bäcklund form (1.125) are written as

\[
X = x \frac{\partial}{\partial x} + ky \frac{\partial}{\partial y} + cu \frac{\partial}{\partial u} \sim \tilde{X} = (cu - xu_x - kyu_y) \frac{\partial}{\partial u} + \cdots.
\]

**Example 1.7.3.** Let \( t, x \) be the independent variables. The generator of the Galilean boost and its canonical Lie-Bäcklund form (1.125) are written as

\[
X = t \frac{\partial}{\partial x} + \frac{\partial}{\partial u} \sim \tilde{X} = (1 - tu_x) \frac{\partial}{\partial u} + \cdots.
\]

The canonical operators leave invariant the independent variables \( x^j \). Therefore, the use of the canonical form is convenient, e.g., for investigating symmetries of integro-differential equations.

**IV.** The following statements describe all Lie-Bäcklund operators equivalent to generators of Lie point and Lie contact transformation groups.

**Theorem 1.7.3.** The Lie-Bäcklund operator (1.120) is equivalent to the infinitesimal operator of a one-parameter point transformation group if and only if its coordinates assume the form

\[
\xi^i = \xi^i_j(x,u) + \xi^i_u, \quad \eta^a = \eta^a_1(x,u) + \left( \xi^i_j(x,u) + \xi^i_u \right) u^a_i,
\]

where \( \xi^i_u \in \mathcal{A} \) is an arbitrary differential function, and \( \xi^i_j, \xi^i_u, \eta^a_1 \) are arbitrary functions of \( x \) and \( u \).

**Theorem 1.7.4.** Let \( m = 1 \). Then operator (1.120) is equivalent to the infinitesimal operator of a one-parameter contact transformation group if and only if its coordinates assume the form

\[
\xi^i = \xi^i_j(x,u,u_{(1)}) + \xi^i_u, \quad \eta = \eta_1(x,u,u_{(1)}) + \xi^i_u u_i,
\]

where \( \xi^i_u \in \mathcal{A} \) is an arbitrary differential function, and \( \xi^i_j, \eta_1 \) are arbitrary first-order differential functions, i.e., depend upon \( x, u \) and \( u_{(1)} \).
1.7 Lie-Backlund transformation groups

1.7.2 Lie-Backlund equations and their integration

Consider the sequence
\[ z = (x, u, u_1, u_2, \ldots), \]  
with the elements \( z^v, v \geq 1, \) were
\[ z^i = x^i, 1 \leq i \leq n, \quad z^{i+\alpha} = u^\alpha, 1 \leq \alpha \leq m. \]

Denote by \([z]\) any finite subsequence of \(z\). Then elements of the space \(\mathcal{A}\) of the differential functions are written as \(f([z])\).

**Definition 1.7.3.** Given an operator (1.120), the following infinite system is called the Lie-Backlund equations:
\[
\frac{d}{da} x^i = \xi^i([z]), \quad \frac{d}{da} u^\alpha = \eta^\alpha([z]),
\]
\[
\frac{d}{da} u_i^\alpha = \zeta_i^\alpha([z]), \quad \frac{d}{da} u_{ij}^\alpha = \xi_{ij}^\alpha([z]), \ldots,
\]
(1.128)

where \(\alpha = 1, \ldots, m\) and \(i,j,\ldots = 1, \ldots, n\).

In the case of canonical operators (1.126), the infinite system of Eqs. (1.128) can be replaced by the finite system
\[
\frac{d}{da} u^\alpha = \eta^\alpha([z]), \quad \alpha = 1, \ldots, m.
\]
(1.129)

Indeed, upon solving the system (1.129), the transformations of the successive derivatives are obtained by the total differentiation:
\[
\bar{u}_i^\alpha = D_i(u^\alpha), \quad \bar{u}_{ij}^\alpha = D_iD_j(u^\alpha), \ldots
\]
(1.130)

We will use the abbreviated form (1.122) of Lie-Backlund operators and write the system (1.128), together with the initial conditions, as follows:
\[
\frac{d}{da} x^i = \xi^i([z]), \quad \bar{x}^i|_{a=0} = x^i,
\]
\[
\frac{d}{da} u^\alpha = \eta^\alpha([z]), \quad \bar{u}^\alpha|_{a=0} = u^\alpha,
\]
\[
\ldots
\]
(1.131)

The formal integrability of the infinite system (1.131) has been proved by Ibragimov (see, e.g. [3], Sect. 15.1. It is also discussed in [5]). For the convenience of the reader, we formulate here the existence theorem. The following notation is convenient for formulating and proving the theorem.

Let \(f\) and \(g\) be two formal power series in one symbol \(a\) with coefficients from the space \(\mathcal{A}\), i.e., let
\[ f(z, a) = \sum_{k=0}^{\infty} f_k([z]) a^k, \quad f_k([z]) \in \mathcal{A}, \] (1.132)

and

\[ g(z, a) = \sum_{k=0}^{\infty} g_k([z]) a^k, \quad g_k([z]) \in \mathcal{A}. \]

Their linear combination \( \lambda f([z]) + \mu g([z]) \) with constant coefficients \( \lambda, \mu \) and product \( f([z]) \cdot g([z]) \) are defined by

\[ \lambda \sum_{k=0}^{\infty} f_k([z]) a^k + \mu \sum_{k=0}^{\infty} g_k([z]) a^k = \sum_{k=0}^{\infty} \left( \lambda f_k([z]) + \mu g_k([z]) \right) a^k, \] (1.133)

and

\[ \left( \sum_{p=0}^{\infty} f_p([z]) a^p \right) \cdot \left( \sum_{q=0}^{\infty} g_q([z]) a^q \right) = \sum_{k=0}^{\infty} \left( \sum_{p+q=k} f_p([z]) g_q([z]) \right) a^k, \] (1.134)

respectively. The space of all formal power series (1.132) endowed with the addition (1.133) and the multiplication (1.134) is denoted by \([\mathcal{A}]\).

Lie point and Lie contact transformations, together with their prolongations of all orders, are represented by elements of the space \([\mathcal{A}]\). Moreover, the utilization of this space is necessary in the theory of Lie-Bäcklund transformation groups. Therefore, \([\mathcal{A}]\) is called the representation space of modern group analysis ([5], Sect. 1.2).

The existence theorem is formulated as follows.

**Theorem 1.7.5.** The Lie-Bäcklund equations (1.110) have a solution in the space \([\mathcal{A}]\). The solution is unique. It is given by formal power series

\[ x^i = x^i + \sum_{k=0}^{\infty} A_k^i([z]) a^k, \quad A_k^i([z]) \in \mathcal{A}, \]

\[ \bar{u}^\alpha = u^\alpha + \sum_{k=0}^{\infty} B_k^\alpha([z]) a^k, \quad B_k^\alpha([z]) \in \mathcal{A}, \] (1.135)

and satisfies the group property.

**Definition 1.7.4.** The group of formal transformations (1.135) is called a one-parameter Lie-Bäcklund transformation group.

Recall that a point transformation group acting in the finite dimensional space of variables \( x = (x^1, \ldots, x^n) \) and generated by an operator \( X \) can be represented by the exponential map (1.22):

\[ \tilde{x}^i = \exp(aX)(x^i), \quad i = 1, \ldots, n, \] (1.136)
where
\[ \exp(aX) = 1 + aX + \frac{a^2}{2!}X^2 + \frac{a^3}{3!}X^3 + \cdots. \]  
(1.137)

Likewise, solution (1.135) to the Lie-Bäcklund equations (1.131) can be represented by the exponential map
\[ \bar{x}^i = \exp(aX)(x^i), \bar{\alpha} = \exp(aX)(\alpha), \bar{\alpha}^i = \exp(aX)(\alpha^i), \ldots, \]
(1.138)
where \(X\) is a Lie-Bäcklund operator (1.120) and \(\exp(aX)\) is given by (1.137).

If we consider canonical operators (1.126) then Eqs. (1.131) reduce to the finite system of equations (1.129) supplemented by the initial conditions, i.e., by the system
\[ \frac{d}{da} \bar{u}^\alpha = \eta^\alpha([z]), \quad \bar{u}^\alpha|_{a=0} = u^\alpha. \]  
(1.139)
Consequently, Lie-Bäcklund transformation groups can be constructed by virtue of the following theorem.

**Theorem 1.7.6.** Given a canonical Lie-Bäcklund operator,
\[ X = \eta^\alpha \frac{\partial}{\partial u^\alpha} + \cdots, \]
the corresponding formal one-parameter group is represented by the series
\[ \bar{u}^\alpha = u^\alpha + a\eta^\alpha + \frac{a^2}{2!}X(\eta^\alpha) + \cdots + \frac{a^n}{n!}X^{n-1}(\eta^\alpha) + \cdots \]  
(1.140)

Together with its differential consequences:
\[ \bar{u}^\alpha_i = u^\alpha_i + aD_i(\eta^\alpha) + \frac{a^2}{2!}X(D_i(\eta^\alpha)) + \cdots + \frac{a^n}{n!}X^{n-1}(D_i(\eta^\alpha)) + \cdots, \]
\[ \bar{u}^\alpha_{i_1 \cdots i_s} = u^\alpha_{i_1 \cdots i_s} + aD_{i_1} \cdots D_{i_s}(\eta^\alpha) + \cdots + \frac{a^n}{n!}X^{n-1}(D_{i_1} \cdots D_{i_s}(\eta^\alpha)) + \cdots. \]

**Example 1.7.4.** Let
\[ X = u_1 \frac{\partial}{\partial u} + u_2 \frac{\partial}{\partial u_1} + \cdots. \]
Here \(\eta = u_1\) and therefore,
\[ X(\eta) = u_2, X^2(\eta) = u_3, \ldots, X^{n-1}(\eta) = u_n. \]
Hence, the transformation (1.140) has the form
\[ \bar{u} = u + \sum_{n=1}^{\infty} \frac{a^n}{n!}u_n. \]

**Example 1.7.5.** Let
Here, $\eta = u_2$ and
\[ X(\eta) = u_4, \quad X^2(\eta) = u_6, \ldots, \quad X^{n-1}(\eta) = u_{2n}. \]

Hence, the transformation (1.140) is given by the power series
\[ \bar{u} = u + \sum_{n=1}^{\infty} \frac{a^n}{n!} u_{2n}. \]

### 1.7.3 Lie-Backlund symmetries

The Lie-Backlund symmetries of differential equations are given by Definition 1.6.7 from Sect. 1.6.4. Thus, we use the following definition.

**Definition 1.7.5.** Let $G$ be a Lie-Backlund transformation group generated by a Lie-Backlund operator (1.120),
\[ X = \xi^i \frac{\partial}{\partial x^i} + \eta^a \frac{\partial}{\partial u^a} + \zeta^a \frac{\partial}{\partial u^a} + \xi_{i_1 i_2} \frac{\partial}{\partial u_{i_1 i_2}} + \cdots. \tag{1.120} \]

The group $G$ is called a group of **Lie-Backlund symmetries** of a system of differential equations
\[ F_k(x,u,u_{(1)},\ldots,u_{(p)}) = 0, \quad k = 1, \ldots, s, \tag{1.141} \]
if the extended frame of Eqs. (1.141) defined by (see Definition 1.6.4)
\[ [F] : \quad F_k = 0, \quad D_i F_k = 0, \quad D_i D_j F_k = 0, \ldots, \tag{1.142} \]
is invariant under $G$. The operator $X$ (1.120) is called an **infinitesimal Lie-Backlund symmetry** for Eqs. (1.141).

The infinitesimal invariance criteria proved in [3] is formulated in the following statements.

**Theorem 1.7.7.** The operator (1.120) is an infinitesimal Lie-Backlund symmetry for Eqs. (1.141) if and only if
\[ XF_k|_{[F]} = 0, \quad XD_i (F_k)|_{[F]} = 0, \quad XD_i D_j (F_k)|_{[F]} = 0, \ldots \quad (k = 1, \ldots, s). \]

Theorem 1.7.7 contains an infinite number of equations. However, it can be simplified and reduced to a finite number of equations by means of the following result.

**Lemma 1.7.1.** The equations
\[ XF_k|_{[F]} = 0 \]
yield the infinite series of equations
\[ XD_i(F_k) \big|_{[F]} = 0, \quad XD_iD_j(F_k) \big|_{[F]} = 0, \ldots. \]

Thus, one arrives at the following finite test for calculating Lie-Bäcklund symmetries of differential equations.

**Theorem 1.7.8.** The operator \((1.120)\) is an infinitesimal Lie-Bäcklund symmetry for Eqs. \((1.141)\) if and only if the following equations hold:
\[ XF_k \big|_{[F]} = 0, \quad k = 1, \ldots, s. \quad (1.143) \]

Eqs. \((1.143)\) are the determining equations for Lie-Bäcklund symmetries.

**Remark 1.7.1.** Every operator of the form \((1.124)\), i.e., \(X_\ast = \xi^i D_i \in L_B\), is an infinitesimal Lie-Bäcklund symmetry for any system of differential equations. Furthermore, all operators \((1.120)\) satisfying the conditions
\[ \xi^i \big|_{[F]} = 0, \quad \eta^\alpha \big|_{[F]} = 0 \quad (1.144) \]
solve the determining equations \((1.143)\). All operators \(X_\ast \in L_B\) and the operators obeying the conditions \((1.144)\) are termed trivial Lie-Bäcklund symmetries ([6], Sect. 1.3.2).

**Example 1.7.6.** The equations of motion of a planet (Kepler’s problem):
\[ m \frac{d^2 x^k}{dt^2} = \mu \frac{x^k}{r^3}, \quad k = 1, 2, 3, \]

have the following three nontrivial infinitesimal Lie-Bäcklund symmetries different from Lie point and contact symmetries (see [3]):
\[ X_i = (2x^j v^k - x^k v^j - (x \cdot v) \delta^k_i) \frac{\partial}{\partial x^i}, \quad i = 1, 2, 3. \]

Here the independent variable is time \(t\), the dependent variables are the coordinates of the position vector \(x = (x^1, x^2, x^3)\) of the planet. The vector \(v = (v^1, v^2, v^3)\) is the velocity of the planet, i.e., \(v = dx/dt\).

**References**


Chapter 2
Approximate Transformation Groups and Symmetries

A detailed discussion of the material presented here as well as of the theory of the multi-parameter approximate groups can be found in [1].

2.1 Approximate transformation groups

2.1.1 Notation and definitions

In what follows, functions \( f(x, \varepsilon) \) of \( n \) variables \( x = (x^1, \ldots, x^n) \) and a parameter \( \varepsilon \) are considered locally in a neighborhood of \( \varepsilon = 0 \). These functions are continuous in the \( x \)'s and \( \varepsilon \), as are also their derivatives to as high an order as enters in the subsequent discussion.

If a function \( f(x, \varepsilon) \) satisfies the condition

\[
\lim_{\varepsilon \to 0} \frac{f(x, \varepsilon)}{\varepsilon^p} = 0,
\]

it is written \( f(x, \varepsilon) = o(\varepsilon^p) \) and \( f \) is said to be of order less than \( \varepsilon^p \). If

\[
f(x, \varepsilon) - g(x, \varepsilon) = o(\varepsilon^p),
\]

the functions \( f \) and \( g \) are said to be approximately equal (with an error \( o(\varepsilon^p) \)) and written as

\[
f(x, \varepsilon) = g(x, \varepsilon) + o(\varepsilon^p),
\]

or, briefly \( f \approx g \) when there is no ambiguity.

The approximate equality defines an equivalence relation, and we join functions into equivalence classes by letting \( f(x, \varepsilon) \) and \( g(x, \varepsilon) \) to be members of the same class if and only if \( f \approx g \).

Given a function \( f(x, \varepsilon) \), let
be the approximating polynomial of degree $p$ in $\epsilon$ obtained via the Taylor series expansion of $f(x, \epsilon)$ in powers of $\epsilon$ about $\epsilon = 0$. Then any function $g \approx f$ (in particular, the function $f$ itself) has the form

$$g(x, \epsilon) = f_0(x) + \epsilon f_1(x) + \cdots + \epsilon^p f_p(x) + o(\epsilon^p).$$

Consequently the function

$$f_0(x) + \epsilon f_1(x) + \cdots + \epsilon^p f_p(x)$$

is called a canonical representative of the equivalence class of functions containing $f$.

Thus, the equivalence class of functions $g(x, \epsilon) \approx f(x, \epsilon)$ is determined by the ordered set of $p+1$ functions

$$f_0(x), \ f_1(x), \ldots, \ f_p(x).$$

In the theory of approximate transformation groups, one considers ordered sets of smooth vector-functions depending on $x$’s and a group parameter $\alpha$:

$$f_0(x, \alpha), \ f_1(x, \alpha), \ldots, \ f_p(x, \alpha),$$

with coordinates

$$f_0^i(x, \alpha), \ f_1^i(x, \alpha), \ldots, f_p^i(x, \alpha), \quad i = 1, \ldots, n.$$

Let us define the one-parameter family $G$ of approximate transformations

$$\tilde{x}^i \approx f_0^i(x, \alpha) + \epsilon f_1^i(x, \alpha) + \cdots + \epsilon^p f_p^i(x, \alpha), \quad i = 1, \ldots, n \quad (2.1)$$

of points $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$ into points $\tilde{x} = (\tilde{x}^1, \ldots, \tilde{x}^n) \in \mathbb{R}^n$ as the class of invertible transformations

$$\tilde{x} = f(x, \alpha, \epsilon), \quad (2.2)$$

with vector–functions $f = (f^1, \ldots, f^n)$ such that

$$f^i(x, \alpha, \epsilon) \approx f_0^i(x, \alpha) + \epsilon f_1^i(x, \alpha) + \cdots + \epsilon^p f_p^i(x, \alpha), \quad i = 1, \ldots, n.$$

Here $\alpha$ is a real parameter, and the following condition is imposed:

$$f(x, 0, \epsilon) \approx x.$$

Furthermore, it is assumed that the transformation (1.79) is defined for any value of $\alpha$ from a small neighborhood of $\alpha = 0$, and that, in this neighborhood, the equation $f(x, \alpha, \epsilon) \approx x$ yields $\alpha = 0$. 

$$f(x, 0, \epsilon) \approx x.$$
2.1 Approximate transformation groups

**Definition 2.1.1.** The set of transformations (2.1) is called a one-parameter approximate transformation group if

\[ f(f(x,a,\varepsilon),b,\varepsilon) \approx f(x,a+b,\varepsilon) \]

for all transformations (2.2).

**Remark 2.1.1.** Here, unlike the classical Lie group theory, \( f \) does not necessarily denote the same function at each occurrence. It can be replaced by any function \( g \approx f \) (see the next example).

**Example 2.1.1.** Let us take \( n = 1 \) and consider the functions

\[ f(x,a,\varepsilon) = x + a \left(1 + \varepsilon x + \frac{1}{2} \varepsilon^2 a\right) \]

and

\[ g(x,a,\varepsilon) = x + a (1 + \varepsilon x) \left(1 + \frac{1}{2} \varepsilon^2 a\right). \]

They are equal in the first order of precision, namely,

\[ g(x,a,\varepsilon) = f(x,a,\varepsilon) + \varepsilon^2 \phi(x,a), \quad \phi(x,a) = \frac{1}{2} a^2 x, \]

and satisfy the approximate group property. Indeed,

\[ f(g(x,a,\varepsilon),b,\varepsilon) = f(x,a + b,\varepsilon) + \varepsilon^2 \phi(x,a,b,\varepsilon), \]

where

\[ \phi(x,a,b,\varepsilon) = \frac{1}{2} a(ax + ab + 2bx + \varepsilon abx). \]

The generator of an approximate transformation group \( G \) given by (2.2) is the class of first-order linear differential operators

\[ X = \xi^i(x,\varepsilon) \frac{\partial}{\partial x^i} \]

such that

\[ \xi^i(x,\varepsilon) \approx \xi^i_0(x) + \varepsilon \xi^i_1(x) + \cdots + \varepsilon^p \xi^i_p(x), \]

where the vector fields \( \xi^i_0, \xi^i_1, \ldots, \xi^i_p \) are given by

\[ \xi^i_{\nu}(x) = \left. \frac{\partial f^i_{\nu}(x,a)}{\partial a} \right|_{a=0}, \quad \nu = 0,\ldots,p; \quad i = 1,\ldots,n. \]

In what follows, an approximate group generator

\[ X \approx (\xi^i_0(x) + \varepsilon \xi^i_1(x) + \cdots + \varepsilon^p \xi^i_p(x)) \frac{\partial}{\partial x^i} \]
is written simply as

\[ X = (\xi_0'(x) + \varepsilon \xi_1'(x) + \cdots + \varepsilon^n \xi_p'(x)) \frac{\partial}{\partial x^i}. \]  \hfill (2.4)

In theoretical discussions, approximate equalities are considered with an error \( o(\varepsilon^n) \) of an arbitrary order \( p \geq 1 \). However, in the most of applications the theory is simplified by letting \( p = 1 \).

### 2.1.2 Approximate Lie equations

Consider the one-parameter approximate transformation groups in the first order of precision, i.e., Eqs. (2.1) of the form

\[ \bar{x}^i \approx f_0^i(x,a) + \varepsilon f_1^i(x,a), \quad i = 1, \ldots, n. \]  \hfill (2.5)

Let

\[ X = X_0 + \varepsilon X_1 \]  \hfill (2.6)

be a given approximate operator, where

\[ X_0 = \xi_0'(x) \frac{\partial}{\partial x^i}, \quad X_1 = \xi_1'(x) \frac{\partial}{\partial x^i}. \]

The corresponding approximate transformation (2.5) of points \( x \) into points \( \bar{x} = \bar{x}_0 + \varepsilon \bar{x}_1 \) with the coordinates

\[ \bar{x}^i = \bar{x}_0^i + \varepsilon \bar{x}_1^i, \]  \hfill (2.7)

where

\[ \bar{x}_0^i = f_0^i(x,a), \quad \bar{x}_1^i = f_1^i(x,a), \]

is determined by the following equations:

\[ \frac{d\bar{x}_0^i}{da} = \xi_0'(\bar{x}_0), \quad \bar{x}_0^i|_{a=0} = x^i, \quad i = 1, \ldots, n, \quad \hfill (2.8)\]

\[ \frac{d\bar{x}_1^i}{da} = \sum_{k=1}^{n} \frac{\partial \xi_0^j(x)}{\partial x^k} \bigg|_{x=\bar{x}_0} \bar{x}_k^j + \xi_1'(\bar{x}_0), \quad \bar{x}_1^i|_{a=0} = 0. \]  \hfill (2.9)

Eqs. (2.8)–(2.9) are called the approximate Lie equations.

**Example 2.1.2.** Let \( n = 1 \) and let

\[ X = (1 + \varepsilon x) \frac{\partial}{\partial x}. \]

Here \( \xi_0(x) = 1, \, \xi_1(x) = x \), and Eqs. (2.8)–(2.9) are written as
2.1 Approximate transformation groups

\[
\frac{d\bar{x}_0}{da} = 1, \quad \bar{x}_0|_{a=0} = x,
\]

\[
\frac{d\bar{x}_1}{da} = \bar{x}_0, \quad \bar{x}_1|_{a=0} = 0.
\]

Its solution has the form

\[
\bar{x}_0 = x + a, \quad \bar{x}_1 = ax + \frac{a^2}{2}.
\]

Hence, the approximate transformation group is given by

\[
\bar{x} \approx x + a + \varepsilon \left(ax + \frac{a^2}{2}\right).
\]

**Example 2.1.3.** Let \( n = 2 \) and let

\[
X = (1 + \varepsilon x^2) \frac{\partial}{\partial x} + \varepsilon xy \frac{\partial}{\partial y}.
\]

Here \( \xi_0(x,y) = (1,0) \), \( \xi_1(x,y) = (x^2,xy) \), and Eqs. (2.8)–(2.9) are written as

\[
\frac{d\bar{x}_0}{da} = 1, \quad \frac{d\bar{y}_0}{da} = 0, \quad \bar{x}_0|_{a=0} = x, \quad \bar{y}_0|_{a=0} = y,
\]

\[
\frac{d\bar{x}_1}{da} = (\bar{x}_0)^2, \quad \frac{d\bar{y}_1}{da} = \bar{x}_0 \bar{y}_0, \quad \bar{x}_1|_{a=0} = 0, \quad \bar{y}_1|_{a=0} = 0.
\]

The integration gives the following approximate transformation group:

\[
\bar{x} \approx x + a + \varepsilon \left(ax^2 + \frac{a^2}{3}\right), \quad \bar{y} \approx y + \varepsilon \left(ax + \frac{a^2}{2}y\right).
\]

**Exercise 2.1.1.** Solve the approximate Lie equations and find the approximate transformation group for the generator

\[
X = \left(t + \frac{\varepsilon}{6} t^2\right) \frac{\partial}{\partial t} - \left(u + \frac{\varepsilon}{3} tu\right) \frac{\partial}{\partial u},
\]

(2.10)

**Exercise 2.1.2.** Solve the approximate Lie equations and find the approximate transformation group for the generator

\[
X = (1 + \varepsilon[(x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2]) \frac{\partial}{\partial x^1} + 2\varepsilon x^1 \left(x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3} + x^4 \frac{\partial}{\partial x^4}\right).
\]
2.1.3 Approximate exponential map

**Theorem 2.1.1.** Given an operator

\[ X = X_0 + \varepsilon X_1, \tag{2.11} \]

with a small parameter \( \varepsilon \), where

\[ X_0 = \xi_0(x) \frac{\partial}{\partial x^i}, \quad X_1 = \xi_1(x) \frac{\partial}{\partial x^i}, \tag{2.12} \]

the corresponding approximate group transformation

\[ \tilde{x}_i = x_0^i + \varepsilon \tilde{x}_1^i, \quad i = 1, \ldots, n, \tag{2.13} \]

are determined by the following equations:

\[ \tilde{x}_0^i = e^{\alpha X_0}(x^i), \quad \tilde{x}_1^i = \langle [a X_0, a X_1] \rangle (\tilde{x}_0^i), \quad i = 1, \ldots, n, \tag{2.14} \]

where

\[ e^{\alpha X_0} = 1 + \alpha X_0 + \frac{\alpha^2}{2!} X_0^2 + \frac{\alpha^3}{3!} X_0^3 + \cdots \tag{2.15} \]

and

\[ \langle [a X_0, a X_1] \rangle = a X_1 + \frac{\alpha^2}{2!} [X_0, X_1] + \frac{\alpha^3}{3!} [X_0, [X_0, X_1]] + \cdots. \tag{2.16} \]

In other words, the approximate operator \( X = X_0 + \varepsilon X_1 \) generates the one-parameter approximate transformation group given by the following approximate exponential map:

\[ \tilde{x}_i = (1 + \varepsilon \langle [a X_0, a X_1] \rangle) e^{\alpha X_0}(x^i), \quad i = 1, \ldots, n. \tag{2.17} \]

**Proof.** (see [4]). The substitution of the operator (2.11),

\[ X_0 + \varepsilon X_1 \]

in to the definition (1.13) of the exponent yields:

\[ e^{\alpha (X_0 + \varepsilon X_1)} = 1 + a (X_0 + \varepsilon X_1) + \frac{\alpha^2}{2!} (X_0 + \varepsilon X_1)^2 + \frac{\alpha^3}{3!} (X_0 + \varepsilon X_1)^3 + \cdots. \]

Now we single out the sum of terms of the first degree in \( \varepsilon \) and obtain
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\[ e^{a(X_0 + \varepsilon X_1)} \approx 1 + aX_0 + \frac{a^2}{2!}X_0^2 + \frac{a^3}{3!}X_0^3 + \cdots + \varepsilon \left\{ aX_1 + \frac{a^2}{2!} (X_0X_1 + X_1X_0) + \frac{a^3}{3!} (X_0^2X_1 + X_0X_1X_0 + X_1X_0^2) + \cdots \right\} \]

By using the identities

\[ X_0X_1 = X_1X_0 + [X_0, X_1] , \]
\[ X_0^2X_1 + X_0X_1X_0 = 2X_1X_0^2 + 3[X_0, X_1]X_0 + [X_0, [X_0, X_1]] , \ldots , \]

we rewrite (2.18) in the form:

\[ e^{a(X_0 + \varepsilon X_1)} \approx 1 + aX_0 + \frac{a^2}{2!}X_0^2 + \frac{a^3}{3!}X_0^3 + \cdots + \varepsilon \left\{ aX_1 \left( 1 + aX_0 + \frac{a^2}{2!}X_0^2 + \frac{a^3}{3!}X_0^3 + \cdots \right) + \frac{a^2}{2!} [X_0, X_1] \left( 1 + aX_0 + \frac{a^2}{2!}X_0^2 + \frac{a^3}{3!}X_0^3 + \cdots \right) + \frac{a^3}{3!} [X_0, [X_0, X_1]] \left( 1 + aX_0 + \frac{a^2}{2!}X_0^2 + \frac{a^3}{3!}X_0^3 + \cdots \right) + \cdots \right\} . \]

Whence, using exponent (1.13) we have

\[ e^{a(X_0 + \varepsilon X_1)} \approx (1 + \varepsilon \langle [aX_0, aX_1] \rangle) e^{aX_0} . \] (2.19)

In other words, the exponential map

\[ \tilde{x}^i = e^{aX}(x^i) \]

written for the operator (2.11) and evaluated in the first order of precision with respect to \( \varepsilon \) has the form (2.17). Taking into account (2.13), one arrives at Eqs. (2.14). This completes the proof of the theorem.

**Example 2.1.4.** Let us apply Theorem 2.1.1 to the operator

\[ X = (1 + \varepsilon x) \frac{\partial}{\partial x} \]

considered in Example 2.1.2. Here
\[ X_0 = \frac{\partial}{\partial x}, \quad X_1 = x \frac{\partial}{\partial x}. \]

Therefore,
\[ X_0(x) = 1, \quad X_0^2(x) = X_0^3(x) = \cdots = 0, \]
and
\[ [X_0, X_1] = \frac{\partial}{\partial x} = X_0, \]
\[ [X_0, [X_0, X_1]] = [X_0, X_0] = 0, \ldots. \]

Consequently,
\[ \bar{x}_0 = e^{aX_0}(x) = x + a, \]
and
\[ \langle \langle aX_0, aX_1 \rangle \rangle = \left( ax + \frac{a^2}{2!} \right) \frac{\partial}{\partial x}, \]
whence
\[ \bar{x}_1 = \langle \langle aX_0, aX_1 \rangle \rangle(\bar{x}_0) = \left( ax + \frac{a^2}{2!} \right) \frac{\partial}{\partial x}(x + a) = ax + \frac{a^2}{2!}. \]

Hence,
\[ \bar{x} \approx x + a + \epsilon \left( ax + \frac{a^2}{2} \right). \]

**Example 2.1.5.** Let us apply Theorem 2.1.1 to the operator from Example 2.1.3. In this case we have
\[ X_0 = \frac{\partial}{\partial x}, \quad X_1 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}. \]

Therefore,
\[ \bar{x}_0 = e^{aX_0}(x) = x + a, \]
and
\[ [X_0, X_1] = 2x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \]
\[ [X_0, [X_0, X_1]] = 2 \frac{\partial}{\partial x}, \]
\[ [X_0, [X_0, [X_0, X_1]]] = 0, \ldots. \]

Consequently,
\[ \langle \langle aX_0, aX_1 \rangle \rangle = aX_1 + \frac{a^2}{2!} \left( 2x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) + \frac{2a^3}{3} \frac{\partial}{\partial x} \]
\[ = \left( ax^2 + a^2x + \frac{a^3}{3} \right) \frac{\partial}{\partial x} + \left( axy + \frac{a^2}{2}y \right) \frac{\partial}{\partial y}. \]
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Whence
\[
\hat{x}_1 = \langle \langle aX_0, aX_1 \rangle \rangle \left( \hat{x}_0 \right) = \left( ax^2 + a^2 x + \frac{a^3}{3} \right) \frac{\partial}{\partial x} (x + a),
\]
\[
\hat{y}_1 = \langle \langle aX_0, aX_1 \rangle \rangle \left( \hat{y}_0 \right) = \left( axy + \frac{a^2}{2} y \right) \frac{\partial}{\partial y} (y).
\]
Hence,
\[
\hat{x}_1 = ax^2 + a^2 x + \frac{a^3}{3}, \quad \hat{y}_1 = axy + \frac{a^2}{2} y.
\]
We thus arrive at the result of Example 2.1.3:
\[
\hat{x} \approx x + a + \varepsilon \left( ax^2 + a^2 x + \frac{a^3}{3} \right),
\]
\[
\hat{y} \approx y + \varepsilon \left( axy + \frac{a^2}{2} y \right).
\]

Example 2.1.6. Consider now the generator from Exercise 2.1.2:

\[
X = \left( 1 + \varepsilon [(x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2] \right) \frac{\partial}{\partial x^1} + 2 \varepsilon x^1 \left( x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3} + x^4 \frac{\partial}{\partial x^4} \right).
\]
Here \( X = X_0 + \varepsilon X_1 \), with
\[
X_0 = \frac{\partial}{\partial x^1},
\]
\[
X_1 = \left( (x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2 \right) \frac{\partial}{\partial x^1}
\]
\[
+ 2 x^1 \left( x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3} + x^4 \frac{\partial}{\partial x^4} \right).
\]
The operator \( X_0 \) generates the translation group:
\[
\hat{x}_0^1 = x^1 + a, \quad \hat{x}_0^j = x^j, \quad j = 2, 3, 4.
\]
We have
\[
[X_0, X_1] = 2 x^1 \left( x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3} + x^4 \frac{\partial}{\partial x^4} \right),
\]
\[
[X_0, [X_0, X_1]] = 2 \frac{\partial}{\partial x^1}, \quad [X_0, [X_0, [X_0, X_1]]] = 0, \ldots.
\]
Consequently, Eq. (2.16) takes the form:
\[
\langle \langle aX_0, aX_1 \rangle \rangle = \left( [(x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2] a + x^1 a^2 + \frac{1}{3} a^3 \right) \frac{\partial}{\partial x^1}.
\]
\[+(2ax^1 + a^2) \left(x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3} + x^4 \frac{\partial}{\partial x^4}\right).\]

Therefore, (2.14) yields

\[\tilde{x}^1 = \langle (aX_0, aX_1) \rangle \langle \tilde{x}^i \rangle = [(x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2]a + x^1a^2 + \frac{1}{3}a^3,\]

\[\tilde{x}^j = (2ax^1 + a^2)x^j, \quad j = 2, 3, 4.\]

We have arrived at the following approximate transformation group:

\[\tilde{x}^i \approx \tilde{x}^0 + \varepsilon x^1 = x^1 + \varepsilon \left([(x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2]a + x^1a^2 + \frac{1}{3}a^3\right),\]

\[\tilde{x}^j \approx x^0 + \varepsilon x^j = x^j + \varepsilon (2ax^1 + a^2)x^j, \quad j = 2, 3, 4.\]

### 2.2 Approximate symmetries

In this section we will carry over the infinitesimal method of Chap. 1 to the approximate symmetries, i.e., approximate transformation groups admitted by differential equations with a small parameter \(\varepsilon\). We will consider the approximation in the first order of precision in \(\varepsilon\). Methods of the approximate symmetries will be illustrated by examples. Readers can find more applications in [3]. See also [5].

#### 2.2.1 Definition of approximate symmetries

**Definition 2.2.1.** Let \(G\) be a one-parameter approximate transformation group:

\[\tilde{z}^i \approx f(z, a, \varepsilon) \equiv f_0^i(z, a) + \varepsilon f_1^i(z, a), \quad i = 1, \ldots, N.\]  

(2.20)

An approximate equation

\[F(z, \varepsilon) \equiv F_0(z) + \varepsilon F_1(z) \approx 0\]  

(2.21)

is said to be *approximately invariant* with respect to \(G\), or *admits* \(G\) if

\[F(\tilde{z}, \varepsilon) \approx F(f(z, a, \varepsilon), \varepsilon) = o(\varepsilon)\]

whenever \(z = (z^1, \ldots, z^N)\) satisfies Eq. (2.21).

If \(z = (x, u, u_{(1)}, \ldots, u_{(k)})\), then (2.21) becomes an approximate differential equation of order \(k\), and \(G\) is an *approximate symmetry group* of the differential equation.
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2.2.2 Determining equations & Stable symmetries

**Theorem 2.2.1.** Eq. (2.21) is approximately invariant under the approximate transformation group (2.20) with the generator

\[ X = X^0 + \varepsilon X^1 \equiv \xi_0(z) \frac{\partial}{\partial z} + \varepsilon \xi^i(z) \frac{\partial}{\partial z^i}, \]  \hspace{1cm} (2.22)

if and only if

\[ [XF(z, \varepsilon)]_{F=0} = o(\varepsilon), \]  \hspace{1cm} or

\[ [X^0F_0(z) + \varepsilon (X^1F_0(z) + X^0F_1(z))]_{(2.21)} = o(\varepsilon). \]  \hspace{1cm} (2.23)

**Proof.** Eq. (2.23) is obtained by substituting (2.21) and (2.22) into the determining equation (1.28) and considering the result in the first-order of precision with respect to \( \varepsilon \).

The operator (2.22) satisfying Eq. (2.23) is called an *infinitesimal approximate symmetry* of, or an *approximate operator admitted* by Eq. (2.21). Accordingly, Eq. (2.23) is termed the *determining equation* for approximate symmetries.

**Remark 2.2.1.** The determining equation (2.23) can be written as follows:

\[ X^0F_0(z) = \lambda(z)F_0(z), \]  \hspace{1cm} (2.24)

\[ X^1F_0(z) + X^0F_1(z) = \lambda(z)F_1(z). \]  \hspace{1cm} (2.25)

The factor \( \lambda(z) \) is determined by Eq. (2.24) and then substituted in Eq. (2.25). The latter equation must hold for all solutions of \( F_0(z) = 0 \).

Comparing Eq. (2.24) with the determining equation of exact symmetries, we obtain the following statement.

**Theorem 2.2.2.** If Eq. (2.21) admits an approximate transformation group with the generator \( X = X^0 + \varepsilon X^1 \), where \( X^0 \neq 0 \), then the operator

\[ X^0 = \xi_0(z) \frac{\partial}{\partial z} \]  \hspace{1cm} (2.26)

is an exact symmetry of the equation

\[ F_0(z) = 0. \]  \hspace{1cm} (2.27)

**Remark 2.2.2.** It is manifest from Eqs. (2.24), (2.25) that if \( X^0 \) is an exact symmetry of Eq. (2.27), then \( X = \varepsilon X^0 \) is an *approximate symmetry* of Eq. (2.21).

**Definition 2.2.2.** Eqs. (2.27) and (2.21) are termed an *unperturbed equation* and a *perturbed equation*, respectively. Under the conditions of Theorem 2.2.2, the operator \( X^0 \) is called a *stable symmetry* of the unperturbed equation (2.27). The corresponding approximate symmetry generator \( X = X^0 + \varepsilon X^1 \) for the perturbed equation
(2.26) is called a deformation of the infinitesimal symmetry \( X^0 \) of Eq. (2.27) caused by the perturbation \( \varepsilon F_1(z) \). In particular, if the most general symmetry Lie algebra of Eq. (2.27) is stable, we say that the perturbed equation (2.21) inherits the symmetries of the unperturbed equation.

2.2.3 Calculation of approximate symmetries

Remark 2.2.1 and Theorem 2.2.2 provide an infinitesimal method for calculating approximate symmetries (2.22) for differential equations with a small parameter. Implementation of the method requires the following three steps.

1st step. Calculation of the exact symmetries \( X^0 \) of the unperturbed equation (2.27), e.g. by solving the determining equation

\[
X^0 F_0(z) \bigg|_{F_0(z)=0} = 0. \tag{2.28}
\]

2nd step. Determination of the auxiliary function \( H \) by virtue of Eqs. (2.24), (2.25) and (2.21), i.e., by the equation

\[
H = \frac{1}{\varepsilon} \left[ X^0 \left( F_0(z) + \varepsilon F_1(z) \right) \bigg|_{F_0(z)+\varepsilon F_1(z)=0} \right] \tag{2.29}
\]

with known \( X^0 \) and \( F_1(z) \).

3rd step. Calculation of the operators \( X^1 \) by solving the determining equation for deformations:

\[
X^1 F_0(z) \bigg|_{F_0(z)=0} + H = 0. \tag{2.30}
\]

Note that Eq. (2.30), unlike the determining equation (2.28) for exact symmetries, is inhomogeneous.

2.2.4 Examples of approximate symmetries

Example 2.2.1. Let us approximate symmetries of the following perturbed nonlinear wave equation:

\[
u_{tt} - (u^2 u_x)_x + \varepsilon u_t = 0. \tag{2.31}
\]

Let us write the approximate group generators in the form

\[
X = X^0 + \varepsilon X^1 \equiv (\tau_0 + \varepsilon \tau_1) \frac{\partial}{\partial t} + (\xi_0 + \varepsilon \xi_1) \frac{\partial}{\partial x} + (\eta_0 + \varepsilon \eta_1) \frac{\partial}{\partial u}, \tag{2.32}
\]

where \( \tau_v, \xi_v, \) and \( \eta_v \) (\( v = 0, 1 \)) are unknown functions of \( t, x, \) and \( u \).
2.2 Approximate symmetries

1st step. Solving the determining equation (2.28) for the exact symmetries $X^0$ of the unperturbed equation

$$u_{tt} - (u^2 u_x)_x = 0,$$

(2.33)

we obtain

$$\tau_0 = C_1 + C_3 t, \quad \xi_0 = C_2 + (C_3 + C_4) x, \quad \eta_0 = C_4 u,$$

(2.34)

where $C_1, \ldots, C_4$ are arbitrary constants. Hence,

$$X^0 = (C_1 + C_3 t) \frac{\partial}{\partial t} + (C_2 + C_3 x + C_4 x) \frac{\partial}{\partial x} + C_4 u \frac{\partial}{\partial u}.$$

(2.35)

In other words, Eq. (2.33) admits the four-dimensional Lie algebra $L_4$ with the basis

$$X^0_1 = \frac{\partial}{\partial t}, \quad X^0_2 = \frac{\partial}{\partial x}, \quad X^0_3 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}, \quad X^0_4 = x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}.$$

(2.36)

2nd step. Substituting the expression (2.35) of the generator $X^0$ into Eq. (2.29) we obtain the auxiliary function

$$H = C_3 u_t.$$

3rd step. Now the determining equation (2.30) for deformations is written as

$$X^1 \left( u_{tt} - u^2 u_{xx} - 2 u u_x^2 \right) \bigg|_{(2.33)} + C_3 u_t = 0,$$

(2.37)

where $X^1$ denotes the prolongation of the operator

$$X^1 = \tau_1 \frac{\partial}{\partial t} + \xi_1 \frac{\partial}{\partial x} + \eta_1 \frac{\partial}{\partial u}$$

to the derivatives of $u$ involved in Eq. (2.31). Upon setting $u_{tt} = (u^2 u_x)_x$, the left-hand side of Eq. (2.37) becomes a polynomial in the variables $u_{tx}, u_{xx}, u_t, u_x$. Equating to zero its coefficients we obtain

$$\tau_1 = \tau_1(t), \quad \xi_1 = \xi_1(x), \quad 3 \tau''_1 = C_3, \quad \xi''_1 = 0, \quad \eta_1 = \left[ \xi'_1(x) - \tau'_1(t) \right] u.$$

Hence,

$$\tau_1 = A_1 + A_3 t + \frac{1}{6} C_3 t^2,$$

$$\xi_1 = A_2 + (A_3 + A_4) x,$$

$$\eta_1 = \left( A_4 - \frac{1}{3} C_3 t \right) u.$$

(2.38)

Substituting (2.34) and (2.38) into (2.32), we obtain the following approximate symmetries for Eq. (2.31):
\[ X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + \frac{\varepsilon}{6} \left( t^2 \frac{\partial}{\partial t} - 2tu \frac{\partial}{\partial u} \right), \quad \text{(2.39)} \]

\[ X_4 = x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}, \quad X_5 = \varepsilon X_1, \quad X_6 = \varepsilon X_2, \quad X_7 = \varepsilon X_4, \quad X_8 = \varepsilon X_3. \]

**Remark 2.2.3.** Note that Eqs. (2.38) yield

\[ X_8 = \varepsilon \left( t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} \right). \]

However, in the first-order of precision, the operator \( X_8 \) can be written in the form given in (2.39).

The following table of commutators, evaluated in the first-order of precision, shows that the operators (2.39) span an eight-dimensional approximate Lie algebra \( L_8 \), and hence generate an eight-parameter approximate transformations group. For the sake of brevity, we write here only the off-diagonal elements (cf. Commutator Table 1.3 on page 29).

**Table 2.1** Approximate commutators

<table>
<thead>
<tr>
<th></th>
<th>( X_1 )</th>
<th>( X_2 )</th>
<th>( X_3 )</th>
<th>( X_4 )</th>
<th>( X_5 )</th>
<th>( X_6 )</th>
<th>( X_7 )</th>
<th>( X_8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_1 )</td>
<td>0</td>
<td>0</td>
<td>( X_1 + \frac{\varepsilon}{3}(X_8 - X_7) )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( X_2 )</td>
<td>0</td>
<td>( X_2 )</td>
<td>( X_3 )</td>
<td>0</td>
<td>0</td>
<td>( X_6 )</td>
<td>( X_6 )</td>
<td></td>
</tr>
<tr>
<td>( X_3 )</td>
<td>0</td>
<td>0</td>
<td>( -X_5 )</td>
<td>( -X_6 )</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( X_4 )</td>
<td>0</td>
<td>0</td>
<td>( -X_6 )</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( X_5 )</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( X_6 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( X_7 )</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( X_8 )</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2.1 shows that although the approximate symmetries \( X_5, X_6, X_7 \) and \( X_8 \) are “trivial” according to Remark 2.2.2, they are necessary for the Lie algebra structure, and hence for constructing multi-parameter approximate transformation groups.

**Remark 2.2.4.** Equations (2.39) show that all symmetries (2.36) of Eq. (2.33) are stable. Hence, the perturbed equation (2.31) inherits the symmetries of the unperturbed equation (2.33).

**Example 2.2.2.** Consider the equation

\[ u_{tt} - \left( u^{-4} u_x \right)_x + \varepsilon u_t = 0. \quad \text{(2.40)} \]

1st step. The unperturbed equation

\[ u_{tt} - \left( u^{-4} u_x \right)_x = 0 \quad \text{(2.41)} \]
admits five-dimensional Lie algebra $L_5$ consisting of the operators

$$X^0 = (C_1 + C_3 t + C_5 t^2) \frac{\partial}{\partial t} + (C_2 + C_3 x + C_4 x) \frac{\partial}{\partial x} + \left( -\frac{1}{2} C_4 + C_5 t \right) u \frac{\partial}{\partial u}.$$ 

A basis of this algebra is

$$X_1^0 = \frac{\partial}{\partial t}, \quad X_2^0 = \frac{\partial}{\partial x}, \quad X_3^0 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x},$$

$$X_4^0 = x \frac{\partial}{\partial x} - \frac{1}{2} u \frac{\partial}{\partial u}, \quad X_5^0 = t^2 \frac{\partial}{\partial t} + tu \frac{\partial}{\partial u}. \quad (2.42)$$

**2nd step.** Equation (2.29) provides the following auxiliary function:

$$H = C_3 u_t + 2C_5 t u_t + C_5 u.$$  

**3rd step.** The determining equation (2.30) for deformations yields that $C_3 = C_5 = 0$. Proceeding further as in the previous example, we obtain the following approximate symmetries of the perturbed equation (2.40):

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \varepsilon \left( t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} \right), \quad X_4 = x \frac{\partial}{\partial x} - \frac{1}{2} u \frac{\partial}{\partial u},$$

$$X_5 = \varepsilon X_1, \quad X_6 = \varepsilon X_2, \quad X_7 = \varepsilon X_4, \quad X_8 = \varepsilon \left( t^2 \frac{\partial}{\partial t} + tu \frac{\partial}{\partial u} \right). \quad (2.43)$$

Equations (2.43) show that not all symmetries (2.42) of Eq. (2.41) are stable. Namely, the operators

$$X_3^0 = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} \quad \text{and} \quad X_5^0 = t^2 \frac{\partial}{\partial t} + tu \frac{\partial}{\partial u}$$

from (2.42) are unstable. Hence, the perturbed equation (2.40) does not inherit the symmetries of the unperturbed equation (2.41).

**Remark 2.2.5.** Equations (2.31) and (2.40) with an arbitrary parameter $\varepsilon \neq 0$ have only three exact symmetries (see [3], page 227, Sect. 9.2.1.1.2). Namely, the exact symmetries of Eq. (2.31) are the operators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_4 = x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u}$$

from (2.39), and the exact symmetries of Eq. (2.40) are the operators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_4 = x \frac{\partial}{\partial x} - \frac{1}{2} u \frac{\partial}{\partial u}$$

from (2.43).
2.2.5 Integration using approximate symmetries

Let us discuss, by way of examples, group methods of integration of differential equations with a small parameter with known approximate symmetries.

**Example 2.2.3.** The second-order equation

\[ y'' - x - \varepsilon y^2 = 0 \]  \hspace{1cm} (2.44)

has no exact point symmetries if \( \varepsilon \neq 0 \) is regarded as a constant coefficient, and hence cannot be integrated by the Lie method. Moreover, this equation cannot be integrated by quadrature. However, it possesses approximate symmetries if \( \varepsilon \) is treated as a small parameter, e.g.

\[ X_1 = \frac{\partial}{\partial y} + \frac{\varepsilon}{3} \left[ 2x^2 \frac{\partial}{\partial x} + \left( 3yx^2 + \frac{11}{20}x^5 \right) \frac{\partial}{\partial y} \right], \]

\[ X_2 = x \frac{\partial}{\partial y} + \frac{\varepsilon}{6} \left[ x^4 \frac{\partial}{\partial x} + \left( 2yx^3 + \frac{7}{30}x^5 \right) \frac{\partial}{\partial y} \right]. \]  \hspace{1cm} (2.45)

The commutator of the operators (2.45) is \([X_1, X_2] \approx 0\). Hence they span a two-dimensional approximate Abelian Lie algebra and can be used for consecutive integration of Eq. (2.44) as follows ([5], Sect. 12.4).

Equations \( X_1(t) \approx 1, X_1(u) \approx 0 \) yield the canonical variables

\[ t = y - \varepsilon \left( \frac{1}{2}x^2y^2 + \frac{11}{60}yx^5 \right), \quad u = x - \frac{2}{3} \varepsilon yx^3 \]  \hspace{1cm} (2.46)

for \( X_1 \) from (2.45). Thus we have \( X_1 \approx \partial/\partial t \), while Eq. (2.44) reads

\[ u'' + uu'^3 + \varepsilon \left[ 3u^2u' + \frac{1}{6}(u^2u')^2 - \frac{11}{60}(u^2u')^3 \right] = 0. \]

Integration by the standard substitution \( u' = p(u) \) yields

\[ p' + up^2 + \varepsilon \left( 3u^2 + \frac{1}{6}u^4p - \frac{11}{60}u^6p^2 \right) = 0. \]  \hspace{1cm} (2.47)

The second operator (2.45) is written as

\[ X_2 = p^2 \frac{\partial}{\partial p} + \varepsilon \left[ \frac{1}{2}u^4 \frac{\partial}{\partial u} + \left( 2u^3p - \frac{13}{15}u^5p^2 \right) \right] \frac{\partial}{\partial p} \]

and becomes the infinitesimal translation

\[ X_2 \approx \frac{\partial}{\partial v} \]

upon introducing the new independent variable \( z \) and dependent variable \( v \) defined by the equations
In the variables (2.48), Eq. (2.47) is written as
\[ v' + z + \frac{11}{60} \varepsilon z^6 = 0 \]
and yields
\[ v = -\frac{11}{60} z^2 - \frac{11}{420} \varepsilon z^7 + C. \]
Substituting this expression for \( v \) into Eqs. (2.48) and eliminating \( z \), one obtains the function \( p(u) \). The quadrature
\[ \int \frac{du}{p(u)} = t + C \]
followed by the substitution into Eqs. (2.46), completes the approximate integration of Eq. (2.44).

2.2.6 Integration using stable symmetries

The van der Pol equation
\[ y'' + y = \varepsilon (y' - y^3), \quad \varepsilon = \text{const.,} \quad (2.49) \]
has only one exact point symmetry, namely the translation of the independent variable \( x \) with the generator
\[ X = \frac{\partial}{\partial x}, \]
if \( \varepsilon \) is treated as an arbitrary constant. On the other hand, it can be shown that Eq. (2.49) with a small parameter \( \varepsilon \) inherits all 8 point symmetries of the unperturbed equation
\[ y'' + y = 0. \quad (2.50) \]
The stability (see Definition 2.2.2) of all symmetries is a necessary condition for existence of an approximate transformation connecting perturbed and unperturbed equations. Let us use this circumstance for the integration of the van der Pol equation in the first order of precision.

The characteristic property of the unperturbed equation (2.50) is that it is homogeneous, i.e., admits the dilation (scaling) generator
\[ X^0 = y \frac{\partial}{\partial y}. \quad (2.51) \]
The reckoning shows that although the perturbed equation (2.49) is not homogeneous, it admits, as an approximate symmetry, the following deformation of the dilation generator (2.51):

\[ X = \left\{ y - \frac{\varepsilon}{4} \left[ y^2y' + 3xy(y^2 + y'^2) \right] \right\} \frac{\partial}{\partial y}. \quad (2.52) \]

We will write Eq. (2.50) and its symmetry (2.51) in the form

\[ z'' + z = 0 \quad (2.53) \]

and

\[ X^0 = z \frac{\partial}{\partial z}, \quad (2.54) \]

respectively, and look for an approximate transformation

\[ y = z + \varepsilon f \]

connecting the van der Pol equation (2.49) with Eq. (2.53).

Our construction will be simplified by using the stability of the dilation generator (2.51). Namely, we will begin with constructing the approximate transformations mapping the operator (2.54) into the operator (2.52). Then we will subject the resulting transformations to the condition that they connect Eqs. (2.53) and (2.49).

Noting that the approximate symmetry (2.52) is a Lie-Backlund operator depending on the first-order derivative \( y' \), we will search the approximate transformation of (2.54) into (2.52) in the form

\[ y = z + \varepsilon f(x, z, z'). \quad (2.55) \]

The operator (2.54) is written in the variable \( y \) given by (2.55) as follows:

\[ \bar{X}^0 = X^0(y) \frac{\partial}{\partial y}, \]

where the prolongation of \( X^0 \) to the derivative \( y' \) is understood. So we have

\[ X^0(y) = \left[ z \frac{\partial}{\partial z} + \varepsilon \frac{\partial}{\partial z'} \right] \left( z + \varepsilon f(x, z, z') \right) = z + \varepsilon \left[ \frac{\partial f}{\partial z} + z' \frac{\partial f}{\partial z'} \right]. \]

Since \( \bar{X}^0 \) should be identical with the operator (2.52), we have

\[ z + \varepsilon \left[ \frac{\partial f}{\partial z} + z' \frac{\partial f}{\partial z'} \right] = y - \frac{\varepsilon}{4} \left[ y^2y' + 3xy(y^2 + y'^2) \right]. \]

Substituting into the right-hand side of this equation the expression (2.55) for \( y \) and noting that \( y = z \) up to terms of order \( \varepsilon \) according to (2.55), we get
2.2 Approximate symmetries

\[ z + \varepsilon \left[ z \frac{\partial f}{\partial z} + z' \frac{\partial f}{\partial z'} \right] = z + \varepsilon \left\{ f - \frac{1}{4} \left[ z^2 z' + 3xz(z^2 + z'^2) \right] \right\}, \]

whence

\[ z \frac{\partial f}{\partial z} + z' \frac{\partial f}{\partial z'} = f - \frac{1}{4} \left[ z^2 z' + 3xz(z^2 + z'^2) \right]. \quad (2.56) \]

Let us solve the non-homogeneous linear first-order partial differential equation (2.56). The first equation of the characteristic system

\[ \frac{dz}{z} = \frac{dz'}{z'} = \frac{df}{f - \frac{1}{4} \left[ z^2 z' + 3xz(z^2 + z'^2) \right]} \]

yields the first integral \( z'/z = \lambda = \text{const.} \) Writing the second equation of the characteristic system in the form

\[ \frac{df}{dz} = \frac{1}{z} \left\{ f - \frac{1}{4} \left[ z^2 z' + 3xz(z^2 + z'^2) \right] \right\}, \]

and substituting \( z' = \lambda z \), we obtain the following linear first-order ordinary differential equation for \( f \):

\[ \frac{df}{dz} = \frac{1}{z} f - \frac{1}{4} \left[ \lambda z^2 + 3x(1 + \lambda^2)z^2 \right]. \]

The integration yields

\[ f = -\frac{1}{8} \left[ K(x)z + \lambda z^3 + 3xz(1 + \lambda^2)z^2 \right], \]

where the “constant of integration” \( K(x) \) is an arbitrary function of \( x \). Substituting \( \lambda z = z' \), we obtain

\[ f = -\frac{1}{8} \left[ K(x)z + z^2 z' + 3xz(z^2 + z'^2) \right] \]

and finally arrive at the following transformation (2.55) mapping the operator (2.54) into (2.52):

\[ y = z - \frac{\varepsilon}{8} \left[ K(x)z + z^2 z' + 3xz(z^2 + z'^2) \right]. \quad (2.57) \]

Since \( y = z \) up to terms of order \( \varepsilon \), the inverse to (2.54) in the first order of approximation is given by

\[ z = y + \frac{\varepsilon}{8} \left[ K(x)y + y^2 y' + 3xy(y^2 + y'^2) \right]. \quad (2.58) \]

Let us determine \( K(x) \) so that (2.57) maps Eq. (2.53) into Eq. (2.49). Let us substitute (2.58) into Eq. (2.53). Since \( y = z \) up to terms of order \( \varepsilon \), the equation \( z'' = -z \) yields that

\[ y'' = -y \]
up to terms of order $\epsilon$. Therefore, from (2.58), we have

$$z' = y' + \frac{\epsilon}{8} [K(x)y' + K'(x)y + 2y^3 + 5yy'^2 + 3x(y^2y' + y'^3)]$$

and

$$z'' = y'' + \frac{\epsilon}{8} [2K'(x)y' + K''(x)y - K(x)y - y^2y' + 8y^3 - 3xy(y^2 + y'^2)].$$

Hence,

$$z'' + z = y'' + y + \frac{\epsilon}{8} [8y^3 + 2K'(x)y' + K''(x)y].$$

It follows that we obtain Eq. (2.49) if we let $K(x) = -4x$. Thus, we have arrived at the following transformation mapping Eq. (2.53) into Eq. (2.49):

$$y = z + \frac{\epsilon}{8} [4xz - z^2z' - 3xz (z^2 + z'^2)]. \quad (2.59)$$

Substituting into (2.59) the general solution $z = A \cos x + B \sin x$ of Eq. (2.53), one arrives at the general approximate solution of the van der Pol equation (2.49):

$$y = A \cos x + B \sin x + \frac{\epsilon}{8} [(4 - 3(A^2 + B^2)) x (A \cos x + B \sin x)$$

$$+ (A \sin x - B \cos x)(A \cos x + B \sin x)^2]. \quad (2.60)$$

Letting here $A = 1, B = 0$ and then $A = 0, B = 1$, i.e., taking a fundamental set of solutions $z_1 = \cos x$ and $z_2 = \sin x$ of the linear equation $z'' + z = 0$, one obtains the following particular approximate solutions of Eq. (2.49):

$$y_1 = \cos x + \frac{\epsilon}{8} [x \cos x + \sin x \cos^2 x],$$

$$y_2 = \sin x + \frac{\epsilon}{8} [x \sin x - \cos x \sin^2 x].$$

### 2.2.7 Approximately invariant solutions

**Example 2.2.4.** Consider Eq. (2.31),

$$u_{tt} - (u^2u_x)_x + \epsilon u_t = 0, \quad (2.61)$$

with known approximate symmetries (2.39) and find an approximately invariant solution based on the approximate symmetry $X = X_3 - X_4$ with $X_3$, $X_4$ from (2.39). We have

$$X = t \frac{\partial}{\partial t} - u \frac{\partial}{\partial u} + \frac{\epsilon}{6} \left( t^2 \frac{\partial}{\partial t} - 2tu \frac{\partial}{\partial u} \right). \quad (2.62)$$
The approximate invariants for operator (2.62) are written in the form

\[ J(t,x,u,\varepsilon) = J^0(t,x,u) + \varepsilon J^1(t,x,u) + o(\varepsilon). \]

They are determined by the equation \( X(J) = o(\varepsilon) \). Using for the operator (2.62) the notation

\[ X = X^0 + \varepsilon X^1, \]

where

\[
X^0 = t \frac{\partial}{\partial t} - u \frac{\partial}{\partial u}, \quad X^1 = \frac{1}{6} \left( t^2 \frac{\partial}{\partial t} - 2tu \frac{\partial}{\partial u} \right),
\]

we will write the determining equation \( X(J) = o(\varepsilon) \) for the approximate invariants in the form

\[ X^0(J^0) + \varepsilon [X^0(J^1) + X^1(J^0)] = 0, \]

whence

\[ X^0(J^0) = 0, \quad X^0(J^1) + X^1(J^0) = 0, \]

or

\[
\begin{align*}
\frac{t}{I} \frac{\partial J^0}{\partial t} - u \frac{\partial J^0}{\partial u} &= 0, \\
\frac{t}{I} \frac{\partial J^1}{\partial t} - u \frac{\partial J^1}{\partial u} &= -\frac{1}{6} \left( t^2 \frac{\partial J^0}{\partial t} - 2tu \frac{\partial J^0}{\partial u} \right). 
\end{align*}
\]

Solving Eqs. (2.63), we will find two functionally independent invariants

\[
\begin{align*}
J_1 &= J^0_1(t,x,u) + \varepsilon J^1_1(t,x,u), \\
J_2 &= J^0_2(t,x,u) + \varepsilon J^1_2(t,x,u) 
\end{align*}
\]

for operator (2.62).

**Remark 2.2.6.** Recall that functions (2.64) are said to be functionally dependent if

\[ J_2 = \Psi(J_1), \]

in other words if the equation

\[ J^0_2(t,x,u) + \varepsilon J^1_2(t,x,u) = \Psi(J^0_1(t,x,u) + \varepsilon J^1_1(t,x,u)) + o(\varepsilon) \]

with a certain function \( \Psi \) holds identically in \( t,x,u \). If such a function \( \Psi \) does not exist, the functions (2.64) are said to be functionally independent. It is manifest that if \( J^0_1(t,x,u) \) and \( J^0_2(t,x,u) \) are functionally independent, then so are the functions (2.64) as well.

The first equation in (2.63) has precisely two functionally independent solutions, namely,

\[ J^0_1 = x, \quad J^0_2 = tu. \]

Substituting \( J^0_1 = x \) into the second equation in (2.63) and taking its simplest solution \( J^1_1 = 0 \), we obtain one invariant in (2.64),
Note that it does not involve the dependent variable \( u \). Now we substitute the solution \( J_2^0 = tu \) of the first equation in (2.63) into the second equation in (2.63) and obtain the following non-homogeneous linear equation:

\[
\frac{\partial J_2^1}{\partial t} - u \frac{\partial J_2^1}{\partial u} = \frac{1}{6} t^2 u.
\]

The first equation of the characteristic system

\[
\frac{dt}{t} = -\frac{du}{u} = 6 \frac{dJ_2^1}{t^2 u}
\]

yields the first integral \( tu = \lambda = \text{const} \). Therefore, the second equation

\[
\frac{dt}{t} = 6 \frac{dJ_2^1}{t^2 u}
\]

of the characteristic system, upon replacing \( tu \) by \( \lambda \), is written as \( \lambda \, dt = 6dJ_2^1 \) and yields

\[
J_2^1 = \frac{1}{6} t \lambda + C = \frac{1}{6} t^2 u + C.
\]

Letting here \( C = 0 \), we obtain the second invariant in (2.64),

\[
J_2 = tu + \frac{\epsilon}{6} t^2 u.
\]

By Remark 2.2.6, invariants (2.65), (2.66) are functionally independent.

Letting \( J_2 = \varphi(J_1) \), i.e.,

\[
\left(1 + \frac{\epsilon}{6} t\right) tu = \varphi(x)
\]

and solving for \( tu \) in the first order of precision,

\[
tu = \left(1 + \frac{\epsilon}{6} t\right)^{-1} \varphi(x) = \left(1 - \frac{\epsilon}{6} t\right) \varphi(x) + o(\epsilon),
\]

we obtain the following form for the approximately invariant solutions:

\[
u = \left(1 - \frac{\epsilon}{6} t\right) \varphi(x).
\]

Now we substitute (2.67) into Eq. (2.61). Differentiation of (2.67) yields:

\[
u_t = -\frac{1}{t^2} \varphi, \quad \nu_{tt} = \frac{2}{t^3} \varphi, \quad \nu_x = \left(\frac{1}{t} - \frac{\epsilon}{6}\right) \varphi'.
\]

Furthermore,
2.2 Approximate symmetries

\[ u^2u_x = \left( \frac{1}{t} - \frac{\varepsilon}{6} \right)^3 \varphi^2 \varphi' = \left( \frac{1}{t^3} - \frac{\varepsilon}{2t^2} \right) \varphi^2 \varphi' + o(\varepsilon), \]

and hence we have in our approximation:

\[ (u^2u_x)_x = \left( \frac{1}{t^3} - \frac{\varepsilon}{2t^2} \right) (\varphi^2 \varphi')'. \]

Therefore,

\[ u_{tt} - (u^2u_x)_x + \varepsilon u_t = \frac{1}{t^3} \left( 1 - \frac{\varepsilon t}{2} \right) \left[ 2\varphi - (\varphi^2 \varphi')' \right]. \]

Thus, Eq. (2.61) yields

\[ (\varphi^2 \varphi')' = 2\varphi. \quad (2.68) \]

Let us integrate Eq. (2.68). We have

\[ \varphi^2 \varphi'' + 2\varphi \varphi'^2 = 2\varphi, \]

or

\[ \varphi \varphi'' + 2\varphi'^2 = 2. \]

The latter equation is reduced to a first-order equation by the standard substitution \( \varphi' = p(\varphi) \). Since \( \varphi'' = pp' \), we have

\[ \varphi pp' + 2p^2 = 2, \]

or

\[ \varphi \frac{dp^2}{d\varphi} + 4p^2 = 4. \]

Now we denote \( p^2 = v \), integrate the resulting linear equation

\[ \varphi \frac{dv}{d\varphi} + 4v = 4 \]

and obtain \( v = 1 + C\varphi^{-4} \). The equation \( p = \pm \sqrt{v} \) yields

\[ p = \pm \sqrt{1 + C\varphi^{-4}}. \]

Thus,

\[ \frac{d\varphi}{dx} = \pm \sqrt{1 + C\varphi^{-4}} \]

and the integration provides the general solution to Eq. (2.68) in the following implicit form involving one quadrature:

\[ \int \frac{\varphi^2 d\varphi}{\sqrt{\varphi^4 + C}} = C \pm x. \quad (2.69) \]
Substituting $\varphi(x)$ given by (2.69) into (2.67) we obtain the invariant solution to Eq. (2.61). For example, setting $C = 0$ we have from (2.69) $\varphi(x) = \pm x$, and the invariant solution (2.67) is

$$u = \pm \left( \frac{x}{t} - \epsilon \frac{x}{6} \right).$$

**Example 2.2.5.** Let us find the *approximate travelling waves* for the following perturbed Korteweg-de Vries equation:

$$u_t = uu_x + u_{xxx} + \epsilon u. \quad (2.70)$$

We will begin with the well-known solitary wave solution $u = U_0$,

$$U_0 = 3\kappa \operatorname{sech}^2 z, \quad (2.71)$$

of the Korteweg-de Vries equation

$$u_t = uu_x + u_{xxx}. \quad (2.72)$$

Here

$$\operatorname{sech} z = \frac{1}{\cosh z} = \frac{2}{e^z + e^{-z}}$$

and

$$z = \frac{\sqrt{\kappa}}{2} (x + \kappa t)$$

is an invariant of the group with the generator

$$X^0 = \frac{\partial}{\partial t} - \kappa \frac{\partial}{\partial x}, \quad \kappa = \text{const.} \quad (2.73)$$

The function $U_0$ given by Eq. (2.71) is invariant under the operator (2.73). Let us check that $u = U_0$ is an invariant solution, i.e., that it solves Eq. (2.72). Using the equations

$$(\operatorname{sech} z)' = -\operatorname{sech}^2 z \cdot \sinh z, \quad (\operatorname{sech}^2 z)' = -2\operatorname{sech}^3 z \cdot \sinh z,$$

where $\sinh z = \frac{e^z - e^{-z}}{2}$, we have

$$
(U_0)_t = -3\kappa^{5/2} \operatorname{sech}^3 z \cdot \sinh z, \\
(U_0)_x = -3\kappa^{3/2} \operatorname{sech}^3 z \cdot \sinh z, \\
(U_0)_{xx} = \frac{9}{2} \kappa^2 \operatorname{sech}^4 z \cdot \sinh^2 z - \frac{3}{2} \kappa^2 \operatorname{sech}^2 z, \\
(U_0)_{xxx} = 3\kappa^{5/2} (3\operatorname{sech}^2 z - 1) \operatorname{sech}^3 z \cdot \sinh z. \quad (2.74)
$$

In the last equation in (2.74) the following identity have been used:
2.2 Approximate symmetries

$$\sinh^3 z = (\cosh^2 z - 1) \sinh z.$$  

It is manifest from Eqs. (2.74) that (2.71) solves Eq. (2.72):

$$(U_0)_t = U_0(U_0)_x + (U_0)_{xxx}.$$  

Using the generators of the Galilean and scaling transformations,

$$X_1 = t \frac{\partial}{\partial x} - \frac{\partial}{\partial t}, \quad X_2 = 3t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - 2u \frac{\partial}{\partial u},$$

admitted by the Korteweg-de Vries equation (2.72) and invoking Remark 2.2.2, we consider the approximate symmetry $X_\varepsilon = X^0 + \varepsilon (\kappa X_1 - X_2)$ of the perturbed equation (2.70). Thus we will take the approximate symmetry

$$X_\varepsilon = \frac{\partial}{\partial t} - \kappa \frac{\partial}{\partial x} - \varepsilon \left[ 3t \frac{\partial}{\partial t} + (x - \kappa t) \frac{\partial}{\partial x} + (\kappa - 2u) \frac{\partial}{\partial u} \right]$$  \hspace{1cm} (2.75)

and use it for finding an approximately invariant solution by looking for the invariant perturbation of the solitary wave (2.71):

$$u = U_0 + \varepsilon v(t, x).$$  \hspace{1cm} (2.76)

Let us write the operator (2.75) in the form

$$X_\varepsilon = X^0 + \varepsilon X^1,$$

where $X^0$ is the operator (2.73) and $X^1$ is given by

$$X^1 = -\left[ 3t \frac{\partial}{\partial t} + (x - \kappa t) \frac{\partial}{\partial x} + (\kappa - 2u) \frac{\partial}{\partial u} \right].$$  \hspace{1cm} (2.77)

Then the invariant equation test for Eq. (2.76),

$$X_\varepsilon (u - U_0 - \varepsilon v) \big|_{(2.76)} = o(\varepsilon),$$

is written as

$$\left[ X^0 (u - U_0) + \varepsilon \left\{ X^1 (u - U_0) - X^0 (v) \right\} \right]_{u = U_0} = 0. \hspace{1cm} (2.78)$$

Note that $X^0 (u - U_0)$ vanishes identically since the operator $X^0$ does not contain the differentiation in $u$ and since the function $U_0$ is an invariant because it depends only on the invariant $z$. Therefore, Eq. (2.78) becomes

$$\left[ X^1 (u - U_0) - X^0 (v) \right]_{u = U_0} = 0,$$
whence, invoking definitions (2.73) and (2.77) of $X^0$ and $X^1$, respectively, we obtain the following differential equation for $v(t,x)$:

$$v_t - \kappa v_x = 3t(U_0)_t + (x - \kappa t)(U_0)_x + 2U_0 - \kappa. \tag{2.79}$$

Since $U_0 = U_0(z)$, it is easy to integrate Eq. (2.79) in the “natural” variables

$$z = \frac{\sqrt{\kappa}}{2}(x + \kappa t), \quad y = t.$$  

Then, denoting by $U'_0$ the derivative of $U_0$ with respect to $z$, we have

$$(U_0)_t = \frac{1}{2} \kappa^{3/2} U'_0, \quad (U_0)_x = \frac{1}{2} \kappa^{1/2} U'_0,$$

and Eq. (2.79) becomes

$$v_y = \left( z + \frac{1}{2} \kappa^{3/2} y \right) U'_0 + 2U_0 - \kappa.$$  

The integration yields

$$v = \left( zy + \frac{1}{4} \kappa^{3/2} y^2 \right) U'_0 + (2U_0 - \kappa)y + g(z).$$

Returning to the variables $t, x$, we have

$$v = \left( xt + \frac{3}{2} t^2 \right) (U_0)_x + (2U_0 - \kappa)t + h(x + \kappa t).$$

Inserting this $v$ in (2.76) and substituting in the perturbed KdV equation (2.70) we obtain $h = x + \kappa t$.

Thus, the approximate symmetry (2.75) provides the following the approximately invariant solution (approximate travelling wave):

$$u = U_0 + \varepsilon \left[ x + 2tU_0 + \left( tx + \frac{3}{2} t^2 \right)(U_0)_x \right], \tag{2.80}$$

or invoking Eqs. (2.71) and (2.74):

$$u = 3\kappa \text{sech}^2 z + \varepsilon \left[ x + 6\kappa t \text{sech}^2 z - 3\kappa^3 \left( tx + \frac{3}{2} \kappa t^2 \right) \text{sech}^3 z \cdot \sinh z \right].$$

### 2.2.8 Approximate conservation laws (first integrals)

Noether’s theorem can be generalized to the equations with a small parameter admitting the approximate transformation groups. For the sake of brevity, we restrict the discussion to the first-order Lagrangians $L(x, u, u(t), \varepsilon) \in \mathcal{A}$ and to the approx-
2.2 Approximate symmetries

imate point transformation groups. Then one can prove the following statement on
the approximate conservation laws [2] (see also [3], Sect. 2.6).

**Theorem 2.2.3.** Let the approximate Euler-Lagrange equations

\[
\frac{\delta L}{\delta u^\alpha} \equiv \frac{\partial L}{\partial u^\alpha} - D_i \left( \frac{\partial L}{\partial u_i^\alpha} \right) = o(\varepsilon)
\]

possess an approximate symmetry

\[
X = \left[ \xi^i_0(x,u) + \varepsilon \xi^i_1(x,u) \right] \frac{\partial}{\partial x^i} + \left[ \eta^\alpha_0(x,u) + \varepsilon \eta^\alpha_1(x,u) \right] \frac{\partial}{\partial u^\alpha},
\]

such that the following holds:

\[
XL + LD_i(\xi^i) = D_i(B^i) + o(\varepsilon), \quad B^i \in \mathcal{A}.
\]

Then the differential functions \(C^i(x,u,u^{(1)},\varepsilon)\) defined by

\[
C^i = L \xi^i + \left( \eta^\alpha - \xi^i u_i^\alpha \right) \frac{\partial L}{\partial u_i^\alpha} - B^i + o(\varepsilon)
\]

satisfy the approximate conservation law for Eq. (2.81), i.e.,

\[
D_i(C^i)\big|_{(2.81)} = o(\varepsilon).
\]

The following notation is used in Eq. (2.82)–(2.83):

\[
\xi^i = \xi^i_0(x,u) + \varepsilon \xi^i_1(x,u), \quad \eta^\alpha = \eta^\alpha_0(x,u) + \varepsilon \eta^\alpha_1(x,u).
\]

An obvious generalization of Theorem 2.2.3 provides approximate conservation
laws for Euler-Lagrange equations with higher-order Lagrangians possessing ap­
proximate point, contact or Lie-Bäcklund symmetries.

**Example 2.2.6.** Consider again equation (2.44),

\[y'' - x - \varepsilon y^2 = 0.\]

Let us take its Lagrangian in the form

\[L = \frac{1}{2} \left( y'' - x y' \right) + \frac{\varepsilon}{3} y^3 \]

and consider an approximate symmetry, e.g. the operator \(X_1\) from (2.45). Then the
condition (2.82) holds. Namely, taking the first prolongation of the operator \(X_1\),

\[
X_1 = \frac{2}{3} \varepsilon x^3 \frac{\partial}{\partial x} + \left[ 1 + \varepsilon \left( y^2 + \frac{11}{60} x^5 \right) \right] \frac{\partial}{\partial y} + \varepsilon \left[ 2xy - x^2 y' + \frac{11}{12} x^4 \right] \frac{\partial}{\partial y'},
\]
we have, ignoring the higher-order terms in $\varepsilon$:

$$X_1L + LD_x(\xi) = \varepsilon \left( 2x y y' + y^2 - x y + \frac{1}{4} x^4 y' - \frac{11}{24} x^6 \right)$$

$$= \varepsilon D_x \left( x y^2 - \frac{1}{4} x^4 y - \frac{11}{24 \cdot 7} x^7 \right).$$

Hence, the condition (2.82) holds with

$$B = \varepsilon D_x \left( x y^2 - \frac{1}{4} x^4 y - \frac{11}{24 \cdot 7} x^7 \right).$$

Furthermore, the reckoning shows that

$$L \xi + (\eta - \xi y') \frac{\partial L}{\partial y'} = y' - \frac{x^2}{2} + \varepsilon \left( x^2 y y' - \frac{1}{3} x^3 y^2 + \frac{11}{60} x^5 y' - \frac{1}{2} x^4 y - \frac{11}{60 \cdot 2} x^7 \right).$$

Thus, the conservation formula (2.83) provides the following approximate first integral for equation (2.44):

$$y' - \frac{x^2}{2} + \varepsilon \left( x^2 y y' - \frac{1}{3} x^3 y^2 - x y^2 - \frac{1}{4} x^4 y + \frac{11}{60} x^5 y' - \frac{11}{35 \cdot 12} x^7 \right) = C. \quad (2.84)$$

In other words, we have reduced the second-order equation $y'' - x - \varepsilon y^2 = 0$ to the first-order equation (2.84) with an arbitrary constant $C$.

References


Chapter 3
Symmetries of Integro-Differential Equations

The major obstacle for the application of the Lie’s infinitesimal techniques to the integro-differential equations or infinite systems of differential equations is that the frames of these equations are not locally defined in the space of differential functions. In consequence, the crucial idea of the splitting determining equations into the over-determined systems, commonly used in the classical Lie group analysis, fails.

There are several known approaches used for calculating the symmetries of the integro-differential equations: the method of moments [13], the method of boundary-differential equations [1], and the direct method [2, 3]. We use in what follows the modification of the direct method given in [7, 8] (see also the review paper [5] and the references therein).

3.1 Definition and infinitesimal test

To extend the classical Lie algorithm to the integro-differential equations it appears necessary to resolve several problems. Firstly, one should define the local one-parameter transformation group $G$ for the nonlocal (integro-differential) equations and formulate the invariance criteria that lead to the determining equations, which appear also nonlocal. Secondly, a procedure of solving the nonlocal determining equations should be described.

3.1.1 Definition of symmetry group

Let an integro-differential equation under consideration be expressed as a zero equality for some functional (here we indicate only one argument for a function $f$), defined for $x_1 \leq x \leq x_2$,

$$F[f(x)] = 0,$$

(3.1)
and let $G$ be a local one-parameter group that transforms $f$ to $\tilde{f}(x,a)$,
\begin{equation}
\tilde{f}(x,a) = f + ax + o(a), \quad x = x. \tag{3.2}
\end{equation}

Here we use the canonical group representation hence independent variables $x$ do not vary. Then the local group $G$ of the point transformations (3.2) is called a symmetry group of the integro-differential equations (3.1) if and only if for any $a$ the function $F$ does not vary [3],
\begin{equation}
F[\tilde{f}(x,a)] = 0. \tag{3.3}
\end{equation}

The differentiating (3.3) with respect to the group parameter $a$ and assuming $a \to 0$ gives the invariance criterion in the infinitesimal form akin to (1.28). In view of the canonical form of the transformations (3.2) the functional $F$ depends upon $a$ via $\tilde{f}$. Therefore to find the infinitesimal invariance criterion we should calculate the derivative $dF/da$.

### 3.1.2 Variational derivative for functionals

Let $f(x,a)$ be a differentiable function with respect to $a$, $f(x,a)$ and $\partial f(x,a)/\partial a$ continuous functions for $a \geq 0$, $x_1 \leq x \leq x_2$. The derivative $dF/da$ [15],
\begin{equation}
\frac{d}{da} F[f(x,a)] = \delta F[f(x,a); f'_a(x,a)], \tag{3.4}
\end{equation}

is given by variation of the functional $\delta F$, defined as a linear in $\delta f$ part of a difference
\begin{equation}
\delta F = F[f + \delta f] - F[f].
\end{equation}

Let $F[f(x,a)]$ be a differentiable functional (recall that the functional $F$, defined on the interval $[x_1,x_2]$, is called a differentiable functional [15] if it has the first derivative in each point of this interval). Then the last formula is rewritten in the following form:
\begin{equation}
\delta F = \int_{x_1}^{x_2} F'[f(x); q] \delta f(q) dq. \tag{3.5}
\end{equation}

Here the variational derivative $F'[f(x); y] = \delta F/\delta f(y)$ of the differentiable functional $F$ with respect to a function $f$ in the point $y$ is defined via the principal (linear) part of an increment of the functional as a limit (if it exists) (see [15]):
\begin{equation}
\frac{\delta F[f]}{\delta f(y)} = \lim_{\varepsilon \to 0} \frac{F[f + \delta f_\varepsilon] - F[f]}{\int_\Delta dy \delta f_\varepsilon(y)}; \quad y \in [x_1, x_2]. \tag{3.6}
\end{equation}

In (3.6) the infinitesimal variation $\delta f_\varepsilon(y) \geq 0$ is a continuously differentiable function given on fixed interval $\Delta = [x_1, x_2]$ which differs from zero only in $\varepsilon$-vicinity of a point $y$, and the norm $\|\delta f_\varepsilon\|_{C^1} \to 0$ at $\varepsilon \to 0$.

**Example 3.1.1.** Let $b(y)$ be a continuous function and $F[f]$ a linear functional
3.1 Definition and infinitesimal test

$$F[f] = \int_{x_1}^{x_2} b(y)f(y)\,dy.$$ 

By $\delta \epsilon f$ denote a variation that differs from zero only in $\epsilon$-vicinity of a point $q$. Then using the mean value theorem

$$F[f + \delta \epsilon f] - F[f] = \int_{x_1}^{x_2} b(y)\delta \epsilon f\,dy = b(q)\int_{x_1}^{x_2} \delta \epsilon f\,dy,$$

we get the variation derivative

$$\frac{\delta F[f]}{\delta f(q)} = \lim_{\epsilon \to 0} \frac{b(q)\int_{\Delta} \delta \epsilon f(y)\,dy}{\int_{\Delta} \delta \epsilon f(y)\,dy} = b(q). \quad (3.7)$$

Choosing $b(y) = 1/(\sqrt{2\pi}\sigma)\exp(-(y-y_0)^2/2\sigma^2)$ we obtain

$$\frac{\delta F[f]}{\delta f(q)} = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y-y_0)^2}{2\sigma^2}\right). \quad (3.8)$$

In the limit $\sigma \to 0$ we have $b(y) \to \delta(y-y_0)$, $F[f] \to f(y_0)$ and hence

$$\frac{\delta f(y_0)}{\delta f(q)} = \delta(y_0 - q). \quad (3.9)$$

3.1.3 Infinitesimal criterion

According to Sect. 3.1.1 to write the infinitesimal criterion for the symmetry group for nonlocal equations one should differentiate (3.3) with respect to the group parameter $a$ and assume $a \to 0$, i.e., calculate the limit of the derivative $dF/da$ for vanishing $a$. Combining (3.4), (3.5) and assuming $a \to 0$ in view of (3.2) we get

$$\left.\frac{dF[F]}{da}\right|_{a=0} = \int \kappa(y) \frac{\delta F[f(x)]}{\delta f(y)}\,dy \equiv YF, \quad (3.10)$$

where we have introduced the generator $Y$ defined by its action on function $F$ as follows:

$$Y(F) = \int \kappa(y) \frac{\delta F}{\delta f(y)}\,dy.$$

We will write this operator formally in the form

$$Y = \int \kappa(y) \frac{\delta}{\delta f(y)}\,dy. \quad (3.11)$$

Hence, the invariance criterion for $F$ with respect to the admitted group can be expressed in an infinitesimal form using the canonical group operator $Y$,
which generalizes the action of a standard canonical group operator \((1.126)\) not only on differential functions but on functionals as well using variational differentiation in the definition of \(Y\) \([7]\). One can verify by direct calculation that the action of \(Y\) on any differential function and its derivatives, e.g., \(f\) and \(f_x, \ldots\) produces the usual result: \(Yf = \varkappa, Yf_x = D_x(\varkappa)\) and so on. Hence, if \(F\) describe usual differential equations then formulas (3.12) lead to standard local determining equations, while for \(F\) having the form of integro-differential equations formulas (3.12) can be treated as nonlocal determining equations as they depend both on local and nonlocal variables.

**Example 3.1.2.** Let \(f(x)\) be a differential function, then \(Yf = \varkappa.\) In view of (3.9) the proof is by direct calculation:

\[
Yf = \int \varkappa(y) \frac{\delta f(x)}{\delta f(y)} \, dy = \int \varkappa(y) \delta(x - y) \, dy = \varkappa(x).
\]

In order to find solutions of determining equations (3.12) one can use different approaches, e.g. expanding coordinates of group generator into formal power series and equating coefficients of various powers \([2, 3]\). However there exists a more traditional way. As we treat local and nonlocal variables in determining equations as independent it is possible to separate these equations into local and nonlocal. The procedure of solving local determining equations is fulfilled in a standard way using Lie algorithm based on splitting the system of over-determined equations with respect to local variables and their derivatives. As a result we get expressions for coordinates of group generator that define the so-called group of intermediate symmetry \([7]\). In the similar manner the solution of nonlocal determining equations is fulfilled using the information borrowed from an intermediate symmetry and by “variational” splitting of nonlocal determining equations using the procedure of variational differentiation. Therefore, the algorithm of finding symmetries of nonlocal equations appears as an algorithmic procedure that consists of a sequence of several steps: a) defining the set of local group variables, b) constructing determining equations on basis of the infinitesimal criterion of invariance, that employs the generalization of the definition of the canonical operator, c) separating determining equations into local and nonlocal, d) solving local determining equations using a standard Lie algorithm, e) solving nonlocal determining equations using the procedure of variational differentiation.

### 3.1.4 Prolongation on nonlocal variables

To complete we describe the procedure of prolongation of a symmetry group on nonlocal variables, say in the form of the integral relation

\[
YF \bigg|_{F=0} = 0,
\]
3.2 Calculation of symmetries illustrated by Vlasov-Maxwell equations

\[ J(u) = \int \mathcal{F}(u(z)) \, dz. \]  \hfill (3.13)

To fulfill this procedure one should first rewrite the operator, say \( Y \), in a canonical form and then formally prolong this operator on the nonlocal variable \( J \),

\[ Y + \varepsilon' \partial_j \equiv \varepsilon \partial_u + \varepsilon' \partial_j. \]  \hfill (3.14)

The integral relation between \( x \) and \( \varepsilon' \) is obtained by applying the generator (3.14) to the definition of \( J \), i.e., to (3.13). Substituting the explicit expression for the coordinate \( x \) of the known operator \( Y \) and calculating integrals obtained gives the desired coordinate \( \varepsilon' \),

\[ \varepsilon' = \int \frac{\delta J(u)}{\delta u(z)} \varepsilon(z) \, dz = \int \frac{\delta \mathcal{F}(u(z'))}{\delta u(z)} \varepsilon(z) \, dz \, dz' = \int \mathcal{F}_u \varepsilon(z) \, dz. \]  \hfill (3.15)

### 3.2 Calculation of symmetries illustrated by Vlasov-Maxwell equations

In this section we present several examples on calculation of the symmetry group for the system of integro-differential equations. In view of the possible applications we choose the system of integro-differential equations, which define the evolution of a collisionless plasma. The macroscopic state of the plasma particles with charge \( e_\alpha \) and mass \( m_\alpha \) is governed by a distribution function \( f^\alpha \), which depend on time \( t \), a radius-vector of particles in coordinate space, \( r \), and the particle velocity \( \mathbf{v} \).

Evolution of the distribution function is described by the Vlasov kinetic equation with a self-consistent electromagnetic field,

\[ f^\alpha_t + \mathbf{v}_t f^\alpha + \frac{e_\alpha}{\gamma m_\alpha} \left\{ \mathbf{E} + \frac{1}{c} \left[ \mathbf{v} \times \mathbf{B} \right] - \mathbf{v} \frac{(\mathbf{v} \cdot \mathbf{E})}{c^2} \right\} f^\alpha = 0, \]  \hfill (3.16)

where the electric and magnetic field vectors \( \mathbf{E} \) and \( \mathbf{B} \) obey the Maxwell’s equations

\[ \mathbf{B}_t + c \, \text{rot} \, \mathbf{E} = 0; \quad \text{div} \, \mathbf{E} = 4\pi \rho; \]
\[ \mathbf{E}_t - c \, \text{rot} \, \mathbf{B} + 4\pi \mathbf{j} = 0; \quad \text{div} \, \mathbf{B} = 0, \]  \hfill (3.17)

and \( \gamma = \sqrt{1 - v^2/c^2} \) is the relativistic factor. Here the charge and current densities, \( \rho \) and \( \mathbf{j} \), are in turn governed by the motion of particles,

\[ \rho = \sum_\alpha e_\alpha m_\alpha^3 \int f^\alpha \gamma^5 \, d\mathbf{v}, \quad j = \sum_\alpha e_\alpha m_\alpha^3 \int f^\alpha \gamma^5 \mathbf{v} \, d\mathbf{v}, \]  \hfill (3.18)

where summation is taken over all plasma particle species, and define electric and magnetic fields in plasma in a self-consistent manner. The system of equations (3.16)–(3.18) is referred to as the Vlasov-Maxwell equations. At present the Vlasov
kinetic equation with a self-consistent field is a basic equation in the theory of a collisionless plasma.

3.2.1 One-dimensional electron gas

Here we describe in details the application of the general algorithm to the most simple one-dimensional non-relativistic model of the one-component charged electron plasma that arises from Eqs. (3.16)–(3.18) while treating only one particle species (electrons) and in one-dimensional plane geometry. We also neglect the relativistic effects here. We consider this model since it is physically simple and informative from the group standpoint. The model has the same characteristic features as the complete three-dimensional system of kinetic equations for the collisionless relativistic electron-ion plasma. The only difference is a smaller amount of calculations necessary for constructing and solving the group determining equations.

In case of the non-relativistic motion of electrons in the self-consistent electric field \( E \), the one-dimensional Vlasov kinetic equation for the distribution function \( f \) is written as follows:

\[
f_t + v f_x + \frac{e}{m} E f_v = 0. \tag{3.19}
\]

Here the potential field \( E \) obeys the Poisson equation and the corresponding Maxwell equation:

\[
E_x = 4\pi \rho, \quad E_t + 4\pi j = 0, \tag{3.20}
\]

and charge density \( \rho \) and current density \( j \) are expressed as the integrals

\[
\rho = e m \int f \, dv, \quad j = e m \int f v \, dv \tag{3.21}
\]

over electron velocities. Momentarily, we will assume that the charge \( e \) and mass \( m \) of the electron (parameters of the system) are invariants. The dependent variables \( E, j, \) and \( \rho \) are functions of two arguments, time \( t \) and coordinate \( x \),

\[
E = E(t,x), \quad j = j(t,x), \quad \rho = \rho(t,x), \tag{3.22}
\]

and the distribution function

\[
f = f(t,x,v) \tag{3.23}
\]

has three arguments, \( t, x \) and electron velocity \( v \). It follows from (3.22) that electric field intensity \( E \), current density \( j \), and charge density \( \rho \) are independent of electron velocity \( v \). Hence we have three additional differential constraints

\[
E_v = 0, \quad j_v = 0, \quad \rho_v = 0, \tag{3.24}
\]

which should be used in group analysis of the system (3.19)–(3.21) as well as compatibility condition for Eqs. (3.20), known as continuity equation,
3.2 Calculation of symmetries illustrated by Vlasov-Maxwell equations

Thus, we will investigate the symmetries of the system of integro-differential equations (3.19)–(3.21) and (3.24):

\[ f_t + v f_x + \frac{e}{m} E f_v = 0, \]
\[ E_x = 4\pi \rho, \quad E_t + 4\pi j = 0, \]
\[ \rho = em \int f \, dv, \quad j = em \int f v \, dv, \]
\[ E_v = 0, \quad j_v = 0, \quad \rho_v = 0. \]

The coordinates \( \xi \) and \( \eta \) of the Lie point symmetry group generator,

\[ X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial v} + \eta^1 \frac{\partial}{\partial f} + \eta^2 \frac{\partial}{\partial E} + \eta^3 \frac{\partial}{\partial j} + \eta^4 \frac{\partial}{\partial \rho}, \]

are considered as functions of the seven group variables

\[ t, x, v, f, E, j, \rho. \]

These coordinates are solutions to the determining equations, which, in turn, appear as necessary and sufficient conditions for the invariance of system (3.19)–(3.21), (3.24) with respect to the group with generator (3.27). Local (differential) determining equations can be stated and solved directly in terms of the generator (3.27). In this section we however use the canonical form

\[ Y = \chi^1 \frac{\partial}{\partial f} + \chi^2 \frac{\partial}{\partial E} + \chi^3 \frac{\partial}{\partial j} + \chi^4 \frac{\partial}{\partial \rho} \]

for generator (3.27), which offers substantial advantages in the group analysis of the Vlasov-Maxwell equations because of non-locality of these equations.

### 3.2.1.1 Solution to the local determining equations

The invariance conditions for the Vlasov kinetic equation (3.19), the field equations (3.20), and (3.24) with respect to the group with canonical generator (3.29) are given by the six local determining equations:

\[ D_t(\chi^1) + v D_x(\chi^1) + \frac{e}{m} E D_v(\chi^1) + \frac{e}{m} \chi^2 f_v = 0, \]
\[ D_x(\chi^2) = 4\pi \chi^4, \quad D_t(\chi^2) = -4\pi \chi^3, \]
\[ D_v(\chi^2) = 0, \quad D_v(\chi^3) = 0, \quad D_v(\chi^4) = 0, \]

which should be solved with taking into account the fact that the group variables (3.28) and the corresponding derivatives are related by manifold (3.19)–(3.21),
Here we use the standard notations for the operator of total differentiation $D_i$ with respect to the group variable indicated by the subscript. For example, the operator $D_v$ of total differentiation with respect to $v$ is given by

$$D_v = \frac{\partial}{\partial v} + f_v \frac{\partial}{\partial f} + f_{vv} \frac{\partial}{\partial f_v} + f_{vvv} \frac{\partial}{\partial f_{vv}} + \ldots$$

The solution of the system of local determining equations (3.30) is given by the following formulas for the coordinates $\xi$ of the generator (3.29):

$$\xi^1 = \eta^1 - f_v \xi^1 - f_x \left[ x \left( A_4 + \frac{1}{2} \xi^1 \right) + \beta \right]$$

$$\xi^2 = E \left[ A_4 - \frac{3}{2} \xi^1 \right] + \frac{m}{e} \left[ \frac{1}{2} x \xi_{tt} + \beta_t \right]$$

$$\xi^3 = j \left[ A_4 - \frac{5}{2} \xi^1 \right] + \rho \left[ \frac{1}{2} x \xi_{tt} + \beta_t \right] + \frac{3}{8\pi} E \xi_{tt}$$

$$\xi^4 = -2\rho \xi^1 + \frac{m}{8\pi e} \xi_{tt} - \rho_t \xi^1 - \rho_x \left[ x \left( A_4 + \frac{1}{2} \xi^1 \right) + \beta \right].$$

The coordinates (3.33) depend on an arbitrary constant $A_4$ and on three arbitrary functions, namely $\xi^1(t)$, $\beta(t)$ and $\eta^1(f)$. The group symmetry that characterizes the system of Eqs. (3.19), (3.20), (3.24) with the help of the generator (3.29) with coordinates (3.33) will be referred to as the intermediate group symmetry of the complete system of Eqs. (3.19)–(3.21). The symmetry is generated only by the differential equations in the integro-differential Vlasov-Maxwell system and does not take into account integral terms in the material equations (3.21), which determine charge and current densities of electrons. The intermediate group symmetry (3.29), (3.33) plays an auxiliary role in obtaining the final equations for the coordinates $\xi$ and $\eta$ of the generator (3.27) of the desired Lie group. The charge density $\rho$ and the current density $j$ have a concrete physical meaning. By introducing them as independent group variables in the set (3.28) along with $t, x, v, f,$ and $E$, we divide the group analysis of the local and the nonlocal part of the Vlasov-Maxwell equations into two stages. The intermediate symmetry (3.29), (3.33) completes the local group analysis of the system. In what follows we shall see that the nonlocal determining equations appearing as invariance conditions for the material equations (3.21) with
3.2 Calculation of symmetries illustrated by Vlasov-Maxwell equations

respect to the sought Lie group eliminate the arbitrary dependence of \( \xi, \beta, \eta^1 \) on \( t \) and \( f \).

3.2.1.2 Solution to the nonlocal determining equations

Since the material equations (3.21) are nonlocal (they involve the integration of the distribution function \( f \) and of the product \( vf \) over the electron velocity \( v \)), the differentiation with respect to \( f \) in the first term of the canonical generator (3.29) should be generalized so as to act not only on functions of \( f \) but also on linear functionals (3.21) of \( f \). Hence, we represent this term as the integral of variational derivative with respect to \( f \) with weight \( \propto^1 \) over electron velocity \( v \):

\[
\propto^1 \frac{\partial}{\partial f} \equiv \int \propto^1(v) \frac{\delta}{\delta f(v)} \, dv.
\]  

For brevity, we indicate only the integration variable \( v \) as an argument of \( f \) and of \( \propto^1 \). Our shorthand notation implies that the coordinate \( \propto^1(v) \) of the canonical generator in (3.33) stands for the following extended expression in (3.33), depending on integration variable \( v \):

\[
\propto^1(v) \equiv \eta^1(f(t,x,v)) - \xi^1 f_i(t,x,v) - \left[ x \left( A_4 + \frac{1}{2} \xi^1_{ir} \right) + \beta_i \right] f_i(t,x,v).
\]  

When applied to functions of \( f \), the operator of the differentiation with respect to \( f \) in (3.33) gives the usual result, i.e., it coincides with the ordinary differentiation with respect to \( f \). When applied to linear functionals of \( f \), i.e., to the charge and current densities (3.21), the derivative in (3.33) permits us to introduce the variational derivative on the right-hand side in (3.33) under the integral over \( v \) together with the coordinate \( \propto^1 \) of the canonical generator (3.29).

Substituting (3.33) into (3.29) and using the well-known identity (3.9),

\[
\frac{\delta f(v)}{\delta f(v')} = \delta (v - v'),
\]  

where \( \delta \) is the Dirac delta-function, we obtain the invariance conditions for the integral material Eq. (3.21) with respect to the Lie group with canonical generator (3.29), the nonlocal determining equations [7, 8]

\[
\propto^1 - em \int \propto^1 \, dv = 0, \quad \propto^2 - em \int v \propto^1 \, dv = 0.
\]  

The integration in (3.36) is over all values of \( v \), just as in (3.21) and (3.33). Let us consider the first of the two determining equations in (3.36) in more detail. Sub-
Substituting the coordinates \( x^1 \) and \( x^4 \) from (3.33) into the determining equation in question and taking into account Eq. (3.21) for charge density \( \rho \), we reduce the determining equation to the simple form

\[
em \int \left[ \eta_1(f(v)) + f(v)K(t) \right] dv - \frac{m}{8\pi\varepsilon_0} \xi_{ttt}^1(t) = 0. \quad (3.37)
\]

The coefficient \( K \) in the product \( Kf \) in the integrand on the left-hand side in (3.37) is independent of \( v \) and \( f \); specifically, we have

\[
K(t) = A_4 + \frac{3}{2} \varepsilon_{tr}^1. \quad (3.38)
\]

The derivation of the determining equation (3.37) involves integrating by parts with respect to \( v \), which removes the derivative \( f_v \) from the integrand in the nonlocal term in the original determining equation. The resultant antiderivative \( f \) is assumed to vanish at the ends of the infinite integration interval, that is,

\[
f \to 0, \quad v \to \pm\infty. \quad (3.39)
\]

The determining equation (3.37) is a linear nonhomogeneous integral equation for \( \eta_1 \), which can easily be solved. According to the general ideas of Lie technique, Eq. (3.37), as well as any determining equation, is an identity with respect to the group variable \( f \). Therefore, it remains valid after differentiating with respect to \( f \). Since the determining equation (3.37) is an integral equation, we should use variational differentiation with respect to \( f \) rather than ordinary differentiation. Taking into account that nonhomogeneous term proportional to \( \xi_{ttt}^1 \) in (3.37) is independent of \( f \), we obtain

\[
\frac{\delta}{\delta f(v')} \int \left[ \eta_1(f(v)) + f(v)K(t) \right] dv = 0. \quad (3.40)
\]

The nonlocal equation (3.37), which is an identity with respect to \( f \), should be combined with its differential corollary (3.40) in the sense that a solution to (3.40) is also a solution to (3.37). Introducing the variational derivative in (3.40) under the integral over \( v \),

\[
\int \left\{ \eta_1 + K \right\} \frac{\delta f(v)}{\delta f(v')} dv = 0, \quad (3.41)
\]

and evaluating the integral over \( v \) with the aid of the delta-function (3.35) that appears in the integrand, as a consequence of (3.37), we obtain a first-order ordinary differential equation for the dependence of the coordinate \( \eta_1 \) of the determining equation (3.27) on \( f \):

\[
\eta_1 + K = 0. \quad (3.42)
\]

Its solution depends on one arbitrary constant

\[
\eta_1 = -Kf + A. \quad (3.43)
\]

Since the coordinate \( \eta_1 \) is independent of \( t \), we immediately obtain the condition
3.2 Calculation of symmetries illustrated by Vlasov-Maxwell equations

\( K_t = 0 \)  

(3.44)

imposed on the coefficient \( K \) of the determining equations (3.37). It follows from (3.38) and (3.44) that

\[ \xi^1_{tt} = 0. \]  

(3.45)

As was mentioned above, it is necessary to consider Eq. (3.43) for \( \eta^1 \), appearing as a direct consequence of variational differentiation of the determining equations (3.37) with respect to the distribution function \( f \), together with Eq. (3.37):

\[ em \int_{-\infty}^{+\infty} A \, dv - \frac{m}{8 \pi e} \xi^1_{tt}(t) = 0. \]  

(3.46)

In view of (3.45), the second term on the left-hand side in (3.46) is zero. Therefore, Eq. (3.46) is reduced to

\[ Aem \int_{-\infty}^{+\infty} dv = 0, \]  

(3.47)

whence follows that the integration constant \( A \) in (3.43) is zero, that is, we have

\[ \eta^1 = -Kf. \]  

(3.48)

The integration of Eq. (3.45) yields

\[ \xi^1(t) = A_1 + 2A_3 t. \]  

(3.49)

We insert the explicit formula (3.49) for the dependence of \( \xi^1 \) on \( t \) into expression (3.38) for the coefficient \( K \) and obtain

\[ K = 3A_3 + A_4, \]  

(3.50)

whence follows the definite expression for the coordinate

\[ \eta^1(f) = -(3A_3 + A_4) f. \]  

(3.51)

Equations (3.49) and (3.51) are the basic result of solving the first nonlocal determining equation in (3.36) and define explicit dependence of \( \xi \) and \( \eta \) on \( t \) and \( f \) in the intermediate group symmetry (3.33). The second nonlocal determining equation in system (3.36) pertains to the invariance of electron current density with respect to the admitted Lie group. By substituting the extended expressions for the coordinates \( \varphi^3 \) and \( \varphi^1 \) from (3.33) into this determining equation, we easily reduce it to the following linear nonhomogeneous integral equation for \( \eta^1 \), which is similar to (3.37):

\[ em \int v (\eta^1 + fK) \, dv + \frac{3}{8 \pi} E \xi^1 - \frac{m x}{8 \pi e} \xi^1_{ttt} - \frac{m}{4 \pi e} \beta_{ttt} = 0. \]  

(3.52)

The passage from (3.36) to (3.52) involves integration by parts with respect to \( v \). Here we take into account conditions (3.39), which state that the electron distribution function \( f \) decays rapidly for large velocities. The coefficient \( K \) in the product...
\( K \psi f \) in the integrand on the left-hand side in (3.52) has the same form (3.38) as in (3.37). Hence, taking into account (3.45) and (3.48), we see that the determining equation (3.52) is reduced to \( \beta_{tt} = 0 \), which implies

\[
\beta(t) = A_2 + A_5 t + \frac{1}{2} A_6 t^2. \tag{3.53}
\]

Substitution (3.49), (3.51) and (3.53) into (3.33) yields canonical coordinates of the group generator (3.27):

\[
\begin{align*}
\mathcal{X}_1 &= -A_1 f_t - A_2 f_x - A_3 (3f + 2 tf_t + xf_x - v f_v) \\
&\quad - A_4 (f + xf_x + v f_v) - A_5 (t f_x + f_v) - A_6 \left( \frac{t^2}{2} f_x + tf_v \right), \\
\mathcal{X}_2 &= -A_1 E_t - A_2 E_x - A_3 (3E + 2 t E_t + x E_x) - A_4 (-E + x E_x) \\
&\quad - A_5 t E_x - A_6 \left( \frac{t^2}{2} E_x - \frac{m}{e} \right), \\
\mathcal{X}_3 &= -A_1 j_t - A_2 j_x - A_3 (5j + 2 t j_t + x j_x) - A_4 (-j + x j_x) \\
&\quad - A_5 (t j_x - \rho) - A_6 \left( \frac{t^2}{2} j_x - \frac{e}{m} \right), \\
\mathcal{X}_4 &= -A_1 \rho_t - A_2 \rho_x - A_3 (4 \rho + 2 t \rho_t + x \rho_x) \\
&\quad - A_4 x \rho_x - A_5 t \rho_x - A_6 \left( \frac{t^2}{2} \right) \rho_x.
\end{align*}
\tag{3.54}
\]

Formulas (3.54) refer to the following six basic generators, written in a non-canonical form:

\[
\begin{align*}
X_1 &= \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \\
X_3 &= 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - v \frac{\partial}{\partial v} - 3f \frac{\partial}{\partial f} - 3E \frac{\partial}{\partial E} - 5j \frac{\partial}{\partial j} - 4 \rho \frac{\partial}{\partial \rho}, \\
X_4 &= x \frac{\partial}{\partial x} + v \frac{\partial}{\partial v} - f \frac{\partial}{\partial f} + E \frac{\partial}{\partial E} + j \frac{\partial}{\partial j}, \\
X_5 &= t \frac{\partial}{\partial x} + \frac{\partial}{\partial v} + \rho \frac{\partial}{\partial j}, \\
X_6 &= \frac{t^2}{2} \frac{\partial}{\partial x} + t \frac{\partial}{\partial v} + \frac{m}{e} \frac{\partial}{\partial E} + \rho \frac{\partial}{\partial j}.
\end{align*}
\tag{3.55}
\]

The set of generators (3.55) span the six-dimensional Lie algebra

\[
L_6 = \langle X_1, X_2, \ldots, X_6 \rangle. \tag{3.56}
\]

Generators (3.55) of the six-parameter symmetry group, admitted by the Vlasov-Maxwell equations (3.19)–(3.21), have clear physical meaning: the operators \( X_1 \)
3.2 Calculation of symmetries illustrated by Vlasov-Maxwell equations

and $X_2$ describe translations along $t$ and $x$-axes, the generator $X_3$ and $X_4$ determine dilations, which can be easily verified, and the generator $X_5$ define the Galilean transformations. The finite transformations corresponding to the generator $X_6$ have the following form for each of six variables (3.28):

$$\begin{align*}
\bar{t} &= t; \quad \bar{x} = x + \frac{at^2}{2}; \quad \bar{v} = v + at; \quad \bar{j} = j; \\
\bar{E} &= E + \frac{ma}{e}; \quad \bar{j} = j + at\rho; \quad \bar{\rho} = \rho.
\end{align*}$$

(3.57)

In mechanics, the one-parameter transformation group with the generator $X_6$ can be interpreted for the first three equations in (3.57) as the transformation of variables due to passing into a coordinate system moving linearly with constant acceleration $a$ with respect to the laboratory frame.

3.2.2 Three-dimensional plasma kinetic equations

To calculate the symmetry group for the system of equations (3.16)–(3.18), we start from the infinitesimal operator of the admitted local group of the point one-parameter transformations in a standard form

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial r} + \xi^3 \frac{\partial}{\partial v} + \sum_\alpha \eta^1_\alpha \frac{\partial}{\partial f^\alpha}$$

$$+ \eta^2 \frac{\partial}{\partial E} + \eta^3 \frac{\partial}{\partial B} + \eta^4 \frac{\partial}{\partial j} + \eta^5 \frac{\partial}{\partial \rho},$$

(3.58)

where coordinates $\xi^i$ and $\eta^k$ depend upon $t, r, v, f^\alpha, E, B, j$ and $\rho$. In the canonical form this operator is given as

$$Y = \sum_\alpha \lambda^\alpha \frac{\partial}{\partial f^\alpha} + \lambda^2 \frac{\partial}{\partial E} + \lambda^3 \frac{\partial}{\partial B} + \lambda^4 \frac{\partial}{\partial j} + \lambda^5 \frac{\partial}{\partial \rho},$$

(3.59)

$$\lambda^{1\alpha} = \eta^{1\alpha} - \xi^1 f^{\alpha}_r - \xi^2 f^\alpha_r - \xi^3 f^\alpha_v,$$

$$\lambda^2 = \eta^2 - \xi ^1 E_v - \left(\xi^2 \cdot \nabla_r\right) E - \left(\xi^3 \cdot \nabla_v\right) E,$$

$$\lambda^3 = \eta^3 - \xi ^1 B_v - \left(\xi^2 \cdot \nabla_r\right) B - \left(\xi^3 \cdot \nabla_v\right) B,$$

$$\lambda^4 = \eta^4 - \xi ^1 j_v - \left(\xi^2 \cdot \nabla_r\right) j - \left(\xi^3 \cdot \nabla_v\right) j,$$

$$\lambda^5 = \eta^5 - \xi ^1 \rho_v - \xi^2 \rho_r - \xi^3 \rho_v,$$

and its action on any function or functional should be understood in generalized sense as in (3.12). Applying the canonical group operator to the joint system of basic equations (3.16)–(3.17), supplemented by additional differential constraints
that are obvious from the physical point of view, gives the system of determining equations:

\[
\begin{align*}
D_t (\mathbf{x}^1) &+ \nu D_r (\mathbf{x}^1) + \frac{e\alpha}{m_\alpha} \left\{ E + \frac{1}{c} [\mathbf{v} \times \mathbf{B}] - \mathbf{v} \frac{(\mathbf{v} \cdot E)}{c^2} \right\} \\
\times D_v (\mathbf{x}^1) &+ \frac{e\alpha}{m_\alpha} \left\{ \mathbf{x}^2 + \frac{1}{c} [\mathbf{v} \times \mathbf{x}^3] - \frac{\mathbf{v}}{c^2} (\mathbf{v} \cdot \mathbf{x}^2) \right\} f^\alpha_v = 0;
\end{align*}
\]

\[
\begin{align*}
c \left[ D_r \times \mathbf{x}^2 \right] + D_t (\mathbf{x}^3) &= 0; \\
c \left[ D_r \times \mathbf{x}^3 \right] - D_t (\mathbf{x}^2) &= 4\pi \mathbf{x}^1; \\
(D_r \cdot \mathbf{x}^2) &= 4\pi \mathbf{x}^5; \\
(D_r \cdot \mathbf{x}^3) &= 0; \\
D_v (\mathbf{x}^3) &= 0; \\
D_v (\mathbf{x}^4) &= 0; \\
D_v (\mathbf{x}^5) &= 0;
\end{align*}
\]

\[
\begin{align*}
\mathbf{x}^5 - \sum_\alpha e_\alpha m_\alpha^3 \int \gamma^5 \mathbf{x}^1 \ dv = 0, \\
\mathbf{x}^4 - \sum_\alpha e_\alpha m_\alpha^3 \int \gamma^5 \mathbf{x}^1 \ dv = 0,
\end{align*}
\]
3.2 Calculation of symmetries illustrated by Vlasov-Maxwell equations

Here $A_0, A_4, b$ and $g$ are constants, and $\eta^{1\alpha}(f^\alpha)$ are arbitrary functions of their arguments.

Now let us turn to the solution of the nonlocal determining equation (3.62) that we rewrite in the following form:

$$\begin{align*}
\eta^5 - \xi^1 \rho_t - \xi^2 \rho_r - \xi^3 \rho_v &= \sum_\alpha e_\alpha m_\alpha \int \gamma^5 \left( \eta^{1\alpha} - \xi^1 f^\alpha_t - \xi^2 f^\alpha_r - \xi^3 f^\alpha_v \right) dv,
\eta^4 - \xi^1 j_t - \left( \xi^2 \cdot \nabla_r \right) j - \left( \xi^3 \cdot \nabla_v \right) j \\
&= \sum_\alpha e_\alpha m_\alpha \int \gamma^5 v \left( \eta^{1\alpha} - \xi^1 f^\alpha_t - \xi^2 f^\alpha_r - \xi^3 f^\alpha_v \right) dv.
\end{align*}$$

(3.65)

As in the case of the local determining equations (3.61), the latter should be solved in view of the original equations (3.16), (3.18). Hence, calculating the derivatives $f^\alpha_t$, $\rho_t$ and $j_t$ from the basic equations (3.16), (3.18) and inserting them into (3.65) and in view of the above expressions (3.64) for coordinates $\xi$ and $\eta$, we obtain the following nonlocal determining equation:

$$\int \gamma^5 \left[ \sum_\alpha e_\alpha m_\alpha \left( \eta^{1\alpha}(f^\alpha) + 2A_4 f^\alpha \right) \right] dv = 0.$$  

(3.66)

As any determining equation, (3.66) is the equality with respect to all group variables that appear in this equation. Therefore, differentiating it with respect to any group variables also leads to equalities. Hence, nonlocal determining equation can be split with respect to independent group variable $f$ using the variational differentiation,

$$\frac{\delta}{\delta f(v')} \int \gamma^5 \left[ \sum_\alpha e_\alpha m_\alpha \left( \eta^{1\alpha}(f^\alpha) + 2A_4 f^\alpha \right) \right] dv = 0.$$  

(3.67)

Introducing the variational derivative $\left( \delta / \delta f(v') \right)$ inside the integral over $v$,

$$\int \gamma^5 \left[ \sum_\alpha e_\alpha m_\alpha \left( \eta^{1\alpha} + 2A_4 \right) \right] \frac{\delta f^\alpha(v)}{\delta f^\alpha(v')} dv = 0,$$

and eliminating integration over $v$ with the help of the Dirac delta-function,

$$\frac{\delta f(v)}{\delta f(v')} = \delta \left( v - v' \right),$$

one comes to the first-order differential equation for $\eta^{1\alpha}$:

$$\eta^{1\alpha}_f + 2A_4 = 0,$$

which gives the linear dependence of $\eta^{1\alpha}$ upon $f^\alpha$,

$$\eta^{1\alpha} = -2A_4 f^\alpha + \frac{1}{e_\alpha m_\alpha} A^a_g,$$  

(3.68)
with some constant $A_s^\alpha$. It is essential that the nonlocal determining equation (3.66) should be solved simultaneously with its differential consequence, i.e., any solution of (3.67) must appear as the solution of (3.66). Substituting this result back into (3.66) links different constants,

$$\sum_{\alpha=1}^n A_s^\alpha = 0, \quad (3.69)$$

provided integral over $v$ has finite value. Solution of the second nonlocal determining equation in (3.62) yields the same result.

Formulae (3.64), (3.68) and (3.69) define the the generators of the symmetry group of the Vlasov-Maxwell equations [6, 9] which we rewrite in the Lie point (non-canonical) form:

$$P_0 = \frac{\partial}{\partial t}; \quad P = \frac{\partial}{\partial r};$$

$$B = r \frac{\partial}{\partial t} + c^2 t \frac{\partial}{\partial r} + c^2 v \left( \frac{\partial}{\partial v} - v \left( \frac{\partial}{\partial v} \right) \right) - c \left[ B \times \frac{\partial}{\partial E} \right] + c \left[ E \times \frac{\partial}{\partial B} \right] + c^2 \rho \frac{\partial}{\partial j} + j \frac{\partial}{\partial \rho};$$

$$R = \left[ r \times \frac{\partial}{\partial r} \right] + \left[ v \times \frac{\partial}{\partial v} \right] + \left[ E \times \frac{\partial}{\partial E} \right];$$

$$D = t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r} - 2 \sum_{\alpha} f^\alpha \frac{\partial}{\partial f^\alpha} - E \frac{\partial}{\partial E} - B \frac{\partial}{\partial B} - 2j \frac{\partial}{\partial j} - 2\rho \frac{\partial}{\partial \rho};$$

$$X^\alpha = (1 - \delta_{\alpha,1}) \left( \frac{1}{e_\alpha m_\alpha^3} \frac{\partial}{\partial f^\alpha} - \frac{1}{e_1 m_1^3} \frac{\partial}{\partial f^1} \right).$$

Thus, if the parameters $e_\alpha, m_\alpha$ and $c$ are not involved in the transformations, the continuous Lie point symmetry group admitted by the Vlasov-Maxwell equations is defined by the $11 + (k - 1)$-dimensional algebra specified by the subalgebra $L_{10}$ of the Poincaré group,

$$L_{10} = \langle P_0, P, B, R \rangle,$$

supplemented by the operator $D$, specifying dilations, and the quasi-neutrality generator $X^\alpha$, defining translations in the space of distribution functions for plasma with two or more particle species.

### 3.2.3 Plasma kinetic equations with Lagrangian velocity

Meanwhile in describing the evolution of the distribution functions frequently it is more convenient to use the non-standard Vlasov equations (3.16) with the Euler
velocity \( \mathbf{v} \), but their hydrodynamic analogue [11, 12] with the *Lagrangian velocity* \( \mathbf{w} \). Then the plasma kinetics is described by hydrodynamic type equations for the density \( N^\alpha(t, r, \mathbf{w}) \) and the velocity \( \mathbf{V}^\alpha(t, r, \mathbf{w}) \), which depend upon \( t, r \) and \( \mathbf{w} \),

\[
N^\alpha_t + \text{div}(N^\alpha \mathbf{V}^\alpha) = 0, \quad \Gamma^\alpha = \left(1 - \frac{(\mathbf{V}^\alpha)^2}{c^2}\right)^{-1/2},
\]

\[
\mathbf{V}^\alpha_t + (\mathbf{V}^\alpha \cdot \nabla)\mathbf{V}^\alpha = \frac{e_\alpha}{m_\alpha c} \Gamma^\alpha \left[ E + \frac{1}{c} (\mathbf{V}^\alpha \times \mathbf{B}) - \frac{\mathbf{V}^\alpha}{c^2} (\mathbf{V}^\alpha \cdot \mathbf{E}) \right].
\]

Here the index \( \alpha \) indicates the plasma particle species with the charge \( e_\alpha \) and mass \( m_\alpha \) and the charge and current densities, \( \rho \) and \( \mathbf{j} \), are in turn determined by the motion of plasma particles:

\[
\rho = \sum_\alpha e_\alpha m_\alpha^3 \int N^\alpha \gamma^5 \, d\mathbf{w},
\]

\[
\mathbf{j} = \sum_\alpha e_\alpha m_\alpha^3 \int N^\alpha \mathbf{V}^\alpha \gamma^5 \, d\mathbf{w}, \quad \gamma = \frac{1}{\sqrt{1 - (\mathbf{w}/c)^2}}.
\]

It is typical that Eqs. (3.71) do not contain the Lagrangian velocity \( \mathbf{w} \) (or *Lagrangian momentum* \( \mathbf{q} \)) in explicit form. In order to find the dependence upon \( \mathbf{q} \) one should solve these equations with the “initial” conditions \( \mathbf{V}^\alpha = \mathbf{w}, N^\alpha = N^\alpha_0(t_0, r, \mathbf{w}) \) which hold for vanishing electric and magnetic fields \( E = B = 0 \) at some \( t = t_0 \). In particular, in homogeneous plasma “initial” conditions for the density \( N^\alpha_0 \) has the form \( N^\alpha_0 = n_0 f_0^\alpha(\mathbf{q}) \), where the stationary and homogeneous function \( f_0^\alpha(\mathbf{q}) \) of the Lagrangian momentum \( \mathbf{q} \) coincides with the function \( f_0^\alpha(\mathbf{p}) \) of the Euler momentum \( \mathbf{p} \).

Given the density \( N^\alpha(t, r, \mathbf{w}) \) and the velocity \( \mathbf{V}^\alpha(t, r, \mathbf{w}) \) that depend upon the Lagrangian momentum the particles distribution function in the Euler representation is restored with the help of the following relations (the index of particles species in these formulas is omitted):

\[
N(t, r, \mathbf{q}) = f(\mathbf{p} = P(q, r, t), r, t) \det \left( \frac{\partial P_i}{\partial q_j} \right),
\]

\[
\mathbf{v} = c^2 \mathbf{p} \left( m^2 c^4 + c^2 \mathbf{p}^2 \right)^{-1/2},
\]

\[
\mathbf{w} = c^2 \mathbf{q} \left( m^2 c^4 + c^2 \mathbf{q}^2 \right)^{-1/2},
\]

\[
\mathbf{V} = c^2 \mathbf{p} \left( m^2 c^4 + c^2 \mathbf{p}^2 \right)^{-1/2}.
\]

The system of Eqs. (3.71)–(3.73) presents the *Lagrangian formulation of the Vlasov-Maxwell equations*, in which (3.72) appear as nonlocal material relations.

The current and charge densities in (3.72) are moments of functions \( N^\alpha \) and \( \mathbf{V}^\alpha \) and, similar to electric and magnetic fields in Maxwell equations (3.17), do not depend upon the plasma particles velocity. This lead to additional differential constraints
\[ E_w = 0; \quad B_w = 0; \quad j_w = 0; \quad \rho_w = 0, \quad (3.74) \]

that are obvious from the physical point of view, however essential for calculating symmetries of the plasma kinetic equations with the Lagrangian velocity.

To calculate the symmetry group for the system of Eqs. (3.71), (3.72), (3.74), we start from the infinitesimal operator of the admitted local group of point one-parameter transformations in the standard form

\[
X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial r} + \xi^3 \frac{\partial}{\partial w} + \sum_{\alpha} \eta^{\alpha} \frac{\partial}{\partial N^\alpha} + \sum_{\alpha} \eta^{2\alpha} \frac{\partial}{\partial V^\alpha} + \eta^3 \frac{\partial}{\partial E} + \eta^4 \frac{\partial}{\partial B} + \eta^5 \frac{\partial}{\partial j} + \eta^6 \frac{\partial}{\partial \rho}, \quad (3.75)
\]

where coordinates \( \xi^i \) and \( \eta^k \) depend upon \( t, r, w, N^\alpha, V^\alpha, E, B, j \) and \( \rho \). Following the procedure, fulfilled in the preceding example, we obtain the continuous Lie point transformation group for the Vlasov-Maxwell equations (3.71), (3.72), (3.74) (with the Lagrangian velocity),

\[
P_0 = \frac{\partial}{\partial t}; \quad P = \frac{\partial}{\partial r};
\]

\[
B = r \frac{\partial}{\partial t} + c^2 t \frac{\partial}{\partial r} - c \left[ B \times \frac{\partial}{\partial E} \right] + c \left[ E \times \frac{\partial}{\partial B} \right] + c^2 \rho \frac{\partial}{\partial j};
\]

\[
R = \left[ r \times \frac{\partial}{\partial r} \right] + \left[ V^\alpha \times \frac{\partial}{\partial V^\alpha} \right] + \left[ E \times \frac{\partial}{\partial E} \right]
\]

\[
D = t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r} - 2 \sum_{\alpha} N^\alpha \frac{\partial}{\partial N^\alpha}
\]

\[
-\frac{E}{\partial E} - \frac{B}{\partial B} - 2 \frac{j}{\partial j} - 2 \frac{\rho}{\partial \rho};
\]

\[
X_\infty = \frac{\xi}{\partial w} - \left( \frac{5 (w \cdot \xi)}{c^2} - \gamma^2 + (V_w \cdot \xi) \right) \sum_{\alpha} N^\alpha \frac{\partial}{\partial N^\alpha},
\]

with \( \xi = \xi(w) \). The operators in (3.76) has a simple physical interpretation: \( P_\mu = (P_0, P) \), where \( \mu = 0,1,2,3 \), specify translation in time and translation along the three components of radius-vector \( r, B \) defines the Lorentz transformations, consisting of hyperbolic rotations (boosts) in the \( \{ct, r\} \) and \( \{cp, j\} \) planes, linear-fractional transformations of the velocity \( V^\alpha \), transformations of the density \( N^\alpha \) and transformations of components of the 4-tensor of the electromagnetic field (see, e.g., §24, 25 in [10]), while \( R \) specifies circular rotations. These ten (scalar) operators define the Poincaré group (compare with (3.71)).
3.2 Calculation of symmetries illustrated by Vlasov-Maxwell equations

\[ L_{10} = \langle P_0, P, B, R \rangle. \]

In (3.76) it is supplemented by the operator \( D \), specifying dilations, and the operator of the infinite subgroup \( X_{\infty} \) (see also [9] and [4] (p.419)), specifying the consistent transformations of the Lagrangian velocity and the density of the plasma particles. Thus, provided parameters \( e_a, m_a \) and \( c \) are not involved in transformations the continuous Lie point group, admitted by the Vlasov-Maxwell equations with the Lagrangian velocity, is defined by the 11-dimensional subalgebra, specified by the algebra \( L_{10} \) of the Poincaré group and the one-dimensional algebra with the dilation operator \( D \), and the infinite-dimensional subalgebra with the operator \( X_{\infty} \).

Symmetries in the space of Fourier variables

In summary we present the formulae for symmetry generators in space of Fourier variables for functions, not dependent upon the Lagrangian velocity \( w \). Specifying of the Fourier transformation, say, of a charge density

\[ \rho(t, r) = \int \rho(t, r) \exp(i\omega t - ikr) \, dt \, dr, \tag{3.77} \]

is equivalent to introduction of a new nonlocal variable. To fulfill the procedure of prolongation of Lie point group operator (3.75) on a nonlocal variable, we follow the receipt (3.14), (3.13) and (3.15), and rewrite down this operator in the canonical form

\[ \begin{align*}
Y = & \sum_{\alpha} x^{1\alpha} \partial_{N^\alpha} + x^{2\alpha} \frac{\partial}{\partial V^\alpha} + x^{3} \frac{\partial}{\partial E} + x^{4} \frac{\partial}{\partial B} + x^{5} \frac{\partial}{\partial j} + x^{6} \frac{\partial}{\partial \rho}, \\
= & \eta^{1\alpha} N^\alpha, \quad x^{2\alpha} = \eta^{2\alpha} V^\alpha, \quad x^{3} = \eta^{3} E, \\
= & \eta^{4} B, \quad x^{5} = \eta^{5} j, \quad x^{6} = \eta^{6} \rho,
\end{align*} \tag{3.78} \]

\[ D \equiv \xi^{1} \partial_t - (\xi^{2} \cdot \nabla_r) - (\xi^{3} \cdot \nabla_w), \]

and formally prolong it on the nonlocal variable \( \tilde{\rho}(\omega, k) \),

\[ \tilde{Y} \equiv Y + x^{6} \partial_{\rho}. \tag{3.79} \]

The integral relation between \( x^{6} \) and \( x^{6} \) results while applying the operator (3.79) to Eq. (3.77). Here we consider it as the definition of the variable \( \tilde{\rho} \),

\[ \tilde{\rho} = \int x^{6} \exp(i\omega t - ikr) \, dt \, dr. \tag{3.80} \]

Substituting \( x^{6} \) from (3.78) into (3.80) and calculating the integrals obtained (integrating by parts), we get the desired coordinate \( \tilde{\rho} \). For example, for the operator of time translations \( P_0 \), the coordinate \( \tilde{\rho} = -\rho \) after substitution into (3.77) yields the following expression for the coordinate \( \tilde{\rho} = i\omega \tilde{\rho} \) in Fourier variables.
Other coordinates of a canonical operator are calculated in a similar way. Inserting these results into (3.79), restricting the group to Fourier variables not containing dependencies upon Lagrangian velocity \( \mathbf{w} \) (i.e., leaving in (3.79) only the contributions responsible for transformation of these variables in Fourier representation) and returning back to non-canonical representation, we obtain the following set of operators for \( \Pi \)-parameter Lie point group in \( \{ \omega, \mathbf{k} \} \) representation:

\[
\begin{align*}
\hat{\mathbf{P}} &= i\omega \left( \frac{\partial}{\partial E} + \frac{\mathbf{B} \cdot \nabla}{\partial \mathbf{B}} + \frac{\mathbf{j} \cdot \nabla}{\partial \mathbf{j}} + \frac{\rho \cdot \nabla}{\partial \rho} \right); \\
\hat{\mathbf{P}} &= -ik \left( \frac{\partial}{\partial E} + \frac{\mathbf{B} \cdot \nabla}{\partial \mathbf{B}} + \frac{\mathbf{j} \cdot \nabla}{\partial \mathbf{j}} + \frac{\rho \cdot \nabla}{\partial \rho} \right);
\end{align*}
\]

\[
\begin{align*}
\hat{\mathbf{B}} &= c^2 k \frac{\partial}{\partial \omega} + \omega \frac{\partial}{\partial k} - c \left[ \mathbf{B} \times \frac{\partial}{\partial \mathbf{E}} \right] + c \left[ \mathbf{E} \times \frac{\partial}{\partial \mathbf{B}} \right] + c^2 \hat{\rho} \frac{\partial}{\partial \rho} + \frac{\mathbf{j} \cdot \nabla}{\partial \mathbf{j}}; \\
\hat{\mathbf{R}} &= \left[ \mathbf{k} \times \frac{\partial}{\partial \mathbf{k}} \right] + \left[ \mathbf{E} \times \frac{\partial}{\partial \mathbf{E}} \right] + \left[ \mathbf{B} \times \frac{\partial}{\partial \mathbf{B}} \right] + \left[ \mathbf{j} \times \frac{\partial}{\partial \mathbf{j}} \right]; \\
\hat{\mathbf{D}} &= -\omega \frac{\partial}{\partial \omega} - \mathbf{k} \frac{\partial}{\partial \mathbf{k}} + 3\mathbf{E} \frac{\partial}{\partial \mathbf{E}} + 3\mathbf{B} \frac{\partial}{\partial \mathbf{B}} + 2\mathbf{j} \frac{\partial}{\partial \mathbf{j}} + 2\hat{\rho} \frac{\partial}{\partial \rho}. \quad (3.81)
\end{align*}
\]

Eqs. (3.81) supplement the group (3.76) by the appropriate transformations of variables in Fourier space. For example, Lorentz transformations with the operator \( \mathbf{B} \) are supplemented with hyperbolic rotations in \( \{ \omega, c\mathbf{k} \} \) and \( \{ c\hat{\rho}, \mathbf{j} \} \) planes and transformations of the 4-tensor of the Fourier-components of the electromagnetic field.

### 3.2.4 Electron-ion plasma equations in quasi-neutral approximation

In describing the plasma flows with the characteristic scale of the density variation \( L \) large as compared to the Debye length \( \lambda_D \ll L \) for plasma particles a so-called quasi-neutral approximation is used. In this approximation charge and current densities in plasma are set equal to zero, that essentially simplifies the initial model with nonlocal terms. Instead of the system of the Vlasov-Maxwell equations (3.16), (3.17) with the corresponding material equations (3.18), only the kinetic equations are used. To make the example obvious we write down the kinetic equations for particle distribution functions in the one-dimensional case:

\[
f_{\mathbf{r}}^{(\alpha)} + v f_{\mathbf{v}}^{(\alpha)} + (e_{\alpha}/m_{\alpha})E(t, x)f_{\mathbf{v}}^{(\alpha)} = 0, \quad (3.82)
\]

with additional nonlocal restrictions imposed on them, which arise from vanishing conditions for the current and the charge densities.
3.2 Calculation of symmetries illustrated by Vlasov-Maxwell equations

\[ \int \sum_{\alpha} e_{\alpha} f^{\alpha} \, dv = 0, \quad \int \sum_{\alpha} e_{\alpha} f^{\alpha} \, v \, dv = 0. \quad (3.83) \]

At that the electric field \( E \) is expressed through the moments of distribution functions:

\[ E(t, x) = \left( \int v^2 \frac{\partial}{\partial x} \sum_{\alpha} e_{\alpha} f^{\alpha} \, dv \right) \left( \int \sum_{\alpha} \frac{e_{\alpha}^2}{m_{\alpha}} f^{\alpha} \, dv \right)^{-1}. \quad (3.84) \]

Eqs. (3.82), (3.83) describe one-dimensional dynamics of a plasma, which is inhomogeneous upon the coordinate \( x \); thus the distribution functions of particles \( f^{\alpha} \) depend upon \( t, x \) and the velocity component \( v \) in the directions of the plasma inhomogeneity.

The peculiarity of calculating the admitted group for Eqs. (3.82), (3.83) is that it qualitatively differs from the method used in Section 3.2.2 in that the electric field \( E \) is treated not as one of the dependent variables but as an unknown function of the variables \( t, x \), \( E = E(t, x) \). This case of finding the symmetry logically follows from the simpler, quasi-neutral model of plasma (3.82), (3.83) in contrast to the complete system of Vlasov-Maxwell equations (3.16)–(3.17).

The group of point Lie transformations admitted by system (3.82) and (3.83) is specified by the following set of infinitesimal operators:

\[ X_1 = \frac{\partial}{\partial t}; \quad X_2 = \frac{\partial}{\partial x}; \quad X_3 = t \frac{\partial}{\partial t} - v \frac{\partial}{\partial v}; \quad X_4 = x \frac{\partial}{\partial x} + v \frac{\partial}{\partial v}; \]

\[ X_5 = \sum_{\alpha} f^{\alpha} \frac{\partial}{\partial f^{\alpha}}; \quad X_6 = t \frac{\partial}{\partial x} + \frac{\partial}{\partial v}; \quad (3.85) \]

\[ X_7 = t^2 \frac{\partial}{\partial t} + tx \frac{\partial}{\partial x} + (x - vt) \frac{\partial}{\partial v}; \quad X_\alpha = \frac{1}{Z_{\alpha+1}} \frac{\partial}{\partial f^{\alpha+1}} - \frac{1}{Z_{\alpha}} \frac{\partial}{\partial f^{\alpha}}, \]

with the general element of the algebra represented by their linear combination

\[ X = \sum_{j=1}^{7} c_j X_j + \sum_{\alpha} b_\alpha X_\alpha. \quad (3.86) \]

In the operators \( X_\alpha \) in system (3.86), \( Z_{\alpha} = e_{\alpha}/|e| \) is the charge number of the particles of type \( \alpha \), and the index \( \alpha + 1 \) denotes the type of particles that follows \( \alpha \). The operators \( X_\alpha \) exist only in plasma with the number of particle types larger than or equal to two and their number is less than the number of plasma components by one. Since here, in contrast to Section 3.2.2, we chose a different normalization of the particle distribution functions, the quasi-neutrality generator \( X_\alpha \) contain factors that do not depend on particle mass.

It is easy to verify that the translation operators \( X_1 \) and \( X_2 \), the Galilean transformation operator \( X_6 \), and the quasi-neutrality operators \( X_\alpha \) are contained in the symmetry group obtained in Section 3.2.2 by a different method without assuming that \( E \) is an arbitrary function of two variables to be determined. The two dilation generators specified for the one-dimensional plasma model in [8] are obtained by combining the three dilation operators \( X_3, X_4, \) and \( X_5 \) from (3.85) and by adding
the contributions responsible for the dilation transformations of the electric field $E$, charge density $\rho$, and electric current density $j$. The projective group operator $X_6$ is new among the generators (3.85).

References

Chapter 4  
Renormgroup Symmetries

The notion “RenormGroup Symmetry” (RGS) originated in mathematical physics at the beginning of the nineties [8] (see also reviews [12, 13, 14]) as a result of joining up the notion “symmetry group” as applied to differential equations and “renormgroup”, i.e., symmetry group of a particular solution. In its turn, the notion of Renormalization Group, or briefly RenormGroup (RG), was imported to mathematical physics from theoretical physics, namely from quantum field theory [16, 17, 4, 1]. In quantum field theory renormgroup was based upon finite (Dyson) transformations and appeared as a continuous group in a usual mathematical sense. This group was then successfully used in developing a regular method of improving approximate perturbation solutions, the renormgroup method [2, 3]. In transferring renormgroup concept to mathematical physics problems [8] the aim was ultimately the same as in quantum field theory – to improve the perturbation theory solutions and to correct the behavior of these solutions in the vicinity of a singularity. In mathematical physics we usually deal with the problems, based on systems of differential equations or integro-differential equations, the symmetry of which can be found using computational methods of modern group analysis. In problems of mathematical physics this feature appeared as decisive in creating renormgroup algorithm (see e.g. [12, 13, 14]) which has united renormgroup ideology of quantum field theory with a regular way of symmetry construction for solutions of boundary value problems.

4.1 Introduction

In mathematical physics, a solution of a physical problem usually appears as a solution of some boundary value problem. Then the corresponding RG transformation can be obtained from the symmetry group related to this boundary value problem with the boundary condition also involved in the group transformation. The key point here is that the relevant symmetry group is calculated by regular procedure of modern group analysis, provided the problem is formulated in terms of differential equations (or integro-differential equations).
Let the Lie group $G$ with generator

$$X = \xi^t \frac{\partial}{\partial t} + \xi^x \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}$$  \hspace{1cm} (4.1)$$

be defined for the system of the first-order partial differential equations

$$y_t = F(t,x,y,y_x).$$  \hspace{1cm} (4.2)$$

The typical boundary value problem for (4.2) is the Cauchy problem with boundary manifold defined by

$$t = 0, \quad y = \psi(x).$$  \hspace{1cm} (4.3)$$

Solution of this Cauchy problem is the $G$-invariant solution iff for any generator (4.1), function $\psi$ satisfies the equation [15, §29]

$$\eta(0,x,\psi) - \xi^t(0,x,\psi)\psi_x - \xi^x(0,x,\psi)F(0,x,\psi,\psi_x) = 0.$$  \hspace{1cm} (4.4)$$

The solution of Cauchy problem (4.2), (4.3) coincides with orbit of the group $G$, and the boundary manifold is not the invariant manifold of the group.

This example gives an instructive idea for constructing generators of RGSs. The milestones here are (a) considering the boundary value problem in the extended space of group variables that involve parameters of the boundary conditions in group transformations, (b) calculating the admitted group using the infinitesimal approach, (c) checking the invariance condition akin to (4.4) to find the symmetry group with the orbit that coincides with the boundary value problem solution, and (d) using the RGS to find the improved (renormalized) solution of the boundary value problem.

The whole algorithm [12, 13, 14] will be described in detail in the next section; here we only give a general grasp of the problem using a trivial example, the boundary value problem for the Hopf equation

$$v_t + vv_x = 0, \quad v(0,x) = \varepsilon U(x),$$  \hspace{1cm} (4.5)$$

where $U$ is an invertible function of $x$ and the parameter $\varepsilon$ defines the initial amplitude at the boundary $t = 0$. For small values of $t \ll 1/\varepsilon$, i.e., near the boundary, $t \to 0$, a perturbation theory solution to (4.5) has the form of a truncated power series in $\varepsilon t$,

$$v = \varepsilon U - \varepsilon^2 t U U_x + O(t^2).$$  \hspace{1cm} (4.6)$$

It is obvious that this solution is invalid for large distances from the boundary, when $\varepsilon t U_x \simeq 1$. The RGS gives a way to improve the perturbation theory result and restore the correct structure of the boundary value problem solution in the vicinity of a singularity (in the event that such singularity appears for some finite value of $t$).

It is convenient to introduce the new function $u = v/\varepsilon$ and rewrite Eqs. (4.5) in the form

$$u_t + \varepsilon u u_x = 0, \quad u(0,x) = U(x).$$  \hspace{1cm} (4.7)$$
In order to calculate the renormgroup symmetries, we add the parameter $\varepsilon$ to the list of the independent variables and consider the manifold (termed in general the basic manifold) given by Eq. (4.7) in the space of variables \{\(t, x, \varepsilon, u_t, u_x\}\). Then we calculate the generator

$$X = \xi^t \frac{\partial}{\partial t} + \xi^x \frac{\partial}{\partial x} + \xi^\varepsilon \frac{\partial}{\partial \varepsilon} + \eta \frac{\partial}{\partial u}$$

(4.8)

of the group admitted by the first equation in (4.7) and obtain the following coordinates of the generator (4.8):

$$\xi^t = \psi^1, \quad \xi^x = \varepsilon u \psi^1 + \psi^2 + x(\psi^3 + \psi^4), \quad \xi^\varepsilon = \psi^4, \quad \eta = u \psi^3,$$

(4.9)

where $\psi^i, i = 2, 3, 4$, are arbitrary functions of $\varepsilon, u,$ and $x - \varepsilon u t$ and $\psi^1$ being an arbitrary function of all the group variables. These formulas define an infinite-dimensional Lie algebra with four generators

$$\begin{align*}
X_1 &= \psi^1 \left( \frac{\partial}{\partial t} + \varepsilon u \frac{\partial}{\partial x} \right), & X_2 &= \psi^2 \frac{\partial}{\partial x}, \\
X_3 &= \psi^3 \left( x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u} \right), & X_4 &= \psi^4 \left( \varepsilon \frac{\partial}{\partial \varepsilon} + x \frac{\partial}{\partial x} \right).
\end{align*}$$

(4.10)

Suppose that a particular solution of boundary value problem (4.7),

$$S \equiv u - W(t, x, \varepsilon) = 0,$$

which defines an invariant manifold of group (4.8), (4.9) is known. The corresponding invariance condition evaluated on frame $S$ is similar to (4.4):

$$X_S \big|_S \equiv (W - x W_x) \psi^3 - W_t \psi^2 - (\varepsilon W_x + x W_x) \psi^4 = 0.$$

(4.11)

The term with $\psi^1$ does not give any input in (4.11) since it is proportional to $W_t + \varepsilon W W_x$ and vanishes on the solutions of (4.7). Equation (4.11) is valid for all $t$. Hence, it remains valid for $t \rightarrow 0$, when $W$ is replaced with approximate solution, which follows from (4.6),

$$W = U - \varepsilon t U U_x + O(t^2).$$

(4.12)

In this limit, $t \rightarrow 0$, condition (4.11) gives a relation between the $\psi^i, i = 2, 3, 4$ (no restrictions are imposed on $\psi^1$), that can be easily prolonged on $t \neq 0$,

$$\psi^2 = -\chi(\psi^3 + \psi^4) + (u/U_\chi) \psi^3, \quad \chi = x - \varepsilon u t,$$

(4.13)

where the derivative $U_\chi$ should be expressed, due to the boundary condition, either in terms of $\chi$ or $u$. By substituting (4.13) into (4.9), we obtain a group of a smaller dimension with generators
The above procedure, which transforms (4.10) to (4.14), is the restriction of the group (4.8) on a particular solution.

The boundary value problem solution defines a manifold, that, by construction, turns to be invariant for any generator $R_i$. Hence, (4.14) defines the desired RGSs. This means that the boundary value problem solution can be constructed by use any of generators in (4.14), the generator $R_3$ for example. Without loss of generality, we choose $\varepsilon = 1$ and obtain the finite RG transformations ($a$ is a group parameter)

$$x' = x + atu, \quad \varepsilon' = \varepsilon + a, \quad t' = t, \quad u' = u,$$

where $t$ and $u$ are invariants of the RG transformations while the transformations of $\varepsilon$ and $x$ are translations, which also depend on $t$ and $u$ in the case of $x$. For $\varepsilon = 0$, in view of (4.6), we have $x = H(u)$, where $H(u)$ is a function inverse to $U(x)$. Eliminating $a, t, u$ from (4.15) and omitting the primes on variables, we obtain the desired solution of boundary value problem (4.7) in the implicit form

$$x - \varepsilon tu = H(u).$$

This in fact is the improved perturbation theory solution (4.6), which is valid not only for small $\varepsilon t \ll 1$, provided dependence (4.16) can be resolved uniquely. Depending upon $H(u)$ it gives either proper singular behavior at some finite $t \to t_{\text{sing}}$ or correct asymptotic behavior at $t \to \infty$.

The peculiarity of the procedure for constructing RGSs is the multi-choice first step, which depends on how the boundary conditions are formulated and the form in which the admitted group is calculated. For example, instead of calculating the Lie point symmetry group, we can consider the Lie-Bäcklund symmetries [6] with the canonical generator $R = \varkappa \partial_u$, where $\varkappa$ depends not only on $t, x, \varepsilon$, and $u$ but also on higher-order derivatives of $u$. We can seek $\varkappa$ in the form of a power series in $\varepsilon$, and invariance condition (4.11) is formulated as vanishing of $\varkappa$ at $t = 0$. Depending on the choice of the zeroth-order term representation, we obtain either an infinite or a truncated power series for $\varkappa$, for example, a form linear in $\varepsilon$,

$$R = \varkappa \frac{\partial}{\partial u}, \quad \varkappa = 1 - \frac{u_x}{U_x(u)} - \varepsilon tu_x.$$

This RG generator (4.17) is equivalent to the Lie point generator $R_2$ in (4.14) and therefore gives the same result.

Another possibility for calculating RGSs for boundary value problem (4.7) is offered by taking some additional differential constraints consistent with boundary
4.2 Renormgroup algorithm

The general construction scheme of the RG algorithm (shown in the Fig. 4.1) is given as four consecutive steps [12, 13, 14]:

I. constructing the basic manifold,
II. calculating the admitted (symmetry) group \( \mathcal{G} \),
III. restricting it on the particular boundary value problem solution and constructing \( \mathcal{RG} \), and
IV. seeking an analytic solution.

Fig. 4.1 Scheme of \( \mathcal{RG} \) algorithm
4.2.1 Basic manifold

The initial issue is to construct the RGS and appropriate transformations that involve the parameters of partial solution. Therefore, the purpose of step I is to include all the parameters, both from the equations and from the boundary conditions on which a particular solution depends, in group transformations in one or another way. This purpose is achieved by constructing a special manifold $\mathcal{RM}$—*basic manifold*—given by a system that consists of $s$ $k$th-order differential equations and $q$ nonlocal relations

$$F_\sigma(z, u, u(1), \ldots, u(k)) = 0, \quad \sigma = 1, \ldots, s, \quad (4.18)$$

$$F_\sigma(z, u, u(1), \ldots, u(r), J(u)) = 0, \quad \sigma = 1 + s, \ldots, q + s. \quad (4.19)$$

The nonlocal variables $J(u)$ here are introduced by integrations,

$$J(u) = \int \mathcal{F}(u(z)) \, dz. \quad (4.20)$$

The presence of relations (4.19) in the system determining $\mathcal{RM}$ characterizes the basic difference between the case of a nonlocal problem and the case of a boundary value problem for differential equations, for which $\mathcal{RM}$ is a differentiable manifold.

4.2.2 Admitted group

Step II is to calculate the widest admitted group $\mathcal{G}$ for system (4.18), (4.19). In application to an $\mathcal{RM}$ defined only by system of differential equations (4.18), the question is about a local group of transformations in a space of differential functions $\mathcal{A}$, for which system (4.18) remains unchanged. This group is defined by the generator of form (4.8) prolonged on all higher-order derivatives,

$$X = \xi^i \frac{\partial}{\partial z^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{ij}^\alpha \frac{\partial}{\partial u_{ij}^\alpha} + \cdots, \quad (4.21)$$

where $\xi^i([z, u]), \eta^\alpha([z, u]) \in \mathcal{A}$ and

$$\zeta_i^\alpha = D_i \xi^\alpha + \xi^j u_{ij}^\alpha, \quad \zeta_{ij}^\alpha = D_{ij} (\eta^\alpha - \xi^i u_i^\alpha) + \xi^j u_{ij}^\alpha.$$

Meanwhile, the classical Lie algorithm using the infinitesimal approach seems to be inapplicable to a manifold $\mathcal{RM}$ set by system (4.18), (4.19). The issue is that the $\mathcal{RM}$ in this case is not determined *locally* in the space of differential functions. Therefore, the main advantage of the Lie computational algorithm, namely, representation of the determining equations as an over-determined system of equations is not realized here. Furthermore, the procedure for prolongation the group operator
of point transformations on nonlocal variables is not defined in the framework of classical group analysis.

In modifying the RG algorithm, we rely on the direct method for calculating symmetries advanced in [5, 11] for finding symmetries for Boltzmann kinetic equation, the equations of motion of viscous-elastic media, and the Vlasov-Maxwell equations in the kinetic theory of plasma and described in Chapter 3. Therefore, constructing the symmetries for the nonlocal equations also appears as an algorithmic procedure. This is the generalization of the second step of the algorithm to the case where \( \mathcal{R}M \) is an integral or integro-differential manifold.

### 4.2.3 Restriction of admitted group on solutions

The group \( \mathcal{G} \) found in step II and determined by operators (4.21) is generally wider than the RG of interest, which is related to a particular solution of a boundary value problem. Hence, to obtain the RGS, we need step III, restricting the group \( \mathcal{G} \) on a manifold determined by this particular solution. From the mathematical standpoint, this procedure consists in checking the vanishing conditions for a linear combination of coordinates \( x_j^\alpha \) of a canonical operator equivalent to (4.21) on some particular boundary value problem solution \( U^\alpha(z) \),

\[
\left\{ \sum_j A^j x_j^\alpha \equiv \sum_j A^j (\eta_j^\alpha - \xi_j^\alpha) \right\} \bigg|_{u^\alpha = U^\alpha(z)} = 0. \tag{4.22}
\]

The form of the condition set by relation (4.22) is common for any solution of the boundary value problem, but how the restriction procedure of a group is realized may differ in each partial case. In the general scheme (given at the beginning of the section), it is related to the dashed arrow connecting the “initial object” (a perturbation theory solution of a particular boundary value problem) to the object arising as a result of step III.

In calculating combination (4.22) on a particular solution \( U^\alpha(z) \), the latter is transformed from a system of differential equations for group invariants to algebraic relations. Note two consequences of step III. First, the restriction procedure results in a set of relations between \( A^j \) and thus “links” the coordinates of various group operators \( X_j \) admitted by \( \mathcal{R}M \) (4.18), (4.19). Second, it (partially or completely) eliminates an arbitrariness that can arise in the values of the coordinates \( \xi_j^i \) and \( \eta_j^\alpha \) in the case of an infinite group \( \mathcal{G} \).

As a rule, the procedure of restricting the group \( \mathcal{G} \) reduces its dimension. After performing this procedure a general element (4.21) of a new group \( \mathcal{RG} \) is represented by a linear combination of new generators \( R_i \) with coordinates \( \hat{\xi}_i^j \) and \( \hat{\eta}_j^\alpha \) and arbitrary constants \( B^j \):

\[
X \Rightarrow R = \sum_j B^j R_j, \quad R_j = \hat{\xi}_j^i \frac{\partial}{\partial x^i} + \hat{\eta}_j^\alpha \frac{\partial}{\partial u^\alpha}. \tag{4.23}
\]
The set of operators $R_j$, each containing the required solution of a problem in the invariant manifold, defines a group of transformations $\mathcal{RG}$, which we also call RenormGroup.

### 4.2.4 Renormgroup invariant solutions

The three steps described above completely form the regular algorithm for constructing the RGS, but to finish a final step is needed. This step IV uses the RGS operators to find analytic expressions for new, improved boundary value problem solutions (compared with the input perturbative solution).

From the mathematical standpoint, realizing this step involves use of $RG$-invariance conditions set by a joint system of Eqs. (4.18) and (4.19) and the vanishing conditions for a linear combination of the coordinates $\hat{x}_j^\alpha$ of the canonical operator equivalent to (4.23),

$$\sum_j R^j_\alpha \hat{x}_j = \sum_j B^j \left( \hat{\eta}_j^\alpha - \hat{\xi}_j^\alpha \right) = 0.$$  

(4.24)

The need to use $\mathcal{RM}$ in constructing the boundary value problem solution is shown in the scheme by the dashed arrow connecting these objects.

Specification of step IV concludes the description of regular algorithm of RGS construction for models with integro-differential equations. We note that last the two steps are basically the same as for models with differential equations. The next sections contains a set of examples showing the ability of the RG algorithm.

### 4.3 Examples

We now present a few examples of the RGS construction with further use of the symmetry for “improving” an approximate solution.

#### 4.3.1 Modified Burgers equation

As the first example, we take the initial value problem for the modified Burgers equation

$$u_t - au_x^2 - vu_{xx} = 0, \quad u(0,x) = f(x).$$  

(4.25)

It is connected to the heat equation

$$\tilde{u}_t = v\tilde{u}_{xx}$$  

(4.26)
by the transformation \( \tilde{u} = \exp(au/v) \) and has an exact solution which therefore allows us to check the validity of our approach. The RGS construction for (4.25) is an apt illustration [9] of the general scheme, shown in 4.2 which may be helpful in understanding other examples of the general algorithm implementation.

The basic manifold \( \mathcal{RM} \) (step I) is given by Eq. (4.25) with the parameters of non-linearity \( a \) and dissipation \( v \) included in the list of independent variables. The Lie calculational algorithm applied to \( \mathcal{RM} \) gives, for the admitted group \( G \) (step II), nine independent terms in the general expression for the group generator

\[
X = \sum_{i=1}^{8} A_i(a,v)X_i + \alpha(t,x,a,v)e^{-au/v} \frac{\partial}{\partial u},
\]

(4.27)

\[
\begin{align*}
X_1 &= 4v^2 \frac{\partial}{\partial t} + 4vt \frac{\partial}{\partial x} - \frac{v}{a} \left( x^2 + 2vt \right) \frac{\partial}{\partial u}, \quad X_2 = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}, \\
X_3 &= \frac{\partial}{\partial t}, \quad X_4 = 2vt \frac{\partial}{\partial x} - \frac{v}{a} \frac{\partial}{\partial u}, \quad X_5 = \frac{\partial}{\partial x}, \quad X_6 = -\frac{v}{a} \frac{\partial}{\partial u}, \\
X_7 &= \frac{\partial}{\partial a} + \left( \frac{v}{a} - u \right) \frac{\partial}{\partial u}, \quad X_8 = 2v \frac{\partial}{\partial v} + x \frac{\partial}{\partial x} + 2 \left( u - \frac{v}{a} \right) \frac{\partial}{\partial u}.
\end{align*}
\]

Here, \( A_i(a,v) \) are arbitrary functions of their arguments and \( \alpha(t,x,a,v) \) is an arbitrary function of four variables, satisfying the heat equation (4.26).

A set of operators \( X_i \) forms an eight-dimensional Lie algebra, \( L_8 \). The first six generators relate to the well-known symmetries of the modified (potential) Burgers equation (see, e.g., Vol.1, p.183 in [7]). They describe projective transformations in the \((t,x)\)-plane \( (X_1) \), dilatations in the same plane \( (X_2) \), translations along the \( t-, x- \) and \( u- \) axes \( (X_3, X_5 \) and \( X_6 \)) and Galilean transformations \( (X_4) \). The last two generators \( X_7 \) and \( X_8 \) relate to dilatations of parameters \( a \) and \( v \), now involved in group transformations.

The procedure of restriction (step III) of the group (4.27) admitted by \( \mathcal{RM} \) (4.25) gives us a check of the invariance condition (4.22) on a particular boundary value problem solution \( u = U(t,x,a,v) \),

\[
\left\{ \eta_\infty + \sum_{i=1}^{8} A_i(a,v) \xi_i \right\}_{u=U(t,x,a,v)} = 0,
\]

(4.28)

\[
\xi_i \equiv \eta_i - \xi_i^1 u_x - \xi_i^2 u_s - \xi_i^3 u_w - \xi_i^4 u_v.
\]

This formula expresses the coordinate \( \alpha \) of the last term in (4.27) in terms of the remaining coordinates of the eight generators \( X_i \) for arbitrary \( t \), and hence for \( t = 0 \), when \( U(0,x,a,v) = f(x) \). As a result, we obtain the "initial" value \( \alpha(0,x,a,v) \) and then, using the standard representation for the solution to the linear parabolic equation (4.26), the value of \( \alpha \) at arbitrary \( t \neq 0 \):

\[
\alpha(t,x,a,v) = -\sum_{i=1}^{8} A_i(a,v) \langle \xi_i(x,a,v) \rangle.
\]

(4.29)
Here, $\mathcal{X}_i(x,a,v)$ denote “partial” canonical coordinates $\mathcal{X}_i$ taken at $t = 0$ and $u = f(x)$. Symbol $\langle F \rangle$ designates the convolution of a function $F$ with the fundamental solution of (4.26), multiplied by the exponential function of $f$ entering into the boundary condition

$$\langle F(x,t,a,v) \rangle \equiv \frac{1}{\sqrt{4\pi vt}} \int_{-\infty}^{\infty} F(y,t,a,v) \exp \left( -\frac{(x-y)^2}{4vt} + \frac{af(y)}{v} \right) \, dy.$$  

Substitution (4.29) in the general expression (4.27) gives the desired RG generators

$$R_i = X_i + \rho_i e^{-au/v} \frac{\partial}{\partial u},$$

$$\rho_1 = \frac{v}{a} \langle x^2 \rangle, \quad \rho_2 = \langle xf_x \rangle, \quad \rho_3 = \frac{1}{v} \langle af_x^2 + vf_xx \rangle, \quad \rho_4 = \frac{v}{a} \langle x \rangle,$$

$$\rho_5 = \langle f_x \rangle, \quad \rho_6 = \frac{v}{a} \langle 1 \rangle, \quad \rho_7 = \langle f - \frac{v}{a} \rangle, \quad \rho_8 = \langle xf_x - f^2 + 2 \frac{v}{a} \rangle.$$  

Operators $R_i$ form an eight-dimensional RG algebra $RL_8$ that has the same tensor of structural constants as $L_8$, i.e., $RL_8$ and $L_8$ are isomorphic. Hence, the group restriction procedure eliminates the arbitrariness presented by the function $\alpha$ and “fits” the boundary conditions into RG generators by means of $\rho_i$.

One can verify that the exact solution of the initial-value problem (4.25),

$$u(t,x;a,v) = \frac{v}{a} \ln(1)$$

is the invariant manifold for any of the above RGS operators. And, vice versa, (4.30) can be reconstructed from an approximate solution with the help of any of the RGS operators or their linear combination. For example, two such operators, $\nu R_3 \equiv R_6$ and $(1/a)(R_6 + R_7) \equiv R_8$ were used in [9] to reconstruct the exact solution from perturbed (in time and in nonlinearity parameter $a$) solutions. Below, we describe this procedure (step IV) using the operator $R_a$,

$$R_a = \frac{\partial}{\partial a} + \frac{1}{a} \left( -u + e^{-au/v}(f(x)) \right) \frac{\partial}{\partial u}.$$  

(4.31)

It is evident that $t,x$ and $v$ are invariants of group transformations with (4.31), whilst finite RG transformations of the two remaining variables, $a$ and $u$, are obtained by solving the Lie equations for (4.31), with $\ell$ the group parameter

$$\frac{da'}{d\ell} = 1, \quad a'|_{\ell=0} = a;$$

$$\frac{du'}{d\ell} = \alpha(t,x,a',v) e^{-a'd'/v} - \frac{d'}{a'}, \quad u'|_{\ell=0} = u.$$  

(4.32)
Combining these equations yields one more invariant $\mathcal{J} = e^{au/v} - \langle 1 \rangle$ for the RGS generator (4.31). Solution of (4.32) along with (4.29) gives the formulae for finite RG transformations of the variables $(t, x, a, v, u)$,

$$
\begin{align*}
t' &= t, \quad x' = x, \quad v' = v, \quad a' = a + \ell, \\
u' &= \frac{v}{a + \ell} \ln \left( e^{au/v} + \langle e^{f(x)/v} - 1 \rangle \right).
\end{align*}
$$

Choosing the value $a$ equal to zero, which is a starting point of the perturbed approximation in $a$, we get $a' = \ell$. Then after excluding $t, x, v$ and $\ell$ from the expression for $u'$ (4.33) and omitting accents over $t', x', v', u'$ and $a'$ the desired boundary value problem solution (4.30) is obtained. It also follows directly from $\mathcal{J}$ in view of the initial condition $\mathcal{J} |_{a=0} = 0$.

A similar procedure can be fulfilled for the other RG operator,

$$R_t = \frac{\partial}{\partial t} + e^{-au/v} \langle af_x^2 + vf_{xx} \rangle \frac{\partial}{\partial u},$$

which is consistent with the perturbed approximation in time $t$. Although invariants for $R_t$ and finite RG transformations differ from that for (4.31), the final result, i.e., the exact solution of boundary value problem (4.25) given by (4.30), is the same. This possibility is the distinct demonstration of the multi-dimensional RGS to reconstruct the unique boundary value problem solution from different perturbed approximations: either in parameter $a$ or in $t$ (though we used only two one-dimensional subalgebras here).

### 4.3.2 Example from geometrical optics

As the second example we consider the boundary value problem for a system of two first-order partial differential equations that are used in a geometrical optics of the collimated beams,

$$
\begin{align*}
\epsilon u_t + \epsilon uu_x &= 0, \quad n_t + \epsilon un_x + \epsilon nu_x = 0; \\
\quad u(0, x) &= U(x), \quad n(0, x) = N(x).
\end{align*}
$$

Here $t$ and $x$ are coordinates, respectively, along and transverse to the direction of the laser beam propagation, $u$ is the derivative of eikonal with respect to $x$, and $n$ is the laser beam intensity. In this case, the functions $U$ and $N$ characterize the curvature of the wave front (up to the factor $\epsilon$) and the beam intensity distribution upon the coordinate $x$ and the entrance of a medium $t = 0$.

The continuous point Lie group admitted by the differentiable manifold (4.34) ($\mathcal{RM}$-manifold) has the generator
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\[ X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial \varepsilon} + \eta^1 \frac{\partial}{\partial u} + \eta^2 \frac{\partial}{\partial n} \equiv \sum_{i=1}^{6} X_i, \quad \text{(4.36)} \]

with six independent operators

\[
X_1 = \frac{1}{\varepsilon} \Delta J^1 \frac{\partial}{\partial t} + (J^1 + u \Delta J^1) \frac{\partial}{\partial x} - n J^1 \frac{\partial}{\partial n}, \\
X_2 = \frac{1}{n} J^2 \left( \frac{\partial}{\partial t} + \varepsilon u \frac{\partial}{\partial x} \right), \quad X_3 = n J^3 \frac{\partial}{\partial n}, \\
X_4 = J^4 \left( -t \frac{\partial}{\partial t} + n \frac{\partial}{\partial n} + \varepsilon \frac{\partial}{\partial \varepsilon} \right), \\
X_5 = \Delta J^5 \left( t \frac{\partial}{\partial t} + \varepsilon t u \frac{\partial}{\partial x} - n \frac{\partial}{\partial n} \right) + J^5 \left( \varepsilon t \frac{\partial}{\partial x} + \frac{\partial}{\partial u} \right), \\
X_6 = -\frac{1}{n} J^6 \left( t \frac{\partial}{\partial t} + \varepsilon t u \frac{\partial}{\partial x} - n \frac{\partial}{\partial n} \right), \quad \Delta J^k = \left( \varepsilon t J^k_x - J^k_u \right). 
\]

The coordinates \( \xi \) and \( \eta \) of the infinite-dimensional generator (4.36) depend upon five functions \( J^i(\chi, u, \varepsilon), \ i = 1, 2, 3, 5, 6 \), which appear as arbitrary functions of their arguments \( \chi = x - vt, u \) and \( \varepsilon \). The sixth one, \( J^4 \), that enters the operator which describes the group transformation of parameter \( \varepsilon \), is an arbitrary function of this parameter only. The restriction of the group admitted by the \( \mathcal{RM} \)-manifold (4.34) on the solution of the boundary value problem \( u = \tilde{u}(t, x, \varepsilon), n = \tilde{n}(t, x, \varepsilon) \) leads to zero equalities for two coordinates of the operator (4.36) in the canonical form:

\[
\eta^1 + \xi^1 \varepsilon \tilde{u}_x - \xi^2 \tilde{u}_x - \xi^3 \tilde{u}_\varepsilon = 0, \quad \eta^2 + \xi^1 \varepsilon (\tilde{n} u)_x - \xi^2 \tilde{n}_x - \xi^3 \tilde{n}_\varepsilon = 0. \quad \text{(4.37)}
\]

These equalities should be valid for any values of \( t \), and certainly for \( t = 0 \), when dependencies \( \tilde{u} \) and \( \tilde{n} \) upon \( x \) are given by boundary conditions (4.35). This yields two linear relations between \( J^i \) and \( J^1_x \):

\[
J^5 = U_x J^1, \quad J^6 = N_x J^1 + N J^1_x - N (U_x) u J^1 - \varepsilon U_x J^2 - N J^3 - NJ^4. \quad \text{(4.38)}
\]

Here, and in what follows functions \( U \) and \( N \) and their derivatives with respect to \( x \) should be expressed either in terms of \( u \) or in terms of \( \chi \). Substituting (4.38) into (4.36) gives the desired RG-symmetries with the RG-operator

\[
R = \sum_{i=1}^{4} R_i, \quad \text{(4.39)}
\]
We see that RG-symmetries for (4.34), (4.35) are presented as a combination of symmetries of infinite-dimensional algebra with the infinitesimal operator (4.36).

Any of the four operators $R_k$ (and their linear combinations with coefficients that are arbitrary functions of $\varepsilon$) contains the boundary value problem solution $u = \tilde{u}(t, x, \varepsilon)$ and $n = \tilde{n}(t, x, \varepsilon)$ in the invariant manifold and is capable of improving a perturbation theory solution. As an example, consider the perturbed solution of (4.34), (4.35) for small value of $\varepsilon t \ll 1$:

$$ u = U(x) - \varepsilon t U_x + O(\varepsilon^2 t^2), $$

$$ n = N(x) - \varepsilon t (U N_x + U U_x) + O(\varepsilon^2 t^2). $$

This approximate solution in the limit $(\varepsilon t) \to 0$ is invariant with respect to RG transformation defined by the operator $R_2$ with arbitrary $\varepsilon J^2 \neq 0$. Assuming $J^2 = 1/\varepsilon$, we obtain the explicit expression for RG-operator

$$ R = \frac{1}{n} (1 + \varepsilon t U_x) \left( \frac{\partial}{\partial t} + \varepsilon u \frac{\partial}{\partial x} \right) - \varepsilon U_x \frac{\partial}{\partial n}, $$

and invariance conditions written in the form of two first-order differential equations:

$$ u_t + \varepsilon u u_x = 0, \quad (1 + \varepsilon t U_x)(n_t + \varepsilon u n_x) + \varepsilon n U_x = 0. $$

Solving Lie equations which correspond to RG-operator (4.41) (and coincide with characteristics equations for (4.42)) enables to reconstruct the desired exact solution of (4.34), (4.35) from the perturbation theory solution (4.40):

$$ u = U(x - \varepsilon u), \quad n = \frac{1}{1 + \varepsilon t U_x} N(x - \varepsilon u), $$

where $U_x$ should be expressed in terms of $u$. For example, in particular case of $N(x) = N_0 \exp(-x^2)$, $U(x) = -x$ and $\varepsilon = 1/T$ the latter formula describe the focusing of Gaussian laser beam in geometrical optics

$$ n = \frac{N_0}{1 - t/T} \exp \left( -\frac{x^2}{(1 - t/T)^2} \right), \quad u = \frac{x}{t/T - 1}, \quad t \leq T. $$

\[ \]
4.3.3 Method based on embedding equations

In this section we present a specific method of constructing RGS, which is based on the embedding equations [10]. It is of prime interest for physical systems described by ordinary differential equations. We demonstrate the idea of this method for the very simple boundary value problem

\[ u_t = f(t,u,a), \quad u(\tau) = x. \]  

(4.45)

The extension of the original differentiable manifold by adding to the original equation, the embedding equation that appears as a linear first-order partial differential equation

\[ u_\tau + f(\tau,x,a)u_x = 0 \]  

(4.46)

gives the desired manifold, where \( u \) is now treated as the function of four variables \( \{t, \tau, x, a\} \). Performing the group analysis for this basic manifold involves the boundary data and parameter \( a \) in group transformations, while the subsequent restriction of the group obtained on any solution of the boundary value problem yields the desired renormgroup symmetries. For concreteness we give an example of such calculations for \( f = au^2 \). In this event the basic manifold (4.45)-(4.46) is given by two equations

\[ u_t = au^2, \quad u_\tau + ax^2u_x = 0 \]  

(4.47)

that admit an infinite-dimensional Lie point algebra with five independent elements:

\[ X = \sum_{i=1}^{5} \alpha_i X_i, \quad X_1 = \frac{\partial}{\partial t} + au^2 \frac{\partial}{\partial u}, \quad X_2 = \frac{\partial}{\partial \tau} + ax^2 \frac{\partial}{\partial x}, \]
\[ X_3 = u^2 \frac{\partial}{\partial u}, \quad X_4 = x^2 \frac{\partial}{\partial x}, \quad X_5 = x^2 \tau \frac{\partial}{\partial x} + u^2 \tau \frac{\partial}{\partial u} + \frac{\partial}{\partial a}. \]  

(4.48)

Here, functions \( \alpha_1 \) and \( \alpha_2 \) depend upon five variables \( \{t, \tau, x, a, u\} \), whereas \( \alpha_i \), \( i = 3, 4, 5 \), are arbitrary functions of three combinations \( at + (1/u), a\tau + (1/x), a \).

The procedure of restriction of the group obtained leads to the invariance condition

\[ U^2(\alpha_3 + a\alpha_1 + \alpha_5\tau) - \alpha_1 U_t - \alpha_2 U_\tau - x^2(\alpha_4 + a\alpha_2 + \alpha_5)U_x - \alpha_5 U_u = 0 \]  

(4.49)

to be fulfilled on an exact or approximate solution \( u = U(t,x,\tau,a) \) of the boundary value problem (4.45)-(4.46); for example, one can take the perturbation theory solution as an expansion in powers of \( a \):

\[ u = U(t,x,\tau,a) \equiv x + ax^2(t-\tau) + O(a^2), \quad a \ll 1. \]  

(4.50)

Substituting (4.50) into (4.49) shows that the invariance condition (4.49) is fulfilled for \( \alpha_3 = \alpha_4 \equiv \alpha \) and arbitrary \( \alpha_1, \alpha_2 \) and \( \alpha_5 \). Assuming \( \alpha_1 = \alpha_2 = \alpha = 0 \) and \( \alpha_5 = 1 \) in (4.48) yields one of the RG-operators.
4.3 Examples

\[ R = x^2 \tau \frac{\partial}{\partial x} + \frac{\partial}{\partial a} + u^2 \frac{\partial}{\partial u}, \]  

(4.51)

which enables us to transform the perturbative solution of (4.45) for small \( a \ll 1 \) to the following exact solution:

\[ u = \frac{x}{1 - ax(t - \tau)}. \]

This result is found by solving the Lie equations, that correspond to the RG-operator (4.51).

4.3.4 Renormgroup and differential constraints

In the previous section basic manifold was obtained by combining an original differential equation and an embedding equation. More generally instead of an embedding equation, an additional differential constraint can be used that satisfy two conditions: firstly, it must be compatible with the original differential equation and, secondly, it should explicitly take boundary conditions into account. This constraint naturally emerges when a coordinate of a canonical operator of the Lie-Bäcklund renormgroup admitted by the boundary value problem is assumed to be equal to zero. Adding this constraint to original equations we obtain the basic manifold.

To illustrate, consider the boundary value problem (4.34)–(4.35) which we rewrite using hodograph transformations in a simple form for new variables \( \tau = nt \) and \( \chi = x - vt \),

\[ \chi_n = 0, \quad \chi_v + \tau_n = 0. \]  

(4.52)

The Lie-Bäcklund symmetries of this system of differential equations are given by a canonical operator

\[ X = f \frac{\partial}{\partial \tau} + g \frac{\partial}{\partial \chi}, \]  

(4.53)

with coordinates \( f \) and \( g \) depending upon \( v \) and derivatives \( \tau_s + n \chi_{s+1}, \chi_s \) of an arbitrary order \( s \geq 0 \):

\[ f = F(v, \chi_s, \tilde{\tau}_s) = n \left[ \frac{\partial}{\partial v} + \sum_{k=0}^{\infty} \left( \tilde{\tau}_{k+1} \frac{\partial}{\partial \tilde{\tau}_s} + \chi_{k+1} \frac{\partial}{\partial \chi_s} \right) \right] G, \]  

(4.54)

\[ g = G(v, \chi_s, \tilde{\tau}_s), \quad \tilde{\tau}_s = \tau_s + n \chi_{s+1}, \quad \tau_s = \frac{\partial_s \tau}{\partial v^s}, \quad \chi_s = \frac{\partial_s \chi}{\partial v^s}. \]

Consider a particular case of a boundary value problem (4.34) with boundary conditions defined by \( U(x) = -x \) and arbitrary \( N(x) \). In terms of the variables \( \tau \) and \( \chi \), these conditions are described, for example, by a pair of differential constraints

\[ \chi_{vv} = 0, \quad \tau_{vv} - N_{vv} \chi_v - N_v \chi_{vv} = 0. \]  

(4.55)
Here the dependence of \( N \) upon \( x \) is given in terms of \( \nu \) with the use of the above boundary condition.

It is easily checked by direct substituting into (4.54) that left-hand sides of these equalities are the corresponding coordinates \( g \) and \( f \) of the second-order Lie-Bäcklund symmetry operator (4.53). Adding differential constraints (4.55) to the original equation (4.52), we obtain the desired \( \mathcal{R}\mathcal{M} \)-manifold:

\[
\chi_n = 0, \quad \chi_\nu + \tau_n = 0, \quad \chi_{\nu\nu} = 0, \quad \tau_{\nu\nu} - N_{\nu\nu}\chi_\nu = 0. \quad (4.56)
\]

The latter admits a 17-parameter group of point transformations given by the operators

\[
X = \sum_{i=1}^{17} c_i X_i, \quad (4.57)
\]

where

\[
\begin{align*}
X_1 &= \nu^2 \frac{\partial}{\partial \nu} + \nu[2(n-N) + \nu N_n] \frac{\partial}{\partial n} + [\chi(N-n) + \nu \tau] \frac{\partial}{\partial \tau} + \nu \chi \frac{\partial}{\partial \chi}, \\
X_2 &= \nu \chi \frac{\partial}{\partial \nu} + [\chi(n-N) + \nu(\chi N_n - \tau)] \frac{\partial}{\partial n} + 2\tau \chi \frac{\partial}{\partial \tau} + \chi^2 \frac{\partial}{\partial \chi}, \\
X_3 &= -\nu \frac{\partial}{\partial n} + (N-n - \nu N_n) \frac{\partial}{\partial n}, \quad X_4 = \nu \chi \frac{\partial}{\partial n} - \chi^2 \frac{\partial}{\partial \chi}, \\
X_5 &= \nu \frac{\partial}{\partial n}, \quad X_6 = (N-n) \frac{\partial}{\partial n} + \chi \frac{\partial}{\partial \chi}, \quad X_7 = (n-N) \frac{\partial}{\partial n} + \tau \frac{\partial}{\partial \tau}, \\
X_8 &= \frac{\partial}{\partial \tau}, \quad X_9 = \nu \frac{\partial}{\partial \tau}, \quad X_{10} = (N-n) \frac{\partial}{\partial n} + \nu \frac{\partial}{\partial \chi}, \quad X_{11} = \frac{\partial}{\partial \chi}, \\
X_{12} &= \chi \frac{\partial}{\partial \tau}, \quad X_{13} = -\nu^2 \frac{\partial}{\partial n} + \nu \chi \frac{\partial}{\partial \tau}, \quad X_{14} = -\frac{\partial}{\partial n} - \nu N_n \frac{\partial}{\partial n}, \\
X_{15} &= \frac{\partial}{\partial n}, \quad X_{16} = \chi \frac{\partial}{\partial \nu}, \quad X_{17} = -\chi \frac{\partial}{\partial \nu} + (\tau - \chi N_n) \frac{\partial}{\partial n}.
\end{align*}
\]

The usual procedure of restriction of the group obtained on a solution of the boundary value problem (4.52) relates different coefficients in the sum (4.57) and gives the desired RG operators

\[
R = \sum_{i=1}^{13} c_i R_i, \quad (4.58)
\]

\[
\begin{align*}
R_1 &= X_1, & R_2 &= X_2, & R_3 &= X_3 + \varepsilon X_{17}, & R_4 &= X_4, \\
R_5 &= X_5 + \varepsilon X_{16}, & R_6 &= X_6 + \varepsilon X_{17}, & R_7 &= X_7 + \varepsilon X_{16}, \\
R_8 &= X_8 + \varepsilon X_{15}, & R_9 &= X_9 - \varepsilon^2 X_{16}, & R_{10} &= X_{10} - \varepsilon^2 X_{17}, \\
R_{11} &= X_{11} + \varepsilon X_{14}, & R_{12} &= X_{12} + \varepsilon X_{16}, & R_{13} &= X_{13}.
\end{align*}
\]

The exact solution of the boundary value problem

\[
\chi = -\nu/\varepsilon, \quad \tau = (1/\varepsilon)(n-N)
\]
is found either by solving Lie equations corresponding to any of these RG-operators, or as the intersection of all invariant manifolds.

References

Chapter 5
Applications of Renormgroup Symmetries

This chapter is aimed at describing feasible applications of the renormgroup symmetries. In demonstrating the efficiency of this approach we mainly use the algorithm based on approximate groups. The method can be applied to the systems described in terms of models based on differential or integro-differential equations with small parameters. These parameters allows us to consider a simple subsystem of the original equations, treated as the zero-order basic manifold $RM$, that usually admits an extended symmetry group inherited by the original equations. Restricting this approximate group on the solution of the boundary value problem yields the desired renormgroup symmetries.

We consider a limited number of examples from nonlinear physics including nonlinear optics and plasma physics. The choice of the particular physical problem is justified by the eligibility to demonstrate the great potential of the RGS approach.

5.1 Nonlinear optics

We begin with the problem, which plays an important role in nonlinear electrodynamics since 1960s, namely the problem of self-focusing a high-power light beam. The detailed quantitative understanding of self-focusing is still missing (see Ref. 45 in [2]), and there is no method which allows to find an analytic solution to the corresponding equations with arbitrary boundary conditions.

Here, we demonstrate the great potential of the RGS approach in constructing analytic solutions of boundary value problem equations with arbitrary boundary conditions. The RGS method allows to consider different types of models for self-focusing process which include the plane and cylindrical beam geometry, nonlinear refraction and diffraction. The merit of the RGS method is that it describes the boundary value problem solutions with one- or two-dimensional singularities in the entire range of variables from the boundary up to the singularity point.

Let us consider a stationary wave beam propagating in a nonlinear medium that occupies a half space $t \geq 0$. The complex amplitude of the beam inside the medium
\( t > 0 \) is defined by the \textit{nonlinear Schrödinger equation}, that is reduced to the following two partial differential equations (see, e.g., Ref. 48 in [1]) which are considered as a basic system of equations:

\[
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} - \alpha \varphi(n) \frac{\partial n}{\partial x} - \beta \frac{\partial}{\partial x} \left( \frac{x^{1-v}}{\sqrt{n}} \frac{\partial}{\partial x} \left( x^{v-1} \frac{\partial}{\partial x} \left( \sqrt{n} \right) \right) \right) = 0,
\]
\[
\frac{\partial n}{\partial t} + n \frac{\partial v}{\partial x} + \frac{\partial n}{\partial x} + (v-1) \frac{mv}{x} = 0.
\]

(5.1)

The dimensionless coordinates \( t \) and \( x \) in (5.1) are used to describe the spatial evolution of the eikonal derivative \( v \) and the normalized intensity \( n \) in the direction inside the nonlinear medium and in the transverse direction, the parameters \( \alpha \) and \( \beta \) define the role of the nonlinear refraction with the characteristic \textit{nonlinearity function} \( \varphi(n) \) and diffraction, \( v = 1 \) for a plane and \( v = 2 \) for a cylindrical beam geometry.

At the entrance of a nonlinear medium \( (t = 0) \) the curvature of the beam wave front and the normalized beam intensity distribution \( n \) upon the transverse coordinate \( x \) are given by

\[
v(0,x) = V(x) = -x/T, \quad n(0,x) = N(x).
\]

(5.2)

These boundary conditions describe a focused light beam with a smooth intensity distribution.

In case of vanishing diffraction, \( \beta \to 0 \), and in the plane beam geometry (at \( v = 1 \)), Eqs. (5.1) are linearized by hodograph transformations and are reduced to the system of basic equations

\[
\tau_w - n \chi_n / \varphi(n) = 0, \quad \chi_w + \alpha \tau_n = 0,
\]

(5.3)

for functions \( \tau = nt \) and \( \chi = x - vt \) of \( w = v/\alpha \) and \( n \) arguments, with boundary conditions

\[
\tau(0,n) = 0, \quad \chi(0,n) = H(n),
\]

(5.4)

where \( H(n) \) is the inverse to \( N(x) \).

For \( \beta \to 0 \) and the three-dimensional beam with \( v = 2 \) the hodograph transformation reduces (5.1) to a nonlinear system as well:

\[
\tau_w - \left( n/\varphi(n) \right) \chi_n = 0,
\]
\[
\chi_w + \alpha \left[ \tau_n + \frac{w}{\chi} \left( \tau_n \chi_w - \frac{n}{\varphi(n)} \chi_n^2 \right) \right] + \frac{\alpha^2 w}{n^2 \chi} \left[ 2n \tau \tau_n - \tau^2 \right] = 0.
\]

(5.5)

In the preceding chapter we employed yet simplified model with \( v = 1 \) and \( \alpha = 0 \), i.e., the system of geometrical optics equations in linear medium. Now we turn to a realistic situation that takes effects of nonlinearity and diffraction into account.
5.1 Nonlinear optics

5.1.1 Nonlinear geometrical optics

Constructing of approximate RGS for plane beam geometry

Let us start with the most simple case of a plane geometry \((v = 1)\) and neglect the diffraction, \(\beta = 0\). Here, the procedure of the RGS constructing makes use of the approximate Lie-Bäcklund symmetry and is described as follows [2]. The manifold \(\mathcal{RM}\) (step I) is defined by Eqs. (5.3) treated in the extended space that include the dependent and independent variables \(\tau, \chi, w, n\) and derivatives of \(\tau\) and \(\chi\) with respect to \(n\) of an arbitrary high order (and possibly to \(\alpha\) as well). The admitted symmetry group \(\mathcal{G}\) is represented by the canonical Lie-Bäcklund operator

\[
X = f \frac{\partial}{\partial \tau} + g \frac{\partial}{\partial \chi}.
\]

To proceed further (step II) we express the coordinates \(f\) and \(g\) of the group canonical operator as a power series with respect to a nonlinearity parameter

\[
f = \sum_{i=0}^{\infty} \alpha^i f^i; \quad g = \sum_{i=0}^{\infty} \alpha^i g^i.
\]

Expansion coefficients \(f^i\) and \(g^i\) in (5.7) are found from determining equations that express the invariance condition for the system (5.3) under group transformations with the generator (5.6) and are formally given by a system of recurrence relations that express higher-order coefficients \(f^{i+1}, g^{i+1}\) in terms of previous ones \(f^i, g^i\). It means that once the zero-order terms are specified, the other terms are reconstructed by the recurrence relations:

\[
f^i = F^i + \int \left\{ (1 - \delta_{i,0}) Z f^{i-1} + \frac{n}{\varphi} Y g^{i-1} \right\} dw,
\]

\[g^i = G^i + (1 - \delta_{i,0}) \int \{ Z g^{i-1} - Y f^{i-1} \} dw.
\]

Here

\[
Y = \frac{\partial}{\partial n} + \sum_{s=0}^{\infty} \left( \tau_{s+1} \frac{\partial}{\partial \tau_s} + \chi_{s+1} \frac{\partial}{\partial \chi_s} \right), \quad Z = \sum_{s=0}^{\infty} \tau_{s+1} \frac{\partial}{\partial \chi_s},
\]

\[
\tau_s = \frac{\partial^s \tau}{\partial n^s}, \quad \chi_s = \frac{\partial^s \chi}{\partial n^s}, \quad \bar{\tau}_s = \tau_s - w \sum_{p=0}^{s} \left( \begin{array}{c} s \\ p \end{array} \right) \frac{\partial^p (n/\varphi)}{\partial n^p} \chi_{s-p+1},
\]

and expressions in brackets before integrating over \(w\) should be given in terms of \(\bar{\tau}_s, \chi_s, n, w\). In (5.9), \(F^i(n, \chi_s, \bar{\tau}_s)\) and \(G^i(n, \chi_s, \bar{\tau}_s)\) are arbitrary functions of their arguments. This arbitrariness is eliminated by the procedure of group restriction (step III), which implies the check of the invariance condition (4.22) that impose limitations on the form of functions \(F^i\) and \(G^i\). It means that they are not arbitrary functions, but should be chosen so that relationships
which appear as the differential constraint (sometimes in particular case simply as algebraic relations), are identically valid on the desired particular solution of the boundary value problem and satisfy at $\tau = 0$ boundary conditions (5.2). Assuming for simplicity the plane phase front at $t = 0$, i.e., $T \to \infty$, these conditions are written in terms of $(\tau, \chi)$ in the following form:

$$w = 0, \quad \tau = 0, \quad \chi = H(n).$$

(5.11)

Provided that functions $F^i, G^i, i \geq 1$, are also equal to zero in this case, boundary conditions are fully correlated with the form of functions $F^0$ and $G^0$. For particular forms of $F^0$ and $G^0$, and, hence, $f^0$ and $g^0$, that is for partial boundary conditions (5.4), infinite series (5.7) are truncated automatically, and we arrive at the exact renormgroup symmetry.

For arbitrary boundary conditions, i.e., for the arbitrary function $H(n)$ in (5.11) infinite series (5.7) are of little utility. However, one can take into account only the finite number of terms provided consideration is restricted to small values of the nonlinearity parameter $\alpha$. If we neglect the higher-order terms in the case of arbitrary boundary conditions (when series (5.7) are not truncated automatically), then we get an approximate RGS, which produces an approximate solution to the boundary value problem.

Examples of RGS for plane beam geometry

Below we present several examples of operators of RGS in a medium with the cubic nonlinearity when the first two expansion coefficients in the series (5.7) are considered that give rise to the binomial form of the coordinates $f$ and $g$,

$$f = f^0 + \alpha f^1, \quad g = g^0 + \alpha g^1.$$

(5.12)

The particular form of the coefficients $f$ and $g$ depends both on the form of the intensity distribution function $N(x)$ at the medium boundary and the choice of the zero-order terms $f^0$ and $g^0$.

Example 5.1.1. $N(x) = \cosh^{-2} x$. For a soliton-like initial beam intensity distribution, two different sets of formulas are presented, that result from different forms of zero-order terms:

a) $f^0 = 2n(1-n)\tau_2 - n \tau_1 - 2nw(\chi_1 + n\chi_2), \quad f^1 = \frac{1}{2} nw^2 \tau_2,\quad g^0 = 2n(1-n)\chi_2 + (2 - 3n)\chi_1,$

$$g^1 = w (2n\tau_2 + \tau_1) + \frac{w^2}{2} (n\chi_2 + \chi_1).$$

(5.13)
b) \( f^0 = 1 + 2n \chi_1 \tanh \chi \),

\[
f^1 = \left( \frac{\tau^2}{n} - 2\tau \tau_1 + 2\tau^2 \tanh \chi \right) \cosh^{-2} \chi ,
\]

\( g^0 = 0, \quad g^1 = -2\tau \chi_1 \cosh^{-2} \chi - 2\tau_1 \tanh \chi . \)  

**Example 5.1.2.** \( N(x) = 1 - x^2 \). For the parabolic initial beam intensity distribution, we again present two sets of formulas:

a) \( f^0 = 2n(1 - n) \tau_2 - n \tau_1 + \tau - 2nw(\chi_1 + n\chi_2), \quad f^1 = \frac{1}{2} nw^2 \tau_2 , \)

\[
g^0 = 2n(1 - n)\chi_2 + (2 - 3n)\chi_1 + \chi ,
\]

\( g^1 = w(2n\tau_2 + \tau_1) + \frac{w^2}{2} (n\chi_2 + \chi_1) . \)

\[
(5.15)
\]

b) \( f^0 = n(1 + 2\chi \chi_1), \quad f^1 = \frac{\tau^2}{n} - 2\tau \tau_1 , \)

\( g^0 = 0, \quad g^1 = -2\tau \chi_1 - 2\tau_1 \chi - w . \)

\[
(5.16)
\]

One can easily check for both examples that in cases a) functions \( f^i, g^i \) for \( i \geq 2 \) vanish, hence formulas (5.13), (5.15) and (5.7) yield RGS that is an exact symmetry of basic equations. In cases b) the situation is different: series (5.7) do not truncate for \( i \geq 2 \) and formulas (5.14), (5.16) and (5.7) present RGS that is an approximate symmetry of basic equations. The dependence of coordinates \( f^i \) and \( g^i \) upon the higher derivatives indicates that in both cases a), (5.6) is the second-order Lie-Bäcklund symmetry operator, while in cases b) RGS operator is equivalent to point symmetry operator.

**Example 5.1.3.** \( N(x) = \exp(-x^2) \). For the Gaussian initial beam intensity profile, we also present two sets of coordinates \( f \) and \( g \) of the RGS operators

a) \( f^0 = 1 + 2n \chi \chi_1, \quad g^0 = 0, \)

\[
f^1 = -2\tau \tau_1 + \tau^2 / n, \quad g^1 = -2(\tau \chi_1 + \chi \tau_1) .
\]

\[
(5.17)
\]

b) \( f^0 = 2n(\tau \chi_1 + \tau_1 \chi), \quad g^0 = 1 + 2n \chi \chi_1, \)

\[
f^1 = 2\chi \tau_\alpha, \quad g^1 = 2(\chi \chi_\alpha + \tau_\tau_1) .
\]

\[
(5.18)
\]

Here, in both cases the approximate RGS is equivalent to point symmetry. The presence of the first-order derivative with respect to nonlinearity parameter \( \alpha \) in the case b) indicates that RGS takes into account transformations of nonlinearity parameter as well. The RGS operator in the case b) was obtained in view of the additional constraint \( \partial_a \chi = \partial_a \tau = 0 \) that is approximately consistent with the form of a solution in the vicinity of a beam axis (for details see Ref. 46 in [2]).

The last step IV is performed in a usual way by solving the joint system of basic equations (5.3) and equations that follow from the RG invariance condition (4.24),
or else, using invariants of associated characteristic equations for RG operator provided that RGS is a Lie point symmetry.

Comparison of exact and approximate solutions

In this section we compare exact and approximate solutions of boundary value problem that result from the RGS obtained above.

Let us consider first the pair of solutions, defined by RGS (5.13)-(5.14)

\[ v = -2\alpha nt \tanh(x - vt), \quad \alpha n^2 t^2 = n \cosh^2(x - vt) - 1; \]  
\[ v = -2\alpha n t \tanh(x - vt), \quad \alpha n^2 = \cosh^2(x - vt) \ln(n \cosh^2(x - vt)). \]  

The first expressions (5.19) is the well-known Khokhlov solution (this solution was first obtained in Refs. 41 in [1], though in this publication it did not result from a regular procedure) that describes the process of self-focusing of a soliton beam: the sharpening of the beam intensity profile with the increase of \( t \) is accompanied by the intensity growth on the beam axis. The solution (5.19) is valid up to the singularity point where the derivatives \( v_x \) and \( n_x \) tend to infinity whilst the beam intensity \( n \) remains finite.

Calculated with the help of formulas (5.19) and (5.20) the plots, illustrating dependencies of functions \( v \) and \( n \) upon \( x \) for different values of \( t \) are presented in Fig. 5.1 for \( \alpha = 0.01 \). The comparison of these plots testifies that there is a good agreement between the exact (solid lines) and approximate (dashed lines) solutions in the off-axis region, and the discrepancy is observed in the near axis region for those values of \( t \), which are close to the point of the solution singularity. The location of the singularity point on the beam axis (i.e., the point where the derivative \( v_x \) tends to infinity) and the intensity value at this point for the exact and approximate solutions are given as follows:

\[ t_{\text{max}}^{\text{sol}} = 1/2\sqrt{\alpha}, \quad n_{\text{max}}^{\text{sol}} = 2; \quad t_{\text{approx}}^{\text{sol}} = 1/\sqrt{\alpha e}, \quad n_{\text{approx}}^{\text{sol}} = e. \]  

Thus, the location of the singularity point for the approximate solution (5.20) is shifted as compared to the exact one inside the medium and the maximum value of
the beam intensity is greater although the qualitative behavior of \( n \) and \( v \) remains the same.

Now let us analyze the pair of solutions defined by RGS (5.15)−(5.16):

\[
v = -2x\sqrt{\alpha}d^{-3/2}(1 - d)^{1/2}, \quad n = (1/d) \left(1 - (x/d)^2\right),
\]

\[
d(t) = \cos^2 \left(\sqrt{d(t)(1 - d(t))} - 2t\sqrt{\alpha}\right);
\]

\[
v = -\frac{x\sqrt{2}\alpha\tanh q}{1 - \sqrt{2}\alpha\tanh q}, \quad q(t) = \frac{1}{\sqrt{2}} \arccosh \frac{1 + \alpha t^2}{1 - \alpha t^2},
\]

\[
n = \frac{1}{1 - \alpha t^2} \left(1 - \left(\frac{x}{\cosh q(1 - 2\alpha t\tanh q)}\right)^2\right).
\]

(5.22)

\[\text{Fig. 5.2} \text{ Plots of } v \text{ and } n \text{ versus } x \text{ for different values of } t. \text{ 1 : } t = 1, 2 : t = 5, 3 : t = 7.\]

The plots calculated with the help of formulas (5.22) (solid lines) and (5.23) (dashed lines), are presented in Fig. 5.2 for \( \alpha = 0.01 \) and different values of \( t \). One can see that as in the previous case the discrepancy between the exact (5.22) (see Ref. 48 in [1]) and approximate (5.23) solutions is observed for the values of \( t \) close to the point of the solution singularity where the derivative \( v_x \) tends to infinity. The location of the singularity point on the beam axis and the intensity value at this point for the exact and approximate solutions are given by formulas:

\[
h_{\text{par max}} = \pi/4\sqrt{\alpha}; \quad h_{\text{par max}} = \infty; \quad n_{\text{par max}} \approx 2.584.
\]

It follows from (5.22)−(5.24) that there is a good correlation in qualitative behavior of functions \( v \) and \( n \) defined by exact and approximate solutions: the parabolic shape of the beam and linear dependence of \( v \) upon \( x \) is preserved. Moreover, the position of the singularity point \( t_{\text{par max}} \approx 0.785/\sqrt{\alpha} \) is very close to \( t_{\text{approx}} \). However, the beam intensity for the approximate solution turns to infinity at a point \( t = 1/\sqrt{\alpha} \) that is shifted inside the medium as compared to \( t_{\text{approx}} \).

The mentioned peculiarities are also observed while comparing another pair of solutions defined by RGS for the Gaussian beam (5.17)−(5.18) (for details see Ref. 46 in [2]).
\( x^2 = (\alpha n^2 - \ln n) \left[ 1 - P(\sqrt{\alpha n^2}) \right]^2, \quad v = -\frac{x}{t} \frac{P(\sqrt{\alpha n^2})}{1 - P(\sqrt{\alpha n^2})} \) \hspace{1cm} (5.25)

\[
P(z) = 2ze^{-z^2/2} \int_0^z e^{t^2/2} dt;
\]

\( x^2 = (1 - 2\alpha n^2)^2 \ln \frac{1}{n(1 - \alpha n^2)}, \quad v = -\frac{2x\alpha n}{1 - 2\alpha n^2} \) \hspace{1cm} (5.26)

Fig. 5.3 demonstrates the plots constructed on basis of formulas (5.25) (dashed lines) and (5.26) (solid lines) for the dependencies of functions \( v \) and \( n \) upon \( x \) for \( \alpha = 0.01 \) and different values of \( t \). These plots show that the self-focusing of the Gaussian beam is qualitatively very similar to the spatial evolution of the soliton beam (5.19). The approximate solution (5.25) yields the same values of the focus point location and the increase of the beam intensity on the axis as the approximate solution (5.20) with another boundary condition. This is also true for solutions (5.19) and (5.26): values of \( t_{\text{max}} \) and \( n_{\text{max}} \) for these solutions coincide. Although formulae (5.26) correspond to an approximate solution of the boundary value problem, they exactly describe the behaviour of \( n \) on the beam axis at \( x = 0 \). To estimate the reliability of result (5.26) in the off-axis region, we compared it with another approximate boundary value problem solution (5.25). An agreement of these plots in the far off-axis region indicates the possibility of applying formulas (5.26) everywhere over the \( x \)-region. An agreement of these plots in the far off-axis region indicates the possibility of applying formulas (5.26) everywhere over the \( x \)-region.

**Example of RGS in cylindrical geometry**

In the above discussion we dealt with the plane beam geometry and took into account only effects of nonlinear beam refraction, neglecting diffraction. The flexibility of RGS algorithm allows one to apply it in a similar way to a more complicated model as compared to (5.3), e.g., for the cylindrical beam geometry, \( \nu = 2 \), i.e., for Eqs. (5.5). Coordinates \( f^0 \) and \( g^0 \) are given by the same formulas (5.8), whereas expressions for operator coordinates \( f^i \) and \( g^i \) for \( i \geq 1 \) result from more complicated
5.1 Nonlinear optics

calculations. Below we present explicit formulas only for coordinates \( f^1 \) and \( g^1 \) for the particular case of a medium with cubic nonlinearity (\( \varphi = 1 \)):

\[
\begin{align*}
\hat{f}^1 &= F^1 + \int \left\{ \hat{\mathcal{Z}} f^0 + n Y g^0 \right\} \, dw, \\
\hat{g}^1 &= G^1 - \frac{n \chi_1^2}{\chi^2} g^0 + 2 \frac{n \omega X_{1}}{\chi} Y g^0 + \int \left\{ \hat{\mathcal{Z}} g^0 - Y f^0 \right\} \, dw.
\end{align*}
\] (5.27)

Here \( F^1(n, \chi, \tilde{\tau}) \) and \( G^1(n, \chi, \tilde{\tau}) \), as before, are arbitrary functions of their arguments, and expressions in brackets before integrating with respect to \( w \) should be expressed in terms of \( \tilde{\tau}, \chi, n, w \).

It follows from Eqs. (5.27) that the symmetry \( f^0 \) and \( g^0 \) is \textit{inherited} by the system of Eqs. (5.5) with \( v = 2 \) as approximate in first order of a small parameter \( \alpha \). Depending upon the form of \( f^0 \) and \( g^0 \) this symmetry may appear either as Lie-Bäcklund or as a point symmetry. The full list of the first-order Lie-point generators of the approximate symmetries are to be found in Ref 46 from [2]. Here, we present two simple illustrations.

**Example 5.1.4.** \( N(x) = \exp(-x^2) \). Defining coordinates \( f^0 = 1 + 2n \chi \chi_1 \) and \( g^0 = 0 \), we obtain the point RG-symmetry for the cylindrical Gaussian beam

\[
R = -2 \chi \frac{\partial}{\partial w} + 4a \tau \frac{\partial}{\partial n} + \left( 1 + \frac{2a \tau^2}{n^2} \right) \frac{\partial}{\partial \tau}. \] (5.28)

It is obvious that as compared with the case of the plane beam (5.17) there is double distinction in the coefficients containing nonlinearity parameter \( \alpha \), and the dependencies that characterize the spatial distribution of functions \( v \) and \( n \) for the cylindrical beam from (5.25) on substituting \( \alpha \rightarrow 2\alpha \) and \( P \rightarrow P/2 \).

**Example 5.1.5.** \( N(x) = 1 - x^2 \). Defining coordinates \( f^0 = n + 2n \chi \chi_1 \) and \( g^0 = 0 \), we obtain the point RG-symmetry for the cylindrical parabolic beam

\[
R = -2 \chi \frac{\partial}{\partial w} + 4a \tau \frac{\partial}{\partial n} + n \left( 1 + \frac{2a \tau^2}{n^2} \right) \frac{\partial}{\partial \tau} - aw \frac{\partial}{\partial \chi}. \] (5.29)

Substituting expressions for coordinates of this operator directly into group determining equations one can check that the obtained RG-symmetry is exact, i.e., it is valid for arbitrary value of the parameter \( \alpha \), and it gives rise to the exact solution of the boundary value problem which describes the sharp focusing of rays of the parabolic beam (see, e.g. Ref. 47 in [2]). This group nature of this solution is easily reproduced if we rewrite the generator (5.29) in original variables \( \{ t, x, n, v \} \),

\[
R_{\text{par}} = \left( 1 - 2\alpha \tau^2 \right) \frac{\partial}{\partial t} - 2\alpha \tau x \frac{\partial}{\partial x} - 2\alpha (x - vt) \frac{\partial}{\partial v} + 4\alpha n \frac{\partial}{\partial n}. \] (5.30)

The solution to the boundary value problem is expressed in terms of group invariants for this generator:
\[ J_1 = \frac{x^2}{\rho}; \quad J_2 = n\rho; \quad J_3 = 2\alpha x^2 - v^2 \rho + \frac{xy}{2} \rho; \quad \rho = (1 - 2\alpha t^2). \quad (5.31) \]

The explicit form of dependencies of \( J_2 = 1 - J_1^2, \) \( J_3 = 2\alpha J_1 \) upon \( J_1 \) follows from the boundary conditions (5.2) and lead to the well-known solution

\[ v = (x/(2\rho))\rho, \quad n = (1/\rho)(1 - (x^2/\rho)) \quad (5.32) \]

that describes the convergence of the beam to the singularity point \( t_{\text{sing}}^{\text{par}} = 1/\sqrt{2\alpha} \) where \( \rho = 0 \) and \( n \to \infty. \) The solution singularity is two-dimensional here: the infinite growth of beam intensity in the vicinity of the singularity \( t \to t_{\text{sing}}^{\text{par}} \) is accompanied by the infinite growth of the derivative \( v_x \) and collapsing of the beam size in the transverse direction.

### 5.1.2 Nonlinear wave optics

**Constructing of RGS for the cylindrical beam geometry**

The procedure of constructing RGS for \( \beta \neq 0 \) with RG algorithm based on approximate groups is in many respects similar to that in the previous section (see [2] and Refs. 45, 46, 48 and 49 therein). The manifold \( \mathcal{RM} \) (step I) is defined by Eqs. (5.1) treated in the extended space that include dependent and independent variables \( \{t, x, v, n\} \) and derivatives of \( v \) and \( n \) with respect to \( x \) of an arbitrary high order. While calculating the admitted symmetry group \( \mathcal{G} \) (step II) with the canonical Lie-Bäcklund operator

\[ X = f \frac{\partial}{\partial v} + g \frac{\partial}{\partial n}, \quad (5.33) \]

we present coordinates \( f \) and \( g \) as series in powers of the nonlinearity and the diffraction parameters \( \alpha \) and \( \beta, \)

\[ f = \sum_{i,j=0}^{\infty} \alpha^i \beta^j f^{(i,j)}, \quad g = \sum_{i,j=0}^{\infty} \alpha^i \beta^j g^{(i,j)}. \quad (5.34) \]

Expansion coefficients \( f^{(i,j)}, g^{(i,j)} \) in (5.34) are functions of the variables \( t, x, v, n \) and of the arbitrary-order derivatives of \( v \) and \( n \) with respect to \( x \) and are defined by an infinite set of determining equations that follow from the invariance of the system of Eqs. (5.1) with respect to transformations with operator (5.33).

We consider here only zero-order terms \( f^0 \equiv f^{(0,0)}, g^0 \equiv g^{(0,0)}, \) which are independent on \( \alpha \) and \( \beta, \) and first order terms, which are linear over \( \alpha \) or \( \beta, \)

\[ f^1 \equiv \alpha f^{(1,0)} + \beta f^{(0,1)}, \quad g^1 \equiv \alpha g^{(1,0)} + \beta g^{(0,1)}. \]

In what follows, we confine ourselves to precisely these contributions to infinite series (5.34) and neglect the bilinear and higher-order contributions,
5.1 Nonlinear optics

\[ f = f^0 + f^1 + O(\alpha^2, \beta^2, \alpha \beta), \quad g = g^0 + g^1 + O(\alpha^2, \beta^2, \alpha \beta). \] (5.35)

This is justified if the coordinates \( f \) and \( g \) contain only linear contributions with respect to the parameters \( \alpha \) and \( \beta \) or the values of these parameters are small. In the latter case, the neglect of higher-order terms means that we are finding an approximate symmetry.

Here, we present the system of linear first-order partial differential equations for calculating the coordinates \( f^0, g^0 \) and \( f^1, g^1 \) of canonical operator (5.33) for a cylindrical wave beam (\( \nu = 2 \)):

\[
\begin{align*}
&M_0 f^0 = 0, \quad M_1 g^0 + M_2 f^0 = 0, \\
&M_0 f^1 + D_1 f^0 - \alpha (\varphi D_x + \varphi_3 n_1) g^0 - \beta \{B_n g^0 \} + B_{n_1} (D_x g^0) + B_{n_2} (D_x^2 g^0) + B_{n_3} (D_x^3 g^0) = 0, \\
&M_1 g^1 + D_1 g^0 + M_2 f^1 = 0.
\end{align*}
\] (5.36)

The differential operators entering these equations have the forms

\[
\begin{align*}
&\frac{D_x}{t} - \sum_{s=0}^{\infty} \left[ D_s (\alpha \varphi n_1 + \beta B) \right] \frac{\partial}{\partial v_s}, \\
&D_x (\alpha \varphi n_1 + \beta B) \frac{\partial}{\partial v_s}, \\
&D_x \left[ v_{s+1} \frac{\partial}{\partial v_s} + n_{s+1} \frac{\partial}{\partial n_s} \right], \\
&M_0 = D_x^0 + \nu D_x + v_1, \quad M_1 = M_0 + \frac{\nu}{x}, \quad M_2 = n D_x + n_1 + \frac{n}{x}, \\
&B = D_x \left( (D_x (x D_x (\sqrt{n})) ) / x \sqrt{n} \right), \quad n_s \equiv \frac{\partial^n}{\partial x^n}, \quad v_s \equiv \frac{\partial^s}{\partial x^s}.
\end{align*}
\] (5.37)

The solutions of the linear differential equations for the functions \( f^1 \) and \( g^1 \) involve the arbitrary functions \( F^1 \) and \( G^1 \) (akin to \( F^i \) and \( G^i \) in (5.8)), which depend on the infinite set of invariants of the operator \( \nu D_x + D_x^0 \). This arbitrariness is eliminated by the group restriction (step III) under which the fulfillment of the invariance condition

\[ f = 0, \quad g = 0 \] (5.38)

for the solution of the boundary value problem and the compatibility between this condition and prescribed boundary data (5.2) is verified. In particular one can choose the following simple representation for \( f^0 \) and \( g^0 \) valid both for \( \nu = 1 \) and \( \nu = 2 \):

\[
\begin{align*}
f^0 &= \frac{1}{2T^2} D_x \left( x + \nu T \left( 1 - \frac{I}{T} \right) \right)^2, \\
g^0 &= \frac{x^{1-\nu}}{T} \left( 1 - \frac{I}{T} \right) D_x \left[ nx^{\nu-1} \left( x + \nu T \left( 1 - \frac{I}{T} \right) \right) \right].
\end{align*}
\] (5.39)
Direct substitution of (5.39) into (5.36) proves that functions $f^0$ and $g^0$ satisfy invariance conditions for arbitrary boundary conditions (5.2). In view of (5.39) the following expressions for the first order terms $f^1$ and $g^1$ in a medium with cubic nonlinearity ($\varphi = 1$) are obtained:

a) $v = 1$,

$$
\begin{align*}
    f^1 &= D_x \left\{ \beta \left( \frac{1}{\sqrt{N}} \right) \left( \sqrt{N} \right) \chi_x - \beta \left( 1 - \frac{t}{T} \right)^2 \left( \frac{1}{\sqrt{n}} \right) D_x (D_x (\sqrt{n})) \\
    &- \alpha n \left( 1 - \frac{t}{T} \right)^2 + \alpha \frac{n}{v_1 T} \left( 1 - \frac{t}{T} + \frac{1}{v_1 T} \right) \ln n - \alpha \frac{n}{v_1 T} \left( 1 - \frac{t}{T} \right) \right\}, \\
    g^1 &= D_x \left\{ \alpha \frac{nn_1}{v_1^2 T} \left( \frac{\ln n}{v_1 T} + \left( 1 - \frac{t}{T} + \frac{1}{v_1 T} \right) (\ln n - 1) \right) \\
    &- \alpha \frac{n^2 v_2}{v_1^3 T} \left( \frac{2 \ln n - 1}{v_1 T} + \left( 1 - \frac{t}{T} + \frac{1}{v_1 T} \right) \left( \ln n - \frac{3}{2} \right) \right) \\
    &- \alpha TNN_x \ln n - \beta \tau n \left( \left( \sqrt{N} \right) \chi_x / \sqrt{N} \right) \chi \right\}. 
\end{align*}
$$

(5.40)

b) $v = 2$,

$$
\begin{align*}
    f^1 &= D_x \left\{ S - \left( 1 - \frac{t}{T} \right)^2 \left( \alpha n + \frac{\beta}{x \sqrt{n}} D_x (xD_x \sqrt{n}) \right) \right\}, \\
    g^1 &= -\frac{1}{x} D_x (x \tau n S_x),
\end{align*}
$$

(5.41)

and the function $S$ depends on $\chi = x - vt$,

$$
S(\chi) = \alpha N(\chi) + \frac{\beta}{\chi \sqrt{N(\chi)}} \frac{\partial}{\partial \chi} \left( \chi \frac{\partial}{\partial \chi} \left( \sqrt{N(\chi)} \right) \right). 
$$

(5.42)

Formulas (5.39) and (5.40) define the third-order Lie-Bäcklund RGS for a plane wave beam, and the canonical group operator with coordinates (5.39) and (5.41) is equivalent to the point RG operator for the cylindrical beam,

$$
\begin{align*}
    R &= \left[ \left( 1 - \frac{t}{T} \right)^2 + t^2 S_{xx} \right] \frac{\partial}{\partial t} + \left[ \frac{x}{T^2} + \frac{v}{T} \left( 1 - \frac{t}{T} \right) + S_x \right] \frac{\partial}{\partial v} \\
    &+ \left[ -\frac{x}{T} \left( 1 - \frac{t}{T} \right) + ts_x + vt^2 S_{xx} \right] \frac{\partial}{\partial x} \\
    &+ \left[ \frac{2n}{T} \left( 1 - \frac{t}{T} \right) - nt \left( 1 + \frac{vt}{x} \right) S_{xx} - \frac{nt}{x} S_x \right] \frac{\partial}{\partial n}.
\end{align*}
$$

(5.43)

Just as in the case $\beta = 0$, there exist specific forms of boundary distribution, $N$, for which the RGS generator (5.43) defines exact (not approximate) symmetry valid for arbitrary values of $\alpha$ and $\beta$. It can be verified directly by searching for the exact symmetry of basic equations (5.1) that series (5.34) contain only zero-order and
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linear in $\alpha$ and $\beta$ terms for

$$S = s_0 + s_2 \chi^2/2, \quad (5.44)$$

and bilinear and higher-order terms vanish.

Constructing a particular solution to the boundary value problem (step IV) is performed in a usual way by solving the joint system of basic equations (5.1) and equations that follow from the RG invariance condition (4.24), or else, using invariants of associated characteristic equations for RG operator provided that RGS is a Lie point symmetry. Namely, for the generator (5.43) it implies the use of related group invariants, and the procedure is similar to that one for the parabolic beam, with the generator (5.16). The desired solution of the boundary value problem has the form

$$v(t,x) = \frac{x - \chi}{t}, \quad n(t,x) = N(\mu) \left(1 - \frac{t}{T}\right)^{-1} \frac{\chi S_x^2}{x S_{\mu}^2}. \quad (5.45)$$

Here, $\chi$ and $\mu$ are defined as functions of $t$ and $x$ via the relations

$$x = \chi \left(1 - \frac{t}{T}\right) \left(1 + \frac{2t^2 S_{x}^2}{(1-t/T)^2}\right), \quad S(\mu) = S(\chi) + \frac{t^2 S_{x}^2}{2(1-t/T)^2}. \quad (5.46)$$

We stress that formulas (5.46) are derived without any a priori conditions on the spatial structure of the beam in the medium and these formulas consequently apply to beams with arbitrary smooth boundary data (5.2). The only condition for formulas (5.46) to be not exact but approximate (except some specific types of functions $S(x)$) relates to the smallness of the parameters $\alpha$ and $\beta$. These conditions are used in constructing the RGS as partial sums of series (5.34), and the RGS is approximate in this sense.

Here we illustrate the solutions (5.45) to the boundary value problem (5.1), (5.2) for several variants of the beam intensity distribution at the boundary of the nonlinear medium.

**Example 5.1.6.** Let $S(x)$ be a binomial (5.44). In this case, expressions (5.45) are exact solutions of original equations (5.1), and are valid for arbitrary values of the parameters $\alpha$ and $\beta$. This form of $S(x)$ corresponds to a quite definite dependence of the beam intensity $N$ at the medium boundary on the coordinate $x$. This dependence is found by solving the second-order differential equation for the function $Q(x) = \sqrt{N(x)}$:

$$\beta \left(Q_{xx} + \frac{1}{x} Q_x\right) + \alpha Q^3 - \left(s_0 + \frac{s_2 x^2}{2}\right) Q = 0. \quad (5.47)$$

Substituting (5.44) into (5.45), (5.46) results in formulas preserving the form of the dependence of $v$ and $n$ on the transverse coordinate:
The form of (5.48) implies that the singularity can appear on the beam axis only for \( s_2 \leq 0 \). For \( s_2 = 0 \) the beam focusing at the point \( t = T \) with a “purely geometric” character due to the curvature of the beam phase front on the medium boundary. For \( s_2 = 0 \) and \( T \to \infty \), beam self-channeling occurs without the formation of singularities (the well-known case studied by Townes (Chiao, R., Garmire, E., and Townes, G., *Phys. Rev. Lett.* **16**, 1966, No. 9, 347—349)). For negative values of \( s_2 \), the intensity of the beam blows up at the axis at the point \( t_{f1} = T / \left( 1 + \sqrt{-s_2 T^2} \right) \) as \( n_{axis} = 1 / \left( (1 - t / t_{f1}) (1 - t / t_{f2}) \right) \), where \( t_{f2} = T / \left( 1 - \sqrt{-s_2 T^2} \right) \).

**Example 5.1.7.** Consider now the evolution of a beam with the Gaussian initial intensity distribution \( N(x) = \exp(-x^2) \) that gives

\[
S(x) = \alpha e^{-x^2} + \beta (x^2 - 2).
\]  

(5.49)

Substituting (5.49) into (5.45), (5.46) leads to formulas determining the spatial structure of the Gaussian beam in a nonlinear medium,

\[
v(t, x) = \frac{x - \chi}{t}, \quad n(t, x) = e^{-\mu^2} \left( 1 - \frac{t}{T} \right)^{-1} \chi \left( \beta - \alpha e^{-\mu^2} \right) \frac{\chi}{\beta - \alpha e^{-\mu^2}}.
\]  

(5.50)

Here, the functions \( \chi \) and \( \mu \) are expressed via \( t \) and \( x \) by the relations

\[
\beta \mu^2 + \alpha e^{-\mu^2} = \beta \chi^2 + \alpha e^{-\chi^2} + \frac{2t^2 \chi^2}{(1 - t/T)^2} \left( \beta - \alpha e^{-\chi^2} \right)^2,
\]

\[
x = Y(t, \chi) \equiv \left( 1 - \frac{t}{T} \right) \chi \left[ 1 + \frac{2t^2}{(1 - t/T)^2} \left( \beta - \alpha e^{-\chi^2} \right) \right].
\]  

(5.51)

The solution (5.50), (5.51) describes the self-focusing of the cylindrical Gaussian beam that gives rise to the two-dimensional singularity: both the beam intensity at the axis \( n(t, 0) = 1 / (1 - t / t_{f1})(1 - t / t_{f2}) \) and derivatives \( v_x, n_x \) go to infinity at the point \( t = t_{f1} \) provided that \( \alpha > \beta \), and \( t_{f1} = T / \left[ 1 + T \sqrt{2(\alpha - \beta)} \right] \), \( t_{f2} = T / \left[ 1 - T \sqrt{2(\alpha - \beta)} \right] \). Note that the coordinate of a singularity point for both Examples 5.1.6 and 5.1.7 results from a unified formula \( t_s = T / \left( 1 + T \sqrt{2S_{\chi^2}(0)} \right) \). A detailed analysis of (5.50) is given in Refs. 45, 48 in [2].
5.1 Nonlinear optics

5.1.3 Renormgroup algorithm using functionals

In analyzing the behavior of a light beam in a nonlinear medium the appearance of a solution singularity on a beam axis represents the most attracting physical effect. The surprising thing is that this behavior can be understood without knowledge of the complete solution via the application of RG algorithm to two functionals of the boundary value problem solution, namely, to the intensity of a laser beam \( n^0(t) = n(t, 0) \) and to the second derivative of the eikonal \( w^0(t) = v_x(t, 0) \), calculated on an axis of a beam and related to this solution by formal relationships

\[
n^0(t) = \int \delta(x)n(t,x)\,dx, \quad w^0(t) = \int \delta(x)v_x(t,x)\,dx.
\]  

(5.52)

Boundary conditions for these functionals with the account of (5.2) are given as

\[
n^0(0) = 1, \quad w^0(0) = 0.
\]  

(5.53)

In spite of the fact that these conditions do not contain data on the dependence of a beam intensity upon the coordinate \( x \), such information is included in the RGS operator which explicit form is defined by the profile of the beam intensity \( N(x) \) at \( t = 0 \). We present two examples, corresponding to cylindrical and plane light beams with various \( N(x) \).

**Example 5.1.8.** For a cylindrical beam \( (\nu = 2) \) with the parabolic intensity distribution \( N(x) = 1 - x^2 \), the RGS generator has the form (5.30). To determine the dependence of \( n^0 \) and \( w^0 \) on the coordinate \( t \) we prolong (5.30) on non local variables \( (solution \text{ functional}) \ n^0 \) and \( w^0 \) that gives the following generator in the reduced space of variables \( \{t, n^0, w^0\} \):

\[
R = \left(1 - 2\alpha t^2 \right) \frac{\partial}{\partial t} + 4\alpha n^0 t \frac{\partial}{\partial n^0} - 2\alpha (1 - 2tw^0) \frac{\partial}{\partial w^0}.
\]  

(5.54)

The use of two invariants of generator (5.54), \( J_1 = (1 - 2\alpha t^2)n^0 \) and \( J_2 = w^0(1 - 2\alpha t^2) + 2\alpha t \) with evident equalities \( J_1 = 1 \) and \( J_2 = 0 \), which follow at the account of boundary conditions (5.53), immediately gives expressions

\[
n^0 = \frac{1}{1 - 2\alpha t^2}, \quad w^0 = -\frac{2\alpha t}{1 - 2\alpha t^2}.
\]  

(5.55)

These formulas describe spatial dependence of variables \( n^0(t) \) and \( w^0(t) \), starting from a boundary of a nonlinear medium \( t = 0 \) up to the point \( t_{\text{sing}} = 1/\sqrt{2\alpha} \), where the solution singularity occur, i.e., where the beam intensity and the eikonal derivative turns to infinity; beyond this point there is an area of rays intersection, where Eqs. (5.3) cannot be applied. Expressions (5.55) also follow from Eqs. (5.24), however, the RGS algorithm applied here to solution functionals presents an elegant way of obtaining these formulas without calculating the complete solution to boundary value problem.
Curves of typical dependencies of variables $n^0(t)$ and $w^0(t)$ upon the dimensionless coordinate $t/t_{\text{sing}}$ at $\alpha = 0.1$ are given in Fig. 5.4. The change of the parameter $\alpha$ does not change a type of the curve for the intensity $n^0$, whilst values of $w^0$ on the right panel vary proportionally to $\sqrt{\alpha}$. Block curves correspond to formulae (5.55), i.e., to the parabolic distribution of an intensity of a cylindrical beam at the medium boundary; dotted curves refer to a plane geometry of the beam, considered below.

![Fig. 5.4 Dependencies of the intensity of a laser beam (at the left) and the second derivative of its eikonal (on the right) on the beam axis $x = 0$ at various distance $t/t_{\text{sing}}$ from the boundary of a nonlinear medium $t = 0$, plotted with the use of formulas for the cylindrical (5.55) (block curves) and plane (5.58) (dotted curves) geometry.](image)

**Example 5.1.9.** The procedure of prolongation of the operator on nonlocal variables uses a canonical form of RG generators (Lie-Bäcklund operators) and is suitable also in that case when this generator is given by a higher-order Lie-Bäcklund symmetry. Such case is realized for a plane laser beam with “soliton” profile of the intensity distribution at the boundary, $N(x) = \cosh^{-2}(x)$, when the appropriate RGS generator has rather cumbersome form, either in $\{\tau, \chi\}$ representation (5.6), (5.13), or in $\{v, n\}$ representation (see Ref. 38 in [1]). Prolongation on nonlocal variables (5.52) gives the more simple operator in space of functionals

$$R = \left(4 - 5n^0 - tn_0^0 + 2(n^0 - 1)\frac{n^0\nu_0}{(n_t^0)^2}\right)\frac{\partial}{\partial n^0} + \left(\frac{n_0^0}{n^0} + t\frac{n_0^0}{n_0^0}\right)\frac{\partial}{\partial n^0}$$

$$-t\left(\frac{n_t^0}{n^0}\right)^2 - 2(n^0 - 1)\left[\frac{n_0^0}{(n_t^0)^2} + 2\frac{n_t^0}{(n^0)^2} - 2\left(\frac{n_0^0}{n^0}\right)^2\right]\frac{\partial}{\partial w^0}. \quad (5.56)$$

While obtaining this formula we used the relation between derivatives of functions $n$ and $\nu_\tau$ with respect to spatial variables on the beam axis (at $x = 0$), which follows from the initial equations,

$$\nu_{xxx}(t, 0) = \frac{1}{\alpha n^0} \left[\frac{n_0^0}{n^0} + 10\left(\frac{n_0^0}{n^0}\right)^3 - 8\frac{n_0^0}{(n^0)^2}\right],$$

$$\nu_\tau(t, 0) = \frac{n_0^0}{n^0}, \quad n_{xx}(t, 0) = \frac{1}{\alpha} \left[2\left(\frac{n_0^0}{n^0}\right)^2 - \frac{n_0^0}{n^0}\right]. \quad (5.57)$$
The beam intensity and its second eikonal derivative on an axis are defined from the RG invariance condition (4.24), which is equivalent to vanishing of the coordinates of the canonical generator (5.56). These conditions gives two ordinary differential equations of the second and the third order respectively. Solving at first ordinary differential equation of the second order with the initial conditions (5.53) and the additional condition on the first derivative, 

\[ (n^0_t/\sqrt{n^0-1})_{r=0} = 2\sqrt{\alpha} \]

which follows from the relation (5.57) at \( t = 0 \), we obtain in the implicit form the law of the variation of \( n^0 \) and \( w^0 \),

\[ t = \frac{\sqrt{n^0-1}}{\sqrt{\alpha n^0}}, \quad w^0 = -\frac{2\alpha n^0}{1-2\alpha \alpha^2 n^0}. \]  

(5.58)

These formulas are valid from the boundary of a nonlinear medium \( t = 0 \) up to a point where the solution singularity occurs. The coordinate of the solution singularity is found in view of the fact that the derivative of \( w^0 \) turns to infinity at this point, that gives \( t_{\text{sing}} = 1/2\sqrt{\alpha} \), and the value of the intensity \( n^0 \) in this point is equal to two. The solution of the remaining ordinary differential equation of the third order gives the same result. Dependencies of \( n^0 \) and \( w^0 \) upon the dimensionless coordinate \( t/t_{\text{sing}} \) at \( \alpha = 0.1 \) are plotted on Fig. 5.4 by dotted curves. Without prolongation of RGS on nonlocal variables the result (5.58) follows from (5.19) obtained earlier though in a more complicated way.

We note, that universality of a procedures of prolongation of RG generators presented either as point group operators, or Lie-Bäcklund group operators has allowed to describe from uniform positions an occurrence of a singularity of the boundary value problem solution to (5.3)–(5.2), using for this purpose the reduced description in terms of solution functionals.

5.2 Plasma physics

5.2.1 Harmonics generation in inhomogeneous plasma

We now apply the RG algorithm to find the approximate analytical solutions to the boundary-value problem for the equations of the nonlinear interaction of a laser radiation with a plasma (see Refs. 15, 16 in [1]). Such an interaction for a \( p \)-polarized electromagnetic wave of frequency \( \omega \), which propagates from the vacuum toward an inhomogeneous plasma, is described by a system of the nonlinear nonstationary equations (hydrodynamic equations for collisionless electron plasma) for six scalar functions: two components of the electron velocity \( V_x \) and \( V_y \), the electron density \( n \), two electric field components \( E_x \) and \( E_y \), and the \( z \) component \( B_z \) of the magnetic induction; these functions depend on time \( t \) and the two coordinates \( x \) and \( y \). In a weak nonlinear limit the solution of this problem is presented by a series in powers of a small amplitude of the incident electromagnetic wave. We are interested in finding the analytical solution in the non-perturbation limit, i.e., for the case of the strong nonlinearity, using the RG algorithm.
When constructing $\mathcal{RM}$ manifold (step I) we take into account that the non-linearity of basic equations is essential in a small space domain near the plasma resonance (at $\omega_k^2 \approx \omega$) where the presence of natural small parameters (such as the smooth inhomogeneity of the ion density $N(x)$ along the $x$ axis and the small incidence angles $\vartheta$ of laser beams at the plasma) implies the appearance of a hierarchy of components of the $p$-polarized light wave at the critical plasma point. When constructing the inherited point RGS, this allows reducing the system of six initial equations to a simpler system of two one-dimensional nonlinear partial differential equations for the normalized components of the electron velocity and the electric field along the density gradient vector,

$$\omega \frac{\partial v}{\partial \tau} + av \frac{\partial v}{\partial x} - p = 0, \quad \omega \frac{\partial p}{\partial \tau} + av \frac{\partial p}{\partial x} + \omega_k^2 v = 0. \quad (5.59)$$

Here, the functions $v = V_x/a$ and $p = E_x/a$ are expressed in units of the dimensionless nonlinearity parameter $a$, which is proportional to the value of the magnetic induction $B_z$ at the critical point at the laser frequency; the coordinate $y$ enters only in combination with time, $\tau = \omega t - (\omega y/c) \sin \theta$.

The system of six initial equations admits only a finite group of point transformations, namely, the group of translations along the $t$ and $y$ axes for arbitrary $N(x)$. At a constant ion density, $N = \text{const}$, the additional group of $x$-axis translations and the group of simultaneous rotations in three planes, which are determined by the coordinates $\{x,y\}$ and the corresponding $x$ and $y$ components of the electron velocity and of the electric field, arise. In contrast to the initial equations, calculating of the admitted symmetry group (step II) for Eqs. (5.59) gives an infinite group of point transformations with the operator containing the three terms

$$X = \sum_{i=1}^{3} X_i, \quad (5.60)$$

$$X_1 = \mu_1 Y, \quad X = \omega \frac{\partial}{\partial \tau} + av \frac{\partial}{\partial x} + p \frac{\partial}{\partial v} - \omega_k^2 v \frac{\partial}{\partial p},$$

$$X_2 = \mu_2 \frac{\partial}{\partial x} + \frac{1}{a} Y(\mu_2) \frac{\partial}{\partial v} + \frac{1}{a} Y^2(\mu_2) \frac{\partial}{\partial p},$$

$$X_3 = \frac{\mu_3}{a} \left( a \frac{\partial}{\partial a} - v \frac{\partial}{\partial v} - p \frac{\partial}{\partial p} \right).$$

Here $\mu_i$ are arbitrary function of independent and dependent variables and of the parameter $a$ with two additional differential constraints imposed on $\mu_2$ and $\mu_3$,

$$Y^3(\mu_2) + Y(\omega_k^2 \mu_2) = 0, \quad Y(\mu_3) = 0. \quad (5.61)$$

The arbitrariness in $\mu_i$ is eliminated by the procedure of restriction of the group (5.60) (step III) on a particular perturbation theory solution of the boundary value problem. This solution is constructed in such a way that the leading approximation for the functions $v$ and $p$ is determined by a solution of the linearized system of six
initial equations with the corresponding boundary conditions (the propagation of
the electromagnetic wave from the vacuum toward the plasma) and with the given
density profile \( N(x) \) in the plasma resonance domain taken into account; the cor-
rections proportional to \( a \) appear when linearizing the system (5.59). Verifying the
invariance conditions (4.22) for the group (5.60), we find that \( \mu_1 = 0, \mu_2 = -p/\omega^2 \),
and \( \mu_3 = 1 \) for this particular solution, which gives the desired RGS operator (in the
first of equations (5.61) we replace \( \omega_L^2 \) by \( \omega^2 \))

\[
R = X_2 + X_3 = -(p/\omega^2) \frac{\partial}{\partial x} + \frac{\partial}{\partial a}.
\]  

(5.62)

Solving the system of Lie equations for the operator (5.62) (step IV) gives the exact
solution of Eqs. (5.59):

\[
ap/\omega^2 \Delta = -\varepsilon (f_1 \sin \tau + f_2 \cos \tau), \quad \varepsilon \equiv (q/q_0)^{1/2},
\]

\[
av/\omega \Delta = \varepsilon (f_1 \cos \tau - f_2 \sin \tau), \quad x = \eta + \varepsilon (f_1 \sin \tau + f_2 \cos \tau),
\]  

(5.63)

which describes the nonlinear structure of the electric field in the vicinity of a plasma
resonance. Here the parameter \( \varepsilon \propto a \propto \sqrt{\gamma} \) depends upon the breakdown flux \( q_0 \) for
plasma waves and does not exceed unity. The choice of the particular form of the
functions \( f_1 \) and \( f_2 \) in Eqs. (5.63) is defined by the structure of the electric field
in the vicinity of a plasma resonance, which varies depend upon the plasma density
profile and the electron temperature. For a cold electron plasma with a linear density
profile the functions \( f_1,2 \) have the form

\[
f_1 = (1 + \eta^2)^{-1}, \quad f_2 = \eta (1 + \eta^2)^{-1}.
\]  

(5.64)

In hot plasma when the electric field structure is defined by the thermal motion of
electrons these formulas are replaced by

\[
f_1 = \int_0^{\infty} \cos(\eta \xi + \xi^3/3) d\xi, \quad f_2 = \int_0^{\infty} \sin(\eta \xi + \xi^3/3) d\xi.
\]  

(5.65)

Eqs. (5.63) give exact solutions to Eqs. (5.59) at \( \omega_L = \omega \). In (5.63), (5.64) and (5.65)
coordinates \( x \) and \( \eta \) are normalized to the plasma resonance width \( \Delta \), which for cold
plasma is defined by the particle collisions, \( \Delta = (v/\omega)L \), while in hot plasma the
resonance width \( \Delta = (3V_1^2L/\omega^2)^{1/3} \) depends on the electron temperature, and \( L \) is
the characteristic inhomogeneity scale of the plasma ion density. A feature of the
formulas for \( v \) and \( p \) in (5.63) with the functions \( f_1 \) and \( f_2 \) from (5.65) is that they
give the exact (at \( \omega_L^2 = \omega^2 \) ) solution of Eqs. (5.59), in which the electron pressure
is neglected but the nonzero electron temperature is nevertheless taken into account.
The freedom in choosing the functions \( f_1 \) and \( f_2 \) in Eqs. (5.63) permits to analyze in
the same manner the nonlinear structure of the electric field in the plasma resonance
domain in both cold and hot plasmas.

We rewrite equations for four remaining variables, the electric field \( E_y \), the mag-
netic field \( B_z \), the \( y \)-component of the electron velocity \( V_y \) and the electron density
that allows to integrate them easily. Formulas (5.63), (5.66) give the desired solution to the boundary value problem of interest. In particular it enables to describe generation of higher harmonics from plasma not only in the weak nonlinear limit, but also in the case when nonlinearity plays an essential role. Strong nonlinearity modifies the efficiency of transformation of laser radiation to plasma harmonics and leads to new dependencies of transformation coefficients upon the laser radiation flux and the plasma temperature (see Refs. 15, 16 in [1] for details).

The solution (5.63), (5.66) of six basic equations satisfies boundary conditions and takes into account the strongest plasma nonlinearity, defined by a simple mathematical model (5.59). The approximate RGS used to construct this solution leads to proper results even in the leading order in which small corrections to RGSs are neglected (although a modification of RG generator (5.62) taking these corrections into account is not complicated; the corresponding expression for corrections in the density gradient was given in Ref. 16 in [1]).

5.2.2 Nonlinear dielectric permittivity of plasma

The nonlinearity of electrodynamics of the real medium is due to the nonlinear relation between the induced current and the charge density inside the medium and the electromagnetic field. Generation of this relation, named the material equation, is reduced to a determination of a dependence upon the electromagnetic field of an electric induction vector $D(t,r)$ (see [4], p. 48), related to electric field $E(t,r)$ and the current density $j(t,r)$ via an equality, which in Fourier representation has the following form (here variables “with tildes” are used for differences of Fourier representation from a usual space-time representation):

$$\tilde{D}(\omega,k) = \tilde{E}(\omega,k) + i\frac{4\pi}{\omega} \tilde{j}(\omega,k).$$

For the description of a weak-turbulent plasma, processes of the particle-wave scattering, the parametric instabilities, the generation of harmonics and etc., the material equation is represented as a series in positive powers of electromagnetic fields, i.e., the current density $\tilde{j}(\omega,k)$ is expressed as a sum:

$$\tilde{j}(\omega,k) = \sum_l \tilde{j}^{(l)}(\omega,k), \quad \tilde{j}^{(l)}(\omega,k) \sim O(\tilde{E}^l).$$
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In view of time and spatial dispersion the relation between the induced current and the field appears integral, nonlocal, that results in the material equation which in the Fourier representation has the following form \[^{4}\]:

\[
\tilde{D}_i(\omega, k) = \varepsilon_{ij}(\omega, k)\tilde{E}_j(\omega, k) + \sum_{n=2}^{\infty} \int \delta(\omega - \omega_1 - \cdots - \omega_n) \\
\times \delta(k - k_1 - \cdots - k_n)\varepsilon_{ij_1 \cdots j_n}(\omega_1, k_1; \cdots; \omega_n, k_n) \\
\times \tilde{E}_{j_1}(\omega_1, k_1) \cdots \tilde{E}_{j_n}(\omega_n, k_n) d\omega_1 dk_1 \cdots d\omega_n dk_n. 
\] (5.69)

Comparison of (5.68) and (5.67) with equation (5.69) establishes a relation between multi-index tensors of the nonlinear dielectric permittivity (NDP) of the plasma \(\varepsilon_{ij_1 \cdots j_n}\), which are kernels of nonlinear (with respect to electromagnetic field) integral terms in series (5.69), and the current density \(j^{(l)}\) of the appropriate order \(l \geq 2\).

Usually, without use of the RG algorithm, NDP for hot plasma is obtained by iterating in powers of a self-consistent electromagnetic field of the Vlasov kinetic equation for the distribution function of the particles \(f(t, r, v)\) (3.16) with a stationary and homogeneous in the coordinate \(r\) background distributions \(f_0(v)\) (we omit an index of particles):

\[
f(t, r, v) = f_0(v) + \sum_{l \geq 1} f^{(l)}(t, r, v), \quad f^{(l)} \sim O(E^l),
\] (5.70)

As for NDP for cold plasma it was obtained by iterations of more simple equations of collisionless hydrodynamics for density \(N(t, r)\) and velocity \(V(t, r)\) of particles (written down here for one sort of particles in non-relativistic approach):

\[
N_t + \text{div}(NV) = 0, \quad V_t + (V \cdot \nabla)V = \frac{e}{m} \left\{ E + \frac{1}{c} [V \times B] \right\},
\] (5.71)

in which the electric \(E\) and the magnetic field \(B\) satisfy Maxwell equations (3.17), and charge \(\rho\) and current \(j\) densities have the form

\[
\rho = eN, \quad j = eNV.
\] (5.72)

In the right-hand part of (5.72) summation upon various species of plasma particles is implied, however for the simplification of notations the index of species is omitted and only one sort of particles, for example electrons is underlined further.

It is commonly accepted, that the formulas for NDP in a hot plasma are more general, than in a cold (see, for example, [4], chap. 2) and they are reduced to the last in that specific case, when the distribution function of plasma particles upon mo-

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\(^1\) Here the bottom index specifies on a corresponding tensor component, instead of designating a derivative.
momentum in the initial equilibrium state is represented by the Dirac delta-function, $f_0(v) = \delta(v)$. With growth of the order of nonlinearity ($l \geq 4$) an algebraic procedure of symmetrization for NDP tensors becomes more cumbersome in the hot plasma, than in the cold. The use of the RG algorithm allows to establish a one-to-one correspondence between NDP tensors in the cold and the hot plasma in any order of nonlinearity $l$ and also specifies a way of obtaining expressions for NDP tensors in hot plasma from appropriate “cold” expressions.

For this purpose we present a current density of the given order $\vec{j}^{(l)}(\omega, k)$ in a hot plasma as convolution of the partial, dependent from the Lagrangian velocity of particles $w$, current density $\vec{j}^{(l)}(\omega, k, w)$ with an equilibrium velocity distribution function of particles in absence of electromagnetic fields $f_0(w)$,

$$\vec{j}^{(l)}(\omega, k) = \int f_0(w)\vec{j}^{(l)}(\omega, k, w)dw. \quad (5.73)$$

Expression for the partial current density for $f_0(w) = \delta(w)$, i.e., in cold plasma ($w = 0$), is obtained iterating with respect to the self-consistent field equations (5.71), (5.72), while a transition from expressions for $\vec{j}^{(l)}(\omega, k, 0)$ to expressions $\vec{j}^{(l)}(\omega, k, w)$ at any $w \neq 0$ is carried out with the help of group of transformations, defined by the appropriate RGS operator.

Since the procedure of constructing the multi-index NDP tensor in the hot plasma from the appropriate expressions in the cold plasma is identical for the tensor of permittivity of any order we illustrate it by using, say, the linear with respect to a self-consistent electric field $E$ material relations in a non-relativistic plasma. In the cold plasma the Fourier-components of the partial current $\vec{j}^{(l)}(\omega, k, 0)$ and the charge $\hat{\rho}^{(l)}(\omega, k, 0)$ densities, which are linear in the field $E(\omega, k)$, are obtained by linearizing equations (5.71), (5.72) on a background of the homogeneous and equilibrium electron density $n_e0$ and are determined by well-known relations

$$\vec{j}^{(1)}(\omega, k, 0) = i\frac{e^2n_e0}{m\omega^2} \vec{E}; \quad \hat{\rho}^{(1)}(\omega, k, 0) = i\frac{e^2n_e0}{m\omega^2} (k \cdot \vec{E}). \quad (5.74)$$

The use of the latter in (5.67) gives a scalar dielectric permittivity for cold homogeneous non-relativistic plasma,

$$\varepsilon(\omega, k) = 1 - \frac{4\pi e^2 n_e0}{m\omega^2}. \quad (5.75)$$

The expressions (5.74) define zero-order terms in expansion of the partial current density $\vec{j}^{(l)}(\omega, k, w)$ in powers of the plasma particles velocity $w$. For obtaining the next terms of this series one should use the kinetic description of the plasma. Here it appears more convenient to use instead of the Vlasov equations (3.16) with the Euler velocity $v$ the non-relativistic hydrodynamic analogue of Eqs. (3.71) of the Vlasov equations with Lagrangian velocity $w$ and the equilibrium distribution function $f_0(w)$. Such (Lagrangian) formulation of the kinetic description of the plasma results from a non-relativistic limit of the equations (3.71), and coincides in the
form with (5.71), with that, however, an essential difference, that as against (5.71) the density \( N(t, r, w) \) and the velocity \( V(t, r, w) \) now depend upon Lagrangian velocity as well and in the homogeneous non-perturbed plasma state obey the “initial” conditions at \( t = t_0 = -\infty \):

\[
N(t_0, r, w) = n_e_0 f_0(w), \quad V(t_0, r, w) = w;
\]

\[
E(t_0, r) = B(t_0, r) = 0, \quad \int f_0 dw = 1.
\]  

(5.76)

In a non-relativistic limit material relations (3.72) also become simpler (we use different normalization for the distribution function here, hence material relations do not contain mass multipliers):

\[
\rho(t, r) = e \int N dw, \quad j(t, r) = e \int N V dw.
\]  

(5.77)

Linearizing the equations of plasma kinetics in Lagrangian variables on the background of the basic state (5.76) results to the following formulas for corrections to the partial current density for small values of \( w \):

\[
\hat{j}^{(1)}(\omega, k, w) = i \frac{e^2 n_e_0}{m \omega} \left\{ \hat{E} + \frac{1}{\omega} (w(k \cdot \hat{E}) + k(w \cdot \hat{E})) \right\} + O(w^2).
\]  

(5.78)

To prolong this formula on any nonzero values of \( w \) we employ the RGS operator which is constructed from the Lie group of point transformations (3.76), admitted by plasma kinetic equations. Two operators of the admitted group are of interest for us, namely, the operator of translations in Lagrangian velocity, which results from the operator \( X_\omega \), and the operator of Galilean transformations, which is a non-relativistic analogue of the operator of Lorentz transformations \( B \) in the set (3.76),

\[
Z_1 = \frac{\partial}{\partial w}, \quad Z_2 = t \frac{\partial}{\partial r} + \frac{\partial}{\partial V} - \frac{1}{c} \left[ B \times \frac{\partial}{\partial E} \right] + \rho \frac{\partial}{\partial j}.
\]  

(5.79)

Let’s proceed in the operator \( Z_2 \) from the velocity \( V \) and the density \( N \) to the partial current and charge densities, \( j \) and \( \rho \), prolong the operator obtained on Fourier variables and combine it with the operator of translations \( Z_1 \). As a result we get the operator that leaves the partial current density (5.78) invariant at \( w \to 0 \), i.e., the required RGS operator

\[
R = k \frac{\partial}{\partial \omega} + \frac{\partial}{\partial w} - \frac{1}{c} \left[ \hat{B} \times \frac{\partial}{\partial \hat{E}} \right] + \hat{\rho} \frac{\partial}{\partial \hat{j}}.
\]  

(5.80)

The operator (5.80) is related to a three-parameter group with the vector parameter \( w \), and its final transformations (the variables with primes here correspond to transformed variables)
\[ \omega' = \omega + kw; \quad (\beta_{ls}'/\omega')\tilde{E}'_s = (1/\omega)\tilde{E}_s; \quad \tilde{\rho}' = \tilde{\rho}; \quad \tilde{j}'_s = \beta_{ls}\tilde{j}_s, \]  
\[ k' = k; \quad \tilde{B}' = \tilde{B} = (c/\omega)[k \times \tilde{E}]; \quad \beta_{ls} = \delta_{ls} + k_i\nu_s/(\omega - kw), \]  
(5.81)
give the required relationship between the value of the partial current density \( \tilde{j}(\omega,k,0) \) at \( w = 0 \) (in cold plasma) and the analogous value \( \tilde{j}(\omega,k,w) \) for any value of \( w \neq 0 \). When integrating over velocity \( w \) with the “weight” \( f_0(w) \), following (5.73), we get an expression for a current density of the given order in hot plasma which defines the appropriate multi-index NDP tensor of plasma.

In particular, in the linear in the electric field approximation the use of (5.74) leads to the relationship

\[ \tilde{j}_i^{(1)}(\omega,k,w) = \frac{ie^2n_e\epsilon_0}{m\omega}\beta_{si}\beta_{sa}\tilde{E}_a(\omega,k). \]  
(5.82)

Substitution of (5.82) into (5.73) and the further use of \( \tilde{j}_i^{(1)}(\omega,k) \) in (5.67) gives the required expression for the tensor of the linear dielectric permittivity for hot homogeneous non-relativistic plasma in the absence of external fields with the equilibrium distribution function \( f_0(w) \):

\[ \varepsilon_{ab}(\omega,k) = \delta_{ab} - \frac{4\pi e^2n_e\epsilon_0}{m\omega^2} \int f_0(w)\beta_{sa}\beta_{sb}dw. \]  
(5.83)

Formula (5.83), which arises from the scalar equality (5.75) as a result of application of RG transformations to a partial current density in the cold plasma with the subsequent integration over the group parameter, illustrates an opportunity of obtaining a tensor of dielectric permittivity of the hot plasma from the appropriate “cold” expression (see Refs. 35 in [1]). The appropriate RGS is constructed from symmetry operators of the plasma kinetic equations with their subsequent prolongation on solution functionals—partial current and a charge densities in Fourier representation. Thus a linear in the electromagnetic field approximation used above is not an essential restriction, as relations between transformed (primed) and non-transformed partial current and a charge density remains linear at transformations (5.81). It means, that it is also possible to apply transformations (5.81) to partial current and a charge densities of any order \( l \), i.e., the offered RG scheme allows to build NDP tensors of any order in hot plasma proceeding from the appropriate “cold” expressions for NDP. Omitting intermediate calculations, we present a result of such construction:

\[ \varepsilon_{i_1 \ldots i_n}(\omega_1,k_1;\ldots;\omega_n,k_n) = \int f_0(w)\tilde{\varepsilon}_{ab_1 \ldots b_n}(\Omega_1,k_1;\ldots;\Omega_n,k_n) \]
\[ \times \frac{\Omega_1 \ldots \Omega_n}{\omega_{\Omega 1} \ldots \omega_{\Omega n}}\beta_{ai}(\omega,k)\beta_{b_{i_1}j_1}(\omega_1,k_1)\ldots\beta_{b_{i_n}j_n}(\omega_n,k_n)\, dw; \quad n \geq 2; \]
\[ \omega = \omega_1 + \cdots + \omega_n; \quad k = k_1 + \cdots + k_n; \]
\[ \Omega \equiv (\omega - kw), \quad \Omega_i \equiv (\omega_i - k_iw), \quad i = 1,\ldots,n. \]
(5.84)

Here \( \tilde{\varepsilon} \) corresponds to the NDP tensor in cold collisionless plasma without external fields. For example, for the nonlinearity of the second order it is determined by the
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The similar result can be obtained and for the relativistic plasma, however thus it is necessary to use not the three-parameter group of Galilean transformations, but the six-parameter group including Lorentz transformations and rotations.

5.2.3 Adiabatic expansion of plasma bunches

Here RGS algorithm is applied to the problem of expansion of plasma bunches and related generation of the accelerated particles. The mechanisms and characteristics of ions triggered by the interaction of a short-laser-pulse with plasma are of current interest because of their possible applications to the novel-neutron-source development and isotope production. In the near future ultra-intense laser pulses will be used for the ion beam generation with energies useful for the proton therapy, fast ignition inertial confinement fusion, radiography, neutron-sources.

The commonly recognized effect responsible for the ion acceleration is charge separation in the plasma due to high-energy electrons, driven by the laser inside the target. During the plasma expansion, the kinetic energy of the fast electrons transforms into the energy of electrostatic field, which accelerates ions and their energy is expected to be at the level of the hot-electron energy. The mathematical model describing this phenomenon is based on plasma kinetic equations with a self-consistent field (3.16)–(3.17), which is rather complicated for analytical treatment. However, to describe plasma flows with characteristic scale of density variation large compared to Debye length for plasma particles, the quasi-neutral approximation is used. In this approximation charge and current densities in plasma are set equal to zero, that essentially simplifies the initial model with nonlocal terms. Instead of the system of Vlasov-Maxwell Eqs. (3.16), (3.17) with the corresponding material equations here we use only the kinetic equations for particle distribution functions for various species (3.82) with additional nonlocal restrictions imposed on them, which arise from vanishing conditions for the current and the charge densities (3.83). Initial conditions for solutions of Eqs. (3.82) and (3.83) correspond to distribution functions for electrons and ions, specified at \( t = 0 \):

\[
f^\alpha |_{t=0} = f_0^\alpha (x, v).
\]  (5.86)

Equations (3.82), (3.83) describe one-dimensional dynamics of a plasma bunch, which is inhomogeneous upon the coordinate \( x \); thus the distribution functions of particles \( f^\alpha \) depend upon \( t, x \) and the velocity component \( v \) in the directions of plasma inhomogeneity. Analytical study of such yet simplified model represents the essential difficulties, but due to application of RG algorithm it is possible not only to construct solution at various initial particle distribution functions but also to find the
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law of variation of particles density without calculations of distribution functions for particles in an explicit form (see Refs. 21,22 in [3]).

For construction of RGS we consider (step I) a set of local (3.82) and nonlocal (3.83) equations as $\mathcal{RM}$, in which the electric field $E(t,x)$ appears as some arbitrary function to be found of its variables. Calculating the Lie group of point transformations admitted by this manifold (step II) is given by (3.85), and in particular contains the generator of time translations and the projective group generator. Precisely these operators enables to construct a class of exact solutions to the initial problem that are of interest, as a linear combination of the operator of time translations and the operator of the projective group leaves the approximate perturbation theory solution of the initial value problem $f^\alpha = f^\alpha_0(x,v) + O(t)$ invariant at $t \to 0$, i.e., it is the RGS operator,

$$R = (1 + \Omega^2 t^2) \frac{\partial}{\partial t} + \Omega^2 t x \frac{\partial}{\partial x} + \Omega^2 (x - vt) \frac{\partial}{\partial v}, \quad (5.87)$$

which results from the group restriction procedure (step III), for spatially symmetric initial distribution functions with the zero average velocity. The constant $\Omega$ is proportional to the ratio of a characteristic sound velocity $c_s$ to initial inhomogeneity scale of the density of electrons, $L_0$.

Invariants of the RG generator (5.87) are two combinations, $x/\sqrt{1 + \Omega^2 t^2}$ and $v^2 + \Omega^2 (x - vt)^2$, and particle distribution functions $f^\alpha$. Hence, solutions of initial value problem at any time $t \neq 0$ (step IV) are expressed via these invariants in terms of initial values (5.86),

$$f^\alpha = f^\alpha_0(I(\alpha)), \quad I(\alpha) = \frac{1}{2} \left( v^2 + \Omega^2 (x - vt)^2 \right) + \frac{e_\alpha}{m_\alpha} \Phi_0(x'). \quad (5.88)$$

Here the dependence of $\Phi_0$ upon the variable $x' = x/\sqrt{1 + \Omega^2 t^2}$ is defined by quasi-neutral conditions (3.83), and the electric field $E = -\Phi_\chi$ is found with the help of the potential

$$\Phi(t,x) = \Phi_0(x') \left( 1 + \Omega^2 t^2 \right)^{-1}. \quad (5.89)$$

Formulas (5.88) give the solution to the initial value problem Eqs. (3.82), (3.83). However, for practical applications we need frequently more rough characteristic of plasma dynamics, for example, a density of particles (ions) of the given species $n^q(t,x)$ which can be calculated using the appropriate distribution function:

$$n^q(t,x) = \int_{-\infty}^{\infty} f^q(t,x,v) \, dv. \quad (5.90)$$

In view of the complex dependence upon the invariant $I(\alpha)$ it is not always possible to carry out direct integration of a distribution function over velocity in the analytical form, therefore here the procedure of prolongation of the operator on solution functionals described in Section 4.2.1 comes to the aid. As the density $n^q(t,x)$ is a linear functional of $f^q$, the prolongation of operator (5.87) on the functional of solution (5.90) in the narrowed space of variables $\{t,x,n^q\}$ gives the following RG
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operator:

$$R = (1 + \Omega^2 t^2) \frac{\partial}{\partial t} + \Omega^2 t x \frac{\partial}{\partial x} - \Omega^2 t n'^n \frac{\partial}{\partial n'^n}. \quad (5.91)$$

The solution of Lie equations for the operator $R$ in view of initial conditions (5.86) gives relations between invariants of this operator, namely one of the combinations $J = x\sqrt{1 + \Omega^2 t^2}$ already given for operator (5.87) and the product $J^q = n\sqrt{1 + \Omega^2 t^2}$ for arbitrary $t \neq 0$ with their values at $t = 0$: $J_{|t=0} = x'$, $J^q_{|t=0} = N_q(x')$. This relationship immediately leads to the formulas that characterize spatial-temporal distribution of the density of ions of a given species in terms of the initial density distribution

$$n^q = \frac{1}{\sqrt{1 + \Omega^2 t^2}} N_q\left(\frac{x}{\sqrt{1 + \Omega^2 t^2}}\right), \quad N_q(x') = \int_{-\infty}^{\infty} f^q_0(I^{(q)}) \, dv. \quad (5.92)$$

Example 5.2.1. We illustrate general results with reference to expansion of a plasma slab, consisting of cold ($\alpha = c$) and hot ($\alpha = h$) electrons and of two ion species ($q = 1, 2$). Let initially (at $t = 0$) ions are characterized by Maxwellian distribution functions with densities $n_{10}, n_{20} \ll n_{10}$ and temperatures $T_1, T_2$, and the distribution function of electrons looks like two-temperature Maxwellian distribution with the appropriate densities $n_{c0}$ and $n_{h0} \ll n_{c0}$ ($n_{c0} + n_{h0} = Z_1 n_{10} + Z_2 n_{20}$) and temperatures $T_c$ and $T_h \gg T_c$ of hot and cold components. From the physical point of view such choice of initial conditions refer to an expansion of the target consisting of heavy ions with a small impurity of light ions adsorbed on a surface (for example, protons) which preliminary was heated quickly by a short pulse of laser radiation with formation of a group of hot electrons. Then the solution of the initial problem (5.88) is represented as

$$f^e = \frac{n_{c0}}{\sqrt{2\pi v_{Tc}}} \exp\left(-\frac{v^2}{2v_{Tc}^2}\right) + \frac{n_{h0}}{\sqrt{2\pi v_{Th}}} \exp\left(-\frac{v^2}{2v_{Th}^2}\right),$$

$$f^q = \frac{n_{q0}}{\sqrt{2\pi v_{Tq}}} \exp\left(-\frac{v^2}{2v_{Tq}^2}\right), \quad v_{Tq} = \frac{T_q}{m_q}, \quad q = 1, 2, \quad (5.93)$$

where invariants $I^{(\alpha)}$ are given by relations

$$\frac{I^{(c)}}{v_{Tc}^2} = \mathcal{E} + \frac{(1 + \Omega^2 t^2)}{2v_{Tc}^2}(v - u)^2, \quad \frac{I^{(h)}}{v_{Tq}^2} = \mathcal{E} + \frac{T_c}{T_h} + \frac{(1 + \Omega^2 t^2)}{2v_{Tq}^2}(v - u)^2,$$

$$\frac{I^{(q)}}{v_{Tq}^2} = -\mathcal{E} \left(\frac{Z_q T_{c0}}{T_{e0}}\right) + \frac{U^2}{2v_{Tq}^2} \left(1 + \frac{Z_q m_e}{m_q}\right) + \frac{(1 + \Omega^2 t^2)}{2v_{Tq}^2}(v - u)^2. \quad (5.94)$$

Here $u = x t \Omega^2 / (1 + \Omega^2 t^2)$ is a local velocity of plasma particles, $U = x \Omega / \sqrt{1 + \Omega^2 t^2}$, and a potential $\Phi$ is expressed via the function $\mathcal{E}$,
that is obtained from the transcendental equation,

\[ n_{c0} = \sum_{q=1,2} Z_q n_{q0} \exp \left[ \left( 1 + \frac{Z_q T_c}{T_q} \right) \mathcal{E} - \frac{U^2}{2v_T^2} \left( 1 + \frac{Z_q m_e}{m_q} \right) \right] - n_{h0} \exp \left[ \left( 1 - \frac{T_c}{T_h} \right) \mathcal{E} \right]. \] (5.96)

Formulas (5.93)–(5.96) completely define the behavior of distribution functions of all particle species considered in the given example when studying the expansion of a plasma slab. At that the space-temporal distribution of the ion density of the given species is determined by formulas (5.92), in which the ion density \( N_q \) for the initial distribution functions specified above has the form

\[ N_q = n_{q0} \exp \left[ \mathcal{E} \left( \frac{Z_q T_{c0}}{T_{q0}} \right) - \frac{U^2}{2v_T^2} \left( 1 + \frac{Z_q m_e}{m_q} \right) \right], \quad q = 1, 2, \] (5.97)

where the relation between the function \( \mathcal{E} \) with the variable \( U \) still is from the Eq. (5.96).

**Fig. 5.5** Left panel: typical experimental setup for registration of fast ions from the foil under ultra short laser pulses (from Ref. 23 in [3]). Right panel: the “universal” density \( N_q \) of plasma ions – carbon ions (curves (C)) and protons (curves (H)) – versus the dimensionless “coordinate” \( \chi^2 = (x/L_0)^2 / (1 + \Omega^2 t^2) \). Dotted curves with short and long strokes show the dependencies of a dimensionless density for hot and cold electrons.

In Fig. 5.5 we illustrate the typical “density” distribution (5.97) for a plasma slab, consisting of cold and hot electrons and two ions species: carbon ions C\(^{+4} \) (\( q = 1 \)) and protons H\(^{+1} \) (\( q = 2 \)). Block curves show dependence of a dimensionless “universal” density of plasma ions.
\[ N_q = \left( \frac{n_{q0}}{n_{e0}} \right) N_q, \]

referred to the maximal density of cold electrons, upon the dimensionless “coordinate”
\[ \chi^2 = \left( \frac{x}{L_0} \right)^2 / \left( 1 + \Omega^2 t^2 \right), \]

referred to the characteristic initial density scale of ions \( L_0 \). “Universality” of this dependencies is the direct consequence of a relation which exists between invariants of the RG operator (5.91). Dotted curves give the distribution of the dimensionless density of cold electrons (short strokes),
\[ \left( \frac{n_c}{n_{e0}} \right) \sqrt{1 + \Omega^2 t^2} \]

and hot electrons (long strokes),
\[ \left( \frac{n_h}{n_{e0}} \right) \sqrt{1 + \Omega^2 t^2}, \]

respectively.

Similar results are obtained for more complex distribution functions and beyond the scope of the model used for the one-dimensional expansion, for example for spherically-symmetric expansion of a plasma bunch (see Refs. 21, 22 in [3]).

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