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METHOD IN MATHEMATICS:

BERNARD LONERGAN'S THEORY OF COGNITION

AND ITS APPLICATION TO MATHEMATICAL EDUCATION

by

Charles John Buckley, O.S.B.

Dissertation Committee:

Professor Paul C. Rosenbloom, Chairman
Professor Philip H. Phenix

Submitted in partial fulfillment of the
requirements for the degree of Doctor of Philosophy
under the Executive Committee of The Graduate
School of Arts and Sciences
Columbia University

1977

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ABSTRACT

METHOD IN MATHEMATICS:
BERNARD LONERGAN'S THEORY OF COGNITION AND ITS APPLICATION
TO MATHEMATICAL EDUCATION

Charles John Buckley, O.S.B.

The context of this study is the recent development of new content and methods in school mathematics programs. Within that context, the purpose of the thesis is to provide for mathematics educators a coherent and critically grounded approach to mathematical education, drawn from the published and unpublished works of Bernard Lonergan and of various commentators on his thought.

The first task of the thesis is to provide a theory of human knowing, primarily as exemplified in mathematics. Lonergan's analysis of the knowing process reveals knowing as a structure composed of activities at three distinct levels. The first is the level of experience; activities at this level are seeing, hearing, tasting, imagining, etc.; at this level the knower finds his entire experience questionable. He asks questions such as "What is it?" and through the act of insight reaches answers. At that point he is operating within the second level of cognitional structure, that of understanding, the level of answers to What-questions; here the activities include understanding and formulating. But understanding can be false; an idea can be an inadequate appropriation of intelligibility in experience, so the inquirer has a further series of questions; he asks of his understanding, "Is it so?"

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Here the knower reflects and weighs the evidence, and in answering this question either Yes or No, he is operating on the third level of knowing, that of judgment. The original desire to know is met when the knower arrives at a judgment.

The nature of knowing has an effect on its product; the second task of the thesis is to clarify the nature of mathematics through this analysis of the knowing process.

The value of Lonergan's theory of knowing in illuminating, clarifying, and evaluating proposals in education is exemplified by an examination of the work of Ausubel, Bruner, Gagné, and Polya, all of whom have had great influence on the interpretation and evaluation of recent reforms in the mathematics curriculum.

An appendix relates Lonergan's theory of knowing to the teacher's role in the classroom and to teaching strategies and methods through examples of approaches to several topics within a junior college mathematics program. Recommendations center on the need to respect the knowing process in its origin in experience and its tripartite nature. A primary means of education is thus to bring the student to work within this process, to stimulate this knowing process into action for the student.

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I thank Father Lonergan and Philosophical Library for permission to use the quotations from Insight (37) and Holt, Rinehart, and Winston for permission to quote the books of Ausubel (2) and Gagné (22).

I owe special thanks to Professors Rosenbloom and Phenix, who let me attempt this strange topic and who tried to make me clarify my ideas and be faithful to them in the preparation of this thesis. If and to whatever extent I have failed to do so, it is my own fault and not that of the many people to whom I am indebted.

CJB

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INTRODUCTION

The purpose of this thesis is to develop an approach to mathematical education from a soundly-based view of the human knowing process. For that I rely on the cognitional theory of Bernard J. F. Lonergan, S.J.

Lonergan is a Canadian philosopher and theologian, at this writing still living. He began his work in an attempt to understand Aquinas' theory of knowing, in his dissertation "A Study of the Speculative Development in the Writings of St. Thomas of Aquin" (35).[1] From this grew a series of articles, published in one volume as Verbum: Word and Idea in Aquinas (41). Based on what he learned in that study, he developed his own approach to man's knowing, set forth in the book Insight: A Study of Human Understanding (37). Finally he reached the goal that underlay his work of many years in Method in Theology (42), in which he brought his methodology based on man's cognitional activity into the field of theology to provide an analyzed method of approach to theological inquiry.

Lonergan's work is important for several reasons. He has rejuvenated interest in Thomist philosophy because of his approach, beginning not with philosophical first principles, nor with psychological data-gathering sessions, nor with some fixed and automatic steps to arrive

---

[1]The author-and-number format for references is taken from Campbell and Ballou (12:88). This means that the material cited appears on page 88 of Form and Style by William Campbell and Stephen Ballou, the book that appears as number 12 in the list of REFERENCES. If the name of the author were not mentioned or obvious from the text, the citation would read: (Campbell and Ballou 12:88).

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at certain knowledge, but descriptively; he examined what we do when we are involved in particular instances of coming to know.

Because of his new approach, Lonergan is exerting a strong influence on a growing circle of Catholic scholars. That the influence is mainly on Catholics is due not to sectarianism in his thought but to the fact that his original exposure has been in Catholic theology. But there are notable exceptions, such as John MacQuarrie (45) and Langdon Gilkey (24) among non-Catholic theologians. Lonergan's influence extends beyond the field of theology, and there have been attempts to apply the methodology of insight to other fields such as quantum physics (Heelan 28) and the philosophy of statistics (McShane 47)

In the book Insight, which will serve as the primary reference for this thesis, Lonergan answers three questions: What do we do when we know? Why is doing that knowing? and What do we know when we do that? Fully elaborated and systematized answers to those questions are, respectively, cognitional theory, epistemology, and metaphysics. The purpose of this thesis will be satisfied by examining Lonergan's answer to the first question, although a more extensive development of the second and third answers might bring some clarity to the differences between the various schools of philosophy of mathematics.

Lonergan uses a phenomenological method; he and others commenting on his work have written in a form of self-discovery for the reader. This method is a discovering of ourselves knowing; it is not mere introspection, as if we could know about our own mental workings in a more simple manner than is required to know other things. Lonergan uses numerous examples from the fields of mathematics and physics as a source of clear examples of this knowing process. This thesis will not utilize

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this self-discovery approach, but the reader should be prepared to reflect upon times he has come to know something in order to provide for himself relevant examples of the structure presented here.

How is Lonergan unique? That is a hard question to answer. It is not so much that he introduces new terms or new concepts. Most of the terms and concepts are familiar ones, especially within the Aristotelian-Thomist tradition. Many of the ideas presented as Lonergan's are not strictly original with him. For example, the analysis of definition is very similar to the presentation by Cohen and Nagel (13:224-33). The power of symbolism in mathematics has been discussed many places, such as in Whitehead's An Introduction to Mathematics (71:39-48). Lonergan's originality comes in the situation of these familiar topics in the context of his cognitional theory; he shows how they emerge as a result of the way men know. Tracing the sources of Lonergan's ideas would be the work of a score of dissertations. This one can only utilize his achievement: working from a phenomenological beginning, yet utilizing some of the classical categories, he gives new life and meaning to dead terms.

The topic of the thesis is developed in this fashion:

Chapter I provides a brief survey of the situation in mathematical education during the past several decades. It examines the trends and shows the need for some overarching view from which to evaluate means and methods and to provide a basis for operating within the classroom. This chapter mentions four authors (Gagné, Ausubel, Bruner, and Polya) whose work has been used to analyze learning processes, methods, and structures in the new mathematics programs. A conspicuous absence in this treatment is the name of Jean Piaget. But Piaget's

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work is so immense and varied, so filled with neologisms and variously-translated terms, and so rooted in a philosophical basis of its own, that his views could not be done justice in a brief space.

Chapter II presents Lonergan's cognitional theory, which can serve as the required basis for teaching. This chapter also examines the nature of mathematics, as one product of man's knowing.

Chapter III returns to the work of Gagné, Ausubel, Bruner, and Polya, and examines it in greater detail in the light of Lonergan's work. Even this summary of their work is influenced by Lonergan's cognitional theory, and discussion develops the place of the work of these four men in the broad view Lonergan provides of the knowing process and resultant approaches to classroom learning. This chapter exemplifies the role that an understanding of human knowing based on Lonergan's work can have in evaluating, illuminating, and situating other theories and recommendations for education. Thus this chapter serves as some proof for the claim that motivated the writing of this thesis: Lonergan's understanding of the cognitional process provides a basis for evaluating and understanding and situating the many theories that are used to justify approaches to mathematical education.

Chapter IV presents areas for further investigation and concludes the thesis with a summary.

An Appendix develops the educational implications of this cognitional theory and of the consequent structure of mathematics. It sets out classroom approaches to particular topics, which provide occasion for comment on the nature of the educational process.

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CHAPTER I

THE SITUATION IN MATHEMATICAL EDUCATION

The entire program of mathematical education has undergone complete revision in the past two decades. The reform was obviously needed. Many pressures in society led to a concern about the quality of mathematical education; mathematics had expanded quantitatively and received new emphases in the twentieth century, causing a reorganization of the older mathematics, yet the new shape of mathematics was hardly reflected in the schools (Commission on Mathematics 14:3). Technological growth, automation, and the rise of computers all created a demand for workers with mathematical sophistication; and recently-developed mathematics, such as linear programming, the theory of games, and statistics and probability provided rich potential for applications (Price 59:5). These new needs indicated that students would be better prepared by approaching mathematics in an inquiry mode (Davis 16:9).

The concern about the inadequacy of the current mathematics program grew through the late 1950's and stimulated a number of calls for change. For example, in 1959 the Commission on Mathematics of the College Entrance Examination Board issued a statement:

> It is important to recognize that such a general concern exists, and that the existence of this concern strongly supports the thesis that the time is ripe for the improvement of the high school mathematics program (14:7).

This group proposed a nine-point program for college-capable students in secondary school, including:

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lege mathematics at the level of calculus and analytic geometry;

2. An understanding of the nature and role of deductive reasoning--in algebra, as well as in geometry;

3. Appreciation of mathematical structure ("pattern")--for example, properties of natural, rational, real, and complex numbers;

4. Judicious use of unifying ideas--sets, variables, functions, and relations (14:iii).

The commission also urged attention to particular content areas neglected in current programs--inequalities, coordinate geometry, and elementary functions--and reorganization of some areas such as trigonometry.

In response to this need, many committees and project groups consisting of mathematicians and mathematics educators set out to develop programs by first answering three questions: What mathematics should we teach? To whom should we teach it? How should we teach it? (Fehr 19:424). The last question asked for a reconciliation of the various views of what constitutes school mathematics. Is the goal a set of manipulative skills, or a set of structures that could be developed either intuitively or deductively, or a physical study that could be developed into an abstract science? The reform groups also weighed whether to treat mathematics as separate fields (geometry, algebra) or as one continuous development.

The concern about the teaching of mathematics and the new type of subject matter has directed attention to the learning processes of students. Learning characteristics of students have been used to justify the manner of presentation in the new programs (Shulman 69:23f). Further, evaluation of the new curriculum efforts required a basis in some conception of mathematics learning in order to establish what should be

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sought in the programs. There follow brief presentations of four of the most prominent approaches to the explication of mathematics learning; the authors have been in contact with the reform in various ways.

George Polya, an eminent mathematician, has been active in teacher education for many years. He is credited with a number of provocative works, dealing chiefly with problem solving (54, 56, 58) or with teacher education (57). In his writing he insists that one goal of teaching mathematics should be to teach the heuristics of mathematics, that is, problem solving. The student learning mathematics should be learning more than routine; he should be learning how to conduct his own investigation into problems that arise in mathematics and how to derive the most benefit from them. Polya's work, oriented to the practical teaching and learning of mathematics, influenced the men involved in the curriculum reforms through his teaching and his influence on teacher education; his emphasis on problem solving found a place in the new programs.

Robert Gagné, a learning theorist long active in the area of mathematics learning, became involved in evaluation of the new programs. Therefore, his viewpoint is strongly influenced by his interest in the evaluation of mathematics learning. Gagné wrote that the key to any step in the learning process is the fulfillment of the conditions requisite for the step to take place; if the preconditions in terms of previous learned skills or capabilities are present, then the new task can be readily learned (22:66). What separates Gagné from the behaviorists is his assertion that there are eight basic varieties of learning. The varieties are distinguished by the requisite conditions, but they are not reducible to any one type, such as the stimulus-response connection

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(22:3). Gagné's theory has extensive educational application: since the prerequisite for a learning task is most often the mastery of previous learning tasks, whole branches of subject matter can be arranged in optimal ordered structures of skills called learning heirarchies (22:239). In these hierarchies, skills are arranged to show which ones have been found necessary, through analysis of the subject matter and tests in educational settings, for students to learn a particular skill. For example, for the skill "specifying sets, intersection of sets, and separations of sets, using points, lines, and curves," Gagné and Bessler (23:125) found four necessary prerequisite skills:

1. "Specify the intersection of a triangle and lines or parts of lines as 0, 1, or 2 points";

2. "Specify the intersection of a triangle and lines or parts of lines as parts of lines";

3. "Specify the intersection of lines or parts of lines"; and

4. "Specify the intersection set of a simple curve and parts of lines."

But the first of these subordinate skills requires the ability to "Identify and draw a triangle," to "Identify and draw the intersection of lines or parts of lines taken two at a time as 0 or 1 point," etc. All of these skills can ultimately be based on the skills "Identify separation of entities into groups" and "Identify and draw a point."

The appeal of the hierarchy is that it specifies a path with clearly indicated steps for a student to follow to master a cognitive goal. Since all steps are specified operationally it becomes a clear matter to determine when a student has mastered a particular step and

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is ready to move on to a higher one. Also, since the entire hierarchy is specified, computers may monitor progress and even provide the instruction to connect the several tasks at one level that are prerequisite to the higher level task.

David Ausubel, another learning theorist, has had less direct contact with the reform projects and has discussed topics of interest in mathematics learning only incidentally, but his work has had some influence on how people examine mathematics learning (Shulman 69:36-39). Ausubel also utilized a structure of knowing, but this is a structure of meaning that a student has achieved rather than of skills he has mastered (2:53f). The task of education is to help the student acquire the most fruitful organizing principles for material already learned and to provide a way to relate new material to existing cognitive structure.

His emphasis on meaningful learning led Ausubel to make a distinction in order to clarify the controversy over discovery learning. There are two different scales, one having as extremes discovery and reception learning, the other having as extremes rote and meaningful learning. Much of the discussion of discovery learning seemed to Ausubel to identify reception learning with rote learning. They are not the same, he wrote; reception learning can be meaningful by providing a link between new material and existing cognitive structure (2:24). One way to provide this is by the use of advance organizers, the initial presentation of the key ideas of a topic, so that when less important and more detailed ideas are introduced, they can be related to the central ideas, and so have a more easily established meaning (2:131f).

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Jerome Bruner's main influence on the new mathematics programs has been as a popularizer and organizer of the most important ideas in learning that can be used to analyze the curriculum reforms in many fields. His presentation of views on education in The Process of Education (8) and other works provided the basis for teaching methods used by many teachers with the revised curricula. Bruner's ideas were applied to mathematical education through his work with Dienes (9).

Among the emphases that Bruner has popularized are: the need for activity on the part of the student, with manipulative materials if the student is not far advanced (8:48-50); structuring a course according to the structure of the discipline, so that principles that unify the discipline are the organized principles of the course (8:17-25); the participation of the student in the field of study so that, for example, the student does not learn about mathematics but does mathematics (8: 26-30); and, to implement this last feature, utilization of the discovery method (8:20-22).

The new programs prepared by the committees and commissions of mathematicians and educators were designed to present the expanded mathematics according to current principles of learning. Although the revision was initiated to improve secondary school and especially college preparatory programs, the reforms reached the lower grades and throughout the schools. The School Mathematics Study Group produced the most prominent of the new programs, although numerous other groups prepared texts. Though the resultant programs vary, they have common elements: An emphasis on unifying themes; a presentation of the structure of mathematics as a system using logical deductions and generalization; and discussion of set theory, measurement, systems of numeration, opera-

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tions, and graphical representations (Brown 5:22-28).

Other important aspects of new mathematics programs are not universal, but they show the direction of change. Now, instruction involves manipulative materials such as attribute blocks for discussion of sets or mirror cards to help students understand symmetry and transformation geometry. Schools have set up mathematics laboratories in which students use these and other materials; there teachers may offer individual help, attempting to understand what the students are thinking and to provide puzzling situations lacking simple correct answers in order to stimulate students in inquiry. Computers are often used to assist learning in many ways. The programs rely especially on discovery learning and attempt an increasing internalization of mathematics (Davis 16:38-45).

Now that the programs are available, a great burden is placed on the teacher. Fehr wrote:

> The focal point of successful education in mathematics is the teacher. . . .
> . . . the teacher must know thoroughly and accurately everything which he must teach. Beyond this, a teacher must believe in what he teaches, as knowledge that is essential, interesting, and vital to the affairs of men today and tomorrow. . . . But he must know more than his subject--he must know how to communicate it in a motivating, satisfying, interesting manner so that his students come into possession of treasured knowledge (18:169).

Allen spelled out this role of teachers as those who implement the new programs:

> We must be the guardians of that which is good in the traditional programs, the judges of clarity of exposition, the evaluators of new concepts from the standpoint of teachability, and, most important of all, the operators who transform theory into action in the classroom. We must not fail to criticize expositions that seem to be ineffective or materials that seem to be unteachable, for it is only through such constructive criticism that these expositions and materials will be improved (1:85).

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The teacher must bring all the powerful array of mathematics and methods into the classroom. He must decide or help to decide which textbooks to use, with which materials and activities and methods. He has the ultimate decision on what to do as he faces the students each day. Since so many of the programs emphasize a teaching for understanding in order to reach the meaning of mathematics, teachers have to have a developed notion of what understanding is. Although the recent upheavals in mathematical education have passed from the public eye, the focus of attention has shifted to the lack of achievement throughout the educational process. Facing all these challenges, the classroom teacher has the difficult task of deciding what to do and how to do it. The revolution in school mathematics has not eliminated that task; it cannot, for each teacher must face his unique situation.

In order to have a basis for evaluation and decision, the teacher needs a more comprehensive idea of his goal. What does he want the student to achieve? How can the student reach this goal? Ultimately, how does the student know? Psychology has provided partial answers, but these parts need to be incorporated into a total view.

Bernard Lonergan's achievement, this thesis asserts, has been to provide a broad and deep understanding of human knowing; this understanding of the knowing process will provide a broader and more adequate basis from which the teacher can approach the teaching of mathematics and judge programs, methods, and means for instruction.

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CHAPTER II

LONERGAN ON KNOWING

The Problem of Knowing

The problem of how we know, how we reach reality in our knowing, has been a tantalizing one for millenia. We begin by thinking that our normal senses reach reality; we look and see the table in front of us and we think we have discovered the real table. But a number of classical arguments have invalidated that simplistic approach; optical illusions and dreams seem to make us aware of something that is not. Gradually we have learned that perception is not directly of an object; many processes mediate the object to me; it is a puzzling problem to try to say what makes a table gray when I perceive a gray table. The difficulty of the simplistic approach is exemplified by a few basic experiments: if I remove my glasses, suddenly the table has gone all fuzzy, along with the room in which it stands; or if I close one eye and with my finger tug at the corner of my open eye, the table starts to bounce wildly up and down. The table that I have found in my seeing is not the one out there in the room, if I can affect it by these operations. I do not reach the real table by seeing.

So another approach is to say that I reach what is real by some sort of "mental look" by which I reach the essence of things. Somehow, this says, my ideas are images of the real; by some sort of spiritual look or intuition I reach the real and leave behind all the faded and misleading images of the real that my experience gives. But Kant can

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be credited with the discovery that our ideas do not mirror the thing itself; the objects of our knowledge always are conformed to the structures of human knowing and so are not things as they exist apart from our knowing them but are appearances--things as they affect us in our sensation and categorizing. Judgments remain strictly mental, either a conjunction of subject with predicate but little more, or the affirmation of a characteristic of our knowing rather than of the known.

Philosophers after Kant have had either to ignore his almost frightening discoveries or to try to accustom man to the inability to know what is truly real.

Lonergan's approach is significant in the light of the Kantian problem. He realizes that if we start to doubt the possibility of knowing we are unable to justify our knowledge. On the one hand he recognizes that in trying to know the real we have only our own mental activities to go by, but on the other hand he knows that we have shown remarkable results in dealing with "the world around us." So he tries to find the key to knowing. Lonergan's first vital contribution is his approach: to accept the fact that we know and then to analyze how we go about it. A second important contribution is his recognition that our knowing involves not only the two levels of activity admitted by others: (1) all the activities through which we receive perceptions and other data that are immediately given to us inasmuch as we are aware--experience, and (2) the activities by which we attempt and succeed in understanding what we have experienced and by which we manipulate the patterns thus found--understanding. Lonergan leads us to acknowledge a third level of activity, the level of judgment.

How is our knowing related to reality? The total orientation of

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our process of inquiry--to find out what is--is related to reality, says Lonergan. But we achieve valid knowledge only piecemeal and only through judgment, by which we assert that the understanding we have of what we have received in our experience is really so. We reach the real in Correctly Understood Experience. This is, in a special way, an assumption, a hypothesis, an understanding of our knowing process. The judgment of this understanding is still required; it would seem that the survival of man, his gradual mastery of the world, and even his ability to discover his failures, all provide evidence for the judgment that we know. But that judgment and the judgment that we know precisely through this process, can only be made by the individual knower.

The three levels of activity can be shown in a simple example. Standing in a room, I feel a draft. I wonder, what might be causing that. Oh, I know, the door must be open. So I turn and look, and find that the door is standing open. I thought so, I think to myself. The three levels of knowing are all present here and they succeed one another rapidly. On the level of experience, I feel a draft. On the level of understanding, in the context of a vast store of previous learning I reach a possible reason for the draft, an intelligibility that can provide some ordering to my experience: the door must be open. And to determine if that is so, I could ask someone facing that way, if it were too difficult for me to turn around, or I could turn around myself; the report or my seeing the door standing open would be sufficient for me to say to myself: That's it, then. That is why there is a draft.

What, then, is the reality we know? Consider the table we were attempting to know originally. The real table is not the gray metal and

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plastic table I see and touch out there in the room--the object of my senses. The real table is not the whirling mass of atoms that is mostly empty space, which would make me fearful of resting my elbow on it--the table as understood. I reach the real table through all my affirmed and correct understandings, so that I can say that the table is gray, it is made of metal and plastic, and it is composed of atoms according to some precise statistical formulae which govern that composition. The real table is the object not of my sensations or of my understanding but of all my judgments--my affirmed understandings. That threefold knowing process is how I reach reality.

This chapter will examine that knowing process in greater detail, isolating the moments of that continuous knowing process somewhat as a stroboscopic light exhibits a dancer at moments in his dance. Then it will examine knowing in mathematics a little more specifically.

An Example

The analysis of knowing will be aided by a source of references in an extended example of the knowing process as it took place in a mathematical development. This will be provided by the following brief history of complex numbers, with specific attention to the achivement of William Rowan Hamilton.

The puzzle of an even root of a negative number persisted for a long time. The first indication of the willingness to consider $\sqrt{-1}$ as a number to enter into calculations occurred in a discussion of roots of cubic equations by Italian Mathematician, Raffael Bombelli, in 1572 (Struick 70:92). Although the computational forms were now accepted,

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using the result that $i^2 = -1$, complex numbers were still suspected of being numbers of a somewhat different sort than the now acceptable rationals or reals. But people continued to use them and a whole field of complex function theory developed. In 1748 Euler published his Introductio which among other topics dealt with infinite series; one result was the relation $e^{ix} = \cos x + i \sin x$ (Struick 70:120). About the beginning of the Nineteenth Century Argand and Wessel independently arrived at graphical representations of the complex numbers (Kramer 33:73). Gauss developed this interpretation further, but one of his major accomplishments was in the algebraic field; he used complex numbers and complex analysis to show that any polynomial with complex coefficients has a root in the complex numbers, thus solving one of the long-standing problems of algebra (Courant and Robbins 15:101). This and other achievements demonstrated the usefulness of complex numbers and so hastened their acceptance.

Still in the Nineteenth Century, the Irish physicist William Rowan Hamilton began to work with complex numbers and to generalize their results. Earlier, complex numbers had received a geometrical interpretation and could be associated with points in a plane; but by Cartesian coordinates ordered pairs of real numbers could also be associated with points in a plane. Hamilton combined those two associations; to him, a complex number was simply an ordered pair of real numbers; that is, to him the complex number $4 + 5i$ was just $(4,5)$ (Kramer 33:72). The problem of handling operations arose. If complex numbers are these order pairs, what would be the definitions of addition and multiplication on these ordered pairs that would be analogous to the usual operations on complex numbers? Hamilton's invention probably did

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not appear so clearly segmented into the notion of the representation and the notion of the operations. A vital insight here is simply to continue the association of a + bi with (a,b). That is, the real part of the result of the usual operations should provide the first coordinate of the answer of the new operations on ordered pairs; the imaginary part of the result should provide the second coordinate under the new operations. The addition operation, being rather straightforward, shows more clearly the meaning of the insight and might even be the source of that insight:

since $(a + bi) + (c + di) = a + c + bi + di$

$= (a + c) + (b + d)i$

then we can write

$$(a,b) + (c,d) = (a + c, b + d).$$

That is, the first coordinate of the sum of ordered pairs is the sum of the first coordinates; the second coordinate of the sum is the sum of the second coordinates.

This association between the two representations yields the insight as to what to do with multiplication:

since $(a + bi) \cdot (c + di) = ac + bci + adi + bdi^2$

$= ac + bci + adi - bd$

$= (ac - bd) + (bc + ad)i$

then we can write

$$(a,b) \cdot (c,d) = (ac - bd, ad + bc).$$

Because of the simple nature of the insight that suggests one should continue this representation, it is almost obvious that such is the correct form of the operations on the ordered pairs. To make certain that this is indeed a correct representation of the arithmetic of complex

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numbers in this new form, a simple check would be to see that in this multiplication $(0,1) \cdot (0,1) = (-1,0)$, corresponding to the basic identity for complex numbers, $i^2 = -1$. And so we can affirm with Hamilton: these are operations on ordered pairs that represent the operations on complex numbers.

Hamilton easily achieved this representation of complex number operations. But development did not stop there; a number of questions arose. Since the pairs of numbers can be associated with points on a plane by means at least of Cartesian coordinates, what are the graphical expressions of these operations on ordered pairs? How are these two representations of complex numbers really associated, and is that a reliable association, and what do we really have to test to be sure of this reliability? How does this complex number system really work anyway? How is it like the real numbers? What do I have to test to make sure it has all the properties I require? The answers to these questions were reached slowly. The first question was answered independently of Hamilton in the same century through the development of vector algebra. The others were answered only with the formulation of more abstract algebra--isomorphisms and the isolation of field properties--and which Hamilton's own further work did much to trigger.

Hamilton himself asked a different question: Having found operations for pairs of numbers, can we find operations for triples, quadruples, etc.? He spent fifteen years attempting to formulate definitions of $(a_1, a_2, a_3) + (b_1, b_2, b_3)$ and $(a_1, a_2, a_3) \cdot (b_1, b_2, b_3)$ that would obey all the fundamental laws of algebra and allow subsets of the set of all ordered triples that would behave like the complex and real numbers (i.e., subfields). Finally, failing at

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that, Hamilton shifted his efforts and began to investigate quadruples of real numbers. He still was unable to formulate definitions of addition and multiplication that would obey all the classical laws of algebra and so he settled for operations with all the properties of real number operations except multiplicative commutativity (Kramer 33:76-78). Hamilton's decision to settle for less than he wanted was proven a wise choice much later. Frobenius and Pierce proved that the only finite-dimensional division algebras with reals as scalars are the real numbers themselves, the complex numbers, and the quaternions that Hamilton developed (Kramer 33:72).

At about the time Hamilton was working with the complex numbers and beyond, members of the British algebraic school (Peacock, Gregory, De Morgan, and Boole) were beginning to treat algebra as a strictly postulational-deductive system (Kramer 33:79f). These men helped to isolate the properties that characterized number systems and to classify systems according to these properties. It was a result of this work that the complex numbers finally could be classified as a number system and the adequacy of Hamilton's original representations assessed.

Knowing as Dynamic Structure

For Lonergan, knowing is a dynamic structure. The terms of that assertion need some explanation.

First, "structure." A whole can be examined through its parts. Some wholes are arbitrary or conventional quantities or collections, and their parts also are determined by an arbitrary or conventional division. "In such a case, for example a gallon of milk, the closed set of relations between whole and parts will be a no less arbitrary jumble of arithmetic

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ratios" (39:222). There is no instrinsic reason to consider a pint of milk as necessarily one-eighth of a gallon; the measures have been arbitrarily set. But there is another kind of a whole, a structure. A structure is an organized whole, whose parts make up the whole because of functional relations between the parts; there is no part that is not determined by the demands of other parts; and the whole possesses a certain inevitability in its unity, so that the removal of any part would destroy the whole, and the addition of any further part would be unnecessary. The famous Rube Goldberg inventions are structures; their parts, though highly unlikely, function as parts of the whole because of the way each is related to the other parts.

Hamilton's work with complex numbers was a structure because each activity involved was related to the others, and together they were the answer to the initial curiosity. The original question of the possibility of ordered-pair treatment caused an examination of the complex number system, but an examination conducted with the goal of the inquiry in mind. These data gave rise to the insight of how to represent the new operations, an understanding based upon what was already known. The understanding provoked the question, "Is this really an alternate representation for the arithmetic of complex numbers?" Checking seemed to answer that. All these activities--the various questions and partial answers, algebraic calculations, the computations, and the checks--were related to each other and together they were the answer to the initial curiosity.

Secondly, Lonergan views knowing, human knowing, as a dynamic structure, and it is dynamic in two ways. In the first sense of "dynamic," it is a structure composed of activities: it is materially

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dynamic. Whereas the parts of a whole may be things, as bricks and glass are parts of a house, the parts also may be activities, as the various movements are the parts of a dance. So the parts of knowing are activities, "many distinct and irreducible activities: seeing, hearing, smelling, touching, tasting, inquiring, imagining, understanding, conceiving, reflecting, weighing the evidence, judging" (41:222). In the example with complex numbers, the process of coming to know the operations on ordered pairs included the acts of questioning, calculating, hypothesizing, checking. This structure of knowing is not the structure of what is known: when we speak of the structure involved in knowing mathematics, we do not refer to the mathematical structures of the number systems. The postulates that characterize the mathematical structures of the natural numbers, for example, are not a mere externalization of the processes we used to reach them.

Knowing is a dynamic structure in a second way: it constructs itself; it is formally dynamic; one operation provokes the next. In the complex-number example, the questioning led to an examination of what happens in computations with complex numbers, which led to a proposal for the forms of computation for ordered pairs, which in turn led to a need to check the results. Each activity seemed to demand going on to the next for completion. In general, knowing "puts itself together, one part summoning forth the next, till the whole is reached" (41:223). This construction is not blind, a sort of stimulus-response connection or a stream of consciousness, but a conscious, intelligent, rational occurrence. The dynamic unity unfolds itself through the series of questions that arise in the mind of an inquirer. Experience activates the intelligence, and intelligence brings itself into action

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by inquiry that leads from experience through imagination to insight; and then inquiry moves from insight to the formulation of concepts that contain at the same time the understanding reached in the insight and the relevant elements of experience and imagination. But the development of concepts calls forth reflection; reflection presents the evidence for and against the adequacy of the concepts. The inquiring intelligence demands this reflection, which can result in a judgment, thereby ending the process (41:223).

Three Levels

Lonergan postulates that all these activities can be grouped into three levels; that is, there are three basic levels of activity in this knowing structure: experiencing, understanding, judging.[1] The distinction between the levels is brought out by the kinds of questions that are asked which result in an advance from one level to the next.

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[1]In his later work, Lonergan writes of these stages of activity as levels of consciousness, and he extends his analysis of man's activity beyond merely knowing. For example, in Method in Theology he makes clearer the presence of a fourth level:

> There is the empirical level on which we sense, perceive, imagine, feel, speak, move. There is an intellectual level on which we inquire, come to understand, express what we have understood, work out the presuppositions and implications of our expression. There is the rational level on which we reflect, marshal the evidence, pass judgment on the truth or falsity, certainty or probability of a statement. There is the responsible level on which we are concerned with ourselves, our own operations, our goals, and so deliberate about possible courses of action, evaluate them, decide and carry out our decisions (42:9).

A more recent development in Lonergan is the level of historical consciousness, which is an understanding and affirming of one's participation in a changing world-picture (44:85). Each of these levels incorporates all that had been reached in the previous levels. However in this thesis we will be concerned only with the first three levels.

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In the complex-number example the level of experience consists in examining or remembering the usual computational forms for complex numbers and the form for representing numbered pairs. We ask, What operations on ordered pairs will embody the complex number operations? The understanding enters in answering that question when we grasp that the first coordinate of the result of the new operation is merely the real part of the usual result, and the second coordinate is the imaginary part. Still on the level of understanding comes the work of formulating that grasp, putting it into some expression. Concerning this formulation, we ask, Is it so? Does it work? The third level, that of verification or judgment, comes in when we can answer the Is-question and say finally, That works; we can perform the operations that way.

The first and most important implication from the fact that human knowing is such a dynamic structure is that knowing cannot be imagined to be like merely a part of it, e.g., some special kind of looking or some powerful insight or an analytic judgment (41:222-24). Knowing takes place in the entire three-fold structure of the activities that comprise it. Each of these cognitional activities could be called knowing, because each is a part of the knowing process, but not properly. Knowing takes place in the completion of the activities at all three levels. Unfortunately, the language used for cognitional activity often implies that knowing is like seeing. "Oh, I see," someone says when he understands something; even the key word used here, "insight," carries the same connotation. This thesis will not be able to avoid that language. But knowing is not mere imagery or an association of images. One does not know the Fundamental Theorem of Calculus by staring at a statement of it or at a diagram illustrating it. The most ingenious diagram of

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the Pythagorean theorem (Friedrichs 21:8-10) may make the proof "intuitively obvious," but that is not reached by seeing but by analyzing what one sees. When we know, we do more than open our eyes.

That knowing is more than sensing is obvious from the common instance of two people's seeing the same thing but only one understanding. They may look at a diagram used in developing a geometrical proof, yet one man will be able to develop the proof from that diagram and the other one simply gape. The crafty Sherlock Holmes and his faithful Doctor Watson saw the same things, but only Holmes went on to understand.

Insight

In this three-leveled knowing process, insight is the act by which a knower moves from the level of experience to the level of understanding; it is the reaching of an answer to a what-question.

Lonergan's standard example of insight is the story of Archimedes. King Hiero of Syracuse had a crown made by "a smith of rare skill and doubtful honesty" (37:3); Archimedes was given the problem of determining whether or not the crown was of pure gold. As Archimedes lowered himself into the baths of Syracuse one day, he hit upon the solution, and ran naked through the streets with the cry: Eureka!

That dramatic example is an illustration that insight

(1) comes as a release to the tension of inquiry,
(2) comes suddenly and unexpectedly,
(3) is a function not of outer circumstances but inner conditions,
(4) pivots between the concrete and the abstract, and
(5) passes into the habitual texture of one's mind (37:3f).

Each of these points needs examination.

1. The first characteristic is another way to speak of the drive to know that is characteristic of human activity.

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> Deep within us all, emergent when the noise of other appetites is stilled, there is a drive to know, to understand, to see why, to discover the reason, to find the cause, to explain. Just what is wanted, has many names. In what precisely it consists, is a matter of dispute (37:14).

This impulse is the basic orientation to being that underlies our entire quest for knowledge. It finds expression in the many questions we can ask. The existence of this drive, whatever it be called, is evident in the whole range of human behavior from the persistent questioning of a child to the patient inquiry of a scholar; this drive can be extinguished or diminished, as the critics of poor education so often point out. But it exists, and insight is the first product of this self-generated pressure to solve a problem, to reach an answer to a question, to find out what is happening.

2. Insight comes suddenly and unexpectedly. It is not reached through definite rules or by following precepts. It is discovery. In its most sweeping form, it is genius, which is creative precisely because it disregards established routines and comes to new understandings that will be the basis for procedures and rules in the future (37:4). There is no guaranteed method to the discoveries of genius. They cannot be scheduled. There are helps to insight which will be analyzed in this thesis, but they are only helps.

Perhaps Lonergan's language is too dramatic here. Many people can recognize a rare exciting moment such as Archimedes', but find it difficult to characterize every instance of knowing in that fashion. However, insight is a normal and regular element in everyday life. In the many pedestrian cases it can be as simple as the (usually unformulated or unexpressed) thought when we feel a draft that "The door must be open" or when we work with complex number operations that "It might

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work if we try . . ."

But insight is an advance; it is something new, unpredictable. Even when the application of a routine is called for, the insight must be there that this situation calls for the routine. When someone fails to reach an "obvious" insight, he later berates himself for missing it, but this negative example supports the position that there are no conditions that make an insight necessary.

3. Insight arises from inner conditions, not outer ones. It is not seeing; seeing is accomplished by opening one's eyes. For those not blind, seeing is something that can be activated or deactivated at will; but insight is not.

In part, the necessary inner conditions consist in "native endowment"; simply, some people have more--and more inclusive--insights than others (37:5). But Lonergan, a philosopher, does not try to discover the dimensions of this endowment or intelligence. Insight depends upon a habitual activation of the desire to know, a readiness to ask the question "Why?" One of the conditions for insight is accurate presentation of definite problems. The clarity of presentation could be considered something inherent in the external situation, such as greater or less "noise" in the experience or data or the presence or absence of formulated questions. Still someone with greater "native endowment" can find the relevant experience and formulate the pertinent question; he has a facility in creating an image that will lead to understanding, and so compensate for whatever lack of organization there may be in the data. These conditions are internal; they can be awakened in the presence of some situation, but they must be present or emerge within the knower.

What could be called a "prepared mind" is also part of these inner

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conditions. A knower does not merely have an insight; he has an insight into something. He must possess sufficient data, a sufficient familiarity with the subject so that he can represent the subject adequately and explore the possibilities. Hamilton did not stumble into his theory of quaternions. He worked for years on the requisite properties; only at the end did he reach the vital insight.

4. Insight pivots between the concrete and the abstract; it occurs in conjunction with a particular problem, possibly particular concrete data, a concrete question; but the understanding reached by insight is an understanding, not something on the same level as the problem, to be handled just as the original question was or to be treated as an item along with the original data. Archimedes' understanding is not of the same significance as a weighing of the crown, yet the notion of a weighing and actual weighings are important parts of the original problem. Hamilton's discovery had to do with the actual manipulation of complex numbers and the more abstract nature of the operations themselves.

The understanding, in these examples, is a shift to a new level; what seemed meaningless before in the concrete experience is now intelligible. Thus insights are linked to the concrete and individual, the sensible, or the imaginable:

> Mathematicians need pen and paper, teachers need blackboards, pupils have to perform experiments for themselves, doctors have to see their patients, trouble-shooters have to travel to the spot, people with a mechanical bent take things apart to see how they work (37:6).

From this dealing with the concrete arise the abstract sciences with their numbers, symbols, technical terms, definitions, postulates, and deductions.

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In an example of insight from other realms, the experience of a scene in a good movie can lead to an insight into the meaning of the conditions of life, something that will be widely applicable.

Insight, by its nature, is the mediator, the pivot. It originates in experience, which is concrete, singular, the field of sense and imagination. The result of insight, its product, is an understanding. Something is added by insight: the universality of understanding, so that the understanding can be expressed only in abstract formulations which are sometimes far removed from experiential referents, as in the highly developed sciences. But the link to the concrete remains in the understanding: "The universal abstracts from the particular, but the intelligibility, grasped by insight, is immanent in the sensible and, when the sensible datum, image, symbol, is removed, the insight vanishes" (44:74f).

5. Insight passes into the habitual texture of the mind. Insight is a vital step, making what had been insoluble--a hopeless jumble--now clear and simple and obvious. The insight is irreversible. When the problem occurs again, it shares that newly-found simplicity. As Rosenbloom pointed out, once one knows what the prime numbers are, then the whole numbers never look the same again (61). If one really has the insight into factoring $x^2 - y^2$, then it is never again a serious problem; but if the insight is lacking, then there occurs each time that problem is faced a long groping until a combination of expressions is reached that will multiply together to yield the desired result. This irreversibility of insight "constitutes the possibility of learning" (37:6). Once an understanding is reached through insight, then it can be relied upon; if unused for a while, it can be recalled quickly; new

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insights can be added to complement and combine with ones already reached, so that learning can be cumulative.

The role of insight in mathematics has received much attention through the years, most familiarly in the work of Poincaré and of Hadamard. Hadamard used the analysis by Poincaré (52) and reports by many other mathematicians on the manner in which they reached their results, in order to discuss The Psychology of Invention in the Mathematical Field (26). Hadamard found four stages in the act of mathematical invention or discovery:

a) Preparation--extended conscious work on the problem (26:45);

b) Incubation--a time when the problem is dropped so that the unconscious can work on the problem (26:18f);

c) Illumination--the occurrence of the solution suddenly and spontaneously (26:15); and

d) Final--for the expression, verification, and "precising" of results and continuation of the work to find related results (26:56-60). Hadamard also mentions the help in thinking which concrete representations provide, as well as the variety of the types of representations that are used, from words to algebraic symbols to less distinct images and even to kinetic or auditive representations (26:66-86).

Hadamard uses the word "insight," however, to refer to a "scientific sensibility," a feel for the correct answer; he says that even if he makes a mistake in proceeding through a problem, by insight it occurs to him that something is wrong.

Hadamard focuses on instances of mathematical discovery in which the answer was not immediate, and that specialization presents several problems. For during the preparation stage, the mathematician goes

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through some work; if he finds an answer at that point he will go on to the fourth stage, to verify and precise his result. If he does not have illumination during the time of his conscious work, he will continue. Not all discoveries need to be made at a time when the unanswered question is being ignored. Further, to attribute the achievement of a discovery to "unconscious thinking" during the incubation period is a position with considerable dangers, because the unconscious is admittedly something not open to investigation. But on the whole, Hadamard's scheme for invention does bear some relationship to Lonergan's:

Preparation	---	Experience
Illumination (Precising)	---	Insight
Verification	---	Judgment

Judgment

Insight is the act of coming to understanding about a dilemma, and it answers a question such as: What is that? What is going on? Why? How? There is a further question that occasions a further step in knowing. Every reader presumably has had bright ideas that do not yield solid results, that sound good at first but do not correspond with facts as they come to light, that are tried but fail in the test against reality or against other understandings already reached. So there is a further type of question that must be asked, not the What-question for understanding, but the question for reflection or judgment: Is that so? Really? Is that true? And these questions for judgment call for an answer Yes or No (37:271f).

> The formulations of understandings yield concepts, definitions, objects of thought, suppositions, considerations. But man de-

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> mands more. Every answer to a question for intelligence raises a further question for reflection. There is an ulterior motive to conceiving and defining, thinking and considering, forming suppositions, hypotheses, theories, systems. That motive appears when such activities are followed by the question, 'Is it so?' We conceive in order to judge. As questions for intelligence, What? and Why? and How often?, stand to insights and formulations, so questions for reflection stand to a further kind of insight and to judgment. It is on this third level that there emerge the notions of truth and falsity, of certitude and the probability that is not a frequency but a quality of judgment (37:273).

Another way of making the notion of judgment explicit is to realize that there are two distinct mental attitudes with regard to propositions: one may merely consider them or one may affirm or deny them. In the first case, the proposition is simply an object of thought; in the second, it is the content of an act of judging (37:271). Or, to approach the matter from another aspect, judgment involves a personal commitment, a responsibility of the one who judges. As long as one is merely thinking about something, his reputation is not at stake. But as soon as he says, That's right, or It's wrong, he has taken a stand. Without loss of face a man can admit he has poor eyesight, but he is usually not ready to confess poor judgment (37:272).

We are aware that we make judgments; we answer questions, Is that so? with a Yes or No or I don't know. But what do we do when we judge? Somehow we examine the situation and determine that all the significant questions have been faced and answered. And then we say: It is so. This affirmation requires another act of insight, the reflective insight, which is a grasp of the sufficiency of the evidence for a judgment upon which we are deliberating. Although it is difficult to say exactly what happens in a reflective insight (as it is difficult to say exactly what happens in an insight for understanding), we do make these judgments, and with confidence: "What we know is that to pronounce judg-

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ment without that reflective grasp is merely to guess; again, what we know is that, once that grasp has occurred, then to refuse to judge is just silly" (37:279).

Sometimes we can neither affirm nor deny; we cannot answer at the moment all the questions we need to answer. But we can argue to a likelihood, as Polya outlines in his treatment of plausible reasoning (55). We can give answers that point in a direction: "Possibly." Often we must deny a judgment for the present: "I don't know."

In this analysis of judgment Lonergan attempts, not to prescribe the way to reach correct judgments, but to examine and understand and verify how we actually do make them. We need to understand what it means to have sufficient evidence for a prospective judgment.

There are some mathematical questions that have been answered, yet lack rigorous proof; Gödel has shown that there can be no strictly formal proof of the consistency of arithmetic, yet its consistency is accepted (Kramer 33:48). Hamilton pored over the operations on number pairs, doing what he found necessary so that he could assert that the new operations were adequate. The mathematician working in a more formal and abstract presentation of mathematics will not be satisfied with such an informal check, but will attempt to prove his conjecture from the axioms of the particular system. In each of these cases, the knower satisfies himself that he has sufficient evidence to assert the statement, but in the last case the sufficient evidence is strict deduction.

In Lonergan's terminology, to have the reflective insight grounding a judgment is to grasp the prospective judgment as virtually unconditioned (37:280). Something is virtually unconditioned when it does have conditions, but these conditions are met. A judgment is virtually

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unconditioned when the statement, the understanding, the proposition to be affirmed has conditions for its truth and these conditions are fulfilled.

The judgment occurs in this way: for an understanding to be affirmable, it will have to be adequate to the experience that originated it and be consistent with other affirmed understandings. There must be some connection between this particular understanding and its possible basis in experience or other judgments. In this connection, the statement is conditioned. But it becomes virtually unconditioned when that possible basis is found to be actual.

> The function of reflective understanding is to meet the question for reflection by transforming the prospective judgment from the status of a conditioned to the status of a virtually unconditioned; and reflective understanding effects this transformation by grasping the conditions of the conditioned and their fulfillment (37:280).

In the case of the complex number operations, the steps of understanding and judgment are not very distinct, but in the process of reaching an answer each new investigator proceeds in this way: I determine if in fact the operations in the new representation give the expected results. Since they do, I can affirm: This is the answer. The conditioned is the judgment "This is the way to represent the operations"; the condition is the fact that these formulations give answers I already know to be correct. Since the condition is fulfilled, I can make the judgment. The reflective insight is the realization that once I have answered this question about these conditions, that is enough. I need to investigate no further. I can proceed to judgment.

Mathematical examples can portray clearly this virtually unconditioned. If I examine a particular mathematical structure and it appears sufficiently developed, I can propose: It is a field. That

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statement will be true and I can affirm that the structure is a field precisely if certain conditions--the field axioms--are satisfied. Once I have shown that the axioms are satisfied, I can immediately make the assertion, It is a field. This judgment is a virtually unconditioned, since it is conditioned and the conditions are known and fulfilled.

It might be necessary to make precise the language used here. The principal notion of judgment in formal mathematics remains the deductive proof. When we read through the proof of a mathematical theorem and say we "understand" it, that uses language at the level of judgment that we have used for language Lonergan uses at the level of insight. But this "understanding" of the proof says in one brief statement that we grasp the meaning of the statements in the proof and, more importantly, we grasp the necessary connections between the statements that makes this series of statements a proof of the original proposition. We understand the proof, and so our reflective insight gives rise to the judgment: The theorem is true. The understanding of the theorem itself is a distinct and usually prior achievement.

An analysis of the conditional syllogism will point out further characteristics of judgment. A syllogism is an example of only one sort of judgment, but it is a special concretization of the notion of judgment. Deductive inference argues:

If A, then B.
But A.
Therefore B.

The conclusion, B, is obviously conditioned, for it is presented as the conclusion of an argument, needing the support of that argument. The major premise (If A, then B) exhibits the conclusion as conditioned

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and shows the link of the conclusion to its conditions, the antecedant, A. The minor premise, the affirmation of the antecedent, affirms the fulfillment of those conditions (37:280f). This deductive form of argument, especially as crystallized in the syllogism, has the function of exhibiting a judgment, representing the conclusion as virtually unconditioned. When this is done, reflective insight grasps the pattern, and the judgment follows by compulsion out of the structure of knowing, the drive of inquiry.

Lonergan's notion of judgment does not demonstrate the basis for certainty of any particular proposition; in the syllogism above, the minor premise demands another assertion, and we would be involved in the problem of infinite regress or of determining certain first principles if we tried to ground the certainty of every judgment. But the problem of certainty is one different from the problem of the form of judgment. In analyzing how we know, Lonergan finds that, as a matter of fact, our judgment is a matter of the virtually unconditioned. We must at least recognize this nature of judgment; the matter of certainty or objectivity can be settled later.

The assertion of the conclusion in the deductive form of argument presupposes two other judgments to be true, the major and minor premises (37:281). These prior judgments had to be affirmed as final products themselves of the cognitional process. Before the link between conditioned and conditions, If A, then B, in any particular case could appear in this act of judgment, it had to be reached in a more rudimentary way as an understanding and an affirmed. The same holds for the affirmation of the conditions, But A. Therefore judgment requires other previously attained elements in the cognitional process

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to reach virtually unconditioned. The deductive form is not the only way to reach a judgment.

But the deductive form does rely on other judgments, so some philosophers have spoken of judgment as merely a combination of propositions. That is what the deductive syllogism might appear to be, if we do not accept Lonergan's recognition of the wider scope of judgment.

The foregoing has argued that the deductive form of inference cannot be the only form of judgment, or even the most basic one, because it relies on the truth of preceding judgment (40:14f). What are the other ways to assert the fulfillment of conditions? How else may we judge that an understanding is correct? Belief in divine revelation or in the judgments of others is one way; common sense judgments or reference to empirical data are others.

To give fuller treatment to the other ways we make judgments, Lonergan distinguishes between vulnerable and invulnerable insights. An insight is vulnerable when there is evidently more to be found out, at least, when there are more questions to be asked in the particular situation. Further questions could lead to insights that complement the original understanding, calling for a modification of the expression or giving a whole new outlook to the situation. When there are no further questions to be asked concerning the adequacy of the understanding, when the questions just seem to run out, the insight is invulnerable (37:284).

We began with questioning as a foundation of the knowing process, and it re-enters here. The notion of an invulnerable insight is an operational one; when an understanding has provided answers to all the perti-

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nent questions, it is invulnerable; all the questions have been dealt with. Then the understanding can be judged correct. And this is the more primitive type of judgment.

> When an insight meets the issue squarely, when it hits the bull's eye, when it settles the matter, there are no further questions to be asked and so there are no further insights to challenge the initial position. But when the issue is not met squarely, there are further questions that would reveal the unsatisfactoriness of the insight and would evoke the further insights that put a new light on the matter.
>
> Such, then, is the basic element in our solution. The link between the conditioned and its conditions is a law immanent and operative in cognitional process. The conditioned is the prospective judgment, this or that direct or introspective insight is correct. The immanent law of cognitional process may be formulated from our analysis. Such an insight is correct, if there are no further, pertinent questions.
>
> At once it follows that the conditions for the prospective judgment are fulfilled when there are no further, pertinent questions (37:284).

Recapitulating: Knowing is a process that involves (1) experiencing, (2) an insight that reaches understanding, expressed in some formulation, and (3) judgment as an affirmation (or denial) of the understanding. Each level has its own characteristics; together they form the process of knowing. If any one level is allowed to serve as the model for knowing, the results are a distorted appreciation of knowledge and of reality.

Important Notions

The remainder of this chapter will be a discussion of particular important elements in Lonergan's cognitional theory. This discussion should also serve to clarify the general structure of knowing.

Description and Explanation We can speak of the sun in two ways. First we speak primitively, as if Galileo had never lived, when we say that the sun rises in the East and sets in the West. But we also speak

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scientifically, when we say that the earth is in orbit around the sun, and go on to express this notion through mathematical formulae. Here we have examples of two characteristic ways we can understand things: first, in their relation to us; and second, in their relations among themselves. The first manner of speaking is descriptive, based on how things appear to us; it is strictly in terms of the given, the data. Explanation, the result of analysis of the given, is an understanding of things as they are related among themselves and often requires a shift in language from the pictorial (imagistic) or experiential terms of description to the positing of a universal law.

Both forms of understanding are necessary even for one another. Description serves to isolate appropriate data, which is then considered in explanation; it provides "the tweezers by which we hold things while explanations are being discovered or verified, applied or revised" (37: 291). But description and explanation, despite this connection, are understandings of things in quite different manners, since description does not move far from the sensible, while explanation attempts to give the reason for the particular conjunction of data.

Definition Lonergan makes his analysis of definition through the example of Euclid's definition of a circle as a locus of coplanar points equidistant from a center.

How might a learner reach this definition?[2] We can all draw a circle, though not a perfect one, or conjure one up in our imagination;

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[2]There are various types of definition, that often overlap. Several types may be found in this example. But the concern here is to find someone in the act of defining, to examine the knowing process in progress. Types of definition will be distinguished later.

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we know it is round and flat. We could define a circle as a round figure on a flat surface but that would not be a great achievement. A more effective definition would capture exactly what is the cause or nature of this roundness. A particularly useful round object to think about in this context is a wagon wheel. Its spokes join the outer rim to the hub. If any one of the spokes is different in length from the others, then the wheel will be out of round. If the spokes are equal in length but sunk differing lengths in the hub or the rim, again the wheel will be out of round. So the roundness seems to have to do with the equality of the length of the spokes and evenness of the sinking of the spokes into the hub and rim. If the rim and hub are narrower, there becomes less possibility of variance because of the uneven sinking.

This "seems to have to do with" is a first hint at a reason for roundness. But to arrive at a definition we need to formulate the understanding hinted at here, because what is required is the treatment of the geometrical figure of a circle that has a perimeter with no thickness and a point as a center, both of which are beyond our imagination. Using the hint of the wagon wheel and the necessity of equality of spokes for roundness, and in our imagination multiplying the spokes until they are almost solid around the imagined wheel, and thinking about the shrinking of the hub and of the width of the outer wheel, we might be able to arrive at the definition of a circle as the locus of points equidistant from a center.

The crucial moment in the movement to definition is the insight that the roundness of the wagon wheel seems to have to do with the equality of the length of the spokes and evenness of the sinking of the spokes into the hub and rim. Here, an insight into the reason for

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roundness occurs. But to arrive at a definition we need to formulate the understanding contained in the insight, because we must treat of the geometrical figure of a circle, which is beyond the drawn or imagined circle. We add the phrase "in a plane" to the definition after further analysis of the possibilities of the definition: a sphere is a locus of points equidistant from a point, the center of the sphere. The phrase "in a plane" eliminates this ambiguity.

There are several elements to note about the resulting definition. The terms we use in the definition, points and distance, do not refer properly to objects of sense or imagination; points and lines and geometrical figures are concepts, and what is important in their meaning must be freed from dependence on what can be sensed. Geometrical points have only location, not dimension. The power of intellectual conception must be emphasized here for, although we can imagine points, at other times we create objects that cannot be imagined (37:8). These concepts are constituted by the mere activity of supposing, thinking, formulating, defining.

Secondly, concepts do not emerge at random; they emerge in thinking, in supposing, in formulating, all of these in conjunction with an insight that responds to basic human wonder.

Here we can see the moments in the genesis of the definition. First comes the awakening of wonder, the arising of intelligence, the question directed at some particular object of experience, with the data of the figure of the wagon wheel. Second is the hint, the clue, when insight has begun--in this case "it seems to have to do with the equality of the length of the spokes." Third comes the cooperation of intelligence with imagination; that is, imagination helps us see what happens

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with variation of length of spokes, with penetration of the spokes more or less into the rim or the hub, with multiplication of spokes to decrease possible flattening of the curve between spokes. The wheel is thus where the outer ends of the spokes lie. At the same time intelligence guides the imagination to form suppositions about the effect of the changes, and the imagination guides the suppositions so they remain imaginable. Together question and insight, image and supposition reach a patterned set of concepts: points, center, equal distances. The changing image of the wheel approximates the concepts but cannot reach them, because a point or a line or the continuum of points along the circumference is beyond the imagination; it is insight which links images to concepts. And setting the standard which insight, images, and concepts must meet is the question, the desire to know, that lets this process cease only because the question was answered.

Here we meet the process of insight's arriving at a response to a question What is it?, and reaching the formulation of a definition. The definition then expresses the understanding.

There can be a further moment here, a moment of judgment, when we ask whether the definition we have reached is adequate to what we intended it to cover, whether we have captured the meaning of roundness and circularity in our definition.

Not every definition expresses such an understanding, such intrinsic intelligibility, the reason why, as does the definition of a circle. There are several kinds of definitions. Nominal definitions merely tell us about the correct usage of names, while explanatory definitions express the intelligibility of the object defined. The difference is not the presence or absence of insight in the definition; we

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need insight to use the tool of language properly and effectively; we need insight to say that a circle is a round flat figure. But a nominal definition requires only an insight into the correct use of words, whereas an explanatory definition requires a further insight, an insight into the object referred to by the words. Thus definition of the circle as a round plane curve does no more than help us point to circles; but with the definition asserting that all radii in a circle are equal, it is no longer a matter of pointing or of talking about names; here the definition gives the understanding of what a circle is, no longer only how it appears (37:11).[3]

This distinction between nominal and explanatory definition has a connection with the distinction between description and explanation made earlier. Description refers to sensible or imaginable referents, and a nominal definition describes how language is used. Explanation and explanatory definition both penetrate the object discussed, referring to why it is the way it is.

A problem concerning definition that presents itself especially in a mathematical context is that of primitive terms. We cannot define every term. But to let such a problem be overwhelming is to think of mathematics as emerging as a full-formed structure and to fail to recognize that mathematics develops first through the emergence of insight and understanding and only then proceeds to formal structure as an expression.

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[3]Some authors speak of stipulative definition, in which a term is equated to a specific meaning. The purpose is heuristic, in order to determine what characteristics could be asserted of that new construct. The initial definition would be, in Lonergan's terms, a nominal definition.

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Definitions do not merely arise; they are a product of the drive to know and are part of a context of experiences, images, questions, and insights. Every definition requires several terms; no insight, likewise, can be expressed in a single term. The multiplicity involved seems confusing, but that is the way it happens. No simple one-to-one correspondence exists between insights and terms, as they emerge. What does happen is that for every basic insight there is a circle of terms and relations in which the terms fix the relations, the relations fix the terms, and the insight binds them all together. Some evidence of this is that at times when the insight is temporarily lost or to someone who has not had the insight, the use of terms and their relations seems a hopeless, nonsensical jumble.

What this mutuality shows, however, is that there is no need to search for a term that needs no definition, whose meaning is self-evident. Whereas the formulation of the definition requires some previous insight, some insights can be primitive; that is, with no previous insight they can yield this mutual constitution of terms and relations.

The insight grounding the mutual constitution of terms and relations yields a third kind of definition, implicit definition (37:12), the sort proposed by Hilbert in his attempts to develop a foundation for geometry (29:3-5). In this geometry, the meaning of both "point" and "line" are fixed by the relation that two and only two points determine a line and two lines determine at most one point. "In terms of foregoing analysis, one may say that implicit definition consists in explanatory definition without nominal definition" (37:12). It involves explanatory definition, since the relation between point and line is a postulational element, a relation, like the equality of radii was in the

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definition of a circle. But this implicit definition omits nominal definition, because there is nothing in particular pointed out as an example; there is no one kind of thing required to be a point. An ordered pair of numbers can be a point and a line a linear equation--with neither of these considered in their relation to their representation on the Euclidean plane. The affine geometries use representations but their real significance is in the structure defined by the axioms.

A familiar whimsical example is the story of the four retired gentlemen who comprise six different clubs, two of them in each club, no two clubs having more than one member in common. Each of the gentlemen, then, can be considered a point and each club a line, and the entire constellation constitutes a geometric structure fulfilling the axioms. Thus a line, defined implicitly, can be considered as many things, from the Euclidean line we can represent by ink on paper to a club composed by the eccentric gentlemen.

These examples exhibit the significance of implicit definition: its complete generality. It allows statements to be made that are not restricted to the particular objects that were originally considered (37:12). It is also helpful in freeing the understanding from particular images, allowing the force of formal expression to enter. If the mathematician relies on familiar objects or images, some insight might go unrecognized; because it was so obvious in diagrams Euclid failed to recognize that he needed an axiom to guarantee that two circles obviously intersecting would meet in a point, and without an additional axiom to provide this guarantee, even his proof of Theorem 1, Book I of the Elements was invalid (Heath 27:241f).

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Necessity of Image The role of the image of the wagon wheel in reaching the definition of a circle exemplifies the contribution of an image in the attainment of an insight. Here the image of the wagon wheel was necessary for the insight; if we eliminate that image, the imagined center, the radii, the curve, then we have also eliminated all grasp of the necessary roundness. But the grasp of the necessary connection between the equality of radii and the roundness of the circle constitutes the insight and makes the definition possible; it makes the difference between parroting a definition and uttering it intelligently, with the ability to make up a new phrasing for the definition.

> It follows that the image is necessary for the insight. Inversely, it follows that the insight is the act of catching on to a connection between imagined equal radii and, on the other hand, a curve that is bound to look perfectly round (37:9).

Lonergan's stress on the need of an image (or a phantasm) for the emergence of insight, has its roots back in the philosophy of Aquinas and Aristotle. Lonergan states his acceptance of their position:

> Human intellect in this life needs phantasms as objects--indeed, as proper objects. Since knowledge requires an object, and since phantasm is the object of intellect, a phantasm is always necessary for intellectual activity. . . In a word, one cannot understand without understanding something; and the something understood, the something whose intelligibility is actuated, is in the phantasm. To understand circularity is to grasp by intellect a necessary nexus between imagined equal radii and imagined uniform curvature. The terms to be connected are sensibly perceived; their relation, connection, unification, is what insight knows in the sensitive presentation (41:28f).

The use of the image with respect to understanding in science or mathematics has two directions, Lonergan says. In the development of science, the phantasm--the empirical data "in conversation with" some imagined figure to simplify and isolate the important--is the source of

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insight, the object in which insight finds regularity and intelligibility. This use is exemplified in the mathematician's habit of writing it down or drawing a diagram or setting up the equations on his way to a solution, to put it in a visual form familiar to him. In the subsequent use of science, the understanding that is already grasped is often expressed in a suitable image to serve as a vehicle for explanation and for provoking further insight. The Bohr model of the atom or diagrams of the energy levels of electrons are examples of this use of phantasm.

Tracy refers to the usefulness of the image of the wagon wheel as only one example of a psychological fact: men do not merely stare at images, but they can understand something in and through these images. This insight into phantasm, says Tracy, brings to notice three characteristics of human consciousness. First, men wonder and question. Second, sometimes they reach answers to their questions. Third, they reach these answers not as purely intellectual creatures but they include their senses; "they learn to notice and form images (e.g. the wheel) by means of which they may grasp the intelligibility they seek" (72:63).

Symbolism One of the prominent forms imaging takes is in the utilization of symbols, which Lonergan calls "an image that is a sign of intelligible and rational contents" (37:723).

Two functions of symbolism are crucial in mathematics. First is the obvious advantage of abbreviation: eight hundred thirty-seven can be written as 837, or the symbol dy/dx represents a complicated expression involving a limit (40:4). A second function of symbolism is to supply a relevant image for an insight. This function has

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several facets. The symbolism can incorporate an insight and allow a routinization of that insight; hence, students who have not acquired an understanding of the structure of arithmetical operations can still get correct answers through processes gained in classroom drill. The problem "Find the sum of eight hundred thirty-seven and four hundred fifty-two" requires some explicit work with processes (or an awareness of the potentialities of symbolism and a conversion to it) that the problem

$$\begin{array}{r} 837 \\ +452 \\ \hline \end{array}$$

avoids. Or pupils can much more easily find $\sqrt{1764}$ than the square root of MDCCLXIV. If students know the square root algorithm, the expression $\sqrt{1764}$ can serve as a reminder of the algorithm: mark off the number into pairs of digits, etc. However, the representation in Roman numerals will not key that process; either the number must first be translated into the decimal notation, or the square root be found by squaring various numbers to see what will give the required result. Or it can be a rather routine procedure to work out that some symbolic expression is a tautology by means of truth tables, without understanding the implications of calling a statement a tautology or fully analyzing the relationship of the symbolic expressions to human mental activity; one can simply write down T or F in accord with the simple rules. These examples help to show that in addition to providing abbreviation, proper symbolism can provide a great help in solution of problems (37:17f).

There are other ways besides routinization that symbols can aid the mathematical process. The symbol can serve as an appropriate image to provoke further insight. The use of Leibniz's symbol dy/dx in the calculus is useful in suggesting theorems like the chain rule which can

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be formally established, while Newton's symbol $\dot{y}$ can be used in fewer instances; the operator symbol D_x is useful in suggesting its own range of results and in simplifying other ranges of expressions; e.g., $R(D_x)$, e^{D_x}. On the other hand, a problem with Leibniz's symbol is that it is liable to suggest too much, to encourage sloppy mathematics, as in simplified or cookbook thinking about the chain rule for derivatives or the change of variable theorem for integration; it can be the source of unacknowledged insights. The lesson here is that symbolism can never supplant understanding or judgment. The symbol dy/dx should express a concept or an understanding of a process already achieved. Then it is used to bring to explicit expression the relevant question in a problem, for example, of fluid flow or the traditional vat-emptying problems.

As knowing begins in experience, mathematical questions begin somewhere, usually in a situation filled with superfluous data. The symbol helps to isolate the significant. For example, in the typical word problems of algebra, the first step to a solution is to express the problem in algebraic expressions, the usual symbolism available to us. This symbolic expression is thus the transformation of the original problem inasmuch as it expresses the problem in a way that admits solution with algebraic manipulation. If we do not understand the verbal problem before us, then the algebraic formula will be so many black marks on paper. If we do not know how to perform the algebraic operations necessary for the solution of the algebraic problem, the formula may be a correct expression of the problem but cannot lead us to an understanding of the problem. But if we do not understand the nature of literal or algebraic expressions, then we will not be able to arrive at the algebraic expression for the particular problem and,

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even if the expression is supplied, the manipulations will be so much hand-waving. Thus symbolism has great use within this flow of knowing, but the rest of the knowing process is vital; symbolism accomplishes nothing apart from its underlying meaning.

As shown in the example of the various symbols for the derivative, some symbols can be more apt than others, because the symbol supplies an image to provoke an insight; this image can be more or less successful in provoking that insight by being more or less suggestive of the meaning of the operations involved in that symbolism:

> Mathematical operations are not merely the logical expression of conceptual premises. Image and question, insight and concepts, all combine. The function of the symbolism is to supply the relevant image, and the symbolism is apt inasmuch as its immanent patterns as well as the dynamic patterns of its manipulation run parallel to the rules and operations that have been grasped by insight and formulated in concepts (37:17f).

The symbol is more or less appropriate inasmuch as the image--which it is--is expressive of the underlying understanding.

The advantages of apt or appropriate symbolism are many. First, the symbolism can itself provide a large part of the solution of a problem. The symbols, combined with the routine of the operations upon these symbols, specify what must be done. "Thus, a mathematician will work at a problem up to a point and then announce that the rest is mere routine" (37:18). For example, an integration problem might have a complicated form, but once the student can set it up in the form $\int f'(g)\,dg$ the work is almost done. From there, the answer is obvious.

Secondly, having the symbolism available gives the necessary help to begin the solution of a problem. The key step in a mathematical problem is often naming the unknowns: the problem-solver assigns them symbols. In light of this advantage of symbolism, a significant

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educational task is to enable the student to utilize the heuristic possibilities of symbolism, to help him realize that the traditional "word-problem" difficulty can be solved only through adequate symbolic expression; when the equations are isomporphic to the stated problem, the solution of the mathematical equations will be a solution to the presented problem.

Thirdly, the symbolism can offer hints or clues as to the right way to develop a problem or to formulate or prove a theorem. A good symbolism can provoke insight into the possibilities of the theory expressed in the symbolism.

> Just as the definition of the circle was approached from the clue of the equality of the spokes, so generally insights do not come to us in their full stature; we begin from little hints, from suspicions, from possibilities; we try them out; if they lead nowhere, we drop them; if they promise success, we push them for all they are worth. But this can be done only if we chance upon the hints, the clues, the possibilities; and the effect of the apt symbolism is to reduce, if not entirely eliminate, this element of chance. Here, of course, the classical example is analytic geometry (37:18).

A fourth advantage of an apt symbolism relates to the important notion of invariance: in mathematics or in knowing generally a statement is true in several situations, in spite of change. The word "man" is applicable no matter what position the exemplar assumes or what color his skin; a theorem in geometry holds independently of any of a group of transformations--the geometry is said to be invariant under this group of transformations. An adequate symbol system (or formalization) can express this invariance, because it can eliminate these irrelevant factors from the expression, admitting only factors relevant to the theory. What the symbolic expression does is truly to effect the implicit definition of mathematical structures; it eliminates any

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reference to particular examples of a kind of structure which would include special properties not necessary for the given structure. We can express symbolically the definition of a group, and not have to refer to the example of the integers under addition, because the countability of the integers is not a property of all groups and because operations other than addition can be the operation for a group. A symbolic expression is better, then, because it serves to "endow the pattern of a mathematical expression with the totality of its meaning" (37:18).

A fifth advantage is that an apt symbolism expressive of understanding at a present stage might provide the image in which insight may grasp the rules for the next stage of development (37:19). For example, the relation between the integers, appropriately expressed, can provide the kind of questions needed to move from groups to rings.

Inverse Insight An insight is the reaching of a sought-for understanding or intelligibility. Another kind of insight exists, an inverse insight: the realization that there is no intelligibility to be found where one was sought. Gödel's great discovery was an inverse insight; he hypothesized and then proved that the long-sought intelligibility of arithmetic as a complete system is not to be found; his insight denied this anticipated understanding of the completeness of arighmetic (25).

These inverse insights occupy the same place in a pattern of knowing that the positive insights do; they have a link in experience; they point to elements of experience, but say that these elements are not part of some expected pattern. These insights constitute, according to Lonergan, a "small but significant class" (37:19), and their scarceness is only one aspect of their difficulty.

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Another example from elementary mathematics is available: the square root of two is not to be understood in the rational number system. The Pythagoreans expected that every number was to be understood as a rational number, and the insight that the square root of two was not among them destroyed that expected result. Once the attempt is made to show that $\sqrt{2}$ is irrational, the rather elegant proof may proceed. The vital step, though, is the move from the position accepting $\sqrt{2}$ as rational and trying to find a proof that it is rational to the conjecture that it might not be a rational number after all and trying to prove it is not rational. The inverse insight here is understanding precisely where there is no intelligibility to be found, discovering that there is no answer to the problem of proving the rationality of $\sqrt{2}$.

> While direct insight grasps the point, or sees the solution, or comes to know the reason, inverse insight appreciates that in some fashion the point is that there is no point, or that the solution is to deny a solution, or that the reason is that the rationality of the real admits distinctions and qualifications. . . . the conceptual formulation of an inverse insight affirms empirical elements only to deny an expected intelligibility (37:19).

Space and Time One interesting feature of the human activity of questioning or knowing is that it spontaneously questions some differences in sensible presentations while neglecting others. In the acts of understanding, a person behaves in this way, but he may not notice that he is systematically ignoring certain elements of experience.

A chemist mixes samples of two compounds and notes the reaction. Some time later at another place he mixes other samples of the same compounds and finds a different reaction occurring. He will immediately perform a test on the purity of his samples. It will not occur to him to explain the different result by the mere difference in time or place

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of the two trials; he may seek an explanation in some factor different in the two settings, but not in the fact that it was a different time and place.

The totality of the elements of experience that are recognized as not pertaining to the understanding is called, by Lonergan, the empirical residue. The characteristics of the empirical residue (distinct from inverse insight) are that it

(1) consists in positive empirical data,
(2) is to be denied any immanent intelligibility of its own, and
(3) is connected with some compensating higher intelligibility of notable importance (37:25f).

These data are positive; the times of the experiments are noted and the locations exactly specified. But a vital activity of insight in the situation is precisely to neglect these data, to assert that there is an absence of system in them. To reach an explanation of a particular chemical reaction is to assert that the location of a mixing of the compounds has nothing to do with the result. To deny intelligibility to the location is not to deny the experience of being in different places, but to assert that it is not part of the explanation.

There is a compensating higher intelligibility to this: there is a law at work in cognitional process that similars are similarly understood (37:288). This could be rephrased as an anticipation that the universe is consistent. Lonergan here helps us to identify the source in the knowing process of that seemingly automatic anticipation.[4]

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[4]Besides particularity of time and place, there are other elements of empirical residue, such as randomness of events and the continuum. An explication of those items lies outside the range of this thesis. There is discussion of these elements in Lonergan (37:25-32) and McShane (47:28-31).

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The notions of time and space can be clarified through this idea of the empirical residue. Newton did not need to put into his explanation of gravitation a factor that would change the prediction for someone doing the experiment at a different time or in a different place, but with factors of mass and relative distances the same. It has been a contribution of modern physics to make this explicit, to formulate laws that are covariant under linear transformations of the four-dimensional space-time structure (Weyl 73:130f).

The particularity of time and place is part of the empirical residue. First, the differences of time and place are given as elements of experience: there is one place and there is a different one. It is a difficult thing to explain this difference:

> Because no one ever asks why one place is not another or why one time is not another, people are apt to be puzzled when the question is put, to imagine that something different from such obvious foolishness must be meant, and to experience a variety of fictitious difficulties before arriving at the simple conclusion that
> (1) particular places and particular times differ as a matter of fact, and
> (2) there is no immanent intelligibility to be grasped by direct insight into that fact (37:26).

Lonergan acknowledges that there might be some attempt to explain the difference between two times, two places. But we do not understand the difference of two things with respect to time and place--the difference is given in the experience: "the difference of particular places and the difference of particular times are given prior to any questioning and prior to any insight" (37:27). This difference is left behind in the search for intelligibility. The three different distances involved in three equi-distant points are really different on the level of experience and presentation, but that difference in no way enters the understanding.

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We can provide reference systems for time and space, but the function of these reference frames, creations of imagination and insight into the possibility of ordering, is simply to order the differences; this order does not explain the differences but presupposes them, as lining up the students in class according to height does not explain the students or either create or account for the differences in height. An intelligibility is present in the ordering of differences, but not in the differences themselves (37:27). This distinction might be exemplified by the different metrics possible on the same space; differences in measure are possible, given the original differences of points. It is possible to say that the points are different, but not why they are different; the given differences can be organized, though, through the various measures.

Abstraction The idea of abstraction often causes many difficulties, but Lonergan's approach, building out of the notion of empirical residue, seems to simplify the issue.

Abstraction is the intellectual process of selecting the important, the significant, from what is unimportant; it is insight into meaning and focusing on that:

> Properly, then, abstraction is not a matter of apprehending a sensible or imaginative Gestalt; it is not a matter of using other tools; finally, it is not even a matter of attending to one question at a time and, meanwhile, holding other questions in abeyance. Properly, to abstract is to grasp the essential and to disregard the incidental, to see what is significant and set aside the irrelevant, to recognize the important as important and the negligible as negligible (37:30).

And we can find the essential or significant or important in two ways, depending on the level at which we consider knowing. On the level of

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experience, the significant is in what is experienced: it is the set of aspects in the data necessary for the occurrence of the insights; on the level of understanding, the significant is in the knower: it consists of the set of related concepts necessary for expression of the insights (37:30f).

We could take two different approaches to abstraction. In a simple and probably the common view, abstraction is simplification and idealization, so that the abstract is an impoverished replica of the concrete; therefore, to call a particular animal a "dog" means to assert that this animal shares what is common to all instances of dogs, but to neglect or deny the long floppy ears, rich brown coat, or appealing manner of this particular dog.

In Lonergan's view, "So far from being a mere impoverishment of the data of sense, abstraction in all its essential moments is enriching" (37:88). In this interpretation, to call an animal a "dog" is to assert of it all those things we do know of dogs, asserting vertebrate structure, mammalian features, etc. Three moments or stages in abstraction can be distinguished, and their development will point out the nature of this enrichment.

The first moment of abstraction is a selection by human inquiry of an object for investigation (Mathews 49:2, 23), an anticipation that what is presented in sensible data can yield an intelligibility (37:88), and this anticipated intelligibility already enriches the data as potentially significant. A person may look, dully or admiringly, at the night sky, noticing among other things the different locations of the planets Mars and Venus on different nights. The first moment of abstraction is when this observer wonders about this motion of the planets, what

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causes it; it is the focusing on part of the field of data in hope of reaching some understanding. In an example from mathematics, this anticipation of intelligibility can happen when one notices that there is something interesting that happens when we add 0 to a number, thus isolating in an addition table the facts involving 0.

The second moment of abstraction is the insight, the arrival at some understanding. The exact occurrence of this insight is hard to pin down, because there are no necessary observable expressions of it. In the example of the motion of planets, if one imagines to oneself a boy swinging a rock on a string, this second moment might be the act of realization, "It's like that."

The third moment of abstraction is the expression of that understanding reached by the insight, "the formulation of the intelligibility that insight has revealed" (37:88f). Abstraction is thus partly negative; in the formulation, abstraction omits from the understanding the insignificant, the negligible, the incidental. But abstraction is still enriching because in the second moment it adds something, the understanding--the structures the knower makes to find a solution to the apprehended unknown.

A point to note here is the relation of the abstract to the concrete. The abstract understanding of the behavior of orbiting masses--or the definition of a circle, within a system of geometry--is still related to the concrete object of inquiry--the observed night sky or the drawn circle--through the questioning and the insight and the formulation. The reference is always back to the source to reach an understanding. That is why the study of abstract systems--orbiting bodies or analytic geometry--is of great significance in dealing with concrete objects of

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experience. The abstract is not something opposed to the concrete; it is the concrete as understood. It is this continuing relation of the abstract to the concrete that distinguishes Lonergan's cognitional theory (44:74f).

This whole account of abstraction is a psychological one; abstraction is the elimination by the understanding of the intellectually irrelevant precisely because it is understood to be irrelevant (41:39).

Higher Viewpoints In the development of mathematics from arithmetic to algebra, a particular process needs to be picked out. Based on the addition and multiplication tables for integers, along with rules for negative numbers, insights into the general properties of addition and multiplication of integers become possible. These insights can be of general properties of the total algebraic structure, such as the distributivity law, or more specific ones such as $(x-y)(x+y) = x^2 - y^2$. This move from arithmetic to algebra is an example of what Lonergan calls an advancement to a higher viewpoint.

At a later stage in mathematics, a student might be working with the operations on matrices and recognize special properties of this system, such as $A(B+C) = AB+AC$, etc. This insight, for a student with a background in the algebra of real numbers, could spur the further insight, "There is something similar in these structures," and abstract algebra, for that investigator, would be born; this is another movement to higher viewpoint.

One further example comes from geometry. Someone may be familiar with Euclidean geometry and realize what happens when the parallel postulate is replaced by its various alternatives. He may know some affine

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and projective geometry. With these achievements before him he might ask, "What am I doing? What is geometry?" It would be a great achievement then to reach the understanding in terms of transformation theory: A geometry is the study of those properties of a set S which remain invariant when the elements of S are subjected to the transformations of some transformation group. This new understanding can be described as reaching a higher viewpoint.

These three examples provide a basis for the analysis of higher viewpoints. Basically a higher viewpoint is an insight into the intelligibility in expressions of previous understandings (37:13-19). Moving to a higher viewpoint is the knowing process circling back on itself. In mathematics, this move happens when, after inquiry has moved from images to insights to definitions and postulates to symbolic operations, the work with these symbolic operations provides another and more general image, which serves as the source of further insight.

The emergence of the higher viewpoint brings into systematic understanding elements that from the lower context, the lower understanding, were coincidental. In our first example, the arithmetic tables listed a series of facts: $2\times3 = 6$, $2\times4 = 8$, $2\times7 = 14$, $3+4 = 7$, $6+8 = 14$, and many like that. It is of no matter to the tables and the achievement expressed in them that $6+8 = 2(3+4) = 2\times7$. It took the insight into this fact and the regularity of facts like that and the formulation $a(b+c) = ab+ac$ to organize these data systematically.

This emergence of higher viewpoints is crucial in the development of mathematics.

Intellectual Pattern of Experience Man has many abilities and activities and he can put them to the service of various goals. Although in

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everyday knowing, for instance, the operations are spontaneous and not methodical, still there is a unifying factor, known variously as conation, interest, attention, purpose, concern (Tracy 72:116). It would be wrong to think of man's activities as a series of atomic conscious moments; there is a "stream of consciousness"--rather, there are several streams. These streams Lonergan calls patterns of experience, which link together the various elements of the knower's activity; Tracy, developing Lonergan's theory, describes a pattern of experience as "a set of intelligible relations that link together sequences of sensations, memories, images, conations, emotions, and bodily movements" (72:116). The pattern can be a biological one; the activities then center on eating, reproduction, or self-preservation. Or the pattern can be esthetic; the person finds joy in experience for the sake of the experience itself. Or the pattern can be an intellectual one; man's abilities are put to the service of his wonder, his desire to know. These patterns are not exclusive; a man can operate within the intellectual pattern in developing his mathematics and then pause to eat a meal or enjoy a symphony.

In the intellectual pattern of experience, the flow of sensations, images, emotions, and bodily movements is freed from the strict demands of bodily drive and gives to experience a flexibility that makes it available to serve man's spirit of inquiry. This subjection is evident in the control of the scholar:

> To the liveliness of youth, study is hard. But in the seasoned mathematician, sensitive process easily contracts to an unruffled sequence of symbolic notations and schematic images. In the trained observer, outer sense forgets its primitive biological functions to take on a selective alertness that keeps pace with the refinements of elaborate and subtle classifications (37:186).

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In this pattern, since wonder is the central concern in consciousness, experience takes on a pattern demanded by that concern. Someone concentrating on a mathematics problem is aware of all that has to do with the problem, and can more successfully ignore anything else clamoring for his attention.

Even the subconscious or the pre-conscious is sensitive to clues that will give the needed insight. Archimedes solved his famous problem and discovered the principles of specific gravity and displacement, not when he was trying to think the problem out, but only later when he was doing something else, attempting to rest from the problem (38:No. 4).

The other capabilities of man supplement his wonder. When one is making a judgment, for instance, often a passionless calm arises in reflection, the imagination runs through all the possibilities that might contradict the understanding being evaluated, the memory shuffles through all possible contradictory facts. While everything in the man is cooperating spontaneously to serve that intellectual wonder, all other concerns fade away. This minimizing of all other concerns is what is called impartiality or intellectual detachment or intellectual interest (38:No. 4). The intellectual pattern can be so strong, the wonder so control the normal spontaneity of sensation, that the memories and imaginings rise above the threshold of consciousness only if they have some possible relevance to the problem to be solved or the judgment to be made.

The frequency, intensity, duration, and purity of the intellectual pattern of experience and the subjection of all the skills and activities of the person to the service of the spirit of inquiry depend on many things, including native aptitude, training, age, development, external

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circumstance, and chance that confronts the person with problems and that supplies at least the occasional opportunity to work towards their solutions. To be intellectually talented is to move easily into this intellectual pattern, but not necessarily to achieve easy results; but when the pattern is entered,

> Insights come readily. Exact formulation follows promptly. Outer sense pounces upon significant detail. Memory tosses out immediately the contrary instance. Imagination devises at once the contrary possibility. Still, even with talent, knowledge makes a slow, if not a bloody, entrance. To learn thoroughly is a vast undertaking that calls for relentless perseverance (37:186).

Meaning and Expression

Meaning

Meaning is the content of the acts of knowing; these acts are the only proper occurrences of meaning. Words or other expressions have no meaning of themselves, but they do have meaning secondarily because they express insights or judgments. In themselves, words are only a sequence of sounds or a series of markings on paper--things to be experienced; but when they are experienced intelligently they become meaningful (Flanagan 20:67). Thus meaning has to do with what happens in the minds of both the speaker and the hearer of the words. This is the basic approach to meaning in Lonergan's theory.

An important conclusion results: what is meant must be distinguished from the expression of that meaning and from the mental acts that develop it or affirm it as true. Meaning occurs only in the cognitional act; and every cognitional act, from sensing through judging and deciding, has meaning because every act has content: one does not merely taste but he tastes something, and one does not merely judge but

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he judges the truth or falsity of some affirmation.

The vital moment in this process or production of meaning seems to be the reaching of an understanding. What we know appears primarily as contained in the act of understanding; judgment transcends this level by affirming that understanding and so transforms the content from an idea to an assertion about reality, while experience can have meaning primarily in terms of the potential intelligibility there--the potential understanding to be reached.

Expression of Meaning

The expression of meaning is one type of act that carries meaning, but the expression is not identified with the meaning. Corresponding to the three levels of knowing, we can find three components of expression.

First, a number of repetitions of the same expression are possible; this instrumental multiplicity corresponds with the basic, repeatable level of knowing, the level of experience and imagination, where there can be a material multiplicity in the occurrence of data. Second, the combination of words that comprises the expression is (or can be) significant, intelligible, revelatory of meaning; this component of expression as bearer of meaning corresponds to insight and understanding. And, thirdly, statements affirm or deny, are positive or negative utterances; this component of expression as indicating facts or reality corresponds to the level of judgment in the knowing process (37:553).

This one-to-one correspondence between the elements of knowledge and expression, which Lonergan calls an isomorphism (37:553), is still not an identity. The expression can be there without the corresponding level of knowing. Thus an assertion can be a lie; or a generalization

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in words can mask a confusion in thought, or a student can parrot a definition. On the other hand, the expression may not be adequate to the knowledge. Thus someone who understands may fail "to hit upon the happy and effective combination of phrases and sentences" (37:553) to express that understanding; or a sentence can suitably express the speaker's understanding, yet the hearer can fail to have the same insight; or a sentence may use the word "is" and seem to express a judgment, yet the speaker may pose it only as an hypothesis, as a possibly adequate understanding.

Still knowledge and expression interpenetrate. Coming to know is a process; it occurs not all in one grand insight, but through small insights that lead to other questions that lead to further insights and then to larger insights that include the previous ones and that lead to further questions. At each stage of this process, it is helpful to formulate in some way the understanding reached and what still remains to be found. So expression becomes an important part of the process of learning; knowledge and expression or language develop together: "the attainment of knowledge tends to coincide with the attainment of the ability to express it" (37:554).

Words and language have many functions. They are sensible; they have a role in linking men together in society; they have power to evoke feelings and emotions. Beyond all this, words bear meaning. Words fit together to give a language; they are combined in typical patterns. To learn a language one must first grasp these patterns and, secondly, let the insights which grasp the patterns be routinized so that one's attention can focus on higher matters.

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> Just as the concert pianist is not thinking of the place of middle C, so the speaker or writer is not thinking of the meaning of his words. . . . But these sensitive routines, these typical patterns, are able to carry the meaning of words only because initially there occurred the insights that linked words intelligibly not only with one another but also with [what is meant and the acts that contain this content] (37:554).

Here is an echo of what was said earlier about symbolization taking over much of the work, routinizing it.

Because expression is isomorphic to knowledge, because words can occur in sentences that express an affirmation, words do not have to be related only to other words. They have meaning. They have reference also to being.

But expression is not identified with knowledge. Because of the interpenetration of knowledge and expression the usual acceptance arises that knowledge and expression are inseparable. But efforts to explain what we mean often end with the statement that our meaning is obvious and does not need any explanation. There are communication gaps, and often we say that we understand but cannot express it. It is only coincidence that people converse who share much of the same common sense or that writing is addressed to people who share much of the same understanding as the writer. When this background is not shared, the difference between knowledge and expression becomes obvious, even painful (37:555f).

The aim of expression is to share knowledge. If the intended reader lacks experience or insights preliminary to the understanding to be communicated, the writer must assess how much necessary background the reader has and what he lacks, and have a practical set of insights as to what manner and order his communication must take in order to enable the

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reader to acquire that necessary background and then the central understanding. This last set of insights is certainly far from the origin of this process, the understanding to be expressed.

Because expression is not identified with knowledge, expressions are not true or false, strictly speaking. Expressions are merely adequate or inadequate instruments of knowledge, related to the truth of the meaning that is expressed (37:556). Truth pertains to the judgment, and that will be the next topic to be discussed. The adequacy of the expression is related to the ability of the particular expression to evoke understanding in the hearer, to induce the relevant insights (37:557). Thus language or expression, as an instrument of meaning, provides a source for the reader or hearer to begin his own process of understanding. It becomes the given, presentation, data; thus expression is not a way to transfer a particular understanding from one mind to another, but a way for one person to enable another to reach the same understanding.

Hence, what an author or speaker does is to provide a multiplicity of experiences in his spoken or written words so that his readers will have a source of experience to provoke their own wonder. It is not as raw a source as nature is, because it presumes a background of understanding in the reader, and it has been chosen and ordered carefully to lead to an understanding. If the author has chosen his phrases intelligently and luckily and the readers are alert, they may have an insight and reach the understanding to be communicated, or one close to it. They can determine the adequacy of this understanding by determining whether the formulation they have been given is an adequate formulation of their own understanding, and then they may come to some formulation

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of the understanding for themselves; "under the power of this understanding they can consciously, carefully and deliberately conceive for themselves your meaning which in turn can be expressed in similar or quite different phrases and paragraphs" (Flanagan 20:66).

While a logician might prefer to think of a one-to-one correspondence between words and concepts, no such neat match between words or paragraphs and insights is at hand. Different ways of expression are often required to provoke the same understanding in different people. Particular phrases may provoke some insight, but insights may build up so that paragraphs are required to express them. Significant, broad insights may require numerous smaller ones as prerequisites, spaced out over an extensive discussion so that readers might be led step by step into a series of insights that will finally end at the central one.

The reader, when he has finally come to understand, might think that the point could have been made in a simpler or more direct fashion, or with fewer or more illustrations. But this fact shows that he has reached the meaning of the passage; he has understood, and he can forget the particular way the writer has phrased his argument; he is then free to express that understanding to himself or to others in whatever way he finds more adequate. Expression is a communication of meaning, not of words; its goal is to provoke in the reader the same knowing experience the author has had.

Truth

It would be presumptuous to assert that a few paragraphs will resolve the difficulties that men have experienced in defining truth for over 3000 years. But it is necessary to point out the direction

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Lonergan goes in meeting this problem; it sounds much like some familiar definitions, because in fact Lonergan had a familiar definition in mind as an ultimate goal; but now this old definition may take on new meaning through its basis in cognitional theory.

The definition of truth enters implicitly when we consider being as our goal in knowing.

> For being was identified with what is to be known through intelligent grasp and reasonable affirmation; but the only reasonable affirmation is the true affirmation; and so being is what is known truly. Inversely, then, knowing is true by its relation to being, and truth is a relation of knowing to being (37:552).

What relation? It is possible, within the structure discussed here, to define implicitly such terms as object, subject, objectivity. Using these terms, we could then reach "the traditional definition of truth as the conformity or correspondence of the subject's affirmations and negations to what is and is not" (37:552).

If this definition seems a roundabout approach to something that should be simple, it is good to remember that we cannot climb outside our head and take a look at reality and at what we know and somehow compare these looks, because our knowing is not merely seeing. But the discussion of truth must end, for the whole attempt to reach a definition and the underlying understanding of the notion of truth requires much more space and subtlety than are here available.

The Appropriation of Truth: Learning and Teaching

This topic was mentioned above in the discussion of expression of meaning, under the activity of the hearer or reader. To appropriate a truth is to make it one's own. The task of the teacher is to bring this

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about. According to Lonergan, this communication of truth is achieved through the teacher's resolution of three problems in the individual case (37:558-62).

The first is the problem of learning, of coming to know, of acquiring the initial totality of insights that combine to form a system of knowledge. The teacher must know in order to teach.

The second is the problem of identifying precisely those elements which are unified and grasped by insight, to acknowledge what experience or image served to provoke the insight. When a person can make such an identification, the effect is that he really possesses the insight as his own. He can apply the resultant understanding with confidence and realize the extent of its relevance.

The third problem in the appropriation of truth Lonergan calls orientation, which means the realization, once a person reaches knowledge, of how he reached it--through the process of the threefold structure of knowing. Because the truth seems obvious, once the learner has reached it, he is tempted to simplify the way it was reached, to make the knowing process one of recognition or of seeing. This over-simplification can occur, for instance, when the scientist who has reached an understanding draws a picture to illustrate that understanding, and thinks he has fully expressed it. He has thus distorted what knowing is.

These problems in the appropriation of the truth are parallel to the knowing structure. The problem of learning, of reaching the structure of knowledge, is met through understanding and formulation. The problem of identifying is met through experience, the personal and intellectual experience of performance of the understanding. The problem of orientation is met on the level of judgment when the knower can say, "This is what it is to know,"

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> when at last we grasp that every issue closes when we can say definitively, It is so, or, It is not so, that the objective of knowing is being, that while being is a protean notion still its content is determined by intelligent grasp and reasonable affirmation and, after affirmation, by nothing else (37:559f).

At least a modest achievement of answers to these problems is necessary for the teacher. First of all, he must understand what he is to teach. And more, the teacher must have met the problem of identification, to know the experiences that apply to the insight he wishes to elicit in his students;

> The understanding that enables one to teach adds identification to insight. By that addition one is able to select and arrange and indicate to others the combination of sensible elements that will give rise to the same insight in them. One is able to vary the elements at the demand of circumstances. One is able to put the questions that elicit from the pupil indications of his blind-spots and, then, to proceed afresh to the task of bringing him to the prior insights he must reach before he can master the present lesson (37:559).

And finally the teacher must have met the problem of orientation; he must know what it is to learn, to reach knowledge. Thus he cannot take shortcuts; he cannot avoid leading his pupils through the sometimes painful process of learning. He can make it easier through his help, but he cannot convert it into something else, hoping his pupils will know by being able to parrot answers or repeat processes without understanding. He cannot merely draw a picture and hope to express the concept, thinking that if the student can duplicate the picture or repeat the operation he must have mastered the concept. For example, he cannot merely advance a heuristic presentation of limit and hope that the students will come up with the exactness and precision required; moreover he cannot merely state the bald definition of limit, and think the students have an idea of the meaning of limit if they can repeat the definition or even use the intricate epsilon-delta terminology in a few simple examples.

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Types of Knowing

The basic pattern of knowing is realized in different ways depending on the circumstances and the goal of inquiry. An analysis of several types of knowing will show how the structure is variously realized.

Concrete Judgment of Fact

Many of our attempts to know or our verifications hinge on "facts." But a fact is not merely raw datum; it is concrete, indeed, but it is intelligible: There is an insight required to establish it as a fact, to give it its precision and its accuracy (37:331). It is subject to judgment.

To explain the nature of fact, Lonergan uses the following example (37:281). After work one day a man returns to his normally well-ordered home to find the door and windows smashed in, smoke in the air, and water covering the floor; whereupon he makes the conservative judgment of fact: Something happened.

Once more, we investigate not whether the man was correct, but how he came to make this affirmation. The elements of this judgment are two sets of data referred to the same thing--the before and after views of his house--and the insight that they differ. What the man knows here is change, by grasping the same identity (the house) at different times in different data. This process can be set up as an extended syllogism:

I. If the same thing exhibits different data at different times, it has changed.

II. If there occurs a change, something has happened.

III. But the same thing (house) exhibits different data at different times.

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IV. Therefore something has happened.

The man walking up to his house does not make these statements and he affirms only the last one as a judgment. But the statements I through III can be present as unanalyzed structures or procedures or acceptances within cognitional process. This kind of unanalyzed structure is what is operative in concrete judgment of fact, linking the conditioned with the fulfilling conditions (37:282).

The situation above exhibits the three levels of the cognitional process. First, on the level of experience or presentations are the two sets of data. On the level of intelligence or understanding is the insight that refers both sets of data to the same thing. These two levels taken together result in the notion of knowing change. Reflective understanding takes the experience, the understanding, and the notion of change together to exhibit the virtually unconditioned as outlined above and to ground the judgment that "Something happened."

In this judgment, knowing reaches fact, which is a statement about reality. The judgment of fact is, by the definition of judgment, virtually unconditioned: it might not have been. Still, it is; based on data, it is known. It has a sort of conditional necessity, because the data bring into being the cognitional process that requires the judgment. In the face of the disrupted house of our example, the judgment "Something happened" is necessary.

Because fact is reached through cognitional process, it has some special characteristics that should be noted to distinguish it from raw data. Since the knowing process begins with experience in a particular instance, fact shares the concreteness of experience; since fact is reached through insight, that is, since it is an understanding, fact

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achieves the coherence of intelligence; since it is affirmed in judgment, fact shares the finality that can be reached through cognitional process. As the product of all these acts, fact is the unity anticipated by original inquiry and developed by sensation, perception, imagination, insight, formulation, reflection, and judgment (37:331).

Perhaps another example would help to free the meaning of the concrete judgment of fact from too close a tie with the notion of change. One type of historical knowing seems to be of the type of the judgment of fact. In trying to establish a fact, for example that the battle of Washington Heights occurred in a particular way on such a day at such a place, a historian has various sources for his investigations. He has documents and artifacts. He reads the contemporary reports on both sides. He examines the kinds of weapons used and walks over the scene as it remains to get some idea of the terrain. And finally after he does all his research and he begins to understand the ebb and flow of the battle, he approaches the judgment; it happened this way. All the data he has examined, his partial insights, and judgment have come together to yield a coherent understanding, and the same data provide justification for the judgment. He speaks of the one concrete occurrence of the battle when he affirms: It was this way.

Common Sense

In much of this discussion so far, the knowing discussed has been knowing in science, the technical, empirical sort. It seems to be a form of knowing much stricter and more formal than the knowing by which we operate on a practical level, day by day, in managing our daily life. Lonergan refers to this latter as common-sense knowing, and he shows that it

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can be analyzed according to the same cognitional structure (37:173ff). The discipline of common sense is best displayed, for example, by practitioners (craftsmen, mechanics, family doctors, businessmen, diplomats) who bring to their tasks an assortment of lore or proverbs that help solve problems in a particular situation. Some of their maxims would seem contradictory if held up against one another, but the man of common sense also has the insight to apply the correct proverb in the concrete situation.

Common sense then is an accumulation of maxims, guidelines, or standard processes, that come together to amount to an intellectual habit; as such, it can resemble a science (38:No. 3). In a science like physics after one has accumulated acts of understanding by understanding numerous things, gradually the learner builds up a habit of learning so that he understands much more easily, has developed ready insight. Common sense builds up in the same fashion, except the accumulation of understanding in common sense results in the individual's grasping how to behave, what to say, what to do, and how to do it in particular concrete situations. The person who lacks this common sense does not know what to do in real situations; he does not know how to cope with the real world in which he lives.

Common sense, as an accumulation and collaboration, is something shared. It is not a list of general truths on which all men agree, but it is a structure of these precepts, etc., to which everyone contributes by adding his experience to the common store, and from which each individual draws, making his own adjustments, to fill out the set of particular standards he will use to guide his actions (37:289ff).

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Lonergan suggests that common sense might be comparable to one of those "universal tools" available in hardware stores, a handle into which the user can fit a hammer or a screwdriver or an awl--one tool that can be adapted to many purposes (38:No. 3). Common sense is like that; armed with the various possibilities and with a clear view of the immediate situation, the man of common sense will decide precisely what the situation is and what is to be done. And so common sense is this collection, this general nucleus of maxims or guidelines, that is relevant to most of the situations in which a man finds himself.

Common sense requires intelligence in using that fund of general approaches. Some people are filled with proverbs but somehow cannot see their appropriateness. On the other hand, in practical affairs, the man who does have common sense is more ready to catch on, to see the point, to acquire know-how. What is crucial here is insight, and so even the smallest instances of this insight are marked by the same characteristics as Archimedes' famous discovery. The necessity of insight for understanding is emphasized when we contrast its occurrence with the frustrating occasions when someone cannot learn what to do even what seems most obvious (37:173).

Common sense, then, is whatever it is that bases the large number of "elementary judgments which everyone makes, everyone relies on, and almost everyone regards as obvious and indisputable" (37:289). It is "a core of habitual understanding that is to be adjusted by further learning in each new situation that arises" (37:297).

There is a difference between common sense and common-sense judgment. Common sense <u>is</u> common; operation within that domain is basically a collective collaboration in learning, the work of the group sharing

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the circumstances of living in some time and place. On the other hand, common-sense judgment is not something that is shared, not something done by some public authority named common sense. This judgment is made by an individual using common sense in his own particular situation, deciding that a maxim applies (37:292f). Even more, the correctness of this judgment is an individual matter because it is based on all that is available to the one learning and acting. The individual uses the common element in common sense, and applies it to the particular case through common-sense judgment.

This element of judgment is seldom explicit. But its presence is often indicated by the pause between remembering a maxim and accepting it as pertinent to a particular situation.

Science vs. Common Sense

A comparison and contrast of common sense and scientific knowing will help to clarify each. First of all, common sense does not aspire to be a science; it does not wish to possess knowledge which is valid universally, nor to express this knowledge absolutely. Its concern is with the immediate, concrete, particular situation; it aims to master each problem as it arises. Since common sense has to do exclusively with the concrete, it cannot explicitly formulate its own nature, domain, logic, and methodology. It must come to appreciate what these are in order to remain within its proper limits; however it must arrive at them in a way which is in accord with its own ways of operating, through instances, examples, and proverbs. This self-definition of common sense will enter the judgments common sense makes, not as premises for deductions but as possible ways of operating (37:297). Science, on the

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other hand, requires the search for an explicit methodology as it moves farther and farther from the bare data and more and more to an understanding of things as related to one another.

Again, unlike science, common sense does not need a technical language nor a formal mode of speech, because this incomplete set of insights cannot be brought into the form of definitions and postulates; it is an understanding neither of a concrete situation nor of any general truth. Thus common sense has its own universe of discourse, separate from that of science. Science attempts to move into explanation, requiring a shift from descriptive terminology, so that all the statements of empirical science contain at least implicitly the reservation "from the viewpoint of explanation." But common sense remains concrete, linked to descriptive language, so that all the statements of common sense contain the implicit reservation "from the viewpoint of ordinary description" (37:395).

Disputes often arise about the superiority of common sense over the technicalities of empirical science. Those familiar with the rigors of a more formal mode of thinking are likely to look with scorn on the man of common sense, as the man of common sense is likely to look with scorn on the ivory-towered intellectual. The reason is that people on each side do not recognize the limits of the proper realms of their own methods. The scientist asks: "How can the world's work be done either intelligently or efficiently, if it is done by men of common sense who never bother their heads a minute about scientific method?" But here the scientist forgets why scientific method was developed. Scientists need scientific method, all the paraphernalia of laboratories, precision of observation, measurements, specialized libraries, logic,

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abstruse technical terms, and abstract reasoning because scientific investigation moves from the familiar to the unfamiliar, from the concrete to the abstract, from an understanding of things as related to us to an understanding of things as related to one another.

> When they reach the universal relations of things to one another, they are straining beyond the native range of insight into sensible presentations and they need the crutches of method to fix their gaze on things as neither sensibly given nor concrete nor particular (37:179).

Since common sense remains within the familiar world of things for us, it is concerned not with how everybody is to or does behave, but how "I" am to behave. Scientific method is superfluous in this world.

Both science and common sense reach their conclusions through the self-correcting process of learning, but these conclusions differ greatly because, within the same knowing process, radically different criteria are used. The primary difference is in what constitutes a further relevant question. "It is this fundamental difference in the criterion of the relevance of further questions that marks the great divide between a scientific attitude and a common-sense attitude" (37:295). The scientist seeks ultimate explanation so he must continually ask "Why?" until he reaches fundamental theory. But the man of common sense is only interested in things as related to him, as they enter the domain of human concerns; therfore, he stops asking "Why?" as soon as further inquiry would cease to make appreciable difference in his daily life. Herein is the source of his skepticism about the goal and use of science; he imposes his own criteria upon science with his further question "What is the good of it?" This practical question for science would eliminate all further progress, for the strict aim of science is to understand, to know its domain, apart from any particular use.

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But the scientist imposes his own standards on common sense when he considers the practical attitude a lack of interest in truth, which he alone seeks. It is a lack of interest in the truth the scientist seeks, but truth can be found in several domains.

These two ways of knowing, then, are complementary (37:297f). Science needs common sense for several reasons. Scientists need common sense to apply their methods to the concrete tasks of particular investigations; they need it again to reach the results of applied science.

The application of scientific results must recognize the abstract nature of the scientific enterprise, the fact that what might be incidental to the theory might be crucial in the application. Therefore the technologist who puts the science to use retains the skepticism of common sense toward science. He knows that the scientist has not taken everything into consideration, has not explained everything, so the technologist is conscious of the necessity of taking safety measures to provide for possibilities of error. For example, although the mathematical procedure of linear programming is perfectly straightforward, its use in a particular application involves an estimate of the significant factors, a presumption that linear relations are involved, etc. (Tintner 71:27-31). The decision-maker must take all this approximation into consideration when he evaluates the optimum solution indicated by the linear programming procedure. He must make a common-sense judgment that this is probably the right thing to do, and weigh that probability: Would you bet your life or your profit on it? If not, how much insurance should you take out? The practical man must use common sense in his judgment, and scientific results are only part of what he must consider.

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Furthermore, while science pursues its investigations at the pace that data yield insights, the work of the world must go on. In a similar context, discussing the difference between craft and technology, Polanyi cites the difficulty of analyzing cotton-spinning, for instance, in order to mechanize it (54:52). The craftsman spinning cotton is a man of common sense, applying this means and that to the situation as he reads it, without really being able to analyze this method; in order to mechanize this process, all the actions and judgments must be formulated. The difficulty of formulation is why some things must continue to get done until science can catch up.

On the other hand, common sense needs science, for common sense does not answer all the questions and science can provide a basis for further additions to common sense or refute current maxims or eliminate prejudices or blindness in current opinions.

The picture of the intellectual component in common sense begins to emerge. Common sense begins with spontaneous questions, accumulations of insights, and collaboration in testing and improving them, leading to development of a habitual but incomplete set of insights that is completed with appropriate variations in each concrete set of circumstances that call for speech or action. Common sense specializes, not by theoretical differences, such as physics and chemistry, but by concrete differences, the empirical differences of place and time, circumstance and environment, so that each geographical area at a particular time can have its own set of proverbial wisdom, its own way of doing things.

Belief

The knowledge required of a civilized man is great indeed. While he can find out much for himself through inner and outer experience,

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insight, and judgment, this immanently generated knowledge is a small fraction of his total store. He relies on the reports of the experience of other men of other places and times, which he can understand, not so much by his own originality, but mostly through repeating the acts of understanding first made and mapped out by others. Even more, the development of his understanding is due to presuppositions he has taken for granted, probably many more than he himself has ever reached. He accepts these propositions because they are commonly accepted and because he has neither the time, inclination, nor perhaps ability to determine them for himself. His judgments of truth and of value, since they depend on experience and understanding, are probably seldom generated entirely by his own efforts. A man's knowledge is not separated into neat compartments of what he knows for himself and what he believes, but the former stands "in symbiotic fusion with a far larger context of beliefs" (42:42).

So the knowledge of a man owes much to his sharing in the common human collaboration in knowing; he contributes his share by generating his own knowledge and expressing it to others, while he most often receives from it through belief.

What is meant by belief here is not something that must go without proof because proof is lacking. Something of that type would be the consistency of arithmetic, which most men would accept but which cannot be formally demonstrated, or the existence of infinitely many prime pairs. What is meant instead is something that someone else has come to know through the normal process of knowing. Logarithmic tables are an example; every engineer could develop his own logarithm for a particular number or his own tables, at much effort, leaving little time for his central

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concern; but he takes the word of the publisher that the particular tables are true, that the value given for $\log_{10} 5$ really is that number accurate to the given number of decimal places. This sort of belief permeates science and really makes it possible. One scientist is not required to perform every experiment in physics, but he can believe that his colleagues report experiments accurately; he can believe their theories, too, knowing that errors in their theories as well as his own will be found out in the continuing process of science.

Lonergan makes an extensive analysis of belief, though much of it lies outside the scope of this thesis. In belief, as in the process for reaching immanently generated knowledge, insight and judgment are both present: insight yields the understanding of what another is expressing; what is vital for belief is the judgment.

This judgment could be phrased: "Since it is good and necessary to share in this collaboration of knowledge through belief, since the man telling me this is a good scientist and always tells things true, and since what he is saying now is clear, then I will accept what he is telling me." This reliance on belief could arise in a mathematical development that would benefit from the use of the assertion that the well-ordering theorem was equivalent to the axiom of choice, but where the machinery to prove it was lacking. Students, basing their judgment on the teacher's reputation for correctness and honesty and his clear expression of what the assertion involved, would easily make the judgment to accept this equivalence.

The Notion of Being

Chapter II so far has been in answer to the question "What do we

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do when we know?" The result is a cognitional theory. The questions that come next in the development of Lonergan's philosophy are "Why is doing that knowing?" and "What do we know when we do that?" Answers to these questions in turn give an epistemology and a metaphysics, which lie outside the range of this thesis. Still it is important to note the direction the answer to that last question will take in order to have a better grasp of the entire process. What do we know? What do we try to know? What is real? In traditional language, What is being?

Lonergan answers: Being is the "objective of the pure desire to know" (37:348). This is an operational definition of being; it does not say what being is in itself or what the word "being" means, but it states how that meaning is to be determined. It says that when someone knows, what he knows is being, and that if he wishes to know, he wishes to know being. It does not specify the content of his knowledge or whether or not he will ever know (37:350).

Lonergan's position seems to agree most readily with what most men assume, whether they are scientists or men of common sense or even philosophers, as they set out on their inquiries, that when we set out to know by asking questions, we will finish by knowing the real. The real, being, is the totality of the goal of questioning; it is what is to-be-known in true judgment. In terms of the scientific enterprise it is what is to-be-known in Theory Verified in Instances. Since it is not what is already known, the real is not dependent on man's knowledge, nor is it a function of what is known either through science or through common sense. The real is not a construct nor a Kantian ideal. The real

> is there to-be-known and we move asymptotically towards its knowledge through the true judgments of science and common sense. Reality is not a function of what is known; rather, knowing is a function of the real, where the word 'function'

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> has a dual sense. It has the mathematical sense of isomorphism. But it also has the biological sense . . . Knowing is a process which takes place in the human organism. It is, one might say, a higher process of digestion with questions for appetite. It is a process which has evident survival value--witness Western science, the arms race, or merely man against beast. Had that process emerged as Plato or Plotinus described it, we would long since have become extinct. It is its adaptation to the environment, material reality, which ensures its organic possessor survival, and that adaptation is the one we described as an isomorphism (McShane 47:145).

This is an appropriate time to introduce a further distinction Lonergan makes, that of spheres of being (42:75ff). We can affirm various things, and we can affirm that various things exist. Using the same word "exists" we say that the moon exists and that the logarithm of $\sqrt{-1}$ exists, but we do not mean that the moon is a nice theoretical construct to make our astronomy fit in with desirable theoretical postulates, nor that $\log\sqrt{-1}$ is something to be observed floating somewhere in the physical universe. It is possible to speak of spheres of being, says Lonergan; the sphere of real being must be distinguished from other, restricted spheres such as mathematical, theological, or hypothetical. These spheres, widely differing, are not completely disparate, because the contents of each sphere are rationally affirmed. Once more, this rational affirmation is the act of reflective understanding that grasps the virtually unconditioned. The great distinction in the spheres of being comes from the great difference in the conditions to be fulfilled. Whereas the conditions to be fulfilled to affirm something as real being are appropriate data from the senses or from consciousness consistent with the understanding, the fulfilling condition for proposing something as a hypothesis isits possible relevance to correct understanding of data, while the fulfilling conditions for a correct mathematical statement need not even include an explicit relevance to data (42:75f).

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Mathematical Knowledge

Lonergan has what amounts to a slogan for the direction of his work:

> Thoroughly understand what it is to understand, and not only will you understand the broad lines of all there is to be understood, but you will possess a fixed base, an invariant pattern, opening up all further developments of understanding (37:xxviii).

Since we have just examined what it is to know, what men do as they come to know something, we will now examine what men do as they come to know mathematics. We will be able to discover from this analysis the broad lines of the structure of mathematics and the pattern of its development.

Example: Extension of the Number System

The child begins by counting; the mathematician begins with a successor-generating operator on sets. From there both develop rich number systems. What follows here is a schematized version of the extension of number systems by the developing child.[5]

From counting we can develop addition and produce a whole array of addition tables. We can proceed two directions from here. First, we can notice various regularities, the commutativity or the associativity of addition. Secondly, we can develop the new operation of subtraction; corresponding to each entry $a + b = c$ of the addition tables we can find the new operation $c - b = a$. But as we try to systematize this new operation we find that we do not have a referent in the addition tables for many subtraction problems we can formulate: $2 - 2 = ?$, $1 - 2 = ?$. The key step in generating a new number system

[5]This example was suggested by Mathews (47).

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is noticing the lack of answers for these and similar problems and asking: What would the answer be, if there must be an answer? Can we imagine some kind of an answer?

It can take a long time to develop a definition of zero and of the negative numbers from these sorts of questions; yet the movement toward definition begins in that kind of question. Then the problem becomes one of incorporating those new numbers in the number system. We have to find answers for problems like $(-2) + 5 = ?$ and $7 - (-4) = ?$.

Development does not stop here; we have just begun. The addition tables can be consolidated into multiplication tables and the regularities of commutativity and associativity noted there. The operation of division can be defined, and this yields another entire series of unanswered problems, so the rational numbers can be introduced. We have to show that these newly-developed rational numbers result in a system consistent with the system with which we began.

Further questions arise: powers and therefore roots, the irrational and complex numbers, etc. All this complicated structure that many grade school students wish would never have been mentioned, seems to push itself forward out of the simple operations with natural numbers and the insistent demands: What if . . . ? Could we . . . ?

The topic of the extension of the number system is approached again much later in a student's mathematical career, usually in a course of real analysis. The natural numbers are first developed through a set successor function and the usual properties of addition (associativity and commutativity) are proved with much difficulty. The integers are defined as equivalence classes of ordered pairs of natural numbers, the operations developed, and the natural numbers shown to be isomorphic

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to a subset of the set of integers. Then rational numbers are defined as equivalence classes of integers, etc. All of this is accomplished through sophisticated notation and careful proof.

Origin of Mathematics

Through this example, we are now in a position to begin to discuss the nature of mathematics by first examining where mathematics begins. It begins with experience and with questions about experience and with attempts to understand experience. Historically it seemes to have begun with counting (Eves 17:7) and to have moved to a notion of number and then to an expansion of the idea of number.

Theoretically, we can discover the origin of number ideas and spatial geometric concepts in the knowing process, somewhat as Kant did (30:41-55). The notion of empirical residue gives us the necessary tool to do this. Whenever we recognize several things to be understood similarly, we still know that they are not the same object; we have a multiplicity of objects that are to be distinguished only (or at least) by their difference in time or space. This difference is the source of the notion of succession or number. The basic empirical given, the bare residue of difference in space-time, seems to be the source of counting. Once the numbers are begun, we move beyond into the series of questions that expands our notion of number. The attempt to regularize the difference found as empirical residue moves the questioner from the given to a system--an understanding--which orders this given, and questioning continues to move him forward. The notion of difference leads to a notion of number and then to addition, to multiplication, to larger and larger number systems; the ordering of spatial difference yields geometry and then measure and then the various possible geometries.

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Mathematics slowly and continually shifts its focus. From the basic given, mathematics first yields operations and then moves as answers to questions about operations previously performed. Counting is the first operation, and addition arises as answers to questions about the systematization of the process of counting; multiplication is a generalization of addition--repeated addition; subtraction and division have meaning in terms of the other operations, and the number system is expanded because of these operations. So each development is the fruit of reflection upon a previous stage. The discussion of the development of complex numbers as ordered pairs presented early abstract algebra as the result of analysis of several kinds of operations and structures.

Another example of mathematics' characteristic of reflecting upon itself is the development of the calculus. The naive notion of a derivative as a quotient of differences was the origin of the idea and allowed some advance. But questions arose as to how far this simple process could be extended, when it works, why it works when it does. These questions led to the new level of discourse, the new horizon of continuity and conditions for differentiability (Kramer 33:528ff). The formal development of calculus is the answer to questions about conditions for differentiability. Analysis moved to new stages when questions were about this previous stage: What is most needed in these conditions? What results are available if conditions are modified? What sort of mathematical structure might fulfill the modified conditions?

Mathematics is a pursuit of answers to questions arising from previous results. It is not simply the totality of the results.

As all knowing does, mathematical thought begins with something experienced, whether it is physical multiplicity or the operations per-

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formed at some level of development. And mathematical thinking then works to answer questions about the ordering, the understanding, the extension of these data.

There follows the level of judgment. Lonergan emphasizes the distinction between the levels of intelligence and reflection as they occur in mathematics, the level of discovering and understanding and seeing connections vs. the level of checking and verifying (37:310f). The goal of checking, obviously, is to marshall the evidence for reflective understanding, to provide a basis for rational judgment. The level of judgment in mathematics can involve extensive formal apparatus, none of which can be applied unintelligently in order really to serve its purpose.

Thus mathematics is like other forms of knowing. But an important question is, What distinguishes mathematical statements and judgments from various other kinds of judgments? Mathematics uses a process of proof which reduces conclusions to premises, but so does deductive reasoning. Definitions and postulates coalesce into a self-justifying meaning in a mathematical statement, as they do in any analytic statement. What distinguishes mathematical judgments?

Based upon the structure imparted by the knowing process, we can conceive of mathematics as a series or succession of departments. Each of these consists of rules governing and so defining some certain operations, and of operations proceeding from some terms to others and so relating and defining them through implicit definition. That is, each department of mathematics consists of operations and elements, which are mutually defined by the rules that govern the action of the operation on the elements.

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In this sequence of departments of developed mathematics, each department is brought to formal expression by being stated in a set of definitions, postulates, and deductions.

Uniqueness of Mathematics

There can be other formalizations, equally or more rigorous and elegant, but not part of this series of mathematical departments. We can be precise in distinguishing mathematical formalizations from others by specifying for mathematics the three elements which are necessary to characterize any field of study: the material, formal, and actual elements (37:311ff).

These three elements of a field correspond to the three stages of knowing. They can be illustrated through the example of scientific investigation. The material element is the level of the experiential, from which learning always moves; in science this element is the pertinent data: the dial readings and measurements and more primitive observations. The formal element in a field of study is abstraction as enriching, the contribution by insight of intelligible correspondences and unities; in science, the development of theory is the formal element. The actual element lies in the conjunction of the material and formal elements; this comes at the level of judgment; in science this is when the theories are found pertinent to and justified by the dial readings and measurements.

To characterize mathematics, then, we will specify the three elements.

Material Element Mathematics, as was said earlier, begins with the most elemental aspects of data; the material element of mathematics is

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ultimately the empirical residue, the aspects of data from which understanding always abstracts, such as the individual, the continuum, particular places and times, and the non-systematic divergence of actual frequency from probable expectations. Mathematics begins with this residue as the stuff from which it works. Other sciences deal with data of determinate kinds; for the mathematician what remains is the empirical residue. As the example illustrated, the development of number is a systematization of numerical difference; the development of geometry is an ordering of spatial differences. These are matters with which no science can deal, but these differences are where mathematics begins; that is the material element.

Formal Element Insight adds "a component to knowledge that does not exist actually on the level of sense or imagination" (37:311). This contribution by insight of unities and order is the formal element, and to the mathematician it is something dynamic. The mathematician has to go through a laborious process of "learning mathematics," which is really learning to do mathematics. It consists not merely in mathematical results, but more in gradually acquiring the insights necessary to understand the kinds of problems and questions that arise in mathematics and to work out mathematical solutions. This is achieved as the mathematician does mathematics, as he reflects upon what he has done and so reaches a succession of higher viewpoints in moving from one department of mathematics to another. The various departments are connected more psychologically or intellectually than logically, for the various departments have their own definitions, postulates, inferences. Yet, intellectually they are "continuous," inasmuch as the symbolic representations of operations in one field provide the images for insight into

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the new rules that govern operations in the higher field. Thus this movement to a higher viewpoint, this abstractive enrichment that finds an intelligibility in the patterns of the previous stage, is the formal element of mathematics, the element that gives it meaning and structure. The questioning and movement of the example of the extension of numbers showed the role of the formal element.

The direction of mathematical development is not arbitrary. Lower viewpoints lead to higher viewpoints, and so there is this intellectual connection. Also, mathematical development seems to move from the particular to the general, from a part to the whole, from the approximate to the ideal. This is the movement of abstraction as enriching.

> If there exist concrete instances of one, two, three, the mathematician explores the totality of positive integers, of real numbers, of complex numbers, of ordered sets. If there exist edges and surfaces, the mathematician works out not merely one geometry but the total series of possible geometries. If there are various fields in which it seems mathematics may be applied, the mathematician sets out to explore the whole of each region in which the fields occur (37:312).

But the material element also seems to restrict the development of mathematical thought. This does not mean that the mathematician is restricted to existents: individuals that exist, places and times that exist, or to any other actual elements in the empirical residue that may be discovered through the introduction of new techniques of abstraction; it is obvious in the behavior of mathematicians and their statements that they are profoundly unconcerned with the existent. Still, mathematics does begin somewhere; the inquiring mind needs something to question, and the empirical residue supplies mathematics with "samples of the type of stuff on which mathematical ideas confer intelligibility and order" (37:312). This refers again to the fact that the material ele-

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ment of mathematics is the empirical residue of all data. The role of the material element in the development of mathematics may be less a restriction of development than an occasion for continuing development. Questions of possibility, of mathematical sense, arise from scientific questions. Coincident with mathematical questions (Leibniz), physical questions (Newton) occasioned a development in the calculus (Struick 70: 110-14). The presence of "difficult" point sets (e.g. for discontinuous functions) in function theory, dynamics, and probability called for an extension of the class of integrable functions and occasioned the development of further definitions of the integral and of integrability by Lebesgue, Young, and Stieltjes (Kramer 33:647). The material element, this empirical residue of all data, then, is basically a source of questions; and the kind of questions asked determines in some way the kind of answers given. That is, the kind of intelligibility sought determines partially, but not completely, the kind of intelligibility found.

Actual Element The actual element of mathematics is the conjunction of the material and formal elements; it is what we have when understanding in mathematics reaches possible orderings or arrangements or understandings of the original differences.

This conjunction imparts to mathematics a special character with respect to science. To show the role of mathematics in science, Lonergan compares the structures of empirical method to the action of a pair of scissors (37:312f). The lower blade of this scissors is the empirical side, rising from data through measurements and curve-fitting to formulae; the upper blade is the formal requirement of the postulates of invariance and equivalence and the accepted mathematical foundations,

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e.g. differential and operator equations. Together, they make empirical method possible, so that the requirements of theory operate in conjunction with data-gathering and interpretation. The question to be answered here is: What is the possibility of the mathematical element of the upper blade? How can it arise? How can it have any meaning?

To see where mathematics fits in this process, let us take a closer look at the process, which includes the two directions of movement, the two blades. First, empirical science moves from description to explanation, from its own proper domain of data to systems of laws that implicitly define the terms they relate; its goal is the ideal explanation in which all aspects of data, except the empirical residue, will have their intelligible counterpart in systems of explanatory conjugates and ideal frequencies. The other movement is that of mathematics, which begins from the empirical residue and endeavors to explore all possible ways that abstraction as enriching can give intelligibility to any materials that resemble the empirical residue. These movements are complementary. The scientist asks: How are we to understand this fact or relation? The mathematician asks: What are the possible systems that include this fact, that give it a meaning, that include this relation, or that give meaning to things like this? For the one fact has no meaning in itself. The scientist will try to approach the most successful explanation of the many facts. The mathematician looks for all possible systems that would include this fact or relation as element or as result. The scientist then uses the tool of the developed mathematics to develop his explanation. Sometimes the mathematics has already been developed; then we marvel at the results. For instance, the explanation of alternating current required a system that included

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the result $i^2 = -1$; of course recognizing that this was exactly what was required was a big step. Fortunately, systems that included that result had already been developed, such as the complex numbers and the quaternions. From these possible systems, the physicists chose the complex numbers as their explanatory system. At other times, the physicist requires a system that includes some relations that have not fully been explored by mathematics, and so he starts a new branch of mathematics. At this point the distinction between the two fields becomes blurred; physics looks like mathematics. But the physicist ultimately seeks one explanatory system for his result, and the physicist will gladly use one of the mathematician's systems, if it fits. But the physicist finally returns to the level of data in order to make a judgment, to see if the system does fit.

The mathematician uses data in another way, as his ultimate source in the empirical residue; if the mathematical exploration of intelligible systems is thorough, if it includes all possibilities, then it will necessarily include the systems of explanatory conjugates that the empirical sciences use and verify as explanations of phenomena in their respective domains. These results may be applicable, but it is not the business of the mathematician to justify his work in terms of applicability.

To sum up this comparison of the ways mathematics and science use this ultimate data:

The empirical residue has no innate intelligibility. Mathematics is the development of all possible ways to add intelligibility. One of these ways should be the way intelligibility is found in the particular

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situation by the sciences. Therefore, the actual situation should coincide with one of the possible intelligibilities found by mathematics.

We are still discussing the actual element of mathematics, the conjunction between the material and formal elements. We need to delve a little further into the nature of mathematical statements to account for both their formal nature and their continuing availability for application:

> It does happen that abstruse regions of mathematics are occasionally pulled out of their cold and airy regions to become the tools of empirical hypotheses and theories and to share with such formulations the probable existential reference that they possess (37:310).

What is there about mathematics that allows this possibility of application? These abstract propositions of mathematics, in order to assume the role of scientific statements and have probable existential reference, must somehow be capable of this. Because empirical applications of mathematics can be discovered, there must be something about mathematics that allows empirical reference for at least some mathematical statements. What can that be?

Nature of Mathematical Statements

We are still seeking to specify the status of mathematical statements or judgments. We can achieve this through a discussion of several types of propositions.

Analytic Propositions In a typical characterization, analytic propositions are propositions whose affirmation is possible strictly because of the form of the statements themselves, without reference to anything actual or even possible; they are conclusions resting on definitions and rules of meaning. An example is the formal statement, "If A and B, then

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A." There is a judgment involved, a case of the virtually unconditioned, an assertion of the connection between the subject and the predicate (37:304-6). But since the proposition is analytic, the judgment is no guarantee that the terms are significant, i.e., that the statement could have meaning as a judgment of fact or of scientific explanation. So analytic propositions remain apart and sterile, unless there is some other kind of validation for them, i.e., unless the same terms occur in another judgment with empirical reference. As a result, analytic propositions can be produced endlessly; they are not significant increments of knowledge; they are tautologies.

Analytic Principles There is a step beyond analytic propositions; an analytic principle is an analytic proposition of which the partial terms are existential, i.e., the terms involved in the analytic principle occur in the same sense as they are defined in some judgment of fact, either a concrete judgment of fact or an established empirical generalization (37:306-8). An example of an analytic principle is, "A mammal is a vertebrate." This is analytic in the sense that a mammal by definition is a particular type of vertebrate. But there are scientific statements and judgments of fact that use the terms "mammal" and "vertebrate as they are used here. So analytic principles are like analytic propositions in the fact that their validation or verification relies only on definitions and rules of logic. But they are unlike analytic propositions in the fact that the parts of the proposition have some reference and so what they assert has the possibility of referring to something outside the mere system of rules and definitions of their formulation. These analytic principles are of a basic, primitive character, for they lie outside the realms of common sense and empirical

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science. They lie outside common sense because, again, common sense regards the particular, while these analytic principles deal with the universal, with anything that fulfills definitions, while common sense avoids formal definitions. They are not part of empirical science because, though the scientist formulates definitions, postulates, and inferences from linking of concepts through insights, he knows what he asserts not as certainly correct but only as probable. His defined terms are open to revision as are the probable judgments of fact that contain them and give rise to them. And so the statements of science are called provisional analytic principles; the terms of these principles occur in probable empirical generalizations.

Serial Analytic Principles The premises of mathematical thought are analytic propositions, says Lonergan, but not all analytic propositions are mathematical premises. Analytic propositions can be recited all day, requiring some effort only if the provision of a coherent system were added, but mathematics is an effort; the premises of mathematical thought are reached "only through the discoveries of genius and the labour of learning what genius has grasped" (37:310).

Lonergan characterizes mathematical statements as serially analytic principles (37:313); they are those propositions which are part of a system of analytic propositions, the development of which involves analytic principles. Somewhere within the system, that is, are analytic propositions with terms or relations that occur in actual or probable judgments of fact.

For example, we can regard the entire system of arithmetic from two aspects, at two different points of its formulation. It can be

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regarded at the outset as a very empirical study, where the number 4, for instance, is represented by this handful of rocks. At that stage, the assertion $3+4=7$ is a judgment of fact pertaining to combinations of actual items; it could be otherwise expressed as: If I combine a pile consisting of three rocks with a pile consisting of four rocks, I obtain a pile of seven rocks. In this statement, "three" and "four" are adjectives. The system becomes abstract when the numbers become nouns.

But the arithmetic becomes formalized mathematics when we begin with the successor function s, define $3 = s(2)$ and $4 = s(3)$, define the operation + and establish its properties, and show from the definitions and properties that $3 + 4 = 7$.

The last statement is an analytic proposition, because from the definitions of 3, 4, 7, and + we can establish the equality. But it has reference to the simple system of counting that preceded all the formal development. Therefore that statement, and those included in the same system, are serially analytic principles.

Another example to show the statements of mathematics are serially analytic principles is the proposition $i^2 = -1$. This statement is immediately true by definition, in an abstract statement of the properties of complex numbers. But it also has possible reference to physical situations through the mathematical study of alternating current. The possible reference of statements within mathematics to judgments of fact constitute the analytic propositions of mathematics as serially analytic principles.

The identification of the propositions of mathematics with serially analytic principles seems to account for several important

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characteristics of mathematics:

1) recognition that not all free formalizations are mathematical;

2) the possibility of isomorphism between mathematical relations and the relations of the empirical sciences, since both sets of relations are products of enriching abstraction and both possess a relevance to the empirical residue in data; and

3) the movement of mathematical thought, beginning in the material element, and tending toward a general, complete, and ideal account of the manners in which enriching abstraction can add intelligibility and order to the material element.

Thus the mathematician is concerned to establish generally, completely, and ideally the range of possible systems that include verifiable scientific systems as particular, fragmentary, or approximate cases.

This analysis reveals the dynamic character of mathematics. It is an invitation to mathematicians to explore the possibility of setting up the series of deductive expansions that could lead to a foundation for other sciences as it has done for physics; the mathematics used may be different from that used in the standard physical sciences. In fact, statistical argument is forming a stronger part of biology; game theory and other areas of mathematics are finding their place in management analysis, for example decision making theory in the area of the social sciences. A key role is for the scientist to understand what he is doing when he uses a mathematical model.

This understanding of mathematics also recognizes the symbiosis between pure and applied mathematics. They have common roots. Applied mathematics can supply the questions that demand development of a whole new field of pure mathematics. And all pure mathematics has potential application (48).

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Still, mathematics is not restricted to being a tool for the physical sciences. Mathematics remains free to do what it has always done. The mathematician takes as his starting point anything that resembles the empirical residue or, most often, any prior development. He is free to add to that as he will, discovering all possibilities; he can explore with full generality, completeness and ideality, the enrichments that the exercise of human intelligence can add. In all of this his creations remain serially analytic, for they are part of a series of systems, to some of which the empirical scientist can currently say "Yes," and others to which some later scientist may also say "Yes," or at least, "probably."

Higher Viewpoints The main vehicle of this development of mathematics is the movement to higher viewpoint. As mathematics advances, it does so in stages, first by achieving a new central understanding and then by a consolidation at that level. One example of this process was the development of extensions of number. Another is the development through arithmetic to algebra. On the level of arithmetic, many insights and understandings can be reached and developed with a rather complete and elaborate treatment. But then there are always further questions and further insights that move beyond the field of arithmetic, based on the patterns found there, and algebra is the level of understanding the operations of arithmetic. This is a case of moving to a higher viewpoint.

Lonergan describes the process this way: When insights occur in related fields or matters, they do not remain individual but contribute to one another; they combine, cluster, coalesce, into the mastery of a subject; they are the insights that find expression in definitions,

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postulates, or deductions, and can be applied to a wide range of cases. That is what constitutes them as a unified subject matter. But the process does not stop when a coherent whole is reached; further insights arise as answers to wider questions. The shortcomings of the previous position are recognized and so new, more fundamental definitions and axioms must be developed; deductions are based on these to give rise to an even wider field of development with wider applications. "Such a complex shift in the whole structure of insights, definitions, postulates, deductions, and applications, may be referred to very briefly as the emergence of a higher viewpoint" (37:13).

This complex of combined and related definitions, postulates, and deductions, all linked by understanding, can be called a viewpoint. When a viewpoint is found insufficient to provide adequate answers to questions that arise, some sort of shift is required to a new basis, a new viewpoint, that will allow the knower to understand all he did before, and to expand his scope of knowledge because of the more basic nature of the insights that constitute the new viewpoint.

Hamilton's development of the system of quaternions and his recognition of the properties of the operations on quaternions was a great achievement. But the subsequent developments of abstract algebra showed how results within the system of quaternions were consequences of particular properties of the operations; the system of quaternions was shown to be a particular case of a more general kind of structure. This wider approach, which treated quaternions along with many other examples, was a higher viewpoint. The questions that led to this higher viewpoint were elicited by the achievement of the quaternions: How is this system like or unlike others? How are the unusual properties related?

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Another example of an achievement of higher viewpoint is in the systematization of the various geometries. First of all, Euclidean geometry seemed to be the only possible understanding in geometry. Then non-Euclidean geometries were developed, the first stage in forcing a new viewpoint. The higher viewpoint was reached in the understanding that unified all these geometries, Euclidean and non-Euclidean, in a single, total grasp. This happened in Klein's demonstration that each of the geometries could be considered as an instance of doing geometry, each distinguished by the particular set of properties that remain unchanged under a particular group of transformations (32:133f). That new viewpoint leaves Euclidean geometry unchanged, but understands it within a wider context that recognizes and answers further questions.

In the paper in which Stefan Banach developed the basic notions of Banach space, he wrote:

> This present work has the object of establishing certain theorems that hold in several different branches of mathematics, which will be specified later. However, in order to avoid proving those theorems for each branch individually, which would be very wearisome, I have chosen a different way, which is this: I consider in a general way sets of elements for which I postulate certain properties. From these I deduce theorems and then I prove for each separate branch of mathematics that the postulates adopted are true of it (3).

This can be cited as an example of reaching a higher viewpoint; Banach reached an intelligibility in otherwise indifferent matters in disparate fields. He took his impetus from the symbolic expressions in the various fields. And he reached a new level of understanding that incorporated what had been done before, but as part of his new and wider understanding.

The reorganization of science in terms of new paradigms, as described by Kuhn (34), can be seen as a description of a moving view-

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point. When the new viewpoint effectively systematizes both old areas of science and new realms in the science, it can be called a higher viewpoint.

This movement to a higher viewpoint is a strong characteristic of mathematics. Mathematics develops; it is dynamic. A student of mathematics cannot go directly from arithmetic to group theory. The gradual acquisition of the insights necessary to understand mathematical problems and mathematical arguments or to work out mathematical solutions comes in the succession of higher viewpoints.

> One department of mathematics follows upon another. Logically, they are discontinuous, for each has its own definitions, postulates, and inferences. But intellectually they are continuous, inasmuch as the symbolic representation of operations in the lower field provides the images in which intelligence grasps the idea of the new rules that govern operations in the higher field (37:312).

This movement to higher viewpoints is what links analytic principles together serially to constitute mathematical statements as serially analytic principles.

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CHAPTER III

COMPARISON WITH OTHER POSITIONS

This chapter compares and contrasts Lonergan's theory of cognition and its use in education with theories of learning which currently influence mathematical education. The theories discussed here, for the most part, are based on psychological studies; this immediately creates conflict with Lonergan's position because of varying language and contrary presuppositions. For instance, current psychology requires measurable results and so defines psychological constructs in terms of possible measures. A consequence of this is that psychological learning theory tends to focus on the simpler, more distinct stimulus-response association, and so learning theory often appears reductionistic. Neither measurability of results nor the simple mechanism of association receive notice in Lonergan's theory; Lonergan's method relies on self-understanding and moves toward the more complicated aspects of knowing. The intent here is not to reconcile these differences. This chapter will point out differences and, more often, use Lonergan's theory to situate the other theories and interpret their findings or restate the theories in order to reconcile them with Lonergan's. The discussion will also note how the various theories complement Lonergan's.

For each author, a presentation of the pertinent points of his theory will precede discussion from the standpoint of Lonergan's theory.

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Gagné

His Theory

In his book The Conditions of Learning (22), Robert Gagné makes two points that are specially pertinent here. First, there are many kinds of learning--eight in fact--and learning should not be identified exclusively with any one of those patterns. Secondly, the key to the occurrence of each kind of learning is the fulfillment of the conditions for that step to occur; therfore, an educator can construct a hierarchy of learning tasks leading to an ultimate goal; each state of the hierarchy is a specification of the conditions required for the higher task.

Several terms must be defined in Gagné's usage. Learning, according to Gagné, is "a change in human disposition or capability, which can be retained, and which is not simply ascribable to the process of growth" (22:2). This change is exhibited as a change in behavior, the only way an experimental psychologist can detect it, and learning is inferred when a subject can do after a "learning situation" what he was unable to do before it. Gagné equates the word "knowledge" with information (22:187f).

Eight Types of Learning Gagné insists that a prime mistake of experimental investigation of learning is the contention that all learning is basically the same. This false apprehension causes a distortion of the actual processes involved.

> To equate the responding of an animal to a warning signal with the learning of a child asking for a doll, or the learning of a student to identify a chromosome or the learning to predict inheritance with the laws of genetics is considered to be a matter of gross disregard for some obvious and simple observation (22:65).

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Among models that he rejects as models for all of learning is the Gestalt tradition's insight model, which Gagné interprets as a suddenly occurring reorganization of the field of experience, as in "having a new idea" or "discovering a solution to a problem." He makes two objections to the wide application of this model. First, some of the original evidence for its occurrence can be more simply explained by previous learning or by learning set. Secondly, there are many instances of learning--for example, in the purely informational areas--where insight cannot remotely be found, such as learning the names of plants or stars, learning to read, or learning to speak a foreign language (22:17).

Therefore Gagné finds it necessary to distinguish eight varieties of learning. Each begins with a different state of the learner and ends with a different capability for performance, and these two differences comprise the basis of categorization.

1. Signal learning is the classical conditioned response, in which the individual learns to make a general, diffuse response to a signal. Examples are the Pavlovian results with dogs, or a horse learning to stop at the cry "Whoa!" by associating the sound of the cry with a painful experience (22:97f).

2. Stimulus-response learning (trial-and-error, operant, or instrumental learning) occurs when the learner acquires a precise response to a discriminated stimulus. In this the stimulus and response seem to become bound together through a single connection. One kind of stimulus-response learning is the adoption of a receptivity set toward verbal communication: learning to attend to verbal stimuli, to respond to verbal requests, or to follow verbal directions (22:118f).

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3. Chaining is the connection of a set of individual stimulus-response connections in a sequence. These may be motor or verbal responses; turning on a washing machine consists in a sequence of motor responses while the pledge of allegiance to the flag is entirely verbal (22:123). Learning a maze is a case of chaining, because it is the learning of a series of individual responses. A key condition for learning of this type is that the individual must be capable of performing each of the individual responses, and these are linked together by some cues, usually verbal, in the learning situation. Gagné uses the example of a child learning to ask for a doll when he lies down. He must first of all have the stimulus-response units (lying down--hugging the doll) and (seeing the doll--saying "doll"). The connection between the two pairs could arise when the mother asks the child lying in bed if he wants his doll and, later, the child seeing the doll or beginning the hugging motions associated with the doll.

4. Verbal association is really a chaining of verbal connections, so it is an instance of the last type of learning, but it is distinguished because of the use of language. Because of the versatility of human processes it has some special characteristics. In this type of chaining the intermediary, internal links may be selected from the previously learned language of the individual (22:63). Examples Gagné uses are simply naming an object, learning a sequence of nonsense symbols, or memorizing a passage (22:135).

5. Discrimination learning is a type of learning in which the individual learns to distinguish various stimuli; he learns to make different indentifying responses to different stimuli which more or less resemble one another. In order to do this, the individual must have

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achieved the whole set of individual chains of responses that identify the different species of stimulus. Gagné's illustration is the young man learning to identify the different models of cars. Since many models are similar, the young man must first learn the particular characteristics that identify each model and then he must learn to distinguish these characteristics from one another (22:48ff). This is a complicated process, because of the many prerequisite chains. Because of the similarity of the characteristics to be distinguished, this learning is very susceptible to interference and so to forgetting.

6. Concept learning means learning to classify stimulus situations in terms of abstracted properties like color, shape, etc. (22:51). The learner develops the "capability of making a common response to a class of stimuli that may differ from each other widely in physical appearance" (22:63f); thus he learns a response to a whole class of objects or events. (These are concrete concepts, concepts learned by observation; there is another kind of concept which is basically an abstraction, learned by definition and involving relations, which is classed under rule learning.)

The achievement of concept learning is demonstrated by the learner's use of the concept, not by formulation. In Gagné's example, an individual has learned the concept of a cube when he identifies it "intuitively"--"on the basis of an internalized representation that does not employ the words of the geometer's definition" (22:51f). The learner possessing the concept is able to classify objects that widely differ in their physical appearance; the controlling factors in this classification are not specifiable particular stimuli but "abstract properties of such stimuli" (22:52). Verbalization of the concept may

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follow and it will be useful for recall and for later learning (22:193f), but it will be useful only if the concept is first known by reference to a class of concrete situations.

What moves concept learning beyond any of the previous types is the characteristic of generalizability. When concepts are learned, the learner is freed from specific stimuli. And so the presence of concepts can be tested by demonstrating that generalizing occurs (22:182). Thus learning concepts frees the individual from the domination of immediate objects in the physical environment (22:187). Possession of concepts gives the individual his main tool in organizing his world and his environment and it provides his main means of communication.

7. Rule learning is the learning of a chain of two or more concepts. An example is the acquisition of the idea contained in the proposition "gases expand when heated." What is important here is the idea, not the words, which can be learned as a verbal chain (type 4). Rule learning may be described as a chaining of concepts, for what is required for this type of learning is that the learner have the two concepts and put them together in the proper order. Since a concept is the ability to consider as one (i.e., to give the same response to) an entire class of stimuli, a rule is defined to be "an inferred capability that enables the individual to respond to a class of stimulus situations with a class of performances" (22:191).

Gagné observes that, while once the stimulus-response connection was considered the unit of mental organization, most psychologists have replaced this by the rule; most human behavior in natural situations is governed by rules (22:191).

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Defined (abstract, relational) concepts are considered rules because they do in fact relate two or more simpler concepts, which is the function of a rule. An example is the definition of a diagonal as "a line connecting opposite corners of a quadrilateral"; this develops the relation between the concepts line and opposite corners of a quadrilateral, which are concrete concepts (22:190).

8. Problem solving is the eighth type of learning and includes the experiences usually classed under that name. Problem-solving is, Gagné says, "a process by which the learner discovers a combination of previously learned rules that he can apply to achieve a solution for a novel problem situation" (22:214). This is not necessarily the mere application of a previously learned rule; it can actually be the development of a new rule as a compilation of old rules, developed to meet a particular situation. This process of combining previously-learned rules into a great variety of new, higher-order rules, Gagné calls thinking (22:59).

Gagné finds in problem-solving the same sequence of events that goes back to Dewey: presentation of the problem, definition of the problem by isolation of the essential features, formulation of hypotheses, and verification; the last three are internal events (22:215). Gagné agrees that solution to the problem is found in a sudden flash of insight.

The capability achieved in this kind of learning is no different from what is achieved through combining rules to reach a higher level rule (type 7). The difference is in the kind and amount of guidance provided during the learning. Thus problem solving requires discovery; the learner must construct the new rule his own way.

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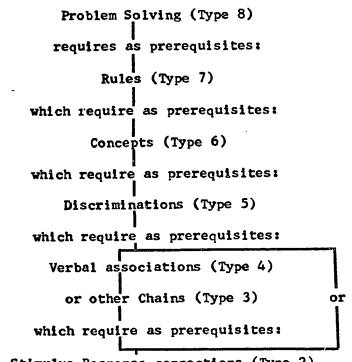
Since what is involved here is a use of rules in particular situations, then the chief prerequisite for this type of learning is that the learner know the pertinent rules. There are individual differences that can influence this problem-solving process. Among the factors that can facilitate the achievement of a solution are the store of rules available to the learner, distinctiveness of required concepts, the ease of recall of relevant rules, the ability to match specific instances to a general class, and fluency of hypothesis. The intellectual fluency factors have also been called creativity.

Learning Hierarchies The main implication of this breakdown of learning into eight types and of the recognition of the requirements for each type is the capability of specifying prerequisites for learning tasks. Gagné schematizes the dependence among the types this way (22:66):



When the type of a learning task is specified, the key step in teaching it is to identify its prerequisites and provide the proper way to combine

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them.

Therefore a learning hierarchy is possible. This hierarchy is a continued analysis of a learning task in terms of prerequisites; it expresses an entire set of capabilities in ordered relation, with prerequisite capabilities subordinate to the task to which they are prerequisite, indicating that they need to be learned previously. A particular problem solving or higher-order rule learning task can be analyzed this way.[1] The analysis can extend the hierarchy through simpler and simpler kinds of learning, but needs to continue only to the level of capabilities that can be assumed present in the group being taught.

When a pair of abilities are shown in superordinate-subordinate relation in the hierarchy, what is indicated is that the superordinate capability or performance will be "more readily learned (on the average, throughout a group of students) if the subordinate capabilities have been previously acquired and are readily available for recall" (22:239). The learning hierarchy, then, is the identification of the set of intellectual skills ordered to indicate "substantial amounds of positive transfer from those skills of lower position to connected ones of higher position" (22:239).

Thus a learning hierarchy will indicate a route of instruction insuring that all the students will be able to achieve a particular goal. It may be that a bright student will not need to follow all the steps, because he might already have acquired subordinate skills or developed some problem solving strategies that can reach both a

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[1] A partial example was given in Chapter I, page 8.

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subordinate skill and a final one at the same time. Therefore, says Gagné, "The learning hierarchy is supposed to provide a basis for finding a suitable learning route for every student" (22:241).

Now this is important for education in several ways. First of all, readiness for any particular task can be defined as possession of the prerequisite capabilities. Secondly, because the task is so broken down, there is room for alternate forms of instruction; taking the learning step by step is the primary characteristic of programmed instruction.

The analysis of the learning conditions of any task must be performed before the actual learning situation is entered and instruction begins. Therefore prior planning, answering the questions "From where does the student begin; and where is he going? What are the specific prerequisites for learning, and what will he be able to learn next?" (22:26) will specify a certain structure in the subject to be learned--the learning hierarchy--by analyzing the types of learning involved. Gagné gives another example of a hierarchy: adding integers. This task has as prerequisite the capability to (1) state and use the the definition of addition of two positive integers, and (2) state and use the definition of the sum of two integers, if at least one of the addends is a negative integer. This last in turn has as a prerequisite the capability to supply other names for positive integers in statements of equality, which itself requires the capability to state and use the definition of addition of an integer and its additive inverse, etc. This analysis descends through the hierarchy until the initial capabilities are specified: use the whole number 0 as the additive identity, perform addition and subtraction of whole numbers, use parentheses to group names

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for the same whole number (22:255). Teaching the addition of integers begins by determining if the group to be taught can do these last-named tasks.

To instruct is to arrange properly the conditions of learning for each of the types of learning, guaranteeing that the conditions are met: ordering the learning in a stage-by-stage fashion, taking account at each stage of the capabilities just learned, allowing for retention, and providing the specific stimulus situation needed for the next stage. Thus instruction is the very intricate activity of orchestration; it requires a verbal cue or statement of the goal of each stage of instruction and reminders to the student of what he has already learned; it involves directing the student's attention and actions and thinking.

> All these events are instituted for the purpose of establishing the proper external conditions for learning. Assuming that the necessary internal capabilities have been previously learned, a suitable arrangement of instructional events will bring about efficient learning (22:29).

Gagné lists the events of instruction which serve to insure that the timing and sequencing of events internal to the learner is optimal for the occurrence of learning, retention, and transferability (22:303-5):

1. Gaining and controlling attention by some external stimulus,
2. Informing the learner of expected outcomes,
3. Stimulating recall of relevant prerequisite capabilities,
4. Presenting the stimuli inherent to the learning task,
5. Offering guidance for learning by direction of thinking through verbal hints or prompts,
6. Providing feedback, by informing the learner of the correctness of his performance,

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7. Appraising performance, by providing opportunity for the learner to verify his achievement in other situations,

8. Making provisions for transferability through additional examples,

9. Insuring retention.

Finally, Gagné says that although verbal information of various kinds is important in learning new intellectual skills, the core of what the individual takes away from education is not the verbal information but intellectual skill; the learning hierarchies are the ways to describe the structure of intellectual skills within the discipline (22:245).

Comment

1. Gagné's discussion gives a needed expansion to the notion of learning and knowing presented here. The possession of information or facts, the simple use of simple words, items like the addition and multiplication tables are foundations, as Gagné says, for higher intellectual achievements. Certainly some of the activities such as the simple connections do not require understanding or insight, and so they have been neglected in the discussion from Lonergan's point of view.

Lonergan discussed ease at arriving at insight as some sort of native endowment. Gagné gives further content to this notion and indicates the factors that might be involved in that native endowment in his discussion of individual differences in the problem-solving process. His listing of factors does not truly indicate what brings about an insight, but it does give more precision to the characterization of "native endowment."

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2. One problem with Gagné's classification of learning into categories is that the ultimate building block seems still to be the stimulus-response connection of either type 1 or type 2, because learning of further types is described as an appropriate connection of the skills of the previous types; and the connection of the higher type often seems to be a response to a stimulus that is itself a connection. However, there is some sort of a shift in the connection in the middle types of learning. With reference to type 4, verbal chaining, Gagné refers to Ausubel's distinction of meaningful and rote verbal learning and discusses the rapidity, retention, and resistance to interference which characterize learning the substance (or ideas) of meaningful verbal material as opposed to the exact words of meaningful material or nonmeaningful verbal material.[2] Gagné has no response to this:

> Since it is a highly developed theory, however, it will serve as an outstanding example of a conception of verbal learning requiring inferred processes which are considerably more elaborate than those involved in the notion of verbal chaining. At the same time, it is well to recognize that some theorists continue to hold a kind of faith that the processes described by Ausbel can ultimately be accounted for in terms of the properties of chains of stimulus-response connections, although, to be sure, no one has yet done so. There the matter must rest for the moment (22:150).

Later, discussing rules, Gagné says that the organized nature of rules seems to promote resistance to interference and forgetting, but "how this works is not known" (22:210).

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[2]A discussion of Ausubel's theory will follow.

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All this seems to make some sort of wide distinction between types of learning that depend on meaning and those that do not. But meaning is difficult to situate in this discussion of learning, primarily because it has no place among observable situations and measurable results. Meaning is the content of an understanding, which is not subject to measure or observation.

3. The crux of the problem is whether or not there are previous conditions that are sufficient to the occurrence of learning in one of the higher (or meaningful) types. For example, it is extremely difficult to speak of a concept without reference to meaning; to speak of it, as Gagné does, as the "capability of making a common response to a class of stimuli that may differ from each other widely in physical appearance" (22:63f), is to confuse two things that are related but not always identical. It is similar to the two ways we may think of a set; a set can be specified either by a rule or by a listing of the elements. Countable sets may be listed, with an understood continuation. But uncountable sets and, usually, countable ones require a rule. As for the latter sets, there is no way to list all the stimulus situations that would belong to a concrete concept; what there must be is a rule, which we would do well to call the meaning of the concept. (The notion of rule here is not the same as Gagné's seventh type of learning.) The concept of red would then not be {this red bike, this red ball,....} but it would be more like $\{x|x$ is similar in color to this red ball$\}$. The point is, the concept must have a meaning, which we have called a rule of definition, and for the learner to possess the concept he must do something--he must master the rule; in his acquisition of the concept he must be an active participant.

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4. Further, the characterization of concept and rule learning is that it is generalizable. But Gagné acknowledges the lack of agreement on a satisfactory answer to the problem of teaching for generalization or transfer (22:336). He suggests that organized group discussion, as well as making sure that mastery of the learning and its subordinate capabilities are mastered, might be one of the most effective means to achieve transfer. Finally, the greater the variety of stimulus situations over which the student is encouraged to generalize his knowledge, the greater should be the ability to transfer this learning (22:339). But the problem is similar to the last one: Does the concept extend to more than those instances that have been shown to be in the list of stimuli to which the concept applies? What is the key to the spontaneous ability to extend the list. The answer is precisely in the notion of generalization or abstraction or meaning. It is true that the lengthening of the roster of the elements included in the class comprising the concept or rule will enable a more adequate conception of the meaning that specifies that class, so that generalization will occur. But unless there is meaning involved, meaning that is more than a list of elements in a class but some sort of intellectual grasp of what is the standard for that set, there can never be generalization. Here is the place where Lonergan's theory of cognition is important; meaning is an achievement of the knower in the act of insight that takes him beyond the immediate and the particular.

5. Verbalization of already attained capabilities seems a problem; Gagné does not seem to indicate how the ability to formulate a meaning might be learned. If the teacher supplies the verbalization, then it may be acquired through verbal chaining (meaningful or not). But the process of development of expression seems neglected in this presentation.

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6. The whole presentation of Gagné does not seem to develop out of natural situations; the learning situation is always one arranged by the teacher or instructor for the learning by the individual. Outside of this deliberately planned situation, one wonders how learning occurs. Does it become a purely random event? It seems that Gagné's approach neglects the natural drive to know, the fact that much of our effort at learning is purposive, arises from inquiry, and has a goal of finding out what is so. Gagné also allows scanty place in the structure for judgment, for determining the validity of what is learned. The only entry of judgment in the process is in the last type, problem-solving, in the teacher's activity to provide corroboration or correction as part of the learning task. But the need for determining whether or not the student is correct is present in the whole learning process. It may be met either by the teacher or by some other source. A good enough student will not be satisfied with the teacher's word that he is right or wrong; he will demand to know how his understanding is inadequate.

The provision of a statement of expected outcomes in the beginning of the learning situation provides a partial remedy for the omission of judgment. This statement is some standard for reference, because the student can compare what he achieves with this statement. But this is a very extrinsic sort of judgment.

This separation of learning situations in educational settings from normal patterns of cognition is harmful in other ways. It enforces the separation of school learning from the "real world"; and it specifies a pattern of learning that is not homogeneous and gives a false view of inquiry conducted at the limits of man's knowing and in personal pursuits.

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7. A final objection to using Gagné's theory as a basis for educational practice is that it seems so narrow; to specify goals for education completely in terms of the measurable and observable seems to miss much that is important. The "higher" mental processes of analysis and evaluation are particularly resistant to reliable measure, but the difficulty in measurement does not mean that these processes are not worthy goals. In fact, those higher-level goals are precisely the ones that are true marks of mathematics achievement. The fuller and deeper theory of Lonergan is needed in education to keep perspective and give school learning the true richness of human knowing.

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Ausubel

His Theory

David Ausubel, in Educational Psychology: A Cognitive View (2), writes in a way much closer to the applications we have made of Lonergan's cognitional theory. Ausubel's main emphasis is on meaning, and he gives an extensive treatment of meaningful learning, the role of language, cognitive structure, and concepts, among other topics.

His first concern is to distinguish two dimensions of learning, the meaningful-rote dimension and the reception-discovery dimension. What has led to many troubles in recent psychological writing and school strategies is the confusion of these dimensions. Ausubel insists that they are separate and distinct. In reception learning, the content of what is learned is presented to the learner in complete and final form; in discovery learning, on the other hand, the principal content of what is to be learned must be discovered, so there are really two learning tasks: to discover what must be learned, and to learn it (2:22). On the other dimension, learning is meaningful "if the learning task can be related in nonarbitrary, substantive (non-verbatim) fashion to what the learner already knows" (2:24). Rote learning is characterized by purely arbitrary associations that compose the learning task; it also occurs when the learner lacks the prior knowledge necessary to make the learning task relatable to previous learning or when the learner sets out to learn something rotely. Because these two dimensions are distinct, says Ausubel, reception learning need not be rote. The reason that much potentially meaningful learning occurs rotely when taught by verbal exposition is not the expository method but the abuses of it which fail to capitalize

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on the meaningful nature of the material.

Meaning Ausubel explains or defines meaning in terms of the meaningful learning process; meaning is a product of that process. The term meaning refers to "the differentiated cognitive content evoked in a given learner by a particular symbol, or group of symbols, after either of these expressions has been meaningfully learned" (2:42). The material with which the learning starts is only potentially meaningful, because it is only relatable to the learner's structure of knowledge; only when the learning has taken place, when the symbolic expression is related to the previously achieved relevant ideas in the learning structure and interacts with them, is this meaning actually constituted; the meaning of the symbolic expression will then be evoked when the symbolic expression is present.

Thus there are different elements of this meaning, considered at different moments of the process. The acquisition of meaning, or actual meaningful learning, requires potentially meaningful material and a meaningful learning set. Logical meaningfulness is a quality of the material itself, and it has to do with the nonarbitrary and substantive relatability of the material to relevant ideas that at least some men could learning meaningfully. What makes this logical meaningfulness potential meaningfulness is the availability of the relevant ideas in the cognitive structure of the individual actually attempting the learning. Finally, idiosyncratic (phenomenological, psychological, or actual) meaning is the product of the meaningful learning process upon the potentially meaningful material; it is wholly individual cognitive experience (2:44f).

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Parallel to the distinction between logical meaning, inherent in symbolic material by its very nature, and psychological meaning, which is wholly individual, there are two structures of knowledge, the logical structure inherent in subject matter and the psychological structure that each man builds as he learns meaningfully.

The psychological structure is the structure of ideas that each man develops as he learns. The clarity, stability, and suitable organization of this cognitive structure has a great deal to do with a person's ability to learn and remember accurate and unambiguous meanings. Therefore, says Ausubel, an important part of the educational process is the development of cognitive structure for most effective learning and retention. One goal of the curriculum reform movements in mathematics has been to clarify and strengthen cognitive structure; the strengthening can be effected either substantively, in the scope and integrative power of the principles used to organize the particular discipline, or programmatically, in the order and means of presenting learning materials and practice (2:127f).

The first variable affecting this structure in the learning of meaningful materials is the availability of relevant anchoring ideas general enough to insure that new ideas can be related to the present structure. If these anchoring ideas are not available, they must be provided, and this is the basis for Ausubel's recommendation of advance organizers. A second factor affecting learning and retention is the extent to which the new learning task is discriminable from old ideas. A third factor is the stability and clarity of the anchoring ideas within the structure (2:131f).

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Meaningful Learning The ability to relate and anchor new material to relevant ideas in established cognitive structure marks the difference between meaningfully- and rotely-learned materials and accounts for the qualitatively different ways that these materials are learned and retained. Learning and retention of rotely learned materials are governed primarily by the properties of the systems of ideas to which the new materials are related.

There are several types of meaningful learning. The most basic is representational learning; in this process, single symbols or words come to represent for the learner the objects or ideas to which they refer; the learner equates the sign with the significate.

In propositional learning the learner comes to know the meaning of composite ideas expressed by propositions or other multiple symbols; the proposition is generated by relating many words, each of which has a single referent; the relations between the ideas here are more than mere lumping together of the meanings of the words (2:43).

Concept learning, the third kind of meaningful learning, involves two types of learning, both a substantive learning that reaches the criterial attributes or meaning of the concept itself and a representational learning of the word which symbolizes the new concept. Thus propositional and concept learning are similar; one difference is that in propositional learning the new idea or meaning is composite while in concept learning the meaning is general but refers to one idea (2:43f).

In meaningful learning of the last two types, the new concept or proposition is connected to the existing structure, either as a particular case of a more general idea, as a generalization, or in a more complicated way that involves some combination of established ideas. When

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the link is a superordination--one idea including the other as a particular--the more inclusive or superordinate idea will be the more stable and in time the less inclusive idea will be forgotten. Because new material is so assimilated to old, a person's knowledge of a particular subject-matter field can be represented as a hierarchically-ordered pyramid of ideas, with the most general and broadly explanatory ideas at the top, while progressively less inclusive ideas are subordinated to these, stretching toward the bottom (2:92-94). The ideas at the top, then, are the most stable, while the more detailed and specific propositions tend to be obliterated. This lower-order knowledge, which constitutes "the flesh, if not the skeleton, of any body of knowledge," is lost while the skeleton or organization of that knowledge is retained (2:94). Learning an academic discipline must somehow consist in mastering both.

Ausubel says that propositions can most easily be learned if they are subsumable as particulars under relevant ideas already learned. In light of this and of the possibility of presenting cognitive structure as a hierarchy ordered by subsumption, Ausubel suggests that the subsumptive mode of meaningful learning should be used whenever possible. In education this means to present first the most general statements in the subject matter and to move toward the most particular; it also supports the use of advance organizers, which present generalizations early in the learning sequence to which later propositions are subordinate (2:53f).

Cognition and Perception In meaningful verbal learning, it is hard to distinguish perceptual and cognitive processes, because both involve an

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interaction between the stimulus input and cognitive structure. We perceive the verbal messages and also learn their meaning in cognitional processes by interpreting them through their relation to existing knowledge. Perception, however, is more immediate and less complex; it involves the immediate content of awareness before even the processes of reception learning come into play. Cognition begins with the processes which relate the new idea to relevant parts of existing knowledge, reconcile apparent contradictions between the new and the old, or rephrase the material in personal terms (2:55). Thus, the acquisition of meaning is a cognitive process; for a child learning the meaning of words, to learn that particular auditory stimuli (words) represent particular objects is a cognitive process, as is to learn the role of words in sentences. After he has mastered both vocabulary and syntax, the child can grasp the meaning of words and their functions in a sentence on a merely perceptual basis (2:57). Thus meaningful reception of symbols begins as a cognitive function but becomes a perceptual one. The same holds for the reception of potentially meaningful propositions; at first encounter the reception of a proposition is a cognitive one, but after repeated encounters with the same proposition the reception is a perceptual process. This shift from cognitive to perceptual processes frees the learner to advance; when he confronts a proposition and wishes to learn it, the syntax and vocabulary of the proposition are already meaningful in perception and so he needs cognitive processes only to relate the idea to cognitive structure (2:57f).

Verbalization Verbalization is not indifferent to learning. Ausubel agrees that a nonverbal or inexpressible understanding does exist as

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precursor to some verbal understanding especially in children and adults that are novices in a field. But for the generation of new insights, verbal expressions of concepts and propositions are almost necessary; "ideas that are not represented by words lack sufficient manipulability to be used in any complex type of thought process" (2:478). The subverbal and intuitive stage is a product of thought, but it is an intermediate stage when the insight or meaning is not clearly and precisely defined. The process of verbalization refines this product and thus gives a greater power for transfer. A still later stage is the representational process of naming these verbalized meanings and making them still more manipulable for purposes of thought (2:478). An example of this could be that after one can name the properties of associativity, unity, inverse, etc., he gives the name of group to the structure that satisfies them.

Although there is a semi-abstract stage of understanding at which a non-verbal awareness contains the whole essence of an emerging idea for which language merely gives a convenient handle, this stage is short for both children working with less difficult abstractions and older learners in less familiar subject-matter areas. But when learners attain sophistication, they by-pass this semi-abstract stage completely; meaning and verbalization are now coincident. Ausubel concludes that secondary or college students well grounded in mathematics could be taught the subject meaningfully and most efficiently through verbal exposition, supplemented by appropriate problem-solving experience; for them use of the discovery method is time-consuming and of no advantage. But a truly meaningful learning must occur; it is not enough to be content with rote, verbatim learning or an ambiguous or marginally compe-

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tant verbally expressed idea (2:480).

Concepts Ausubel is concerned with concept learning and the relation of concepts to reality. Man lives in a world of concepts, not in a world of objects, events, and situations. He experiences physical properties of his environment or his own sensible responses only indirectly; both are transmitted through a conceptual or categorical filter. When he hears or reads a word or group of words, the cognitive content called forth is highly simplified, compared to the realities in the physical world to which they refer or compared to the experiences he has had. When one sees that proverbial long-eared, brown, scampering dog and says that he sees a "dog," he is communicating not what he actually experiences, but only an interpretation that reflects the accepted identifying attributes of what we all call "dog" (2:505). The man's actual experience includes all those elements of size, color, etc., which he could never fully express. Even his own conscious experience is not a complete sensory representation of reality but only a cognitive content molded by the existing concept of "dog" he already possesses. Thus the concepts a man has achieved serve to filter his experience, so that his conscious experience of reality is a simplified and generalized representation of reality, not a complete sensory catalogue of it. His conscious experience remains more detailed and particular than the standardized meanings that the culture shares--the dictionary definition of "dog" --which is to be assumed in communicating or acquiring new knowledge.

These standardized and shared concepts are what makes language and communication possible, by making available relatively uniform meanings for all members of a culture. They give a structuring of knowledge by arranging subsuming and subsumed concepts and by combinations of

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concepts in propositions; these ideas may be learned as part of a meaningful cognitive structure. This formation of conceptual equivalence relations simplifies and standardizes the environment by grouping related "items of experience" into categories; reception learning, problem solving, and communication are all facilitated (2:506). As concepts become more abstract and freed from representational images, they may be used more to generate relations among the concepts than to make statements discovered directly from the reality to which they refer. This abstraction allows manipulation, interrelation, and reorganization of ideas by means of the concepts themselves, the generation and testing of hypotheses, and meaningful problem-solving.

This is not to say that this world processed by concepts is not at all related to the real world, says Ausubel; rather, conceptual reality may be characterized as a "selectively schematic version" of the real world. The similarity of meanings and syntax in different languages shows the strong influence exerted "by similar physical, functional, and relational properties of objects and events in the real world and by the inherent logic of classification" (2:507). Therefore, conceptual reality is not an entirely arbitrary representation of the physical world. The adequacy or "veridicality" of a concept, the extent to which it selects and identifies significant elements of experience with objective reality, is important in generating concepts. Concepts may be formulated using more or less central or characteristic criterial attributes; the more adequate the insight into criterial attributes and the formulation of these, the greater will be the usefulness of the concept in structuring knowledge, learning, communicating, and solving problems.

Since concepts are abstractions, they represent one way to select

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criterial attributes or to define a class, and there may be others; concepts as such do not exist in the physical world. But concepts do achieve some sort of reality, in that they can be acquired, "perceived," manipulated as if they were independently existent (2:508).

Thus the relation between concepts and physical reality is two-fold; reality is a determining factor in the formation of concepts, and concepts are the main tools by which man experiences reality.

Concepts are acquired in two main ways (2:509-11). Concept formation is the characteristic way in which the preschool child inductively and spontaneously acquires generic ideas from concrete-empirical experience. The child inductively identifies the criterial attributes of a class of stimuli; he learns the concept "dog" by meeting many different sizes, shapes, and colors of both dogs and other animals, saying "dog," and being told "Yes, dog" or "No, not dog." Concept formation is made easier by the basic insight into the notion of categorization, which is similar and related to the basic insight that everything has a name (2:518).

In older children, adolescents, and adults, the characteristic type of concept acquisition is the process of concept assimilation, in which the learner begins with the criterial attributes of a new concept and relates them to the relevant established ideas in cognitive structure. Since definitions and context are presented and not discovered here, this is a form of meaningful reception learning; it involves active cognitive processes relating the new concept to the established cognitive structure, so it is neither passive nor simply a perceptual phenomenon. In fact, if the meaning of the new word is not immediately evident in its context, then processes of

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abstracting, differentiating, generating and testing hypotheses, and generalizing may become involved, bringing the process close to concept formation.

Meaningful Learning Set The acquisition of meanings is an active process, whether it is receptive or consists of independent data-gathering and interpretation. So the student must intend and want to take part in this activity. An unmotivated student could go through discovery procedures, gathering and interpreting data, with no more personal participation than by one who receives expository instruction. But a motivated student will consider, rework and reword, and integrate new propositions into his cognitive structure no matter how he obtains them. This reworking is not simply a cataloguing of formed concepts within the existing structure; it involves a judgment of relevance to determine which of the ideas in the existing structure are most relatable to a new learning task; it requires some differentiation and reconciliation between similar new and established ideas; the new propositions usually require reformulation to be brought into the personal frame of reference of the learner, which depends on his background of experience, vocabulary, and structure of ideas. Sometimes the activity must be carried further; if the new idea has no basis for being reconciled with previous ideas, the learner might be inspired to attempt a more radical synthesis or reorganization of his knowledge around different explanatory principles, either his own or from an alternative or more sophisticated exposition of the field.

Except for the last of the above, all of this need not occur in problem-solving or discovery situations. The basic activities there are all ways to assimilate new meanings and integrate them

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into cognitive structure. Problem solving or discovery is of a qualitatively different order, for it requires the further activity of autonomously reorganizing and restructuring new information and ideas already in cognitive structure to fit the given problem.

So the activity involved in meaningful reception learning can vary, depending on the need of the learner to integrate the meaning fully and on the extent of his self-critical faculty. He can relate his new knowledge to only a part of what he already knows or integrate it with all his previous knowledge. He may push to reformulate what he has received in his own personal language or be content with the original way it was presented. He may strive for precise formulation or be content with vague, more intuitive notions (2:88).

But that last series of choices indicates a real danger; although the student may not set out to learn rotely and he may not memorize, he could go wrong by thinking he has really reached the meaning when all he has is a vague and confused set of empty verbalisms. This can be due to a lack of self-critical ability and a reluctance to perform all the activity that this learning requires.

> He finds it easy enough to manipulate words glibly so as to create a spurious impression of knowledgeability, and thereby to delude himself and others into thinking that he truly understands when he really does not (2:89).

Therefore teaching must foster the activity of true meaningful reception learning, including a truly independent and critical approach to understanding; it can do this by motivating self-critical attitudes and fostering precise meanings, by presenting first the central unifying ideas of a discipline before more peripheral concepts and information, by emphasizing accurate definition, and by demanding personal reformulation.

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Advance Organizers Advance organizers are introductory materials that are very general and inclusive, presented in the learning situation before the specific learning material that is the goal of instruction (2:132).

> These organizers are introduced in advance of the learning material itself and are also presented at a higher level of abstraction, generality, and inclusiveness; and since the substantive content of a given organizer or series of organizers is selected on the basis of their appropriateness for explaining, integrating, and interrelating the material they precede, this strategy simultaneously satisfies the substantive as well as the programming criteria specified above for enhancing the organizational strength of cognitive structure (2:148).

The organizer bridges the gap between what the learner already knows and what he needs to know in order to learn the next lesson. It gives the ideational scaffolding for learning and remembering the more detailed material that follows it.

These externally supplied organizers aid in meaningful learning in three ways. They refer to and activate the potentially relevant anchoring concepts already established in the learner's cognitive structure, helping the learner use his extant knowledge most efficiently. Secondly, since the organizers provide specifically relevant ideas that include later concepts or propositions as more particular cases, the learning and memory of the later material will be eased. Finally the use of organizers can eliminate much of the rote memorization of details and information in a new discipline that must occur without any guiding generalities (2:137f).

An organizer should be constructed and provided for each unit of new material, says Ausubel. This gives the learner an overview of what is to be learned and an idea in his cognitive structure relevant

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to the particular content to be learned. As far as possible, the organizers should be the more general and inclusive ideas of the discipline itself (2:149).

Ausubel admits that the organizers are more helpful in the learning of factual material such as history than of abstract material, since abstractions "contain their own built-in organizers." He recommends organizers only for material that includes a substantial amount of differentiated or factual content. The value of organizers really depends on how organized the material itself is; if it is well organized and moves of itself from areas of lesser to greater differentiation, then the material already includes its own organizers and no further need be provided. "To be useful, however, organizers themselves must obviously be learnable and must be stated in familiar terms" (2:149). The actual teaching of the organizers depends on the developmental level of the learner, but can include appropriate analogies and illustrations when needed.

The ideal order of presentation of a discipline is to begin with the most general and inclusive ideas and to move through progressively more differentiated ideas.

> This order of presentation presumably corresponds to the natural sequence of acquiring cognitive awareness and sophistication when human beings are spontaneously exposed either to an entirely unfamiliar field of knowledge or to an unfamiliar branch of a familiar body of knowledge. It also corresponds to the postulated way in which this knowledge is represented, organized, and stored in the human cognitive system (2:152).

This recommendation is based on two principles stated earlier. First, a man can more easily grasp the particulars of a previously-learned and more inclusive whole than he can formulate the more inclusive

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whole from the previously-learned particulars. Secondly, the organization of a discipline in the individual's mind conforms to this hierarchical ordering.

This is not the usual order of textbooks or teaching procedures, complains Ausubel. Instead, textbooks usually separate homogeneous materials into chapters and subchapters on the basis of topical relatedness while ignoring their relative level of abstraction, generality, and inclusiveness. Details are the first material encountered in new and unfamiliar disciplines, before students have an adequate body of organizing ideas available. So students are required to treat material that is potentially meaningful as rote, because they lack the necessary elements in cognitive structure to relate to the new material.

> The teaching of mathematics and science, for example, still relies heavily on rote learning of formulas and procedural steps, on rote recognition of stereotyped "type problems," and on mechanical manipulation of symbols. In the absence of clear and stable ideas which can serve as anchoring points and organizing foci for the incorporation of new logically meaningful material, students are trapped in a morass of confusion and have little choice but rotely to memorize learning tasks for examination purposes (2:153).

Once the more general idea has been presented as an organizer, the more particular ideas should be related back, showing how they exemplify it. This may cause some redundancy, but it provides reinforcement of the general principles.

Comment

1. Ausubel's system is evidently more consonant with the cognitional theory of Lonergan than was Gagné's. This is primarily because Ausubel admits meaning; his is a theory of meaningful learning. Gagné was not willing to postulate anything about the processes inside

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the learner besides a minimal mechanism; for him, the learner is some sort of black box and only behavior can be considered, so Gagné says little about what happens inside that black box. Ausubel, on the other hand, finds it necessary to attribute to human knowing the ability to deal with meaning. Ausubel contrasts his own position with Gagné's on knowledge: whereas Gagné views knowledge as a capability, the capability of performing kinds of problem-solving tasks, Ausubel views knowledge as an achievement, a substantive, ideational phenomenon (2:130). Obviously, this latter is more in keeping with Lonergan's theory.

2. While the center of agreement between Lonergan and Ausubel is the focus on meaning, the core of their difference is that Ausubel seems to have no room for truth; he acknowledges the act of understanding but not the act of judgment. This affects many things, including the structure and nature of knowledge itself. Ausubel acknowledges the relationship of concepts to the physical world, but his notion of "reality" is a naive identification with physical existents, of which sense perception is the chief contact and seemingly the only standard. It is hard to understand how he could ever consider whether propositions about "reality" were true or not.

The absence of judgment seems to surface primarily in terms of Ausubel's chief educational tool, advance organizers. He wishes to foster a self-critical attitude in challenging implicit and unwarranted assumptions, but he asks the student to accept initially the most general and most questionable statements, the advance organizers. The organizers are given to students first, prior to the more particular details that are really the foundation or justification for stating the more general propositions. This order of presentation leads to the

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problem that a science or history or anything else will be learned as a doctrine and not as a science, not as an achievement of men but only as something pre-known, pre-structured. This creates a discontinuity between school learning and any learning that a man achieves otherwise.

It might be debatable what really occurs in the process Ausubel describes; is the more general, more inclusive, more abstract idea really learned first--learned meaningfully, with understanding? Or is it actually learned through the entire process of first presenting that generalization, showing how it is true in the more specific cases, so that it is fully understood and judged correct only through and after the entire process? If it is the latter, it seems inadequate to say that the more abstract statement was learned first.

3. There is some problem with Ausubel's notion of meaning. For him, a proposition is not primarily meaningful in itself, but only through relation to some structure of propositions, whether that structure is a formal one in a formulated body of a subject matter or the personal one that a learner has built up. For Lonergan, meaning is the content of some act of the human knower; a proposition has meaning when it is understood and/or affirmed by someone: that understanding or judgment constitutes the meaning. We can use the classical paradox, the blackboard with the sole sentence written on it: "The only statement on this blackboard is false." Is that statement true or false? What is the meaning of it? The point is that what is on the blackboard is only chalk; it has no meaning of itself; only as considered as content of human understanding is there meaning involved. While Lonergan's and Ausubel's notions of meaning might amount to the same thing when the conceptual structure is complete, while it is in process there is a dif-

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ference.

Further, meaning is not transparent. If words are not understood, then meaning is not there. If meaning needs to be dug out, can be misinterpreted, then it is not so simple. The teacher cannot presume that this communication of meaning will be obvious, even if the idea is present in the student and relevant anchoring ideas are available. The burden on the student is always great; his activity is always vital; he remains close to the problem-solving mode, because he always needs to sort out what really is to be learned and what is really relevant and what is the meaning of the teacher's statement. Even first learning of concepts--concept formation--is an active search for understanding, an insight into the similarity of objects. The reception of new words and concepts and propositions requires insight into the meaning communicated, an insight that can never be completely prepared for. The insight may not be as major as the discovery of electricity or the development of a new theorem in mathematics, but it is still an insight into the meaning from the data of the patterns of the printed words or heard sounds. The understanding that the student reaches is then usually tested; he is required to work some exercises or express this understanding and thus he verifies that his understanding was the one intended to be communicated. Therefore meaning is an achievement.

4. Both authors examined so far treat the structure of knowledge. Gagné presents a structure of skills. Ausubel presents a structure of propositions and concepts. Lonergan presents a structure of cognitional operations, and from them he develops two structures of knowledge.

According to Lonergan, there are actually two structures

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involved in a discipline, the order of discovery and the order of explanation (42:142-44). The first is the structure involved, for example, in coming to know mathematics; it follows the order of development of the material. The second is a deductive order, the order of most efficient presentation of material already grasped, ordered according to the structure within the materials.

Mathematicians working on the edge of their developing field, extending their knowledge, operate within the first structure, examining the particular and the odd individual result and developing a more general statement and trying to see if that statement is adequate and if it can be extended further. Pascal and Fermat worked that way in developing a mathematical theory of probability; they began with odd results and puzzling problems; they noticed odd rules about theoretical probability that they could develop; and finally mathematicians worked until they developed a definition of a probability measure (Kasner and Newman 31:240f). The second order is followed in a theoretical treatise on probability, which begins with the axioms of a probability space, proves the intermediate theorems from the axioms, and explains puzzling problems in terms of this developed theory. The orders are reversed; the order of inquiry begins with the most immediate to the knower and works toward what is fundamental to explanation; the order of presentation (explanation, deduction) begins with first principles and works from there to result, finally, in an explanation of the problems that were the first to raise questions. Textbooks usually follow the latter path; the chemistry textbook does not start with all the difficulties and the several million compounds that are at hand; to explain all these the chemistry textbook begins with the 100 or so elements and the possi-

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ble ways of combining them and then goes on to discuss properties of compounds and binding forces. A modern algebra textbook usually starts with careful definition, because the author and other mathematicians have already isolated conditions that will avoid problems in later development; the text moves from a set of axioms or premises that were formulated long after the initial results in algebra were found.

These two orderings or structures are present in any body of knowledge. In historical development, first comes the discovery phase, beginning from the particular and gradually reaching the basis for explanatory system; then comes the formulation of the subject as a deductive system.

In the learning of a subject, especially in the mathematics program which occupies so many years of a student's life, both of these orders become involved. The whole of mathematics need not, indeed cannot, be rediscovered by each new student. Therefore the explanatory structure must dictate what happens in the mathematics program, at least as a goal. But the student comes to the work naive, with no background in the experiences that are so nicely ordered by explanatory system. By the combination of the two orders, the student moves in the order of development but with his path smoothed by instructors guided by the order of explanation. The sequence of textbooks and courses thus follows a somewhat erratic path in relation to these two structures of mathematics. Yet somehow this erratic path is very much in keeping with the nature of mathematics; for the sequence of topics covered from year to year corresponds to the order of discovery, in the fact that new realms of mathematics are gradually brought under system as the student advances, corresponding to the sequence of higher viewpoints that com-

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prise mathematics. The ordering principles to organize the material at the lower level are the insights that constitute the next higher level, and that higher level operates in a sequence that provides an explanatory presentation of the lower level.

Many of the new programs in the schools intend to present mathematics as a deductive science. However, this is to risk losing contact with what is actually understood in the mathematical system. The structure of mathematics must be respected, but that structure should be the twofold structure of discovery and explanation.

We can now contrast Lonergan's idea of the structure of knowledge with that of Ausubel. Lonergan's orders of discovery and explanation are not quite the same as Ausubel's logical and psychological orders. The orders Lonergan proposes have a temporal connotation. The order of discovery is the order of developing knowledge, from the data to understanding and to higher and higher viewpoints, the succession of understandings by which the field was developed, until finally the highest, most general and most abstract propositions are reached. The order of explanation proceeds in the opposite direction, in an order of deduction, exhibiting the particular as a case of the general; this development is subsequent to and dependent upon the development of the order of discovery. In mathematics, Euclidean geometry was discovered, then the non-Euclidean geometries, and finally the principle that constitutes geometry: the preservation of structures under families of transformations. Consequently, geometry could be explained by specifying various families of transformations and studying properties preserved under each. This shows the two orders. But it does not seem possible to use either of them purely as the structure for education. The first takes

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too long, to build up the whole achievement of man in any one man's lifetime. Neither is it possible to start the beginning geometer with the transformation schemes, because the real meaning of the definition of geometry as related to transformations is so tightly bound up with the abstract formulation, and so dependent on experience in the practice of geometry, that it would be hard for the novice to grasp the definition or any of the resultant geometry as an understanding of anything.

The beginning student of chemistry will not begin with the most general and most adequate presentation of that field; before he can do that he must make some beginning. He might start with something a little more general, B (Bohr's model of the atom), and see how that can be used to explain the more familiar, A (the formation of compounds). However, that would be in order to understand B, where B is already the expression of an understanding of A. The next step would be to move on to something a little more general, C (quantum levels), which includes B as a partial realization, all as an understanding of the original data A. This is how learning seems to take place; this is how mathematics can be taught, even though we use Ausubel's notion of advance organizers. Ausubel mentions a spiral development, returning to a subject again and again, each time with a little more sophisticated approach. From Lonergan's analysis we can see that the purpose of spiral development is to teach abstract notions by recalling the more particular ideas which had been treated the previous time the topic was covered. The more particular now serves as data to found the more general understanding.

The basic problem with Ausubel's position is that advance organizers are each higher abstractions. But an abstraction is an under-

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standing of a given. The strategy proposed by Ausubel presents abstractions first, before the data. How is it possible for the student really to learn the general first?

In all of this, the cognitional structure must be operating, so that school learning is not something completely apart from learning in other situations when abstraction and generalization is not already done for the student. The contact of knowledge with experience, personal understanding, and judgment must not be severed.

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Bruner

His Theory

Jerome Bruner has been an influential figure in the recent curriculum reform movements both through his participation and through the psychological basis or apologia he has provided.

Categorization Bruner's early work on the concept is a key to his later, fuller development. To categorize, says Bruner, is to render discriminably different things equivalent by grouping the objects and events and people around us into classes and responding to them in terms of their class membership rather than their uniqueness (11:1). A concept is the category that we use, or invent, for their grouping.

Categorization can be found on two levels, says Bruner; at the perceptual level it "consists of the process of identification, literally an act of placing a stimulus input by virtue of its defining attributes into a certain class" (11:9). On the "conceptual" level, categorization "also involves the fit of a set of instances to the specifications of a category" (11:9). There are also two kinds of categorization: the first infers identity of objects, the other equivalence. We recognize a house either as Jim's house seen once again or as a house or another instance of our category house.

On the nature of categories and how they relate to knowable reality, Bruner is an explicit nominalist. He says that science's history is marked by naive realism, that for Newton it was a voyage in an uncharted sea to discover the islands of truth that already exist in nature. But the actual situation, says Bruner, is the

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opposite: "Science and common-sense inquiry alike do not discover the ways in which events are grouped in the world; they invent ways of grouping" (11:7). And so the test of hypotheses or inventions is the predictive benefits that arise from following these hypotheses. But do categories exist; do categories such as tomatoes, lions, snobs, atoms, and mammalia exist? Bruner answers: "In so far as they have been invented and found applicable to instances of nature, they do. They exist as inventions, not as discoveries" (11:7).

Categorizing achieves several things. First, it enables us to anticipate fuller results, to "go beyond the information given." We can learn to use certain attributes as reliable indicators of full membership in a category, and so we can alter our definition of conceptual category so that we can determine membership by a more perceptual cue. Also, it enables the knower to reduce the complexity of his environment. Third, it provides the means by which we identify the objects of the world, so that we need not travel through a world filled with unknown objects. Fourth, categorizing reduces the necessity of constant learning; it immediately gives much information about a newly-met member of a known category. Next, it provides direction for instrumental activity; that is, it gives clues for further activity with respect to a newly-met or not quite familiar object. Finally, categorizing allows us to order and relate entire classes of events, when intellectual activity begins to operate with systems of categories, so eventually we give meaning to our environment not only by relating individual things or events among themselves but by relating the categories that include them (11:12f).

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Bruner discusses the process of concept attainment--the act of grasping a conceptual distinction--in terms that this thesis has called insight:

> It is, if you will, an enigmatic process and often a sudden process. The psychologist's "aha experience" singles out this suddenness as does the literary man's "shock of recognition." Something happens quickly and one thinks one has found something. Concept attainment seems almost an intrinsically unanalyzable process from an experimental point of view: "Now I understand the distinction, before there was nothing, and in between was only a moment of illumination" (11:50).

The remainder of that first work is dedicated to answering the following experimental questions about that experience of concept attainment.

> How do people achieve the information necessary for isolating and learning a concept? How do they retain the information gained from encounters with possibly relevant events so that it may be useful later? How is retained information transformed so that it may be rendered useful for testing an hypothesis still unborn at the moment of first encountering new information? (11:51)

These questions have to do with elements of the process of knowing that surround the central insight experience.

Validation This early work also dealt with the process of validation. There are, Bruner says, four general procedures people use to assure themselves that their categorizations are "valid": tests by consistency, by consensus, by affective congruence, and by recourse to an ultimate criterion. Recourse to an ultimate criterion is a determination whether the object possesses the defining characteristic of the category: to distinguish between mushrooms that are safe and toadstools that will make you sick, the final test is to eat the questionable plant; if it makes you sick, it was a toadstool. The test by affective congruence refers to an act of categorizing that is

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accompanied by a feeling of subjective certainty or even necessity. The nature of the other two tests is obvious (11:19f).

These are ways to verification. When is that reached? When does a categorization become certain? When and how does a person know he has learned a concept in a way that will be usable? "A sufficiently high degree of certainty" in predicting whether something new is in the category or not might be the key, but individual standards of that sufficiently high degree vary, from one person to another. Bruner concludes:

> In fact it is very difficult to describe what it is that leads a subject to state that he has now learned the concept. For simplicity's sake, it is often better to by-pass the question and to ask instead whether the attributes that are criterial for the subject in his categorizing judgment are also the attributes that are defining of the concept (11:53).

Transfer In later work, Bruner extended and applied this study of categorization and concepts. For example, a long-continuing argument among cognitional theorists and other students of education concerns the validity and extent of transfer of training. Bruner treats this in terms of the ability of the one who possesses a concept to go beyond the information given.

> Nothing is being transferred, really. The organism is learning codes that have narrower or wider applicability... We propose that when one goes beyond the information given, one does so by virtue of being able to place the present given in more generic coding system and then one essentially "reads off" from the coding system additional information either on the basis of learned contingent probabilities or learned principles of relating material. Much of what has been called transfer of training can be fruitfully considered a case of applying learned coding systems to new events (6:47,49).

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Curriculum Reform Bruner, moving farther from his laboratory work, has been involved in extensive planning for education in cooperation with leaders in various subject matter fields and in curriculum reform movements. In his report as chairman of the Woods Hole Conference, he stressed four themes emerging from that conference, concerning the curriculum of schools (8). First, the structure of the subject matter field should be emphasized. Second, schools may have postponed teaching certain subjects through a false idea of readiness for learning, whereas the form of expression of these subjects might actually be the factor that hinders learning. Third, various kinds of thinking, especially the intuitive, should be fostered in school learning. Fourth, means should be found to stimulate the desire to learn. Several of these themes need further specification.

The conference urged that in the teaching of subjects like mathematics the structure of the subject should be taught. This approach makes the subject itself more comprehensible (8:23); it makes the material more rememberable; "unless detail is placed into a structured pattern, it is rapidly forgotten" (8:24). Also, teaching structure and hence the most fundamental ideas of a subject is the main way to achieve transfer. Finally, continual focus on the structure and primary principles of a subject in the material at each level of education will make various levels of sophistication, from elementary to advanced, more similar and more continuous. All this argues that "the curriculum of a subject should be determined by the most fundamental understanding that can be achieved of the underlying principles that give structure to that subject (8:31).

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Bruner contrasts intuitive thinking with analytic thinking, and urges the development of intuitive thinking, difficult as that may be, along with analytic thinking. Although "intuition" can have two different meanings, says Bruner, intuition in both senses is important and should be sought in education. One meaning makes intuition like the more radical experiences of insight: "an individual is said to think intuitively when, having worked for a long time on a problem, he rather suddenly achieves the solution, one for which he has yet to provide a formal proof" (8:55f). The other use refers to the ability to make good, quick guesses whether something is so, or which possible approach to a problem might prove most fruitful; a mathematician with good intuition will make a guess whether it is easier to set up a volume-finding integral by building up concentric cylindrical shells or by stacking up circular slabs. Intuition is not just an ability to go from nothing to something; the effectiveness of intuition is based upon a sure knowledge of the subject which provides the material for intuition.

The contrast with analytic thinking is one of great relevance in this thesis. Analytic thinking, says Bruner, characteristically proceeds a step at a time, in explicit and reportable steps. The analytic thinker is relatively aware of the information and operations he uses, often involving deductive reasoning and an explicit plan of attack, or through a methodical process of induction and experiment guided by principles of research design and statistical analysis (8:57f).

In contrast to this, intuitive thinking usually advances not in ordered steps but through operations that seem to rely on an "implicit perception of the total problem" (8:58); the process used to reach an

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answer is often not clear even to the thinker, and the aspects of the problem that contributed to the answer are also not very specifiable.

> Usually intuitive thinking rests on familiarity with the domain of knowledge involved and with its structure, which makes it possible for the thinker to leap about, skipping steps and employing short cuts in a manner that requires a later rechecking of conclusions by more analytic means, whether deductive or inductive (8:58).

Bruner stresses the complementarity of these two styles of thinking:

> Through intuitive thinking the individual may often arrive at solutions to problems which he would not achieve at all, or at best more slowly, through analytic thinking. Once achieved by intuitive methods, they should if possible be checked by analytic methods, while at the same time being respected as worthy hypotheses for such checking. Indeed, the intuitive thinker may even invent or discover problems that the analyst would not. But it may be the analyst who gives these problems the proper formalism (8:58).

Bruner developed this contrast between the two modes of thinking in order to point out the problem that school learning up to that writing had emphasized formalism and devalued intuition. This defect is probably due primarily to an uncertainty as to the nature of intuitive thinking and consequently the means to develop it, compared to the clarity of the analytic style and the consequent ease in teaching it. But it is much easier to lose sight of the nature of the subject matter in the midst of the analytic or formal processes.

Bruner refers a number of times to what I have called here the desire to know but which may be called a motive to learn or an urge to formalization. "What impels one to formalization we cannot say. That the urge is strong is unquestionable" (11:6). Bruner and the others at the Woods Hold Conference recognized that a primary means of education is to capitalize upon and intensify this drive to know, to develop in the learner an interest in what he is learning along with

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positive attitudes toward intellectual activity in general (8:72). Tentative recommendations of means to do this included increasing the inherent interest of materials taught, developing in the student a sense of discovery, and translating the material into thought forms appropriate to the learner. In Lonergan's development, this desire is there as a basic component of man's intellect, his capacity to know; the task of education with respect to that desire is to direct it to school subjects--to show that they are actually responses to that desire for knowledge.

Theory of Instruction Bruner has carried his findings and method even more directly into education by formulating the beginnings of a theory of instruction; with the aid of Zoltan Dienes and others he has attempted to illustrate some theorems from this theory through examples in mathematics education (9).

A theory of instruction, says Bruner, should specify four main features of the educational process:

1. The experiences which will most successfully foster in the individual the desire to learn;
2. The ways in which a body of material should be structured so that the learner can most easily learn;
3. The most effective order of presentation of the material;
4. The pacing and kind of rewards and punishments in the learning process (9:307f).

So Bruner takes up the topic of structure that we have examined before. The possible structures of a body of knowledge may be characterized in three different ways, each affecting the ability of the

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learner to master it. First, the mode of representation can vary; the material can be given an enactive representation--a specification of actions appropriate for achieving the particular results, or an ikonic representation--a set of images that represent a concept, without complete definition, or a symbolic representation--"a set of symbolic or logical propositions drawn from a symbolic system that is governed by rules or laws for forming and transforming propositions" (9:310). These various modes of representation can be more or less appropriate to given learning situations, depending on the age and background of the learner and the nature of the material. Thus mathematics is adaptable to alternative modes of representation.

Second, the economy in the representation or structure of a domain of knowledge has to do with the amount of information that must be mastered and retained to allow comprehension. Economy varies with the mode of representation, but it also depends upon the sequence in which the material is presented.

The third standard of structures is the effective power of the particular structure; a structure is better according to this standard if the set of learned propositions can generate more conclusions and more correct ones. This power may be described as the capacity of the representation to allow the learner to connect matters that are apparently quite separate (9:312f).

Sequencing of learning materials, which seems really to be the order of presentation of the representations of the subject matter already structured according to the standards above, depends on several factors. While the normal sequence would be to move progressively through the enactive, ikonic, and symbolic forms of representation of

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the matter, the first two might be omitted if the learner has already developed a strong ability to operate at the formal level. Successful sequencing avoids overloading the capacities of the individual to process information, says Bruner; a sequence that begins with economical use of information would be best, and thus a simple model learned first will aid in reducing the complexity of later materials. But a sequence can be judged good or bad only in terms of the criteria used to evaluate the final learning.

> A classification of such criteria will include at least the following: (a) speed of learning, (b) resistance to forgetting, (c) transferability of what has been learned to new instances, (d) form of representation in terms of which what has been learned in terms of its generativeness for new hypotheses and combinations (9:314).

Since achievement on all these scales cannot be reached through the same means--in fact, focus on achievement in speed, for instance, may minimize transfer--a sequence must be arranged to meet the particular goals chosen for the learning.

Abstraction and Images Bruner's work with Dienes in classrooms was for students at a developmental level outside the scope of the applications of Chapter III, but one series of comments is of special interest.

> What was so striking in the performance of the children was their initial inability to represent things to themselves in a way that transcended immediate perceptual grasp. The achievement of more comprehensive insight requires, we think, the building of a mediating representational structure that transcends such immediate imagery, that renders a sequence of acts and images unitary and simultaneous. . . .
>
> We have already commented upon the fact that by giving the child multiple embodiments of the same general idea expressed in a common notation we lead him to "empty" the concept of specific sensory properties until he is able to grasp its abstract properties. But surely this is not the best way of describing the child's increasing development of insight. The growth of such abstractions is important. But what struck us about the children as we observed them is that they had not

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> only understood the abstractions they had learned but also had a store of concrete images that served to exemplify the abstractions. . . .
>
> We reached the tentative conclusion that is probably necessary for a child, learning mathematics, to have not only a firm sense of the abstraction underlying what he was working on but also a good stock of visual images for embodying them. For without the latter, it is difficult to track correspondences and to check what one is doing symbolically (9:328-29).

This comment corroborates Lonergan's position that abstraction is not simply the "emptying" of a concept, but the achievement of an understanding, so that all the images that are carried along by the child who has mastered the subject are images that are now understood through this abstraction. If the child has an abstraction that is not an understanding of anything, he may be able to parrot the words but he will not be able to use them intelligibly and in new situations. It also brings out the fact that some original images assume the role of paradigm or special symbol and so enable the learner to operate on new material by relating it to those special images. These images enable the student to track correspondences and check in actual examples the work that is being done symbolically, in the abstract (9:319). The symbol is an expression of what understanding has grasped in these images.

Bruner also deals with other aspects of what this thesis has called the necessity of images. One of the conditions of creativity, says Bruner, is the freedom to be dominated by the object. What happens in symbolism or creating an image is that we externalize the object, which is a product of our thoughts, and treat it as something "out there." He refers to Freud's comment on projection that humans seem better able to deal with stimuli that come from without than from within.

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> So it is with the externalizing of a creative work, permitting it to develop its own being, its own autonomy coming to serve it. It is as if it were easier to cope with there, as if this arrangement permitted the emergence of more unconscious impulse, more material not readily accessible (7:26).

Bruner also refers to the "economical significance of a good mode of representing things to oneself" in mathematics. He cites an example from group theory; it is difficult to determine the closure of a set of transformations, so the easiest thing to do is form the table of compositions on paper or a blackboard. This will exhibit the group structure as a whole and allow the student to investigate more than closure; he can find out what other properties hold. "Good representation, then, is a release from intellectual bondage" (7:26).

Role of Teacher Bruner emphasizes the role of the teacher. There are many devices to aid the teacher to extend the student's range of experience, to help him understand the underlying structure of the material, to dramatize the significance of the material, even to take some of the load of teaching from the teacher (8:84). But these are only aids; they cannot dictate their own purpose; the teacher remains the "principal aid in the teaching process as it is practiced in our schools" (8:88). The teacher is a communicator of knowledge and so must be able to respond to students' ways of knowing. The teacher is also a model of the product of learning. "Somebody who does not see anything beautiful or powerful about mathematics is not likely to ignite others with a sense of the intrinsic excitement of the subject" (8:90). And so the teacher is also "an immediately personal symbol of the educational process, a figure with whom students can identify and compare themselves" (8:90). Therefore the role of the teacher in education has much to do with stimulation and direction of the drive to know, as well as actual instruction.

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Comment

A number of elements in Bruner's theories are very similar to elements in Lonergan's, from insight and verification to the desire to know and the place of the teacher. One reason for this is that through his work, Bruner has moved farther from the psychological laboratory toward the classroom than the previous two authors. A number of points in particular deserve comment.

1. A key to any differences that might arise is in the basic assertion of the relation between the known and reality. Bruner uses as his key to the existence of categories the question: Do categories such as tomatoes exist? Bruner answers that they exist insofar as they have been invented and found applicable to instances of nature; they exist as inventions, not as discoveries. But there are further questions to clarify the meaning of that answer, and the answers to these questions are important. Do tomatoes exist? That is, do some things exist of which it can truly be said that they are tomatoes; do they have something in common with a number of other things also called tomatoes? Or do just a number of things exist, and what we might say about them is truly up to us, not dependent on what the things are themselves? Bruner's question will distinguish a nominalist from an idealist, but it takes the further questions to isolate Lonergan, a critical realist. The nominalist will say that categories are our own invention, done purely for the convenience of ordering all the array of sensations we receive. All definitions, then, are nominal, because no definition reaches what something is in itself. A realist, wherever he is on the spectrum of opinion, holds that there is something in reality that is intelligible and this is what he is approaching when he knows; there is something correspond-

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ing to his knowledge. This thesis has sided with Lonergan in saying that there is something to be known; judgment has the role of saying whether or not an understanding we have reached is really so. Through our knowing we approach being, being truly understood.

The distinction between the nominalist and the realist has a relevance to what takes place in the classroom and to the nature of mathematics: principally, mathematics is to be learned not as dogma, but as something true, something to be understood, that has some sort of "existence" prior to our understanding. It is not that numbers "exist," which might be the position that Bruner attributes to the realist; but certainly the operations in arithmetic performed by people exist--as actions--and we make statements about these operations that can be true or false, not because we have so structured our notion of true and false or because we so rearrange our operations to fit the statements, but because the statements are are or are not adequate to the reality of the operations.

The nominalist position of Bruner has fruit in his basic principles for education. First, it is a foundation for the position that the function of knowing and learning is to enable the individual to construct his own reality. Knowing, to Bruner, is principally a process, not a product, because the product has little validity in itself; it is a convenience, an hypothesis, a structure that the knower finds most useful. Following Bruner too closely on the nature of knowing would accentuate a problem already prevalent in the mathematics classroom. Students seem to feel that what they are trying to do or are required to do is to come up with some answers that will fit whatever pattern is in the mind of the teacher; any sort of discovery, to them, is the discov-

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ery of some sort of arbitrary notion in the teacher's head, rather than the discovery of what is inherent in the operation that they are discussing. The precise difficulty of algebra is that for many students it is a system of arbitrary rules, rather than a study of the regularities of arithmetic. Even when the nature of algebra is clarified, it is difficult enough. Without that clarification, the rules may be no harder or easier to master, but the nature of the subject that is being studied is certainly distorted. The nature of knowing also alters the sort of preceding experience that a teacher gives to the learner and the manner and order of presentation. The distinction in classroom practice is between "Guess what I'm thinking" and "Let me see if I can help you find out what is happening." The nature of mathematics, and certainly its relation to the physical sciences, become distorted in Bruner's view of the nature of knowing. The problem, in Bruner's view, is that we are trapped on our own side of the gulf between knowledge and reality, and can never reliably cross that gulf. Then mathematics, as the study of possible orderings of reality, is all that can ever be done, and its futility is pronounced.

2. Bruner recognizes insight and judgment, but the nature of this judgment seems to remain a mystery to him. It is a puzzling problem that an insight that turns out to be true and one that eventually turns out to be inadequate are phenomenologically the same. The point Lonergan makes is that there is no difference; insight itself cannot establish its own validity. The further step of judgment is required. The key to verification, says Bruner, is a sufficiently high degree of certainty; he is assisted by Lonergan in the definition of this sufficient degree: the virtually unconditioned. That is what Bruner

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requires in describing what it is that leads a subject to state that he has now learned the concept.

Bruner said it is usually better to avoid the question of verification entirely and to determine whether the attributes the individual uses to determine category membership are actually the characteristics that determine or define membership. But this simplification only lands us in further trouble: Defines or determines membership for whom? Concepts, from the nominalist stance, are nothing in themselves but only some convention established, individually or collectively, about the use of concept words. And it then becomes hard to align defining characteristics among a large number of people.

3. Bruner's contrast between the two modes of thinking is good; it has to do with the two structures of knowledge in Lonergan's theory; it also has much to do with the two activities of insight and judgment. Intuitive thinking is the process by which we can arrive at the ideas, which we then accept or reject through analytic processes. The reliance of the analytic upon the intuitive for the generation of ideas and of the intuitive upon the analytic for verification of these ideas seems to repeat what Lonergan says about the stages of knowing.

The slightly different meaning of mathematical induction might illustrate the parallel between intuitive/analytic thinking and insight/judgment. Mathematical induction is a neat, strictly formal process, to prove a formula holds for all natural numbers; it provides a form of argument:

$$P(1)\wedge(P(n)\rightarrow P(n+1)) \rightarrow (\forall n)\, P(n).$$

For example, we can argue that the sum of the first n natural numbers, for any natural number n, is $\frac{1}{2}n(n+1)$ by showing that $1=\frac{1}{2}\cdot 1(1+1)$ and

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$\frac{1}{2}n(n+1) + (n+1) = \frac{1}{2}(n+1)((n+1)+1)$. But nowhere in the strict procedure of mathematical induction is there a step to follow to obtain the general form $\frac{1}{2}n(n+1)$. Unless this formula is explicitly given, then what is required is a lot of intelligent guesswork--intuitive thinking--that finally yields an insight that this formula might be the right one. Then and only then formal, analytic thinking takes over to verify the adequacy of the formula.

These two types of thinking and the parallel steps in knowing are complementary, as Bruner argues. But we can carry that further. To omit either one is not only to deprive a student of some necessary exposure and some necessary skills; it is to distort the nature of knowing.

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Polya

His Theory

Problem Solving Polya's work has been extensively concerned with problem solving. He outlines the method of solution of a problem as a series of questions, not of answers. These questions are ordered to meet four goals. To solve a problem, the solver should:

1. Understand the problem. Drawing a diagram or translating the problem to suitable mathematical notation helps the student to understand what the problem really is.

2. Develop a plan to establish a connection between the known and the unknown. This might involve recall of previous problems that might be related or reformulation of the problem to allow this recall; it could involve reducing to definition some terms in the problem. All of this is an attempt to discover the place of the problem within the structure of understandings already reached.

3. Carry out the plan. This is a verification that this understanding is adequate to the problem. The previous step reached the insight into the place of the problem with respect to previous knowledge and a plan for proof of this; this step is to carry out the proof.

4. Look back over the solution. Polya says that for the problem-solving experience to be most fruitful it should not be terminated the moment the solution is reached. Instead, the most fruitful way to use problems is, after a solution has been reached, to move back into the pattern of questioning and to vary the conditions and to ask what new results will be possible; this way the learner can isolate relationships between conditions and results, move towards generalizations or

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recognition of specific cases as examples of the solution just reached, or examine how the method of solution could be generalized and to what it would apply (56:xvi-xvii).

Questions The questions which Polya provides to direct the attention of the problem solver through his inquiry are, he says, natural, simple, plain common sense, but phrased in general terms to meet a variety of situations. He makes them explicit because, though the inquirer might usually be guided by the same questions, he could be unconcerned or unable to express these guiding questions in explicit terms (56:3).

Polya assembled his list of questions (What is the unknown? What do you know? Do you know any similar problems? . . .) from the activity of the teacher in aiding or in "teaching" problem-solving. In that situation, the teacher's role is to aid the student unobtrusively and naturally by guiding his inquiry as indirectly as possible, through questions that really help the student to understand the problem and otherwise lead him into the problem (56:1f).

The questions the teacher asks should begin with the most general, and only gradually come down to more specific and concrete questions or suggestions, until the student has an idea--an insight. If the student needs more help to exploit this insight, questions must start again from the general. The role of the generality of the questioning is to develop the student's ability to solve problems, so that he learns to ask himself a series of questions that will be most applicable to the widest sort of problem situations (55:21). The role of the teacher, then, is to teach problem-solving, not solutions to problems. The series of questions makes explicit the activity of coming to know, and it is the chief thing to be learned, so that later the student can use his

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own knowledge of how he knows to help him in problem solving.

Experience Polya acknowledges the role of experience in mathematics. He uses a quotation of Euler to head the first chapter of the volume Induction and Analogy in Mathematics (54:3). In that passage Euler argues that observation or experience is a starting place even in "pure mathematics"; in number theory, for instance, properties of numbers are first discovered by observation and only much later confirmed by rigid demonstrations. What mathematicians learn by observations but have not yet confirmed by demonstration "must be carefully distinguished from the truth . . ." (54:3) for, until confirmed, it is only a hypothesis reached through induction. Polya cites the Goldbach conjecture that every positive even integer can be written as the sum of two primes as something most probable, arrived at by experience and observation, but which has not yet yielded to proof. This comment witnesses to the role of observation and experience in mathematics yet distinguishes what is reached by observation from true mathematical knowledge.

Polya, later, again acknowledges the role of experience and conjecture with respect to a problem he has developed on the number of regions of three-space separated by a given number of planes. Verification of this law in particular cases does not really help to establish the law in general, but it does lead to a conjecture of the answer in the general case and it can suggest a method of verification: "the careful observation of special cases that leads us to a general mathematical result may also suggest its proof" (54:51).

The role of this experience or observation is further spelled out in Polya's procedural steps in solving problems; he offers suggestions

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to stimulate insight that Lonergan would call disposing the image. Polya says that in solving a problem the investigator should start with the statement of the problem, attempt to visualize the problem as a whole and to understand the question and then to isolate the principal parts of the problem (hypothesis and conclusion; or unknown, data, and conditions). Especially, the student should seek a similarity between the problem and his store of already solved problems; the similarity should provoke an insight, which he should then follow and develop (56: 33f). The experience in mathematics in many cases may be this similarity in initial form to something done before; therefore the whole of the previously accomplished mathematics serves as the experiential ground for later work.

The recognition of similarity may be only: "There is a rectangle involved in this problem and I already have some knowledge of rectangles," without the insight how any of this previous knowledge might contribute to the solution of this problem. Then the imagination takes over, giving expression to the first hypotheses; in a geometric problem it may involve the introduction of a dotted line between two previously unconnected points. In an algebraic problem it may require setting up equations in terms of a quantity that was not even mentioned in the problem, or solving for something that was not initially specified as an object of interest. For instance, a maximization problem of a type in many texts for calculus for business applications is: A ski club can get a chartered airplane flight to Denver for $50 a person, as long as there are at least 70 passengers. For every 3 extra passengers, the fare will be reduced $2 per passenger, up to a total of 130 passengers. How many ski club members must make the trip to maximize the revenue of

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the airline? The simplest computations result from specifying the variable x= the number of 3-person increments in the trip load. This is a rather simple form of what Polya calls introducing auxiliary elements, but it does help solve the problem.

Understanding Polya recalls to us the suspicion we and other students of mathematics feel at a proof that seems to arise out of nowhere or at a line drawn without apparent motivation, but making the solution obvious, or at the strange-looking number multiplied and divided by the same expression. An example is the expression used to verify the chain rule for derivatives, where we equate

$$\frac{f(g(x+h)) - f(g(x))}{h} \quad \text{and} \quad \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h} .$$

This is a tricky step, and it works. But if the goal is to enable the student to do mathematics himself, and not merely to verify the results already attained, it seems that the instructor should take some time to develop the reason why that might be an interesting and good way to rewrite the first expression, before its actual use:

> Mathematics is interesting in so far as it occupies our reasoning and inventive powers. But there is nothing to learn about reasoning and invention if the motive and purpose of the most conspicuous step remain incomprehensible. To make such steps comprehensible by suitable remarks . . . or by carefully chosen questions and suggestions . . . takes a lot of time and effort; but it may be worthwhile (56:50).

Provocation of the necessary insight is the goal; there is nothing that can lead infallibly to that insight but there are means, such as questions and suggestions, that are frequently helpful and that is what How to Solve It is intended to provide. Repeated suggestions in that book are, for example, "Look at the unknown" and "Vary the problem" (56:53).

Polya acknowledges the role and nature of insight and the recog-

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nition of this common experience back even in a passage in Aristotle (56:58). What insight involves, how it is provoked, when we can say it has occurred are developed in Polya's discussion of the progress of an inquiry. Solution of a problem involves various activities: recollection of essential facts, formerly solved problems, definitions, and theorems; adaptation and combination of these facts for the problem at hand; conversion of the mode of conception of the problem or varying the problem to get a "view" of the problem from various standpoints. All of these aid our movement toward a solution; in fact they constitute that movement.

> As we progress toward our final goal we see more and more of it, and when we see it better we judge that we are nearer to it. As our examination of the problem advances, we foresee more and more clearly what should be done for the solution and how it should be done . . . . We do not foresee such things with certainty, only with a certain degree of plausibility (56:158).

All of these steps are an approach to the solution.

> What is progress toward the solution? Advancing mobilization and organization of our knowledge, evolution of our conception of the problem, increasing prevision of the steps which will constitute the final argument. We may advance steadily, by small imperceptible steps, but now and then we advance abruptly, by leaps and bounds. A sudden advance toward the solution is called a bright idea, a good idea, a happy thought, a brainwave (in German there is a more technical term, Einfall). What is a bright idea? An abrupt and momentous change of our outlook, a sudden reorganization of our mode of conceiving the problem, a just emerging confident prevision of the steps we have to take in order to attain the solution (56:159).

Polya also recommends guessing, but useful guessing. Guesses do not yield proven truth; only proof will do that. But guesses are useful if they are made after a serious study of a problem; they can often contain at least a fragment of the truth. An examination of these guesses may provide a way to reach the entire truth. The key point here is that the guesses are in the middle of the process; if they are examined cri-

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tically, they can lead to fuller understanding (56:99).

Verification After an insight has been attained, whether it be labeled a guess or an understanding, it is only a good idea; before it can be accepted as the solution to a problem or the explanation of a puzzling situation it must be tested; it must be judged true or false.

There are two kinds of reasoning or argument that can be involved in verification, says Polya: heuristic and formal. The heuristic sort is involved in the movement to understanding, using plausible and provisional arguments. These are the insights into the connection between parts of the problem that are made explicit in formal argument. The plausible argument is of service only in advancing the solution, in finding the unknown or in formulating the more formal proof. After that, the formal proof serves to support the judgment that the solution is correct or the theorem is true. Heuristic reasoning is valuable, as long as it is appreciated for what it is, only a device to aid understanding. The formal proof is the means of judgment. As long as these functions are remembered, as long as the likely argument is later replaced by the necessary argument, then they both serve their function. But the heuristic argument alone is like the bright idea; it may or may not hold. In classroom use, the heuristic argument is often a very necessary element of the presentation, helping the student first to reach an understanding, before that understanding is verified.

The formal method--which Polya calls the Euclidean way of exposition--is to start with the given and move toward the unknown or the conclusion, through correct steps, systematically, step by step. Any subproofs are mastered as they are met, and only finally, at the end of the entire process, is the conclusion reached (56:68f). This is a good

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method if the student already has a broad perspective and an understanding of how the proof will proceed in general. But if the general movement of the proof is unfamiliar, the argument often seems to reach its result by chance, like an idle whittler suddenly arriving at a complete sculpture. So Polya recommends:

> It is not so desirable that the teacher should present many proofs in the pure Euclidean manner, although the Euclidean presentation may be very useful after a discussion in which, as is recommended by the present book, the students guided by the teacher discover the main idea of the solution as independently as possible. Also desirable seems to be the manner adopted by some textbooks in which an intuitive sketch of the main idea is presented first and the details in the Euclidean order of exposition afterwards (56:71).

Heuristic reasoning is actually part of the method of arriving at understanding, assisting the learner to make a likely guess. Therefore it is nothing to be ashamed of; it can prepare for rigorous argument by indicating a total view of the proof.

Demonstrative reasoning is the means of securing our knowledge in mathematics and demonstrative logic. Outside these two fields, says Polya, all our knowledge consists of conjectures, which are supported by plausible reasoning:

> the inductive evidence of the physicist, the circumstantial evidence of the lawyer, the documentary evidence of the historian, and the statistical evidence of the economist belong to plausible reasoning (54:v).

Polya takes the position that demonstrative reasoning is safe, certain, but unable to yield essentially new knowledge about the world; plausible reasoning, on the other hand, is hazardous, controversial, provisional, and the only way of achieving new knowledge. While demonstrative reasoning obeys strict standards, plausible reasoning is fluid; there is no field of heuristic logic comparable to demonstrative logic in either clarity or agreement on its basis.

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As Aristotle noted that particular forms of argument occur repeatedly, isolated them, and thus proposed the syllogism as the basis for demonstrative reasoning, Polya has attempted to isolate the syllogisms that occur in heuristic or plausible reasoning. This systematized plausible reasoning is used to examine a guess or an understanding in order to determine its status, short of having a rigorous, demonstrative proof (55:113).

Teaching Finally, Polya offers some rules for teaching mathematics. His first rule is, know what you are supposed to teach. The second rule is, know a little more than what you are supposed to teach. The teacher who wishes to teach mathematics and impart a right attitude toward that subject should both be a master of the mathematics he wishes to teach and have a right attitude toward it (56:173). Principally, he is teaching the method of inquiry and solution at the heart of mathematics, so that the questions that guide him in his own searches should be the questions that he uses to direct his students' searches. Also, the knowledge the teacher must have is a thorough one, acquired not through mere memorization but by application to interesting problems (57:209).

Polya states that although demonstrative proof should have a role, heuristic proof or plausible reasoning, the art of guessing, should be a central part of mathematics teaching. It is more valuable than any mathematical fact for the student to learn, first, to distinguish a proof from a guess and, second, to distinguish a more reasonable guess from a less reasonable one. Polya gives as an example of the area in which plausible reasoning is more valuable than deductive, the teaching of calculus to engineers. It would be wrong on pragmatic grounds, he says, to teach calculus to engineering students rigorously, in terms

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of epsilon-delta proofs, because this rigor is certainly beyond their interest and sometimes beyond their ability. It would also be wrong to teach calculus as something arbitrarily imposed, a set of arbitrary and dogmatic rules. Plausible argument is the best method for this level of calculus, says Polya, argument for the likelihood and meaning of the calculus without the rigorous detail of demonstrative proof. If this is done explicitly and consciously, there need be no apology for fake proofs, and this approach will give some meaning to the operations of calculus, so that the content of the calculus course may endure past the final examination (56:219).

Although Polya has said it is desirable to teach guessing, he admits it is not easy: "There is no foolproof method for guessing, and therefore there cannot be a foolproof method to teach guessing" (55: 159). Still, he has given in his book Patterns of Plausible Reasoning (55) several models of guessing to serve as methods to follow in teaching guessing. The normal procedure is the "guess-and-prove" problem, in which mathematical facts are first guessed and then proved.

> If the learning of mathematics has anything to do with the discovery of mathematics, the student must be given some opportunity to do problems in which he first guesses and then proves some mathematical fact on an appropriate level (55:160).

Another type of guessing is involved in the "test-consequences" problem, the inductive development, in which a general statement is examined by testing its particular consequences. There are also "you-may-guess-wrong" problems; the students should be given the experience that even the best, most respectable, most reasonable guesses may be wrong. A final type Polya mentions is the "small-scale-theory" problem in which a relatively elementary problem is found whose solution might suggest a method of solution for a more general or more difficult problem. The

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elementary problems may be hard to find, because they must clearly involve the relevant features of plausible or inventive reasoning necessary for the larger-scale problem (55:161). These are various ways Polya suggests the teaching of mathematics proceed; they are intended to help the student become a mathematician by performing mathematical inquiry and mathematical knowing.

Comment

Polya's problem-solving orientation naturally aligns his position on knowing with that of Lonergan, because problem solving has been the traditional place to find the activity of insight. Therefore the similarities with Lonergan's position are many. Because Polya does not attempt to state a complete position on all of knowing, there are few points on which there are contradictions of Lonergan.

1. A correspondence of Polya's four goals in problem-solving with the knowing process outlined by Lonergan seems possible. Polya says the first step is to understand the problem, using diagrams or reduction to mathematical notation to achieve this; this is activity that Lonergan classifies on the level of experience, amassing the data of the problem, attaining an image through diagram or symbolism to provoke an insight.

Polya's second goal is to develop a plan to establish a connection between the known and the unknown; the plan is on the level of understanding, reached by insight into the problem, requiring all the wiles of intelligence. During this process, there are repeated small insights and judgments: This has to do with . . ., No, that does not help . . ., Well, it has to do with . . ., Yes, and since that is so it implies that . . . Gradually the complete plan for solution of the

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problem is reached. But it is only a plan.

Polya's third step is to carry out the plan; this establishes the ground for the judgment that the connection is there. The learner must actually carry out the proof, if that is what is required, or actually find the solution and verify that the answer is correct or appropriate.

Finally, Polya says to look back over the solution; this corresponds to making this particular achievement of knowledge only one part of the cumulative effort to know. Once something is known, the knower re-enters the knowing process, impelled by his drive to know more and more. Polya's final step is thus a call to use the solved problem as data to move to a higher viewpoint.

Therefore, Polya's instructions are a specification of the knowing process, as Lonergan has presented it, and a direction to act within the knowing process in order to reach knowledge.

2. Polya emphasizes the role of questioning throughout his work. In fact, the questioning gives structure to inquiry. Most of the questions he gives are What? How? Why? questions for understanding; since he writes for a mathematically-educated reader for whom the manner of answering Is-questions is the standardized form of mathematical proof, he does not emphasize that aspect, although the Is-question is present in the stage of carrying out the plan for the solution of the problem. But in total Polya helps to make clear the dynamism of human inquiry as it gives form to itself and constructs the act of knowing through its questioning.

3. Polya helps to make clear how previous learning serves as an experiential base for continuing inquiry. He says that a new problem must be investigated against the background of all previous knowledge,

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of all problems solved before and all definitions made. In similarities of wording, ground for an understanding of the new problem may be hinted at. Polya's work thus supplements Lonergan's insistence on experience as the basis of all knowing by specifying what experience is in mathematical inquiry.

4. Polya also insists that in mathematics instruction it is important to utilize and present both kinds of reasoning, heuristic and formal. This is another way to state a prominent point of instructional method drawn from Lonergan's cognitional theory: remember that learning takes place in stages, and it is important to respect the differences of the stages. There must be an understanding of theorems and definitions, so that the formal apparatus is a true judgment upon an understanding that has been reached.

5. Polya's description of progress toward a solution is a magnificent description of the knowing process. Polya asserts that this progress, whether by small and imperceptible steps or by sudden and dramatic leaps, is all of a piece; this supports Lonergan's insight model of knowing. So Polya's authority and study support some of the central features of the process of knowing outlined in this thesis.

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CHAPTER IV

CONCLUSION

Directions of Further Work

The material of this thesis opens up prospects for development in several areas.

First and most obvious, it demands an attempt to make these rather abstract considerations more concrete for mathematical education. The theories of Gagné, Ausubel, Bruner, and Polya are not merely descriptive of the learning process; they are prescriptive, in the sense that they give a basis on which to rely, at least partially, in developing curricula or classroom approaches. This thesis has shown to some extent how Lonergan's cognitional theory gives a broad and consistent understanding of the human knowing process and a basis for evaluating other, partial, theories of knowing. In that, it has been descriptive. What remains is the translation of the theory into practice: in the specification and ordering of topics in the mathematics curriculum, in the development of materials, and in classroom approach. Some brief examples of applications of this theory in the classroom, with discussion from the viewpoint of Lonergan's theory, appear in Appendix I. But that is only a hint of what could and should be done. The larger goal is to enable teachers to approach their teaching from an understanding of the knowing process of their students. Consequently, this thesis challenges teachers to awaken the drive to inquiry in their students and to guide and focus it on lines that will lead them to mathematical knowledge.

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This awakening of students' inquiry has been understood previously as a motivational need; the analysis of the knowing process here discovers that it is more of an intellectual need: when it is met, then students possess their knowledge as a personal achievement; they then have some notion of what their knowledge is. Teachers are called on in practice to provide the answer to the second of Lonergan's three basic problems for teachers: They must know the experience that is met as questionable and which is adequately understood in the mathematics they teach. Therefore the necessary further development of this thesis is an assembling of puzzling experience that requires mathematical understanding for a solution--further work of which Appendix I only provides examples. The complete accomplishment of this task could not be neatly encompassed in a textbook or a series of textbooks, for what is required will vary from classroom to classroom and student to student. But the experiential sources of mathematics can be assembled so that the teacher will have resources from which to draw in order to awaken his students' drive to know and to supply sufficient experience upon which the student can base an adequate understanding.

The analysis here of how things happen when someone comes to know might provide a basis for another direction of development: an analysis of how someone goes wrong when he fails to know. This thesis has done nothing to explore the diagnostic and remedial possibilities of Lonergan's cognitional theory. It is conceivable that students' reports and the pattern of their incorrect answers would be able to reveal false understandings and thus provide a way to correct them. This is a bit different from the diagnostic procedures of programmed learning because Lonergan is not so performance-oriented. That relation could be spelled

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out further and tested in practice.

Theoretical areas only briefly mentioned in this thesis are worth further development. For example, Lonergan discusses the nature of formal processes (35) and McShane has shown how Lonergan's approach clarifies some issues in the philosophy of mathematics (46). Using the knowing process as a source of definition of mathematics itself provides a means to analyze disputes over the nature of mathematics.

It is even possible to find the differences between the schools of the foundations of mathematics rooted in differing apprehensions of the knowing process. For example, logicism is the view that mathematics is a purely deductivist system and the task of mathematics is to achieve the systematic presentation in which one can pass by deduction to all the theorems of mathematics. Thus, logicism treats only fully axiomatized and systematized mathematics (McShane 46:381f). Logicism stresses the act of formulation of mathematics, which is on the level of understanding, almost to the exclusion of a recognition of any of the other acts involved in the creation of mathematics. It fails to recognize the role of insight in understanding and formulation; it sees deduction as a tautological manipulation of concepts and not as a particular form of judgment. Through this restriction of mathematical activity, logicism ignores the course of development of mathematics in history and in individuals, because this development involves advancing through image to insight and then to formulation and the process then is compounded in the movement from lower to higher viewpoints. Part of the process of mathematics is to render understanding in adequate formulation through axiomatized structures, but that is only a part; another part is to develop the insight that at one time may be only casual, not yet crys-

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tallized in expression. Logicism is thus a static view of mathematics, focusing too exclusively on formulation at the level of understanding as the entire mathematical process. This examination of the schools of mathematics from the aspect of their implicit notion of the knowing process could be continued.

Another contribution to metamathematics of the theory presented in this thesis could be a clarification of some of the paradoxes that have stimulated concern about the nature of mathematics and resulted in the various schools of the philosophy of mathematics. Among other paradoxes, McShane discusses the liar paradox, in one of its various forms (46:384f). Someone states, "I am a liar." Is that statement true or false? If it is true, the speaker is a liar; but that means he is telling the truth, which makes the statement false, etc. But the statement "I am a liar," as it is, is only a sequence of sounds or a pattern of black marks on a white background. It is only matter available at the experiential level of knowing. The reader or listener may also think about or consider the statement, without advancing a judgment about it. But it can only be true or false as the content of a judgment. The reader cannot use "I am a liar" as the statement of a judgment, for judgments must be based on intelligent grasp of evidence; for the present proposition, the only sort of evidence would consist in one's actually lying, and in that case the correct judgment is "I have lied" (46:385). What allows the resolution of the paradox is the distinction between the three levels of knowing, the resultant distinction between the products of these levels, and the recognition of truth as a property of personal judgment.

These are all topics for further work with the methodological

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approach to mathematics set forth here. What has been presented is a fundamental tool for examining methods and results in mathematical education.

Summary

The thesis began with an analysis of the knowing process, following Bernard Lonergan. It set forth the threefold pattern of experience, understanding, and judgment in knowing, the three levels of activity that constitute the knowing process and which are required if one would know reality. The thesis then explored important aspects of knowing: the place of symbolism, the occurrence of higher viewpoints, etc. Finally, mathematics emerged as a particular realization of this knowing process, in which the knower reaches possible intelligible orderings of the empirical residue.

Finally, a summary and analysis of several theories of knowing currently prominent in mathematical education showed the power of Lonergan's theory to evaluate and situate other approaches to knowing in mathematics.

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APPENDIX I

CLASSROOM APPLICATIONS

One conclusion from this thesis is that knowing what our knowing process is will affect what we do in teaching mathematics in a classroom. This appendix will present approaches to the teaching of several topics in mathematics, with comments basing the approaches in the nature of the knowing process and of mathematics. Particular topics used arose in courses I have been teaching at St. Gregory's College, a small, private junior college in Shawnee, Oklahoma. The school's program is transfer-oriented and most graduates of the college do continue their academic work in Bachelor's degree programs.

It could be objected that many of the uses of Lonergan's theory here are not startling: "They are just the things a good teacher does naturally, so what is the use of developing this elaborate cognitional theory?" That is the common-sense question in the face of emerging science, but one of the subcurrents throughout this theses has been the investigation of what makes the good teacher good. Why does what he does naturally work? The body of the thesis was an attempt to answer these questions, by investigating the learner. This appendix returns to the level of application in order to examine where the theory leads in practice.

Algorithm for Exponents

Instructional Approach

A first example of instructional method based on Lonergan's cog-

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nitional theory is a type of problem in an early algebra course which is very routine but still presents a struggle even to students exposed to algebra for several years. In this example, the goal is to teach the students to solve problems that involve use of the algorithm $x^a \cdot x^b = x^{a+b}$, assuming natural number exponents. A first step is to remind the students of the meaning of the exponential expression, if that has not been a recent topic--that x^a means that x is used a times as a factor. Next I work with simple examples of the law: $x^3 \cdot x^4$ can be expressed, by the meaning of the exponents, as $(x \cdot x \cdot x) \cdot (x \cdot x \cdot x \cdot x) = x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x = x^7$. Then I ask a question that cannot be reduced so easily: $x^{23} \cdot x^{45} = ?$ It is difficult to express the meaning of the exponents here in written form, but the same meaning--the same pattern of operations--is involved and the students are able to deal with that meaning: How many times is x used as a factor in x^{23}? In x^{45}? In total? How can we express that? The students are required to use the same basis for answering this series of questions as they used in multiplying $x^3 \cdot x^4$. I try to lead them to attend to what they are doing and to express what they are doing in some way. Only at this point does it seem appropriate to introduce the formal expression of the principle $x^a \cdot x^b = x^{a+b}$.

Next, I ask the students to test if the understanding expressed in this form is correct; do they get the same answer in the simple exercises through the use of this formal expression as they did with the more primitive means? Does the formula express the operations they use?

But the initial goal has not yet been addressed, that of enabling the students to solve problems of that general form. The algorithm first had to be developed as the expression of an understanding that unites many possible cases. To make explicit the relation of the

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abstract formula to the more concrete problem, I now ask the students to create problems that this algorithm will enable them to solve. But the real source of confusion comes in when not only this algorithm but also the other algorithms for working with exponents are available and the students have to decide what to do when they are confronted with an involved exponential expression to simplify. So once again, when all the laws are available, I ask the students to prepare problems involving the several properties of exponents. When students approach a problem that could involve any of the several laws, I ask them--and remind them to ask themselves--to make explicit the process involved: What is the operative algorithm? What is this a case of? Finally, there is a series of questions and problems I can pose to utilize the laws that the students have now learned (Rosenbloom 61):

1. How many miles are there in a light year? Of what order of magnitude is this result? We can answer by using, roughly, powers of 10 and the algorithm for the exponent of a product.

2. Examine the expression $10^{10^{100}}$. If I write the number out, taking one second to write each digit, how long will it take? If each "0" occupies $\frac{1}{4}$ inch, how far will the written number extend? If I stand at the "1" in that expression, how long will it take the light from the last "0" to reach me? These questions explore the compactness of the exponential form and the last question requires use of exponential laws.

3. How many digits are there in 2^{100}? Here we use the power law and some basic inequalities: $2^3 < 10$ so $2^{99} < 10^{33}$ so $2^{100} < 10^{34}$. This means that the decimal expression has no more than 34 digits. On the other hand, $10^3 < 2^{10}$ so $10^{30} < 2^{100}$. This means that the decimal expression of 2^{100} has at least 31 digits. Therefore by this simple means of

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approximation, the decimal expression of 2^{100} has from 31 to 34 digits.

Comment

1. Although this is a simple approach to a standard topic, it owes much to the cognitional theory in this thesis. The prominence of questions as an element of the presentation is no mistake. The questions are intended to be real ones, actually provoking a response from the students. In a classroom situation, with various levels of achievment and various rates of response, an immediate answer should be discouraged, until everyone has had the chance to perceive the question as a question and attempt or reach an answer. The reason for basing the procedure on questions as steps to an answer is that for the unaccomplished student of mathematics it is difficult to see the final mathematical expression as the end product of an inquiry; by developing the matter through explicit questioning the final result is more evident as the result of the questioning drive to know. Though the questioning might not be the students' own, still it is a line of questioning, and the mathematical expression then has meaning as an answer to a problem, rather than some arbitrarily imposed procedure. Too many poor mathematics performers regard mathematics as some sort of magic; if the proper but meaningless rituals are gone through, somehow there will emerge an answer that is acceptable to the teacher; the approach here regards algorithms as useful when they can be shown to embody the meaning of the mathematics. This is a reflection of the development of mathematics as an investigation of prior operations.

2. Another point that the prominent use of question emphasizes is that the learning process is something the student does. The student must approach the experience as something questionable, have the

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insight, formulate his understanding, judge if it is true. He can have assistance; in fact, if he is to do all that he is required to do in his few years of education, he must have help. But the help cannot substitute for his own learning; the teacher cannot put something inside the student's head. Learning is the student's activity; education is not something done to the student.

3. The approach oscillates between mastered examples of the algorithm and the rule which is a higher-level understanding encompassing the examples; it alternates between the more concrete and the more abstract and so attempts to provoke the insight. The goal of this lesson is the understanding of this particular relation that when two exponential expressions with the same base are multiplied, the product can be expressed as the base with exponent equal to the sum of the original exponents. That is an extremely complicated statement, which is much simpler looking in the algebraic form; but the algebraic form does no good if it is not known as an expression of the understanding. So the topic starts with examples, an advertence to the experience of operating with the exponential expressions. Insight into the principle behind all this is provoked in the students' thinking by means of the questions. The algorithm is shown to be an expression of this understanding. Finally the adequacy of this expression and understanding is verified--the level of judgment--in checking that this expression provides satisfactory answers at least to the problems students could originally solve.

4. Experience, understanding, and judgment are all made explicit parts of this brief lesson. The threefold pattern of knowing should generally be the basis of instruction, and the teacher should point out this constant feature to the students.

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5. The further step of having the students provide examples of the algorithms has more to do with the role of abstract expression than with the particular law. To too many students the abstractness of mathematics means that it has no reference, that it is a series of arbitrary routines. Mathematical education should enable students to grow in understanding the nature of mathematics--where it begins and how it progresses and how it is expressed. Thus I include this segment in order to make students understand the layering effect in mathematics--the role of higher viewpoint--and the freedom that abstract expression gives by focusing upon the core of intelligibility. By providing examples, students can learn from the opposite direction the multiplicity of problems whose inner workings are understood by the simple algorithm.

The nature of mathematical abstraction and expression are important topics that must be developed first through laying a broad experiential base, of which the activity of providing instances forms a part. It might be in a much later class that we talk about solution sets of problems, for example, and then we can reflect on the problems of this lesson as examples; moving from the expression $x^a \cdot x^b = x^{a+b}$ to the expression $5^9 \cdot 5^7 = 5^{16}$ is an act that can be systematized by speaking of the replacement set for the unknowns $\underline{x}$, $\underline{a}$, and $\underline{b}$.

6. Along with abstract expression, this topic gives occasion for reflection upon the power of symbols; when students gain familiarity with the rules of operations, they can perform the symbolic operations automatically. But the teacher should be careful from the first to make clear the nature of symbol as an expression, and so base the manipulation of symbols on the underlying meaning.

7. The sequence of operations that often characterizes the dis-

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covery method was used in this sample instruction: examples followed by rule. It is not proper to demand, on the basis of cognitional theory, that this sequence be used in every instance; I do think that the example-rule sequence should be used sufficiently, and sufficiently early, that the students do learn the meaningfulness of mathematics--that a formula crystallizes an understanding. When student have been led to understand this use of formal expression, they will be more able to relate a new formal expression to the cases that are subsumed. For example, because of the order of instruction used in this topic, it might be easier in a later case to present initially the formula $(x^a)^b = x^{ab}$ as a property of exponents and then show its use in particular cases. But for this the students need sufficient background. That is to say, the example-insight-rule order of presentation is not inviolable; however it must be used and emphasized frequently enough that students learn that a mathematical formulation is the expression of an understanding and even if the general statement comes first, they still have to work to achieve the understanding it expresses. It is important that the student appreciate even the most abstract mathematics as a structure of understanding.

8. The use of the final applications of the various laws of exponents is really an attempt to show the value of the mathematical enterprise, to show that the understandings reached are really understandings that can be used as the realization of intelligibility in a large number of situations. It shows the value of the general expression, so that what is captured is free from particularity. It shows that the question of common sense "What is it worth?" is not one that can be immediately answered by the more theoretical and abstract science

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that is mathematics, but that does not destroy its ultimate worth. It shows, in a low level example, that the results of definition and abstract investigation can have results in more concrete problems, because what belongs to the understanding of the more concrete has already been captured in the theoretical investigation.

Division of Fractions

Instructional Approach

The problem of division of fractions and complex fractions is often troublesome. To reach the usual algorithm, I have a silly routine I use, my "desert island" example. I use this after acquainting students with multiplication of fractions and with the term "multiplicative inverse"--its meaning and the way to find it. I usually focus on an admitted heavy smoker in the class and present this situation: "You are trapped on a South Seas island with only 20 cigarettes. You decide to allow yourself 2 cigarettes a day, in hope that help will arrive before the cigarettes and you expire. How many days will the cigarettes last? What operation do you use in answering?" The answers are easy, so I write them on the board: $20 \div 2 = 10$. "Actually, you think it will take a long time to be rescued, so you decide to allow yourself only $\frac{1}{2}$ cigarette a day. How long will you last?" The student can answer, but is not usually sure why. Since the structure of the question is the same, so must be the structure of the answer: $20 \div \frac{1}{2} = 40$. Now we reach a crisis: "You have carefully rationed your cigarettes, but now have only $\frac{1}{2}$ cigarette left; wishing to extend your resources further, you pledge to smoke only $\frac{1}{4}$ cigarette a day. How many days can you last now?" The answer is 2, so I write $\frac{1}{2} \div \frac{1}{4} = 2$. With a number of these operations

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on the board, I transfer to multiplication: From $20 \div 2 = 10$, I ask "$20 \cdot ? = 10$"; from $20 \div \frac{1}{2} = 40$, I ask "$20 \cdot ? = 40$." And I look for the relations of the new numbers to something in the former problems. In this way we discover that we can obtain the same result as dividing by multiplying by the multiplicative inverse of the divisor.

The students know the rule "Invert and multiply," but too often they use it in a ritual fashion and quote it on many occasions, only on some of which is it appropriate. Here the rule is presented only after its meaning and appropriateness are drawn out; even then the more precise formulation of the procedure is used. Once the procedure to follow has been developed, we can apply it to not so obvious situations.

Comment

1. Fractions are ambiguous. In fact, extensions of a number system often yield ambiguous results; the forms of expression of these extensions may be efficient, but they can also be confusing. For example, the number -3 is the result of subtracting in a situation when we did not have a number to provide an answer: $0 - 3 = ?$ We express the result in a form that carries the traces of the problem we were trying to solve. When we do have numbers sufficient to answer a problem (e.g., $9 - 5 = 4$) we are not embarrassed by the minus sign; but when we have to leave our original system and create new numbers to provide a result, it is very simple to associate the minus sign with the result. However this leaves the minus sign with two meanings; on one hand it indicates the operation of subtraction; on the other hand it indicates a negative number. This example exhibits the fact that our symbolic systems (e.g., our form of representing negative numbers) are choices and may sometimes involve

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difficulties; in those cases we have to dig into the symbol system in order to unearth the meaning that it is intended to express. The resultant clarification will then help us to perform the symbolic operations with less confusion. Lonergan's analyses of symbol and expression help us to be alert to the reasons for confusion and to find ways to achieve clarification.

Similar to the minus sign, fractions have a dual meaning. The fraction 2/3 represents the act of dividing 2 by 3 and it also represents the result, since no integer could represent the result of the division. In using this new number, 2/3, we admit that we are having to extend our notion of number. But we still intend in the development to show that if we do operations correctly we can have many of the nice properties of the old number system, and more. This introduction to the meaning of a fraction is something students often miss; insight into the inbuilt ambiguity of fractions can help them at least to situate the complexity of the operations that arise and isolate the sources of confusion.

The knowledge of the dual meaning helps students to simplify "complex fractions"; to remember that the vinculum also means division is a great help in removing the frightening aspect of these expressions.

2. The reason for the whimsical example of the cigarettes is to provide an image from which to work. If students forget the rule "Invert and multiply" or what it applies to or what they are supposed to invert, they now have something to refer to again in order to recapture the meaning of the formulation. This instructional device portrays the role of image in leading to the insight and, again, the role of insight as pivoting between concrete and abstract, between example and understanding.

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Factoring

Instructional Approach

Factoring has always been a troublesome topic in introductory algebra. A good time to introduce factoring might be after some simple work with solution of equations. If a student knows how to solve $3x - 12 = 0$, and he wishes to solve $x^2 + x - 12 = 0$, factoring provides a very helpful tool in accomplishing this detective work: What values for x make that a true statement?

The main difficulty for students is usually not the form of factoring in any one pattern. If students know they are factoring a difference of squares, they can proceed rather easily, but their main problem is in reaching the insight that this is a difference of squares type of problem. So my colleagues and I have developed an approach to factoring that makes explicit a series of questions that reaches the pattern of factoring involved.[1] The series of questions is an obvious one: (1) Are there common factors? (2) How many terms? If two, is the expression a difference of squares, sum of cubes, difference of cubes? If three, can this expression be factored by the trinomial method? If four or more, can the terms be grouped together into smaller wholes that can be factored so that the whole expression will have factors? (3) Will any of the factors already found factor again?

This is the core of my approach to factoring. First I go through the several individual forms of factoring in conjunction with special forms of products. After many examples of the form $(x + y)(x - y) =$

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[1]See Appendix B. Dr. Joseph Barrick was largely responsible for formulating these questions.

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$x^2 - y^2$, students are prepared to reverse the equality: $x^2 - y^2 = (x + y)(x - y)$.

Then the class deals with problems in which the prototype is not specified and students must sort out what approach to attempt. It is at this point that I rely on the schema of questions. When a student has a difficulty with a factoring problem, I lead him through the questions, and ask him to lead himself through the same series. Once he does this, he can usually perform the factoring.

Comment

1. The purpose of situating the factoring in the discussion of solution of equations is to provide factoring as a way to achieve needed answers--to provide its place in a line of inquiry. The key here is to activitate the student's drive to know, to move toward the topic, be it as mundane as factoring, as an answer to required intelligence. Sometimes this background in line of inquiry is difficult to provide. Often it arose historically from obscure directions in search for answers that were never found, or out of some private curiosities. But the importance of anchoring mathematics in the knowing process, as a real result, that gives intelligibility to situations where it was sought but lacking before, is an important part of the teaching process.

2. This series of questions in this approach presupposes the prior ability to factor within each of the patterns. The procedure outlined above is geared to aid the insight as to which of the factoring patterns is applicable. It is not an algorithm--it is not something that can be performed without knowing the basis behind it, like the algorithm for long division. But it is a procedure for "disposing the

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image," for carrying on the investigation into the expression in order to facilitate the insight which type of factoring should be attempted. Although this insight is almost automatic for an experienced or accomplished student, it can be incomprehensible for a less accomplished one. Simply asking the question, Is it a difference of squares? will help to provoke the insight. And so this questioning process can make the insight more readily available, but not automatic.

3. This process proposed as an aid in factoring makes explicit a process that Lonergan calls a concrete inference from classical laws (35:46). While Lonergan cites the example of scientific explanation, using classical laws to predict a result in a particular situation, the entire process is embodied in this rather simple example of factoring. A concrete inference from classical laws presumes: (1) information on some concrete situation, (2) knowledge of the pertinent laws, and (3) insight into the given situation. The insight should give, first, what laws are to be selected for the inference, second, how the selected laws are to be combined to represent the spatial and dynamic configuration of the concrete situation, and third, what dimensions in the situation are to be measured to supply numerical values that particularize the selected and combined laws.

An example of concrete inference is calculating the orbit of a planet around the sun. The first requirement is the series of observations of the motion of the planet and some calculation of the velocities, sizes, and masses of the planet, the sun, and the other bodies of the solar system. The law involved is the formula for the gravitational attraction between point-masses. The insight needed is that for a first approximation the sun and planet could be considered as point-

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masses and the only data required to calculate the orbit are distances, masses, and the initial velocity of the planet with respect to the sun. For closer approximations, other factors need to be considered: the variation in density of the planet, the action of a planet and its satellites together as an orbiting body, the effect of tidal action, the effect of gravitational attraction of other planets. All of these allow more complete understanding of the action of the planet and so a closer and closer approximate calculation of the orbit.

That is a much more complicated situation than factoring polynomials, but that factoring might help to prepare students to face more complicated situations later. For the student to factor $a^4 + 3a^2 - 4$, he needs the three basic prerequisites that the procedure helps him to realize: (1) he has the information presented to him on the particular problem, (2) he is required to know the various possible forms of factoring and how they work in particular instances, (3) he needs the insight that connects the particular form of factoring of trinomials to this problem.

The fact that this procedure serves as an example of an approach to a wide range of mathematics problems should be emphasized, and referred to again later.

This seems to be a specification of Polya's instructions when facing a problem to be solved (56:xvi). Polya asks a general series of questions: What do you know in this situation; what is the given? What laws or theorems do you know that apply to something like this? Do they apply here? Lonergan's requirement of insight refers to the last two questions from Polya, because the insight is required to be able to say, "This problem is <u>like</u> the other."

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4. The role of understanding in elementary algebraic operations and the nature of mathematical development is often made painfully clear in the topic of factoring. This is one of the particular difficulties that students experience when the topic they are studying involves an action on operations, not on numbers. This occurs, for example, in the chaining of properties: in the performance of some operations that involves more than one operation, in composition of functions, or in inverse operations. That is what is happening in factoring; it seems to be much more difficult for the struggling student than multiplication; it even seems harder for the student to conceive what he is doing, because he is somehow reversing the operation of multiplication.

Although multiplication in arithmetic is defined as repeated addition and particular answers may be verified that way, multiplication soon assumes a kind of identity of its own and it can easily be modeled by familiar processes. Factoring polynomials, on the other hand, is initially specified in terms of multiplication and never achieves a meaning apart from that; to understand the process of factoring requires not only a performance of multiplication but some conceptual grasp of the operation of multiplication itself.

Similarly, the basic routines of integration are more difficult than the corresponding differentiation problems, because the reliability of the one operation is rooted in the reliability of the one learned first.

One reason for the difficulty of factoring is that it is a more creative activity; it involves writing down something that is not there. But a second reason is that factoring is verified in terms of some other operation; the definition of the process of factoring requires reference

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to a previous operation. Here is an indication of the development of mathematics as a series of higher viewpoints.

Fundamental Theorem of Calculus

Instructional Approach

The Fundamental Theorem of Calculus is exactly what is says, the central unifying understanding that links together the two basic operations of the calculus. Whereas differentiation is defined through limit of a particular function and integration is defined through the limit of a particular sum, the fundamental theorem explicates how these two limits are related. By the time it is reached in my course, students know how to perform numerous differentiations and integrations; they have a definition of an integral by sums, either in the Riemann definition or a modified Stieltjes one in which the integral is defined to be the unique number between all upper and lower sums. And probably we will have found some very simple integrals by definition.

Next, as part of the practice of applying the derivative and integral rules, students perform several homework problems of these sorts: For a specified $f(x)$, find $F(x)=\int_a^x f(t)dt$ and find $F'(x)$. Given a particular $F(x)$, find $F'(x)$ and then $\int F'(x)\,dx$. By this point I can elicit from the students a conjecture as to the relation between the operations of taking a derivative and finding an antiderivative. Then, before I attempt a formal proof of the fundamental theorem, I do more work to establish what is happening and why that is a likely result.

The first part of the attempt at a heuristic proof is to use a diagram like the one suggested by Bers (4:244). If we let $F(x) =$

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$\int_a^x f(t)dt$ and attempt to examine what $F'(x_o)$ might yield, we use a diagram (Fig. 1a). We note that

$$F(x_o + h) - F(x_o)$$

is very close to $f(x_o)\cdot h$ when h is small enough. By definition of derivative

$$F'(x_o) = \lim_{h \to 0} \frac{F(x_o + h) - F(x_o)}{h},$$

which is approximately equal, by our argument based on the diagram, to

$$\frac{f(x_o)\cdot h}{h}$$

which gives: $F'(x_o) = f(x_o)$.

But there is another part to the fundamental theorem, the one that inverts the order of integration and differentiation, namely:

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

That formula can be proved from previous results; the proof is good mathematics but it is basically unsatisfying. The meaning of the theorem is developed better through a heuristic approach and another diagram (Fig. 1b) suggested by Owen and Munroe (51:309). We partition the interval and determine the slope at the initial point of each subinterval. This provides a way to approximate the change in the function through the subinterval: e.g., $F(x_1) - F(x_o)$ is approximately equal to $F'(x_o)\cdot(x_1 - x_o)$. By adding up these approximate changes we can approximate the total change; that is, F(b) - F(a) is approximately equal to $\sum_{i=0}^{4} F'(x_i)\cdot(x_{i+1} - x_i)$. A further partitioning

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Figure 1. The Fundamental Theorem of Calculus

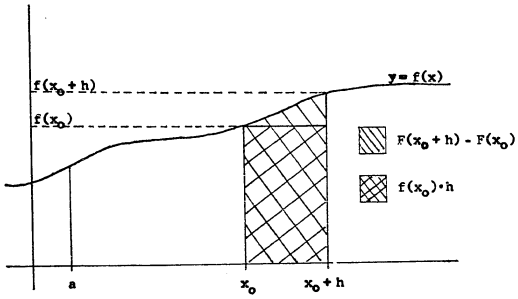

Fig. 1a

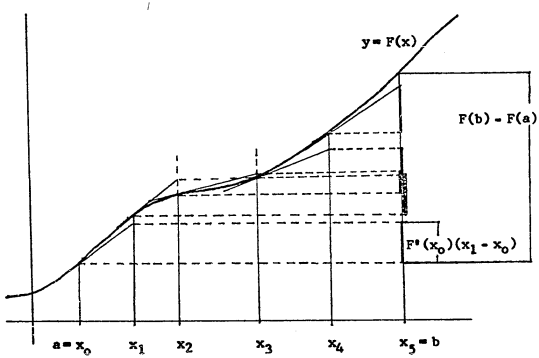

Fig. 1b

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of the interval [a,b] would produce a closer approximation to the desired F(b)-F(a). We can see from the way the integral is originally defined that an extension of the partitions to the limit will result in the integral $\int_a^b F'(x)\,dx$ and the desired result $\int_a^b F'(x)\,dx = F(b) - F(a)$.

Now that the meaning of the fundamental theorem has been made a likely result, and a feel for the method of proof attempted, now is the time for the formal statement and the formal proof. I use a standard statement and a standard proof.

The formal statement of the proof contains conditions upon the kind of function that will give the desired result; the first part of the theorem requires a bounded and piecewise continuous function f. Thus after the formal proof is complete, then is a good time to return to those conditions and through examples of functions that violate the conditions show why the conditions are required. Although the required conditions are used in the proof, it is still good to have a way to ground this formal requirement in an image; this image aids the insight into the requirement of the conditions.

Comment

1. Many commentators make the point that theorems and devices in proofs should be approached heuristically; they should not seem to spring from nowhere (Polya 57:50). That point is emphasized even more strongly by the analysis of human knowing in this thesis. Since the goal of a mathematics class is that students come to know the mathematics taught them, the students themselves will have to go through the threefold process of knowing. If parts are omitted in the presentation,

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then the students have that much more work to do themselves, because to learn truly they will have to supply the omitted stages. If a theorem is revealed, already formulated in its precise language, followed by a formal proof, a conscientious student, as he pores over his notes, will have to go back and say, What could that mean? He might try to think of examples that fit the hypothesis and see what the conclusion will mean for them, and how examples violating the hypothesis fail to fit the conclusion. By this return to an experiential level, he will reach an insight into the intelligibility that is expressed in the formulated theorem. The proof has been supplied, but the student will have to pore over the proof to discover how it supplies the answers to all relevant questions; thus he makes the proof his own judgment. This is a difficult project, if the student does not realize what he must do to master the material. Therefore mathematics instruction should be self-conscious; it should lay open the process of inquiry and knowing, so that the student will know just what he knows as he does mathematic . If the three levels are made explicit, then the student who has not mastered a topic will know what he must still do.

2. Diagram--a utilization of images--shows its importance here; for each part of the fundamental theorem of calculus we have used a diagram to allow insight into the relationship of the hypothesis to the conclusion. Diagram has proved a vital tool in mathematics at many stages. Even higher algebra has found it useful in generating insights, for example, in presenting a lattice of subgroups. The goal of a diagram is to generate insights, to provoke them. No particular diagram is essential, for no diagram is mathematics, but the understanding that arises from confronting and questioning a diagram is mathematics.

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3. Proof is not where mathematics begins. It took extensive development of geometry before Euclid could formulate his rigorous system, and then it turned out that his system was not so rigorous, because he omitted the formulation of several important assumptions because they were so automatic in his diagrams (Heath 27:242f, 249f, 280f). It took a longer time to produce algebra or even arithmetic as a system that could admit so satisfactory a proof system. Therefore the role of proof in mathematics instruction has to be somewhat balanced. It is necessary to move the student to some sort of formal proof, so that he will not be misled by an unverifiable insight; but the proof is the way to verify an understanding he has reached, and so mathematics is much more than mere proof. And proof can have degrees of rigor. Whereas the original heuristic arguments of my presentation of the fundamental theorem of calculus would be more than enough to satisfy most calculus students that the theorem is true--indeed my mere saying it is so would be sufficient--they must also learn the role of proof to establish the resulting structure of mathematics, and the use of proof is to exhibit the place of a proposition within that structure:

> Proof in any serious meaning of the term presupposes the erection of a system, in which all terms and relations have an exact meaning, and all procedures from some propositions to others are rigorous. But the system itself, in turn has its presuppositions. It presupposes a horizon, a worldview, that has unfolded under the conditions and circumstances of a particular culture and a particular historical development (Lonergan 43:12).

This situates proof within the system, related to some original propositions (axioms) which we accept either as satisfactory or as suppositions. Proof is rigorous within those bounds. But it has only to do with the system of argument that is involved; arguments the set construction proponent will consider rigorous the intuitionist will consider

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gratuitous. So we need to develop a view of proof as locating a proposition within a system, whose presuppositions are not proved in the same way but accepted as part of the way the arguer looks at the world or his mathematics, as part of the equipment he needs to understand the meaning of terms or to grasp the validity of arguments or to situate the goal of the argument.

Though we can recognize its ultimate relativity, still formal proof has a role in mathematical discourse, since what is presented in mathematical structure is a possible ordering, possible intelligibility.

Mathematics requires formal language and formal argument. We cannot let the heuristic argument suffice, for the mathematics is abstract and it moves rapidly away from what is experienced and it deals with possible structures. Therefore it needs the control of technical language and symbolism: piecewise continuous functions, $\int_a^b F'(x)$, etc. It needs the rigor of formal argument to avoid implicit assumptions that might falsify the conclusions.

This formal and abstract nature of mathematics must be appreciated; it is necessary to allow mathematics to be a study of possible intelligibility. This nature is hard to comprehend from the common-sense point of view, which constantly asks "What use is this?" The misunderstanding has resulted in the shibboleth by which the newer programs in school mathematics have been found wanting: "Is 2 plus 2 still equal to 4?" The presumption is that there is only one system of numbers and operations possible, the one that occurs "out there, in the real world, that we can see." On the contrary, the answer could be 0 in a modulo system base 4, which might or even does have an

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interpretation in what we know about the real world. Formalism and abstraction and proof allow this exploration of possibilities.

4. This recognition of the role of proof and formal statement has application at the higher reaches of mathematics instruction, where the definition-theorem-proof format is followed more rigorously. Here the students might be more able to fill in the steps that precede the formal statement and formal proof, but only if they can come to realize the essential role of this completion. To help students do this would make them more a part of the continuing process of mathematical inquiry that occurs in mathematical research. This participation of students in mathematical inquiry is an explicit goal of mathematics programs in elementary and secondary school programs; it should be considered as essential for college and university programs, also.

Introduction to Linear Algebra

Instructional Approach

St. Gregory's College offers a course in linear algebra, intended for students who have reached some sophistication in mathematics, either in the ten-hour sequence of calculus for engineers and science majors or in a course of calculus for the non-physical sciences and a finite mathematics course. For the most part, even in their high school backgrounds, students entering the course have had no experience with modern algebra. They have heard the language of algebraic structure in their high school algebra courses--the properties of commutativity and distributivity and identity; they can identify the use of axioms in various real-number operations, but they understand the properties only as proper names, basically as names of particular properties of the real number

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system. They do not know that these properties can be used to characterize various mathematical structures; they lack sufficient experience to ground these properties as true understandings of operations on mathematical structures.

Therefore in the linear algebra course, before I treat the set of axioms that characterize a general real vector space, it is necessary to help the students reach the understanding that the properties they studied in algebra are properties of certain operations on certain sets of elements--that they are truly general statements that may be true in some cases and not true in others.

Since students need first to be able to characterize structures in terms of the algebraic properties of the operations, the first step in the course is to use the field axioms to examine various familiar and unfamiliar structures: the set of natural numbers with respect to addition and/or multiplication, the integers, the rationals, the reals, several of the modulo arithmetics, and the set of transformations of several regular polygons. We also investigate more general binary operations, e.g., $a*b$ defined to be a^b on $R^+ \times R^+$.

This is usually new material to the students and it is needed to demonstrate that the properties of the real number system are actually useful in distinguishing types of mathematical structures.

The relation of these axioms to other properties of the real number system is also examined, e.g., the connection between cancellation properties and the existence of inverses, or multiplicative cancellation and the absence of zero divisors. From there we move to a discussion of the nature of abstract algebra: when we start with a particular axiom system, then certain other properties will follow. I point out that

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this connection is why it is helpful to prove certain properties strictly from the axioms, so that the same properties may be asserted of any structure that behaves in accordance with the same axioms.

The last point is aided in the subsequent development of the course by the textbook (O'Nan 50). Before the author discusses vector spaces in general, he treats operations with vectors as n-tuples of real numbers and operations with m x n real matrices. In each section, he proves or assigns as exercises the axioms of a vector space and key properties that might be proved from the axioms; he does not state that these are the general properties defining a structure as a vector space, at least at this point. Finally, when he reaches the topic of real vector spaces from an algebraic point of view, he has provided some familiar examples for reference in order to show that this general structure is truly common to the previous systems and isolates common properties; he can then move to discover what might be true about all systems that obey these primary properties. However the author fails to make clear the procedure he follows; he does not point out, first, that the n-tuples and matrices are indeed particular examples of the general structure, and, second, that he now proves for all vector spaces, on the basis of the axioms, the results he or the student had shown for the two particular examples. Although he does not make this procedure explicit, he does use a fruitful procedure; in teaching this material, I point out with great emphasis the procedure he has followed.

Comment

1. This method of approaching linear algebra makes clear the development of mathematics as a series of higher viewpoints. The insight that grounds the characterization of the general vector space

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reaches a higher viewpoint; it grasps the previous mathematics (R^n and the m x n matrices) in a system that leaves behind what is particular in these embodiments of the vector space and is true for these and many other examples. This insight achieves an intelligibility in activities that seemed coincidental at the former level: the similar properties. I point out to students the character of this insight as I begin to formulate the new order of understanding.

2. Symbolism plays a large role in this movement to higher viewpoint. Whether we speak of vector addition $\vec{a} + \vec{b}$ or matrix addition A + B, it is a conscious effort of textbook authors and classroom teachers to keep the symbolic expression of parallel meanings parallel. A mastery of the higher viewpoint--the understanding that the two operations are actually parallel--is the reason for the instructor's use of the parallel notations. And in turn it is the use of parallel notation that allows the student to reach the insight that yields the higher viewpoint.

3. Discussion of the new mathematics programs finds one of the primary goals to be an emphasis on the structure of the discipline. Lonergan's analysis of the structure of knowledge was discussed earlier. The approach to linear algebra described here utilizes that two-fold structure of knowledge: it begins in the order of discovery, examining various systems, until it focuses on the definition of a vector space. From there, it moves in the order of explanation through the proofs of the usual properties of that abstract system.

Examinations

We can also examine testing practice from the viewpoint of Loner-

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gan's cognitional theory. A successful test can do more than test individual skills; it can be a learning situation. Preparation for a test, especially for a final examination in a course, provides occasion for the student to look back over the material of the course, which might seem only a string of isolated concepts, and find the connection between them. It is a time to find the mathematics presented in the course is a structure developed in response to continuing inquiry.

As an aid in preparing my students for examinations, my usual practice is to provide a sample test, consisting of problems of the type that will be on the test; it is usually longer than the actual test. Its purpose is to familiarize students with the wording of the test and especially to provide an instrument through which they may be able to relate the topics of the course.

An example of the kind of problem on a test that can bring together many topics is curve sketching in calculus. For example, for the functions given by the relations

$$y = x^2 - x - 1 \qquad \text{or} \qquad y = \frac{(2x+1)(x-1)^2}{(x-2)(x+1)^2},$$

I ask students to give the domain, range, critical values, points of inflection, regions in which y is increasing or decreasing and concave up or down, the limit of y at points of discontinuity and at infinity, and finally a sketch of the graph. These questions are interlocking and involve about half of the topics covered in differential calculus.[2]

Another example of the kind of test that can be used to evoke the organizing intelligibility of a course is used in the Linear Algebra

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[2]The sample functions and interesting methods for graphing are found in Riddle (60).

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course.[3] Topics covered in that course include solutions of systems of linear equations by Gaussian elimination, the algebra of matrices, inverses, determinants, vector algebra, and linear transformations. The final examination is aimed at a practical demonstration of mathematics as finding a common intelligibility in apparently different situations. Therefore the examination reduces computation to a minimum and stresses the meaning of mathematics as a way to understand diverse situations.

Summary: Educational Practice

The most important result for education that comes from Lonergan's learning theory is that the learning process is something the student does. He must have the experience, get the insight, formulate his understanding, judge if it is true. He may have help; if he is to do all that he is required to do in his few years of education he must have help.

And that is the teacher's task. It can be a frustrating one, because the teacher is neither the necessary nor sufficient cause of learning occurring. The student must learn for himself, and the teacher can only help. Lonergan compares the teacher's role to the role Aquinas saw for the doctor in minor illnesses: nature heals, while the doctor helps; the doctor removes impediments to healing, allowing nature and the condition of the patient to carry on the healing process (Lonergan 38:No. 5). In education it is the student who wonders and desires to know what is at work; the teacher must encourage this activity, help it and direct it and remove impediments. But he cannot form images in the

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[3]See Appendix C.

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student's imagination or have the insight for him, though he can suggest the image and provide appropriate symbols and expressions so that the student might reach the image. The teacher can approach the matter from several directions and provide alternate diagrams and give opportunities for questions so that the student might be helped to isolate the important and detect and correct elements in his understanding that are not adequate. The teacher can assist in proper formulation or help to bring out how a formal expression does get at the meaning of mathematics.

The teacher must therefore provide orchestration. In a classroom, insight occurs at varying rates. Some students catch on immediately, and find all the teacher's subsequent efforts boring, for after one has reached the understanding, further instances are mere repetition. Other students understand after the teacher has gone through all his available repertoire. Others understand after some more work with examples in homework. Some begin to understand at a later review. Some never understand, although they can mechanically perform some operations, acting within the power of the symbolism but not understanding what this symbolism expresses. Others do not catch on at all; possibly they could with the right images, with the elements of the insight separated and specified and a careful process of small insights made possible for them. But the debate will continue to rage whether or not everyone could be taught any topic. The problem with this assertion, though, is that insight is never necessary; it can never be forced or guaranteed.

The classroom teacher must meet the three problems cited earlier; he must know the mathematics, he must know the appropriate experiences that mathematics understands, and he must know what it is to know so he

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may best help his students come to the same knowledge.

Since education is a process of communication, it must begin where the student is. But part of where the student is includes his desire to know; the teacher must appeal to his interest and awaken questions. This itself has a number of implications.

Part of education is the broadening of the student's world, the one he constructs, his world of meaning. But this broadening cannot be done if the appeal of education is only to interests the student already has.

> If he is already interested in mathematics, my teaching him more mathematics does not broaden his horizon; it merely fills out and further organizes the horizon he has already attained. But suppose he is not interested in mathematics, how do I broaden his horizon to include a concern for learning mathematics? (Sanders 62:146)

This appeal cannot be to something within mathematics, its beauty, its generality, or some particular area like calculus. The appeal then must be to something more basic: to the initial wonder and desire to know. Providing this appeal can be one of the teacher's most difficult tasks; it could happen that each student might need a separate entry into mathematics. Often it can be done through some problematic situation, one that demands more than a common sense orientation. To support this continuing inquiry, it must be rich enough to raise the questions Why? and How? and not provide immediate answers; yet it must be a situation bordering on levels the student has already mastered.

One way of forcing a student out of his present horizon is to present him with a situation in which his present equipment will not suffice, will not produce an answer. This is the kind of thing that will stimulate the student's own inquiry, by making something puzzling.

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W. W. Sawyer in his many books (63, 64, 65, 66, 67, 68) has many examples of puzzling questions that yield fruitful results in addition to giving many examples of images helpful in provoking insights.

At the end of the long process of learning mathematics there is a wisdom, not the wisdom of common sense but a wisdom proper to mathematics. It is an ease in insight, the ability to ask the right questions for understanding and draw the right diagrams that lead most effectively to insight. It is an ease in judgment that knows the right questions for judgment and so knows when all the relevant questions have been answered. It is a structure of insights that provides sufficient background so that any expression in mathematics will be tied in somewhere to this background. This wisdom is only reached gradually and painfully as the learner develops that connection of understandings and repeated judgments and gradually moves through higher and higher viewpoints until grasps of larger and larger wholes are attained. Insights and understandings will build up; understandings once reached can be recaptured easily.

> This, too, is a universal characteristic of insight and, indeed, it constitutes the possibility of learning. For we can learn inasmuch as we can add insight, inasmuch as the new does not extrude the old but complements and combines with it. Inversely, inasmuch as the subject to be learned involves the acquisition of a whole series of insights, the process of learning is marked by an initial period of darkness in which one gropes about insecurely, in which one cannot see where one is going, in which one cannot grasp what all the fuss is about; and only gradually, as one begins to catch on, does the initial darkness yield to a subsequent period of increasing light, confidence, interest, absorption. Then, the infinitesimal calculus or theoretical physics or the issues of philosophy cease to be the mysterious and foggy realms they had seemed. Imperceptibly we shift from the helpless infancy of the beginner to the modest self-confidence of the advanced student. Eventually we become capable of taking over the teacher's role and complaining of the remarkable obtuseness of pupils that fail to see what, of

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> course, is perfectly simple and obvious to those that understand (Lonergan 37:6).

All of this is a fruit of the spontaneous desire to know, the drive to inquiry, the thirst for knowledge, that begins with experience and the questions What? Why? How? and the next series Really? Is that so? Education is the process of directing that inquiry into the intellectual pattern of knowing, the desire to understand possible systems that would give meaning to the structures of our experience. Beginning simply with the fact of multiplicity, mathematicians throughout history have created the impressive edifice that is mathematics; beginning there, in a more directed fashion, every student in each new generation builds his own partial edifice of mathematics, no one ever mastering all, but extending sections of it through individual efforts, through each new mathematician's spontaneous and insatiable questioning as he moves to his own higher viewpoints. Every teacher of mathematics, from kindergarten to university, must respect this pattern of the development of knowledge and perform his function of assisting and directing this inquiry most efficiently.

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APPENDIX B

PROCEDURE FOR FACTORING

The following is a listing of steps which will help you to factor most expressions.

I. Check for a factor or factors common to each term and write that product.

example 1 $x^2y + ybx + yx = yx(x + b + 1)$

II. Now count the number of terms in each factor:

1. If there is one term, it is prime (as yx in example above).

2. If there are two terms, the expression will be:

   a. the difference of squares and will factor as the product of the sum and difference of two numbers

      example 2 $x^2 - y^2 = (x + y)(x - y)$

   b. the sum of two cubes which will factor by the formula $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

   c. the difference of two cubes which will factor by the formula $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

   d. prime.

3. If there are three terms, the expression will be:

   a. a perfect square trinomial such as $a^2 + 2ab + b^2 = (a + b)^2$ or $a^2 - 2ab + b^2 = (a - b)^2$

   b. factorable by the method of checking the middle term.

      example 3 $x^2 - 6x + 8 = (x - 4)(x - 2)$

      example 4 $3x^2 + 4x - 15 = (3x - 5)(x + 3)$

   c. prime.

4. If there are four or more terms, you may be able to factor by grouping the terms.

   example 5 $ax + ay + bx + by = (ax + ay) + (bx + by) = a(x + y) + b(x + y) = (a + b)(x + y)$

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example 6 $x^2 + 2x + 1 - y^2 = (x^2 + 2x + 1) - y^2 =$
$(x + 1)^2 - y^2 = [(x + 1) - y][(x + 1) + y]$

III. Repeat this procedure until all factors are prime.

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APPENDIX C

LINEAR ALGEBRA
FINAL EXAMINATION

I. Given the system: $3x_1 - 2x_2 + 6x_3 - x_4 = 0$

$x_1 + 3x_2 + 2x_3 + x_4 = 0$

$x_2 + 2x_4 = 0$

1. Find all solutions.

2. Consider the linear transformation $L: R^4 \to R^3$ for which

$$L(e_1) = \begin{bmatrix}3\\1\\0\end{bmatrix} \quad L(e_2) = \begin{bmatrix}-2\\3\\1\end{bmatrix} \quad L(e_3) = \begin{bmatrix}6\\2\\0\end{bmatrix} \quad L(e_4) = \begin{bmatrix}-1\\7\\2\end{bmatrix}$$

a. Find the matrix of transformation.

b. Find $L\begin{bmatrix}3\\1\\-2\\4\end{bmatrix}$

c. Find the range space of L. What is the rank of L?

d. Find the nullspace of L. What is the nullity of L?

II. Given the system: $x_1 + x_2 + x_3 = 1$

$x_1 + x_3 = 2$

$x_2 + x_3 = 3$

1. Solve by Gaussian elimination.

2. Prove $\begin{bmatrix}1&1&1\\1&0&1\\0&1&1\end{bmatrix}^{-1} = \begin{bmatrix}1&0&1\\1&-1&0\\-1&1&1\end{bmatrix}$

3. Find $\begin{vmatrix}1&1&1\\1&0&1\\0&1&1\end{vmatrix}$

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4. Set up two other ways to solve the original system of equations:

   a. Cramer's rule,

   b. using the inverse of the coefficient matrix.

5. Show $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis of R^3.

6. Give the coordinates of $\vec{e}_1$ and $\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$ with respect to this basis.

III. 1. If V is a vector space and H and K are subspaces of V, prove that $H + K = \{ \vec{u} + \vec{v} \mid \vec{u} \in H \text{ and } \vec{v} \in K \}$ is a subspace of V.

2. Prove, using the properties of a vector space, that if V is a vector space, $\vec{x}$, $\vec{a}$, and $\vec{b} \in V$, and $\vec{x} + \vec{a} = \vec{b} + \vec{a}$, then $\vec{x} = \vec{b}$.

IV. Determine if the following are linear transformations. If a transformation is linear, find its matrix of transformation from the standard basis.

1. $F: R^2 \to R^3$ where $F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ y \\ 2x + 3y \end{bmatrix}$.

2. $F: R^4 \to R^2$ where $F\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} x + y + 1 \\ z + w + 1 \end{bmatrix}$

V. If $M = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 4 & 7 & -2 \\ -1 & 3 & 0 \end{bmatrix}$ determine the linear transformation associated with M.

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