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Birkhoffian Generalization of Hamiltonian Mechanics

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Questo volume è dedicato ai miei genitori

Ida ed Ermanno Santilli

con devozione ed affetto

Contents

Preface

In the preceding volume,[1] I identified necessary and sufficient conditions for the existence of a representation of given Newtonian systems via a variational principle, the so-called *conditions of variational self-adjointness.*

A primary objective of this volume is to establish that *all* Newtonian systems satisfying certain locality, regularity, and smoothness conditions, whether conservative or nonconservative, can be treated via conventional variational principles, Lie algebra techniques, and symplectic geometrical formulations. This volume therefore resolves a controversy on the representational capabilities of conventional variational principles that has been lingering in the literature for over a century, as reported in Chart I.3.1.[2]

The primary results of this volume are the following. In Chapter 4,[3] I prove a *Theorem of Direct Universality of the Inverse Problem.* It establishes the existence, via a variational principle, of a representation for all Newtonian systems of the class admitted (universality) in the coordinates and time variables of the experimenter (direct universality). The underlying analytic equations turn out to be a generalization of conventional Hamilton equations (those without external terms) which: (a) admit the most general possible action functional for first-order systems; (b) possess a Lie algebra structure in the most general possible, regular realization of the product; and (c)

[1] Santilli (1978a). As was the case for Volume I, the references are listed at the end of this volume, first in chronological order and then in alphabetic order.

[2] All references to the preceding volume have the prefix "I", e.g., Section I.1.1, Equation (I.1.1.5). Script letter $\mathscr{I}$ is used to refer to elements within the Introduction to the present volume.

[3] To stress the continuity with the three chapters of Volume I, those of this volume are numbered 4, 5, and 6.

characterize a symplectic two-form in its most general possible local and exact formulation. For certain historical reasons, indicated in the text, I have called these equations *Birkhoff's equations.*

In Chapter 5 I present the *transformation theory of Birkhoff's equations.* Essentially, it emerges that, while Hamilton's equations preserve their structure only under special classes of transformations (the canonical and the canonoid), Birkhoff's equations preserve their structure under arbitrary, generally noncanonical, transformations. I then present a step-by-step generalization of the Hamiltonian transformation theory. In addition, I point out that Birkhoff's equations can be obtained from Hamilton's equations via the use of noncanonical transformations. The inverse reduction occurs instead via the use of Darboux's transformations of the symplectic geometry. This allows the proof in Chapter 6 of the *Theorem of Indirect Universality of Hamilton's Equations*, according to which conventional Hamilton equations are unable to represent Newtonian systems at large in the reference frame of their experimental observation; nevertheless, a representation can always be achieved via use of the transformation theory.

As has been known since Galilei's time, physics requires that abstract mathematical algorithms admit a realization in the frame of the observer. The inability of Hamilton's equations to satisfy this fundamental requirement confirms the need for their Birkhoffian generalization.

The analysis presented in these volumes therefore establishes that the treatment in the frame of the observer of Newtonian systems with unrestricted dynamical conditions requires the use of *generalized analytic formulations* for the most general possible first-order Pfaffian action and of *generalized geometric formulations* for the most general possible local and exact two-forms. These occurrences render inevitable a reinspection of Lie's theory (enveloping associative algebras, Lie algebras, and Lie groups) to achieve a form which is directly compatible with the generalized analytic and geometric formulations—that is, a form which is classically of noncanonical character and quantum mechanically of predictable nonunitary character. This study is conducted in the final stage of a program where the existence of *generalized algebraic formulations* is shown. These formulations essentially consist of a reformulation of Lie's theory that is directly applicable to the most general possible associative envelopes, the most general possible non-Hamiltonian/Birkhoffian realizations of the Lie product, and the most general possible noncanonical/nonunitary structures of the Lie groups. By keeping in mind that Lie's theory was developed for the simplest possible associative product $X_i X_j$ of the envelope, the simplest possible form $X_i X_j - X_j X_i$ of the Lie product, and the simplest possible structure $\exp \theta^i X_i$ of the Lie groups, the need for the reformulation under consideration is self-evident. I have called the emerging formulations *isotopic generalizations*, where the term "isotopic" expresses the preservation of the primary analytic, Lie, or symplectic character.

In this way, we see the emergence of the foundations of a *Birkhoffian Generalization of Hamiltonian Mechanics* which

1. applies to a class of physical systems broader than that for which Hamiltonian Mechanics was conceived—systems with action-at-a-distance, potential, self-adjoint forces, as well as contact, non-potential, non-self-adjoint forces;
2. is based on an isotopic generalization of the analytic, algebraic, and geometric methods of Hamiltonian Mechanics; and
3. is capable of recovering Hamiltonian Mechanics identically when all non-self-adjoint forces are null.

A number of applications to systems of ordinary differential equations in Newtonian Mechanics, Space Mechanics, Statistical Mechanics, Engineering, and Biophysics are presented during the course of our analysis, with more specific treatment appearing in Chapter 6. With the understanding that quantum mechanical profiles are beyond the scope of this volume, I have briefly indicated the existence of an isotopic generalization of Heisenberg's equations, as well as of a number of related quantum mechanical aspects, for the description of particles under action-at-a-distance, potential interactions, as well as contact, nonpotential interactions, which are conceivable under mutual wave penetration, and overlap. The rather old (and currently dormant) problem of the generalization of Quantum Mechanics is therefore brought to life in an intriguing and direct way by the Birkhoffian Generalization of Hamiltonian Mechanics. Regrettably, for the sake of brevity I have been forced to ignore several additional, equally intriguing developments such as the extension of Birkhoffian Mechanics to field theory—a study which has already been initiated in the literature.[4]

The mathematically inclined reader should be informed from the outset that I have given priority of presentation to methods and insights, not only in local coordinates but also within a single fixed system of variables, those relative to the observer. The use of transformation theory is presented only as a second phase of study. Finally, generalization via coordinate-free, global, and geometric approaches is presented as a more advanced approach. This style of presentation implies a reversal of the priorities of contemporary mathematical studies, particularly those of geometric character, but it is dictated by specific pedagogical and technical needs.

On pedagogical grounds, my teaching experience has suggested that it is best to expose students first to geometric structures in specific local variables and show that the essential geometric properties persist under arbitrary (but smoothness- and regularity-preserving) transformations of the local variables. Then the students may be brought, in a progressive motivated way, to advanced coordinate-free techniques.

The technical reasons for giving priority to formulating the methods in local variables are even more pressing than the pedagogical ones. In fact, the crucial inability of conventional Hamilton equations to represent Newtonian systems in the frame of the observer can be identified only via the local formulation of the theory because, at the abstract, coordinate-free level,

[4] Kobussen (1979).

Hamilton's and Birkhoff's equations are indistinguishable. At any rate, a primary function of the Inverse Problem is to provide methods for the computation of an analytic representation of specific differential equations in specific local variables. Clearly, this task can be accomplished most effectively via the local formulation of the theory.

These priorities should not be interpreted as denying the need for global techniques. On the contrary, these techniques will be quite useful throughout our study, particularly in proving the main theorems.

This work reflects the organization of Volume I: a main text, a series of charts,[5] illustrative examples, and problems. The main text is devoted to the simplest possible presentation of the techniques in local variables. The charts complement the presentation through more advanced topics in Abstract Algebras, Functional Analysis, Differential Geometry, and other disciplines. The examples are intended to illustrate only the most important aspects. Finally, the problems are designed to test the student's understanding of the basic ideas and methods and to evaluate the student's capability for practical applications.

The relevance of the analysis presented in this and the preceding volume can be indicated essentially as follows. Within the context of *Theoretical Physics*, the methods presented permit the identification and treatment of a new class of interactions called "closed non-self-adjoint" (Chapter 6). These interactions verify the conventional conservation laws of total quantities (closure), yet the internal forces are partially of action-at-a-distance, potential type and partially of contact, nonpotential type (non-self-adjointness).[6] The interactions of primary interest in contemporary physics (e.g., electromagnetic interactions and the unified gauge theories of weak and electromagnetic interactions) turn out to be of the closed self-adjoint type upon the extension of the methods to relativistic and field theoretical settings (Santilli 1977a,b,c, and 1978b).

In essence, the transition from closed self-adjoint to non-self-adjoint interactions is given by the replacement of *pointlike* constituents with *extended* constituents under sufficiently small mutual distances. Points can only interact at a distance, thus admitting only self-adjoint interactions. Extended objects, on the contrary, whether particles or waves, admit the additional contact interactions for which the notion of potential energy has no physical basis. Thus they are of the non-self-adjoint type. The former interactions are of Lagrangian/Hamiltonian type, while a necessary condition for the latter interactions to be truly non-self-adjoint is that they *are not* of Lagrangian/Hamiltonian type in the frame of the observer; yet they can be treated via the Birkhoffian Mechanics and related isotopic generalization

[5] As in Volume I, the term "chart" is used in its nautical sense of "guiding" the student through the main ideas of a more advanced topic, while providing selected references for subsequent studies.

[6] For the reader's convenience, some of the theorems of self-adjointness of Volume I are reviewed in the Introduction and in Section 4.1.

of analytic, algebraic, and geometric methods. The conceptual and technical advances are self-evident.

A rather forceful Newtonian example of closed non-self-adjoint interactions is given by our Earth. If considered isolated from the rest of the universe, the Earth verifies the conservation of total physical quantities, but the interior motions are of the non-self-adjoint type, as is the case for satellites during reentry into our atmosphere, spinning tops with drag torques, etc.

It is often argued that nonpotential forces are due to the "immaturity" of the Newtonian description, and that the local/potential/Lagrangian or Hamiltonian nature is recovered in full when passing to elementary constituents of matter. This view has been criticized in recent times because it is based on the pointlike abstraction of the elementary constituents and because it ignores the experimentally established conditions of mutual wave overlappings for all interior problems under strong interactions, such as the structures of nuclei, of strongly interacting particles (hadrons), and of stars. At any rate, the idea that the experimentally established nonpotential Newtonian interactions can be reduced to a large collection of potential interactions has no practical computational value (owing to the large number of constituents of macroscopic bodies). It has no experimental support at this time, is therefore merely a scientific belief, and when subjected to an actual mathematical study, is afflicted by a host of consistency problems such as the need to recover nonpotential dynamics via a large collection of potential ones.

In different terms, the Newtonian description of the structure of our Earth with its established potential and nonpotential forces in local or nonlocal[7] treatment, is a model of invaluable guidance in the study of the more complex structures of nuclei, hadrons, and stars, rather than knowledge to be bypassed via pointlike abstractions of the elementary constituents. To put it quite candidly, I have conducted most of these studies because of the possibility that our Earth can be viewed as a Newtonian image of the structure of hadrons, in the same way as our solar system is seen as a Newtonian image of the structure of atoms.

Once the closed non-self-adjoint interactions are acknowledged either as an experimentally established reality (classical mechanics) or as a possibility (particle physics), the relevance of the methods of these volumes becomes self-evident. In fact, by recalling that the broader interactions considered, by conception, cannot be directly treated via Hamilton's (or Heisenberg's) Mechanics, the methods employed in these volumes permit the use of rigorous analytic, algebraic, and geometric techniques which would otherwise be precluded. Besides the evident classical applications, the methods are potentially useful for the future experimental resolution of the problem of the structure of strong interactions; that is, whether the ultimate structure of the

[7] It should be stressed that this volume in general, and Birkhoff's equations in particular, treat *local* non-self-adjoint interactions. The *nonlocal* non-self-adjoint interactions demand a generalization of Birkhoff/Lie/symplectic formulations, e.g., into the so-called *Lie-admissible formulations* (see Chart 4.7).

universe can be reduced to a collection of points, or a substantially more complex reality must sooner or later be acknowledged.

The relevance of the methods in *Engineering* is equally self-evident. Engineering systems are non-self-adjoint as a rule and are self-adjoint only in very special cases. I am referring to computer or electric systems inclusive of internal losses, trajectory problems with follower forces, etc. The techniques presented in these volumes allow the computation of an action functional for all these systems, by treating them via well-established methods such as the canonical perturbation theory and the Hamilton–Jacobi theory.

The reader can now see the relevance of the Inverse Problem for other disciplines, such as *Space Mechanics*, *Statistical Mechanics*, *Biophysics*, etc.

This volume originated in the following way. In Volume I, I reported on studies of the integrability conditions for the existence of a Lagrangian as available in the literature and presented my own work on the independent existence of a Hamiltonian (i.e., existence of a Hamiltonian without prior knowledge of a Lagrangian). I also presented my methods for the computation of these functions from the equations of motion, when the integrability conditions are verified. I identified the capability of these functions to represent both potential and non-potential forces and treated a number of complementary aspects.

While conducting these studies, I became aware that the *violation* of the integrability conditions for the existence of a Lagrangian, or independently, of a Hamiltonian, is the rule in practical cases and that their verification is the exception. Although I have no available evidence, I believe that this restrictive character of the conditions of self-adjointness has been known since the early studies on the Inverse Problem in the last century, and this resulted in the subsequent lack of significant attention to the problem in both the mathematical and physical literature, as reported in the Introduction to Volume I.

Clearly, in order to reach a level of practical effectiveness, I had to "solve the Inverse Problem." That is, I had to identify methods capable of turning *all* non-self-adjoint systems of the class admitted into equivalent self-adjoint forms for which an action functional can (at least formally) be computed.

After a considerable library search (in addition to that reported in Volume I), I succeeded in tracing efforts back to Mayer (1896); additional relevant contributions were made by Davis (1931) and Havas (1957). All these contributions deal specifically with the Indirect Lagrangian Problem and are reported in the Appendix along with the Newtonian reduction of my field theoretical studies on the topic (Santilli (1977c)).

Even though the methods permitted the construction of Lagrangian representations for genuine nonconservative nonpotential systems, the lack of direct universality of the Inverse *Lagrangian* Problem was soon established.[8] This situation called for additional efforts. At this point the

[8] I should indicate from the outset that these limitations refer specifically to the Lagrangians of contemporary use in theoretical physics, those depending at most on the velocities (first-order Lagrangians).

Independent Inverse *Hamiltonian* Problem became crucial. In essence, in Volume I, I had established the symbiotic character of the conditions of self-adjointness for first-order systems by ensuring: 1) the derivability from a variational principle, 2) the Lie algebraic character, and 3) the symplectic geometric structure. As indicated in Volume I, Hamilton's equations are only a *particular case* of these conditions of self-adjointness. The existence of more general equations preserving the analytic, algebraic, and geometric character was then ensured. I therefore proceeded to the identification of these broader equations as the most general possible form permitted by the conditions of self-adjointness. In this way, I "rediscovered" equations which had been proposed by Birkhoff (1927) without the algebraic and geometric techniques of the Inverse Problem. Their direct universality for systems of first-order ordinary differential equations resulted from an unexpected property identified by Havas (1973). Unlike second-order systems, first-order systems always admit (under sufficient topological conditions) a regular matrix of integrating factors which produces an equivalent self-adjoint form. The Birkhoffian representation of the systems is then consequential, and its explicit form can be computed via the Converse of the Poincaré Lemma (Section I.1.2). These studies are presented in Chapter 4, jointly with a number of complementary topics such as the indirect Birkhoffian representation of Hamilton's equations, the algebraic significance of the self-adjointness inducing and preserving transformations of the equations of motion, etc.

The next step of my studies was predictable and consisted of reducing a direct Birkhoffian representation into an indirect Hamiltonian form through the transformation theory. By noting that the former is characterized by a general exact symplectic structure, while the latter is characterized by the fundamental one, the reduction is done simply by a Darboux's transformation. This is, in essence, the Theorem of Indirect Universality of the Hamiltonian Representations, presented in Chapter 6 by using variational self-adjointness and its algebraic/geometric structures. Not surprisingly, the theorem was conceived first by Lie (1871) who, as part of the pioneering studies on the algebras and groups carrying his name, also probed the universality of their application to ordinary[9] differential equations. Subsequently, the problem was reinspected by Koenigs (1895), and the theorem is sometimes referred to in specialized literature as the *Lie–Koenigs Theorem.* This theorem is presented in this volume, however, for its direct geometric content (a manifestation of Darboux's theorem), rather than for its algebraic interpretation, as generally presented in the existing literature.

Needless to say, seeing in this way that the Inverse Problem always admits a solution was rewarding for me, but, as the mature researcher well knows, whenever primary research objectives have been achieved, it is time to provide the utmost possible *critical* examination of the results. My subsequent efforts have been devoted to the implications of the indirect nature of

[9] I am not aware of attempts by Sophus Lie to apply his theory to nonlocal/integro-differential systems.

the universality of Hamiltonian representations. The results are presented as an application of the Theorem of Indirect Universality and consist of a technical identification of the fact that Galilei's relativity *does not* hold for Newtonian systems at large (closed non-self-adjoint systems of extended particles) and does hold for only a special class of systems (closed self-adjoint systems of particles admitting an effective pointlike approximation). This identifies the problem of the possible generalization of Galilei's relativity for closed non-self-adjoint systems via the isotopic generalization of conventional Hamiltonian, Lie, and symplectic techniques, which is, perhaps, the most intriguing aspect of our analysis.

My studies therefore confirm the traditional pattern of a continuing scientific process. The Theorems of Direct and Indirect Universality of Analytic Representations do indeed solve the most crucial aspects of the Inverse Problem. Jointly, however, the theorems identify new, rather fundamental, open problems, with particular reference to the relativity and underlying physical laws which are applicable to contact/nonpotential/non-self-adjoint interactions.

I can therefore conclude by saying that Newtonian mechanics, rather than having reached a terminal stage, is still open to new, potentially fundamental advances.

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Almost needless to say, I am solely responsible for the content of this volume, including several modifications implemented in the final version of the manuscript.

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Introduction

The study of the systems of particles occurring in our environment has established the existence of a considerable variety of forces, such as Newtonian and non-Newtonian, potential and nonpotential, or local and non-local forces.

I.a Newtonian and Non-Newtonian Forces

Forces in Euclidean space E_3 with local coordinates $\mathbf{r}$ (or in configuration space with generalized coordinates, q^k, $k = 1, 2, \ldots, n$)[1] are called Newtonian when they are independent of the accelerations and depend in general on time t, coordinates $\mathbf{r}$, and velocities $\dot{\mathbf{r}}$, $\mathbf{F} = \mathbf{F}(t, \mathbf{r}, \dot{\mathbf{r}})$ (or $F = F(t, q, \dot{q})$). The following are familiar examples of Newtonian forces:

$$\mathbf{F} = -k\mathbf{r}; \qquad \mathbf{F} = -\gamma\dot{\mathbf{r}}; \qquad \mathbf{F} = \pm \frac{QQ'}{r^3}\mathbf{r}; \qquad \mathbf{r} \in E_3 \qquad (\mathscr{I}.1)$$

Forces with an explicit dependence on the accelerations, $\mathbf{F} = \mathbf{F}(t, \mathbf{r}, \dot{\mathbf{r}}, \ddot{\mathbf{r}})$, are called non-Newtonian because they generally violate some of the principles of Newtonian mechanics (e.g., the principle that total acceleration is given by the vectorial sum of the accelerations produced by each

[1] The notations of the preceding volume will be maintained throughout this volume (boldface letters for vectors, sum of repeated indices, etc.). They will be defined in footnotes only when necessary. Script letter $\mathscr{I}$ is used to refer to elements within the Introduction of the present volume.

individual force). Examples of acceleration-dependent forces occur in radiation damping, or in systems of coupled oscillators (where they are sometimes called *acceleration couplings*).[2]

I.b Potential and Nonpotential Forces

A Newtonian force $\mathbf{F}(t, \mathbf{r}, \dot{\mathbf{r}})$ is said to be potential (nonpotential) when it verifies (does not verify) the integrability conditions for the existence of a potential energy $U(t, \mathbf{r}, \dot{\mathbf{r}})$ according to the rule

$$\mathbf{F} = -\frac{\partial U}{\partial \mathbf{r}} + \frac{d}{dt}\frac{\partial U}{\partial \dot{\mathbf{r}}}. \qquad (\mathscr{I}.2)$$

Potential forces represent *action-at-a-distance interactions* (e.g., Coulomb interactions). Nonpotential forces represent instead *contact interactions* (e.g., interactions occurring for motion within a resistive medium.[3]

I.c Local and Nonlocal Forces

Consider a system of particles moving in vacuum at large mutual distances. In this case the shape and structure of the objects do not affect the dynamics. The objects can then be approximated as *massive points*. Under these conditions, the forces are local, in the sense that they occur at a number of isolated points. Nonlocal forces occur instead in the motion of extended objects within a resistive medium (e.g., a satellite in Earth's atmosphere) and in other systems whose dynamic evolution is affected by the shape and structure of the objects. In this case the forces call for a suitable integral form which represents the action occurring at all points of the surface (or volume) of the objects; e.g.,

$$\mathbf{F} = -\iiint d\mathbf{r}'\mathbf{K}(t, \mathbf{r}, \mathbf{r}', \dot{\mathbf{r}}, \dot{\mathbf{r}}', \ldots), \qquad (\mathscr{I}.3)$$

with the understanding that the non-locality can also be in time (see Mittelstaedt (1970) and, more recently, Trostel (1982)).

It is evident that local forces are often an *approximation* of nonlocal forces. In fact, forces of type ($\mathscr{I}$.3) are often approximated in mechanics via power-

[2] For more details see Volume I, Appendix A.6.

[3] The reader should keep in mind the classification of Newtonian systems into *conservative*, *dissipative*, and *dynamical* (or *nonconservative*), given in Volume I, Appendix I.A, for which the total mechanical energy is conserved, monotonically non-increasing in time, and arbitrarily (but continuous) varying in time, respectively. Only the conservative systems of this classification admit potential forces. The forces of all the remaining systems are generally nonpotential.

series expansions in the velocities truncated at a power selected on the basis of experimental information,

$$\iint ds\ \mathbf{K}(t, \mathbf{r}, \dot{\mathbf{r}}, \ldots) \cong \gamma_1 \dot{\mathbf{r}} + \gamma_2 |\dot{\mathbf{r}}| \dot{\mathbf{r}} + \gamma_3 \dot{\mathbf{r}}^2 \dot{\mathbf{r}} + \cdots . \qquad (\mathscr{I}.4)$$

For instance, the simplest conceivable resistive force is given by the expression linear in the velocity $\mathbf{F} = -\gamma_1 \dot{\mathbf{r}}$. A first improvement of the approximation can be accomplished by adding a term quadratic in the velocity, $\mathbf{F} = -\gamma_1 \dot{\mathbf{r}} - \gamma_2 |\dot{\mathbf{r}}| \dot{\mathbf{r}}$. The subsequent improvement, which is needed particularly for high speed, is given by adding a term cubic in the velocity, $\mathbf{F} = -\gamma_1 \dot{\mathbf{r}} - \gamma_2 |\dot{\mathbf{r}}| \dot{\mathbf{r}} - \gamma_3 \dot{\mathbf{r}}^2 \dot{\mathbf{r}}$, thus yielding a truncated power series of type ($\mathscr{I}$.4).

The analysis of the preceding volume has been restricted to local Newtonian forces which are either potential or nonpotential. The study of non-Newtonian and/or nonlocal forces will be excluded from this volume as well (apart from occasional mention). In Chapters I.2 and I.3 we established the integrability conditions for a force to be of potential type, which became known as the *conditions of variational self-adjointness*. The methods for the computation of a potential from the force, when all integrability conditions are verified, were also established.

Reviewing the following elemental properties may be advantageous.

Theorem $\mathscr{I}$.1 (Self-Adjoint Newtonian Forces, Theorem I.2.2.2, p. I.67, and Charts I.3.8 and I.3.9, pp. I. 192–I.195). *A necessary and sufficient condition for a local class $\mathscr{C}^1$ Newtonian force* $\mathbf{F}(t, \mathbf{r}, \dot{\mathbf{r}})$ *to be derivable from a potential* $U(t, \mathbf{r}, \dot{\mathbf{r}})$ *according to Rule* ($\mathscr{I}$.2), *is that the force is at most linear in the velocity, i.e., it is of the type*

$$F_i = \rho_{ij}(t, \mathbf{r}) \dot{r}^j + \sigma_i(t, \mathbf{r}); \qquad i, j = x, y, z, \qquad (\mathscr{I}.5)$$

and all the following conditions of variational self-adjointness

$$\rho_{ij} + \rho_{ji} = 0, \qquad (\mathscr{I}.6a)$$

$$\frac{\partial \rho_{ij}}{\partial r^k} + \frac{\partial \rho_{jk}}{\partial r^i} + \frac{\partial \rho_{ki}}{\partial r^j} = 0, \qquad (\mathscr{I}.6b)$$

$$\frac{\partial \rho_{ij}}{\partial t} = \frac{\partial \sigma_i}{\partial r^j} - \frac{\partial \sigma_j}{\partial r^i}, \qquad (\mathscr{I}.6c)$$

are identically verified in a star-shaped neighborhood of a point $(t, \mathbf{r})$. *In this case the potential can be computed from the force according to the rule*

$$U = -r^k \int_0^1 d\tau\ F_k(t, \tau \mathbf{r}, \tau \dot{\mathbf{r}}) \qquad (\mathscr{I}.7)$$

$$= B_k(t, \mathbf{r}) \dot{r}^k + C(t, \mathbf{r}).$$

The relativistic extension of the theorem has been studied in Santilli (1978b) and the field-theoretic generalization in Santilli (1977a,b,c).

As clearly established by Theorem $\mathscr{I}.1$, the conditions of self-adjointness constitute a mathematical tool for the rigorous treatment of the physical notion of potential. When all the conditions of the theorem are verified, we have a *self-adjoint force* F_{SA}, and a potential exists. When at least *one* of the conditions of the theorem is violated, we have a *non-self-adjoint force* F_{NSA}, and a potential does not exist.[4]

Corollary $\mathscr{I}$.1a. *A necessary condition for a function $U(t, \mathbf{r}, \dot{\mathbf{r}})$ to be the potential of a Newtonian force is that it is at most linear in the velocity.*

This property is clearly a consequence of condition ($\mathscr{I}.5$), and it is expressed via the last form of ($\mathscr{I}.7$). It has been recalled here because of a tendency in the contemporary literature of theoretical physics (particularly in high energy physics) to call "potential" any sufficiently smooth function $U(t, \mathbf{r}, \dot{\mathbf{r}})$. As we shall see, unless the condition of linearity in the velocity is met, the potential $U(t, \mathbf{r}, \dot{\mathbf{r}})$ is really representative of nonpotential forces.

When all forces are self-adjoint, the equation of motion for an unconstrained Newtonian particle in Euclidean space admits the so-called *ordered direct Lagrangian representations* (Section I.3.4)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^k} - \frac{\partial L}{\partial r^k} \equiv m\ddot{r}_k - F_k^{\text{SA}}(t, \mathbf{r}, \dot{\mathbf{r}}), \qquad k = 1, 2, 3; \tag{$\mathscr{I}$.8a}$$

$$L = \frac{1}{2} m\dot{\mathbf{r}}^2 - U(t, \mathbf{r}, \dot{\mathbf{r}}). \tag{$\mathscr{I}$.8b}$$

with a natural generalization to a system of particles.

The integrability conditions for the existence of these representations were studied in detail in the preceding volume for the more general case of local Newtonian systems with holonomic constraints, the so-called *fundamental form* of the equations of motion in configuration space (Section I.A.7)

$$A_{ki}(t, q, \dot{q})\ddot{q}^i + B_k(t, q, \dot{q}) = 0, \qquad k = 1, 2, \ldots, n. \tag{$\mathscr{I}$.9}$$

Unconstrained Newtonian systems are a particular case of this form obtainable via the identification of the generalized coordinates q^k with the Cartesian coordinates r^i in a given ordering and of the term A_{ij} with the mass tensor $m\delta_{ij}$.

[4] One of the most general self-adjoint forces possible is the Lorentz force $\mathbf{F} = e(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B})$. In fact, besides being linear in the velocity, the force has the most general possible structure verifying all the conditions of Theorem $\mathscr{I}.1$ (see Example I.2.7, p. I.105, for detail). Non-self-adjoint forces are structurally more general than the Lorentz force because they are generally nonlinear in the velocity. Notice, however, that *linear* velocity-dependent drag forces $\mathbf{F} = -\gamma\dot{\mathbf{r}}$ are non-self-adjoint because they violate conditions ($\mathscr{I}.6a$). Notice also that velocity-independent non-self-adjoint forces are conceivable.

Lagrange's equations[5] in configuration space are given by the familiar form

$$\frac{d}{dt}\frac{\partial L(t,q,\dot{q})}{\partial \dot{q}^k} - \frac{\partial L(t,q,\dot{q})}{\partial q^k}$$

$$= \frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^i}\ddot{q}^i + \frac{\partial^2 L}{\partial \dot{q}^k \partial q^i}\dot{q}^i + \frac{\partial^2 L}{\partial \dot{q}^k \partial t} - \frac{\partial L}{\partial q^k} = 0. \qquad (\mathscr{I}.10)$$

[5] The reader should recall from the preceding volume that the equations customarily referred to as "Lagrange's equations" (and "Hamilton's equations") in the contemporary physical and mathematical literature *are not* the equations originally conceived by Lagrange and Hamilton. The latter are those *with external forces*, i.e.,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} = F_k$$

$$\dot{q}^k = \frac{\partial H}{\partial p_k}, \qquad \dot{p}_k = -\frac{\partial H}{\partial q^k} + F_k.$$

Only since the beginning of this century have the "true" Lagrange's and Hamilton's equations been "truncated" via the removal of the external terms, by acquiring the form of conventional contemporary use. The methodological implications of this "truncation" are considerable. Hamilton's equations *without* external terms possess a Lie algebra structure, while those *with* external terms violate the conditions for a Lie algebra and verify instead those for a more general algebra called Lie-admissible (Santilli (1978c), Myung, Okubo, and Santilli (1978 a,b)). This volume (as well as the preceding one) is devoted to the *Inverse Lie Problem*, that is, to the methods for the construction of a representation of given, generally nonconservative systems, via equations possessing a Lie algebra structure. The use of the equations originally conceived by Lagrange and Hamilton characterizes instead the more general *Inverse Lie-Admissible Problem.* This latter problem will not be considered in this volume, apart from a few incidental comments (see Chart 4.7).

Finally we should indicate here that the extension of ($\mathscr{I}$.10) to field theory which is rather universally used in the contemporary physical literature,

$$\frac{\partial}{\partial x^\mu}\frac{\partial \mathscr{L}}{\partial \varphi^{k;}_\mu} - \frac{\partial \mathscr{L}}{\partial \varphi^k} = 0, \qquad \mu = 0, 1, 2, 3, \qquad k = 1, 2, \ldots, N \qquad \varphi^{k;}_\mu = \frac{\partial \varphi^k}{\partial x^\mu}$$

is *erroneous* under the known, internationally accepted meaning of the symbol $\partial/\partial x^\mu$ as representing *partial* derivatives. The correct equations are those with *total* derivatives d/dx^μ as in ($\mathscr{I}$ 10) and are explicitly given by

$$\frac{d}{dx^\mu}\frac{\partial \mathscr{L}}{\partial \varphi^{k;}_\mu} - \frac{\partial \mathscr{L}}{\partial \varphi^k} = \left\{\varphi^{i;}_{\mu\nu}\frac{\partial}{\partial \varphi^{i;}_\nu} + \varphi^{i;}_\mu \frac{\partial}{\partial \varphi^i} + \frac{\partial}{\partial x^\mu}\right\}\frac{\partial \mathscr{L}}{\partial \varphi^{k;}_\mu} - \frac{\partial \mathscr{L}}{\partial \varphi^k}$$

$$= \frac{1}{2}\left(\frac{\partial^2 \mathscr{L}}{\partial \varphi^{k;}_\mu \partial \varphi^{i;}_\nu} + \frac{\partial^2 \mathscr{L}}{\partial \varphi^{k;}_\nu \partial \varphi^{i;}_\mu}\right)\varphi^{i;}_{\mu\nu} + \frac{\partial^2 \mathscr{L}}{\partial \varphi^{k;}_\mu \partial \varphi^i}\varphi^{i;}_\mu + \frac{\partial}{\partial x^\mu}\frac{\partial \mathscr{L}}{\partial \varphi^{k;}_\mu} - \frac{\partial \mathscr{L}}{\partial \varphi^k} = 0.$$

The erroneous character of the equations with a partial derivative has been established by Santilli (1977a,b,c) by proving that the equations *are not* self-adjoint, and therefore *they are not derivable from a variational principle*, contrary to a rather popular belief. Note that for quadratic Lagrangians the equations formally yield correct equations of motion. However, for sufficiently nontrivial Lagrangian structures, the equations with partial derivatives yield *wrong field equations*, trivially, because of the omission of several terms. Predictably, this is an occurrence of the *physical* literature without a counterpart in the mathematical literature. For instance, the mathematical literature in the calculus of variations for multiple integral path functions unanimously uses and stresses the need of total derivatives in the Euler's necessary condition.

A crucial property for the existence of a representation of (*I*.9) with (*I*.10) is given by the following theorem.

Theorem *I*.2 (Self-Adjointness of Lagrange's Equations, Theorem I.3.3.1, *p*.I.118). *Regular class $\mathscr{C}^2$ Lagrange's equations* (*I*.10) *are always self-adjoint (that is, they are self-adjoint for all possible Lagrangians* $L(t, q, \dot{q})$).

A main result of the preceding volume was the following property.

Theorem *I*.3 (Fundamental Analytic Theorem for Configuration Space Formulations, Theorem I.3.5.1, p. I.131, and Chart I.3.11 p.I.196). *A necessary and sufficient condition for a local system* (*I*.9) *which is well-defined, of at least class $\mathscr{C}^2$, and regular in a star-shaped region $\mathscr{R}^*$ of the variables* $(t, q, \dot{q})$, *to admit an ordered*[6] *direct analytic representation in terms of Lagrange's equations in $\mathscr{R}^*$,*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} \equiv A_{ki}(t, q, \dot{q})\ddot{q}^i + B_k(t, q, \dot{q}), \qquad (I.11)$$

[6] The ordering refers to the equations of motion and to Lagrange's equations, under the condition that these independently selected orderings verify identities (*I*.11). To elaborate on this important point, consider the equations of motion. The first step for the construction of a Lagrangian (or a Hamiltonian) is the selection of the ordering in which the individual equations will be treated. This ordering is quite important for the Inverse Problem, because the self-adjointness or non-self-adjointness of a system *is not* necessarily invariant under permutations of the ordering. This property was illustrated in Volume I a number of times. For instance, system (I.3.4.13) or (I.3.4.19), pp. I.125–I.126, i.e.,

$$\begin{cases} \ddot{q}_1 + b\dot{q}_1 + \omega^2 q_1 = 0 \\ \ddot{q}_2 - b\dot{q}_2 + \omega^2 q_2 = 0 \end{cases}$$

$$\begin{cases} \ddot{q}_2 - b\dot{q}_2 + \omega^2 q_2 = 0 \\ \ddot{q}_1 + b\dot{q}_1 + \omega^2 q_1 = 0 \end{cases}$$

is *non-self-adjoint* in the ordering (q_1, q_2) and *self-adjoint* in the permuted ordering (q_2, q_1). A fully similar sitation occurs for Lagrange's equations which are self-adjoint *in the ordering* $k = 1,2,3,4,5$, etc., but generally non-self-adjoint in an arbitrarily selected ordering, e.g., $k = 5,4,1,2,3$, etc. In conclusion, the left-hand side of Identities (*I*.11) has the natural ordering $k = 1,2,3$, etc., while the right-hand side has, in general, a different ordering, always selected in such a way that Identitie (*I*.11) hold for a given Lagrangian. This situation was illustrated in Section I.3.4 with the example

$$L = \dot{q}_1\dot{q}_2 + \tfrac{1}{2}b(q_1\dot{q}_2 - \dot{q}_1 q_2) + \omega^2 q_1 q_2,$$

$$\begin{pmatrix} \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_1} - \dfrac{\partial L}{\partial q_1} \\ \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_2} - \dfrac{\partial L}{\partial q_2} \end{pmatrix} \equiv \begin{pmatrix} \ddot{q}_2 - b\dot{q}_2 + \omega^2 q_2 \\ \ddot{q}_1 + b\dot{q}_1 + \omega^2 q_1 \end{pmatrix},$$

where Lagrange's equations have the ordering (q_1, q_2), and the equations of motion have the inverted ordering (q_2, q_1). It is hoped that these remarks provide more details on the notion of ordering introduced in Section I.3.4.

is that all the following conditions of variational self-adjointness of the equations of motion are verified in $\mathscr{R}^*$:

$$A_{ij} = A_{ji}, \tag{$\mathscr{I}$.12a}$$

$$\frac{\partial A_{ij}}{\partial \dot{q}^k} = \frac{\partial A_{kj}}{\partial \dot{q}^i}, \tag{$\mathscr{I}$.12b}$$

$$\frac{\partial B_i}{\partial \dot{q}^j} + \frac{\partial B_j}{\partial \dot{q}^i} = 2\left\{\frac{\partial}{\partial t} + \dot{q}^k \frac{\partial}{\partial q^k}\right\} A_{ij}, \tag{$\mathscr{I}$.12c}$$

$$\frac{\partial B_i}{\partial q^j} - \frac{\partial B_j}{\partial q^i} = \frac{1}{2}\left\{\frac{\partial}{\partial t} + \dot{q}^k \frac{\partial}{\partial q^k}\right\}\left(\frac{\partial B_i}{\partial \dot{q}^j} - \frac{\partial B_j}{\partial \dot{q}^i}\right). \tag{$\mathscr{I}$.12d}$$

In this case a Lagrangian can be computed from the equations of motion according to the rule

$$L = -q^k \int_0^1 d\tau [A_{ki}(t, \tau q, \tau \dot{q}) \tau \ddot{q}^i + B_k(t, \tau q, \tau \dot{q})]$$

$$+ \frac{d}{dt} \int_0^1 d\tau \int_0^1 d\tau' \, \tau q^k A_{ki}(t, \tau q, \tau\tau' \dot{q}) \dot{q}^i. \tag{$\mathscr{I}$.13}$$

Alternative methods for the computation of a Lagrangian were provided in Section I.3.6. An interpretation of the structure of the Lagrangian was conducted in Section I.3.7. A number of additional properties and examples completed the study.

In order to complete the analysis of Volume I, in the Appendix of this volume we shall identify the *limitations* of Lagrange's equations, with particular reference to their inability to represent a sufficiently broad class of local, Newtonian, non-potential systems in the frame of the observer. These limitations have motivated the search for a generalization of the Hamiltonian Mechanics reported in the main chapters of this volume.

It should be indicated from the outset that the limitations considered refer to Lagrange's equations of the contemporary literature, those in *first-order Lagrangians* (i.e., Lagrangians for which the maximal total derivative of the dependent variable is of first-order, $L = L(t, q, \dot{q})$). As we shall see in Chapter 4, if this restriction is lifted, and second-order Lagrangians ($L = L(t, q, \dot{q}, \ddot{q})$) are admitted, new possibilities arise. However, the physical implications for acceleration-dependent generalizations of conventional Lagrangians are predictably non-trivial.

Also, the limitations considered exclude the use of velocity-dependent transformations, $q_k \to q'_k(t, q, \dot{q})$, and imply the restriction of the transformation theory to the conventional point transformations of the contemporary literature. This restriction is suggested by a number of open problems inherent in velocity-dependent transformations, including the possible loss of the second-order character of the conventional Lagrange's equations.

The following introductory remarks may be helpful for a better identification of the limitations under consideration.

Consider an unconstrained Newtonian system with self-adjoint forces, such as the one-dimensional harmonic oscillator. The system verifies the conditions of Theorem $\mathscr{I}.3$, a Lagrangian exists, and we can write

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} \equiv (m\ddot{r} + kr)_{\mathrm{SA}}, \qquad (\mathscr{I}.14)$$

$$L = \tfrac{1}{2}(m\dot{r}^2 - kr^2).$$

Suppose now that the system is represented more realistically by adding a drag force linear in the velocity. In this case the system becomes non-self-adjoint, and a Lagrangian for its direct representation does not exist; i.e.,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} \neq [(m\ddot{r} + kr)_{\mathrm{SA}} + \gamma\dot{r}]_{\mathrm{NSA}}. \qquad (\mathscr{I}.15)$$

However, a Lagrangian can still exist for the representation of an equivalent equation of motion, provided that it is self-adjoint. In this way, the problem of the existence of a Lagrangian is reduced to whether a given non-self-adjoint system admits an equivalent self-adjoint form.

The latter problem can be studied with or without the transformation theory. We shall study it first without the transformation theory in order to identify the limits of representational capabilities of Lagrange's equations in the coordinates and time variables actually used in experiments (Section A.1). Once this has been achieved, we shall study the generalization of the methods with the transformation theory (Section A.3).

The condition that the local variables $(t, \mathbf{r}, \dot{\mathbf{r}})$ are not transformed restricts the possibilities of constructing equivalent systems to those provided by the multiplication of a regular matrix of factor functions, or *integrating factors*. The *indirect Lagrangian problem within a fixed system of local variables* can therefore be written in Euclidean space[7]

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^k} - \frac{\partial L}{\partial r^k} \equiv \{h_k^i(t, \mathbf{r}, \dot{\mathbf{r}})[(m\ddot{r}_i - f_i(t, \mathbf{r}, \dot{\mathbf{r}}))_{\mathrm{SA}} - F_i(t, \mathbf{r}, \dot{\mathbf{r}})]_{\mathrm{NSA}}\}_{\mathrm{SA}}, \qquad (\mathscr{I}.16a)$$

$$\det(h_{ij})(t, r, \dot{r})(\mathscr{R}) \neq 0, \qquad (\mathscr{I}.16b)$$

with a self-evident generalization in configuration space under holonomic constraints. The regularity[8] of the matrix (h_{ij}) ensures its invertibility and thus the capability of recovering the equations of motion as they originate from the second law. The equivalence between the original and the transformed equations is then trivial.

[7] Notice that the multiplicative functions $h_i^j(t, \mathbf{r}, \dot{\mathbf{r}})$ *do not* depend on the highest derivative (accelerations) to preserve not only the original solutions but also the structure of Lagrange's equations. The same rule will be used for other types of representations considered later on.

[8] The notion of regularity was introduced in Section I.1.1 and will be elaborated upon shortly.

For the case of the linearly damped harmonic oscillator we have the following solution (Example I.3.2, p.I.210)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} \equiv \{e^{(\gamma/m)t}[(m\ddot{r} + kr)_{SA} + \gamma\dot{r}]_{NSA}\}_{SA}, \qquad (\mathscr{I}.17a)$$

$$L = e^{(\gamma/m)t}\tfrac{1}{2}(m\dot{r}^2 - kr^2). \qquad (\mathscr{I}.17b)$$

As we shall see, despite the capability of representing a considerable class of Newtonian systems, the indirect Lagrangian representations *are not* universal, that is, capable of representing *all* Newtonian systems of the class admitted. This limitation motivates the Birkhoffian generalization of Hamiltonian mechanics to be studied in the subsequent chapters.

The general assumptions used in this volume are the following. The analysis of the preceding volume was conducted for systems of second- (first-) order differential equations which are of class $\mathscr{C}^2$ ($\mathscr{C}^1$), as recalled in Theorem $\mathscr{I}$.1 ($\mathscr{I}$.3). Unless otherwise specified, all systems considered in this volume will be assumed to be *analytic*, that is, admitting a convergent multiple power-series expansion in the neighborhood of a point of the local variables.[9] The condition is essentially suggested by the existence theory of *partial* differential equations which will be used in the proof of the main theorems.[10] For the reader's convenience, we have reviewed in Chart A.1 (A.2) the notion of real (complex) analyticity and in Chart A.3 the Cauchy–Kovalevski theorem on the solution of partial differential equations. As we shall see, this theorem is often useful for the solution of the conditions of self-adjointness in the integrating functions h_i^j.

The smoothness condition above will be referred to a *region* in the space of the local variables, that is, an open neighborhood of a given point. Unless otherwise specified, the open region will be assumed to be *star-shaped* (and denoted with the symbol $\mathscr{R}^*$) in order to comply with the converse of the Poincaré lemma (Section I.1.2). All points considered will be assumed to be *regular points* in the sense reviewed in Chart A.1.

Finally, all systems considered will be assumed to be *regular* in $\mathscr{R}^*$, that is, their functional determinant (Section I.1.1) is non-null in the region conconsidered. The possible existence of a countable number of isolated zeros was considered in the preceding volume and will be ignored here. Recall that the functional determinant of fundamental form ($\mathscr{I}$.9) is given by

$$\mathscr{H}(\mathscr{R}^*) \triangleq |A_{ij}|(\mathscr{R}^*), \qquad (\mathscr{I}.18)$$

[9] Clearly, the condition of analyticity includes that of class $\mathscr{C}^\infty$, but the converse is not necessarily true.

[10] As presented in detail in Volume I, the equations to be represented (Newton's equations of motion) are *ordinary* differential equations, but the equations used for the representation (Lagrange's equations) are *partial* differential equations, as expressed by explicit form ($\mathscr{I}$.10) as well as by the conditions of self-adjointness ($\mathscr{I}$.12). Equivalently, we can say that when a Lagrangian in identities ($\mathscr{I}$.11) is known, the equations are ordinary, but when a Lagrangian must be computed from the equations of motion, the system to be solved consists of partial differential equations.

and that of Lagrange's equations is

$$\mathscr{H}(\mathscr{R}^*) = \left| \frac{\partial^2 L}{\partial \dot{q}^i \, \partial \dot{q}^j} \right| (\mathscr{R}^*). \qquad (\mathscr{I}.19)$$

These determinants are functions of the local variables $(t, q, \dot{q})$ (or $(t, \mathbf{r}, \dot{\mathbf{r}})$). The condition that they are non-null in $\mathscr{R}^*$ has a number of consequences, such as the applicability of the *theorem on implicit functions* (Theorems I.1.1.1 and I.1.1.2 p. I.18–I.20). The existence of the *implicit functions* for all systems ($\mathscr{I}.9$) then follows; they are unique and can be written

$$f^k = (\|A_{lm}\|^{-1})^{ki} B_i. \qquad (\mathscr{I}.20)$$

The corresponding form of the equations of motion is given by

$$\ddot{q}^k - f^k(t, q, \dot{q}) = 0 \qquad (\mathscr{I}.21)$$

and was called the *kinematical form* (Section I.1.1).[11]

Two or more functionally different systems in the same variables will be said to be *equivalent* when their implicit functions f^k coincide. This definition of equivalence is sufficient for the analysis of this volume. In fact, the existence theory for ordinary differential equations as presented in the mathematical literature (see Section I.1.1 for a review) is based on the computation of the implicit functions, reduction of the system to a first-order form, and use of the various techniques for the solution. The identity of the systems of implicit functions then ensures the identity of the solutions.

Two or more systems in different variables are said to be equivalent when the transformations connecting these variables are invertible, single-valued, smoothness-preserving, and leading to the same implicit functions for each considered set of variables. The preservation of the uniqueness of the implicit functions under a change of variables then ensures the equivalence of the systems considered.

Throughout our analysis we shall use the notation SA (self-adjoint) or NSA (non-self-adjoint) for Newtonian forces (systems of differential equations) to denote the verification or lack of verification, respectively, of the integrability conditions for the existence of a potential (or a Lagrangian). Therefore, F_{SA} implies the existence of a potential U according to Theorem $\mathscr{I}.1$, while F_{NSA} implies the violation of at least one condition of this theorem. Similarly, the notation

$$(A_{ki}\ddot{q}^i + B_k)_{\text{SA}} = 0 \qquad (\mathscr{I}.22)$$

[11] The kinematical form of unconstrained Newtonian systems is given by

$$\ddot{\mathbf{r}}_k - \mathbf{F}_k/m_k = 0, \qquad k = 1, 2, \ldots, N.$$

Recall from Section I.2.2 that when the forces are self-adjoint, but the masses are different, the kinematical form is *non-selfadjoint* and, as such, cannot be directly represented via Lagrange's equations. This confirms the importance of the second law in Newtonian mechanics and the related form of the equations of motion, $m\ddot{\mathbf{r}} - \mathbf{F} = 0$.

indicates the property (or condition) that the system verifies Theorem $\mathscr{I}.3$ and that Lagrangian ($\mathscr{I}.13$) exists. Similarly, the notation

$$(A_{ki}\ddot{q}^i + B_k)_{\text{NSA}} = 0 \qquad (\mathscr{I}.23)$$

indicates that the system violates at least one of the conditions of Theorem $\mathscr{I}.3$.

The analysis will be primarily conducted in terms of the independent variable t, the dependent variables $q^k = q^k(t)$, and their derivatives. Cartesian coordinates $\mathbf{r}$ will be used whenever useful to illustrate physical aspects.

CHAPTER 4

Birkhoff's Equations

4.1 Statement of the Problem

As is well-known, the study of Newtonian systems in first-order form permits the achievement of a remarkable symbiosis among *analytic techniques* (e.g., canonical formulations of variational principles), *algebraic techniques* (e.g., theory of Lie algebras), and *geometric techniques* (e.g., the symplectic and contact geometries). The availability of these powerful mathematical tools then renders the study important for several aspects of mechanics, ranging from practical applications (e.g., treatment of systems via the Hamilton–Jacobi theory) to formal problems (e.g., coordinate-free globalizations).

Within the context of the Inverse Problem, the study of systems in first-order form has an additional relevance for achieving *universality*, that is, the representation of "all" systems of the class admitted via a conventional variational principle.[1] A study of the problem reveals that the universality of the Inverse Problem is (at least) threefold. We have first a *direct universality*, that is, universality in the coordinates of the experimenter. We then have an *indirect universality*, that is, universality achieved via transformation theory.

[1] A considerable variety of "variational principles" exists in the literature, ranging from those constructed as particularization of "variational problems" (see Section I.1.3 for details), to special versions whose variations satisfy subsidiary constraints (see footnote[76] of Chart 5.7 for an example). The phrase "conventional variational principles" is used here to stress the fact that the *variations* are the conventional ones of Hamiltonian mechanics. The *actions*, on the contrary, have integrands with unrestricted functional dependence in the local variables.

Finally, we have a *coordinate-free universality* achieved through coordinate-free global geometric techniques.

In Volume I we established the integrability conditions for the existence of a Hamiltonian representation without necessary prior knowledge of a Lagrangian (*Independent Inverse Hamiltonian Problem*); we worked out methods for the computation of a Hamiltonian from the equations of motion; and we identified the analytic, algebraic, and geometrical meanings of the integrability conditions which, predictably, turned out to be the conditions of variational self-adjointness for first-order systems.

In this chapter we establish the direct universality of the Inverse Problem for Newtonian systems which, besides being local, analytic, and regular, are otherwise unrestricted. This includes a large variety of systems of contemporary use in mechanics, such as trajectory problems, spinning and oscillatory motions with damping terms, etc.[2] The direct universality is then extended to systems of arbitrary (but finite) order and dimensionality, as well as arbitrary (i.e., not necessarily Newtonian) interpretation. Indirect universality is studied in the next chapter, and that of coordinate-free type is pointed out in the geometric parts of our analysis.

In this section we review, for the reader's convenience, the main results of the preceding volume on first-order systems, and then reach a more detailed statement of the problem for each of the analytic, algebraic, and geometric profiles. The interrelations between these seemingly different aspects are pointed out too, to illustrate the unity of thought in mechanics.

4.1.1 Reduction of Lagrange's Equations to the Hamiltonian Form.

The reduction (studied in detail in Section I.3.8) is trivial for conservative Lagrangians but not so for arbitrary Lagrangians, in which case a sound knowledge of the Theorem on Implicit Functions (Section I.1.1) and its applications is essential.[3]

[2] Note that the analysis of this volume *excludes* the more general nonlocal (integral) nonpotential (non-self-adjoint) systems. These latter systems call for methods correspondingly more general than those of Lie-symplectic type, such as those of Lie-admissible type (see Chart 4.7).

[3] During my experience as a teacher of mechanics for graduate students, I have found the best opening test of the students' knowledge of Lagrange's and Hamilton's equations is the following.

Assign a Lagrangian with a structure more general than $L = T - V$ and ask the students to compute the equations of motion or the Legendre transform. Unless the students have been specifically exposed to the full form ($\mathscr{I}.10$) of Lagrange's equations or to the methods reviewed in Section I.1.1 for the construction of the implicit functions, they often fail this seemingly simple test. The failure rate on corresponding tests in field theory was even greater, owing to the erroneous way that Lagrange's equations for continuous systems are often written in contemporary literature, as pointed out in footnote[5] of the Introduction. This is not surprising, owing to a rather widespread tendency to remain at the level of a Lagrangian and ignore the equations of motion. The Inverse Problem is intended to prevent or otherwise minimize fundamental deficiencies of this type, because an in-depth knowledge of the structure of Lagrange's and Hamilton's equations is f cused rather naturally.

The reduction is centered in the well-known prescriptions for the characterization of the new independent canonical momenta

$$p_k = \frac{\partial L}{\partial \dot{r}^k} \stackrel{\text{def}}{=} M_k(t, \mathbf{r}, \dot{\mathbf{r}}), \qquad k = 1, 2, \ldots, n, \tag{4.1.1}$$

under the regularity condition

$$\det\left(\frac{\partial M_i}{\partial \dot{r}^j}\right)(\mathscr{R}) = \det\left(\frac{\partial^2 L}{\partial \dot{r}^i \, \partial \dot{r}^j}\right)(\mathscr{R}) \neq 0, \tag{4.1.2}$$

which, together with the assumed smoothness conditions, assures the existence of a unique set of implicit functions in the velocities

$$\dot{r}^k \stackrel{\text{def}}{=} N^k(t, \mathbf{r}, \mathbf{p}). \tag{4.1.3}$$

Once (and only once) implicit functions (4.1.3) have been computed *explicitly*, the Hamiltonian can be expressed in the canonical variables, according to the rule

$$\begin{aligned} H(t, \mathbf{r}, \mathbf{p}) &= \dot{r}^k p_k - L(t, \mathbf{r}, \dot{\mathbf{r}}) \\ &= N^k(t, \mathbf{r}, \mathbf{p}) p_k - L[t, \mathbf{r}, \mathbf{N}(t, \mathbf{r}, \mathbf{p})], \end{aligned} \tag{4.1.4}$$

with underlying (invertible) properties

$$\dot{r}^k = N^k = \frac{\partial H}{\partial p_k} \tag{4.1.5a}$$

$$\frac{\partial L}{\partial r^k} = -\frac{\partial H}{\partial r^k} \tag{4.1.5b}$$

$$\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t} \tag{4.1.5c}$$

Lagrange's equations then become

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^k} = \dot{p}_k = \frac{\partial L}{\partial r^k} = -\frac{\partial H}{\partial r^k}. \tag{4.1.6}$$

The combination of Equations (4.1.5a) and (4.1.6) yields the celebrated *Hamilton's equations without external terms*[4]

$$\begin{cases} \dot{r}^k = \dfrac{\partial H}{\partial p_k}, \\ \dot{p}_k = -\dfrac{\partial H}{\partial r^k}, \end{cases} \qquad k = 1, 2, \ldots, n \tag{4.1.7}$$

[4] The reader should keep in mind from footnote 5 of the Introduction that Equations (4.1.7) are the "truncated Hamilton's equations."

which can be written in the unified form

$$\dot{a}^{\mu} - \omega^{\mu\nu} \frac{\partial H(t, a)}{\partial a^{\nu}} = 0, \qquad \mu = 1, 2, \ldots, 2n, \tag{4.1.8}$$

where

$$a^{\mu} = \begin{cases} r^{\mu}, & \mu = 1, 2, \ldots, n, \\ p_{\mu}, & \mu = n + 1, n + 2, \ldots, 2n, \end{cases} \tag{4.1.9a}$$

$$(\omega^{\mu\nu}) = \begin{pmatrix} 0_{n\times n} & +1_{n\times n} \\ -1_{n\times n} & 0_{n\times n} \end{pmatrix}. \tag{4.1.9b}$$

Equations (4.1.8) are of the *contravariant*[5] type. The equivalent *covariant*[5] form is given by

$$\omega_{\mu\nu} \dot{a}^{\nu} - \frac{\partial H(t, a)}{\partial a^{\mu}} = 0, \tag{4.1.10}$$

where

$$(\omega_{\mu\nu}) = (\|\omega^{\alpha\beta}\|_{\mu\nu}^{-1}) = \begin{pmatrix} 0_{n\times n} & -1_{n\times n} \\ +1_{n\times n} & 0_{n\times n} \end{pmatrix}. \tag{4.1.11}$$

Throughout our analysis we shall ignore conventional form (4.1.7), as we did in the preceding volume, and consider Hamilton's equations only in their unified covariant and contravariant forms. This is done for several reasons. First, the conditions of variational self-adjointness can be readily formulated in the unified a notation, while they become rather impractical to handle in the disjoint (r, p) notation, as we shall see. Second, the use of the unified notation turns out to be particularly valuable for the identification of the desired generalization of Hamilton's equations (Birkhoff's equations). Third, Hamilton's equations in the unified notation exhibit in a rather transparent way the interrelation between the analytic, algebraic, and geometrical profiles according to the following lines.

1. Analytic Profile. The well-known derivability of the equations from *Hamilton's variational principle in phase space* (Section I.1.3) can be written in the unified notation as follows. Introduce the action.

$$\mathscr{A}(\tilde{E}) = \int_{t_1}^{t_2} dt[p_k \dot{r}^k - H(t, \mathbf{r}, \mathbf{p})](\tilde{E})$$

$$\stackrel{\text{def}}{=} \int_{t_1}^{t_2} dt[R_{\nu}^{0}(a)\dot{a}^{\nu} - H(t, \mathbf{r}, \mathbf{p})](\tilde{E})$$

$$R_{\nu}^{0} = \begin{cases} p_{\nu}, & \nu = 1, 2, \ldots, n \\ 0, & \nu = n + 1, n + 2, \ldots, 2n \end{cases} \tag{4.1.12b}$$

[5] Since we have not yet introduced the symplectic geometry, the terms "contravariant" and "covariant" are used, for example, in the sense of the affine geometry of Charts I.A.12 and I.A.13.

where $\tilde{E}$ is a possible path in phase space. The first-order contemporaneous variations with fixed end points then provides the following form of Hamilton's principle,

$$\begin{aligned}\delta\mathscr{A}(\tilde{E}) &= \int_{t_1}^{t_2} dt\left\{\delta a^\mu \frac{\partial}{\partial a^\mu} + \delta\dot{a}^\mu \frac{\partial}{\partial \dot{a}^\mu}\right\}(R^0_a \dot{a}^\nu - H)(\tilde{E}) \\ &= \int_{t_1}^{t_2} dt\left[\left(\frac{\partial R^0_\nu}{\partial a^\mu}\dot{a}^\nu - \frac{\partial H}{\partial a^\mu}\right)\delta a^\mu + R^0_\mu \delta\dot{a}^\mu\right](\tilde{E}) \\ &= \int_{t_1}^{t_2} dt\left[\left(\frac{\partial R^0_\nu}{\partial a^\mu}\dot{a}^\nu - \frac{\partial H}{\partial a^\mu}\right)\delta a^\mu - \dot{R}^0_\mu \delta a^\mu\right](\tilde{E}) \\ &= \int_{t_1}^{t_2} dt\left[\left(\frac{\partial R^0_\nu}{\partial a^\mu} - \frac{\partial R^0_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \frac{\partial H}{\partial a^\mu}\right]\delta a^\mu(\tilde{E}) = 0 \qquad (4.1.13)\end{aligned}$$

which can hold identically if and only if Hamilton's equations hold, i.e.,

$$\left(\frac{\partial R^0_\nu}{\partial a^\mu} - \frac{\partial R^0_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \frac{\partial H}{\partial a^\mu} \equiv \omega_{\mu\nu}\dot{a}^\nu - \frac{\partial H}{\partial a^\mu} = 0, \qquad (4.1.14)$$

where we have used the easily verifyable identities

$$\frac{\partial R^0_\nu}{\partial a^\mu} - \frac{\partial R^0_\mu}{\partial a^\nu} \equiv \omega_{\mu\nu}; \qquad \mu, \nu = 1, 2, \ldots, 2n. \qquad (4.1.15)$$

Note that the equations originating from the variational principle are those of the *covariant* type (4.1.10).

2. Algebraic Profile. The contravariant tensor $\omega^{\mu\nu}$ is called the *fundamental Lie tensor* (or *fundamental cosymplectic tensor*) because it characterizes a fundamental realization of the Lie algebra product, that via the *conventional Poisson brackets*, according to the structure of the *time evolution law* for functions A(a) in phase space

$$\begin{aligned}\dot{A}(a) &= \frac{\partial A}{\partial a^\mu}\dot{a}^\mu = \frac{\partial A}{\partial a^\mu}\omega^{\mu\nu}\frac{\partial H}{\partial a^\nu} \\ &\equiv \frac{\partial A}{\partial r^k}\frac{\partial H}{\partial p_k} - \frac{\partial A}{\partial p_k}\frac{\partial H}{\partial r^k} \stackrel{\text{def}}{=} [A, H]. \qquad (4.1.16)\end{aligned}$$

Note that the form of Hamilton's equations characterizing the algebraic profile is *contravariant*.

3. Geometric Profile.[6] The covariant tensor $\omega_{\mu\nu}$ is called the *fundamental symplectic tensor*, because it characterizes the *fundamental symplectic structure* on the cotangent bundle T^*M with local charts (coordinates) $^\mu$

$$\omega_2 = \frac{1}{2}\omega_{\mu\nu}\, da^\mu \wedge da^\nu \equiv dp_k \wedge dr^k. \qquad (4.1.17)$$

[6] A more technical treatment of the geometric aspect is presented in Charts 4.4–4.6.

Note that the form of Hamilton's equations characterizing the geometric profile is *covariant.*

The deep interrelation between the analytic, algebraic, and geometric profiles is now self-evident. For instance, one can consider the *canonical one-form*

$$R_1^0 = R_\nu^0 \, da^\nu = p_k \, dr^k \tag{4.1.18}$$

as (a component of) the integrand of the action

$$\mathscr{A}(\tilde{E}) = \int_{t_1}^{t_2} (R_\nu^0 \, da^\nu - H \, dt)(\tilde{E}) \tag{4.1.19}$$

and thus of the variational profile. The fundamental symplectic structure is nothing other that the exterior derivative of one-form (4.1.18), i.e.,

$$\begin{aligned} dR_1^0 &= \frac{1}{2}\left(\frac{\partial R_\nu^0}{\partial a^\mu} - \frac{\partial R_\mu^0}{\partial a^\nu}\right) da^\mu \wedge da^\nu \\ &\equiv \tfrac{1}{2}\omega_{\mu\nu} \, da^\mu \wedge da^\nu. \end{aligned} \tag{4.1.20}$$

The fundamental Lie tensor is then given by the elements of the inverse of the matrix of the fundamental symplectic tensor, i.e.,

$$\begin{aligned} \omega^{\mu\nu} &= \left(\left\| \frac{\partial R_\beta^0}{\partial a^\alpha} - \frac{\partial R_\alpha^0}{\partial a^\beta} \right\|^{-1} \right)^{\mu\nu} \\ &= (\|\omega_{\alpha\beta}\|^{-1})^{\mu\nu}. \end{aligned} \tag{4.1.21}$$

As we shall see in the next section, the formulation of Hamilton's equations according to Equations (4.1.8)–(4.1.21) sets the way for a quite natural generalization which is capable of preserving the underlying analytic, algebraic, and geometric characters.

4.1.2 Reduction of Newton's Equations to a First-Order Form.

Suppose that a Newtonian system is assigned as originating from the second law, i.e.,[7]

$$m\ddot{r}_k - F_k(t, \mathbf{r}, \dot{\mathbf{r}}) = 0, \qquad k = 1, 2, \ldots, n. \tag{4.1.22}$$

[7] The generalization to a system of Newtonian particles with different masses is trivial and will not be presented here to avoid unnecessarily complex notations. Theorem 4.5.1 on the direct universality of the Inverse Problem, however, will be formulated and proved for Newtonian systems with different masses.

The Independent Inverse Hamiltonian Problem consists of the *direct* computation of a Hamiltonian, that is, without the intermediate computation of a Lagrangian and subsequent use of the Legendre transform. The problem (studied in detail throughout the preceding volume) is crucially dependent in the reduction of systems (4.1.22) to equivalent first-order forms. The reduction (studied in Sections I.2.4 and I.2.5) is based on the doubling of the number of equations which, in turn, demands the introduction of new independent variables. Since a Lagrangian is not necessarily known, canonical prescriptions (4.1.1) are not necessarily known. However, new independent variables, say, y_k, $k = 1, 2, \ldots, n$, can be introduced via an arbitrary selection of n functions $M_k(t, \mathbf{r}, \dot{\mathbf{r}})$

$$y_k \equiv M_k(t, \mathbf{r}, \dot{\mathbf{r}}), \tag{4.1.23}$$

under the regularity condition

$$\det\left(\frac{\partial M_i}{\partial \dot{r}^j}\right)(\mathscr{R}) \neq 0 \tag{4.1.24}$$

which, together with sufficient smoothness conditions, ensures the existence and uniqueness of the implicit functions in the velocities,

$$\dot{r}^k = N^k(t, \mathbf{r}, \mathbf{y}). \tag{4.1.25}$$

Prescriptions (4.1.23) are conceived to have the same functional dependence of canonical forms (4.1.1). The new variables are denoted with a symbol other than the traditionally used "p" to stress the fact that they are not necessarily canonical. The Independent Inverse Hamiltonian Problem can then be reduced to the integrability conditions for prescriptions (4.1.23) to be canonical. When this property has been ensured (and only then), the variables y_k are canonical, and one can use the identifications $y_k = p_k$ without risking errors in elaborations or applications of the theory.

Once prescriptions (4.1.23) have been selected (and understood), the reduction of system (4.1.22) to an equivalent first-order form is straightforward. In fact, we can write

$$\begin{aligned} m\ddot{r}_k - F_k(t, \mathbf{r}, \dot{\mathbf{r}}) &= m\frac{d}{dt}N_k(t, \mathbf{r}, \mathbf{y}) - F_k[t, \mathbf{r}, \mathbf{N}(t, \mathbf{r}, \mathbf{y})] \\ &= m\left(\frac{\partial N_k}{\partial y_i}\dot{y}_i + \frac{\partial N_k}{\partial r^i}N^i + \frac{\partial N_k}{\partial t}\right) - \tilde{F}_k(t, \mathbf{r}, \mathbf{y}) = 0 \end{aligned} \tag{4.1.26}$$

In view of the assumed regularity conditions, we have

$$\begin{aligned} \dot{y}_k &= \frac{1}{m}\frac{\partial y_k}{\partial N_i}\left(\tilde{F}_i - m\frac{\partial N_i}{\partial r^j}N^j - m\frac{\partial N_i}{\partial t}\right) \\ &\stackrel{\text{def}}{=} O_k(t, \mathbf{r}, \mathbf{y}) \end{aligned} \tag{4.1.27}$$

The combination of Equations (4.1.25) and (4.1.27) then yields the desired form which, in unified notation, can be written

$$\dot{a}^\mu - \Xi^\mu(t, a) = 0, \qquad \mu = 1, 2, \ldots, 2n, \tag{4.1.28a}$$

$$\Xi^\mu = \begin{cases} N^\mu, & \mu = 1, 2, \ldots, n, \\ O_\mu, & \mu = n+1, n+2, \ldots, 2n. \end{cases}$$

$$a^\mu = \begin{cases} r^\mu, \\ y^\mu, \end{cases} \tag{4.1.28b}$$

The equivalence between forms (4.1.22) and (4.1.28) is ensured by the fact that the reduction is everywhere uniquely invertible, which the reader is encouraged to verify.

Equations (4.1.28) are called the *normal first-order form* of the equations of motion, or *Newtonian vector-field form.*[8] A more general form can be achieved by multiplying a regular matrix of functions

$$\begin{pmatrix} h^{(1)}(t, \mathbf{r}, \mathbf{y})_i^k & h^{(2)}(t, \mathbf{r}, \mathbf{y})^{ki} \\ h^{(3)}(t, \mathbf{r}, \mathbf{y})_{ki} & h^{(4)}(t, \mathbf{r}, \mathbf{y})_k^i \end{pmatrix} \begin{bmatrix} \dot{r}^i - N^i(t, \mathbf{r}, \mathbf{y}) \\ \dot{y}_i - O_i(t, \mathbf{r}, \mathbf{y}) \end{bmatrix} = 0, \tag{4.1.29a}$$

$$\det \begin{pmatrix} h^{(1)} h^{(2)} \\ h^{(3)} h^{(4)} \end{pmatrix} (\tilde{R}) \neq 0, \tag{4.1.29b}$$

which can be written in the self-explanatory unified notation

$$C_{\mu\nu}(t, a)\dot{a}^\nu + D_\mu(t, a) = 0, \qquad \mu = 1, 2, \ldots, 2n, \tag{4.1.30a}$$

$$D_\mu = -C_{\mu\nu}\Xi^\nu, \tag{4.1.30b}$$

$$\det(C_{\mu\nu})(\tilde{\mathscr{R}}) \neq 0. \tag{4.1.30c}$$

The above equations are called the *general first-order form* of the equations of motion. Their equivalence with Equations (4.1.22) is trivially ensured by regularity condition (4.1.30c), and the equivalence of form (4.1.28) with the original system. Note that *the normal form is contravariant*, while the *general form is covariant.*[9] This is already sufficient to establish that *only the general first-order form can be derived from a variational principle in a direct way* (that is, without equivalence transformations), while the normal form cannot.

Clearly, among all possible general forms, that which is important for the direct representation via Hamilton's principle is characterized by the identifications $C_{\mu\nu} \equiv \omega_{\mu\nu}$. The corresponding form can be written[10]

$$\omega_{\mu\nu}\dot{a}^\nu - \Xi_\mu(t, a) = 0, \qquad \mu = 1, 2, \ldots, 2n, \tag{4.1.31}$$

[8] We shall call the quantities Ξ^μ *Newtonian vector fields* to distinguish them from the geometric vector fields which are given instead by $\Xi = \Xi^\mu \partial/\partial a^\mu$. See in this latter respect Chart 4.4.

[9] Additional forms were considered in Volume I, such as the *covariant normal form* and the *contravariant general form.* For brevity, these latter forms will not be considered here.

[10] The *fundamental symplectic tensor is not necessarily the lowering tensor of vector field* Ξ^μ. Equivalently (Chart 4.5), *the vector field is not necessarily Hamiltonian.* Equations (4.1.31) represents a scripture which is useful to see, in practice, whether or not the vector field considered is Hamiltonian.

and it is called the *covariant normal form.* In fact, Hamilton's principle in its conventional formulation is capable of recovering only the fundamental symplectic tensor $\omega_{\mu\nu}$, and not the more general tensors $C_{\mu\nu}$.

To summarize, the reduction of second-order Newton's equations to an equivalent first-order form suitable for the Independent Inverse Hamiltonian Problem calls for the implementation of the following steps: (a) select prescriptions (4.1.23); (b) compute the corresponding normal form (4.1.28); and (c) write it in the "*Hamiltonian-type*" form (4.1.31). Explicit examples are given at the end of Chapter I.3.

The degrees of freedom in reaching a general first-order form are clearly essential for the Inverse Hamiltonian Problem. Note that each given system (4.1.22) admits an infinite variety of equivalent normal forms, one per each selected set of prescriptions (4.1.23). As a result, each given Newtonian system admits a double infinity of equivalent general first-order forms, the first characterized by prescriptions (4.1.23) and the second by a multiplicative matrix.

As we shall see, achieving the universality of the Inverse Problem is crucially dependent on these degrees of freedom. Notice that equations of motion in the second-order form admit only a simple infinity of equivalent forms, those characterized by a regular matrix of multiplicative functions, as in ($\mathscr{I}$.16). The reduction to a first-order form therefore doubles the degrees of freedom in writing the equations of motion, with self-evident advantages for the Inverse Problem.

The reduction of third- (and higher) order systems to an equivalent first-order form will be considered in Chart 4.3. The reduction for the case of equations with an arbitrary interpretation in Mechanics, Engineering, Biology, and other branches of science is self-evident, and it will be left to the interested reader.

4.1.3 Conditions of Variational Self-Adjointness for First-Order

The conditions were studied, apparently for the first time, by Santilli (1978c), and then considered in detail in Volume I of the present work (Sections I.2.6, I.2.7, and I.2.8). The quotation of the following theorem may assist the reader in avoiding excessive consultations of the literature.

Theorem 4.1.1 (Self-Adjointness of the Covariant General Form, Theorem I.2.7.2, p. I.87). *A necessary and sufficient condition for a class* $\mathscr{C}^1$ *system* (4.1.30) *to be self-adjoint in a region* $\tilde{\mathscr{R}}$ *of points* (t, a) *is that all the following conditions*

$$C_{\mu\nu} + C_{\nu\mu} = 0, \tag{4.1.32a}$$

$$\frac{\partial C_{\mu\nu}}{\partial a^{\tau}} + \frac{\partial C_{\nu\tau}}{\partial a^{\mu}} + \frac{\partial C_{\tau\mu}}{\partial a^{\nu}} = 0, \tag{4.1.32b}$$

$$\frac{\partial C_{\mu\nu}}{\partial t} = \frac{\partial D_\mu}{\partial a^\nu} - \frac{\partial D_\nu}{\partial a^\mu}, \qquad (4.1.32c)$$

$$\mu, \nu, \tau = 1, 2, \ldots, 2n$$

are identically verified in $\tilde{\mathcal{R}}$.

If at least *one* condition in Equation (4.1.32) is violated, system (4.1.30) is called *non-self-adjoint.* Note that the continuity conditions $C_{\mu\nu}$, $D_\mu \in \mathscr{C}^1$ are sufficient for the formulation and proof of the theorem. Nevertheless, its use throughout this volume will refer to the more restrictive condition of analyticity. Note also that the conditions of self-adjointness are formulated for the *covariant* form because (as indicated earlier), this is the form derivable from a variational principle. Finally, note that conditions (4.1.32) *do not* require linearity of the equations in the local variables.

The fundamental symplectic tensor $\omega_{\mu\nu}$ verifies identically conditions (4.1.32a) and (4.1.32b), as becomes transparent when written in form (4.1.15). The study of the self-adjointness of forms (4.1.31), either in a way independent from the preceding ones (as presented below), or via suitable particularizations of Theorems 4.1.1 or $\mathscr{I}$.1, yields the following result.

Theorem 4.1.2 (Self-Adjointness of the Covariant Normal Form, Theorem I.2.7.3, p. I.88). *A necessary and sufficient condition for a class* $\mathscr{C}^1$ *system* (4.1.31) *to be self-adjoint in a region* $\tilde{\mathcal{R}}$ *of points* (t, a) *is that all the conditions*

$$\frac{\partial \Xi_\mu}{\partial a^\nu} - \frac{\partial \Xi_\nu}{\partial a^\mu} = 0, \qquad \mu, \nu = 1, 2, \ldots, 2n, \qquad (4.1.33)$$

are identically verified in $\tilde{\mathcal{R}}$.

We should stress that, at this stage, the tensor $\omega_{\mu\nu}$ is selected, independently from any geometric consideration, as a solution with constant elements of conditions (4.1.32a) and (4.1.32b). It is understood that this is a very special solution, and that more general solutions exist. This was pointed out in Volume I for the intent of studying it in more detail in this volume.

4.1.4 The Independent Inverse Hamiltonian Problem

This problem was studied in Section I.3.10 through I.3.12. The first step was the characterization of Hamilton's equations via the variational approach to self-adjointness.

Theorem 4.1.3 (Self-Adjointness of Hamilton's equations, Theorem I.3.10.1, p. I.170). *Under the assumptions that the Hamiltonian* $H(t, a)$ *is*

of at least class $\mathscr{C}^2$ and regular[11] *in a region $\tilde{\mathscr{R}}$ of points (t, a), the covariant normal form* (4.1.10) *of Hamilton's equations is always self-adjoint in $\tilde{\mathscr{R}}$ (that is, self-adjoint for all possible Hamiltonians).*

In essence, Hamilton's equations turn out to be self-adjoint in exactly the same measure as Lagrange's equations (Theorem $\mathscr{I}$.2). This confirmed the expectation that the conditions of variational self-adjointness are the integrability conditions for the derivation of a system from a variational principle, regardless of its order and dimensionality.

The second step was the introduction of the notion of *ordered direct representation of a covariant normal form via Hamilton's equations* (see Section I.3.11 for detail)

$$\omega_{\mu\nu}\dot{a}^{\nu} - \frac{\partial H}{\partial a^{\mu}} \equiv \omega_{\mu\nu}\dot{a}^{\nu} - \Xi_{\mu}, \qquad \mu = 1, 2, \ldots, 2n. \tag{4.1.34}$$

Theorem 4.1.4 followed by recalling (1) the methods for the construction of the right-hand side of the identities; (2) the variational self-adjointness of left-hand-side; and (3) the use of the Calculus of Differential Forms for the computation of a Hamiltonian from the equations of motion.

Theorem 4.1.4 (Fundamental Analytic Theorem for Phase Space Formulations, Theorems I.3.12.1 and I.3.12.2, p. I.176). *A necessary and sufficient condition for a local holonomic generally nonconservative Newtonian system in a covariant normal form* (4.1.31), *which is well defined and of (at least) class $\mathscr{C}^1$ in a star-shaped region $\tilde{\mathscr{R}}^*$ of the variables (t, a), to admit an ordered*

[11] Recall from Section I.3.8 that a Hamiltonian is *regular* when

$$\det\left(\frac{\partial^2 H}{\partial p_i\, \partial p_j}\right)(\tilde{\mathscr{R}}) \neq 0$$

and that

$$\det\left(\frac{\partial^2 H}{\partial p_i\, \partial p_j}\right) = \left[\det\left(\frac{\partial^2 L}{\partial \dot{q}^i\, \partial \dot{q}^j}\right)\right]^{-1}.$$

Thus the Legendre transform preserves the regularity of the functions. This notion of *regularity of the Hamiltonian function* should be differentiated from the *regularity of Hamilton's equations* when defined in terms of the *functional determinant* (Section I.1.1). In fact, Hamilton's equations can be written

$$K_{\mu}(t, a, \dot{a}) \overset{\text{def}}{=} \omega_{\mu\nu}\dot{a}^{\nu} - \frac{\partial H}{\partial a^{\mu}} = 0.$$

The functional determinant is then given by

$$\det\left(\frac{\partial K_{\mu}}{\partial \dot{a}^{\nu}}\right)(\tilde{\mathscr{R}}) = \det(\omega_{\mu\nu}) = 1,$$

and it is always regular, regardless of the regularity or degeneracy of the Hamiltonian. This is no contradiction but only a differentiation of objectives. The regularity of the Hamiltonian is conceived for the construction of equivalent second-order forms in the sense of Section I.3.8, while the regularity of Hamilton's equation is conceived in the functional sense of Section I.1.1.

direct analytic representation in terms of Hamilton's equations is that form (4.1.31) *is self-adjoint in* $\tilde{\mathscr{R}}^*$*—that is, all of conditions* (4.1.33) *are identically verified in* $\tilde{\mathscr{R}}^*$*. Under these conditions, a Hamiltonian exists and can be explicitly computed from the equations of motion according to the method*[12]

$$H(t, a) = a^\mu \int_0^1 d\tau \, \Xi_\mu(t, \tau a). \tag{4.1.35}$$

A more general result was achieved via the use of the Cauchy method (rather than the converse of the Poincaré Lemma), and can be written (see Section I.3.12, and footnote 79, p. I.179, in particular)

$$H(t, a) = (a^\mu - a_0^\mu) \int_0^1 d\tau \, \Xi_\mu(t, \tau a + (1 - \tau)a_0), \tag{4.1.36}$$

where the a_0's are constants.

Clearly, whenever Theorem 4.1.4 is verified, prescriptions (4.1.23) are canononical and $y_k = p_k$. Thus the theorem does provide the integrability conditions for the selection of canonical prescriptions as desired. This can be practically implemented according to the following steps: (i) select prescriptions (4.1.23) with n arbitrary functions (and verify that the points under consideration are regular in the sense of Chart A.1); (ii) impose the conditions of variational self-adjointness (4.1.33) on the resulting covariant normal form to identify the n arbitrary functions; and, in case of a positive answer, (iii) compute a Hamiltonian via method (4.1.35) or (4.1.36). For an illustrative case worked out in detail, the reader may consult Example I.3.1, p. I.206.

An important point is that *Theorem* 4.1.4 *does not ensure the existence of a Hamiltonian.* The theorem merely provides the integrability conditions for its existence.

It can be shown that Theorem 4.1.4 *does not* necessarily admit a solution. In fact, if such a solution would always exist, the Indirect Lagrangian Problem ($\mathscr{I}$.16) always admits a solution, which is not the case (see the Appendix).

A moment of reflection is appropriate here. Recall that the existence of a Hamiltonian implies the applicability of an articulated body of established, analytic, algebraic, and geometric tools, ranging from the Hamilton-Jacobi equations (and related quantization) to the canonical realization of Lie's theory (and related symmetries), etc. If a Hamiltonian does not exist, all these formulations are not applicable in the coordinate and time variables of the observer. In turn, this has a rather profound physical meaning regarding the structure of the systems considered and the applicable relativity, as we shall see.

[12] The geometrical interpretation of Theorem 4.1.4 will first be reviewed in Section 4.3 and then treated in more technical details in Chart 4.6.

The following definition permits a classification of systems with respect to the integrability conditions for the existence of a (Lagrangian or a) Hamiltonian

Definition 4.1.1. Local, analytic, regular, Newtonian, second-order systems are subdivided into the following three classes of increasing structural complexity and methodological needs.

1. *Essentially Self-Adjoint Systems* (ESA). These are systems which verify the integrability conditions for the existence of a Direct Lagrangian Representation ($\mathscr{I}.8$) (Theorem $\mathscr{I}.3$) in their form originating from Newton's second law. The verification of the integrability conditions for the existence of a Hamiltonian (Theorem 4.1.4) can then be trivially proved.
2. *Non-Essentially Non-Self-Adjoint Systems* (NENSA). These are systems which, as originating from the second law, violate the integrability conditions of Theorem $\mathscr{I}.3$. Nevertheless, they admit an Indirect Lagrangian Representation ($\mathscr{I}.16$) (see Theorem A.1.1 for details). The existence of suitable prescriptions (4.1.23) for a Hamiltonian representation can be proved, and Theorem 4.1.4 is verified.
3. *Essentially Non-Self-Adjoint* (ENSA). These are systems for which the integrability conditions for the existence of a Lagrangian (Theorem A.1.1) or, equivalently, of a Hamiltonian (Theorem 4.1.4) are inconsistent within the coordinate and time variables of the experimenter.

The increase in structural complexity can be illustrated as follows. ESA systems admit the conventional Lagrangian $L = \frac{1}{2}m\dot{\mathbf{r}}^2 - V \stackrel{\text{def}}{=} L_{\text{free}} + L_{\text{int}}$. The corresponding conventional Hamiltonian can be written $H = H_{\text{free}} + H_{\text{int}}$.

In the transition to the NENSA systems, these conventional functions are insufficient, and structurally more general ones are needed. They have been written (see Section I.3.7) $L = L_{\text{int,I}} L_{\text{free}} + L_{\text{int,II}}$ and $H = H_{\text{int,I}} H_{\text{free}} + H_{\text{int,II}}$ where the multiplicative interaction terms originate from the matrix of integrating factors in (I.16).

In the transition to the ENSA systems, even these generalized Lagrangians and Hamiltonians are insufficient to represent the motion. This is a sign that we have gone beyond the capabilities of conventional Lagrangian and Hamiltonian techniques, and that more general techniques are needed.

It is hoped that the term "essentially self-adjoint," referred to as a *variational* property of systems of ordinary differential equations, does not create confusion with the same term used in the context of the theory of linear *operators* on vector spaces. Actually, this term has been selected precisely because of the parallelism between the variational and operational approaches to self-adjointness, as indicated in Section I.2.8. Furthermore, the variational approach to self-adjointness, though ignored of late, is older than the corresponding operational approach; and the use of "essentially self-adjoint" for a purely Newtonian setting is also intended to stress this historical aspect.

As a final remark, let us note that a number of systems which are essentially non-self-adjoint in our terminology have been identified by Douglas (1941). Some of them will be considered in Example 4.6. We shall then show that, for these systems, a Hamiltonian does not exist, yet the systems admit a Birkhoffian representation.

4.1.5 Analytic, Algebraic, and Geometric Meaning of the Conditions of Variational Self-Adjointness

Earlier in this section we reviewed the analytic, algebraic, and geometric properties of Hamilton's equations. Now, we review the result of the preceding volume according to which all these properties are expressed in a rather symbiotic way by the conditions of variational self-adjointness. In this chapter we show that the same conditions, and therefore, the same properties, are actually shared by equations structurally more general than Hamilton's equations.

With reference to aspects (1), (2), and (3) considered earlier in this section, we have the following results.

1′. Analytic Profile. The conditions of variational self-adjointness are the integrability conditions for systems of differential equations to be derivable from a variational principle, as established by the Fundamental Analytic Theorems in Configurations and Phase Space.

A point which is particularly important for the analysis of this volume is that the conditions admit equations which are structurally more general than Hamilton's equations, as clearly illustrated by the difference between Theorems 4.1.1 and 4.1.2.

In this way we reach the *analytic statement of our problem*, consisting of:

1. the identification, via the conditions of variational self-adjointness, of a generalization of Hamilton's equations capable of preserving their derivability from a variational principle;
2. the formulation of methods for the computation of the most general possible integrand of an action for equations of motion in first-order self-adjoint form; and
3. the proof that the approach achieves the desired direct universality.

2′. Algebraic Profile. Recall that the *conventional Poisson brackets*

$$[A, B] = \frac{\partial A}{\partial a^{\mu}} \omega^{\mu\nu} \frac{\partial B}{\partial a^{\nu}}$$

$$\equiv \frac{\partial A}{\partial r^{k}} \frac{\partial B}{\partial p_{k}} - \frac{\partial B}{\partial r^{k}} \frac{\partial A}{\partial p_{k}}, \qquad \det(\omega^{\mu\nu}) = 1, \tag{4.1.37}$$

are the simplest conceivable realization in Newtonian mechanics of a bilinear product satisfying the *Lie algebra axioms*

$$[A, B] + [B, A] = 0, \tag{4.1.38a}$$

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0. \tag{4.1.38b}$$

The most general known (regular) realization is given by the brackets[13]

$$[A, B]^* = \frac{\partial A}{\partial a^\mu}\,\Omega^{\mu\nu}(t, a)\,\frac{\partial B}{\partial a^\nu}, \qquad \det(\Omega^{\mu\nu})(\tilde{\mathscr{R}}) \neq 0 \tag{4.1.39}$$

where the $\Omega^{\mu\nu}$ tensor is a generalization of the Hamiltonian form $\omega^{\mu\nu}$ such that

$$[A, B]^* + [B, A]^* = 0, \tag{4.1.40a}$$

$$[[A, B]^*, C]^* + [[B, C]^*, A]^* + [[C, A]^*, B]^* = 0, \tag{4.1.40b}$$

in which case the brackets are called *generalized Poisson brackets.*

Similarly, the *conventional Lagrange brackets*

$$\{A, B\} = \frac{\partial a^\mu}{\partial A}\,\omega_{\mu\nu}\,\frac{\partial a^\nu}{\partial B}$$

$$= \frac{\partial p_k}{\partial A}\frac{\partial r^k}{\partial B} - \frac{\partial p_k}{\partial B}\frac{\partial r^k}{\partial A}, \qquad (\omega_{\mu\nu}) \equiv (\omega^{\mu\nu})^{-1} \tag{4.1.41}$$

are the simplest conceivable realization of conditions

$$\{A, B\} + \{B, A\} = 0, \tag{4.1.42a}$$

$$\frac{\partial}{\partial A}\{B, C\} + \frac{\partial}{\partial B}\{C, A\} + \frac{\partial}{\partial C}\{A, B\} = 0, \tag{4.1.42b}$$

and they can be interpreted as the "inverse" of brackets (4.1.37) in the sense of the properties expressed in terms of $2n$ independent functions $A_i(a)$

$$\sum_{k=1}^{2n} [A_i, A_k]\{A_k, A_j\} = \delta_{ij}. \tag{4.1.43}$$

The most general known (regular) realization is given by the brackets

$$\{A, B\}^* = \frac{\partial a^\mu}{\partial A}\,\Omega_{\mu\nu}(t, a)\,\frac{\partial a^\nu}{\partial B}, \qquad \det(\Omega_{\mu\nu})(\tilde{\mathscr{R}}) \neq 0 \tag{4.1.44}$$

under the conditions

$$\{A, B\}^* + \{B, A\}^* = 0, \tag{4.1.45a}$$

$$\frac{\partial}{\partial A}\{B, C\}^* + \frac{\partial}{\partial B}\{C, A\}^* + \frac{\partial}{\partial C}\{A, B\}^* = 0. \tag{4.1.45b}$$

[13] Note the appearance of the explicit dependence of the algebraic tensor $\Omega^{\mu\nu}$ not only on the local coordinates $a = (\mathbf{r}, \mathbf{p})$ but also, in general, on time, while no such dependence appears for the conventional Poisson brackets. As we shall see, this functional generalization is the basis for the direct universality of the Inverse Problem for local systems.

It is possible to prove that, if the generalized Lagrange and Poisson brackets are related by the rule

$$(\Omega_{\mu\nu}) = (\Omega^{\mu\nu})^{-1}, \tag{4.1.46}$$

they verify the generalized version of Equations (4.1.43).

$$\sum_{k=1}^{2n} [A_i, A_k]^*\{A_k, B_j\}^* = \delta_{ij}, \tag{4.1.47}$$

by therefore preserving the property of one being the "inverse" of the other. Also, it is possible to prove that, if brackets (4.1.39) verify axioms (4.1.40), the generalized Lagrange's brackets constructed via rule (4.1.46) automatically verify condition (4.1.45), and vice versa. Put differently, under rule (4.1.46), conditions (4.1.40) and (4.1.45) are equivalent.

These aspects were studied in detail in Section I.2.9. Here we restrict ourselves to recalling the following properties: (a) that the integrability conditions for Lie's axioms (4.1.40) are given by

$$\Omega^{\mu\nu} + \Omega^{\nu\mu} = 0, \tag{4.1.48a}$$

$$\Omega^{\tau\rho}\frac{\partial\Omega^{\mu\nu}}{\partial a^\rho} + \Omega^{\mu\rho}\frac{\partial\Omega^{\nu\tau}}{\partial a^\rho} + \Omega^{\nu\rho}\frac{\partial\Omega^{\tau\mu}}{\partial a^\rho} = 0; \tag{4.1.48b}$$

(b) those for conditions (4.1.45) are

$$\Omega_{\mu\nu} + \Omega_{\nu\mu} = 0, \tag{4.1.49a}$$

$$\frac{\partial\Omega_{\mu\nu}}{\partial a^\tau} + \frac{\partial\Omega_{\nu\tau}}{\partial a^\mu} + \frac{\partial\Omega_{\tau\mu}}{\partial a^\nu} = 0; \tag{4.1.49b}$$

and (c) conditions (4.1.48) and (4.1.49) are equivalent because each set can be reduced to the other via simple algebraic manipulations.

All these algebraic properties are contained in the conditions of self-adjointness, as expressed by the following theorem.

Theorem 4.1.5 Direct and Indirect Algebraic Significance of the Conditions of Self-Adjointness, Theorems I.2.9.1 and I.2.9.2, p. I.94). *The direct significance of the self-adjointness conditions for first-order systems is that conditions* (4.1.32*a*) *and* (4.1.32*b*) *coincide with the integrability conditions* (4.1.49) *for brackets* (4.1.44) *to be generalized Lagrange brackets, that is, to verify axioms* (4.1.45). *The indirect significance is that conditions* (4.1.32*a*) *and* (4.1.32*b*) *are equivalent to the integrability conditions* (4.1.48) *for brackets* (4.1.39) *to be generalized Poisson brackets, that is, to verify Lie algebra axioms* (4.1.40).

Stated in different terms, *the self-adjointness of a covariant general form ensures its Lie algebra character*, in the sense that the brackets characterized by the inverse $(C^{\mu\nu})$ of the matrix $(C_{\mu\nu})$ are Lie. This algebraic meaning has been called "indirect" in Theorem 4.1.5 to emphasize that the conditions of self-adjointness are formulated for the *covariant* form, while a Lie tensor is of *contravariant* type.

The *algebraic statement of the problem* can now be formulated as follows. Recall that Hamilton's equations are expected to be one of the simplest possible forms of equations derivable from a variational principle (Profile (1′)), and they characterize one of the simplest possible realizations of the Lie algebra product (Profile (2′)). Upon identifying the largest possible class of equations that can be derived from a variational principle (analytic problem), we shall study its algebraic structure via the most general possible (regular) realization of the Lie algebra product in Newtonian mechanics. We shall then study the problem of whether or not such a generalization demands a corresponding reformulation of Lie's theory. This latter issue is created by the fact, as we shall see later, that the abstract treatments of Lie's theory have been historically patterned along conventional realizations of the Lie product. A nontrivial generalization of the product then raises the question of whether central theorems of Lie's theory (such as Lie's first, second, and third theorem, the Poincaré–Birkhoff–Witt theorem, Ado's theorem, etc.) apply in their currently available formulation to the generalized realization of the product also, or if they need suitable reformulations.[14]

3′. Geometric Profile. In a way fully parallel to the algebraic profile, the fundamental symplectic structure (4.1.17) is one of the simplest possible closed and exact two-forms on T*M. The most general two-form verifying these properties can be written in local coordinates a^μ at a fixed time t[15]

$$\Omega_2 = \tfrac{1}{2}\Omega_{\mu\nu}(t, a)da^\mu \wedge da^\nu = dR_1, \tag{4.1.50a}$$

$$d\Omega_2 = 0, \qquad \det(\Omega_{\mu\nu}) \neq 0. \tag{4.1.50b}$$

The geometric significance of the conditions of self-adjointness (identified in Chart I.2.5) is that the integrability conditions for a two-form to be an exact symplectic form coincide with conditions (4.1.32a) and (4.1.32b). The ultimate symbiosis between geometric, algebraic, and analytic aspects is further illustrated by the following implications of the existence of a primitive one-form R_1 ($\neq R_1^0$): (i) the exact symplectic character of the two-form; (ii) the Lie algebra character of the brackets of the time evolution; and (iii) the derivability of the equations of motion from a variational principle.

These aspects can be better expressed by performing the extension from the *symplectic geometry* on the cotangent bundle T^*M with local coordinates a^μ to the *contact geometry* on the manifold $\mathbb{R} \times T^*M$ with local coordinates

$$\hat{a}^\mu = \begin{cases} t, & \mu = 0 \\ a^\mu, & \mu = 1, 2, \ldots, 2n. \end{cases} \tag{4.1.51}$$

[14] As we shall see in Chapter 5, the generalization of the Lie algebra product we are considering implies nontrivial generalizations of a truly central part of Lie's theory: the universal enveloping associative algebra. A reinspection then of Lie's theory is rather natural.

[15] The reader with a background in geometry will have noted an explicit *time dependence* in the *symplectic* structure. Such a dependence is not admitted, in general, in the current local formulations of the contact geometry. Nevertheless, as we shall see at the end of Section 4.2, the emergence of such a dependence is rather natural in practical applications. The geometrical implications will be indicated in Chart 4.6.

The local formulations[16] of contact two-forms which become necessary are of the exact type[17]

$$\hat{C}_2 = \tfrac{1}{2}\hat{C}_{\mu\nu}(\hat{a})d\hat{a}^\mu \wedge d\hat{a}^\nu = d\hat{R}_1, \tag{4.1.52a}$$

$$(\hat{C}_{\mu\nu}) = \left(\begin{array}{c|c} \hat{C}_{00} & \hat{C}_{0\mu} \\ \hline \hat{C}_{\mu 0} & \hat{C}_{\mu\nu} \end{array}\right) = \left(\begin{array}{c|c} 0 & -D_\mu \\ \hline D_\mu & C_{\mu\nu} \end{array}\right), \tag{4.1.52b}$$

under the condition that the attached two-forms on T^*M are closed, non-degenerate, and exact (thus symplectic), but otherwise arbitrary, i.e.,

$$\begin{aligned} \hat{C}_2|_{T^*M} = \Omega_2 &= \tfrac{1}{2}C_{\mu\nu}(t, a)da^\mu \wedge da^\nu \\ &= dR_1(t, a). \end{aligned} \tag{4.1.53}$$

The integrability conditions for a tensorial two-form on $\mathbb{R} \times T^*M$ to be an (exact) contact form are given by

$$\delta^{\mu_1\mu_2}_{\nu_1\nu_2}\hat{C}_{\mu_1\mu_2} = 0, \qquad \delta^{\mu_1\mu_2\mu_3}_{\nu_1\nu_2\nu_3}\frac{\partial \hat{C}_{\mu_1\mu_2}}{\partial \hat{a}^{\mu_3}} = 0, \tag{4.1.54}$$

$$\nu_1, \nu_2, \nu_3 = 0, 1, 2, \ldots, 2n$$

where the *generalized Kronecker symbols* (Section I.1.2) are given by

$$\delta^{\mu_1\mu_2}_{\nu_1\nu_2} = \begin{vmatrix} \delta^{\mu_1}_{\nu_1} & \delta^{\mu_1}_{\nu_2} \\ \delta^{\mu_2}_{\nu_1} & \delta^{\mu_2}_{\nu_2} \end{vmatrix}, \qquad \delta^{\mu_1\mu_2\mu_3}_{\nu_1\nu_2\nu_3} = \begin{vmatrix} \delta^{\mu_1}_{\nu_1} & \delta^{\mu_1}_{\nu_2} & \delta^{\mu_1}_{\nu_3} \\ \delta^{\mu_2}_{\nu_1} & \delta^{\mu_2}_{\nu_2} & \delta^{\mu_2}_{\nu_3} \\ \delta^{\mu_3}_{\nu_1} & \delta^{\mu_3}_{\nu_2} & \delta^{\mu_3}_{\nu_3} \end{vmatrix}. \tag{4.1.55}$$

The ultimate geometric meaning of the conditions of self-adjointness is given by the fact that integrability condition (4.1.54), once written explicitly in disjoint coordinates t and a^μ, coincide with the entirety of conditions of self-adjointness (4.1.32), as the reader is encouraged to verify.

The *geometric statement of the problem* can now be made more precise. It essentially consists of establishing the direct universality of the contact geometry for local, analytic,[18] and regular systems. By recalling the lack of direct universality of Theorem 4.1.4, this statement of the problem implies the search for a suitable generalization of Hamilton's equations, with corresponding reinspection of Lie's theory.

[16] Equations (4.1.52) illustrate rather clearly the *local–differential character of the contact (and symplectic) geometry* and the need for more general, *nonlocal/integro–differential geometries* for the treatment of systems of type ($\mathscr{I}$.3).

[17] From now on we shall tacitly assume that, when the symbols at hand are written with an upper hat, e.g., $\hat{a}^\mu$, $\hat{C}_{\mu\nu}$, etc., the greek indices run from 0 to $2n$. When the upper hat is absent, the greek indices run from 1 to $2n$.

[18] The smoothness condition used rather universally in the contemporary literature of the symplectic and contact geometries is that of class $\mathscr{C}^\infty$. The study of systems which are of class $\mathscr{C}^\infty$ but are not analytic is expected to be complex as well as of limited practical value, and we shall ignore it.

4.2 Birkhoff's Equations

The generalization of Hamilton's equations we shall study is given by

$$\left[\frac{\partial R_\nu(t, a)}{\partial a^\mu} - \frac{\partial R_\mu(t, a)}{\partial a^\nu}\right]\dot{a}^\nu - \left[\frac{\partial B(t, a)}{\partial a^\mu} + \frac{\partial R_\mu(t, a)}{\partial t}\right] = 0,$$
$$a = (\mathbf{r}, \mathbf{y}), \qquad \mu = 1, 2, \ldots, 2n. \tag{4.2.1}$$

The following terminology suggested by Santilli (1978c) will be used in this volume. Equations (4.2.1) are called *Birkhoff's equations* for certain historical reasons reviewed at the end of this section. The function $B(t, a)$ is called the *Birkhoffian*, because of certain physical differences with the Hamiltonian which will be indicated in the next sections. Finally, a representation of Newton's equations via Birkhoff's equations is called a *Birkhoffian representation* when certain conditions, identified in detail in the next section, are met.

Birkhoff's equations are clearly a generalization of Hamilton's equations because the latter are recovered from the former as in the particular case of

$$\left\{\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \left(\frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t}\right)\right\}\Bigg|_{\substack{y=p \\ R=R_0=(\mathbf{p},\mathbf{O}) \\ B=H=H(t,\mathbf{r},\mathbf{p})}} \equiv \omega_{\mu\nu}\dot{a}^\nu - \frac{\partial H}{\partial a^\mu} = 0. \tag{4.2.2}$$

In this section we shall prove that Birkhoff's equations

1. originate from the most general possible linear first-order[19] variational principle;
2. characterize the most general possible regular realization of the Lie algebra product via the brackets of a classical time evolution; and
3. admit the most general possible exact symplectic (or contact) structures in local coordinates.

In order to study these important properties in the necessary detail, the introduction of the following terminology is advantageous.

Definition 4.2.1. Birkhoff's equations (4.2.1) are called *autonomous* when the R_μ and B functions do not depend explicitly on time, in which case the equations assume the simplified form

$$\Omega_{\mu\nu}(a)\dot{a}^\nu - \frac{\partial B(a)}{\partial a^\mu} = 0, \tag{4.2.3}$$

where

$$\Omega_{\mu\nu} = \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu} \tag{4.2.4}$$

[19] A *linear first-order* variational principle occurs when the integrand depends at most on first-order derivatives and the dependence is linear.

is called *Birkhoff's tensor*. They are called *semi-autonomous* when the R_μ functions (the Birkhoffian) do not (do) depend explicitly on time, in which case we have the more general form

$$\Omega_{\mu\nu}(a)\dot{a}^\nu - \frac{\partial B(t, a)}{\partial a^\mu} = 0. \tag{4.2.5}$$

Birkhoff's equations are called *nonautonomous* when both the R_μ and B functions have an explicit dependence on time, in which case we have form (4.2.1), which we rewrite

$$\Omega_{\mu\nu}(t, a)\dot{a}^\nu - \frac{\partial B(t, a)}{\partial a^\mu} - \frac{\partial R_\mu(t, a)}{\partial t} = 0. \tag{4.2.6}$$

They are called *regular* when their functional determinant is not null in the region considered:

$$\det(\Omega_{\mu\nu})(\tilde{\mathscr{R}}) \neq 0. \tag{4.2.7}$$

They are called *degenerate* when their functional determinant is identically null in the region considered:

$$\det(\Omega_{\mu\nu})(\tilde{\mathscr{R}}) \equiv 0. \tag{4.2.8}$$

They are said to be *covariant* when they are of type (4.2.3), (4.2.5), or (4.2.6), in which case the corresponding tensor $\Omega_{\mu\nu}$ is called the *covariant Birkhoff's tensor*; and they are said to be *contravariant*, when the nonautonomous equations are written in the equivalent form

$$\dot{a}^\mu - \Omega^{\mu\nu}(t, a)\left[\frac{\partial B(t, a)}{\partial a^\nu} + \frac{\partial R_\nu(t, a)}{\partial t}\right] = 0, \tag{4.2.9}$$

where the tensor

$$\Omega^{\mu\nu} = (\|\Omega_{\alpha\beta}\|^{-1})^{\mu\nu} = \left(\left\| \frac{\partial R_\beta}{\partial a^\alpha} - \frac{\partial R_\alpha}{\partial a^\beta} \right\|^{-1}\right)^{\mu\nu} \tag{4.2.10}$$

is called the *contravariant Birkhoff's tensor*. Finally, Birkhoff's equations are called *strictly regular* when, in addition to condition (4.2.7), the underlying contravariant normal form

$$(\dot{a}^\mu) = \begin{pmatrix} \dot{r}^k \\ \dot{y}_k \end{pmatrix} = (\Xi^\mu(t, a)) = \begin{pmatrix} N^k(t, \mathbf{r}, \mathbf{y}) \\ O_k(t, \mathbf{r}, \mathbf{y}) \end{pmatrix}, \tag{4.2.11a}$$

$$\Xi^\mu = \Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t}\right), \tag{4.2.11b}$$

verifies the regularity condition

$$\det\left(\frac{\partial N_i}{\partial y_j}\right)(\tilde{\mathscr{R}}) \neq 0. \tag{4.2.12}$$

In this volume we shall study only *strictly regular* Birkhoff's equations in their various forms (autonomous, semi-autonomous, nonautonomous,

covariant, and contravariant). Recall that Hamilton's *equations* are always regular when the definition of regularity is based on the functional determinant. Yet the equations can be transformed into an equivalent second-order form only when the *Hamiltonian* is regular.[11]

A similar situation occurs for Birkhoff's equations. In fact, the determinant of the covariant Birkhoff's tensor is the functional determinant of system (4.2.6). The condition of regularity (4.2.7) is, therefore, a generalization of the nondegeneracy of Hamilton's equations. However, condition (4.2.7) *does not* ensure the capability to transform Birkhoff's equations into an equivalent second-order form. In fact, this transformation demands the existence of implicit functions of type (4.1.23) which can exist (and be unique) only under the additional regularity condition (4.2.12).[20]

The transformation of strictly regular Birkhoff's equations to their equivalent second-order form[21] is then straightforward. One can reduce the equations to the contravariant normal form (4.2.11) and compute the implicit functions $y_k = M_k(t, \mathbf{r}, \dot{\mathbf{r}})$ of the first set of equations $\dot{r}^k = N^k(t, \mathbf{r}, \mathbf{y})$. The second-order equations are then given by eliminating the y-dependence from the second set of equations, $\dot{y}_k = O_k(t, \mathbf{r}, \mathbf{y})$. For an illustration of the degenerate case, see Example 4.5.

By recalling that we are primarily interested in the representation of second- (or higher) order equations of Newtonian (or arbitrary) interpretation, the need for the strict regularity as per Definition 4.2.1 is now self-evident.

The following Birkhoffian generalization of Hamiltonian properties (1), (2), and (3) which was pointed out in Section 4.1 holds.

A. Generalized Analytic Formulations. The Hamiltonian action functional

$$\mathscr{A}(\tilde{E}) = \int_{t_1}^{t_2} dt[R_\nu^0(a)\dot{a}^\nu - H(t, a)](\tilde{E}), \qquad R^0 = (\mathbf{p}, 0) \tag{4.2.13}$$

has a rather special integrand. The most general possible linear first-order action functional is given by the *Pfaffian action*

$$\mathscr{A}(\tilde{E}) = \int_{t_1}^{t_2} dt[R_\nu(t, a)\dot{a}^\nu - B(t, a)](\tilde{E}), \tag{4.2.14}$$

which can be obtained by simply lifting all Hamiltonian restrictions on the functional dependence and physical interpretation of the functions R_ν^0 and H.

[20] The degenerate Birkhoffian case *is not* related to Dirac's treatment for systems with subsidiary constraints. In fact, in the latter formulation, the Hamiltonian is *regular*. Degenerate Hamiltonians and Birkhoffians (the latter in the sense of breakdown of condition (4.2.12)) may express the presence of subsidiary constraints according to a different approach, that via Lagrange's multiplier rule (which is more developed in the literature of the calculus of variations, rather than that of analytic mechanics).

[21] The inverse transformation is the basis of the notion of Birkhoffian representation of second-order Newtonian systems and is studied in the next section.

The first-order contemporaneous variations with fixed end-points then reproduce Birkhoff's equations, via the following generalization of Hamilton's principle

$$\delta\mathscr{A}(\tilde{E}) = \int_{t_1}^{t_2} dt \left\{ \delta a^\mu \frac{\partial}{\partial a^\mu} + \delta \dot{a}^\mu \frac{\partial}{\partial \dot{a}^\mu} \right\} (R_\nu \dot{a}^\nu - B)(\tilde{E})$$

$$= \int_{t_1}^{t_2} dt \left[\left(\frac{\partial R_\nu}{\partial a^\mu} \dot{a}^\nu - \frac{\partial B}{\partial a^\mu} \right) \delta a^\mu - \dot{R}_\mu \delta a^\mu \right] (\tilde{E})$$

$$= \int_{t_1}^{t_2} dt \left[\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu} \right) \dot{a}^\nu - \left(\frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t} \right) \right] \delta a^\mu (\tilde{E}) = 0. \quad (4.2.15)$$

As a result, Birkhoff's equations can be derived from a linear first-order variational principle in the same measure as that of Hamilton's equations, although in the most general possible way. This property indicates the existence of a Birkhoffian generalization of Hamiltonian formulations based on variational principles, such as the theory of canonical transformations, Hamilton–Jacobi theory, perturbation theory, etc.

The state of the art on the latter studies is presented in the next chapters.

B. Generalized Algebraic Formulations. The *Birkhoffian time evolution* for the semi-autonomous case is given by

$$\dot{A}(a) = \frac{\partial A}{\partial a^\mu} \dot{a}^\mu = \frac{\partial A}{\partial a^\mu} \Omega^{\mu\nu}(a) \frac{\partial B}{\partial a^\nu}$$

$$\stackrel{\text{def}}{=} [A, B]^*. \quad (4.2.16)$$

The Birkhoffian tensor $\Omega^{\mu\nu}$ verifies integrability conditions (4.1.48), as the reader is encouraged to verify (Problem 4.1). Therefore, the brackets $[A, B]^*$ verify the Lie algebra axioms (4.1.40).

As we shall see, the tensor $\Omega^{\mu\nu}$ turns out to be the most general possible tensor which verifies conditions (4.1.48). Thus Birkhoff's equations not only preserve the Lie algebra character of Hamilton's equations, but actually realize the Lie product in its most general possible regular form.

The transition from the conventional to the generalized Poisson brackets

$$[A, B] = \frac{\partial A}{\partial a^\mu} \omega^{\mu\nu} \frac{\partial B}{\partial a^\nu} \to [A, B]^* = \frac{\partial A}{\partial a^\mu} \Omega^{\mu\nu}(a) \frac{\partial B}{\partial a^\nu} \quad (4.2.17)$$

has such nontrivial implications as to suggest the reformulation of Lie's theory in a form which is directly applicable to unrestricted realizations of the product (and of the enveloping algebra).

The state of the art on these algebraic aspects will be presented in the charts of the next chapter. The intriguing case of the nonautonomous equations is studied in Chart 4.1.

C. Generalized Geometric Formulations. In a predictable way, fully parallel to the algebraic case, the (regular[22]) Birkhoff's equations characterize the most general possible, exact symplectic form in local coordinates. In fact, the exact character of the two-form implies the structure on T^*M

$$\Omega_2 = dR_1 = d[R_\nu(a)da^\nu] = \frac{\partial R_\nu}{\partial a^\mu} da^\mu \wedge da^\nu$$

$$= \frac{1}{2}\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right) da^\mu \wedge da^\nu = \tfrac{1}{2}\Omega_{\mu\nu}(a)da^\mu \wedge da^\nu \qquad (4.2.18)$$

which is characterized by Equations (4.2.5).

More generally, Birkhoff's equations characterize the most general possible local formulation of an exact contact two-form. In turn, this is sufficient to establish that the contravariant Birkhoff's tensor characterizes the most general possible regular realization of the Lie algebra product in mechanics, owing to the known interplay between the Lie algebras and the contact (or symplectic) geometry.

In fact, action (4.2.14) can be written in unified notation (4.1.51)

$$\mathscr{A}(\tilde{E}) = \int_{t_1}^{t_2} \hat{R}_1(\hat{a}), \qquad \hat{a} \in \mathbb{R} \times T^*M \qquad (4.2.19a)$$

$$\hat{R}_1(\hat{a}) = \hat{R}_\nu(\hat{a})d\hat{a}^\nu, \qquad \hat{R}_\nu = \begin{cases} -B, & \nu = 0 \\ R_\nu, & \nu = 1, 2, \ldots, 2n. \end{cases} \qquad (4.2.19b)$$

The exterior derivative of the one-form $\hat{R}_1$ characterizes the two-form on $\mathbb{R} \times T^*M$

$$\hat{\Omega}_2 = d[\hat{R}_\nu(\hat{a})d\hat{a}^\nu] = \frac{1}{2}\left(\frac{\partial \hat{R}_\nu}{\partial \hat{a}^\mu} - \frac{\partial \hat{R}_\mu}{\partial \hat{a}^\nu}\right) d\hat{a}^\mu \wedge d\hat{a}^\nu \stackrel{\text{def}}{=} \tfrac{1}{2}\hat{\Omega}_{\mu\nu}(\hat{a})d\hat{a}^\mu \wedge d\hat{a}^\nu \qquad (4.2.20)$$

which verifies the following properties:

1. $\hat{\Omega}_2$ is the largest possible local formulation of exact two-forms on $\mathbb{R} \times T^*M$, *clearly, because* $\hat{R}_1$ *has the largest admissible functional dependence*;
2. $\hat{\Omega}_2$ is a contact two-form because (Chart I.2.4) it is of covariant type, of maximal rank $2n$, and its restriction to T^*M (that is, form (4.2.18)) is symplectic; and

[22] It should be stressed that the notion of strict regularity of Definition 4.2.1 is redundant for a symplectic two-form. In fact, the condition of regularity alone is sufficient for the characterization of symplectic two-forms. This point is important in illustrating the fact, somewhat obscured in the abstract coordinate-free treatment of geometry, that *the conventional non-degeneracy of a two-form does not guarantee the geometric characterization of a (regular, unconstrained) Newtonian system*, because of the need for an additional regularity condition of type (4.2.12).

3. $\hat{\Omega}_2$ is the geometrical structure of Birkhoff's equations. To see this latter point, one can write the tensor $\hat{\Omega}_{\mu\nu}$ in the disjoint coordinates (t, a)

$$(\hat{\Omega}_{\mu\nu}) = \left(\begin{array}{c|c} 0 & \dfrac{\partial B}{\partial a^\mu} + \dfrac{\partial R_\mu}{\partial t} \\ \hline -\dfrac{\partial R_\mu}{\partial t} - \dfrac{\partial B}{\partial a^\mu} & \dfrac{\partial R_\nu}{\partial a^\mu} - \dfrac{\partial R_\mu}{\partial a^\nu} \end{array}\right). \tag{4.2.21}$$

Birkhoff's equations can then be written in the unified notation on $\mathbb{R} \times T^*M$

$$\hat{\Omega}_{\mu\nu}(\hat{a})\dot{\hat{a}}^\nu = 0, \qquad \mu = 0, 1, 2, \ldots, 2n \tag{4.2.22}$$

or, explicitly,

$$\hat{\Omega}_{\mu\nu}(\hat{a})\dot{\hat{a}}^\nu = \begin{cases} \left(\dfrac{\partial B}{\partial a^\nu} + \dfrac{\partial R_\nu}{\partial t}\right)\dot{a}^\nu = 0, \qquad \mu = 0 & \text{(4.2.23a)} \\ \left(\dfrac{\partial B}{\partial a^\mu} + \dfrac{\partial R_\mu}{\partial t}\right) - \left(\dfrac{\partial R_\nu}{\partial a^\mu} - \dfrac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu = 0, & \\ \qquad\qquad \mu = 1, 2, \ldots, 2n. & \text{(4.2.23b)} \end{cases}$$

The first term is identically null (along a possible or actual path) because of the self-evident property that

$$\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t}\right)\dot{a}^\nu = \left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t}\right)\Omega^{\nu\alpha}\left(\frac{\partial B}{\partial a^\alpha} + \frac{\partial R_\alpha}{\partial t}\right) \equiv 0. \tag{4.2.24}$$

The last $2n$ terms of Equations (4.2.23) coincide with Equations (4.2.1).

By recalling the symbiotic characterization of analytic, algebraic, and geometrical aspects by the conditions of self-adjointness, *all* the above listed properties of Birkhoff's equations can be synthetically expressed via the following property.

Proposition 4.2.1 (Self-Adjointness of Birkhoff's Equations). *Necessary and sufficient condition for a general nonautonomous first-order system*

$$C_{\mu\nu}(t, a)\dot{a}^\nu + D_\mu(t, a) = 0, \qquad \mu = 1, 2, \ldots, 2n \tag{4.2.25}$$

which is analytic and regular in a star-shaped region $\tilde{\mathcal{R}}^*$ of points of $\mathbb{R} \times T^*M$ to be self-adjoint in $\tilde{\mathcal{R}}^*$ is that it is of the Birkhoffian type, i.e.,

$$C_{\mu\nu}\dot{a}^\nu + D_\mu \equiv \left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \left(\frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t}\right) = 0. \tag{4.2.26}$$

PROOF. Conditions (4.1.32) are the integrability conditions for two-forms (4.1.52), i.e.,

$$\hat{C}_2 = \tfrac{1}{2}\hat{C}_{\mu\nu}(\hat{a})d\hat{a}^\mu \wedge d\hat{a}^\nu, \qquad \hat{C}_{\mu\nu} = \begin{cases} D_\mu, & \mu = 1, 2, \ldots, 2n, \nu = 0 \\ C_{\mu\nu}, & \mu, \nu = 1, 2, \ldots, 2n \end{cases} \tag{4.2.27}$$

to be closed, i.e.,

$$d\hat{C}_2 = 0 \tag{4.2.28}$$

(see the review at the end of Section 4.1). The regularity condition implies that $\hat{C}_2$ is of maximal rank and, therefore, is a contact form. The applicability of the converse of the Poincaré Lemma I.1.2.2 implies that form $\hat{C}_2$ is exact, that is, a primitive one-form $\hat{R}_1$ on $\mathbb{R} \times T^*M$ exists such that

$$\hat{C}_2 = d\hat{R}_1. \tag{4.2.29}$$

The use of Equations (4.2.20)–(4.2.24) completes the proof that Equation (4.2.25), under the conditions of self-adjointness, necessarily have Birkhoffian structure (4.2.26). The sufficiency is trivially established by the Direct Poincaré Lemma I.1.2.1 (*Q.E.D.*).

Thus Birkhoff's equations are self-adjoint in a way parallel to the self-adjointness of Hamilton's equations. However, while Hamilton's equations are a *particular form* admitted by the conditions of self-adjointness, Birkhoff's equations are the *most general possible form.* A direct verification that Birkhoff's equations verify all of conditions (4.1.32) is instructive (Problem 4.2).

The Calculus of Differential Forms, as used for the proof of Proposition 4.2.1, provides not only the *integrability conditions* for a two-form on $\mathbb{R} \times T^*M$ to be an exact contact form, but also a *solution* for the primitive one-form. A straightforward use of the techniques reviewed in Section I.1.2 (Equations (I.1.2.30) in particular) permits the proof of the following corollary.

Corollary 4.2.1a (First[23] Method for Computing the Birkhoffian Functions from the Equations of Motion). *Under the condition of Proposition* 4.2.1, *the Birkhoffian functions* $\hat{R} = (-B,R_\mu)$ *can be expressed in terms of Equations* (4.2.25) *according to the rules*

$$\hat{R}_\mu(\hat{a}) = \left[\int_0^1 d\tau\, \tau \hat{C}_{\mu\nu}(\tau\hat{a})\right]\hat{a}^\nu, \qquad \mu = 0, 1, 2, \ldots, 2n. \tag{4.2.30}$$

We proved earlier in this section the contact geometric character of Birkhoff's equations. For completeness, we must also point out the following difference between contact two-forms of Birkhoffian type and those most commonly treated in the contemporary literature. The former possess, in general, an *explicit time dependence* in their *symplectic* substructure, while such a dependence is generally absent in the latter.

The difference originates from the fact that contemporary treatments of contact two-forms have been usually patterned along the structure of Hamilton's equations. In this case, one starts from the fundamental symplectic structure on T^*M

$$\omega_2 = \tfrac{1}{2}\omega_{\mu\nu}\, da^\mu \wedge da^\nu = \frac{1}{2}\left(\frac{\partial R_\nu^0}{\partial a^\mu} - \frac{\partial R_\mu^0}{\partial a^\nu}\right) da^\mu \wedge da^\nu = dp_k \wedge dr^k$$
$$= 1, 2, \ldots, 2n; k = 1, 2, \ldots, n \tag{4.2.31}$$

[23] Additional methods will be presented in Sections 4.4 and 4.5.

and then performs the prolongation (Chart 4.4) into the contact two-form on $\mathbb{R} \times T^*M$

$$\hat{\omega}_2 = \tfrac{1}{2}\hat{\omega}_{\mu\nu} d\hat{a}^\mu \wedge d\hat{a}^\nu = \frac{1}{2}\left(\frac{\partial \hat{R}^0_\nu}{\partial \hat{a}^\mu} - \frac{\partial \hat{R}^0_\mu}{\partial \hat{a}^\nu}\right) d\hat{a}^\mu \wedge d\hat{a}^\nu, \qquad \mu, \nu = 0, 1, 2, \ldots, 2n \tag{4.2.32a}$$

$$(\hat{\omega}_{\mu\nu}) = \left(\begin{array}{c|c} 0 & \dfrac{\partial H}{\partial a^\mu} \\ \hline -\dfrac{\partial H}{\partial a^\mu} & \omega_{\mu\nu} \end{array}\right), \qquad \hat{a} = (t, a) \tag{4.2.32b}$$

which is the geometric structure of Hamilton's equations, as the reader can verify by particularizing Equations (4.2.21)–(4.2.24) for the canonical case $R = R^0 = (\mathbf{p}, \mathbf{0})$. The point is that the symplectic structure (4.2.31) *does not* possess an explicit time dependence, and this feature persists under prolongation to form (4.2.32).

The situation is altered by Birkhoff's equations. As we shall see in the next sections and in the examples at the end of this chapter, the computation of a Birkhoffian representation for given Newtonian systems can be generally achieved in practice via functions R_μ with an explicit time dependence. Specific applications in mechanics therefore demand, in general, the initiation of the geometric study via symplectic two-forms with an explicit dependence on time

$$\Omega_2 = \tfrac{1}{2}\Omega_{\mu\nu}(t, a) da^\mu \wedge da^\nu = \frac{1}{2}\left[\frac{\partial R_\nu(t, a)}{\partial a^\mu} - \frac{\partial R_\mu(t, a)}{\partial a^\nu}\right] da^\mu \wedge da^\nu, \tag{4.2.33}$$

and this dependence clearly persists after prolongation to contact form (4.2.20).

This difference between Hamiltonian and Birkhoffian contact two-forms is not trivial. In fact, it has a number of rather delicate technical implications which will be pointed out throughout our analysis. At this point, it is sufficient to recall that the computation of the primitive one-form R_1 of an exact symplectic two-form Ω_2 via the converse of the Poincaré lemma demands the use of a star-shaped region of local variables, or a topologically equivalent region (Section I.1.2). However, if a region of the variables $(t, a(t))$ is star-shaped at a fixed value of time, this topological character is not necessarily preserved at a later time. This problem is solved in Chart 4.6 via the parametric interpretation of symplectic forms (4.2.33) and their definition in a region deformable to a curve. Additional technical aspects emerge within the context of the transformation theory, and they will be pointed out in the next chapter.

Let us now consider the Lagrangian image of the Birkhoffian representations or, more precisely, the transformation of the Pfaffian action (4.2.14) on T^*M into an equivalent action on TTM. This can be easily done (for strictly

regular Birkhoffian representations) via the knowledge of functional dependences (4.2.11a), under which we write

$$\begin{aligned}
\mathscr{A} &= \int dt[R_\mu(t, a)\dot{a}^\mu - B(t, a)] \\
&\stackrel{\text{def}}{=} \int dt[R_k(t, \mathbf{r}, \mathbf{y})\dot{r}^k + S^k(t, \mathbf{r}, \mathbf{y})\dot{y}_k - B(t, \mathbf{r}, \mathbf{y})] \\
&= \int dt\Big\{R_k[t, \mathbf{r}, \mathbf{M}(t, \mathbf{r}, \dot{\mathbf{r}})]\dot{r}^k \\
&\quad + S^k[t, \mathbf{r}, \mathbf{M}(t, \mathbf{r}, \dot{\mathbf{r}})]\left(\frac{\partial M_k}{\partial t} + \frac{\partial M_k}{\partial r^i}\dot{r}^i + \frac{\partial M_k}{\partial \dot{r}^i}\ddot{r}^i\right) - B[t, \mathbf{r}, \mathbf{M}(t, \mathbf{r}, \dot{\mathbf{r}})]\Big\} \\
&\stackrel{\text{def}}{=} \int dt\ V_k(t, \mathbf{r}, \dot{\mathbf{r}})\ddot{r}^k + W(t, \mathbf{r}, \dot{\mathbf{r}})\Big] \stackrel{\text{def}}{=} \int dt\ L(t, \mathbf{r}, \dot{\mathbf{r}}, \ddot{\mathbf{r}})
\end{aligned} \tag{4.2.34}$$

where the M's are the implicit functions in the y's, Equation (4.1.23), as characterized by the first set of Equations (4.2.11a) under regularity condition (4.2.12). One can see in this way that the Lagrangians are of *second-order* type (i.e., dependent on the accelerations), although of the totally degenerate type.[24]

The variation of action (4.2.34) then yields equations

$$\begin{aligned}
&-\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{r}^k} + \frac{d}{dt}\frac{\partial L}{\partial \dot{r}^k} - \frac{\partial L}{\partial r^k} \\
&= -\left\{\frac{\partial}{\partial t} + \dot{r}^i\frac{\partial}{\partial r^i} + \ddot{r}^i\frac{\partial}{\partial \dot{r}^i}\right\}^2 V_k \\
&\quad + \left\{\frac{\partial}{\partial t} + \dot{r}^i\frac{\partial}{\partial r^i} + \ddot{r}^i\frac{\partial}{\partial \dot{r}^i}\right\}\left(\frac{\partial V_j}{\partial \dot{r}^k}\ddot{r}^j + \frac{\partial W}{\partial \dot{r}^k}\right) \\
&\quad - \frac{\partial V_j}{\partial r^k}\ddot{r}^j - \frac{\partial W}{\partial r^k} = 0,
\end{aligned} \tag{4.2.35}$$

which characterize a system of *second-order* differential equations, contrary to the expectation of their being of third-order. Furthermore, the system is *linear in the acceleration* whenever the V's are independent of the velocities.

These results permit inspection of the Inverse Lagrangian Problem in a new light. In fact, the lack of direct universality of first-order Lagrangians may be due to the restrictions imposed by the first-order character. The direct universality of Birkhoff's equations and images (4.2.34) and (4.2.35) then make it possible for the Inverse Lagrangian Problem to become directly

[24] Recall that a *totally degenerate* Lagrangian occurs when each element of the Hessian is identically null. This is the case for second-order Lagrangians when they are linear in second-order derivatives.

universal for systems of second-order differential equations under the enlargement of the Lagrangian to those of the second-order totally degenerate type, according to the structure

$$-\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}^k}+\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k}-\frac{\partial L}{\partial q^k}\equiv A_{ki}(t, q, \dot{q})\ddot{q}^i + B(t, q, \dot{q}), \tag{4.2.36a}$$

$$L = V_k(t, q)\ddot{q}^k + W(t, q, \dot{q}). \tag{4.2.36b}$$

The study of this problem is left to the interested reader (Problem 4.9).

We conclude this section with a few historical remarks. Equations (4.2.1) have been studied, either in a direct or indirect/implicit way, by several authors. First, the equations coincide, as far as their structure is concerned, with Lagrange's equations in first-order, totally degenerate Lagrangians. In fact, by assuming for "Lagrangian" the expression

$$L(t, a, \dot{a}) = -R_\nu(t, a)\dot{a}^\nu + B(t, a) \tag{4.2.37}$$

Lagrange's equations coincide with equations (4.2.1),

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{a}^\mu}-\frac{\partial L}{\partial a^\mu}\equiv\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu-\left(\frac{\partial B}{\partial a^\mu}+\frac{\partial R_\mu}{\partial t}\right). \tag{4.2.38}$$

However, the use of the terms "Lagrange's equations" for Equations (4.2.1) would be misleading, particularly for the analysis of this volume. In fact, our objective is to seek a generalization of *phase space formulations*, while Lagrange's equations were conceived for *configuration space formulations*, and this spirit has persisted to this day.[25]

Additional studies more directly related to Equations (4.2.1) are those by Pfaff (1814). In fact, the primitive one-form leading to the equations is Pfaff's form (or action), as recalled in regard to Equation (4.2.14). However, it does not appear that Pfaff identified the true meaning of Equations (4.2.1) as bonafide analytic equations of mechanics.

These latter properties were identified in full by Birkhoff (1927) who also provided explicit examples of applications to mechanical systems. Perhaps the best way to illustrate this historical point is through Birkhoff's original words (*loc. cit.*, p. 89):

"Suppose now that we take an extended Pfaffian variational problem

$$\delta\int_{t_0}^{t_1}\left[\sum_{j=1}^{2m}X_j(x_1, \ldots, x_{2m})x_j' + Z(x_1, \ldots, x_{2m})\right]dt = 0, \tag{12}$$

which leads at once to the system of ordinary differential equations of order $2m$

$$\sum_{j=1}^{2m}\left(\frac{\partial X_i}{\partial x_j}-\frac{\partial X_j}{\partial x_i}\right)\frac{dx_j}{dt}-\frac{\partial Z}{\partial x_i}=0 \qquad (i = 1, \ldots, 2m). \tag{13}$$

[25] Lagrangians of type (4.2.37) represent *first-order systems*, and as such, they are along the phase space (rather than the configuration space) formulation of mechanics.

We propose to consider these equations in the case when there is an equilibrium point at the origin, under the assumption that the $2m$ analytic functions X_i are such that the schew-symmetric determinant

$$\left| \frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i} \right|$$

is not 0 at the origin. The constant terms in the series for the functions X_i may obviously be omitted throughout.

It is clear that the Hamiltonian equations appear as a particular case of these Pfaffian equations (12). As will be shown in the following chapter, this generalization of the Hamilton's equations possesses the same property of automatically fulfilling all of the conditions for complete stability, once the obvious conditions for first order stability are satisfied. Hence, from this point of view, the Pfaffian equations seem as significant for dynamics as the Hamiltonian equations, although more general in type. Moreover, they possess the additional advantage of maintaining their Pfaffian form under an arbitrary transformation of the formal group."

Notice the clear identification of Equations (4.2.1) by Birkhoff as being (1) derivable from a variational principle; (2) a generalization of Hamilton's equations; and (3) "as significant for dynamics as Hamilton's equations".[26] Specific illustrative applications were provided later on in Birkhoff's memoir and they are still recommendable for study. The extension to the nonautonomous case was provided soon after the quoted passage.

For these reasons, Equations (4.2.1) were called "Birkhoff's equations" by Santilli (1978c) and this terminology was subsequently adopted by a number of authors. Additional studies on the equations which deserve mention are presented here. The equations were briefly indicated by Whittaker (1904) (which is the only reference known to the author for the period between the studies of Pfaff and Birkhoff). After 1927, the equations were studied in more detail by Feraud (1930). Lee (1945), Pauli (1953), and Martin (1959) studied them to a considerable extent, but primarily for quantum mechanical considerations. More recently, Hughes (1961) considered the equations for relativistic treatments.

All the references quoted above treat *equations* of type (4.2.1). The algebraic-geometric character of *tensor* (4.2.4) has been studied by numerous authors, beginning with De Donder (1927) and Cartan (1971).

The variational self-adjointness of Equations (4.2.1) was identified by Santilli (1978c) by reaching the first unified treatment of the analytic, algebraic, and geometric properties of the equations. This author also initiated the first study (see Santilli (1978b)) of (a) the applications of the equations to the representation of local nonconservative Newtonian systems; (b) the

[26] Notice also Birkhoff's emphasis on the regularity of the equations. The reader can now see that Definition 4.2.1, apart a number of geometric and technical implementations, has been conceived to coincide with Birkhoff's original view as closely as possible. Notice also Birkhoff's mathematical elegance in expressing the regularity condition. In fact, it is expressed at the *origin*, with the tacit understanding that its preservation at other points is guaranteed under regular transformations.

equations' consequential role for the Inverse Problem; and (c) the identification of their direct universality. The variational self-adjointness of the equations was subsequently studied by Sarlet and Cantrijn (1978) and Sarlet (1979) (following a private communication by Santilli). These authors also identified a Birkhoffian generalization of the Hamilton–Jacobi equations which will be reviewed in Chapter 6. Additional studies, e.g., on the transformation theory of Birkhoff's equations, were conducted by Kobussen (1978 and 1979).

The application of Birkhoff's equations to Space Mechanics was studied by Broucke (1979), while the application to Biophysics was studied by Lumsden and Trainor (1979). These and other applications are reviewed in Chapter 6.

Furthermore, a quantum mechanical generalization of Heisenberg's equations which leads to Birkhoff's equations under the correspondence principle has been proposed by Santilli (1978d), and it is also reviewed in Chapter 6.

Finally, the reader should keep in mind that Birkhoff's equations have been studied by the author and presented in this volume for the treatment of *local non-potential forces* (or *interactions*). In this way, the condition of derivability from a potential inherent in most of the Hamiltonian treatment is removed, but the locality condition persists.

In principle, Birkhoff's equations might be studied for the possible representation of *non-local* nonpotential systems (or interactions), via integro-differential functions of the type

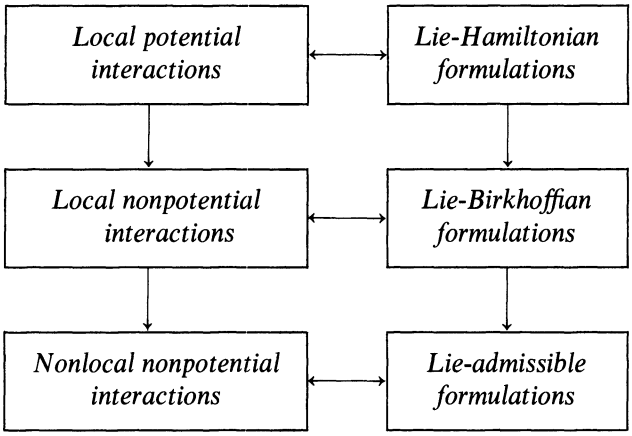
$$\begin{aligned} \mathscr{R}_\mu &= R_\mu(t, a) + \int_D da' \mathbb{R}_\mu(t, a, a'), \\ \mathscr{B} &= B(t, a) + \int_D da' \mathbb{B}'(t, a, a'), \end{aligned} \tag{4.2.39}$$

which, via the reduction to second-order form presented earlier in this section, can represent nonlocal nonpotential Newtonian systems of the type

$$m\ddot{\mathbf{r}} - \mathbf{F}(t, \mathbf{r}, \dot{\mathbf{r}}) - \int_D d\mathbf{r}' \, K(t, \mathbf{r}, \mathbf{r}', \dot{\mathbf{r}}, \dot{\mathbf{r}}', \ldots) = 0. \tag{4.2.40}$$

This use of Birkhoff's equations, even though computationally conceivable, is not recommended here for a number of reasons. A first reason is the nature of the underlying geometry, the symplectic or contact geometry, which has been developed historically as a *local–differential geometry*. No formulation of Birkhoff's equations of integro-differential type can acquire a true scientific value without the prior achievement of an integro-differential formulation of the underlying geometry. Additional reasons are pragmatic. The Inverse Birkhoffian Problem, as we shall stress in Section 4.5, already has a quite difficult practical solution for local systems, and these practical difficulties are expected to multiply for possible integro-differential generalizations.

For these (and other) reasons, it appears advisable that nonlocal nonpotential systems (or interactions) be treated with still more general equations, such as those of Lie-admissible type, according to the following chain of progressive, physical, and mathematical implementations.[27]



(4.2.41)

In fact, the direct universality of the Lie-admissible formulations for all systems of type (4.2.40) has been established. The need for a suitable integro-differential generalization of the underlying geometry emerges rather naturally in the approach. Last, but not least, the explicit computation of a Lie-admissible representation is truly simple for all systems considered because it is based on *algebraic* equations with known solution (see Chart 4.7 for more details).

Different criteria for selections (based on the need for antisymmetric or nonantisymmetric products and their relationship to closed or open systems) are indicated in Chapter 6.

4.3 Birkhoffian Representations of Newtonian Systems

In Section 4.1 we reviewed the method for reducing a second-order system to an equivalent first-order form. In Section 4.2 we introduced Birkhoff's equations as the most general possible equations derivable from a linear

[27] The reader should keep in mind that virtually all studies in contemporary theoretical physics are done along the first line of classification (4.2.41) (or its Lagrangian image). In fact, the contemporary theoretical models currently receiving the majority of attention (particularly in high-energy physics) are all dominated by the notion of local potential forces or interactions and corresponding Lagrangians or Hamiltonians. At this writing (late 1980), the restrictive character of the condition of potentiality appears to be propagating in both mathematical and physical circles, and the number of papers on local nonpotential interactions, along the second line of classification (4.2.41), is increasing considerably. However, the need for the still more general nonlocal nonpotential interactions and their treatment along the third line of classification (4.2.41) (or some alternative possibility) is just beginning to be felt by mathematicians and physicists.

first-order variational principle. To proceed in our program, we must define the notion of "representation" of given equations of motion in terms of Birkhoff's equations. For this purpose, we first review the concept of representation via Hamilton's equations and then point out its Birkhoffian generalization.

Suppose that a Newtonian system (4.1.22) is given, prescriptions (4.1.23) have been selected, and the corresponding *contravariant normal form* (4.1.28), i.e.,

$$\dot{a}^\nu = \Xi^\nu(t, a), \tag{4.3.1}$$

has been computed. The Newtonian vector field Ξ^ν is said to possess a *representation in terms of Hamilton's equations* (or be of *Hamiltonian type*) in the neighborhood $\tilde{\mathcal{R}}$ of a (regular) point (t, a) of the variables, when there exist a function $H(t, a)$, the Hamiltonian, such that all the following equations are identically verified in $\tilde{\mathcal{R}}$[28]

$$\omega_{\mu\nu}\Xi^\nu(t, a) = \frac{\partial H(t, a)}{\partial a^\mu}, \qquad \mu = 1, 2, \ldots, 2n, \tag{4.3.2a}$$

$$\omega_{\mu\nu} = \frac{\partial R^0_\nu}{\partial a^\mu} - \frac{\partial R^0_\mu}{\partial a^\nu}, \qquad R^0 = (\mathbf{p}, \mathbf{o}). \tag{4.3.2b}$$

Under these conditions, the covariant normal version of equations (4.3.1) is self-adjoint (Theorem 4.1.2), and the following *direct representation via the conventional Hamilton's principle in phase space* holds

$$\begin{aligned}\delta \int_{t_1}^{t_2} dt(R^0_\nu \dot{a}^\nu - H)(\tilde{E}) &= \int_{t_1}^{t_2} dt\left[\left(\omega_{\mu\nu}\dot{a}^\nu - \frac{\partial H}{\partial a^\mu}\right)_{\mathrm{SA}} \delta a^\mu\right](\tilde{E}) \\ &\equiv \int_{t_1}^{t_2} dt[(\omega_{\mu\nu}\dot{a}^\nu - \Xi_\mu)_{\mathrm{SA}}\,\delta a^\mu](\tilde{E}) = 0, \qquad \Xi_\mu = \omega_{\mu\nu}\Xi^\nu.\end{aligned} \tag{4.3.3}$$

Conditions (4.3.2) are the local-analytic version of a corresponding geometrical notion of the symplectic (and contact) geometry, that of the *Hamiltonian vector field*. The latter is expressed in coordinate-free form via the inner product (Chart 4.5)

$$\Xi \lrcorner\, \omega_2 = -dH \tag{4.3.4}$$

and can be explicitly written in local coordinates

$$\begin{aligned}\Xi \lrcorner\, \omega_2 &= \tfrac{1}{2}\delta^{\nu_1\nu_2}_{\mu_1\mu_2}\omega_{\nu_1\nu_2}\Xi^{\mu_2}\, da^{\mu_1} \\ &= \omega_{\mu_1\mu_2}\Xi^{\mu_1}\, da^{\mu_2} = -dH = -\frac{\partial H}{\partial a^{\mu_2}}\, da^{\mu_2}.\end{aligned} \tag{4.3.5}$$

The equivalence of the geometric notion (4.3.4) with the analytic version (4.3.2) is then self-evident. Equation (4.3.5) also expresses the meaning of the

[28] For topological conditions, see Charts 4.4 and 4.5.

fundamental symplectic tensor $\omega_{\mu\nu}$ as the geometrical tensor of the Hamiltonian vector field, that is, as the proper tensor for the lowering of the indices of Ξ^ν.[29]

The above definition of Hamiltonian representation is often restrictive in practical applications in the following sense. Suppose that, rather than a *normal* form, a *general* first-order form (4.1.30) is given,

$$C_{\mu\nu}(t, a)\dot{a}^\nu + D_\mu(t, a) = 0. \tag{4.3.6}$$

By following Definition I.3.11.1, we say that Equation (4.3.6) admits an *indirect analytic representation in terms of Hamilton's equations* when there exist $(4n^2 + 1)$ functions, a Hamiltonian $H(t, a)$, and a regular matrix of multiplicative functions $(h^\nu_\mu(t, a))$, such that the following identities hold:

$$\omega_{\mu\nu}\dot{a}^\nu - \frac{\partial H}{\partial a^\mu} \equiv h^\alpha_\mu(C_{\alpha\nu}\dot{a}^\nu + D_\alpha), \qquad \mu = 1, 2, \ldots, 2n. \tag{4.3.7}$$

Clearly, the identities can hold if and only if

$$h^\alpha_\mu C_{\alpha\nu} \equiv \omega_{\mu\nu}, \qquad h^\alpha_\mu D_\alpha = -\Xi_\mu = -\frac{\partial H}{\partial a^\mu}. \tag{4.3.8}$$

In this sense, definitions (4.3.7) and (4.3.2) are equivalent. However, definition (4.3.7) illustrates more clearly the generally indirect nature of the Hamiltonian representations.

Note that the integrability conditions for all representations (4.3.2), and (4.3.7) are those of Theorem 4.1.4, and, as stressed in Section 4.1, *a Hamiltonian for a given system does not necessarily exist within a fixed system of local coordinates.*

The Birkhoffian generalization of the Hamiltonian notions introduced above is straightforward. Consider a vector field $\Xi^\nu(t, a)$, and suppose that *it is not* Hamiltonian, that is, a functions $H(t, a)$ verifying equations (4.3.2) does not exist. This means, geometrically, that the fundamental symplectic tensor $\omega_{\mu\nu}$ is not the lowering tensor of the vector field Ξ^ν.

We shall say that the vector field Ξ^ν possesses a *representation in terms of Birkhoff's equations* (or is of *Birkhoffian type*) in a neighborhood $\tilde{\mathcal{R}}$ of a point (t, a), when Birkhoffian functions $R_\mu(t, a)$ and $B(t, a)$ exist such that all the following equations are identically verified in $\tilde{\mathcal{R}}$[30]

$$\Omega_{\mu\nu}(t, a)\Xi^\nu(t, a) = \frac{\partial B(t, a)}{\partial a^\mu} + \frac{\partial R_\mu(t, a)}{\partial t}, \tag{4.3.9a}$$

$$\Omega_{\mu\nu} = \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}. \tag{4.3.9b}$$

[29] The fundamental Lie tensor $\omega^{\mu\nu}$ is the raising tensor for the covariant form Ξ_ν.

[30] The preservation of the notation "a" in the transition from Hamilton to Birkhoff's equations may be misleading unless properly understood. Strictly speaking, one should use different notations, say, $a = (\mathbf{r}, \mathbf{p})$ for the former and $b = (\mathbf{r}, \mathbf{y})$ for the latter, to stress the differentiation between the independent variables $\mathbf{p}$ and $\mathbf{y}$. This differentiation has not been implemented here to illustrate in a way as direct as possible the preservation of the analytic, algebraic, and geometric character in the transition from Hamilton's to Birkhoff's equations.

The vector field $\Xi^\nu(t, a)$ is nonautonomous. Representation (4.3.9) is therefore *non-autonomous* as well. However, following Definition 4.2.1, there may exist also the following *semi-autonomous* representation

$$\Omega_{\mu\nu}(a)\Xi^\nu(t, a) = \frac{\partial B(t, a)}{\partial a^\mu}, \tag{4.3.10}$$

in which the explicit time dependence is restricted to the Birkhoffian. Clearly, an *autonomous* Birkhoffian representation of a *nonautonomous* vector field does not exist.

The situation for autonomous vector fields $\Xi^\nu(a)$ is different. First, these vector fields may admit an *autonomous* Birkoffian representation which we write

$$\Omega_{\mu\nu}(a)\Xi^\nu(a) = \frac{\partial B(a)}{\partial a^\mu}. \tag{4.3.11}$$

However, a non-autonomous representation in this case cannot be excluded, because of the possibility that the explicit time dependences of the tensor $\Omega_{\mu\nu}$ and of the Birkhoffian B "cancel out" in such a way to yield consistent equations (4.3.9).[31]

To summarize, when the *fundamental* symplectic tensor $\omega_{\mu\nu}$ does not permit the achievement of consistent identities (4.3.2), the transition to the *general* symplectic tensor $\Omega_{\mu\nu}$ allows a solution. The vector field, however, is not Hamiltonian. It has been called here a "Birkhoffian vector field," following the terminology introduced by Santilli (1978c), to stress the lack of Hamiltonian character while preserving a perfectly acceptable symplectic or contact structure. Illustrative examples are given at the end of this and the next chapter.

When contravariant form (4.3.1) is Birkhoffian, it admits the covariant general form

$$\Omega_{\mu\nu}(t, a)\dot{a}^\nu + \Gamma_\mu(t, a) = 0, \qquad \Gamma_\mu = \frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t} \tag{4.3.12}$$

which is clearly self-adjoint (from Proposition 4.2.1). The following *direct representation of form* (4.3.12) *via the conventional Pfaffian principle* then follows

$$\delta \int_{t_1}^{t_2} dt(R_\nu \dot{a}^\nu - B)(\tilde{E}) = \int_{t_1}^{t_2} dt\left[\left(\Omega_{\mu\nu}\dot{a}^\nu - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t}\right)_{SA} \delta a^\mu\right](\tilde{E})$$

$$\equiv \int_{t_1}^{t_2} dt[(\Omega_{\mu\nu}\dot{a}^\nu + \Gamma_\mu)_{SA}\delta a^\mu](\tilde{E}) = 0. \tag{4.3.13}$$

The Birkhoffian representations are the analytic version of a corresponding geometric notion (studied in Charts 4.4–4.6) called the *Birkhoffian vector field.* This notion can also be introduced as a direct geometric generalization of the

[31] As we shall see, this case occurs rather frequently in practical applications.

notion of Hamiltonian vector field, via the inner product for the autonomous case

$$\Xi \lrcorner \Omega_2 = -\mathrm{d}B \tag{4.3.14}$$

with corresponding generalizations for the semi-autonomous and the non-autonomous cases.

In the local coordinates needed for practical applications to specific systems, Equations (4.3.14) read[32]

$$\begin{aligned}\Xi \lrcorner \Omega_2 &= \tfrac{1}{2}\delta^{\nu_1\nu_2}_{\mu_1\mu_2}\Omega_{\nu_1\nu_2}\Xi^{\mu_2}\, da^{\mu_1} \\ &= \Omega_{\mu_1\mu_2}\Xi^{\mu_1}\, da^{\mu_2} = -dB = -\frac{\partial B}{\partial a^{\mu_2}}\, da^{\mu_2}.\end{aligned} \tag{4.3.15}$$

The generalization of Hamiltonian structure (4.3.5) so as to preserve the underlying geometry (and therefore, the algebra) is self-evident.

As recalled earlier, first-order systems are generally given in form (4.3.6), in which case the Birkhoffian definitions given above do not apply directly. This limitation is resolved via the following definition.

Definition 4.3.1. A general covariant form (4.3.6), which is well defined, analytic, and regular in a neighborhood $\tilde{\mathscr{R}}$ of a regular point of the variables, admits a *representation in terms of Birkhoff's equations* when a regular matrix $(h^2_\mu(t, a))$ of multiplicative functions which are analytic in $\tilde{\mathscr{R}}$ and a set of Birkhoffian functions $R_\mu(t, a)$ and $B(t, a)$ which are also analytic in $\tilde{\mathscr{R}}$ exist such that the following identities hold in $\tilde{\mathscr{R}}$ in a given ordering:

$$\left[\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \left(\frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t}\right)\right]_{\mathrm{SA}} \equiv [h^\alpha_\mu(C_{\alpha\nu}\dot{a}^\nu + D_\alpha)_{\dot{\mathrm{NSA}}}]_{\mathrm{SA}},$$
$$\mu = 1, 2, \ldots, 2n. \tag{4.3.16}$$

The representation is called *direct* when (h^α_μ) is the unit matrix; otherwise, it is called *indirect*. Finally, the representation is called *nonautonomous*, *semi-autonomous*, or *autonomous* when Birkhoff's equations are of the corresponding type in the sense of Definition 4.2.1.

It is understood that the integrability conditions for representations (4.3.9) and (4.3.16) are equivalent.

4.4 Isotopic and Genotopic Transformations of First-Order Systems

As indicated in the preceding section, particularly in Deffnition 4.3.1, the construction of a Birkhoffian representation is essentially dependent on the capability of writing first-order systems in a self-adjoint form, and, more

[32] It is important to stress even at this introductory geometric level that *the differentiation between the Hamiltonian and Birkhoffian vector fields is lost for the global coordinate-free formulations of geometry*. This point illustrates the need of local formulations and the *insufficiency* of the coordinate-free approach in mechanics, if considered alone. For more details, see the geometric charts at the end of this chapter.

specifically, on the capability of turning a given non-self-adjoint system into an equivalent self-adjoint form.

A technical context for the rigorous treatment of these transformations is provided by the theory of Abstract Algebras and, in particular, by the so-called isotopies and genotopies. In this section we shall deal with the simplest possible part of the topic, that dealing explicitly with first-order systems. A more technical treatment within the context of Abstract Algebras will be provided in Chart 5.2.

Definition 4.4.1.[33] An equivalence transformation of a system of differential equations (or first- or higher-order) is called *self-adjoint* (*non-self-adjoint*) *isotopic* or *self-adjoint* (*non-self-adjoint*) *genotopic*, depending on whether the transformation preserves, in the isotopic case, or induces, in the genotopic case, the self-adjointness (non-self-adjointness).

This definition has been given for all possible equivalence transformations,[34] including those characterized by the transformation theory. If we restrict the transformations to a fixed system of local variables, the only possible transformations are those characterized by the multiplication of a regular matrix of factor functions. The classification of all possible cases from the viewpoint of the variational self-adjointness, then leads to the following possibilities.

$$(C_{\mu\nu}\dot{a}^{\nu} + D_{\mu})_{\mathrm{SA}} = 0 \rightarrow \quad (C^{*}_{\mu\nu}\dot{a}^{\nu} + D^{*}_{\mu})_{\mathrm{SA}} = 0, \tag{4.4.1a}$$

$$(C_{\mu\nu}\dot{a}^{\nu} + D_{\mu})_{\mathrm{NSA}} = 0 \rightarrow \quad (C^{*}_{\mu\nu}\dot{a}^{\nu} + D^{*}_{\mu})_{\mathrm{SA}} = 0, \tag{4.4.1b}$$

$$(C_{\mu\nu}\dot{a}^{\nu} + D_{\mu})_{\mathrm{NSA}} = 0 \rightarrow (C^{*}_{\mu\nu}\dot{a}^{\nu} + D^{*}_{\mu})_{\mathrm{NSA}} = 0, \tag{4.4.1c}$$

$$(C_{\mu\nu}\dot{a}^{\nu} + D_{\mu})_{\mathrm{SA}} = 0 \rightarrow (C^{*}_{\mu\nu}\dot{a}^{\nu} + D^{*}_{\mu})_{\mathrm{NSA}} = 0, \tag{4.4.1d}$$

$$C^{*}_{\mu\nu} = h^{\alpha}_{\mu}C_{\alpha\nu}, \qquad D^{*}_{\mu} = h^{\alpha}_{\mu}D_{\alpha}. \tag{4.4.1e}$$

According to Definition 4.4.1, the above cases can be identified as follows:

— transformations (4.4.1a) are *self-adjoint isotopic* because they preserve the self-adjointness of the original system;
— transformations (4.4.1b) are *self-adjoint genotopic* because they transform a non-self-adjoint system into a self-adjoint one, by therefore inducing the self-adjointness;
— transformations (4.2.1c) are *non-self-adjoint isotopic* because they preserve the original non-self-adjointness of the systems; and, finally,

[33] R. M. Santilli (1977c). With a minimal amount of linguistic license, the Greek for "isotopic" is $\iota'\sigma o\varsigma\ \tau o'\pi o\varsigma$, which means "same configuration." The term "genotopic" has been suggested to me by Mrs. Carla Santilli from the Greek $\gamma\varepsilon\nu\nu\alpha'\omega\ \tau o'\pi o\nu$, meaning "induce configuration."

[34] Note that the term "equivalence" does not possess the traditional mathematical meaning within the context of transformations (4.4.1) because, once an analytic and regular matrix h is selected, the equivalence character of the transformation also depends on the system considered. For instance, $(h^{j}_{i}(\dot{q}))$ may generate an equivalence transformation of the second-order system $F_k(t, q, \dot{q}, \ddot{q}) = 0$ but not of the first-order system $F_k(t, q, \dot{q}) = 0$. Nevertheless, all transformations considered in this volume are equivalence transformations by a construction. This implies all the necessary restrictions on multiplicative functions, ensuring the equivalence of the original and transformed systems. We shall therefore apply the term "equivalence transformation" to transformations of type (4.4.1) as well.

transformations (4.4.1d) are *non-self-adjoint genotopic* because they transform a self-adjoint system into a non-self-adjoint form, by therefore inducing the non-self-adjointness.

A few examples may assist the reader in identifying the type of transformations under consideration. They are given below for the second-order case, with the understanding that the first-order case follows similar patterns.

The equivalence transformation of the radial equation of a particle in a central force field (studied in Example I.3.5, p. I.212)

$$\left[m^2r^3\ddot{r} - M^2 + mr^3\frac{\partial V(r)}{\partial r}\right]_{\text{NSA}} = 0$$

$$(r \neq 0)$$

$$\rightarrow \left\{\frac{1}{mr^3}\left[m^2r^3\ddot{r} - M^2 + mr^3\frac{\partial V(r)}{\partial r}\right]_{\text{NSA}}\right\}_{\text{SA}}$$

$$= \left[m\ddot{r} - \frac{M^2}{mr^3} + \frac{\partial V(r)}{\partial r}\right]_{\text{SA}} = 0, \tag{4.4.2}$$

is a self-adjoint genotopic transformation. The inverse transformation is then of non-self-adjoint genotopic type. What is important for this analysis is that the multiplication of the equation of motion by the term $1/mr^3$ clearly leaves the solution of the system unaffected (equivalence transformation), yet it is not trivial from the viewpoint of the existence of a Lagrangian representation. In fact, this representation exists (and is well-known) for the self-adjoint form of the equations of motion, yet *does not exist* for the equivalent non-self-adjoint form. Notice that the form of the equation of motion originating from Newton's second law ($m\ddot{r} - F = 0$) is self-adjoint in this case.

The equivalence transformation of the equation of motion of a particle subject to a linear velocity-dependent drag force (studied in Example I.3.1, p. I.206)

$$[\ddot{r} + \gamma\dot{r}]_{\text{NSA}} = 0 \rightarrow \left\{\frac{1}{\dot{r}}[\ddot{r} + \gamma\dot{r}]_{\text{NSA}}\right\}_{\text{SA}} = 0, \tag{4.4.3}$$

$$(\dot{r} \neq 0, m = 1)$$

is a self-adjoint genotopic transformation. In this case the transformation inducing self-adjointness is provided by the factor $1/\dot{r}$ (which the reader can verify through (*I*.12)). Notice that, in this case, the equation of motion originating from Newton's second law is non-self-adjoint.

For additional illustrations of the self-adjoint genotopic transformations, we refer the reader to the examples at the end of this chapter (as well as those at the end of Chapter 5 and of the Appendix).

The case of self-adjoint isotopy is similar to that of self-adjoint genotopy. For instance, the equivalence transformation of the one-dimensional harmonic oscillator (conservative case),

$$[\ddot{r} + r]_{\text{SA}} = 0 \rightarrow \{(\dot{r}^2 + r^2)[\ddot{r} + r]_{\text{SA}}\}_{\text{SA}} = 0, \tag{4.4.4}$$

preserves self-adjointness. The reader is encouraged to verify this. The equivalence transformation of the particle with linear velocity damping (nonconservative case),

$$\left[\frac{1}{\dot{r}}\ddot{r} + \gamma\right]_{SA} = 0 \rightarrow \left\{e^{\gamma t}\dot{r}\left[\frac{1}{\dot{r}}\ddot{r} + \gamma\right]_{SA}\right\}_{SA} = 0, \tag{4.4.5}$$

also preserves self-adjointness. According to Definition 4.4.1, (4.4.4) and (4.4.5) are therefore cases of self-adjoint isotopy.

Equivalence transformations which induce the non-self-adjointness are particularly useful in generalizing variational principles so as to represent directly non-self-adjoint systems, and they will be considered later. The same transformations are also useful for the non-Lie study of Newtonian systems (e.g., that of Lie-admissible type), but this latter approach will not be considered here.

In conclusion, the behavior of a given system of differential equations under conditions of variational self-adjointness is highly sensitive to the way in which the system is written. In particular, the multiplication by a regular (and thus invertible) matrix of functions, while leaving the implicit functions (and thus the solutions) unaffected, generally alters the variational character of the system.

Clearly, the transformations that are important for the Birkhoffian representations are those of self-adjoint (isotopic and genotopic) type. Their integrability conditions are identified in the following theorem where the terms "self-adjoint transformations" represent both the isotopic and the genotopic ones.

Theorem 4.4.1 (Self-Adjoint Transformations of First-Order Systems). *Consider a first-order system*

$$C_{\alpha\nu}(t, a)\dot{a}^{\nu} + D_{\alpha}(t, a) = 0, \qquad \alpha = 1, 2, \ldots, 2n \tag{4.4.6}$$

which is well-defined, analytic, regular, and either non-self-adjoint or self-adjoint in a region $\tilde{\mathscr{R}}$ of the variables (t, a). A necessary and sufficient condition for the transformation in $\tilde{\mathscr{R}}$

$$\{h^{\alpha}_{\mu}(t, a)[C_{\alpha\nu}(t, a)\dot{a}^{\nu} + D_{\alpha}(t, a)]\} = [C^{*}_{\mu\nu}(t, a)\dot{a}^{\nu} + D^{*}_{\mu}(t, a)] = 0, \tag{4.4.7a}$$

$$C^{*}_{\mu\nu} = h^{\alpha}_{\mu}C_{\alpha\nu}, \qquad D^{*}_{\mu} = h^{\alpha}_{\mu}D_{\alpha}, \qquad \det(h^{\alpha}_{\mu})(\tilde{\mathscr{R}}) \neq 0 \tag{4.4.7b}$$

to be self-adjoint is that all the conditions

$$C^{*}_{\mu\nu} + C^{*}_{\nu\mu} = 0, \tag{4.4.8a}$$

$$\frac{\partial C^{*}_{\mu\nu}}{\partial a^{\tau}} + \frac{\partial C^{*}_{\nu\tau}}{\partial a^{\mu}} + \frac{\partial C^{*}_{\tau\mu}}{\partial a^{\nu}} = 0, \tag{4.4.8b}$$

$$\frac{\partial C^{*}_{\mu\nu}}{\partial t} = \frac{\partial D^{*}_{\mu}}{\partial a^{\nu}} - \frac{\partial D^{*}_{\nu}}{\partial a^{\mu}}, \tag{4.4.8c}$$

$$\mu, \nu, \tau = 1, 2, \ldots, 2n,$$

are identically verified in $\tilde{\mathscr{R}}$.

As indicated in Section 4.1, in practice one often constructs first a covariant normal form (4.1.31) for the possible identification of a Hamiltonian. When this form is not self-adjoint (and a Hamiltonian does not exist), one can search for a Birkhoffian representation. In this case the following particularization of Theorem 4.4.1 is useful.

Corollary 4.4.1a. *When system* (4.4.6) *is the* (*not necessarily self-adjoint*) *covariant normal form*

$$\omega_{\alpha\nu}\dot{a}^{\nu} - \Xi_{\alpha}(t, a) = 0, \tag{4.4.9}$$

conditions (4.4.8) *for the construction of a self-adjoint general form*

$$\{h_{\mu}^{\alpha}(t, a)[\omega_{\alpha\nu}\dot{a}^{\nu} - \Xi_{\alpha}(t, a)]\}_{\mathrm{SA}} = [C_{\mu\nu}(t, a)\dot{a}^{\nu} + D_{\mu}(t, a)]_{\mathrm{SA}}, \tag{4.4.10a}$$

$$C_{\mu\nu} = h_{\mu}^{\alpha}\omega_{\alpha\nu}, \quad D_{\mu} = -h_{\mu}^{\alpha}\Xi_{\alpha}, \tag{4.4.10b}$$

reduce to

$$h_{\mu}^{\alpha}\omega_{\alpha\nu} + h_{\nu}^{\alpha}\omega_{\alpha\mu} = 0, \tag{4.4.11a}$$

$$\frac{\partial h_{\mu}^{\alpha}}{\partial a^{\tau}}\omega_{\alpha\nu} + \frac{\partial h_{\nu}^{\alpha}}{\partial a^{\mu}}\omega_{\alpha\tau} + \frac{\partial h_{\tau}^{\alpha}}{\partial a^{\nu}}\omega_{\alpha\mu} = 0, \tag{4.4.11b}$$

$$\frac{\partial h_{\mu}^{\alpha}}{\partial t}\omega_{\alpha\nu} = \frac{\partial}{\partial a^{\mu}}(h_{\nu}^{\alpha}\Xi_{\alpha}) - \frac{\partial}{\partial a^{\nu}}(h_{\mu}^{\alpha}\Xi_{\alpha}). \tag{4.4.11c}$$

Sometimes, self-adjoint transformations admit a tensor h_{μ}^{α} with constants elements. The following particular case is then useful.

Corollary 4.4.1b. *When all elements of the matrix* (h_{μ}^{α}) *are constants, conditions* (4.4.11) *become*

$$(h_{\mu}^{\alpha}) \stackrel{\mathrm{def}}{=} \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix}, \tag{4.4.12a}$$

$$h_1 = h_4^{\mathrm{T}}, \qquad h = -h_2^{\mathrm{T}}, \qquad h_3 = -h_3^{\mathrm{T}}, \tag{4.4.12b}$$

$$h_{\mu}^{\alpha}\frac{\partial \Xi_{\alpha}}{\partial a^{\nu}} - h_{\nu}^{\alpha}\frac{\partial \Xi_{\alpha}}{\partial a^{\mu}} = 0. \tag{4.4.12c}$$

The algebraic and geometrical implications of the self-adjoint transformations can be pointed out, in a preliminary way, via the following theorem. Its proof is a direct consequence of the algebraic and geometric meaning of the conditions of self-adjointeness (Section 4.1) and, as such, is ignored here.

Theorem 4.4.2 (Lie and Symplectic Character of the Self-Adjoint Transformations). *Under integrability conditions* (4.4.8), *transformed systems* (4.4.7) *have a Lie algebraic and a symplectic geometric structure, in the sense that the brackets*

$$[A, B]^{*} = \frac{\partial A}{\partial a^{\mu}} C^{*\mu\nu}(t, a)\frac{\partial B}{\partial a^{\nu}}, \qquad C^{*\mu\nu} = (\|C_{\alpha\beta}^{*}\|^{-1})^{\mu\nu} \tag{4.4.13}$$

are Lie, and the two-forms

$$C_2^* = \tfrac{1}{2}C_{\mu\nu}^*(t, a)da^\mu \wedge da^\nu \tag{4.4.14}$$

are symplectic.

Recall that Theorem 4.4.1 applies whether or not the original system is self-adjoint. The following classification of Theorem 4.4.2 then follows.

(A) *Self-adjoint isotopic transformations.* In this case the original brackets

$$[A, B] = \frac{\partial A}{\partial a^\mu} C^{\mu\nu} \frac{\partial B}{\partial a^\nu}, \qquad C^{\mu\nu} = (\|C_{\alpha\beta}\|^{-1})^{\mu\nu}, \tag{4.4.15}$$

are Lie, and the exterior two-form

$$C_2 = \tfrac{1}{2}C_{\mu\nu}\, da^\mu \wedge da^\nu \tag{4.4.16}$$

is symplectic. We therefore have the *Lie algebra preserving transformation of the brackets*

$$[A, B] = \frac{\partial A}{\partial a^\mu} C^{\mu\nu} \frac{\partial B}{\partial a^\nu} \to [A, B]^* = \frac{\partial A}{\partial a^\mu} C^{*\mu\nu} \frac{\partial B}{\partial a^\nu}, \tag{4.4.17}$$

with corresponding *symplectic preserving transformation of the two-form*

$$C_2 = \tfrac{1}{2}C_{\mu\nu}\, da^\mu \wedge da^\nu \to C_2^* = \tfrac{1}{2}C_{\mu\nu}^*\, da^\mu \wedge da^\nu. \tag{4.4.18}$$

(B) *Self-adjoint genotopic transformations.* In this case the original brackets

$$(A, B) = \frac{\partial A}{\partial a^\mu} C^{\mu\nu} \frac{\partial B}{\partial a^\nu} \tag{4.4.19}$$

are not Lie, (e.g., the tensor $C^{\mu\nu}$ is not totally antisymmetric). Consequently, the *tensorial* two-form

$$C_2 = C_{\mu\nu}\, da^\mu \otimes da^\nu \tag{4.4.20}$$

cannot be reduced entirely to the exterior form (4.4.16)[35] and, as such, is not symplectic. In this case we have the *Lie algebra inducing transformation of the brackets*

$$(A, B) = \frac{\partial A}{\partial a^\mu} C^{\mu\nu} \frac{\partial B}{\partial a^\nu} \to [A, B]^* = \frac{\partial A}{\partial a^\mu} C^{*\mu\nu} \frac{\partial B}{\partial a^\nu} \tag{4.4.21}$$

[35] Recall from Section I.1.2 that the tensorial product $\otimes$ is neitheir symmetric nor antisymmetric, while the exterior product $\wedge$ is totally antisymmetric. It then follows that, whenever the tensor $C_{\mu\nu}$ is antisymmetric, the tensorial two-forms reduce automatically to the exterior one, according to the rule

$$C_{\mu\nu}\, da^\mu \otimes da^\nu = \tfrac{1}{2}(C_{\mu\nu} - C_{\nu\mu})da^\mu \wedge da^\nu + \tfrac{1}{2}(C_{\mu\nu} + C_{\nu\mu})da^\mu \times da^\nu \equiv C_{\mu\nu}\, da^\mu \wedge da^\nu.$$

If the tensor $C_{\mu\nu}$ is not antisymmetric, this reduction is not possible, and the underlying geometry is not symplectic.

with corresponding *symplectic-inducing transformation of the (tensorial) two-form*

$$C_2 = C_{\mu\nu}\, da^\mu \otimes da^\nu \to C_2^* = \tfrac{1}{2} C_{\mu\nu}^*\, da^\mu \wedge da^\nu. \tag{4.4.22}$$

Note that all transformations (4.4.17), (4.4.18), (4.4.21), and (4.4.22) occur, by construction, within one single fixed system of local variables. As a result, the transformations express the algebraic and geometric degrees of freedom of the specific reference frame of the observer. Also, since no change of variables is involved, the transformations are a new algebraic and geometric type, which will be studied in more detail in Charts 4.2.

It is remarkable that the identification and treatment of these new transformations is a direct result of the conditions of variational self-adjointness.

By comparing Theorems A.1.1 and 4.4.1, we see a considerable similarity in the construction of self-adjoint transformations for second- and first-order systems. Nevertheless, a deeper study reveals a rather profound difference at the basis of the universality of the Inverse Problem, as well as of a number of important properties.

As stressed in the Introduction, *second-order systems do not necessarily admit a self-adjoint transformation within a fixed system of local variables.* As a result, a Lagrangian or a Hamiltonian for the representation of a Newtonian system in the coordinate and time variables of the experimenter does not necessarily exist.

The situation for first-order systems is different. (Havas (1973)). In fact, as we shall show, *first-order systems always admit a self-adjoint transformation within fixed local variables.* As a result, whenever a Hamiltonian does not exist, a Birkhoffian representation can be established.

Theorem 4.4.3 (Universality of the Self-Adjoint Transformations of First-Order Systems). *Local, analytic, regular and even-dimensional systems of first-order ordinary differential equations always admit at least one self-adjoint transformation in the neighborhood of a regular point of their variables.*

PROOF. To prove the theorem it is sufficient to consider the case when the functions h_μ^α of Equations (4.4.10)[36] possess an explicit time dependence. Equations (4.4.11) always admits a solution in the neighborhood of a regular point because they can be written in the equivalent Cauchy–Kovalevski form

$$\frac{\partial h_\mu^\alpha}{\partial t} = \omega^{\alpha\nu} \frac{\partial}{\partial a^\nu}(h_\mu^\rho \Xi_\rho) - \omega^{\alpha\nu} \frac{\partial}{\partial a^\mu}(h_\nu^\rho \Xi_\rho), \tag{4.4.23}$$

and the functions Ξ_ρ are analytic. Moreover, the intial conditions can be chosen in such a way that the tensor $h_\mu^\alpha \omega_{\alpha\nu}$ has a curl structure, say, $C_{\mu\nu}^0$ at the initial time $t = t_0$. Thus the Cauchy–Kovalevski theorem is verified. This ensures the existence of an (analytic)

[36] The proof deals with the particular case of Equation (4.4.10). Its extension to the general case (4.4.7) is left to the interested reader (Problem 4.3).

solution h_μ^α such that $h_\mu^\alpha \omega_{\alpha\nu}$ has a curl structure at all times, as can be seen in the formal expression

$$\Omega_{\mu\nu}(t, a) = h_\mu^\alpha(t, a)\omega_{\alpha\nu} = \int_{t_0}^{t} dt \frac{\partial \Gamma_\mu'}{\partial a^\nu} - \int_{t_0}^{t} dt \frac{\partial \Gamma_\nu'}{\partial a^\mu} + C_{\mu\nu}^0. \tag{4.4.24}$$

All tensors of rank two with a curl structure verify Equations (4.4.11a) and (4.4.11b) (see Problem 4.2), and this completes the proof of the theorem. (*Q.E.D.*).

Theorem 4.4.3 is remarkable inasmuch as it establishes that system (4.4.11) of partial differential equations in the unknown functions h_μ^α for fixed terms $\omega_{\alpha\nu}$ and Ξ_α is always consistent, despite its overdetermined character, with a similar case occurring for the more general system (4.4.8).

The geometric implications of the solutions are also intriguing. Recall that the systems considered are nonautonomous and that, as such, they can be more properly described via the contact geometry. Recall from Section 4.2 that contact two-forms, in their current general formulation, have attached symplectic forms without an explicit time dependence. Theorems 4.4.1, 4.4.2, and 4.4.3 establish instead that the presence of an explicit time dependence in the symplectic form is rather natural in mechanics.

The following method for the explicit construction of a self-adjoint form due to Hojman (1981) is important on both formal and practical grounds.

Proposition 4.4.1 (A Method for the Construction of a Self-Adjoint First-Order Form). *Consider a contravariant, first-order, normal form,*

$$\dot{a}^\mu = \Xi^\mu(t, a), \qquad \mu = 1, 2, \ldots, 2n, \tag{4.4.25}$$

which is analytic in the neighborhood $\tilde{\mathscr{R}}$ of a regular point of the variables, and suppose that $2n$ independent first integrals $I^\mu(t, a)$,

$$\dot{I}^\mu(t, a) = \frac{\partial I^\mu}{\partial t} + \frac{\partial I^\mu}{\partial a^\nu} \Xi^\nu = 0, \tag{4.4.26a}$$

$$\det(\partial I^\mu / \partial a^\nu)(\tilde{\mathscr{R}}) \neq 0 \tag{4.4.26b}$$

are known. Consider $2n$ functions $G_\mu(I(a))$ such that

$$\det\left(\frac{\partial G_\mu}{\partial I^\nu} - \frac{\partial G_\nu}{\partial I^\mu}\right)(\tilde{\mathscr{R}}) \neq 0. \tag{4.4.27}$$

Then the covariant general form

$$C_{\mu\nu}(t, a)\dot{a}^\nu + D_\mu(t, a) = 0, \tag{4.4.28a}$$

$$C_{\mu\nu} = \left(\frac{\partial G_\alpha}{\partial I^\beta} - \frac{\partial G_\beta}{\partial I^\alpha}\right) \frac{\partial I^\alpha}{\partial a^\mu} \frac{\partial I^\beta}{\partial a^\nu}, \tag{4.4.28b}$$

$$D_\mu = -C_{\mu\nu}\Xi^\nu, \tag{4.4.28c}$$

is self-adjoint in $\tilde{\mathscr{R}}$.

The proof of Proposition 4.4.1 is left as an instructive exercise for the interested reader (Problem 4.4). The construction of the Birkhoffian functions via the method of the proposition will be presented in the next section.

The reader should keep in mind that (as was the case for Theorems 4.4.1 and 4.4.2) Theorem 4.4.3 also applies whether or not the original system is self-adjoint. In the former case, an analytic representation already exists, while in the latter case, it is induced by the transformation.

By recalling the remarks following Theorem 4.4.2, we can conclude this section by stating that the *self-adjoint isotopic (genotopic) transformations preserve (induce) the derivability of the system from a variational principle, its Lie algebra character, and its symplectic geometric structure.*

4.5 Direct Universality of Birkhoff's Equations

Definition 4.3.1 and Theorem 4.4.3 are sufficient for establishing the direct universality of Birkhoff's equations. Nevertheless, for the sake of completeness, we shall give below a more direct proof based on the Cauchy–Kovalevski theorem to establish the existence of the Birkhoffian functions.

Theorem 4.5.1[37] (Direct Universality of Birkhoff's Equations for Local Newtonian Systems). *All local, analytic, regular, finite-dimensional, unconstrained or holonomic, conservative or non-conservative, and self-adjoint or non-self-adjoint systems in first-order form always admit, in a star-shaped neighborhood of a regular point of their variables, a representation in terms of Birkhoff's equations in the coordinate and time variables of the experimenter.*

PROOF. Unconstrained systems of the class admitted are given by the essentially non-self-adjoint systems (Definition 4.1.1). In the (Cartesian) coordinate and time variables of the experimenter, they can be written[38]

$$\{[(m_a\ddot{r}_{ka} - f_{ka}(t, \mathbf{r}, \dot{\mathbf{r}}))_{\text{ESA}} - F_{ka}(t, \mathbf{r}, \dot{\mathbf{r}})]_{\text{NENSA}} - \mathscr{F}_{ka}(t, \mathbf{r}, \dot{\mathbf{r}})\}_{\text{ENSA}} = 0,$$
$$k = 1, 2, 3, \qquad a = 1, 2, \ldots, N, \qquad \mathbf{r} \in E_{3N} \tag{4.5.1}$$

and they do not admit a Lagrangian (or a Hamiltonian) representation in the local variables considered.

To reduce the systems to equivalent first-order forms, introduce the physical (generally non-canonical) linear momentum

$$p_{ka} = m_a\dot{r}_{ka} \tag{4.5.2}$$

[37] The *Lagrangian* version of the theorem (see Chart 4.3) was formulated and proven by Havas (1973). Theorem 4.5.1 in the given Birkhoffian version, was given by Santilli (1978c). Note that the systems need not necessarily be Newtonian (e.g., they may have acceleration-dependent forces). The theorem is extendable to systems of order higher than one via the reduction to first-order form presented in Chart 4.3.

[38] In Equations (4.5.1), no summation on the repeated a index exists.

as a realization of prescriptions (4.1.23). The contravariant normal forms (4.1.28) are then given by

$$\begin{pmatrix} \dot{r}_a^k \\ \dot{p}_{ka} \end{pmatrix} - \begin{pmatrix} p_{ka}/m_a \\ f_{ka}^{\mathrm{ESA}}(t, \mathbf{r}, \mathbf{p}/m) + F_{ka}^{\mathrm{NENSA}}(t, \mathbf{r}, \mathbf{p}/m + \mathscr{F}_{ka}^{\mathrm{ENSA}}(t, \mathbf{r}, \mathbf{p}/m) \end{pmatrix} = 0 \qquad (4.5.3)$$

and can be written in unified notation

$$\dot{a} - \Xi(t, a) = 0, \qquad (4.5.4a)$$

$$(\dot{a}) = \begin{pmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{p}} \end{pmatrix}; \qquad (\Xi) = \begin{pmatrix} \mathbf{p}/m \\ \mathbf{f}^{\mathrm{ESA}} + \mathbf{F}^{\mathrm{NENSA}} + \mathscr{F}^{\mathrm{ENSA}} \end{pmatrix}. \qquad (4.5.4b)$$

Our proof of the theorem consists of showing that, under the assumed smoothness and regularity conditions, the fundamental equations (4.3.9) for a Birkhoffian representation, i.e.,

$$\left[\frac{\partial R_\nu(t, a)}{\partial a^\mu} - \frac{\partial R_\mu(t, a)}{\partial a^\nu}\right]\Xi^\nu(t, a) = \frac{\partial B(t, a)}{\partial a^\mu} + \frac{\partial R_\mu(t, a)}{\partial t}, \qquad \mu = 1, 2, \ldots, 2n \qquad (4.5.5)$$

always admit a solution in the functions R_μ and B, that is, a solution exists for an arbitrary functional dependence of the vector field Ξ^ν. As such, the proof automatically extends to arbitrary prescriptions (4.1.23) other than the physical selection (4.5.2), as well as to arbitrary second-order systems of the class admitted, e.g., holonomic systems in the general form (*.9). By recalling particularization (4.2.2), the conventional Hamiltonian representation of systems with potential forces is a trivial subcase.

Case 1. The Functions R_μ have an Explicit Dependence on Time (Non-autonomous Case). For any given function B, equations (4.5.5) *are* of the Cauchy–Kovalevski type, as one can see by writing them in the form

$$\frac{\partial R_\mu}{\partial t} = \left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\Xi^\nu - \frac{\partial B}{\partial a^\mu}. \qquad (4.5.6)$$

Then, under the assumed smoothness, regularity, and locality conditions, Theorem 1 of Chart A.3 holds, and a solution always exists.

Notice that this case applies also when the vector field is autonomous, by therefore being sufficient *per se* to prove the theorem. Nevertheless, to be in line with contemporary formulations of contact two-forms,[39] the case of autonomous functions $R_\mu(a)$, when applicable, is relevant.

Case 2. The Functions R_μ do not have an Explicit Dependence on Time (Semi-autonomous and Autonomous Cases). The fundamental equations for a Birkhoffian representation are now given by Equation (4.3.10), for the semiautonomous case, and by Equation (4.4.11), for the autonomous case. However, both sets of equations are of the Cauchy–Kovalevski form, as one can see by writing them in the form

$$\frac{\partial R_\mu}{\partial a^1} = -(\Xi^1)^{-1}\left\{\sum_{n=2}^{2n} \frac{\partial R_\mu}{\partial a^\nu}\Xi^\nu + \frac{\partial \Xi^\nu}{\partial a^\mu} R_\nu\right\} \qquad (4.5.7)$$

under identification

$$B = R_\nu \Xi^\nu \qquad (4.5.8)$$

and this completes the proof of the theorem. (*Q.E.D.*).

[39] See the remarks in Section 4.2 following Corollary 4.5.1a.

A number of physical and mathematical properties deserve inspection. First, we would like to identify the nature of the "direct universality" of Birkhoff's equations. This can be done by identifying all the mathematical algorithms at hand, that is, the local coordinates t and $a = (\mathbf{r}, \mathbf{p})$, and the function B (the meaning of the functions R_μ will be identified shortly).

Corollary 4.5.1a. *The direct universality of Birkhoff's equations for local unconstrained Newtonian systems in Euclidean space is characterized by the following properties.*

1. *The local variables t and* $\mathbf{r}$ *can be the time and Cartesian coordinates actually used by the experimenter,*
2. *The variables* $\mathbf{p}$ *can be the physical linear momenta* $m\dot{\mathbf{r}}$,
3. *The function* $B(t, a)$ *can be the physical energy* E_{tot}, *that is, the sum of the kinetic energy and of the potential energy of all self-adjoint forces.*

Note that the *physical energy* can be equivalently defined as the total energy of the maximal self-adjoint subsystem of (4.5.1). Needless to say, the total energy is generally *non-conserved* because of the presence of contact non-potential interactions. The definition is also introduced to stress the distinction from the familiar *canonical Hamiltonian* (which, as pointed out in Chart A.11, is often "conserved" while the system is nonconservative).

The reader can now appreciate the importance of the direct universality of Corollary 4.5.1a. In fact, lacking a precise physical identification for all the mathematical symbols at hand, one risks drawing mathematically correct conclusions which are physically meaningless. This situation becomes even more pronounced when one confronts the problem of quantizing non-potential interactions, as is expected for mutual penetration of wave packets.[40] In this case the canonical momentum "$\mathbf{p}$," the canonical angular momentum "$\mathbf{M}$" $= \mathbf{r} \times \mathbf{p}_{\text{can}}$, and the canonical Hamiltonian "H" *do not* represent the physical linear momentum, the physical angular momentum, and the physical energy, respectively, as a *necessary condition* for the existence of a Hamiltonian representation. But then, the attempt to preserve conventional quantum mechanical settings (e.g., the spectrum of "H" interpreted as "energy levels," or the spread "Δp" interpreted as "uncertainty in the momentum," etc.) risks being sterile.

It is hoped that the reader will begin to see a reason for this volume's emphasis on achieving analytic representations of Newtonian systems first in the *variables and functions of direct physical meaning.* Once this has been achieved, one can then study mathematical topics (such as nonlinear, experimentally unrealizable transformations of the coordinates), by minimizing possible physical inconsistencies.

Corollary 4.5.1a essentially states that one can first identify the quantities t, $\mathbf{r}$, $\mathbf{p}$, and $B = E_{\text{tot}}$ directly with physical quantities, and then search for a Birkhoffian representation. But the quantity B represents, in this case, only

[40] This problem will be touched in the charts of Chapter 6.

potential forces. This creates the need for identifying the ways in which Birkhoff's equations represent the remaining nonpotential forces. At this point the geometric or algebraic structure of Birkhoff's equations acquires a direct *dynamic* content.

Corollary 4.5.1b. *Under the conditions of direct universality of Corollary* 4.5.1*a, all nonpotential (non-self-adjoint) forces are represented by the covariant symplectic tensor*

$$\Omega_{\mu\nu}(t, a) = \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}, \tag{4.5.9}$$

or, equivalently, by the contravariant Lie tensor

$$\Omega^{\mu\nu}(t, a) = (\|\Omega_{\alpha\beta}\|^{-1})^{\mu\nu}. \tag{4.5.10}$$

In particular, when the Birkhoffian represents kinetic energy only, all acting forces are entirely represented by the geometric or algebraic tensor.

The difference between the Hamiltonian and Birkhoffian time evolutions (say, for the autonomous case)

$$\dot{A}(a) = \frac{\partial A}{\partial a^\mu}\,\omega^{\mu\nu}\,\frac{\partial H}{\partial a^\nu} = [A, H], \tag{4.5.11a}$$

$$\dot{A}(a) = \frac{\partial A}{\partial a^\mu}\,\Omega^{\mu\nu}(a)\,\frac{\partial B}{\partial a^\nu} = [A, B]^*, \tag{4.5.11b}$$

can now be understood. In the conventional Hamiltonian case, all forces (whether potential or not) are represented by the Hamiltonian. In fact, the fundamental Lie tensor $\omega^{\mu\nu}$ has constant elements and therefore does not carry a direct dynamic content. In the transition to the Birkhoffian case the situation is different insofar as the *Birkhoffian represents only part of the acting forces, while the remaining forces are directly embedded in the structure of the brackets of the time evolution.*

Equivalently, we can say that the direct universality of Birkhoff's equations, in the final analysis, is a consequence of the utmost possible use of the underlying geometry and algebra. When the realizations of symplectic two-forms and Lie brackets are restricted to canonical forms, the capacity to represent unrestricted systems in the coordinates of the observer is lost.

For an explicit illustration of this important function of the geometry and algebra, we recommend that the reader consult the examples at the end of this and the next chapters, with particular reference to Example 4.1 on the Hamiltonian *and* Birkhoffian representations of the (Newtonian) electromagnetic interactions.

After having identified the admissible physical meaning of the local variables, the Birkhoffian functions, and the underlying geometric/algebraic tensors, the next objective is to characterize physically the space in which Birkhoff's equations act. The relevance of the characterization will be pointed

out soon. Note that, on mathematical grounds, the problem has been solved in Section 4.2 via the identification of the local variables a^μ with a chart of the cotangent bundle T^*M. It is advisable to compare the most salient physical properties of the carrier space of Birkhoff's equations with those for the Hamiltonian case.

Corollary 4.5.1c. *While the variables* $\mathbf{r}$ *and* $\mathbf{p}$ *of a Hamiltonian representation are canonically conjugated, i.e., they verify the canonical rule*

$$\dot{\mathbf{r}} = \frac{\partial H}{\partial \mathbf{p}} \tag{4.5.12}$$

and span a phase space, the variables $\mathbf{r}$ *and* $\mathbf{p}$ *of a Birkhoffian representation are not canonically conjugated because, in general,*

$$\dot{\mathbf{r}} \neq \frac{\partial B}{\partial \mathbf{p}} \,. \tag{4.5.13}$$

As a result, the space of the Birkhoffian variables, $a = (\mathbf{r}, \mathbf{p})$*, is not necessarily a phase space; it will be referred to as a "dynamic space." In particular, while the phase space can be equipped with a fundamental Lie algebra structure* $\omega^{\mu\nu}$ *which represents directly the fundamental Poisson brackets*

$$(\omega^{\mu\nu}) = ([a^\mu, a^\nu]) = \begin{pmatrix} ([r^i, r^j])([r^i, p_j]) \\ ([p_i, r^j])([p_i, p_j]) \end{pmatrix} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}, \tag{4.5.14}$$

this structure is inapplicable to the dynamic space and must be replaced with the general Lie algebra structure, $\Omega^{\mu\nu}(a)$*,*[41] *which now represents the generalized fundamental brackets*

$$(\Omega^{\mu\nu}(a)) = ([a^\mu, a^\nu]^*) = \begin{pmatrix} ([r^i, r^j]^*)([r^i, p_j]^*) \\ ([p_i, r^j]^*)([p_i, p_j]^*) \end{pmatrix} \neq \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}. \tag{4.5.15}$$

As a result, components of coordinates and moments with different indices commute in the phase space,

$$[r^i, r^j] = [p_i, p_j] = [r^i, p_j] = 0, \qquad i \neq j, \tag{4.5.16}$$

but they do not generally commute in the dynamic space

$$[r^i, r^j]^* \neq 0, \qquad [p_i, p_j]^* \neq 0, \qquad [r^i, p_j]^* \neq 0, \qquad i \neq j. \tag{4.5.17}$$

The loss of the conventional phase space and its replacement with a more general space has a rather deep impact in mechanics. For an idea, consider the problem of quantizing non-potential interactions when the classical equations are given by *Birkhoff's* (rather than Hamilton's) equations. Under

[41] We exclude the nonautonomous case because of the lack of algebraic character of Birkhoff's equations (Chart 4.1).

these circumstances, conventional physical laws for potential interactions, such as Heisenberg's uncertainty principle,

$$\Delta r \Delta p \geq \tfrac{1}{2}\hbar, \tag{4.5.18}$$

cannot even be consistently formulated, let alone applied, trivially, because of the loss of the quantum mechanical version of fundamental brackets (4.5.14).

The differences between the dynamic space and the conventional phase space constitute one of the best mathematical formulations of the *physical differences* between the potential and nonpotential interactions. Jointly, the differences illustrate the essentially misleading nature of the Hamiltonian formulations when applied to nonpotential interactions, unless proper care is used for the physical interpretation of the algorithms at hand.

In fact, if Hamiltonian representations are used for non-potential systems, conventional quantum mechanical images are expected to apply, leading to principle (4.5.18) (because now commutation rules (4.5.14) hold). This conclusion is mathematically correct, but its physical interpretation is in doubt because, as indicated earlier, a necessary condition for the existence of the Hamiltonian representation is that the quantity "p" *does not* represent the physical linear momentum. In the transition to the Birkhoffian representation of the *same* system, according to the direct universality of Corollary 4.5.1a, insidious physical occurrences of the type considered are removed by construction. However, a generalization of basic physical laws which is more directly compatible with the underlying generalized algebra and geometry appears unavoidable.

Occurrences of this type should not be surprising. Hamilton's equations (without external terms[42]) have been developed throughout this century for the study of potential interactions. Birkhoff's equations have been rediscovered for the study of fundamentally more general interactions. The fate of the underlying physical laws is then predictable.

After having identified some preliminary physical aspects of Birkhoffian representations, the next objective is to determine methods for computing the Birkhoffian functions from the equations of motion. Notice that a first method is provided by Corollary 4.2.1a. However, the method is of somewhat formal inspiration and, as such, calls for reformulation into an operational version. The identification of additional methods and their interpretation is also desirable.

Corollary 4.5.1d. *Suppose that a second-order Newtonian system is given, and its equivalent contravariant first-order form* (4.5.4) *has been constructed via physical prescriptions* (4.5.2). *Some methods for the construction of the Birkhoffian functions* $R_\mu(t, a)$ *and* $B(t, a)$ *from the equations of motion are the following.*

[42] See footnote 5 of the Introduction.

Method 1.[43] *Identify B with the total energy* E_{tot} *in the sense of Corollary* 4.5.1*a, and then solve the Cauchy–Kovalevski equations* (4.5.6) *in the functions* R_μ.

Method 2.[43] *Construct a self-adjoint covariant general form*

$$[\Omega_{\mu\nu}(t, a)\dot{a}^\nu + \Gamma_\mu(t, a)]_{SA} = 0 \tag{4.5.19}$$

via the methods of Section 4.4. *The functions* R_μ *are then given by*

$$R_\mu(t, a) = \left[\int_0^1 d\tau\, \tau\Omega_{\mu\nu}(t, \tau a)\right]a^\nu, \tag{4.5.20}$$

and the Birkhoffian is provided by the rule

$$B(t, a) = -\left[\int_0^1 d\tau\left(\Gamma_\mu + \frac{\partial R_\mu}{\partial t}\right)(t, \tau a)\right]a^\mu. \tag{4.5.21}$$

Method 3.[44] *Suppose that* $2n$ *independent first integrals* $I^\mu(t, a)$ *(in the sense of Proposition* 4.4.1*) are known. Then functions* R_μ, *from Equations* (4.4.28*b*), *are given by*

$$R_\mu(t, a) = G_\alpha \frac{\partial I^\alpha}{\partial a^\mu}, \tag{4.5.22}$$

and the Birkhoffian is given by

$$B(t, a) = -G_\alpha \frac{\partial I^\alpha}{\partial t}. \tag{4.5.23}$$

A few comments are in order. The first method is clearly inspired by the desire to have a direct physical meaning for all local variables and functions. However, the method leads to functions R_μ possessing, in general, an explicit dependence on time, as evident from Equations (4.5.6). The geometric implications of this dependence have been indicated in Section 4.2, and the algebraic implications are pointed out in Chart 4.1. The reader should be fully aware of these implications before passing to applications (e.g., quantization). A method for attempting the elimination of the explicit dependence on time will be worked out shortly.

The second method is recommended when no physical condition is imposed on the meaning of the Birkhoffian and on the prescriptions for the construction of the first-order form. It is often preferable in practice, clearly, because of the greater freedom in the Birkhoffian functions. Notice that, compared with Proposition 4.2.1, the method stresses the need to compute first the R-functions and then the Birkhoffian, as clearly expressed by the contribution of the former to the latter according to Equation (4.5.21). This

[43] Santilli (1978c).
[44] Hojman (1981).

necessary procedure is somewhat hidden in formal method (4.2.30).[45] This second method is more readily set for the semi-autonomous case, that is, for the representation of nonautonomous vector fields by autonomous R-functions and the consequential elimination of the problematic aspects of Chart 4.1.

The third method is conceived to provide a first interpretation of the results. In fact, it essentially emerges that the *Birkhoffian functions are functions of a maximal independent set of first integrals.*

It should be stressed that *none of the methods guarantees the capability of actually constructing the Birkhoffian functions in the needed explicit form.* In fact, the solutions are often expressed by power-series expansions. Theorem 4.5.1 guarantees their convergence and therefore, the *existence* of a solution, but the computation of the sums in the needed explicit closed form may often turn out to be beyond practical computational capabilities.

Example 4.4 has been included to illustrate the practical *difficulties* in the construction of the Birkhoffian functions. The possibility of constructing *approximate Birkhoffian representations* (that is, representations provided by the first terms of convergent power-series expansions) should not be overlooked. In fact, Physics is intrinsically an approximation of nature. The important point is to identify the degree of approximation which can be accepted for the case at hand.[46] This line of study is left to the interested reader (Problem 4.5).

After the identification of the methods for the construction of the Birkhoffian functions, the next problem is to study their degrees of freedom, that is, their functional arbitrariness for fixed implicit functions (or solutions).

Corollary 4.5.1e. *A given second-order Newtonian system verifying the conditions of Theorem* 4.5.1 *always admits infinite varieties of equivalent Birkhoffian representations, all in the same time and coordinates of the experimenter. Some of the functional degrees of freedom are the following.*

Class 1. *An infinite variety of prescriptions* (4.1.23) *exists, i.e.,*

$$y_{ka} = M_{ka}(t, \mathbf{r}, \dot{\mathbf{r}}), \qquad k = 1, 2, 3, \qquad a = 1, 2, \ldots, N, \quad (4.5.24)$$

for the construction of equivalent normal forms, one for each selection of the arbitrary functions M_{ka} *(subject to regularity conditions* (4.1.24)*). For each of these infinitely different possibilities, Theorem* 4.5.1 *applies and the corresponding Birkhoffian representations are equivalent, in the sense that they can all be reduced to the same second-order system.*

[45] Another important difference between methods (4.2.30) and (4.5.20)-(4.5.21) is that in the former, the τ-factorization includes that of time, while such factorization is absent for the latter.

[46] For instance, the approximation of Newtonian systems via the functions $L = T - V$ and $H = T + V$ should be rejected because it literally implies the existence of perpetual motion in our environment. Along these lines, a (local, nonpotential) Birkhoffian approximation is more acceptable because it provides a quantitative treatment of the nonconservative character of the systems. The understanding is that by no means should such a Birkhoffian representation be considered terminal in character, owing to the more realistic non-local/integral and non-potential/non-self-adjoint nature of the systems, as indicated at the end of Section 4.2.

Class 2. For each prescription (4.5.24) *and each self-adjoint covariant general form, an infinite variety of different isotopic transformations exists*

$$\{h^{\alpha}_{\mu}(t, a)[\Omega_{\alpha\nu}(t, a)\dot{a}^{\nu} + \Gamma_{\mu}(t, a)]_{SA}\}_{SA} = 0, \qquad \det(h^{\alpha}_{\mu})(\tilde{\mathscr{R}}) \neq 0 \quad (4.5.25)$$

originating from the degrees of freedom of the solutions of system (4.4.8). *For each of these self-adjoint forms, Theorem* 4.5.1 *holds. All corresponding Birkhoffian representations are then equivalent in the sense that they all admit the same vector field* Ξ^{ν}.

Class 3. For each prescription (4.5.24), *isotopy* (4.5.25), *and corresponding Birkhoffian functions* R_{μ} *and* B, *an infinite variety of functions exists characterized by the gauge transformations*

$$R_{\mu}(t, a) \to R'_{\mu}(t, a) = R_{\mu}(t, a) + \frac{\partial G(t, a)}{\partial a^{\mu}}, \qquad (4.5.26a)$$

$$B(t, a) \to B'(t, a) = B(t, a) - \frac{\partial G}{\partial t}. \qquad (4.5.26b)$$

All the corresponding Birkhoffian representations are equivalent in the sense that Birkhoff's equations are the same for all possible functions (4.5.26), *i.e.*,

$$\left(\frac{\partial R'_{\nu}}{\partial a^{\mu}} - \frac{\partial R'_{\mu}}{\partial a^{\nu}}\right)\dot{a}^{\nu} - \left(\frac{\partial B'}{\partial a^{\mu}} + \frac{\partial R'_{\mu}}{\partial t}\right)$$
$$\equiv \left(\frac{\partial R_{\nu}}{\partial a^{\mu}} - \frac{\partial R_{\mu}}{\partial a^{\nu}}\right)\dot{a}^{\nu} - \left(\frac{\partial B}{\partial a^{\mu}} + \frac{\partial R_{\mu}}{\partial t}\right). \qquad (4.5.27)$$

Note that degrees of freedom for Class 1 actually imply the initiation of the transformation theory. In fact, starting from the physical variables $a = (\mathbf{r}, \mathbf{p})$, $\mathbf{p} = m\dot{\mathbf{r}}$, the degrees of freedom imply the transition to the *different* variables $a' = (\mathbf{r}, \mathbf{y})$ where y now is no longer subject to the condition of direct physical meaning. Clearly, the transition $a \to a'$ is a particular type of transformation in the cotangent bundle. As such, it will be studied in the next chapter. We have included the case here to stress the property that Theorem 4.5.1 is consistent for *all* possible prescriptions (4.5.24), whether physically inspired or not.

The degrees of freedom of Class 2 are a direct result of the methodology of the Inverse Problem and can be constructed via the following rule of *Birkhoffian isotopy*

$$\left(\frac{\partial R^{*}_{\nu}}{\partial a^{\mu}} - \frac{\partial R^{*}_{\mu}}{\partial a^{\nu}}\right)\dot{a}^{\nu} - \left(\frac{\partial B^{*}}{\partial a^{\mu}} + \frac{\partial R^{*}_{\mu}}{\partial t}\right)$$
$$\equiv \left\{h^{\nu}_{\mu}\left[\left(\frac{\partial R_{\rho}}{\partial a^{\nu}} - \frac{\partial R_{\nu}}{\partial a^{\rho}}\right)\dot{a}^{\rho} - \left(\frac{\partial B}{\partial a^{\nu}} + \frac{\partial R_{\nu}}{\partial t}\right)\right]_{SA}\right\}_{SA}. \qquad (4.5.28)$$

Clearly, here we have a cotangent bundle image of the Lagrangian isotopies of the Appendix. Intriguingly, this image permits the achievement of the following new interpretation of the isotopic degrees of freedom (whose

Lagrangian counterpart is still unknown at this time). Recall from Section 4.4 the construction of a self-adjoint form through the use of arbitrary functions of independent first integrals (Proposition 4.4.1). Recall also from Corollary 4.5.1d that the construction results in a method for the computation of Birkhoffian functions (Equations (4.5.22) and (4.5.23)). By reinspecting these results we see that *different Birkhoffian functions which can be constructed via rule* (4.5.28) *can represent the arbitrariness of functions* $G_\mu(I)$ *in the first integrals* I_α *as well as the functional degrees of freedom of the first integrals themselves.*

The gauge degree of freedom of Class 3 is trivial and can be best proved by writing Birkhoff's equations in the Lagrangian form (4.2.32), where we have a situation similar to that of Equations (A.2.3), i.e.,

$$L = -R_\nu(t, a)\dot{a}^\nu + B(t, a) \to L^\dagger = -R_\nu^\dagger(t, a)\dot{a}^\nu + B^\dagger(t, a)$$
$$= -R_\nu \dot{a}^\nu + B - \dot{G}(t, a) \tag{4.5.29a}$$

$$\frac{d}{dt}\frac{\partial L^\dagger}{\partial \dot{a}^\mu} - \frac{\partial L^\dagger}{\partial a^\mu} \equiv \frac{d}{dt}\frac{\partial L}{\partial \dot{a}^\mu} - \frac{\partial L}{\partial a^\mu} \tag{4.5.29b}$$

The applications of the degrees of freedom of the Birkhoffian representations are intriguing. Below, a few representative cases are given.

In Chart I.3.6 we recalled a rather old and controversial aspect of mechanics. It consists of the fact that, on one side, Hamilton's equations possess a symplectic structure in a rather clear and direct way while, on the other side, the variational principle from which the equations are derived, the conventional Hamilton's principle in phase space

$$\delta \int_{t_1}^{t_2} dt(p_k \dot{r}^k - H)(\tilde{E}) = 0, \tag{4.5.30}$$

does not appear to possess a symplectic character in an equally clear way. Apparently, this is the basis of a tendency in contemporary circles of mathematicians to ignore the treatment of mechanics via variational principles and restrict the geometrical study to the analytic equations themselves.

Two seemingly independent resolutions of this controversy were proposed in the preceding volume. The first is given by the reformulation of Principle (4.5.30) via the unified notation $a = (\mathbf{r}, \mathbf{p})$ after which the geometric character of the integrand of the principle (as the contact canonical one-form) is expressed more transparently, together with the corresponding character of the analytic equations, i.e.,

$$\delta \int_{t_1}^{t_2} dt(p_k \dot{r}^k - H)(\tilde{E}) = \delta \int_{t_1}^{t_2} (R_\nu^0\, da^\nu - H\, dt)(\tilde{E})$$
$$= \int_{t_1}^{t_2} dt\left[\left(\omega_{\mu\nu}\dot{a}^\nu - \frac{\partial H}{\partial a^\mu}\right)\delta a^\mu\right](\tilde{E}) = 0 \tag{4.5.31}$$

Independently from that, we introduced in Chart I.3.6 a reformulation of Principle (4.5.31) with an explicit symplectic structure in the intengrand of the action itself, according to the equation

$$\delta \int_{t_1}^{t_2} dt[\tfrac{1}{2}a^\mu \omega_{\mu\nu}\dot{a}^\nu - H(t, a)](\tilde{E}) = \int_{t_1}^{t_2} dt\left[\left(\omega_{\mu\nu}\dot{a}^\nu - \frac{\partial H}{\partial a^\mu}\right)\delta a^\mu\right](\tilde{E}) = 0. \tag{4.5.32}$$

Evidently, even though the integrands of principles (4.5.31) and (4.5.32) are *different*, the underlying analytic equations are the *same*, and we can write

$$\delta \int_{t_1}^{t_2} dt(R_\nu^0 \dot{a}^\nu - H)(\tilde{E}) \equiv \delta \int_{t_1}^{t_2} dt(\tfrac{1}{2}a^\mu \omega_{\mu\nu}\dot{a}^\nu - H)(\tilde{E}). \tag{4.5.33}$$

Inspected within the context of Corollary 4.5.1e, this degree of freedom results in being trivially given by the gauge

$$R_\nu^0 \to R_\nu^{0\dagger} = R_\nu^0 + \frac{\partial G}{\partial a^\nu}, \qquad G = -rp \tag{4.5.34}$$

Another application of Corollary 4.5.1e is given by the possible removal of the explicit time dependence in the R_μ functions. This can be done by *transforming non-autonomous representations $R_\mu(t, a)$ and $B(t, a)$ into an equivalent semiautonomous forms $R'_\mu(a)$ and $B'(t, a)$*. A formal solution can be obtained via the degrees of freedom of Class 3, and reads

$$R_\mu(t, a) \to R'_\mu(a) = R_\mu + \frac{\partial G}{\partial a^\mu}, \qquad G = \int_0^t dt(B' - B). \tag{4.5.35}$$

In this way, one can first compute the functions R_μ and B as they originate naturally from Corollary 4.5.1a (with an explicit dependence on time), and then attempt to eliminate such a dependence via rule (4.5.35). Note that if the original Birkhoffian represents total energy, the new Birkhoffian *cannot* preserve this physical meaning under transformation (4.5.35).

The algebraic implications of this are nontrivial. As we shall see in detail in Chart 4.1, if B is identified with the physical energy E_{tot}, the Birkhoffian time evolution for non-autonomous systems *cannot* have a Lie algebra character. If such algebraic character is desired, the Birkhoffian *cannot* represent physical energy. To state it in different terms, the Lie algebra character of the evolution and the direct physical meaning of the Birkhoffian, rather surprisingly, turn out to be mutually exclusive in a number of cases of Birkhoffian representations.

We end this section with the *representation of Hamilton's equations in terms of Birkhoff's equations*. Recall from Section 4.4 that the isotopic transformations are universal for first-order systems. As a result, they exist also for Hamilton's equations in all possible Hamiltonians. More explicitly, consider Corollary 4.4.1a, and suppose that covariant form (4.4.9) is that of Hamilton's equations with $\Xi_\alpha = \partial H/\partial a^\alpha$. Theorem 4.4.3 establishes that

Equations (4.4.11) always admit a non-trivial solution in the isotopic functions; that is, they admit a solution (h_μ^α) other than the identity for all possible Hamiltonians. In this way we reach the following property.

Proposition 4.5.1 (Representation of Hamilton's Equations in Terms of Birkhoff's Equations). *Hamilton's equations in all possible analytic and regular Hamiltonians $H(t, a)$ always admit an indirect Birkhoffian representation in a star-shaped neighborhood of a regular point of their variables*

$$\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \left(\frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t}\right) \equiv \left\{h_\mu^\alpha(t, a)\left[\omega_{\alpha\beta}\dot{a}^\beta - \frac{\partial H(t, a)}{\partial a^\alpha}\right]_{SA}\right\}_{SA}. \tag{4.5.36}$$

Upon computation of the isotopic functions for each given Hamiltonian via the solution of Equation (4.4.11), *the Birkhoffian functions are given by*

$$R_\mu = \left[\int_0^1 d\tau\, \tau h_\mu^\alpha(t, \tau a)\right]\omega_{\alpha\beta}a^\beta, \tag{4.5.37a}$$

$$B = -\left[\int_0^1 d\tau\left(h_\mu^\alpha \frac{\partial H}{\partial a^\alpha} + \frac{\partial R_\mu}{\partial t}\right)(t, \tau a)\right]a^\mu. \tag{4.5.37b}$$

We should stress that the universality of the isotopy exists for the transition from Hamilton's to Birkhoff's equations. The inverse case is not universal; that is, *for arbitrarily given functions R_μ and B, the decomposition of Birkhoff's equations into the Hamiltonian form according to rule* (4.5.36) *does not necessarily exist*. In fact, the direct universality of Birkhoff's equations for local Newtonian systems, as compared to its absence for Hamilton's equations, is due precisely to the *lack* of general existence of a reduction (4.5.36).

In Corollary 4.5.1c we have stressed the fact that, in general, the conventional phase space character of the carrier space of Birkhoff's equations is lost. Proposition 4.5.1 permits the identification of the following property. *Whenever Birkhoff's equations with a tensor $\Omega_{\mu\nu}(t, a)$ other than the fundamental symplectic tensor $\omega_{\mu\nu}$ admit a factorization of Hamilton's equations according to rule* (4.5.36), *the variables $a = (\mathbf{r}, \mathbf{p})$ span a phase space, that is, the variables* **r** *and* **p** *are canonically conjugated.*

Proposition 4.5.1 opens up, at least in principle, new possibilities of research for *conservative* systems such as the three-body system. In fact, as the study of these systems via Hamilton's equations can be considered as virtually exhausted at this time, the representation of the same systems via Birkhoff's equations permits the application of new, more general methods ranging from new first integrals to generalized perturbation techniques.

In conclusion, the Birkhoffian generalization of Hamiltonian mechanics is useful not only for the non-potential systems for which it was conceived, but also for the more conventional potential systems.

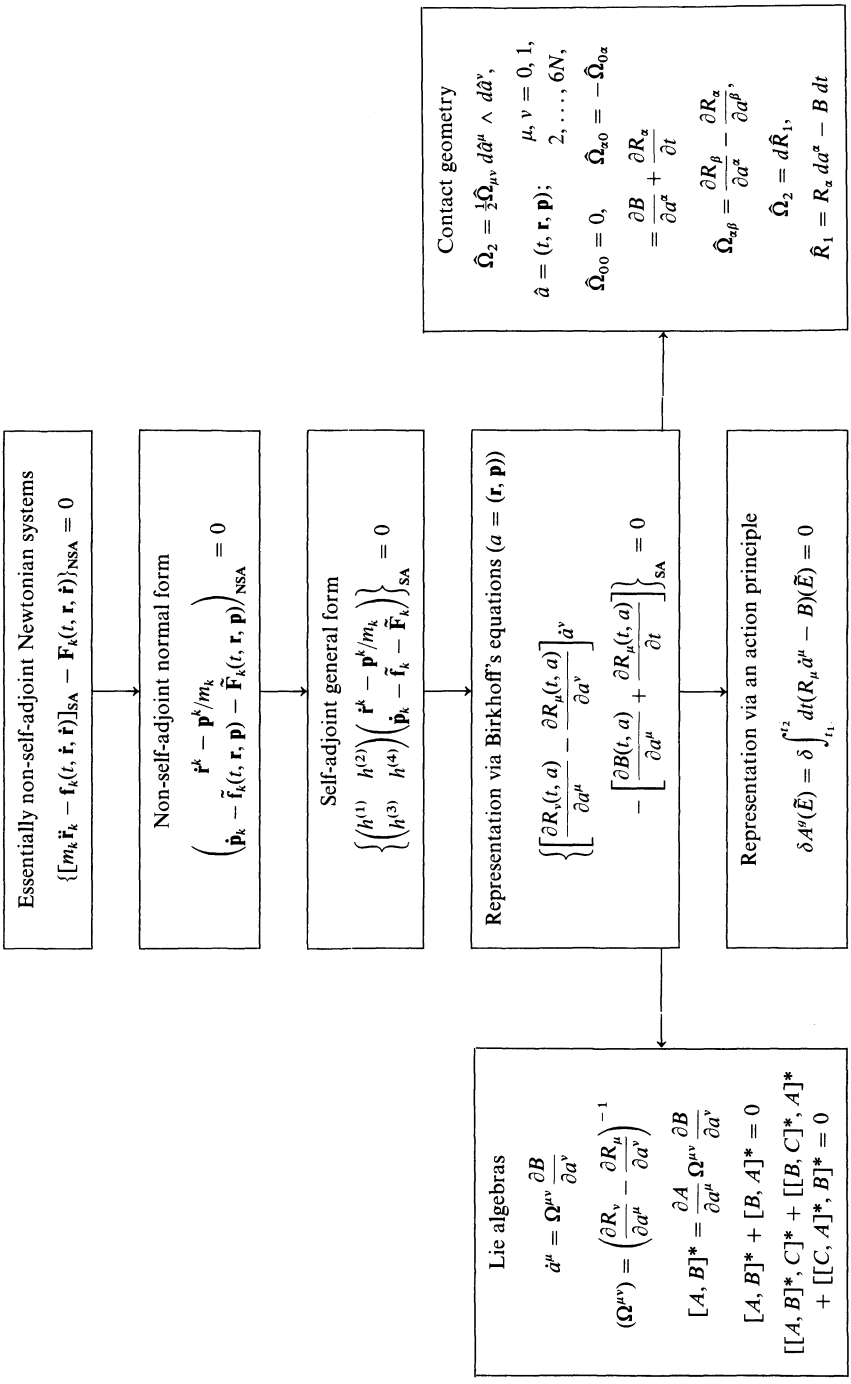

Essentially non-self-adjoint Newtonian systems
$\{[m_k\ddot{\mathbf{r}}_k - \mathbf{f}_k(t, \mathbf{r}, \dot{\mathbf{r}})]_{SA} - \mathbf{F}_k(t, \mathbf{r}, \dot{\mathbf{r}})\}_{NSA} = 0$
Non-self-adjoint normal form
$\begin{pmatrix} \dot{\mathbf{r}}^k - \mathbf{p}^k/m_k \\ \dot{\mathbf{p}}_k - \tilde{\mathbf{f}}_k(t, \mathbf{r}, \mathbf{p}) - \tilde{\mathbf{F}}_k(t, \mathbf{r}, \mathbf{p}) \end{pmatrix}_{NSA} = 0$
Self-adjoint general form
$\left\{ \begin{pmatrix} h^{(1)} & h^{(2)} \\ h^{(3)} & h^{(4)} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{r}}^k - \mathbf{p}^k/m_k \\ \dot{\mathbf{p}}_k - \tilde{\mathbf{f}}_k - \tilde{\mathbf{F}}_k \end{pmatrix} \right\}_{SA} = 0$
Representation via Birkhoff's equations ($a = (\mathbf{r}, \mathbf{p})$)
$\left\{ \left[\frac{\partial R_\nu(t, a)}{\partial a^\mu} - \frac{\partial R_\mu(t, a)}{\partial a^\nu} \right] \dot{a}^\nu - \left[\frac{\partial B(t, a)}{\partial a^\mu} + \frac{\partial R_\mu(t, a)}{\partial t} \right] \right\}_{SA} = 0$
Representation via an action principle
$\delta A^a(\tilde{E}) = \delta \int_{t_1}^{t_2} dt(R_\mu \dot{a}^\mu - B)(\tilde{E}) = 0$
Contact geometry
$\hat{\Omega}_2 = \frac{1}{2}\hat{\Omega}_{\mu\nu}\, d\hat{a}^\mu \wedge d\hat{a}^\nu,$
$\hat{a} = (t, \mathbf{r}, \mathbf{p}); \quad \mu, \nu = 0, 1, 2, \ldots, 6N,$
$\hat{\Omega}_{00} = 0, \quad \hat{\Omega}_{\alpha 0} = -\hat{\Omega}_{0\alpha} = \frac{\partial B}{\partial a^\alpha} + \frac{\partial R_\alpha}{\partial t}$
$\hat{\Omega}_{\alpha\beta} = \frac{\partial R_\beta}{\partial a^\alpha} - \frac{\partial R_\alpha}{\partial a^\beta},$
$\hat{\Omega}_2 = d\hat{R}_1,$
$\hat{R}_1 = R_\alpha\, da^\alpha - B\, dt$
Lie algebras
$\dot{a}^\mu = \Omega^{\mu\nu} \frac{\partial B}{\partial a^\nu}$
$(\Omega^{\mu\nu}) = \left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu} \right)^{-1}$
$[A, B]^* = \frac{\partial A}{\partial a^\mu} \Omega^{\mu\nu} \frac{\partial B}{\partial a^\nu}$
$[A, B]^* + [B, A]^* = 0$
$[[A, B]^*, C]^* + [[B, C]^*, A]^* + [[C, A]^*, B]^* = 0$

Figure 4.1 (A schematic view of the direct universality of Birkhoff's equations for local Newtonian systems). The physical systems in our environment (such as motions in atmosphere, spinning tops with drag torques, damped oscillations, etc.) generally violate the integrability conditions for the existence of a Hamiltonian representation in the coordinate and time variables of the observer. This fact, established in Section 4.1, is clear evidence of the *insufficiency* in mechanics of Hamilton's equations in their contemporary formulation (that without external terms). A central objective of this volume is the identification of a generalization of Hamilton's equations which preserves their analytic, algebraic, and geometric character, while achieving direct universality, that is, the representation of all systems of the class admitted in the coordinate and time variables of the experimenter. The generalized equations are identified via the conditions of variational self-adjointness for first-order systems, owing to their symbiotic representation of the analytic character (derivability from a variational principle), algebraic character (Lie brackets of the time evolution), and geometric character (underlying two-form of contact type). The explicit form of the equations is that studied originally by Birkhoff (1927). The equations were therefore called *Birkhoff's equations* by Santilli (1978c); the terminology was subsequently adopted by a number of authors, and it has been used for this work. The analysis presented in the chapter establishes that Birkhoff's equations are the most general possible equations possessing the indicated analytic, algebraic, and geometric features. Their direct universality results from a rather surprising mathematical property. Unlike the case of second-order systems, first-order systems always admit equivalent self-adjoint forms, or, more geometrically, a vector field $\hat{\Xi}$ in a star-shaped neighborhood of a regular point of $\mathbb{R} \times T^*M$ always admits an exact contact form $\hat{C}_2 = d\hat{R}_1$ verifying the property $\hat{\Xi} \lrcorner \hat{C}_2 = 0$ (see Chart 4.6). This table depicts the simplest possible method for practical applications. A given second-order system is turned into an equivalent first-order form in the variables: t (time of the observer), $\mathbf{r}$ (coordinates of the system in the reference frame of the observer), and $\mathbf{p} = m\dot{\mathbf{r}}$ (linear momomentum of the system also in the reference frame of the observer). This first-order form is then turned into an equivalent self-adjoint form via multiplicative functions verifying the integrability conditions of the theory. A number of methods presented in the text permit the computation of a Birkhoffian representation, that is, of a generalized (Pfaffian) action functional whose conventional variations reproduce the system. This Birkhoffian generalization of familiar Hamiltonian settings has rather deep physical and mathematical implications. In particular, it implies the existence of a suitable generalization of virtually all the classical canonical formulations (Hamilton–Jacobi theory, transformation theory, etc.), with the expectation of a corresponding generalized image at the quantum mechanical level. The state of the art on these intriguing studies is presented throughout the analysis. Most intriguing is the expected existence of a suitable generalization of the (classical and quantum mechanical) physical laws in the transition from potential interactions to the more general nonpotential form, ranging from a (classical) Birkhoffian generalization of Galilei's relativity for nonconservative and Galilei form noninvariant systems to a (quantum mechanical) generalization of Heisenberg's equations and uncertainty relations for wave packets in conditions of mutual penetration, as occurring, say, in strong interactions. The occurrence is not, after all, surprising, given the fundamental character of Hamilton's equations in contemporary theoretical physics and the predictably fundamental character which can be acquired by their Birkhoffian generalization. Having indicated these possibilities of mathematical and physical advancements, the analysis presented in this volume identifies a number of *limitations* of Birkhoff's equations. The most conspicuous one is given by the fact that the explicit computation of a Birkhoffian representation from given equations of motion is often so complex that it discourages the scholar most devoted to Lie algebra and the symplectic or contact geometry. Also, Birkhoff's equations do not appear suitable for the more general nonlocal/integral and nonpotential/non-self-adjoint interactions. For these (and other) reasons, the existence of a generalization of Birkhoff's equations of the so-called Lie-admissible and symplectic-admissible type is recommendable (Chart 4.7).

Chart 4.1 Lack of Algebraic Character of Nonautonomous Birkhoff Equations

As commonly understood in the contemporary theory of Abstract Algebras, an *algebra U* is a vector space of elements $a, b, c, \ldots$, over a field F of characteristic p (=0, or prime) with elements $\alpha, \beta, \gamma, \ldots$, equipped with a bilinear (abstract) *product ab* satisfying the *right* and *left distributive laws*

$$a(b + c) = ab + ac, \tag{1a}$$

$$(a + b)c = ac + bc, \tag{1b}$$

and the *scalar laws*[47]

$$(\alpha a)b = a(\alpha b) = \alpha(ab) \tag{2}$$

for all elements $a, b, c \in U$, and $\alpha \in F$. Additionally, when the *associative law*,

$$[a, b, c] \stackrel{\text{def}}{=} a(bc) - (ab)c = 0, \tag{3}$$

is verified for all elements $a, b, c \in U$, we have an *associative algebra*; otherwise, we have a *nonassociative algebra*. In the contemporary liliterature, the term "algebras" generally represents "nonassociative algebras," and the same usage is adopted in this volume (unless the adjective "associative" is explicitly used). A truly large variety of algebras have been identified in the mathematical and physical literature. They are characterized first by the distributive and scalar laws (for the product to characterize an algebra), and then by additional, specific laws (also called *identities* or *axioms*).

The algebras playing a relevant role for the analysis of this volume are the following.

1. *Lie Algebras*. They are algebras L over F characterized by the laws

$$ab + ba = 0, \tag{4a}$$

$$a(bc) + b(ca) + c(ab) = 0. \tag{4b}$$

2. *Lie-Admissible Algebras*. They are algebras U over F such that the attached algebra U^-, which is the same vector space as U equipped with the product

$$[a, b]_U = ab - ba \tag{5}$$

is a Lie algebra. Associative algebras A are clearly the simplest possible realizations of Lie-admissible algebras. Lie algebras L are also Lie-admissible because $[a, b]_L = 2[a, b]_A$. However, there exists a large number of Lie-admissible algebras which are not Lie. Thus the Lie-admissible algebras constitute a generalization of the Lie algebras.

3. *Jordan Algebras*. They are algebras J over F characterized by the laws

$$ab - ba = 0, \tag{6a}$$

$$[a^2, b, a] = 0. \tag{6b}$$

[47] The additional properties

$$(a\alpha)b = a(b\alpha) = (ab)\alpha = \alpha(ab)$$

can be proved to be a consequence of law (2).

4. *Jordan-Admissible Algebras.* They are algebras U over F such that the attached algebra U^+, which is the same vector space as U equipped with the product

$$\tfrac{1}{2}\{a, b\}_U = \tfrac{1}{2}(ab + ba), \tag{7}$$

is a Jordan algebra. Again, associative and Jordan algebras are Jordan-admissible, but the inverse does not necessarily hold. Thus the Jordan-admissible algebras are a bona fide generalization of the Jordan algebras.

5. *Alternative Algebras.* They are algebras U over F verifying the laws

$$a^2b = a(ab), \qquad ba^2 = (ba)a. \tag{8}$$

For recent mathematical and physical studies of these algebras, as well as for an extensive bibliography, we refer the interested reader to the proceedings of the second (1979) and third workshops (1981) on Lie-admissible formulations.

In physics, the abstract elements $a, b, c, \ldots$ are realized via specific quantities, such as functions $A(a)$ on T^*M or operators $\tilde{A}$ on a Hilbert space $\mathscr{H}$; the field is usually assumed to have characteristic zero (e.g., the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers); and the abstract product ab assumes an explicit form depending on the selected realization of the elements. Different realizations of the product then yield generally different algebras. An important (and often overlooked) point is that all possible realizations of the product must verify laws (1) and (2) to qualify as the product of an algebra.

As an example, the product

$$A(a) \circ B(a) \stackrel{\text{def}}{=} \frac{\partial A}{\partial a^\mu} C^{\mu\nu}(a) \frac{\partial B}{\partial a^\nu} \tag{9}$$

is a fully acceptable classical realization of an algebra of functions on T^*M because it verifies laws (1) and (2), i.e.,[48]

$$A \circ (B + C) = A \circ B + A \circ C, \tag{10a}$$

$$(A + B) \circ C = A \circ C + B \circ C, \tag{10b}$$

$$\alpha \circ A = A \circ \alpha \equiv 0. \tag{10c}$$

To have a more specific algebra, suitable integrability conditions must be imposed on the tensor $C^{\mu\nu}$. For instance, for the product $A \circ B$ to characterize a Lie algebra, the tensor $C^{\mu\nu}$ must verify integrability conditions (4.1.48).

Consider now the autonomous or semi-autonomous Birkhoff's equations (4.2.3) or (4.2.5). The brackets $[A, B]^*$ of the time evolution of functions $A(a)$ on T^*M

$$\dot{A}(a) = \frac{\partial A}{\partial a^\mu} \Omega^{\mu\nu}(a) \frac{\partial B}{\partial a^\nu} \stackrel{\text{def}}{=} [A, B]^* \tag{11}$$

first, verify laws (1) and (2) and second, are Lie. We can then say that the autonomous and semi-autonomous Birkhoff's equations possess a consistent algebraic structure and that such structure turns out to be that of Lie algebras.

[48] Under property (10c), law (2) (as well as those of footnote 47) is trivially verified.

This important algebraic property is lost for the non-autonomous Birkhoff's equations (4.2.6). In fact, time evolution (11) now takes the form

$$\dot{A}(a) = \frac{\partial A}{\partial a^\mu}\,\Omega^{\mu\nu}(t, a)\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t}\right) \stackrel{\text{def}}{=} A \circ B. \tag{12}$$

It is easy to see that the "product" $A \circ B$ does not characterize an algebra because it violates the right distributive and scalar laws

$$A \circ (B + C) \neq A \circ B + A \circ C, \tag{13a}$$

$$(A + B) \circ C = A \circ C + B \circ C, \tag{13b}$$

$$(\alpha \circ A) \circ B \neq (A \circ B) \circ \alpha. \tag{13c}$$

As a result, the *nonautonomous Birkhoff's equations do not have an algebraic structure* in the sense that the brackets of their time evolution do not qualify as the product of an algebra. Note that *it is not a breakdown of Lie algebras but, more profoundly, the breakdown of the very definition of algebras*[49].

The occurrence is perhaps the most significant difference between the (nonautonomous) Hamilton's and Birkhoff's equations. In fact, the former have a consistent Lie algebra structure.

As is familiar from the analysis of Sections 4.2 and 4.5, the occurrence originates from the explicit time-dependence of the R_μ functions. Intriguingly, the occurrence provides an algebraic motivation for the contemporary semi-autonomous form of contact two-forms (that is, without an explicit time dependence in the attached symplectic structure).

A method has been presented in Section 4.5 (see Equations (4.5.35)) for transforming the nonautonomous (nonalgebraic) time evolution (12) into the semi-autonomous (algebraic) form (11). Regrettably, however, the transformation does not allow the function B to represent the total energy. This can be easily seen by noting that, in general, the energy does not depend explicitly on time, while the nonpotential forces can have such a dependence. In this case, the only possible representation is the non-autonomous one with nonalgebraic time evolution (12).

This illustrate the statement of Section 5.5 to the effect that *the direct physical meaning of the Birkhoffian function, and the algebraic character of the time evolution are, in general, mutually incompatible.*[50]

Chart 4.2 Algebraic Significance of Isotopic and Genotopic Transformations

Let U be an algebra with elements $a, b, c, \ldots$ over a field F of characteristic p verifying the set of axioms

$$I_k(ab) = 0, \qquad k = 1, 2, \ldots, \tag{1}$$

[49] This property was identified by Santilli (1979 b).

[50] In passing to the Lie-admissible generalization of Birkhoff's (and of Hamilton's) equations, this incompatibility is resolved. See in this respect Chart 4.7. Finally, note that, despite the breakdown of the algebraic character of law (12), the tensor $\Omega^{\mu\nu}$ is Lie, that is, it verifies integrability conditions (4.1.48). To state this in different terms, brackets (11) are algebraic and Lie regardless of whether or not the tensor $\Omega^{\mu\nu}$ depends explicitly on time. This algebraic character is then lost in brackets (12) because of the additive term in $\partial R_\mu/\partial t$ and *not* because of the explicit time dependence of $\Omega^{\mu\nu}$.

where ab is the product. Construct the new algebra U^* which is the same vector space as U but equipped with the new product

$$a * b = (ac)b \tag{2}$$

where c is a fixed element of U. U^* is called an *isotopic extension* or more simply, an *isotope* of U, when the new product $a * b$, besides preserving the distributive and scalar laws, also verifies the identities of U, i.e.,

$$I_k(a * b) = 0. \tag{3}$$

The isotopy is called *regular* (*singular*) when the element c is (is not) invertible. A regular isotopy is invertible in the sense that

$$a *^{-1} b = (ac)c^{-1}b = ab. \tag{4}$$

More generally, *we shall define isotopy as any*[51] *transformation of the product of an algebra via elements of the algebra itself and/or of the field which preserves* 1) *the original algebra as vector field,* 2) *the distributive and scalar laws, and* 3) *the identities of the original algebra.*

As a simple example, let $\mathscr{A}$ be the associative algebra of matrices $A, B, C, \ldots$ over the field $\mathbb{R}$ of real numbers, equipped with the conventional product of matrices AB. Let C be an element of $\mathscr{A}$. The transformation of the product

$$AB \to A * B = ACB \tag{5}$$

for all $A, B \in \mathscr{A}$ and fixed C characterizes an *isotope $\mathscr{A}^*$ of the associative algebra $\mathscr{A}$* (Santilli (1978d)) because the new product $A * B$ is still associative. In this case there is no need to specify the association $(AC)B$ or $A(CB)$ because, from the associativity law, $(AC)B = A(CB)$.

As an example of *nonassociative isotopy*, let L be a Lie algebra of matrices $A, B, C, \ldots$ over $\mathbb{R}$ and product

$$[A, B]_{\mathscr{A}} = AB - BA. \tag{6}$$

Let C be also an element of L. Then the transformation of the product

$$[A, B]_{\mathscr{A}} = AB - BA \to [A, B]_{\mathscr{A}^*} = ACB - BCA \tag{7}$$

characterizes an *isotope L^* of the Lie algebra L.*

An example of algebraic isotopy in Newtonian mechanics is given by the transition from the conventional to the generalized Poisson brackets (Santilli 1978c)

$$[A, B]_{(a)} = \frac{\partial A}{\partial a^\mu}\,\omega^{\mu\nu}\,\frac{\partial B}{\partial a^\nu} \to [A, B]^*_{(a)} = \frac{\partial A}{\partial a^\mu}\,\Omega^{\mu\nu}(a)\,\frac{\partial B}{\partial a^\nu} \tag{8a}$$

$$(\omega^{\mu\nu}) = (\omega_{\mu\nu})^{-1} = \left(\frac{\partial R^0_\nu}{\partial a^\mu} - \frac{\partial R^0_\mu}{\partial a^\nu}\right)^{-1},$$

$$R^0 = (\mathbf{p}, \mathbf{0}),\ (\Omega^{\mu\nu}) = (\Omega_{\mu\nu})^{-1} = \left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)^{-1} \tag{8b}$$

[51] Several possibilities are conceivable, the first through the assumption of an association different than that of Equation (2), i.e., $a^*b = a(cb)$. Other possibilities are given by combinations of the type $a^*b = (ac)b + b(da)$, with c and d fixed elements either of the algebra, or of the field, or of both.

Indeed, the generalized Lie tensor can be decomposed into

$$\Omega^{\mu\nu}(a) = h^{\mu}_{\alpha}(a)\omega^{\alpha\nu} \tag{9}$$

and therefore obtained as a modification of the original tensor via elements $h^{\mu}_{\alpha}(a)$ of the algebra.

Recall that the covariant tensors $\omega_{\mu\nu}$ and $\Omega_{\mu\nu}$ are the geometric tensors of corresponding general first-order systems. When both the original and the final systems are self-adjoint, the preservation of the Lie algebra is ensured by Theorem 4.4.2.

The algebraic significance of the self-adjoint isotopic transformations

$$[\omega_{\mu\nu}\dot{a}^{\nu} + \Gamma_{\mu}(t, a)]_{SA} = 0 \rightarrow \{h^{\alpha}_{\mu}(a)[\omega_{\alpha\nu}\dot{a}^{\nu} + \Gamma_{\alpha}(t, a)]_{SA}\}_{SA} = 0 \tag{10}$$

is therefore that of characterizing a regular Lie algebra isotopy. However, isotopy (10) is the basis of the generalization of Hamilton's into Birkhoff's equations. In this way we reach the following result.

Proposition 1. *The generalization of Hamilton's equations into the (autonomous or semiautonomous*[52]*) Birkhoff's equations*

$$\left[\omega_{\mu\nu}\dot{a}^{\nu} - \frac{\partial H(t, a)}{\partial a^{\mu}}\right]_{SA} = 0 \rightarrow \left[\Omega_{\mu\nu}(a)\dot{a}^{\nu} - \frac{\partial B(t, a)}{\partial a^{\mu}}\right]_{SA} = 0 \tag{11}$$

is the analytic counterpart of the algebraic notion of Lie isotopy.

As we shall see in Chapter 6, Proposition 1 turns out to be crucial for the identification of a possible quantum mechanical image of Birkhoff's equations.

After identifying the analytic meaning of isotopy, our next task is to identify its meaning for symmetries, first integrals, and conservation laws. Recall from Chart A.10 that two Lie algebras L and L^* are called isotopically related when they are symmetry algebras of two isotopically related Lagrangians leading to the same first integrals (or conservation laws). The algebras L and L^* are generally nonisomorphic; but (a) they have the same dimension r; (b) they are defined on the same carrier space (the configuration or phase space); and (c) they coincide as vector spaces; that is, the generators of the two algebras are the same. The only possibility for the two algebras L and L^* to be generally non-isomorphic therefore occurs when *the products are different.* Their isotopic relationship in the algebraic sense introduced in this chart is then consequential.

An example is useful here. Consider Lagrangian (3) of Chart A.10, i.e..

$$L = \tfrac{1}{2}[(m\dot{x}^2 + m\dot{y}^2 + m\dot{z}^2) - (kx^2 + ky^2 + kz^2)]. \tag{12}$$

It possesses the **SO**(3) symmetry algebra with conserved generators

$$M_k = (\mathbf{r} \times m\dot{\mathbf{r}})_k, \qquad \mathbf{r} \equiv (x, y, z), \qquad k = x, y, z, \tag{13}$$

and commutation rules

$$[M_x, M_y] = M_z, \qquad [M_y, M_z] = M_x, \qquad [M_z, M_x] = M_y. \tag{14}$$

Consider now the isotopically mapped Lagrangian (4) of Chart A.10, i.e.,

$$L^* = \tfrac{1}{2}[m\dot{x}^2 - m\dot{y}^2 + m\dot{z}^2) - (kx^2 - ky^2 + kz^2)]. \tag{15}$$

[52] We exclude the nonautonomous case because of the loss of the algebra in the time evolution (Chart 4.1).

L^* breaks the **SO**(3) symmetry and possesses instead Lorentz symmetry **SO***(3) = **SO**(2.1) isotopically related to **SO**(3); that is, **SO**(2.1) is the symmetry of the new Lagrangian L^* which leads, via Noether's theorem, to the conservation of the generators of **SO**(3).

Now, the carrier space (the space of the Cartesian coordinates x, y, z and momenta p_x, p_y, p_z) has remained unchanged by construction, and the Lorentz algebra **SO**(2.1), to be consistently defined for the case at hand, must be defined in terms of the generators of **SO**(3) (that is, via the *angular momentum components*). This is possible if and only if **SO**(2.1) is realized via an isotopy of the product of **SO**(3).

A study of the case (Santilli (1979a)) indicates that a solution exists, and it is given by the commutation rules of **SO**(2.1)

$$[M_x, M_y]^* = M_z, \qquad [M_y, M_z]^* = -M_x, \qquad [M_z, M_x]^* = M_y \tag{16}$$

defined via the Lie isotopy

$$\mathbf{SO}(3)\colon [M_i, M_j] = \frac{\partial M_i}{\partial a^\mu}\,\omega^{\mu\nu}\,\frac{\partial M_j}{\partial a^\nu} \to \mathbf{SO}^* = \mathbf{SO}(2.1)\colon [M_i, M_j]^*$$

$$= \frac{\partial M_i}{\partial a^\mu}\,\Omega^{\mu\nu}\,\frac{\partial M_j}{\partial a^\nu}, \qquad a = (\mathbf{r}, \mathbf{p}) \tag{17a}$$

$$(\omega^{\mu\nu}) = \begin{pmatrix} 0_{3\times 3} & \begin{pmatrix} +1 & 0 \\ & +1 \\ 0 & +1 \end{pmatrix} \\ \begin{pmatrix} -1 & 0 \\ & -1 \\ 0 & -1 \end{pmatrix} & 0_{3\times 3} \end{pmatrix} \to (\Omega^{\mu\nu}) = \begin{pmatrix} 0_{3\times 3} & \begin{pmatrix} +1 & 0 \\ & -1 \\ 0 & +1 \end{pmatrix} \\ \begin{pmatrix} -1 & \\ & +1 \\ & & -1 \end{pmatrix} & 0_{3\times 3} \end{pmatrix}. \tag{17b}$$

The reader can now see the nontrivial implications of isotopic generalizations of the Lie product in regard to Lie's theory, as well as the need for a suitable reformulation of the theory itself. In fact, the needed broader theory must permit the formulation of a Lie group, say SO(2.1), in terms of the generators, the base manifold, and the parameters[53] of a generally nonisomorphic group, say SO(3). This is not readily permitted by the available conventional formulations of Lie's theory, as we shall see in the charts of Chapter 5.

As stressed in Section 4.4, the self-adjoint isotopic transformations are only part of the transformations permitted by the conditions of self-adjointness. A second important class is given by the self-adjoint genotopic transformations. The algebraic meaning of the latter transformations is the following.

Let U be an algebra over a field F verifying axioms (1), where ab is the product. Construct a new algebra $U^\#$ which is the same vector space as U equipped with one of the following new products

$$a \# b = (ac)b,\ a(cb),\ (ac)b + a(db), \text{ etc.} \tag{18}$$

[53] As we shall see in Chapter 5, the Lorentz group SO(2.1) not only must be defined in terms of the angular momentum components, but the parameters remains those of the group of rotations, that is, the Euler angles!

where $c, d, \ldots$ are fixed elements of U. $U^{\#}$ is called a *genotopic extension* or simply a *genotope* of U when the new product $a \# b$ verifies the distributive and scalar laws, but violates axioms (1) and verifies instead a different set of axioms

$$I'_k(a \# b) = 0, \qquad k = 1, 2, \ldots . \tag{19}$$

When the genotopy is invertible in the sense of Equation (4), it is called *regular*; otherwise it is called *singular*.

More generally, *we define as genotopy any transformation of the product of an algebra via elements of the algebra itself and/or of the field, which* (1) *preserves the original algebra as vector field*; (2) *verifies the distributive and scalar laws*; *and* (3) *satisfies a set of axioms different than those of the original algebra.*

As an example, consider a Lie algebra L of matrices A, B, C, and product (6). The following transformation of the product

$$L\colon [A, B] = AB - BA \to L^{\#}\colon (A, B) = ACB - BDA \tag{20}$$

where C and D are fixed elements of L, characterizes a genotope $L^{\#}$ of L. In fact, $L^{\#}$ coincides with L as vector space; the product (A, B) verifies the distributive and scalar laws, but, unlike the case of isotopy (7), the product (A, B) now *violates* the Lie algebra laws and verifies instead the conditions for a Lie-admissible algebra, because the algebra $(U^{\#})^-$ with product

$$[A, B]^* = (A, B) - (B, A) = ATB - BTA, \qquad T = C + D, \tag{21}$$

is Lie. Thus Equation (20) characterizes a *genotopy of a Lie algebra into the more general Lie-admissible algebra.*

An example of algebraic genotopy in Newtonian mechanics is given by the transition from the generalized Poisson (Birkhoff) brackets to non-Lie, Lie admissible brackets (Santilli, 1978c,e)

$$L\colon [A, B]^*_{(a)} = \frac{\partial A}{\partial a^\mu}\,\Omega^{\mu\nu}(a)\frac{\partial B}{\partial a^\nu} \to L^{\#}\colon (A, B)_{(a)} = \frac{\partial A}{\partial a^\mu}\,S^{\mu\nu}(a)\,\frac{\partial B}{\partial a^\nu}, \tag{22a}$$

$$(\Omega^{\mu\nu}) = \left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)^{-1}, \qquad (S^{\mu\nu} - S^{\nu\mu}) = \left(\frac{\partial R'_\nu}{\partial a^\mu} - \frac{\partial R'_\mu}{\partial a^\nu}\right)^{-1}. \tag{22b}$$

Note that the brackets (A, B) are Lie-admissible because, according to the definition of Chart 4.1, they are non-Lie, yet the attached anti-symmetric part is Lie. This can be equivalently seen by noting that the product (A, B) can be decomposed into the sum of two terms: a first term which is antisymmetric and Lie, and a second term which is symmetric and arbitrary,

$$(A, B) = \frac{\partial A}{\partial a^\mu}\,S^{\mu\nu}\,\frac{\partial B}{\partial a^\nu} = \frac{\partial A}{\partial a^\mu}\,\Omega^{\mu\nu}\,\frac{\partial B}{\partial a^\nu} + \frac{\partial A}{\partial a^\mu}\,T^{\mu\nu}\,\frac{\partial B}{\partial a^\nu}, \tag{23a}$$

$$\Omega^{\mu\nu} = \text{Lie}, \qquad T^{\mu\nu} = T^{\nu\mu}. \tag{23b}$$

Note also that the notion of genotopy is a generalization of that of isotopy. In fact, isotopies (7) and (8) are particular cases of corresponding genotopies (20) and (22).

Within the context of abstract algebras the genotopy is, essentially, a transformation of an arbitrary (nonassociative) algebra U into a different (generally non-associative) algebra $U^{\#}$ under the condition that U and $U^{\#}$ coincide as vector spaces. Within the context of the Inverse Problem,

a most important case occurs when the original algebra U is arbitrary, and the genotope $U^{\#}$ is a Lie algebra.

The algebraic significance of the self-adjoint genotopic transformations

$$[C_{\mu\nu}(t, a)\dot{a}^{\nu} + D_{\mu}(t, a)]_{\text{NSA}} = 0 \rightarrow \{h^{\alpha}_{\mu}(t, a)[C_{\alpha\beta}(t, a)\dot{a}^{\beta} + D_{\alpha}(t, a)]_{\text{NSA}}\}_{\text{SA}} = 0 \tag{24}$$

is therefore that of characterizing a regular Lie algebra genotopy, that is, of inducing a Lie algebra structure. In fact, as is now familiar, the original tensor $C^{\mu\nu}$, $(C^{\mu\nu}) = (C_{\mu\nu})^{-1}$, is not Lie, while the final tensor $C^{*\mu\nu}(C^{*\mu\nu}) = (h^{\alpha}_{\mu} C_{\alpha\nu})^{-1}$, is Lie. This illustrates the reason why the *Inverse Problem* has been sometimes referred to as the *Inverse Lie Problem.*

We are now in a position to identify the algebraic significance of all possible equivalence transformations of first-order systems which can be characterized via the conditions of self-adjointness. Besides the self-adjointness-preserving and the self-adjointness-inducing transformations considered earlier in this chart, we have two additional transformations, according to

$$\begin{aligned} \textit{Self-adjointness-preserving} &= \textit{Lie algebra isotopy} \\ \textit{Self-adjointness-inducing} &= \textit{Lie algebra genotopy} \\ \textit{Non-self-adjointness-preserving} &= \textit{non-Lie algebra isotopy} \\ \textit{Non-self-adjointness-inducing} &= \textit{non-Lie algebra genotopy} \end{aligned} \tag{25}$$

The direct universality of the Birkhoffian representation has the following algebraic counterpart. It essentially implies that a Lie algebra genotopy exists for all possible brackets (23) with a *local* tensor $S^{\mu\nu}(a)$.

Equivalently, we can say that all local, analytic, and regular first-order systems can be treated via Lie algebra genotopies and isotopies. When considering the more general, non-local, integro—differential systems, the Lie-admissible genotopies and isotopies turn out to be possible, as we indicate in Chart 4.7.

The notion of genotopy was introduced by Santilli (1978d), and was not found to be treated in the literature of Abstract Algebra, despite a rather laborious search. The notion of isotopy is considerably neglected in the contemporary mathematical literature. However, a search revealed that the notion is treated in the early literature of set theory and linear algebra. See in this respect Bruck (1958, Chapter III). As Bruck puts it (*loc. cit.,* page 56), the notion of algebraic isotopy is "so natural to creep in unnoticed." For additional treatments by mathematicians, see McCrimmon (1965, Section III.1), Myung (1982a), and Osborn (1982). The notion of isotopy was introduced in physics by Santilli (1978c,d,e and 1979a,b) and subsequently considered by a number of authors, such as Sarlet and Cantrijn (1978), Kobussen (1979), and others.

I hope that the term isotopy will not create confusion with other terms used in the physical literature, such as "isotopic spin." After due consideration, I elected to preserve the term because the notion of "algebraic isotopy" was born considerably earlier than that of "spin isotopy."

Chart 4.3 Havas's Theorem of Universality of the Inverse Problem for Systems of Arbitrary Order and Dimensionality

In Section 4.5 we presented the universality of the Inverse Problem for analytic and regular systems of second-order ordinary differential equations.

This universality was essentially achieved by reducing the system considered to an equivalent first-order form and then proving that this form can always be written in a self-adjoint version via the multiplication of a matrix of genotopic functions. The conditions of analyticity and regularity were necessary to ensure the applicability of the underlying existence theory for implicit functions, partial differential equations, etc.

The objective of this chart is to indicate that the universality of the Inverse Problem proved in the text is, actually, a particular case of a more general universality holding for systems of ordinary differential equations of arbitrary (but finite) order and dimensionality.

Consider a system of differential equations of order s in r unknowns which is analytic and regular in a region of their variables

$$F_k(t, Q^{(0)i}, Q^{(1)i}, \ldots, Q^{(s)i}) = 0, \qquad i, k = 1, 2, \ldots, r, \tag{1}$$

$$Q^{(l)i} = \frac{d^l Q^i}{dt^l}, \qquad l = 0, 1, \ldots, s.$$

Then, the implicit functions in the maximal derivatives exist (and are unique), and we write

$$Q^{(s)i} = f^i(t, Q^{(0)i}, Q^{(1)i}, \ldots, Q^{(s-1)i}). \tag{2}$$

Introduce now r new variables via the prescriptions

$$Q_k^{(1)} = Y_k^{(0)}, \tag{3}$$

or any more general form of type (4.1.23). System (2) is now reduced to

$$Q_k^{(1)} = Y_k^{(0)}, \; Y_k^{(s-1)} = f_k(t, Q_k^{(0)}, Y_k^{(0)}, Y_k^{(1)}, \ldots, Y_k^{(s-2)}). \tag{4}$$

The iteration of this procedure results in the reduction of the original system (1) into an equivalent first-order system of the normal type

$$\dot{q}^k - \Xi^k(t, q) = 0, \qquad k = 1, 2, \ldots, 2N, \qquad q = (Q, Y), \tag{5}$$

where the dimension $2N$ is rs, if rs is even, or $rs + 1$ if rs is odd, in which case the last equation can be of the type $Y_0^{(1)} = 0$. However, Equations (5) always admit an equivalent self-adjoint form of the type (Theorem 4.4.3)

$$\{h_{ki}(t, q)[\dot{q}^i - \Xi^i(t, q)]_{\text{NSA}}\}_{\text{SA}}. \tag{6}$$

Thus an action functional

$$J(E) = \int_{t_1}^{t_2} dt[L(t, q, \dot{q})](E) \tag{7}$$

always exists for the representation of system (1). In this way we reach the following universality theorem proved by Havas (1973, Appendix A).

Theorem 1. *Any system* (1) *of ordinary differential equations of order s and dimension r which is analytic and regular in a neighborhood of a regular point of its variables can always be transformed into an equivalent self-adjoint system of first-order equations of dimension* $2N = rs$, *if rs is even, or* $2N = rs + 1$, *if rs is odd, admitting the representation via the variational principle*

$$\begin{aligned} \delta \int_{t_1}^{t_2} dt\, L(t, q, \dot{q}) &= -\int_{t_1}^{t_2} dt \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} \right) \delta q^k \\ &= -\int_{t_1}^{t_2} dt \{h_{ki}(t, q)[\dot{q}^i - \Xi^i(t, q)]\}_{\text{SA}}\, \delta q^k = 0. \end{aligned} \tag{8}$$

Note that the theorem holds without transforming the original variables. Thus it is in line with the general objectives of this chapter. Let us recall from Section 4.2 that the action functional under consideration here is totally degenerate (the integrand is linear in the first-order derivative). Thus, when this action is interpreted as being Lagrangian, it violates Legendre's necessary condition for an extremum, by exibiting a number of problematic aspects for the applicability of conventional tools of the calculus of variations or the optimal control theory. This is what stimulated the work in this chapter on the effect of interpreting (8) as a generalized *Hamiltonian* rather than Lagrangian type (as in the original Havas's approach).

Chart 4.4 Rudiments of Differential Geometry[54]

In Charts I.2.1–I.2.5 we reviewed a few basic notions of the symplectic and contract geometries. In this and in the subsequent two charts we shall continue the study of geometry with particular emphasis on the local formulation of coordinate-free, global, geometric techniques; the global treatment of Hamilton's and Birkhoff's equations; and the coordinate-free treatment of the Inverse Problem.

In these charts we shall also point out some rather fundamental *physical* differences which emerge when the same geometric algorithms are used for the treatment of *conservative* and *nonconservative* systems. Recall that the symplectic and contact geometries were developed much along the lines of the conservative Hamiltonian mechanics. Basic geometrical tools, such as Lie's derivative, were then conceived and applied by and large to represent *conservation laws*. In these volumes we establish the universality of the contact geometry for local, analytic, and regular Newtonian systems. Within such a setting, the geometrical algorithms remain essentially the same, as we shall see, particularly in the coordinate-free formulation. The physical emphasis, however, is now in their use to characterize *nonconservation laws* (time rate of variation of physical quantities). Unless due care is provided for this physical aspect, one risks achieving mathematically correct coordinate-free global formulations of mechanics which literally imply the perpetual motion in our environment.

Let M be an n-dimensional $\mathscr{C}^\infty$ manifold with atlas $\{(U_i, \varphi_i)\}$, $\varphi_i: U_i \to \varphi_i(U_i) \in \mathbb{R}^n$ (Chart I.2.1). A set of local coordinates will be denoted by $x = (x^1, \ldots, x^n)$. For a point $m \in U \subset M$, $\varphi(m) = (x^1, \ldots, x^n)$, where each x^i is considered as a map from M to $\mathbb{R}$.

Definition 1. A *tangent vector* X_m at a point $m \in M$ is a linear function from $\mathscr{C}^\infty_m(M)$ (the space of $\mathscr{C}^\infty$ functions defined in the neighborhood of m) to $\mathbb{R}$, satisfying the rules

$$\begin{aligned} X_m(\alpha f + \beta g) &= \alpha X_m(f) + \beta X_m(g), \\ X_m(fg) &= f(m)X_m(g) + g(m)X_m(f), \\ \forall f, g \in \mathscr{C}^\infty_m(M); &\qquad \alpha, \beta \in \mathbb{R}. \end{aligned} \tag{1}$$

[54] The literature on differential geometry is rather extensive. The interested reader may consult, for instance, Dedecker (1957), Sternberg (1964), Abraham and Marsden (1967), Loomis and Sternberg (1968), Souriau (1970), Spivak (1970–73), Edelen (1977), Guillemin and Sternberg (1977), Thirring (1978), Arnold (1978) and Sniatycki (1979).

A number of alternative definitions of tangent vectors exist in the literature. The following (equivalent) definition is relevant for our objectives.

Definition 2. A *tangent vector* X_m at $m \in M$ is an equivalence class of curves which are tangent to each other at m, i.e., $X_m = [\gamma]_m$ and $\gamma_1 = \gamma_2$ if and only if

$$\frac{d}{dt}(\varphi \circ \gamma_1)(t)\bigg|_{t=0} = \frac{d}{dt}(\varphi \circ \gamma_2)(t)\bigg|_{t=0}. \tag{2}$$

By combining Definitions 1 and 2, we can interpret X_m as an operator performing the mapping $\mathscr{C}^\infty_m(M) \to \mathbb{R}$ for which

$$X_m(f) = \frac{d}{dt}(f \circ \gamma)(t)\bigg|_{t=0}, \qquad \gamma \in [\gamma]_m. \tag{3}$$

The *tangent space* $T_m M$ at $m \in M$ is the vector space of all tangent vectors at m. The $2n$-dimensional manifold $TM = \bigcup_{m \in M} T_m M$ can be equipped with a $\mathscr{C}^\infty$ structure in a natural way, and constitutes the *tangent bundle* over M, where the fiber at each point is the tangent space at that point.

The *cotangent space* $T^*_m M$ at m is the dual of $T_m M$, and it is the space of all linear functionals on $T_m M$. The $2n$-dimensional manifold $T^*M = \bigcup_{m \in M} T^*_m M$ can again be equipped with a $\mathscr{C}^\infty$ structure, and it is called the *cotangent bundle.*

If $f \in \mathscr{C}^\infty_m(M)$, we can define an element of $T^*_m M$, called *differential* of f at m, by

$$df(m) = \frac{df}{\partial x^k}(m)\, dx^k(m) \tag{4}$$

It can be shown that, if $(x^1, \ldots, x^n)$ are local coordinates defined in a neighborhood of m, then $dx^i|_m$ form a *basis* of $T^*_m M$. The basis of $T_m M$, the dual of $dx^i|_m$, is then given by (in the sense of Definition 1) $\partial/\partial x^i|_m$.

Let $\theta_m \in T^*_m M$ and $X_m \in T_m M$, then their local form can be given by

$$\theta_m = \theta^i(x)dx^i, \qquad X_m = X^i(x)\frac{\partial}{\partial x^i} \tag{5}$$

where the reference to the point m is understood. We also have

$$\theta_m(X_m) = \theta_i X^i \in \mathbb{R}, \qquad X_m(f) = X^i \frac{\partial f}{\partial x^i} \in \mathbb{R}, \qquad f \in \mathscr{C}^\infty_m(M). \tag{6}$$

Let $F: M \to N$ be a $\mathscr{C}^\infty$ map from a manifold M to another manifold N. We can associate to this map the *tangent map* $TF: TM \to TN$ by

$$TF(X_m) = Y_{F(m)}, \qquad Y_{F(m)}(g) = X_m(F \circ g), \qquad g \in \mathscr{C}^\infty_m(M). \tag{7}$$

A tensor of contravariant order r and covariant order s, i.e., of type (r, s) (Chart I.2.1), on a vector space V is an $(r + s)$-multilinear map

$$\pi^r_s: \overbrace{V^* \times \cdots \times V^*}^{r\text{-times}} \times \overbrace{V \times \cdots \times V}^{s\text{-times}} \to \mathbb{R} \tag{8}$$

where V^* is the dual of V.

When V is identified with $T_m M$, we have the *tensor bundle* $T^r_s(M) = \bigcup_{m \in M} T^r_s(T_m M)$, where T^r_s is the set of all (r, s) tensors over $T_m M$. Clearly, $T^1_0(M) = TM$, $T^0_1 = T^*M$, and $T^0_0(M) = \mathbb{R}$.

An (r, s)-*tensor field* is a $\mathscr{C}^\infty$ map

$$b: M \to T^r_s(M), \qquad b(m) \in T^r_s(T_m M). \tag{9}$$

Most important for our analysis are the (1.0)-tensor fields on M, called *vector fields*. They characterize a map from M to TM. Equally important are the (0, 1)-tensor fields on M called *one-forms*. They characterize a mapping from M to T^*M.

The set of all p-*forms* on M (Section I.1.2) is denoted by $\Lambda^p(M)$. Hence $\alpha \in \Lambda^p(M)$ mean $\alpha(m) \in \Lambda^p(T^*_m M)$. Equations (5) therefore give a local expression for vector fields X and one-forms θ. In particular

$$\theta(X) = X^i \theta_i \in \mathscr{C}^\infty(M) \tag{10}$$

A curve $\gamma: I \to M$ at m is an *integral curve* of a vector field X at m if $X(\gamma(t)) = T_\gamma(t, I)$, for all $t \in I$, where T_γ is the tangent map of γ. Suppose that (U, φ) is a chart at m, $\varphi(m) = (x^1_0, \ldots, x^n_0)$, and $(\varphi \circ \gamma)(t) = (x^1(t), \ldots, x^n(t))$, then x is an integral curve of X at m if and only if $(x^1(t), \ldots, x^n(t))$ satisfies the system of first-order ordinary differential equations

$$\dot{x}^i = X^i(x), \qquad x^i(0) = x^i_0. \tag{11}$$

In this way we reach the first contact with the analysis of this volume. As worked out in Volume I and reviewed in Section 4.1, Newtonian systems can always be written in the *normal form*

$$\dot{a}^\mu = \Xi^\mu(a), \qquad a(0) = a_0, \qquad a = (\mathbf{r}, \mathbf{y}) \tag{12}$$

via arbitrary prescriptions for the characterization of n new variables y. The quantities Ξ^μ have been referred to in the main text as vector fields, merely to express the Newtonian character of transforming as contravariant vectors. We now learn that they can be interpreted as vector fields in the geometric sense, that is

$$\Xi = \Xi^\mu(a) \frac{\partial}{\partial a^\mu}. \tag{13}$$

Equation (10) is then the differential equation corresponding to the (geometrical definition of) vector field Ξ.

We should keep in mind that, whether the Newtonian or the geometrical definition is used, these vector fields characterize *autonomous nonconservative systems*. The extension to the nonautonomous case is self-evident. Autonomous conservative systems are, of course, not excluded, *as a particular case* of this broader physical context.

From the existence theory of ordinary differential equations (Section I.1.1) we can see that, at every point $m \in M$, there exists a unique integral curve of a vector field X at m. For all $m \in U \subset M$, these integral curves define a *local, one-parameter pseudogroup* of transformations on M which becomes a *local, one-parameter group* if the interval of time in which it is defined is independent of m, or it is the whole real line. The vector field X then acquires the meaning of *generator* of this pseudogroup.

Suppose that a $\mathscr{C}^\infty$-map $F: M \to N$ is given. Then, a natural map from $T^0_p(N)$ to $T^0_p(M)$ exists called the *pull-back* of F and given by

$$F^*: T^0_p(N) \to T^0_p(M), \qquad \alpha \to F^*(\alpha), \qquad \alpha \in \Lambda^p(M), \tag{14}$$

$$(F^*_\alpha)(m)(X_{1m}, \ldots, X_{pm}) = \alpha(F(m))(TF(X_{1m}), \ldots, TF(X_{pm})).$$

F^* maps p-forms into p-forms, and it is an (algebra) homomorphism with respect to the exterior product (also called *wedge product* in differential geometry). Notice in particular that, if F is a diffeomorphism, then $(F^*)^{-1} = (F^{-1})^*$.

Given two vector fields X and Y on M, one can define a third vector field via the associative composition law

$$(XY)(f) = X(Y(f)), \qquad \forall f \in \mathscr{C}^\infty(M). \tag{15}$$

The (nonassociative) Lie rule

$$[X, Y] = XY - YX \tag{16}$$

then also defines a vector field which can be written in local coordinates

$$[X, Y] = \left(X_i \frac{\partial Y_j}{\partial x^i} - Y_i \frac{\partial X_j}{\partial x^i}\right) \frac{\partial}{\partial x^j}. \tag{17}$$

We see in this way that the set of all vector fields on a manifold forms a *Lie algebra* under brackets (16).

Let X be a vector field on M and b an (r, s)-tensor field. Also, let F_t be the local one-parameter pseudogroup given by the integral curves of X for $t \in I$. Then $F_t: U \to M$ maps every point m of U onto the point of M lying in the integral curve through m at t.

The *Lie derivative* of the tensor field b with respect to the vector field X is defined by

$$L_X b(m) = \lim_{t \to 0} \frac{(F_t^*(b))(m) - b(m)}{t}. \tag{18}$$

If the tensor b is a scalar f, we have

$$L_X f = df(X), \qquad \forall f \in \mathscr{C}^\infty(M); \tag{19}$$

if b is a vector field Y we have

$$L_X Y = [X, Y]; \tag{20}$$

and for the case of one-forms we have (locally)

$$L_X \theta = \left(X^k \frac{\partial \theta_j^k}{\partial x^k} + \theta_k \frac{\partial X^k}{\partial x^j}\right) dx^j. \tag{21}$$

We also have the property

$$L_X(\theta(Y)) = (L_X \theta)(Y) + \theta([X, Y]). \tag{22}$$

Finally, by using the global formulation of the exterior derivative (Section I.1.2), we have

$$dL_X \alpha = L_X(d\alpha), \qquad \forall \alpha \in \Lambda^p(M). \tag{23}$$

The inner product of a vector field X and a p-form α, denoted by $i_X \alpha$ or by $X \lrcorner \alpha$ (Section 4.3) is the $(p-1)$-form

$$i_X \alpha(X_1, \ldots, X_p) = p\alpha(X, X_1, \ldots, X_{p-1}) \tag{24}$$

verifying the properties

$$i_X^2 = 0, \; i_X(\alpha \wedge \beta) = (i_X \alpha) \wedge \beta + (-1)^p \alpha \wedge (i_X \beta),$$
$$i_X df = L_X f. \tag{25}$$

In this way we reach the following important property of the Lie derivative

$$\begin{aligned} L_X\alpha &= i_X(d\alpha) + d(i_X\alpha) \\ &\equiv X \lrcorner\, d\alpha + d(X \lrcorner\, \alpha), \qquad \forall\alpha \in \Lambda^p(M). \end{aligned} \tag{26}$$

To establish a link with the treatment of this volume, it is significant to identify the explicit form of these properties in local coordinates. From the definitions given above, we have

$$(i_X\alpha)Y = 2\alpha(X, Y), \qquad \alpha \in \Lambda^2(M). \tag{27}$$

Thus

$$X = X^\mu \frac{\partial}{\partial a^\mu}, \qquad Y = Y^\mu \frac{\partial}{\partial a^\mu}, \tag{28}$$

$$\alpha = \alpha_{\mu\nu}(a)da^\mu \wedge da^\nu,$$

and

$$(i_X\alpha)Y \equiv (X \lrcorner\, \alpha)Y = 2\alpha_{\mu\nu}X^\mu Y^\nu. \tag{29}$$

Therefore,

$$i_X\alpha \equiv X \lrcorner\, \alpha = 2\alpha_{\mu\nu}X^\mu da^\nu. \tag{30}$$

This yields the expression of the inner product in local coordinates of Section 4.3, Equations (4.3.5) and (4.3.17), i.e.,

$$\begin{aligned} \Xi \lrcorner\, \omega_2 &= \tfrac{1}{2}\delta^{\nu_1\nu_2}_{\mu_1\mu_2}\omega_{\nu_1\nu_2}\Xi^{\mu_2}\, da^{\mu_1} \\ &= \omega_{\mu_1\mu_2}\Xi^{\mu_1}\, da^{\mu_2}, \end{aligned} \tag{31a}$$

$$\begin{aligned} \tilde{\Xi} \lrcorner\, \Omega_2 &= \tfrac{1}{2}\delta^{\nu_1\nu_2}_{\mu_1\mu_2}\Omega_{\nu_1\nu_2}\{a\}\tilde{\Xi}^{\mu_2}\, da^{\mu_1} \\ &= \Omega_{\mu_1\mu_2}\{a\}\tilde{\Xi}^{\mu_1}\, da^{\mu_2}. \end{aligned} \tag{31b}$$

The reader will recall that the following contractions

$$\omega_{\mu_1\mu_2}\Xi^{\mu_2} = \Xi_{\mu_1}, \qquad \Omega_{\mu_1\mu_2}(a)\tilde{\Xi}^{\mu_2} = \tilde{\Xi}_{\mu_1}, \tag{32}$$

have played a rather crucial role in the main text for the construction of a Hamiltonian and a Birkhoffian representation, respectively. In this case, the forms ω_2 and Ω_2 are the fundamental symplectic form, and a general (but local and exact) symplectic form, respectively.

Chart 4.5 Global Treatment of Hamilton's Equations

Let M be an n-dimensional, $\mathscr{C}^\infty$ manifold with local coordinates $(q^1, \ldots, q^n)$ (the configuration space of Newtonian Mechanics), and let TM and T^*M be its tangent and cotangent bundles, respectively. The bundles TM and T^*M are customarily used for the characterization of the Lagrangian and Hamiltonian formulations of mechanics, respectively. We are here interested in the latter case.

A point of T^*M consists of a couple (m, l), where m is a point of M, and l belongs to T^*M. The projection map (Chart I.2.2) $\pi\colon T^*M \to M$ maps the whole fiber (m, T^*_mM) onto m. As a $2n$-dimensional manifold, T^*M can be equipped with a fundamental one-form, called a *canonical form*, as follows. Consider the mapping

$$\theta\colon T^*M \to T^*(T^*M), \qquad \theta \in \Lambda^1(T^*M), \tag{1}$$

defined by the following properties. Let $X_{(m,l)}$ be an arbitrary tangent vector to T^*M at (m, l). Then

$$\theta(m, l)(X_{(m,l)}) = l(T\pi(X_{(m,l)})) \tag{2}$$

where $T\pi$: $T(T^*M) \to TM$ is the tangent map of the projection π. Let $(a^\mu) = (q^k, p_k)$ be local coordinates for T^*M. The vector field $X_{(m,l)}$ then acquires the local form

$$X_{(m,l)} = A^i(q, p)\frac{\partial}{\partial q^i} + B_j(q, p)\frac{\partial}{\partial p_j} \equiv \Xi^\mu(a)\frac{\partial}{\partial a^\mu} \tag{3}$$

for which

$$T\pi(X_{(m,l)}) = A^i\frac{\partial}{\partial q^i}, \tag{4}$$

that is,

$$l(T\pi(X_{(m,l)})) = p_i A^i. \tag{5}$$

The canonical form is then given by

$$\theta(m, l) = p_i\, dq^i = R^0_\mu(a)da^\mu = R^0_1, \qquad (R^0_\mu) = (\mathbf{p}, \mathbf{O}), \tag{6}$$

where θ is the notation generally used in the literature of differential geometry,[54] while R^0 is that used in the literature of the Birkhoffian mechanics. Clearly, the two-form

$$\omega = d\theta = dp_i \wedge dq^i = \tfrac{1}{2}\omega_{\mu\nu}\, da^\mu \wedge da^\nu = dR^0_1$$

$$(\omega_{\mu\nu}) = \begin{pmatrix} 0_{n\times n} & -1_{n\times n} \\ 1_{n\times n} & 0_{n\times n} \end{pmatrix} \tag{7}$$

is nowhere degenerate and closed. It is the familiar *fundamental symplectic form*. The $2n$-dimensional manifold T^*M, equipped with the form ω is a *symplectic manifold* (Chart I.2.5).

Let $H = H(a)$ be a function defined on T^*M. A *Hamiltonian vector field* is a vector field X verifying the condition

$$i_X\omega \equiv X \lrcorner\, \omega = -dH. \tag{8}$$

We recover in this way the definition of Hamiltonian vector field of Section 4.3, i.e.,

$$\Xi \lrcorner\, \omega_2 = \omega_{\mu_1\mu_2}\Xi^{\mu_1}\, da^{\mu_2} = \Xi_\mu\, da^\mu = -dH. \tag{9}$$

Equations (8) constitute a *global treatment of Hamilton's equations for autonomous systems*. Indeed, they are the coordinate-free version of our local formulation (9).

The explicit form of Hamilton's equations in local coordinates is recovered as follows. Recall from Chart 4.4 that

$$i_\Xi\omega = B_i\, dq^i - A^i\, dp_i \equiv \tfrac{1}{2}\omega_{\mu\nu}\Xi^\nu\, da^\mu. \tag{10}$$

Thus, if Ξ is a Hamiltonian vector field, it admits the local form

$$\Xi = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} \equiv \omega^{\mu\nu}\frac{\partial H}{\partial a^\nu}\frac{\partial}{\partial a^\mu}, \qquad (\omega^{\mu\nu}) = (\omega_{\mu\nu})^{-1}. \tag{11}$$

But, from Equations (11) of Chart 4.4, we have

$$\dot{a}^\mu = \Xi^\mu(a). \tag{12}$$

Thus we reach Hamilton's equations in the conventional notation of differential geometry

$$\dot{q}^k = \frac{\partial H}{\partial p_k}, \qquad \dot{p}_k = -\frac{\partial H}{\partial q^k} \tag{13}$$

or, in our unified notation,

$$\dot{a}^\mu = \omega^{\mu\nu} \frac{\partial H}{\partial a^\nu}. \tag{14}$$

Let us recall a few basic properties of Hamiltonian vector fields from Abraham and Marsden (1967; see 1978 edition).

Proposition 1. *Let X be an autonomous Hamiltonian vector field on a symplectic manifold T^*M with fundamental form ω, and let F_t be the one-parameter pseudogroup characterized by its integral curve (the flow of X). Then, for each t, $F_t^*\omega = \omega$; that is, F_t is simplectic and preserves the phase space volume (Liouville's theorem).*

Proposition 2. *Let X be an autonomous Hamiltonian vector field on a symplectic manifold (T^*M, ω). Then H is constant along the integral curve of X, i.e.,*

$$L_X H = [H, H] = \frac{\partial H}{\partial a^\mu} \omega^{\mu\nu} \frac{\partial H}{\partial a^\nu} \equiv 0. \tag{15}$$

Recall here that Equation (15) is often referred in the mathematical literature as the "conservation of the energy." This interpretation is generally erroneous because the Hamiltonian does not necessarily represent the total energy[55] of the system.

In fact, for *conservative systems* the Hamiltonian can be an *isotopic image* of the total energy (Section A.2). For instance, the conventional linear harmonic oscillator $\ddot{r} + r = 0$ $(m = K = 1)$ can be represented either via the conventional Hamiltonian $H = \frac{1}{2}(p^2 + r^2)$ or via its isotopic image (Example 4.1)

$$H^* = \ln | r \sec \tfrac{1}{2} r p^* | . \tag{16}$$

It is clear that Equation (15) *does not* represent the conservation law of the energy when used for Hamiltonian H^*.

For *nonconservative systems*, the situation is created by the fact that the canonical Hamiltonian *cannot* represent the energy, as a necessary condition for the existence of a Hamiltonian representation. In particular, non-conservative systems can be autonomous, thus admitting a Hamiltonian which does not dependent explicitly on time. In this case, Equation (15) does not possess the meaning of energy conservation.

Let us look at the global characterization of Hamilton's equations of nonautonomous type (i.e., $H = H(t, r, p)$). This is customarily done by extending the cotangent bundle T^*M to the $(2n + 1)$-dimensional manifold $\mathbb{R} \times T^*M$, where $\mathbb{R}$ represents time, and the canonical form (6) to the form[56]

$$\theta_H = \theta - H(t, \mathbf{r}, \mathbf{p})dt = \hat{R}^0_\mu(\hat{a})d\hat{a}^\mu = \hat{R}^0_1, \tag{17}$$

$$\mu = 0, 1, 2, \ldots, 2n, \qquad (\hat{a}^\mu) = (t, a), \qquad (\hat{R}_\mu) = (-H, R^0_\mu).$$

[55] The notion of total energy was given in Appendix I.A and reviewed after Corollary 4.5.1.a.

[56] The notation θ_H is often used in the mathematical literature, while the notation $\hat{R}^0_1(\hat{a})$ is used in the main text of this volume.

The two-form

$$\omega_H = d\theta_H = d\hat{R}_1^0 \tag{18}$$

is then a closed (and exact) two-form of maximal rank. Thus it is a *contact form* (Chart I.2.5). The $2n + 1$-dimensional manifold $\mathbb{R} \times T^*M$ equipped with form (18) is a *contact manifold.*

A *nonautonomous globally Hamiltonian vector field* can then be defined as a *characteristic vector field* $\tilde{X}$ of ω_H, that is, a vector field verifying the properties

$$i_{\tilde{X}}\omega_H \equiv \tilde{X} \lrcorner \omega_H = 0 \tag{19a}$$

$$dt(\tilde{X}) = 1. \tag{19b}$$

The equations above constitute a global treatment of our local formulation of nonautonomous Hamilton's equations, i.e.,

$$\hat{\omega}_{\mu\nu} d\hat{a}^\mu = 0, \qquad \mu = 0, 1, 2, \ldots, 2n, \tag{20a}$$

$$\{\hat{a}^\mu\} = \{t, a\} = \{t, \mathbf{r}, \mathbf{p}\}, \tag{20b}$$

$$\hat{\omega}_{\mu\nu} = \omega_{\mu\nu}, \qquad \mu, \nu = 1, 2, \ldots, 2n, \tag{20c}$$

$$\hat{\omega}_{0\nu} = -\hat{\omega}_{\nu 0} = \frac{\partial H}{\partial a^\nu}. \tag{20d}$$

Proposition 3. *If $\tilde{X}$ is a nonautonomous Hamiltonian vector field in a contact manifold $\mathbb{R} \times T^*M$ with structure* (22), *then*

$$L_{\tilde{X}} H = \frac{\partial H}{\partial t}. \tag{21}$$

This property does not necessarily express a physical law. This is due to the fact that conservative systems may admit an explicitly time-dependent Hamiltonian. For instance, the conventional harmonic oscillator ($\ddot{r} + r = 0$) admits the Hamiltonian (Example A.1)

$$H = r(\tan t)\sec t\,|\,(e^{p'} \cot \tfrac{1}{2}t)^{\cos t} - \ln(e^{p'} \cot \tfrac{1}{2}t)^{\cos t} - 1\,|. \tag{22}$$

In this case the Hamiltonian is not a first integral, i.e., $\dot{H} \neq 0$, yet the total physical energy is conserved.

If the system is nonconservative, Equation (21) also does not express the nonconservation of the energy. Indeed, for the damped oscillator we may have the Hamiltonian

$$\mathrm{H} = e^{-\gamma t}\tfrac{1}{2}p^2 + e^{\gamma t}\tfrac{1}{2}r^2, \tag{23}$$

and the Lie derivative (with respect to the Hamiltonian vector field) of this quantity *is not* equal to the variation of the energy in time (Example I.A.6):

$$\frac{d}{dt} E_t = \frac{d}{dt}\tfrac{1}{2}(\dot{r}^2 + r^2) = F_{\mathrm{NSA}}\dot{r} = -2\gamma\dot{r}^2. \tag{24}$$

A study of this occurrence indicates that this is the case for *all* Hamiltonian representations of nonconservative systems, whether essentially or nonessentially non-self-adjoint.

Thus we conclude that, *within the context of the global geometrical treatment of Hamiltonian mechanics, the Lie derivative is representative of a*

physical law, the conservation of energy (15), *only under certain restrictions on the nature of the systems, the physical meaning of the local coordinates, and the nature of the representation.*

Chart 4.6 Global Treatment of Birkhoff's Equations

Theorem 4.5.1 establishes that *all* local Newtonian systems which are analytic and regular in a star-shaped neighborhood of a regular point of their variables

$$\{[m_k \ddot{\mathbf{r}}_k - \mathbf{f}_k(t, \mathbf{r}, \dot{\mathbf{r}})]_{SA} - \mathbf{F}_k(t, \mathbf{r}, \dot{\mathbf{r}})\}_{NSA} = 0, \qquad k = 1, 2, \ldots, N, \quad (1)$$

admit an equivalent first-order self-adjoint form

$$[\Omega_{\mu\nu}(t, a)\dot{a}^\nu + \Gamma_\mu(t, a)]_{SA} = 0, \qquad \begin{matrix} \mu = 1, 2, \ldots, 2n; \; n = 3N, \\ a = (\mathbf{r}, \mathbf{p}), \; \mathbf{p} = m\dot{\mathbf{r}} \end{matrix} \quad (2a)$$

$$\Omega_{\mu\nu} + \Omega_{\nu\mu} = 0, \quad (2b)$$

$$\frac{\partial \Omega_{\mu\nu}}{\partial a^\tau} + \frac{\partial \Omega_{\nu\tau}}{\partial a^\mu} + \frac{\partial \Omega_{\tau\mu}}{\partial a^\nu} = 0, \quad (2c)$$

$$\frac{\partial \Omega_{\mu\nu}}{\partial t} = \frac{\partial \Gamma_\mu}{\partial a^\nu} - \frac{\partial \Gamma_\nu}{\partial a^\mu} = 0, \quad (2d)$$

characterizing the closed and exact two-form of maximal rank on the $(2n + 1)$-dimensional manifold $\mathbb{R} \times T^*M$

$$\hat{\Omega}_2 = \hat{\Omega}_{\mu\nu}(\hat{a}) d\hat{a}^\mu \wedge d\hat{a}^\nu, \qquad d\hat{\Omega}_2 = 0, \qquad \hat{\Omega}_2 = d\hat{R}_1, \quad (3a)$$
$$\mu, \nu = 0, 1, 2, \ldots, 2n$$

$$\hat{\Omega}_{\mu\nu} = \Omega_{\mu\nu}, \qquad \mu, \nu = 1, 2, \ldots, 2n, \quad (3b)$$

$$\hat{\Omega}_{0\nu} = \Gamma_\nu = -\hat{\Omega}_{\nu 0}, \qquad \nu = 1, 2, \ldots, 2n, \quad (3c)$$

Thus *Theorem* 4.5.1 *can be equivalently formulated by saying that all Newtonian systems of the class admitted can be treated via the global contact geometry, in general, and by the symplectic geometry, in particular.*

In order to identify the analytic character of systems (2a), that is, their derivability from a variational principle we have represented them in the main text of this volume via Birkhoff's equations. In this chart, we are interested in outlining the global treatment of these equations. For the reader's convenience, we shall considered separately, in the terminology of Definition 4.2.1, Birkhoff's equations in the *autonomous form*:

$$\left\{\left[\frac{\partial R_\nu(a)}{\partial a^\mu} - \frac{\partial R_\mu(a)}{\partial a^\nu}\right]\dot{a}^\nu - \frac{\partial B(a)}{\partial a^\mu}\right\}_{SA} = 0; \quad (4)$$

the *semi-autonomous form*:

$$\left\{\left[\frac{\partial R_\nu(a)}{\partial a^\mu} - \frac{\partial R_\mu(a)}{\partial a^\nu}\right]\dot{a}^\nu - \frac{\partial B(t, a)}{\partial a^\mu}\right\}_{SA} = 0; \quad (5)$$

and the general *nonautonomous form*:

$$\left\{\left[\frac{\partial R_\nu(t, a)}{\partial a} - \frac{\partial R_\mu(t, a)}{\partial a^\nu}\right]\dot{a}^\nu - \left[\frac{\partial B(t, a)}{\partial a^\mu} + \frac{\partial R_\mu(t, a)}{\partial t}\right]\right\}_{SA} = 0. \quad (6)$$

Let M be a n-dimensional manifold with cotangent bundle T^*M, and denote the local coordinates q^i, with $i = 1, 2, \ldots, n$ ($q = r$ for systems (1)), and a^μ, with $\mu = 1, 2, \ldots, 2n$, respectively ($a = (r, p)$ for systems (1)). We shall call T^*M an *exact symplectic manifold* when equipped with a nowhere-degenerate, closed, and exact two-form Ω, and we shall write (T^*M, Ω). This implies that $\Omega = dR$ and thus $d\Omega = 0$.

We shall call a *globally Birkhoffian vector field* any vector field X on T^*M verifying the property

$$i_X\Omega \equiv X \lrcorner \Omega = -dB \tag{7}$$

for some function B on T^*M. The equations above are a global characterization of autonomous Birkhoff's equations. In fact, Equation (7) is a global version of our local treatment of Section 4.3

$$\Xi \lrcorner \Omega_2 = \Omega_{\mu_1\mu_2}\Xi^{\mu_1}\, da^{\mu_2} = \Xi_\mu\, da^\mu = -dB. \tag{8}$$

The explicit form of the equations is recovered via a straightforward generalization of the Hamiltonian case of Chart 4.5. In fact, we have now the expression

$$i_\Xi\Omega = \tfrac{1}{2}\Omega_{\mu\nu}\Xi^\nu\, da^\mu. \tag{9}$$

Thus, if the vector field Ξ is Birkhoffian, it must admit the explicit form

$$\Xi = \Omega^{\mu\nu}\frac{\partial B}{\partial a^\nu}\frac{\partial}{\partial a^\mu}, \qquad (\Omega^{\mu\nu}) = (\Omega_{\mu\nu})^{-1} \tag{10}$$

by therefore characterizing the autonomous Birkhoff's equations in their contravariant form

$$\dot{a}^\mu = \Omega^{\mu\nu}(a)\frac{\partial B(a)}{\partial a^\nu}. \tag{11}$$

A comparison of Equation (7) above, with Equation (8) of the preceding chart establishes that *in the transition from the local to the global coordinate-free formulation of geometry, all distinctions between Hamiltonian and Birkhoffian vector fields are lost.* In fact, the notion of a "Birkhoffian vector field" introduced in this volume coincides with the notion of a "Hamiltonian vector field" of the contemporary mathematical literature in symplectic geometry. This should not be surprising because the Birkhoffian generalization of Hamilton's equations have been conceived so as to preserve the underlying geometry, which is possible if and only if all distinctions are lost at the abstract, coordinate-free level.

This illustrates quite clearly the physical differences of abstract mathematical algorithms when realized in specific systems of local variables. In fact, we can introduce *one* abstract, coordinate-free notion and *different* realizations in local coordinates. For instance, we can call *globally symplectic vector fields* all vector fields satisfying Equation (7) above (or, equivalently, Equation (8) of the preceding chart). We then have Hamiltonian *or* Birkhoffian vector fields depending on whether the symplectic structure in a local chart is the fundamental one or a general exact one.

Clearly, the Hamiltonian case is a *particular case* of the more general Birkhoffian one, as expressed by the fact that the fundamental symplectic (Hamiltonian) structure *is* of Birkhoff's type

$$\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)_{R=R^0} = (\omega_{\mu\nu}), \qquad R^0 = (\mathbf{p}, \mathbf{0}). \tag{12}$$

To outline the global treatment of Equation (5), perform the extension of the T^*M manifold to $\mathbb{R} \times T^*M$. We shall say that such a $(2n + 1)$-dimensional manifold is an *exact contact manifold* when equipped with a closed and exact two-form of maximal rank. If Ω is an exact symplectic form, the contact form can be constructed by using the projection map π: $\mathbb{R} \times T^*M \to T^*M$ for which one can define $\pi^*\Omega = \tilde{\Omega}$. The form

$$\Omega_B = \tilde{\Omega} - dB \wedge dt \tag{13}$$

for some function B on $\mathbb{R} \times T^*M$ can then be proved to be an exact contact form.

A *semi-autonomous globally Birkhoffian vector field* is then any vector field $\tilde{X}$ on $(\mathbb{R} \times T^*M, \Omega^B)$ verifying the properties

$$i_{\tilde{X}}\Omega_B \equiv \tilde{X} \lrcorner \Omega_B = 0, \tag{14a}$$

$$dt(\tilde{X}) = 1. \tag{14b}$$

This is the desired global treatment of Equations (5). Indeed, according to Equation (13), the Birkhoffian has an explicit time dependence, but the substructure $\tilde{\Omega}$ *does not* possess, in local coordinates, such a dependence.

Clearly, Equation (14) above and Equation (19) of the preceding chart are equivalent. No distinction can be made therefore between semi-autonomous Birkhoffian vector fields and the nonautonomous Hamiltonian ones at the coordinate-free level.

We consider now the general case of Equation (6), which includes all preceding cases, whether Hamiltonian or Birkhoffian, and introduce an arbitrary one-form $\hat{R}$ on $\mathbb{R} \times T^*M$ subject to the condition that the associated two-form via exterior derivative

$$\hat{\Omega} = d\hat{R} \tag{15}$$

is of maximal rank. We shall call a *general, global, Birkhoffian vector field* any nonautonomous vector field $\tilde{X}$ on $\mathbb{R} \times T^*M$ verifying the properties

$$i_{\tilde{X}}\, d\hat{R} \equiv \tilde{X} \lrcorner d\hat{R} = 0, \tag{16a}$$

$$dt(\tilde{X}) = 1. \tag{16b}$$

The equations above provide the desired global treatment of Equation (6). Indeed, the one-form $\hat{R}$ can be written in local coordinates

$$\hat{R} = \hat{R}_\mu(\hat{a})d\hat{a}^\mu = R_\mu(t, a)da^\mu - B(t, a)dt, \qquad B = R_0, \tag{17}$$

and characterizes precisely the integrand of the variational principle for systems (1) (Section 4.2). The inner product of the vector field with the exterior derivative of form (15) then yields precisely Birkhoff's equations (6) in our unified notation

$$\hat{\Omega}_{\mu\nu}(\hat{a})d\hat{a}^\nu = 0, \qquad \mu = 0, 1, 2, \ldots, 2n, \qquad \hat{a} = (t, a), \tag{18a}$$

$$\hat{\Omega}_{\mu\nu} = \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}, \qquad \mu, \nu = 1, 2, \ldots, 2n, \tag{18b}$$

$$\hat{\Omega}_{0\nu} = -\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t}\right) = -\hat{\Omega}_{\nu 0}, \qquad \nu = 1, 2, \ldots, 2n. \tag{18c}$$

The structure $\mathbb{R} \times T^*M$ has been introduced for definition (16) mainly to keep in touch with the physical insight, that is, to associate time with $\mathbb{R}$. On more general geometric grounds, such an association is lost, in the

sense that the equations can be defined in an arbitrary $(2n + 1)$-dimensional manifold M equipped with a closed and exact two-form of maximal rank. Time would be associated then with the space of the (null) codeterminant of maximal rank, as we shall see better in Section 5.3.

We shall now study the nonconservative nature of systems (1). The problem consists of identifying a geometric characterization of the *energy rate of variation in time*. This is achieved through the Lie derivative (Chart 4.4). In general, the Lie derivative of the Birkhoffian $B(t, a)$ with respect to a globally Birkhoffian vector field $\tilde{X}$ is given, in local coordinates, by

$$L_{\tilde{X}} B = \frac{\partial B}{\partial a^{\mu}} \Omega^{\mu\nu} \frac{\partial R_{\nu}}{\partial t} + \frac{\partial B}{\partial t}. \tag{19}$$

Now construct a Birkhoffian representation of systems (1) according to Corollary 4.5.1a, whereby the Birkhoffian is the total energy

$$B(t, a) = E_{\text{tot}}(t, a) = E_{\text{tot}}(t, \mathbf{r}, \mathbf{p}) = T(\mathbf{p}) + U(t, \mathbf{r}, \mathbf{p}), \tag{20}$$

that is, the Birkhoffian is the Hamiltonian of the maximal self-adjoint subsystems of systems (1). This energy is necessarily nonconserved owing to the presence of nonconservative forces. The desired *geometric characterization of the energy rate of variation in time* is then given by the particularization of rule (19)

$$\dot{E}_{\text{tot}}(t, a) = L_{\tilde{X}} E_{\text{tot}} = \frac{\partial E_{\text{tot}}}{\partial a^{\mu}} \Omega^{\mu\nu}(t, a) \frac{\partial R_{\nu}}{\partial t} + \frac{\partial E_{\text{tot}}}{\partial t}, \tag{21}$$

$$\tilde{X} = \Omega \frac{\partial E_{\text{tot}}}{\partial a} + \frac{\partial}{\partial t}.$$

An instructive exercise for the interested reader is to verify that law (21) may provide a description of the energy rate of variation.[57] However, the reader should keep in mind that *law* (21) *does not possess a Lie algebra structure for the general non-autonomous case* (Chart 4.1).

Evidently, *nonconservation law* (21) admits, as a particular case, *conservation law*

$$\dot{E}_{\text{tot}}(a) = L_X E_{\text{tot}} = [E_{\text{tot}}, E_{\text{tot}}]^{*}_{(a)} = 0. \tag{22}$$

In this case, the law does possess a Lie algebra structure, but we are dealing with truly particular Newtonian systems (the autonomous, conservative, essentially self-adjoint systems in direct analytic representations).

We consider now the peculiar aspect of the Birkhoffian realizations of contact two-forms mentioned in the text, that is, the explicit time dependence of the symplectic substructure. This occurrence creates a number of technical problems, such as 1) the region of definition of the two-form; 2) the applicable version of the Poincaré lemma, and 3) the proper formulation of the transformation theory. We consider here problems 1) and 2). Problem 3) is studied in the next chapter.

Consider contact structure (15) in realization (18). It is rather natural to think of a star-shaped region in the variables a^{μ} at each fixed value of time, but then one needs a mechanism whereby, as time varies, different regions at different values of time are smoothly connected. Also, as recalled in Section 4.2, a star-shaped region does not necessarily remain this type under an arbitrary transformation.

[57] This can be verified, for instance, by using the Birkhoffian representation of Example 4.2.

In order to overcome these difficulties, Sarlet and Cantrijn (1978a) introduced the notion of a "region deformable to a curve"—that is, one which (a) is topologically equivalent to a star-shaped region, (b) allows a smooth connection between regions at different values of time, and (c) preserves its topological character under arbitrary diffeomorphisms (transformations).

Note from the outset that the approach by Sarlet and Cantrijn is a natural generalization of the "deformability to a point" by Flanders (1963). We shall therefore take this opportunity to review the formulation of the direct and of the converse of the Poincaré Lemma which apply to structure (15). This provides an alternative approach to that by Lovelock and Rund (1975) reviewed in Section I.1.2 and which, as is now familiar, is based on the notion of star-shaped region.

Let O denote an open subset of $\mathbb{R}^m$, and $F^p(O)$ the set of all $\mathscr{C}^\infty$ p-forms (Section I.1.2) on O. An element A^p of $F^p(O)$ assigns to every $a \in O$ a p-linear alternating mapping

$$A^p = A^p(a) = A_{\mu_1, \ldots, \mu_p}(a) da^{\mu_1} \wedge \cdots \wedge da^{\mu_p}. \tag{23}$$

Let $\mathscr{G}$ be a subset of $\mathbb{R} \times \mathbb{R}^m$, and put

$$\Sigma_t = \{a \in \mathbb{R}^m \mid (t, a) \in \mathscr{G}\}. \tag{24}$$

Definition 1. $\mathscr{G}$ is smoothly deformable to a curve (monotonically increasing in the t-direction) if a family of mappings

$$\begin{aligned} \varphi_t &: I \times \Sigma_t \to \Sigma_t, \qquad I = [0, 1] \\ (\tau, a) &\to \varphi_t(\tau, a) = b \in \Sigma_t \end{aligned} \tag{25}$$

exists such that (i) $\varphi_t(1, a) = a$, $\varphi_t(0, a) = a_0$, for all $a \in \Sigma_t$ and where $a_0 = a_0(t)$ is fixed on Σ_t; (ii) the map $\varphi: I \times \mathscr{G} \to \mathscr{G}$ is of class $\mathscr{C}^\infty$ with respect to all arguments.

Clearly, the property of being smoothly deformable to a curve is preserved by all images $\mathscr{G}'$ of $\mathscr{G}$ under class $\mathscr{C}^\infty$, invertible transformations. Also, regions at different values of time are smoothly connected. Finally, the topological equivalence of Definition 1 with the notion of star-shape is also ensured.

Starting from a family of p-forms $A_t^p \in F^p(\Sigma_t)$ given by

$$A_t^p = A_t^p(a) = A_{\mu_1, \ldots, \mu_p}(t, a) da^{\mu_1} \wedge \cdots \wedge da^{\mu_p} \tag{26}$$

for each t such that $\Sigma_t \neq \varphi$, where $A_{\mu_1, \ldots, \mu_p}(t, a)$ are given $\mathscr{C}^\infty$ functions on $\mathscr{G}$, we can define a *parametric p-form* on $\mathscr{G}$, $\mathscr{A}^p \in \mathscr{T}^p(\mathscr{G})$, by

$$\mathscr{A}^p(t, a) = A_t^p(a), \qquad \forall (t, a) \in \mathscr{G}. \tag{27}$$

As a straightforward extension of the definitions of Section I.1.2, we have

$$\delta: \mathscr{T}^p(\mathscr{G}) \to \mathscr{T}^{p+1}(\mathscr{G}), \tag{28}$$

that is, the *exterior derivative* of parametric p-forms can be written (actually, can be defined by)

$$(\delta \mathscr{A}^p)(t, a) = (dA_t^p)(a), \tag{29}$$

where d is the exterior derivative in $F^p(\Sigma_t)$. A parametric p-form $\mathscr{A}^p$ is *exact* when a parametric $(p-1)$ form $\mathscr{A}^{p-1}$ exists, called *primitive form*, such that

$$\mathscr{A}^p = \delta\mathscr{A}^{p-1}. \tag{30}$$

Similarly, $\mathscr{A}^p$ is *closed* if and only if

$$\delta\mathscr{A}^p = 0. \tag{31}$$

The following property is a simple, direct generalization of Lemma I.1.2.1.

Lemma 1 (Direct Poincaré Lemma). *Every exact parametric p-form is closed.*

The proof of the following property is, on the contrary, nontrivial. For brevity, we refer the interested reader to Sarlet and Cantrijn (*loc. cit.*).

Lemma 2 (Converse of the Poincaré Lemma). *Let $\mathscr{G}$ be a subset of $\mathbb{R} \times \mathbb{R}^m$ that is smoothly deformable to a curve. Let $\mathscr{A}^p$ be a parametric p-form of $\mathscr{G}$ that is closed. Then $\mathscr{A}^p$ is exact on $\mathscr{G}$.*

In conclusion, the existence theorems of the Inverse Problem studied in this volume, particularly those for Birkhoffian representations, can be subjected to a dual approach. One can first use conventional star-shaped regions in $(2n+1)$-dimension for a contact approach to the forms considered. This is the case, for instance of Corollary 4.2.1a. Alternatively, we can consider the forms at a fixed value of time. In this latter case the parametric approach outlined in this chart applies. This is the case of Theorem 4.5.1. This latter approach will be tacitly implemented throughout our analysis whenever considering symplectic structures with an explicit time dependence.

Chart 4.7 Lie-Admissible/Symplectic-Admissible Generalization of Birkhoff's Equations for Nonlocal Nonpotential Systems

In the text of this chapter we established the *universality* of the Lie algebras and of the symplectic (or contact) geometry for local Newtonian systems. A few words on the *limitations* of these mathematical tools in physics are now in order to prevent a possible expectation of their terminal methodological character. Stated explicitly, after having identified rather substantial capabilities, it is important to point out that the Lie algebras and the symplectic geometry *do not* provide the final formulation of mechanics. On the contrary, they characterize only one stage of an ever-continuing process of mathematical and physical advances.

Consider the problem of interactions. The effectiveness of the Lie algebra and of the symplectic geometry for the treatment of the electromagnetic interactions is well-known. In Example 4.1 we shall show that the formulations considered apply to the characterization of the electromagnetic interactions, not only in their conventional (Hamiltonian) form, but also in their most general possible (Birkhoffian) form. This effectiveness persists for the more general interactions of contemporary physics, such as the unified gauge theories of weak and electromagnetic interactions.

All these interactions, whether Newtonian or quantum field theoretical, are of local/differential and potential/self-adjoint type. In fact, all these

interactions are characterized by a conventional, local, Lagrangian function or operator-valued distribution $L_{tot} = L_{free} + L_{int}$.

Recent studies[58] have identified a number of insufficiencies of these interactions in several branches of physics. In fact, the systems of *Newtonian Mechanics* are, more properly, of the nonlocal type (Section 4.1), in the sense that they demand integro–differential equations to represent the interactions at all points of a surface volume. The systems of *Statistical Mechanics*, whether classical or quantum mechanical, are also nonlocal whenever the extended character of the constituents is taken into account, together with their inelastic collisions. Along quite similar lines, the systems of *Particle Mechanics* are also nonlocal in their more adequate treatment. This is the case in particular for the strong interactions, because of the need for mutual penetration of the wave packets of particles.[59]

When the Lie algebra and the symplectic geometry are considered in this context, they emerge possessing rather precise limitations. In fact, the symplectic geometry is, in the final analysis, a local/differential geometry, that is, a geometry which, when realized in a local chart, admits *ordinary* (or *partial*) differential equations. As a result, no possibility is known at this time for an effective treatment of nonlocal systems via the symplectic geometry in its current formulation.

We can therefore say that *the Lie algebra and the symplectic (or contact) geometry, rather than providing the ultimate formulations of mechanics, provide instead a mere approximation of the local/nonpotential type, with the understanding that more general algebraic and geometric structures are expected to exist for nonlocal/nonpotential treatments.*

In this chart we review the main ideas of the possibility of generalizing the Lie algebra and the symplectic geometry into the so-called Lie-admissible algebras and the symplectic-admissible geometry.

The most general form of unconstrained Newtonian systems in Euclidean space known at this time is given by the so-called integro–differential, variationally non-self-adjoint systems. These are systems with a superposition of local/differential and nonlocal/integral forces which are derivable and nonderivable from a potential. By using a self-evident notation, the systems can be written[60]

$$\left\{\left[m_a \ddot{r}_{ka} - f_{ka}(t, \mathbf{r}, \dot{\mathbf{r}}) - \iiint d\mathbf{r}'\, k_{ka}(t, \mathbf{r}, \mathbf{r}', \dot{\mathbf{r}}, \dot{\mathbf{r}}', \ldots)\right]_{SA} - F_{ka}(t, \mathbf{r}, \dot{\mathbf{r}}) - \iiint d\mathbf{r}'\, K_{ka}(t, \mathbf{r}, \mathbf{r}', \dot{\mathbf{r}}, \dot{\mathbf{r}}', \ldots)\right\}_{NSA} = 0, \quad (1)$$

$$a = 1, 2, \ldots, N, \qquad k = x, y, z,$$

[58] See the *Proceedings of the Second Workshop (1979)* and of *the Third Workshop (1980) on Lie-admissible Formulations*, and the *Proceedings of the First International Conference on Nonpotential Interactions and their Lie-admissible Treatment (1982)*. An extended presentation of this chart is also provided in the monograph Santilli (1982d). For a mathematical study of flexible Lie-admissible algebras, see the monograph Myung (1982b). For an introductory mathematical account, see Benkart (1982). For an historical mathematical account, see Tomber (1982).

[59] See Chart 6.1.

[60] Several ways exist of writing nonlocal forces. In Equation (1) we selected "bilocal" form with a kernel. For another form, see Equation (4.1.3).

and represent the motion of extended objects in a resistive medium with center of mass coordinates **r**, under local and nonlocal forces of action-at-a-distance/potential type as well as contact/non-potential type. As indicated in Sections 4.1, the local nonpotential systems considered in this volume (and in the preceding one) are an *approximation* of systems (1).

Hamilton's equations (without external terms) are clearly *insufficient* in representing the systems considered because of their inability to achieve direct universality at the level of local approximations, let alone the full nonlocal treatment. Birkhoff equations are equally *insufficient* to represent systems (1) for a number of reasons, ranging from the lack of integro–differential character of the underlying geometry, to the practical impossibility of computing a Birkhoffian representation.

To overcome these (and other) difficulties, Santilli (1978c and e) proposed a generalization of Birkhoff's equations (and thus, of Hamilton's equations) which achieves a direct universality for all systems (1) via a generalization of the underlying algebra and geometry.

I. Generalization of Lie Algebras into Lie-Admissible Algebras. In Chart 4.1 we pointed out the property that the Lie-admissible algebras constitute a genuine generalization of the Lie algebras. At the mathematical level, this can be seen in a number of ways, such as the fact that the axioms characterizing a Lie-admissible algebra are a direct generalization of the Lie algebra axioms, or that the Lie algebras are contained in the classification of all Lie-admissible algebras, or that all Lie algebras are Lie-admissible, but the opposite property is not necessarily true.

At the physical level, the generalized character can be seen by noting that the product of a Lie-admissible algebra is neither antisymmetric nor symmetric. Thus it can always be decomposed into an antisymmetric and a symmetric part. The product verifies the conditions of Lie-admissibility when the antisymmetric part is Lie. Finally, Lie algebras are recovered as a particular case when the symmetric part is null.

The realization in Newtonian Mechanics of the product of Lie-admissible algebras is given by

$$(A, B) = \frac{\partial A}{\partial a^\mu} S^{\mu\nu}(t, a) \frac{\partial B}{\partial a^\nu} = \frac{\partial A}{\partial a^\mu} \Omega^{\mu\nu} \frac{\partial B}{\partial a^\nu} + \frac{\partial A}{\partial a^\mu} T^{\mu\nu} \frac{\partial B}{\partial a^\nu},$$

$$\Omega^{\mu\nu} = \left(\left\| \frac{\partial R_\alpha}{\partial a^\beta} - \frac{\partial R_\beta}{\partial a^\alpha} \right\|^{-1} \right)^{\mu\nu}, \qquad \det(S^{\mu\nu}) \neq 0, \qquad \det(\Omega^{\mu\nu}) \neq 0,$$

$$T^{\mu\nu} = T^{\nu\mu}, \tag{2}$$

where the underlying carrier space is the same as that of Birkhoff's equations (e.g., the cotangent bundle T^*M), with the understanding that the use of suitable generalizations is not only possible but encouraged.

Product (2) is Lie-admissible because its antisymmetric part is Birkhoffian and therefore Lie. The Poisson brackets are recovered via a double simplification. First, one simplifies the product via the restriction $T^{\mu\nu} = 0$, and second, one assumes the further simplification of the general Lie tensor $\Omega^{\mu\nu}$ into the canonical form $\omega^{\mu\nu}$.

The separation of the product into an antisymmetric and a symmetric part is along rather precise physical motivations. The antisymmetric part can represent all possible forces and dynamic conditions which are treatable via Lie's theory along the existence theorems of this volume. The symmetric part can then represent all forces which are outside the

capability of Lie's theory, such as the nonlocal nonpotential forces. This separation of Lie and non-Lie dynamics subsequently results in being important in the quantitative treatment of the physical consequences due to the presence of non-Lie forces.[61]

The fact that product (1) is the most general possible regular realization of Lie-admissible algebras on T^*M can be seen as follows. The axiom for the product (A, B) to be Lie-admissible is given by[62]

$$[A, B, C] + [B, C, A] + [C, A, B] - [C, B, A] - [B, A, C] - [A, C, B] = 0, \quad (3)$$

where

$$[A, B, C] = ((A, B), C) - (A, (B, C)), \quad (4)$$

and result into the conditions on the tensor $S^{\mu\nu}$

$$(S^{\mu\rho} - S^{\rho\mu}) \frac{\partial}{\partial a^\rho} (S^{\nu\tau} - S^{\tau\nu}) + (S^{\nu\rho} - S^{\rho\nu}) \frac{\partial}{\partial a^\rho} (S^{\tau\mu} - S^{\mu\tau}) + (S^{\tau\rho} - S^{\rho\tau}) \frac{\partial}{\partial a^\rho} (S^{\mu\nu} - S^{\nu\mu}) = 0, \quad (5)$$

with the general solution

$$S^{\mu\nu} = \left(\left\| \frac{\partial R_\alpha}{\partial a^\beta} - \frac{\partial R_\beta}{\partial a^\alpha} \right\|^{-1} \right)^{\mu\nu} + T^{\mu\nu}, \qquad T^{\mu\nu} = T^{\nu\mu}. \quad (6)$$

Recent studies have indicated that the generalization of Lie algebras into Lie-admissible algebras occur at a central methodological level of Lie's theory, that of the universal enveloping associative algebra. In turn, this permits a consistent exponentiation into a connected *Lie-admissible group of transformations*, i.e., a set of transformations which constitutes a connected group in the conventional sense, yet whose reduction in the

[61] One of the implications of the possible existence of a nonlocal, nonpotential component in the strong interactions is a departure from the electromagnetic characteristics of particles,[58] such as magnetic moments, spin, parity, etc. (which, of course, can only be an internal effect of a closed system under strong interactions not detectable from the outside under long-range interactions). The separation in the theory of the local/potential part from the nonlocal/nonpotential component is for the computation of the deviations from the physical characteristics under the former part, due to the presence of the latter part. For consistency, it is important to begin the separation at the primitive classical level and to preserve it at all subsequence levels of study.

[62] A first axiom indicating the antisymmetry of the attached product, i.e.,

$$[A, B]^* - [B, A]^* = 0, \qquad [A, B]^* = (A, B) - (B, A)$$

has been ignored here because the field has characteristic zero. Note that axiom (3) is a generalization of the Jacobi law. In fact, when $(A, B) = [A, B]$, axiom (3) reduces to four times the Jacobi law. This illustrates the remark made earlier that the Lie-admissible axioms are a generalization of the Lie axioms.

neighborhood of the identity exhibits a non-Lie, Lie-admissible algebra. This can be seen in the following exponentiation of product (2):

$$a' = \exp\left(\theta S^{\mu\nu} \frac{\partial X}{\partial a^{\nu}} \frac{\partial}{\partial a^{\mu}}\right) a = \text{connected group} \tag{7a}$$

$$\frac{da}{d\theta} \cong \left.\frac{a' - a}{\theta}\right|_{\theta \approx 0} = \frac{\partial a}{\partial a^{\mu}} S^{\mu\nu} \frac{\partial X}{\partial a^{\nu}} = (a, X) = \text{Non-Lie, Lie-admissible algebra.} \tag{7b}$$

These features indicate the possibility of constructing, in due time, a Lie-admissible generalization of Lie's theory, including generalizations of the Poincaré–Birkhoff–Witt Theorem, Lie's theorems, the representation theory, etc., and considerable research activity is currently under way along these lines.[63]

II. Generalization of the Symplectic Geometry into the Symplectic-Admissible Geometry. The symplectic geometry is clearly unable to "geometrize" Lie-admissible algebras, e.g., because of the total antisymmetric character of the symplectic two-forms. To bypass this difficulty, Santilli (*loc. cit.*) proposed the development of the *symplectic-admissible geometry* as the geometry of manifolds equipped with tensorial two-forms whose antisymmetric part is symplectic. By assuming for the manifold the cotangent bundle T^*M, and for the local chart the variables $a = (\mathbf{r}, \mathbf{p})$, the tensorial two-form under consideration can be written[64]

$$S_2 = S_{\mu\nu}(t, a) da^{\mu} \otimes da^{\nu} = \Omega'_{\mu\nu}\, da^{\mu} \wedge da^{\nu} + T'_{\mu\nu} da^{\mu} \times da^{\nu}, \tag{8a}$$

$$dS_2 \neq 0, \qquad d(\Omega'_{\mu\nu}\, da^{\mu} \wedge da^{\nu}) = 0, \qquad \det(S_{\mu\nu}) \neq 0, \qquad \det(\Omega'_{\mu\nu}) \neq 0, \tag{8b}$$

where $\otimes$ is the tensorial product, $\wedge$ is the exterior product, and $\times$ is the symmetric product. The geometrization of product (2) is achieved when

$$S^{\mu\nu} = (\|S\|^{-1})^{\mu\nu}. \tag{9}$$

The integrability conditions are given by

$$\frac{\partial}{\partial a^{\tau}} (S_{\mu\nu} - S_{\nu\mu}) + \frac{\partial}{\partial a^{\mu}} (S_{\nu\tau} - S_{\tau\nu}) + \frac{\partial}{\partial a^{\mu}} (S_{\tau\mu} - S_{\mu\tau}) = 0 \tag{10}$$

with the general solution

$$S_{\mu\nu} = \left(\frac{\partial R'_{\nu}}{\partial a} - \frac{\partial R'_{\mu}}{\partial a^{\nu}}\right) + T'_{\mu\nu}, \qquad T'_{\mu\nu} = T'_{\nu\mu}. \tag{11}$$

[63] See the Bibliography by Tomber *et al.* (1979 and 1981).

[64] In Equations (8) we have distinguished the covariant tensors $\Omega'_{\mu\nu}$ and $T'_{\mu\nu}$ from their contravariant counterparts $\Omega^{\mu\nu}$ and $T^{\mu\nu}$ because, in general,

$$S^{\mu\nu} = (\|S_{\alpha\beta}\|^{-1})^{\mu\nu}, \qquad S^{\mu\nu} - S^{\nu\mu} \neq (\|S_{\alpha\beta} - S_{\beta\alpha}\|^{-1})^{\mu\nu},$$
$$S^{\mu\nu} + S^{\nu\mu} \neq (\|S_{\alpha\beta} + S_{\beta\alpha}\|^{-1})^{\mu\nu}.$$

Notice the appearance of an explicit time dependence in the two-form which is similar to the corresponding Birkhoffian case. This is an indication that the more adequate treatment is that on $\mathbb{R} \times T^*M$ with corresponding contact-admissible extension. This latter aspect will not be considered here for brevity.

Tensorial two-forms whose antisymmetric part is symplectic were called *symplectic-admissible two-forms* in order to attempt a geometric counterpart of the algebraic definition of a Lie-admissible product (via the Lie character of the antisymmetric part).[65] The manifold T^*M equipped with such a form was then called a *symplectic-admissible manifold*. The main motivation was parallel to that of product (2). In fact, the implementation of a symplectic two-form (or manifold) into the broader symplectic-admissible structure permits the representation of local/differential equations via the symplectic part, and the possible treatment of the nonlocal/integral terms via the symmetric part. A condition for studying this objective was the loss of the notion of (geometric) closure (that is, $dS_2 \neq 0$)[66].

A few comments are in order. The full geometrization of nonlocal interactions calls for an "integro–differential geometry" which is expected to be considerably more complex than the "symplectic-admissible geometry," beginning with its topological foundations. In fact, the needed geometry calls for abandoning the local notion of points in favor of suitable nonlocal/integral generalizations. The symplectic-admissible geometry has been suggested as an intermediary step, prior to such a full nonlocal treatment. In fact, the geometry is patterned along the pragmatic formulation of equations (1) whereby the center of mass coordinates $\mathbf{r}$ are purely local, and the nonlocal effects are represented via additive forces. At the geometric level, this results in structures (8) consisting of local/differential exterior two-forms plus nonlocal/integral symmetric two-forms.

It should be indicated here for completeness, that the symplectic-admissible geometry is apparently needed for reasons independent of those considered here. As is well-known, the canonical symplectic two-form

$$\omega_2 = \tfrac{1}{2}\omega_{\mu\nu}\, da^\mu \wedge da^\nu = dp_k \wedge dr^k \tag{12}$$

has been historically conceived for the geometrization of the Poisson's brackets

$$[A, B] = \frac{\partial A}{\partial a^\mu}\,\omega^{\mu\nu}\,\frac{\partial B}{\partial a^\nu} = \frac{\partial A}{\partial r^k}\,\frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial r^k}\,\frac{\partial A}{\partial p_k}. \tag{13}$$

However, as is also well-known, the primitive algebraic product in Lie's theory is that of the enveloping associative algebra. The Lie product is merely an attached product.

The ultimate geometrization of the Poisson's brackets therefore demands its realization at the level of the envelope. This problem was

[65] If a contravariant tensor $S^{\mu\nu}$ is Lie-admissible, its covariant version $S_{\mu\nu}$, defined by Equation (9), is not necessarily symplectic-admissible, and vice versa. As a result, the joint condition of Lie-admissibility and symplectic-admissibility must be imposed. As we shall review in a moment (Equations (22) later on), a joint solution of this type exists, and it is directly universal for all systems (1). This is sufficient on physical grounds. On mathematical grounds, the situation is different, and much remains to be done. For instance, if a tensor $\Omega^{\mu\nu}$ is (regular and) Lie, its covariant form defined by $\Omega_{\mu\nu} = (\|\Omega^{\alpha\beta}\|^{-1})_{\mu\nu}$ is *always* symplectic, and vice versa. Studies are currently in progress via grading and other mechanisms to see whether or not the corresponding property at the more general Lie-admissible/symplectic-admissible level can be recovered.

[66] For additional geometrical studies, see refs. 58 [in particular, the contributions by Oehmke (1982), and Sagle (1982)]. See also the Index of the bibliography by Tomber *et al.* (1981).

formulated by Santilli (1978e), who pointed out that the envelope of brackets (13) is given by the *nonassociative Lie-admissible product*

$$A \circ B = \frac{\partial A}{\partial r^k} \frac{\partial B}{\partial p_k}. \tag{14}$$

The primitive geometry, therefore, *is not* the symplectic geometry, but rather that geometrizing product (14). This leads in a rather natural way to the symplectic-admissible geometry as the primitive geometry of the envelope of Poisson's brackets with tensorial two-forms

$$S_2 = S_{\mu\nu}\, da^\mu \otimes da^\nu = \tfrac{1}{2}\omega_{\mu\nu}\, da^\mu \wedge da^\nu + t_{\mu\nu}\, da^\mu \times da^\nu, \qquad t_{\mu\nu} = t_{\nu\mu}. \tag{15}$$

The symplectic geometry then acquires a derived meaning, in the sense that symplectic form (12) is merely the attached antisymmetric form of the fundamental form (15).

Finally, the independence of the symplectic-admissible character from the selected local variables should be indicated. As we show in detail in the next chapter, symplectic two-forms remain symplectic under arbitrary (but smoothness- and regularity-preserving) transformations of the variables. Explicitly, if the form $\Omega_2 = \Omega_{\mu\nu}\, da^\mu \wedge da^\nu$ is symplectic, the transformed forms under all possible new variables $a'(a)$

$$\Omega'_2 = \Omega'_{\mu\nu}\, da'^\mu \wedge da'^\nu, \qquad \Omega'_{\mu\nu} = \frac{\partial a^\rho}{\partial a'^\mu}\, \Omega_{\rho\sigma}\, \frac{\partial a^\sigma}{\partial a'^\nu} \tag{16}$$

are symplectic. In turn, the independence of the symplectic character from the local variables is at the foundations of the coordinate-free globalization of the symplectic geometry.

Santilli (*loc. cit.*) proved that the property above is, in actuality, a particular case of the more general property that the symplectic-admissible character of a tensorial two-form is independent of the selected local variables. Explicitly, if (8a) is symplectic-admissible, all possible transformed forms

$$S'_2 = S'_{\mu\nu}\, da'^\mu \otimes da'^\nu, \qquad S'_{\mu\nu} = \frac{\partial a^\rho}{\partial a'^\mu}\, S_{\rho\sigma}\, \frac{\partial a^\sigma}{\partial a'^\nu} \tag{17}$$

are also symplectic-admissible. In turn, this feature gives hope of achieving, in due time, a coordinate-free globalization of the symplectic-admissible geometry.[63,66]

III. Generalization of Birkhoff's Equations into Lie-Admissible Symplectic-Admissible Forms. After having identified the notions of Lie-admissible algebras and of symplectic-admissible geometry, Santilli (*loc. cit.*) proposed a generalization of Birkhoff's equations which can be written in the contravariant/algebraic form

$$\dot{a}^\mu - S^{\mu\nu}(t, a)\, \frac{\partial H(t, a)}{\partial a^\nu} = 0, \qquad \mu = 1, 2, \ldots, 2n, \tag{18}$$

or in the covariant/geometric form

$$S_{\mu\nu}(t, a)\dot{a}^\nu - \frac{\partial H(t, a)}{\partial a^\mu} = 0, \tag{19}$$

where the tensors $S^{\mu\nu}$ and $S_{\mu\nu}$ are Lie-admissible and symplectic-admissible in the sense of Equations (5) and (10), respectively, and interrelationship (9) holds.

The generalized nature of Equations (18) or (19) with respect to both Birkhoff's and Hamilton's equations is now trivial.

The direct universality for the most general systems known at this time is also trivial. Equations (1) can be written in the first-order form

$$\dot{a}^\mu = \hat{\Xi}^\mu(t, a); \qquad (\hat{\Xi}^\mu) = \begin{pmatrix} p/m \\ f_{\text{LOCAL}} \end{pmatrix} + \begin{pmatrix} 0 \\ F_{\text{NONLOCAL}} \end{pmatrix}, \tag{20}$$

and the integro–differential vector fields $\hat{\Xi}$ are always *Birkhoff-admissible*. That is, a function H and a tensor $S_{\mu\nu}$ always exist such that

$$S_{\mu\nu}\hat{\Xi}^\nu = (\Omega_{\mu\nu} + T_{\mu\nu})\hat{\Xi}^\nu = \frac{\partial H}{\partial a^\mu}. \tag{21}$$

This property can be better seen by recalling that Equations (18) or (19) were proposed in order to reach a consistent algebraic and geometric characterization of the equations originally conceived by Hamilton, those with external terms. One of the simplest possible form of the generalized equations occurs when the Birkhoffian part reduces to Hamiltonian form, in which case we have the equations

$$\dot{a}^\mu = S^{\mu\nu}\frac{\partial H}{\partial a^\nu} = \omega^{\mu\nu}\frac{\partial H}{\partial a^\nu} + T^{\mu\nu}\frac{\partial H}{\partial a^\nu}, \tag{22a}$$

$$(T^{\mu\nu}) = \begin{pmatrix} 0 & 0 \\ 0 & -s \end{pmatrix}, \qquad s = \text{diag}(F^{\text{NSA}}/(p/m)), \tag{22b}$$

which can be written in the disjoint **r** and **p** coordinates

$$\begin{cases} \dot{r}^k = \dfrac{\partial H}{\partial p_k}, \\ \dot{p}_k = -\dfrac{\partial H}{\partial r^k} + F_k^{\text{NSA}}, \qquad F_k^{\text{NSA}} = -s_{ki}\dfrac{\partial H}{\partial p_i}, \end{cases} \tag{23}$$

by therefore coinciding with the equations originally conceived by Hamilton.

The equations are written in form (22) rather than (23) because the latter do not admit a consistent algebraic structure, in that the product of the time evolution

$$\dot{A}(a) \stackrel{\text{def}}{=} A * H = \frac{\partial A}{\partial r^k}\frac{\partial H}{\partial p_k} - \frac{\partial A}{\partial p_k}\frac{\partial H}{\partial r^k} + \frac{\partial A}{\partial p_k}F_k \tag{24}$$

violates the right distributive and the scalar laws, by therefore being unable to characterize a consistent algebra (Chart 4.1). On the contrary, Equations (22) admit the product of the time evolution

$$\dot{A}(a) \stackrel{\text{def}}{=} (A, H) = \frac{\partial A}{\partial a^\mu}S^{\mu\nu}\frac{\partial H}{\partial a^\nu} = \frac{\partial A}{\partial r^k}\frac{\partial H}{\partial p_k} - \frac{\partial A}{\partial p_k}\frac{\partial H}{\partial r_k} + \frac{\partial A}{\partial p_i}s_{ij}\frac{\partial H}{\partial p_j} \tag{25}$$

which does indeed satisfy the right and left distributive and scalar laws. Thus the product (A, H) characterizes an algebra, and this algebra turns out to be a Lie-admissible generalization of a Lie algebra.

Notice that simplified Equations (22) already provide the explicit solution of the representation of all systems (1). The simplicity of representing non-local systems via Lie-admissible equations should be compared with the complexity of the construction of a representation of the

simpler local systems via Birkhoff's equations. Notice also that Equations (18) possess a consistent algebra for the most general possible nonautonomous equations, which is not the case for Birkhoff's equations (Chart 4.1). Finally, the reader should keep in mind that all symbols of Equations (18) or (19) readily permit a direct physical meaning; that is, t is the time of the observer, the $\mathbf{r}$'s are the center of mass coordinates with respect to the observer, the $\mathbf{p}$'s are the physical linear momenta $m\dot{\mathbf{r}}$, H is the total (generally nonconserved) mechanical energy, and $S^{\mu\nu}$ represents all nonconservative, non-potential forces (or contact interactions).

Generalized equations (18) or (19) appear to have rather intriguing implications for a number of open problems of mechanics, such as the relativity which is applicable to a Newtonian particle under unrestricted forces and dynamical conditions. Also, the equations have been extended to statistical mechanics, classical field theory, quantum mechanics, and other branches of physics, in each of which they have resulted in being directly universal. For these and related studies, we refer the interested reader elsewhere. (See footnote 58 on page 91.)

EXAMPLES

Example 4.1

In this example we shall identify *the Hamiltonian and Birkhoffian representations of the Newtonian electromagnetic interactions* (charged particles under the Lorentz force). The idea is to indicate that the local formulation of the electromagnetic interactions is not only compatible with the conventional analytic/Lie/symplectic formulations but is actually compatible with these formulations in their most general possible (Birkhoffian) form. This sets the foundations of the methodological treatment of the electromagnetic interactions which persists, upon due technical implementations, at different levels of treatment (such as quantum mechanical), as well as for other interactions which are similar in structure to the electromagnetic ones (such as the weak interactions but not necessarily the strong[58]).

In Example I.2.7 we proved the *variational self-adjointness of the Lorentz force* which we write in this example in the form for one charged particle

$$\{m\ddot{r}_i - e[E_i - (\mathbf{B} \times \dot{\mathbf{r}})_i]_{\mathrm{SA}}\}_{\mathrm{SA}}$$

$$= \left\{ m\ddot{r}_i - e\left[\left(\frac{\partial \varphi}{\partial r^i} - \frac{\partial A_i}{\partial t}\right) - \delta_{ij}^{mn} \frac{\partial A_n}{\partial r^m} \dot{\mathbf{r}}^j\right]_{\mathrm{SA}} \right\}_{\mathrm{SA}} = 0, \qquad (1)$$

$$i, j, m, n = x, y, z.$$

We shall first review the conventional Hamiltonian formulation. We shall then identify the broader Birkhoffian approach. Later, in Example 6.1, we shall study the reduction of the Birkhoffian representation to the Hamiltonian form. In order to identify more clearly the implications in the transition from the Hamiltonian to the Birkhoffian representation (and vice versa), we shall consider the analytic, algebraic, and geometric profiles separately.

A. Conventional Hamiltonian representation. As presented in numerous treateses in Newtonian Mechanics, system (1) admits the Hamiltonian representation

$$H_{\text{Lorentz}} = \frac{1}{2m}(\mathbf{P} - e\mathbf{A})^2 + e\varphi, \tag{2a}$$

$$\mathbf{P} = m\dot{\mathbf{r}} + e\mathbf{A}, \tag{2b}$$

often referred to as characterized by the so-called *minimal coupling rule* $\mathbf{P} \to \mathbf{P} + e\mathbf{A}$. Within the context of the Inverse Problem, this occurrence is trivial, owing to the self-adjointness of the system.

Conventional Lie structure. The brackets of the time evolution law for the Hamiltonian representation are the conventional Poisson brackets[67]

$$[A, B] = \frac{\partial A}{\partial b^\mu}\omega^{\mu\nu}\frac{\partial B}{\partial b^\nu} = \frac{\partial A}{\partial \mathbf{r}}\cdot\frac{\partial B}{\partial \mathbf{P}} - \frac{\partial A}{\partial \mathbf{P}}\cdot\frac{\partial B}{\partial \mathbf{r}}, \tag{3}$$

$$b = (\mathbf{r}, \mathbf{P}).$$

Conventional geometric structure. The self-adjointness of system (1) implies the existence of the vector field

$$\Xi_{\text{Lorentz}} = \Xi^\mu(b)\frac{\partial}{\partial b^\mu} \tag{4}$$

which is Hamiltonian, in the sense of Chart 4.5, i.e., verifies the rule

$$\Xi_{\text{Lorentz}} \lrcorner \omega_2 = -dH_{\text{Lorentz}} \tag{5}$$

with respect to the fundamental symplectic structure

$$\omega_2 = \tfrac{1}{2}\omega_{\mu\nu}\, db^\mu \wedge db^\nu = dP_i \wedge dr^i. \tag{6}$$

The underlying symplectic manifold is the cotangent bundle T^*E_3 of the three-dimensional Euclidean space E_3 with local coordinates $\mathbf{r}$.

B. Birkhoffian representation. Rather than represent the Lorentz force via the Hamiltonian, we can represent it via the generalized Birkhoff's tensor. Among the possibilities at hand, we select that for which the Birkhoffian represents the total energy

$$B_{\text{Lorentz}} = \frac{1}{2m}\mathbf{p}^2 + e\varphi, \tag{7}$$

$$\mathbf{p} = m\dot{\mathbf{r}}.$$

while the part of the Lorentz force originating from the vector potential is represented via Birkhoff's tensor (Sarlet and Cantrijn (1978a and b))

$$\Omega_{\mu\nu} = \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}, \tag{8a}$$

$$\{R_\mu\} = \{\mathbf{p} + e\mathbf{A}, 0\}, \qquad a = (\mathbf{r}, \mathbf{p}). \tag{8b}$$

[67] We use the variables $b = (\mathbf{r}, \mathbf{P})$ for the Hamiltonian representation to stress their differences with the variables $a = (\mathbf{r}, \mathbf{p})$ to be used later for the Birkhoffian representations. The different variables also indicate that the two representations can be connected via suitable noncanonical transformations (Darboux's diffeomorphisms) as we shall see in Chapter 6.

The Birkhoffian representation then explicitly reads

$$\begin{pmatrix} e\left(\frac{\partial A_i}{\partial r^j} - \frac{\partial A_j}{\partial r^i}\right) & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} e\left(\frac{\partial \varphi}{\partial r^i} - \frac{\partial A_i}{\partial t}\right) \\ \frac{1}{m} p_i \end{pmatrix}. \tag{9}$$

Clearly, the representation above is fully equivalent on analytic grounds with representation (2). We simply have the transition from the conventional action principle

$$\delta \int_{t_1}^{t_2} dt(\mathbf{P} \cdot \dot{\mathbf{r}} - H_{\text{Lorentz}})(\tilde{E}_0) = 0, \tag{10}$$

to the generalized one

$$\delta \int_{t_1}^{t_2} dt[(\mathbf{p} + e\mathbf{A}) \cdot \dot{\mathbf{r}} - B_{\text{Lorentz}}](\tilde{E}_0) = 0. \tag{11}$$

Generalized Lie structure. The brackets of the time evolution law are now of the generalized type

$$\begin{aligned} [A, B]^* &= \frac{\partial A}{\partial a^\mu} \Omega^{\mu\nu} \frac{\partial B}{\partial a^\nu} \\ &= \frac{\partial A}{\partial \mathbf{r}} \cdot \frac{\partial B}{\partial \mathbf{p}} - \frac{\partial A}{\partial \mathbf{p}} \cdot \frac{\partial B}{\partial \mathbf{r}} + e \frac{\partial A}{\partial p_i} \delta_{ij}^{mn} \frac{\partial A_m}{\partial r^n} \frac{\partial B}{\partial p_j} \end{aligned} \tag{12}$$

$$(\Omega^{\mu\nu}) = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & e(\partial A_i/\partial r^j - \partial A_j/\partial r^i) \end{pmatrix} = (\Omega_{\mu\nu})^{-1}.$$

Nevertheless, they are fully Lie in algebraic character, that is, they verify the Lie algebra laws. What is physically and mathematically significant is that the component of the Lorentz force originating from the vector potential enters directly into the algebraic structure of the approach. This feature is absent in brackets (3). Also, in the former case the local variables $\mathbf{r}$ and $\mathbf{p} = m\dot{\mathbf{r}}$ are the physical variables, while in the latter case only $\mathbf{r}$ is a physical coordinate (that is, used for the experimental detection of the system), while the canonical momentum $\mathbf{P} = m\dot{\mathbf{r}} + e\mathbf{A}$ does not coincide with the physical linear momentum. This can be equivalently expressed by saying that, when the variables $\mathbf{r}$ and $\mathbf{p}$ of the algebraic brackets of system (1) represent the Euclidean coordinates and the physical linear momentum, respectively, these variables *are not* canonically conjugate (Corollary 4.5.1c). In conclusion, the reformulation of the local variables

$$\mathbf{r} \to \mathbf{r}, \qquad \mathbf{P} = m\dot{\mathbf{r}} + e\mathbf{A} \to \mathbf{p} = m\dot{\mathbf{r}} \tag{13}$$

implies the reformulation of the algebraic tensor

$$\omega^{\mu\nu} \to \Omega^{\mu\nu}. \tag{14}$$

The local variables do not span a phase space under such a reformulation. Yet the algebraic structure remains Lie, although expressed via generalized brackets. Most importantly, the Lie algebra product itself becomes representative of the Lorentz force by acquiring a direct dynamic content.

Generalized geometric structure. System (1) admits an alternative representation as vector field

$$\tilde{\Xi}_{\text{Lorentz}} = \tilde{\Xi}^\mu(a) \frac{\partial}{\partial a^\mu} \tag{15}$$

which is now Birkhoffian (in the sense of Chart 4.5), i.e., it verifies the rule

$$\Xi_{\text{Lorentz}} \lrcorner\, \Omega_2 = -dB_{\text{Lorentz}} \tag{16}$$

with respect to the nonfundamental, symplectic, exact, two-form

$$\begin{aligned} \Omega_2 &= \Omega_{\mu\nu}(a) da^\mu \wedge da^\nu \\ &= dR_1, \qquad R_1 = R_\mu\, da^\mu. \end{aligned} \tag{17}$$

The underlying symplectic manifold is still the cotangent bundle T^*E_3 equipped with form (17). This broader geometric characterization of system (1) is, in essence, a local formulation of a global, non-Hamiltonian, approach to the electromagnetic interactions recently advocated by a number of mathematicians (Souriau (1970) and others).

A number of generalization then become possible. For instance, the replacement of E_3 with the Minkowski space $M_{3.1}$ yields a geometric characterization of the Lorentz force in special relativity. Quantization can then be performed by *geometric quantization* (e.g., via a linear associated bundle from a principal bundle). See in this latter respect, Abraham and Marsden (1967) and Sniatycki (1979). Also, the extension of the electric charge to other "charges" currently used in high-energy physics (e.g., the isospin) can be performed via the non-Abelian gauge groups. Finally, the extension to field theory remains structurally the same, although now in infinite dimension. As such, the geometrical treatment becomes considerably more delicate and technically involved.

Example 4.2

In this example we illustrate how a known Hamiltonian representation of a nonconservative system can be turned into a Birkhoffian representation. The one-dimensional, analytic, regular, non-self-adjoint Newtonian system

$$[(\ddot{r} + r)_{\text{SA}} + \gamma\dot{r}]_{\text{NSA}} = 0, \qquad m = 1, \omega^2 = 1, \tag{1}$$

describes the linear damped oscillator. A Hamiltonian representation of this system has been computed in Example I.3.2, and it is given by

$$\omega_{\mu\nu}\dot{b}^\nu - \frac{\partial H}{\partial b^\mu} = 0, \qquad \mu = 1, 2, \tag{2}$$

where

$$\begin{aligned} &\{b^\mu\} = \{r, P\}, \qquad P = e^{\gamma t}\dot{r}, \qquad H = \tfrac{1}{2}(e^{-\gamma t}P^2 + e^{+\gamma t}r^2), \\ &(\omega_{\mu\nu}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{3}$$

Even though this representation is mathematically rigorous, it is not immune from problematic physical aspects. For instance, upon application of the conventional canonical quantization techniques, the physical meaning of the expectation values of the canonical operators P and H is in doubt owing to the fact that these quantities do not coincide with conventional physical quantities already at the classical level. The joint representation of system (1) via Hamilton's and Birkhoff's equations may conceivably be of assistance in studying these issues.

The Birkhoffian representation of interest is therefore that for which the algorithms at hand have a direct physical significance, that is, 1) the local variables $\{a^\mu\} = \{r, p\}$

represent the coordinates of the experimental detection of system (1) and the physical linear momentum, $p = m\dot{r}$ ($m = 1$, for simplicity); 2) the Birkhoffian $B(a)$ represents the total physical energy, $E_{\text{tot}} = \frac{1}{2}(\dot{r}^2 + r^2)$, i.e., the total energy of the maximal associated self-adjoint sybsystem as in Corollary 4.5.1a; and 3) the symplectic tensor $\Omega_{\mu\nu}$ of Birkhoff's equations represents the nonconservative, non-self-adjoint force $F = -\gamma\dot{r}$.

Such a representation can be constructed as follows. First, assume the prescriptions $p = \dot{r}$ for the reduction to an equivalent first-order system, under which Equation (1) assumes the non-self-adjoint form

$$\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \dot{r} \\ \dot{p} \end{pmatrix} - \begin{pmatrix} r + \gamma p \\ p \end{pmatrix}\right]_{\text{NSA}} = 0. \tag{4}$$

The simplest possible matrix of self-adjoint genotopic function according to Theorem 4.4.3 is given by $(\exp\{\gamma t\}\delta^{\nu}_{\mu})$. This yields the self-adjoint normal form

$$\left[\begin{pmatrix} 0 & -e^{\gamma t} \\ e^{\gamma t} & 0 \end{pmatrix}\begin{pmatrix} \dot{r} \\ \dot{p} \end{pmatrix} - \begin{pmatrix} e^{\gamma t}(r + \gamma p) \\ e^{\gamma t} p \end{pmatrix}\right]_{\text{SA}} = 0. \tag{5}$$

The construction of a Birkhoffian representation is now straightforward. Equations (4.5.20) and (4.5.21) give the expressions

$$(R_{\mu}) = (e^{\gamma t}p, 0), \qquad B = \tfrac{1}{2}(p^2 + r^2)e^{\gamma t}. \tag{6}$$

This is not yet the desired representation because the Birkhoffian does not represent the total energy. Nevertheless, the use of the degrees of freedom (4.5.26) provides the desired result, which is

$$\{R'_{\mu}\} = \left\{e^{\gamma t}\left(p + \frac{1}{\gamma}r\right) - rt; \quad \frac{1}{\gamma}pe^{\gamma t} - pt\right\},$$

$$B' = \tfrac{1}{2}(p^2 + r^2) = \tfrac{1}{2}(\dot{r}^2 + r^2) = E_{\text{tot}} \tag{7}$$

$$R'_{\mu} = R_{\mu} + \frac{\partial G}{\partial a^{\mu}}, \qquad B' = B - \frac{\partial G}{\partial t}, \qquad G = \tfrac{1}{2}(p^2 + r^2)\left(\frac{1}{\gamma}e^{\gamma t} - t\right), \qquad \{a^{\mu}\} = \{r, p\}.$$

Notice that Birkhoff's tensor replaces the fundamental symplectic tensor with one of the simplest possible generalized form, that induced by a multiplicative function of time, i.e.,

$$\Omega_{\mu\nu} = \frac{\partial R'_{\nu}}{\partial a^{\mu}} - \frac{\partial R'_{\mu}}{\partial a^{\nu}} = e^{\gamma t}\omega_{\mu\nu}. \tag{8}$$

This mechanism allows the representation of non-conservative system (1) under the conditions that the Birkhoffian represents the total physical energy and the algorithm p represents the physical linear momentum.[68]

[68] Recovering the direct physical meaning of algorithms r and p has nontrivial implications, e.g., for a possible quantum mechanical description. For instance, it would imply the transition from the conventional fundamental commutation rules to suitable generalized form, much like the generalization of the classical rules (4.5.14) into the Birkhoffian form (4.5.15). In turn, this has far reaching implications, such as the need to generalize Heisenberg's principle. This illustrates the viewpoint expressed by Santilli (1978d) [see also Schober, Ed. (1982)] that quantum mechanics needs a suitable generalization in the transition from the arena for which it was conceived (local potential forces) to the different physical arena of the mutual wave overlapping of particles (strong interactions) and their local nonpotential approximation.

An alternative Birkhoffian representation of system (1) is given by

$$\{R''_\mu\} = \left\{\tfrac{1}{2}p + \frac{1}{\gamma}r(e^{\gamma t} - 1) - tr; -\tfrac{1}{2}r + \frac{1}{\gamma}p(1 - e^{-\gamma t})\right\} \tag{9a}$$

$$B = \tfrac{1}{2}(p^2 + r^2), \tag{9b}$$

in which case the symplectic tensor is still the fundamental one, i.e.,

$$\frac{\partial R''_\nu}{\partial a^\mu} - \frac{\partial R''_\mu}{\partial a^\nu} = \omega_{\mu\nu}, \qquad \{a^\mu\} = \{r, p\}. \tag{10}$$

Nevertheless, Birkhoffian (9b) *does not* represent the total physical energy because the algorithm "p" does not coincide with the linear momentum.

Another Birkhoffian representation can be constructed via the self-adjoint isotopic transformation of representation (2). This provides also an illustration of Proposition 4.5.1. A matrix of isotopic functions in the equivalence transformation

$$\left\{\left(\frac{\partial R'''_\nu}{\partial b^\mu} - \frac{\partial R'''_\mu}{\partial b^\nu}\right)\dot{a}^\nu - \left(\frac{\partial B'''}{\partial b^\mu} + \frac{\partial R'''_\mu}{\partial t}\right)\right\}_{\text{SA}}$$
$$\equiv \left\{(h^\rho_\mu)\left[\omega_{\rho\nu}\dot{b}^\nu - \frac{\partial H}{\partial b^\rho}\right]_{\text{SA}}\right\}_{\text{SA}}, \qquad b = (r, P) \tag{11}$$

for the case $\gamma^2 - 4 > 0$ is given by

$$(h^\rho_\mu) = (\alpha e^{-1/2t}r - e^{\alpha t})(\delta^\rho_\mu) \tag{12}$$

under the condition

$$\alpha + \frac{1}{\alpha} = \gamma. \tag{13}$$

The use, again, of Equations (4.5.20) and (4.5.21) then yields the Birkhoffian representation:

$$\begin{aligned}\{R'''_\mu\} &= \{\tfrac{1}{2}e^{\alpha t}\mathbf{P}^2; \tfrac{1}{2}\alpha e^{1/\gamma t}r^2\}\\ B''' &= \tfrac{1}{3}e^{(2\alpha+(1/\alpha))t}\mathbf{P}^3 - \tfrac{1}{2}\alpha e^{\gamma t}\mathbf{P}^2 r\\ &\quad + \tfrac{1}{2}e^{-(1/\alpha)t}\mathbf{P}r^2 - \tfrac{1}{3}\alpha e^{-(\alpha+(2/\alpha))t}r^3.\end{aligned} \tag{14}$$

Notice, however, that in this case, the Birkhoffian does not represent the total energy.

As a final remark, note that system (1) is autonomous, while all Birkhoffian representations (7), (9), and (14) depend explicitly on time. The question then arises of whether or not a Birkhoffian representation without an explicit dependence on time can be found for system (1) via Equations (4.5.35). This problem turns out to be rather involved in practice, because it calls for the solution of a parabolic, second-order partial differential equation. This case illustrates the statement of Sections 4.4 and 4.5 to the effect that Birkhoffian representations $\{R_\mu(t, a), B(a)\}$ with an explicit dependence on time for autonomous (nonconservative) Newtonian systems are, in general, easier to compute than those which do not depend explicitly on time.

Example 4.3

Consider the nonlinear, nonconservative, non-self-adjoint system in two-dimensions:

$$\left[m\begin{pmatrix}\ddot{r}_x\\ \ddot{r}_y\end{pmatrix} - k\begin{pmatrix}\dfrac{\dot{r}_x}{r_y^2}(r_x\dot{r}_y - \dot{r}_x r_y) - \dfrac{\dot{r}_x\dot{r}_y}{r_y}\\[2ex] \dfrac{\dot{r}_y}{r_x^2}(r_y\dot{r}_x - \dot{r}_y r_x) - \dfrac{\dot{r}_x\dot{r}_y}{r_x}\end{pmatrix}\right]_{\text{NSA}} = 0, \qquad \frac{k}{m} = 1. \tag{1}$$

To construct a Birkhoffian representation, we search for prescriptions (4.1.23) which are capable of yielding a symplectic tensor other than the fundamental one. The presence of the terms $(r_x\dot{r}_y - \dot{r}_x r_y)$ with opposite signs in the non-self-adjoint force suggests the study of the prescriptions

$$p_x = r_y\dot{r}_x, \qquad p_y = r_x\dot{r}_y. \tag{2}$$

Theorem 4.4.1 applies by yielding the self-adjoint general form

$$\left[\begin{pmatrix} 0 & (p_y - p_x) & -r_y & 0 \\ (p_x - p_y) & 0 & 0 & -r_x \\ r_y & 0 & 0 & 0 \\ 0 & r_x & 0 & 0 \end{pmatrix}\begin{pmatrix} \dot{r}_x \\ \dot{r}_y \\ \dot{p}_x \\ \dot{p}_y \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ p_x \\ p_y \end{pmatrix}\right]_{\text{SA}} = 0. \tag{3}$$

Theorem 4.5.1 applies, too, by yielding in this case the closed solution

$$\{R_\mu\} = \{p_x r_y, p_y r_x; 0, 0\}, \qquad B = \tfrac{1}{2}(p_x^2 + p_y^2). \tag{4}$$

Example 4.4

In this example we illustrate the technical difficulties for the practical construction of a Birkhoffian representation of Newtonian systems, even for the case of one (space) dimension.

The known *Van der Pol equation*

$$[(\ddot{r} + r)_{\text{SA}} - \varepsilon(1 - r^2)\dot{r}]_{\text{NSA}} = 0, \qquad m = 1, k = 1 \tag{1}$$

characterizes, in the language of the Inverse Problem, a non-self-adjoint extension of the one-dimensional harmonic oscillator. We are searching here for a Birkhoffian representation of such a system under the condition that the prescriptions (4.1.23) characterize a physical quantity, the linear momentum $p = \dot{r}$, ($m = 1$ for simplicity), and that the Birkhoffian represents the total physical energy, $B = \frac{1}{2}(\dot{r}^2 + r^2)$. The construction of a Birkhoffian representation is then reduced to the search for a solution in R_μ of the quasilinear system of first-order partial differential equations:

$$\begin{aligned} \left(\frac{\partial R_1}{\partial p} - \frac{\partial R_2}{\partial r}\right)[r - \varepsilon(1 - r^2)p] - r - \frac{\partial R_1}{\partial t} = 0 \\ \left(\frac{\partial R_1}{\partial p} - \frac{\partial R_2}{\partial r}\right)p - p - \frac{\partial R_2}{\partial t} = 0, \end{aligned} \tag{2}$$

which can be equivalently written

$$\begin{aligned} p[r - \varepsilon(1 - r^2)p] + [r - \varepsilon(1 - r^2)p]\frac{\partial R_2}{\partial t} = pr + p\frac{\partial R_1}{\partial t} \\ p\frac{\partial P_1}{\partial p} = p\frac{\partial R_2}{\partial z} + p + \frac{\partial R_2}{\partial t}. \end{aligned} \tag{3}$$

Thus, in terms of an arbitrary function f,

$$pR_1 = -\varepsilon(1 - r^2)p^2 t + [r - \varepsilon(1 - r^2)p]R_2 + f(q, p), \tag{4}$$

and we reduce the problem considered to the computation of a solution of the quasi-linear partial differential equation of the first-order

$$\frac{\partial R_2}{\partial t} + p\frac{\partial R_2}{\partial r} - \frac{\partial R_2}{\partial p}[r - \varepsilon(1 - r^2)p] + R_2\frac{r}{p}$$

$$-\frac{\partial f}{\partial p} + \frac{1}{p}f + p + \varepsilon(1 - r^2)pt = 0. \tag{5}$$

To illustrate the practical difficulties in computing such a solution in the desired closed form, suppose that R_2 is of the form

$$R_2 = g(q, p)t \tag{6}$$

with g an unknown function. By substituting this into Equation (5) and with some manipulation, the problem reduces to the solution of the characteristic equations

$$\frac{dr}{p} = \frac{dp}{-[r - \varepsilon(1 - r^2)p]} = \frac{dg'}{-(r/p)g'}, \qquad g' = g + p. \tag{7}$$

The point is that the solution of this latter equation is equivalent to the solution of the original system. Thus the construction of a Birkhoffian representation of system (1), under the assumptions $p = m\dot{r}$ ($m = 1$) and $B = \frac{1}{2}(\dot{r}^2 + r^2)$ and for time dependence (6) of the R_2 function, calls for a solution of the non-linear equation of motion.

Additional studies are left to the interested reader (Problems 4.5 and 4.6).

Example 4.5

In this example we illustrate the following important property (in the language of Definition 4.2.1): the condition of *strict regularity* is necessary for Birkhoff's equations to represent Newtonian systems. This will be demonstrated via given Birkhoff's equations which are *regular* and which cannot be turned into an equivalent second-order form.

Consider the autonomous covariant Birkhoff's equations

$$\Omega_{\mu\nu}(a)\dot{a}^\nu - \frac{\partial B(a)}{\partial a^\mu} = 0, \qquad \Omega_{\mu\nu} = \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}, \qquad \begin{matrix}\mu = 1, 2, 3, 4,\\ a = (\mathbf{r}, \mathbf{y})\end{matrix} \tag{1}$$

for the case

$$(R_\mu) = (\tfrac{1}{2}y_1^2, \tfrac{1}{2}y_2^2; 0, 0), \qquad B = \tfrac{1}{2}(y_1^2 + y_2^2 + r_1^2 + r_2^2). \tag{2}$$

Birkhoff's tensor is then given explicitly by

$$(\Omega_{\mu\nu}) = \begin{pmatrix} 0 & 0 & \begin{pmatrix} -y_1 & 0 \\ 0 & -y_2 \end{pmatrix} \\ 0 & 0 & \\ \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} & & 0 & 0 \\ & & 0 & 0 \end{pmatrix}, \tag{3}$$

and it is regular, i.e.,

$$\det(\Omega_{\mu\nu})(\tilde{\mathscr{R}}) = (y_1^2 y_2^2)(\tilde{\mathscr{R}}) \neq 0. \tag{4}$$

Equations (1) are therefore regular and admit the contravariant form

$$\dot{a}^\mu - \Omega^{\mu\nu}(a)\frac{\partial B(a)}{\partial a^\nu} = 0$$

$$(\Omega^{\mu\nu}) = (\Omega_{\mu\nu})^{-1} = \begin{pmatrix} 0 & 0 & \frac{1}{y_1} & 0 \\ 0 & 0 & 0 & \frac{1}{y_2} \\ -\frac{1}{y_1} & 0 & 0 & 0 \\ 0 & -\frac{1}{y_2} & 0 & 0 \end{pmatrix}. \tag{5}$$

This form violates the condition of strict regularity, Equation (4.2.12). In fact, it can be explicitly written

$$\begin{cases} \dot{r}_1 - 1 = 0; \\ \dot{r}_2 - 1 = 0; \\ \dot{y}_1 + r_1/y_1 = 0; \\ \dot{y}_2 + r_2/y_2 = 0. \end{cases} \tag{6}$$

The implicit functions of the variables y_k cannot, therefore, be constructed from the first two equations, and system (1) cannot be turned into a second-order form. It is then a bona fide *first-order* system of four equations in four variables which does not admit an equivalent second-order form in the r-variables.

Example 4.6

In Volume I we reported (see the Introduction and Chart I.3.14) the *negative* results by Douglas (1941) on the Inverse Problem, with particular reference to the proof of the existence of second-order two-dimensional systems of ordinary differential equations which *do not* admit a (first-order) Lagrangian. We also indicated that, perhaps, Douglas' results were responsible for the lack of subsequent interest on the Inverse Problem for a considerable period of time.

In this example, we would like to report the result by Hojman and Urrutia (1981) according to which the following system,

$$\ddot{x} + \dot{y} = 0, \qquad \ddot{y} + y = 0, \tag{1}$$

admits a Birkhoffian representation (in our language), while the system is essentially non-self-adjoint according to Douglas' proof and thus does not admit a Lagrangian representation.

The construction of a Birkhoffian representation is, in this case, rather simple because the system is linear, therefore admitting an easily computable solution. Once a solution is known, it can be turned into first integrals. The Birkhoffian representation, in turn, can be computed from the first integrals via Method 3 of Corollary 4.5.1d.

System (1) admits the equivalent first-order form

$$\begin{aligned} &\dot{a}^\mu = \Xi^\mu(a), \qquad a^1 = x, \qquad a^2 = y, \qquad a^3 = \dot{x}, \qquad a^4 = \dot{y}, \\ &\Xi^1 = a^3, \qquad \Xi^2 = a^4, \qquad \Xi^3 = -a^4, \qquad \Xi^4 = -a^2, \end{aligned} \tag{2}$$

which can be easily proved non-Hamiltonian. It is equally easy to see that the vector field is Birkhoffian. In fact, the general solution is given by

$$\begin{aligned} a^1 &= -c_1 \sin t + c_2 \cos t + c_3 t + c_4, \\ a^2 &= c_1 \cos t + c_2 \sin t, \\ a^3 &= -c_1 \cos t - c_2 \sin t + c_3, \\ a^4 &= -c_1 \sin t + c_2 \cos t. \end{aligned} \tag{3}$$

The use of the theorem on implicit functions permits the computation of the c–constants in terms of the t and a variables, which assume the meaning of first integrals, i.e.,

$$\begin{aligned} I_1 &= a^2 \cos t - a^4 \sin t, \\ I_2 &= a^2 \sin t + a^4 \cos t, \\ I_3 &= a^2 + a^3, \\ I_4 &= a^1 - a^4 - (a^2 + a^3)t. \end{aligned} \tag{4}$$

Equations (4.5.22) and (4.5.23) then yield the following Birkhoffian representation via simple manipulations:

$$\begin{gathered} R_1 = a^2 + a^3, \qquad R_2 = 0, \qquad R_3 = a^4, \qquad R_4 = 0, \\ B = \tfrac{1}{2}[(a^3)^2 + 2a^2a^3 - (a^4)^2]. \end{gathered} \tag{5}$$

The first-order Pfaffian action is then given by

$$\mathscr{A} = \int dt\{(a^2 + a^3)\dot{a}^1 + a^4\dot{a}^3 - \tfrac{1}{2}[(a^3)^2 + 2a^2a^3 - (a^4)^2]\}. \tag{6}$$

Douglas' result can now be easily illustrated. In fact, the transformation of the integrand of action (6) to the original variables (x, y) yields a *second-order* action.

Example 4.7

The two-dimensional system

$$\begin{aligned} \ddot{x} - \tfrac{1}{4}y &= 0, \\ \ddot{y} + \tfrac{4}{3}y - \tfrac{1}{3}x &= 0, \end{aligned} \tag{1}$$

can be proved essentially non-self-adjoint. Their Birkhoffian representation has been computed by Hojman and Urrutia (1981) also via the method of Example 4.6, and it is given by

$$\begin{aligned} \mathscr{A} = \int dt\{ & e^t[(6a^4 - a^2)\dot{a}^1 + (2\dot{a}^3 + 12a^4 - 3a^1)\dot{a}^2 \\ & + (18a^4 - 4a^4)\dot{a}^3 + (6a^3 + 3a^2)\dot{a}^4 + (a^1)^2 + \tfrac{1}{4}(a^2)^2 \\ & - 2(a^3)^2 - \tfrac{9}{2}(a^4)^2] + e^{t/3}[(3a^2 + 2a^4)\dot{a}^1 \\ & + (a^1 + 6a^3 - 4a^4)\dot{a}^2 + (4a^1 - 2a^4)\dot{a}^3 \\ & - (6a^3 - 3a^2)\dot{a}^4 + \tfrac{1}{3}(a^1)^2 + \tfrac{3}{4}(a^2)^2 + 2(a^3)^2 + \tfrac{1}{2}(a^4)^2]\}, \\ & a^1 = x, \qquad a^2 = y, \qquad a^3 = \dot{x}, \qquad a^4 = \dot{y}. \end{aligned} \tag{2}$$

Note that system (1) of this example is *linear* while Equation (1) of Example 4.4 is *nonlinear*. The computation of the general solution is therefore readily achievable in the first case but is a rather complex undertaking in the second. In turn, the capability of computing a Birkhoffian representation is possible in the first case but is difficult to achieve in the second case.

Problems

4.1 Prove that the contravariant Birkhoff's tensor defined via Equations (4.2.10) verifies integrability conditions (4.1.48) for the characterization of Lie brackets.

4.2 Prove that Birkhoff's equations (4.2.1) verify *all* conditions (4.1.32) of variational self-adjointness. In particular, prove that the solution of Conditions (4.1.32a) and (4.1.32b) can always be cast into a curl structure like Birkhoff's tensor (4.2.4).

4.3 Extend the proof of Theorem 4.4.3 from particular case (4.4.10) to general case (4.4.7).

4.4 Prove Proposition 4.4.1.

4.5 Consider *Duffing's equation*,

$$[(\ddot{r} + r)_{SA} + wr^3]_{SA} = 0, \qquad w \approx 0,$$

and search for an *approximate Birkhoffian representation* as follows. Assume for B the total energy of the unperturbed oscillator, $B = \frac{1}{2}(\dot{r}^2 + r^2)$, and search for a solution in the R functions via the multiple power-series expansion in the parameter w

$$R_\mu = R_\mu^0 + wR_\mu^1 + w^2R_\mu^2 + \cdots.$$

Prove that a solution for R_μ^0 is $(p, 0)$, $p = \dot{r}$, and compute a solution for R_μ^1. Compute the equations of motion characterized by the approximate solution $R_\mu = R_\mu^0 + wR_\mu^1$, and elaborate on the corresponding approximate character of the representation.

4.6 Prove that Duffing's equation from the preceding problem admits the (exact) Birkhoffian representation

$$B = \tfrac{1}{2}y^2(y^2 + r^2)$$

$$R = \left[-\frac{1}{y} + y(\tfrac{1}{2}y^2 + \tfrac{1}{2}r^2 + \tfrac{1}{4}wr^4); \qquad \frac{r}{y^2} - (\tfrac{1}{2}y^2r + \tfrac{1}{6}r^3 + \tfrac{1}{20}wr^5) \right].$$

4.7 The following Kepler system in a dissipative medium with nonlinear damping term

$$\left[\left(\ddot{r} + \frac{1}{r^2} \right)_{SA} - \frac{\dot{r}^2}{r} \right]_{NSA} = 0$$

admits the Birkhoffian representation

$$B = \tfrac{1}{2}p^2 + r,$$

$$\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu} \right) = \begin{pmatrix} 0 & -\dfrac{1}{r} \\ \dfrac{1}{r} & 0 \end{pmatrix}.$$

Since the system is in one dimension, a Hamiltonian exists (Corollary A.1.1a). Prove that the Birkhoffian representation above cannot be factored into a Hamiltonian form according to rule (4.5.36). Compute a Hamiltonian for the system and an isotopic transformation of Hamilton's equations, with corresponding Birkhoffian form, by therefore illustrating that rule (4.5.36), when properly treated, does indeed hold.

4.8 Prove that all *self-adjoint symmetry breakings* of Chart A.12 i.e.,

$$\left[\left(\omega_{\mu\nu}\dot{a}^{\nu} - \frac{\partial H}{\partial a^{\mu}}\right)^{\mathrm{ES}}_{\mathrm{SA}} - F_{\mu}\right]^{\mathrm{BS}}_{\mathrm{SA}} = 0, \qquad F_{\mu} = \frac{\partial H^{\mathrm{BS}}}{\partial a^{\mu}}$$

can be represented via the Birkhoffian gauge (Corollary 4.5.1e):

$$R_{\mu} = R^{0}_{\mu} + \frac{\partial G}{\partial a^{\mu}}, \qquad B = H - \frac{\partial G}{\partial t}$$

$$R^{0} = (\mathbf{p}, \mathbf{0}), \qquad G = \int H^{\mathrm{BS}} dt.$$

4.9 Identify the foundations of the *second-order Lagrangian mechanics* with particular reference to (A) the study of the possibility that the Inverse Lagrangian Problem is directly universal, as is expected from the direct universality of Birkhoff's equations and Lagrangian images of types (4.2.35) and (4.2.36). In particular, work out the methods for the computation of a Lagrangian from the equations of motion, as well as a few representations of known systems (e.g., the Kepler problem). (B) Work out the transformation theory, as well as the theory of symmetries and first integrals, including the reformulation of Norther's theorem. In particular, see whether the isotopic transformations of second-order Lagrangians coincide with those of Birkhoffian representations. (C) Identify the generalization of the Legendre transform for second-order Lagrangians which leads to Birkhoff's equations, that is, which preserves the Lie and symplectic character of Hamilton's equations.

CHAPTER 5

Transformation Theory of Birkhoff's Equations

5.1 Statement of the Problem

As is now familiar, an objective of this volume is to establish methodological foundations for the treatment of the most general known class of local interactions, those of the variational non-self-adjoint type. The interactions can be essentially reduced to a superposition of action-at-a-distance, potential forces F^{SA}, and contact forces F^{NSA} for which the notion of potential energy is inapplicable, according to the systems

$$m_a \ddot{r}_{ka} - f^{\mathrm{SA}}_{ka}(t, \mathbf{r}, \dot{\mathbf{r}}) - F^{\mathrm{NSA}}_{ka}(t, \mathbf{r}, \dot{\mathbf{r}}) = 0, \qquad (5.1.1)$$

$$a = 1, 2, \ldots, N, \qquad k = x, y, z.$$

In Chapter 4, we established the *insufficiency* of conventional (Lagrangians and) Hamiltonian formulations for the treatment of the systems considered, because of their lack of direct universality, that is, their general inability to provide a description in the coordinate and time variables of the observer.

We therefore reduced the systems to an equivalent first-order form of the type

$$\dot{a}^{\mu} = \Xi^{\mu}(t, a), \quad (a^{\mu}) = \begin{pmatrix} \mathbf{r} \\ \mathbf{p} \end{pmatrix}, \qquad (\Xi^{\mu}) = \begin{pmatrix} \mathbf{p}/m \\ f^{\mathrm{SA}} + F^{\mathrm{NSA}} \end{pmatrix}, \qquad (5.1.2)$$

$$\mu = 1, 2, \ldots, 2n = 6N,$$

and established the direct universality of Birkhoff's equations

$$\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \left(\frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t}\right) = 0. \tag{5.1.3}$$

In the same Chapter 4 we also established that the universality of Equations (5.1.3) implies that of the Pfaffian action principle (analytic profile)

$$\delta \int_{t_1}^{t_2} dt[R_\mu(t, a)\dot{a}^\mu - B(t, a)](\tilde{E}_0) = 0; \tag{5.1.4}$$

the Lie algebras realized via the most general possible regular product (algebraic profile)

$$[A, B]^* = \frac{\partial A}{\partial a^\mu}\Omega^{\mu\nu}\frac{\partial B}{\partial a^\nu}, \qquad \Omega^{\mu\nu} = \left(\left\| \frac{\partial R_\beta}{\partial a^\alpha} - \frac{\partial R_\alpha}{\partial a^\beta} \right\|^{-1}\right)^{\mu\nu}; \tag{5.1.5}$$

and the contact geometry realized via the most general possible, exact, contact two-form on $\mathbb{R} \times T^*M$ (geometric profile)

$$\hat{\Omega}_2 = d(\hat{R}_\nu(\hat{a})d\hat{a}^\nu) = \frac{1}{2}\left(\frac{\partial \hat{R}_\nu}{\partial \hat{a}^\mu} - \frac{\partial \hat{R}_\mu}{\partial \hat{a}^\nu}\right)d\hat{a}^\mu \wedge d\hat{a}^\nu, \tag{5.1.6}$$

$$(\hat{a}^\mu) = \begin{pmatrix} t \\ a^\mu \end{pmatrix}, \qquad (\hat{R}_\nu) = (-B, R^\nu), \qquad \mu, \nu = 0, 1, 2, \ldots, 2n.$$

In turn, these results established the applicability of rigorous analytic, algebraic, and geometric methods for the treatment of systems (5.1.1) in the reference frame of the observer.

In this chapter we study the transformation theory of Birkhoff's equations. An objective is to establish that the derivability of the systems considered from a Pfaffian principle, their Lie algebraic character, and their contact geometric structure are independent of the selected reference frame (that is, they persist under the most general possible (but smoothness- and regularity-preserving) transformations of the local variables). The frame independence of the primitive analytic, algebraic, and geometric characteristics then clears the way for coordinate-free globalizations.

The single most important aspect of this chapter is that the transformation theory of local non-self-adjoint interactions in general and that of Birkhoff's equations in particular is *noncanonical*. This notion originates at the dynamic foundations of the theory, via the property that the time evolution of the systems

$$a^\mu(t) = e^{t\Xi^\alpha \partial/\partial a^\alpha} a^\mu(0) \tag{5.1.7}$$

does not preserve the conventional fundamental Poisson brackets. It is then confirmed by the noncanonical character of the transformations preserving the Birkhoffian form of the equations of motion. Finally, the same notion reemerges in a number of diversified aspects.

By keeping in mind that contemporary theoretical physics has been mainly patterned (classically and quantum mechanically) along the theory of canonical transformations, the noncanonical character of the Birkhoffian transformation theory has a number of fundamental implications. For example, it implies the need for:

1. a generalization of virtually all methodological tools of Hamiltonian mechanics;
2. a generalization of conventional formulations of Lie's theory;
3. a generalization of Galilei's relativity.

At the quantum mechanical level, the implications are equally fundamental. In fact, the noncanonical character of the classical transformation theory is sufficient, per se, to render inevitable a generalization of quantum mechanics for the treatment of nonpotential interactions, such as those which are possible for one (particle) wave packet under conditions of penetration within other wave packets. Indeed, for evident consistency, the noncanonical character of time evolution (5.1.7) must result in the nonunitary character of the corresponding "quantum mechanical"[1] description. In turn, this demands the construction of a new theory which is form-invariant under nonunitary transformations, in the same way as Birkhoffian mechanics is form-invariant under noncanonical transformations.

Needless to say, the problem of generalizing quantum mechanics goes beyond the objectives of this volume. We therefore limit ourselves to the indication of the algebraic notions which are expected to be common to both the macroscopic and the microscopic descriptions and refer the interested reader to the specialized literature on the subject for technical details.

The three classes of transformations we consider are the following:

(*I*) *contemporaneous transformations on* T^*M, i.e.,

$$t \to t' \equiv t, \qquad a^\mu \to a'^\mu(a), \qquad \mu = 1, 2, \ldots, 2n; \tag{5.1.8}$$

(*II*) *contemporaneous transformations on* $\mathbb{R} \times T^*M$, i.e.;

$$t \to t' \equiv t, \qquad a^\mu \to a'^\mu(t, a), \qquad \mu = 1, 2, \ldots, 2n; \tag{5.1.9}$$

and

(*III*) *noncontemporaneous transformations on* $\mathbb{R} \times T^*M$, i.e.,

$$t \to t'(t, a), \qquad a^\mu \to a'^\mu(t, a), \qquad \mu = 1, 2, \ldots, 2n \tag{5.1.10}$$

or, in the unified notation of Equations (4.1.51),

$$\hat{a}^\mu \to \hat{a}'^\mu(\hat{a}), \qquad \mu = 0, 1, 2, \ldots, 2n. \tag{5.1.11}$$

[1] The apparent departures from conventional ideas which are implied by the contact effects due to mutual penetration of particles are so deep as to render questionable the same terms "quantum mechanics." For these reasons Santilli (1978d) proposed the name *Atomic Mechanics* for the current mechanics (that is, the mechanics for the structure of atoms), and the name *Hadronic Mechanics* for the new mechanics (that is, the mechanics for the structure of hadrons as well as, more generally, closed systems under strong internal forces).

All transformations considered will be *analytic* in their region of definition, i.e., the new variables $a'(a)$ or $\hat{a}'(\hat{a})$ are analytic functions of the old variables. However, recall that this is due to the existence theory of partial differential equations used in the analysis and that the transformation theory can be consistently formulated and often applied under weaker smoothness conditions (e.g., $a' \in \mathscr{C}^2$).

All regions of definition of the transformations, usually denoted with the symbol $\tilde{\mathscr{R}}$ ($\hat{\mathscr{R}}$) for transformations on T^*M ($\mathbb{R} \times T^*M$), will be tacitly assumed to be *star-shaped* or to satisfy topologically equivalent conditions (e.g., deformability to a curve, as indicated at the end of Chart 4.6). As now familiar, this is due to the need to apply the converse of the Poincaré lemma, in order to ensure the existence of a Birkhoffian representation of the systems considered. Nevertheless, the transformation theory can be formulated and applied also in regions verifying weaker topological properties.

Finally, all transformations considered will be assumed to be *regular*, that is, their Jacobian is non-null as a function, e.g.,

$$J(a) = \det\left(\frac{\partial a'^{\mu}}{\partial a^{\nu}}\right)(\tilde{\mathscr{R}}) \neq 0. \tag{5.1.12}$$

In particular, we shall tacitly assume that all points in whose neighborhood the transformations are considered are not isolated zeros of the Jacobian, that is, they are not solutions of the *equation* $J(a) = 0$. As a consequence, all transformations considered are *invertible* in their regions of definition, i.e., whenever transformations (5.1.8), (5.1.9), and (5.1.10) are assigned, their corresponding inverses

$$t' \to t \equiv t', \qquad a'^{\mu} \to a^{\mu}(a') \tag{5.1.13a}$$

$$t' \to t \equiv t', \qquad a'^{\mu} \to a^{\mu}(t, a') \tag{5.1.13b}$$

$$t' \to t(t', a'), \qquad a'^{\mu} \to a^{\mu}(t', a') \tag{5.1.13c}$$

always exist.

For the reader's convenience, as well as for notational and subsequent reference needs, we begin our analysis with a review of the theory of canonical transformations (Section 5.2). The transformation theory of Birkhoff's equations will then be constructed (Section 5.3) as a step-by-step generalization of that of Hamilton's equations. Our subsequent analysis will be devoted to a number of related aspects, such as the underlying formulation of Lie's theory.

The analysis will be primarily conducted for *essentially non-self-adjoint systems*, namely (Definition 4.1.1), systems which do not admit a Hamiltonian representation in the coordinate and time variables of the observer and for which the need of Birkhoff's equations is more transparent. Nevertheless, we shall not exclude the class of nonconservative systems admitting a Hamiltonian in the variables indicated (which are called *non-essentially non-self-adjoint systems*, also from Definition 4.1.1). The reader should keep in mind that the Inverse Hamiltonian (or Lagrangian) Problem has a particular methodological function for these systems. In fact, the knowledge of a

Hamiltonian permits the use of the entire body of methods of the canonical transformation theory, which would be otherwise precluded.[2]

In the final analysis, this is the spirit of these monographs: rather than assuming a given methodological setting and restricting the dynamics to simplified, compatible forms, we prefer to consider unrestricted dynamic conditions as the foundations of the theory (classically and quantum mechanically), and then seek compatible methodological tools. However, in doing so, the researcher should be prepared to abandon some familiar fundamental notions of contemporary physics and search for suitable generalizations.[3]

5.2 Transformation Theory of Hamilton's Equations

One of the most salient properties of canonical transformations is that of preserving the *structure* of Hamilton's equations, i.e.,

$$\omega_{\mu\nu}\dot{a}^{\nu} - \frac{\partial H}{\partial a^{\mu}} = 0 \rightarrow \omega_{\mu\nu}\dot{a}'^{\nu} - \frac{\partial H'}{\partial a'^{\mu}} = 0. \tag{5.2.1}$$

As a matter of fact, this can be assumed as one of the possible definitions of canonical transformations.

A deeper study reveals that canonical transformations preserve the form of Hamilton's equations for *all* possible Hamiltonians. This suggests the definition of canonical transformations without any reference to Hamilton's equations and by using only the fundamental algebraic tensor $\omega^{\mu\nu}$ or, equivalently, its geometric counterpart $\omega_{\mu\nu}$. By recalling that these tensors transform according to the general rules for contravariant and covariant tensors, respectively (Chart I.A.13), we have the following definition.

Definition 5.2.1.[4] Contemporaneous transformations (5.1.8) are called *canonical* when they preserve the value of the fundamental Lie tensor, i.e.,

$$\omega^{\mu\nu} \rightarrow \Omega'^{\mu\nu} = \frac{\partial a'^{\mu}}{\partial a^{\rho}}\,\omega^{\rho\sigma}\,\frac{\partial a'^{\nu}}{\partial a^{\sigma}} \equiv \omega^{\mu\nu} \tag{5.2.2}$$

or, equivalently, of the fundamental symplectic tensor, i.e.,

$$\omega_{\mu\nu} \rightarrow \Omega'_{\mu\nu} = \frac{\partial a^{\rho}}{\partial a'^{\mu}}\,\omega_{\rho\sigma}\,\frac{\partial a^{\sigma}}{\partial a'^{\nu}} \equiv \omega_{\mu\nu}. \tag{5.2.3}$$

[2] As an example, lacking the use of the Inverse Hamiltonian Problem, the only possible treatment of the spinning top via the Hamilton–Jacobi equations is that under the perpetual-motion approximation of conserved angular momentum.

[3] Heisenberg's vivid and inspiring words, stated in his memoir (1971, page 70), come to mind here: "*In science, ... it is impossible to open up new territory unless one is prepared to leave the safe anchorage of established doctrines and run the risk of a hazardous leap forward.*" To this he added soon thereafter: "*However, when it comes to entering new territory, the very structure of scientific thought may have to be changed, and that is far more than most men are prepared to do.*"

[4] A considerable variety of definitions of canonical transformations exists in the literature. Some of them are given in Chart 5.6 along with a number of references. The reader should be aware that they are not all equivalent.

The equivalence of Definitions (5.2.2) and (5.2.3) is easily seen from the properties

$$(\omega^{\mu\nu}) = (\omega_{\mu\nu})^{-1}, \tag{5.2.4a}$$

$$\left(\frac{\partial a'^{\mu}}{\partial a^{\rho}}\,\omega^{\rho\sigma}\,\frac{\partial a'^{\nu}}{\partial a^{\sigma}}\right) = \left(\frac{\partial a^{\rho}}{\partial a'^{\mu}}\,\omega_{\rho\sigma}\,\frac{\partial a^{\sigma}}{\partial a'^{\nu}}\right)^{-1}. \tag{5.2.4b}$$

Thus whenever one of the two conditions is verified, the other follows.

Definition 5.2.1 implies the preservation of the conventional Poisson brackets, i.e.,

$$[A, B]_{(a)} = \frac{\partial A}{\partial a^{\mu}}\,\omega^{\mu\nu}\,\frac{\partial B}{\partial a^{\nu}} = \frac{\partial A'}{\partial a'^{\mu}}\,\omega^{\mu\nu}\,\frac{\partial B'}{\partial a'^{\nu}} = [A', B']_{(a')}, \tag{5.2.5a}$$

$$A'(a') = A(a(a')), \qquad B'(a') = B(a(a')). \tag{5.2.5b}$$

In fact, the equations above can be assumed as (necessary and sufficient) conditions for a transformation to be canonical. In particular, property (5.2.5a) implies that a time evolution which is Hamiltonian in one reference frame remains Hamiltonian under all possible *canonical* transformations.

Recall from Section I.2.9 (see also Equations (4.5.14)) that the fundamental Lie tensor represents in a unified way all fundamental Poisson brackets. Thus Definition 5.2.1 is based on the preservation of these brackets in the transition from the old to the new variables, and we can write

$$[a^{\mu}, a^{\nu}]_{(a)} = [a'^{\mu}, a'^{\nu}]_{(a)} = \omega^{\mu\nu}, \qquad \mu, \nu = 1, 2, \ldots, 2n. \tag{5.2.6}$$

Another implication of Definition 5.2.1 is the preservation of the conventional Lagrange's brackets, i.e.,

$$\{A, B\}_{(a)} = \frac{\partial a^{\mu}}{\partial A}\,\omega_{\mu\nu}\,\frac{\partial a^{\nu}}{\partial B} = \frac{\partial a'^{\mu}}{\partial A'}\,\omega_{\mu\nu}\,\frac{\partial a'^{\nu}}{\partial B'} = \{A', B'\}_{(a')}, \tag{5.2.7}$$

and this can be assumed as yet another definition of canonical transformations.

Similarly, by recalling that the fundamental symplectic tensor $\omega_{\mu\nu}$ represents in a unified way all fundamental Lagrange's brackets, Definition 5.2.1 is based on the preservation of these brackets, and we can write

$$\{a^{\mu}, a^{\nu}\}_{(a)} = \{a'^{\mu}, a'^{\nu}\}_{(a)} = \omega_{\mu\nu}, \qquad \mu, \nu = 1, 2, \ldots, 2n. \tag{5.2.8}$$

These properties imply the following *transformation rule of Hamilton's equations under canonical transformations without an explicit time dependence*

$$\left(\omega_{\mu\nu}\dot{a}^{\nu} - \frac{\partial H}{\partial a^{\mu}}\right)_{\mathrm{SA}} = \frac{\partial a'^{\rho}}{\partial a^{\mu}}\left(\omega_{\rho\sigma}\dot{a}'^{\sigma} - \frac{\partial H'}{\partial a'^{\rho}}\right)_{\mathrm{SA}} = 0, \tag{5.2.9a}$$

$$\omega_{\mu\nu} \to \Omega'_{\mu\nu} = \frac{\partial a^{\rho}}{\partial a'^{\mu}}\,\omega_{\rho\sigma}\,\frac{\partial a^{\sigma}}{\partial a'^{\nu}} \equiv \omega_{\mu\nu}, \tag{5.2.9b}$$

$$H(t, a) \to H'(t, a') = H(t, a(a')). \tag{5.2.9c}$$

The self-adjointness of Hamilton's equations in both the old and new variables is a consequence of Theorem I.3.10.1 (see Theorem 4.1.3 for a review). The non-self-adjointness of the right-hand side is a consequence of the arbitrariness of the functional dependence of the new variables in the old, as the reader can verify through conditions (4.1.32).

Note that scalar rule (5.2.9c) *does not* apply when the transformations depend explicitly on time, as we shall soon see.

We now move to the study of more general transformations (5.1.9) which are still contemporaneous, yet possess an explicit dependence on time. For this purpose, we assume a *definition of canonical transformation which is different than that of Equations* (5.2.2) *and* (5.2.3).

Definition 5.2.2.[4] Contemporaneous time-dependent transformations (5.1.9) are called *canonical* when they preserve Hamilton's principle in the transition from the old phase space variables

$$\delta \int_{t_1}^{t_2} dt[p_k \dot{q}^k - H(t, q, p)](\tilde{E}_0) = 0, \tag{5.2.10}$$

to the new variables

$$\delta \int_{t_1}^{t_2} dt[p'_k \dot{q}'^k - H'(t, q', p')](\tilde{E}'_0) = 0. \tag{5.2.11}$$

Definition 5.2.2 is broader than Definition 5.2.1 in that the former admits the latter as a particular case and, in addition, permits transformations such as the *dilations*,

$$q^k \to q'^k = e^c q^k, \quad p_k \to p'_k = e^{-c} p_k, \tag{5.2.12}$$

and the *reciprocity transformations*,

$$(q, p) \to (q', p') = (p, q), \tag{5.2.13}$$

which are canonical for Definition 5.2.2 but not for Definition 5.2.1.[5] This is a good illustration of the subtle differences between the geometric approach (Definition 5.2.1) and the analytic approach (Definition 5.2.2).[6]

Even though variations (5.2.10) and (5.2.11) are individually null, the difference between their integrand is not null. Nevertheless, such a difference can at most equal the total differential of a function $F(t, q, p, q', p')$ that is analytic in all its variables (under our general smoothness conditions). In this way we reach the following fundamental identity:

$$p_k \dot{q}^k - H(t, q, p) - p'_k \dot{q}'^k + H'(t, q', p') = \dot{F}(t, q, p, q', p'), \tag{5.2.14}$$

[5] The reader can see now the differences between the definitions of canonical transformations of Chart 5.6.

[6] Transformations of type (5.2.12) and (5.2.13) are fully acceptable on analytic grounds. Yet on geometric grounds, they imply a *change of the fundamental symplectic structure*. As such, they have nontrivial technical implications.

that provides the means for the explicit construction of canonical transformations.

The function F, called a *generating function*, generally depends on the $4n + 1$ variables (t, q, p, q', p').[7] However, only $2n + 1$ of them can be independent, owing to transformations (5.1.9). These $2n + 1$ independent variables can be arbitrarily selected via any $2n$-dimensional subset of the variables (q, p, q', p') and time. Thus many different cases of generating functions are possible. The most significant ones are the following six.[8,9]

Case 1: $F = F_1(t, q, q')$. Indentity (5.2.14) in this case reads

$$p_k \dot{q}^k - H - p'_k \dot{q}'^k + H' = \frac{\partial F_1}{\partial q^k} \dot{q}^k + \frac{\partial F_1}{\partial q'^k} \dot{q}'^k + \frac{\partial F_1}{\partial t}, \tag{5.2.15}$$

yielding the transformation laws

$$p_k = \frac{\partial F_1}{\partial q^k}, \qquad p'_k = -\frac{\partial F_1}{\partial q'^k}, \tag{5.2.16a}$$

$$H' = H + \frac{\partial F_1}{\partial t}. \tag{5.2.16b}$$

Case 2: $F = F_2(t, q, p')$. The use of the Legendre transform reduces F_2 to F_1 (Problem 5.2)

$$F_2 = F_1 + p'_k q'^k, \tag{5.2.17}$$

resulting in the new transformation laws

$$p_k = \frac{\partial F_2}{\partial q^k}, \qquad q'^k = \frac{\partial F_2}{\partial p'_k}, \tag{5.2.18a}$$

$$H' = H + \frac{\partial F_2}{\partial t}. \tag{5.2.18b}$$

Case 3: $F = F_3(t, q', p)$. The reduction via a Legendre transform

$$F_3 = F_1 - p_k q^k, \tag{5.2.19}$$

yields the transformation laws

$$q^k = -\frac{\partial F_3}{\partial p_k}, \qquad p'_k = -\frac{\partial F_3}{\partial q'^k}, \tag{5.2.20a}$$

$$H' = H + \frac{\partial F_3}{\partial t}. \tag{5.2.20b}$$

[7] Note that one (necessary and sufficient) condition for a canonical transformation to depend explicitly on time is that the generating function exhibits such a functional dependence. This dependence, however, *does not* imply that the transformation is noncontemporaneous (that is, time is also transformed. This occurrence implies that *Definitions* 5.2.1 *and* 5.2.2, *as well as all Definitions reviewed in Chart* 5.6 *do not incorporate the full Galilei's transformations.* Nevertheless, the definitions can be enlarged into $\mathbb{R} \times T^*M$) to include Galilei's transformations (see Chart 5.6).

[8] The existing literature generally presents only Cases 1–4.

[9] The more general construction of canonical transformations via Holder's principle is left as an exercise for the interested reader (Problem 5.1).

Case 4: $F = F_4(t, p, p')$. The reduction

$$F_4 = F_1 - p_k q^k + p'_k q'^k, \tag{5.2.21}$$

characterizes the laws

$$q^k = -\frac{\partial F_4}{\partial p_k}, \qquad q'^k = \frac{\partial F_4}{\partial p'_k}, \tag{5.2.22a}$$

$$H' = H + \frac{\partial F_4}{\partial t}. \tag{5.2.22b}$$

Case 5: $F = F_5(t, q', p')$. In this case identity (5.2.14) yields

$$p_i\left(\frac{\partial q^i}{\partial q'^k}\dot{q}'^k + \frac{\partial q^i}{\partial p'_k}\dot{p}'^k\right) + p_i\frac{\partial q_i}{\partial t} - H - p'_k q'^k + H' = \frac{\partial F_5}{\partial q'^k}\dot{q}'^k + \frac{\partial F_5}{\partial p'_k}\dot{p}'^k + \frac{\partial F_5}{\partial t}, \tag{5.2.23}$$

by characterizing the transformation laws[10]

$$p'_k - p_i\frac{\partial q^i}{\partial q'^k} = -\frac{\partial F_5}{\partial q'^k}, \qquad p_i\frac{\partial q^i}{\partial p'_k} = \frac{\partial F_5}{\partial p'_k}, \tag{5.2.24a}$$

$$H' = H + \frac{\partial F_5}{\partial t} - p_i\frac{\partial q^i}{\partial t}. \tag{5.2.24b}$$

Case 6: $F = F_6(t, q, p)$. In this case we have

$$p_k\dot{q}^k - H - p'_i\left(\frac{\partial q'^i}{\partial q^k}\dot{q}^k + \frac{\partial q'^i}{\partial p^k}\dot{p}^k\right) - p'_i\frac{\partial q'^i}{\partial t} + H' = \frac{\partial F_6}{\partial q^k}\dot{q}^k + \frac{\partial F_6}{\partial p_k}\dot{p}_k + \frac{\partial F_6}{\partial t}, \tag{5.2.25}$$

with the corresponding transformation laws

$$p_k - p'_i\frac{\partial q'^i}{\partial q^k} = \frac{\partial F_6}{\partial q^k}, \qquad p'_i\frac{\partial q'^i}{\partial p_k} = -\frac{\partial F_6}{\partial p_k} \tag{5.2.26a}$$

$$H' = H + \frac{\partial F_6}{\partial t} + p_i\frac{\partial q^i}{\partial t}. \tag{5.2.26b}$$

The use of the transformation laws given above is twofold. First, it is possible to assign a generating function F to any of the classes outlined. The corresponding canonical transformation can be then computed via the application of the theorem on implicit functions (Theorem I.1.1.1) to the transformation laws of the class considered. This is due to the fact that, for

[10] Notice the appearance of new rules for the transformation of the Hamiltonian. In fact, we have scalar rule (5.2.9c) under contemporaneous transformations (5.1.8); we have the more general rules (5.2.16b) and (5.2.24b) under the more general, but still contemporaneous transformations (5.1.9); and, as we shall see in the next section, we have still more general rules for non-contemporaneous transformations (5.1.10).

instance, transformation (5.2.16a) contains the complete sets of transformations $q'(t, q, p)$ and $p'(t, q, p)$ only implicitly.

The second use of the transformation laws given above is the opposite of the preceding one. In certain instances, a canonical transformation is assigned, and the knowledge of the corresponding generating function is requested. In principle, such a generating function can be computed via the use of any of the cases above. A solution is given by reversing the procedure for the construction of a canonical transformation via a generating function, according to the following steps:

(a) select a type of generating function to be computed (e.g., F_1);
(b) turn the given canonical functions $q'(t, q, p)$ and $p'(t, q, p)$ into the corresponding form (e.g., for F_1, one must write $p(t, q, q')$ and $p'(t, q, q')$); and
(c) solve the corresponding transformation laws, now interpreted as partial differential equations in the unknown generating function.

Within such a context, the integrability conditions for the existence of a generating function are relevant. The now familiar application of the converse of the Poincaré lemma yields the following *integrability conditions for the existence of a generating function.*

Case 1:
$$\frac{\partial p_i}{\partial q'^j} = -\frac{\partial p'_j}{\partial q^i}. \tag{5.2.27a}$$

Case 2:
$$\frac{\partial p_i}{\partial p'_j} = \frac{\partial q'^j}{\partial q^i}. \tag{5.2.27b}$$

Case 3:
$$\frac{\partial q^i}{\partial q'^j} = \frac{\partial p'_j}{\partial p_i}. \tag{5.2.27c}$$

Case 4:
$$\frac{\partial q^i}{\partial p'_j} = -\frac{\partial q'^j}{\partial p_i}. \tag{5.2.27d}$$

Case 5:
$$\{q'^i, q'^j\}_{(q,p)} = \{p'_i, p'_j\}_{(q,p)} = 0, \qquad \{q'^i, p'_j\}_{(q,p)} = \delta^i_j. \tag{5.2.27e}$$

Case 6:
$$\{q^i, q^j\}_{(q',p')} = \{p_i, p_j\}_{(q',p')} = 0, \qquad \{q^i, p_j\}_{(q',p')} = -\delta^i_j. \tag{5.2.27f}$$

It is possible to prove that these conditions are automatically verified by canonical transformations as per Definition 5.2.2[11] (Problem 5.3).

Equations (5.2.27a)–(5.2.27d) are better known in the existing physical literature as *inversion formulae* (see, for instance, Pars (1965)). Indeed,

[11] This is a remarkable property inasmuch as the conditions constitute an *overdetermined* system of partial differential equations, that is, a type of system whose consistency study is, in general, rather complex. The remarkable point is that the preservation of the fundamental Lie or symplectic tensor or, more generally, of a variational principle, readily provide the integrability conditions of these difficult systems.

these equations produce the conversion of the fundamental Poisson brackets into the fundamental Lagrange brackets, and vice versa, i.e.,

$$\{p'_i, p'_j\}_{(q,p)} = \frac{\partial p_k}{\partial p'_i}\frac{\partial q^k}{\partial p'_j} - \frac{\partial q^k}{\partial p'_i}\frac{\partial p_k}{\partial p'_j} = \frac{\partial q'^i}{\partial q^k}\left(-\frac{\partial q'^j}{\partial p_k}\right) - \left(-\frac{\partial q'^i}{\partial p_k}\right)\left(\frac{\partial q'^j}{\partial q^k}\right)$$
$$= -[q'^i, q'^j]_{(q,p)}. \tag{5.2.28}$$

In our unified notation, all integrability conditions (5.2.27) can be written

$$\frac{\partial a^\mu}{\partial a'^\rho} = \omega_{\rho\alpha}\frac{\partial a'^\alpha}{\partial a^\beta}\omega^{\beta\mu}, \tag{5.2.29}$$

with inversion rule

$$\{a'^\mu, a'^\nu\}_{(a)} = \omega_{\mu\alpha}\omega_{\nu\beta}[a'^\alpha, a'^\beta]_{(a)}. \tag{5.2.30}$$

We now study *the integrability conditions for the existence of a new Hamiltonian*, for simplicity but without loss of generality, for transformations without an explicit time dependence.[12] For this purpose, we interpret the variables $a'^\mu(a)$ as ordinary functions in a-space, as well as new independent variables, yielding the expressions

$$\dot{a}'^\alpha = \omega^{\gamma\delta}\frac{\partial a'^\alpha}{\partial a^\gamma}\frac{\partial H}{\partial a^\gamma} = \omega^{\alpha\beta}\frac{\partial H'}{\partial a'^\beta} \tag{5.2.31}$$

which can be written

$$\frac{\partial H'}{\partial a'^\mu} = \omega_{\mu\alpha}\omega^{\gamma\delta}\frac{\partial a'^\alpha}{\partial a^\gamma}\frac{\partial H}{\partial a^\delta}. \tag{5.2.32}$$

By using the converse of the Poincaré lemma (Example I.1.4 p. I.50, in particular), the integrability conditions for the existence of H' are given by

$$\frac{\partial^2 H'}{\partial a'^\mu\,\partial a'^\nu} - \frac{\partial^2 H'}{\partial a'^\nu\,\partial a'^\mu} = 0, \qquad \mu, \nu = 1, 2, \ldots, 2n \tag{5.2.33}$$

and, when expressed in the space of the original variables, can be written

$$\frac{\partial a^\rho}{\partial a'^\mu}\frac{\partial}{\partial a^\rho}\left(\omega_{\nu\alpha}\omega^{\gamma\delta}\frac{\partial a'^\alpha}{\partial a^\gamma}\frac{\partial H}{\partial a^\delta}\right) - \frac{\partial a^\rho}{\partial a'^\nu}\frac{\partial}{\partial a^\rho}\left(\omega_{\mu\alpha}\omega^{\gamma\delta}\frac{\partial a'^\alpha}{\partial a^\gamma}\frac{\partial H}{\partial a^\delta}\right) = 0. \tag{5.2.34}$$

By multiplying both terms by $(\partial a'^\mu/\partial a^\sigma)(\partial a'^\nu/\partial a^\tau)$ and summing up the repeated indices, we have

$$\frac{\partial a'^\nu}{\partial a^\tau}\frac{\partial}{\partial a^\sigma}\left(\omega_{\nu\alpha}\omega^{\gamma\delta}\frac{\partial a'^\alpha}{\partial a^\gamma}\frac{\partial H}{\partial a^\delta}\right) - \frac{\partial a'^\mu}{\partial a^\sigma}\frac{\partial}{\partial a^\tau}\left(\omega_{\mu\alpha}\omega^{\gamma\delta}\frac{\partial a'^\alpha}{\partial a^\gamma}\frac{\partial H}{\partial a^\delta}\right) = 0. \tag{5.2.35}$$

[12] These transformations are studied by a number of authors. See, for instance, Sudarshan and Mukunda (1974).

By adding the identically null term,

$$\frac{\partial^2 a'^{\nu}}{\partial a^{\sigma}\,\partial a^{\tau}}\,\omega_{\nu\alpha}\omega^{\gamma\delta}\frac{\partial a'^{\alpha}}{\partial a^{\gamma}}\frac{\partial H}{\partial a^{\delta}} - \frac{\partial^2 a'^{\mu}}{\partial a^{\tau}\,\partial a^{\sigma}}\,\omega_{\mu\alpha}\omega^{\gamma\delta}\frac{\partial a'^{\alpha}}{\partial a^{\gamma}}\frac{\partial H}{\partial a^{\delta}} \equiv 0; \qquad (5.2.36)$$

finally, integrability conditions (5.2.35) can be written in the form

$$\frac{\partial}{\partial a^{\sigma}}\left[\left(\frac{\partial a'^{\nu}}{\partial a^{\tau}}\,\omega_{\nu\alpha}\frac{\partial a'^{\alpha}}{\partial a^{\gamma}}\right)\omega^{\gamma\delta}\frac{\partial H}{\partial a^{\delta}}\right] - \frac{\partial}{\partial a^{\tau}}\left[\left(\frac{\partial a'^{\mu}}{\partial a^{\sigma}}\,\omega_{\mu\alpha}\frac{\partial a'^{\alpha}}{\partial a^{\gamma}}\right)\omega^{\gamma\delta}\frac{\partial H}{\partial a^{\delta}}\right] = 0 \qquad (5.2.37)$$

or, by introducing the Lagrange brackets, in the more concise form

$$\frac{\partial}{\partial a^{\sigma}}\left[\{a^{\tau}, a^{\gamma}\}_{(a')}\omega^{\gamma\delta}\frac{\partial H}{\partial a^{\delta}}\right] - \frac{\partial}{\partial a^{\tau}}\left[\{a^{\sigma}, a^{\gamma}\}_{(a')}\omega^{\gamma\delta}\frac{\partial H}{\partial a^{\delta}}\right] = 0. \qquad (5.2.38)$$

By inspecting these equations, we see that *sufficient* conditions for the existence of a new Hamiltonian are given by

$$\{a^{\alpha}, a^{\beta}\}_{(a')} = N\omega_{\alpha\beta}, \qquad \alpha, \beta = 1, 2, \ldots, 2n \qquad (5.2.39)$$

where N is a numerical constant. Indeed, in this case, Equations (5.2.38) reduce to

$$N\left[\frac{\partial}{\partial a^{\sigma}}\left(\omega_{\tau\gamma}\omega^{\gamma\delta}\frac{\partial H}{\partial a^{\delta}}\right) - \frac{\partial}{\partial a^{\tau}}\left(\omega_{\sigma\gamma}\omega^{\gamma\delta}\frac{\partial H}{\partial a^{\delta}}\right)\right] = N\left(\frac{\partial^2 H}{\partial a^{\sigma}\,\partial a^{\tau}} - \frac{\partial^2 H}{\partial a^{\tau}\,\partial a^{\sigma}}\right) = 0, \qquad (5.2.40)$$

namely, they reduce to the continuity property $H \in \mathscr{C}^2$ up to a multiplicative constant.

The differences between Definitions 5.2.1 and 5.2.2 now become clear. In fact, when $N = 1$ we have the former, while for $N \neq 1$ we have the latter. Evidently, transformations of type (5.2.12) and (5.2.13) are admitted under the condition $N \neq 1$.

To see the *necessity* of conditions (5.2.40), we recall the crucial property indicated earlier that, for a transformation to be canonical, it must be so for all possible Hamiltonians. The necessity of conditions (5.2.40) originates from this property. In fact, when integrability conditions (5.2.38) are interpreted for *one given Hamiltonian H*, they characterize a different class of transformations (the so-called canonoid transformations to be introduced later in this section).

We therefore conclude by saying that the use of the converse of the Poincaré lemma within the context of the canonical transformation theory permits the identification of new meanings of the fundamental symplectic tensor $\omega_{\mu\nu}$ and fundamental Lie tensor $\omega^{\mu\nu}$. The former characterizes the integrability conditions for the existence of a new Hamiltonian, Equations (6.2.39), while the latter characterizes those for the existence of a generating function, Equations (5.2.28), up to multiplicative constants.

For completeness, we now reinterpret from a Hamiltonian viewpoint a number of known transformations of Lagrange's equations that are reviewed in Section A.3 for the reader's convenience.

Consider first the *point transformations of Lagrange's equations*, Equations (A.3.15). The direct Legendre transform applies to both L and L', yielding the Hamiltonians

$$p_k = \frac{\partial L}{\partial \dot{q}^k}, \qquad H = p_k \dot{q}^k - L = H(t, q, p), \tag{5.2.41a}$$

$$p'_k = \frac{\partial L'}{\partial \dot{q}'^k}, \qquad H' = p'_k \dot{q}'^k - L' = H'(t, q', p'). \tag{5.2.41b}$$

It is easy to see that *the phase space images of point transformations are canonical transformations.* Indeed, in view of the identities $L = L'$ and $H = H'$, we have

$$p_k dq^k \equiv p'_k dq'^k, \tag{5.2.42a}$$

$$p_k \frac{\partial q^k}{\partial q'^i} = p'_i, \qquad p_k \frac{\partial q^k}{\partial p'_i} \equiv 0. \tag{5.2.42b}$$

The reader can verify by inspection that the underlying transformation $\{a\} = \{q, p\} \to \{a'\} = \{q'(q), p'(q, p)\}$ is a particular case of the canonical transformations. Indeed, the new coordinates q'^k depend only on the old ones by the very definition of point transformations in configuration space; for a canonical transformation, the new coordinates generally depend on both the old coordinates and momenta. Also, the new momenta, from Equations (5.2.42b), depend linearly on the old momenta, which is not necessarily the case for a canonical transformation.

The transformations verifying rule (5.2.42) were called *homogeneous contact transformations* by Lie and subsequently renamed *Mathieu's transformations*, or *extended point transformations.*[13]

On similar grounds, it is easy to see that *the phase space images of the Newtonian gauge transformations are canonical.* Indeed, the transformations considered are given by (Section A.3):

$$L(t, q, \dot{q}) \to L^\dagger(t, q, \dot{q}) = L(t, q, \dot{q}) + \dot{G}(t, q), \tag{5.2.43}$$

and their phase space image is characterized by

$$p_k = \frac{\partial L}{\partial \dot{q}^k}, \quad H = p_k \dot{q}^k - L = H(t, q, p), \tag{5.2.44a}$$

$$p_k^\dagger = \frac{\partial L^\dagger}{\partial \dot{q}^k}, \qquad H^\dagger = p_k^\dagger \dot{q}^k - L^\dagger = H^\dagger(t, q, p^\dagger). \tag{5.2.44b}$$

The reader can then see that the underlying transformation $\{a\} = \{q, p\} \to \{a^\dagger\} = \{q, p^\dagger(t, q, p)\}$ verifies conditions (5.2.2) or (5.2.3). Again, this time we have a particular subclass of canonical transformations in which the space

[13] See, for instance, Whittaker (1904).

coordinates are not transformed at all, while the new momenta are given by the rule

$$p_k^{\dagger} = p_k + \frac{\partial G}{\partial q^k}. \tag{5.2.45}$$

We remain with the problem of the phase space image of the *isotopic transformations of a Lagrangian* (Section A.2), i.e.,

$$\left(\frac{d}{dt}\frac{\partial L^*}{\partial \dot{q}^k} - \frac{\partial L^*}{\partial q^k}\right)_{SA} \equiv \left[h_k^i\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i}\right)_{SA}\right]_{SA}. \tag{5.2.46}$$

These transformations can be subjected to a dual phase space interpretation. First, we can apply the direct Legendre transform to both L and L^*, yielding the rules

$$p_k = \frac{\partial L}{\partial \dot{q}^k}, \qquad H = p_k \dot{q}^k - L = H(t, q, p), \tag{5.2.47a}$$

$$p_k^* = \frac{\partial L^*}{\partial \dot{q}^k}, \qquad H^* = p_k^* \dot{q}^k - L^* = H^*(t, q, p^*). \tag{5.2.47b}$$

It is readily seen that *the phase space images of the isotopic transformations of a Lagrangian are not canonical.* Indeed, the reader can determine by simple inspection that the underlying transformation $\{a\} = \{q,\ p\} \to \{a^*\} = \{q, p^*(t, q, p)\}$ *does not* verify conditions (5.2.2) or (5.2.3). Also, the Hamiltonian $H^*(a^*)$ *cannot* be obtained, in general, from the old Hamiltonian $H(a)$ via scalar rule (5.2.9c) or (5.2.24b), i.e.,

$$H^*(t, a^*) \neq H(t, a(a^*)) + \frac{\partial F}{\partial t} + \frac{\partial q^k}{\partial t} p_k, \tag{5.2.48}$$

and thus transformation law (5.2.9a) of Hamilton's equations does not apply.

In this way we learn the remarkable property that *canonical transformations do not exhaust the class of all possible transformations capable of preserving Hamilton's equations.*[14] Also, it is our first exposure to noncanonical transformations of direct meaning in analytic mechanics. It is therefore important to study the transformations under consideration in more detail.

For simplicity, consider analytic and invertible transformations without an explicit time dependence,

$$a^\mu \to a^{*\mu} = a^{*\mu}(a), \qquad \mu = 1, 2, \ldots, 2n, \tag{5.2.49}$$

and interpret them as functions in a-space. The time evolution law then reads

$$\dot{a}^{*\mu} = \frac{\partial a^{*\mu}}{\partial a^\alpha}\dot{a}^\alpha = \omega^{\alpha\beta}\frac{\partial a^{*\mu}}{\partial a^\alpha}\frac{\partial H}{\partial a^\beta}. \tag{5.2.50}$$

[14] Another remarkable property which will be pointed out in Section 6.3 is that the symmetries of Hamilton's equations (which are a subclass of the class of canonical transformations) *do not* exhaust all possible symmetries of the vector field represented by Hamilton's equations.

Suppose now that a new Hamiltonian $H^*(a^*)$ exists such that

$$\left(\omega^{\alpha\beta}\frac{\partial a^{*\mu}}{\partial a^{\alpha}}\frac{\partial H}{\partial a^{\beta}}\right)(a^*) = \omega^{\mu\nu}\frac{\partial H^*}{\partial a^{*\nu}}. \tag{5.2.51}$$

Then transformations (5.2.49) are called *canonoid* (i.e., not quite canonical), according to Saletan and Cromer (1971., page 187).

It is easy to see that the phase space image of the isotopic transformations of a Lagrangian are precisely (a particular form of[15]) the canonoid transformations. Indeed, Hamilton's equations in the a^*-coordinate system exist, while the Hamiltonian does not verify the conventional scalar rule, under these transformations.

The integrability conditions for the existence of a canonoid transformation for a given Hamiltonian are easily computed via simple generalization of the integrability conditions for canonical transformations, and they are given by

$$\frac{\partial}{\partial a^{*\nu}}\left(\omega_{\mu\rho}\omega^{\alpha\beta}\frac{\partial a^{*\rho}}{\partial a^{\alpha}}\frac{\partial H}{\partial a^{\beta}}\right)(a^*) - \frac{\partial}{\partial a^{*\mu}}\left(\omega_{\nu\rho}\omega^{\alpha\beta}\frac{\partial a^{*\rho}}{\partial a^{\alpha}}\frac{\partial H}{\partial a^{\beta}}\right)(a^*) = 0. \tag{5.2.52}$$

The use of the Inverse Hamiltonian Problem then yields the new Hamiltonian according to the familiar rule

$$H^*(a^*) = a^{*\mu}\int_0^1 d\tau\left(\omega_{\mu\rho}\omega^{\alpha\beta}\frac{\partial a^{*\rho}}{\partial a^{\alpha}}\frac{\partial H}{\partial a^{\beta}}\right)(\tau a^*). \tag{5.2.53}$$

A few comments are in order. It should be indicated that integrability conditions (5.2.52) are, in actuality, the conditions for the variational self-adjointness of the normal form

$$\omega_{\mu\nu}\dot{a}^{*\nu} - \Xi^*_{\mu}(a^*) = 0, \qquad \Xi^*_{\mu} = \omega_{\mu\rho}\omega^{\alpha\beta}\frac{\partial a^{*\rho}}{\partial a^{\alpha}}\frac{\partial H}{\partial a^{\beta}}, \tag{5.2.54}$$

expressed in the a^*-variables, i.e.,

$$\frac{\partial \Xi^*_{\mu}}{\partial a^{*\nu}} - \frac{\partial \Xi^*_{\nu}}{\partial a^{*\mu}} = 0. \tag{5.2.55}$$

This is also the case of the integrability conditions for canonical transformations, Equations (5.2.38). However, a fundamental difference exists between the integrability conditions for canonoid and canonical transformations. In the former case the conditions hold *for one given Hamiltonian*, while in the latter they hold for all Hamiltonians, as indicated earlier. It is precisely this difference that renders the canonoid transformations generally noncanonical.

Intriguingly, the canonoid transformations can clearly be canonical when they are canonoid with respect to all Hamiltonians. This is a first indication

[15] The canonoid transformations generally imply the transformations of both, coordinates and momenta, while the phase space image of the isotopic transformations of a Lagrangian does not transform, by assumption, the space coordinates.

of the existence of more general transformations admitting the canonical transformations as a subclass. Additional, noncanonical transformations of this type will be identified in Section 5.3. This situation confirms the expectation that the transformation theory of Hamilton's equations should not be restricted to canonical transformations.

We shall now inspect the behavior of the regularity of the Hamiltonian under canonical transformations. Recall that a necessary condition for the applicability of the inverse Legendre transform is that the Hamiltonian verifies the regularity conditions (Section I.3.8)

$$\det\left(\frac{\partial^2 H}{\partial p_i\, \partial p_j}\right)(\tilde{\mathscr{R}}) \neq 0, \tag{5.2.56}$$

and this condition is equivalent to the corresponding regularity condition for the Lagrangian

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^i\, \partial \dot{q}^j}\right)(\mathscr{R}) = \left\{\det\left(\frac{\partial^2 H}{\partial p_i\, \partial p_j}\right)(\tilde{\mathscr{R}})\right\}^{-1}. \tag{5.2.57}$$

Regularity property (5.2.57) *is not preserved by canonical transformations.* This is easily seen if one considers the property of canonical transformations of reducing a Hamiltonian $H = \frac{1}{2}\mathbf{p}^2 + V$ into a form which is *linear* in the momentum or a *constant*. When a Hamiltonian is degenerate, the inverse Legendre transform, as presented in Section I.3.8, is *inapplicable*. In this case, the construction of an equivalent Lagrangian representation is rather involved and belongs to the *theory of systems with subsidiary constraints*. As such, it will not be considered here (the problem is treated in the specialized mathematical literature of the canonical treatment of the calculus of variations, but it does not appear to be treated in the physical literature, to the author's knowledge[16]).

Notice that a fully equivalent situation occurs at the Lagrangian level, provided that the transformation theory is extended to include the configuration space image of the canonical transformations, that is, the velocity-dependent transformations. This occurrence confirms the equivalence of the Lagrangian and Hamiltonian approaches also with respect to the transformation theory.

We can therefore say that the transformation theory indicates the existence of the possibility of *transforming given regular determined systems into equivalent degenerate systems with subsidiary constraints.*

In this section we reviewed the contemporary approach to the theory of canonical transformations, which is rather universally restricted to transformations of type (6.1.9). The reader should keep in mind, however, the

[16] The approach well-known in the physical literature as *Dirac's mechanics* transforms a *degenerate* Lagrangian into a Hamiltonian which can be proved to be (generally) *regular* in the sense of (5.2.56). The problem referred to in the text is the opposite of Dirac's, that is, the transformation of a *degenerate* Hamiltonian into an equivalent, generally regular, Lagrangian image.

need to consider the more general transformations (6.1.9), as established for instance by the structure of Galilei's transformations

$$\begin{cases} t \to t' = t + t_0 \\ \mathbf{r} \to \mathbf{r}' = R\mathbf{r} + \mathbf{v}_0 t + \mathbf{r}_0, \qquad R \in SO(3), \\ \mathbf{p} \to \mathbf{p}' = R\mathbf{p} + m\mathbf{v}_0 \end{cases} \tag{5.2.58}$$

as well as by other symmetries of systems in first-order form. The behavior of Hamilton's equations under these more general transformations will be studied in the next section, as a particular case of the transformation theory of Birkhoff's equations.[17]

In closing this section we note that *the restriction of the transformation theory to canonical transformations prohibits the existence of indirect Hamiltonian representations.* In fact, by their very definition, canonical transformations preserve the Hamiltonian character of a vector field. As we shall see in Section 6.4, this implies the inability to transform a given non-Hamiltonian vector field into an Hamiltonian form, by therefore preventing the construction of a Hamiltonian. The generalization of the transformation theory to arbitrary, generally non-canonical transformations is therefore mandatory for the Inverse Hamiltonian Problem.

5.3 Transformation Theory of Birkhoff's Equations

In this section we shall first establish the property that noncanonical transformations transform Hamilton's equations into Birkhoff's equations. The preservation of the structure of Birkhoff's equations under unrestricted transformations will then be consequential. The generalization of the canonical transformation theory will be considered thereafter.

For clarity, we shall consider first the contemporaneous transformations without an explicit time dependence, Equations (5.1.8), and then extend the results to transformations (5.1.9) and (5.1.10). Also, we shall study first the behavior of the Lie and symplectic tensors under the transformations considered, and then extend our findings to the complete analytic equations. We hope that in this way the reader can see the implications of each aspect of the theory.

Let us begin our study by showing that an autonomous Lie tensor $\Omega^{\mu\nu}(a)$ and its associated symplectic form $\Omega_{\mu\nu}(a) = (\|\Omega^{\alpha\beta}\|^{-1})_{\mu\nu}$ preserve their Lie and symplectic character, respectively, under arbitrary transformations (5.1.8). In the language of Definition 4.4.1 and Chart 4.2, this important property can be formulated and proved as follows.

[17] The generalization indicated in Chart 5.6 is sufficient for the inclusion of the trivial translations $t \to t' = t + t_0$. The generalization we are referring to in the text is that for the *maximal possible functional dependence* on $\mathbb{R} \times T^*M$ of the new variables in the old, i.e., $t' = t'(t, \mathbf{r}, \mathbf{p})$, $\mathbf{r}' = \mathbf{r}'(t, \mathbf{r}, \mathbf{p})$, and $\mathbf{p}' = \mathbf{p}'(t, \mathbf{r}, \mathbf{p})$.

Lemma 5.3.1. *All possible smoothness-preserving*[18] *and regular transformations* $a \to a'(a)$ *of the local variables* $a = (\mathbf{r}, \mathbf{p})$ *of a cotangent bundle* T^*M *are jointly Lie isotopic and symplectic isotopic.*

PROOF. Suppose that a rank two tensor $\Omega^{\mu\nu}(a)$ on T^*M is regular, in the sense that

$$\det(\Omega^{\mu\nu})(\tilde{\mathscr{R}}) \neq 0 \tag{5.3.1}$$

and Lie, in the sense of verifying integrability conditions (4.1.48), i.e.,

$$\Omega^{\mu\nu} + \Omega^{\nu\mu} = 0, \tag{5.3.2a}$$

$$\Omega^{\mu\rho}\frac{\partial\Omega^{\nu\tau}}{\partial a^\rho} + \Omega^{\nu\rho}\frac{\partial\Omega^{\tau\mu}}{\partial a^\rho} + \Omega^{\tau\rho}\frac{\partial\Omega^{\mu\nu}}{\partial a^\rho} = 0. \tag{5.3.2b}$$

Then, under all possible transformations which are regular and of the same continuity class of $\Omega^{\mu\nu}$,

$$a^\mu \to a'^\mu(a), \qquad \det\left(\frac{\partial a'^\mu}{\partial a^\rho}\right)(\tilde{\mathscr{R}}) \neq 0, \tag{5.3.3}$$

the transformed tensor

$$\Omega'^{\mu\nu}(a') = \frac{\partial a'^\mu}{\partial a^\rho}\,\overline{\Omega}^{\rho\sigma}(a')\,\frac{\partial a'^\nu}{\partial a^\sigma}, \qquad \overline{\Omega}^{\rho\sigma} = \Omega^{\rho\sigma}(a(a')) \tag{5.3.4}$$

is still regular, in view of the properties

$$\det(\Omega'^{\mu\nu}) = \det\left(\frac{\partial a'^\mu}{\partial a^\rho}\right)\det(\Omega^{\rho\sigma})\det\left(\frac{\partial a'^\nu}{\partial a^\sigma}\right) \neq 0 \tag{5.3.5}$$

and it is still Lie, that is, it verifies conditions (5.3.2) in the new coordinate systems, because of the properties

$$\Omega'^{\mu\nu} + \Omega'^{\nu\mu} = \frac{\partial a'^\mu}{\partial a^\rho}(\Omega^{\rho\sigma} + \Omega^{\sigma\rho})\frac{\partial a'^\nu}{\partial a^\sigma} \equiv 0, \tag{5.3.6a}$$

$$\begin{aligned}
&\Omega'^{\mu\rho}\frac{\partial\Omega'^{\nu\tau}}{\partial a'^\rho} + \Omega'^{\nu\rho}\frac{\partial\Omega'^{\tau\mu}}{\partial a'^\rho} + \Omega'^{\tau\rho}\frac{\partial\Omega'^{\mu\nu}}{\partial a^\rho} \\
&\quad = \left[\frac{\partial a'^\mu}{\partial a^\alpha}\frac{\partial}{\partial a^\beta}\left(\frac{\partial a'^\nu}{\partial a^\gamma}\frac{\partial a'^\tau}{\partial a^\delta}\right) + \frac{\partial a'^\nu}{\partial a^\alpha}\frac{\partial}{\partial a^\beta}\left(\frac{\partial a'^\tau}{\partial a^\gamma}\frac{\partial a'^\mu}{\partial a^\delta}\right)\right. \\
&\qquad \left. + \frac{\partial a'^\tau}{\partial a^\alpha}\frac{\partial}{\partial a^\beta}\left(\frac{\partial a'^\mu}{\partial a^\gamma}\frac{\partial a'^\nu}{\partial a^\delta}\right)\right]\Omega^{\alpha\beta}\Omega^{\gamma\delta} \\
&\qquad + \left(\frac{\partial a'^\mu}{\partial a^\alpha}\frac{\partial a'^\nu}{\partial a^\gamma}\frac{\partial a'^\tau}{\partial a^\delta} + \frac{\partial a'^\nu}{\partial a^\alpha}\frac{\partial a'^\tau}{\partial a^\gamma}\frac{\partial a'^\mu}{\partial a^\delta} + \frac{\partial a'^\tau}{\partial a^\alpha}\frac{\partial a'^\mu}{\partial a^\gamma}\frac{\partial a'^\nu}{\partial a^\delta}\right)\Omega^{\alpha\beta}\frac{\partial\Omega^{\gamma\delta}}{\partial a^\beta} \equiv 0,
\end{aligned} \tag{5.3.6b}$$

[18] With the terms "smoothness preserving" we express the condition that a class $\mathscr{C}^\infty$ or analytic manifold (Chart I.2.1) is transformed into a manifold of the same continuity class.

which are ensured by their original form (5.3.2) (Problem 5.4). Therefore, all possible transformations (5.3.3) are *Lie-isotopic*, that is, the brackets characterized by the tensor $\Omega^{\mu\nu}$ remain always Lie, and we write[19]

$$[A, B]^{*}_{(a)} = \frac{\partial A}{\partial a^{\mu}} \Omega^{\mu\nu}_{(a)} \frac{\partial B}{\partial a^{\nu}} = \frac{\partial A'}{\partial a'^{\rho}} \frac{\partial a'^{\rho}}{\partial a^{\mu}} \overline{\Omega}^{\mu\nu} \frac{\partial a'^{\sigma}}{\partial a^{\nu}} \frac{\partial B'}{\partial a'^{\sigma}}$$

$$= \frac{\partial A'}{\partial a'^{\rho}} \Omega'^{\rho\sigma}(a') \frac{\partial B'}{\partial a'^{\sigma}} = [A', B']^{*'}_{(a')}. \tag{5.3.7}$$

The proof of the second part follows from the property that the tensor $\Omega_{\mu\nu}$ associated with a *Lie* tensor $\Omega^{\mu\nu}$ via the rule

$$(\Omega_{\mu\nu}) = (\Omega^{\mu\nu})^{-1} \tag{5.3.8}$$

is always *symplectic* (and vice versa). This property therefore persists for the transformed tensor $\Omega'^{\mu\nu}$. Aside from that, suppose that a covariant rank-two tensor $\Omega_{\mu\nu}(a)$ on $T^{*}M$ is regular, in the sense that

$$\det(\Omega_{\mu\nu})(\mathscr{H}) \neq 0, \tag{5.3.9}$$

and symplectic, in the sense of verifying conditions (4.1.49), i.e.,

$$\Omega_{\mu\nu} + \Omega_{\nu\mu} = 0, \tag{5.3.10a}$$

$$\frac{\partial \Omega_{\mu\nu}}{\partial a^{\tau}} + \frac{\partial \Omega_{\nu\tau}}{\partial a^{\mu}} + \frac{\partial \Omega_{\tau\mu}}{\partial a^{\nu}} = 0. \tag{5.3.10b}$$

Then, the transformed tensor

$$\Omega'_{\mu\nu}(a') = \frac{\partial a^{\rho}}{\partial a'^{\mu}} \overline{\Omega}_{\rho\sigma}(a') \frac{\partial a^{\sigma}}{\partial a'^{\mu}}, \qquad \overline{\Omega}_{\rho\sigma}(a') = \Omega_{\rho\sigma}(a(a')), \tag{5.3.11}$$

is still regular, in view of the properties

$$\det(\Omega'_{\mu\nu}) = \det\left(\frac{\partial a^{\rho}}{\partial a'^{\mu}}\right) \det(\Omega_{\rho\sigma}) \det\left(\frac{\partial a^{\sigma}}{\partial a'^{\nu}}\right) \neq 0, \tag{5.3.12}$$

and it verifies conditions (5.3.10) in the new reference frame in view of the properties

$$\Omega'_{\mu\nu} + \Omega'_{\nu\mu} = \frac{\partial a^{\rho}}{\partial a'^{\mu}} (\Omega_{\rho\sigma} + \Omega_{\sigma\rho}) \frac{\partial a^{\sigma}}{\partial a'^{\nu}} \equiv 0, \tag{5.3.13a}$$

$$\frac{\partial \Omega'_{\mu\nu}}{\partial a'^{\tau}} + \frac{\partial \Omega'_{\nu\tau}}{\partial a'^{\mu}} + \frac{\partial \Omega'_{\tau\mu}}{\partial a'^{\nu}} = \left[\frac{\partial}{\partial a'^{\tau}} \left(\frac{\partial a^{\rho}}{\partial a'^{\mu}} \frac{\partial a^{\sigma}}{\partial a'^{\nu}}\right) + \frac{\partial}{\partial a'^{\mu}} \left(\frac{\partial a^{\rho}}{\partial a'^{\nu}} \frac{\partial a^{\sigma}}{\partial a'^{\tau}}\right) + \frac{\partial}{\partial a'^{\nu}} \left(\frac{\partial a^{\rho}}{\partial a'^{\tau}} \frac{\partial a^{\sigma}}{\partial a'^{\mu}}\right)\right] \Omega_{\rho\sigma}$$

$$+ \left(\frac{\partial a^{\rho}}{\partial a'^{\mu}} \frac{\partial a^{\sigma}}{\partial a'^{\nu}} \frac{\partial a^{\alpha}}{\partial a'^{\tau}} + \frac{\partial a^{\rho}}{\partial a'^{\nu}} \frac{\partial a^{\sigma}}{\partial a'^{\tau}} \frac{\partial a^{\alpha}}{\partial a'^{\mu}} + \frac{\partial a^{\rho}}{\partial a'^{\tau}} \frac{\partial a^{\sigma}}{\partial a'^{\mu}} \frac{\partial a^{\alpha}}{\partial a'^{\nu}}\right) \frac{\partial \Omega_{\rho\sigma}}{\partial a^{\alpha}} \equiv 0, \tag{5.3.13b}$$

[19] The symbol $[\cdots, \cdots]^{*'}$ indicates brackets *different* than $[\cdots, \cdots]^{*}$ although still of Lie type. (Recall that in our notation the symbol $[\cdots, \cdots]$ denotes the conventional Poisson brackets, while the symbol $[\cdots, \cdots]^{*}$ denotes the generalized ones).

which are ensured by their original form (5.3.10) (Problem 5.4). As a result, all transformations of the class admitted are *symplectic-isotopic*, that is, the local symplectic two-forms characterized by the tensor $\Omega_{\mu\nu}$ remain locally symplectic,

$$\Omega_2 = \tfrac{1}{2}\Omega_{\mu\nu}(a)da^\mu \wedge da^\nu = \frac{1}{2}\frac{\partial a^\mu}{\partial a'^\rho}\overline{\Omega}_{\mu\nu}(a')\frac{\partial a^\nu}{\partial a'^\sigma}da'^\rho \wedge da'^\sigma = \Omega'_2, \tag{5.3.14a}$$

$$d\Omega_2 = d\Omega'_2 \equiv 0, \tag{5.3.14b}$$

and this completes the proof. (*Q.E.D.*)

The remarkable property expressed by Lemma 5.3.1 *is not new*. In fact, it can be considered at the foundation of the coordinate-free globalization of the symplectic geometry.[20] The property has been merely expressed here in local variables. This also illustrates the pedagogical and technical significance of the local formulation of the theory, prior to passing to abstract, more advanced geometric approaches.[21]

The Lie and symplectic tensors of Lemma 5.3.1 are arbitrary. When they are the fundamental tensors we have the following particular case.

Corollary 5.3.1a. *The fundamental Lie tensor $\omega^{\mu\nu}$ and the fundamental symplectic tensor $\omega_{\mu\nu}$ preserve their Lie and symplectic character, respectively, under all possible transformations* (5.3.3).

In this way we reach another important result. Recall from Definition 5.2.1 that canonical transformations not only preserve the *Lie* character of the conventional Poisson brackets, but they actually preserve the *value* of the fundamental tensor $\omega^{\mu\nu}$, i.e.,

$$[A, B]_{(a)} = \frac{\partial A}{\partial a^\mu}\omega^{\mu\nu}\frac{\partial B}{\partial a^\nu} = \frac{\partial A'}{\partial a'^\rho}\frac{\partial a'^\rho}{\partial a^\mu}\omega^{\mu\nu}\frac{\partial a'^\sigma}{\partial a^\nu}\frac{\partial B'}{\partial a'^\sigma}$$

$$= \frac{\partial A'}{\partial a'^\rho}\omega^{\rho\sigma}\frac{\partial B'}{\partial a'^\sigma} = [A', B']_{(a')}. \tag{5.3.15}$$

We learn from Corollary 5.3.1a that, while the value of the fundamental tensor is not preserved, noncanonical transformations preserve in full the Lie character of the product

$$[A, B]_{(a)} = \frac{\partial A}{\partial a^\mu}\omega^{\mu\nu}\frac{\partial B}{\partial a^\nu} = \frac{\partial A'}{\partial a'^\rho}\frac{\partial a'^\rho}{\partial a^\mu}\omega^{\mu\nu}\frac{\partial a'^\sigma}{\partial a^\nu}\frac{\partial B'}{\partial a'^\sigma}$$

$$= \frac{\partial A'}{\partial a'^\rho}\Omega'^{\rho\sigma}(a')\frac{\partial B'}{\partial a'^\sigma} = [A', B']^*_{(a')}. \tag{5.3.16}$$

[20] A representative list of references in this field is given in footnote 54 of Chart 4.4.

[21] Santilli (1978e) has shown that Lemma 5.3.1 is actually a particular case of the more general property that Lie-admissible tensors $S^{\mu\nu}(a)$ or symplectic-admissible tensors $S_{\mu\nu}(a)$ on T^*M (Chart 4.7) preserve their Lie-admissible or symplectic-admissible character, respectively, under all possible transformations of the class considered.

The geometric counterpart of this algebraic result is immediate. Canonical transformations (according to Definition 5.2.1) preserve the symplectic character of the fundamental structure via the preservation of the value of the tensor $\omega_{\mu\nu}$, and we write

$$R_1^0 = R_\mu^0(a)da^\mu = p_k dr^k = R_\mu^0 \frac{\partial a^\mu}{\partial a'^\sigma} da'^\sigma = R_\sigma'^0(a')da'^\sigma = p'_k dr'^k,$$

$$(R_\mu^0) = (\mathbf{p}, \mathbf{0}), \qquad R_\sigma'^0(a') = R_\mu^0(a(a')) \frac{\partial a^\mu}{\partial a'^\sigma},$$

$$(R_\sigma'^0) = (\mathbf{p}', \mathbf{0}) \tag{5.3.17a}$$

$$\omega_2 = dR_1^0 = \frac{1}{2}\left(\frac{\partial R_\nu^0}{\partial a^\mu} - \frac{\partial R_\mu^0}{\partial a^\nu}\right) da^\mu \wedge da^\nu = dp_k \wedge dr^k$$

$$= d[R_\sigma'^0(a')da'^\sigma] = \frac{1}{2}\left(\frac{\partial R_\sigma'^0}{\partial a'^\rho} - \frac{\partial R_\rho'^0}{\partial a'^\sigma}\right) da'^\rho \wedge da'^\sigma = dp'_k \wedge dr'^k. \tag{5.3.17b}$$

We learned from Corollary 5.3.1a that noncanonical transformations do not preserve the fundamental character of the two-form. Nevertheless, the form remains fully symplectic, and actually acquires the most general possible (but still exact, local, and autonomous) structure, i.e.,

$$R_1^0 = R_\mu^0(a)da^\mu = p_k dr^k = R_\mu^0 \frac{\partial a^\mu}{\partial a'^\sigma} da'^\sigma = R'_\sigma(a')da'^\sigma \neq p'_k dr'^k,$$

$$R'_\sigma(a') = R_\mu^0(a(a')) \frac{\partial a^\mu}{\partial a'^\sigma} \neq R_\sigma'^0(a'), \tag{5.3.18a}$$

$$\omega_2 = d[R_\mu^0(a)da^\mu] = \frac{1}{2}\left(\frac{\partial R_\nu^0}{\partial a^\mu} - \frac{\partial R_\mu^0}{\partial a^\nu}\right) da^\mu \wedge da^\nu$$

$$= d[R'_\sigma(a')da'^\sigma] = \frac{1}{2}\left(\frac{\partial R'_\sigma}{\partial a'^\rho} - \frac{\partial R'_\rho}{\partial a'^\sigma}\right) da'^\rho \wedge da'^\sigma = \Omega'_2. \tag{5.3.18b}$$

The notion of *Lie isotopy* was introduced in Chart 5.2 to express any invertible modification of a given Lie product which preserves its Lie character. We have learned here that all possible modifications characterized by (regular) transformations of the variables are always isotopic. The notion has then been extended to that of (regular) *symplectic isotopy* as a geometric counterpart, with the understanding that the notion is a local realization of a corresponding global property of the symplectic geometry. The deep interrelation between algebraic property (5.3.16) and the geometric one (5.3.18) is remarkable.

Once the transformation properties of the algebraic or geometric tensors have been identified, the extension of the results to the analytic equations is straightforward. In this way we reach the following *transformation rule of nonautonomous Hamilton's equations into the semi-autonomous Birkhoff's*

equations (*Definition* 5.2.1) *under noncanonical, contemporaneous transformations without time dependence*:

$$t \to t' \equiv t, \qquad a^\mu \to a'^\mu(a), \tag{5.3.19a}$$

$$\left[\omega_{\mu\nu}\dot{a}^\nu - \frac{\partial H(t,a)}{\partial a^\mu}\right]_{\mathrm{SA}}$$

$$\equiv \left\{\left[\frac{\partial R^0_\nu(a)}{\partial a^\mu} - \frac{\partial R^0_\mu(a)}{\partial a^\nu}\right]\dot{a}^\nu - \frac{\partial H(t,a)}{\partial a^\mu}\right\}_{\mathrm{SA}} \tag{5.3.19b}$$

$$= \left\{\frac{\partial a'^\rho}{\partial a^\mu}\left\{\left[\frac{\partial R'_\sigma(a')}{\partial a'^\rho} - \frac{\partial R'_\rho(a')}{\partial a'^\sigma}\right]\dot{a}'^\sigma - \frac{\partial B'(t,a')}{\partial a'^\rho}\right\}_{\mathrm{SA}}\right\}_{\mathrm{NSA}} = 0,$$

$$(R^0_\mu) = (\mathbf{p}, \mathbf{0}), \qquad R'_\rho(a') = \left(\frac{\partial a^\alpha}{\partial a'^\rho} R^0_\alpha\right)(a'), \tag{5.3.19c}$$

$$B'(t,a') = H(t, a(a')). \tag{5.3.19d}$$

Equivalently, we can write the following *transformation rule of Hamilton's principle into Pfaff's principle* under the same class of transformations[22]

$$\begin{aligned}
\delta\mathscr{A} &= \delta\int_{t_1}^{t_2} dt[p_k r^k - H(t,a)](\tilde{E}_0) \\
&= \delta\int_{t_1}^{t_2} [R^0_\mu(a)da^\mu - H(t,a)dt](\tilde{E}_0) \\
&= \delta\int_{t_1}^{t_2}\left[R^0_\mu(a(a'))\frac{\partial a^\mu}{\partial a'^\rho}da'^\rho - H(t,a(a'))dt\right](\tilde{E}'_0) \\
&= \delta\int_{t_1}^{t_2}[R'_\rho(a')da'^\rho - B'(t,a')dt](\tilde{E}'_0) = 0.
\end{aligned} \tag{5.3.20}$$

The transformation rule of Birkhoff's equations is then a trivial consequence, and it is given by rule (5.3.19) via only the replacement of the canonical functions $(R^0_\mu) = (\mathbf{p}, \mathbf{0})$ in Equations (5.3.19b) with arbitrary functions $R_\mu = R_\mu(a)$. In this way we reach the following important result.

Lemma 5.3.2. *The semiautonomous Birkhoff's equations preserve their structure*[23] *under all possible smoothness-preserving and regular transformations of the local variables* $a = (\mathbf{r}, \mathbf{p}) \to a'(a) = (\mathbf{r}'(\mathbf{r}, \mathbf{p}), \mathbf{p}'(\mathbf{r}, \mathbf{p}))$.

To express the result in different terms, we can say that, *while Hamiltonian Mechanics demands the restriction of the transformation theory to certain*

[22] Note that the integrand of the action in Equations (5.3.20) transforms identically, without the appearance of the Jacobian $\partial a'^\rho/\partial a^\mu$ as in Equations (5.3.19). The equivalence of the two approaches is established by the property $\delta a'^\rho = (\partial a'^\rho/\partial a^\mu)\delta a^\mu$.

[23] It should be stressed here that the transformations under consideration *are not* symmetries of Birkhoff's equations. We therefore have a preservation of the "structure" of the equations, but we do not have their "form invariance." This latter problem will be studied in the next chapter.

special classes of transformations to preserve its structure (the canonical and canonoid transformations), the transition to the covering[24] *Birkhoffian Mechanics implies the removal of all restrictions on transformations for the preservation of its structure, except conventional smoothness and regularity restrictions.*

We shall soon discover that this important property is actually a particular case of a general property on $\mathbb{R} \times T^*M$ with rather intriguing characteristics, particularly from the viewpoint of the relativity which is applicable in Newtonian mechanics for unrestricted dynamic conditions. First, however, a study of the intermediary step of the contemporaneous, explicitly time-dependent transformations (5.1.9) is recommended.

It is easy to see that Lemma 5.3.1 also applies for transformations (5.1.9), apart from delicate topological aspects due to the explicit time dependence which can be handled, e.g., via the parametric approach to symplectic forms of Chart 4.6. For instance, a general symplectic tensor transforms according to the rule

$$\Omega_{\mu\nu}(t, a) = \frac{\partial R_\nu(t, a)}{\partial a^\mu} - \frac{\partial R_\mu(t, a)}{\partial a^\nu} \to \Omega'_{\mu\nu}(t, a')$$

$$= \frac{\partial a^\rho}{\partial a'^\mu}\overline{\left(\frac{\partial R_\sigma}{\partial a^\rho} - \frac{\partial R_\rho}{\partial a^\sigma}\right)}\frac{\partial a^\sigma}{\partial a'^\nu} = \frac{\partial R'_\nu(t, a')}{\partial a'^\mu} - \frac{\partial R'_\mu(t, a')}{\partial a'^\nu}, \quad (5.3.21a)$$

$$R'_\mu(t, a') = \left(\frac{\partial a^\rho}{\partial a'^\mu} R_\rho\right)(t, a'), \quad (5.3.21b)$$

where the upper bar indicates computation in the new variables. In particular, when the original tensor is time-independent, it generally acquires such a dependence under the transformations admitted, while preserving its exact symplectic character.

The transition to the analytic equations is trivial. In this way we reach the following *transformation rule of Hamilton's equations into the non-autonomous Birkhoff's equations via noncanonical, time-dependent, contemporaneous transformations*:

$$t \to t' \equiv t, \qquad a^\mu \to a'^\mu(t, a) \quad (5.3.22a)$$

$$\left\{\left[\frac{\partial R^0_\nu(a)}{\partial a^\mu} - \frac{\partial R^0_\mu(a)}{\partial a^\nu}\right]\dot{a}^\nu - \frac{\partial H(t, a)}{\partial a^\mu}\right\}_{SA}$$

$$= \left\{\frac{\partial a'^\rho}{\partial a^\mu}\left\{\left[\frac{\partial R'_\sigma(t, a')}{\partial a'^\rho} - \frac{\partial R'_\rho(t, a')}{\partial a'^\sigma}\right]\dot{a}'^\sigma - \left[\frac{\partial B'(t, a')}{\partial a'^\rho} + \frac{\partial R'_\rho(t, a')}{\partial t}\right]\right\}_{SA}\right\}_{NSA}$$

$$= 0, \quad (5.3.22b)$$

[24] The terms *covering mechanics* or *theory* are intended to express the generalization of an old theory into a new one under the conditions: (a) the new theory refers to a class of physical systems and dynamical conditions more general than those for which the old theory was conceived; (b) the new theory is based on a suitable generalization of the methods of the old theory; and, last, but not least, (c) the new theory recovers the old one identically when the physical systems considered are restricted to those of the old class. The Birkhoffian Mechanics verifies all these conditions with respect to the Hamiltonian Mechanics and thus is a covering of the latter.

$$(R_\mu^0) = (\mathbf{p}, \mathbf{0}), \qquad R'_\rho(t, a') = \left(\frac{\partial a^\alpha}{\partial a'^\rho} R_\alpha^0\right)(t, a'), \tag{5.3.22c}$$

$$B'(t, a') = \left(H - \frac{\partial a^\alpha}{\partial t} R_\alpha\right)(t, a'). \tag{5.3.22d}$$

with a corresponding transformation of the variational principle.

The *transformation rule of the nonautonomous Birkhoff's equations* under the same transformations is then given by

$$\begin{aligned}
&\left\{\left[\frac{\partial R_\nu(t, a)}{\partial a^\mu} - \frac{\partial R_\mu(t, a)}{\partial a^\nu}\right]\dot{a}^\nu - \left[\frac{\partial B(t, a)}{\partial a^\mu} + \frac{\partial R_\mu(t, a)}{\partial t}\right]\right\}_{\mathrm{SA}} \\
&\quad = \left\{\frac{\partial a'^\rho}{\partial a^\mu}\left\{\left[\frac{\partial R'_\sigma(t, a')}{\partial a'^\rho} - \frac{\partial R'_\rho(t, a')}{\partial a'^\sigma}\right]\dot{a}'^\sigma - \left[\frac{\partial B'(t, a')}{\partial a'^\rho} + \frac{\partial R'_\rho(t, a')}{\partial t}\right]\right\}_{\mathrm{SA}}\right\}_{\mathrm{NSA}} \\
&\quad = 0,
\end{aligned} \tag{5.3.23a}$$

$$R'_\rho(t, a') = \left(\frac{\partial a^\alpha}{\partial a'^\rho} R_\alpha\right)(t, a'), \tag{5.3.23b}$$

$$B'(t, a') = \left(B - \frac{\partial a^\alpha}{\partial t} R_\alpha\right)(t, a'). \tag{5.3.23c}$$

The implications of an explicit time dependence in the symplectic structure have been indicated in Section 4.2 and Charts 4.2 and 4.6. Methods for eliminating this dependence without altering the underlying dynamics have been identified in Section 4.5. Note, however, that even when the original symplectic structure in Birkhoff's equations does not depend explicitly on time, the corresponding structure under rule (5.3.23) generally acquires such a dependence. In order to prevent problematic aspects such as those of Chart 4.1 (lack of algebraic structure of the time evolution), the explicit time dependence in the transformed symplectic structure can be eliminated by again using the methods of Section 4.5.

Recall that structure (5.3.21) is invariant under the Birkhoffian gauge

$$R'_\mu(t, a') \to R'^\dagger_\mu(t, a') = R'_\mu(t, a') - \frac{\partial G'(t, a')}{\partial a'^\mu}, \tag{5.3.24a}$$

$$B'_\mu(t, a') \to B'^\dagger(t, a') = B'(t, a') + \frac{\partial G'(t, a')}{\partial t}. \tag{5.3.24b}$$

Under the conditions

$$\frac{\partial \Omega'_{\mu\nu}}{\partial t} = \frac{\partial^2 R'_\nu(t, a')}{\partial t\, \partial a'^\mu} - \frac{\partial^2 R'_\mu(t, a')}{\partial t\, \partial a'^\nu} \equiv 0, \tag{5.3.25}$$

an (analytic) function $A'(t, a')$ always exists such that

$$\frac{\partial R'_\mu}{\partial t} = \frac{\partial A'(t, a')}{\partial a'^\mu}. \tag{5.3.26}$$

The formal solution of gauge $G'(t, a')$ to verify condition (5.3.25) is then given by[25]

$$G'(t, a') = \int_0^t d\tau A'(\tau, a'). \tag{5.3.27}$$

In this way we reach the following *transformation rule of semi-autonomous Birkhoff's equations into semi-autonomous forms under noncanonical, contemporaneous, time-dependent transformations*:

$$t \to t' \equiv t, \qquad a^\mu \to a'^\mu(t, a), \tag{5.3.28a}$$

$$\left\{\left[\frac{\partial R_\nu(a)}{\partial a^\mu} - \frac{\partial R_\mu(a)}{\partial a^\nu}\right]\dot{a}^\nu - \frac{\partial B(t, a)}{\partial a^\mu}\right\}_{SA}$$

$$= \left\{\frac{\partial a'^\rho}{\partial a^\mu}\left\{\left[\frac{\partial R^\dagger_\sigma(a')}{\partial a'^\rho} - \frac{\partial R^\dagger_\rho(a')}{\partial a'^\sigma}\right]\dot{a}'^\sigma - \frac{\partial B^\dagger(t, a')}{\partial a'^\rho}\right\}_{SA}\right\}_{NSA} = 0, \tag{5.3.28b}$$

$$R^\dagger_\rho(a') = R'_\rho(t, a') - \frac{\partial G'(t, a')}{\partial a'^\rho}, \qquad R'_\rho(t, a') = \left(\frac{\partial a^\alpha}{\partial a'^\rho} R_\alpha\right)(t, a'), \tag{5.3.28c}$$

$$B^\dagger(t, a') = B'(t, a') + \frac{\partial G'(t, a')}{\partial t}, \qquad B'(t, a') = \left(B - \frac{\partial a^\alpha}{\partial t} R_\alpha\right)(t, a) \tag{5.3.28d}$$

$$G'(t, a') = \int_0^t d\tau A'(\tau, a'), \qquad \frac{\partial A'(t, a')}{\partial a'^\mu} = \frac{\partial R'_\mu}{\partial t}. \tag{5.3.28e}$$

We switch now to the study of Hamilton's and Birkhoff's equations under the most general possible transformations, those of type (5.1.10). The study can be essentially carried out via the generalization of the symplectic framework of Lemma 5.3.1 into the broader contact geometric setting.

Lemma 5.3.3. *All possible smoothness preserving and regular transformations*

$$\begin{aligned}(\hat{a}^\mu) = (t, a) \to (\hat{a}'^\mu(\hat{a})) &= (t'(t, a), a'(t, a)) \\ &= (t'(t, \mathbf{r}, \mathbf{p}), \mathbf{r}'(t, \mathbf{r}, \mathbf{p}), \mathbf{p}'(t, \mathbf{r}, \mathbf{p})) \\ \mu &= 0, 1, 2, \ldots, 2n\end{aligned} \tag{5.3.29}$$

[25] Sarlet and Cantrijn (1978a).

of the local variables of the $(2n + 1)$*-dimensional manifold* $\mathbb{R} \times T^*M$ *are contact isotopic, that is, a contact two-form*[26]

$$\hat{\Omega}_2 = \tfrac{1}{2}\hat{\Omega}_{\mu\nu}(\hat{a})d\hat{a}^\mu \wedge d\hat{a}^\nu, \qquad \operatorname{rank}(\hat{\Omega}_{\mu\nu}) = 2n, \tag{5.3.30a}$$

$$\delta^{\mu_1\mu_2}_{\nu_1\nu_2}\hat{\Omega}_{\mu_1\mu_2} = 0, \tag{5.3.30b}$$

$$\delta^{\mu_1\mu_2\mu_3}_{\nu_1\nu_2\nu_3}\frac{\partial\hat{\Omega}_{\mu_1\mu_2}}{\partial\hat{a}^{\mu_3}} = 0, \tag{5.3.30c}$$

$$\mu, \nu, \mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3 = 0, 1, 2, \ldots, 2n,$$

preserves its contact character under all transformations of the class admitted,

$$\hat{\Omega}_2 = \tfrac{1}{2}\hat{\Omega}_{\mu\nu}(\hat{a})d\hat{a}^\mu \wedge d\hat{a}^\nu = \frac{1}{2}\frac{\partial\hat{a}^\mu}{\partial\hat{a}'^\rho}\bar{\hat{\Omega}}_{\mu\nu}(\hat{a}')\frac{\partial\hat{a}^\nu}{\partial\hat{a}'^\sigma}d\hat{a}'^\rho \wedge d\hat{a}'^\sigma$$

$$\stackrel{\text{def}}{=} \tfrac{1}{2}\hat{\Omega}'_{\rho\sigma}(\hat{a}')d\hat{a}'^\rho \wedge d\hat{a}'^\sigma = \hat{\Omega}'_2, \qquad \operatorname{rank}(\hat{\Omega}'_{\rho\sigma}) = 2n, \tag{5.3.31a}$$

$$\delta^{\rho_1\rho_2}_{\sigma_1\sigma_2}\hat{\Omega}'_{\rho_1\rho_2} = 0, \tag{5.3.31b}$$

$$\delta^{\rho_1\rho_2\rho_3}_{\sigma_1\sigma_2\sigma_3}\frac{\partial\hat{\Omega}'_{\rho_1\rho_2}}{\partial\hat{a}'^{\rho_3}} = 0. \tag{5.3.31c}$$

PROOF. Preservation of the maximal rank is ensured by the regularity of the transformations, while the preservation of properties (5.3.30b) and (5.3.30c) can be proved via the same argument as that for Lemma 5.3.1. (*Q.E.D.*)

Lemma 5.3.3 provides the desired rules for the transformation of Birkhoff's equations, as well as of Hamilton's into Birkhoff's equations, under the desired most general possible transformations. However, in order to avoid insidious technical and conceptual aspects, it is important first to identify the "new time," that is, the variable which corresponds to t under transformations (5.3.29).

Recall that contact two-form (5.3.30a) has the matrix structure

$$(\hat{\Omega}_{\mu\nu}) = \left(\begin{array}{c|ccc} 0_{00} & \hat{\Omega}_{01} & \cdots & \hat{\Omega}_{02n} \\ \hline \hat{\Omega}_{10} & & & \\ \vdots & & (\cdots) & \\ \hat{\Omega}_{2n0} & & & \end{array}\right), \tag{5.3.32}$$

where the $(2n \times 2n)$-matrix $(\cdots)$ is symplectic and, as such, carries the maximal rank $2n$. In particular, *time is the variable associated with the diagonal element whose comatrix has maximal rank.* For the case of structure (5.3.32), time is the variable $\hat{a}^0$.

Now, a symplectic structure remains nondegenerate under regular transformations in $2n$-dimensions, but this is no longer necessarily the case when

[26] The generalized Kronecker's symbols were introduced in Section I.1.2, and are reviewed (in part) via Equations (4.1.55).

the transformations are in $(2n + 1)$ dimension. Thus the new matrix $(\Omega'_{\mu\nu}(t', a'))$ under transformations (5.3.29) is generally degenerate. In fact, the only regularity property ensured is that of the preservation of the maximal rank $2n$, but there is no guarantee that this rank is necessarily preserved in the image of the original $(2n \times 2n)$ matrix. It then follows that the new two-form (5.3.31a) has the structure

$$(\hat{\Omega}'_{\mu\nu}) = \left(\begin{array}{c|c|c} \begin{pmatrix} 0_{00} & \cdots \\ \vdots & \ddots \end{pmatrix} & \hat{\Omega}'_{0\mu} \begin{matrix} \\ \vdots \end{matrix} & \begin{pmatrix} \cdots & \hat{\Omega}'_{02n} \\ & \vdots \end{pmatrix} \\ \hline \hat{\Omega}'_{\mu 0} \quad \cdots & 0_{\mu\mu} & \cdots \quad \hat{\Omega}'_{\mu 2n} \\ \hline \begin{pmatrix} \vdots & \\ \hat{\Omega}'_{2n0} & \cdots \end{pmatrix} & \hat{\Omega}'_{2n\mu} \begin{matrix} \\ \vdots \end{matrix} & \begin{pmatrix} \ddots & \vdots \\ \cdots & 0_{2n2n} \end{pmatrix} \end{array}\right), \tag{5.3.33}$$

where the new symplectic substructure is given by the four matrices $(\cdots)$.

If we preserve the original definition of time, for consistency, we reach the following property.[27]

Corollary 5.3.3a. *The new time under transformations* (5.3.29) *can be any component* $\hat{a}'^{\mu}$ *of the new variables, where* μ *can assume any one of the values* $0, 1, 2, \ldots, 2n$.

The relevance of the result for the problem of relativity in Newtonian mechanics is self-evident and will be elaborated upon in Chapter 6. At this point we mention only that Corollary 5.3.3a identifies a form of equivalence of space and time variables which until now has been considered only within the context of the special relativity. In fact, the corollary establishes that, when one considers

(a) the most general possible dynamic equations on $\mathbb{R} \times T^*M$,
(b) the most general possible analytic equations, and
(c) the most general possible transformations,

the equivalence between space and time occur also in a purely *Newtonian* setting, although according to a structure considerably more complex than that of the special relativity.

Consider now an exact contact structure, in which case we can write[28]

$$\begin{aligned} \hat{\Omega}_2 &= d\hat{R}_1 = d(\hat{R}_\mu(\hat{a})d\hat{a}^\mu) \\ &= \frac{1}{2}\left(\frac{\partial \hat{R}_\nu(\hat{a})}{\partial \hat{a}^\mu} - \frac{\partial \hat{R}_\mu(\hat{a})}{\partial \hat{a}^\nu}\right) d\hat{a}^\mu \wedge d\hat{a}^\nu. \end{aligned} \tag{5.3.34}$$

[27] We imply here the use of dimensionless variables. When this is not the case, a dimensional scaling factor must be taken into account.

[28] As studied in Volume I and as reviewed in Section 4.1, two-form (5.3.20a) is an exact contact form whenever integrability conditions (5.3.30b) and (5.3.30c) hold in a star-shaped region of the variables.

The following property then holds.

Corollary 5.3.3b. *The transformation rule of a Pfaffian action on* $\mathbb{R} \times T^*M$ *under the most general possible transformations is given, in unified notation* $\hat{a}$, *by*

$$\hat{R}_\mu(\hat{a})d\hat{a}^\mu = \hat{R}'_\alpha(\hat{a}')d\hat{a}'^\alpha, \tag{5.3.35a}$$

$$\hat{R}'_\alpha(\hat{a}') = \left(\hat{R}_\mu \frac{\partial \hat{a}^\mu}{\partial \hat{a}'^\alpha}\right)(\hat{a}) \tag{5.3.35b}$$

$$\mu, \alpha = 0, 1, 2, \ldots, 2n$$

or, in disjoint notation $\hat{a} = (t, a)$, *by*

$$\begin{aligned} &R_\mu(t, a)da^\mu - B(t, a)dt \\ &\quad = \left[\left(R_\mu \frac{\partial a^\mu}{\partial a'^\alpha}\right)(t', a') - \left(B \frac{\partial t}{\partial a'^\alpha}\right)(t', a')\right]da'^\alpha \\ &\qquad - \left[\left(B \frac{\partial t}{\partial t'}\right)(t', a') - \left(R_\mu \frac{\partial a^\mu}{\partial t'}\right)(t', a')\right]dt' \\ &\quad \stackrel{\text{def}}{=} R'_\alpha(t', a')da'^\alpha - B'(t', a')dt' \\ &\qquad \mu, \alpha = 1, 2, \ldots, 2n \end{aligned} \tag{5.3.36}$$

where the time variable in the new coordinate system is the element $\hat{a}'^\mu$ *according to Corollary* 5.3.3*a, and the new Birkhoffian is the corresponding element* $\hat{R}'_\mu(\hat{a}')$.

The last part of the corollary has been presented to stress the fact that familiar symbols such as t', H', B', even though *mathematically* well-defined, do not necessarily carry their familiarly expected *physical* meaning.

To make the point more precise, let us consider the transformation of Birkhoff's equations. The following property is a trivial consequence of Lemma 5.3.3.

Corollary 5.3.3c. *The transformation rule of Birkhoff's equations under the most general possible transformations on* $\mathbb{R} \times T^*M$ *is given, in unified notation* (4.2.23), *by*[29]

$$\hat{a}^\mu \to \hat{a}'^\mu(\hat{a}), \tag{5.3.37a}$$

$$\hat{\Omega}_{\mu\nu}(\hat{a})d\hat{a}^\nu = \frac{\partial \hat{a}'^\rho}{\partial \hat{a}^\mu} \hat{\Omega}'_{\rho\sigma}(\hat{a}')d\hat{a}'^\sigma = 0, \tag{5.3.37b}$$

$$\hat{\Omega}'_{\rho\sigma}(\hat{a}) = \frac{\partial \hat{a}^\mu}{\partial \hat{a}'^\rho} \hat{\Omega}_{\mu\nu}(\hat{a}(\hat{a}')) \frac{\partial \hat{a}^\nu}{\partial \hat{a}'^\sigma}, \tag{5.3.37c}$$

$$\mu, \nu, \rho, \sigma = 0, 1, 2, \ldots, 2n,$$

[29] The treatment here is in terms of differentials $d\hat{a}'^\mu$, rather than derivatives $d\hat{a}'^\mu/dt'$, for two reasons. First of all, t' is not necessarily the "new time," as indicated earlier. Secondly, the approach permits a better focusing of the fact that, for noncontemporaneous transformations, the integrands of action functionals transform as densities, rather than scalars.

or, in disjoint notation (t, a), *by*

$$(\hat{\Omega}_{\mu\nu}(\hat{a})d\hat{a}^{\nu}) = \begin{pmatrix} \left(\frac{\partial B}{\partial a^{\nu}} + \frac{\partial R_{\nu}}{\partial t}\right)da^{\nu} \\ \left(\frac{\partial R_{\nu}}{\partial a^{\mu}} - \frac{\partial R_{\mu}}{\partial a^{\nu}}\right)da^{\nu} - \left(\frac{\partial B}{\partial a^{\mu}} + \frac{\partial R_{\mu}}{\partial t}\right)dt \end{pmatrix} = 0, \quad (5.3.38a)$$

$$(\hat{\Omega}'_{\rho\sigma}(\hat{a}')d\hat{a}'^{\sigma}) = \begin{pmatrix} \left(\frac{\partial B'}{\partial a'^{\sigma}} + \frac{\partial R'_{\sigma}}{\partial t'}\right)da'^{\sigma} \\ \left(\frac{\partial R'_{\sigma}}{\partial a'^{\rho}} - \frac{\partial R'_{\rho}}{\partial a'^{\sigma}}\right)da'^{\sigma} - \left(\frac{\partial B'}{\partial a'^{\rho}} + \frac{\partial R'_{\rho}}{\partial t'}\right)dt' \end{pmatrix} = 0, \quad (5.3.38b)$$

$$R'_{\rho}(t', a') = \left(R_{\mu}\frac{\partial a^{\mu}}{\partial a'^{\rho}} - B\frac{\partial t}{\partial a'^{\rho}}\right)(t', a'), \quad (5.3.38c)$$

$$B'(t', a') = \left(B\frac{\partial t}{\partial t'} - R_{\mu}\frac{\partial a^{\mu}}{\partial t}\right)(t', a'), \quad (5.3.38d)$$

$$\mu, \nu, \rho, \sigma = 1, 2, \ldots, 2n,$$

where the new time and Birkhoffian are given by the elements $\hat{a}'^{\mu}$ *and* $\hat{R}'_{\mu}$, *respectively, whose comatrix in two-form* (5.3.33) *has rank* $2n$.

The last statement can be proved as follows. Recall from Section 4.2 that the first term in Equations (5.3.38a) is identically null along all possible paths (which are not necessary solutions of the equations) owing, first of all, to the existence of the inverse

$$\Omega^{\mu\nu} = \left(\left\| \frac{\partial R_{\beta}}{\partial a^{\alpha}} - \frac{\partial R_{\alpha}}{\partial a^{\beta}} \right\|^{-1}\right)^{\mu\nu} \quad (5.3.39)$$

and, secondly, to the trivial identities

$$\left(\frac{\partial B}{\partial a^{\nu}} + \frac{\partial R_{\nu}}{\partial t}\right)da^{\nu} = \left(\frac{\partial B}{\partial a^{\nu}} + \frac{\partial R_{\nu}}{\partial t}\right)\Omega^{\nu\alpha}\left(\frac{\partial B}{\partial a^{\alpha}} + \frac{\partial R_{\alpha}}{\partial t}\right)dt \equiv 0. \quad (5.3.40)$$

The point is that the first term in the new equations (5.3.38b) *is not* necessarily identically null along all possible paths because of the lack of necessary existence of the inverse of the transformed matrix $(\Omega'_{\mu\nu}(t', a'))$. Out of the $(2n + 1)$ terms of Equations (5.3.38b), the only term which verifies properties corresponding to (5.3.40) is therefore that whose complement in two-form (5.3.33) has rank $2n$. This identically null term will consist of a sum of terms in $d\hat{a}'^{0}, d\hat{a}'^{1}, \ldots, d\hat{a}'^{2n}$ *less* only one term, say that in $d\hat{a}'^{\mu}$ for one given (fixed) μ. The new time is then $\hat{a}'^{\mu}$ and the new Birkhoffian is $R'_{\mu}(t', a')$.[30]

[30] Note that, despite a contrary appearance, all the $(2n + 1)$ equations of the column (5.3.38b) have the same number of terms $2n$, trivially, because of the antisymmetry of Birkhoff's tensor. As a result, Equations (5.3.38b) are fully "symmetric" in all variables $\hat{a}' = (t', a') = (t', \mathbf{r}', \mathbf{p}')$, and this symmetry is at the origin of the arbitrariness of the new time.

This confirms that the new variable $t'(t, \mathbf{r}, \mathbf{p})$ is not necessarily the new time. This is remarkable on relativity grounds for a number of reasons. As we shall see more clearly in the next chapter, Birkhoff's equation are generally form non-invariant under Galilei transformations (this is also a general property of Newton's equations of motion (Chart 4.12)). Their form invariance therefore calls for the identification of more general symmetries which must be a subclass of transformations (5.3.38). Now, dependences of the type $t'(t, \mathbf{r}, \mathbf{p})$ $(= t''(t, \mathbf{r}, \dot{\mathbf{r}}))$ are typical of the special relativity, but *not* of Galilean relativity (for which $t' = t + t_0$). Their occurrence in Newtonian mechanics is therefore new. Second, in both the Galilean (and the special) relativity, the new time is simply the image t' of t as characterized by the transformations *only*. In the covering Birkhoffian Mechanics, the new time is characterized by both the transformations *and* the underlying dynamics, that is, the $\hat{R}$-functions, and this is an additional novel feature. Last, but not least, Equations (5.3.38b) are fully symmetric in all variables $\hat{a}' = (t', \mathbf{r}', \mathbf{p}')$, and this is also remarkably new in Newtonian mechanics.

Lemma 5.3.3 clearly contains the transformation rule of Hamilton's equations which is given below for the reader's convenience.

Corollary 5.3.3d. *Under the most general possible transformations on* $\mathbb{R} \times T^*M$, *Hamilton's equations transform into Birkhoff's equations with the indicated prescription for the identification of the new time and Birkhoffian, and we write*

$$
\begin{aligned}
(\hat{\omega}_{\mu\nu} d\hat{a}^\nu) &= \begin{pmatrix} \dfrac{\partial H}{\partial a^\nu} da^\nu \\ \omega_{\mu\nu} da^\nu - \dfrac{\partial H}{\partial a^\mu} dt \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{\partial H}{\partial a^\nu} \omega^{\nu\alpha} \dfrac{\partial H}{\partial a^\alpha} dt \\ \left(\dfrac{\partial R^0_\nu}{\partial a^\mu} - \dfrac{\partial R^0_\mu}{\partial a^\nu} \right) da^\nu - \dfrac{\partial H}{\partial a^\mu} dt \end{pmatrix} = \left(\frac{\partial \hat{a}^\mu}{\partial \hat{a}'^\rho} \hat{\Omega}'_{\rho\sigma}(\hat{a}') d\hat{a}'^\sigma \right) \\
&= \begin{pmatrix} \dfrac{\partial t}{\partial t'} \dfrac{\partial t}{\partial a^\rho} 0 \\ 0 \dfrac{\partial a^\mu}{\partial t} \dfrac{\partial a^\mu}{\partial a^\rho} \end{pmatrix} \begin{pmatrix} \left(\dfrac{\partial B'}{\partial a'^\sigma} + \dfrac{\partial R'_\sigma}{\partial t'} \right) da'^\sigma \\ \left(\dfrac{\partial R'_\sigma}{\partial a'^\rho} - \dfrac{\partial R'_\rho}{\partial a'^\sigma} \right) da'^\sigma - \left(\dfrac{\partial B'}{\partial a'^\rho} + \dfrac{\partial R'_\rho}{\partial t'} \right) dt' \end{pmatrix} = 0,
\end{aligned}
\tag{5.3.41}
$$

where B' *and* R'_μ *are given by Equations* (5.3.38*c*) *and* (5.3.38*d*), *respectively.*

The property above is also remarkable inasmuch *it does not* admit a Lagrangian counterpart in the following sense. Whether Hamiltonian or Pfaffian, first-order action principles on $\mathbb{R} \times T^*M$ remain first-order under

the most general possible transformations. On the contrary, first-order Lagrangian principles on $\mathbb{R} \times TM$ *do not* preserve their first-order character under the most general possible transformations. In fact, the transformations are velocity-dependent and, as such, transform first-order Lagrangians (second-order Lagrange's equations) into second-order Lagrangians (third-order Lagrange's equations[31]).

The property expressed by Corollary 5.3.3d also indicates the (rather unpredictable) fact that Hamilton's equations preserve their analytic, algebraic, and geometric characters under the most general possible transformations.[32] This aspect can be made mode precise by noting that Equations (5.3.30b) and (5.3.30c) *are* the conditions of variational self-adjointness for first-order systems.[33] Lemma 5.3.3 therefore expresses the preservation of the variational self-adjointness under arbitrary transformations of the class considered. By keeping in mind the analytic, algebraic, and geometric meaning of the conditions of self-adjointness, we have the following property.

Corollary 5.3.3e. *All possible transformations on* $\mathbb{R} \times T^*M$ *are self-adjoint isotopic when the transformed system is that defined without the Jacobian, i.e.,*

$$(\hat{\Omega}_{\mu\nu} d\hat{a}^{\nu})_{\mathrm{SA}} = \frac{\partial \hat{a}^{\rho}}{\partial a'^{\mu}} (\hat{\Omega}'_{\rho\sigma} d\hat{a}'^{\sigma})_{\mathrm{SA}} = 0. \tag{5.3.42}$$

As a consequence, Hamilton's and Birkhoff's equations preserve their derivability from a variational principle, and their Lie algebraic character and contact geometric structure under the most general possible transformations.

The frame independence of the analytic/algebraic/geometric characteristics then turns out to be at the basis of the coordinate-free globalization of the contact geometry, as presented in the specialized literature on this topic.[20]

Notice that the transformations of the equations of motion according to Corollary 5.3.3e *preserve* the self-adjointness, as well as the non-self-adjointness. Thus they are not intended for the Inverse Birkhoffian/Hamiltonian Problem, which demands the use of self-adjointness-*inducing* transformations. These latter transformations are readily given by a subclass of the transformations inclusive of the Jacobian, i.e.,

$$\begin{aligned} (\hat{C}_{\mu\nu}(\hat{a}) d\hat{a}^{\nu})_{\mathrm{NSA}} &= (\hat{C}^{*}_{\mu\sigma}(\hat{a}') d\hat{a}'^{\sigma})_{\mathrm{SA}} \\ &= \left[\frac{\partial \hat{a}^{\rho}}{\partial \hat{a}'^{\mu}} (\hat{C}'_{\rho\sigma}(\hat{a}') d\hat{a}'^{\sigma})_{\mathrm{NSA}} \right]_{\mathrm{SA}} = 0. \end{aligned} \tag{5.3.43}$$

[31] See, in this respect, Equations (4.2.35).

[32] A moment of reflection on the unified notation $\hat{a}$ is important here. In fact, the *intuition* of this result in the disjoint variables t, $\mathbf{r}$, and $\mathbf{p}$ (let alone its proof) would be virtually impossible.

[33] See Section 4.1 for a review of the studies of Volume I on the topic.

These transformations, in particular, turn out to be "universal" in the sense that, given a system in the $\hat{a}$-variables which is not Hamiltonian, transformations $\hat{a} \to \hat{a}'(\hat{a})$ always exist under which the transformed system defined via the inclusion of the Jacobian is Hamiltonian (see the Section 6.2.).

We pass now to the study of the property that *the canonical transformation theory admits a consistent step-by-step generalization of Birkhoffian type.* It should be indicated that the property has been studied in local coordinates either implicitly or explicitly by a number of authors, such as De Donder (1927), Lee (1945), Pauli (1953), Martin (1959), Hughes (1961), Cartan (1971), Sudarshan and Mukunda (1974), Santilli (1978c), Sarlet and Cantrijn (1978a,b), Kobussen (1979), and others. From a global viewpoint, the property can be studied via the transformation theory on a contact manifold.[20]

For this purpose, it is recommended that we reinspect the notion of canonical transformations within the context of the preceding analysis and return to the study of autonomous systems under contemporaneous transformations without an explicit dependence on time.

Definition 5.3.1.[34] Transformations $a \to a'(a)$ of the local variables of T^*M are *canonical* when they are *Lie identity isotopic* with respect to the fundamental Lie tensor $\omega^{\mu\nu}$ or, equivalently, when they are *symplectic identity isotopic* with respect to the fundamental symplectic tensor $\omega_{\mu\nu}$.

As is now familiar (see the remarks regarding Equation (5.3.15)), the canonical transformations according to Definition 5.2.1 not only preserve the Lie algebra (Lie isotopy), but actually preserve the value of the fundamental brackets; that is, they preserve identically the realization of the Lie algebra product. These properties are expressed by Definition 5.3.1 via the notion of "Lie identity isotopy." A fully equivalent situation exists for the case of the "symplectic identity isotopy."

Once these algebraic or geometric aspects have been understood, their generalization to the Birkhoffian case is straightforward.

Definition 5.3.2.[34] A *generalized canonical transformation* is a *Lie identity isotopic transformation* of the contravariant Birkhoff's tensor $\Omega^{\mu\nu}$ or, equivalently, a *symplectic identity isotopic transformation* of the covariant tensor $\Omega_{\mu\nu}$.

Explicitly, this definition implies the transformation rule

$$\Omega^{\mu\nu}(a) \to \Omega'^{\mu\nu}(a') = \frac{\partial a'^{\mu}}{\partial a^{\rho}} \Omega^{\rho\sigma}(a(a')) \frac{\partial a'^{\nu}}{\partial a^{\sigma}} \equiv \Omega^{\mu\nu}(a') \tag{5.3.44}$$

as a natural generalization of rule (5.2.2), with the understanding that the Lie isotopy is always ensured by Lemma 5.3.1 and that the Lie identity isotopy

[34] Santilli (*loc. cit.*).

is obtained via the *condition* $\Omega'^{\mu\nu} \equiv \Omega^{\mu\nu}$. Thus generalized canonical transformations are a *subclass* of all possible transformations on T^*M. Also, *a canonical transformation is not necessarily a generalized one, and vice versa.* This is clearly the result of the fact that the Lie identity isotopy is obtained via conditions (5.3.44) for the former, and different conditions $\Omega'^{\mu\nu} \equiv \omega^{\mu\nu}$ for the latter.

The *transformation rule of the covariant, semi-autonomous, Birkhoff's equations under symplectic identity isotopic transformations* is therefore given by the following particular case of rule (5.3.2)

$$t \to t' \equiv t, \qquad a^\mu \to a'^\mu(a), \tag{5.3.45a}$$

$$\left\{\left[\frac{\partial R_\nu(a)}{\partial a^\mu} - \frac{\partial R_\mu(a)}{\partial a^\nu}\right]\dot{a}^\nu - \frac{\partial B(t,a)}{\partial a^\mu}\right\}_{SA}$$
$$= \left\{\frac{\partial a'^\rho}{\partial a^\mu}\left\{\left[\frac{\partial R_\sigma(a')}{\partial a'^\rho} - \frac{\partial R_\rho(a')}{\partial a'^\sigma}\right]\dot{a}'^\sigma - \frac{\partial B'(t,a')}{\partial a'^\rho}\right\}_{SA}\right\}_{NSA} = 0, \tag{5.3.45b}$$

$$R_\mu(a) \to R'_\mu(a') = \left(\frac{\partial a^\alpha}{\partial a'^\mu} R_\alpha\right)(a') \equiv R_\mu(a'), \tag{5.3.45c}$$

$$B(t,a) \to B'(t,a') = B(t,a'(a)). \tag{5.3.45d}$$

In this way we see that the notion of symplectic (and Lie) identity isotopy can be reduced to that of the preservation of the functional dependence of the primitive one-form, i.e., to the following particular case of rule (5.3.35)

$$R_1 = R_\mu(a)da^\mu = R_\mu(a(a'))\frac{\partial a^\mu}{\partial a'^\alpha}da'^\alpha \equiv R_\alpha(a')da'^\alpha. \tag{5.3.46}$$

However, at the level of these one-forms, the notion is always defined up to Birkhoffian gauges. Clearly, the notion of generalized canonical transformations is a *covering* of that of conventional canonical transformations, in the sense that all conditions of footnote 24 are verified.

As indicated in the general assumptions of Section 5.1, all transformations are considered in a given region of the variables. It can be proved that the topological properties of these regions are preserved from the general smoothness and regularity conditions assumed in this work. However, the "range" of the new and old regions may be different, by therefore creating problematic aspects for the construction of realizations of Lie groups via both conventional and generalized canonical transformations. This situation suggests the following refinement of Definition 5.3.2.

Definition 5.3.3.[35] A transformation $a \to a'(a)$ of the local variables of T^*M which is analytic and regular in a region $\tilde{\mathcal{R}}$ is called *strictly Lie identity*

[35] Sarlet and Cantrijn (*loc. cit.*).

isotopic with respect to Birkhoff's tensor $\Omega^{\mu\nu}(a)$ when Lie identity isotopy (5.3.44) holds and, in addition, the range of the old variables and its image under the transformations coincide, $\tilde{\mathscr{R}} \equiv \tilde{\mathscr{R}}'$.

The condition of preservation of the range here essentially refers to the condition that the numerical values admitted by the old and new variables coincide. For additional studies, we refer the interested reader to the work of Sarlet and Cantrijn.[35]

The extension of the results to the case of contemporaneous transformations with an explicit time dependence is straightforward and will be tacitly assumed from this point on.

We move now to the study of the integrability conditions for the existence of generalized canonical transformations.

Proposition 5.3.1.[35] *A necessary and sufficient condition for a contemporaneous, time-dependent transformation* $t \to t' = t$, $a \to a'(t, a)$ *to be a Lie identity isotopic transformation with respect to Birkhoff's tensor* $\Omega^{\mu\nu}(t, a)$ *is that a smoothness-preserving function* $F(t, a)$ *exists such that*

$$R'_\mu(t, a') = R_\mu(t, a') + \frac{\partial F(t, a')}{\partial a'^\mu} \tag{5.3.47}$$

for all points of the region of definition.

PROOF. By recalling the Birkhoffian gauges, Condition (5.3.47) implies that

$$\frac{\partial}{\partial a'^\mu}[R'_\nu(t, a') - R_\nu(t, a')] - \frac{\partial}{\partial a'^\nu}[R'_\mu(t, a') - R_\mu(t, a')] = 0. \tag{5.3.48}$$

Thus the one-form

$$\theta_1 = [R'_\mu(t, a') - R_\mu(t, a')]da'^\mu \tag{5.3.49}$$

is a closed parametric one-form (Chart 4.6), therefore implying Equations (5.3.47). The necessity of the conditions then follows from the exact character of the Birkhoffian two-forms, while the sufficiency is trivially proved from the same character. (*Q.E.D.*)

The particularization of Proposition 5.3.1 to the case of canonical transformations is instructive (Problem 5.5). Notice that conditions (5.3.47) do not characterize the function F uniquely.

We move now to the study of the methods for the construction of generalized canonical transformations, via a step-by-step generalization of those for conventional canonical transformations. This also serves the purpose of illustrating the fact that Hamiltonian Mechanics admits a consistent covering of Birkhoffian type.

Let us begin by reformulating the definition of generalized canonical transformations via a variational principle. The following generalization of Weiss's (or Holder's) principle for Birkhoff's equations

$$\hat{\delta}\int_{t_1}^{t_2}[R_\mu(t, a)da^\mu - B(t, a)dt](\tilde{E}_0)$$
$$= (EPC)^{(1)} = |R_\mu(t, a)\hat{\delta}a^\mu - B(t, a)\hat{\delta}t|_{t_1}^{t_2}(\tilde{E}_0), \tag{5.3.50}$$

can be proved via a straightforward application of Equations (I.1.3.39) page I.43 (with the identification $q = a$). Principle (5.3.50) is remarkable inasmuch it shows that the total differential of Birkhoff's action is equal to the *integrand* computed at end points. In turn, this property is relevant for a number of applications, e.g., the construction of the Hamilton–Jacobi Theory for Birkhoff's equations, the problem of symmetries and first integrals, etc.

We now call a contemporaneous transformation $a \to a'(t, a)$ identity isotopic with respect to Birkhoff's equations when principle (5.3.50) holds in the new coordinates *without altering the functional dependence of the R functions*, i.e.,

$$\hat{\delta} \int_{t_1}^{t_2} [R_\mu(t, a')da'^\mu - B'(t, a')dt](\tilde{E}_0) = |R_\mu(t, a')\hat{\delta}a'^\mu - B'(t, a')dt|_{t_1}^{t_2}(\tilde{E}_0). \tag{5.3.51}$$

This latter condition is clearly essential in achieving the identity isotopy.[36]

As for the canonical case, the difference between the integrands of principles (5.3.50) and (5.3.51) is not identically null, but can at most be equal to the total differential of a function $F(t, a, a')$. Thus we reach the following *fundamental identity* for the construction of generalized canonical transformations ($\hat{\delta} \approx d$)

$$R_\mu(t, a)da^\mu - R_\mu(t, a')da'^\mu - B(t, a)dt + B'(t, a')dt = dF(t, a, a'), \tag{5.3.52}$$

which is clearly a direct generalization of Equations (5.2.14). Indeed, identity (5.3.52) is expressed in terms of Birkhoff functions, while it trivially recovers the canonical identity (5.2.14) for $R = (\mathbf{p}, \mathbf{0})$.

The function F of Equation (5.3.52) can thus be called the *generating function* of the generalized canonical transformations. This function, in particular, can be a function of any $2n$-dimensional subset of the variables (a, a') and time. Assume first that $F = F(t, a')$. Then identity (5.3.52) can be explicitly written

$$R_\alpha \frac{\partial a^\alpha}{\partial a'^\mu} da'^\mu - R_\mu da'^\mu - \left(B + B' + R_\alpha \frac{\partial a^\alpha}{\partial t}\right)dt = \frac{\partial F}{\partial a'^\mu} da'^\mu + \frac{\partial F}{\partial t} dt, \tag{5.3.53}$$

yielding the transformation laws

$$R'_\mu = R_\alpha \frac{\partial a^\alpha}{\partial a'^\mu} = R^\mu + \frac{\partial F}{\partial a'^\mu}, \tag{5.3.54a}$$

$$B' = B + \frac{\partial F}{\partial t} + R_\alpha \frac{\partial a^\alpha}{\partial t}. \tag{5.3.54b}$$

[36] As was the case for conventional canonical transformations, we expect the existence of transformations which are generalized canonical in the sense of preserving variational principle (5.3.50), but not in the sense of a Lie identity isotopy (5.3.44). This aspect is left to the interested reader (Problem 5.6).

In this way we recover Equations (5.3.47). In particular, we see that relations (5.3.54) are a generalization of the case $F = F_5$ of the canonical transformations (Equations (5.2.24)).

Note that, besides case (5.3.47), there are numerous other possible cases, depending on the selected $2n$-dimensional subset of the variables (a, a').[37]

For the reader's convenience, as well as for further needs, we compute here the case of $F = F_1(t, q, q')$. Identity (5.3.52) in this case is, for $R = (R_k, R^k)$,

$$R_k dq^k + R^k\left(\frac{\partial p_k}{\partial q^i} dq^i + \frac{\partial p_k}{\partial q'^i} dq'^i + \frac{\partial p_k}{\partial t} dt\right)$$
$$- \bar{R}_k dq'^k - \bar{R}^k\left(\frac{\partial p'_k}{\partial q^i} dq^i + \frac{\partial p'_k}{\partial q'^i} dq'^i + \frac{\partial p'_k}{\partial t} dt\right) - B\, dt + B'\, dt$$
$$= \frac{\partial F_1}{\partial q^k} dq^k + \frac{\partial F_1}{\partial q'^k} dq'^k + \frac{\partial F_1}{\partial t} dt,$$
$$\bar{R} = R(t, q', p') \tag{5.3.55}$$

yielding the transformation laws

$$R_k + R^i \frac{\partial p_i}{\partial q^k} = \bar{R}'^i \frac{\partial p'_i}{\partial q^k} + \frac{\partial F_1}{\partial q^k}, \tag{5.3.56a}$$

$$\bar{R}_k + \bar{R}^i \frac{\partial p'_i}{\partial q'^k} = R^i \frac{\partial p_i}{\partial q'^k} - \frac{\partial F_1}{\partial q'^k}, \tag{5.3.56b}$$

$$B' = B + \frac{\partial F_1}{\partial t} - R^i \frac{\partial p_i}{\partial t} + \bar{R}^i \frac{\partial p'_i}{\partial t}, \tag{5.3.56c}$$

which are indeed a generalization of rules (5.2.16) for the Hamiltonian case. In particular, the latter rules are recovered identically for $R = (\mathbf{p}, \mathbf{0})$, that is, for the Hamiltonian subcase. The generalization of the other Hamiltonian cases F_2, F_3, F_4, and F_6 is then straightforward, and it is left here to the interested reader, along with the study of other aspects (Problem 5.7).

Note that the generating functions of canonical and generalized canonical transformations may coincide. Nevertheless, the corresponding transformations are different.

This completes our study of the Birkhoffian generalization of the *canonical* transformation theory. We shift now to the generalization of the theory of *canonoid* transformations of the *reduced* type, which permits a Hamiltonian image of the isotopic transformations of a Lagrangian (Section A.2).

[37] All these dependences, however, can be reduced to the a' dependence through a generalization of the Legendre transform of the canonical case. This reduction has been assumed in Proposition 5.3.1.

Proposition 5.3.2. *The phase space image of the isotopic transformations of a Lagrangian, $L(q, \dot{q}) \to L^*(q, \dot{q})$, given by the reduced canonoid transformations*

$$\omega_{\mu\nu}\dot{a}^\nu - \frac{\partial H}{\partial a^\mu} = 0 \to \omega_{\mu\nu}\dot{a}^{*\nu} - \frac{\partial H^*}{\partial a^{*\mu}} = 0, \tag{5.3.57a}$$

$$\{a^\mu\} \to \{a^{*\mu}\} = \{q^k, p^*(q, p)\}, \qquad H^*(a^*) \neq H(a), \tag{5.3.57b}$$

verfies the chain rule

$$\begin{aligned}\left[\omega_{\mu\nu}\dot{a}^{*\nu} - \frac{\partial H^*}{\partial a^{*\mu}}\right]_{SA} &= \left\{\frac{\partial a^\rho}{\partial a^{*\mu}}\left[\left(\frac{\partial R^\sigma}{\partial a^\rho} - \frac{\partial R^\rho}{\partial a^\sigma}\right)\dot{a}^\sigma - \frac{\partial B}{\partial a^\rho}\right]_{SA}\right\}_{NSA} \\ &= \left\{\frac{\partial a^\rho}{\partial a^{*\mu}}\left[h^\sigma_\rho\left(\omega_{\sigma\alpha}\dot{a}^\alpha - \frac{\partial H}{\partial a^\sigma}\right)_{SA}\right]_{SA}\right\}_{NSA} = 0,\end{aligned} \tag{5.3.58}$$

for some matrix (h^ν_μ) of isotopic functions with respect to H.

PROOF. A reduced canonoid transformation $a \to a^*$ (which is not the identity) is noncanonical, as is its inverse $a^* \to a$. Thus Hamilton's equations transform into Birkhoff's equations under the inverse transformation, according to general rule (5.3.19). This proves the first step of rule (5.3.58), but the systems admit a Hamiltonian representation $H(a)$ by assumption. The second step of rule (5.3.58) then follows from the self-adjointness of Birkhoff's equations. The matrix (h^ν_μ) is'then necessarily isotopic with respect to H. (*Q.E.D.*)

The proposition establishes a rather natural emergence of Birkhoff's equations via the degrees of freedom of a Lagrangian induced by the integrability conditions for its existence. In turn, this has rather intriguing implications. In essence, Proposition 5.3.2 establishes that, under the integrability conditions for the existence of isotopically mapped Lagrangians, the *same* system can be represented with *both*, Hamilton's and Birkhoff's equations in the *same* local variables. This means that the acting forces are such to allow the following redefinitions at a fixed point of the a space

$$\omega^{\mu\nu}\frac{\partial H}{\partial a^\nu} = \Omega^{\mu\nu}(a)\frac{\partial B}{\partial a^\nu}, \tag{5.3.59a}$$

$$H(a) \neq B(a). \tag{5.3.59b}$$

In turn, this implies the *lack* of uniqueness of the time evolution law and related brackets, in the sense that, whenever a Lagrangian $L(q, \dot{q})$ admits an isotopic image $L^*(q, \dot{q})$, the time evolution law admits the *dual* characterization

$$\dot{A}(a) = \frac{\partial A}{\partial a^\mu}\dot{a}^\mu = \begin{cases}\dfrac{\partial A}{\partial a^\mu}\omega^{\mu\nu}\dfrac{\partial H}{\partial a^\nu} = [A, H]_{(a)}, \\ \dfrac{\partial A}{\partial a^\mu}\Omega^{\mu\nu}(a)\dfrac{\partial B}{\partial a^\nu} = [A, B]^*_{(a)}.\end{cases} \tag{5.3.60}$$

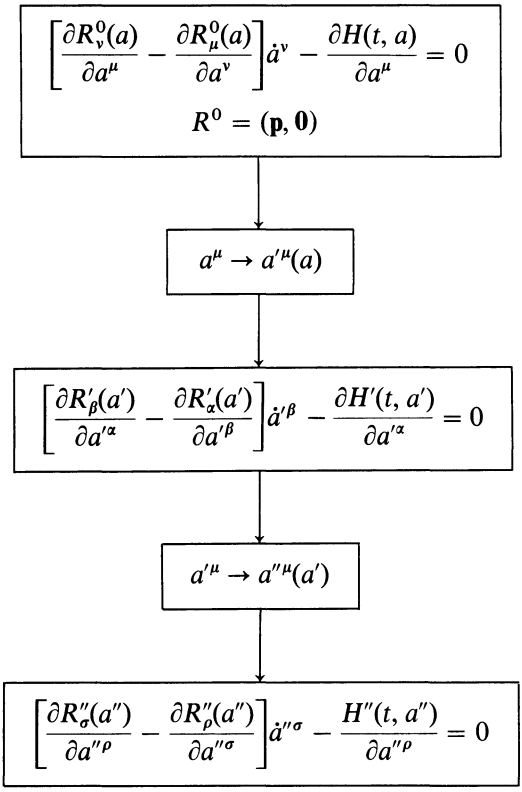


Figure 5.1. *Construction of the Birkhoffian Mechanics via the transformation of the Hamiltonian Mechanics.* This figure schematically represents the main idea of this chapter: that virtually all aspects of Birkhoffian Mechanics can be constructed via noncanonical transformations of the corresponding aspects of Hamiltonian Mechanics. The application of the rule begins with the birth of Birkhoff's equations, as schematically represented above. The rule then applies for the construction of the transformation theory of the new mechanics, as shown in Section 5.3. The rule will also apply for the construction of other aspects of the new mechanics, such as the generalization of the Hamilton–Jacobi theory, as we shall see in the next chapter. This rule should be kept in mind because it can be of considerable guidance in the construction of other aspects of Birkhoffian Mechanics, such as the generalization of the canonical perturbation theory. Particularly significative is the aspect established by the transformation theory that *Birkhoffian Mechanics is the most general possible mechanics that can be constructed from Hamiltonian Mechanics via the transformation theory.* In fact, Birkhoff's equations preserve their structure under the most general possible transformations (Lemma 5.3.1). The point serves also to illustrate the fact that the Birkhoff-admissible equations and related mechanics (Chart 4.7) are truly novel in the sense that they cannot be constructed from Birkhoff's equations via the transformation theory. This is a sign indicating the existence of new mathematical tools needed in the transition from Birkhoff's to Birkhoff-admissible equations. These tools have been interpreted in this volume as being of Lie-admissible genotopic type. It should be indicated here that the idea schematically expressed in this figure has implications far beyond Newtonian Mechanics. In fact, the idea is currently being applied to the construction of generalizations of other branches of physics, such as Statistical Mechanics and Atomic Mechanics, into forms compatible with the Birkhoffian Mechanics. For a review of these latter studies, see Santilli (1982). For a general treatment see the *Proceedings of the First International Conference on Nonpotential Interactions and their Lie-admissible Treatment* (1982). For an indication of the main ideas, see the charts of the next chapter.

Redefinition (5.3.60) then implies the transition

$$[A, H]_{(a)} \to [A, B]^{*}_{(a)} \tag{5.3.61}$$

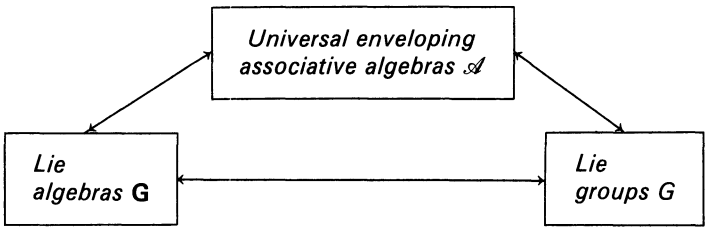
which is precisely a *Newtonian realization of the Lie isotopy* of Chart 4.2—this time, *within a fixed system of local variables.*

Thus we confirm that, beside the general case of Lie isotopy under arbitrary transformations (Lemma 5.3.1), there is also a particular type of Lie isotopy of the brackets of the time evolution law within a fixed system of local variables.

This situation has rather intriguing algebraic as well as group theoretical implications for Lie's theory, to be indicated in the charts of this chapter. At this point we simply remark that, a classical realization of the generators does *not* imply a unique Lie algebra, trivially, because the *same* generators can be equipped with *different* Lie products in the *same* space of the local variables. We therefore expect the possibility of characterizing *non-isomorphic* Lie algebras via the *same* generators and the use of *different* Lie products. As we shall see, this occurrence is an application of the isotopic generalization of Lie's theory and has a particular meaning within the context of symmetries and first integrals.

Chart 5.1 Need to Generalize the Contemporary Formulation of Lie's Theory

The terms "Lie's theory" are referred today to an articulated body of sophisticated mathematical tools encompassing several diversified disciplines. Whether in functional analysis or in the theory of linear operators, the structure of the contemporary formulation of Lie's theory can be reduced to the following three parts



As duly emphasized in the mathematical literature (see, for instance, Jacobson (1962), Dixmier (1977), and others), a truly fundamental part of Lie's theory is the enveloping algebra $\mathscr{A}$. In fact, algebra $\mathscr{A}$ provides a symbiotic characterization of both the Lie algebras and the Lie groups. This is due to the fact that the basis of $\mathscr{A}$ (which is constructed via the Poincaré–Birkhoff–Witt Theorem, to be reviewed in the next chart) is given by an infinite number of suitable polynomial powers of the generators X_i of **G** such as

$$\mathscr{A}: \mathbb{1} \in \mathbb{F}; X_i; X_iX_j (i \leq j); X_iX_jX_k (i \leq j \leq k); \cdots \tag{1}$$

where the products $X_i X_j$, etc., are associative. It then follows that the Lie algebra **G**

$$\mathbf{G}\colon [X_i, X_j] = X_i X_j - X_j X_k = C_{ij}^k X_k, \tag{2}$$

is (homomorphic to) the attached algebra $\mathscr{A}^-$ of $\mathscr{A}$. The Lie group G of **G** is then the infinite power series

$$G\colon e^{\theta^k X_k} = \mathbb{1} + \frac{\theta^k}{1!} X_k + \frac{\theta^i \theta^j}{2!} X_i X_j + \cdots \tag{3}$$

which, evidently, can be properly defined and treated only in the enveloping algebra (note that all terms from $X_i X_j$ on are *outside* the Lie algebra). One can then see why fundamental aspects of Lie *algebras* (such as the representation theory) are treated by mathematicians within the context of its *enveloping algebra*.

On physical grounds, the role of the enveloping algebra is equally crucial, even though not sufficiently emphasized in the current literature. For instance, a frequent physical problem is the computation of the magnitude of physical quantities, such as the magnitude (eigenvalue) of the angular momentum (operator) $M = \|\mathbf{M}^2\|^{1/2}$. While the components M_i of **M** are elements of the Lie algebra **SO**(3), the quantity $\mathbf{M}^2$ is *outside* **SO**(3) and can only be defined in the (center of) the enveloping algebra $\mathscr{A}(\mathbf{SO}(3))$. Thus, while the Lie algebra **SO**(3) essentially characterizes the components of the angular momentum and their commutation rules, the envelope $\mathscr{A}(\mathbf{SO}(3))$ characterizes: 1) the components M_i; 2) their commutation relations via the attached rule $\mathscr{A}^- \approx \mathbf{SO}(3)$; 3) the magnitude of the angular momentum $\mathbf{M}^2$; 4) the exponentiation to the Lie group of rotations; 5) the representation theory, etc. In short, we can state that a *truly primitive part of the contemporary formulation of Lie's theory is its universal enveloping associative algebra*.

Once the mathematical and physical motivations of this occurrence are understood in full, the need for a suitable generalization of Lie's theory becomes unavoidable. Lie algebras emerge in Physics at the truly fundamental part, the brackets of the time evolution. The above remarks then imply that the primitive algebraic structure of the time evolution is the enveloping algebra. Santilli (1978e, pp. 1330–1334) points out that the enveloping algebra of the time evolution of Hamiltonian Mechanics is *nonassociative*, by therefore being not directly compatible with the contemporary formulation of Lie's theory. In fact, the author essentially indicated that the conventional Poisson brackets

$$\mathbf{G}\colon [X_i, X_j] = \frac{\partial X_i}{\partial a^\mu}\, \omega^{\mu\nu}\, \frac{\partial X_i}{\partial a^\nu} = \frac{\partial X_i}{\partial r^k}\frac{\partial X_j}{\partial p_k} - \frac{\partial X_j}{\partial r^k}\frac{\partial X_i}{\partial p_k} \tag{4}$$

are the attached brackets of the algebra

$$\mathscr{U}\colon (X_i, X_j) = \frac{\partial X_i}{\partial r^k}\frac{\partial X_j}{\partial p_k} \tag{5}$$

which is *nonassociative*; that is, the vector space $\mathscr{A}$ of elements X_i and their polynomial powers, over the field $\mathbb{R}$ of real numbers equipped with product (5), is first of all an algebra in the sense of Chart 4.1: it verifies the left and right distributive laws and the scalar law. Secondly, this algebra turns out to be nonassociative because of properties

$$((X_i, X_j), X_k) \neq (X_i, (X_j, X_k)). \tag{6}$$

Since associative and nonassociative algebras are *different algebras*, without a known interconnecting mapping, Santilli (*loc. cit.*) argues that the insistence on the associative character of the envelope would literally prohibit the conventional formulation of Hamiltonian Mechanics, that according to time evolution (4). He therefore advocates a dual generalization of Lie's Theory (see the preceding paper, Santilli (1978c, pp. 298–375) as well as the memoir (1979a, Section 1.2) according to the following classification.

I. *Contemporary Formulation of Lie's Theory*. This is the formulation available in the contemporary literature, and it is expressed via an envelope with *conventional associative product* X_iX_j (e.g., the conventional product of matrices or operators).
II. *Lie-Isotopic Generalization of Lie's Theory*. This is a first generalization based on envelopes which are still associative yet are formulated via *the most general possible associative product*, say, $X_i * X_j$, whose attached product $X_i * X_j - X_j * X_i$ is Lie.
III. *Lie-Admissible Generalization of Lie Theory*. This is the largest possible generalization of Lie's theory conceivable at this time. It is based on envelopes that are Lie-admissible (Chart 4.1), that is, on envelopes with *the most general possible nonassociative product*, say, (X_i, X_j), whose attached product $(X_i, X_j) - (X_j, X_i)$ is Lie.

In subsequent charts we shall outline the state of the art on the Lie-isotopic generalization of Lie's theory. The Lie-admissible generalization is currently under study at the yearly Workshops on Lie-Admissible Formulations (see the proceedings (1979–1981)) and will not be reviewed here.

A few introductory remarks may help the reader to reach a better mathematical and physical understanding of the generalizations under consideration, as well as the truly intriguing (and substantial) research yet to be done.

A difficulty generally experienced by mathematicians trying to see the need for a generalization of Lie's theory is that simple Lie algebras over a field of characteristic zero have been classified and are given by the well-known Cartan classification.[38] In fact, the Poincaré–Birkhoff–Witt theorem essentially ensures that *all* Lie algebras over a field of characteristic zero can be obtained as the attached algebras of enveloping algebras with the *conventional* associative product X_iX_j. Thus the classification of Lie algebras has been already achieved by Formulation I. The point is that *generalizations II and III are not intended for the classification. Instead, they are intended for the formulation of Lie's theory in the most general possible (rather than simplest possible) form, as a necessary condition for its direct applicability in physics*. Generalizations II and III are, of course, expected to recover Cartan classification. But this is a minor aspect of the issue. The issue is that of abandoning the conventional mathematical treatment of Lie algebra,

$$[X_i, X_j] = X_iX_j - X_jX_i, \tag{7}$$

[38] It is appropriate to recall here that the classification of Lie algebras over a field of characteristics $p \neq 0$ is far from complete. The generalizations of Lie's theory here referred to are intended primarily for the conventional case of characteristic zero which is the most important for current physical applications (in fact, no physical application is known at this time for algebras and/or fields of characteristic $p \neq 0$).

where X_iX_j is the conventional associative product, in favor of the most general conceivable product:

$$[X_i, X_j]^* = (X_i, X_j) - (X_j, X_i) \tag{8}$$

where (X_i, X_j) is a nonassociative Lie-admissible product. Only in this way does the theory acquire a form suitable for direct application to mechanics while possessing trivial realization (7) as a particular case. At any rate, while the formulation of Lie's theory for structure (8) includes that of structure (7) as a particular case, the opposite is not necessarily true.[39] As an example, the current formulation of the representation theory is inapplicable to Lie algebras (8) beginning from its foundations (necessary and sufficient conditions for a representation to be faithful, Ado's theorem). At a deeper analysis, it soon emerges that the alteration of the associative character of the envelope into a nonassociative form demands the reformulation of the entire theory.

Perhaps an effective way for a mathematician to see the need of reformulating Lie's theory is through a comparative analysis with the corresponding situation in the symplectic and contact geometries, for which no reformulation is needed. In essence, these geometries, in their most abstract and general form (the coordinate-free form), present a body of notions, properties, and theorems which *preserve their validity under all possible realizations of the symplectic and contact forms*. For instance, all the parts of the symplectic geometry dealing with exact symplectic two-forms

$$\Omega_2 = d\theta \tag{9}$$

preserve their validity regardless of whether the two-form is the canonical form,

$$\omega_2 = dp_k \wedge dr^k, \tag{10}$$

or the most general possible Birkhoffian form,

$$\Omega_2 = \frac{1}{2}\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right) da^\mu \wedge da^\nu, \qquad a = (r, p). \tag{11}$$

The crucial character of the theory, that of being applicable to all possible realizations, is lost for the contemporary formulation of Lie's theory. In fact, if the enveloping algebra is generalized from the trivial product X_iX_j to a more general product $X_i * X_j$ (e.g., $X_i * X_j = X_iTX_j$, with T fixed and nonsingular; see Chart 4.1), then the notion of Lie group (3) is generalized into structures, for instance, of the type

$$G^*\colon e^{\theta^k X_k}\big|_{\mathscr{A}^*} = \mathbb{1} + \frac{\theta^k}{1!}X_k + \frac{\theta^i\theta^j}{2!}X_i * X_j + \cdots. \tag{12}$$

The fact that the notions, properties, and theorems developed for the conventional structure (3) *are not* necessarily applicable to the more general structure (12) is established, for instance, by the fact that $\mathbb{1}$ is no longer the unit of the envelope, trivially, because now $\mathbb{1} * X_i \neq X_i \neq X_i * \mathbb{1}$.

[39] As will be soon evident, nonassociative products exist which can be trivially reduced to an associative form. However, an associative product can never be reduced to a nonassociative form.

Remarkably, while the symplectic and contact geometries have been developed by keeping the most general possible realizations of the two-forms in mind, the theory of Lie algebras has been developed for the simplest possible realization of the Lie algebra product. The Lie-isotopic generalization of Lie's theory is advocated here in order to recover the compatibility of formulation with the symplectic and contact geometries, that is, to reach algebraic notions, properties, and theorems which are directly applicable to the most general possible realizations, in exactly the same case as it occurs for the geometric counterparts. The Lie-admissible generalization of Lie's theory, instead, is intended as the algebraic counterpart of the symplectic-admissible generalization of the symplectic geometry (Chart 4.7).

A further point which should be clarified is that *the Lie-isotopic generalization of Lie's theory is not directly applicable to the Hamiltonian as well as the Birkhoffian Mechanics.* In fact, the envelope is still associative by conception, while algebra (5) is already nonassociative for Hamiltonian mechanics, and this algebraic character clearly persists for the covering Birkhoffian Mechanics.[40] The theory under consideration is merely an intermediate step prior to the full treatment of type III. Nevertheless, a possibility exists that the theory is applicable in a specific case, on account of the following property. Often, when structure (8) is worked out, it implies the possible reformulation

$$[X_i, X_j]^* = (X_i, X_j) - (X_i, X_j) = X_i * X_j - X_j * X_i. \tag{13}$$

An example (Charts 4.1 and 4.2) is given by the product $(X_i, X_j) = X_i R X_j - X_j S X_i$, with R and S fixed and nonsingular, and $X_i R$, $R X_j$, etc., associative). Then we have

$$\begin{aligned}(X_i, X_j) - (X_j, X_i) &= (X_i R X_j - X_j S X_i) - (X_j R X_i - X_i S X_j) \\ &= X_i T X_j - X_j T X_i = X_i * X_j - X_j * X_i, \qquad T = R + S,\end{aligned} \tag{14}$$

where $X_i T X_j$ is clearly isotopic associative. Thus, in certain instances, the intermediary Lie-isotopic generalization may be sufficient.

For the case of the Hamiltonian Mechanics, one can attempt modifications of product (5) into more general Lie-admissible forms of the type

$$\begin{aligned}(X_i, X_j)^* &= \frac{\partial X_i}{\partial r^k}\frac{\partial X_j}{\partial p_k} + \frac{\partial X_i}{\partial r^i}\,\alpha^{ij}\,\frac{\partial X_j}{\partial r^i} + \frac{\partial X_j}{\partial p_i}\,\beta_{ij}\,\frac{\partial X_j}{\partial p_j} \\ \alpha^{ij} &= \alpha^{ji}, \qquad \beta_{ij} = \beta_{ji}\end{aligned} \tag{15}$$

[40] This point is self-evident from the generalization of Hamilton's into Birkhoff's tensor

$$\omega^{\mu\nu} = \left(\left\|\frac{\partial R_\beta^0}{\partial a^\alpha} - \frac{\partial R_\alpha^0}{\partial a^\beta}\right\|^{-1}\right)^{\mu\nu} \to \Omega^{\mu\nu}(a) = \left(\left\|\frac{\partial R_\beta}{\partial a^\alpha} - \frac{\partial R_\alpha}{\partial a^\beta}\right\|^{-1}\right)^{\mu\nu}$$

$$R^0 = (\mathbf{p}, \mathbf{0}) \to R = R(a) \neq R^0$$

under which the property

$$[X_i, X_j] = (X_i, X_j) - (X_j, X_i);\ (X_i, X_j) = \frac{\partial X_i}{\partial r^k}\frac{\partial X_j}{\partial p^k} = \text{Nonassociative product,}$$

necessarily implies the generalized one

$$[X_i, X_j]^* = (X_i, X_j)^* - (X_j, X_i)^*;\ (X_i, X_j)^* = \text{Nonassociative product.}$$

In fact, $[X_i, X_j]$ is a particular case of $[X_i, X_j]^*$ if and only if (X_i, X_j) is a particular case of $(X_i, X_j)^*$.

that is, modifications which are such as to preserve the conventional Poisson brackets as the attached Lie brackets. With the understanding that modifications (15) remain nonassociative in general,[41] it may be that the associative law is regained in particular cases. The important point is that, even when the associative character of the envelope is regained via extensions of type (15), the enveloping algebra is not of the trivial type $X_i X_j$, but rather of the most general possible type $X_i * X_j$. As a result, assuming that the associative character of the envelope of classical mechanics is regained via (still unknown) methods, the isotopic generalization of Lie's theory remains mandatory for its direct applicability. Lacking the generalization, one risks the application of existing theorems conceived for formulations I which are actually meaningless for physical models belonging to case II or III.

Some of the most remarkable and intriguing implications are those for particle physics. The *only* time evolution known at this point with a structure truly of type I (that is, with an associative envelope with trivial product $X_i X_j$) is that of *Heisenberg's equations* in quantum mechanics:[42]

$$\dot{\tilde{A}} = \frac{1}{i}\,[\tilde{A}, \tilde{H}] = \frac{1}{i}\,(\tilde{A}\tilde{H} - \tilde{H}\tilde{A}), \qquad \tilde{A}\tilde{H} = \text{Associative product} \tag{16}$$

$$(\hbar = 1)$$

with fundamental brackets (in our unified notation $\tilde{a} = (\tilde{\mathbf{r}}, \tilde{\mathbf{p}})$, of course, now referred to as operators in a Hilbert space)

$$[\tilde{a}^\mu, \tilde{a}^\nu] = \tilde{a}^\mu\tilde{a}^\nu - \tilde{a}^\nu\tilde{a}^\mu = i\tilde{\omega}^{\mu\nu} = i\left(\left\| \frac{\partial R_\nu^0}{\partial a^\mu} - \frac{\partial R_\mu^0}{\partial a^\nu} \right\|^{-1}\right)^{\mu\nu}, \qquad \tilde{R}^0 = (\tilde{\mathbf{p}}, \tilde{\mathbf{0}}). \tag{17}$$

The mere identification of the possibilities of generalizing Lie's theory according to types II and III immediately implies the possibility of generalizing Heisenberg's equations accordingly.

In fact, Santilli (1978d, pp. 725 and 752) proposed the following Lie-isotopic generalization of Heisenberg's equations

$$\dot{\tilde{A}} = \frac{1}{i}\,[\tilde{A}, \tilde{H}]^* = \frac{1}{i}\,(\tilde{A}\hat{T}\tilde{H} - \tilde{H}\hat{T}\tilde{A}) \tag{18a}$$

$$[\tilde{a}^\mu, \tilde{a}^\nu]^* = i\tilde{\Omega}^{\mu\nu}(\tilde{a}) = i\left(\left\| \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu} \right\|^{-1}\right)^{\mu\nu} \tag{18b}$$

and the following Lie-admissible generalization (*loc. cit.*, pp. 719 and 746)

$$\dot{\tilde{A}} = \frac{1}{i}\,(\tilde{A}, \tilde{H}) = \frac{1}{i}\,(\tilde{A}\hat{R}\tilde{H} - \tilde{H}\hat{S}\tilde{A}) \tag{19a}$$

$$(\tilde{a}^\mu, \tilde{a}^\nu) = i\mathscr{S}^{\mu\nu}(\tilde{a}) = i(\tilde{\Omega}^{\mu\nu} + \tilde{T}^{\mu\nu}), \qquad \tilde{T}^{\mu\nu} = \tilde{T}^{\nu\mu}. \tag{19b}$$

[41] One can easily see that the associative law *cannot* in general be verified for product (15) because, for instance, the expression $((X_i, X_j), X_k)$ implies only first-order derivatives for X_k, while the expression $(X_i, (X_j, X_k))$ implies second-order derivatives for X_k. Nevertheless, restrictions on the functional dependence of the generators are conceivable under which $((X_i, X_j), X_k) = (X_i, (X_j, X_k))$.

[42] From now on all quantum mechanical operators will be denoted with an upper tilde, e.g., $\tilde{A}$, $\tilde{H}$, etc.

As a matter of fact, generalizations of Lie's theory of types II and III were intended as mathematical tools for the proper treatment of the corresponding generalized equations of type (18) and (19). The Lie-isotopic generalization of Lie's theory will therefore be outlined in the subsequent charts according to the motivations for which it was originally conceived, that of directly characterizing Equations (18).

Independently from these studies, Okubo (1981a) has proposed a generalization of Heisenberg's equations of the type

$$\dot{\tilde{A}} = \frac{1}{i}\,[\tilde{A}, \tilde{H}]^* = \frac{1}{i}\,[(\tilde{A}, \tilde{H}) - (\tilde{H}, \tilde{A})], \tag{20}$$

where the product $(\tilde{A}, \tilde{H})$ is nonassociative, flexible, and Lie-admissible. The generalization was motivated by the fact that the replacement of the associative envelope of Heisenberg's mechanics with a nonassociative one (but of flexible Lie-admissible type to preserve the differential rule) permits the avoidance of some of the problems of consistency of conventional quantization. This was, after all, expected from the nonassociativity of the product (A, H) in Poisson's brackets (4).[43] Also intriguing is the expected relationship between Equations (20) and (18) via rules of type (14) which, for the case of linear operator algebras, appears possible.

As an example, one could assume in equations (20) the flexible Lie-admissible envelope

$$(\tilde{A}, \tilde{H}) = \tilde{A}\tilde{H} + \tfrac{1}{2}\{\tilde{A}, \tilde{H}\} = \tfrac{3}{2}\tilde{A}\tilde{H} + \tfrac{1}{2}\tilde{H}\tilde{A} \tag{21}$$

The reformulation of Equations (20) into Equations (18), and vice versa, is then given by

$$[\tilde{A}, \tilde{H}]^* = (\tilde{A}, \tilde{H}) - (\tilde{H}, \tilde{A}) = \tilde{A}\tilde{H} - \tilde{H}\tilde{A} \equiv [\tilde{A}, \tilde{H}] \tag{22}$$

Following this, we therefore use Equations (16) when dealing with generalizations of Lie's theory of type II, and Equations (19) when dealing with those of type III, with interconnecting mechanisms of type (14).[44]

Chart 5.2 Isotopic Generalization of the Universal Enveloping Associative Algebra

In this chart we shall first review the definition of universal enveloping associative algebra, and the methods for the construction of its basis according to the Poincaré–Birkhoff–Witt theorem. We shall then present their *isotopic* generalizations, that is,[45] generalizations which preserve

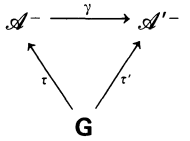
[43] Even though the algebras characterized by Hamilton's and Heisenberg's brackets are both Lie, their envelopes are different because that of the former is nonassociative, while that of the latter is associative. Under these circumstances, the rigorous definition of quantization via a mapping is virtually impossible. For these and numerous other problematic aspects related to the quantization of Hamilton's into Heisenberg's equations, one may consult Santilli (1980a) and cited references. Additional problematic aspects (different than those related to the envelopes) are treated by Chernoff (1981). A theorem establishing the lack of the so-called full quantization is proved by Abraham and Marsden (1978, p. 435).

[44] See also the outlines of applications of the charts of Chapter 6.

[45] The Greek meaning of the work "isotopy" was related in footnote 33 of Chapter 4.

the associative character of the product. By keeping in mind the primitive character of the enveloping algebra in Lie's theory (Chart 5.1), the generalizations presented in this chart render inevitable a corresponding reinspection of Lie algebras and of Lie groups (Charts 5.3 and 5.4).

Definition 1. The *universal enveloping associative algebra* of a Lie algebra **G** is the set $(\mathscr{A}, \tau)$ where $\mathscr{A}$ is an associative algebra[46] and τ a homomorphism of **G** into the attached algebra $\mathscr{A}^-$ of $\mathscr{A}$ satisfying the following properties. If $\mathscr{A}'$ is another associative algebra and τ' a homomorphism of **G** into $\mathscr{A}'$, a unique homomorphism γ of $\mathscr{A}$ into $\mathscr{A}'$ exists such that $\tau' = \tau\gamma$; i.e., the following diagram is commutative.



(1)

Whenever an algebra $\mathscr{A}$ belongs to the context of the definition above, we shall write $\mathscr{A}(\mathbf{G})$. All Lie algebras are assumed, for simplicity, to be finite-dimensional. Also, all algebras and fields are assumed to have characteristic zero, and the basis of all Lie algebras is ordered.

The construction of the enveloping algebra $\mathscr{A}(\mathbf{G})$ is conducted as follows. Consider the algebra **G** as a (linear) vector space with basis given by the (ordered set of) generators X_i, $i = 1, 2, \ldots, m$. The *tensorial product* $\mathbf{G} \otimes \mathbf{G}$ is the ordinary Kronecker (or direct) product of **G** with itself as a vector space. Such a tensorial product constitutes an algebra because it satisfies the distributive and scalar laws (Chart 4.1). Also, the algebra is associative because the Kronecker product is associative. The most general possible, associative, *tensor* algebra which can be constructed on **G** as vector space is given by

$$\mathscr{T} = \mathbf{F}\mathbb{1} \oplus \mathbf{G} \oplus \mathbf{G} \otimes \mathbf{G} \oplus \mathbf{G} \otimes \mathbf{G} \otimes \mathbf{G} \oplus \cdots, \tag{2}$$

where F is the base field and $\oplus$ denotes the direct sum. Let $\mathscr{R}$ be the ideal generated by all elements of the form

$$[X_i, X_j] - (X_i \otimes X_j - X_j \otimes X_i) \tag{3}$$

where $[X_i, X_j]$ is the product of **G**. Then the universal enveloping algebra $\mathscr{A}(\mathbf{G})$ of **G** is given (or, equivalently, can be defined) by the quotient

$$\mathscr{A}(\mathbf{G}) = \mathscr{T}/\mathscr{R}. \tag{4}$$

It is possible to prove that the algebra (4) satisfies all the conditions of Definition 1 (see, for instance, Jacobson (1962)).

Of utmost importance for mathematical and physical considerations is the identification of the basis of $\mathscr{A}(\mathbf{G})$. The quantities

$$M_s = X_{i_1} \otimes X_{i_2} \otimes \cdots \otimes X_{i_s} \tag{5}$$

are called *standard* (*nonstandard*) *monomials* of order s depending on whether the ordering

$$i_1 \leq i_2 \leq \cdots \leq i_s \tag{6}$$

[46] Note that no restriction is placed on the associative character of the algebra. Nevertheless, the definition is restrictive for reasons which will be identified via Definition 2 below and subsequent comments.

is verified (not verified). It is possible to prove that every element of $\mathscr{A}(\mathbf{G})$ can be reduced to a linear combination of standard monomials and (cosets of) 1. This yields the following fundamental theorem on enveloping associative algebras (Jacobson, *loc. cit.*).

Theorem 1 (Poincaré–Birkhoff–Witt Theorem).[47] *The cosets of* 1 *and the standard monomials form a basis of the universal enveloping associative algebra* $\mathscr{A}(\mathbf{G})$ *of a Lie algebra* $\mathbf{G}$.

The associative envelope $\mathscr{A}(\mathbf{G})$, as presented, is still abstract in the sense that the product of $\mathscr{A}(\mathbf{G})$ is the tensorial product $X_i \otimes X_j$, while the product used in physical (e.g., quantum mechanical) applications is the conventional associative product $X_i X_j$. Consider then the algebra

$$A(G) = F\mathbb{1} \oplus A^{(1)} \oplus A^{(2)} \oplus \cdots$$
$$A^{(s)} = \{X_{i_1} X_{i_2} \cdots X_{i_s}\}, \qquad i_1 \leq i_2 \leq \cdots \leq i_s. \tag{7}$$

It is possible to prove that $\mathscr{A}(\mathbf{G})$ is homomorphic to $A(\mathbf{G})$, in line with Definition 1. Thus the algebra $A(\mathbf{G})$ can be assumed as the universal enveloping associative algebra of $\mathbf{G}$ with basis

$$\mathbb{1}, \quad X_i, \quad X_{i_1} X_{i_2}, \quad X_{i_1} X_{i_2} X_{i_3}, \ldots, \tag{8}$$
$$i_1 \leq i_2, \quad i_1 \leq i_2 \leq i_3,$$

and arbitrary elements

$$X^k_{i_1} X^k_{i_2} \cdots X^k_{i_s}, \tag{9}$$

where the X's are the generators of $\mathbf{G}$. Notice that $A(\mathbf{G})$ is infinite-dimensional. The *center* of $A(\mathbf{G})$ is the set of all polynomials $P(X)$ verifying the property

$$[P(X), X_i]_A = 0, \tag{10}$$

for all elements $X_i \in \mathbf{G}$. Most important elements of the center are the so-called *Casimir invariants* of $\mathbf{G}$. For additional study, we refer the interested reader to the mathematical literature on the topic, such as Jacobson (*loc. cit.*) or Dixmier (1977).

We move now to the identification of the desired associative-isotopic generalization.

Definition 2 (Santilli 1978c). The *isotopically mapped universal enveloping associative algebra* of a Lie algebra $\mathbf{G}$ is the set $((\mathscr{A}, \tau), \mathscr{A}^*, i, \tau^*)$ where

(i) $(\mathscr{A}, \tau)$ is the universal enveloping associative algebra as per Definition 1;

(ii) i is an isotopic mapping of $\mathbf{G}$, $i\mathbf{G} = \mathbf{G}^*$, as per Chart 4.2;

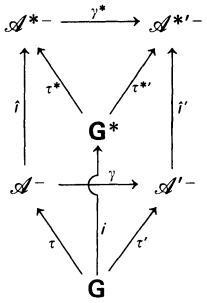
(iii) $\mathscr{A}^*$ is an associative algebra generally nonisomorphic to $\mathscr{A}$; and

(iv) τ^* is a homomorphism of $\mathbf{G}^*$ into $\mathscr{A}^{*-}$, such that the following properties are verified.

If $\mathscr{A}^{*\prime}$ is still another associative algebra and $\tau^{*\prime}$ a homomorphism of $\mathbf{G}^*$ into $\mathscr{A}^{*\prime -}$, a unique homomorphism γ^* of $\mathscr{A}^*$ into $\mathscr{A}^{*\prime}$ exists such that

[47] It should be indicated that the name "Birkhoff" in "Birkhoff's equations" and in "Poincaré–Birkhoff-Witt theorem" refers to father (G. D. Birkhoff) and son (G. Birkhoff), respectively.

$\tau^{*\prime} = \gamma^* \tau^*$, and two unique isotopies $\hat{\imath}$ and $\hat{\imath}'$ exist for which $\hat{\imath}\mathscr{A} = \mathscr{A}^*$ and $\hat{\imath}'\mathscr{A}' = \mathscr{A}^{*\prime}$, i.e., the following diagram is commutative.



(11)

Whenever an algebra $\mathscr{A}^*$ verifies the conditions of the definition above, we write $\mathscr{A}^*(\mathbf{G})$. Again, for simplicity, we assume that all Lie algebras are finite-dimensional, all algebras and fields have characteristic zero, and all Lie algebra bases are ordered.

We are now in a position to elaborate on the insufficiency of Definition 1, and the need of Definition 2 for the physical and mathematical studies under consideration in these volumes. We shall indicate first the mathematical aspect and then point out the physical profile.

The main idea of Definition 1 is, beginning with the basis of a Lie algebra $\mathbf{G}$, to construct an enveloping algebra $\mathscr{A}(\mathbf{G})$ such that $[\mathscr{A}(\mathbf{G})]^- \approx \mathbf{G}$. The more general idea of Definition 2 is, beginning also with the basis of a Lie algebra $\mathbf{G}$, to construct an enveloping algebra $\mathscr{A}^*(\mathbf{G})$ such that the attached algebra $[\mathscr{A}^*(\mathbf{G})]^-$ *is not*, in general, isomorphic to $\mathbf{G}$ but rather is isomorphic to an isotope $\mathbf{G}^*$ of $\mathbf{G}$, and we write[48]

$$[\mathscr{A}^*(\mathbf{G})]^- \approx \mathbf{G}^* \not\approx \mathbf{G}. \tag{12}$$

The lack of unique association of a given basis with the envelope then ensures freedom in the realization of the associative product. Equivalently, we can say that within the context of Definition 1, a given basis essentially yields a single unique enveloping algebra and thus a single unique attached Lie algebra. On the contrary, within the context of Definition 2, a given basis yields all possible enveloping algebras and thus all possible Lie algebras. Still equivalently, we can say that, as is conventional in the contemporary formulation of Lie's theory, nonisomorphic Lie algebras are expressed via the use of different generators and the same realization of the Lie product. On the contrary, within the context of the isotopic formulation of Lie's theory, nonisomorphic Lie algebras can be obtained via the use of the same basis and different realizations of the Lie product. We can therefore state that all possible enveloping associative algebras can indeed be introduced according to Definition 1, which is therefore suitable for the classification of Lie algebras (Chart 5.1). Definition 2 is more general inasmuch as, besides permitting the introduction of all possible enveloping algebras, it also permits us to construct nonisomorphic algebras via the same basis, by therefore rendering necessary the use of the most general possible realizations of the associative product.

On physical grounds, these mathematical mechanisms are at the foundation of the Lie isotopic generalization of Hamilton's and Heisenberg's equations for closed non-self-adjoint interactions (Section 6.3).

[48] Note that the scripture $\mathscr{A}^*(\mathbf{G})$ (rather than $\mathscr{A}^*(\mathbf{G}^*)$) is intended to stress precisely properties (12).

As now familiar, the definition of physical quantities is independent of whether or not the system possesses nonpotential interactions. When these interactions are admitted by the theory, they are represented via an alteraction of the Lie algebra product. As a result, when the Hamiltonian description of a closed self-adjoint system

$$\dot{A}(a) = [A, E_{tot}] = \frac{\partial A}{\partial a^{\mu}} \omega^{\mu\nu} \frac{\partial E_{tot}}{\partial a^{\nu}}, \tag{13}$$

is generalized into a Birkhoffian form to represent the additional presence of internal, contact, nonpotential, interactions

$$\dot{A}(a) = [A, E_{tot}]^* = \frac{\partial A}{\partial a^{\mu}} \Omega^{\mu\nu}(a) \frac{\partial E_{tot}}{\partial a^{\nu}}, \tag{14}$$

the basis of the original Lie algebra remains unchanged, together with the underlying carrier space ($\mathbb{R} \times T^*M$) and the field, and only the realization of the Lie algebra product (that is, the realization of the envelope) is permitted to change. As a result, the original Lie algebra **G** with basis $X_i(a)$ over T^*M equipped with conventional Poisson brackets is mapped into the isotope **G***, which preserves the original basis $X_i(a)$ in the same local coordinates of T^*M, although it is now equipped with the generalized Poisson brackets, i.e.,

$$\mathbf{G}\colon [X_i, X_j]_{(a)} = (X_i, X_j)_{(a)} - (X_j, X_i)_{(a)}$$
$$\rightarrow \mathbf{G}^*\colon [X_i, X_j]^*_{(a)} = (X_i, X_j)^*_{(a)} - (X_j, X_i)^*_{(a)}. \tag{15}$$

In the transition to the case of Heisenberg's equations, the situation is essentially the same and actually turns out to be more directly compatible with Definition 2. In fact, for consistency of the theory with its classical image, during the generalization of Heisenberg's equations

$$\dot{\tilde{A}}(\tilde{a}) = \frac{1}{i} [\tilde{A}, \tilde{H}] \tag{16}$$

into the Lie-isotopic form

$$\dot{\tilde{A}}(\tilde{a}) = \frac{1}{i} [\tilde{A}, \tilde{H}]^*, \tag{17}$$

the nonpotential forces due to charge overlapping are expressed via the Lie-isotopic generalization of the product

$$\mathbf{G}\colon [\tilde{X}_i, \tilde{X}_j] = \tilde{X}_i X_j - \tilde{X}_j, \tilde{X}_i \rightarrow \mathbf{G}^*\colon [\tilde{X}_i, \tilde{X}_j]^* = \tilde{X}_i \tilde{T} \tilde{X}_j - \tilde{X}_j \tilde{T} \tilde{X}_i. \tag{18}$$

Mechanism (18) is clearly along Definition 2 rather than 1.

The alternative approach would be that of preserving the original simplest possible product and changing the basis in order to reach direct compatibility with Definition 1. However, this approach has a number of problematic aspects. First of all, it is centered on the loss of the direct physical meaning of the generators (e.g., the physical linear momentum in one dimension, $p = m\dot{r}$, is replaced by abstract objects of the type $p = \alpha \exp \beta r\dot{r}$). Secondly, the approach does not permit the achievement of the direct universality, as established in the preceding chapters. The removal of unnecessary restriction on the realization of the enveloping algebras is clearly preferable, both mathematically and physically.

Owing to the relevance of mechanisms (15) and (18) for the program of these volumes, it is important to give an explicit example. To stress the

fact that the ideas are not necessarily restricted to nonpotential interactions, we select an example of isotopy for the harmonic oscillator in a three-dimensional Euclidean space.

In Chart 4.2 we pointed out that the nonisomorphic groups SO(3) and SO(2.1) are isotopic symmetries with respect to the Hamiltonians

$$H(a) = \tfrac{1}{2}(p_x^2 + p_y^2 + p_z^2) + \tfrac{1}{2}(x^2 + y^2 + z^2), \qquad a = (r, p), \quad m = k = 1, \tag{19a}$$

$$H^*(a) = \tfrac{1}{2}(p_x^2 - p_y^2 + p_z^2) + \tfrac{1}{2}(x^2 - y^2 + z^2), \tag{19b}$$

that is, they are symmetries leading to the same conservation laws of the components M_b, $b = x, y, z$, of the angular momentum via the use of Noether's theorem. Let us review the case again and reinterpret it in light of Definitions 1 and 2.

The Hamiltonian realization of the symmetry SO(3) of $H(a)$ is based on the Lie algebra of conserved quantities

$$\mathbf{SO}(3)\text{: } [M_x, M_y] = M_z, \; [M_y, M_z] = M_x, \; [M_z, M_x] = M_y \tag{20}$$

which is defined in terms of the conventional Poisson brackets

$$[M_b, M_c] = (M_b, M_c) - (M_c, M_b) \tag{21a}$$

$$(M_b, M_c) = \frac{\partial M_b}{\partial r^i}\,\delta_j^i\,\frac{\partial M_c}{\partial p_j}; \quad (\delta_j^i) = \begin{pmatrix} +1 & & 0 \\ & +1 & \\ 0 & & +1 \end{pmatrix}. \tag{21b}$$

In the transition to the equivalent Hamiltonian $H^*(a)$, the conserved quantities M_b clearly remain conserved, but the SO(3) symmetry is broken and is replaced by the nonisomorphic symmetry SO(2.1). The problem now is the construction of a realization of the **SO**(2.1) algebra (the *Lorentz* algebra in (2 + 1)-dimensions) whose generators are those of the nonisomorphic **SO**(3) algebra (the *rotational* algebra in three-dimensions). This can clearly be achieved if and only if one alters the Lie algebra product. An explicit realization has been identified by Santilli (1979a) and is given by the well-known commutation rules

$$\mathbf{SO}(2.1)\text{: } [M_x, M_y]^* = M_z, \; [M_y, M_z]^* = -M_x, \; [M_z, M_x]^* = \mathrm{M}_y, \tag{22}$$

which are now expressed in terms of the generalized Poisson (Birkhoffian) brackets[49]

$$[M_b, M_c]^* = (M_b, M_c)^* - (\mathrm{M}_c, M_b)^* \tag{23a}$$

$$(M_b, M_c)^* = \frac{\partial M_b}{\partial r^i}\,\alpha_j^i\,\frac{\partial M_c}{\partial p_j}, \qquad (\alpha_j^i) = \begin{pmatrix} +1 & & 0 \\ & -1 & \\ 0 & & +1 \end{pmatrix}. \tag{23b}$$

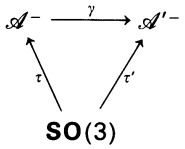
Note that the insistence in the preservation of the same realization of the Lie algebra product, in this case, would prohibit the representation of the conservation of the angular momentum via a symmetry of the Hamiltonian $H^*(a)$.

The example considered therefore establishes that one given basis (the components of the angular momentum $\mathbf{M} = \mathbf{r} \times \mathbf{p}$, $\mathbf{p} = m\dot{\mathbf{r}}$) can define a hierarchy of enveloping algebras and attached Lie algebras,

[49] See the Birkhoffian interpretation of the Hamiltonian isotopies of Section 5.3.

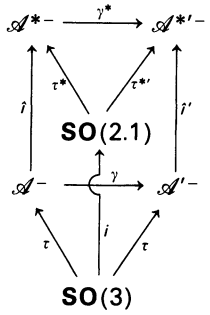
depending on the selected realizations of the products, which is fully in line with diagram (11) and Definition 2. The example actually establishes not only the insufficiency of Definition 1 but also that of Definition 2 itself. In fact, the algebras (M_b, M_c) and $(M_b, M_c)^*$ are *nonassociative*, therefore demanding further generalization of Definition 1 for nonassociative enveloping algebras of type III (see the classification of the preceding chart), even though the existence of a realization within the context of the Lie-isotopic generalization is expected to exist.

Stated in different terms, the example establishes the generalization of the conventional definition of the envelope of the Lie algebra of the group of rotations as per diagram (1)



(24)

into the Lie-isotopic form as per diagram (11)[50]



(25)

which is expected for operator-type realizations (18). In addition, the example establishes that generalization (25) is only an intermediate step, prior to a more general nonassociative realization which is not considered here for the sake of brevity (see Definition 3.7.3 of Santilli (1978a, page 354), as well as the more recent review (1979b, p. 1602)).

With a clear understanding of the new capabilities, as well as limitations, of the Lie-isotopic generalization, we pass now to the study of the generalization of Theorem 1.

The construction of an isotope $\mathscr{A}^*(\mathbf{G})$ of $\mathscr{A}(\mathbf{G})$ can be conducted as follows. Perform an *isotopic mapping of the tensorial product* $X_i \otimes X_j$ of $\mathscr{A}(\mathbf{G})$,

$$X_i \otimes X_j \to X_i * X_j \tag{26}$$

that is, any invertible modification of the product $\otimes$ via elements of $\mathscr{A}(\mathbf{G})$, of the base manifold, and of the field, which preserves the distributive and scalar laws (to qualify as an algebra), as well as the associativity of the product (to qualify as an isotopy), i.e.,

$$(X_i * X_j) * X_k = X_i * (X_j * X_k). \tag{27}$$

[50] By no means does diagram (25) exhaust all possible isotopies of the group of rotations. In fact, by recalling the properties of the isotopically related symmetry algebras from Chart A.10, we know that an isotope SO*(3) of SO(3) can even, in principle, be Abelian.

The product of two elements $X_i * X_j$ and $X_r * X_s$ is then given by

$$(X_i * X_j) * (X_r * X_s) = X_i * X_j * X_r * X_s, \tag{28}$$

and no ordering ambiguity arises because of the preservation of the associative character of the original product.[51]

The isotope of the associative tensorial algebra $\mathscr{T}$ can then be written

$$\mathscr{T}^* = F\mathbb{1} \oplus \mathbf{G} \oplus \mathbf{G} * \mathbf{G} \oplus \mathbf{G} * \mathbf{G} * \mathbf{G} \oplus \cdots. \tag{29}$$

Let $\mathscr{R}^*$ be the ideal of $\mathscr{T}^*$ generated by

$$[X_i, X_j]^* - (X_i * X_j - X_j * X_i), \tag{30}$$

where $[X_i, X_j]^*$ is the product in $\mathbf{G}^*$. An *isotopically mapped universal enveloping associative algebra* $\mathscr{A}^*(\mathbf{G})$ of a Lie algebra $\mathbf{G}$ can then be written.

$$\mathscr{A}^*(\mathbf{G}) = \mathscr{T}^*/\mathscr{R}^*. \tag{31}$$

Structure (31) is, by construction, the universal enveloping associative algebra of $\mathbf{G}^*$, where $\mathbf{G}^*$ is realized via an isotopic mapping of $\mathbf{G}$.

The remaining aspects of the theory of $\mathscr{A}^*(\mathbf{G})$ are essentially given by an isotopic mapping of the corresponding steps for $\mathscr{A}(\mathbf{G})$ outlined above.

The quantities

$$M_s^* = X_{i_1} * X_{i_2} * \cdots * X_{i_s}, \tag{32}$$

are called *isotopically mapped standard* (*nonstandard*) *monomials* depending on whether the following ordering condition

$$i_1 \leq i_2 \leq \cdots \leq i_s \tag{33}$$

is verified (not verified). In the reduction of an arbitrary element of $\mathscr{A}^*(\mathbf{G})$

$$X^k_{i_1 1} * X^k_{i_2 2} * \cdots * X^k_{i_r 2}, \tag{34}$$

to standard monomials, a new feature arises, due to the fact that the emerging combinations of these latter monomials may occur via *functions on the base manifold*. This, in turn, is because the isotopy $\otimes \to *$ can be realized via functions of this type. We call these combinations F^*-linear, to differentiate them from the F-linear combinations for the conventional case, that is, combinations only via elements of the field. As we shall see in the next chart, these F^*-linear combinations have a precise interpretation within the context of the isotopic Lie's theory. Despite this generalization, the construction of the basis of $\mathscr{A}^*(\mathbf{G})$ parallels that for $\mathscr{A}(\mathbf{G})$, because $\mathscr{A}^*(\mathbf{G})$ is a conventional envelope for $\mathbf{G}^*$. The (inverse) isotopy then simply reduces $\mathbf{G}^*$ to $\mathbf{G}$.

Theorem 2 (Isotopic Generalization of the Poincaré–Birkhoff–Witt Theorem).[52] *The cosets of* $\mathbb{1}$ *and the standard isotopically mapped monomials form a basis of the isotopically mapped universal enveloping associative algebra* $\mathscr{A}^*(\mathbf{G})$ *of a Lie algebra* $\mathbf{G}$.

[51] Note that, for the more general nonassociative Lie-admissible generalization, the left- and right-hand sides of quantities (27) would be different. In this case all possible different orderings of the product must be taken into account.

[52] Santilli (1978c, Theorem 3.7.2, page 353) reprinted in Myung *et al.* (1978-I). See also Santilli (1979b, page 1580).

The basis is thus given by

$$\mathbb{1}, \quad X_i, \quad X_{i_1} * X_{i_2}, \quad X_{i_1} * X_{i_2} * X_{i_3} \tag{35}$$
$$i_1 \leq i_2, \quad i_1 \leq i_2 \leq i_3$$

The distinction between the tensorial realization and that used in practical applications is now lost. Indeed the mapping $X_i \otimes X_j \to X_i X_j$ can be considered, in the final analysis, a particular form of isotopy.

The explicit form of the basis depends on the assumed type of isotopy $\otimes \to *$. In turn, this depends on the realization of the basis X_i of **G**, whether via matrices, quantum mechanical operators, or classical functions on phase space, etc.

Suppose that the X's are realized via matrices. Then an isotopy is provided by Equation (18). Let T be a polynomial on the X's (not necessarily on the center of $\mathscr{A}^*(\mathbf{G})$).[53] Then the explicit form of basis (35) is given by

$$\mathbb{1}, \quad X_i, \quad X_{i_1} T X_{i_2}, \quad X_{i_1} T X_{i_2} T X_{i_3}, \ldots \tag{36}$$
$$i_1 \leq i_2 \qquad i_1 \leq i_2 \leq i_3, \qquad T = \text{fixed}.$$

Needless to say, the isotopy $X_i X_j \to X_i T X_j$ is only one example of possible associativity-preserving modifications of the product, and numerous additional forms exist. For instance, if W is an idempotent matrix ($W^2 = W$), then another associative isotopy is given by[54]

$$X_i * X_j = W X_i W X_j W. \tag{37}$$

The identification of additional isotopies is left to the interested researcher.

A comment on the quantity $\mathbb{1}$ of Theorem 2 is in order here. As anticipated in the preceding chart, *the element* $\mathbb{1} \in \mathbb{F}$ *is no longer the unit element of the enveloping algebra under an isotopic mapping of the product.* In fact, for isotopic envelope (36) the unit element (when it exists) is given by

$$\mathbb{1}^* = T^{-1} \tag{38}$$

because only this quantity verifies the (left and right) rules $\mathbb{1}^* * X_i = X_i * \mathbb{1}^* = X_i$. Nevertheless, Theorem 2 has been formulated for the element $\mathbb{1}$ of $\mathscr{A}$. This is to preserve the general rule of isotopy according to which the basis of the original algebra is preserved, including its unit element. The new mathematical (and physical) structure is represented via an isotopic alteration of the product. A reformulation of Theorem 2 in terms of the unit $\mathbb{1}^*$ is, of course, expected to exist, but its study is left to the interested researcher. For additional studies (within the context of the Lie-admissible generalization of Theorem 1) we refer the reader to Myung and Santilli (1979), where unit $\mathbb{1}$ is called the *weak unit* of the algebra.

The mathematical aspect conveyed in this chart is that *the knowledge of a given set of generators does not uniquely characterize a Lie algebra* because of the freedom in the selection of the enveloping algebra. The physical aspect treated is that established in the text, that *the knowledge of a Hamiltonian does not uniquely characterize the physical system*

[53] In a number of applications, the element T cannot actually be expressed via F^*-linear combinations of polynomials of the original basis, and as such, it is outside the original envelope.

[54] Intriguingly, isotopy (37) was introduced within the context of the studies for a possible isotopic generalization of Heisenberg's indeterminancy principle for strong interactions (see Chart 6.1).

because such a characterization also depends on the explicit form of the brackets of the time evolution. As we shall see, the implications are rather intriguing. For instance, the assumption of a *Hermitian* Hamiltonian $\tilde{H}$ contrary to popular belief, does not ensure that the time evolution is unitary and thus does not guarantees that $\tilde{H}$ is observable.[55]

Chart 5.3 Isotopic Generalization of Lie's First, Second, and Third Theorems

As is well-known, an effective historical, and technical way of presenting Lie groups and Lie algebras is according to their original derivation by Sophus Lie via his celebrated First, Second, and Third Theorems. In this chart we shall first review these theorems and then show that they admit a consistent Lie isotopic generalization which is compatible with the isotopic generalization of the enveloping algebra of the preceding chart. More specifically, the objective of this chart is to show that the notion of connected Lie transformation group admits a generalization such that, when reduced in the neighborhood of the identity, admits Lie algebras in their most general possible realizations of the product.

The emerging isotopic generalization[56] of Lie's theory (that is, of the enveloping algebra, the Lie algebras, and the Lie groups) was used for the construction of the isotopic generalization of Galilei's relativity for closed non-self-adjoint systems of Section 6.3. Since the theory also admits operator-type realizations (Chart 6.1), its abstract formulation is expected to permit the joint treatment of closed, classical and quantum mechanical, nonpotential interactions, in much of the same way as the conventional abstract formulation of Lie's theory permits a joint treatment of closed classical and quantum mechanical interactions of potential/Hamiltonian type. The underlying physical objective is therefore to achieve, in due time, the generalization of the contemporary notion of interactions, with corresponding generalization of relativities and physical laws (Appendix A.1).

Definition 1. Let M be a Hausdorff, second-countable, analytic, N-dimensional manifold with local coordinates a^μ, $\mu = 1, 2, \ldots, N$ (e.g., $M = T^*M$ or $\mathbb{R} \times T^*M$). The set of transformations on M depending on r-independent parameters θ^i, $i = 1, 2, \ldots, r$

$$a \to a' = f(a; \theta) = \{f^\mu(a^\alpha; \theta^i)\} \tag{1}$$

[55] The reader should keep in mind the physical arena of these volumes, that is, the study of systems with contact, non-self-adjoint forces. The Hamiltonian $\tilde{H}$ can then represent the energy for, say, a proton in the core of a star. To avoid paradoxical situations (e.g., the setting up of the measuring apparatus in the core of a star), the theory should prevent conventional observability criteria, and call for a more adequate approach to measures under non-self-adjoint forces.

[56] Since the theory is fully Lie by conception, a number of researchers object at the term "generalization" and prefer different terms such as "reformulation." Others, on the contrary, believe that the mathematical and, most of all, physical implications are considerable and prefer the term "generalization." With the understanding that this is a question of semantics, and as such, immaterial for the objectives of these volumes, this author prefers the latter term, if for no other reason than to attract the researcher's attention to the need to formulate Lie's theory via rules more general than the simplest possible one of current use, $[A, B] = AB - BA$, where AB is the conventional associative product.

is called a *Lie transformation group*[57] when the following conditions are verified.

1. All functions f^{μ} are analytic in their variables.
2. For any given two transformations

$$a' = f(a; \theta), \qquad a'' = f(a'; \theta'), \tag{2}$$

a set of parameters exists

$$\theta''^{i} = g^{i}(\theta, \theta') \tag{3}$$

characterized by analytic functions g^{i} called *group composition laws*, such that

$$a'' = f(a; \theta''). \tag{4}$$

3. Transformations (1) recover the identity transformation at the null value of the parameters, i.e.,

$$a = f(a; 0). \tag{5}$$

4. Corresponding to each transformation (1), there is a unique inverse transformation

$$a = f(a'; \theta^{-1}), \tag{6}$$

and thus the transformations are regular.

5. The combination of any transformation (1) with its inverse (6) yields the identity transformation.

The number r of independent parameters is called the *dimension* of the Lie group.

A central property of Lie transformation groups is that they are *connected*; that is, they can be continuously connected to the identity. The primary idea of Lie's theorems is that, under the conditions indicated, the groups can be studied via their infinitesimal transformations, because a finite transformation can be recovered via infinite successions of infinitesimal transformations.

We shall review these ideas by following as closely as possible their original derivation.[57] Consider transformations (1) and their identity

$$a' = f(a; \theta), \qquad a = f(a; 0) \tag{7}$$

and perform the infinitesimal variations

$$a' = a + da = f(a; \theta + d\theta); \; a + \delta a = f(a; \delta\theta), \tag{8}$$

where $d\theta$ and $\delta\theta$ represent two independent variations of the parameters. We can then write

$$da = \frac{\partial f(a; \theta)}{\partial \theta} d\theta, \tag{9a}$$

$$\delta a = \left[\frac{\partial f(a; \theta)}{\partial \theta}\right]_{\theta = 0} \delta\theta. \tag{9b}$$

[57] The literature on Lie's theory is so large that it discourages even a partial outline. The most inspiring reading is the original work, e.g., Lie (1891, 1893, and 1896). For a recent account, see Sagle and Walde (1973).

The transformation $\theta + d\theta$ can be interpreted as the product of transformations relative to θ and $\delta\theta$, i.e.,

$$\theta^i + d\theta^i = \varphi^i(\theta, \delta\theta), \tag{10}$$

for which

$$\theta^i + d\theta^i = \varphi^i(\theta, 0) + \left[\frac{\partial\varphi^i(\theta, \alpha)}{\partial\alpha^j}\right]_{\alpha=0} \delta\theta^j + \cdots. \tag{11}$$

Thus we can write

$$d\theta^i = \mu^i_j(\theta)\delta\theta^j, \qquad \mu^i_j = \left[\frac{\partial\varphi^i(\theta, \alpha)}{\partial\alpha^j}\right]_{\alpha=0}. \tag{12}$$

The formula above represents a relation between $d\theta$ and $\delta\theta$ which can also be written

$$\delta\theta^j = \lambda^j_i(\theta)d\theta^i, \qquad \lambda^j_k\mu^k_i = \mu^k_i\lambda^j_k = \delta^j_i. \tag{13}$$

By putting

$$u^\mu_i(a) = \left[\frac{\partial f^\mu(a; \theta)}{\partial\theta^i}\right]_{\theta=0}, \tag{14}$$

and by using Equation (13), Equation (4b) can be written

$$da^\mu = u^\mu_k(a)\lambda^k_j(\theta)d\theta^j. \tag{15}$$

In this way we reach *Lie's first theorem.*

Theorem 1.[57] *When transformations* (1) *form a connected, m-dimensional, Lie group, then*

$$\frac{\partial a^\mu}{\partial\theta^j} = u^\mu_k(a)\lambda^k_j(\theta), \tag{16}$$

where the functions u^μ_k *are analytic.*

Let $A(a)$ be an (analytic) function of the a variables. The infinitesimal Lie transformation $a \to a + da$ induces a variation of $A(a)$ which can be written

$$dA = \frac{\partial A}{\partial a^\mu} u^\mu_j\delta\theta^j = \delta\theta^k u^\mu_k \frac{\partial}{\partial a^\mu} A$$

$$= \delta\theta^k X_k A. \tag{17}$$

The m-independent quantities

$$X_k = X_k(a) = u^\mu_k(a)\frac{\partial}{\partial a^\mu} = \left[\frac{\partial f^\mu(a; \theta)}{\partial\theta^k}\right]_{\theta=0} \frac{\partial}{\partial a^\mu} \tag{18}$$

are called the *infinitesimal generators* of the transformations (or of the group). For our later needs, we refer to the X's defined by Equations (18) as the *standard generators.*

We are now interested in the (necessary and sufficient) conditions for transformations (1) to constitute a Lie group. By using the converse of the Poincaré lemma, they can be written

$$\frac{\partial^2 a'^\mu}{\partial\theta^i\,\partial\theta^j} = \frac{\partial^2 a'^\mu}{\partial\theta^j\,\partial\theta^i}, \tag{19}$$

that is

$$\frac{\partial u_k^\mu}{\partial \theta^i}\lambda_j^k + u_k^\mu \frac{\partial \lambda_j^k}{\partial \theta^i} = \frac{\partial u_k^\mu}{\partial \theta^j}\lambda_i^k + u_k^\mu \frac{\partial \lambda_i^k}{\partial \theta^j}. \tag{20}$$

Thus

$$\begin{aligned} u_k^\mu\left(\frac{\partial \lambda_j^k}{\partial \theta^i} - \frac{\partial \lambda_i^k}{\partial \theta^j}\right) &= \lambda_j^k \frac{\partial u_k^\mu}{\partial \theta^i} - \lambda_i^k \frac{\partial u_k^\mu}{\partial \theta^j} \\ &= \lambda_j^k \frac{\partial u_k^\mu}{\partial a^\nu}\frac{\partial a^\nu}{\partial \theta^i} - \lambda_i^k \frac{\partial u_k^\mu}{\partial a^\nu}\frac{\partial a^\nu}{\partial \theta^j} \\ &= \lambda_j^r u_l^\nu \lambda_i^l \frac{\partial u_r^\mu}{\partial a^\nu} - \lambda_i^k u_l^\nu \lambda_j^l \frac{\partial u_k^\mu}{\partial a^\nu}. \end{aligned} \tag{21}$$

Therefore,

$$u_i^\nu \frac{\partial u_j^\mu}{\partial a^\nu} - u_j^\nu \frac{\partial u_i^\mu}{\partial a^\nu} = C_{ij}^k u_k^\mu, \tag{22}$$

where

$$C_{ij}^k = \mu_i^r \mu_j^s \left(\frac{\partial \lambda_r^k}{\partial \theta^s} - \frac{\partial \lambda_s^k}{\partial \theta^r}\right). \tag{23}$$

The m^3 quantities C_{ij}^k are independent from θ. This can be seen by differentiating Equation (22) with respect to θ. After some simple calculations, one then see that

$$\frac{\partial C_{ij}^k}{\partial \theta^l} = 0, \qquad i, j, k, l = 1, 2, \ldots, m. \tag{24}$$

In this way we reach *Lie's second theorem.*

Theorem 2. *If X_i, $i = 1, 2, \ldots, m$, are the generators of an m-dimensional Lie group, they satisfy the closure relations*

$$[X_i, X_j]_A = X_i X_j - X_j X_i = C_{ij}^k X_k, \tag{25}$$

where the quantities C_{ij}^k are called structure constants.

The symbol A in Equation (25) denotes an associative algebra with a conventional, associative product of operators $X_i X_j$. At closer inspection, this algebra emerges as being the *universal enveloping associative algebra* of the Lie algebra characterized by rule (25).

The *fundamental Lie's rule* (25) can be explicitly written

$$[X_i, X_j]_A = \left[u_i^\mu \frac{\partial}{\partial a^\mu}, u_j^\nu \frac{\partial}{\partial a^\nu}\right]_A = C_{ij}^k u_k^\alpha \frac{\partial}{\partial a^\alpha}, \tag{26}$$

where the product $[X_i, X_j]_A$ is Lie; that is, it satisfies the identities

$$[X_i, X_j]_A + [X_j, X_i]_A = 0, \tag{27a}$$

$$[[X_i, X_j]_A, X_k]_A + [[X_j, X_k]_A + X_i]_A + [[X_k, X_i]_A, X_j]_A = 0. \tag{27b}$$

By substituting into these expressions the explicit form of the Lie product in terms of the structure constants, *Lie's third theorem* is reached.

Theorem 3.[57] *The structure constants of a Lie group in standard realization* (18) *obey the relations*

$$C_{ij}^{k} + C_{ji}^{k} = 0, \tag{28a}$$

$$C_{ij}^{k} C_{kl}^{r} + C_{jl}^{k} C_{ki}^{r} + C_{li}^{k} C_{kj}^{r} = 0. \tag{28b}$$

Theorems 1, 2, and 3 essentially provide the correspondence between a given (connected) Lie group G and its Lie algebra $\mathbf{G}$. In particular, they allow the characterization of a Lie group in the neighborhood of the identity via the structure constants. We have here tacitly implied that different Lie groups may exist all admitting the same Lie algebra, that is, the same structure constants. However, among all Lie groups with the same Lie algebra only one is simply connected, called the *universal covering group*. For instance, group $SU(2)$ ($SL(2.C)$) is the universal covering group of the group of rotations $SO(3)$ (the homogeneous Lorentz group $SO(3.1)$).

The inverse transition from a Lie algebra to a corresponding Lie group can be characterized via the inverses of Lie's first, second, and third theorems. We urge the interested reader to study the specialized literature on this topic, such as Gilmore (1974) and cited references. For the reader's convenience, we have outlined one of the simplest approaches, known as the *exponential mapping*. Write Equations (15) in the form

$$\frac{\partial a^{\mu}}{\partial \theta^{i}} = u_{k}^{\mu}(a)\lambda_{i}^{k}(\theta) = \lambda_{i}^{k}(\theta)X_{k}(a)a^{\mu}, \tag{29}$$

and introduce the one-dimensional parametrization

$$\theta^{k} = \tau\alpha^{k}, \quad a'^{\mu} = a'^{\mu}(\theta(\tau)) = a''^{\mu}(\tau). \tag{30}$$

Then we write

$$a''^{\mu}(\tau) = T_{\nu}^{\mu}(\tau)a^{\nu}, \qquad a^{\nu} = [a''^{\nu}(\tau)]_{\tau=0}. \tag{31}$$

To compute the elements $T_{\nu}^{\mu}(\tau)$, consider the equations

$$\frac{da^{\mu}}{d\tau} = \frac{\partial a^{\mu}}{\partial \theta^{i}}\frac{d\theta^{i}}{d\tau} = \alpha^{k}\lambda_{k}^{r}(\theta)X_{r}(a)a''^{\mu}(0), \tag{32a}$$

$$\frac{d}{d\tau}T_{\nu}^{\mu}(\tau)a^{\nu} = \alpha^{k}\lambda_{k}^{r}(\theta)X_{r}(a)T_{\nu}^{\mu}(\theta)a''^{\nu}(0). \tag{32b}$$

However, $a''^{\nu}(0)$ are arbitrary initial values. Thus the solutions of the total differential equations

$$\frac{d}{d\tau}T_{\nu}^{\mu}(\tau) = \alpha^{k}\lambda_{k}^{r}(\theta)X_{r}(a(\tau))T_{\nu}^{\mu}(\tau) \tag{33}$$

with initial conditions

$$T_{\nu}^{\mu}(0) = \delta_{\nu}^{\mu}, \qquad \frac{d}{d\tau}T_{\nu}^{\mu}(\tau)\Big|_{\tau=0} = \alpha^{k}\lambda_{k}^{r}(\theta)X_{r}(a(0))\delta_{\nu}^{\mu} \tag{34}$$

can be written

$$T_{\nu}^{\mu}(\tau) = \sum_{n=0}^{\infty}\frac{1}{n!}[\theta^{k}X_{k}(a(0))\delta_{\nu}^{\mu}]^{n} \tag{35}$$

yielding the exponential mapping

$$a'^{\mu} = e^{\theta^{k}X_{k}(a)}\Big|_{A}a^{\mu}. \tag{36}$$

If, instead of the variables of the base manifold, we have a function of the same variables, the procedure above also applies, and we can write

$$A(a') = e^{\theta^k X_k(a)}|_A A(a). \tag{37}$$

In particular, the infinitesimal (standard) generators can be recovered by the rule

$$X_k = \left[\frac{\partial}{\partial\theta^k} e^{\theta^i X_i}|_A\right]_{\theta=0}. \tag{38}$$

Notice that the standard realization (36) of the group of transformations (1) is manifestly connected. The verification of the conditions to qualify as a Lie group is a simple but instructive exercise for the interested reader. Here we restrict ourselves to recalling that the product of two elements of group (36),

$$e^{X_\alpha} e^{X_\beta} = e^{X_\rho}, \tag{39}$$

is characterized by the so-called *Baker–Campbell–Hausdorff formula*:

$$X_\rho = X_\alpha + X_\beta + \tfrac{1}{2}[X_\alpha, X_\beta]_A + \tfrac{1}{12}[(X_\alpha - X_\beta), [X_\alpha, X_\beta]_A]_A + \cdots. \tag{40}$$

It is significant for our program to indicate that a Lie algebra does not necessarily admit a corresponding Lie group. For specific examples of Lie algebras of this type, the reader may consult, for instance, Hurst (1968). In essence, the applicability of the exponential mapping in general, or the "integration" of a Lie algebra to a Lie group must satisfy certain (convergence) conditions of the underlying infinite series, known as *integrability conditions*. We also refer the reader in this respect to the specialized literature in the subject and, in particular, to Nelson (1959).

We pass now to the Lie-isotopic generalization of Definition 1 and Theorems 1, 2, and 3. The prior identification of the main objective may be useful here. Lie's crucial result is fundamental rule (25). This rule essentially characterizes Lie algebras via the conventional associative product $X_i X_j$ of vector fields $X_i = u_i^\mu(a)\,\partial/\partial a^\mu$ on a manifold M. Our main objective is to generalize Definition 1 and Theorems 1, 2, and 3 in such a way as to characterize a Lie algebra via the most general possible associative product $X_i * X_j$ of vector fields on a manifold.

Of utmost importance is the condition that *the base manifold M with local coordinates a^μ, the parameters θ^i, and the generators X_i of the conventional formulation of Lie's theorems are not changed in their isotopic generalization.* This is due to physical requirements which are uncompromisable for the description under consideration. As we recalled earlier, the local coordinates of M customarily have a direct physical meaning such as the coordinates of the frame of the experimental setup; the parameters carry a direct physical meaning as measurable quantities such as time, angle, etc., and the generators directly represent physical characteristics such as energy, angular momentum, etc. When the conventional description of self-adjoint interactions via Theorems 1, 2, and 3 is broadened to permit the additional presence of the non-self-adjoint interactions, the frame of the experimental observer must be preserved; measurable quantities such as time and angles must be preserved; and physical characteristics such as energy and angular momentum must also be preserved unaltered.

These objectives can be realized as follows.

Definition 2.[58] Let

$$G: a^\nu \to a'^\nu = f^\nu(a; \theta) \tag{41}$$

be an r-dimensional Lie transformation group G as per Definition 1. A *Lie isotopic image* or, simply an *isotope* G^* of G is a set of transformations characterizable via a regular $(N \times N)$ matrix of analytic functions $(g^\mu_\nu(a; \theta))$ acting on (41)

$$G^*: a^\mu \to a^{*\mu} = g^\mu_\nu(a; \theta) f^\nu(a, \theta) = f^{*\mu}(a; \theta) \tag{42a}$$

$$\det(g^\mu_\nu) \neq 0, \qquad g^\mu_\nu|_{\theta=0} = \delta^\mu_\nu \tag{42b}$$

which verify the following properties. (a) The transformations $a^* = f^*(a; \theta)$ constitute a Lie transformation group, by therefore verifying conditions 1–5 of Definition 1. (b) The group G^* is realized via the same base manifold, the same parameters and the same generators of G. (c) When reduced in the neighborhood of the identity transformation,[59] the group G^* can be characterized by a Lie algebra isotope $\mathbf{G}^*$ of $\mathbf{G}$.

Condition (c) is introduced to avoid non-Lie, Lie-admissible algebras in the neighborhood of the identity transformations. As a matter of fact, it is precisely this possibility that permits the further generalization of Lie's theory of type III.[58]

Since the group of transformations $f^{*\mu}(a; \theta)$ is a conventional, connected Lie group by assumption, it can be studied in the neighborhood of the identity as in the conventional case. The repetition of the analysis of $f(a; \theta)$ then yields the expressions

$$da^\mu = u^{*\mu}_k(a) \lambda^k_i(\theta) d\theta^i \tag{43a}$$

$$u^{*\mu}_k(a) = \left| \frac{\partial}{\partial \theta^k} g^\mu_\nu(a; \theta) f^\nu(a; \theta) \right|_{\theta=0}. \tag{43b}$$

In order to realize the isotopy, we then introduce the following reformulation in terms of the quantities of G for given $g^i_k(a)$ functions

$$u^{*\mu}_k(a) = g^i_k(a) u^\mu_i(a), \qquad \det(g^i_k) \neq 0. \tag{44}$$

Note that the other possibility $u^{*\mu}_k = g_\nu{}^\mu u^\nu_k$, even though conceivable (and actually more in line with Equations (43)), is excluded here because it would imply the redefinition of the generators $X_k = u^\mu_k(\partial/\partial a^\mu) \to X^*_k = g^\mu_\nu u^\nu_k(\partial/\partial a^\mu)$ which is *contrary* to the notion of isotopy under study. The analyticity of the transformations then trivially implies the following generalization of Lie's First Theorem.

Theorem 4.[58] *If transformations* (42) *characterize an isotopic image* G^* *of the Lie group* G *of transformations* (41), *then analytic functions* $g^i_k(a)$ *exist such that*

$$\frac{\partial a^{*\mu}}{\partial \theta^i} = g^k_i(a) u^\mu_k(a) \lambda^i_j, \qquad \det g \neq 0, \tag{45}$$

and the $u^\mu_k(a)$ *functions are analytic.*

[58] Santilli (1978c, Section 3.6, pp. 329–348), reprinted in Myung *et al.* (1978-I). See also Santilli (1979b, Section 1.2).

[59] The *identity transformation* of a Lie group should not be confused with the *unit element* of the universal enveloping associative algebra. As we shall see, the identity transformation of G^* is preserved in a way compatible with the loss of the unit character of the element $\mathbb{1}$ for $A^*(\mathbf{G})$.

This theorem, though analytically trivial, has nontrivial implications. Indeed, it implies a modification of the structure of the group in the neighborhood of the identity, i.e.,

$$G\colon a'^{\mu} \approx a^{\mu} + \theta^{i} u_i^{\mu}(a) \rightarrow G^*\colon a^{*\mu} \approx a^{\mu} + \theta^{i} g_i^{j}(a) u_j^{\mu}(a), \tag{46}$$

which is precisely the desired situation. We must now identify the integrability conditions under which such a behavior is still Lie in algebraic character, when expressed in terms of the generators and parameters of the original group. Under these conditions, we say that the quantities g_i^j of Equations (45) or (46) are *isotopic functions* with respect to G.

The group G is Lie and thus admits the standard realization worked out earlier in this chart

$$u_i^{\nu} \frac{\partial}{\partial a^{\nu}} u_j^{\mu} - u_j^{\nu} \frac{\partial}{\partial a^{\nu}} u_i^{\mu} = C_{ij}^{k} u_k^{\mu} \frac{\partial a}{\partial a^{\mu}}, \tag{47a}$$

$$C_{ij}^{k} = \mu_i^{r} \mu_j^{s} \left(\frac{\partial \lambda_r^{k}}{\partial \theta^{s}} - \frac{\partial \lambda_s^{k}}{\partial \theta^{r}} \right), \tag{47b}$$

$$[X_i, X_j]_A = X_i X_j - X_j X_i = C_{ij}^{k} X_k \tag{47c}$$

$$X_k = u_k^{\mu}(a) \frac{\partial}{\partial a^{\mu}}. \tag{47d}$$

The group G^* is also Lie and thus can be realized in the standard form

$$u_i^{*\nu} \frac{\partial}{\partial a^{\nu}} u_j^{*\mu} - u_j^{*\nu} \frac{\partial}{\partial a^{\nu}} u_i^{*\mu} = C_{ij}^{*k} u_k^{*\mu} \frac{\partial}{\partial a^{\mu}}, \tag{48a}$$

$$C_{ij}^{*k} = \mu_i^{*r} \mu_j^{*s} \left(\frac{\partial \lambda_r^{*k}}{\partial \theta^{s}} - \frac{\partial \lambda_s^{*k}}{\partial \theta^{r}} \right), \tag{48b}$$

$$[X_i^*, X_j^*]_A = X_i^* X_j^* - X_j^* X_i^* = C_{ij}^{*k} X_k^*, \tag{48c}$$

$$X_k^* = u_k^{*\mu} \frac{\partial}{\partial a^{\mu}}. \tag{48d}$$

However, this realization generally implies a change of the generators in the transition from G to G^*

$$\mathbf{G}\colon X_k = u_k^{\mu} \frac{\partial}{\partial a^{\mu}} \rightarrow G^*\colon X_k^* = u_k^{*\mu} \frac{\partial}{\partial a^{\mu}} \tag{49}$$

and, as such, does not verify the conditions for isotopy. To achieve the objective under consideration, we introduce the following isotopy of the universal enveloping associative algebra, according to Chart 5.2, this time realized via *functions on the base manifold.*

$$A(\mathbf{G})\colon X_i X_j \rightarrow A^*(\mathbf{G})\colon X_i * X_j = g_i^{r} X_r g_j^{s} X_s. \tag{50}$$

Notice that this mapping does verify the conditions of isotopy, in the sense that it is realized via the generators of the original algebra, while preserves the associativity of the product,

$$(g_i^{r} X_r g_j^{s} X_s) g_k^{t} X_t = g_i^{r} X_r (g_j^{s} X_s g_k^{t} X_t). \tag{51}$$

The fundamental Lie rule (47c) can now be rewritten

$$u_i^\nu \frac{\partial}{\partial a^\nu} * u_j^\mu - u_j^\nu \frac{\partial}{\partial a^\nu} * u_i^\mu = \tilde{C}_{ij}^k u_k^\mu \tag{52a}$$

$$\tilde{C}_{ij}^k = C_{ij}^{*r} g_r^k(a). \tag{52b}$$

The integrability conditions for the functions $g_k^j(a)$ to be isotopic, that is, to yield rule (52), can then be readily computed. Thus we reach the following generalization of Lie's second theorem.

Theorem 5.[58] *Under the integrability conditions*

$$g_i^k u_k^\nu \frac{\partial}{\partial a^\nu} g_j^l - g_j^k u_k^\nu \frac{\partial}{\partial a^\nu} g_i^l = g_j^r g_i^s C_{rs}^l + C_{ij}^{*r} g_k^l \tag{53}$$

the generators X_i of an isotope G^* of a Lie group G satisfy the isotopic rule of associative Lie-admissibility

$$[X_i, X_j]_{A^*} = X_i * X_j - X_j * X_i = \tilde{C}_{ij}^k(a) X_k, \tag{54a}$$

$$A^*(\mathbf{G}): X_i * X_j = g_i^r X_r g_j^s X_s, \tag{54b}$$

$$X_k = u_k^\mu(a) \frac{\partial}{\partial a^\mu}, \tag{54c}$$

where the quantities $\tilde{C}_{ij}^k(a)$, here called structure functions, are generally dependent on the (local) coordinates of the base manifold of the original group.

In this way we reach an interpretation of the F^*-linear combination of the isotopically mapped standard monomials of Chart 5.2. While in the standard realization (47c) the quantities C_{ij}^{*k} are constants (the structure *constants* of a Lie group), the corresponding quantities which emerge after the reformulation of the same group G^* in terms of the base manifold, the parameters, and the generators of G, acquire an explicit dependence on the local coordinates (the structure *functions* $\tilde{C}_{ij}^k(a)$). This situation has numerous technical implications (e.g., from the viewpoints of the representation and classification theory) which are not considered here.

The reformulation of Lie's third theorem is now straightforward. Indeed, the use of the Lie algebra laws for the isotopically mapped product (54a) yields the following property.

Theorem 6.[58] *The structure functions $\tilde{C}_{ij}^k(a)$ of the isotopic realization of a Lie group G^* verify the identities*

$$\tilde{C}_{ij}^k + \tilde{C}_{ji}^k = 0, \tag{55a}$$

$$\tilde{C}_{ij}^k \tilde{C}_{kl}^r + \tilde{C}_{jl}^k \tilde{C}_{ki}^r + \tilde{C}_{li}^k \tilde{C}_{kj}^r + [\tilde{C}_{ij}^r, X_l]_{A^*} + [\tilde{C}_{jl}^r, X_i]_{A^*} + [\tilde{C}_{li}^r, X_j]_{A^*} = 0. \tag{55b}$$

The exponentiation from the Lie algebra to the Lie group can now be formulated in terms of the *isotopic image of the exponential law* (37), i.e.,[60]

$$G: e^{\theta^i X_i}|_A \to G^*: e^{\theta^i X_i}|_{A^*}, \tag{56}$$

[60] The proof that Equations (29)–(36) admit a consistent isotopic generalization is left to the interested researcher (Problem 5.10).

which is based on the following *rule of Lie isotopy*

$$\mathbf{G}\colon [X_i, X_j]_A = C_{ij}^k X_k \to \mathbf{G}^*\colon [X_i, X_j]_{A^*} = \tilde{C}_{ij}^k(a) X_k \tag{57}$$

with consequential *isotopically mapped Baker–Campbell–Hausdorff formula*

$$e^{X_\alpha^*} e^{X_\beta^*} = e^{X_j^*}, \qquad X^* = gX, \tag{58a}$$

$$X^* = X_\alpha^* + X_\beta^* + \tfrac{1}{2}[X_\alpha, X_\beta]_{A^*} + \tfrac{1}{12}[(X_\alpha - X_\beta), [X_\alpha, X_\beta]_{A^*}]_{A^*} + \cdots, \tag{58b}$$

whose existence is ensured by that of the standard realization. The reader can now see the emergence of the F^*-linear combination of the basis directly in the group composition law. Clearly, the enveloping algebra underlying expressions (57) is the isotope $A^*(\mathbf{G})$ of $A(\mathbf{G})$.

A simple example may be useful in illustrating the analysis of this chart. Consider the one-parameter group of dilations

$$r' = f(r; \theta) = e^\theta r. \tag{59}$$

The standard generator for this group is given by

$$X = r \frac{\partial}{\partial r}. \tag{60}$$

Indeed

$$e^{\theta r(\partial/\partial r)} r = \left[1 + \frac{\theta}{1!}\left(r\frac{\partial}{\partial r}\right) + \frac{\theta^2}{2!}\left(r\frac{\partial}{\partial r}\right)^2 + \cdots\right] r = e^\theta r. \tag{61}$$

The group composition law is, in this case, trivial, i.e.,

$$r'' = f(r'; \theta') = e^{\theta'} r' = e^{\theta' + \theta} r. \tag{62}$$

Consider now the one-parameter connected Lie group of *nonlinear* transformations

$$r^* = f^*(r; \theta) = \frac{r}{1 - \theta r} = g(r, \theta) f(r, \theta), \qquad g = \frac{e^\theta}{1 - \theta r}, \tag{63}$$

with composition law

$$r^{**} = f^*(r^*; \theta') = \frac{r^*}{1 - \theta' r^*} = \frac{r/(1 - \theta r)}{1 - \theta'(1/r - \theta r)} = \frac{r}{1 - (\theta' + \theta) r}. \tag{64}$$

We are interested in realizing this group, as a necessary condition of isotopy, via the generator (60) of the different group (59). This implies the search for an isotopic function, that is, a function which multiplies generator (60) to yield the correct transformation law of f^* as a solution of integrability conditions (53). Such a solution, in the case at hand, is simple and is given by r. Indeed, the isotopically mapped exponential law (56) yields the correct result

$$e^{\theta r(r(\partial/\partial r))} = \left[1 + \frac{\theta}{1!}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\theta^2}{2!}\left(r^2\frac{\partial}{\partial r}\right)^2 + \cdots\right] r$$

$$= \frac{r}{1 - \theta r}. \tag{65}$$

Thus group (63) can be realized as an isotopic image of group (59).

The case considered above is trivial in the sense that all connected one-dimensional Lie groups are (locally) isomorphic. Thus, to activate the truly nonisomorphic character of the isotope with respect to the original group, one needs more than one dimension. Such a case is already provided by the realization of $SO(2.1)$ as an isotope of $SO(3)$, in Equations (23) of Chart 5.2.

Chart 5.4 Isotopic Generalizations of Enveloping Algebras, Lie Algebras, and Lie Groups in Classical and Quantum Mechanics

In this chart we shall first review the conventional realizations of enveloping associative algebras, Lie algebras, and Lie groups via Hamiltonian formulations on a cotangent bundle T^*M (classical mechanics) and on a Hilbert space $\mathscr{H}$ (quantum mechanics). We shall then present their Lie-isotopic generalizations and show that they constitute a realization of the generalized theory of Charts 5.1–5.3.

The techniques of the Inverse Lie Problem presented in these volumes have established the universality of the applicability of Lie's theory to *all* local Newtonian systems satisfying certain topological conditions, without any restriction on their dynamics. In particular, the universality resulted in being of twofold nature. The first, essentially along Lie's original intuition, is established by Theorem 6.2.1 on the Indirect Universality of Hamiltonian Formulations. The second is that established by Theorem 4.5.1 on the Direct Universality of Birkhoffian Formulations, in which the universality is achieved without the use of the transformation theory. In this latter case, the underlying mechanics is necessarily of the generalized type, according to the following main lines

$$\overbrace{\underbrace{\dot{a}^\mu - \Xi^\mu(t, a)}_{\text{Hamiltonian representation}} - F^\mu(t, a)}^{\text{Birkhoffian representation}} = 0, \tag{1a}$$

$$\begin{pmatrix} \dot{r}^{ka} \\ \dot{p}_{ka} \end{pmatrix} - \begin{pmatrix} p_{ka/m_k} \\ f_{ka}^{SA}(t, \mathbf{r}, \mathbf{p}) \end{pmatrix} - \begin{pmatrix} 0 \\ F_{ka}^{NSA}(t, \mathbf{r}, \mathbf{p}) \end{pmatrix} = 0, \tag{1b}$$

$$\Xi^\mu(t, a) = \omega^{\mu\nu} \frac{\partial H(t, a)}{\partial a^\nu}, \tag{1c}$$

$$\omega^{\mu\nu} = \left(\left\| \frac{\partial R_\beta^0}{\partial a^\alpha} - \frac{\partial R_\alpha^0}{\partial a^\beta} \right\|^{-1} \right)^{\mu\nu}, \qquad R^0 = (\mathbf{p}, \mathbf{0}) \tag{1d}$$

$$\Gamma^\mu(t, a) = \Xi^\mu(t, a) + F^\mu(t, a) = \Omega^{\mu\nu}(a) \frac{\partial B(t, a)}{\partial a^\nu}, \tag{1e}$$

$$\Omega^{\mu\nu}(a) = \left(\left\| \frac{\partial R_\beta}{\partial a^\alpha} - \frac{\partial R_\alpha}{\partial a^\beta} \right\|^{-1} \right)^{\mu\nu}. \tag{1f}$$

$k = 1, 2, \ldots, N; \quad a = x, y, z; \quad \mu, \nu = 1, 2, \ldots, 2n = 6N; \quad a \in T^*M.$

A primary objective of this chart *is not* the indication that the Birkhoffian formulations constitute a realization of Lie's theory (which would be a trivial task). Instead, the objective is to indicate that the formulations constitute a Lie-algebra-preserving generalization of the Hamiltonian realization of Lie's theory. A fully similar situation exists at the level of operators in a Hilbert space, as we shall see.

Part A: *Classical Hamiltonian Realization of Lie's Theory*. As reviewed in Chart 5.3, Lie's theory in its original conception is a *theory of vector fields on a manifold*. The classical Hamiltonian realization of the theory can therefore be identified via the following steps.

(A.1). The manifold M is assumed to be the cotangent bundle T^*M (phase space) with local coordinates $a = (\mathbf{r}, \mathbf{p}), a = \{a^\mu\}, \mu = 1, 2, \ldots, 2n$.

(A.2) The vector fields are assumed to have the following Hamiltonian structure

$$X_k = u_k^\mu(a) \frac{\partial}{\partial a^\mu} \equiv \omega^{\mu\nu} \frac{\partial G_k(a)}{\partial a^\nu} \frac{\partial}{\partial a^\mu} \tag{2}$$

where the G's are given functions on T^*M.

(A.3) The universal enveloping associative algebra of the vector fields has therefore the basis[61]

$$\mathscr{A}: \mathbb{1}; \quad \underset{i \le j}{X_i X_j}; \quad \underset{i \le j \le k}{X_i X_j X_k}; \ldots . \tag{3}$$

All other features of the theory can be derived as consequences of the above assumptions. In fact, from Lie's First Theorem, one can derive the following *canonical realization of the infinitesimal Lie transformations*

$$a^\mu \dashrightarrow a'^\mu = a^\mu + \theta^k \omega^{\mu\nu} \frac{\partial G_k}{\partial a^\nu}. \tag{4}$$

From Equation (36) of Chart 5.3, one then has the *canonical realization of the Lie transformation groups*

$$G: e^{\theta^k X_k} \equiv \exp\left(\theta^k \omega^{\mu\nu} \frac{\partial G_k}{\partial a^\nu} \frac{\partial}{\partial a^\mu} \right) \tag{5}$$

which, in view of the convergent power series expansion,

$$e^{\theta^k X_k} = \mathbb{1} + \frac{\theta^k}{1!} X_k + \frac{\theta^i \theta^j}{2!} X_i X_j + \cdots, \tag{6}$$

is clearly defined in the enveloping algebra $\mathscr{A}$. The vector field character of realizations (2), (4), and (5) is self-evident.

Not equally self-evident is the *realization of the Lie-algebra product in terms of the conventional Poisson brackets*

$$\mathbf{G}: [G_i, G_j]_{PB} = (G_i, G_j) - (G_j, G_i) = C_{ij}^k G_k \tag{7a}$$

$$(G_i, G_j) = \frac{\partial G_i}{\partial r^k} \frac{\partial G_j}{\partial p_k} = \text{nonassociative} \tag{7b}$$

[61] Note that the element $\mathbb{1}$ is the *unit*, trivially, because $\mathbb{1} X_k = X_k$, and $X_k \mathbb{1} = 0 + X_k$. Needless to say, the field of Hamiltonian realizations is the real $\mathbb{R}$.

which, as indicated in Chart 5.1, has a *nonassociative envelope*. Thus it is not of the same algebraic type as the conventional Lie rule (25) of Chart 5.3, i.e.,

$$\mathbf{G}\colon [X_i, X_j]_{\mathscr{A}} = X_i X_j - X_j X_i = C_{ij}^k X_k, \tag{8a}$$

$$X_i X_j = \text{associative}, \tag{8b}$$

$$X_i = \omega^{\mu\nu} \frac{\partial G_i}{\partial a^\nu} \frac{\partial}{\partial a^\mu}. \tag{8c}$$

Realization (7) clearly calls for the Lie-admissible generalization of Lie's theory indicated in Chart 5.1 (that based on the nonassociative generalization of the enveloping algebra). With the understanding that a considerable amount of research remains to be done in order to reach a true technical understanding of the replacement of realization (8) with the conventional (7), we limit ourselves to the indication that the replacement appears to be in line with general property (14) of Chart 5.1, that, given a realization of a Lie algebra via a nonassociative envelope, an equivalent realization exists via an associative envelope, and vice versa.[62]

For completeness, as well as subsequent needs, we move now to a more detailed consideration of realization (4). Consider the one-parameter, contemporaneous transformations on T^*M,

$$a^\mu \to a'^\mu = a^\mu + \delta\theta G^\mu(t, a), \tag{9}$$

where the G's are analytic functions. The conditions that the transformations are identity isotopic with respect to the canonical Lie tensor (that is, canonical as per Definition 5.3.1) are given by

$$\omega^{\mu\beta} \frac{\partial G^\nu}{\partial a^\beta} + \frac{\partial G^\mu}{\partial a^\alpha} \omega^{\alpha\nu} = [a^\mu, G^\nu] + [G^\mu, a^\nu] = 0. \tag{10}$$

The use of the converse of the Poincaré lemma allows the computation of an explicit solution in the G^μ functions,

$$G^\mu = \omega^{\mu\rho} \frac{\partial G}{\partial a^\rho} = [a^\mu, G], \tag{11}$$

as well as the identification of the integrability conditions for transformations (9) to be identity isotopic, which can be written[63]

$$[a^\mu, [a^\nu, G]] + [a^\nu, [G, a^\mu]] + [G, [a^\mu, a^\nu]] = 0. \tag{12}$$

By inspecting these latter equations, we conclude that *the Jacobi identity can be interpreted as the integrability conditions for infinitesimal transformations to be canonical*. This establishes an additional significance of the Jacobi identity in Newtonian Mechanics, besides characterizing the algebraic structure of Hamilton's equations. Integrability conditions (12) also indicate that the Lie algebra laws are at the very foundation of the infinitesimal canonical transformations.

[62] This is a *conjecture* at this time; that is, we know of a number of cases in which the property is true, but we do not know whether it is a general property for all possible realizations of Lie algebras.

[63] See Problem 5.12.

Each given infinitesimal canonical transformation characterizes a variation of a dynamical quantity in phase space, say $A(a)$, given by

$$\delta A = A(a + \delta a) - A(a) = \frac{\partial A}{\partial a^{\mu}} \delta a^{\mu} = \delta\theta[A, G]. \tag{13}$$

However, infinitesimal first-order variations are first-order differentials (Section I.1.3). We can therefore write

$$\frac{dA}{d\theta} = [A, G]. \tag{14}$$

The scalar function G of Equations (13) or (14) has a rather crucial methodological function. It is called the *generator* of the infinitesimal canonical transformation, in the sense that, whenever such a function is assigned, a (unique) infinitesimal canonical transformation is then characterized via either rule (11) or (13).

We now review the following most important cases of infinitesimal canonical transformations.

(a) *Infinitesimal time evolutions.* Assume that

$$\delta\theta = \delta t, \qquad G = H. \tag{15}$$

Then we can write

$$\begin{aligned} \delta q^k &= \delta t[q^k, H] = \delta t\dot{q}^k, \\ \delta p_k &= \delta t[p_k, H] = \delta t\dot{p}_k, \\ \delta A &= \delta t[A, H] = \delta t\dot{A}. \end{aligned} \tag{16}$$

Thus *the infinitesimal evolution in time of a Newtonian system represented by the Hamiltonian H can be described via an infinitesimal canonical transformation with the Hamiltonian as the generator and δt as the parameter.*

(b) *Infinitesimal space translations.* Assume that

$$\delta\theta = \delta q^i, \qquad G = p_i, \quad i = \text{fixed}. \tag{17}$$

Then we can write

$$\begin{aligned} \delta q^k &= \delta q^i[q^k, p_i] = \delta q^k, \\ \delta p_k &= \delta q^i[p_k, p_i] = 0, \\ \delta A &= \delta q^i[A, p_i] = \delta q^i \frac{\partial A}{\partial q^i}. \end{aligned} \tag{18}$$

Thus *an infinitesimal translation in the q^i component can be described via an infinitesimal canonical transformation with the generalized momentum p_i as generator and δq^i as parameter.*

(c) *Infinitesimal space rotations.* Assume now that

$$\delta\theta = \delta\alpha, \qquad G = M_z = xp_y - yp_x. \tag{19}$$

Equation (13) produces the known expressions of infinitesimal rotations in a plane

$$\begin{aligned} \delta x &= -y\delta\alpha, & \delta p_x &= -p_y\delta\alpha, \\ \delta y &= x\delta\alpha, & \delta p_y &= p_x\delta\alpha. \end{aligned} \tag{20}$$

More generally, we can state that *an infinitesimal rotation $\delta\alpha$ along an axis* **n** *in a three-dimensional Euclidean space can be described via an infinitesimal canonical transformation whose generator is the component of the angular momentum along* **n**, $\mathbf{M} \cdot \mathbf{n}$, *and whose parameter is* $\delta\alpha$.

We move now to a more detailed study of realization (5). Under infinitesimal transformations (4), the local variables transform according to the rule

$$\frac{da^\mu}{d\theta} = [a^\mu, G]_{(a)}, \qquad \mu = 1, 2, \ldots, 2n. \tag{21}$$

Consider a (finite) canonical transformation to a new set of variables, say, a_0^μ. By recalling that these transformations are identity isotopic with respect to the conventional Poisson brackets, we can write (for details, see Sudarshan and Mukunda (1974, pp. 51–54)):

$$\frac{da^\mu}{d\theta} = [a^\mu(a_0), G'(a_0)]_{(a_0)}, \qquad G'(a_0) = G(a(a_0)). \tag{22}$$

However,

$$\frac{dG}{d\theta} = [G, G] \equiv 0, \qquad G'(a_0) = G(a_0). \tag{23}$$

We can therefore write

$$\frac{da^\mu}{d\theta} = [a^\mu(a_0), G(a_0)]_{(a_0)}. \tag{24}$$

The above equations can be interpreted as a system of $2n$, first-order ordinary differential equations in the unknowns a^μ with a_0^μ as the initial conditions. Under our smoothness assumptions, a formal solution can be written via the power-series expansion

$$a^\mu = a_0^\mu + \frac{\theta}{1!}[a_0^\mu, G]_{(a_0)} + \frac{\theta^2}{2}[[a_0^\mu, G]_{(a_0)}, G]_{(a_0)} + \cdots, \tag{25}$$

which represents the construction of a finite canonical transformation, $a_0^\mu \to a^\mu$, via an infinite number of successive infinitesimal transformations.[64]

Expansion (25) is customarily written in the closed form (5), i.e.,

$$a^\mu = \exp\left(\theta\omega^{\alpha\beta}\frac{\partial G}{\partial a_0^\beta}\frac{\partial}{\partial a_0^\alpha}\right)a_0^\mu, \tag{26}$$

The extension of Equation (26) to the transformation of an arbitrary quantity in phase space is immediate and looks like

$$A(a) = \exp\left(\theta\omega^{\alpha\beta}\frac{\partial G}{\partial a^\beta}\frac{\partial}{\partial a^\alpha}\right)A(a_0). \tag{27}$$

Suppose now that

$$\theta = t, \qquad G = H, \qquad a_0^\mu = a^\mu|_{t=t_0}. \tag{28}$$

[64] On a comparative basis with expansion (6), note that expansion (25) is on a nonassociative algebra, while (6) is on an associative algebra. It is remarkable that, despite this difference, the final exponential form is the same.

Equations (26) and (27) then become

$$a^{\mu}(t) = \exp\left(t\omega^{\alpha\beta}\frac{\partial H}{\partial a^{\beta}}\frac{\partial}{\partial a^{\alpha}}\right)a_0^{\mu}, \tag{29a}$$

$$A(a) = \exp\left(t\omega^{\alpha\beta}\frac{\partial H}{\partial a^{\beta}}\frac{\partial}{\partial a^{\alpha}}\right)A(a_0). \tag{29b}$$

Thus, besides infinitesimal time evolutions (16), a *finite time evolution of a Newtonian system represented by the Hamiltonian H can be obtained via an infinite succession of infinitesimal canonical transformations with the Hamiltonian as generator and time as parameter.* The extension of this occurrence to other finite transformations of physical relevance is immediate.

Part 2: Classical Lie-Isotopic Realization of Lie's Theory. As indicated in Chart 5.3, the Lie-isotopic generalization of Lie's theory is a *theory in which the vector fields on manifolds of the conventional formulation remain unchanged, and only the associative product of the envelope is altered in an associativity preserving way.* With respect to steps A.1, A.2, and A.3, we can therefore characterize the theory via the following assumptions.

(A*.1) The manifold T^*M and, most importantly, the local variables of the Hamiltonian formulation remain unchanged.

(A*.2) The vector fields X_k and the parameters θ^k of the Hamiltonian formulation also remain unchanged.

(A*.3) The universal enveloping associative algebra $\mathscr{A}$ of the Hamiltonian description is changed into the isotope characterized by Theorem 2 of Chart 5.2 and Theorems 4, 5, and 6 of Chart 5.3, i.e.,[65]

$$\mathscr{A}^*: \mathbb{1};\quad X_i;\quad \underset{i\le j}{X_i * X_j};\quad \underset{i\le j\le k}{X_i * X_j * X_k};\quad \cdots, \tag{30a}$$

$$X_i * X_j = X_i g_j^s X_s, \tag{30b}$$

$$\det(g_i^r) \neq 0;\quad X_i = \omega^{\mu\nu}\frac{\partial G_i}{\partial a^{\nu}}\frac{\partial}{\partial a^{\mu}}. \tag{30c}$$

The fact that the Birkhoffian generalization of Hamiltonian formulations is a realization of the theory is established by the following isotopic interpretation of the Birkhoffian vector fields according to rule (44) of Chart 5.3, i.e.,

$$X_k^* = \Omega^{\mu\nu}(a)\frac{\partial G_k}{\partial a^{\nu}}\frac{\partial}{\partial a^{\mu}} \equiv g_k^i\omega^{\mu\nu}\frac{\partial G_i}{\partial a^{\nu}}\frac{\partial}{\partial a^{\mu}} = g_k^i X_i, \tag{31}$$

where, of course, the g's verify the integrability conditions of Theorem 5 of Chart 5.3.

[65] Note that we have omitted the functions g_i^j on the left of the elements X_j, $X_i * X_j$, etc. This is permitted because the basis is defined via F^*-linear combinations, that is, combinations via functions on the base manifold (Chart 5.2). Note that the element $\mathbb{1}$ *is not* the unit, but rather the *weak unit* (Myung and Santilli (1979)).

We shall say that assumptions A*.1, A*.2, and A*.3, with the isotopic functions characterized by Equations (31), characterize a *classical Lie-isotopic (that is, Birkhoffian) realization of Lie's theory*. All other features can be obtained as a consequence of the specified assumptions. For instance, from Theorem 4, one can derive the following *realization of infinitesimal Lie-isotopic transformations via generalized canonical transformations*

$$a^\mu \to a'^\mu = a^\mu + \delta\theta^k \Omega^{\mu\nu} \frac{\partial G_k}{\partial a^\nu}. \tag{32}$$

Similarly, from Equation (56) of Chart 5.3, one has the following *Birkhoffian realization of the Lie-isotopic transformation groups*[66]

$$G^*: e^{\theta^k * X_k}|_{\mathscr{A}^*} = \mathbb{1} + \frac{\theta^k * X_k}{1!} + \frac{(\theta^k * X_k)^{2*}}{2!} + \cdots \tag{33}$$

which is clearly defined in $\mathscr{A}^*$.

We do not exclude the reinterpretation of Equations (30) or (33) as providing a realization of the conventional formulation of the enveloping algebra and of Lie's theory, that based directly on the Birkhoffian vector fields X_k^*. In fact, this interpretation is quite natural. However, we are primarily interested in identifying tools for the theoretical treatment of the physical implications of non-Hamiltonian forces, and this can best be done by identifying the departures from Hamiltonian formulations, as indicated earlier in this chart.

The Lie-isotopy applied to rule (7) yields the *Birkhoffian realization of Lie algebras*

$$\mathbf{G}^*: [G_i, G_j]^*_{\text{GPB}} = (G_i, G_j)^* - (G_j, G_i)^* = C^k_{ij}(a) G_k, \tag{34a}$$

$$(G_i, G_j)^* = \text{nonassociative isotopic} \tag{34b}$$

which, again, has a nonassociative envelope. The corresponding associative isotopic rule is given by

$$\mathbf{G}^*: [X_i, X_j]_{\mathscr{A}^*} = X_i * X_j - X_j * X_i = C^k_{ij}(a) X_k \tag{35a}$$

$$X_i * X_j = \text{associative isotopic.} \tag{35b}$$

The argument is now clear. Since rule (34) is characterized by the nonassociative isotopy $(G_i, G_j) \to (G_i, G_j)^*$, we must have, for consistency, a corresponding associative isotopy $X_i X_j \to X_i * X_j$. Nevertheless, let us state again that a full understanding of the transition from rule (35) to (34) will be achieved only after the development of the Lie-admissible generalization of Lie's theory.

We move now to a more detailed study of the infinitesimal transformations (32) for the intent of showing that they provide a step-by-step generalization of the infinitesimal canonical transformations.

[66] Expansion (33) can be written in a number of ways. Explicitly, the expansion reads

$$\mathbb{1} + \frac{\theta^i g^r_i X_r}{1!} + \frac{\theta^i g^r_i X_r \theta^j g^s_j X_s}{2!} + \cdots$$

The conditions that transformations (9) are identity isotopic with respect to Birkhoff's tensor $\Omega^{\mu\nu}(a)$ (that is, generalized canonical as per Definition 5.3.2 or 5.3.3) can be written

$$[a'^{\mu}, a'^{\nu}]^{*}_{(a)} = \Omega^{\mu\nu}(a), \tag{36}$$

and, to first-order in the parameters, are given by

$$\Omega^{\mu\rho}\frac{\partial G^{\nu}}{\partial a^{\rho}} + \frac{\partial G^{\mu}}{\partial a^{\rho}}\Omega^{\rho\nu} + G^{\rho}\frac{\partial \Omega^{\mu\nu}}{\partial a^{\rho}} = 0. \tag{37}$$

Thus, compared with the Hamiltonian case (10), we see the appearance of an additional term due to the a dependence of the Ω—tensor. Despite that, the generalization of the Hamiltonian case is straightforward. The use of the converse of the Poincaré Lemma yields the solution

$$G^{\mu} = \Omega^{\mu\nu}\frac{\partial G}{\partial a^{\nu}}. \tag{38}$$

The function G can therefore also be called the (infinitesimal) *generator* of the transformation.

The integrability conditions are then predictably given by the Jacobi * law for the generalized product, i.e.,

$$[a^{\mu}, [a^{\nu}, G]^{*}]^{*} + [a^{\nu}, [G, a^{\mu}]^{*}]^{*} + [G, [a^{\mu}, a^{\nu}]^{*}]^{*} = 0. \tag{39}$$

As a direct generalization of the Hamiltonian case, we have the following variation of a function $A(a)$ on $T^{*}M$

$$\delta A = \frac{\partial A}{\partial a^{\mu}}\delta a^{\mu} = \delta\theta[A, G]^{*}, \tag{40}$$

with the following important cases.

1. The Birkhoffian B can be interpreted as the generator of the infinitesimal translation in time

$$\delta A = \delta t[A, B]^{*}. \tag{41}$$

2. The linear momentum component p_k can be interpreted as the generator of translations in the q^k component

$$\delta A = \delta q^{k}[A, p_k]^{*} \text{ (no sum)}. \tag{42}$$

3. The angular momentum component $\mathbf{M}\cdot\mathbf{n}$ can be interpreted as the generator of a rotation along the axis $\mathbf{n}$

$$\delta A = \delta\alpha[A, \mathbf{M}\cdot\mathbf{n}]^{*}. \tag{43}$$

The Birkhoffian generalization of the finite transformations (26) is also straightforward. Transformations (32) imply

$$\frac{da^{\mu}}{d\theta} = [a^{\mu}, G]^{*}_{(a)}. \tag{44}$$

By repeating the analysis according to Equations (23)–(25), one reaches the formal solution

$$a^{\mu} = a^{\mu}_{0} + \frac{\theta}{1!}[a^{\mu}_{0}, G]^{*}_{(a_0)} + \frac{\theta^2}{2!}[[a^{\mu}_{0}, G]^{*}_{(a)}, G]^{*} + \cdots \tag{45}$$

which can be written in closed form (33), i.e.,

$$a^{\mu} = \exp\left(\theta * \omega^{\alpha\beta} \frac{\partial G}{\partial a^{\beta}} \frac{\partial}{\partial a^{\alpha}}\right)\Big|_{\mathscr{A}^*} a_0^{\mu} = \exp\left(\theta g(a) \omega^{\alpha\beta} \frac{\partial G}{\partial a^{\beta}} \frac{\partial}{\partial a^{\alpha}}\right)\Big|_{\mathscr{A}} a_0^{\mu} \quad (46)$$

$$g(a)\omega^{\alpha\beta} = \Omega^{\alpha\beta}.$$

Thus, as it occurs for Hamiltonian formulations, *the finite time evolution of Birkhoffian vector fields can be obtained via an infinite succession of infinitesimal generalized canonical transformations with time as the parameter and the Birkhoffian as the generators,* i.e.,

$$a^{\mu}(t) = \exp\left(t * \omega^{\alpha\beta} \frac{\partial B}{\partial a^{\beta}} \frac{\partial}{\partial a^{\alpha}}\right)\Big|_{\mathscr{A}^*} a_0^{\mu} \quad (47a)$$

$$A(a) = \exp\left(t * \omega^{\alpha\beta} \frac{\partial B}{\partial a^{\beta}} \frac{\partial}{\partial a^{\alpha}}\right)\Big|_{\mathscr{A}^*} A(a_0). \quad (47b)$$

The direct universality of the above time evolution from Theorem 4.5.1 should be kept in mind.

We move now to an aspect of our analysis which is particularly important, both classically and quantum mechanically. It is given by the identification of the implications of nonpotential forces in the algebra of physical quantities.

We shall conduct the analysis for the algebra which is at the foundation of Galilei's (as well as Einstein's special) relativity: the algebra of the group of rotations $SO(3)$ in an Euclidean three-dimensional space $M = E(3)$. First, let us consider the conventional Hamiltonian case for *closed self-adjoint systems i.e., conservative systems* (*see Section* 6.3 *for detail*). In this case the Lagrangian and Hamiltonian have the conventional structure

$$L_{\text{tot}}^{\text{conv}} = L_{\text{free}}(\dot{\mathbf{r}}) + L_{\text{int}}(\mathbf{r}); \quad H_{\text{tot}}^{\text{conv}} = H_{\text{free}}(\mathbf{p}) + H_{\text{int}}(\mathbf{r}); \quad (48)$$

the canonical momentum coincides with the physical linear momentum

$$\mathbf{p} = \mathbf{p}^{\text{can}} = \partial L_{\text{tot}}^{\text{conv}}/\partial \dot{\mathbf{r}} \equiv \mathbf{p}^{\text{phys}} = m\dot{\mathbf{r}}; \quad (49)$$

the canonical angular momentum also coincides with the physical one,

$$\mathbf{M} = \mathbf{M}^{\text{can}} = \mathbf{r} \times \mathbf{p}^{\text{can}} \equiv \mathbf{M}^{\text{phys}} = \mathbf{r} \times m\dot{\mathbf{r}} \quad (50)$$

yielding the transformation rule

$$\delta A = \delta\theta \cdot [A, \mathbf{M}]_{\text{PB}}. \quad (51)$$

The canonical realization of the Lie algebra and of the Lie group of rotations are then given by the familiar rules

$$\mathbf{SO}(3): [M_i, M_j]_{\text{PB}} = \varepsilon_{ijk} M_k, \qquad i, j = x, y, z \quad (52a)$$

$$SO(3): \exp\left(\theta^k \omega^{\alpha\beta} \frac{\partial M_k}{\partial a^{\beta}} \frac{\partial}{\partial a^{\alpha}}\right), \qquad a = (\mathbf{r}, \mathbf{p}) \quad (52b)$$

where the θ's are Euler's angles.

However, as indicated a number of times, the insistence on the preservation of equations of conservative type is literally equivalent to the acceptance of the perpetual motion in our environment. When excessive approximations are avoided, the systems are *closed, essentially non-self-adjoint*

for which the canonical formalism is not directly applicable, yet conventional total conservation laws are valid (Section 6.3).

We are interested in achieving a theory of rotations for the latter systems which, while being still of Lie type, is formulated in terms of quantities possessing a direct physical meaning. Both conditions are achieved by the Lie-isotopic/Birkhoffian formulations under which the transformation of physical quantities is given by

$$\delta A = \delta\theta \cdot [A, \mathbf{M}]^*_{GPB}, \tag{53}$$

while the isotopically mapped Lie algebra and Lie group are

$$\mathbf{SO}^*(3): [M_i, M_j]^*_{GPB} = C^k_{ij}(a) M_k, \tag{54a}$$

$$SO^*(3): \exp\left(\theta^k \Omega^{\alpha\beta}(a) \frac{\partial \mathbf{M}_k}{\partial a^\beta} \frac{\partial}{\partial a^\alpha}\right). \tag{54b}$$

The important point is that, in the transition from the theory of rotations of closed self-adjoint systems to that of more general closed non-self-adjoint ones, all physical quantities, **r**, **p**, **M**, etc., remain unchanged. Once this point is understood, one can see the implications of the nonpotential forces for the theory of rotations. In fact, isotopes (54) depend explicitly on the nonpotential forces and vary from system to system. Also, recall that, in general, $\mathbf{SO}^*(3)$ *is not* isomorphic to $\mathbf{SO}(3)$, as expected. Finally, recall that $\mathbf{SO}^*(3)$ is a covering of $\mathbf{SO}(3)$ in the sense of footnote 24, as desired.

The extension of isotopes (54) to other transformations, such as translations in space and time, boosts, etc., then leads to the notion of *isotopic generalization of Galilei's group*. The further conditions that such a group leaves invariant the system leads to the *isotopic generalization of Galilei's relativity* of Section 6.3.

The relativistic extension of these ideas is under study.

Part 3: Quantum Mechanical/Heisenberg's Realization of Lie's Theory. For completeness, we briefly indicate the main ideas of Heisenberg's Mechanics when seen from the viewpoint of the preceding analysis. With the understanding that the rules given below are insufficient for the characterization of quantum mechanics, those relevant for Lie's theory and its physical interpretation are the following.

(B.1) The carrier space is given by a Hilbert space $\mathscr{H}$ and Hermitian operators $\tilde{\mathbf{r}}$, $\tilde{\mathbf{p}}$, $\tilde{H}$, $\tilde{\mathbf{M}}$, etc., possessing a unique, direct, physical interpretation (position operator, linear momentum operator, energy operator, angular momentum operator, etc.).[67]

(B.2) The vector fields of Lie's theory are realized via said Hermitian operators

$$X_k \to \tilde{X}_k \tag{55}$$

[67] No rigorously established rule exists for identifying the physical meaning of given operators. The issue is settled either via an *ad hoc* assumption, or, more credibly, via *arguments of similarity* with the classical case. In this latter case, the complete set of observable must be taken into account. For instance, the operator $\tilde{p} = i\partial/\partial r$ cannot be claimed to represent the physical linear momentum unless the Hamiltonian operator has the conventional structure $H = \frac{1}{2}\tilde{p}^2 + V$. In fact, if the Hamiltonian has a generalized structure, say $H = \tilde{p}f(\tilde{r})\tilde{p} + \tilde{C}(\tilde{r})$, one can establish that the classical quantity p does not represent the linear momentum. Regardless of whether or not one uses the realization $p = i\partial/\partial r$, the mathematical algorithm $\tilde{p}$ cannot represent, for consistency, the quantum mechanical linear momentum.

(B.3) The universal enveloping associative algebra is then given by[68]

$$\tilde{\mathscr{A}}: \mathbb{1}; \quad \tilde{X}_k; \quad \underset{i \leq j}{\tilde{X}_i \tilde{X}_j}; \quad \underset{i \leq j \leq k}{\tilde{X}_i \tilde{X}_j \tilde{X}_k}; \ldots . \tag{56}$$

Unlike the Hamiltonian case, the enveloping algebra identifies directly, without redefinitions, the Lie algebra of the observables

$$\tilde{\mathbf{G}}: [\tilde{X}_i, \tilde{X}_j]_{\tilde{\mathscr{A}}} = C_{ij}^k \tilde{X}_k . \tag{57}$$

Explicitly, in this case we do not have to reinterpret Lie's rule of associative Lie-admissibility, Equations (8), with Hamilton's rule of nonassociative Lie-admissibility, Equations (7). In fact, the universal enveloping associative algebra $\tilde{\mathscr{A}}$ truly sets the entire realization of Lie's theory, beginning with the Lie algebra **G** as the attached algebra $\tilde{\mathscr{A}}^-$.

The realization of Lie groups is then given by the following unitary transformations

$$\tilde{G}: \tilde{A}' = e^{i\theta^k \tilde{x}_k} |_{\tilde{\mathscr{A}}} \tilde{A} e^{-i\theta^k \tilde{x}_k} |_{\tilde{\mathscr{A}}} . \tag{58}$$

Again, a subtle but important difference with the Hamiltonian case exists. The carrier space on which Hamiltonian group (27) acts is a conventional, *one-sided module* (a left module, according to the conventional application of expansion (5) to the right). The carrier space on which Heisenberg's group (58) acts, is actually a bona fide *two-sided module*.[69] In fact, both the left and right actions are needed to reach, in this case, a Lie algebra in the neighborhood of the identity.

We hope that these remarks indicate the *structural differences* between Hamilton's and Heisenberg's realizations of Lie's theory. In turn, these differences may be of value in understanding the lack of achievement until now of a resolution of the problem of quantization.[43]

Keeping an open mind on this, we can say that the conventional realizations of physically relevant Lie algebras in Hamilton's and Heisenberg's mechanics are *similar*. For instance, for the case of the Lie algebra of rotations, we have

$$\mathbf{SO}(3): [\tilde{M}_i, \tilde{M}_j]_{\tilde{\mathscr{A}}} = i\varepsilon_{ijk} \tilde{M}_k; \quad \tilde{\mathbf{M}} = \tilde{\mathbf{r}} \times \tilde{\mathbf{p}} \tag{59}$$

which should be compared with rules (52a).

Part 4: Lie-Isotopic Generalization of Heisenberg's Realization. For completeness, we will briefly indicate the rule of Lie-isotopy which was used by Santilli (1978c) for the generalization of Heisenberg's equations given by Equations (18) of Chart 5.1 (description of particles in mutual penetration). Evidently, the isotopy is algebraically similar to that for the transition from Hamilton's to Birkhoff's equations and can be expressed according to the following rules.

(B*.1) The Hilbert space of Heisenberg's formulation, and, more particularly, the local operators of direct physical meaning $\tilde{\mathbf{r}}$, $\tilde{\mathbf{p}}$, are preserved, although in a predictable generalized way (Chart 6.1).

(B*.2) The generators $\tilde{X}_k$ and the parameters θ^k of Heisenberg's formulation are also preserved in a generalized meaning (Chart 6.1).

(B*.3) The enveloping associative algebra $\tilde{\mathscr{A}}$ of Heisenberg's formulation is changed into the isotope

$$\tilde{\mathscr{A}}^*: \mathbb{1}; \quad \tilde{X}_k; \quad \tilde{X}_i * \tilde{X}_j; \quad \tilde{X}_i * \tilde{X}_j * \tilde{X}_k; \cdots$$
$$\tilde{X}_i * \tilde{X}_j = \tilde{X}_i \tilde{T} \tilde{X}_j, \qquad \tilde{T} = \text{fixed}. \tag{60}$$

[68] The field is now, evidently, that of complex number $\mathbb{C}$.

[69] Santilli (1979c).

Unlike the Birkhoffian case, the enveloping algebra $\tilde{\mathscr{A}}^*$ now directly characterizes the Lie algebra via the rule of isotopic associative Lie-admissibility

$$\tilde{\mathbf{G}}^*\colon [\tilde{X}_i, \tilde{X}_j]_{\tilde{\mathscr{A}}^*} = \tilde{C}_{ij}^k(a)\tilde{X}_k. \tag{61}$$

The isotopic realization of Lie groups is then given by

$$\tilde{G}^*\colon \tilde{A}' = e^{i\tilde{X}_k * \theta^k} |_{\tilde{\mathscr{A}}^*} \tilde{A} e^{-i\theta^k * \tilde{X}_k} |_{\tilde{\mathscr{A}}^*}, \tag{62}$$

which clearly preserves the two-sided structure on $\tilde{\mathscr{H}}$. The isotopic realization of the Lie algebra of rotations is then given by

$$\mathbf{SO}^*(3)\colon [\tilde{M}_i, \tilde{M}_j]_{\tilde{\mathscr{A}}^*} = \tilde{C}_{ij}^k(\tilde{a})\tilde{M}_j \tag{63}$$

which should be compared to rule (59).

The nontriviality of the generalization can be indicated via the fact, for instance, that transformations (62) are not necessarily unitary, trivially because the operators $\tilde{T}$ and $\tilde{X}_k$ do not necessarily commute. In turn, the nonunitarity of the theory confirms that it is the desired "quantum mechanical image" of the Birkhoffian Mechanics. In fact, as stressed earlier, this mechanics is of the noncanonical type.

The state of the art on the studies for a possible generalization of quantum mechanics along the lines under consideration here was presented at the *First International Conference on Nonpotential Interactions*, held at the Université d'Orléans, France, from January 5 to 9, 1982. We refer the interested reader to the proceedings of the conference (1982), as well as to those of the preparatory workshops on Lie-Admissible Formulations (1979, 1981). For a brief and rudimentary review of the main ideas see Chart 6.1.

Chart 5.5 Darboux's Theorem of the Symplectic and Contact Geometries

For the reader's convenience, we formulate here (without proof) *Darboux's Theorem*, which plays an important role for Theorem 6.2.1 of indirect universality of Hamilton's equations (as well as for quantum mechanical considerations). We shall present this theorem, first, within the context of the symplectic geometry, and then within the context of the broader contact geometry.

Theorem 1. (*See, for instance, Abraham and Marsden* (1967, 1978 *edition, page* 175.) *Suppose* Ω_2 *is a nondegenerate two-form on a* $2n$*-dimensional* (*analytic, for the context of this volume*) *manifold* M. *Then the form* Ω_2 *is closed,* $d\Omega_2 = 0$ (*and thus symplectic*) *if and only if a chart* (U, φ) *exists at each* $m \in M$ *such that* $\varphi(m) = 0$, *and with* $\varphi = (x^1, \ldots, x^n, y_1, \ldots, y_n)$ *we have*

$$\Omega_2 | U = \omega_2 = dy_k \wedge dx^k = \tfrac{1}{2}\omega_{\mu\nu} da^\mu \wedge da^\nu \tag{1}$$

$$k = 1, 2, \ldots, n; \quad \mu, \nu = 1, 2, \ldots, 2n; \quad a = (x, y),$$

$$(\omega_{\mu\nu}) = \begin{pmatrix} 0_{n\times n} & -1_{n\times n} \\ 1_{n\times n} & 0_{n\times n} \end{pmatrix}$$

The formulation of this theorem in local coordinates has also been studied by Pauli (1953), and called *Pauli's theorem* in Jost (1964). In the language of this volume, this formulation can be expressed as follows.

Theorem 2. *Given an analytic and regular Birkhoff's (symplectic) tensor on a $2n$-dimensional manifold with local coordinates $a = (\mathbf{r}, \mathbf{p})$*

$$\Omega_{\mu\nu}(a) = \frac{\partial R_\nu(a)}{\partial a^\mu} - \frac{\partial R_\mu(a)}{\partial a^\nu} \tag{2}$$

smoothness- and regularity-preserving transformations of the local variables

$$a^\mu \to a'^\mu = a'^\mu(a) \tag{3}$$

always exist under which tensor (2) reduces to the fundamental form, i.e.,

$$\Omega_{\mu\nu}(a) \to \Omega'_{\mu\nu} = \frac{\partial a^\rho}{\partial a'^\mu}\,\Omega_{\rho\sigma}(a(a'))\,\frac{\partial a^\sigma}{\partial a'^\nu} = \omega_{\mu\nu}. \tag{4}$$

It is significant to point out that Darboux's charts, or reductions (4), are not unique.

Corollary 2a. *The transformations of reduction (4) are always defined up to the infinite family of all possible canonical transformations.*

Indeed, canonical transformations are identity isotopic with respect to ω_2; that is, they preserve the values of the tensor $\omega_{\mu\nu}$ (Definition 5.2.1). This implies the existence of an infinite family of possible transformations, all capable of performing reduction (4), for each given tensor (2).

Also, the reader should keep in mind that Darboux's theorem is *local* in character. This is precisely in line with Theorem 6.2.1, as we shall see. We consider now the extension of Darboux's theorem to the contact geometry.

Theorem 3. (*See, for instance, Abraham and Marsden* (1967, *loc. cit.*, 1978 *edition, page* 372). *Let $(M, \hat{\Omega}_2)$ be an exact contact manifold (Chart 4.4) with primitive form $\hat{R}_1$, $d\hat{R}_1 = \hat{\Omega}_2$. Then, at each point $m \in M$, a chart (U, φ) always exists with local coordinates $\varphi = (t', x^1, \ldots, x^n, y_1, \ldots, y_n)$ and a function H on M, such that*

$$\hat{R}_1 \mid U = y_k\,dx^k - H(t', x, y)dt'. \tag{5}$$

We have formulated the theorem for *exact* contact manifolds. As pointed out in Section 1.1, the condition is essential to compute the action functional of self-adjoint first-order systems.

Also, the reduction customarily considered in the available treatises of differential geometry is of the particular type

$$\hat{R}_1 \mid U = y_k\,dx^k - dt', \tag{6}$$

where the Hamiltonian is clearly $H = 1$. The discrepancy with Equation (5), however, is only apparent. Forms (5) and (6) are indeed related, by a canonical transformation.

Finally, we should indicate the fact that (whether or not the contact manifold is exact), the transformations (charts) of Theorem 3 treat time and the a variables on equal footing; that is, they are of the type

$$\hat{a} = (t, a) = (t, \mathbf{r}, \mathbf{p}) \to \hat{a}' = \hat{a}'(a) = (t'(t, \mathbf{r}, \mathbf{p}), \mathbf{x}(t, \mathbf{r}, \mathbf{p}), \mathbf{y}(t, \mathbf{r}, \mathbf{p})). \tag{7}$$

The reformulation of Theorem 3 in the language of this volume is then self-evident.

Theorem 4. *Given an analytic Birkhoff's tensor in* $(2n + 1)$*-dimension and of maximal rank*

$$(\hat{\Omega}_{\mu\nu}) = \left(\begin{array}{c|c} 0 & \dfrac{\partial B}{\partial a^{\nu}} + \dfrac{\partial R_{\nu}}{\partial t} \\ \hline -\dfrac{\partial B}{\partial a^{\mu}} & \\ & \\ -\dfrac{\partial R_{\mu}}{\partial t} & \dfrac{\partial R_{\nu}}{\partial a^{\mu}} - \dfrac{\partial R_{\mu}}{\partial a^{\nu}} \end{array}\right) \tag{8}$$

analytic and invertible transformations (7) *always exist under which the tensor assumes the Hamiltonian form*

$$\begin{aligned} (\hat{\Omega}'_{\mu\nu}) &= \left(\frac{\partial \hat{a}^{\rho}}{\partial \hat{a}'^{\mu}} \hat{\Omega}_{\rho\sigma}(\hat{a}(\hat{a}')) \frac{\partial \hat{a}^{\sigma}}{\partial \hat{a}'^{\nu}}\right) \\ &= (\hat{\omega}_{\mu\nu}) = \left(\begin{array}{c|c} 0 & \dfrac{\partial H}{\partial a'^{\nu}} \\ \hline -\dfrac{\partial H}{\partial a'^{\mu}} & \omega_{\mu\nu} \end{array}\right). \end{aligned} \tag{9}$$

Theorem 4 above, once matched with Theorem 4.5.1 (Direct Universality of Birkhoffian Formulations) provides the proof of the indirect universality of Hamiltonian formulations given in Section 6.2.

In this way we confirm that, given a system which is non-Hamiltonian in the local variables $(t, \mathbf{r}, \mathbf{p})$ of the observer, the system can be reduced to a Hamiltonian form in new variables $(t', \mathbf{r}', \mathbf{p}')$ via the use of a Darboux's transformation and "intermediate use" of the Birkhoffian representations. In particular, the new Hamiltonian, if needed, can assume a conventional form in the new variables, including the "free" form $H' = \frac{1}{2}\mathbf{p}'^2$. As a result, Darboux's transformations are useful for the quantization of non-self-adjoint systems, in the sense that they permit the use of the conventional Heisenberg's equations (as well as other equations of quantum mechanics) in the new variables.

However, let us stress that the approach implies the necessary loss of direct physical meaning of the local variables and the functions defined on them (e.g., *H would not* represent the energy of the system). If the experimenter insists in the preservation of the local variables actually

used in the measures, one is forced to transform the Hamiltonian representations in $(t', \mathbf{r}', \mathbf{p}')$ into an equivalent form in the original variables $(t, \mathbf{r}, \mathbf{p})$, via the inverse of Darboux's transformation, but this transformation is *noncanonical*, as we know from this analysis, and the return to the Birkhoffian representation is then unavoidable. Needless to say, the situation which is expected at the quantum level is the use of a *nonunitary* transformation under which the conventional associative product of quantum mechanical operators is mapped into an isotopic form of the type presented in the preceding charts. In turn, this would confirm that the quantum mechanical treatment of non-self-adjoint systems appears to call for a suitable generalization of quantum mechanics, perhaps along the line of the Lie-isotopic generalization of Lie's theory indicated earlier.

Chart 5.6 Some Definition of Canonical Transformations

A considerable variety of definitions of canonical transformations exists in the physical and mathematical literature. A few representative definitions are collected here for the reader's convenience.

(1) A transformation

$$q \to q'(t, q, p), \qquad p \to p'(t, q, p) \tag{1}$$

is canonical when a new Hamiltonian $H'(t, q', p')$ exists for which

$$\dot{q}' = \frac{\partial H'}{\partial p'}, \qquad \dot{p}' = -\frac{\partial H'}{\partial q'}.$$

This definition was adopted, for instance, by Goldstein (1950, page 239) and by Landau and Lifshitz (1960, page 146).

(2) A transformation (1) is canonical when the difference $p'_k dq'^k - p_k dq^k$ is the total differential of a function. This definition was adopted, for instance, by Whittaker (1904, page 234).

(3) A transformation (1) is canonical when the identity

$$\delta p' dq' - dp' \delta q' = \delta p dq - dp \delta q$$

holds for any two independent variations δ and d. This definition was adopted by Pars (1965, page 434).

(4) A transformation (1) is canonical when it is canonoid with respect to all Hamiltonians. This definition was adopted by Saletan and Cromer (1971, page 188).

(5) A transformation (1) is canonical when the fundamental Poisson (Lagrange) brackets transform contravariantly (covariantly) and invariantly. This definition was adopted, for instance, by Sudarshan and Makunda (1974, page 40).

(6) A transformation (1) is canonical when it characterizes a Lie identity isotopy (symplectic identity isotopy) of the fundamental Lie tensor $\omega^{\mu\nu}$ (of the fundamental symplectic tensor $\omega_{\mu\nu}$). This definition was adopted by Santilli (1978c) (see also Section 5.3).

(7) Let (M, ω_2) be a symplectic manifold and $(\mathbb{R} \times M, \hat{\omega}_2)$ the corresponding contact manifold. A smooth mapping $F: \mathbb{R} \times M$

$\to \mathbb{R} \times N$ is called a *canonical transformation* if and only if each of the following holds: (i) F is a diffeomorphism; (ii) F preserves the time, i.e., $F_* t = t$; and (iii) a function $K_F \in \mathscr{C}^\infty(\mathbb{R} \times M)$ exists such that $F_* \hat{\omega}_2' = \hat{\omega}_2 + dK_F \wedge dt$. This definition was adopted, for instance, by Abraham and Marsden (1978, page 138), or Loomis and Sternberg (1968, page 560).

(8) A diffeomorphism $T^*(M) \supset U_1 \xrightarrow{\psi} U_2 \subset T^*(N)$ that takes the fundamental symplectic form $\omega_2|_{U_1}$ to $\omega_2|_{U_2}$ is called a *local canonical transformation*. If $U_1 = U_2 = T^*(M)$, then ψ is called a *canonical transformation*. This definition was adopted by Thirring (1978, p. 78).

It should be stressed that *these definitions are not equivalent among themselves.*

For instance, Definitions (1) and (5) *are not* equivalent, but Definitions (5) and (7) are equivalent. Note that, to incorporate Galilei's transformations,[7] one can generalize Definition (7) above to the form $F_* dt = dt$ on $\mathbb{R} \times T^*M$. Similar generalizations hold for other definitions.

Chart 5.7 Isotopic and Genotopic Transformations of Variational Principles

As is now familiar, the Fundamental Analytic Theorem establishes that the conditions of variational self-adjointness are the integrability requirements allowing a (quasilinear) second-order system of differential equations to be directly represented with a conventional action principle, such as *Hamilton's principle*. For the case of unconstrained Newtonian systems in Euclidean space, we have the ordered direct representations of systems with potential forces[70]

$$\begin{aligned}
\delta A(E_0) &= \left[\delta \int_{t_1}^{t_2} dt L_{\text{tot}}^{\text{conv}}(t, \mathbf{r}, \dot{\mathbf{r}})\right](E_0) \\
&= -\int_{t_1}^{t_2} dt \left[\left(\frac{d}{dt}\frac{\partial L_{\text{tot}}^{\text{conv}}}{\partial \dot{r}^k} - \frac{\partial L_{\text{tot}}^{\text{conv}}}{\partial r^k}\right)_{\text{SA}} \delta r^k\right](E_0) \qquad (1) \\
&= -\int_{t_1}^{t_2} dt [m\ddot{r}_k - f_k(t, \mathbf{r}, \dot{\mathbf{r}})]_{\text{SA}}(E_0)\, \delta r^k(E_0) = 0,
\end{aligned}$$

$$L_{\text{tot}}^{\text{conv}} = L_{\text{free}}(\dot{\mathbf{r}}) + L_{\text{int}}(t, \mathbf{r}, \dot{\mathbf{r}}) = T(\dot{\mathbf{r}}) - U(t, \mathbf{r}, \dot{\mathbf{r}}).$$

The idea that all forces are potential implies an excessive approximation of nature.[71] The inclusion of nonpotential forces is therefore necessary for more closely representing the Newtonian physical reality. Theorem A.1.1, which covers indirect Lagrangian representations, permits the

[70] A review of the calculus of variations with particular reference to variational principles is provided in Section I.1.3.

[71] The restriction of mechanics to descriptions of type 1) is essentially equivalent to the acceptance of perpetual motion in our environment.

representation via a conventional action principle of a class of systems with potential *and* nonpotential forces according to the structure[72]

$$\begin{aligned}
\delta A^*(E_0) &= \left[\delta \int_{t_1}^{t_2} dt L_{\text{tot}}^{\text{gen}}(t, \mathbf{r}, \dot{\mathbf{r}})\right](E_0) \\
&= -\int_{t_1}^{t_2} dt \left[\left(\frac{d}{dt}\frac{\partial L_{\text{tot}}^{\text{gen}}}{\partial \dot{r}^k} - \frac{\partial L_{\text{tot}}^{\text{gen}}}{\partial r^k}\right)_{\text{SA}} \delta r^k\right](E_0) \\
&= -\int_{t_1}^{t_2} dt \{h_k^i(t, \mathbf{r}, \dot{\mathbf{r}})[(m\ddot{r}_i - f_i(t, \mathbf{r}, \dot{\mathbf{r}}))_{\text{SA}} - F_i(t, \mathbf{r}, \dot{\mathbf{r}})]_{\text{NSA}}\}_{\text{SA}} \\
&\quad \times (E_0)\delta r^k(E_0) = 0 \\
&\quad L_{\text{tot}}^{\text{gen}} = L_{\text{int, I}}(t, \mathbf{r}, \dot{\mathbf{r}}) L_{\text{free}}(\dot{\mathbf{r}}) + L_{\text{int, II}}(t, \mathbf{r}, \dot{\mathbf{r}}).
\end{aligned} \tag{2}$$

The equations emerging from these representations, however, are not in the form originating from Newton's second law but in an equivalent form, characterized by a regular matrix of multiplicative functions (called the self-adjoint genotopic functions in Section 4.4). This often creates uneasiness in students without a sufficient exposure to nonpotential interactions.

In this chart we shall show that the indirectness of (2) results from unnecessary restrictions on the δ–variations. If these restrictions are lifted, then more general variations exist (denoted by δ^*, to be identified shortly) under which all non-self-adjoint systems verifying the integrability conditions of Theorem A.1.1 admit the ordered direct representations

$$\begin{aligned}
\delta^* A^*(E_0) &= \left[\delta^* \int_{t_1}^{t_2} dt L_{\text{tot}}^{\text{gen}}(t, \mathbf{r}, \dot{\mathbf{r}})\right](E_0) \\
&= -\int_{t_1}^{t_2} dt \left[\left(\frac{d}{dt}\frac{\partial L_{\text{tot}}^{\text{gen}}}{\partial \dot{r}^k} - \frac{\partial L_{\text{tot}}^{\text{gen}}}{\partial r^k}\right)_{\text{SA}} \delta^* r^k\right](E_0) \\
&= -\int_{t_1}^{t_2} dt [(m\ddot{r}_k - f_k(t, \mathbf{r}, \dot{\mathbf{r}}))_{\text{SA}} - F_k(t, \mathbf{r}, \dot{\mathbf{r}})]_{\text{NSA}} \\
&\quad \times (E_0)\delta r^k(E_0) = 0
\end{aligned} \tag{3}$$

proposed by Santilli (1977c) for the field theoretical case, and presented here for its Newtonian version.

The transition from principle (2) to the generalized form (3) implies a transition from a self-adjoint variational principle[72] to a non-self-adjoint form. For this reason the transition is called here a *non-self-adjoint genotopic mapping of Hamilton's principle.*

To present a derivation of principle (3), we return to the generalized coordinates q^k, with the understanding that they can represent the Cartesian coordinates in a given ordering, as well as any other needed set of variables in applied mathematics, physics, and engineering.

The (first-order) variations δq^k customarily used in Analytic Mechanics are of the simple type

$$\delta q^k = \varepsilon \eta^k(t), \qquad \varepsilon \approx 0 \tag{4}$$

[72] Principles (1) and (2) are called *self-adjoint variational principles* (Chart I.3.3).

and are often called *weak variations* in the literature of the calculus of variation.[73] Actually, the variations of a given path q^k can have either an implicit or an explicit dependence in the independent variable, or both, provided that the desired continuity and end-point conditions are met (Section I.1.3).

The explicit functional dependence in which we are interested is of the type

$$\delta q^k = (\delta q^k)(t, q, \dot{q}), \tag{5}$$

under the condition that the variations verify the fixed end-point properties

$$\delta q^k|_{t_s} = 0, \qquad s = 1, 2, \tag{6}$$

as well as our continuity assumptions (analyticity in the indicated local variables[74]).

Using the language of Section I.1.3, variations (6) are *infinitesimal, first-order, abstract, and admissible with fixed end points*. Among all their possible realizations, we are interested in the particular form

$$\delta^* q^k = g_i^k(t, q, \dot{q})\delta q^i, \qquad \det(g_i^j)(\mathscr{R}) \neq 0, \qquad \delta q^i = \varepsilon \eta^i(t). \tag{7}$$

When these variations are computed along a possible path, they recover the customary dependence in the independent variable only, but now of the "transformed" type

$$\begin{aligned} \delta^* q^k(E_0) &= g_i^k(t, q, \dot{q})|_{E_0}\delta q^i(t) = h_i^k(t)\delta q^i(t) \\ &= \varepsilon g_i^k(t)\eta^i(t) = \varepsilon \rho^k(t). \end{aligned} \tag{8}$$

It is easy to see that variations (8) verify the fixed end-point conditions whenever variations δq^k are those of the conventional Hamilton's principle. Also, under the assumed regularity condition of the g matrix, variations (8) are "invertible" in the sense that they allow the formulation of the conventional (weak) variations

$$\delta q^k(t) = h_i^k(t, q, \dot{q})\delta^* q^i(t, q, \dot{q}), \qquad (h) = (g)^{-1}. \tag{9}$$

The proof of the following reformulation of the fundamental lemma of the calculus of variations (Lemma I.1.3.1) is then trivial.

Lemma 1. *If the functions $\beta_i(t)$ and $g_k^i(t)$, $i, k = 1, 2, \ldots, n$, are of (at least) class $\mathscr{C}^0$ and* $\det(g) \neq 0$ *in the (closed) interval $[t_1, t_2]$, and if*

$$\int_{t_1}^{t_2} dt g_k^i(t)\beta_i(t)\eta^k(t) = 0 \tag{10}$$

for all functions $\eta^k(t)$ of (at least) class $\mathscr{C}^0$ in the same interval, which are identically null at end points

$$\eta^k(t_s) = 0, \qquad s = 1, 2, \tag{11}$$

[73] See, for instance, Ewing (1969, page 90).

[74] These smoothness properties are highly rendundant from the viewpoint of the calculus of variations. Indeed, the proper treatment of the so-called *Weierstrass necessary condition* for an extremum demands the use of variations of only class $\mathscr{C}^0$, in order to allow for the presence of corner points. Nevertheless, the analyticity of the variations in their local variables is in line with the general treatment of this volume.

then

$$g^i_k(t)\beta_i(t) = 0, \qquad k = 1, 2, \ldots, n, \tag{12}$$

for all values $t \in (t_1, t_2)$.

The following condition for all variational principles with fixed end-points will be tacitly implemented:

$$\delta^*\dot{q}^k \stackrel{\text{def}}{=} \frac{d}{dt}\,\delta^* q^k. \tag{13}$$

The lemma permits the non-self-adjoint genototopic mapping of Hamilton's principle (Santilli, *loc. cit.*)

$$\delta^* A(E_0) = -\int_{t_1}^{t_2} dt \left[g^i_k \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} \right)_{SA} \right]_{NSA} (E_0)\delta q^k(E_0) = 0. \tag{14}$$

The underlying analytic equations are Lagrange's equations, not in their conventional (self-adjoint) form (14), but rather in the *equivalent non-self-adjoint* form[75]

$$\left\{ g^i_k(t, q, \dot{q}) \left[\frac{d}{dt}\frac{\partial L(t, q, \dot{q})}{\partial \dot{q}^i} - \frac{\partial L(t, q, \dot{q})}{\partial q^i} \right]_{SA} \right\}_{NSA} = 0. \tag{15}$$

The representation of non-self-adjoint systems as in (3) then follows from the identification of the matrix (g^i_k) with the inverse $(h^k_i)^{-1}$ of the matrix solution of Principle (2), according to the rule

$$\begin{aligned} \left[g^i_k \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} \right)_{SA} \right]_{NSA} &= \{ g^i_k [h^j_i (A_{jl}\ddot{q}^l + B_j)_{NSA}]_{SA} \}_{NSA} \\ &= (A_{kl}\ddot{q}^l + B_k)_{NSA} = 0. \end{aligned} \tag{16}$$

An application of the techniques of this chart which deserves mention is the transition from the Inverse Problem (based on Lagrange's or Hamilton's equations *without* external terms) to the equations originally conceived by Lagrange and Hamilton, *with* external terms (see footnote 5 of the Introduction). As the reader can verify, these latter equations are non-self-adjoint for all external terms $\mathbf{F}_k$ which cannot be derived from a potential, i.e.,

$$\left[\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^{ka}} - \frac{\partial L}{\partial r^{ka}} \right)_{SA} - F_{ka} \right]_{NSA} = 0. \tag{17}$$

[75] The case of *degenerate* equations of this type is intriguing from the viewpoint of the theory of systems with subsidiary constraints. Indeed, the degenerate character can be achieved by

1. a degenerate Lagrangian L and a regular matrix (g);
2. a regular Lagrangian L and a degenerate matrix (g); or
3. a degenerate Lagrangian L and a degenerate matrix (g).

These cases are not studied in this volume. Equivalence transformations of Lagrange's equations have been studied by several authors; nevertheless, their analytic character (that is, their derivability from a variational principle) has been pointed out, apparently for the first time, by Santilli (1977c).

As result, the transition under consideration can best be studied via the formulation of the Inverse Problem in terms of the *non-self-adjoint* equations (15), according to the direct representations

$$\left[g_{ka}^{jb}\left(\frac{d}{dt}\frac{\partial L'}{\partial \dot{r}^{jb}} - \frac{\partial L'}{\partial r^{jb}}\right)_{SA}\right]_{NSA} \equiv \left[\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^{ka}} - \frac{\partial L}{\partial r^{ka}}\right)_{SA} - F_{ka}\right]_{NSA}$$
$$\equiv [(m_k \ddot{r}_{ka} - f_{ka})_{SA} - F_{ka}]_{NSA}. \qquad (18)$$

For a study of this transition see Santilli (1978c). In this way we reach the conclusion that *Lagrange's equations with external terms can be derived via a non-self-adjoint, genotopically mapped principle*

$$\delta^* \int_{t_1}^{t_2} dt\, L_{tot}^{gen} = -\int_{t_1}^{t_2} dt\left[\frac{d}{dt}\frac{\partial L_{tot}^{conv}}{\partial \dot{q}^k} - \frac{\partial L_{tot}^{conv}}{\partial q^k} - F_k\right]_{NSA} \delta q^k = 0 \qquad (19)$$

provided that they are non-essentially non-self-adjoint. This derivation should be compared with the alternative forms available in the current literature[76].

The generalized principle (14) does indeed achieve the desired objective. In fact, the principle permits the representation of (a class of) Newtonian systems with nonpotential forces as they originate from the second law, in the coordinate and time variables of the experimenter.[77] It should be stressed that (14) *does not* enlarge the class of systems verifying Theorem A.1.1. In fact, the integrability conditions of this theorem *are* the integrability conditions for the existence of principle (3).[78]

The direct universality can, of course, be reached by applying the techniques of this chart to Birkhoff's equations. Let a_μ δ = 1, 2, . . . , 2*n*, be the local coordinates of the cotangent bundle T^*M (or, equivalently, the

[76] For instance, Goldstein (1950, pp. 38–40), proposes the following principle

$$\delta \int_{t_1}^{t_2} dt(L + W) = -\int_{t_1}^{t_2} dt\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} - F_k\right)\delta q^k = 0, \qquad W = F_k q^k \quad \text{(a)}$$

which however holds only under the *subsidiary condition*

$$\delta W = F_k \delta q^k. \qquad \text{(b)}$$

Thus, principle (a) is not a conventional variational principle as commonly understood. On the contrary, principle (2) achieves the representation of a nonconservative system via a conventional variational procedure. Only the structure of the Lagrangian is generalized in line with the calculus of variations (Section I.1.3). The objective of this chart was therefore to achieve direct analytic representations of nonconservative systems without recourse to procedures, such as subsidiary constraints (b), outside the conventional formulation of the calculus of variations.

[77] Variational principle (14) illustrates more clearly an aspect of Section I.3.4 to the effect that analytic representations of Newtonian systems of *N* particles in a three-dimensional Euclidean space demand, in general, the knowledge of $(3N)^2 + 1$ functions, $(3N)^2$ factor functions, and a Lagrangian. The proof that the matrix of the factor functions of principle (14) induces, in general, a non-self-adjoint structure, is an instructive exercise for the interested reader.

[78] The reader interested in *variational problems* (rather than variational principles) should keep in mind that in this chart we have simply proved the equivalence of the genotopic or isotopic images of Hamilton's principle with the conventional version of this principle. The extremal aspect of the analysis, particularly the implications of variations (8) for the necessary or sufficient conditions for an extremum, will not be considered here.

dynamic space defined in Section 4.5), and let $A(\tilde{E}_0)$ be the Pfaffian action (4.2.14). Then principle (14), when reformulated for Birkhoff's equations, reads

$$\delta^* A(\tilde{E}_0) = \delta^* \int_{t_1}^{t_2} dt[R_\mu(t, a)\dot{a}^\mu - B(t, a)](\tilde{E}_0)$$
$$= \int_{t_1}^{t_2} dt \left\{ g_\mu^\nu \left[\left(\frac{\partial R_\alpha}{\partial a^\nu} - \frac{\partial R_\nu}{\partial a^\alpha} \right) \dot{a}^\alpha - \left(\frac{\partial B}{\partial a^\nu} - \frac{\partial R_\nu}{\partial t} \right) \right]_{SA} \right\}_{NSA} \delta a^\mu(\tilde{E}_0) = 0 \tag{20}$$

and implies the following *direct universality for first-order systems*:

$$\delta^* A(\tilde{E}_0) = \int_{t_1}^{t_2} dt[\dot{a}^\mu - \Xi^\mu(t, a)]\delta a_\mu(\tilde{E}_0)$$
$$= \int_{t_1}^{t_2} dt \underbrace{\begin{matrix} \dot{r}_{ka} - p_{ka}/m_k \\ \dot{p}_{ka} - f_{ka}^{SA}(t, \mathbf{r}, \mathbf{p}) - F_{ka}^{NSA}(t, \mathbf{r}, \mathbf{p}) \end{matrix}}_{} \begin{pmatrix} \delta r_{ka} \\ \delta p_{ka} \end{pmatrix} (\tilde{E}_0) = 0, \tag{21}$$

that is, the direct representation of non-self-adjoint first-order systems in their contravariant form, originating in the reduction from the second-order form (Section 4.1). Representation (21) evidently occurs when matrix (g_μ^ν) is the inverse of Birkhoff's matrix $(\Omega_{\mu\nu})$.

The non-self-adjoint character of (20) permits a number of additional applications. We mention here the *direct representation of the Birkhoff-admissible equations via a Pfaffian action principle* which we write in the notation of Chart 4.7:

$$\delta^* A(\tilde{E}_0) = \int_{t_1}^{t_2} dt \left\{ g_\mu^\nu \left[\Omega_{\nu\alpha}(t, a)\dot{a}^\alpha - \frac{\partial B(t, a)}{\partial a^\nu} - \frac{\partial R_\nu(t, a)}{\partial t} \right]_{SA} \right\}_{NSA} \delta a^\mu(\tilde{E}_0)$$
$$= \int_{t_1}^{t_2} dt \left\{ S_{\mu\nu}(t, a)\dot{a}^\nu - \frac{\partial H(t, a)}{\partial a^\mu} \right\}_{NSA} \delta a^\mu(\tilde{E}_0) = 0. \tag{22}$$

The algebraic implications are as follows. The conventional δ-variations in Pfaff's principle (4.2.15) characterize covariant equations (Birkhoff's equations) whose contravariant form has a *Lie algebra structure* (when the R functions do not depend explicitly on time). The generalized δ^*-variations in (22) characterize covariant equations (the Birkhoff-admissible ones) whose contravariant forms have instead a more general *Lie-admissible algebraic structure*. Actually, we see that all Lie-admissible equations can be derived via a Pfaffian principle, provided that they verify the locality, continuity, and regularity conditions considered here.

As concluding remark, recall that the existence of a representation for a Newtonian system within a fixed system of local variables demands that the implicit functions of both the equations of motion and the analytic representation coincide. This condition is satisfied by all analytic representations considered here and ensures the preservation of the solutions in the transition from the equations of motion to their analytic representation. This is the mathematically essential part of the Inverse Problem. The way in which the equations of motion are written (whether in the form originating from the second law or in an equivalent form) is purely a matter of personal preference, and not of mathematical rigor. In fact, replacing a direct representation with an indirect one and vice versa, can be accomplished through the degrees of freedom of the variations.

EXAMPLES

Example 5.1

For later need, it is useful to recall the following examples of canonical transformations identified in currently available textbooks on the subject.

(a) *Identity transformation* $q^k \to q'^k = q^k$ and $p_k \to p'_k = p_k$. The generating function (Section 5.2) is

$$F = F_2 = q^k p'_k. \tag{1}$$

(b) *Total Inversions* $q^k \to q'^k = -q^k$ and $p_k \to p'_k = -p_k$. The generating function is, in this case,

$$F = F_2 = -q^k p'_k. \tag{2}$$

(c) *Born reciprocity transformations* $q^k \to q'^k = p_k$ and $p_k \to p'_k = -q^k$ in which case the generating function is

$$F = F_1 = q^k q'_k \qquad (q'_k \equiv q'^k). \tag{3}$$

(d) *Born counter-reciprocity transformations* $q^k \to q'^k = -p_k$ and $p_k \to p'_k = q^k$ with generating function

$$F = F_1 = -q^k q'_k. \tag{4}$$

(e) *Scale transformations* $q^k \to q'^k = e^{\lambda} q^k$ and $p_k \to p'_k = e^{-\lambda} p_k$ with generating function is

$$F = F_2 = e^{-\lambda} q^k p'_k. \tag{5}$$

(f) *Born scaled reciprocity transformations* $q^k \to q'^k = e^{\lambda} p_k$ and $p_k \to p'_k = e^{-\lambda} q^k$ with generating function is

$$F = F_1 = e^{-\lambda} q^k q'_k. \tag{6}$$

Note that the transformation

$$q' = \mathscr{F}(q)\cos p, \quad p' = \mathscr{F}(q)\sin p \tag{7}$$

is canonical if

$$\mathscr{F}\frac{\partial \mathscr{F}}{\partial q} = 1, \qquad \mathscr{F}^2 = 2(q + c), \quad c = \text{const.}, \tag{8}$$

in which case

$$q' = [2(q + c)]^{1/2} \cos p, \quad p' = [2(q + c)]^{1/2} \sin p, \tag{9}$$

and the generating function is

$$F = F_3 = -\tfrac{1}{2} q'^2 \tan p + cp. \tag{10}$$

The transformation

$$q' = e^{\lambda p}\mathscr{F}(q), \quad p' = e^{-\lambda p} g(q) \tag{11}$$

is canonical if

$$-\lambda\left[\mathscr{F}\frac{\partial g}{\partial q} + g\frac{\partial \mathscr{F}}{\partial q}\right] = 1, \quad -\lambda \mathscr{F} g = q + c, \quad c = \text{const.}, \tag{12}$$

with a solution

$$q' = e^{\lambda p}(q + c)^{1/2}, \quad p' = -\frac{1}{\lambda} e^{-\lambda p}(q + c)^{1/2}. \tag{13}$$

Other cases of canonical transformations can be constructed via the methods of Section 5.2. Needless to say, an in-depth knowledge of these techniques is essential before initiating a serious study of the generalized canonical transformations of the Birkhoffian mechanics, according to the methods of Section 5.3 (see also Problem 5.7).

Example 5.2

In this example we illustrate (a) the construction of Hamiltonian representations via the transformation theory, (b) the need that the transformations are not canonical whenever the original system is non-self-adjoint, and (c) the capability of Hamiltonian representations, when achieved, of preserving the derivability from a variational principle under noncanonical transformations.

The equation of motion in configuration space

$$\ddot{r} + \frac{r^2}{\dot{r}} = 0, \qquad m = 1 \tag{1}$$

is non-self-adjoint, and the equivalent vector field under the prescription $p = \dot{r}$, i.e.,

$$\Xi_\mu(a) \frac{\partial}{\partial a^\mu}, \quad (\Xi_\mu) = \begin{pmatrix} p \\ -r^2/p \end{pmatrix}, \qquad (a^\mu) = (r, p) \tag{2}$$

is non-Hamiltonian.

To construct an indirect Hamiltonian representation we need a transformation of the local coordinates

$$(a^\mu) = (r, p) \to (a'^\mu) = (r'(r, p), p'(r, p)) \tag{3}$$

such that the transformed vector field

$$\Xi'_\mu(a') = \left(\frac{\partial a'^\alpha}{\partial a^\mu} \Xi_\alpha\right)(a') \tag{4}$$

is self-adjoint, i.e.,

$$\frac{\partial \Xi'_\nu}{\partial a'^\nu} \equiv \frac{\partial \Xi'_\nu}{\partial a'^\mu}. \tag{5}$$

A study of the case indicates that a solution is given by

$$a'^\mu: \quad r = r'p', \qquad p = r'^{3/2} p'^{1/2} \tag{6}$$

under which the self-adjoint vector field is given by

$$(\Xi'_\mu(a')) = \begin{pmatrix} -r'^{1/2} p'^{3/2} \\ -r'^{3/2} p'^{1/2} \end{pmatrix}. \tag{7}$$

The Hamiltonian is then computed via rule (5.1.35), yielding the expression

$$H'(a') = a'^\mu \int_0^1 d\tau\, \Xi'_\mu(\tau a') = -\tfrac{2}{3} r'^{3/2} p'^{3/2}. \tag{8}$$

It is easy to see that transformations (6) are (necessarily) noncanonical, e.g.,

$$[r, p]_{(r', p))} = -r'^{3/2}p'^{1/2} \neq 1. \tag{9}$$

In fact, only a noncanonical transformation can turn a non-Hamiltonian vector field into an equivalent Hamiltonian form.

Now that a Hamiltonian representation has been achieved, it is also easy to see that its derivability from a variational principle persists under arbitrary noncanonical transformations (of the admitted class of continuity and regularity conditions). It is understood here that in the process the Hamiltonian character of the variational principle is lost in favor of the Birkhoffian (Pfaffian) one.

Consider the variational principle for Hamiltonian (8)

$$\delta \int_{t_1}^{t_2} dt[R_\mu^0(a')\dot{a}'^\mu - H'(a')](\tilde{E}_0')$$

$$= \delta \int_{t_1}^{t_2} dt\left[p'\dot{r}' + \frac{2}{3}r'^{3/2}p'^{3/2}\right](\tilde{E}_0') = 0. \tag{10}$$

Under the inverse transformation (6), this principle becomes

$$\delta \int_{t_1}^{t_2} dt[R_\alpha(a)\dot{a}^\alpha - B(a)](E) = 0$$

$$(R_\alpha) = (r^{3/2}p, r^{5/2}p^{1/2})$$

$$B = -\tfrac{2}{3}r^{3/2}$$

by therefore preserving the derivability of the system from a variational principle.

Example 5.3

In this example we shall illustrate how the construction of canonical transformations via generating functions admits a simple and direct generalization into the generalized canonical transformations of Birkhoffian Mechanics.

Born reciprocity transformation (Example 5.1) can be constructed via the generating function

$$F_1 = qq', \tag{1}$$

and Hamiltonian rules (5.2.16), i.e.,

$$p = \frac{\partial F_1}{\partial q} = q', \quad p' = -\frac{\partial F_1}{\partial q'} = -q. \tag{2}$$

In the transition to the covering Birkhoffian formulations, we have rules (5.3.56). Suppose, for simplicity, that $(R_\mu) = (p, r)$, r, $p \in \mathfrak{R}_1$. Then, the latter rules become, for the same generating function (1),

$$qp = \frac{\partial F_1}{\partial q} = q', \qquad q'p' = -\frac{\partial F_1}{\partial q'} = -q' \tag{3}$$

yielding the generalized canonical transformation

$$q' = qp, \qquad p' = -\frac{1}{p}. \tag{4}$$

A similarly straightforward generalization occurs for all other canonical transformations, as the reader is encouraged to verify.

Problems

5.1 The analytic treatment of the theory of canonical transformations is generally done in the existing literature via Hamilton's principle. Prove that the replacement of principle (5.2.10) with Holder's principle (Section I.1.3) in phase space, i.e.,

$$\hat{\delta} \int_{t_1}^{t_2} dt[p_k \dot{q}^k - H(t, q, p)](\tilde{E}^0) = |p_k \hat{\delta} q^k - H \hat{\delta} t|_{t_1}^{t_2}(\tilde{E}^0)$$

not only permits a consistent theory of canonical transformations, but that the underlying generalization of Identity (5.2.14) permits the joint derivation of (a) transformation laws such as those for F_1, as well as (b) Hamilton's equation in both the old and new coordinates (which are not derivable via the conventional theory reviewed in Section 5.2).

5.2 Prove that a Legendre transform reduces the generating function $F_1(t, q, q')$ to (5.2.17).

5.3 Prove that canonical transformations as per Definition 5.2.2 verify the inversion formulas (5.2.27a)–(5.2.27d).

5.4 Prove Equations (5.3.6) and (5.3.13).

5.5 Reformulate the integrability conditions for the existence of canonical transformations via a particularization of Proposition 5.3.1.

5.6 Identify the transformations which are generalized canonical in the sense of preserving generalized variational principle (5.3.50), but are not identity isotopic in the sense of equations (5.3.44). Show that these transformations contain as particular cases, transformations (5.2.12) and (5.2.13).

5.7 Construct the Birkhoffian generalization of the Hamiltonian generating functions F_2, F_3, F_4, F_5, and F_6 along the lines of the method (5.3.56) for F_1.

5.8 Prove that the time component of Galilei's relativity (Chart I.A.1)

$$a' = \exp\left(t_0 \omega^{\alpha\beta} \frac{\partial H}{\partial a^\beta} \frac{\partial}{\partial a^\alpha}\right) a$$

is always canonical, that is, it is canonical for all possible Hamiltonians.

5.9 Prove the following property.

Theorem. *Necessary and sufficient condition for the time evolution*

$$a' = \exp\left(t_0 \Xi^\alpha(a) \frac{\partial}{\partial a^\alpha}\right) a$$

to be noncanonical, that is, to violate the condition of preservation of the fundamental Poisson brackets, is that the vector field Ξ *is non-Hamiltonian*

5.10 Prove that equations (29)–(36) of Chart 5.3 admit a consistent generalization for the isotopic generalization of Lie's theory, resulting in exponentiation (56) of the same chart.

5.11 The **SU**(2) Lie algebra of the Pauli's matrices, when realized on the conventional associative envelope, admits the commutation rules

$$\mathbf{SU}(2): \quad [\sigma_1, \sigma_2]_A = 2i\sigma_3, \quad [\sigma_2, \sigma_3]_A = 2i\sigma_1, \quad [\sigma_3, \sigma_1] = 2i\sigma_2.$$

Find the operator T of equations (36) of Chart 5.2 as a polynomial expression of the σ's. under which the following isotope **SU***(2) of **SU**(2) holds

$$\mathbf{SU}^*(2): \quad [\sigma_1, \sigma_2]_{A^*} = 0, \quad [\sigma_2, \sigma_3]_{A^*} = -2i, \quad [\sigma_3, \sigma_1]_{A^*} = 0.$$

5.12 Prove that equations (39) of Chart 5.4 are the integrability conditions for transformations (9) to be infinitesimal generalized canonical transformations. Discuss the particular canonical case.

CHAPTER 6

Generalization of Galilei's Relativity

6.1 Generalization of Hamilton–Jacobi Theory

One of the most speculative yet intriguing implications of Birkhoffian Mechanics is the possible generalization of *Atomic Mechanics* (the ordinary quantum mechanics) into a form specifically conceived for strong interactions and known as *Hadronic Mechanics.*

As is well known, Hamiltonian and Atomic Mechanics can be considered, in the final analysis, as two different realizations of Lie's theory, the first via functions in phase space and the second via operators on a Hilbert space.

In the preceding chapter we showed that Birkhoffian Mechanics is a classical realization of the more general Lie-isotopic theory. Hadronic Mechanics is therefore predicted as the operator realization of the same Lie-isotopic theory. At any rate, until the identification of an operator mechanics which admits Birkhoffian Mechanics as a classical image has not been accomplished, our description of the microscopic world will be incomplete because the atomic theory is unable to reach Birkhoffian Mechanics under the correspondence principle.

The proposal to construct Hadronic Mechanics was submitted by Santilli (1978d). The studies conducted since that time have been collected in the reprint volumes edited by Schober (1982). The identification of the state of the art in the experimental, theoretical, and mathematical studies of the new mechanics was conducted at the First International Conference on Non-potential Interactions and their Lie-Admissible Treatment (see the *Proceedings* (1982)).

This volume on Birkhoffian Mechanics would be incomplete without the indication of the basic ideas which led to these developments. Predictably, these ideas are of algebraic character and are centered on the *Lie-isotopic generalization of Heisenberg's equations* (see equations (18) of Chart 5.1), as the operator image of Birkhoff's equations. However, for Hadronic Mechanics to be a genuine covering of the atomic one, it must admit consistent and compatible generalizations of other dynamic equations of Atomic Mechanics.

Thus, we come to the problem of generalizing Schrödinger's equations so as to achieve compatibility with the Lie-isotopic generalization of Heisenberg's equations, on the operator side, and with Birkhoff's equations, on the classical side. This problem was studied by Santilli (1982a) via a Birkhoffian generalization of the Hamilton–Jacobi theory we shall review in this section. These studies were inspired by work by Mignani (1981 and 1982). The identification of the structure of the underlying Hilbert space, and a first axiomatization, were achieved by Myung and Santilli (1982a and b), including the proof of the equivalence of the hadronic generalizations of Heisenberg's and Schrödinger's equations. Additional research can be found in Schober (*loc cit.*).

The conceptual foundation of the theory is the Newtonian property that the potential energy has no physical basis for contact interactions. Since all strongly interacting particles (called *hadrons*) have a size (charge radius) which is of the order of magnitude of the range of the strong interactions (about 10^{-13} cm $= 1F$), the possible existence in the strong interactions of a component of contact non-Hamiltonian type is then rather natural. In turn, this brings to the assumption of Birkhoff's equations as the basic classical equations representing a superposition of conventional, potential, action-at-a-distance forces, as well as contact, non-Hamiltonian ones.

The classical theoretical foundation is provided by the transformation theory of Section 5.3. In fact, this theory has permitted the construction of the desired generalization of the Hamilton–Jacobi equations in a way fully parallel to the conventional Hamiltonian case.

The operator foundation of the theory is given by a suitable reformulation of the Hilbert space in a way which is directly compatible with the Lie-isotopic theory (Chart 6.1). Note that this aspect will not be considered in this section, and we limit ourselves to presenting the hadronic generalization of Schrödinger's equations, in much the same historical (rather than contemporary) way that the original equations were presented during the first part of this century.

Regrettably, in the interest of brevity, we are unable to treat a number of additional aspects, such as the Birkhoffian generalization of the canonical perturbation theory, and the corresponding operator image expected within the context of Hadronic Mechanics. We hope, however, that the methods identified in this volume for constructing the Birkhoffian generalization of specific aspects of Hamiltonian Mechanics are applicable also to other aspects. The same methods, based essentially on noncanonical transformations, are applicable to the construction of the hadronic generalization of Atomic

Mechanics upon suitable operator reformulation in terms of nonunitary transformations. An example of this latter aspect is indicated in the next section.

Part A: Hamilton–Jacobi Theory

Let us begin by reviewing, for notational purposes as well as for comparison, the conventional formulation of the Hamilton–Jacobi theory. Suppose that Hamilton's equations are known,

$$\omega_{\mu\nu}\dot{a}^{\nu} - \frac{\partial H(t,a)}{\partial a^{\mu}} = \left(\frac{\partial R^0_{\nu}}{\partial a^{\mu}} - \frac{\partial R^0_{\mu}}{\partial a^{\nu}}\right)\dot{a}^{\nu} - \frac{\partial H}{\partial a^{\mu}} = 0,$$
$$(R^0_{\mu}) = (\mathbf{p}, \mathbf{0}), \qquad (a^{\mu}) = \begin{pmatrix}\mathbf{r}\\ \mathbf{p}\end{pmatrix}, \qquad \mu = 1, 2, \ldots, 2n. \tag{6.1.1}$$

The *Hamilton–Jacobi problem* consists of the identification of a canonical transformation which is invertible, is of the same continuity class of H, and has a generating function F under which the transformed Hamiltonian

$$t \to t' \equiv t, \qquad a^{\mu} \to a^{\mu}_0(t, a), \tag{6.1.2a}$$

$$H \to H_0 = H(t, a(t, a_0)) + \frac{\partial F}{\partial t} \equiv 0, \tag{6.1.2b}$$

is identically null. Equations (6.1.1) in the new frame become

$$\omega_{\mu\nu}\dot{a}^{\nu}_0 = 0 \tag{6.1.3}$$

with general solution

$$a^{\mu}_0 = a^{\mu}_0(t, a) = \text{const.} \tag{6.1.4}$$

The solution of the original equations (6.1.1) is then given by the inverse transformation,

$$a^{\mu} = a^{\mu}(t, a_0), \tag{6.1.5}$$

in which the a_0's play the role of the arbitrary constants.

A formal solution for the generating function is given by the action

$$F = F_1 = A(\tilde{E}_0) = \int_{t_0}^{t} dt[R^0_{\mu}(a)\dot{a}^{\mu} - H(t,a)](\tilde{E}_0)$$
$$= \int_{t_0}^{t} dt[p_k\dot{r}^k - H(t, \mathbf{r}, \mathbf{p})](\tilde{E}_0). \tag{6.1.6}$$

Weiss's principle (see Section I.1.3) then yields the end point contribution

$$dA = d\int_{t_0}^{t} dt[p_r\dot{r}^k - H](\tilde{E}_0) = |p_k dr^k - Hdt|_{t_0}^{t}(\tilde{E}_0)$$

$$= p_k dr^k - Hdt - p_{0k}dr_0^k, \qquad \begin{matrix}\mathbf{r}_0 = \mathbf{r}(t_0)\\ \mathbf{p}_0 = \mathbf{p}(t_0).\end{matrix} \tag{6.1.7}$$

The use of partial derivatives finally yields the celebrated *Hamilton–Jacobi equations*

$$\frac{\partial A}{\partial t} + H(t, \mathbf{r}, \mathbf{p}) = 0, \tag{6.1.8a}$$

$$p_k = \frac{\partial A}{\partial r^k}, \qquad p_{0k} = -\frac{\partial A}{\partial r_0^k}. \tag{6.1.8b}$$

We should indicate, for completeness, that form (6.1.8) of the Hamilton–Jacobi equations *is not unique*, and several additional forms exist. This is clearly due to the fact that the desired canonical transformation can be constructed via any generating function, not necessarily $F_1 = A$. By recalling the existence of a large number of possible generating functions, a corresponding number of different Hamilton–Jacobi equations follows.

For instance, for the case of a generating function $F = F_5 = F_5(t, \mathbf{r}_0, \mathbf{p}_0)$, one can prove that (6.1.8) is replaced by

$$\frac{\partial F_5}{\partial t} + H(t, \mathbf{r}, \mathbf{p}) + \frac{\partial r^k}{\partial t} p_k = 0, \tag{6.1.9a}$$

$$p_{0k} - \frac{\partial r^i}{\partial r_0^k} p_i = \frac{\partial F_5}{\partial r_0^k}; \quad \frac{\partial r^i}{\partial p_0^k} p_i = -\frac{\partial F_5}{\partial p_0^k}. \tag{6.1.9b}$$

The study of additional forms of the equations for some other type of generating function is instructive but is left to the interested reader (Problem 6.1). Needless to say, all these possible different forms of the Hamilton–Jacobi equations are equivalent, because they are related by the same Legendre transforms which interconnect different generating functions (Section 5.2).

Before passing to the identification of the Birkhoffian generalization of (6.1.8), it is important to point out a "reformulation" which is permitted by the techniques presented in these volumes.

The following generalization of action (6.1.6)

$$A^+(E) = \int_{t_0}^{t} dt[R_\mu^{0+}(a)\dot{a}^\mu - H(t, a)](E), \qquad R_\mu^{0+} = -\tfrac{1}{2}\omega_{\mu\nu}a^\nu \tag{6.1.10}$$

was introduced in Chart I.3.6. We can easily see that the equations characterized by contemporary variations with fixed end points are *exactly* given by Hamilton's equations (6.1.1). Thus action (6.1.10) permits generalized

variational principles while leaving the underlying analytic equations unchanged.

In Section 4.5 we pointed out that the transition from action (6.1.6) to generalized form (6.1.10) is given by the Birkhoffian gauge for the particular Hamiltonian case

$$R_\mu^0 \to R_\mu^{0+} = R_\mu^0 + \frac{\partial G}{\partial a^\mu}; \qquad G = -\mathbf{r}\cdot\mathbf{p}. \tag{6.1.11}$$

The preservation of the original equations (6.1.1) is then trivial. However, the generalization of (6.1.6) into (6.1.10) *is not* trivial from the viewpoint of the Hamilton–Jacobi theory and, inevitably, from the viewpoint of Schrödinger's Mechanics. This can be seen by noting that the original action (6.1.6) is *independent* of the momenta (velocities), e.g.,

$$\frac{\partial A}{\partial p_k} \equiv 0, \qquad k = 1, 2, \ldots, n, \tag{6.1.12}$$

while the new action (6.1.10) is indeed dependent on $\mathbf{p}$, i.e.,

$$\frac{\partial A^+}{\partial p_k} \neq 0. \tag{6.1.13}$$

A reformulation of (6.1.8) is then expected, with nontrivial quantum mechanical implications, as we shall see.

The construction of the desired reformulation of (6.1.8) is straightforward. When (6.1.10) is subjected to the same variations of principle (6.1.7) (non-contemporaneous variations with variables end points—see Section I.1.3 for details), we obtain the principle

$$\begin{aligned} dA^+ &= d\int_{t_0}^{t} [R_\mu^{0+}(a)da^\mu - H(t,a)dt](\tilde{E}_0) = |R_\mu^{0+}(a)da^\mu - H(t,a)dt|_{t_0}^{t}(\tilde{E}_0) \\ &= R_\mu^{0+}(a)da^\mu - H(t,a)dt - S_\mu^{0+}(a_0)da_0^\mu. \end{aligned} \tag{6.1.14}$$

The use of partial derivatives then yields the equations

$$\frac{\partial A^+}{\partial t} + H(t,a) = 0 \tag{6.1.15a}$$

$$R_\mu^{0+}(a) = \frac{\partial A^+}{\partial a^\mu}; \quad S_\mu^{0+}(a_0) = -\frac{\partial A^+}{\partial a_0^\mu} \tag{6.1.15b}$$

which are a *reformulation of the Hamilton–Jacobi equations induced by the gauge degrees of freedom* (6.1.11).

Some important differences and similarities between (6.1.8) and (6.1.15) are the following. Under the condition that the Hamiltonian depends explicitly on *all* the values $\partial A/\partial r^k = p_k$, $k = 1, 2, \ldots, n$, equations (6.1.8) can be reduced to a single, generally nonlinear, partial differential equation in

$\partial A/\partial t$ and $\partial A/\partial r^k$, plus subsidiary conditions given by the second of equations (6.1.8b),

$$\frac{\partial A}{\partial t} + H\left(t, \mathbf{r}, \frac{\partial A}{\partial \mathbf{r}}\right) = 0 \tag{6.1.16a}$$

$$p_{0k} = -\frac{\partial A}{\partial r_0^k}. \tag{6.1.16b}$$

Note that if the Hamiltonian does not depend on one of the p's, say $\partial H/\partial p_i = 0$, i = fixed, then the equations $p_i = \partial A/\partial r^i$ must be kept as a subsidiary condition.

We can see that reduction (6.1.16) is fully applicable also to (6.1.15) under similar conditions. In fact, suppose that H depends explicitly on *all* values $\partial A/\partial a^\mu$, $\mu = 1, 2, \ldots, 2n$, then (6.1.15) can be reduced to a single, generally nonlinear, partial differential equation in $\partial A^+/\partial t$ and $\partial A^+/\partial a^\mu$ plus subsidiary conditions given by the second set of equations (6.1.15b),

$$\frac{\partial A^+}{\partial t} + H(t, a^\mu) = \frac{\partial A^+}{\partial t} + H\left(t, -2\omega^{\mu\nu}\frac{\partial A}{\partial a^\nu}\right) = \frac{\partial A^+}{\partial t} + H(t, r, p)$$

$$= \frac{\partial A^+}{\partial t} + H\left(t, -2\frac{\partial A}{\partial \mathbf{p}}, 2\frac{\partial A}{\partial \mathbf{r}}\right) = 0 \tag{6.1.17a}$$

$$S_\mu^{0+}(a_0) = -\frac{\partial A^+}{\partial a_0^\mu}. \tag{6.1.17b}$$

If $\partial H/\partial a^\nu = 0$ for ν fixed, then the expression $R_\nu^{0+} = \partial A^+/\partial a^\nu$ must be kept as a subsidiary constraint in exactly the same way as it occurs for the conventional Hamiltonian case.

Thus, on methodological grounds, the primary difference between the conventional formulation of the Hamilton–Jacobi equations and reformulation (6.1.17) is the extension of the partial differential equation to include the terms $\partial A^+/\partial p_k$.

The reader should keep in mind that (6.1.15) and (6.1.17) are a direct consequence of the *Birkhoffian* generalization of the Hamiltonian formulations. As a matter of fact, equations (6.1.1) are written in the version which is the Hamiltonian particularization of Birkhoff's equations.

The classical relevance of (6.1.15)–(6.1.17) will be self-evident in a monent. The quantum mechanical relevance can be anticipated here via the following remark. While quantization of (6.1.16), as is well-known, is based on a wave function depending only on time and coordinates $\psi(t, \mathbf{r})$, the quantization of equations (6.1.17) is expected to imply the existence of a reformulation based on a "wave function" which depends also on the generalized momenta, $\psi(t, \mathbf{r}, \mathbf{p})$. In turn, the existence of the reformulation is expected to be useful to study still open problems of (conventional) quantum mechanics, such as the problematic spreading of the wave packets of particles,

the equivalence (or nonequivalence) of Heisenberg's and Schrödinger's representations, the still controversial issues of quantization and classical limits, etc. We should stress that the reformulation of Schrödinger's equations referred to here is intended specifically for Atomic and not for Hadronic Mechanics (i.e., for electromagnetic and not strong interactions).

Part B: Birkhoffian Generalization of the Hamilton–Jacobi Theory

We can show that the Hamilton–Jacobi theory generalizes in its entirety into a consistent Birkhoffian form. Consider the semiautonomous Birkhoff's equations

$$\Omega_{\mu\nu}(a)\dot{a}^{\nu} - \frac{\partial B(t, a)}{\partial a^{\mu}} = \left(\frac{\partial R_{\nu}(a)}{\partial a^{\mu}} - \frac{\partial R_{\mu}(a)}{\partial a^{\nu}}\right)\dot{a}^{\nu} - \frac{\partial B(t, a)}{\partial a^{\mu}} = 0. \quad (6.1.18)$$

The Birkhoffian generalization of the Hamilton–Jacobi problem[1] consists of identifying an identity isotopic transformation (generalized canonical transformation) under which the transformed Birkhoffian is identically null, i.e.,

$$t \to t' \equiv t, \qquad a^{\mu} \to a_0^{\mu}(t, a), \quad (6.1.19a)$$

$$B(t, a) \to B'_0(t, a'_0) = \left(B - \frac{\partial a_0^{\alpha}}{\partial t} R_{\alpha}\right)(t, a_0) \equiv 0. \quad (6.1.19b)$$

Equations (6.1.18) then reduce to

$$\Omega_{\mu\nu}(a_0)\dot{a}_0^{\nu} = 0. \quad (6.1.20)$$

By assuming that Birkhoff's tensor $\Omega_{\mu\nu}$ is regular, i.e.,

$$\det\left(\frac{\partial R_{\nu}}{\partial a^{\mu}} - \frac{\partial R_{\mu}}{\partial a^{\nu}}\right) \neq 0, \quad (6.1.21)$$

[1] A second statement of the problem can be reached through the Birkhoffian formulations and consists of the search of a transformation $a \to a_0(t, a)$ under which the $2n$-vector R_{μ} becomes identically null, i.e.,

$$R'_{\mu}(t, a_0) = \left(\frac{\partial a^{\alpha}}{\partial a_0^{\mu}} R_{\alpha}\right)(t, a_0) \equiv 0.$$

Note that we can use $2n + 1$ independent functions (the transformations $a_0(t, a)$ and a gauge function). Thus one can ask for the additional condition that the Birkhoffian becomes also identically null,

$$B'(t, a_0) = B(t, a(t, a_0)) - \left(\frac{\partial a^{\alpha}}{\partial t} R_{\alpha}\right)(t, a_0) \equiv 0.$$

This alternative formulation, which is applicable also for Hamilton's equations, will not be explored here for brevity. Note that, since the transformation does not preserve the Birkhoffian (or the Hamiltonian) tensor, *it is not* generalized canonical.

one can see that this regularity property is preserved by the transformation theory. The solution of (6.1.20) is also given by constants a_0^μ as in the Hamiltonian case. The solution of the original equations (6.1.18) is then given by the inverse transformation $a^\mu(t, a_0)$, also in full analogy with the canonical case. As a result, the generalized problem considered can provide a solution of the equations of motion, at least on formal grounds, in a way fully parallel to the conventional case.

It can be proved that a formal solution for the generating function of transformation (6.1.19) is given by the Pfaffian action

$$F = F_1 = A^g(\tilde{E}) = \int_{t_0}^{t} dt[R_\mu(a)\dot{a}^\mu - B(t, a)](\tilde{E})$$

$$\stackrel{\text{def}}{=} \int_{t_0}^{t} dt[P_k(t, \mathbf{r}, \mathbf{p})\dot{r}^k + Q^k(t, \mathbf{r}, \mathbf{p})\dot{p}_k - B(t, \mathbf{r}, \mathbf{p})](\tilde{E}) \quad (6.1.22)$$

which is clearly a generalization of actions (6.1.6) and (6.1.10). The corresponding generalization of (6.1.7) and (6.1.14) is given by[2]

$$\begin{aligned} dA^g(\tilde{E}_0) &= d\int_{t_0}^{t} [R_\mu(a)da^\mu - B(t, a)dt](\tilde{E}_0) \\ &= |R_\mu(a)da^\mu - B(t, a)dt|_{t_0}^{t}(\tilde{E}_0) \\ &= R_\mu(a)da^\mu - B(t, a)dt - R_\mu(a_0)da_0^\mu; \quad a_0^\mu = a^\mu|_{t_0}. \end{aligned} \quad (6.1.23)$$

[2] Owing to the importance of principle (6.1.23) for the hadronic generalization of Schrödinger's Mechanics, it may be valuable here to indicate its derivation. The principle is a particular form of a well-known property of the calculus of variation reviewed in detail in Section I.1.3. Given a Euler function

$$L = R_\mu(a)\dot{a}^\mu - B(t, a), \qquad a \in R_m, \quad \text{(a)}$$

the first-order variation of the action functional in L with variable endpoints, when computed along an arbitrary path $\tilde{E}$ (of the topological conditions admitted), characterizes the variational problem

$$\delta A(\tilde{E}) = \int_{t_1}^{t_2} dt L_\mu(\tilde{E})\delta a^\mu + \left|\frac{\partial L}{\partial a^\mu}\delta a^\mu - \left(\frac{\partial L}{\partial a^\mu}\dot{a}^\mu - L\right)\delta t\right|_{t_1}^{t_2}(\tilde{E}). \quad \text{(b)}$$

When path $\tilde{E}$ is a possible or an actual path $\tilde{E}_0$, the Euler's equation $L_\mu(\tilde{E}_0) = 0$ *coincide* with Birkhoff's equations (6.1.18), and therefore they are identically null. *Variational problem* (b) then yields the *variational principle* (6.1.23), i.e.,

$$\begin{aligned} \hat{\delta}A(\tilde{E}_0) &= \left|\frac{\partial L}{\partial a^\mu}\hat{\delta}a^\mu - \left(\frac{\partial L}{\partial a^\mu}\dot{a}^\mu - L\right)\hat{\delta}t\right|_{t_1}^{t_2}(\tilde{E}_0) \\ &\equiv |R_\mu(a)\hat{\delta}a^\mu - B(t, a)\hat{\delta}t|_{t_1}^{t_2}(\tilde{E}_0). \end{aligned} \quad \text{(c)}$$

The property is fundamental, classically and "quantum mechanically." Classically, we learn that the total differential of an action functional is equal to the integrand computed at end points under the condition that such integrand is of first-order type (that is, Pfaffian). In turn, this property is at the foundation of a number of aspects of the Birkhoffian Mechanics, such as the Birkhoffian generalization of the canonical transformation theory, Noether's theorem, etc. The quantum mechanical relevance of the property will be self-evident in a moment.

The use of partial derivatives finally yields the desired *Birkhoffian generalization of the Hamilton–Jacobi equations* which can be written in the form

$$\frac{\partial A^g}{\partial t} + B(t, a) = 0, \tag{6.1.24a}$$

$$R_\mu(a) = \frac{\partial A^g}{\partial a^\mu}, \qquad R_\mu(a_0) = -\frac{\partial A^g}{\partial a_0^\mu}. \tag{6.1.24b}$$

We can also prove that (6.1.24), under the conditions $\partial B/\partial a^\mu \neq 0$, $\mu = 1, 2, \ldots, 2n$, can *always* be reduced to a single partial differential equation in $\partial A^g/\partial t$ and $\partial A^g/\partial a^\mu$ in a way fully parallel, although generalized, to that of (6.1.16) and (6.1.17).

To see it, note that regularity property (6.1.21) *does not* imply that of the matrix $(\partial R_\mu/\partial a^\nu)$. Consider, then, the case in which

$$\det\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right) \neq 0, \qquad \det\left(\frac{\partial R_\mu}{\partial a^\nu}\right) = 0. \tag{6.1.25}$$

However, a Birkhoffian gauge transformation

$$R_\mu \to R_\mu^+ = R_\mu + \frac{\partial G}{\partial a^\mu} \tag{6.1.26}$$

always exists under which

$$\det\left(\frac{\partial R_\nu^+}{\partial a^\mu} - \frac{\partial R_\mu^+}{\partial a^\nu}\right) \equiv \det\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right) \neq 0, \qquad \det\left(\frac{\partial R_\mu^+}{\partial a^\nu}\right) \neq 0. \tag{6.1.27}$$

In fact, for this purpose, selecting an arbitrary function $G(a)$ such that

$$\det\left(\frac{\partial^2 G}{\partial a^\mu\, \partial a^\nu}\right) \neq 0. \tag{6.1.28}$$

is sufficient. Once the regularity property

$$\det\left(\frac{\partial R_\mu}{\partial a^\nu}\right) \neq 0 \tag{6.1.29}$$

has been ensured, one can perform the change of coordinates of (6.1.24a) from the Birkhoffian ones a^μ to the new ones $R_\mu(a)$

$$a^\mu \to a'^\mu(a) \stackrel{\text{def}}{=} R_\mu(a) \tag{6.1.30}$$

under which we have

$$\frac{\partial A^g}{\partial t} + B(t, a(R)) = \frac{\partial A^g}{\partial t} + \mathscr{B}(t, R) = \frac{\partial A^g}{\partial t} + \mathscr{B}(t, \mathbf{P}(t, \mathbf{r}, \mathbf{p}), \mathbf{Q}(t, \mathbf{r}, \mathbf{p})) = 0. \tag{6.1.31}$$

Thus *the 2n-components of the vector* $R_\mu(a)$ *appearing in the original equations* (6.1.18) *are assumed as the new variables of* (6.1.31), of course, upon selection of the gauge in which regularity property (6.1.29) holds. Note that this was exactly the case for the Hamiltonian form (6.1.17). In fact, the original vector $R^0_\mu(a)$ does not verify property (6.1.29), trivially because of its structure $(R^0_\mu) = (\mathbf{p}, \mathbf{0})$. However, the "gauged" vector $R^{0+}_\mu(a) = -\frac{1}{2}\omega_{\mu\nu}a^\nu$ does verify property (6.1.29), by therefore permitting the change of coordinates $a^\mu \to R^0_\mu$. This, in turn, permits the achievement of the single partial differential equation (6.1.17a).

The reduction of (6.1.24) to a single partial differential equation in A^g (plus subsidiary conditions) is now self-evident. It is given by

$$\frac{\partial A^g}{\partial t} + \mathscr{B}\left(t, \frac{\partial A^g}{\partial a^\mu}\right) = 0, \tag{6.1.32a}$$

$$\frac{\partial A^g}{\partial a^\mu_0} = -R_\mu(a_0). \tag{6.1.32b}$$

Needless to say, if the Birkhoffian does not depend on some of the a variables (e.g., when $B = \frac{1}{2}\mathbf{p}^2$), the missing terms must be kept as subsidiary conditions, in exactly the same way as it occurs for (6.1.16) and (6.1.17).

Notice that *the Hamiltonian particularization of* (6.1.32) *is given by the gauge Hamilton–Jacobi form* (6.1.17) *and not by the original form* (6.1.8). In fact, the particularization $R_\mu(a) = (\mathbf{p}, \mathbf{0})$ implies the violation of regularity condition (6.1.29), under which (6.1.32a) becomes singular. On the contrary, the Hamiltonian particularization

$$R_\mu(a) \equiv R^{0+}_\mu(a) = -\tfrac{1}{2}\omega_{\mu\nu}a^\nu \tag{6.1.33}$$

preserves regularity property (6.1.29), as indicated earlier.

By no means do equations (6.1.24) exhaust all possible Birkhoffian generalizations of Hamilton–Jacobi equations. In fact, a class of equations equivalent to (6.1.24) can be constructed via the Legendre transform of the generating function, in a way fully parallel to the conventional case.

To illustrate this, we recall that the generalization under consideration was studied by Sarlet and Cantrijn (1978b) who reached the equations

$$\frac{\partial F}{\partial t} + B(t, a(t, a_0)) + \frac{\partial a^\alpha}{\partial t} R_\alpha(a(t, a_0)) = 0$$

$$\left(\frac{\partial a^\alpha}{\partial a^\mu_0} R_\alpha\right)(t, a_0) - \frac{\partial F}{\partial a^\mu_0}(t, a_0) = R_\mu(a_0). \tag{6.1.34}$$

The generating function of these equations can be shown to be *Class* 5, while that of (6.1.24) is of *Class* 1. As a result, equations (6.1.34) are a generalization of (6.1.9) rather than (6.1.8). Our preference of generalized form (6.1.24) is due to quantum mechanical considerations. In fact, the classical equations at the foundations of Schrödinger's Mechanics are equations (6.1.8). It is therefore important to achieve a Birkhoffian generalization of the

Hamilton–Jacobi equations in their form directly used for quantum mechanical purposes (see Problem 6.2 for additional forms).

Furthermore, (6.1.34) *cannot* be reduced to a single partial differential equation in F (Sarlet and Cantrijn, *loc. cit.*, p. 1597). This implies severe technical difficulties in attempting the construction of a generalization of Schrödinger's equations via form (6.1.34). Equations (6.1.24), on the contrary, bypass this problem by permitting reduction to form (6.1.32a).

As a final remark, let us note that the Birkhoffian generalization of the Hamilton–Jacobi theory for the case of the nonautonomous equations

$$\left[\frac{\partial R_\nu(t, a)}{\partial a^\mu} - \frac{\partial R_\mu(t, a)}{\partial a^\nu}\right]\dot{a}^\nu - \left[\frac{\partial B(t, a)}{\partial a^\mu} + \frac{\partial R_\mu(t, a)}{\partial t}\right] = 0 \qquad (6.1.35)$$

will not be considered here for a number of reasons. The first is that (6.1.35) can be reduced to an equivalent semiautonomous form (6.1.18) in the same local variables via the use of a gauge transformation (see Section 4.5). Therefore, the study of (6.1.18) is sufficient for our purposes. Deeper reasons also exist. Our primary objective is to indicate a conceivable "Schrödinger-type" analog of the isotopic generalization of Heisenberg's equations. These equations, in turn, are an image of the *semiautonomous* equations (6.1.18) and *not* of (6.1.35), because the latter equations do not admit a consistent algebraic structure in the time evolution (see Chart 4.1).

Part C: Schrödinger's Equation

Consider a conservative system in the contravariant form

$$\dot{a}^\nu = \Xi^\mu(a), \qquad (\Xi^\mu) = \begin{pmatrix} p_{ka}/m_a \\ f_{ka}^{SA}(r) \end{pmatrix}, \qquad k = 1, 2, \ldots, N; \quad a = x, y, z \qquad (6.1.36)$$

such as a Kepler system in vacuum. The construction of Hamilton–Jacobi equations for these systems is trivial. It is based on the *conventional Hamiltonian structure*

$$H = \sum_{k=1}^{N} \frac{1}{2m_k}\mathbf{p}_k^2 + V(\mathbf{r}) \qquad (6.1.37)$$

with the direct physical meaning of total energy. Since the potential energy does not depend on the velocity (and no contact interaction exists by assumption), one can prove that *the canonical momentum coincides with the physical momentum*

$$\mathbf{p}_k = m_k \dot{\mathbf{r}}_k. \qquad (6.1.38)$$

Under the conditions considered, the *canonical angular momentum* $\mathbf{M}_k$ *coincides with the physical angular momentum.*

Note that representation (6.1.37) is not unique. Among all possible analytic representations, we have selected the *unique*, direct, canonical representation

of the system in the frame of the observer (no transformation theory!), under which all canonical quantities have a direct physical meaning.

The celebrated *canonical quantization rules*

$$-H = \frac{\partial A}{\partial t} \to i\frac{\partial}{\partial t} = -\tilde{H}, \qquad \mathbf{p}_k = \frac{\partial A}{\partial \mathbf{r}^k} \to \frac{1}{i}\nabla_k = \tilde{\mathbf{p}}_k, \qquad \hbar = 1, \quad (6.1.39)$$

readily turn Hamilton–Jacobi equation

$$\frac{\partial A}{\partial t} + H(\mathbf{r}, \mathbf{p}) = \frac{\partial A}{\partial t} + \sum_{k=1}^{N} \frac{1}{2m_k} \frac{\partial A}{\partial \mathbf{r}^k} \cdot \frac{\partial A}{\partial \mathbf{r}^k} + V(\mathbf{r}) = 0 \qquad (6.1.40)$$

into the familiar *Schrödinger's equation*

$$i\frac{\partial}{\partial t}\psi(t, \mathbf{r}) = \tilde{H}(\tilde{\mathbf{r}}, \tilde{\mathbf{p}})\psi(t, \mathbf{r}) = \left[-\sum_{k=1}^{N} \frac{1}{2m_k}\Delta_k + V(\mathbf{r})\right]\psi(t, \mathbf{r}) \quad (6.1.41)$$

Without any claim of mathematical rigor,[3] the quantization satisfies the *correspondence principle* in the sense that, under the wave function

$$\psi \doteq Ne^{iA} = Ne^{i\int(\mathbf{p}_k \cdot d\mathbf{r}^k - Hdt)}, \qquad N \in \mathbb{R} \qquad (6.1.42)$$

and for large (e.g., macroscopic) values of the action,

$$\frac{1}{A} \approx 0, \qquad \frac{1}{i}\nabla_k \ll \frac{\partial A}{\partial \mathbf{r}^k}, \qquad i\frac{\partial}{\partial t} \ll \frac{\partial A}{\partial t}, \qquad (6.1.43)$$

the zero-order term of the expansion of Equation (6.1.41) in terms of $1/A$ coincides with the Hamilton–Jacobi equation (6.1.40) identically. First-order terms then yield continuity equations and other properties which are ignored here for brevity.

A fundamental feature of conventional wave equation (6.1.41) *is that it complies with the quantum mechanical Galilei's relativity.* Intriguingly, this condition is *necessary* to verify the correspondence principle because the original system is compatible with Galilei's relativity to begin with.

This feature has numerous direct or indirect implications at virtually all levels of treatment. It can be expressed initially by noting that the eigenfunction $\psi(t, \mathbf{r})$ must be a Galilei scalar. In turn, this sets the structure

$$\psi(t, \mathbf{r}) = \int dE\varphi(E)e^{i\int(p_k dr^k - Hdt)} \qquad (6.1.44)$$

which characterizes a fundamental notion of quantum mechanics, that of *wave packets* (or *Green function*). In turn, (6.1.44) constitutes one way to establish the indeterministic nature of quantum mechanics,

$$\Delta r \Delta p \geq \tfrac{1}{2}\hbar. \qquad (6.1.45)$$

[3] See footnote 43 of Chapter 5.

Indeed, the widths of the amplitude φ and that of the wave packet ψ are inversely related, yielding (6.1.45) after simple elaborations.

Part D: Hadronic Generalization of Schrödinger's Equations

As is well known, the conventional notion of wave packets according to (6.1.44) is only a crude approximation, because it applies only under the condition that the particle can be considered as nearly free over a distance of a number of wavelengths. In an attempt to improve the approximation, particularly for particles under intense forces within the distance of *one* wavelength (as expected for the strong interactions), we search for a generalization of (6.1.44) into the form

$$\psi(t, \mathbf{r}, \mathbf{p}) = \psi(t, a) = \int dB\varphi(B)e^{\int (R_\mu da^\mu - Bdt)}$$

$$= \int dB\varphi(B)e^{\int [P_k(t, \mathbf{r}, \mathbf{p})dr^k + Q^k(t, \mathbf{r}, \mathbf{p})dp_k - B(t, \mathbf{r}, \mathbf{p})dt]}. \qquad (6.1.46)$$

The physical implications of the Pfaffian generalization of the action functional now come to light. In fact, the transition from wave packet (6.1.44) to generalized form (6.1.46) is clearly based on the replacement of the canonical action with the Pfaffian one. The transition from a wave function depending only on time and coordinates to one depending also on momenta is then self-evident (as anticipated earlier in this section).

The hadronic generalization of Schrödinger's equations is attempted in this section in such a way as to generalize the historical process which lead to structure (6.1.44), that is, so as to admit a classical limit into the Birkhoffian generalization of the Hamilton–Jacobi equations. The formal solution is the following.

Consider a nonconservative, non-self-adjoint implementation of system (6.1.36)

$$a^\mu - \Xi^\mu(a) - \Gamma^\mu(t, a) = 0; \quad (\Gamma^\mu) = \begin{pmatrix} 0 \\ F_{ka}^{\mathrm{NSA}}(t, a) \end{pmatrix} \qquad (6.1.47)$$

in which the non-self-adjoint forces can be considered, for instance, as corrections to (6.1.36) due to the extended nature of the particles. Represent the system with equations (6.1.32), i.e.,

$$\frac{\partial A^g}{\partial t} + B(t, a) = 0, \qquad (6.1.48a)$$

$$a^\mu(R_\mu) = a^\mu\left(\frac{\partial A^g}{\partial a^\mu}\right). \qquad (6.1.48b)$$

We can easily see that the following *Birkhoffian quantization rules*[4]

$$-B = \frac{\partial A^g}{\partial t} \to i\frac{\partial}{\partial t} = -\tilde{B}, \qquad R_\mu = \frac{\partial A^g}{\partial a^\mu} \to \frac{1}{i}\frac{\partial}{\partial a^\mu} = \tilde{R}_\mu \tag{6.1.49}$$

yield the desired *hadronic generalization of Schrödinger's equations*[5]

$$i\frac{\partial}{\partial t}\psi(t, a) = \tilde{B}(t, \tilde{a})\psi(t, a) = \mathscr{B}\left(t, \frac{1}{i}\frac{\partial}{\partial a}\right)\psi(t, a) \tag{6.1.50}$$

where a suitable symmetrization of the Birkhoffian functions B or $\mathscr{B}$ and $a^\mu(R)$ appearing in (6.1.31) is understood.

Under the wave function

$$\psi = Ne^{iA^g} \tag{6.1.51}$$

and values

$$\frac{1}{A^g} \approx 0, \qquad \frac{1}{i}\frac{\partial}{\partial a^\mu} \ll \frac{\partial A^g}{\partial a^\mu}, \qquad \frac{1}{i}\frac{\partial}{\partial t} \ll \frac{\partial A^g}{\partial t}, \tag{6.1.52}$$

the zero-order term of the expansion of (6.1.50) in $1/A^g$ reproduces the classical (6.1.48) identically, as the reader is encouraged to verify. The first-order term then yields a continuity equation of equally easy derivation.

We should indicate that *hadronic wave packet* (*or Green function*) (6.1.46) *is the general solution of* (6.1.50) *under the most general possible combination of* (*local, analytic, regular*) *potential and nonpotential interactions.* By comparison, no general solution of Schrödinger's equation of Atomic Mechanics is known under arbitrary *potential* forces in such a simple way.

A few concluding remarks are in order. First, conventional Schrödinger's equations (6.1.41) *are not* a particular case of (6.1.50) owing to the use of the Birkhoffian gauge which is absent in the former equations. The atomic particularization of (6.1.50) (i.e., the particularization when all nonpotential forces are identically null) is given by a suitable reformulation of (6.1.41)

[4] To avoid possible misrepresentations, we point out that the commutativity of the operators $\tilde{R}_\mu$ is illusory for Hadronic Mechanics. In fact, the conventional associative product $\tilde{R}_\mu\tilde{R}_\nu$ must be replaced by the isotopic one $\tilde{R}_\mu * \tilde{R}_\nu = \tilde{R}_\mu\tilde{T}(a)\tilde{R}_\nu$. As a result, conventional commutators $[\tilde{R}_\mu, \tilde{R}_\nu] = \tilde{R}_\mu\tilde{R}_\nu - \tilde{R}_\nu\tilde{R}_\mu$ must be replaced by the isotopic commutators of Chart 5.1, $[\tilde{R}_\mu, \tilde{R}_\nu]^* = \tilde{R}_\mu\tilde{T}\tilde{R}_\nu - \tilde{R}_\nu\tilde{T}\tilde{R}_\mu$. It is then easy to see that the operators $\tilde{R}_\mu$ are generally noncommuting for Hadronic Mechanics, in the sense that, in general, $[\tilde{R}_\mu, \tilde{R}_\nu]^* \neq 0$. To put it in different terms, the isotopic product has no meaning in Atomic Mechanics, in that the conventional product must be used for the computation of magnitudes, eigenvalues, etc. By the same token, the conventional product has no meaning for Hadronic Mechanics, and the isotopic one must be used unless one desires the atomic particularization.

[5] An empirical rule for reaching hadronic equations (6.1.50) (as well as several other hadronic aspects) is the following; it is based on the identification of the generalization occurring in the transition from the canonical to the Pfaffian actions. As is now familiar, the rule is characterized by the replacement of the Hamiltonian quantities r^k, p_k, and H with the corresponding Birkhoffian ones a^μ, R_μ, and B. At the operator level, this is given by the replacement of the atomic operators $\tilde{\mathbf{r}}^k$, $\tilde{\mathbf{p}}_k = (1/i)(\partial/\partial r^k)$, and $\tilde{H}(t, \tilde{\mathbf{r}}, \tilde{\mathbf{p}})$ with the hadronic ones $\tilde{a}^\mu$, $\tilde{R}_\mu = (1/i)(\partial/\partial a^\mu)$, and $\tilde{B}(t, \tilde{a}(\tilde{R}))$. Several atomic properties can then be readily generalized into a hadronic form via this simple rule, once properly implemented (e.g., by keeping in mind that, while the atomic operators $\tilde{r}^k$ commute, this is not the case for the hadronic ones a^μ—see footnote 4).

obtained via quantization of (6.1.15). This reformation is identified in Problem 6.3, and its study is left here to the interested researcher.

We therefore have the following implication for conventional potential forces. The structure of the contemporary formulation of Atomic Mechanics can be expressed via the dynamic equations of Figure 1 and their interrelations with the understanding that several additional approaches exist, e.g., that of path-integral type, Lagrange type, etc. Another understanding is that the achievement of consistent quantization-correspondence processes is still open at this moment and that the equivalence of the two representations of the figure has been proved only in very special cases.

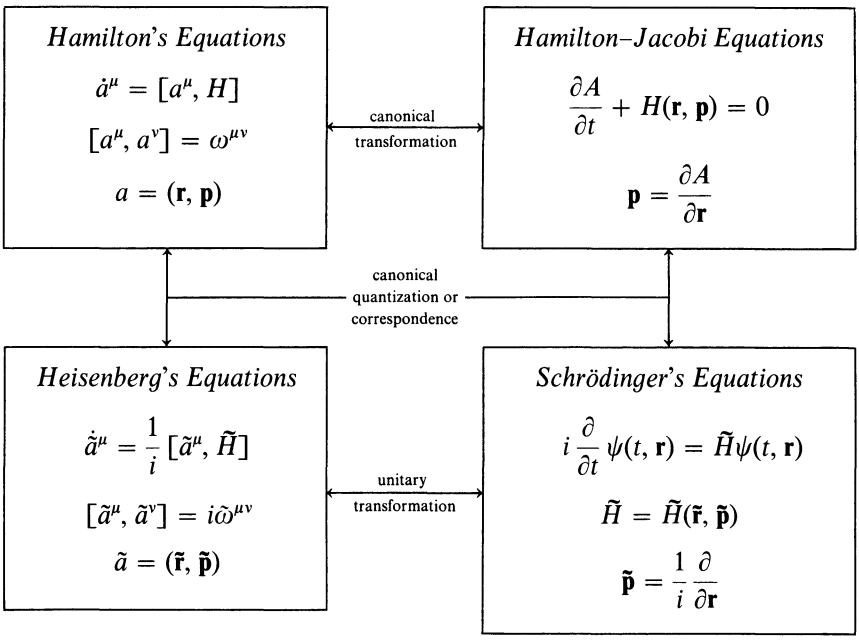


Figure 1

When the function H represents a conservative system $H = T(\mathbf{p}) + V(\mathbf{r})$ and all quantities have a direct physical meaning, a set of deeply interrelated and mutually compatible atomic laws emerges as valid, in full agreement with experimental data on electromagnetic interactions. These laws can be depicted according to the scheme in Figure 2. Galilei's relativity is considered of fundamental character not only for the impact of any relativity in the physical description of nature, but also because, out of all the principles of Figure 2, it is the only setting which persists at the Newtonian limit.

By inspecting the formulations of Figure 1 the following aspect soon emerges. Hamilton and Hamilton–Jacobi formulations are defined in the cotangent bundle T^*M with local coordinates $\mathbf{r}$ and $\mathbf{p}$. Heisenberg's

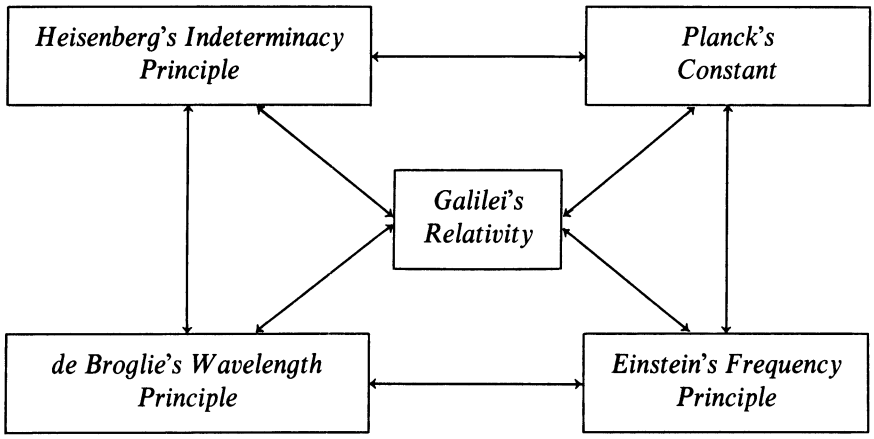


Figure 2

formulations are defined in terms of (Hermitian) operators which can be considered as polynomial expressions in $\tilde{\mathbf{r}}$ and $\tilde{\mathbf{p}}$ while the states are corresponding elements of the underlying Hilbert space. In the transition to the Schrödinger representation, the situation is somewhat altered, inasmuch

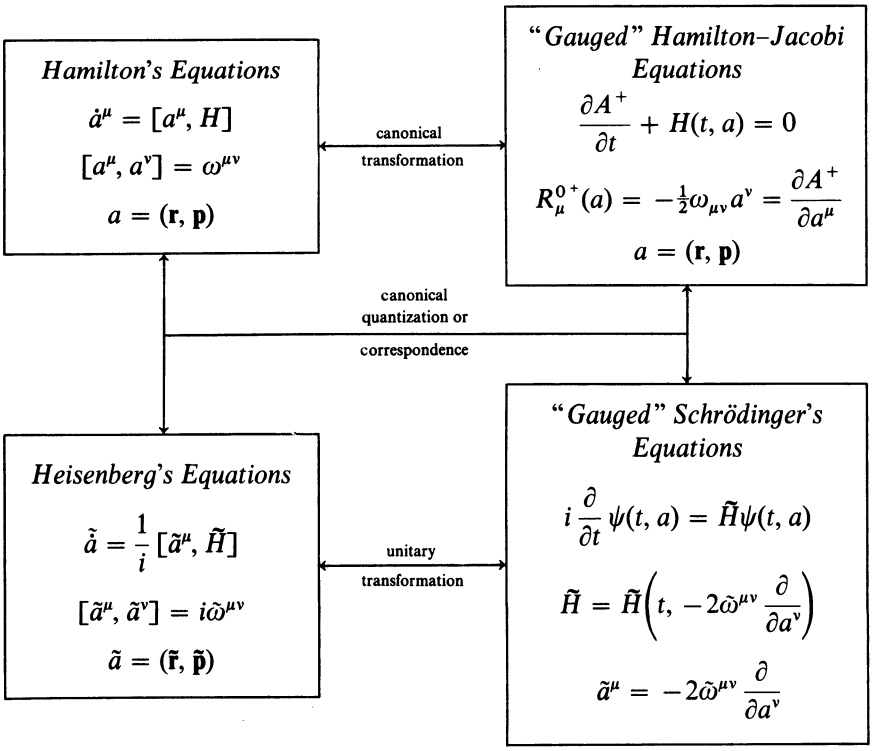


Figure 3

as the states (wave functions) are now dependent only on **r** (and time). The consistency of the formulation is out of the question, as indicated earlier, and at any rate is permitted by the fact that the second set of Hamilton–Jacobi equations implies only the **p** variable (see property (6.1.12)).

However, the form of the Hamilton–Jacobi equations used historically in the construction of Schrödinger's mechanics is *by far nonunique*, and several equivalent forms are possible, as indicated in this section. This situation opens the problem, indicated earlier, which we now reformulate in Figure 3. As one can see, Hamilton's and Heisenberg's equations are left unchanged, and only the Hamilton–Jacobi and Schrödinger equations are "gauged" in the Birkhoffian sense. Needless to say, all the basic physical laws, principles, and relativities of Figure 2 are expected to preserve their validity under the "gauged" reformulations of Figure 3.

When passing from the atomic–electromagnetic setting to that of strong interactions, the forces may become more complex than those representable by the simplistic Hamiltonian $H = T + V$, because of the mutual penetration of hadrons one within the space occupied by others. A generalization of Atomic Mechanics onto Hadronic Mechanics is then conceivable.

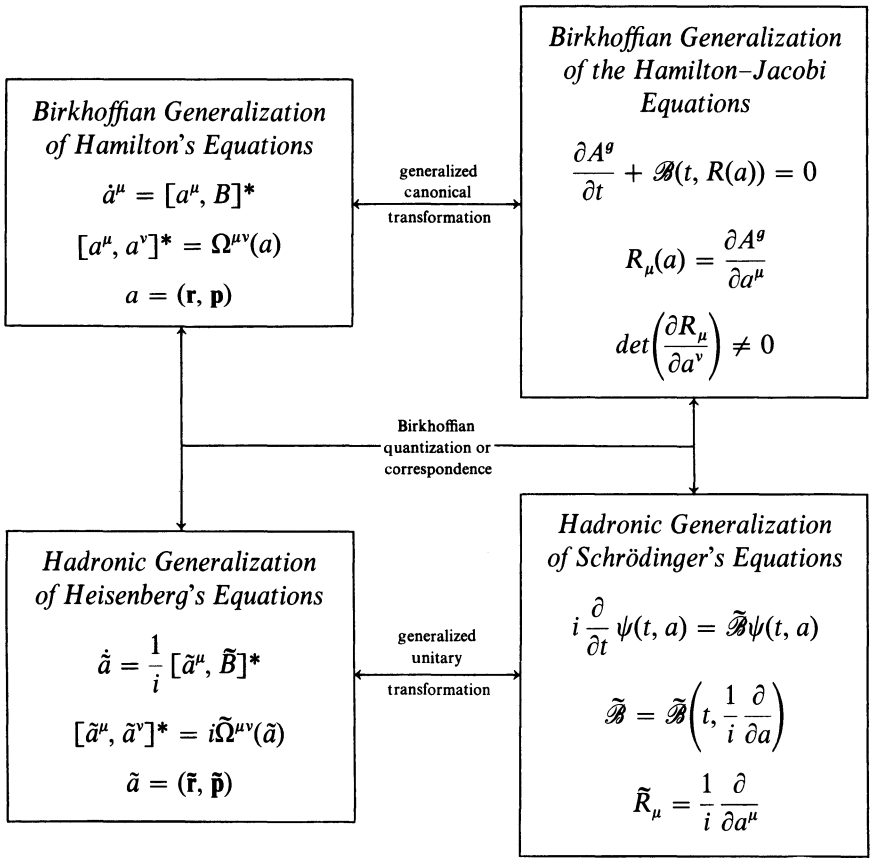


Figure 4

In this and in the preceding section, we have indicated only two aspects of the current efforts to construct the Hadronic Mechanics, those of Birkhoffian/Lie-isotopic type, with the understanding that additional efforts (e.g., of Birkhoffian-admissible/Lie-admissible type) are under way. The efforts considered can be summarized as in Figure 4, where the *nonunitary* (generalized unitary) transformation interconnecting the formulations has been studied by Myung and Santilli (*loc. cit.*) via the isotopic generalization of the Hilbert space and the operations defined on it (including unitarity).

Despite their tentative character, a number of aspects related to the hadronic formulations has emerged quite clearly. In particular the mathematical structure turns out to be based on the isotopic generalization $AB \to A * B = ATB$ of the envelope of Atomic Mechanics (Chart 5.1).

An aspect which may appeal to researchers interested in the pursuit of novel physical knowledge is that the generalization $AB \to A * B$ inevitably implies the possible existence of a hadronic generalization of *all* physical laws, principles, and relativities of Atomic Mechanics, which we can schematically depict as in Figure 5.

This occurrence is evident from a mere inspection of the hadronic wave-packets (6.1.46) on a comparative basis with the atomic ones (6.1.44). Its ultimate roots are, predictably, of *Newtonian* character and can be identified with the fact that the systems of our environment, when restricted to the frame of the observer, break Galilei's relativity according to one or the other of the mechanisms classified in Chart A.12.

We reach in this way one of the most important objectives of these volumes: the attempt to construct a generalization of Galilei's relativity in Newtonian Mechanics which is directly universal, that is, applicable to all systems of the

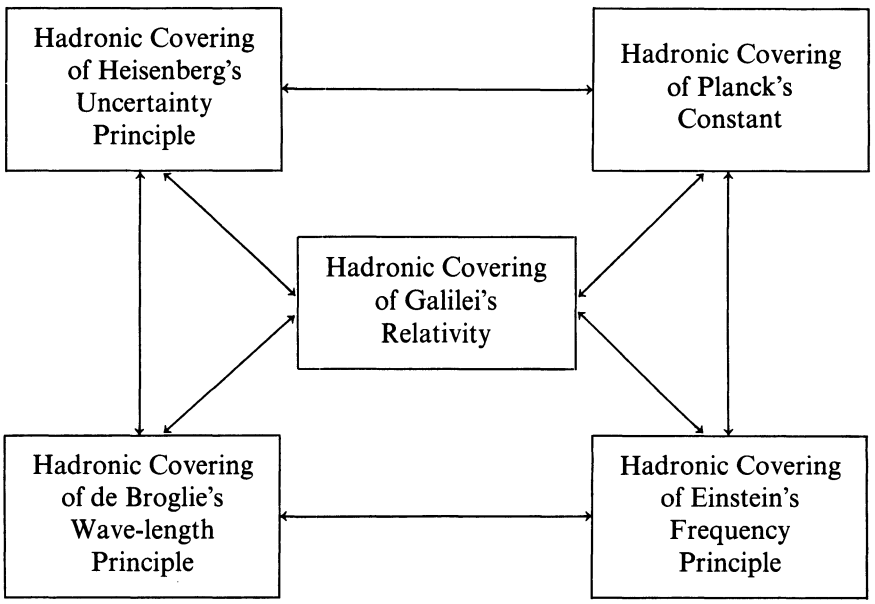


Figure 5

class admitted (local, non-Hamiltonian, analytic, and regular) in the frame of the observer. The hope is that the generalized classical relativity may be valuable in the study of its hadronic image, as well as of all other hadronic generalizations of Figure 5.

The construction of the generalized relativity will be conducted as follows. In the next section we identify the Hamiltonian reduction of Birkhoff's equations via the use of the transformation theory. The reduction is clearly useful to identify an hypothetical frame in which non-Hamiltonian and Galilei-noninvariant systems acquire a Hamiltonian and Galilei-invariant form. The use of the inverse transition studied in Chapter 5 will then permit the identification of the structure of the relativity which is applicable in the frame of the observer. This latter task will be conducted in Section 6.3. By keeping in mind that detailed treatments of quantum mechanical aspects are beyond the scope of this volume, the problem of the expected operator image of the generalized relativity will be referred to the existing literature, except a few incidental remarks.

6.2 Indirect Universality of Hamilton's Equations

In Section 5.3 we showed that noncanonical transformations map Hamilton's equations into Birkhoff's equations. In this section we show that, under certain topological conditions, Birkhoff's equations can always be reduced to the Hamiltonian form via the transformations of Darboux's theorem of the contact geometry (Chart 5.5). The direct universality of Birkhoff's equations (Section 4.5) therefore permits the establishing of the indirect universality of Hamilton's equations. These results can be expressed via the formulation and proof of the following theorem.

Theorem 6.2.1 (The Theorem of Indirect Universality of Hamilton's Equations). *All analytic and regular systems of ordinary differential equations of first- (or higher) order admit, in a star-shaped neighborhood of a regular point of the variables, an indirect Hamiltonian representation.*

GEOMETRIC PROOF. In Section 4.1 and Chart 4.3, we showed that systems of differential equations of second (or higher) order can be reduced to an equivalent, regular first-order form. Theorem 4.5.1 establishes that all first-order systems of the type considered admit, in a star-shaped neighborhood of a regular point of their variables, a representation via Birkhoff's equations

$$\hat{\Omega}_{\mu\nu} d\hat{a}^{\nu} = 0, \qquad \mu = 0, 1, 2, \ldots, 2n, \tag{6.2.1a}$$

$$\{\hat{a}^{\mu}\} = \{t, a^{\mu}\} = \{t, q^{k}, p_{k}\}, \tag{6.2.1b}$$

$$\hat{\Omega}_{0\nu} = \frac{\partial B}{\partial a^{\nu}} + \frac{\partial R_{\nu}}{\partial t} = -\hat{\Omega}_{\nu 0}, \qquad \nu = 1, 2, \ldots, 2n, \tag{6.2.1c}$$

$$\hat{\Omega}_{\mu\nu} = \frac{\partial R_{\nu}}{\partial a^{\mu}} - \frac{\partial R_{\mu}}{\partial a^{\nu}} = \Omega_{\mu\nu}, \qquad \mu, \nu = 1, 2, \ldots, 2n, \tag{6.2.1d}$$

$$B = B(\hat{a}) = B(t, a), \qquad R_{\mu} = R_{\mu}(\hat{a}) = R_{\mu}(t, a), \tag{6.2.1e}$$

where $\hat{\Omega}_{\mu\nu}$ characterizes the closed two-form of maximal rank (except contact form)

$$\hat{\Omega}_2 = \tfrac{1}{2}\hat{\Omega}_{\mu\nu}(\hat{a})d\hat{a}^\mu \wedge d\hat{a}^\nu, \qquad \mu, \nu = 0, 1, 2, \ldots, 2n. \tag{6.2.2}$$

From Darboux's theorem for the contact geometry we know that, under the assumed smoothness and regularity conditions, an analytic and regular transformation always exists,

$$\hat{a}^\mu \to \hat{a}'^\mu = \hat{a}'^\mu(\hat{a}), \qquad \mu = 0, 1, 2, \ldots, 2n, \tag{6.2.3}$$

under which form (6.2.2) reduces to the canonical form (see Chart 5.5)

$$\hat{\omega}_2 = \tfrac{1}{2}\hat{\omega}_{\mu\nu}d\hat{a}'^\mu \wedge d\hat{a}'^\nu, \qquad \mu, \nu = 0, 1, 2, \ldots, 2n, \tag{6.2.4a}$$

$$\hat{\omega}_{0\nu} = \frac{\partial H'}{\partial a'^\nu} = -\hat{\omega}_{\nu 0}, \qquad \nu = 1, 2, \ldots, 2n, \tag{6.2.4b}$$

$$\hat{\omega}_{\mu\nu} = \omega_{\mu\nu}, \qquad \mu, \nu = 1, 2, \ldots, 2n, \tag{6.2.4c}$$

$$\hat{\omega}_{\mu\nu} = \left[\frac{\partial \hat{a}^\rho}{\partial \hat{a}'^\mu}\hat{\Omega}_{\rho\sigma}(\hat{a})\frac{\partial \hat{a}^\sigma}{\partial \hat{a}'^\nu}\right](\hat{a}'). \tag{6.2.4d}$$

This ensures the reduction of Equations (6.2.1) to the Hamiltonian form

$$\hat{\omega}_{\mu\nu}d\hat{a}'^\nu = 0, \qquad \mu = 0, 1, 2, \ldots, 2n \tag{6.2.5}$$

and completes the geometric proof of the theorem. (*Q.E.D.*)

Theorem 6.4.1 was first studied by Lie (1871) and Koenigs (1895) (see also Whittaker (1904, pp. 275–276). Subsequently, the theorem has been studied by a number of authors. See, for instance, Kerner (1964).

It may be advantageous for the applications of the Inverse Problem to have an alternative proof of Theorem 6.2.1. In this way, the interested reader is equipped with alternative approaches for attempting the explicit computation of a Hamiltonian for a given system.

PFAFFIAN PROOF. Theorem 4.5.1 establishes that the systems considered, under the conditions assumed, admit an analytic representation in terms of action functional (4.2.14) with integrand

$$\begin{aligned}\hat{R}_1 &= R_\mu(t, a)da^\mu - B(t, a)dt \\ &= P_k(t, q, p)dq^k + Q^k(t, q, p)dp_k + B(t, q, p)dt,\end{aligned} \tag{6.2.6}$$

but the *Pfaffian problem* of reducing form (6.2.6) to the canonical form

$$\begin{aligned}&\hat{R}_1^0 = p'_k dq'^k - H'dt, \\ &p'_k = p'_k(t, q, p), \qquad q'^k = q'^k(t, q, p), \qquad H' = H'(t, q', p'),\end{aligned} \tag{6.2.7}$$

always admits (at least) one solution. This is sufficient to establish the existence of an indirect Hamiltonian representation for all systems considered. (*Q.E.D.*)

The literature on Pfaff's problem is quite extensive. For historical as well as detailed accounts, the reader may consult for instance, Forsyth (1890) and

Goursat (1922). A number of approaches have been studied for the solution of Pfaff's problem. For the reader's convenience, we have outlined a method originally due to Clebsh (see Forsyth, *loc. cit.*, pp. 210–214), and more recently reelaborated by Hill (1966)).

The solution of the Pfaff's problem consists of the identification of the invertible, contemporaneous, but explicitly time-dependent transformations

$$\begin{aligned} t &\to t' \equiv t, \\ \{a^\mu\} = \{q^k, p_k\} &\to \{a'^\mu\} = \{a'^\mu(t, a)\} \\ &= \{q'^k(t, q, p), p'_k(t, q, p)\}, \end{aligned} \tag{6.2.8}$$

under which

$$R_\mu da^\mu = P_k(t, q, p)dq^k + Q^k(t, q, p)dp_k = p'_k dq'^k. \tag{6.2.9}$$

This implies that

$$R_\mu = p'_k \frac{\partial q'^k}{\partial a^\mu}. \tag{6.2.10}$$

As a result, Birkhoff's tensor $\Omega_{\mu\nu}$ can be interpreted as representing the Lagrange brackets

$$\begin{aligned} \Omega_{\mu\nu}(t, a) &= \frac{\partial p'_k}{\partial a^\mu}\frac{\partial q'^k}{\partial a^\nu} - \frac{\partial q'^k}{\partial a^\mu}\frac{\partial p'_k}{\partial a^\nu} \\ &= (\{a^\mu, a^\nu\}_{(q', p')})(t, a). \end{aligned} \tag{6.2.11}$$

It then follows that the contravariant form

$$(\Omega^{\mu\nu}) = (\Omega_{\mu\nu})^{-1} \tag{6.2.12}$$

yields the conventional Poisson brackets

$$\Omega^{\mu\nu}(t, a) = \frac{\partial a^\mu}{\partial q'^k}\frac{\partial a^\nu}{\partial p'_k} - \frac{\partial a^\mu}{\partial p'_k}\frac{\partial a^\nu}{\partial q'_k} = ([a^\mu, a^\nu]_{(q', p')})(a). \tag{6.2.13}$$

The Poisson brackets between any two functions in a–space, say, $C(a)$ and $D(a)$, can be reinterpreted as follows

$$([C, D]_{(a')})(a) = \frac{\partial C}{\partial a^\mu}\Omega^{\mu\nu}(t, a)\frac{\partial D}{\partial a^\nu}. \tag{6.2.14}$$

The necessary and sufficient conditions for the solution of Pfaff's problem can be obtained via the use of Equations (6.2.10) and (6.2.14) and can be

written (for brevity, we refer the interested reader to the quoted references for the rather lengthy proof):

$$\Omega^{\mu\nu} R_\nu \frac{\partial q'^k}{\partial a^\mu} = 0, \tag{6.2.15a}$$

$$\Omega^{\mu\nu} \frac{\partial q'^i}{\partial a^\nu} \frac{\partial q'^k}{\partial a^\mu} = 0, \tag{6.2.15b}$$

$$\Omega^{\mu\nu} R_\nu \frac{\partial p'_k}{\partial a^\mu} = p'_k, \tag{6.2.15c}$$

$$\Omega^{\mu\nu} \frac{\partial p'_i}{\partial a^\nu} \frac{\partial p'_k}{\partial a^\mu} = 0, \tag{6.2.15d}$$

$$\Omega^{\mu\nu} \frac{\partial q'^i}{\partial a^\mu} \frac{\partial p'_j}{\partial a^\nu} = \delta^i_j. \tag{6.2.15e}$$

The integration of these equations yields the desired solution. Specifically, the integration of Equations (6.2.15a) and (6.2.15b) yields the functions $q'^k(t, q, p)$, while the integration of Equations (6.2.15c), (6.2.15d), and (6.2.15e) yields the functions $p'_k(t, q, p)$ under which the general symplectic tensor $\Omega_{\mu\nu}$ reduces to the fundamental form $\omega_{\mu\nu}$, i.e.,

$$\Omega_{\mu\nu}(t, a) \to \Omega'_{\mu\nu} = \frac{\partial a^\rho}{\partial a'^\mu} \Omega_{\mu\nu}(t, a(t, a')) \frac{\partial a^\sigma}{\partial a'^\nu} = \omega_{\mu\nu} \tag{6.2.16}$$

or, more explicitly,

$$\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu} \to \frac{\partial a^\rho}{\partial a'^\mu} \frac{\overline{\partial R_\sigma}}{\partial a^\rho} \frac{\partial a^\sigma}{\partial a'^\nu} - \frac{\partial a^\rho}{\partial a'^\mu} \frac{\overline{\partial R_\rho}}{\partial a^\sigma} \frac{\partial a^\sigma}{\partial a'^\nu}$$

$$= \frac{\partial R^{0'}_\nu}{\partial a'^\mu} - \frac{\partial R^{0'}_\mu}{\partial a'^\nu} = \omega_{\mu\nu}. \tag{6.2.17}$$

A rather crucial aspect of the Pfaffian problem is the proof that Equations (6.2.15) always admit solutions under the conditions considered. Regrettably, this historical proof is rather lengthy and involved. We shall therefore omit it here and content ourselves with the geometrical proof given above.

The reduction of Birkhoff's equations in the (t, a) variables to Hamilton's equations in the (t, a') variables is now completed via the rule

$$\left[\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \left(\frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t}\right)\right]_{\text{SA}} = \left[\frac{\partial a'^\rho}{\partial a^\mu}\left(\omega_{\rho\sigma}\dot{a}'^\sigma - \frac{\partial H'}{\partial a'^\rho}\right)_{\text{SA}}\right]_{\text{NSA}} = 0, \tag{6.2.18a}$$

$$H' = H'(t, a') = B(t, a(t, a')) - \left(\frac{\partial a^\alpha}{\partial t} R_\alpha\right)(t, a'), \tag{6.2.18b}$$

which is the desired inverse reduction of transformations of type (5.3.22). Notice that the Birkhoffian *does not* transform into the Hamiltonian according to a scalar rule, but transforms instead according to rule (6.2.18b), due to the explicit time-dependence of the transformations.

Almost needless to say, the geometric and the Pfaffian proofs are ultimately equivalent. The former deals with the reduction of exact, contact, two-forms to the canonical form, while the latter deals with the same reduction, but for primitive one-forms. Nevertheless, these proofs are based on different methods, and as such, they can be of assistance for practical applications.

A difference exists in the proofs given above that should be indicated. The transformations via the use of Darboux's theorem for contact geometry, Equations (6.2.3), imply that the "time" of Hamilton's equations depends on the time, coordinates, and velocities of the original Newtonian system as experimentally detected. Indeed, these transformations can be explicitly written

$$t \to t' = t'(t, q, p) = t'(t, q, p(t, q, \dot{q})), \tag{6.2.19a}$$

$$q^k \to q'^k = q'^k(t, q, p) = q'^k(t, q, p(t, q, \dot{q})), \tag{6.2.19b}$$

$$p_k \to p'_k = p'_k(t, q, p) = p'_k(t, q, p(t, q, \dot{q})). \tag{6.2.19c}$$

On the contrary, the transformations via the Pfaffian problem, Equations (6.2.8), are contemporaneous, even though explicitly time-dependent (for the non-autonomous case). Clearly, this latter approach may be preferred over the former in practical cases.

When the Pfaffian proof is reinspected within the context of the symplectic (rather than contact) geometry, it emerges dealing with the canonical reduction of simplectic forms *with an explicit time dependence*, i.e.,

$$\Omega_2 = \tfrac{1}{2}\Omega_{\mu\nu}(t, a)da^\mu \wedge da^\nu \to \omega_2 = \tfrac{1}{2}\omega_{\mu\nu}da'^\mu \wedge da'^\nu \tag{6.2.20}$$

which, strictly speaking, should belong to the contact geometry under proper prolongation into $2n + 1$ dimension.

The fact that reduction (6.2.20) can be properly treated within the context of the symplectic geometry is established by the parametric approach to symplectic forms of Chart 4.6. Consider a representation of the systems admitted according to Theorem 6.2.1, and select the region $\tilde{R}$ of definition of Birkhoff's equations to be smoothly deformable to a curve monotonically increasing in time. Under transformations (6.2.8), Birkhoff's equations transform according to Equations (5.3.23), i.e.,

$$R_\mu(t, a) \to R'_\mu(t, a') = \left(\frac{\partial a^\alpha}{\partial a'^\mu} R_\alpha\right)(t, a'), \tag{6.2.21a}$$

$$B(t, a) \to H'(t, a') = B(t, a(t, a)) - \left(\frac{\partial a^\alpha}{\partial t} R_\alpha\right)(t, a'). \tag{6.2.21b}$$

When the transformations verify all the needed smoothness requirements, the image region $\tilde{R}'$ preserves the topological character of $\tilde{R}$. Then transformations (6.2.8) always exist under which Birkhoff's equations acquire the Hamiltonian form, that is, such that

$$R'_k = \left(\frac{\partial a^\alpha}{\partial a'^k} R_\alpha\right)(t, a') = p'_k, \tag{6.2.22a}$$

$$R'_{n+k} = \left(\frac{\partial a^\alpha}{\partial a'^{n+k}} R_\alpha\right)(t, a') = 0, \tag{6.2.22b}$$

as guaranteed by the application of Darboux's theorem of the symplectic geometry to nondegenerate, closed, and parametric forms.

We conclude this section with a number of remarks. First, it may be of some significance to indicate that *Theorem* 6.2.1 *admits an infinite number of different solutions.* Indeed, the transformations which reduce a contact structure to a fundamental structure are always defined up to an infinite number of possible identity isotopic (that is, canonical) transformations of the fundamental tensors $\omega^{\mu\nu}$ or $\omega_{\mu\nu}$. In conclusion and as anticipated earlier in our analysis, *canonical transformations constitute a sort of "degree of freedom" of the Theorem of Indirect Universality of the Inverse Problem,*[6] although, they have no constructive role.

The condition of analyticity of Theorem 6.2.1 is due largely to the methods we have selected for the proof of Theorem 4.5.1 on Birkhoffian representations, while the condition of infinite differentiability is sufficient for Darboux's theorem. Thus, in principle, Theorem 6.2.1 could be reformulated and proved for systems of class $\mathscr{C}^\infty$ only.

It should be recalled that the point of the local variables of Theorem 6.2.1 must be regular in the sense of Chart A.1 and must not be a possible zero of the determinant of the matrix $(\Omega_{\mu\nu})$. Also, a neighborhood of such a point must be star-shaped (or topologically equivalent) to ensure the applicability of the converse of the Poincaré lemma.

A comparison of the *nonlinearity* inherent in the geometric and Pfaffian approaches is instructive. The geometric approach demands the solution of non-linear systems of partial differential equations (6.2.4d) or (6.2.17). In the transition to the Pfaffian approach, such non-linearity generally persists. Indeed, Clebsh's Equations (6.2.15) are also nonlinear, though of a different type.

Notice that the systems of partial differential equations for the Hamiltonian reduction *are not*, in general, of the Cauchy–Kovalevsky type, nor can they

[6] Note that, on purely formal grounds, these degrees of freedom can be used for the solution of the equations of motion. In fact, one can attempt to identify a Darboux's transformation plus a canonical transformation, under which the new Hamiltonian is identically null, by therefore implying that $a' = c_0 =$ constant. The use of the inverse transformation $a' \to a(a')$ would then produce the solution of the system.

be readily transformed to a Cauchy–Kovalevsky form. The lack of consequent applicability of the contemporary existence theory for partial differential equations confirms the rather crucial methodological function of the symplectic and contact geometries.

As an historical note, the reader may be interested in knowing that Darboux's theorem is sometimes called *Pauli's theorem* in the literature (see Jost (1964)). As a matter of fact, a study of the original paper by Pauli (1953) (as well as its elaboration by Jost (*loc. cit.*)) is recommended, because it is directly relevant for the Inverse Problem, although understandably not intended for such a purpose. In essence, "Pauli's theorem" can be interpreted as a reformulation of Darboux's theorem, and this is sufficient to indicate the relevance of Pauli's studies for the Inverse Problem.

Needless to say, Theorem 6.2.1 is an *existence theorem.* As such, it guarantees that a Hamiltonian exists under the conditions indicated, but it does not guarantee that such a Hamiltonian can be computed in the needed closed form. In fact, the technical difficulties related to the Hamiltonian reduction of Birkhoff's equations can be rather considerable, as we shall illustrate in the examples at the end of this chapter.

Despite this restrictive character, Theorem 6.2.1 has an important meaning for mechanics. In fact, the theorem establishes that, on formal grounds, *all* possible Newtonian systems verifying the conditions of the theorem can be treated via the canonical version of analytic, algebraic, and geometric formulations.

On more explicit grounds, the systems represented by Theorem 6.2.1 are of the following three classes: (a) essentially self-adjoint; (b) non-essentially non-self-adjoint; and (c) essentially non-self-adjoint. For class (a), the theorem is actually redundant because the systems admit a conventional Hamiltonian representation (although the use of the techniques of the theorem may be equally useful for the problem of symmetries and first integrals). For class (b), the theorem is applicable, although only in the simplified version without the intermediary use of Birkhoff's equations. Clearly, for the most general possible class of systems, those of class (c), the theorem is applicable in its most general possible formulation, including the necessary intermediate use of Birkhoff's equations.

As indicated in the Preface, achieving the primary research objectives by no means allows the relaxation of the critical examination of the results. Part of the next section will therefore be devoted to the critical examination of the physical implications of the theorem.

We conclude this section by pointing out that Theorem 6.2.1 suggests rather forcefully the Lie-isotopic structure of the hadronic generalization of Atomic Mechanics (Section 6.1). The property was identified by Santilli (1978d, 1979b, and 1982b). A simple presentation of the argument is the following.

The objective is to show that Heisenberg-type treatments of contact/non-Hamiltonian interactions among extended particles in conditions of mutual

penetration (as in the strong interactions) do not admit the conventional associative enveloping algebra of operators $\tilde{A}, \tilde{B}, \ldots$, with product $\tilde{A}\tilde{B}$, but rather its isotopic generalization, e.g., of the type $\tilde{A} * \tilde{B} = \tilde{A}\tilde{T}\tilde{B}$, with $\tilde{T}$ a fixed nonsingular operator satisfying all needed conditions (Hermiticity, positivity, etc.).

For this purpose, consider a classical, essentially non-self-adjoint first-order form

$$\begin{pmatrix} \dot{\mathbf{r}} - \mathbf{p} \\ \dot{\mathbf{p}} - \mathbf{f}^{SA} - \mathbf{F}^{NSA} \end{pmatrix} = 0, \qquad m = 1, \tag{6.2.23}$$

which, as is now familiar, is non-Hamiltonian by assumption. Theorem 6.2.1 establishes that, under the assumed topological conditions, the systems can be transformed into an equivalent form in new variables $t' = t$, $\mathbf{r}'$, and $\mathbf{p}'$, which not only is Hamiltonia, but is actually "free," e.g., it admits the trivial Hamiltonian $H' = \frac{1}{2}\mathbf{p}'^2$, i.e.,

$$\begin{pmatrix} \dot{\mathbf{r}}' - \mathbf{p}' \\ \dot{\mathbf{p}}' \end{pmatrix} = 0. \tag{6.2.24}$$

Now quantize this system into Heisenberg's equations

$$i\frac{d\tilde{A}'}{\partial t} = [\tilde{A}', \tilde{H}'] = \tilde{A}'\tilde{H}' - \tilde{H}'\tilde{A}' \tag{6.2.25a}$$

$$\tilde{A}' = \tilde{A}'(\tilde{\mathbf{r}}', \tilde{\mathbf{p}}'), \qquad \tilde{H}' = \tfrac{1}{2}\tilde{\mathbf{p}}'^2, \qquad \hbar = 1 \tag{6.2.25b}$$

by conventional techniques. However, variables $\mathbf{r}'$ and $\mathbf{p}'$ are not realizable via experiments (because they are nonlinear functions of the physical coordinates r^k and linear momenta p_k actually used by the experimenter). Thus, in order to achieve an operator description in the frame of the observer, one must identify the inverse transform from system (6.2.24) to (5.2.23),

$$\mathbf{r}' \to \mathbf{r}(\mathbf{r}', \mathbf{p}'), \qquad \mathbf{p}' \to \mathbf{p}(\mathbf{r}', \mathbf{p}'), \tag{6.2.26}$$

and the corresponding operator form

$$\tilde{\mathbf{r}}' \to \tilde{\mathbf{r}}\left(\tilde{\mathbf{r}}', \frac{1}{i}\nabla_{r'}\right), \qquad \tilde{\mathbf{p}}' \to \tilde{\mathbf{p}}\left(\tilde{\mathbf{r}}', \frac{1}{i}\nabla_{r'}\right). \tag{6.2.27}$$

Since the original system is essentially non-self-adjoint, transformations (6.2.26) are necessarily noncanonical (Section 5.3). For the consistency of the theory, the operator image (6.2.27) must therefore be nonunitary.

Our objective is then achieved by noting that, under a nonunitary transformations, Heisenberg's equations (6.2.25) transform into the isotopic form (18) of Chart 5.1. To see it, suppose for simplicity, but without loss of generality, that transformations (6.2.27) are expressible via the nonunitary operator

$\exp(i\theta\tilde{Z})$, where θ is the parameter and $\tilde{Z}$ is a non-Hermitian operator ($\tilde{Z}^\dagger \neq Z$). Then we have the formal rule

$$i\dot{\tilde{A}} = i\frac{d\tilde{A}}{dt} = [A, B]^* = ATB - BTA, \tag{6.2.28a}$$

$$A = e^{i\theta Z}A'e^{-i\theta Z^\dagger}, \qquad B = e^{i\theta Z}H'e^{-i\theta Z^\dagger} \tag{6.2.28b}$$

$$[A, B]^* = e^{i\theta Z}[A', H']e^{-i\theta Z^\dagger} \tag{6.2.28c}$$

$$T = e^{i\theta Z^\dagger}e^{-i\theta Z} = T^\dagger. \tag{6.2.28d}$$

Equations (6.2.28) confirm the existence of an operator realization of the Lie-isotopic theory. In addition, they confirm the apparent, rather general, physical meaning of isotopy at the various levels of mechanics (Newtonian, statistical, particle, etc.). We are referring here to the capability of the isotopic mapping of the enveloping algebra or of the Lie product to represent contact/non-Hamiltonian interactions.

Intriguingly, the hadronic generalization (6.1.50) of Schrödinger's equations is equivalent to the isotopic generalization (6.2.28a) of Heisenberg's equations (Myung and Santilli, 1982a).

6.3 Generalization of Galilei's Relativity

In this section we review the canonical foundations of the contemporary formulation of Galilei's relativity. We then show that such relativity is applicable to a rather restricted class of systems. Finally, we identify the rudiments of a possible generalization of Galilei's relativity of Lie-isotopic and symplectic-isotopic type which is applicable to local Newtonian systems with potential and non-potential forces. A good knowledge of the Lagrangian treatment of symmetries and first integrals (e.g., as reviewed in Charts A.6 through A.12) is assumed.

Definition 6.3.1. A first-order system of ordinary differential equations[7]

$$(\dot{\hat{a}}^\mu) = \begin{pmatrix} \dfrac{dt}{dt} \\ \\ \dfrac{da^\mu}{dt} \end{pmatrix} = \begin{pmatrix} 1 \\ \dot{\mathbf{r}}^k \\ \dot{\mathbf{p}}_k \end{pmatrix} = (\hat{\Xi}^\mu) = \begin{pmatrix} 1 \\ \Xi^\mu \end{pmatrix} = \begin{pmatrix} 1 \\ \mathbf{p}_k/m_k \\ \mathbf{F}_k(t, \mathbf{r}, \mathbf{p}) \end{pmatrix} \tag{6.3.1}$$

in the vector-field form on $\mathbb{R} \times T^*M$

$$\hat{\Xi}(\hat{a}) = \hat{\Xi}^\mu(\hat{a})\frac{\partial}{\partial \hat{a}^\mu} = \Xi^\mu(t, a)\frac{\partial}{\partial a^\mu} + \frac{\partial}{\partial t} \tag{6.3.2}$$

[7] We continue to use the notation whereby the index μ runs from 1 to $2n$ for a^μ and from 0 to $2n$ for $\hat{a}^\mu$. The same notation is used for other quantities, such as R_μ and $\hat{R}_\mu$. We pass liberally from one notation to the other, depending on whether or not the separation of the time dependence is important.

is said to possess a *symmetry* under smoothness preserving and regular transformations

$$\hat{a}^\mu \to \hat{a}'^\mu(\hat{a}) \tag{6.3.3}$$

when it is form-invariant according to the rule

$$\hat{\Xi}^\mu(\hat{a}) \frac{\partial}{\partial \hat{a}^\mu} = \hat{\Xi}^\mu(\hat{a}(\hat{a}')) \frac{\partial \hat{a}'^\alpha}{\partial \hat{a}^\mu} \frac{\partial}{\partial \hat{a}'^\alpha}$$

$$\stackrel{\text{def}}{=} \hat{\Xi}'^\alpha(\hat{a}') \frac{\partial}{\partial \hat{a}'^\alpha} \equiv \hat{\Xi}^\alpha(\hat{a}') \frac{\partial}{\partial \hat{a}'^\alpha}. \tag{6.3.4}$$

The symmetries of a vector field can be classified into *manifest*, *nonmanifest*, *discrete*, *connected*, *finite*, *infinitesimal*, *contemporaneous*, *noncontemporaneous*, etc., in essentially the same way as that of the symmetries of second-order systems (Chart A.6).

Definition 6.3.2. A function $I(\hat{a}) = I(t, a) = I(t, \mathbf{r}, \mathbf{p})$ is called a *first integral* of vector field (6.3.2) when its total time derivative along the direction of the vector field is identically null, i.e.,

$$\frac{dI}{dt} = \frac{\partial I}{\partial a^\mu} \dot{a}^\mu + \frac{\partial I}{\partial t} = \frac{\partial I}{\partial a^\mu} \Xi^\mu + \frac{\partial I}{\partial t} = \frac{\partial I}{\partial \hat{a}^\mu} \hat{\Xi}^\mu(\hat{a}) \equiv 0. \tag{6.3.5}$$

A first integral is called a *conservation law* when the quantity $I(\hat{a})$ directly represents a physical quantity, such as the total energy, the total linear momentum, etc.

Several differences between first integrals and conservation laws were presented in Chart A.8 for the second-order case, and they are readily adapted to the first-order one.

Theorem 6.3.1 (Invariance Property of First Integrals). *A first Integral $I(\hat{a})$ of a vector field $\hat{\Xi}$ is invariant under infinitesimal transformations with $\hat{\Xi}^\mu$ as generators and $\hat{\delta}t$ as parameter.*

PROOF.

$$\hat{\delta}I = I(\hat{a}^\mu + \hat{\delta}t\hat{\Xi}^\mu) - I(\hat{a}^\mu) = \frac{\partial I}{\partial \hat{a}^\mu} \hat{\Xi}^\mu \hat{\delta}t \equiv 0. \tag{6.3.6}$$

(*Q.E.D.*)

This property was identified by Sophus Lie. For historical notes as well as a presentation of the topic and related aspects, the reader may consult Hagihara (1970, pp. 291–293). To restate the property in different terms, we can say that the infinitesimal transformations

$$t \to t' = t + \hat{\delta}t, \qquad a^\mu \to a'^\mu = a^\mu + \hat{\delta}t\Xi^\mu \tag{6.3.7}$$

constitute a symmetry of the vector field on account of the properties

$$I(t, a) = I(t(t'), a(t', a')) = I'(t', a') \equiv I(t', a') \tag{6.3.8}$$

which are ensured by rule (6.3.5).

Under the assumed topological conditions, it is also possible to prove the inverse property, that is, if a function $I(t, a)$ is form-invariant under transformations (6.3.7), then it is a first-integral with respect to $\hat{\Xi}$.

Definition 6.3.3. A set of functions $V_k(t, a)$, $k = 1, 2, \ldots, m \leq 2n$ are called *invariant relations* with respect to the vector field $\hat{\Xi}$ when the identities

$$V_k(t_0, a_0(t_0)) = 0, \tag{6.3.9}$$

hold along the solution a_0 of the system at one given value of time t_0 and can be satisfied for all values of time.

The difference between first integrals and invariant relations is instructive, as well as important for the objectives of this section. In essence, for the case of first integrals, the relation $\dot{I} \equiv 0$ holds identically; that is, it holds for all possible paths a which are not necessarily solutions of the system. An invariant relation, on the other hand, holds only along the solution of the system. As a result, the quantity I is not necessarily an invariant relation i.e., $\dot{I} \neq V$. The $(2n + 1 - m)$-dimensional hypersurface on $\mathbb{R} \times T^*M$ characterized by Equations (6.3.9) is called the *hypersurface of the invariant relations*. For additional properties, one may consult, for instance, Hagihara (*loc. cit.*).

A set of physical quantities $X_k(t, a), k = 1, 2, \ldots,$ can therefore be conserved in more than one way. First, the total time derivatives can be identically null along the direction of the vector field, i.e.,

$$\dot{X}_k(\hat{a}) = \frac{\partial X_k}{\partial \hat{a}^\mu} \hat{\Xi}^\mu(\hat{a}) \equiv 0, \tag{6.3.10}$$

in which case they are first integrals. Secondly, a regular matrix of functions $\lambda_k^i(t, a)$ may also exist such that

$$\dot{X}_k(\hat{a}_0) = \left(\frac{\partial X_k}{\partial \hat{a}^\mu} \hat{\Xi}^\mu\right)(\hat{a}_0) = (\lambda_k^i V_i)(\hat{a}_0) \triangleq 0, \tag{6.3.11}$$

in which case the X's are conserved by virtue of the invariant relations. In the former case we say that quantities X_k are *strongly conserved*, while in the latter case we shall say they are *weakly conserved*. Also, the *strong equality* will be denoted with the symbol $\equiv$ used in Equations (6.3.10), while the *weak equality* will be denoted with the symbol $\triangleq$ used in Equations (6.3.11).

A simple example is given by the total energy $E_{\text{tot}}(a)$ of a conservative system. When the equations considered have no initial conditions, the energy can assume an arbitrary constant value C; $E_{\text{tot}}(a)$ is a first integral; and we can write the strong equality $E_{\text{tot}}(a) \equiv C$. However, if we assume a given

fixed value C_0 of C, then the relation $E_{\text{tot}}(a) = C_0$ can only hold weakly, that is, $V = E_{\text{tot}}(a) - C_0$ is an invariant relation.[8] In fact, the assumption of the value C_0 of the energy at one given value of time causes the system to preserve the same energy at all subsequent times.

Note that invariant relations can occur for all possible vector fields and not necessarily only for conservative ones. The understanding is, however, that the physical interpretation of relations (6.3.11) becomes considerably more abstract for nonconservative systems.

We now *restrict* the vector field to be Hamiltonian in the sense of Equations (4.3.5), i.e.,

$$\Xi^{\mu}(t, a) = \omega^{\mu\nu} \frac{\partial H(t, a)}{\partial a^{\nu}} \tag{6.3.12}$$

and review the conventional definition of symmetry within the context of canonical formulations.

Definition 6.3.4.[9] A contemporaneous smoothness-preserving regular transformation.

$$t \to t' \equiv t, \qquad a^{\mu} \to a'^{\mu}(a) \tag{6.3.13}$$

is a *symmetry of Hamilton's equations* when it is, first, Lie identity isotopic (that is, canonical),

$$\omega^{\mu\nu} \to \Omega^{\mu\nu} = \frac{\partial a'^{\mu}}{\partial a^{\rho}} \omega^{\rho\sigma} \frac{\partial a'^{\nu}}{\partial a^{\sigma}} \equiv \omega^{\mu\nu} \tag{6.3.14}$$

and, in addition, leaves the Hamiltonian form-invariant, i.e.,

$$H(t, a) \to H'(t, a') = H(t, a(a')). \tag{6.3.15}$$

Consider the case when the symmetry is constituted by an r-dimensional Lie group of infinitesimal transformations G_r. The condition that these transformations are canonical demands that they have the structure (see Chart 5.4 for detail)

$$G_r\colon a'^{\mu} = a^{\mu} + w^{k} \omega^{\mu\nu} \frac{\partial X_k}{\partial a^{\nu}} \tag{6.3.16}$$

where the w's are the infinitesimal parameters and the X's are the generators of G_r.

[8] The case $E_{\text{tot}} = C_0$ is also referred to as a *particularized first integral.*

[9] The extension of the definition to the noncontemporaneous case is given later as a particularization of the more general notion of symmetry of Birkhoff's equations. *Note that the symmetries of Hamilton's equations do not recover all possible symmetries of the represented vector field.* The proof of this property is left as an instructive exercise for the interested reader (Problem 6.4).

However, the image of the Hamiltonian under transformations (6.3.16) is given by

$$H'(t, a') = H(t, a) + w^k \frac{\partial H}{\partial a^\mu} \omega^{\mu\nu} \frac{\partial X_k}{\partial a^\nu} = H + w^k[H, X_k]. \quad (6.3.17)$$

The following (well-known) important property for the autonomous case then follows. Its extension for the non-autonomous case is not considered for brevity.

Theorem 6.3.2 (Integrability Conditions for Hamiltonian Symmetries). *A necessary and sufficient condition for transformations* (6.3.16) *to be symmetries of a Hamiltonian $H(a)$ is that the conventional Poisson brackets of the Hamiltonian with all the generators $X_k(a)$ are identically null, i.e.,*

$$[H, X_k] = 0, \qquad k = 1, 2, \ldots, r. \quad (6.3.18)$$

The evident distinction between *Lie transformation groups* and *Lie symmetry groups* should be kept in mind. Also, one should remember that, if a given Lie group is a symmetry group for one given Hamiltonian, the same group is not necessarily a symmetry group for another Hamiltonian.

The use of Lie's theory, with particular reference to Lie's theorems and the universal enveloping associative algebra (reviewed in the charts of Chapter 5 for the reader's convenience) then permits the following important consequence of Theorem 6.3.2.

Corollary 6.3.2a. *The Lie algebra $\mathbf{G}_r$ of an r-dimensional Lie symmetry group G_r of a Hamiltonian H is given by the vector space (over the field $\mathbb{F}$ of real numbers) of the generators X_k on T^*M verifying conditions* (6.3.18), *equipped with the conventional Poisson brackets as the realization of the Lie product, and obeying the following closure rules expressed in terms of the structure constants C_{ij}^k (from Lie's second theorem)*

$$[X_i, X_j] = C_{ij}^k X_k. \quad (6.3.19)$$

It is understood that H can be one element of G_r. It is also understood that $\mathbf{G}_r$ can be infinite-dimensional. Nevertheless, most Lie algebras of symmetry groups relevant in physics are finite-dimensional. This is the case particularly for space-time symmetry groups such as the ten-dimensional algebra of the Galilei's group on $\mathbb{R} \times T^*E(3)$[10]

$$\mathbf{G}(3.1) = [\mathbf{SO}(3) \oplus \mathbf{T}_r(3)] \oplus [\mathbf{T}_b(3) + \mathbf{T}_t(1)] \quad (6.3.20)$$

[10] We restrict ourselves for simplicity here and in the following, to presenting the simplest possible form of Galilei's group, that without scalar extension. For a study of broader structures, see, for instance, Levy-Leblong (1971). See also Sudarshan and Mukunda (1974, Chap. 19). The reader must be aware that a Poisson brackets realization of algebra (6.3.20) exists for the case of null mass, and that the use of the scalar extension is needed to treat the case of non-null mass. For similar reasons, the subsequent exponentiation (6.3.28) must be interpreted as occurring for subgroups and conditions not demanding the scalar extension. It should be stressed that similar occurrences are expected for the isotopic generalization of Galilei's relativity.

of Galilei's group of transformations

$$G(3.1): \begin{cases} t \to t' = t + t_0, \\ \mathbf{r} \to \mathbf{r}' = R\mathbf{r} + \mathbf{v}_0 t + \mathbf{r}_0, \\ \mathbf{p} \to \mathbf{p}' = R\mathbf{p} + m\mathbf{v}_0, \end{cases} \tag{6.3.21}$$

where $\mathbf{SO}(3)$, $\mathbf{T}_r(3)$, $\mathbf{T}_b(3)$, and $\mathbf{T}_t(1)$ are the Lie algebras of the groups of rotations, translations in space, Galilei's boosts, and translations in time, respectively; and the symbols $+$ and $\oplus$ denote direct and semidirect sums, respectively.

The preceding elements are sufficient to illustrate that the notions expressed by Definiton 6.3.4, Theorem 6.3.2, and Corollary 6.3.2a are of fundamental relevance in contemporary theoretical physics. In fact, the notions are the basis of Galilei's relativity in Newtonian mechanics as well as, upon a number of technical implementations, Galilei's relativity in quantum mechanics, Einstein's special relativity in classical discrete mechanics, quantum mechanics, or quantum field theory, etc.

The following definition has been conceived to focus attention on some of these methodological foundations.

Definition 6.3.5. Consider a local, analytic, regular, unconstrained, conservative, Newtonian system of N particles in the unique, normal, first-order form expressed in the local variables of its experimental observation

$$(\dot{a}^\mu) = \begin{pmatrix} \dot{r}^{ka} \\ \dot{p}_{ka} \end{pmatrix} = (\Xi^\mu(a)) = \begin{pmatrix} p_{ka}/m_k \\ f^{\mathrm{SA}}_{ka}(\mathbf{r}) \end{pmatrix} \tag{6.3.22}$$

$$\mu = 1, 2, \ldots, 2n = 6N, \qquad k = 1, 2, \ldots, N, \qquad a = x, y, z, \qquad \mathbf{p} = m\dot{\mathbf{r}}$$

with the ten total conservation laws

$$E_{\mathrm{tot}} = T(\mathbf{p}) + V(\mathbf{r}) = X_1, \tag{6.3.23a}$$

$$\mathbf{P}_{\mathrm{tot}} = \sum_{k=1}^{N} \mathbf{p}_k = \sum_{k=1}^{N} m_k \mathbf{p}_k = \{X_2, X_3, X_4\}, \tag{6.3.23b}$$

$$\mathbf{M}_{\mathrm{tot}} = \sum_{k=1}^{N} \mathbf{r}_k \times \mathbf{p}_k = \{X_5, X_6, X_7\}, \tag{6.3.23c}$$

$$\mathbf{G}_{\mathrm{tot}} = \sum_{k=1}^{N} (m_k \mathbf{r}_k - t\mathbf{p}_k) = \{X_8, X_9, X_{10}\}. \tag{6.3.23d}$$

Then, *Galilei's relativity*[11] can be defined as a[12] form-invariant description of the closed self-adjoint character of the system, that is, as the symmetry of

[11] A number of references on Galilei's relativity have been given in Chart I.A.1, beginning with Galilei's historic work.

[12] As indicated in Chart A.12 (see also Problem A.10) Galilei's symmetry is not necessarily the sole symmetry capable of characterizing conservation laws (6.3.23) via Noether's theorem, owing to the existence of the isotopically mapped symmetries.

the equations of motion under the ten-parameter Lie transformation group $G(3.1)$ (form-invariance):

$$G(3.1): \hat{a}^{\mu} \to \hat{a}'^{\mu}(\hat{a}), \qquad \hat{a} = (t, a) \tag{6.3.24a}$$

$$\begin{aligned} \hat{\Xi}(\hat{a}) &= \hat{\Xi}^{\mu}(\hat{a}) \frac{\partial}{\partial \hat{a}^{\mu}} = \Xi^{\mu}(a) \frac{\partial}{\partial a^{\mu}} + \frac{\partial}{\partial t} \\ &= \hat{\Xi}^{\mu}(\hat{a}(\hat{a}')) \frac{\partial \hat{a}'^{\alpha}}{\partial \hat{a}^{\mu}} \frac{\partial}{\partial \hat{a}'^{\alpha}} = \hat{\Xi}'^{\alpha}(\hat{a}') \frac{\partial}{\partial a'^{\alpha}} \\ &\equiv \hat{\Xi}^{\alpha}(\hat{a}') \frac{\partial}{\partial \hat{a}'^{\alpha}} = \Xi^{\alpha}(a') \frac{\partial}{\partial a'^{\alpha}} + \frac{\partial}{\partial t'} = \hat{\Xi}(\hat{a}') \end{aligned} \tag{6.3.24b}$$

whose ten generators X_k represent the conservation laws of total quantities (6.3.23), i.e., (closed self-adjoint character):

$$\dot{X}_k(\hat{a}) = \frac{\partial X_k}{\partial \hat{a}^{\mu}} \hat{\Xi}^{\mu}(\hat{a}) \equiv 0, \qquad k = 1, 2, \ldots, 10. \tag{6.3.25}$$

The relativity is characterized by the following formulations.

I. *Analytic formulations* essentially consist of the representation of the equations of motion via the conventional Hamilton's equations

$$\left[\frac{\partial R^0_{\nu}(a)}{\partial a^{\mu}} - \frac{\partial R^0_{\mu}(a)}{\partial a^{\nu}}\right] \dot{a}^{\nu} - \frac{\partial H(a)}{\partial a^{\mu}} = 0, \tag{6.3.26a}$$

$$(\omega_{\mu\nu}) = \left(\frac{\partial R^0_{\nu}}{\partial a^{\mu}} - \frac{\partial R^0_{\mu}}{\partial a^{\nu}}\right) = \begin{pmatrix} 0_{n\times n} & -1_{n\times n} \\ 1_{n\times n} & 0_{n\times n} \end{pmatrix}, \qquad R^0 = (\mathbf{p}, \mathbf{0}) \tag{6.3.26b}$$

and related canonical formulations (canonical transformation theory; canonical perturbation theory; Hamilton–Jacobi equations; etc.).

II. *Algebraic formulations* essentially consist of the universal enveloping associative algebra $\mathscr{A}(\mathbf{G}(3.1))$ of Galilei's algebra

$$\mathscr{A}(\mathbf{G}(3.1)) = \frac{\mathscr{T}}{\mathscr{R}}, \tag{6.3.27a}$$

$$\mathscr{T} = \mathbb{F} \oplus \mathbf{G} \oplus \mathbf{G} \otimes \mathbf{G} \oplus \cdots, \tag{6.3.27b}$$

$$\mathscr{R}: [X_i, X_j] - (X_i \otimes X_j - X_j \otimes X_i), \tag{6.3.27c}$$

$$\mathbf{G}(3.1) \approx [\mathscr{A}(\mathbf{G}(3.1))]^{-}: [X_i, X_j] = C^k_{ij} X_k, \tag{6.3.27d}$$

the canonical realization of Galilei's group[13]

$$G(3.1)\colon a^{\mu} \to a'^{\mu} = \exp\left(\theta^{k}\omega^{\alpha\beta}\frac{\partial X_{k}}{\partial a^{\beta}}\frac{\partial}{\partial a^{\alpha}}\right)a^{\mu}, \tag{6.3.28a}$$

$$\{\theta^{k}\} = \{t_0; \mathbf{r}_0; \alpha_0, \beta_0, \gamma_0; \mathbf{v}_0\}, \tag{6.3.28b}$$

and related Lie's theory (representation theory, etc.).

III. *Geometric formulations* essentially consist of the characterization of the (autonomous) equations of motion as a Hamiltonian vector field

$$\Xi \lrcorner\, \omega_2 = -dH \tag{6.3.29}$$

with respect to the fundamental symplectic structure

$$\omega_2 = \frac{1}{2}\left(\frac{\partial R^0_{\nu}}{\partial a^{\mu}} - \frac{\partial R^0_{\mu}}{\partial a^{\nu}}\right)da^{\mu} \wedge da^{\nu} = dp_{ka} \wedge dr^{ka} \tag{6.3.30}$$

and related symplectic and contact geometric formulations (Lie's derivatives, etc.).

A few comments are in order. First, we should stress the restriction of the applicability of Galilei's relativity *only* to closed self-adjoint systems. This restriction is based on the notion of (physically) *exact symmetry* of Chart A.12 applied to the case at hand. In fact, we have the combination of the mathematical condition of Hamiltonian form-invariance and related first integrals, with the physical condition that the first integrals directly represent laws of nature. The conservative character of the forces is then a consequence, e.g., of the conservation of the energy.

We can say in different terms that Definition 6.3.5 applies only for systems of Newtonian particles verifying the following conditions.

(1) *Closure condition*: The system can be considered as isolated from the rest of the universe in order to permit the conservation laws of the total mechanical energy, the total physical linear momentum, the total physical angular momentum, and the uniform motion of the center of mass.

[13] The "time component" of canonical realization (6.3.28) of Galilei's relativity

$$a' = \exp\left(t\omega^{\alpha\beta}\frac{\partial H}{\partial a^{\beta}}\frac{\partial}{\partial a^{\alpha}}\right)a$$

characterizes the time evolution of the system and should not be confused with the time translation. In particular, the latter acts on time, $t \to t' = t + t_0$, while the former acts on the a variables, $a(t) \to a(t + t_0)$. Also, the latter is unique, while the former depends explicitly on the Hamiltonian, and therefore its explicit form is different for different systems. The proof of the canonicity of the time component has been left as an exercise for the interested reader (Problem 5.8). (See also footnote 10 on page 229.)

(2) *Selfadjointness condition*: The particles can be well approximated as massive points moving in vacuum along stable orbits without collisions,[14] in order to restrict all possible forces to those of action-at-a-distance, potential type.

The existence of physical systems obeying these conditions is unequivocal. For instance, our solar system in Newtonian approximation is indeed a system of this type, and, as such, obeys *all* conditions for the applicability of Galilei's relativity.

Nevertheless, the applicability of Galilei's relativity is the exception, and its violation is the rule in Newtonian mechanics for several reasons. The most important is that Newtonian "particles" can be well approximated as "massive points" only under very special conditions. In fact, Newtonian systems generally imply motions of extended objects (e.g., a satellite) in a resistive medium (e.g., Earth's atmosphere), in which case their reduction to massive points would imply excessive approximations (e.g., the approximation of the satellite orbiting in our atmosphere with a conserved angular momentum).

When the extended character of the objects is represented together with their general motion within media, the dynamic conditions become unrestricted. As a result, the equations of motion break the Galilei's symmetry according to one of the mechanisms of the classification of Chart A.12 (*isotopic, self-adjoint, semicanonical, canonical, and essentially self-adjoint breakings*).

Equivalently, we can say that, if Galilei's relativity is imposed in the exact meaning of Chart A.12, it generally implies an excessive restriction of the acting forces, with consequentially excessive approximations of the perpetual-motion type.[15]

[14] A few rudimentary remarks on the problem of the global stability of the system and that of the orbits of each constituent will be presented momentarily.

[15] The considerations suggesting a generalization of Galilei's relativity for nonpotential interactions are numerous, both within the context of Newtonian Mechanics, as well as in relation to other disciplines, such as statistical mechanics or quantum mechanics. Within the context of Newtonian Mechanics, the breaking of Galilei's symmetry by the systems of daily life (recalled earlier) is only one aspect, and several additional considerations exist. For instance, it can be proved (Problem 5.9) that systems with nonpotential forces evolve according to a *noncanonical law*. In turn, this implies the inapplicability of virtually all methodological foundations of Galilei's relativity, as reviewed in Definition 6.3.5. It can also be proved that, if one imposes the canonical character of the time evolution in the variables t, $\mathbf{r}$, and $\mathbf{p} = m\dot{\mathbf{r}}$ of the experimental observation, all non-self-adjoint forces are identically null. This is, perhaps, one of the most direct ways to see that Galilei's relativity does not permit the representation of contact forces. The need for a suitable generalization is then consequential. In the transition to other disciplines, the need reemerges perhaps even more forcefully. For instance, a statistical system of particles obeying Galilei's relativity, in the strict sense of Chart A.12, prohibits a consistent formulation of thermodynamics, e.g., because of the invariance of the equations of motion under time inversion, with consequential inability to account for the entropy (see Chart I.A.4 on the arrow of time and the entropy). One can therefore see the need to reach a relativity which, while preserving the conventional description of total conservation laws and global stability, is consistent with experimentally established thermodynamic laws. Additional arguments exist within the context of quantum mechanics because of the inability of Galilei's relativity to describe effectively wave packets in mutual penetration and overlapping. The generalization of Galilei's relativity presented later on in this section has been conceived to solve or at least alleviate these problems.

In the following, we shall identify the rudiments of a possible generalization of Galilei's relativity which is more generally applicable to local Newtonian systems. The research attitude needed for this task is the opposite of the conventional one. Customarily, one first assumes an established relativity, and then restricts the dynamics to that compatible with the relativity assumed. On the contrary, we advocate here first the assumption of dynamic conditions as identifiable in nature, and then the search for a compatible relativity. This research attitude can be implemented according to the following three steps: the identification of the largest possible class of systems with unrestricted dynamics, the identification of the methods for the treatment of the systems considered and of their symmetries, and the identification of the covering relativity.

Step I: *Closed Non-Self-Adjoint Systems.* When a system of particles is isolated from the rest of the universe, it must necessarily obey the ten conservation laws (6.3.25); that is, it must be closed. However, this does not necessarily imply that all internal forces are of the potential, action-at-a-distance type. In fact, closure conditions (6.3.25) are compatible with internal forces of contact, nonpotential, non-self-adjoint type due to internal collisions and/or motion within resistive media. This leads in a natural way to the notion of *closed non-self-adjoint systems*[16] reviewed in Chart A.8 for the case of second-order systems. Their formulation for first-order systems can be presented as follows.

Implement closed self-adjoint systems (6.3.22) with an unrestricted collection of local, analytic, Newtonian forces. These additive forces can be classified into self-adjoint[17] and non-self-adjoint, resulting in the following systems

$$(\dot{a}^\mu) = \begin{pmatrix} \dot{r}^{ka} \\ \dot{p}_{ka} \end{pmatrix} = (\Gamma^\mu(t, a)) = (\Xi^\mu(a)) + (F^\mu(t, a))$$

$$= \begin{pmatrix} p_{ka}/m_k \\ f^{SA}_{ka}(\mathbf{r}) \end{pmatrix} + \begin{pmatrix} 0 \\ F^{SA}_{ka}(t, \mathbf{r}, \mathbf{p}) + \mathscr{F}^{NSA}_{ka}(t, \mathbf{r}, \mathbf{p}) \end{pmatrix}. \tag{6.3.31}$$

The total energy will be modified in this implementation, trivially, because of the additional presence of potential forces,

$$E_{tot} = T(\mathbf{p}) + V(\mathbf{r}) + U(t, \mathbf{r}, \mathbf{p}), \tag{6.3.32a}$$

$$T(\mathbf{p}) = \sum_{k=1}^{N} \frac{1}{2m_k} \mathbf{p}_k \cdot \mathbf{p}_k, \qquad \mathbf{f}^{SA}_k = -\frac{\partial V}{\partial \mathbf{r}^k}, \tag{6.3.32b}$$

$$\mathbf{F}^{SA}_k\left(t, \mathbf{r}, \frac{\mathbf{p}}{m}\right) = -\frac{\partial U}{\partial \mathbf{r}^k} + \frac{d}{dt}\frac{\partial U}{\partial \dot{\mathbf{r}}^k}. \tag{6.3.32c}$$

However, all the other total quantities (6.3.23b)–(6.3.23d) remain unchanged.

[16] Santilli (1978d).

[17] As implicit in the treatment of Chart A.12, a Galilei form-noninvariant force need not be non-self-adjoint. In fact, several *self-adjoint* forces of common use in mechanics break Galilei's symmetry either in part or in full.

In fact, as indicated during the course of our analysis, physical quantities such as the total linear momentum $\mathbf{P}_{\text{tot}}$ are defined in a way independent from the acting forces which, clearly, can only affect their behavior in time.

Definition 6.3.6. The most general possible class of *local, analytic, closed, non-self-adjoint Newtonian systems* is given by the class of all possible, consistent, generally overdetermined and constrained systems

$$\begin{cases} (\dot{a}^\mu) = \begin{pmatrix} \dot{r}^{ka} \\ \dot{p}_{ka} \end{pmatrix} = (\Gamma^\mu(t, a)) = \begin{pmatrix} p_{ka}/m_k \\ f^{\text{SA}}_{ka}(\mathbf{r}) + F^{\text{SA}}_{ka}(t, \mathbf{r}, \mathbf{p}) + \mathscr{F}^{\text{NSA}}_{ka}(t, \mathbf{r}, \mathbf{p}) \end{pmatrix}, & (6.3.33a) \\ \dot{X}_i(t, a) = \dfrac{\partial X_i}{\partial a^\mu}\,\dot{a}^\mu - \dfrac{\partial X_i}{\partial t} = 0, & (6.3.33b) \\ X_1 = E_{\text{tot}} = T(\mathbf{p}) + V(\mathbf{r}) + U(t, \mathbf{r}, \mathbf{p}), & (6.3.33c) \\ \{X_2, X_3, X_4\} = \mathbf{P}_{\text{tot}} = \sum_{k=1}^{N} m_k \mathbf{p}_k, & (6.3.33d) \\ \{X_5, X_6, X_7\} = \mathbf{M}_{\text{tot}} = \sum_{k=1}^{N} \mathbf{r}_k \times \mathbf{p}_k, & (6.3.33e) \\ \{X_8, X_9, X_{10}\} = \mathbf{G}_{\text{tot}} = \sum_{k=1}^{N} (m_k \mathbf{r}_k - t\mathbf{p}_k), & (6.3.33f) \\ \mu = 1, 2, \ldots, 6N, \qquad k = 1, 2, \ldots, N, \qquad a = x, y, z \\ i = 1, 2, \ldots, 10. \end{cases}$$

The primary difference between closed self-adjoint and non-self-adjoint systems is the same as that for the second-order case; namely, the conservation laws of total quantities are first integrals of the equations of motion for the former, while they are, *in general*, subsidiary constraints for the latter.

The *physical existence* of closed non-self-adjoint systems is established by a simple observation of nature. For instance, the Earth, when considered as isolated from the rest of the universe and inclusive of its atmosphere, is precisely a closed systems with unrestricted internal forces.

The *mathematical existence* of the systems is established by the existence theory of overdetermined systems. As indicated by Santilli (*loc. cit.*) in his original proposal, a hierarchy exists of classes of consistent systems (6.3.33) with a dynamics of increasing complexity and methodological needs.

Definition 6.3.7. Closed non-self-adjoint systems can be classified into:

Class α: when the conserved total physical quantities are first integrals of the vector field;

Class β: when the conserved total physical quantities constitute invariant relations of the vector field;

Class γ: when the conserved total physical quantities constitute bona fide subsidiary constraints of the vector field.

For brevity, we limit ourselves to the illustration of class α. The existence of the more general classes β and γ will be only indicated.

Assume for simplicity that the additive self-adjoint forces in Equations (6.3.33a) are null. This implies that the original total energy (6.3.23a) persists during the implementation of the systems with internal contact forces. We now impose the conservation laws to be first integrals of the new systems according to the strong equality

$$\begin{aligned}\dot{X}_i(t, a) &= \frac{\partial X_i}{\partial a^\mu}\Gamma^\mu + \frac{\partial X_i}{\partial t} \\ &= \left[\frac{\partial X_i}{\partial a^\mu}\Xi^\mu + \frac{\partial X_i}{\partial t}\right) + \frac{\partial X_i}{\partial a^\mu}F^\mu \equiv 0, \end{aligned} \tag{6.3.34}$$

but the original equations (6.3.25) are verified by assumption. Thus conditions (6.3.34) reduce to

$$\frac{\partial X_i}{\partial a^\mu}F^\mu = \frac{\partial X_i}{\partial p_{ka}}\mathscr{F}_{ka} \equiv 0; \tag{6.3.35}$$

that is, the non-self-adjoint forces must be null eigenvectors of the matrix $(\partial X_i/\partial p_{ka})$. When all ten conservation laws are worked out in detail, they imply the following conditions on the non-self-adjoint forces

$$\sum_{k=1}^{N} \mathbf{p}_k \cdot \mathbf{F}_k^{\mathrm{NSA}} \equiv 0, \tag{6.3.36a}$$

$$\sum_{k=1}^{N} \mathbf{F}_k^{\mathrm{NSA}} \equiv 0, \tag{6.3.36b}$$

$$\sum_{k=1}^{N} \mathbf{r}_k \times \mathbf{F}_k^{\mathrm{NSA}} \equiv 0. \tag{6.3.36c}$$

Note that these are conditions on non-self-adjoint forces for total physical quantities to be first integrals. As a result, conditions (6.3.36) are *only sufficient* for the consistency of systems (6.3.33) and *not necessary*.

It is now trivial to see that consistent systems of class α do indeed exist. In fact, the consistency of systems (6.3.33) has been reduced to that of systems (6.3.36). These are functional systems of seven equations in $3N$ unknown functions $\mathscr{F}_{ka}^{\mathrm{NSA}}$ violating the integrability conditions of Theorem A.1.1. Solutions in the functions $\mathscr{F}_{ka}^{\mathrm{NSA}}$ exist beginning with $N = 3$. The case $N = 2$ is a special one, inasmuch as the closure forces the orbit to be in a plane. The number of equations (6.3.36) therefore reduces to five, while the number of functions $\mathscr{F}_{ka}^{\mathrm{NSA}}$ is four. Despite the lack of sufficient degrees of freedom, a solution still exists, and it is presented in Example 6.3. It essentially demands the abandonment of the restriction that the contact forces are of *Newtonian* type and the acceptance of more general, acceleration-dependent, contact forces. As a result, the case of the *two-body, closed, non-self-adjoint system* is particularly instructive in Newtonian mechanics, as well as for possible

quantum mechanical and quantum field theoretical generalizations (Chart 6.1).

The *N-body, closed, non-self-adjoint systems of class* α ($N \geq 3$) are equally instructive at all levels of study. For instance, conditions (6.3.36) might conceivably be derived via arguments of *global stability of the system achieved via unstable orbits of the constituents.* More explicitly, the global stability of a closed self-adjoint system is essentially achieved via the stability of the orbits of each constituent, as is typically the case, say, in our solar system or in atomic structure. In the transition to the closed non-self-adjoint systems, the situation is fundamentally different inasmuch as global stability is achieved without prohibiting internal collisions with the consequential instability of the orbits of each constituent, as is evidently the case with the sun, for instance.

In fact, condition (6.3.36a) (which ensures the conservation of the total energy) is clearly a first condition for global stability via unrestricted internal exchanges of energy; condition (6.3.36b) (which ensure the conservation of the total linear momentum and the uniform motion of the center of mass) is a clear expression of the additional condition of global stability via unrestricted action and reaction effects with null total value; and condition (6.3.36c) (which ensure the conservation of the total angular momentum) is clearly the last expectable condition for global stability.[18]

However, as indicated earlier, conditions (6.3.36) are only sufficient for the systems considered. When the broader class β is admitted, equations (6.3.34) are generalized into the weak equality

$$\dot{X}_i(t, a_0) = \lambda_i^j(t, a_0) V_j(t, a_0) \triangleq 0; \tag{6.3.37}$$

that is, they are expressed via invariant relations according to Definition 6.3.3. In turn, conditions (6.3.37) themselves are only sufficient, inasmuch as the most general class of the systems (class γ) is that for which the conservation laws are bona fide subsidiary constraints of the equations of motion. The study of these latter systems is left here to the interested researcher.

In closing step 1, the reader may recall (Chart A.8) that closed non-self-adjoint systems were proposed as structure models of hadrons with extended internal constituents and non-Hamiltonian structural dynamics.

Step II: Symmetries, First Integrals, and Conservation Laws of Birkhoff's Equations. As is well-known, Galilei's relativity in its contemporary interpretation is an expression of some of the most advanced analytic, algebraic, and geometric techniques of Hamiltonian Mechanics. But a necessary condition for a closed system to be non-self-adjoint is that the vector field is not Hamiltonian in the variables $(t, \mathbf{r}, \mathbf{p})$, $\mathbf{p} = m\dot{\mathbf{r}}$, of its experimental observation. This implies that, for systems (6.3.33), not only do we have the general lack of Galilei form-invariance, but we actually have the lack of applicability of the methodological foundations of the relativity. In turn, this creates the need to

[18] A first statistical study of closed non-self-adjoint systems has been conducted by Tellez-Arenas, Fronteau, and Santilli (1979).

identify covering methods before any attempt at the construction of a covering relativity can acquire scientific value.

The direct universality of Birkhoff's equations for the representation of *all* closed non-self-adjoint systems was established in Chapter 4, together with the methods for the construction of the Birkhoffian representation from the equations of motion, as well as the identification of the underlying degrees of freedom. The representation can be constructed according to the equations

$$\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\Gamma^\nu(t, a) = \frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t}, \qquad \mu = 1, 2, \ldots, 6N, \quad (6.3.38)$$

where the Birkhoffian can be the total energy,

$$B = T(\mathbf{p}) + V(\mathbf{r}) + U(t, \mathbf{r}, \mathbf{p}), \quad (6.3.39)$$

and the R-functions are obtained via the solution of one of the Cauchy–Kovalevski equations (4.5.6) or (4.5.7), or via one of the three methods of Corollary 4.5.1d. In this way, while all self-adjoint forces are represented by the Birkhoffian (as it occurs for the Hamiltonian), all non-self-adjoint forces are represented via the generalization of the canonical tensor $\omega_{\mu\nu}$ into the Birkhoffian form $\Omega_{\mu\nu}$ (which is not possible in Hamiltonian formulations).

In the preceding chapter we have established that the Birkhoffian transformation theory is a bona fide covering of the Hamiltonian one. Thus Birkhoffian Mechanics is a natural candidate for attempting a generalization of Galilei's relativity. In order to conduct this task, the problem remains of generalizing the methods underlying symmetries, first integrals, and conservation laws.

Definition 6.3.8. The most general possible transformations on $\mathbb{R} \times T^*M$,

$$\hat{a}^\mu \to \hat{a}'^\mu(\hat{a}), \qquad \mu = 0, 1, 2, \ldots, 6N, \quad (6.3.40)$$

are said to constitute *symmetries of Birkhoff's equations* (5.3.38), i.e.,

$$\hat{\Omega}_{\mu\nu}(\hat{a})da^\nu = 0, \quad (6.3.41)$$

when they are *identity contact isotopic* with respect to the $(2n + 1) \times (2n + 1)$ tensor $\hat{\Omega}_{\mu\nu}(\hat{a})$. By recalling that all transformations (6.3.40) are contact-isotopic (Lemma 5.3.3), we have a symmetry when the following particularization of transformation laws (5.3.31) holds

$$\hat{\Omega}_{\mu\nu}(\hat{a})d\hat{a}^\nu = \frac{\partial \hat{a}'^\alpha}{\partial \hat{a}^\mu}\hat{\Omega}'_{\alpha\beta}(\hat{a}')d\hat{a}'^\beta$$

$$\equiv \frac{\partial \hat{a}'^\alpha}{\partial \hat{a}^\mu}\hat{\Omega}_{\alpha\beta}(\hat{a}')d\hat{a}'^\beta = 0, \quad (6.3.42a)$$

$$\hat{\Omega}'_{\alpha\beta}(\hat{a}') = \frac{\partial \hat{a}^\rho}{\partial \hat{a}'^\alpha}\hat{\Omega}_{\rho\sigma}(\hat{a}(\hat{a}'))\frac{\partial \hat{a}^\sigma}{\partial \hat{a}'^\beta}, \quad (6.3.42b)$$

or, more explicitly, when the following particularization of transformation rules (5.3.38) holds,

$$(\hat{\Omega}_{\mu\nu}(\hat{a})d\hat{a}^{\nu}) = \begin{pmatrix} \left(\frac{\partial B}{\partial a^{\nu}} + \frac{\partial R_{\nu}}{\partial t}\right)da^{\nu} \\ \left(\frac{\partial R_{\nu}}{\partial a^{\mu}} - \frac{\partial R_{\mu}}{\partial a^{\nu}}\right)da^{\nu} - \left(\frac{\partial B}{\partial a^{\mu}} + \frac{\partial R_{\mu}}{\partial t}\right)dt \end{pmatrix}$$

$$= \left(\frac{\partial \hat{a}'^{\alpha}}{\partial \hat{a}^{\mu}}\hat{\Omega}_{\alpha\beta}(\hat{a}')d\hat{a}'^{\beta}\right)$$

$$= \left(\frac{\partial \hat{a}'^{\alpha}}{\partial \hat{a}^{\mu}}\right)\begin{pmatrix} \left(\frac{\partial B}{\partial a'^{\rho}} + \frac{\partial R}{\partial t'}\right)da'^{\rho} \\ \left(\frac{\partial R_{\beta}}{\partial a'^{\alpha}} - \frac{\partial R_{\alpha}}{\partial a'^{\beta}}\right)da'^{\beta} - \left(\frac{\partial B}{\partial a'^{\alpha}} + \frac{\partial R_{\alpha}}{\partial t'}\right)dt' \end{pmatrix} = 0, \tag{6.3.43a}$$

$$R_{\alpha}(t', a') = \left(R_{\mu}\frac{\partial a^{\mu}}{\partial a'^{\alpha}} - B\frac{\partial t}{\partial a'^{\alpha}}\right)(t', a'), \tag{6.3.43b}$$

$$B(t', a') = \left(B\frac{\partial t}{\partial t'} - R_{\mu}\frac{\partial a^{\mu}}{\partial t}\right)(t', a'). \tag{6.3.43c}$$

Equivalently, we have a symmetry when the primitive one-form of Birkhoff's equations (the integrand of the Pfaff's action) is form-invariant up to Birkhoffian gauges,

$$\hat{R}_{\mu}(\hat{a})d\hat{a}^{\mu} = \hat{R}'_{\alpha}(\hat{a}')d\hat{a}'^{\alpha} \equiv \left[\hat{R}_{\alpha}(\hat{a}') + \frac{\partial G(\hat{a}')}{\partial \hat{a}'^{\alpha}}\right]d\hat{a}'^{\alpha}, \tag{6.3.44a}$$

$$\hat{R}'_{\alpha}(\hat{a}') = \left(R_{\mu}\frac{\partial \hat{a}^{\mu}}{\partial \hat{a}'^{\alpha}}\right)(\hat{a}'). \tag{6.3.44b}$$

The covering nature of Definition 6.3.8 over 6.3.4 is established by the fact that the symmetries of Hamilton's equations are a particular case of the symmetries of Birkhoff's equations, in exactly the same way as the transformation rule of Hamilton's equations is a particular case of that of Birkhoff's equations (Corollary 5.3.3d).

Most important is the property that the new time t', in general, can be not only a function of all old variables $t'(t, \mathbf{r}, \mathbf{p})$, but also the image of any old variable (Corollaries 5.3.3a and 5.3.3c).

We move now to the identification of the generalized methods for the construction of first integrals from known symmetries of Birkhoff's equations. For this purpose we suppose that given Birkhoff's equations possess

the following Lie symmetry group of infinitesimal transformations

$$G_r^*: (\hat{a}^\mu) = \begin{pmatrix} t \\ a^\mu \end{pmatrix} \to (\hat{a}'^\mu) = \begin{pmatrix} t' \\ a'^\mu \end{pmatrix} = (\hat{a}^\mu + \hat{\delta}\hat{a}^\mu) = (\hat{a}^\mu + w^i \alpha_i^\mu(\hat{a}))$$

$$= \begin{pmatrix} t + w^i \hat{\rho}_i(t, a) \\ a^\mu + w^i \hat{\eta}_i^\mu(t, a) \end{pmatrix}, \tag{6.3.45}$$

where, again, the w's are the infinitesimal parameters.

Then, by recalling rules (6.3.44) or, equivalently, via the direct use of the variational techniques of Section I.1.3, the Pfaffian action under transformations (6.3.45) transforms according to

$$\hat{\delta}A = \int_{D_t} \hat{R}_\mu(\hat{a}) d\hat{a}^\mu - \int_{D_{t'}} \hat{R}_\mu(\hat{a}') d\hat{a}'^\mu = -\int_{D_{t'}} d[\hat{\delta}G(\hat{a})], \tag{6.3.46}$$

where D_t is the original (closed) interval of time, and $D_{t'}$ is its image under the transformations.

By recalling generalized variational principle (5.3.50), we can write along a possible or actual path

$$\begin{aligned} \hat{\delta} \int_{D_t} dt \hat{R}_\mu(\hat{a}) \dot{\hat{a}}^\mu &= \int_{D_t} dt \Omega_{\mu\nu}(\hat{a}) \dot{\hat{a}}^\nu \hat{\delta}\hat{a}^\mu \\ &= -\int_{D_t} dt \frac{d}{dt} [\hat{R}_\mu(\hat{a}) \hat{\delta}\hat{a}^\mu + \hat{\delta}G(\hat{a})](\tilde{E}^0) \\ &= -w^i \int_{D_t} dt \frac{d}{dt} [\hat{R}_\mu(\hat{a}) \hat{\alpha}_i^\mu(\hat{a}) + G_i(\hat{a})](\tilde{E}^0) \\ &= -w^i \int_{D_t} dt \frac{d}{dt} \\ &\quad \times [R_\mu(t, a) \hat{\eta}_i^\mu(t, a) - B(t, a) \hat{\rho}_i(t, a) - G_i(t, a)](\tilde{E}^0). \end{aligned} \tag{6.3.47}$$

In this way we reach the following result.

Theorem 6.3.3 (Noether's Theorem for Birkhoff's Equations). *If Birkhoff's equations admit a symmetry under an r-dimensional connected Lie Group G_r^* of infinitesimal transformations, then r linear combination of Birkhoff's equations exist along an admissible path which are exact differentials, i.e.,*

$$\frac{d}{dt} I_i(\hat{a}) = \hat{\Omega}_{\mu\nu}(\hat{a}) \dot{\hat{a}}^\nu \hat{\alpha}_i^\mu, \tag{6.3.48a}$$

$$\begin{aligned} I_i(\hat{a}) &= \hat{R}_\mu(\hat{a}) \hat{\alpha}_i^\mu(\hat{a}) + G_i(\hat{a}) \\ &= R_\mu(t, a) \hat{\eta}_i^\mu(t, a) - B(t, a) \hat{\rho}_i(t, a) + G_i(t, a), \qquad i = 1, 2, \ldots, r. \end{aligned} \tag{6.3.48b}$$

A quite simple, alternative proof can be formulated via (a) the property that Noether's theorem (Chart A.9) also applies to first-order totally degenerate Lagrangians $L(t, a, \dot{a})$; (b) the property that Birkhoff's equations coincide with Lagrange's equations in $L(t, a, \dot{a})$ according to Equations (4.2.38); and (c) the specialization of Equations (6b) of Chart A.9 to the case at hand. This alternative approach gives rise to the quantities

$$L(t, a, \dot{a}) = R_\mu(t, a)\dot{a}^\mu - B(t, a), \tag{6.3.49a}$$

$$\begin{aligned} I &= \frac{\partial L}{\partial \dot{a}^\mu}\hat{\delta}a^\mu - \left(\frac{\partial L}{\partial \dot{a}^\mu}\dot{a}^\mu - L\right)\hat{\delta}t + \hat{\delta}G(t, a) \\ &= R_\mu \hat{\delta}a^\mu - (R_\mu \dot{a}^\mu - R_\mu \dot{a}^\mu + B)\hat{\delta}t + \hat{\delta}G \\ &= w^i[R_\mu(t, a)\hat{\eta}_i^\mu(t, a) - B(t, a)\hat{\rho}_i(t, a) + G_i(t, a)], \end{aligned} \tag{6.3.49b}$$

which are equivalent to those of Equations (6.3.48b).

Corollary 6.3.3a. *The quantities* (6.3.48b) *are first integrals of Birkhoff's equations.*

In fact, the properties along a possible or actual path

$$\frac{d}{dt} I_i(\hat{a})|_{\hat{a}_0} = \hat{\Omega}_{\mu\nu}(\hat{a})\dot{a}^\nu \hat{\alpha}_i^\mu(\hat{a})|_{\hat{a}_0} \equiv 0 \tag{6.3.50}$$

are equivalent to Equations (6.3.5), where $\hat{\Xi}$ is the vector field represented by Birkhoff's equations.

The covering character of Theorem 6.3.3 over Hamiltonian formulations is expressed by the fact that, when the Pfaffian form becomes the canonical one (i.e., for $R = R^0 = (\mathbf{p}, \mathbf{0})$ and $B = H$), we have

$$\begin{aligned} I_i &= p_{ka}\hat{\eta}_i^{ka} - H\hat{\rho}_i + G_i \\ &= \frac{\partial L}{\partial \dot{r}^{ka}}\hat{\eta}_i^{ka} - \left(\frac{\partial L}{\partial \dot{r}^{ka}}\dot{r}^{ka} - L\right)\hat{\rho}_i + G_i \end{aligned} \tag{6.3.51}$$

which is the Hamiltonian formulation of the conventional Noether's theorem. Additional properties (such as the lack of necessary independence of the r first integrals (6.3.48b), the lack of their necessary direct physical meaning, etc.) can be obtained via the extension to a Birkhoffian context of the analysis of Chart A.9.

We now progress to the identification of the Lie algebra structure of an r-dimensional symmetry G_r^* of Birkhoff's equations. By recalling the lack of algebraic structure of the general nonautonomous case (Chart 4.1), we must restrict ourselves for this purpose to semi-autonomous equations (Definition 5.2.1). The capability of reducing all nonautonomous equations to this form was proved in Section 4.5 (see also Section 5.3) and will be tacitly assumed here. Also, we assume the reader is familiar with the problematic aspects related to the physical meaning of the Birkhoffian under the reduction considered. Finally, we shall assume that Theorem 6.3.3 is applied to the reduced

semi-autonomous form (rather then the original nonautonomous form), because symmetries are not necessarily preserved under the reduction considered.

An inspection of the notion of symmetries of Birkhoff's equations soon reveals that they are a particular form of the generalized canonical transformations; that is, in general, they *are not* canonical transformations. The canonical structure (6.3.16) is therefore generally not applicable. Instead, the necessary and sufficient condition for infinitesimal transformations (6.3.45) to be generalized canonical transformations is that they have the form

$$a'^{\mu} = a^{\mu} + w^{i}\Omega^{\mu\nu}(a)\frac{\partial X_i}{\partial a^{\nu}}(t, a), \tag{6.3.52a}$$

$$\Omega^{\mu\nu} = \left(\left\| \frac{\partial R_{\nu}}{\partial a^{\mu}} - \frac{\partial R_{\mu}}{\partial a^{\nu}} \right\|^{-1}\right)^{\mu\nu}, \tag{6.3.52b}$$

where the w's are, again, the infinitesimal parameters and the X's the generators of G_r^* (see Chart 5.4 for details).

The necessary and sufficient condition for a transformation of this type to be a symmetry is therefore that it leaves the Birkhoffian invariant, i.e.,

$$B'(t, a') = B(a) + \frac{\partial B}{\partial a^{\mu}} w^{i}\Omega^{\mu\nu}\frac{\partial X_i}{\partial a^{\nu}} = B(a) + w^{i}[B, X_i]^* \equiv B(a). \tag{6.3.53}$$

Thus we reach the following covering of Theorem 6.3.2.

Theorem 6.3.4 (Integrability Conditions for Birkhoffian Symmetries). *Necessary and sufficient conditions for infinitesimal, generalized canonical transformations to be symmetries of the autonomous Birkhoff's equations are that the generalized Poisson brackets of the Birkhoffian with all the generators $X_i(a)$ of the transformations are identically null, i.e.*

$$[B, X_i]^* \equiv 0, \qquad i = 1, 2, \ldots, r. \tag{6.3.54}$$

The use of the isotopic generalization of Lie's theory worked out in the charts of this chapter then yields the following covering of Corollary 6.3.2a (see, in particular, the generalization of Lie's structure *constants* C_{ij}^k into the structure *functions* $C_{ij}^k(a)$ of Chart 5.3).

Corollary 6.3.4a. *The Lie algebra $\mathbf{G}_r^*$ of an r-dimensional Lie symmetry group G_r^* of Birkhoff's equations is given by the vector space (over the field $\mathbb{F}$ of real numbers) of the generators X_i verifying Equations (6.3.54) equipped with the generalized Poisson brackets as the applicable realization of the Lie product, and verifying the following closure rules expressed in terms of the structure functions $C_{ij}^k(a)$*

$$[X_i, X_j]^* = C_{ij}^k(a)X_k. \tag{6.3.55}$$

In closing step II, we can therefore say that each and every aspect of the Hamiltonian formulation of symmetries, first integrals, and conservation laws can be consistently generalized into a Birkhoffian form.

Step III: *Construction of the Covering Relativity.* At this point we define the intended covering relativity and then identify the additional methods needed for its construction.

Definition 6.3.9. The *isotopic covering of Galilei's relativity*[19] is a description of physical systems verifying the following primary conditions:

1. the relativity provides a form-invariant description of closed systems of extended particles under action-at-a-distance self-adjoint interactions as well as contact non-self-adjoint interactions;
2. the relativity is based on the isotopic generalization of the methodological formulations of Galilei's relativity, that is, on the Birkhoffian generalization of Hamiltonian mechanics, on the isotopic generalization of Lie theory, and on the symplectic and contact geometries in their most general possible local and exact realizations; and
3. the generalized relativity recovers the conventional one identically when the systems are reduced to pointlike constituents with consequential lack of contact non-self-adjoint interactions.

By keeping in mind the conditions for a new theory to qualify as the covering of an existing one (see footnote 24 of Chapter 5), property 1 ensures that the new relativity applies to a physical arena broader than that of the conventional one; property 2 ensures that the new relativity is based on a

[19] We present here a Lie-isotopic particularization of the more general *Lie-admissible covering of Galilei's relativity* proposed by Santilli (1978c, pp. 390–394; see also 1978e and 1982d) for *open* non-self-adjoint interactions. The particularization has been made possible by the *mathematical* property that Lie-admissible formulations contain the Birkhoffian formulations as a particular case, as well as by the *physical* property that closed systems are a particular subclass of the open ones, trivially, when the time rate of variation of total quantities is identically null. The main difference between the Lie-admissible covering and the Lie-isotopic covering of Galilei's relativity is that *symmetries are used in the former case to represent time rate of variations of physical quantities*, while in the latter case symmetries are used to represent their conservation. The transition from the Lie-isotopic to the more general Lie-admissible treatment of mechanics therefore implies a rather profound departure from contemporary conceptual settings, including those of the generalized relativity presented here. The Lie-isotopic and the Lie-admissible coverings of Galilei's relativity turn out to be considerably more compatible and complementary than expected. In fact, the Lie-isotopic relativity is currently used for the description of *a closed system as a whole*, in which case the emphasis must be on the total conservation laws, while the Lie-admissible relativity is currently used for the characterization of *each individual constituent* of the said closed systems, in which case the emphasis must be on the time rate of variation of physical quantities. The complementarity and mutual compatibility of the two relativities is then self-evident. The need of *both* relativities, one for the global (exterior) treatment and one for the constituent (interior) treatment, does not exist for closed self-adjoint systems (because the same relativity can trivially characterize both the state as a whole and the constituents), but it becomes *mandatory* for the more general class of closed non-self-adjoint systems. In fact, a variety generally exists of dynamic effects (e.g., those of nonlocal type) which dominate each constituent, while their total effect is null, much along Equations (6.3.36b). As a result, the use of the *exterior* relativity for the characterization of the *constituents*, even though conceivable, is generally restrictive and potentially erroneous. This duality of mutually compatible relativity was proposed by Santilli (*loc. cit.*).

generalization of the methods of the conventional one; and property 3 ensures the compatibility of the new relativity with the conventional one.

On more specific grounds, property 1 is realized via the construction of a ten-parameter Lie transformation group $G^*(3.1)$ which verifies the form invariance of systems (6.3.33a), i.e.,

$$G^*(3.1): \hat{a} \to \hat{a}'(a), \qquad \hat{a} = (t, a) \tag{6.3.56a}$$

$$\hat{\Gamma}(\hat{a}) = \hat{\Gamma}^{\mu}(\hat{a}) \frac{\partial}{\partial \hat{a}^{\mu}} = \hat{\Gamma}^{\mu}(\hat{a}(\hat{a}')) \frac{\partial \hat{a}'^{\alpha}}{\partial \hat{a}^{\mu}} \frac{\partial}{\partial \hat{a}'^{\alpha}}$$

$$= \hat{\Gamma}'^{\alpha}(\hat{a}') \frac{\partial}{\partial \hat{a}'^{\alpha}} \equiv \hat{\Gamma}^{\alpha}(\hat{a}') \frac{\partial}{\partial \hat{a}'^{\alpha}}, \tag{6.3.56b}$$

$$\hat{\Gamma} = (1, \Gamma^{\mu}(t, a)),$$

and whose generators $X_i(\hat{a})$ represent directly the conservation laws of total quantities (6.3.33c)–(6.5.33f), i.e.,

$$\dot{X}_i(\hat{a}) = 0, \qquad i = 1, 2, \ldots, 10. \tag{6.3.57}$$

Property 2 is realized via the following formulations.

I. *Isotopic generalization of Hamiltonian formulations* essentially consist of the representation of the equation of motion via the semi-autonomous Birkhoff's equations

$$\left\{\left[\frac{\partial R_{\nu}(a)}{\partial a^{\mu}} - \frac{\partial R_{\mu}(a)}{\partial a^{\nu}}\right]\dot{a}^{\nu} - \frac{\partial B(t, a)}{\partial a^{\mu}}\right\}_{SA} = 0 \tag{6.3.58}$$

and related Birkhoffian generalization of Hamiltonian formulations (generalized canonical transformations, generalized Hamilton–Jacobi equations, etc.).

II. *Isotopic generalization of Lie's theory* essentially consist of the isotopically mapped universal enveloping associative algebra $\mathscr{A}^*(\mathbf{G}(3.1))$ of Galilei's algebra $\mathbf{G}(3.1)$ and attached isotopic algebra $\mathbf{G}^*(3.1)$

$$\mathscr{A}^*(\mathbf{G}(3.1)) = \frac{\mathscr{T}^*}{\mathscr{R}^*}, \tag{6.3.59a}$$

$$\mathscr{T}^* = \mathbb{F} \oplus \mathbf{G} \oplus \mathbf{G} * \mathbf{G} \oplus \cdots, \tag{6.3.59b}$$

$$\mathscr{R}^* = [X_i, X_j]^* - (X_i * X_j - X_j * X_i), \tag{6.3.59c}$$

$$\mathbf{G}^*(3.1) \approx [\mathscr{A}^*(\mathbf{G}(3.1))]^- : [X_i, X_j]^* = C_{ij}^k(a) X, \tag{6.3.59d}$$

the Lie isotopic realization of the symmetry group $G^*(3.1)$[10],

$$G^*(3.1): a^\mu \to a'^\mu = \exp\left(\theta^k \Omega^{\alpha\beta}(a) \frac{\partial X_k}{\partial a^\beta} \frac{\partial}{\partial a^\alpha}\right) a^\mu, \tag{6.3.60a}$$

$$\Omega^{\alpha\beta} = \left(\left\| \frac{\partial R_\beta}{\partial a^\alpha} - \frac{\partial R_\alpha}{\partial a^\beta} \right\|^{-1} \right)^{\alpha\beta}, \qquad \{\theta^k\} = \{t_0; \mathbf{r}_0; \alpha_0, \beta_0, \gamma_0; \mathbf{v}_0\}, \tag{6.3.60b}$$

and related theory (generalized representation theory, etc.).

III. *Isotopic generalization of canonical geometries* essentially consist of the characterization of the (autonomous) equations of motion as a Birkhoffian vector field

$$\Gamma \lrcorner \Omega_2 = -dB \tag{6.3.61}$$

with respect to the exact but otherwise unrestricted symplectic structure

$$\Omega_2 = \frac{1}{2} \left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu} \right) da^\mu \wedge da^\nu, \tag{6.3.62}$$

and related symplectic as well as contact geometric formulations (Birkhoffian realization of Lie's derivatives, etc.).

Finally, property 3 is realized via the additional condition that, together with the reduction of systems (6.3.33a) to the self-adjoint and Galilei form-invariant form

$$(\Gamma^\mu)|_{\mathcal{F}^{\mathrm{NSA}}=0} = \begin{pmatrix} p_{ka}/m_k \\ f_{ka}^{\mathrm{SA}} + \mathcal{F}_{ka}^{\mathrm{NSA}} \end{pmatrix}_{\mathcal{F}^{\mathrm{NSA}}=0} = \begin{pmatrix} p_{ka}/m_k \\ f_{ka}^{\mathrm{SA}} \end{pmatrix} = (\Xi^\mu), \tag{6.3.63}$$

we have the reduction of the group $G^*(3.1)$ to Galilei's group $G(3.1)$, i.e.,

$$G^*(3.1)|_{\mathcal{F}^{\mathrm{NSA}}=0} \equiv G(3.1), \tag{6.3.64a}$$

$$\exp\left(\theta^k \Omega^{\alpha\beta}(a) \frac{\partial X_k}{\partial a^\beta} \frac{\partial}{\partial a^\alpha}\right) \Big|_{\mathcal{F}^{\mathrm{NSA}}=0} \equiv \exp\left(\theta^k \omega^{\alpha\beta} \frac{\partial X_k}{\partial X^\beta} \frac{\partial}{\partial a^\alpha}\right). \tag{6.3.64b}$$

When *all* these conditions are met, group $G^*(3.1)$ is called the *isotopic covering of Galilei's group.*

A rather direct way of arriving at the covering relativity is the following. When confronted with equations of motion violating Galilei's form-invariance, a frequent attitude is that of transforming the equations in a new coordinate system in which the applicability of familiar notions is recovered. In the preceding sections of this chapter we have established that this is always possible. In fact, Theorem 6.2.1 on the Indirect Universality of Hamilton's equations has the following consequence (which can be proved via the superposition of a Daurboux's and a canonical transformation).

Corollary 6.2.1a. *Consider a non-self-adjoint and Galilei form-non-invariant system* (6.3.33a). *Then a transformation always exists under which the transformed system is Galilei form-invariant.*

Consider, for simplicity, the case of autonomous equations (6.3.33a). Then, Corollary 6.2.1a establishes that a transformation

$$a^{\mu} \to a^{*\mu}(a) \tag{6.3.65}$$

always exists under which the new system acquires the "free" structure

$$(\Gamma^{*\mu}(a^*)) = \begin{pmatrix} \mathbf{p}^*/m \\ \mathbf{0} \end{pmatrix}, \qquad \Gamma'^{*\mu} = \left(\Gamma^{\alpha} \frac{\partial a^{*\mu}}{\partial a^{\alpha}}\right)(a^*), \tag{6.3.66}$$

with consequential form-invariance under Galilei's group

$$G(3.1)_{(a^*)}\colon a^{*\mu} \to a^{*\prime\mu} = \exp\left(\theta^k \omega^{\alpha\beta} \frac{\partial X_k^*}{\partial a^{*\beta}} \frac{\partial}{\partial a^{*\alpha}}\right) a^{*\mu}. \tag{6.3.67}$$

However, this way of recovering Galilei's relativity is mathematically consistent but physically illusory. In fact, one of the uncompromisable conditions for the physical meaning of abstract mathematical algorithms is that they admit a realization in the frame of the experimental observation. It is easy to see that *the variables* $\mathbf{r}^*(\mathbf{r}, \mathbf{p})$ *and* $\mathbf{p}^*(\mathbf{r}, \mathbf{p})$ *in which relativity* (6.3.67) *holds are generally nonrealizable experimentally.* In fact, the functional dependence of the new variables in the old is generally nonlinear (Section 6.2), therefore implying the inability of setting measuring apparata along trajectories of the type $\mathbf{r}^* = \boldsymbol{\alpha} \exp \beta \mathbf{r} \cdot \mathbf{p}$, etc.

This deficiency can be easily bypassed by transforming symmetry (6.3.67) from the mathematical coordinates $\mathbf{r}^*$, $\mathbf{p}^*$ to the original physical ones $\mathbf{r}$, $\mathbf{p}$ via the inverse $a^* \to a(a^*)$ of trasformations (6.3.65). However, these transformations must be necessarily *noncanonical*, trivially, because the original vector field is non-Hamiltonian by assumption. We can then easily prove that, under such an inverse transformation, the conventional relativity (6.3.67) in the mathematical coordinates transforms into the isotopic covering relativity in the physical coordinates. In fact, under noncanonical transformations, Hamilton's equations transform into Birkhoff's equations; the conventional Poisson brackets transform into the generalized ones; and the conventional canonical realization of Galilei's group transforms exactly into its isotope (6.3.60a) according to the formal rules

$$\theta^k \omega^{\alpha\beta} \frac{\partial X_k^*}{\partial a^{*\beta}} \frac{\partial}{\partial a^{*\alpha}} \equiv \theta^k \Omega^{\alpha\beta}(a) \frac{\partial X_k}{\partial a^{\beta}} \frac{\partial}{\partial a^{\alpha}} \tag{6.3.68a}$$

$$\Omega^{\alpha\beta}(a) = \frac{\partial a^{\alpha}}{\partial a^{*\mu}} \omega^{\mu\nu} \frac{\partial a^{\beta}}{\partial a^{*\nu}}, \qquad X_k = X_k^*(t, a^*(a)). \tag{6.3.68b}$$

We can therefore conclude by saying that the covering relativity emerges rather naturally from the analysis of these volumes, provided that excessive

approximations of perpetual-motion type are avoided, and the local variables are those of the frame of the experimental observation.

The preceding remarks also provide a *first method* for the formal construction of the generalized relativity according to the following procedure.
(a) Identify the closed non-self-adjoint system under consideration (whether of class α, β, or γ), by making sure that the local variables have a direct physical meaning (e.g., "p" is "$m\dot{r}$"), that the vector field in these physical variables is non-Hamiltonian, and that the total physical quantities are indeed conserved.
(b) Construct a semiautonomous Birkhoffian representation of the vector field via the methods of Chapter 4.
(c) Construct a Darboux's transformation of the type of Corollary 6.4.1a, by therefore reducing Birkhoff's equations to a Hamiltonian, Galilei form-invariant form.
(d) Construct the canonical realization of Galilei's relativity in the transformed reference frame, via the explicit computation of the infinite series of Equations (6.3.67).
(e) Transform this Galilean symmetry $G(3.1)_{(a^*)}$ into its isotope $G^*(3.1)_{(a)}$ via the inverse of Darboux's transformation, and see whether it does indeed provide a form-invariant description of the system considered in the sense of Definition 6.3.9.[20]

A *second method* can be identified via the use of the "inverse Noether's Problem" within the context of Theorem 6.3.3, that is, the construction of a Birkhoffian symmetry from known conserved quantities.[21] A *third method* can be identified via Lie's construction of the symmetries of given equations of motion (Chart A.7), of course, upon its suitable reformulation for non-Hamiltonian/Birkhoffian vector fields, as well as for the selection of the symmetries obeying the crucial condition (6.3.64).[22] The study of these, as well as other conceivable methods, is left here to the interested researcher. Particularly recommended is the geometric analysis by Schober (1981 and 1982).

We would like to close this section with a few remarks. First, the researcher should keep in mind that, while Galilei's relativity applies to structurally simple forces and is a manifest symmetry, this is not the case for the generalized relativity. In fact, one of the first examples provided by this author following the original proposal of the generalized relativity was that the generalized symmetry transformations can be nonmanifest to the point of being expressed via *transcendental functions.* This is a reflection of the fact

[20] In studying this problem, the reader should keep in mind that a vector field is always form-invariant under the time evolution induced by itself (Problem 6.5). As a result, the form-invariance of the closed non-self-adjoint systems under the time component of group (6.3.60a) is *always* verified. As a matter of fact, the computation of a Birkhoffian representation of the system, and its exponentiation, is sufficient to provide the desired form-invariance for the time component, without necessarily going through all of steps (a)–(e). The situation for the remaining nine components of group (6.3.60a) is different, owing to the current lack of solution of the so-called Inverse Noether's Problem (see below).

[21] See Problem 6.6.

[22] The initiation of this study for the more general *Lie-admissible* (rather than Birkhoffian) realization of symmetries has been conducted by G. E. Prince *et al.* (1979).

that the complexity of the admitted dynamics carries over to the applicable relativity.

A second new aspect is related to the contemporary tendency to express a relativity via a unique symmetry, e.g., (6.3.21). This is no longer possible under contact nonpotential interactions. In fact, these interactions are represented via the Birkhoffian generalization $\Omega^{\mu\nu}$ of the canonical tensor $\omega^{\mu\nu}$. As a result, they appear directly in the structure of the covering symmetry, equations (6.3.60a). In turn, this means that *different non-self-adjoint forces generally imply different relativities.* As a result, Definition 6.3.9 actually treats an *infinite variety of possible generalizations of Galilei's relativity.* The coordinate-free globalization of all these generalized relativity is expected to be unique (and actually to coincide with that of Galilei's relativity[23]). However, the reduction of the infinite local symmetries to only one generalized form would imply, again, unnecessary restrictions on the dynamics, or the abandonment of the form-invariant description of physical systems (which is at the basis of all relativities, whether conventional or generalized).

A further new aspect is related to another contemporary tendency, that of expressing relativities in inertial reference frames. As is well-known, inertial frames are of conceptual more than experimental value because they are not available to the contemporary experimenter, nor are they expected to be in the foreseable future. The covering relativity as per Definition 6.3.9 was conceived for *noninertial frames*, as one way to represent the actual noninertial character of all available experimental frames. This feature is necessary whenever one imposes the condition that the local variables are those of the experimenter and persists under transformations (6.3.60). To stress this important point, one should not restrict the study to the observation that transformations (6.3.60) are noninertial. Instead, one should begin the study with the observation that the experimental frame in which the relativity is constructed must necessarily be noninertial to comply with physical reality, and then take the necessary precautions that this noninertial character is preserved by the class of symmetries admitted.[24]

A number of additional aspects (such as the apparent characterization by the covering relativity of a class of privileged frames, those at rest with the closed non-self-adjoint system considered) are under study at this moment, and we refer the interested reader to the literature on the subject.

[23] As indicated in the geometric charts of Chapter 4, *the coordinate-free globalization of the symplectic and contact geometry implies the loss of distinction between Hamilton's and Birkhoff's equations.* The globalization of Galilei's relativity has been studied by a number of authors (see, for instance, Souriau (1970)). Even though these studies were specifically and strictly intended for conventional Galilean/Hamiltonian/Lie settings, they may be more suitably expressed to include all possible relativities of the same mathematical class, that is, the isotopic covering of Galilei's relativity.

[24] All frames on Earth are known to be noninertial due to the gravitational, Coriolis and other forces inherent in the Earth's rotations. The situation clearly persists for laboratory frames in orbit around the Earth as well as on the Moon. When interplanetary travel becomes feasible in the future, even this will not provide a laboratory frame which is truly inertial. In fact, our entire galaxy (let alone our solar system) is in apparent *accelerated* motion in our universe.

The most intriguing implication of the isotopic covering of Galilei's relativity is clearly the possibility of stimulating the construction of a corresponding generalization of Einstein's special relativity in classical and quantum mechanics, and of Einstein's general relativity for the *interior* problem of gravitation.

Einstein's special relativity in its contemporary formulation is compatible with the conventional Galilei's relativity, as is well-known, but *not* with its isotopic covering. This can be seen, for instance, from the fact that closed non-self-adjoint systems demand noncanonical time evolutions at the classical level and nonunitary time evolutions at the quantum mechanical level. This creates the need of generalizing Einstein's special relativity so as to recover the isotopic covering of Galilei's relativity for low speeds. Expectedly, such covering relativity could permit the description of contact nonpotential interactions which, being instantaneous by nature, cannot be described via existing relativistic formulations.

Einstein's general relativity for the interior problem appears to be in equally pressing needs of generalization. In fact, the systems studied in these volumes (such as satellites in Earth's atmosphere; spinning tops with drag torques; etc.), strictly speaking, *are* interior systems of the problem of gravitation. As is familiar from our analysis, a necessary condition for avoiding excessive approximations of the perpetual-motion type is that these systems admit unrestricted forces. It is possible to show that these forces are outside the technical possibilities of Einstein's general theory of gravitation, as well as of other existing generalizations, e.g., of so-called gauge or supersymmetric type. This is established by the fact that all these relativities do not permit unrestricted non-self-adjoint forces in their Newtonian limit. Equivalently, the situation can be seen from the fact that all existing theories of the interior problem of gravitation are locally Lorentz in character, while a direct representation of satellites with nonconserved angular momenta can best be achieved via theories which are not locally Lorentz in character, evidently in order to permit the local breaking of the symmetry under the group of rotations. More generally, the situation can be seen by noting that interior problems of the Earth, sun, and physical systems at large are nonlocal, thus requiring an integro–differential geometry for their adequate treatment. On the contrary, all geometries currently used for the interior problem of gravitation are of local–differential character.

In the transition to quantum mechanics and quantum field theory, the need to construct covering relativities appears to be even more pressing. In fact, as stressed earlier, the interior of stars, hadrons, and nuclei is actually constituted by extended particles in necessary conditions of mutual penetration and overlapping, resulting in the most natural as well as most general possible class of closed, integro–differential, variationally non-self-adjoint interactions. The complexity of these systems is such that they are simply outside the technical capabilities of existing relativities for the microscopic world, such as Einstein's special relativity in quantum kinematics and quantum field theory.

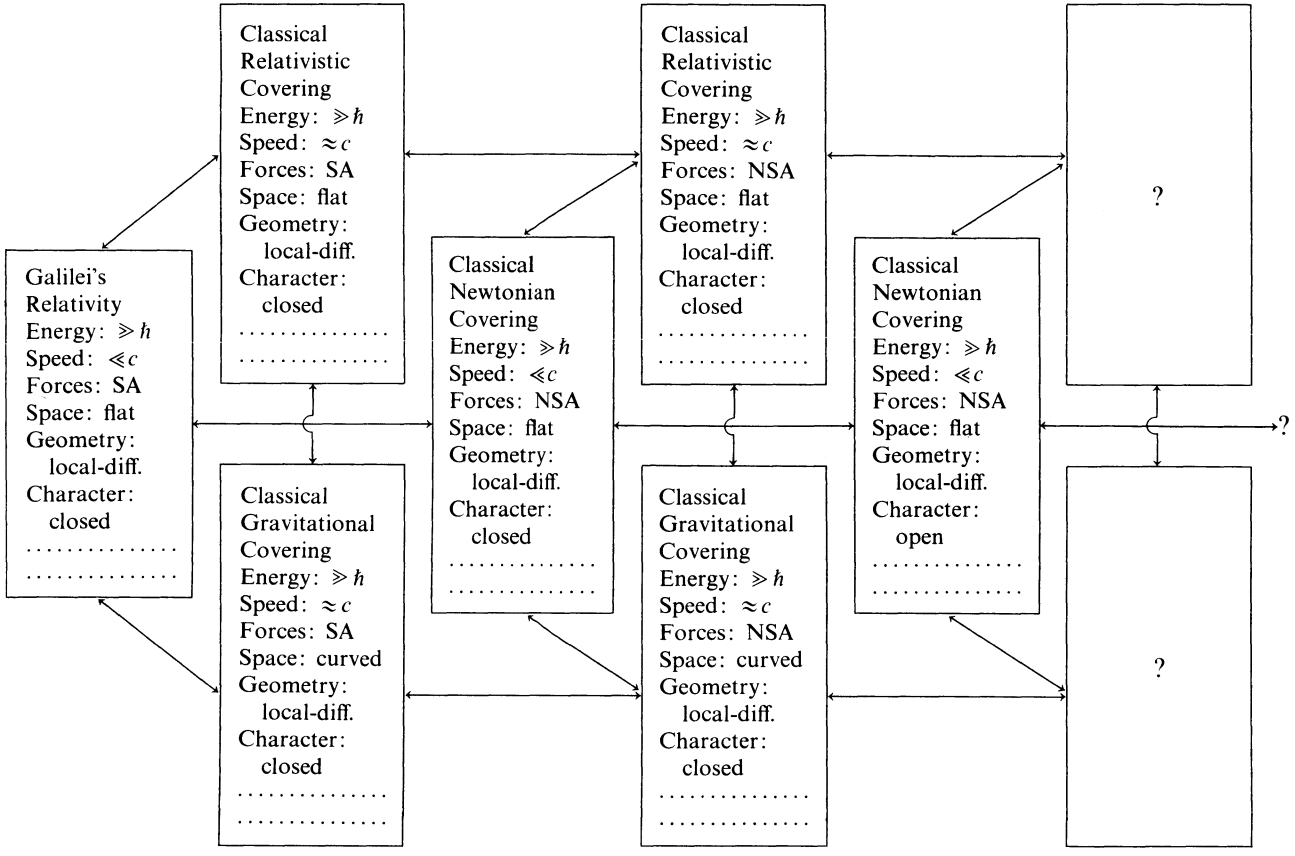

Galilei's
Relativity
Energy: ≫ ħ
Speed: ≪ c
Forces: SA
Space: flat
Geometry:
local-diff.
Character:
closed
Classical
Relativistic
Covering
Energy: ≫ ħ
Speed: ≈ c
Forces: SA
Space: flat
Geometry:
local-diff.
Character:
closed
Classical
Gravitational
Covering
Energy: ≫ ħ
Speed: ≈ c
Forces: SA
Space: curved
Geometry:
local-diff.
Character:
closed
Classical
Newtonian
Covering
Energy: ≫ ħ
Speed: ≪ c
Forces: NSA
Space: flat
Geometry:
local-diff.
Character:
closed
Classical
Relativistic
Covering
Energy: ≫ ħ
Speed: ≈ c
Forces: NSA
Space: flat
Geometry:
local-diff.
Character:
closed
Classical
Gravitational
Covering
Energy: ≫ ħ
Speed: ≈ c
Forces: NSA
Space: curved
Geometry:
local-diff.
Character:
closed
Classical
Newtonian
Covering
Energy: ≫ ħ
Speed: ≪ c
Forces: NSA
Space: flat
Geometry:
local-diff.
Character:
open
?
?
?

Figure 6.1. A schematic illustration of the absence in Physics of terminal theories, via an expansion of the diagram of Santilli (1978c, page 229). The first column depicts the fundamental relativity, Galilei's relativity, from which all others can be constructed via suitable conceptual and mathematical generalizations. The relativity is identified via the specialization of a number of characters, such as energy $\gg \hbar$, speed $\ll c$, forces SA, etc., plus a number of additional unlisted aspects, some of which are known and others conceivably unknown at this time. The second column depicts the contemporary generalizations of classical type into the special and the general relativity which preserve the local, closed, and self-adjoint character of Galilei's relativity. All other columns depict further generalizations, some of which have been identified while others are only expected. They are characterized via a chain of sequential generalizations of the characteristics of the original Galilei's relativity. In fact, the third column depicts the covering relativity presented in Section 6.3 which generalizes the self-adjoint character of the original relativity into a non-self-adjoint form, while leaving all remaining characters unchanged. The fourth column depicts further relativistic and gravitational extensions which are expected to be identifiable via techniques similar to those producing the relativities of the second column. The fifth column depicts the covering relativity of Lie-admissible type, which is based on the relaxation of the closed character of the systems, the representation of the time rate of variation of physical quantities, and the loss of antisymmetric character of the underlying algebra. Additional relativities can then be expected with a minimum of scientific vision, e.g., via the relaxation of the local–differential character of the geometry. Note that the conditions for a theory to be the covering of an existing one are verified by all relativities listed in the table, and must be verified by any new formulation to qualify as a truly "new" relativity. In particular, this demands not only the identification of the new physical arena (which can be essentially done via a continuation of the data appearing in this table), but the actual construction of *new mechanics*, as it has been the case for the chain of covering relativities of the central row (Hamiltonian Mechanics $\leftrightarrow$ Birkhoffian Mechanics $\leftrightarrow$ Lie-Admissible Mechanics). Also, note that the transition from one relativity to its covering is based on the improvement of the approximation of nature. The need to abandon inertial frames as well as transformations preserving such an inertial character has been stressed in the text as a necessary condition for complying with nature. On similar grounds, one can see the need to abandon the closed character of the systems. In fact, no system in nature (other than, perhaps, the entire universe) can be considered truly closed. The same situation occurs for all other characteristics. The understanding is that the time in which a specific improvement of our approximation of nature will be felt by the scientific community at large depends on unpredictable elements, such as technological developments. Perhaps a most intriguing aspect of the diagram is that, by no means, can it be considered exhaustive. In fact, a conceivable, corresponding chain of quantum mechanical relativities is not included (see, in this latter respect, Chart 6.1).

The need to generalize existing relativities appears rather natural if one meditates a moment on the limitations of their conceptual foundations. In fact, apart from technical differences, all existing relativities are based on the notion of the *pointlike constituent*, as conceived by Galilei and Newton and, more recently, embraced by Lorentz, Poincaré, Einstein, and others. The originators of the notion argued that when extended objects move in vacuum, their actual shape and structure do not affect the dynamics. Under these conditions, the objects can be well approximated as massive points. This produces the Galilean or Newtonian approximations of the sun, Earth, and all other planets as points, which were subsequently used by Lorentz, Poincaré, and Einstein, for the relativistic description of massive and charged particles.[25]

The development of contemporary relativities from this primitive notion can be understood via the techniques of these volumes. In fact, systems of pointlike constituents, by their very conception, demand that the interaction is localized at a collection of isolated points. This implies the local–differential character of the geometries, from the symplectic to the Riemannian geometry. Furthermore, pointlike particles, also by their very conception, demand that all admissible interactions are of action-at-a-distance, potential type. This implies the derivability from a conventional variational principle of all admitted systems, whether in Euclidean or Riemannian space. In short, the assumption of pointlike constituents implies the restriction of physical systems to those of closed self-adjoint type. The transition from one relativity to another is then performed on the basis of data which do not depend on the structure of the constituents and on the acting forces (see Figure 6.1 for more details).

However, while the Earth can be well approximated as a massive point for the description of its trajectory in the solar system, the same approximation becomes excessive with respect to a satellite in Earth's atmosphere or to a proton in the core of a star. In fact, the dynamic evolution of these latter systems is directly affected by their actual shape and structure. The fundamental notion of the isotopic covering of Galilei's relativity is therefore that of *extended constituent*. The generalization of the closed self-adjoint systems into the closed non-self-adjoint ones is a mere consequence of the extended character of the constituents. In fact, the moment the constituents acquire a dimension in space, contact effects due to collisions and other interactions become unavoidable. But the notion of potential energy has no physical basis for contact forces. The existence of closed non-self-adjoint interactions then becomes inevitable. The understanding, stressed a number of times during our analysis, is that our local non-self-adjoint treatment must be

[25] The reader is urged to study the *original contributions* by Lorentz (1904), Poincaré (1905), and Einstein (1905). He will discover that the restriction to pointlike particles moving in vacuum under long-range, action-at-a-distance interactions is expressed quite clearly in these limpid writings. These fundamental restrictions have often been omitted in subsequent studies by other authors.

considered a first approximation of an expected sequence of technical improvements of the description of extended particles.

The isotopic covering of Galilei's relativity has therefore been presented in these pages as the first, most rudimentary possible treatment of the generalization of the Galilean/Newtonian conception of point-like particles into extended particles. The existence of a chain of generalizations of other contemporary relativities then becomes rather natural.

Put explicitly, the covering relativity has been presented in these pages not as the end, but rather as the beginning of new advancement. It is a manifestation of my conviction that Physics is a science that will never admit terminal theories.

Chart 6.1. Applications to Hadron Physics

As is well known, Hamiltonian Mechanics is at the foundation of a number of branches of contemporary physics, ranging from statistical mechanics to field theory, to particle physics, etc. The existence of a Birkhoffian generalization of Hamiltonian Mechanics is therefore of fundamental relevance inasmuch as it implies the possible existence of corresponding generalized formulations in all branches of physics (as well as of science at large) currently treated with Hamiltonian methods. The advantage of the generalization can be anticipated from the analysis in these volumes; it consists of removing unnecessary restrictions on the structure of the systems represented, with consequential possibility of representing nature more realistically.

This volume would therefore be incomplete without an indication of the implications of Birkhoffian Mechanics for non-Newtonian branches of science. This chart is devoted to the basic ideas regarding applications to strongly interacting particles (*hadrons*). Needless to say, this is a volume on *Newtonian* mechanics and no in-depth treatment of other fields should be expected. The ideas are presented in their simplest possible current understanding and are supplemented by primary references for a technical study of the issues.

To begin, let us recall that quantum mechanics was conceived for the structure of atoms and for electromagnetic interactions at large. For this reason, the mechanics will be referred to herein as *Atomic Mechanics*. Its validity in the arena where it was conceived has been established by an impressive amount of experimental evidence and is assumed here.

Nevertheless, authoritative doubts on the final character of the mechanics have been expressed since its inception, and a number of them remain unresolved. We cite here the historical doubts by Einstein on the terminal character of Heisenberg's indeterminacy, Jordan's doubts on the associative character of the enveloping algebra of operators, Fermi's doubts on the applicability of conventional geometries (and relativities) within the region of space occupied by a proton or a neutron, etc. A review of these doubts can be found in Section 2.1 of the memoir (Santilli, 1979b).

With the passing of time, Atomic Mechanics has shown more and more limitations in effectively representing physical conditions which are increasingly different from those it originally described (point-like particles under mutual, long-range, electromagnetic interactions). We limit ourselves here to the observation that in the transition from the atomic

to the nuclear two-body problem, *suppression of the energy spectrum* occurs. In fact, while the hydrogen atom (or the positronium) has the well-known infinite spectrum of energy, no exited level has been identified to date for the deuteron (a minimum of five nucleons are needed to attain a nuclear structure admitting an energy spectrum reminiscent of the atomic one).

This fact alone could be sufficient to motivate the construction of a new mechanics specifically conceived for the structure of nuclei, under the condition that Atomic Mechanics is admitted not only as a particular case, but also for (the peripheral states of) heavy nuclei. In much the same way, Atomic Mechanics recovers its classical origin not only under the correspondence principle, but also for (the peripheral states of) heavy atoms (or for sufficiently large orbits). Deeper scrutiny allows one to see that Atomic Mechanics has been unable to achieve a quantitative, satisfactory representation of several aspects of nuclear physics (such as the total magnetic moments or even the total values of spin), even though the emergence of a meaningful first approximation is not denied. Most of all, despite over half a century of research, Atomic Mechanics has failed to produce the solution of the ultimate problem of nuclear dynamics: the nature of the nuclear forces.

In the transition to the deeper level of hadron structure, the limitations of Atomic Mechanics have emerged more clearly. For instance, even though the atomic two-body system is generally unphysical for the deuteron, at least it admits positive energies. In the transition to a hadronic two-body system of the type needed for the lightest known hadrons, the π^0 meson, even the positivity of the energy is generally lost. In fact, in this case we need very light constituents as compared to the total energy. Under these circumstances, no negative binding energy is generally possible via the (nonrelativistic) Schrödinger's equation, and the admissible values of the total energy become generally complex.[26] This is only the beginning. When the current status of hadron physics is examined objectively, a host of unresolved fundamental problems emerge. In nuclear physics, we can say that the use of Atomic Mechanics, while leaving unresolved the problem of the nuclear force, at least has permitted the final identification of the nuclear constituents. When applied to hadron physics, Atomic Mechanics has left unresolved not only the basic dynamics, but—essentially—the problem of the hadronic constituents, despite one of the most massive (financially and humanly) efforts in the history of physics. In fact, the conjecture that hypothetical particles called *quarks* are the constituents of hadrons, faces a number of still unresolved basic problems.[27] At any rate, on sound scientific grounds, we cannot claim today that the problem of the structure of hadrons has been resolved in a way comparable to that of the structure of nuclei and atoms.

The construction of a new mechanics, specifically conceived for the hadronic structure, is therefore advocated. This mechanics has been tentatively called *Hadronic Mechanics* by Santilli (1978d, p. 756) and the same terminology will be adopted here. By conception, the new mechanics must admit a simpler specialization called by the same author *Nuclear Mechanics* (*loc. cit.*). In turn, Nuclear Mechanics must admit

[26] See R. M. Santilli (1974, Appendix C) and quoted references.

[27] For a review of some of the problematic aspects of contemporary hadron physics, the reader may consult R. M. Santilli (1981c).

conventional Atomic Mechanics as a particular case, according to the enclosure properties

$$\textit{Atomic Mechanics} \subset \textit{Nuclear Mechanics} \subset \textit{Hadronic Mechanics}. \quad (1)$$

In the study of this chain of generalizations, several new, rather intriguing, mathematical, physical, and experimental problems emerge. The first problem is, predictably, of conceptual nature and consists of the identification of the physical differences between the atomic, nuclear, and hadronic forces. If Atomic Mechanics is used, all these forces are structurally the same, i.e., they are all of potential type. If a hierarchy of covering mechanics is advocated, this simplistic condition can be relaxed, opening the way to the study of broader physical structures. The analysis in these volumes, even though (or perhaps because) classical, can be valuable for this problem. In fact, our methods permit the identification of the following hierarchy of interactions of increasing structural complexity and methodological needs.

Class I: Closed, Local, Self-Adjoint Interactions: These are interactions which verify the conventional conservation laws of total quantities (closure), which occur at a finite number of isolated points (locality), and which verify the theorems of the Inverse Problem as being of action-at-a-distance, potential type (self-adjointness). The great majority of interactions of contemporary physics are of this type, of course, upon extension of the techniques of the Inverse Problem to relativistic and field theoretical settings. For instance, the electromagnetic interactions at their various levels of study, as well as the unified gauge theories of weak and electromagnetic interactions, are of closed, local, and selfadjoint type (Santilli (1978b)).

Class II: Closed, Local, Non-Self-Adjoint Interactions: These are interactions which are closed and local as those of Class I, but whose internal forces are structurally more general than those of the first class, inasmuch as they admit contact, nonpotential forces (non-self-adjointness), besides conventional, potential forces. A rather forceful example is given by Earth when considered as isolated from the rest of the universe. The system is closed, but the internal forces are generally nonderivable from a potential.

Class III: Closed, Nonlocal, Non-Self-Adjoint Interactions: These are interactions which are closed and non-self-adjoint as those of Class II, but which generally occur at all points of a volume or surface (nonlocality), therefore demanding the transition from conventional, ordinary (or partial) differential equations (Classes I and II) to integro-differential generalizations. The interactions occur whenever the extended size of the constituents cannot be ignored, e.g., for the motion of a satellite in Earth's atmosphere. Therefore, with deeper study, the Earth is a system of the more general Class III. Its treatment under Class II is a good approximation because, even though the non-locality is lost, the existence of contact, nonpotential forces is preserved. On the other hand, the use of interactions of Class I would lead to excessive approximations of perpetual-motion-type (e.g., motion of a satellite in our atmosphere with a conserved angular momentum).

In other words, the methods of the Inverse Problem identify the fact that the contemporary characterization of interactions via the selection of a Lagrangian or a Hamiltonian implies a fundamental, often excessive simplification of nature.

By recalling that atoms are systems of Class I, one can readily see that nuclei could be studied within the setting of Class II, while hadrons may

likely demand the still more general treatment of Class III. The argument is quite simple. Clear experimental evidence establishes that protons and neutron (as well as all hadrons) have a finite charge radius which is of the order of 10^{-13} cm ($=1F$). Once they are members of a nuclear structure, protons and neutrons are in average conditions of mutual penetration of the order of 10^{-3} units of their volume. This clearly suggests contact interactions for which the notion of potential energy has no physical basis.

The fundamental physical hypothesis for the construction of Nuclear Mechanics is therefore that *the nuclear forces have a non-Hamiltonian component which, as such, is outside the technical possibilities of Atomic Mechanics*. The construction of a covering theory then becomes mandatory. The fundamental approximation is that the non-Hamiltonian component is still local. The transition to full Hadronic Mechanics can be anticipated and consists of assuming a dynamics which is not only non-Hamiltonian, but also nonlocal, with the understanding that a local approximation may be meaningful for the structure of light hadrons.

Once the basic physical conditions have been identified, the next logical step is the identification of the mathematical tools for their treatment. As recalled in Section 6.1, Atomic Mechanics is, in essence, an operator realization of Lie's theory. The need to achieve a generalization of Lie's theory for the treatment of non-Hamiltonian forces is then inevitable.

This problem has been studied at the yearly Workshops on Lie-admissible Formulations of 1978–1981 (see the *Proceedings* of 1979 and 1981) by a number of pioneering mathematicians identified in Chart 4.7. The studies have resulted in two progressive generalizations of Lie's theory, one of *Lie-isotopic type* (which has been indicated in the charts of Chapter 5), and a more general one of *Lie-admissible type* (touched on in Chart 4.7).

The next problem is to identify the arena of applicability of these mathematical tools. At this point, new features emerge without counterpart in Atomic Mechanics. Within that theory, one single formulation is sufficient for the characterization of a bound state as a whole as well as its individual constituents (the point was elaborated upon in this volume for the case of closed, variationally self-adjoint, classical systems). This is not so in the more general class of closed, variationally non-self-adjoint systems, because specific dynamic effects at the constituent level (e.g., due to nonlocal forces) may "cancel out" in the treatment of the system as a whole (see equations (6.3.36b)). As a result, the mechanics which is effective for the *exterior treatment* of a bound state is not expected to be equally effective for the *interior treatment*, i.e., the description of the constituents. The use of two different but complementary mechanics is then advocated, one for the "global" treatment and one for characterizing the constituents.

The primary algebraic character of the two mechanics can be identified as follows. Assume that the time evolution for both mechanics is expressed by an algebra with product $A \times H$ where the operator H represent the energy,

$$\dot{A} = c(A \times H), \qquad c \in \mathbb{C}. \tag{2}$$

When the exterior description of a closed system is considered, the primary emphasis is on the conservation of total quantities, such as the total energy $H = H_{tot}$. In this case, the product of law (2) must necessarily be antisymmetric, that is, a product $A * H_{tot}$ must exist such that

$$\dot{A} = c(A \times H_{tot}) = c(A * H_{tot} - H_{tot} * A). \tag{3}$$

In fact, only an antisymmetric product is capable of permitting the conservation of the total energy

$$\dot{H}_{tot} = c(H_{tot} * H_{tot} - H_{tot} * H_{tot}) = 0. \tag{4}$$

Further arguments (e.g., related to the integrability conditions for the transformation theory, much along the lines of Chart 5.4) suggest the use of the Jacobi law as an additional condition. As a result, the product $A * H_{tot}$ must be Lie-admissible, that is (Chart 4.1), it must be such that the attached product $A * H_{tot} - H_{tot} * A$ is Lie. In this simple way we reach the conclusion that *the exterior treatment of all mechanics, whether for the structure of atoms, nuclei, or hadrons, is expected to have a Lie algebraic character.* However—and this is equally important—*the Lie algebraic character suggested by total conservation laws is not required to be of the conventional type* $A \times H_{tot} = [A, H_{tot}] = AH_{tot} - H_{tot}A$, *where* AH_{tot} *is the conventional associative product. One can* therefore see naturally the possibility of constructing a hierarchy of generalizations of Atomic Mechanics along hypothesis (1) via the use of a corresponding hierarchy of enveloping Lie-admissible algebras, as we shall indicate below.

When passing to the interior description, the situation becomes fundamentally different. In this case, the energy of the particle considered, $H = H_{part}$, is now strictly nonconserved. A necessary condition of consistency is therefore that the product $A \times H_{part}$ is not antisymmetric, that is, it must be *non-Lie*:

$$\dot{A} = c(A \times H_{part}) = \text{NON-LIE}, \qquad c \in \mathbb{C}. \tag{5}$$

In fact, only an algebra for which $A \times H_{part} \neq -H_{part} \times A$ can account for the time rate of variation of the energy

$$\dot{H}_{part} = c(H_{part} \times H_{part}) \neq 0, \tag{6}$$

but the interior and the exterior treatment of the same closed system must be compatible. This condition can be expressed, at the algebraic level, through the requirement that the antisymmetric part of the interior product coincide with the exterior product, i.e., $A \times B - B \times A \equiv A * B - B * A$. Thus, *for the case of the interior description* (5), *the product* $A \times H_{part}$ *is expected to be a nonassociative, non-Lie, Lie-admissible product* (some classical forms were presented in Chart 4.7). It is then easy to see that a hierarchy of interior hadronic mechanics can be constructed via a hierarchy of the algebras indicated.

In this chart, we cannot possibly review all the studies dealing with the application of this dual algebraic approach to Hadron Mechanics. These studies include (directly or indirectly) work by an increasing number of physicists:

- *experimental physics* (team leaders): R. J. Slobodrian (Université Laval, Québec), H. E. Conzett (Lawrence Berkeley Laboratory, Berkeley), H. Rauch (Atominstitut, Wien, Austria), and others;
- *theoretical physicists* (besides this author): G. Eder (Atominstitut, Wien, Austria), R. Mignani (Universitá di Roma, Roma, Italy), S. Okubo (University of Rochester, Rochester), E. Kapusćik (I.N.F. Warsaw, Poland), Chun-Xuan Jiang (Peking, China), A. Schober (I.B.R., Cambridge), J. Kobussen (Universität Zürich, CH), R. Trostel (Technische Universität, Berlin, W. Germany), D. P. K. Ghikas (University of Patras, Greece), J. Lõhmus, M. Kôiv, and L. Sorgsepp (Estonian Academy of Science, USSR), J.Fronteau

and A. Tellez-Arenas (Université d'Orléans, France), S. Guiaşu (Université de Québec), J. Salmon (Conservatoire Nationale, Paris), and others.

The interested reader may consult the four volumes of the *Proceedings of the First International Conference on Nonpotential Interactions and their Lie-admissible treatment* (1982) held at the Université d'Orléans, France, as well as the volumes of the reprint series edited by A. Schober (1982).

In this chart we shall limit ourselves to the review of few basic ideas underlying the branch of the Hadronic Mechanics that is more in line with the Birkhoffian Mechanics of the main text. By recalling that the latter is a classical realization of the Lie-isotopic theory, the selection of the (local) Lie-isotopic branch of Hadronic Mechanics is evident.

We have thus narrowed our objective to the second line of Diagram 1.

Diagram 1

Theory	Classical Realization	Operator Realization
Lie's Theory	Hamiltonian Mechanics	Atomic Mechanics
Lie-isotopic Theory	Birkhoffian Mechanics	Exterior branch of Hadronic Mechanics
Lie-Admissible Theory	Birkhoffian-Admissible Mechanics	Interior branch of Hadronic Mechanics

On more specific grounds, we shall indicate the generalization of the Hilbert space structure which seems advisable in order to represent closed non-Hamiltonian systems, according to the axiomatic studies by Myung and Santilli (1982 a and b).

Consider a Hamiltonian description of particle interactions as provided by Atomic Mechanics, with Hilbert space $\mathscr{H}$, unit $I = \hbar^{-1}$; basis $|a\rangle$; normalization $\langle a|a'\rangle = \delta_{aa'}$; enveloping algebra $\mathfrak{A}$ of operators $A, B, \ldots$ with conventional associative product AB; attached Lie algebra $\mathfrak{A}^-$ with product $[A, B] = AB - BA$, etc.

The construction of the exterior closed treatment of Hadronic Mechanics is based on the selection of a suitable isotopy operator $T(\mathbf{r}, \mathbf{p}, \ldots)$ verifying all needed topological conditions (positivity, Hermiticity, etc.), under which the algebra $\mathfrak{A}$ is mapped into the isotope $\mathfrak{A}^*$ with product $A * B = ATB$, with T fixed. The attached Lie algebra is then given by the isotope (Charts 5.1–5.5)

$$[\mathfrak{A}^*]^-: [A, B]_{\mathfrak{A}^*} = A * B - B * A = ATB - BTA, \tag{7}$$

with the understanding that possible, more general isotopes are not excluded.

The generalization of $\mathfrak{A}$ into $\mathfrak{A}^*$ essentially implies the generalization of Planck's unit $I = \hbar^{-1}$ into a bona fide (left and right) operator unit

$$I^* * A = A * I^* = A, \qquad I^* = T^{-1} \tag{8}$$

with the understanding that its space–time average can approach $\hbar^{-1}$ as closely as desired. The physical objective is to attempt a representation of the increased physical complexity of the processes of absorption and emission of energy in the transition from the structure of atoms (where electrons can freely "jump" from one orbit to the other), to nuclei (where nucleons cannot freely "jump" from one orbit to the other owing to the densely occupied volume of nuclei).

The alteration of the unit has rather profound implications for virtually all physical and mathematical aspects. We mention here only a few generalizations that are consequential. Since the unit is no longer I, normalization must be generalized accordingly, e.g., to the form

$$\langle a \mid * \mid a' \rangle = \langle a \mid T \mid a' \rangle = \delta^*_{aa'} = I^* \delta_{aa'}. \tag{9}$$

A similar generalization occurs for the decomposition of probability, expectation values, etc.

Under certain restrictions (particularly, the positivity of T) the generalized product $\langle a \mid * \mid a' \rangle$ is still an inner product, and the underlying space is still a Hilbert space we shall call $\mathcal{H}^*$. However, $\mathcal{H}^*$ does not act linearly on the conventional field of Atomic Mechanics, that of the complex numbers $\mathbb{C}$. In order to preserve the crucial linearity (clearly necessary to preserve the Hilbert character of the space), the field $\mathbb{C}$ is generalized into the operator form

$$\mathbb{C}^* = \{c^* \mid c^* = I^* c, \qquad c \in \mathbb{C}\} \tag{10}$$

where the "numbers" c^* are called *T-scalars*.

The isotopic generalization of all conventional operators of Atomic Mechanics (Hermitean, anti-Hermitean, unitary, antiunitary, etc.) is then predictable. Here we mention only the conditions for an operator U to be nonunitary but *T-unitary*:

$$U * U^{-1} = U^{-1} * U = I^*. \tag{11}$$

The admittance of the conventional atomic case as a particular case is evident.

The picture of basic generalizations is completed by that of traces

$$\mathrm{tr}_T A = I^* t_r A \tag{12}$$

and of determinants

$$\det_T A = (\det AT) I^*. \tag{13}$$

In this way we reach the following *isotopic generalization of the eigenvalue equations*

$$H * |\rangle = HT|\rangle = c^* * |\rangle = c|\rangle \tag{14}$$

also proposed by Myung and Santilli (*loc. cit.*). Its capability to represent non-Hamiltonian forces has been established as follows. One first notes that the *hadronic-isotopic generalization of Schrödinger's equations* can be written

$$i \frac{\partial}{\partial t} | \rangle = B * | \rangle \tag{15}$$

and constitutes a reformulation of equations (6.1.50) in $\mathcal{H}^*$. Non-Hamiltonian forces then follow from the fact that the classical image of (6.1.50) is given by the Birkhoffian generalization of Hamilton–Jacobi equations (6.1.24).

Since the enveloping algebra of operators $\mathfrak{A}^*$ is still associative, suitable generalizations of the various theorems of Lie's theory related to exponentiation are possible (Poincaré–Birkhoff–Witt theorem, Nelson theorem, etc.). The time evolution can therefore be characterized also via the finite, *Lie-isotopic group of T-unitary operators*[28]

$$A(t) = I^* e^{iTH} * A(0) * e^{-iHT} I^* \tag{16}$$

whose local expression is the *hadronic-isotopic generalization of Heisenberg's equations* (equations (18) of Chart 5.1), i.e.,

$$i\dot{A} = [A, B]_{\mathfrak{A}^*} = A * B - B * A. \tag{17}$$

Under certain conditions, (15) and (17) are equivalent in that they are connected by a T-unitary transformation of the type

$$U = e^{-itBT}. \tag{18}$$

Particularly significant is the fact that *unitary (and antiunitary) operators of Atomic Mechanics do not constitute, in general, symmetries of Hadronic Mechanics because they alter the structure of the isotopic product,* i.e.,

$$\begin{aligned} U[A, B]_{\mathfrak{A}^*} U^{-1} &= U(ATB - BTA)U^{-1} \\ &= [A', B']_{\mathfrak{A}^{*'}} = A'T'B' - B'T'A'. \end{aligned} \tag{19}$$

On the contrary, *a necessary condition for operators to constitute symmetries of Hadronic-Isotopic Mechanics is that they are T-unitary (or T-antiunitary)*. In fact, under this condition, we have the rule

$$\begin{aligned} U * [A, B]_{\mathfrak{A}^*} * U^{-1} &= UT(ATB - BTA)TU^{-1} \\ &= [A', B']_{\mathfrak{A}^*} = A'TB' - B'TA'. \end{aligned} \tag{20}$$

The construction of the hadronic-isotopic symmetries then follows the same conceptual pattern as the Birkhoffian symmetries introduced in Section 6.3. Particularly important is the emergence of a possible hadronic-isotopic generalization of Galilei's relativity via T-unitary operators as the operator image of the generalized relativity of Section 6.3:

$$A' = I^* e^{i\theta k T X k} * A * e^{-i\theta k X k T} I^*, \qquad [X_k, B]_{\mathfrak{A}^*} = 0. \tag{21}$$

This confirms that Hadronic Mechanics demands a generalization not only of the basic structure of Atomic Mechanics, but also (and perhaps

[28] We refer here to a particular case of the Lie-admissible generalization of Heisenberg's equations proposed by Santilli (1978d). A more recent formulation of the generalized equations is as follows. First, consider the open interior problem of strongly interacting particles, that is, the study of *one* hadron under external strong interactions, in much the same way Dirac conceived his equation for the electron. Second, differentiate the isotopies depending on whether the motion is forward or backward in time, and denote them with the time symbols $\rhd$ and $\lhd$, respectively. This implies the differentiation of all notions indicated in the text, including forward and backward Hilbert spaces $\mathcal{H}^{\rhd}$ and ${}^{\lhd}\mathcal{H}$. The generalization of Heisenberg's equations under consideration can then be written

$$A(t) = I^{\rhd} e^{itT^{\rhd}H} \rhd A(0) \lhd e^{-iH^{\lhd}T^{\lhd}} I, \tag{a}$$

and its local form is given by the Lie-admissible product (19) of Chart 5.1, i.e.,

$$i\dot{A} = (A, H) = A \lhd H - H \rhd A. \tag{b}$$

Note for subsequent needs the intrinsically irreversible character of the law, that is, its violation of the time-reversal symmetry regardless of the symmetry properties of the total energy operator. For more detail, see Santilli (1982c).

more importantly) of the very *notion of symmetry*. In fact, the structure of the symmetry groups is generalized into the Lie-isotopic form, predictably, in order to hold under non-Hamiltonian forces.

These ideas are sufficient to indicate the existence of a suitable isotopic generalization of the basic axioms and postulates of Atomic Mechanics, such as those related to states, observables, probabilities, etc. The corresponding generalization of the basic laws and principles is then consequential, with particular reference to Heisenberg's uncertainty principle, Pauli's exclusion principle, Einstein's frequency law, etc.

The nuclear and atomic particularizations are now self-evident and can be expressed as follows. Hadronic Mechanics reduces to Nuclear Mechanics whenever space–time averages of the isotopy operator T are possible, and it recovers Atomic Mechanics when these averages not only are possible, but yield the value $\hbar$.

The experimental verification of the new mechanics is encouraging at this writing, although understandably tentative, with particular reference to

- a violation of the time-reversal invariance recently measured by Slobodrian, Conzett, *et al.*, which is in remarkable agreement with the structure of Hadronic Mechanics;
- a small deformation of the charge distribution of hadrons under impact (and strong interactions) with nuclei, recently measured by Rauch and associates, which is predicted by Hadronic Mechanics; and
- a very small penetration of the wave packets of incident neutrons within the neutron core of the tritium, also indicated by Rauch and associates, which would imply a departure from Pauli's exclusion principle much in agreement with Hadronic Mechanics.

For these and additional experimental studies, we refer the interested reader to the specialized literature cited herein.

With an open mind toward the pursuit of new knowledge, we can say that the Hamiltonian restriction of the systems considered, which has conditioned science for over three quarters of a century, appears to be lifted, and the way is open to a variety of refreshingly new developments.

Chart 6.2. Applications to Statistical Mechanics

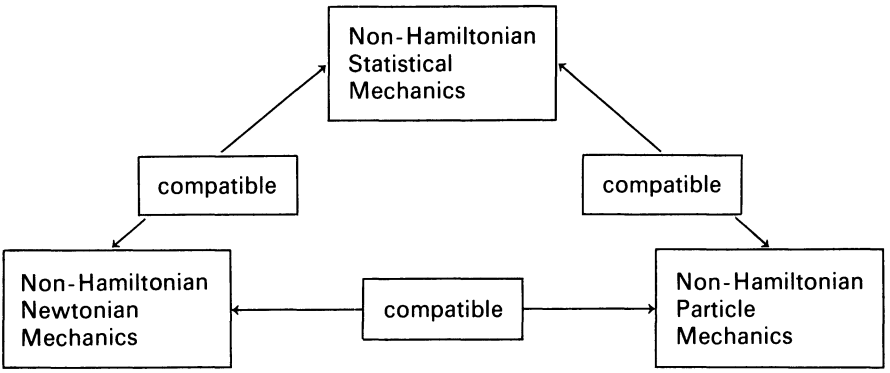
Some of Atomic Mechanics' most serious problems of consistency (when applied to physical arenas different from those for which it was conceived) are perhaps outside the realm of particle physics and rest in the current lack of unity in physics, as well as of science at large. In fact, all systems except particle physics are non-Hamiltonians. We are referring to systems in Newtonian Mechanics, Statistical Mechanics, Plasma Physics, Solid-State Physics, Engineering, Biophysics, etc.

The reduction of these *experimentally established* non-Hamiltonian systems to the *conjectured* Hamiltonian character of its particle components, according to Atomic Mechanics, is plagued with rather serious problems whenever quantitative studies are conducted. For instance, recall that the time evolution of Newtonian systems in our environment is *noncanonical* (Problem 5.9). If Atomic Mechanics holds for the description of the particle constituents of the Newtonian system, the noncanonical time evolution must be reduced to a large collection of *unitary* time

evolutions. No serious theoretical study of the consistency of this reduction, assuming that it can be established, is available at this moment.[29]

Clearly, the most natural idea suggested by the unity of physics is that the interactions of particles, even though Hamiltonian under electromagnetic interactions, may be non-Hamiltonian under contact, short-range interactions, that is, actual contact among molecules, atoms, nuclei, and hadrons. In fact, this idea permits a self-evident compatibility of different branches of science, as depicted in Diagram 2, that would be otherwise lacking.

Diagram 2.



Under these conditions, we would regain unity in science not only on the nature of the forces, but also on their mathematical structure, to the point that different theories would merely be different realizations of the same abstract mathematical structure.[29]

At any rate, the unequivocal irreversibility of the macroscopic reality sees its most natural origin in the nonpotentiality of systems. This view is embraced today by a number of researchers in Statistical Mechanics.

I. Prigogine (University of Texas at Austin and Université Libre de Bruxélles, Belgium) and his associates B. Misra, C. George, F. Henin, F. Mayné, and others (Université Libre de Bruxélles, Belgium) have established the nonconservative character of Statistical Mechanics at both the classical and operator level.[30] The mathematical structure of their time evolution is unknown at this writing. Nevertheless, it is likely to be of Lie-isotopic type. In fact, the operator structure of the theory can be expressed via a *nonunitary* transform of a conventional (atomic) time evolution of densities. This would yield the Lie-isotopic structure via the use of rule (6.2.28).

The advancement of Prigogine's statistics over preceding ones is remarkable and self-evident. Nevertheless, the problem of whether the nonconservativity is a collective property of systems or it originates at

[29] For a recent analysis of the problem of the unity of physics, consult Santilli (1982a).

[30] See, for instance, I. Prigogine (1977) and cited references, and B. Misra, I. Prigogine, and M. Courbage (1979).

the level of each individual particles, is left essentially unresolved in the theory.

A second group of statisticians, including J. Fronteau and A. Tellez-Arenas (Université d'Orléans, France), S. Guiaşu (Université du Québec, Canada), J. Salmon (Conservatoire National. Paris, France), M. Grmela (École Polytéchniques, Montréal, Canada), and others,[31] have independently studied the problem. This group begins the study from the experimentally established non-Hamiltonian character of Newtonian systems

$$(\dot{a}^\mu) = \begin{pmatrix} \dot{r}^k \\ \dot{p}_k \end{pmatrix} = (\Xi^\mu(a)) = \begin{pmatrix} p_k/m \\ f_k^{\mathrm{SA}}(t, \mathbf{r}, \mathbf{p}) + F_k^{\mathrm{NSA}}(t, \mathbf{r}, \mathbf{p}) \end{pmatrix} \tag{1}$$

and their representation, not with Hamilton's and Liouville's equations of the contemporary literature, but rather with the equations conceived by their originators, that is, the "true" Hamilton's equations

$$(\dot{a}^\mu) = \begin{pmatrix} \dot{r}^k \\ \dot{p}_k \end{pmatrix} = \left(\omega^{\mu\nu} \frac{\partial H}{\partial a^\nu} + \Gamma^\mu_{\mathrm{NSA}} \right) = \begin{pmatrix} \partial H/\partial p_k \\ -(\partial H/\partial r^k) + F_k^{\mathrm{NSA}} \end{pmatrix} \tag{2}$$

and the "true" Liouville's theorem

$$\frac{d}{dt} \log |J_t| = [\mathrm{div}_p F^{\mathrm{NSA}}]_{r_t, p_t} = \left[\frac{\partial F^{\mathrm{NSA}}}{\partial p} \right]_{r_t, p_t}, \tag{3}$$

$$J_t = D(\Xi_t)/D(\Xi_0).$$

The emerging statistical mechanics is therefore non-Hamiltonian by conception, in the sense that the time evolution of densities

$$\frac{\partial \rho}{\partial t} + [\rho, H] + F_k^{\mathrm{NSA}} \frac{\partial \rho}{\partial p_k} + \rho \, \mathrm{div}_p F^{\mathrm{NSA}} = 0 \tag{4}$$

cannot be entirely represented via the Poisson brackets but demands more general algebras. Intriguingly, the use of Lie-admissible algebras turns out to be directly universal for the statistical case considered, in full analogy with the Newtonian and particle cases, according to the rules

$$\frac{\partial \rho}{\partial t} + (\rho, H) = 0, \qquad (\rho, H) = \frac{\partial \rho}{\partial a^\mu} S^{\mu\nu}(t, a) \frac{\partial H}{\partial a^\nu}$$

$$(S^{\mu\nu}) = \begin{pmatrix} 0 & 1 \\ -1 & S \end{pmatrix}, \qquad S = \mathrm{diag}(\text{non-Hamiltonian terms}/(\mathrm{p/m})) \tag{5}$$

Irreversibility, entropy, and other aspects of Statistical Mechanics and Thermodynamics are then derived accordingly. The compatibility of the theory with physical reality is remarkable. Equally remarkable is the compatibility of such (Lie-admissible) Statistical Mechanics with the current experimental indications of the irreversible character of nuclear interactions (Chart 6.1). Clearly, the possible experimental finalization of nuclear irreversibility would imply a profound revision, not only of Atomic Mechanics, but also of Statistical Mechanics. Finally, note that the statistical studies by the two groups considered here are likely to be compatible, owing to the compatibility of the Lie-isotopic and Lie-admissible approaches (Chart 6.1).

[31] See the memoir Fronteau (1979) and the papers by the same authors in the *Proceedings of the First International Conference on Nonpotential Interactions* (1982).

Chart 6.3 Applications to Space Mechanics

The accuracy with which spaceships today can travel in the solar system and reach distant planets at the expected time and position is remarkable. Equally remarkable is our inability to make accurate predictions for the much shorter trajectory of a spaceship within our atmosphere.

We hope that the analysis in these volumes has clarified such differences. The former system is Hamiltonian and, as such, treatable with the body of methodological tools of Lie's theory. The second system, on the contrary, is non-Hamiltonian. The space mechanicist, therefore, simply does not possess directly applicable Lie-type tools.

We also hope that Birkhoffian Mechanics can fill this methodological gap, once developed up to the diversification needed for space applications including the Birkhoffian generalization of the canonical perturbation theory. This need has been anticipated by R. Broucke (1979), who has worked out several Birkhoffian generalizations of conventional Hamiltonian formulations for Space Mechanics (called Pfaffian by this author). Particularly remarkable is the completely identical rule of time and space coordinates emerging from these studies (see Section 5.3). The effectiveness with which nonpotential forces can be incorporated in the theory is also remarkable, as is the diversification of its applications to solar wind problems, optimization of flight paths, Galissot problem, etc.

Chart 6.4 Applications to Engineering

In this chart we shall outline applications to modern engineering. Let us emphasize from the outset that these studies are rather numerous. The objective of this chart, therefore, is mainly to outline some of the most representative contributions of which I am aware. Also, the review will be mainly conceptual, and the interested reader is urged to study the literature cited for the technical profile. Finally, the techniques developed by engineers appear to be relevant for fields other than engineering. We hope that this chart will be of some value in promoting a dialogue between engineering, physics, and applied mathematics.

Let us begin by outlining the studies independently conducted by H. H. E. Leipholz. Engineering systems are generally nonconservative in the sense of mechanics, and non-self-adjoint in the sense of the calculus of variations. Typical examples are fast-moving objects in viscous media or fast-moving viscous media in containing bodies (e.g., aircrafts, submarines, transportation vehicles, pipelines, etc.). The forces rendering the systems nonconservative are called *follower forces*, in the sense that, being frictional forces tangential to the surface, they follow the surface itself. For general treatments on the follower forces the reader may consult Bolotin (1963), Ziegler (1968), and Leipholz (1970).

Leipholz realized the power of the techniques of classical mechanics, calculus of variations, and optimal control theory. He conducted a series of studies aimed at rendering non-self-adjoint engineering systems treatable via variational techniques.

A modification of classical variational principles was studied by Leipholz (1977 and 1978a) and consisted of adding non-holonomic virtual work terms caused by the follower forces to the variation of kinetic and potential energy. Even though advantageous from the viewpoint of practical

engineering calculations, this modification remained unsatisfactory because, as known from mechanics, variational principles modified in this way cease to be stationary principles.

A first way for constructing equivalent self-adjoint forms of non-self-adjoint systems was identified by Bateman (Chart I.3.13) and consists of adding to the system considered its adjoint, expressed in terms of new variables. This technique was identified in engineering circles by a number of authors, including van Dungen (1945), Ballio (1967), and Prasad and Hermann (1969). A systematic study of this approach for the case of follower forces was conducted by Leipholz (1972). The approach was based on generalized Lagrangians and Hamiltonians, providing a joint representation of the system considered and its adjoint. It allowed the direct variational treatment of non-self-adjoint systems. The approach also allowed the development of hybrid equations of Ritz–Galerkin type (Leipholz, 1977), as well as the extension of classical stability theory (Leipholz, 1972).

This second approach remained unsatisfactory because of the duplication of the number of variables, with consequential duplication of the boundary conditions. Additional difficulties emerged within the context of a Liapunov-type stability theory because of the general lack of sign-definite character of the action functional.

Owing to this situation, Leipholz initiated a third stage of studies consisting of a generalized notion of self-adjointness (Leipholz, 1974a) which allowed the treatment of follower forces via a generalized Rayleigh quotient, with such applications as that to the Pflüger's rod. Further studies (Leipholz, 1974-b) pointed out the preservation, for non-self-adjoint systems which are self-adjoint in a generalized sense, of a number of features typical of conservative systems, such as the property that the systems become unstable by divergence. The possibility of having sign-definite functionals under generalized self-adjointness was pointed out by Leipholz (1976).

A comprehensive comparative analysis between conventional and generalized self-adjointness was studied by Leipholz (1974c). Conventional self-adjoint and conservative systems were first considered with particular reference to the following properties. (1) Their eigenvalues are real so that they become unstable by divergence. (2) Their energy is conserved. (3) Their energy functional may be used as a Liapunov functional. (4) They possess a Rayleigh quotient with extremum properties. These systems were called *conservative systems of the first kind*. Secondly, nonconservative non-self-adjoint systems with follower forces were selected so as to be generalized self-adjoint and to possess a sign-definite functional as a generalization of the energy functional. In particular, it was shown that properties (1)–(4) can all be preserved via the replacement of the energy functional with the generalized functional. Under these conditions, the systems were called *conservative systems of the second kind*. If the generalized functional is sign indefinite, the systems were called *conservative systems of the third kind* (Leipholz, 1980). In this latter case, some of the properties of conservative systems are preserved and others not. For instance, the system may have complex eigenvalues and may thus become unstable by flutter. For an outline, as well as a detailed presentation of the generalized variational principles under consideration, the reader may consult Leipholz (1978a).

Further studies based on generalized adjointness and self-adjointness with the treatment of the convolution theorem and of the follower forces, are given in Leipholz (1978b and c). An interpretation of the new theory

as basic to the finite element method and the final, most abstract, presentation of the new theory, involving the notion of linear function spaces with semi-scalar products, is given in Leipholz (1980).

Next, we would like to bring to the physicist's attention the research well known in engineering circles by K. Huseyin (see the recent paper by this author (1982) and cited references, as well as the monograph by the same author (1975)). These studies have brought into focus the relationships existing between instabilities, bifurcations, and catastrophes for some of the most general possible classes of systems, those of nonlinear and nonpotential type. The underlying methods appear to be applicable to a number of situations in Newtonian, Statistical, and particle mechanics.

The studies by Huseyin also cover a considerable number of practical cases, including conservative, pseudo-conservative, gyroscopic, and circulatory systems under the presence of dampings. The critical divergence conditions that lead to instabilities are analyzed via general and critical points within the context of a unified theory.

Additional aspects are related to the study of fold, cusp, etc., catastrophes, as well as flutter instability, Hopf bifurcation, etc. which have been identified in conservative systems, and extended by Huseyin to more general systems. (The unity of the analytic methods underlying all sciences which emerges in the study of engineering research is remarkable.)

We would like to indicate also the studies conducted by a group of engineers at Drexel University, including L. Y. Bahar, H. G. Kwatny, F. M. Massimo, and others. See the publications by these authors of (1977, 1978a and b, 1979a and b). The main line of these studies has been the reduction of a non-self-adjoint system into an equivalent self-adjoint form (see Appendix A). The main application has been to interconnected electrical power systems.

The need to bring dissipative systems into the framework of the classical theory applicable to conservative systems stems from the fact that, in the reduction of the original large-scale physical system to several subsystems of simpler nature, the reduced order system must retain the essential physical features of the original system.

While several alternative methods for preserving physical structure have been proposed, the studies considered here adopted the definition of retention of physical structure as being synonymous with preserving the canonical representation throughout the reduction process. Thus a reduced system retains the physical structure of the original canonical system if it can be represented by a set of Hamiltonian canonical equations. The Hamiltonian characterizing the original systems is, in general, different from that of the reduced system, but they must both include the dissipative effects that are ever present in the large-scale interconnected electric power systems.

Finally, we would like to bring to the reader's attention the studies by V. M. Fatić and W. A. Blackwell (1979a and b, and references cited therein). These studies were applied to network theory and were centered on the construction of variational principles for non-self-adjoint systems according to the following specializations:

(i) generalization of the image method to the linear discrete systems and networks with time-varying parameters;
(ii) extension of the multiplier method to a class of nonlinear discrete systems with one degree of freedom;
(iii) derivation of variational principles for the lossy transmission line with constant parameters by the multiplier and the image method;

(iv) broadening of the conventional framework of variational principles to include Lagrangians containing path-dependent integrals.

Particularly instructive for the physicist is the analysis of nonlocal (integro-differential) models by a number of engineering studies indicated in this chart.

Chart 6.5 Applications to Biophysics

In this final chart, we would like to indicate the related studies in biophysics by C. J. Lumsden, E. H. Trainor, and E. O. Wilson. In this way the reader can see that the applications of the methods considered in these volumes go beyond physical science as commonly understood and involve biophysics as well as other fields (such as economics) which are not reviewed here.

There are many biophysical and biological systems which can be effectively treated by local, first-order, ordinary (or partial) differential equations (vector fields)

$$\dot{x}_k = X_k(\mathbf{x}), \qquad k = 1, 2, \ldots, N \tag{1}$$

For a general study, the reader may consult the recent monograph by C. J. Lumsden and E. O. Wilson (1981).

Phenomenological models like these are applied routinely in many disciplines with which biophysics makes contact, including biochemistry, molecular biology, physiology, ecology, and the social sciences. For example, chemical concentration variables in the Lotka biochemical oscillator follow rate laws of the form

$$\begin{aligned} \dot{x}_1 &= \alpha_1 + \beta_1 e^{x_2} \\ \dot{x}_2 &= \alpha_2 + \beta_2 e^{x_1}, \end{aligned} \tag{2}$$

while the number of animals alive in certain two-species ecosystems can be modeled by the dynamics

$$\begin{aligned} \dot{x}_1 &= \alpha_1 x_1 + \beta_1 \gamma_2 e^{x_2} \\ \dot{x}_2 &= \alpha_2 x_2 + \beta_2 \gamma_1 e^{x_1}. \end{aligned} \tag{3}$$

Since most organisms display a high degree of self-regulation, the vector fields $\mathbf{X}(\mathbf{x})$ of greatest interest in models like (1) are nonconservative and, in fact, highly dissipative. This dissipation can express itself in simple forms, as in isolated attractor points (biologic "thermostats") or limit cycles (biologic "clocks") but, models with two or more degrees of freedom, can easily slip into a complex nonconservative dynamic "chaos."

To date, the exposition of rate law models has proceeded in mathematical biology and biophysics more or less independently of advances in Newtonian dynamics. An *ésprit* even exists to the effect that such models can have no contact with Newtonian or Hamiltonian structures because the latter deal only with conservative systems and have nothing to say about problems of biological relevance. Such a position must, of course, be fundamentally revised.

In a new set of papers on Hamiltonian structures in biology, Lumsden and Trainor (1977, 1979a, b, and 1980) have identified several major benefits to be gained from systematic study of biological equations of motion using the Inverse Problem approach. Until recently, the Inverse Problem and its generalizations have been left unexplored in all but a few conservative biological models. It is becoming clear, however, that analysis of a broad spectrum of such models using the inverse problem approach can contribute directly to the classification and enumeration of biologically relevant vector fields $\mathbf{X}(\mathbf{x})$. A major enterprise in mathematical biology, the production of these vector fields in local coordinate form, has somewhat outstripped the set of techniques available for their qualitative analysis. The Inverse Problem, by connecting such vector fields to Lagrangian and Hamiltonian dynamics, is clearly a useful new tool.

A second key advantage recalls that for many applications the complexity of real organic systems cannot be ignored and that N, the number of equations in (1), is very large. Lumsden and Trainor are especially concerned with this problem of biophysical complexity and with adapting methods from many-body theory to predict the collective properties which, in analogy to physical properties like temperature and pressure, define an organic system's overall structure and function. These properties correspond to the model's *phenotype* and are of crucial biological significance. Although many of the tenets of Statistical Mechanics do not require a Hamiltonian framework, such a frame of reference is useful in deciding the correct handling of biophysical models which combine great complexity with dissipation, self-regulation, and self-reproduction.

Lumsden and Trainor also point out that mathematical biology has a direct, as well as an inverse problem. In the direct problem, questions about optimal design and efficiency in living systems lead to important applications of variational principles and Hamiltonian structures, usually within the framework of optimal control theory. Given the established significance of this type of reasoning in the life sciences, the generalized inverse problem takes on a further role. For every successful inverse construction from (1) ending in a variational principle, a new candidate for a principle of optimal organic design has been discovered.

Lumsden and Trainor start from the suggestion that in biological applications a natural "Lagrangian" for (1) which has local existence properties and is attached to the variational principle

$$\delta \int_{t_1}^{t_2} L \, dt = 0 \qquad \text{(fixed end points)} \tag{4}$$

would be first-order in the rates $\dot{x}_k$:

$$L = U_k(\mathbf{x})\dot{x}_k - U_0(\mathbf{x}) \qquad \text{(summation convention)}. \tag{5}$$

Explicit time dependence of the functions $U_0(\mathbf{x})$ and $U_k(\mathbf{x})$, $k = 1, \ldots, N$ can occur in dissipative systems. For example, a linear Lagrangian corresponding to the conservative biochemical oscillator (2) is

$$L = \tfrac{1}{2}(\dot{x}_1 x_2 - x_1 \dot{x}_2) - (\alpha_1 x_1 + \beta_1 e^{x_1}) + (\alpha_2 x_2 + \beta_2 e^{x_2}) \tag{6}$$

while that for the dissipative, self-regulating ecology (3) is

$$L = \tfrac{1}{2} e^{-(\alpha_1+\alpha_2)t}[\dot{x}_1 x_2 - x_1 \dot{x}_2 + (\alpha_1 - \alpha_2) x_1 x_2 + 2\beta_2 \gamma_1 e^{x_2} - 2\beta_1 \gamma_2 e^{x_1}]. \tag{7}$$

Due to the everywhere singular structure of the linear Lagrangian (recall Chart I.3.8),

$$\left\| \frac{\partial^2 L}{\partial \dot{x}_k \partial \dot{x}_j} \right\| = \mathbb{O}, \tag{8}$$

a standard canonical Hamiltonization, and Statistical Mechanics has remained an elusive goal for mathematical biophysicists. Lumsden and Trainor have now shown that considerable explicit Hamiltonian content can be synthesized for such systems by directly using the singular properties of L. Two Hamiltonian structures have been obtained and used to extend many-body theory to biological applications.

Both structures are generalizations of the standard Hamiltonian algorithm. Lumsden and Trainor point out that there is a Lie bracket induced by the linear Lagrangian that makes (1) equivalent to a generalized Poisson bracket (GPB) dynamics:

$$\begin{aligned} \dot{x}_k &= [\![x_k, U_0(\mathbf{x})]\!] \\ [\![\cdot, \cdot]\!] &= \Gamma^{lm}\, \partial_l \cdot \partial_m \cdot \\ \Gamma^{lm} &= -\left\| \frac{\partial U_l}{\partial x_m} - \frac{\partial U_m}{\partial x_l} \right\|^{-1} \end{aligned} \tag{9}$$

which is exactly the Birkhoffian time evolution, here denoted (BIR), of this volume.

An attempt to "Hamiltonize" (5) using the standard algorithm ends abruptly with the momenta p_k related not to the velocities $\dot{x}_j$ but to the configuration coordinates x_j by the set of phase space constraints

$$p_k = U_k(\mathbf{x}), \qquad k = 1, \ldots, N \tag{10}$$

so that in a $2N$-dimensional phase space of coordinates $(x_1, \ldots, x_N, p_1, \ldots, p_N)$ the system must move on a hypersurface $\mathscr{M}$ defined by the vanishing of the functions

$$\varphi_k = p_k - U_k(\mathbf{x}). \tag{11}$$

Lumsden and Trainor have shown that the linear Lagrangian (5) and the constraint system (11) are sufficiently simple to be treated by Dirac's theory of generalized Hamiltonian dynamics (DIR). In their work the Dirac theory has its first biophysical and many-body applications.

For the linear Lagrangian, a Dirac *total Hamiltonian*

$$\mathscr{H}_T = U_0 + \varphi_k \dot{x}_k \tag{12}$$

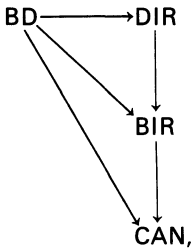
induces the canonical dynamics

$$\begin{aligned} \dot{x}_k &\approx [x_k, \mathscr{H}_T] \\ \dot{p}_k &\approx [p_k, \mathscr{H}_T] \end{aligned} \tag{13}$$

on the $2N$-dimensional $(\mathbf{x}, \mathbf{p})$-phase space. In this symbolism $[\cdot, \cdot]$ is the *standard* Poisson bracket and $\approx$ are the weak equalities. They mean that

the phase flow is constrained to $\mathcal{M}$ by allowing the φ_k to vanish after the *PB*'s have been fully evaluated.

Because there can be other Hamiltonization procedures (call them CAN) leading to canonicalization of the biodynamics (BD) (1) without a singular Lagrangian, DIR and GPB do not exhaust the range of possibilities. Lumsden and Trainor have shown, however, that such possibilities are linked in a framework



(14)

where the diagram is commutative. Mathematical biology has thus been equipped with a structure which brings the known routes from (1) to explicit Hamiltonian models to bear in a unified way.

These dynamic results prepare the ground for models which are large-N complex. Lumsden is currently studying the vector fields induced by (14) on sets of macrovariables $A_1(\mathbf{x}), \ldots, A_M(\mathbf{x})$, $M \ll N$, which model the collective properties of (1). In an ensemble theory, (14) yields generalized Liouville equations, such as

$$\partial_t \rho \approx \mathcal{L}_\rho \qquad \mathcal{L} = [\mathcal{H}_T, \cdot] \tag{15}$$

and generalized Langevin equations with the formal structure

$$\dot{A}_m = \mathrm{K}_m(\mathbf{A}) + \int_{-\infty}^{t} Q_m(\mathbf{x}) A_m(\mathbf{x}) d\mathbf{x} + F_m(t) \tag{16}$$

in the macrovariables. One must deduce for which types of organic system and under what boundary conditions the **A**-dynamics in (16) *closes* into $\dot{A}_m = Y_m(\mathbf{A})$, sealing off the **A**-level of the system from all but residual stochastic influence from the **x**-variables. When this occurs, the macrovariables and boundary conditions define a *new regime* of lawful pattern and order in the system. Biological systems are characterized by many such levels of pattern and order, and one of the great unsolved mysteries of modern biology is to understand why and through what mechanisms these various levels of organization arose during the long course of evolution by natural selection.

EXAMPLES

Example 6.1

Consider the equation

$$\left[\left(\ddot{r} + \frac{1}{r^2}\right)_{\mathrm{SA}} - \frac{\dot{r}^2}{r}\right]_{\mathrm{NSA}} = 0 \tag{1}$$

which can represent a hadron (with unit mass and charge) under a self-adjoint Coulomb force $1/r^2$ and a resistive non-self-adjoint force $\dot{r}^2/r$ due, for instance, to the penetration of the hadron within the region of space occupied by other hadrons (e.g., for a proton moving in the core of a star).

Birkhoffian representation (6.1.24) can be written

$$(R_\mu) = \left(\frac{p}{r}, 0\right), \qquad B = \tfrac{1}{2}p^2 + r \tag{2}$$

but it is not suitable for the representation of the system via the hadronic generalization of Schrödinger's equations (Section 6.1) because it violates regularity conditions (6.1.29). This deficiency is soon remedied via the Birkhoffian gauge

$$(R_\mu) \to (R_\mu^+) = \left(R_\mu + \frac{\partial G}{\partial a^\mu}\right) = \left(p\left(1 + \frac{1}{r}\right), r\right) \tag{3a}$$

$$G = rp \tag{3b}$$

under which (1) remains unchanged, as the interested reader is encouraged to verify. The representation of the system via generalization (6.1.24) of the Hamilton–Jacobi equation is then given by (ignoring subsidiary conditions)

$$\frac{\partial A^g}{\partial t} + \tfrac{1}{2}p^2 + r = 0 \tag{4a}$$

$$(R_\mu) = \begin{pmatrix} p\left(1 + \dfrac{1}{r}\right) \\ r \end{pmatrix} = \begin{pmatrix} \dfrac{\partial A^g}{\partial r} \\ \dfrac{\partial A^g}{\partial p} \end{pmatrix} = \left(\frac{\partial A^g}{\partial a^\mu}\right). \tag{4b}$$

To reach a form which is better suited for "Birkhoffian quantization," one can reduce the equations to a single partial differential form. This is accomplished via the change of coordinates (6.1.30),

$$Q = r, \qquad P = p\left(1 + \frac{1}{r}\right) \tag{5}$$

under which (4) becomes

$$\frac{\partial A^g}{\partial t} + \frac{1}{2}p^2 + r = \frac{\partial A^g}{\partial t} + \frac{1}{2}\left(\frac{PQ}{Q+1}\right)^2 + Q = \frac{\partial A^g}{\partial t} + \frac{1}{2}\left[\frac{\dfrac{\partial A^g}{\partial r}\dfrac{\partial A^g}{\partial p}}{\dfrac{\partial A^g}{\partial p} + 1}\right]^2 + \frac{\partial A^g}{\partial p}. \tag{6}$$

The use of rules (6.1.49) then yields the formal hadronic representation (Santilli (1982b))

$$i\frac{\partial}{\partial t}\psi(t, r, p) = \left[\frac{1}{2}\left(\frac{\dfrac{\partial}{\partial r}\dfrac{\partial}{\partial p}}{\dfrac{\partial}{\partial p} + 1}\right)^2 + \frac{\partial}{\partial p}\right]\psi(t, r, p) \tag{7}$$

where proper symmetrication is understood. Note that the zero-order term of (7) in $1/A^g$ under conditions (6.1.52) reproduces (6) identically.

It is important to note that hadronic equation (7) *does not* admit the conventional Schrödinger's equation (6.1.41) as a particular case. This can be seen from the property indicated earlier that classical equation (6.1.32a) does not recover the conventional equation (6.1.17a) at the canonical particularization $(R_\mu) = (p, 0)$.

Example 6.2

The reduction of Birkhoffian into Hamiltonian representations (Theorem 6.2.1) can be illustrated in a simple but important way in the case of *Newtonian electromagnetic interactions*. Consider the Hamiltonian and Birkhoffian representations of a charged particle under the Lorentz force as identified in Example 5.1

$$\begin{cases} H_{\text{Lorentz}} = \dfrac{1}{2m}[\mathbf{P} - e\mathbf{A}(t, \mathbf{r})]^2 + e\varphi(t, \mathbf{r}), \\ (R^0_\mu) = (\mathbf{P}; \mathbf{0}), \quad \mathbf{P} = m\dot{\mathbf{r}} + e\mathbf{A}, \end{cases} \tag{1a}$$

$$\begin{cases} B_{\text{Lorentz}} = \dfrac{1}{2m}\mathbf{p}^2 + e\varphi(t, \mathbf{r}) \\ (R_\mu) = (\mathbf{p} + e\mathbf{A}; \mathbf{0}), \qquad \mathbf{p} = m\dot{\mathbf{r}}. \end{cases} \tag{1b}$$

It is easy to see that the familiar transformation characterizing the minimal coupling rule

$$(a^\mu) = (\mathbf{r}, \mathbf{p}) \to (b^\mu) = (\mathbf{r}, \mathbf{P}), \qquad \mathbf{P} = \mathbf{p} + e\mathbf{A} \tag{2}$$

is a *Darboux's transformation* because it transforms the Birkhoffian into the Hamiltonian representation

$$R'_\mu = \left(\frac{\partial a^\alpha}{\partial b^\mu} R_\alpha\right)(b) \equiv R^0_\mu \tag{3a}$$

$$B' = B_{\text{Lorentz}}(a(b)) = H_{\text{Lorentz}}(b). \tag{3b}$$

Note that transformation (2) *is not* canonical.

Example 6.3

We illustrate here the notion of closed non-self-adjoint systems (Section 6.3) for the particular case of two-particle constituents. The equations of motion under consideration are given by a non-self-adjoint generalization of conventional two-body systems

$$\begin{cases} \dot{\mathbf{r}}_k = \mathbf{p}_k/m_k, \qquad k = 1, 2 \\ \dot{\mathbf{p}}_k = \mathbf{f}^{\text{SA}}_k(\mathbf{r}) + \mathbf{F}^{\text{NSA}}_k(t, \mathbf{r}, \mathbf{p}), \end{cases} \tag{1}$$

under closure conditions (6.3.36), i.e.,

$$\sum_{k=1}^{2} {}_k\mathbf{F}_k^{\mathrm{NSA}} \equiv 0, \tag{2a}$$

$$\sum_{k=1}^{2} \mathbf{p}_k \cdot \mathbf{F}_k^{\mathrm{NSA}} \equiv 0, \tag{2b}$$

$$\sum_{k=1}^{2} \mathbf{r}_k \times \mathbf{F}_k^{\mathrm{NSA}} \equiv 0. \tag{2c}$$

The model was proposed by Santilli (1978c, pp. 623–633) and the results can be reviewed as follows. Constraint (2a) implies that $\mathbf{F}_1^{\mathrm{NSA}} = -\mathbf{F}_2^{\mathrm{NSA}} \stackrel{\mathrm{def}}{=} \mathbf{F}^{\mathrm{NSA}}$. Thus the motion is in a plane as for conventional self-adjoint two-body systems. The systems can then be written

$$M\ddot{\mathbf{R}} = 0, \qquad \mu\ddot{\mathbf{r}} - \mathbf{f}^{\mathrm{SA}}(\mathbf{r}) - \mathbf{F}^{\mathrm{NSA}}(\mathbf{r}, \dot{\mathbf{r}}) = 0 \tag{3a}$$

$$\dot{\mathbf{r}} \cdot \mathbf{F}^{\mathrm{NSA}} \equiv 0 \tag{3b}$$

$$\mathbf{r} \times \mathbf{F}^{\mathrm{NSA}} \equiv 0 \tag{3c}$$

$$M = m_1 + m_2, \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}, \qquad \mathbf{R} = \mathbf{r}_1 + \mathbf{r}_2, \qquad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2. \tag{3d}$$

The conservation of the total linear momentum and the uniform motion of the center of mass are now ensured. Equations (3b) and (3c) then ensure the conservation of the total energy and of the total angular momentum, respectively.

It is easy to see that the force

$$\mathbf{F}^{\mathrm{NSA}} = g\ddot{\mathbf{r}}, \qquad g = \text{const.} \tag{4}$$

is one of the simplest possible solutions of conditions (3b) and (3c). Note that *force* (4) *is non-Newtonian* and that *the only admissible orbit is the circle.*

The model was also studied by Tellez–Arenas, Fronteau, and Santilli (1979) who examined the solution.

$$\mathbf{F}^{\mathrm{NSA}} = \mu\dot{\mathbf{r}}\varphi(\tfrac{1}{2}\mu\dot{\mathbf{r}}^2 + V(r)) = \mu\dot{\mathbf{r}}\varphi(E) \tag{5}$$

where φ is a continuous function with simple zeros. It is easy to see that the constraints restrict the admissible φ-functions and the trajectories to those for which $\varphi(E) = 0$. The statistical implications are intriguing but, for brevity, will not be reviewed here.

Example 6.4

We illustrate here the form-invariance of a vector field (or, equivalently, of a system of first-order differential equations) under the Hamiltonian and the Birkhoffian time evolutions which are at the basis of the time component of Galilei's relativity and of its isotopic generalization, respectively.

Consider first the free particle of unit mass

$$\Xi_0 = \Xi_0^{\mu}(a)\frac{\partial}{\partial a^{\mu}}, \qquad (a^{\mu}) = \begin{pmatrix} r \\ p \end{pmatrix}, \qquad (\Xi_0^{\mu}) = \begin{pmatrix} p \\ 0 \end{pmatrix}, \qquad \mu = 1, 2 \tag{1}$$

with Hamiltonian representation

$$\Xi_0^\mu = \omega^{\mu\nu} \frac{\partial H}{\partial a^\nu}, \qquad H = \tfrac{1}{2}p^2. \tag{2}$$

The canonical realization of the time component of Galilei's relativity in this simple case reads

$$\begin{aligned}(\hat{a}^\mu(\hat{t})) &= \exp\left[\hat{t}\omega^{\mu\nu} \frac{\partial H}{\partial a^\nu} \frac{\partial}{\partial a^\mu}\right](a^\mu(t)) \\ &= \begin{pmatrix} r + \dfrac{\hat{t}}{1!}[r, H] + \dfrac{\hat{t}^2}{2!}[[r, H], H] + \cdots \\ p + \dfrac{\hat{t}}{1!}[p, H] + \dfrac{\hat{t}^2}{2!}[[p, H], H] + \cdots \end{pmatrix} = \begin{pmatrix} r + \hat{t}p \\ p \end{pmatrix}\end{aligned} \tag{3}$$

and implies the trivial form-invariance

$$\begin{pmatrix} \dot{r} \\ \dot{p} \end{pmatrix}_t - \begin{pmatrix} p \\ 0 \end{pmatrix}_t = 0 \to \begin{pmatrix} \dot{\hat{r}} \\ \dot{\hat{p}} \end{pmatrix}_{\hat{t}} - \begin{pmatrix} \hat{p} \\ 0 \end{pmatrix}_{\hat{t}} = 0. \tag{4}$$

The understanding (indicated in Section 6.3) is that the explicit form of the time evolution is different for different Hamiltonians. Thus the form-invariance under consideration is characterized by a variety of transformations, one per each given (conservative) system.

We consider now the following non-self-adjoint generalization of system (1),

$$(\Xi^\mu) = \begin{pmatrix} p \\ F^{\text{NSA}}(r, p) \end{pmatrix}, \qquad \mu = 1, 2, \tag{5}$$

where possible self-adjoint forces are ignored for simplicity (but without loss of generality in the results). The Birkhoffian representation of systems (5) reads (Section 4.3)

$$\Xi^\mu \equiv \Omega^{\mu\nu}(a) \frac{\partial B(a)}{\partial a^\nu}. \tag{6}$$

Galilei's time component (3) is then generalized into the isotopic form

$$\begin{aligned}(\hat{a}^\mu(\hat{t})) &= \exp\left[\hat{t}\hat{\Omega}^{\mu\nu}(a) \frac{\partial B}{\partial a^\nu} \frac{\partial}{\partial a^\mu}\right](a(t)) \\ &= \begin{pmatrix} r + \dfrac{\hat{t}}{1!}[r, B]^* + \dfrac{\hat{t}^2}{2!}[[r, B]^*, B]^* + \cdots \\ p + \dfrac{\hat{t}}{1!}[p, B]^* + \dfrac{\hat{t}^2}{2!}[[p, B]^*, B]^* + \cdots \end{pmatrix}\end{aligned} \tag{7}$$

and verifies the form-invariance

$$\begin{pmatrix} \dot{r} \\ \dot{p} \end{pmatrix}_t - \begin{pmatrix} p \\ F^{\text{NSA}}(r, p) \end{pmatrix}_t = 0 \to \begin{pmatrix} \dot{\hat{r}} \\ \dot{\hat{p}} \end{pmatrix}_{\hat{t}} - \begin{pmatrix} \hat{p} \\ F^{\text{NSA}}(\hat{r}, \hat{p}) \end{pmatrix}_{\hat{t}} = 0 \tag{8}$$

under sufficient topological conditions here ignored (Problem 6.5).

Illustrations of law (8) have been given by Santilli (1978e) for a number of specific cases. Consider first the particle with linear velocity damping force

$$(\Xi^{\mu}) = \begin{pmatrix} p \\ -\gamma p \end{pmatrix}. \tag{9}$$

Then series (7) converge to the closed form

$$\begin{aligned} \hat{r} &= r - \frac{1}{\gamma} p(e^{-\gamma \hat{t}} - 1) \\ \hat{p} &= p e^{-\gamma \hat{t}} \end{aligned} \tag{10}$$

under which we have the form-invariance

$$\begin{pmatrix} \dfrac{dr}{dt} - p \\ \dfrac{dp}{dt} + \gamma p \end{pmatrix} = 0 \rightarrow \begin{pmatrix} \dfrac{d\hat{r}}{d\hat{t}} - \hat{p} \\ \dfrac{d\hat{p}}{d\hat{t}} + \gamma \hat{p} \end{pmatrix} = 0. \tag{11}$$

Consider next the case of the particle with a nonlinear (quadratic) damping force

$$(\Xi^{\mu}) = \begin{pmatrix} p \\ -\gamma p^2 \end{pmatrix}. \tag{12}$$

Then series (7) converge in this case to the closed form

$$\begin{aligned} \hat{r} &= r + \frac{1}{\gamma} \ln(1 + \gamma \hat{t} p) \\ \hat{p} &= \frac{p}{(1 + \gamma \hat{t} p)} \end{aligned} \tag{13}$$

for which we have again the form-invariance

$$\begin{pmatrix} \dfrac{dr}{dt} - p \\ \dfrac{dp}{dt} + \gamma p^2 \end{pmatrix} = 0 \rightarrow \begin{pmatrix} \dfrac{d\hat{r}}{d\hat{t}} - \hat{p} \\ \dfrac{d\hat{p}}{d\hat{t}} + \gamma \hat{p}^2 \end{pmatrix} = 0. \tag{14}$$

A similar case occurs for other examples of systems (5).

Note the variation of the symmetry with the system. Note also that *this is not new*, inasmuch as it is inherent in the canonical realization of Galilei's relativity.

As a final note, it should be indicated that the examples recalled here were worked out for the still more general Lie-admissible generalization of Galilei's relativity proposed by the author.

Example 6.5

It is well known that the meaning of relativity is a form-invariant description of the physical characteristics of the systems admitted. For the case of Galilei's relativity, Galilei's form-invariance is an expression of the closed self-adjoint character of the systems (Section 6.3). In this example we illustrate the fact that the Galilei-isotopic relativity carries exactly the same physical content. A main difference is the removal of

unnecessary restrictions on the nature of the forces. The understanding is that, under such condition, conventional space–time symmetries are broken (Chart A.12). In the final analysis, this breaking is a prerequisite for a genuinely new relativity.

Let us consider the canonical realization of the component of Galilei's relativity dealing with the rotation of a system in $E(3)$ around the third axis. By denoting with M_3 the generator (third component of the conventional angular momentum), the component under consideration can be written

$$
\begin{aligned}
(\hat{a}^\alpha) &= \exp\left[\theta\omega^{\mu\nu}\frac{\partial M_3}{\partial a^\nu}\frac{\partial}{\partial a^\mu}\right](a^\alpha) \\
&\quad \times \begin{pmatrix} r_k + \dfrac{\theta}{1!}[r_k, M_3] + \dfrac{\theta^2}{2!}[[r_k, M_3], M_3] + \cdots \\ \\ p_k + \dfrac{\theta}{1!}[p_k, M_3] + \dfrac{\theta^2}{2!}[[p_k, M_3], M_3] + \cdots \end{pmatrix} \\
&= \begin{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} \\ \\ \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \end{pmatrix}, \qquad k = 1, 2, 3, \qquad \alpha = 1, 2, \ldots, 6.
\end{aligned} \tag{1}
$$

The underlying form-invariance is then similar to that of Example 6.4 and reads

$$
\begin{pmatrix} \dot{r}_k \\ \dot{p}_k \end{pmatrix} - \begin{pmatrix} p_k \\ F_k^{SA}(\mathbf{r}) \end{pmatrix} = 0 \to \begin{pmatrix} \dot{\hat{r}}_k \\ \dot{\hat{p}}_k \end{pmatrix} - \begin{pmatrix} \hat{p}_k \\ F_k^{SA}(\hat{\mathbf{r}}) \end{pmatrix} = 0. \tag{2}
$$

As stressed in the charts of the Appendix, symmetry (1) is broken under non-self-adjoint forces, as necessary, say, for spinning tops under drag torques. In this case, the breaking is a manifestation of the nonconservation of the angular momentum.

However, the generalized relativity proposed in Section 6.3 is intended to express the following more general situation:

(a) conservation of the total angular momentum;
(b) breaking of the rotational symmetry under non-Hamiltonian internal forces; and
(c) validity of the isotopic covering of the rotational symmetry.

The objective of this example is to illustrate the last case.

One of the simplest classes of Birkhoffian tensors is given by the factorization into the canonical form

$$
\Omega^{\mu\nu} = K\omega^{\mu\nu} \tag{3}
$$

where, of course, the quantity K can be a function of the local coordinates, as well as any other physical quantity (pressure, density, viscosity, etc.).

Hamiltonian symmetry (1) is now generalized into the isotopic form

$$
\begin{aligned}
\hat{a}^\alpha &= \exp\left[\theta\Omega^{\mu\nu}\frac{\partial M_3}{\partial a^\nu}\frac{\partial}{\partial a^\mu}\right]a^\alpha \\
&= a^\alpha + \frac{\theta}{1!}[a^\alpha, M_3]^* + \frac{\theta^2}{2!}[[a^\alpha, M_3]^*, M_3]^* + \cdots.
\end{aligned} \tag{4}
$$

The case under consideration here is when the non-Hamiltonian generalization of the Galilean system (2) is form-invariant under the covering transformations (4), according to the rule

$$\begin{pmatrix} \dot{r}_k \\ \dot{p}_k \end{pmatrix} - \begin{pmatrix} p_k \\ F_k^{\mathrm{SA}}(\mathbf{r}) + F_k^{\mathrm{NSA}}(\mathbf{r}, \mathbf{p}) \end{pmatrix} = 0 \to \begin{pmatrix} \dot{\hat{r}}_k \\ \dot{\hat{p}}_k \end{pmatrix} - \begin{pmatrix} \hat{p}_k \\ F_k^{\mathrm{SA}}(\hat{\mathbf{r}}) + F_k^{\mathrm{NSA}}(\hat{\mathbf{r}}, \hat{\mathbf{p}}) \end{pmatrix} = 0. \quad (5)$$

To see this, it is sufficient to consider the case when the quantity K of (3) does not depend on the r- and p-variables (but can depend on other quantities). Then, it is easy to see that transformations (4) are no longer a pure rotation. Instead, they are given by a combination of a rotation and a dilatation of the type

$$\begin{pmatrix} \hat{r}_1 \\ \hat{r}_2 \\ \hat{r}_3 \end{pmatrix} = \begin{pmatrix} K & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}. \quad (6)$$

The underlying physical situation is then given by the case when the angular momentum is conserved while the spherical symmetry is broken due to deformations. This is one of the simplest conceivable cases for which the generalized relativity of Section 6.3 is intended.

The example has been proposed as a classical analog of the current studies in particle physics according to which the charge distribution of strongly interacting particles could be deformed under impact with other particles, resulting in a breaking of the rotational symmetry, while the third component and the magnitude of the total angular momentum remain the conventional ones. For details, see Santilli (1978d), Eder (1981), and the third volume of reprints edited by Schober (1982).

Problems

6.1 Recall that the conventional form of the Hamilton–Jacobi equations, form (6.1.8), is characterized by an F_1 generating function, while form (6.1.9) is characterized by an F_5 function. Construct the form of the equations for the remaining generating functions F_2, F_3, F_4, and F_6 (Section 5.2). Identify the form of all these equivalent Hamilton–Jacobi equations under Birkhoffian gauges (Section 4.5).

6.2 The Birkhoffian generalizations of the Hamilton–Jacobi equations for the cases of generating functions F_1 and F_5 are given by equations (6.1.24) and (6.1.34), respectively. Identify the generalizations for the cases of generating functions F_2, F_3, F_4, and F_6 (Section 5.3). Identify the form of all these equations under Birkhoffian gauges.

6.3 Study the following reformulation of the conventional Schrödinger's equations

$$\begin{aligned} i\frac{\partial}{\partial t}\psi(t, \mathbf{r}, \mathbf{p}) &= i\frac{\partial}{\partial t}\psi(t, a) = \tilde{H}(-2\omega^{\mu\nu}\tilde{R}_\nu^{0\dagger})\psi(t, a) \\ &= \left(\sum_{k=1}^{N} \frac{2}{m_k}\left(\frac{\partial}{\partial \mathbf{r}}\right)^2 + V\left(-2\frac{\partial}{\partial \mathbf{p}}\right)\right)\psi(t, r, p) \end{aligned} \quad (a)$$

which can be obtained via canonical quantization rules

$$\frac{\partial A}{\partial t} \to i\frac{\partial}{\partial t} = -\tilde{H}, \quad R_\mu \to \tilde{R}_\mu = \frac{1}{i}\frac{\partial}{\partial a^\mu} = -\tfrac{1}{2}\tilde{\omega}_{\mu\nu}\tilde{a}^\nu = \tilde{R}_\mu^{0\dagger} \quad (b)$$

applied to gauged Hamilton–Jacobi equations (6.1.15) or (6.1.17). By recalling that the classical equations (6.1.15) are equivalent to the conventional form (6.1.8), see whether the operator equations (a) are equivalent to the conventional Schrödinger's form (6.1.41). In particular, identify the implications of the Birkhoffian gauge for basic atomic notions such as states, expectation values, probability, energy spectra, etc.

6.4 Prove the following theorem, e.g., via the Birkhoffian representation of Hamilton's equations (Section 4.5).

Theorem. *Consider a Hamiltonian vector field*

$$\Xi^{\mu}(t, a) = \omega^{\mu\nu} \frac{\partial H(t, a)}{\partial a^{\nu}}.$$

Then the contemporaneous symmetries of Hamilton's equations according to Definition 6.3.4

$$\omega^{\mu\nu} \frac{\partial H(t, a)}{\partial a^{\nu}} = \frac{\partial a^{\mu}}{\partial a'^{\alpha}} \left[\frac{\partial a'^{\alpha}}{\partial a^{\rho}} \omega^{\rho\delta} \frac{\partial a'^{\beta}}{\partial a^{\delta}} \right] \frac{\partial H(t, a(a'))}{\partial a'^{\beta}}$$

$$\equiv \frac{\partial a^{\mu}}{\partial a'^{\alpha}} \omega^{\alpha\beta} \frac{\partial H(t, a')}{\partial a'^{\beta}}$$

do not recover all the possible contemporaneous symmetries of the equations of motion according to Definition 6.3.1, *i.e.*,

$$\Xi^{\mu} = \frac{\partial a^{\mu}}{\partial a'^{\alpha}} \left[\frac{\partial a'}{\partial a^{\rho}} \Xi^{\rho} \right] (t, a')$$

$$\equiv \frac{\partial a^{\mu}}{\partial a'^{\alpha}} \Xi^{\alpha}(t, a').$$

6.5 Prove the following property.

Theorem. *Under sufficient topological conditions, a vector field* $\Xi(a)$ *always verifies the form-invariance rule*

$$\Xi(a) = \Xi^{\mu}(a) \frac{\partial}{\partial a^{\mu}} = \Xi^{\mu}(a(a')) \frac{\partial a'^{\alpha}}{\partial a^{\mu}} \frac{\partial}{\partial a'^{\alpha}}$$

$$= \Xi'^{\alpha}(a') \frac{\partial}{\partial a'^{\alpha}} \equiv \Xi^{\alpha}(a') \frac{\partial}{\partial a'}$$

where

$$a' = e^{t\Xi(a)} a.$$

Note that the property holds whether the vector-field is Hamiltonian, Birkhoffian, or Lie-admissible. It therefore establishes the form-invariance of the time component of Galilei's relativity as well as of its Lie-isotopic and Lie-admissible coverings.

6.6 Establish whether the following conjecture is correct or erroneous.

Conjecture. (*Construction of Symmetries from First Integrals of Birkhoff's Equations*). *If r functions* $X_i(a)$ *are first integrals of autononomous Birkhoff's equations, i.e.,*

$$\dot{X}_i = \frac{\partial X_i}{\partial a^{\mu}} \Omega^{\mu\nu} \frac{\partial B}{\partial a^{\nu}} = [X_i, B]^* = 0, \qquad i = 1, 2, \ldots, r$$

then an infinitesimal symmetry G_r^* *always exists which leads to quantities* X_i *via Theorem* 6.3.3 *and this symmetry is explicitly given by*

$$G_r^*: a^\mu \to a'^\mu = a^\mu + w^i \Omega^{\mu\nu} \frac{\partial X_i}{\partial a^\nu}.$$

Apply the results to the construction of the isotopic covering of Galilei's relativity.

APPENDIX A

Indirect Lagrangian Representations

A.1 Indirect Lagrangian Representations within Fixed Local Variables

In this appendix we study methods for the possible reduction of a quasi-linear, second-order, non-self-adjoint system of ordinary differential equations in the *fundamental form*

$$[A_{ki}(t, q, \dot{q})\ddot{q}^i + B_k(t, q, \dot{q})]_{\text{NSA}} = 0, \qquad k = 1, 2, \ldots, n, \qquad \text{(A.1.1)}$$

into an equivalent self-adjoint form, under the general regularity and continuity conditions assumed in this volume (see the Introduction). A number of complementary aspects are also considered, such as the degrees of freedom of a Lagrangian, the existence theory of partial differential equations, Lagrangian symmetries and their breakings in mechanics, etc. These aspects can be considered as a continuation of the analysis in Volume I and are recommended as an introduction to the Birkhoffian generalization of Hamiltonian Mechanics presented in this volume.

In this first section we are interested in transforming systems (A.1.1) into an equivalent self-adjoint form *without altering the local coordinates* (the inclusion of the transformation theory will be done in Section A.3). This essentially restricts the analysis to the use of regular matrices of multiplicative functions, according to the equivalence transformation

$$h_k^i(t, q, \dot{q})[A_{ij}(t, q, \dot{q})\ddot{q}^j + B_i(t, q, \dot{q})] = 0. \qquad \text{(A.1.2)}$$

Since we are interested in the possible construction of a Lagrangian, the transformations of primary relevance here are the *self-adjoint genotopic* ones, that is, in the language of Definition 4.4.1, transformations (A.1.2) which induce the self-adjointness. One should keep in mind the existence also of the *self-adjoint isotopic* transformations (see next section), as well as others (see Charts 4.1 and 4.2).

Consider an (analytic and regular) system (A.1.1) which, as given, is non-self-adjoint. According to Theorem $\mathscr{I}.3$, a Lagrangian for its ordered *direct* representation *does not exist*, but ($\mathscr{I}.11$) is a particular case of the more general *indirect* representations ($\mathscr{I}.16$) (see Section I.3.4 for details). Thus one can attempt the construction of a Lagrangian for the non-self-adjoint system considered by searching for the self-adjointness-inducing equivalence transformations. Theorem $\mathscr{I}.3$ can then be reformulated for the equivalent form of the system. Finally, Theorem I.3.6.1 provides a method for the explicit computation of a Lagrangian when the integrability conditions are verified. This procedure is summarized in Theorem 4.2.1 for the reader's convenience.

Theorem A.1.1[1] (Generalization of the Fundamental Analytic Theorem 1.3 to Indirect Lagrangian Representations within Fixed Local Variables).

A necessary and sufficient condition for a finite-dimensional second-order system of ordinary differential equations,

$$A_{ki}(t, q, \dot{q})\ddot{q}^i + B_k(t, q, \dot{q}) = 0, \qquad k = 1, 2, \ldots, n \tag{A.1.3}$$

which is analytic and regular in a region $\mathscr{R}$ of points $(t, q, \dot{q})$ to admit an ordered indirect representation in terms of Lagrange's equations without transforming the coordinates and time variables

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} \equiv h_k^i[A_{ij}\ddot{q}^j + B_i], \qquad det(h_k^i)(\mathscr{R}) \neq 0 \tag{A.1.4}$$

is that the system admits at least one self-adjoint transformation which verifies the smoothness conditions of the Converse of the Poincaré Lemma (Lemma I.1.2.2 and Chart 4.6). That is, all the following conditions of self-adjointness on the functions h_i^j (for fixed A_{ij} and B_i terms)

$$A_{ij}^* = A_{ji}^*, \tag{A.1.5a}$$

$$\frac{\partial A_{ik}^*}{\partial \dot{q}^j} = \frac{\partial A_{jk}^*}{\partial \dot{q}^i}, \tag{A.1.5b}$$

$$\frac{\partial B_i^*}{\partial \dot{q}^j} + \frac{\partial B_j^*}{\partial \dot{q}^i} = 2\left\{\frac{\partial}{\partial t} + \dot{q}^k \frac{\partial}{\partial q^k}\right\} A_{ij}^*, \tag{A.1.5c}$$

[1] The study of indirect Lagrangian representations has been conducted by a number of authors, including Mayer (1896), Davis (1931), Havas (1957), Santilli (1977b,c), and others (see the Introduction of Volume 1 for a comprehensive bibliography). We follow the Newtonian reduction of the field theoretical studies by Santilli (1977c).

$$\frac{\partial B_i^*}{\partial q^j} - \frac{\partial B_j^*}{\partial q^i} = \frac{1}{2}\left\{\frac{\partial}{\partial t} + \dot{q}^k \frac{\partial}{\partial q^k}\right\}\left(\frac{\partial B_i^*}{\partial \dot{q}^j} \quad \frac{\partial B_j^*}{\partial \dot{q}^i}\right), \tag{A.1.5d}$$

$$A_{ij}^* = h_i^k A_{kj}, \qquad B_i^* = h_i^k B_k, \tag{A.1.5e}$$

are identically verified in a star-shaped subregion $\mathscr{R}^* \subset \mathscr{R}$. *When these conditions are met, a Lagrangian exists and is given by*[2]

$$L^*(t, q, \dot{q}) = K^*(t, q, \dot{q}) + D_k^*(t, q)\dot{q}^k + C^*(t, q), \tag{A.1.6a}$$

$$K^* = \dot{q}^i \int_0^1 d\tau' \left\{\left[\int_0^1 d\tau A_{ij}^*(t, q, \tau\dot{q})\right]\dot{q}^j\right\}(t, q, \tau'\dot{q}), \tag{A.1.6b}$$

$$D_k^* = \left[\int_0^1 d\tau\tau Z_{ki}^*(t, \tau q)\right]q^i, \tag{A.1.6c}$$

$$C^* = \left[\int_0^1 d\tau W_k^*(t, \tau q)\right]q^k, \tag{A.1.6d}$$

$$A_{ij}^* = h_i^k A_{kj}, \qquad B_i^* = h_i^k B_k \tag{A.1.6e}$$

$$Z_{ij}^* = \frac{1}{2}\left(\frac{\partial B_i^*}{\partial \dot{q}^j} - \frac{\partial B_j^*}{\partial \dot{q}^i}\right) + \left(\frac{\partial^2 K^*}{\partial q^i\, \partial \dot{q}^j} - \frac{\partial^2 K^*}{\partial \dot{q}^i\, \partial q^j}\right) \tag{A.1.6f}$$

$$W_i^* = \frac{\partial D_i^*}{\partial t} - B_i^* - \frac{\partial K^*}{\partial q^i} + \frac{\partial^2 K^2}{\partial \dot{q}^i\, \partial t} + \left[\frac{\partial^2 K^*}{\partial q^i\, \partial \dot{q}^j} + \frac{1}{2}\left(\frac{\partial B_i^*}{\partial \dot{q}^j} - \frac{\partial B_j^*}{\partial \dot{q}^i}\right)\right]\dot{q}^j. \tag{A.1.6e}$$

On practical grounds, this theorem can be implemented according to the following steps.

1. Verify that the system is analytic (in the sense of Chart A.1) and regular. Then select, for the region of definition, a neighborhood of a regular point (also in the sense of Chart A.1). Notice that this smoothness condition *excludes* the presence of discontinuous forces, such as the impulsive forces (Chart I.A.3), and is more restrictive than the condition of class $\mathscr{C}^\infty$.

2. See whether the system admits a self-adjoint genotopic transformation; that is, whether (A.1.9) admit a solution in the functions h_i^j[3].

3. In the case of an affirmative answer, compute a Lagrangian according to (A.1.10). Alternative methods (such as those of Chart I.3.11) can be used in case of difficulties in computing the integrals in the needed closed form.

This tacitly assumes that the reader is familiar with the applications of Theorem $\mathscr{I}$.3 to direct Lagrangian representations, as well as with the methods for the computation of a Lagrangian (Section I.3.6, in particular).

Clearly, the major difficulty for the practical application of Theorem A.1.1 is whether (A.1.9) in the unknowns h_i^j is consistent or not, and, if so, whether a solution can be explicitly computed in the needed closed form.

[2] Notice that method (A.1.10) is different than that of ($\mathscr{I}$.13).

[3] The conditions of self-adjointness of Theorems $\mathscr{I}$.3 and A.1.1 are the same. Nevertheless, their interpretation and use are different. For Theorem $\mathscr{I}$.3, conditions ($\mathscr{I}$.12) are restrictions on the equations of motion—that is, on the A_{ki} and B_k terms. For Theorem A.1.1, conditions (A.1.9) are restrictions on the integrating factors h_k^i for given fixed terms A_{ij} and B_i.

The following remarks are in order in this respect. First, (A.1.9) constitutes, in general, an *overdetermined*[4] *system of partial differential equations* in the unknowns h_i^j. These systems are not, in general, consistent. Thus the Inverse Lagrangian Problem (A.1.8) *is not* "universal"; that is, a solution of (A.1.9) for a given system (A.1.7) *does not necessarily exist.*

The methods used to ascertain whether a given system of differential equations is consistent or not are often referred to as *existence theory.* The case of determined systems has been studied extensively in the mathematical literature. In Chart A.3 we present one of the most important theorems for the case of determined systems of partial differential equations, the *Cauchy–Kovalevski theorem.* The case of overdetermined systems is still open in the mathematical literature to a considerable extent. An interested reader may consult, for instance, Goldschmidt (1967), Spencer (1969), and Gasqui (1975).[5]

In a number of cases of practical interest, system (A.1.9) can be reduced to a determined system by the appropriate selection of a subset of the functions h_i^j. In this case, according to Theorem 1 of Chart A.3, the reduced system is consistent if it can be written equivalently in the *Cauchy–Kovalevski form*, i.e., in the form of Equations (3) of Chart A.3. At this point the condition of analyticity of Theorem A.1.1 acquires its true significance. Indeed, the Cauchy–Kovalevski Theorem applies specifically to *analytic* systems of partial differential equations. Thus, even though Theorem A.1.1 can indeed be formulated and proven under weaker smoothness conditions (as for Theorem $\mathscr{I}$.3), practical applications outside the class of analytic systems are generally confronted with the nontrivial extension of the Cauchy–Kovalevski Theorem to nonanalytic systems.

Notice that when the Cauchy–Kovalevski Theorem is applicable to system (A.1.9), the solutions are also analytic, and therefore the needed smoothness conditions for the equivalent self-adjoint form of the equations of motion are automatically verified.

Finally, it is appropriate to recall that, even when system (A.1.9) is consistent, the possibility of readily computing a solution h_i^j in a closed form is not guaranteed. This creates additional difficulties. Indeed, the methods of the Inverse Problem refer to the computation of a Lagrangian, when it exists, in an *explicit form.* When the solution h_i^j of systems (A.1.9) can only be established via multiple power-series expansions which are convergent but of an unknown sum, a Lagrangian cannot be computed explicitly.

In conclusion, the practical application of Theorem A.1.1 is complicated by a number of technical difficulties which are typical of overdetermined

[4] A system of differential equations is called *determined*, *overdetermined*, or *undetermined* when the number of equations is equal, bigger, or smaller, respectively, than the number of unknowns.

[5] The outline presented in Section I.1.2 on the calculus of differential forms and the converse of the Poincaré Lemma can also be used to study the consistency of *overdetermined* systems of *partial* differential equations (see Examples I.1.3–I.1.6). Thus, whenever system (A.1.9) cannot be reduced to a determined form, one can still attempt a study of its consistency via the methods of Section I.1.2. The bibliography on the theory of partial differential equations is rather vast. A partial listing is given in Chart A.3.

systems of partial differential equations.[6] Nevertheless, Theorem A.1.1 is significant for a number of cases in which it does indeed provide an explicit Lagrangian representation.

For one-dimensional systems, conditions (A.1.9) reduce to a single partial differential equation in only one unknown, h, which can be written in the Cauchy–Kovalevski form[7] as

$$\frac{\partial h}{\partial \dot{q}} = B^{-1}\left\{\frac{\partial}{\partial t} + \dot{q}\frac{\partial}{\partial q}\right\}(hA) - B^{-1}\frac{\partial B}{\partial \dot{q}}h \tag{A.1.7}$$

which originated from (A.1.9c). The condition of analyticity then ensures the existence of a solution via Theorem 1 of Chart A.3.

Corollary A.1.1a. *Analytic, regular, and one-dimensional systems always admit a self-adjoint transformation. Thus a Lagrangian always exists for their indirect representation.*

This is the *universality for the existence of a Lagrangian for one-dimensional systems* which was proven for the first time by Darboux (1894). A more detailed presentation of this case is given in Chart A.4.

Another significant subcase of Theorem A.1.1 occurs when the original system is in kinematical form.

Corollary A.1.1b. *When system* (*A*.1.7) *is of the kinematical form*

$$\ddot{q}_k - f_k(t, q, \dot{q}) = 0; \qquad k = 1, 2, \ldots, n; \qquad q_k = q^k, \tag{A.1.8}$$

the integrability conditions for the existence of a self-adjoint transformation, Equations (*A*.1.9), *reduce to*

$$h_{ij} = h_{ji}, \tag{A.1.9a}$$

$$\frac{\partial h_{ik}}{\partial \dot{q}^j} = \frac{\partial h_{jk}}{\partial \dot{q}^i}, \tag{A.1.9b}$$

$$\frac{\partial}{\partial t}h_{ij} + \dot{q}^k\frac{\partial}{\partial q^k}h_{ij} + \frac{1}{2}\left[\frac{\partial}{\partial \dot{q}^j}(h_{ik}f^k) + \frac{\partial}{\partial \dot{q}^i}(h_{jk}f^k)\right] = 0, \tag{A.1.9c}$$

$$\frac{\partial}{\partial q^j}(h_{ik}f^k) - \frac{\partial}{\partial q^i}(h_{jk}f^k) = \frac{1}{2}\left\{\frac{\partial}{\partial t} + \dot{q}^l\frac{\partial}{\partial q^l}\right\}\left[\frac{\partial}{\partial \dot{q}^j}(h_{ik}f^k) - \frac{\partial}{\partial \dot{q}^i}(h_{jk}f^k)\right]. \tag{A.1.9d}$$

[6] This is one of the motivations for the search for an alternative approach to the Inverse Problem that is capable of providing universality, that is, capable of providing a representation through a conventional variational principle for *all* systems of the class admitted. In this way, half of the difficulties of Theorem A.1.1 (the study of whether the integrability conditions are verified or violated for each given system) would be absent. The remaining difficulties—those related to the explicit computation of a solution in the desired closed form—would persist however. As we shall see in Section 4.5, this is precisely the case for Birkhoff's equations.

[7] When the function B is analytic, the function B^{-1} is locally analytic only under certain technical conditions (see Chart A.1 and quoted references). Clearly, for the Cauchy–Kovalewski theorem to be applicable to Equation (A.1.11), the function B^{-1} must be analytic together with all other functions appearing in the right-hand side (i.e., B and A). The verification of these (and other) conditions is tacitly assumed here.

In particular, when the implicit functions are all independent of $\dot{q}^k$, a necessary condition for the existence of a self-adjoint transformation is that the multiplicative functions h_i^j are conserved along the solution of the system, i.e., are first integrals.[8] *That is,*

$$\frac{d}{dt} h_{ij}(t, q, \dot{q}) = 0; \qquad i, j = 1, 2, \ldots, n. \tag{A.1.10}$$

The last part of the corollary can be easily proved by using (A.1.13b) and (A.1.13c), as well as the identification along a solution of the system $\ddot{q}^k \equiv f^k$.

Corollary A.1.1b proves useful in illustrating the interplay between the fundamental and kinematical forms of the equations of motion or, for that matter, any system of ordinary second-order differential equations. In fact, the corollary is centered in the transition from the unique kinematical form to *one* equivalent fundamental form, i.e.,

$$\ddot{q}_k - f_k(t, q, \dot{q}) = 0 \rightarrow h_{ki}(t, q, \dot{q})[\ddot{q}^i - f^i] \stackrel{\text{def}}{=} A_{ki}\ddot{q}^i + B_k = 0. \tag{A.1.11}$$

This transition is trivial from the viewpoint of the existence theory for ordinary differential equations but not trivial from the viewpoint of the existence of a Lagrangian.

In essence, the kinematical form of a system is *necessarily* non-self-adjoint when the implicit functions are nonlinearly dependent on the velocities.[9] As such, they *do not* admit, a direct Lagrangian representation. The only possibility is to search for an indirect representation. The factor functions h_i^j then produce a fundamental form. Even though the existence of an indirect Lagrangian representation is not ensured, the representation is in principle, possible. In particular, the transition removes the restriction of the linearity in the velocities because the conditions of self-adjointness for the fundamental form *do not* demand the linearity of the A_{ij} and B_i terms in the velocities.

In conclusion, Corollary A.1.1b deals essentially with the transition from the unique but generally non-self-adjoint kinematical form to at least one equivalent self-adjoint fundamental form.[10]

This analytic procedure has nontrivial physical implications. Let us recall that, apart from the multiplication of the mass tensor, the kinematical form originates from Newton's second law (for the case of unconstrained systems). When the implicit functions (the Newtonian forces) are self-adjoint, the kinematical form can be directly represented by Lagrange's equations and is written in Cartesian coordinates

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^k} - \frac{\partial L}{\partial r^k} \equiv [\ddot{r}_k - f_k(t, r, \dot{r})]_{\text{SA}}, \qquad m = 1. \tag{A.1.12}$$

[8] The notion of the first integral is reviewed in Chart A.8.

[9] This is explicitly stated in Theorem $\mathscr{I}.1$; see Section I.2.2 for details.

[10] Another significant subcase of Theorem A.1.1 occurs when the matrix of factor functions is diagonal, i.e., of the type

$$h_{ij} = h_i \delta_{ij} \text{ (no sum).}$$

This case has been investigated in detail by Havas (1957).

This is the conventional way of representing systems with forces derivable from a potential and produces the conventional Lagrangian (as well as Hamiltonian) structure[11]:

$$L = L_{\text{tot}}^{\text{conv}} = L_{\text{free}}(\dot{r}) + L_{\text{int}}(t, r, \dot{r}) \tag{A.1.13a}$$

$$H = H_{\text{tot}}^{\text{conv}} = H_{\text{free}}(p) + H_{\text{int}}(t, r, p) \tag{A.1.13b}$$

$$L_{\text{free}} = \tfrac{1}{2}\dot{r}^i\delta_{ij}\dot{r}^j, \qquad H_{\text{free}} = \tfrac{1}{2}p_i\delta^{ij}p_j, \tag{A.1.13c}$$

$$L_{\text{int}} = B_i(t, r)\dot{r}^i + C(t, r), \qquad H_{\text{int}} = D^i(t, r)p_i + E(t, r), \tag{A.1.13d}$$

$$H = \dot{r}^k p_k - L, \qquad p_k = \frac{\partial L}{\partial \dot{r}^k}. \tag{A.1.13e}$$

The physical context of Corollary A.1.1b is fundamentally different. We are referring here to systems which are assumed to violate the integrability conditions for direct representations (A.1.16). Under conditions (A.1.13), the only Lagrangian representations possible without transforming the local coordinates are those of the indirect type

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^k} - \frac{\partial L}{\partial r^k} \equiv \{h_{ki}(t, r, \dot{r})[\ddot{r}^i - f^i(t, r, \dot{r})]_{\text{NSA}}\}_{\text{SA}}. \tag{A.1.14}$$

In this case, the functions h_i^j become *acceleration couplings* and produce a generalization of the conventional Lagrangian structure of the type identified in Section I.3.7; i.e.,[12]

$$L = L_{\text{tot}}^{\text{gen}} = L_{\text{int, I}}(t, r, \dot{r})L_{\text{free}}(\dot{r}) + L_{\text{int, II}}(t, r, \dot{r}) \tag{A.1.15a}$$

$$H = H_{\text{tot}}^{\text{gen}} = H_{\text{int, I}}(t, r, p)H_{\text{free}}(p) + H_{\text{int, II}}(t, r, p) \tag{A.1.15b}$$

[11] Upon suitable quantum mechanical and quantum field theoretical extensions, Lagrangian structures (A.1.17) persist in some of the most advanced parts of contemporary theoretical physics, such as quantum electrodynamics and quantum chromodynamics (a theory currently under study for strong interactions). In fact, in all these theories the forces (or couplings) are of the potential type. With a deeper analysis under the conditions of variational self-adjointness and upon inclusion of symmetries and conservation laws, the theories indicated here emerge as part of the so-called *closed self-adjoint interactions*, as we shall see in Chart A.8, Section 6.3 and Chart 6.1.

[12] Generalized Lagrangian or Hamiltonian structures (A.1.19) have been proposed by Santilli (1977a,b,c, 1978a,b,c, 1979b) for the study of strong interactions. According to clear experimental evidence, all strongly interacting particles (*hadrons*) have a charge size of the order of 10^{-13} cm (1 F); they are composed of wave packets, and their size coincides with the range of strong interactions. As a result, a necessary condition for the activation of strong interactions is that the wave packets of hadrons penetrates one within the other, at least up to their charge radius. Strong interactions are therefore expected to call for nonlocal/integral and nonpotential forces. The nonlocal nature is needed to ensure the existence of the interaction at *all points* of the volume of overlapping (rather than at a few isolated points, as in theories currently preferred). The nonpotential character is due to the fact, stressed throughout our analysis, that the notion of potential has no physical foundation for contact interactions. This does not exclude the possible existence in the strong interactions of action-at-a-distance, potential terms. The important point is the existence of at least one non-local nonpotential term. Also, approximations of the local power-series type ($\mathscr{I}$.4) are not excluded, provided that, again, the expansion contains nonpotential terms. When all terms realizing the strong interactions are reduced to those of the potential type, we have only action-at-a-distance and no genuine representation of the contact effects due to mutual penetration. Notice that models of type (A.1.19) are *open* (nonconservative). Therefore, they are particularly suited for the study of *one* strongly interacting particle, while the rest of the system is considered external. The extension of the system to include all particles leads to a generalization of the contemporary notion of interaction called *closed non-self-adjoint interactions*, which will be reviewed in Chart A.8, Section 6.3 and Chart 6.1.

where the *multiplicative* interaction terms to the term representing free motion are *necessary* whenever the kinematical form is non-self-adjoint. In turn, these multiplicative interaction terms have a number of nontrivial physical implications, both classically (e.g., for symmetries and conserved quantities[13]) and quantum mechanically (e.g., for the emergence of spin–spin and spin–orbit interactions which *multiply* the free term[14]).

In conclusion, the structure of the Lagrangian or Hamiltonian which emerges through the Inverse Problem techniques is of the *generalized* type (A.1.19) and *not* of the conventional type (A.1.17).

Until now we have been interested in illustrating Theorem A.1.1 for the indirect representations of non-self-adjoint systems. Conditions of self-adjointness (A.1.9), however, do not necessarily demand that the original system is of this type. Theorem A.1.1 therefore also applies to *indirect* representations of *self-adjoint* systems, which are useful for the study of the "degrees of freedom" of Lagrangian representations (Section A.2). In this way we reach the following classification of the representational capabilities of Theorem A.1.1.

Corollary A.1.1c. *Under integrability conditions* (*A*.1.9), *the indirect analytic representations* (*A*.1.8) *can be either self-adjoint genotopic or self-adjoint isotopic*; *i.e., by the variational approach to self-adjointness, representations* (*A*.1.8) *can be classified according to the following*:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} \equiv [h_k^i(A_{ij}\ddot{q}^j + B_i)_{\text{NSA}}]_{\text{SA}}, \tag{A.1.16a}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} \equiv [h_k^i(A_{ij}\ddot{q}^j + B_i)_{\text{SA}}]_{\text{SA}}. \tag{A.1.16b}$$

It seems wise to introduce a specific name for the "integrating factors" h_i^j considered in this section, so as to distinguish them from other factors we shall encounter during the course of our analysis. From now on we shall call the solutions h_i^j of Equations (A.1.9) *self-adjoint isotopic* or *self-adjoint genotopic functions* (or, simply, *isotopic* or *genotopic functions*) depending on whether the original system is self-adjoint or not, respectively.

[13] See Charts A.6–A.12.

[14] In conventional quantum mechanics (studies based on models of type (A.1.17)) spin–spin and spin–orbit interactions are represented by additive terms in the Hamiltonian. The realization of the same interactions in the same electromagnetic way for the different case of strong systems is faced with a number of problematic aspects (e.g., the inability to reach a *real* value of the mass of a bound state when the masses of the constituents are smaller than the total masses, as expected in the structure of the pion and other hadrons). These and other problems appear to be resolved if one realizes spin–spin and spin–orbit interactions via terms which *multiply* the kinetic energy (upon suitable symmetrization, of course). In turn, this has several implications. First, one reaches spin–spin and spin–orbit interactions of "strength" and dynamical implications different than those of the additive electromagnetic type. Second, the multiplicative nature of the term ensures a true representation of contact interactions. Finally, the approach appears to resolve the problem of real total mass indicated earlier and permits Bohr-type consistent structure equations for the light hadrons (e.g., the mesons). For these quantum mechanical aspects, the interested reader may consult Chart 6.1.

A.2 Isotopic Transformations of a Lagrangian

After having identified the integrability conditions for the existence of indirect Lagrangian representations, the next problem is to study their "degrees of freedom."

The equivalence transformations of a Lagrangian can be classified as follows:

I. those occurring within a fixed system of local variables,
II. those induced by coordinate transformations at a fixed value of time, and
III. those induced by more general transformations (e.g., involving time),

as well as by any combination of these transformations in a given ordering. In this section we shall study the transformation of type I. Those of type II will be studied in Section A.3, and those of type III will be indicated in Chapter 5. Our study should not be interpreted as exhausting all possible equivalence transformations. We are merely interest in identifying the transformations which are important for our program.

A first subclass of the equivalence transformations of a Lagrangian within a fixed system of local variables is expressed by the mappings

$$L(t, q, \dot{q}) \to L^{\dagger}(t, q, \dot{q})$$
$$= L(t, q, \dot{q}) + \dot{G}(t, q) = L(t, q, \dot{q}) + \frac{\partial G}{\partial q^k}\dot{q}^k + \frac{\partial G}{\partial t}, \qquad \text{(A.2.1a)}$$

$$\frac{d}{dt}\frac{\partial \dot{G}}{\partial \dot{q}^k} - \frac{\partial \dot{G}}{\partial q^k} \equiv 0 \qquad \text{(A.2.1b)}$$

which are often referred to as *Newtonian gauge transformations.*[15]

Mappings (A.2.1) imply a change in the functional dependence of a Lagrangian, illustrated by the following "gauging" of the conventional Lagrangian for a one-dimensional harmonic oscillator:

$$L = \tfrac{1}{2}(\dot{q}^2 - q^2) \to L^{\dagger} = \tfrac{1}{2}[\dot{q}^2 + (t + q)\dot{q} - q^2 + q], \qquad \text{(A.2.2)}$$
$$G = \tfrac{1}{2}(tq + \tfrac{1}{2}q^2).$$

Nevertheless, mappings (A.2.1) are trivial from the viewpoint of the equations of motion and of the integrability conditions for the existence of a Lagrangian.

[15] The term "gauge transformation" is customarily associated with the transformations of the electromagnetic four-potential in Minkowski space

$$A^{\mu} \to A'^{\mu} = A^{\mu} + \partial\chi/\partial x^{\mu}, \qquad \chi = \chi(x^0, x^1, x^2, x^3), \qquad \mu = 0, 1, 2, 3$$

which leave invariant the electromagnetic tensor $F_{\mu\nu} = \partial A_{\mu}/\partial x^{\nu} - \partial A_{\nu}/\partial x^{\mu}$. Additional gauge transformations exist in field theory within the context of the unification of weak and electromagnetic interactions. It should be stressed here that gauge transformations customarily used in Newtonian Mechanics *are not* the Newtonian limit of these field theoretical transformations.

Indeed, Lagrange's equations in L and in $L^\dagger$ *coincide* for all gauge transformations,[16]

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} \equiv \frac{d}{dt}\frac{\partial L^\dagger}{\partial \dot{q}^k} - \frac{\partial L^\dagger}{\partial q^k}. \tag{A.2.3}$$

As a matter of fact, one can *define* two functionally different Lagrangians $L(t, q, \dot{q})$ and $L^\dagger(t, q, \dot{q})$ to be related by an equivalence transformation of gauge type when Lagrange's equations in L and those in $L^\dagger$ coincide.

On physical grounds, mappings (A.2.1) are sometimes interpreted as induced by *forces which do not work.*[17] This can be seen, for instance, by interpreting $\dot{G}(t, q)$ as a (generalized) potential. The (generalized) work is then identically null, i.e.,

$$\delta W = \left(-\frac{\partial U}{\partial q^k} + \frac{d}{dt}\frac{\partial U}{\partial \dot{q}^k}\right)\delta q^k \equiv 0, \qquad U = -\dot{G}(t, q). \tag{A.2.4}$$

The Newtonian gauge transformations do not exhaust the equivalence transformations of class I. This can be seen by the following transformation, also for a one-dimensional harmonic oscillator [18]

$$L = \tfrac{1}{2}(\dot{q}^2 - q^2) \to L^* = \tfrac{1}{6}\dot{q}^3 \cos t + \tfrac{1}{2}q\dot{q}^2 \sin t - q^2\dot{q} \cos t \tag{A.2.5}$$

which is clearly of the nongauge type; that is, the difference $L^* - L$ *cannot* be expressed as a total time derivative of a function G which depends on time and coordinates only. Equivalently, Lagrange's equations in L and in L^* *do not* coincide. Nevertheless, these two equations are equivalent because they characterize the same implicit function (𝒻.20). The reader is encouraged to verify this.

Another example is provided by Lagrangians for a one-dimensional particle with linear velocity damping (identified in Example I.3.1):

$$L = \tfrac{1}{2}e^{\gamma t}\dot{q}^2 \to L^* = \dot{q} \ln \dot{q} - \gamma q. \tag{A.2.6}$$

Again, these Lagrangians are not related by a gauge transformation, yet they are equivalent because they both yield the (unique) implicit function of the system and thus the same solution. Nevertheless, the two Lagrangians are profoundly different in structure (as well as in symmetry, as we shall see later on).

In conclusion, the examples presented should be sufficient to indicate that, besides the known Newtonian gauge transformations, an additional class of equivalence transformations within fixed local variables exists, which produces a nontrivial change in the structure of a Lagrangian, with a consequential nontrivial change of the manifest symmetries. Since these symmetries

[16] The maximal admissible functional dependence for a function G to be a "gauge function" is understood to be that in time and (generalized) coordinates q^k. Also, the function G in mappings (A.2.1) is understood to preserve the continuity of the original Lagrangian. The preservation of the regularity is ensured from the linearity of $\dot{G}$ in the derivatives $\dot{q}^k$.

[17] See Wintner (1941, page 112).

[18] See Example A.1 for the construction of this transformation.

yield first integrals via Noether's theorem (Chart A.9), the study of the latter transformations is significant for our program.

Definition A.2.1. Two analytic and regular but functionally different Lagrangians, $L(t, q, \dot{q})$ and $L^*(t, q, \dot{q})$, are said to be *isotopically related.*[19] and the mapping $L \to L^*$ is called an *isotopic transformation,* when a (smoothness- and regularity-preserving) matrix of isotopic functions $h_i^j(t, q, \dot{q})$ exists with respect to L (or equivalently, h_i^{-1j} with respect to L^*, $(h_i^{-1j}) = (h_i^j)^{-1}$), such that all the following identifications

$$\left[\frac{d}{dt}\frac{\partial L^*}{\partial \dot{q}^k} - \frac{\partial L^*}{\partial q^k}\right]_{SA} \equiv \left[h_k^i\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i}\right)_{SA}\right]_{SA}, \qquad k = 1, 2, \ldots, n, \quad \text{(A.2.7)}$$

or their equivalent forms[20]

$$\left[\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k}\right]_{SA} \equiv \left[h_k^{-1i}\left(\frac{d}{dt}\frac{\partial L^*}{\partial \dot{q}^i} - \frac{\partial L^*}{\partial q^i}\right)_{SA}\right]_{SA}, \qquad \text{(A.2.8)}$$

hold in the (common) region of definition for L and L^* in a given ordering.

By recalling the analysis of Section 4.4, the reason for selecting the term "isotopic transformations" is self-evident. Indeed, these transformations are based on a self-adjointness-preserving equivalence transformation of Lagrange's equations, rule (A.2.7) or (A.2.8).

The integrability conditions are *already provided* by Theorem A.1.1. They need only to be better identified for the reader's convenience.

Recall from Equations (A.1.20b) that Theorem A.1.1 can also provide an indirect Lagrangian representation for self-adjoint systems. Thus integrability conditions (A.1.9) constitute necessary and sufficient conditions for the existence of an isotopically mapped Lagrangian through the identifications

$$A_{ij}^* = h_i^k \frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^j}, \qquad \text{(A.2.9a)}$$

$$B_i^* = h_i^k\left(\frac{\partial^2 L}{\partial \dot{q}^k \partial q^j}\dot{q}^j + \frac{\partial^2 L}{\partial \dot{q}^k \partial t} - \frac{\partial L}{\partial t}\right). \qquad \text{(A.2.9b)}$$

[19] The equivalence transformations under consideration were introduced, apparently for the first time, by Saletan and his collaborators with conventional techniques (other than those of the Inverse Problem) and under the name of "fouling transformations." See Currie and Saletan (1966), Gelman and Saletan (1973), and Marmo and Saletan (1978). See also, Kilmister (1967, page 119). We follow here the studies by Santilli (1977c, 1978c and 1979a) in the use of the Inverse Problem. This approach provides the necessary and sufficient conditions for the existence of the equivalence transformations under consideration, as well as methods for the explicit computation of the new Lagrangian (or Hamiltonian), and a methodological perspective for identifying the implications and possible applications.

[20] The isotopic transformations, when they exist, are always invertible. In particular, if a matrix (h_k^i) is isotopic with respect to a Lagrangian L, its inverse $(h_k^i)^{-1} = (h_k^{-1i})$ is isotopic with respect to L^*, as clearly indicated in Equations (A.2.7) and (A.2.8).

A Lagrangian L^* can then be computed by method (A.1.10). If method ($\mathscr{I}$.13) is used instead, we can write

$$L^*(t, q, \dot{q}) = \frac{d}{dt}\int_0^1 d\tau' \int_0^1 d\tau \tau q^i \left[\left(h_i^k \frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^j}\right)(t, \tau q, \tau\tau' \dot{q})\right]\dot{q}^j$$
$$- q^k \int_0^1 d\tau \left[h_k^i \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i}\right)\right](t, \tau q, \tau\dot{q}, \tau\ddot{q}). \quad \text{(A.2.10)}$$

The mechanism of the Lagrangian isotopy can be made more precise by the following lemma, whose proof is left to the interested reader (Problem A.2).

Lemma A.2.1. *A necessary and sufficient condition for two functionally different Lagrangians $L(t, q, \dot{q})$ and $L^*(t, q, \dot{q})$ within a fixed system of local variables to be isotopically related is that the systems of implicit functions of Lagrange's equations in L and in L^* coincide in a given ordering. That is, all the following equations*

$$f^k = P^{ki}\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\dot{q}^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial t} - \frac{\partial L}{\partial q^i}\right)$$
$$\equiv P^{*ki}\left(\frac{\partial^2 L^*}{\partial \dot{q}^i \partial q^j}\dot{q}^j + \frac{\partial^2 L^*}{\partial \dot{q}^i \partial t} - \frac{\partial L^*}{\partial q^i}\right) \quad \text{(A.2.11a)}$$

$$(P^{ki}) = \left(\frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^i}\right)^{-1}; \quad (P^{*ki}) = \left(\frac{\partial^2 L^*}{\partial \dot{q}^k \partial \dot{q}^i}\right)^{-1} \quad \text{(A.2.11b)}$$

are identically verified in the (common) region of definition.

The preservation of the implicit functions then ensures the property that the Lagrangian isotopies leave the dynamics of the system unaffected. In fact, not only the solution but also the first integrals and the conservations laws are unchanged, as we shall see.

Needless to say, *a given Lagrangian does not necessarily admit an isotopic image.* The techniques of the Inverse Problem merely provide the integrability conditions for its existence. Nevertheless, Lagrangians for one-dimensional systems always admit an isotopic image because, under the smoothness conditions admitted, Equation (A.2.11) must always admit a solution.

Corollary A.2.1a. *Analytic and regular Lagrangians for one-dimensional systems always admit at least one isotopic transformation other than the the identity. In particular, a first integral always constitutes an isotopic function for these transformations.*

The above property can be called the *universality of the Lagrangian isotopy for one-dimensional systems.* It is lost in the transition to systems of more than one dimension. However, when an n-dimensional system admits a (direct or indirect) Lagrangian representation, the Lagrangian is never

unique, and the isotopic mappings become possible. This is due to the fact that the conditions for the existence of a Lagrangian are given by a system of *partial* differential equations. These systems, when consistent, are known to admit considerable degrees of functional freedom in their solutions.

When reinspected from the viewpoint of self-adjointness, these degrees of freedom imply the existence of different, yet equivalent, self-adjoint forms. More specifically, the techniques of the Inverse Lagrangian Problem are capable of producing not only one Lagrangian representation (when it exists), but *all possible Lagrangian representations.*

Lemma A.2.2. *Under the smoothness conditions for the applicability of the converse of the Poincaré Lemma, the class of all possible isotopic transformations of a Lagrangian exhausts the class of all possible equivalent self-adjoint forms of the system.*

This property implies that conditions (A.1.9) can admit a *family* of solutions in the integrating functions, say $h^j_{(1)i}, h^j_{(2)i}, \ldots$, each of which induces a Lagrangian, say, $L_{(1)}, L_{(2)}, \ldots$. All these possible equivalent (but different) Lagrangians can be related by a rule of chain isotopy; that is, isotopic functions must exist such that

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} &\equiv \delta^i_k\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i}\right)\\ &\equiv h^{*i}_k\left(\frac{d}{dt}\frac{\partial L^*}{\partial \dot{q}^i} - \frac{\partial L^*}{\partial q^i}\right)\\ &\equiv h^{**i}_k\left(\frac{d}{dt}\frac{\partial L^{**}}{\partial \dot{q}^i} - \frac{\partial L^{**}}{\partial q^i}\right)\\ &\equiv \cdots \end{aligned} \tag{A.2.12}$$

where we have assumed $L = L_{(1)}$, $L^* = L_{(2)}$, etc. The first identities are generated by the trivial solution $\delta^j_i = h^{*j}_i$. The corresponding mapping $L \to L^* = L$ is called the *identity isotopic transformation.*[21] For an illustration of this occurrence, see Example A.1.

We are now equipped to study some physical implications of the Lagrangian isotopies. As indicated earlier, the techniques of the Inverse Lagrangian Problem produce a Lagrangian in the abstract mathematical form (A.1.10). The same techniques produce the isotopic degrees of freedom of a Lagrangian according to additional mathematical forms (A.2.9) or (A.2.10). In order to identify the implications of these structure for the problem of interactions, the use of generalized Lagrangians (A.1.19) is advantageous. In fact, these

[21] The possible existence of a family of isotopic matrices admitting the identity and an inverse is strongly suggestive of the possible existence of a group structure. The study of this aspect is left to the interested reader.

structures clarify the point that the interactions of the physical universe can be represented not only by additive terms to the free Lagrangian term (as generally used in contemporary physics), but also by a combination of additive *and* multiplicative interactions terms.[12,14]

We must now study the degrees of freedom of these additive and multiplicative interactions terms. For this purpose, the first task is the reinterpretation of generalized Lagrangians (A.1.19) through mathematical structures (A.2.9). This is easily accomplished by noting that, to avoid unnecessary degrees of freedom, the additive interaction term must be linear in the velocities,

$$L_{\text{int, II}} = \mathbf{D}_k(t, \mathbf{r}) \cdot \dot{\mathbf{r}}^k + C(t, \mathbf{r}). \tag{A.2.13}$$

This is a *necessary* condition for conventional Lagrangians (A.1.17) (Corollary $\mathscr{I}$.1a). Since the term representing the free motion is unique,

$$L_{\text{free}} = \tfrac{1}{2} m_k \dot{\mathbf{r}}^{k2}, \tag{A.2.14}$$

restriction (A.2.13) yields a *unique* reformulation of *each* structure (A.2.9) according to the rule

$$\begin{aligned} L_{\text{tot}}^{\text{gen}} &= K(t, \mathbf{r}, \dot{\mathbf{r}}) + \mathbf{D}_k(t, \mathbf{r}) \cdot \dot{\mathbf{r}}^k + C(t, \mathbf{r}) \\ &= L_{\text{int, I}}(t, \mathbf{r}, \dot{\mathbf{r}}) L_{\text{free}}(\dot{\mathbf{r}}) + L_{\text{int, II}}(t, r, \dot{\mathbf{r}}), \\ L_{\text{int, I}} &= K / L_{\text{free}}. \end{aligned} \tag{A.2.15}$$

Our problem is the identification of the degrees of freedom of the multiplicative and additive interaction terms within a fixed system of local variables. By again ignoring gauge transformations (which involve trivial degrees of freedom of $L_{\text{int, II}}$), the problem consists of reinterpreting the notion of Lagrangian isotopy for generalized Lagrangians (A.2.15). By assuming that the Lagrangian isotopies do not change the free term, we have the following property.

Lemma A.2.3. *The class of all possible isotopic transformations of a Lagrangian exhausts, up to Newtonian gauge transformations, the class of all possible pairs of multiplicative and additive interaction terms of a generalized total Lagrangian characterizing the same system of implicit functions. We shall then write*[22]

$$L_{\text{tot}}^{\text{gen}} = L_{\text{int, I}} L_{\text{free}} + L_{\text{int, II}} \to L_{\text{tot}}^{*\text{gen}} = L_{\text{int, I}}^{*} L_{\text{free}} + L_{\text{int, II}}^{*}. \tag{A.2.16}$$

[22] The quantization of isotopically mapped Lagrangians (or, more precisely, their Hamiltonian images) has rather nontrivial implications. For these, see Marmo and Saletan (1978) and Santilli (1978d, Section 4). At a quantum field theoretical level, the implications appear to be even more intriguing. Indeed, at least in principle, a Lagrangian (density) which apparently cannot be renormalized might be reduced via mechanism (A.2.16) to a simpler structure which can.

As an illustration, we have the following Lagrangian isotopy for the linearly damped particle in one dimension.

$$L_{\text{tot}}^{\text{gen}} = e^{\gamma t}\tfrac{1}{2}\dot{q}^2, \qquad L_{\text{int,I}} = e^{\gamma t}, \qquad L_{\text{int,II}} = 0, \tag{A.2.17a}$$

$$L_{\text{tot}}^{*\text{gen}} = \dot{q}\ln\dot{q} - \gamma q, \qquad L_{\text{int,I}}^{*} = \frac{2}{\dot{q}}\ln\dot{q}, \qquad L_{\text{int,II}}^{*} = -\gamma q. \tag{A.2.17b}$$

Lemma A.2.3 refers to the largest possible class of Newtonian systems verifying the integrability conditions of Theorem A.1.1. For the case of the conventional systems with forces which can be derived from a potential, we have the following corollary expressed in the language of Definition 4.1.1.

Corollary A2.3b. *When the system represented is essentially self-adjoint, the class of all admissible multiplicative and additive interaction terms of a generalized total Lagrangian admits the case* $L_{\text{int,I}} = 1$, *i.e., the following reduction is always possible*:

$$L_{\text{tot}}^{*\text{gen}} = L_{\text{int,I}}^{*}L_{\text{free}} + L_{\text{int,II}}^{*} \to L_{\text{tot}}^{\text{conv}} = 1L_{\text{free}} + L_{\text{int,II}}. \tag{A.2.18}$$

It is understood that such a reduction is prohibited when the system considered possesses forces not derivable from a potential (but is non-essentially non-self-adjoint).

As we shall see, Lagrangian isotopy (A.2.18) can be used in more than one way. First, when a conventional structure is known, a generalized structure is useful to, say, search for new first integrals (see Chart A.10). On the other hand, when a generalized Lagrangian structure is known, one can use reduction (A.2.18) to search for a simpler structure, with self-evident physical implications at classical, quantum mechanical, and quantum field theoretical levels.

Stated explicitly, *when a given Lagrangian has a generalized structure, it is not sufficient to guarantee the presence of forces which cannot be derived from a potential.* In this respect, the reader may consult Example A.1. The most effective way to see whether or not non-self-adjoint forces are present is to *compute explicitly the equations of motion*, construct the form originating from Newton's second law, and verify its behavior under the conditions of self-adjointness.

In conclusion, *the Inverse Problem suggests a sort of return "ad originem" in Newtonian Mechanics. The fundamental dynamic quantities are Newton's equations of motion, while the admissible Lagrangians for their analytic representation have a primarily methodological function of the type indicated in the Preface. In particular, the selection of any given specific Lagrangian from among all possible Lagrangians which can be constructed through the techniques of the Inverse Problem is merely a question of personal preference, rather than mathematical consistency, whether or not the forces are self-adjoint.*

We pass now to a reinspection of the isotopic transformations of a Lagrangian within the context of generalized principle (14) of Chart 5.7. As pointed out Theorem A.1.1 may also apply to Newtonian systems with potential

forces. The theorem therefore establishes the integrability conditions for the following generalized variational principle

$$\delta A^*(E_0) = \left[\delta \int_{t_1}^{t_2} dt L_{\text{tot}}^{*\text{conv}}(t, \mathbf{r}, \dot{\mathbf{r}})\right](E_0)$$

$$= -\int_{t_1}^{t_2} dt\left[\left(\frac{d}{dt}\frac{\partial L_{\text{tot}}^{*\text{conv}}}{\partial \dot{r}^k} - \frac{\partial L_{\text{tot}}^{*\text{conv}}}{\partial r^k}\right)_{\text{SA}} \delta r^k\right](E_0)$$

$$= -\int_{t_1}^{t_2} dt\{h_k^i(t, \mathbf{r}, \dot{\mathbf{r}})[m\ddot{r}_i - f_i(t, \mathbf{r}, \dot{\mathbf{r}})]_{\text{SA}}\}_{\text{SA}}(E_0)\delta r^k(E_0) = 0$$

$$L_{\text{tot}}^{*\text{conv}} = L_{\text{tot}}^{\text{gen}} \tag{A.2.19}$$

in which the Lagrangian is the generalized type and the representation is indirect. If a *direct* representation of a *self-adjoint* system is desired via a *generalized* Lagrangian, Lemma A.2.1 allows the following principle

$$\delta^* A^*(E_0) = \left[\delta^* \int_{t_1}^{t_2} dt\, L_{\text{tot}}^{*\text{conv}}(t, \mathbf{r}, \dot{\mathbf{r}})\right](E_0)$$

$$= -\int_{t_1}^{t_2} dt\left\{\left[g_k^i\left(\frac{d}{dt}\frac{\partial L_{\text{tot}}^{*\text{conv}}}{\partial \dot{r}^i} - \frac{\partial L_{\text{tot}}^{*\text{conv}}}{\partial r^i}\right)_{\text{SA}}\right]_{\text{SA}} \delta r^k\right\}(E_0)$$

$$= -\int_{t_1}^{t_2} dt[m_k \ddot{r}_k - f_k(t, \mathbf{r}, \dot{\mathbf{r}})]_{\text{SA}}(E_0)\delta r^k(E_0) = 0 \tag{A.2.20}$$

$$\times\ \delta^* r^i = g_k^i \delta r^k$$

called a *self-adjoint isotopic mapping of Hamiltons' principle.* In this way, the *Lagrangian's isotopic degrees of freedom can be interpreted as a form of the degrees of freedom of the variations.*

We note, incidentally, that conventional (self-adjoint) variational principles can be turned into non-self-adjoint form by the simple rule

$$\delta A(E_0) = -\int_{t_1}^{t_2} dt\left[\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^k} - \frac{\partial L}{\partial r^k}\right)_{\text{SA}} \delta r^k\right](E_0)$$

$$= -\int_{t_1}^{t_2} dt\left\{\left[g_k^i\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^i} - \frac{\partial L}{\partial r^i}\right)_{\text{SA}}\right]_{\text{NSA}} \delta_* r^k\right\}(E_0) = 0$$

$$\delta_* r^k = h_i^k \delta r^i, \qquad (h) = (g)^{-1}. \tag{A.2.21}$$

Similarly, when the Lagrangian considered admits an isotopic image, we have the self-explanatory reformation

$$\delta \int_{t_1}^{t_2} dt\, L(t, \mathbf{r}, \dot{\mathbf{r}}) = \delta_* \int_{t_1}^{t_2} dt\, L^*(t, \mathbf{r}, \dot{\mathbf{r}}) \tag{A.2.22}$$

as well as the equivalent form

$$\delta^* \int_{t_2}^{t_2} dt\, L(t, \mathbf{r}, \dot{\mathbf{r}}) = \delta \int_{t_2}^{t_2} dt\, L^*(t, \mathbf{r}, \dot{\mathbf{r}}). \tag{A.2.23}$$

A.3 Indirect Lagrangian Representations via the Use of the Transformation Theory

The transformation theory in configuration space studies the possibility of representing the equations of motion in an infinite variety of different coordinates, say, q'^k, q''^k, ..., and times, say, t', t'', ... A central requirement is that the equations of motion in the original variables and those in the new variables are equivalent. This equivalence is ensured by a number of conditions on the transformations, comprising the invertibility and the preservation of the continuity properties of the original system. Since the systems studied in this volume are analytic, we shall assume from here on that all transformations considered are analytic in the region assumed.[23] Nevertheless, the reader should keep in mind that most of the results can be formulated under weaker smoothness conditions. Also, the transformations considered will be assumed to be single-valued (namely, one-to-one), although this requirement too can often be relaxed (see the Introduction for additional aspects).

The following classes of transformations are particularly relevant for the Inverse Problem:

A. velocity-independent transformations of the coordinates at a fixed value of time

$$t \to t' \equiv t, \qquad q^k \to q'^k = q'^k(q), \tag{A.3.1a}$$

$$|J_i^j|(\mathscr{R}) = \left|\frac{\partial q'^j}{\partial q^i}\right|(\mathscr{R}) \neq 0, \tag{A.3.1b}$$

often called *point transformations*,[24] to stress their local character, or *contemporaneous transformations*,[24] to stress the lack of time transformation.

B. velocity-independent transformations of the coordinates and time variables, i.e.,

$$t \to t' = t'(t, q), \qquad q^k \to q'^k = q'^k(t, q), \tag{A.3.2}$$

sometimes called *noncontemporaneous transformations*.[24]

[23] All regions of definitions for the transformations considered are assumed to be topologically equivalent to a star-shaped region to ensure the applicability of the converse of the Poincaré lemma. This is due to the fact that, unlike conventional treatments, the transformation theory is used in this volume for the construction of the analytic representations of Newtonian systems and, as such, must possess the needed topology. Further details will be provided later.

[24] The literature on point transformations in configuration space is rather vast. See, for instance, Lanczos (1949) and Pars (1965).

C. velocity-dependent transformations of the coordinates and time variables, i.e.,

$$t \to t' = t'(t, q, \dot{q}), \qquad q^k \to q'^k = q'^k(t, q, \dot{q}). \tag{A.3.3}$$

The analysis of this section will be restricted to point transformations. The most general possible noncontemporaneous, velocity-dependent transformations is studied in Chapter 5 within the context of Hamiltonian formulations and related Birkhoffian generalizations.

Velocity-dependent transformations in configuration space do not appear in the literature of Newtonian mechanics, while they are only occasionally treated in the literature of differential geometry[25] and the calculus of variations.[26] This is due to a number of technical difficulties, in both the formal study and applications, which are either absent or become manageable in the transition from second-order to first-order systems.

The reader should keep in mind that a comprehensive velocity-dependent transformation theory of second-order systems and Lagrange's equations remains an important problem of mechanics for the following (and other) reasons.

1. Galilei transformations (Chart I.A.1) depend explicitly on velocities as a parameter. Thus, the inclusion of the velocity dependence is important for relativity profiles.
2. The symmetries of the equations of motion leading to first integrals via Noether's theorem (Charts A.6–A.11) are generally dependent on velocities in an explicit way. Thus the Lagrangian formulation of the problem of symmetries and first integral demands, for completeness, the use of velocity-dependent transformations.
3. The configuration space image of the familiar canonical transformations has an explicit dependence, in general, on the velocities, trivially because the canonical transformations generally depend on the momenta. As a consequence, the Lagrangian image of the transformation theory of Hamilton's equations is crucially dependent on an explicit dependence on the velocities.

We should indicate from the outset that point transformations are important for the practical application of the Inverse Lagrangian Problem, although they *do not* enlarge the representational capabilities of Theorem A.1.1. In order to achieve an analytic representation of essentially non-self-adjoint systems via the transformation theory, the use of velocity-dependent transformations is essential. However, a generalization of Lagrange's equations ($\mathscr{I}.10$) will be essential too, as shall be seen.[27]

The reader can now anticipate the technical difficulties of velocity-dependent transformations. In fact, *the transformations do not preserve, in general, the second-order Lagrangian character of the equations*, apart from a special

[25] See, for instance, Caratu *et al.* (1976).

[26] See, for instance, Gelfand and Fomin (1963, page 81).

[27] At this point the reader may keep in mind third-order Lagrange's equations in second-order Lagrangians (see Section 4.2). The velocity dependence of the transformations is expected to increase the order of the Lagrangian by producing, in general, higher order equations.

class which expectedly consists of the Lagrangian image of the canonical transformations.

We shall now study the transformation of Newton's equations of motion under point transformations. Let the system considered be of the fundamental form, i.e.,

$$A_{ki}(t, q, \dot{q})\ddot{q}^i + B_k(t, q, \dot{q}) = 0. \tag{A.3.4}$$

Since the point transformations do not depend explicitly on velocities and time, the velocities in the old and new system of coordinates are linearly related, i.e.,

$$\dot{q}^k = \frac{\partial q^k}{\partial q'^i}\dot{q}'^i. \tag{A.3.5}$$

The following properties then hold

$$\frac{\partial \dot{q}^k}{\partial \dot{q}'^i} = \frac{\partial q^k}{\partial q'^i} \tag{A.3.6a}$$

$$\frac{d}{dt}\frac{\partial q^k}{\partial q'^i} = \frac{\partial \dot{q}^k}{\partial q'^i} = \frac{\partial^2 q^k}{\partial q'^i\,\partial q'^j}\dot{q}'^j. \tag{A.3.6b}$$

The accelerations therefore transform according to the rules

$$\begin{aligned}\ddot{q}^k &= \frac{d}{dt}(\dot{q}^k) = \frac{d}{dt}\left(\frac{\partial q^k}{\partial q'^i}\right)\dot{q}'^i + \frac{\partial q^k}{\partial q'^i}\ddot{q}'^i \\ &= \frac{\partial^2 q^k}{\partial q'^i\,\partial q'^j}\dot{q}'^i\dot{q}'^j + \frac{\partial q^k}{\partial q'^i}\ddot{q}'^i,\end{aligned} \tag{A.3.7}$$

but the A and B terms of the equations of motion transform according to (Chart I.A.13)

$$\hat{A}_{ij} = \bar{A}_{rs}\frac{\partial q^r}{\partial q'^i}\frac{\partial q^s}{\partial q'^j}, \tag{A.3.8a}$$

$$\bar{A}_{rs} = \bar{A}_{rs}(t, q', \dot{q}') = A_{rs}(t, q(q'), \dot{q}(q', \dot{q}')), \tag{A.3.8b}$$

$$\hat{B}_i = \bar{B}_r\frac{\partial q^r}{\partial q'^i}, \tag{A.3.8c}$$

$$\bar{B}_r = \bar{B}_r(t, q', \dot{q}') = B_r(t, q(q'), \dot{q}(q', \dot{q}')). \tag{A.3.8d}$$

In this way we reach the following *tranformation law of the fundamental form of Newtonian systems under point transformations*:

$$A_{ki}(t, q, \dot{q})\ddot{q}^i + B_k(t, q, \dot{q}) = \frac{\partial q'^i}{\partial q^k}[A'_{ij}(t, q', \dot{q}')\ddot{q}'^j + B'_i(t, q', \dot{q}')] = 0 \tag{A.3.9}$$

where

$$A'_{ij} = \hat{A}_{ij}, \tag{A.3.10a}$$

$$B'_i = \hat{B}_i + \bar{A}_{rs}\frac{\partial q^r}{\partial q'^i}\frac{\partial^2 q^s}{\partial q'^j\,\partial q'^l}\dot{q}'^j\dot{q}'^l. \tag{A.3.10b}$$

Notice that the B' terms possess a dependence also on the A terms, according to Equations (A.3.10b), whenever the point transformations are nonlinear. Notice also that the existence of the inverse transformations

$$A'_{ij}(t, q', \dot{q}')\ddot{q}'^j + B'_i(t, q', \dot{q}') = \frac{\partial q^k}{\partial q'^i}\left[A_k(t, q, \dot{q})\ddot{q} + B_k(t, q, \dot{q})\right] = 0 \quad \text{(A.3.11)}$$

is ensured by the regularity of the Jacobian (A.3.1b).

The transformation law of the kinematical form of Newton's equations of motion is a particular case of that of law (A.3.9), i.e.,[28]

$$\ddot{q}_k - f_k(t, q, \dot{q}) = \frac{\partial q'^i}{\partial q^k}\left[\frac{\partial q^r}{\partial q'^i}\delta_{rs}\frac{\partial q^s}{\partial q'^j}\ddot{q}'^j - f'_i(t, q', \dot{q}')\right] = 0, \quad \text{(A.3.12)}$$

where the new implicit functions are given by

$$f'_i = f_j(t, q(q'), \dot{q}(q', \dot{q}'))\frac{\partial q^j}{\partial q'^i} - \frac{\partial q^r}{\partial q'^i}\delta_{rs}\frac{\partial^2 q^s}{\partial q'^j\,\partial q'^l}\dot{q}'^j\dot{q}'^l. \quad \text{(A.3.13)}$$

As we shall see in Chart A.6, transformation laws (A.3.9) and (A.3.12) will be particularly useful in identifying the notion of symmetry of Newton's equations of motion and forces, respectively.

Our next objective is to identify the behavior of Lagrange's equations under points transformations. This is easily accomplished by the scalar rule[29]

$$L(t, q, \dot{q}) = L(t, q(q'), \dot{q}(q', \dot{q}')) = L'(t, q', \dot{q}') \quad \text{(A.3.14)}$$

and the use of Equations (A.3.6), under which we have the following *transformation law of Lagrange's equations under point transformations*

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} &= \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}'^i}\frac{\partial \dot{q}'^i}{\partial \dot{q}^k}\right) - \frac{\partial L'}{\partial q'^i}\frac{\partial q'^i}{\partial q^k} - \frac{\partial L'}{\partial \dot{q}'^i}\frac{\partial \dot{q}'^i}{\partial q^k} \\ &= \frac{\partial q'^i}{\partial q^k}\left[\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}'^i} - \frac{\partial L'}{\partial q'^i}\right] = 0. \end{aligned} \quad \text{(A.3.15)}$$

[28] Notice that Equations (A.3.12) can be written

$$\ddot{q}_k - f_k(t, q, \dot{q}) = \frac{\partial q^k}{\partial q'^j}\ddot{q}'^j - f_k(t, q(q'), \dot{q}(q', \dot{q}')) + \frac{\partial^2 q_k}{\partial q'^j\,\partial q'^i}\dot{q}'^i\dot{q}'^j.$$

However, this alternate does not indicate the proper transformation of the *metric tensor* of configuration space according to the rule

$$\delta_{ij} \to g_{ij}(q) = \frac{\partial q^r}{\partial q'^i}\delta_{rs}\frac{\partial q^s}{\partial q'^j}.$$

This rule, in turn, indicates the existence of a number of geometrical implications of point transformations. some of which will be indicated later in this section.

[29] It should be anticipated from Chart A.6 that rule (A.3.14) *does not* hold for noncontemporaneous transformations. In this case a Lagrangian transforms as a *density* rather than as a scalar. For details, see Definition 2 of Chart A.6.

As expected, we have the same transformation law as that for Newton's equations. The corresponding transformation of variational principles is consequential.

A few comments are in order. Recall that, for the case of unconstrained Newtonian systems in a three-dimensional Euclidean space, laws (A.3.9) or (A.3.12) read

$$m_k \ddot{r}_{ka} - F_{ka}(t, \mathbf{r}, \dot{\mathbf{r}}) = \frac{\partial r'^{ib}}{\partial r^{ka}} \left[\frac{\partial r^{jc}}{\partial r'^{ib}} \delta_{jl} \delta_{cd} \frac{\partial r^{ld}}{\partial r'^{sf}} m_s \ddot{r}'^{sf} - F'_{ib}(t, \mathbf{r}', \dot{\mathbf{r}}') \right] = 0, \tag{A.3.16a}$$

$$F'_{ib} = F_{jc}(t, \mathbf{r}(\mathbf{r}'), \dot{\mathbf{r}}(\mathbf{r}', \dot{\mathbf{r}}')) \frac{\partial r^{jc}}{\partial r^{ib}} - \frac{\partial r^{jc}}{\partial r'^{ib}} \delta_{jl} \delta_{cd} \frac{\partial^2 r^{ld}}{\partial r'^{ue} \partial r'^{tf}} \dot{r}'^{ue} \dot{r}'^{tf}, \tag{A.3.16b}$$

$$k, i, j, l, s, t, u = 1, 2, \ldots, N; \qquad a, b, c, d, e, f = x, y, z$$

by therefore providing the *transformation law of Newtonian forces under point transformations.* Notice in this respect that, if the original forces $\mathbf{F}_k$ are conservative and the transformations are nonlinear, the transformed forces $\mathbf{F}'_k$ are quadratically dependent on the velocities, by therefore acquiring the structure of "non-conservative" forces in the transformed reference frame. Likewise, nonconservative forces F_k which are quadratically dependent on the velocities may, in principle, be reduced to forces $\mathbf{F}'_k$ of "conservative type" via nonlinear point transformations.[30]

These dynamic effects of point transformations on the structure of the Newtonian forces are due to the geometry of nonlinear point transformations. Indeed, starting from a region $\mathscr{R}$ in a (flat) Euclidean space, the image region $\mathscr{R}'$ under nonlinear point transformations belongs to a curved space (Chart I.A.14). In particular, the transformations considered *do not* map straight lines into straight lines. The emergence of velocity-dependent terms in the transformation of Newtonian forces is then consequential. As we shall see later in this section, these effects of point transformations on the structure of Newtonian forces are, in essence, the basis for the effectiveness of the transformation theory for the Inverse Problem. Indeed, under the integrability conditions of the Inverse Problem, the theory is capable of transforming a class of non-self-adjoint forces in the reference frame of their experimental identification, into structurally different forces which can be derived from a potential, but in a new reference frame.

The structure of an admissible Lagrangian under point transformations also deserves a comment. In essence, if one starts with a Lagrangian with the *conventional* structure

$$L = L_{\text{free}}(\dot{\mathbf{r}}) + L_{\text{int}}(t, \mathbf{r}, \dot{\mathbf{r}}), \tag{A.3.17a}$$

$$L_{\text{free}} = \sum_{k=1}^{N} \tfrac{1}{2} m_k \dot{\mathbf{r}}^k \cdot \dot{\mathbf{r}}^k, \qquad L_{\text{int}} = -U(t, \mathbf{r}, \dot{\mathbf{r}}), \tag{A.3.17b}$$

[30] Intriguingly, cases exist in which *the transformed force is identically null* (see Example A.6).

the transformed Lagrangian according to scalar rule (A.3.14) can be interpreted as being of the *generalized* structure (A.1.19a) in the new reference frame, i.e.,[31]

$$L' = L'_{\text{int, I}}(\mathbf{r}')L'_{\text{free}}(\dot{\mathbf{r}}') + L'_{\text{int, II}}(t, \mathbf{r}', \dot{\mathbf{r}}') \tag{A.3.18a}$$

$$L'_{\text{free}} = \sum_{k=1}^{N} \tfrac{1}{2} m_k \dot{\mathbf{r}}'^k \cdot \dot{\mathbf{r}}'^k. \tag{A.3.18b}$$

This occurrence also indicates the capability of the transformation theory of reducing a generalized Lagrangian into a conventional one, under the integrability conditions identified below.

Another aspect which deserves comment is the reinterpretation of the generalized Lagrange's equations (15) of Chart 5.7 within the context of the transformation theory. An inspection of the transformation law of Lagrange's equations indicates that such a law actually transforms the conventional Lagrange's equations into generalizations of type (15) of Chart 5.7; that is, those with a regular matrix of factor terms. In particular, such a factor matrix assumes the meaning of the Jacobian of the transformation. Furthermore, the generalized variations and action principles of Chart 5.7 and Section A.2 emerge quite naturally within the context of the transformation theory.

In order to identify the relevance of the transformation theory for the Inverse Problem, we must now reinspect the preceding analysis from the viewpoint of variational self-adjointness. Our primary objective is to see whether a non-self-adjoint system can be transformed into an equivalent self-adjoint form through point transformations. This task depends on the definition of a transformed system. In particular, we have the alternative of applying the conditions of variational self-adjointness to the right-hand side of transformation law (A.3.9) without and with the Jacobian matrix. The following property, expressed in the language of Definition 4.4.1, is useful for the resolution of this alternative.

Lemma A.3.1. *Under the assumption that the transformed system is defined by*

$$A'_{ij}(t, q', \dot{q}')\ddot{q}'^j + B'_i(t, q', \dot{q}') = 0, \tag{A.3.19a}$$

$$A'_{ij} = \bar{A}_{rs} \frac{\partial q^r}{\partial q'^i} \frac{\partial q^s}{\partial q'^j}, \qquad B'_i = \bar{B}_r \frac{\partial q^r}{\partial q'^i} + \bar{A}_{rs} \frac{\partial q^r}{\partial q^i} \frac{\partial^2 q^s}{\partial q'^j \partial q'^l} \dot{q}'^j \dot{q}'^l, \tag{A.3.19b}$$

all point transformations of the class admitted here are self-adjoint isotopic.[32]

PROOF. Under the conditions of the lemma, the original system is self-adjoint and thus of Lagrangian type. The self-adjointness-preserving character of the point trans-

[31] Interpretation (A.3.18) is *purely formal* and inspired by the simple evidence of lack of curvature in the Newtonian physical reality. However, when a curved space is admitted, Lagrangian (A.3.18b) *is not* admissible for free motion because, in this case, the trajectories are not geodesic.

[32] That is, the self-adjoint isotopic character occurs for all possible functional dependences of the transformations.

formations then follows from the fact that, from law (A.3.15), all transformed systems (A.3.19) are self-adjoint. (*Q.E.D.*)

Lemma A.3.1 describes the variational character of the transformation law of Lagrange's equations, which we now rewrite

$$\left[\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k}\right]_{\text{SA}} = \left\{\frac{\partial q'^i}{\partial q^k}\left[\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}'^i} - \frac{\partial L'}{\partial q'^i}\right]_{\text{SA}}\right\}_{\text{NSA}} = 0. \qquad \text{(A.3.20)}$$

Evidently, this law coincides with that of self-adjoint systems under the condition (here tacitly implemented) of the applicability of the converse of the Poincaré lemma.

On equivalent grounds, it is possible to prove that, if the original system is non-self-adjoint and the transformed system is that without the Jacobian of the transformations, the point transformation *do not* induce a self-adjoint structure. Thus the definition of the transformed system according to Equations (A.3.19) *is not* effective for the Inverse Lagrangian Problem.

However, when the transformed system is defined to include the Jacobian matrix,

$$A^{*}_{ij}(t, q', \dot{q}')\ddot{q}'^j + B^{*}_i(t, q', \dot{q}') \equiv \frac{\partial q'^k}{\partial q^i}\left[A'_{kj}(t, q', \dot{q}')\ddot{q}'^j + B'_k(t, q', \dot{q}')\right] \qquad \text{(A.3.21)}$$

the situation is different. In this case, the Jacobian matrix performs a role equivalent to that of matrix (h^j_i) of the integrating factors of Section A.1. In particular, under definition (A.3.21) of the transformed system, point transformations can induce an equivalent self-adjoint form, starting from an originally given, non-self-adjoint system. This possibility is illustrated by Equations (A.3.20). Lagrange's equations in the Lagrangians L and L' are self-adjoint from Theorem $\mathscr{I}$.2. Nevertheless, the right-hand side of Equations (A.3.20) is generally non-self-adjoint.

Thus the characterization of the transformation law of Lagrange's equations via the conditions of self-adjointness brings into focus the following remarkable property: *the self-adjoint or non-self-adjoint character of a system is not necessarily preserved in the transition from one reference frame to another, provided that the transformed system is defined according to rule* (A.3.21). No contradiction arises in Equations (A.3.20) from the self-adjointness of the left-hand side and the non-self-adjointness of the right-hand side, because these two members refer to different reference frames, as do the conditions of self-adjointness. For illustrations, the reader may consult the examples at the end of this chapter.

Of course, the particular case in which the right-hand side of Equation (A.3.20) is self-adjoint is not excluded. In this way, we reach the following classification of the variational character of point transformations:

$$[A_{ki}\ddot{q}^i + B_k]_{\text{SA}} = [A^{*}_{ki}\ddot{q}'^i + B^{*}_k]_{\text{SA}} = 0, \qquad \text{(A.3.22a)}$$

$$[A_{ki}\ddot{q}^i + B_k]_{\text{NSA}} = [A^{*}_{ki}\ddot{q}'^i + B^{*}_k]_{\text{SA}} = 0, \qquad \text{(A.3.22b)}$$

$$[A_{ki}\ddot{q}^i + B_k]_{\text{NSA}} = [A^{*i}_{ki}\ddot{q}'^i + B^{*}_k]_{\text{NSA}} = 0, \qquad \text{(A.3.22c)}$$

$$[A_{ki}\ddot{q}^i + B_k]_{\text{SA}} = [A^{*}_{ki}\ddot{q}'^i + B^{*}_k]_{\text{NSA}} = 0. \qquad \text{(A.3.22d)}$$

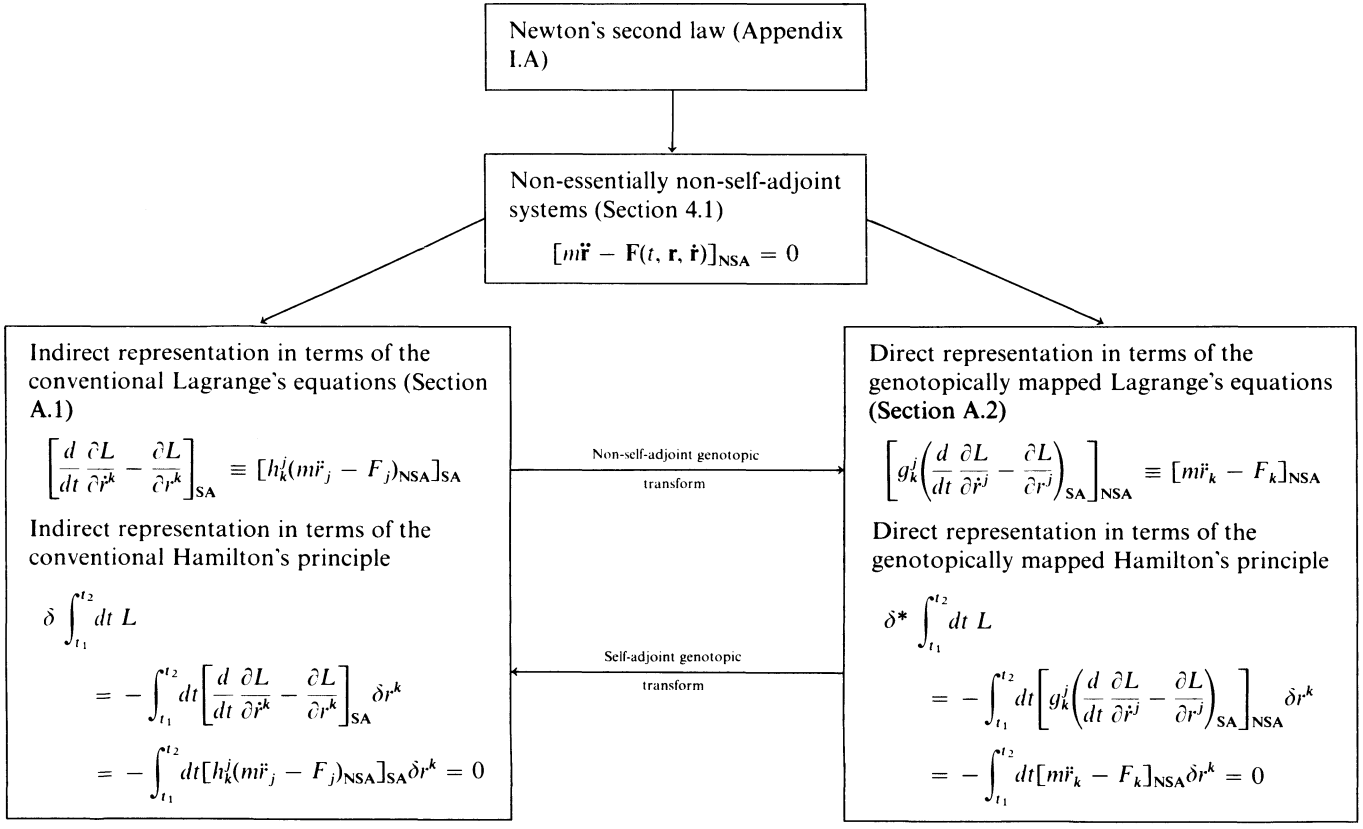

Newton's second law (Appendix I.A)
Non-essentially non-self-adjoint systems (Section 4.1)
$[m\ddot{\mathbf{r}} - \mathbf{F}(t, \mathbf{r}, \dot{\mathbf{r}})]_{\mathrm{NSA}} = 0$
Indirect representation in terms of the conventional Lagrange's equations (Section A.1)
$\left[\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^k} - \frac{\partial L}{\partial r^k}\right]_{\mathrm{SA}} \equiv [h_k^j(m\ddot{r}_j - F_j)_{\mathrm{NSA}}]_{\mathrm{SA}}$
Indirect representation in terms of the conventional Hamilton's principle
$\delta \int_{t_1}^{t_2} dt\, L$
$= -\int_{t_1}^{t_2} dt \left[\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^k} - \frac{\partial L}{\partial r^k}\right]_{\mathrm{SA}} \delta r^k$
$= -\int_{t_1}^{t_2} dt [h_k^j(m\ddot{r}_j - F_j)_{\mathrm{NSA}}]_{\mathrm{SA}} \delta r^k = 0$
Non-self-adjoint genotopic transform
Self-adjoint genotopic transform
Direct representation in terms of the genotopically mapped Lagrange's equations (Section A.2)
$\left[g_k^j\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^j} - \frac{\partial L}{\partial r^j}\right)_{\mathrm{SA}}\right]_{\mathrm{NSA}} \equiv [m\ddot{r}_k - F_k]_{\mathrm{NSA}}$
Direct representation in terms of the genotopically mapped Hamilton's principle
$\delta^* \int_{t_1}^{t_2} dt\, L$
$= -\int_{t_1}^{t_2} dt \left[g_k^j\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^j} - \frac{\partial L}{\partial r^j}\right)_{\mathrm{SA}}\right]_{\mathrm{NSA}} \delta r^k$
$= -\int_{t_1}^{t_2} dt [m\ddot{r}_k - F_k]_{\mathrm{NSA}} \delta r^k = 0$

Figure A.1 A schematic view of the *Indirect Lagrangian Representations*. The Newtonian systems of everyday life (particle motion in atmosphere, spinning tops, oscillators, etc.) contain nonpotential nonconservative forces and are non-self-adjoint in the form originating from the second law. As such, these systems violate the integrability conditions for the existence of the direct Lagrangian representations studied in Volume I. The table presents a schematic view of the main elements for the construction of indirect Lagrangian representations, that is, representations of equivalent systems rather than the systems in their original form. The table also presents a schematic view of the possibility of turning these indirect representations into direct ones, by using simple degrees of freedom of the variations. Essentially, when a system is non-self-adjoint in the form originating from the second law, a regular matrix of integrating factors may exist that is capable of turning the system into an equivalent self-adjoint form. This yields the *indirect* representations in the left-hand part of the table. However, the same matrix of integrating factors can be interpreted as a form of degree of freedom of the variations. This yields the *direct* representations of the right-hand part of the table. The transformation theory establishes the equivalence of the two approaches. In fact, the matrices of integrating factors turn out to be the Jacobian of the transformations. The approach can provide Lagrangian representations for a considerable class of non-conservative Newtonian systems (called non-essentially non-self-adjoint). However, the Indirect Lagrangian Problem is not universal, that is, it does not permit a representation of all systems of the class admitted. The lack of universality persists with the addition of transformations without an explicit dependence of the velocities. But velocity-dependent transformations generally alter the structure of Lagrange's equations as commonly understood in contemporary physics. The search for universality therefore leads naturally to the study of more general analytic equations, as presented in the main text of this volume.

Thus, in full analogy with Equations (4.4.1), point transformations can be either self-adjoint or non-self-adjoint, and each of these types can be either isotopic or genotopic. Notice that, as was the case for the genotopic functions of Section A.1, if a given point transformation is self-adjoint genotopic for one system, the same transformation does not necessarily exhibit the same character for a different system.

The integrability conditions for point transformations to be self-adjoint are given, again, by the conditions of variational self-adjointness and are provided in Theorem A.3.1 for the reader's convenience.

Theorem A.3.1[33] (Generalization of the Fundamental Analytic Theorem in Configuration Space with the Inclusion of Point Transformations). *Necessary and sufficient conditions for a holonomic non-conservative Newtonian system*

$$A_{ki}(t, q, \dot{q})\ddot{q}^i + B_k(t, q, \dot{q}) = 0; \qquad k = 1, 2, \ldots, n \tag{A.3.23}$$

which is analytic and regular in a region $\mathscr{R}$ of the variables $(t, q, \dot{q})$ to admit an ordered indirect analytic representation in terms of Lagrange's equations in a new system of variables $(t, q', \dot{q}')$

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}'^k} - \frac{\partial L'}{\partial q'^k} \equiv A^{*}_{ki}(t, q', \dot{q}')\ddot{q}'^i + B^{*}_{k}(t, q', \dot{q}') \tag{A.3.24a}$$

$$A^{*}_{ki} = \frac{\partial q'^j}{\partial q^k}\bar{A}_{rs}\frac{\partial q^r}{\partial q'^j}\frac{\partial q^s}{\partial q'^i} = \bar{A}_{ks}\frac{\partial q^s}{\partial q'^i}, \tag{A.3.24b}$$

$$B^{*}_{k} = \bar{B}_k + \bar{A}_{ks}\frac{\partial^2 q^s}{\partial q'^i\,\partial q'^j}\dot{q}'^i\dot{q}'^j, \tag{A.3.24c}$$

is that the transformed system is defined and self-adjoint in a star-shaped region $\mathscr{R}^{*\prime}$ of the variables $(t, q', \dot{q}')$. That is, it verifies conditions (A.1.9) *in $\mathscr{R}^{*\prime}$, in which case a Lagrangian is provided by Equations* (A.1.10).

Theorem A.3.1 essentially states that, under integrability conditions (A.1.9), the Jacobian matrix of point transformations can play the role of the matrix of integrating factors of Theorem A.1.1. In this sense, Theorem A.3.1 is a reformulation of Theorem A.2.1 *for the case when the integrating factors h^i_k depend only on the coordinates.* Indeed, when the conditions of Theorem A.3.1 are verified, a Lagrangian L' exists in the new coordinates. This implies the existence of a Lagrangian L in the old coordinates via the inverse of transformation (A.3.15) and thus the verification of the conditions of Theorem A.1.1.

In conclusion, Theorem A.3.1 as formulated *does not* broaden the arena of applicability of Theorem A.1.1. Nevertheless, the former theorem often has a pragmatic value because of the technical difficulties in solving partial

[33] Santilli (1977). This paper treats the corresponding formulation of the theorem within the context of field theories in Euclidean or Minkowski space.

differential equations (A.1.9). In fact, when these conditions have a difficult solution in one given reference frame, their solution can be attempted in a different reference frame via point transformations. In particular, one can attempt the joint use of Theorems A.1.1 and A.3.1, that is, the use of point transformations and integrating factors according to Equation (A.3.3).[34]

Notice that Theorem A.3.1 is formulated for both self-adjointness-inducing and preserving transformations. Thus it includes the case when the original system is self-adjoint. In this case, the theorem provides a relevant alternative for the possible construction of the isotopic transformations of a Lagrangian (Section A.2).

It should be stressed that Theorem A.1.1 is *broader* than Theorem A.3.1. This is due trivially to the fact that the integrating factors of the former can have an arbitrary functional dependence on $(t, q, \dot{q})$, while the elements of the Jacobian matrix of the latter can depend at most on (q').

In turn, this situation indicates the *restrictive* character of point transformations from the viewpoint of the Inverse Problem and suggests in a rather natural way the broadening of the transformation theory to noncontemporaneous velocity-dependent transformations of type (A.3.3).[35]

Chart A.1 Analytic Newtonian Systems

In Volume I we introduced the notion of class $\mathscr{C}^m$ Newtonian system, $(m = 0, 1, 2, \ldots, \infty)$. In particular, the minimal continuity conditions for the existence of a direct Lagrangian (Hamiltonian) representation emerged as those of class $\mathscr{C}^2$ $(\mathscr{C}^1)$. In order to formulate and prove the existence theorems for indirect analytic representations (of Lagrangian, Hamiltonian and Birkhoffian type), we need continuity and smoothness restrictions stronger than those of class $\mathscr{C}^\infty$.

We shall say that a *Newtonian force* $F(t, q, \dot{q})$ is *analytic* at the point $P_0 = (t_0, q_0, \dot{q}_0)$ when it admits the convergent, multiple, power series expansion in the *neighborhood*[36] $(P_0)_\varepsilon$ of P_0

$$F(t, q, \dot{q}) = \sum_{\substack{i_j=1 \\ j=1, 2, \ldots, 2n+1}}^{\infty} a_{i_1, \ldots, i_{2n+1}} (t - t_0)^{i_1} \times (q^1 - q_0^1)^{i_2} \cdots (q^n - q_0^n)^{i_{n+1}} (\dot{q}^1 - \dot{q}_0^1)^{i_{n+2}} \cdots \times (\dot{q}^n - \dot{q}_0^n)^{i_{2n+1}}. \tag{1}$$

An *unconstrained Newtonian system* is called *analytic* when all its forces are analytic. A *constrained holonomic Newtonian system* in the fundamental form of ($\mathscr{I}$.9) is called *analytic* when all the functions A_{ij} and B_j are

[34] The joint use of Theorems A.3.1 and A.1.1 is not "commutative." That is, if a self-adjoint form is induced via a) the use of a point transformation, and b) a matrix (h) of integrating factors, the inverse procedure using the same transformation and the same matrix (h) (but now computed along the old coordinates) does not necessarily produce a self-adjoint system.

[35] For a treatment of the transformation of Lagrange's equations under the contemporaneous transformations $t \to t' = t, q^k \to q'^k(t, q)$, see Kilmister (1967, page 129).

[36] For a definition of the neighborhood of a point, see page $\mathscr{I}$.16.

analytic in the sense indicated above. Of course, the analyticity or non-analyticity property of a given Newtonian system *does not* depend on whether or not the system is self-adjoint.

For a review of the convergence conditions of power series particularly suited for practical applications, we refer the reader in particular to Rektorys (1969, Chapter 15). Notice that series (1) can be absolutely and uniformly convergent in $(P_0)_\varepsilon$. Nevertheless, the nature of the convergence, as well as the largest possible domain in which it occurs, are not required by our analysis. This is because the existence theorems for partial differential equations we shall use (see Chart A.3) apply to any type of convergence and are local in character—that is, they hold in the neighborhood of a given point.

Quite often, we shall assume analyticity in the neighborhood of the origin, in which case Equation (1) becomes

$$F(t, q, \dot{q}) = \sum_{\substack{i_j=1 \\ j=1, 2, \ldots, 2n+1}}^{\infty} a_{i_1, \ldots, i_{2n+1}} t^{i_1} q^{1 i_2} \cdots q^{n i_{n+1}} \dot{q}^{1 i_{n+2}} \cdots \dot{q}^{n i_{2n+1}}. \tag{2}$$

It is easy to prove that

$$a_{i_1, \ldots, i_{2n+1}} = \frac{1}{i_1 ! i_2 ! \cdots i_{2n+1} !} \frac{\partial^k F}{(\partial t)^{i_1} (\partial q^1)^{i_2} \cdots (\partial \dot{q})^{i_{2n+1}}},$$
$$k = \sum_{j=1}^{2n+1} i_j. \tag{3}$$

Thus analyticity implies continuity of the partial derivatives of all orders (class $\mathscr{C}^\infty$). Nevertheless, the inverse property does not necessarily occur. In this sense, the analyticity condition is stronger than that of class $\mathscr{C}^\infty$.

The sum, product, and the quotient of analytic functions are analytic (under certain conditions). Also, analytic functions of analytic functions are analytic. Thus, if a Newtonian system in any second-order form is analytic, its equivalent first-order form (Section 4.1)

$$\dot{a}^\mu = \Xi^\mu(t, a), \qquad a = (q, y), \qquad \mu = 1, 2, \ldots, 2n \tag{4}$$

constructed via prescriptions for the introduction of n new independent variables, e.g.,

$$\dot{q}^k = Z^k(t, q, y) \tag{5}$$

is also analytic at a point $P = (t, a)$, provided that prescriptions (5) are analytic at P. Equivalently, we shall say that the *Newtonian vector field* Ξ^μ is *analytic* at $P_0 = (t_0, a_0)$ when it admits the convergent, multiple power-series expansion in the neighborhood $(P_0)_r$

$$\Xi^\mu(t, a) = \sum_{\substack{i_j=1 \\ j=1, \ldots, 2n+1}}^{\infty} a^\mu_{i_1, \ldots, i_{2n+1}} (t - t_0)^{i_1} \times (a^1 - a^1_0)^{i_2} \cdots (a^{2n} - a^{2n}_0)^{i_{2n+1}}. \tag{6}$$

Throughout our analysis we shall only consider analytic vector fields.

The notion of analyticity introduced in this chart is that of *real analyticity* (also called *Weierstrass analyticity*), in the sense that it deals with the the analyticity of real valued functions of real variables. This is clearly needed for the case of Newtonian systems. A sell-known treatise on this

subject is that by Bochner and Martin (1948). For a more recent treatment, the interested reader may consult, for instance, Gunning and Rossi (1965), Hörmander (1966), and Griffits and Harris (1978).

The need for an analytic continuation of the vector fields $\Xi^\mu(t, a)$ to complex variables $z = t + it'$ and $w^\mu = a^\mu + ia'^\mu$ may originate for a number of technical aspects of the Inverse Problem. In this case the notion of *complex analyticity* (also called *holomorphy*) is needed, that is, the existence of an absolutely convergent multiple power series expansion in the neighborhood of the point considered

$$\Xi^\mu(z, w) = \sum_{\substack{i_j=1 \\ j=1, 2, \ldots, 2n+1}}^{\infty} a^\mu_{i_1, \ldots, i_{2n+1}} (z - z_0)^{i_1} \times (w^1 - w^1_0)^{i_2} \cdots (w^{2n} - w^{2n}_0)^{i_{2n+1}}. \tag{7}$$

It is understood that such an analytic extension can be performed in only some of the variables (say, some of the a's) while leaving the others real (e.g., time), and that

$$\lim_{\substack{\mathrm{Im}\, z=0, \\ \mathrm{Im}\, w=0}} \Xi^\mu(z, w) = \Xi^\mu(t, a) \tag{8}$$

The point $P = (t, q, \dot{q})$ or $P = (t, a)$ in the neighborhood where the techniques of the Inverse Problem are applied must be selected with care. In particular, it must be a *regular point* in the sense of the theory of differential equations (see, for instance, Coddington and Levinson (1955)). This important restriction will be implemented, often tacitly, throughout our analysis.

As an indication of the type of *undesiderable* points for the application of the Inverse Problem, consider the case of *linear vector fields* in their holomorphic extension

$$\frac{dw^\mu}{dz} = A^\mu_\nu(z) w^\nu. \tag{9}$$

A point z_0 of z is called a singular point when the A functions have a *pole* in it. When the *Poincaré rank* of the singularity is 0 or ≥ 1, z_0 is called a *singular point of the first* or *second kind*, respectively. When the A's have an isolated singularity at z, the *fundamental matrix* $\Phi(z)$ of systems (9) (that is, the $2n \times 2n$ matrix composed by $2n$ rows of independent solutions) admits the decomposition

$$\Phi(z) = S(z)(z - z_0)^M = S(z)\exp M \log(z - z_0) \tag{10}$$

where S is (single-valued and) holomorphic at z_0 and M is a constant matrix. If S has at most a pole at z_0, then z_0 is called a *regular singular point*; otherwise it is called an *irregular singular point.*

All these singular points, as well as their extension to the case of nonlinear equations, together with any other type of singularity (or irregularity) are excluded from the analysis of this volume.

Chart A.2 Analytic Extensions of Lagrangian and Hamiltonian Functions to Complex Variables

In Chart A.1 we introduced the simplest and most conventional notion of analytic extension to complex variables. In this chart we shall indicate the existence of a broader notion which is particularly significant from

the viewpoint of symmetries and conserved quantities. We shall therefore deal specifically with a Lagrangian, with the understanding that the results trivially extend to a Hamiltonian (as well as a vector field). The presentation of this chart is also intended to be elementary; more advanced treatments will be indicated in the selected references.

Let $L(q, \dot{q})$ be a Lagrangian of a (non-essentially non-self-adjoint) Newtonian system in a region $\mathscr{R}$ of the variables $(q, \dot{q})$, assumed to be all *nonignorable*.[37] We introduce the complex extensions

$$z^k = q^k + iq'^k, \qquad \bar{z}^k = q^k - iq'^k, \qquad \dot{z}^k = \dot{q}^k + i\dot{q}'^k, \qquad \dot{\bar{z}} = \dot{q}^k - i\dot{q}'^k, \tag{1}$$

of variables $(q, \dot{q})$ in a complex extension $\mathbb{C}$ of $\mathscr{R}$.

Definition 1. An *extension* to complex variables of a Lagrangian $L(q, \dot{q})$ is a function $\bar{L}(z, \bar{z}; \dot{z}, \dot{\bar{z}})$ of the variables (1) such that

$$\lim_{\mathrm{Im}\, z = 0,\ \mathrm{Im}\, \dot{z} = 0} \bar{L} = L \tag{2}$$

where the limit holds if and only if, for every $\varepsilon > 0$, a $\delta > 0$ exists such that $|L(Z) - L(Q)| < \varepsilon$ for all values of Z such that $|Z - Q| < \delta$ and $Z \neq Q$, and where Z and Q represent the sets of variables $(z, \bar{z}; \dot{z}, \dot{\bar{z}})$ and $(q, \dot{q})$, respectively.

Notice that the extension $\bar{L}$ of L can also be considered a function of the $4n$ *real* variables $(q, \dot{q}; q', \dot{q}')$ in a $4n$-dimensional prolongation $\mathscr{R}'$ of $\mathscr{R}$.

Definition 2. A complex-valued function $F(z, \dot{z}; \bar{z}, \dot{\bar{z}})$ in the complex variables $(z, \dot{z}; \bar{z}, \dot{\bar{z}})$ is *analytic* in a region $\mathbb{C}$ of their variables when it is the sum of an absolutely convergent, multiple power-series expansion

$$\begin{aligned} F(z, \dot{z}; \bar{z}, \dot{\bar{z}}) &= F'(q, \dot{q}; q', \dot{q}') \\ &= \sum_{\substack{i_j = 1 \\ j = 1, 2, \ldots, 4n}}^{\infty} a_{i_1, \ldots, i_{4n}} (p^1 - p_0^1)^{i_1} \cdots \\ &\quad \times (p^{4n} - p_0^{4n})^{i_{4n}}, \qquad p = (q, \dot{q}; q', \dot{q}') \end{aligned} \tag{3}$$

in the neighborhood of every point of $\mathbb{C}$.

It should be stressed that Definitions 1 and 2 above are *broader* than the corresponding ones of the contemporary literature in the field, because the latter are generally restricted to the dependence on z and $\dot{z}$ only.

A convenient type of neighborhood for a point is the family of *polycircles*

$$C(p, p_0; r) = \{p \mid |p^k - p_0^k| < r^k; r^k > 0, k = 1, 2, \ldots, 4n\}. \tag{4}$$

This definition is preferred to the one on page $\mathscr{I}$.16 because it introduces a basis of open sets and thus a topology in the space of the local variables.

Definition 3. Let $L(q, \dot{q})$ be a Lagrangian which is (regular and) analytic in a region $\mathscr{R}$ of the variables $(q, \dot{q})$. Then an extension $\bar{L}(z, \dot{z}; \bar{z}, \dot{\bar{z}}) = \bar{L}'(q, \dot{q}; q', \dot{q}')$ of L to complex variables is an *analytic extension* of L when it is analytic in the sense of Definition 2.

[37] For a definition of ignorable and nonignorable coordinates, see page I.239.

From the viewpoint of complex variables, the analytic extensions can be classified as follows.

CLASS I: *Extensions analytic in the complex variables* $(z, \dot{z})$ *only*. If all the conditions

$$\frac{\partial \bar{L}}{\partial \bar{z}^k} = \frac{1}{2}\left(\frac{\partial \bar{L}'}{\partial q^k} + i\frac{\partial \bar{L}'}{\partial q'^k}\right) = 0, \qquad \frac{\partial \bar{L}}{\partial \dot{\bar{z}}^k} = \frac{1}{2}\left(\frac{\partial \bar{L}'}{\partial \dot{q}^k} + i\frac{\partial \bar{L}'}{\partial \dot{q}'^k}\right) = 0,$$

$$k = 1, 2, \ldots, n, \tag{5}$$

hold in $\mathbb{C}$, then $\bar{L}$ is analytic in $\mathbb{C}$, independent of $(\bar{z}, \dot{\bar{z}})$ (that is, $\bar{L} = \bar{L}(z, \dot{z})$), it admits the conventional decomposition of a complex analytic (i.e., holomorphic) function

$$\bar{L}(z, \dot{z}) = \bar{L}_1(q, \dot{q}; q', \dot{q}') + i\bar{L}_2(q, \dot{q}; q', \dot{q}'), \tag{6}$$

and the following rearrangement of the power-series expansion exists:

$$\begin{aligned}\bar{L}' &= \sum a_{i_1, \ldots, i_{4n}} q^{1 i_1} \cdots q^{n i_n} \dot{q}^{1 i_{n+1}} \cdots \dot{q}^{n i_{2n}} \\ &\quad \times q'^{1 i_{2n+1}} \cdots q'^{n i_{3n}} \dot{q}'^{1 i_{3n+1}} \cdots \dot{q}'^{n i_{4n}} \\ &= \sum b_{i_1, \ldots, i_{2n}} z^{1 i_i} \cdots z^{n i_n} \dot{z}^{1 i_{n+1}} \cdots \dot{z}^{n i_{2n}} = \bar{L}(z, \dot{z}).\end{aligned} \tag{7}$$

These extensions will be called *complex analytic extensions*.

CLASS 2. Extensions analytic in the real variables $(q, \dot{q}; q', \dot{q}')$ *only*. If all the conditions

$$\frac{\partial \bar{L}}{\partial z^k} \neq 0, \qquad \frac{\partial \bar{L}}{\partial \bar{z}^k} \neq 0, \qquad \frac{\partial \bar{L}}{\partial \dot{z}^k} \neq 0, \qquad \frac{\partial \bar{L}}{\partial \dot{\bar{z}}^k} \neq 0, \qquad k = 1, 2, \ldots, n \tag{8}$$

hold, then $\bar{L}$ has an essential functional dependence in all the $4n$ complex variables $(z, \dot{z}; \bar{z}, \dot{\bar{z}})$, decomposition (6) does not hold, and instead of Equation (7), we have the rearrangement

$$\begin{aligned}\bar{L}' &= \sum a_{i_1, \ldots, i_{4n}} q^{1 i_i} \cdots q^{n i_n} \dot{q}^{1 i_{n+1}} \cdots \dot{q}^{n i_{2n+1}} \\ &\quad \times q'^{1 i_{2n+1}} \cdots q'^{n i_{3n}} \dot{q}'^{1 i_{3n+1}} \cdots \dot{q}'^{n i_{4n}} \\ &= \sum b_{i_1, \ldots, i_{4n}} z^{1 i_i} \cdots z^{n i_n} \dot{z}^{1 i_{n+1}} \cdots \dot{z}^{n i_{2n}} \\ &\quad \times \bar{z}^{1 i_{2n+1}} \cdots \bar{z}^{n i_{3n}} \dot{\bar{z}}^{1 i_{3n+1}} \cdots \dot{\bar{z}}^{n i_{4n}} = \bar{L}.\end{aligned} \tag{9}$$

These extensions will be called *real analytic extensions*. The difference between a real analytic and a complex extension is meant to stress the property that the notion of analyticity in the real and imaginary components of the local variables is broader than that of analyticity in the complex variables only (holomorphy). Indeed, the former implies the latter as a particular case, but the converse is generally not true.

CLASS III. Extensions of the mixed analytic type. This is the case when $\bar{L}(z, \dot{z}; \bar{z}, \dot{\bar{z}})$ is real analytic in some variables and complex analytic in others.

The implications of the analytic extensions introduced in this chart from the viewpoint of symmetries and conserved quantities are self-evident. Suppose, for instance, that the original Lagrangian $L(q, \dot{q})$ is invariant under the orthogonal group in n-dimensions, $O(n)$. Then, the complex analytic extension $\bar{L}(z, \dot{z})$ of $\bar{L}$ will be invariant under the complex

analytic extension $O(n, \mathbb{C})$ of $O(n)$, while the real analytic extension $\bar{L}(z, \dot{z}; \bar{z}, \dot{\bar{z}}) = \bar{L}'(q, \dot{q}; q', \dot{q}')$ of L will be invariant under the unitary group in n dimensions, $U(n)$. The nonequivalence of the real and complex analytic extensions then follows from the fact that the symmetry groups $O(n, \mathbb{C})$ and $U(n)$ are *nonisomorphic* (actually, they have different dimensionalities and connectivity properties).

The classification of analytic extension to complex variables of this chart was introduced by Santilli and Roman (1971) for the case of the vacuum expectation values of the product of field operators, although the methodology trivially applies to Lagrangians (or Hamiltonians) of Newtonian systems. For further studies along these lines, see also Santilli *et al.* (1972), and Santilli and Ktorides (1973). This latter reference may be consulted, together with the specialized references quoted therein, in case the need arises to reinterpret the region of definition of the analytic extension as an algebraic variety.

Chart A.3 The Cauchy–Kovalevski[38] Theorem

In Section I.1.1 we outlined the *existence theory of ordinary differential equations*. In Section I.1.2 we then reviewed one of the central methods for the study of the integrability conditions for partial differential equations, the *calculus of differential forms*, and the converse of the Poincaré lemma. These mathematical tools were sufficient for the studies of the preceding volume, but they are insufficient for the studies of this volume, owing to the more direct need of the *existence theory of partial differential equations*.

For earlier textbooks on this subject, we recommend Goursat (1891) and Forsyth (1906, Volumes V and VI). For more recent treatments see, for instance, Bernstein (1950), Garabedian (1969), and Rektoris (1969). For more advanced approaches see, for instance, Langer (1961), Carroll (1969), and John (1975).

For the reader's convenience we recall, without proof one of the central theorems of the existence theory for partial differential equations, the *Cauchy–Kovalevski Theorem*, according to the presentation by Goursat (*loc. cit.*, page 2); (see also Rektoris *loc. cit.*, page 862).

Theorem 1. *Suppose that the system of n first-order partial differential equations in n unknown functions* $u^k, k = 1, 2, \ldots, n$, *and in m independent variables* $r^i, i = 1, 2, \ldots, m, m \gtrless n$

$$S_k\left(r^1, \ldots, r^m, \frac{\partial u^1}{\partial r^1}, \ldots, \frac{\partial u^1}{\partial r^m}, \ldots, \frac{\partial u^n}{\partial r^1}, \ldots, \frac{\partial u^n}{\partial r^m}\right) = 0,$$

$$k = 1, 2, \ldots, n \tag{1}$$

subject to the n initial conditions

$$u^k(0, r^2, \ldots, r^m) = h^k(r^2, \ldots, r^m) \tag{2}$$

admits, in the neighborhood P_ε *of the point* $P = (r^1, \ldots, r^m)$, *the equivalent form*

$$\frac{\partial u^k}{\partial r^1} = \sum_{i=1}^{n} \sum_{j=2}^{m} A_i^{kj}(r) \frac{\partial u^i}{\partial r^j} + \sum_{i=1}^{n} B_i^k(r) u^i + C(r), \tag{3}$$

[38] In western literature, the name of Madame Kovalevski is often presented in different versions, such as Kovalewski or Kowalewski.

called the Cauchy–Kovalevski form. Suppose also that functions A, B, C, and h are real analytic at P. Then a unique solution $u^1, \ldots, u^n$ *of the initial value problem* (1) *and* (2) *exists which is real analytic at P.*

The above theorem is frequently used in the text (e.g., Theorems A.1.1 and 4.5.1, although in the simpler version without initial conditions. Once a system of partial differential equations is assigned and the study of the consistency is necessary, a pragmatic way of using Theorem 1 is the following: a) select a regular point (Chart A.1); b) see whether the system admits an equivalent Cauchy–Kovalevski form; and, in case of affirmative answer, c) see whether the *A*- *B*- and *C*-functions are analytic. When these conditions are met, the system is consistent; that is, it admit a solution. From Theorem 1 we also learn that the number of arbitrary functions of such solution is equal to the order of the system and that these arbitrary functions involve one less independent variable than the number appearing in the system (for the case with initial conditions).

A few remarks are in order here. The proofs of Theorem 1 most commonly presented in the literature are those based on convergent, multiple power-series expansions in the sense of Chart A.1. This is the reason we restrict the application of Theorem 1 to the case where the condition of real analyticity is met. We assume that the reader is familiar with the fact that Theorem 1 does not ensure that the solution is readily computable in a closed form.

If the system under consideration involves partial derivatives of an order higher than first, it can be reduced to a system involving only first-order partial derivatives via techniques similar to those used in the reduction of second-order Newtonian systems to first-order forms (Section 4.1). However, a crucial new aspect must be taken into consideration. The invertibility of the reduction for the partial differential case does not readily occur as for the case of ordinary differential equations. In turn, this may affect the equivalence of the original system with the reduced one. (For a treatment of this aspect we refer the reader to Garabedian (*loc. cit.*, page 11). For the reader's convenience, we quote the following extension of the Cauchy–Kovalevski Theorem according to Petrovski (1954).

Theorem 2. *Suppose that a system of partial differential equations can be written in the Cauchy–Kovalevski form*

$$\frac{\partial^{n_i} u_i}{(\partial r^1)^{n_i}} = F_i\left(r^1, \ldots, r^m, u_1, \ldots, u_n, \frac{\partial^k u_j}{(\partial r^1)^{k_1} \cdots (\partial r^m)^{k_m}}, \ldots\right), \quad (4)$$

$$i, j = 1, 2, \ldots, n, \quad k_1 + \cdots + k_m = k \leq n_j, \quad k_1 < n_j, \quad n \gtrless m$$

under the initial conditions

$$\frac{\partial^k u_i}{(\partial r^1)^k} = \varphi_i^k(r^2, \ldots, r^m), \qquad k = 1, 2, \ldots, n_i - 1. \quad (5)$$

If all the functions F_i *are analytic in the neighborhood of a point* $(r^1, \ldots, r^m, \varphi^0_{j, k_1, \ldots, k_m}, \ldots)$ *and if all the functions* φ_i^k *are analytic in the neighborhood of the point* $(r^2, \ldots, r^m)$, *then a unique analytic solution* $u_1, \ldots, u_n$ *exists in the neighborhood of the point* $(r^1, \ldots, r^m)$.

The theorems presented in this chart (as well as a number of reformulations in the literature) are the result of a fundamental contribution by Madame Kovalevski of 1875 (certain textbooks, such as Rektorys (1969,

page 862), refer to the *Kovalevski Theorem*); the presentation adopted here is crucially dependent on the Cauchy initial value problem, (hence the name *Cauchy–Kovalevski Theorem*). For almost three quarter of a century, the approach remained the *only* existence theory for partial differential equations. Nevertheless, the limitation of the theory (originating from the restriction to analytic functions) more recently stimulated the study of extensions applicable under less restrictive continuity conditions. This recent effort, conducted by a number of mathematicians, has resulted in the contemporary existence theory of partial differential equations. However, it does not achieve the technical maturity and completeness of the corresponding theory for ordinary differential equations and calls for technical aspects beyond the level of presentation intended for these monographs.

In essence, these are the reasons behind the formulation and proof of the central theorems of this second volume for analytic systems; the extension of the theorems to less restrictive continuity conditions is encouraged.

Chart A.4 Kobussen's Treatment of Darboux's Theorem of Universality for One-Dimensional Systems

In Section A.1 we proved the universality of the Inverse Lagrangian Problem for one-dimensional systems. This result was obtained, apparently for the first time, by Darboux (1894, page 53). In this chart we outline Darboux's theorem according to the reformulation by Kobussen (1979) because of its considerable value for our applications.

Darboux studied the Inverse Lagrangian Problem for one-dimensional systems in the kinematic form possessing an explicit dependence on velocity. This dependence renders the system non-self-adjoint (Corollary I.2.2.2a), and we write

$$[\ddot{q} - f(t, q, \dot{q})]_{\text{NSA}} = 0. \tag{1}$$

Rather than seeking indirect analytic representations (A.1.8), Darboux looked for the solution in L of the one-dimensional, quasilinear, second-order partial differential equation

$$\frac{\partial^2 L}{\partial \dot{q}^2} f + \frac{\partial^2 L}{\partial \dot{q}\, \partial q} \dot{q} + \frac{\partial^2 L}{\partial \dot{q}\, \partial t} - \frac{\partial L}{\partial q} = 0. \tag{2}$$

He proved that such a solution (under the needed smoothness conditions expressed here in terms of the analyticity of the implicit function) *always* exists; that is, it exists for *all* possible implicit functions f.

Darboux's proof[39] was formulated in the mathematical language of his time[40] and is presented in Example A.5 for the general case of explicit dependence on time. In this chart we present the treatment by Kobussen (*loc. cit.*) formulated, as we shall see, in the current physical language via the use of constants of motion (or, more technically, first integrals). Our analysis will be restricted to implicit functions f which do not depend

[39] For an alternative treatment of Darboux's theorem in its original formulation, the interested reader should consult Akhiezer (1962, page 165).

[40] For instance, Darboux used the expression of the solution of the system in the implicit form characterized by two first integrals.

explicitly on time. The extension to the case of an explicit time dependence is left to the interested reader (Problem A.8).

A general solution of Equation (1) in L can be written

$$L = \dot{q}F(q, \dot{q}). \tag{3}$$

By writing x instead of q and y instead of $\dot{q}$, and by denoting the partial derivatives of F with respect to q and y as F_q and F_y, respectively, Equation (2) with "ansatz" (3) can be written

$$-y^2F_{xy} - f(x, y)(2F_y + yF_{yy}) = 0 \tag{4}$$

which is a first-order, linear, partial differential equation of hyperbolic type in $G = F_y$, and can be rewritten

$$yG_x + f(x, y)G_y = -\frac{2}{y} f(x, y)G. \tag{5}$$

A standard method for solving this equations is called the "characteristic method."[41] Consider the transformation

$$x \to I(x, y), \qquad y \to S(x, y). \tag{6}$$

Then

$$\begin{aligned} \frac{\partial}{\partial x} &= I_x \frac{\partial}{\partial I} + S_x \frac{\partial}{\partial S} \\ \frac{\partial}{\partial y} &= I_y \frac{\partial}{\partial I} + S_y \frac{\partial}{\partial S}, \end{aligned} \tag{7}$$

and Equation (5) becomes

$$(yI_x + f(x, y)I_y)G_x + (yS_x + f(x, y)S_y)G_y = -\frac{2}{y} f(x, y)G. \tag{8}$$

Assume now that $I(x, y)$ is a solution of the characteristic equation

$$yI_x + f(x, y)I_y = 0. \tag{9}$$

Equation (8) becomes a partial differential equation in two variables, and we have

$$(yS_x + f(x, y)S_y)G_y = -\frac{2}{y} f(x, y)G. \tag{10}$$

Characteristic equation (9) does not fix transformation (6). We can therefore assume that y is not transformed, i.e.

$$S(x, y) = y. \tag{11}$$

Then $S_x = 0$, $S_y = 1$, and Equation (10) becomes

$$f(x, y)\left(G_y + \frac{2}{y} G\right) = 0, \tag{12}$$

with the solution

$$G(I, y) = \frac{1}{y^2} E(I), \tag{13}$$

[41] See, for instance, Rektorys (1969).

where $E(I)$ is an arbitrary function. One can easily see that, if $I(x, y)$ is a solution of characteristic equation (9), so is $E(I(x, y))$. Thus, without loss of generality, we can say

$$G(I, y) = \frac{I(x, y)}{y^2}. \tag{14}$$

Also, Equation (9) implies

$$\frac{d}{dt} I(q, \dot{q}) = I_q \dot{q} + I_{\dot{q}} f = I_q \dot{q} + I_{\dot{q}} \ddot{q} = 0. \tag{15}$$

Thus $I(q, \dot{q})$ is a first integral of the equations of motion. The desired solution of Equation (2) is then given by

$$L(q, \dot{q}) = \dot{q} \int_0^{\dot{q}} dy \frac{I(q, y)}{y^2}. \tag{16}$$

In this way, according to Kobussen's method, the solution of the Inverse Problem for one-dimensional systems is turned into the knowledge of one first integral. Indeed, the construction of a Lagrangian for the system considered is provided by Equation (16), in which the only unknown is a first integral.

Notice that the Lagrangian representations are indirect. The use of different first integrals then yields *isotopically mapped Lagrangians* (Section A.2). The use of a lower limit of integral (16) greater than zero yields *gauge transformations*. The proof of the equivalence of this approach with that of the main text is left to the interested reader (Problem A.1). The following differentiation between these two methods is relevant, particularly for quantum mechanical considerations. Recall that the system considered is non-self-adjoint by assumption. Thus it is *nonconservative*. The method outlined in this chart essentially yields Lagrangians *without an explicit dependence on time* for the representation of autonomous, nonconservative systems. The method of Section A.1 provides, in general, a Lagrangian *with an explicit time dependence* for the representation of the same systems. All these different Lagrangians are equivalent (Lagrangian isotopy). As such, they can all be useful in *classical* mechanics. The situation in *quantum* mechanics appears to be different. Here the problems are (at least) twofold: first is the problem of quantizing *one* analytic representation so that it complies with the correspondence principle; then there is the problem of quantizing isotopically related Lagrangians (or, more precisely, Hamiltonians, when the quantization is done via the Hamilton–Jacoby equation). When the problem is seen from this profile, it appears that Lagrangians and Hamiltonians with explicit time dependences for the representation of autonomous non-conservative systems are preferable for a first quantization to those without such explicit dependence. Indeed, the former imply a breaking of the canonical realization of the symmetry under translations in time which becomes directly representative of the *non-conservative* nature of the system. In turn, this sets the proper methodological context for quantization, in order to avoid a quantum mechanical treatment which is formally conservative and, as such, would *violate* the correspondence principle. However, additional reasons exist for preferring an explicit time dependence in the analytic representation, expressed by the need to reach a *covering* quantum mechanical description—that is, one which identically recovers conventional conservative quantum mechanical settings at the limit of null non-conservative forces.

Lagrangians without and with an explicit time dependence behave differently under this limit. For instance, for the case of the damped oscillator,

$$(\ddot{r} + \gamma\dot{r} + \omega^2 r)_{NSA} = 0, \qquad \left(m = 1, \omega^2 - \frac{\gamma^2}{4} > 0, r\dot{r} > 0\right), \tag{17}$$

we have two isotopically mapped Lagrangians (Example I.3.2)

$$L(t, r, \dot{r}) = e^{\gamma t}(\tfrac{1}{2}\dot{r}^2 - \tfrac{1}{2}\omega^2 r^2), \tag{18a}$$

$$L^*(r, \dot{r}) = \frac{2\dot{r} - \gamma r}{2r\sqrt{\omega^2 - \gamma^2/4}}\, tg^{-1}\left(\frac{2\dot{r} + \gamma r}{2r\sqrt{\omega^2 - \gamma^2/4}}\right) - \tfrac{1}{2}\ln(\dot{r}^2 + \gamma r\dot{r} + \omega^2 r^2). \tag{18b}$$

Lagrangian L does indeed recover trivially that of the conservative linear oscillator under the value $F^{NSA} = -\gamma\dot{r} = 0$, that is, under $\gamma = 0$. However, the isotopic image L^* does not.[42] In any case, the problem of the quantization of Lagrangian L^* appears to be substantially more complex than that of Lagrangian L. As a result, Lagrangians L and L^* both have significance on classical analytic grounds. However, for quantum mechanics, the Lagrangian L appears preferable to its isotopic image L^*. Also, the non-conservative character of system (17) is transparently exhibited by Lagrangian L, while the technical characterization of this physical property via the isotopically mapped Lagrangian L^* is not that trivial. In fact, in the latter case, we have a Lagrangian which is invariant under translations in time; nevertheless, the total energy is not conserved by assumption (see Charts 4.10–4.12).

Chart A.5 Vanderbauwhede's Functional Approach to the Inverse Problem

As indicated in Volume I (Introduction and Charts I.3.16 and I.3.17), the branch of Functional Analysis dealing with nonlinear operators on Banach spaces provides a rigorous mathematical ground for the study of the Inverse Problem. Such a functional approach constitutes a valuable alternative to the variational approach adopted for the main lines of these volumes. Even though the explicit forms of the integrability conditions for the existence of a Lagrangian identified by these two approaches coincide, the functional approach allows the rigorous mathematical treatment of aspects which are not naturally focused upon by the variational approach. On the other hand, the variational approach enjoys a pragmatic effectiveness which is not possessed by the abstract structure

[42] For $F^{NSA} = 0$, L^* becomes an *isotope* of the conventional Lagrangian for the harmonic oscillator. The point is that the quantization of the isotopic images is unknown at this time for the conservative case. Also, the transition from a Lagrangian to one of its isotopes is expressed, at the quantum mechanical level, by a non-unitary transformation. As is well-known, conventional quantum mechanical laws (those for conservative systems) *are not* preserved by non-unitary transformations (see Santilli (1979b) for a detailed analysis). This confirms the non-trivial character of the problem considered here.

of nonlinear operators on Banach spaces. The reader with a serious interest in the Inverse Problem is urged to study it from all relevant aspects, including the functional.

The functional approach to the Inverse Problem was initiated by Vainberg (1964) as part of what is, by now, a fundamental contribution to the theory of non-linear operators. However, it took, a number of years for these studies to propagate in mathematical and physical circles. As mentioned in the Preface, this was partly due to the fact that Vainberg's approach was so abstract that it remained inaccessible to many applied mathematicians and engineers (Atherton and Homsy, 1975). This situation was remedied by Tonti who, in a series of contributions (1968, 1969a and b, 1982) brought the methodology to a level suitable for a broader audience. Additional contributions were made by a number of authors, such as Atherton and Homsy (*loc. cit.*) and Magri (1976). More recently, Vanderbauwhede (1978) reinspected the problem from a number of yet open aspects, including the methods for explicit computation of a Lagrangian. This chart consists of a review of Vanderbauwhede's analysis.

Let X, Y, ... denote (real) Banach spaces and X^*, Y^*, ... their dual, with canonical pairing $\langle x^*, x\rangle$, $\langle y^*, y\rangle$, ..., $x \in X$, $x^* \in X^*$, etc. Let $F: X \to Y$ be a map between two Banach spaces induced by a nonlinear operator. The Gateaux differential (Chart I.3.16) of F at the point $x \in X$ in the direction $h \in X$ will be denoted with $DF(x; h)$, while the symbol $DF(x)$ is reserved for a Gateaux derivative for which the map $h \to DF(x; h)$ is continuous and linear from X to Y. In this latter case we shall write $DF(x; h) = DF(x) \cdot h$. The Gateaux differential of a functional $f(x): X \to R$ will be written as an element $Df(x)$, of X^*, and we shall use the notation $Df(x) \cdot h = \langle Df(x), h\rangle$.

Consider a map $F: X \to X^*$, and let X_0 be a closed subspace of X. F is called a *potential operator* with respect to X_0, if a functional $f: X \to R$ exists such that i) $Df(x)$ exists for all $x \in X$ and ii) $\langle Df(x), h\rangle = \langle F(x), h\rangle$ for all $x \in X$ and all $h \in X_0 \subset X$. Thus the notion of potential operator is the equivalent (in the language of the functional approach to the Inverse Problem) of the notion of a self-adjoint system of ordinary differential equations (in the language of the variational approach to the same problem), as we shall discuss more fully later.

Let $X = C^{2n}([t_1, t_2], R^m)$ be a Banach space referring to a given compact time interval with n and m given non-negative indices, and introduce the following notations:

$$
\begin{aligned}
&q = \{q_i(t)\}, \qquad i = 1, 2, \ldots, m; \quad q^{(\alpha)} = \frac{d^\alpha q}{dt^\alpha}, \qquad \alpha = 0, 1, \ldots, \\
&\tilde{q} = \{q, q^{(1)}, \ldots, q^{(n)}\}, \qquad \tilde{\tilde{q}} = \{q^{(n+1)}, \ldots, q^{(2n)}\}, \\
&\bar{q} = \{q, q^{(1)}, \ldots, q^{(2n)}\}, \qquad \bar{\bar{q}} = \{q^{(2n+1)}, \ldots, q^{(4n)}\}, \\
&\hat{q} = \{q, q^{(1)}, \ldots, q^{(n-1)}\}, \qquad \hat{\hat{q}} = \{q^{(n)}, \ldots, q^{(2n)}\}.
\end{aligned}
\tag{1}
$$

The Inverse Problem for systems of ordinary differential equations of arbitrary (finite) order can be defined, within the context of the functional approach, as follows. Let f be a class $\mathscr{C}^\infty$ functional

$$
f = f(q) = \int_{t_1}^{t_2} dt\, L(t, \tilde{q}), \qquad q \in X. \tag{2}
$$

Its Gateau derivative at $q \in X$, in the direction of a function

$$\varphi \in X_0 = \{q \in X \mid q^{(\alpha)}(t_s) = 0,\ s = 1, 2,\ \alpha = 1, \ldots, n-1\} \tag{3}$$

gives rise to the (Euler-) Lagrange equations, since we have

$$\langle D\mathscr{F}(q), \varphi\rangle \overset{\text{def}}{=} \int_{t_1}^{t_2} dt \left\{ \sum_{\alpha=0}^{n} (-1)^{\alpha} \left(\frac{d^{\alpha}}{dt^{\alpha}} \frac{\partial L}{\partial q_k^{(\alpha)}} \right)(t\bar{q}) \right\} \varphi_k(t). \tag{4}$$

Consider now an element

$$F \in C^{\infty}([t_1, t_2] \times \mathbb{R}^{(2n+1)m}, \mathbb{R}^m), \qquad (t, \bar{q}) \to F(t, \bar{q}) \in \mathbb{R}^m \tag{5}$$

and the associated operator

$$\mathscr{T}: X \to C^0([t_1, t_2], \mathbb{R}^m), \qquad \mathscr{T}(q)(t) = F(t, \bar{q}(t)). \tag{6}$$

Then, $\mathscr{T}(q)$ can be identified with an element of X^*, via the introduction of the following canonical pairing between X and $C^0([t_1, t_2], \mathbb{R}^m)$

$$\langle \mathscr{T}(q), \varphi\rangle = \int_{t_1}^{t_2} dt\, \mathscr{T}_k(q)(t)\varphi_k(t). \tag{7}$$

The *functional approach to the existence theory of the Inverse Problem* consists of finding necessary and sufficient conditions on F such that $\mathscr{T}$, considered as operator from X to X^*, is potential with respect to X^*, i.e.,

$$\langle D\mathscr{F}(q), \varphi\rangle = \langle \mathscr{F}(q), \varphi\rangle, \qquad \forall \varphi \in X_0 \tag{8}$$

and so that $\langle D\mathscr{F}(q), \varphi\rangle$ is of form (4) for some

$$L \in \mathscr{C}^{\infty}([t_1, t_2] \times \mathbb{R}^{(n+1)m}, \mathbb{R}). \tag{9}$$

This latter condition emphasizes that L does not depend on $q^{(\alpha)}$, $\alpha = n+1$, $n+2, \ldots$. This is the generalization to higher orders of the condition that second-order differential equations, Equations ($\mathscr{I}$.3), are represented via fist-order Lagrangians, Equations ($\mathscr{I}$.18). Incidentally, this condition was not met by Vainberg's original formulation.

Theorem 1. *Let* $F \in \mathscr{C}^{\infty}([t_1, t_2] \times \mathbb{R}^{(2n+1)m}, \mathbb{R}^m)$, *and define* $\mathscr{T}$ *via Equations* (6). *Then the following statements are equivalent*:

1. $\mathscr{T}$ *is potential with respect to* X_0;
2. $\langle D\mathscr{T}(q) \cdot \psi, \varphi\rangle = \langle D\mathscr{T}(q) \cdot \varphi, \psi\rangle, \qquad \forall \varphi, \psi \in X_0,\ q \in X;$ (10)
3. *the following conditions*

$$\sum_{\beta=\alpha}^{2n} (-1)^{\beta} C_{\beta}^{\alpha} \left[\frac{d^{\beta-\alpha}}{dt^{\beta-\alpha}} \left(\frac{\partial F_k}{\partial q_i^{(\alpha)}} \right) \right] (t, \bar{q}, \bar{\bar{q}}) = \frac{\partial F_i}{\partial q_k^{(\alpha)}} (t, \bar{q}) \tag{11}$$

are indentically verified for all $t \in [t_1, t_2]$, $\bar{q} \in \mathbb{R}^{(2n+1)m}$, $\bar{\bar{q}} \in \mathbb{R}^{2nm}$, $\alpha = 0, 1, \ldots, 2n$; $i, k = 1, \ldots, m$.

This theorem clearly shows that the functional and variational approaches to the Inverse Problem produce *exactly the same* integrability conditions, Equations (11), as first pointed out in Chart I.3.17. Indeed, for second-order equations, conditions (11) coincide with conditions ($\mathscr{I}$.12) (see Section I.2.2 for details). The higher order case can be proved by simply extending to higher order systems the variational approach to self-adjointness of Section I.2.1. Notice the simplicity of the variational techniques, compared to the abstract language of the functional approach. The latter approach is preferred on the grounds of formal mathematical rigor.

The problem of the explicit construction of a Lagrangian, under conditions (10) or (11), can be studied by searching for the action functional

$$\mathscr{F}(q) = \int_{t_1}^{t_2} dt\, L(t, \bar{q}(t)) \tag{12}$$

such that

$$\langle D\mathscr{F}(q), \varphi \rangle = \langle \mathscr{T}(q), \varphi \rangle \tag{13}$$

not for all $\varphi \in X_0$, but instead for φ satisfying the more restrictive boundary conditions,

$$\varphi^{(\alpha)}(t_s) = 0; \quad s = 1, 2; \quad \alpha = 0, 1, \ldots, 2n - 1 \tag{14}$$

(from the $\bar{q}$-dependence of L). Unfortunately, this implies that $\mathscr{F}$ is insufficient to characterize a potential operator $\mathscr{T}$ with respect to X_0. The proof of the following property calls for a considerable amount of further study which, for brevity's sake, is not reported here (see Vanderbauwhede, *loc. cit.*).

Theorem 2. *Under the hypotheses of Theorem* 1, *if any of the statements of that theorem is equivalent to Equation* (4), *then a functional depending only on* $\tilde{q}$ *exists*

$$L \in \mathscr{C}^{\infty}([t_1, t_2] \times \mathbb{R}^{(n+1)m}, \mathbb{R}) \tag{15}$$

such that

$$F_k(t, \bar{q}) = \sum_{\alpha=0}^{n} (-1)^{\alpha} \left(\frac{d^{\alpha}}{dt^{\alpha}} \frac{\partial L}{\partial q_k^{(\alpha)}} \right)(t, \bar{q}) \tag{16}$$

for all $k = 1, 2, \ldots, m$, $t \in [t_1, t_2]$, $\bar{q} \in \mathbb{R}^{(2n+1)m}$. *Such a Lagrangian functional is explicitly given by*

$$L(t, \tilde{q}) = \int_0^1 F_k(t, \tau\bar{q}) q_k \, d\tau - \frac{d}{dt} \left\{ \sum_{\alpha=n}^{2n-1} \sum_{\beta=0}^{2n-1-\alpha} \int_0^1 d\tau' \int_0^1 d\tau\, \tau A_{ki}^{(\alpha+1,\beta)}(t, \tau\hat{q}, \tau\tau'\hat{\hat{q}}) q_k^{(\beta)} q_i^{(\alpha)} \right\}, \tag{17}$$

where

$$A_{ki}^{(\alpha,\beta)} = \sum_{\gamma=\beta}^{2n-\alpha} (-1)^{\gamma} C_{\gamma}^{\beta} \frac{d^{\gamma-\beta}}{dt^{\gamma-\beta}} \left(\frac{\partial F_k}{\partial q_i^{(\alpha+\beta)}} \right). \tag{18}$$

Note that, for the case $n = 1$ (corresponding to second-order differential equations), Equations (17) coincide with the formula proposed by Engels (1978):

$$L(t, q, \dot{q}) = q^k \int_0^1 d\tau\, F_k(t, \tau q, \tau\dot{q}, \tau\ddot{q}) - \frac{d}{dt} \int_0^1 \int_0^1 d\tau\, d\tau'\, \tau q^i A_{ij}(t, \tau q, \tau\tau'\dot{q}) \dot{q}^j \tag{19}$$

(see Equation ($\mathscr{I}$.13)). Thus Theorem 2 of this chart provides a natural generalization to the case of arbitrary order of the methods presented in these volumes for computation of Lagrangians.

The interested researcher may wish to note that studies on the functional approach to the Inverse Problem are far from over. What appears to be well established is a mere existence theory in functional language. Yet, as stressed in the Preface, physical systems generally *violate* such an existence theory. Thus additional studies are essential before the functional approach is of true value in direct, practical applications. We are referring here to the methods studied in this chapter for turning a non-self-adjoint system into an equivalent self-adjoint form, as well as the reformulation in the language of functional analysis of the proof of the universality of the Inversa Problem studied in the Chapters 4, 5, 6. Additional problems, not yet studied in the functional context, are associated with the degrees of freedom of the Lagrangian representations, particularly with the isotopic transformations.

Chart A.6 Symmetries

Undoubtedly, the studies of symmetries of physical systems constitutes one of the most physically significant, mathematically instructive, and aesthetically appealing topics in physics. Symmetries often represent physical laws and, as such, are at the foundations of the relativities of mechanics. Mathematically symmetries permit the computation of first integrals which are important for the possible solution of equations of motion via quadratures. Aesthetically, our efforts to represent physical reality are often guided by simple beauty. In short, symmetries are important for the reduction of physical systems to primitive group theoretic/algebraic notions.

Owing to these (and other) features, many studies have been devoted to the topic[43], particularly to Noether's celebrated theorem (reviewed in Chart A.9). These studies, however, have been restricted mainly to systems admitting conventional Lagrangian (and Hamiltonian) representations.

The Inverse Problem is clearly useful in the study of symmetries. In fact, knowledge of a Lagrangian for given equations of motion permits the use of Noether's theorem and other techniques which would otherwise be precluded. The Inverse Lagrangian Problem is therefore a *prerequisite* for the practical applicability of Noether's theorem. At a deeper level, the Inverse Problem permits an *enlargement* of the formulation of symmetries and first integrals, as the attentive reader already has been able to ascertain by inspecting the isotopic transformations of a Lagrangian (Section A.2).

In this and the remaining charts of this chapter we shall present the rudiments of the problem of symmetries, first integrals, and conservation laws for all Newtonian systems verifying the integrability conditions for the existence of an indirect Lagrangian representation (Theorem A.1.1), whether conservative or not. We shall attempt to clarify a number of misconceptions lingering in the current literature, which become apparent when the application of Noether's theorem is enlarged from conservative systems to all possible systems admitting a Lagrangian. For this purpose, we shall attempt as much as possible to separate the mathematical aspect of the theory from its physical interpretation. A prior minimal knowledge of the topic is assumed.

[43] The number of references in the field is large enough to discourage even a partial outline.

Definition 1. Newton's equations of motion for an (unconstrained, regular, and analytic) system in Euclidean space

$$m\ddot{r}_k - F_k(t, \mathbf{r}, \dot{\mathbf{r}}) = 0, \qquad k = 1, 2, 3 \tag{1}$$

is said to possess a *symmetry* under point transformations

$$t \to t' \equiv t, \qquad \mathbf{r} \to \mathbf{r}' = \mathbf{r}'(\mathbf{r}), \tag{2}$$

when the equations are *form-invariant* under these transformations; that is, the equations of motion preserve their form under the transformations considered, up to the Jacobian of the transformation. By recalling transformation law (A.3.16) we shall then write

$$m\ddot{r}_k - F_k(t, \mathbf{r}, \dot{\mathbf{r}}) = \frac{\partial r'^i}{\partial r^k}\left[m\ddot{r}'_i - F_i(t, \mathbf{r}', \dot{\mathbf{r}}')\right] = 0, \tag{3a}$$

$$g_{ij} = \frac{\partial r^l}{\partial r'^i}\,\delta_{lm}\,\frac{\partial r^m}{\partial r'^j} \equiv \delta_{ij}, \tag{3b}$$

$$\begin{aligned} F'_i(t, \mathbf{r}', \dot{\mathbf{r}}') &= \frac{\partial r^j}{\partial r'^i} F_j(t, \mathbf{r}(\mathbf{r}'), \dot{\mathbf{r}}(r', \dot{\mathbf{r}}')) - \frac{\partial r^j}{\partial r'^i}\,\delta_{jl}\,\frac{\partial^2 r^l}{\partial r'^m\,\partial r'^n}\,\dot{r}'^m\dot{r}'^n \\ &\equiv F_i(t, \mathbf{r}', \dot{\mathbf{r}}'). \end{aligned} \tag{3c}$$

More generally, a quasilinear, second-order system of ordinary differential equations (say, Newtonian systems with holonomic constraints)

$$A_{ki}(t, q, \dot{q})\ddot{q}^i + B_k(t, q, \dot{q}) = 0, \qquad k = 1, 2, \ldots, n \tag{4}$$

is said to possess a *symmetry* under the point transformations

$$t \to t' \equiv t, \qquad q \to q' = q'(q) \tag{5}$$

when the system preserves its form under the transformations up to the Jacobian. By recalling transformation law (A.3.9), we write

$$A_{ki}(t, q, \dot{q})\ddot{q}^i + B_k(t, q, \dot{q}) = \frac{\partial q'^i}{\partial q^k}\left[A_{ij}(t, q', \dot{q}')\ddot{q}'^j + B_i(t, q', \dot{q}')\right] = 0, \tag{6a}$$

$$\begin{aligned} A'_{ij}(t, q', \dot{q}') &= \frac{\partial q^r}{\partial q'^i}\,\frac{\partial q^s}{\partial q'^j}\,A_{rs}(t, q(q'), \dot{q}(q', \dot{q}')) \\ &\equiv A_{ij}(t, q', \dot{q}') \end{aligned} \tag{6b}$$

$$\begin{aligned} B'_i(t, q', \dot{q}') &= \frac{\partial q^r}{\partial q'^i}\,B_r(t, q(q'), \dot{q}(q', \dot{q}')), \\ &\quad + A_{rs}(t, q(q'), \dot{q}(q', \dot{q}'))\,\frac{\partial q^r}{\partial q'^i}\,\frac{\partial^2 q^s}{\partial q'^m\,\partial q'^n}\,\dot{q}'^m\dot{q}'^n \\ &\equiv B_i(t, q', \dot{q}'). \end{aligned} \tag{6c}$$

An equivalent definition holds for transformations more general than (2) and (5) although the transformation laws are predictably more involved.[44]

[44] See Chapter 6.

Definition 1 illustrates the difference between *transformation laws* (A.3.16) or (A.3.9) and *symmetry laws* (3) or (6). Both transformations are *equivalence transformations* (in the sense of Section A.1). Nevertheless, symmetries comprise only a subset of the equivalence transformations because of the condition of form-invariance of the system. Notice that Definition 1 is purely mathematical in that it deals with a property which is independent of the possible physical nature of the system considered and its transformations.

Definition 2. A Lagrangian $L(t, q, \dot{q})$ is said to possess a *symmetry* under the transformations

$$t \to t' = t'(t), \qquad q \to q' = q'(q), \tag{7}$$

when the new Lagrangian constructed according to the rule

$$L(t, q, \dot{q})dt = L'(t', q', \dot{q}')dt' \tag{8}$$

preserves its original functional dependence up to the possible addition of terms with null Lagrange's equations, i.e.,

$$L'(t', q', \dot{q}') = L(t', q', \dot{q}') + \frac{d}{dt'} G(t', q'), \tag{9a}$$

$$\frac{d}{dt'} \frac{\partial \dot{G}}{\partial \dot{q}'^k} - \frac{\partial \dot{G}}{\partial q'^k} \equiv 0. \tag{9b}$$

Notice that we have enlarged the class of transformations by including time (although in a way independent from that of the generalized coordinates). This has been done to recall that a Lagrangian, in general, *does not* transform according to a scalar rule $L(t, q, \dot{q}) = L'(t', q', \dot{q}')$, but rather according to *density* rule (8). This condition ensures the *equivalence of the action principles* in the new and old coordinates.

Under the integrability conditions of Theorem A.1.1 (or A.3.1), Definitions 1 and 2 are equivalent. In fact, by assuming for simplicity $t' = t$, symmetry law (6) implies law (9) and vice versa; i.e.,

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} &\to A_{ki}(t, q, \dot{q})\ddot{q}^i + B_k(t, q, \dot{q}) \\
&\equiv \frac{\partial q'^i}{\partial q^k}\left(\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}'^i} - \frac{\partial L'}{\partial q'^i}\right) \\
&\equiv \frac{\partial q'^i}{\partial q^k}\left[A'_{ij}(t, q', \dot{q}')\ddot{q}'^j + B'_i(t, q', \dot{q}')\right] \\
&\equiv \frac{\partial q'^i}{\partial q^k}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}'^i} - \frac{\partial L}{\partial q'^i}\right) \\
&\equiv \frac{\partial q'^i}{\partial q^k}\left[A_{ij}(t, q', \dot{q}')\ddot{q}'^j + B_i(t, q', \dot{q}')\right].
\end{aligned} \tag{10}$$

The "gauge" degree of freedom in law (9) is clearly a consequence of the fact that all functions $\dot{G}(t', q')$ have null Lagrange's derivatives according to Equations (A.2.1b).

Definition 3. Symmetries of Newton's equations of motion, or of Lagrange's equations, can be classified as follows:

1. *manifest symmetries*, which can be identified by simple means (usually a visual inspection) and generally have a simple functional dependence of the new variables in the old (e.g., of linear type);
2. *non-manifest symmetries*, which demand special mathematical techniques for their identification and generally possess a complex functional dependence (e.g., of the nonlinear type);
3. *discrete symmetries*, when the transformations are discrete (e.g., space and time inversion);
4. *connected symmetries*, when the transformations can be continuously connected to the identity (e.g., rotations);
5. *finite symmetries*, when the transformations are finite;
6. *infinitesimal symmetries*, when the transformations are infinitesimal (of the first-order);
7. *contemporaneous* (*noncontemporaneous*) *symmetries*, when the transformations do not (do) include that of time ($t \to t' \neq t$);
8. *without* (*with*) *gauge*, when the function $G(t', q')$ of law (9) is identically null (non-null);
9. *pure space, pure time, and mixed space–time symmetries*, which are characterized by transformations involving space coordinates only, time only, and mixed space and time variables.

As a simple example, the conventional two-body Lagrangian

$$L = \tfrac{1}{2}(m_1\dot{\mathbf{r}}_1^2 + m_2\dot{\mathbf{r}}_2^2) + \frac{k}{|\mathbf{r}_1 - \mathbf{r}_2|} \tag{11}$$

possesses the following symmetries: a) a symmetry under rotations $\mathbf{r}_i' = R\mathbf{r}_i$, $R \in SO(3)$, $i = 1, 2$, which is manifest, connected, contemporaneous, finite, without gauge, and of pure space character; b) a symmetry under time inversion $t' = -t$, which is manifest, discrete, noncontemporaneous, and of pure time character; and c) a symmetry under Galilei boosts $\mathbf{r}_i' = \mathbf{r}_i + \mathbf{v}t$ (Chart I.A.1) which is manifest, connected, contemporaneous, with gauge, and of mixed space–time character.

Additional examples of symmetries will be provided later, jointly with the important concepts of "exact symmetry" and "broken symmetry."

Chart A.7 Lie's Construction of Symmetries of Given Equations of Motion

Clearly, manifest symmetries are the simplest possible symmetries possessed by physical systems. These symmetries generally do not permit the identification of the first integrals needed for solution by quadrature, which creates the problem of finding a method for the general construction of symmetries of given equations of motion without any restriction on the type of symmetries (as well as equations) admitted.

To my best knowledge, this problem was identified, studied, and solved by Lie (1891). Regrettably, Lie's method was subsequently ignored to a considerable extent, apart from rare exceptions (such as the treatment by Ince (1926)). Only recently has a systematic study of Lie's method been initiated by C. J. Eliezer and associates (for a review and a complete list of these latter studies see Eliezer (1979)). In the following, I shall review

Lie's method along Eliezer's presentation and then implement it via the techniques of the Inverse Lagrangian Problem in order to render it applicable to Noether's theorem.

As we shall review in the charts of Chapter 5, a one-parameter connected Lie group of finite transformations of the generalized coordinates and time can be written in the exponential form

$$t' = e^{\varepsilon X} t, \qquad q'^k = e^{\varepsilon X} q^k, \tag{1}$$

where ε is the group parameter and X is the so-called generator. When X does not depend explicitly on the velocities, it can be written in the vector-field form[45]

$$X(t, q) = \hat{\rho}(t, q) \frac{\partial}{\partial t} + \hat{\eta}^k(t, q) \frac{\partial}{\partial q^k}, \tag{2}$$

where $\hat{\rho}$ and $\hat{\eta}^k$ are functions satisfying certain smoothness conditions (e.g., being of class $\mathscr{C}^\infty$).

The infinitesimal first-order transformations can then be written

$$t' = t + \hat{\rho}(t, q)\varepsilon, \qquad q'^k = q^k + \hat{\eta}^k(t, q)\varepsilon, \qquad \varepsilon \approx 0. \tag{3}$$

When these transformations are known (that is, when the functions $\hat{\rho}$ and $\hat{\eta}$ are known), one can construct finite transformations (1) by the techniques of Lie's theory ("exponentiation" of an infinitesimal transformation to a finite form) or, more empirically, via the solution of the system of differential equations

$$\frac{dt'}{\hat{\rho}(t, q)} = \frac{dq'^1}{\hat{\eta}^1(t, q)} = \cdots = \frac{dq'^n}{\hat{\eta}^n(t, q)} = \varepsilon. \tag{4}$$

Variations in derivatives higher than the first can be constructed via the extended generators

$$X^{(m)} = \hat{\rho} \frac{\partial}{\partial t} + \hat{\eta}^k_{(0)} \frac{\partial}{\partial q^{(0)k}} + \hat{\eta}^k_{(1)} \frac{\partial}{\partial q^{(1)k}} + \cdots + \hat{\eta}^k_{(m)} \frac{\partial}{\partial q^{(m)k}}, \tag{5a}$$

$$\hat{\eta}^k_{(m)}(t, q^{(0)}, \ldots, q^{(m)}) = \frac{d\hat{\eta}^k_{(m-1)}}{dt} - q^{(m)k} \frac{d\hat{\rho}}{dt}, \tag{5b}$$

$$\hat{\eta}^k_{(0)} = \hat{\eta}^k, \qquad q^{(0)k}, \qquad q^{(m)k} = \frac{d^m q^k}{dt^m}. \tag{5c}$$

Lie's method is essentially based on the following condition for transformations (3) to be symmetric of the given equations of motion in the kinematic form

$$X^{(2)}[\ddot{q}^k - f^k(t, q, \dot{q})] = 0, \tag{6}$$

which is a condition on the unknown functions $\hat{\rho}$ and $\hat{\eta}^k$. More specifically, the implicit functions f^k are known. Thus the only unknown quantities in Equation (6) are functions $\hat{\rho}$ and $\hat{\eta}^k$.

The extension to the case of an r-dimensional symmetry group G_r is straightforward. We write the infinitesimal transformations in the form

$$t' = t + \hat{\rho}_i \varepsilon^i, \qquad q'^k = q^k + \hat{\eta}^k_i \varepsilon^i, \qquad i = 1, 2, \ldots, r \tag{7}$$

[45] The notation of Section I.1.3 is used here, according to which noncontemporaneous variations are denoted with the symbols $\hat{\rho}$ and $\hat{\eta}^k$, while the contemporaneous variations ($\hat{\rho} \equiv 0$) are denoted with η^k.

with the corresponding group form

$$t' = e^{\varepsilon_i X_i} q, \qquad q'^k = e^{\varepsilon_i X_i} t. \tag{8}$$

The underlying r generators X_i, each with corresponding extensions of type (5), follow.

Note that condition (6) constitutes a system of *partial* differential equations in functions $\hat{\rho}$ and $\hat{\eta}$. These systems (when consistent) admit solutions with considerable degrees of functional freedom. As a result, r functionally different sets of solutions $\hat{\rho}_i$ and $\hat{\eta}_i^k$, $(i = 1, 2, \ldots, r)$ yield an r-dimensional symmetry group G_r.

As an explicit example, Lie's method for the case of the Kepler problem in the plane (x, y)

$$\ddot{\mathbf{r}} + \frac{\mathbf{r}}{|r|^3} = 0, \qquad \mathbf{r} \equiv (x, y) \tag{9}$$

permits the identification of the following generators for a three-dimensional symmetry group (Eliezer (*loc. cit.*))

$$\begin{aligned} X_1 &= \frac{\partial t}{\partial t'} \quad X_2 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \\ X_3 &= t\frac{\partial}{\partial t} + \frac{2}{3}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right). \end{aligned} \tag{10}$$

One easily recognizes X_1 as the generator of translations in time and X_2 as the generator of rotations. Simple elaborations show that X_3 is the generator associated to the Runge–Lenz vector.

We refer the interested reader to the review by Eliezer (*loc. cit.*) and quoted papers for numerous additional examples worked out in detail.

As we shall see in Chart A.9, a necessary condition for the applicability of Noether's theorem is that the equations of motion are in the Lagrangian form. This situation creates something of a gap between Lie's method and its application to the computation of first integrals via Noether's theorem. In fact, Lie's method was conceived for arbitrary (not necessarily Lagrangian) systems of differential equations. It is tempting to speculate that perhaps this general lack of Lagrangian character is responsible for the lack of interest in Lie's method which prevailed until recently.

The Inverse Lagrangian Problem clearly provides the technical means for establishing a continuity of thought in the transition from Lie's method to Noether's theorem. This can be achieved by imposing Lie's condition (6) on an equivalent system in the self-adjoint form (Section A.1)

$$\begin{aligned} &X^{(2)}\{h_{ki}(t, q, \dot{q})[\ddot{q}^i - f^i(t, q, \dot{q})]_{\text{NSA}}\}_{\text{SA}} \\ &\qquad = X^{(2)}[A_{ki}(t, q, \dot{q})\ddot{q}^i + B_k(t, q, \dot{q})]_{\text{SA}} = 0, \\ &A_{ki} = h_{ki}, \qquad B_k = -h_{ki}f^i. \end{aligned} \tag{11}$$

Equivalently, when a Lagrangian has been computed via Equations (A.1.10), one can impose the symmetry condition

$$X^{(1)}L + \left(\frac{d}{dt}\hat{\rho}\right)L + \frac{d}{dt}G = 0 \tag{12}$$

where G is the gauge function from Definition 3 of the preceding chart (see Chart A.9 for more details).

Notice that, when L is the Lagrangian for the representation of the self-adjoint equations of motion in conditions (11), Lie's methods (11) and (12) are equivalent.

Let us stress here that the class of *manifest* symmetries of Newton's equations of motion is generally *larger* than that of *one* Lagrangian for its analytic representation. This is clearly due to the fact that, in the transition from condition (6) to (11), some of the original manifest symmetries can be lost because of the genotopic functions h_{kj}. As a specific example, the equation for a particle in one dimension,

$$(\ddot{r} + \gamma\dot{r})_{\text{NSA}} = 0, \tag{13}$$

has two manifest symmetries: those under translations in time and those under translation in space. Two self-adjoint forms provided by the Inverse Problem (Example I.3.1) are given by

$$[e^{\gamma t}(\ddot{r} + \gamma\dot{r})_{\text{NSA}}]_{\text{SA}} = 0, \tag{14a}$$

$$\left[\frac{1}{\dot{r}}(\ddot{r} + \gamma\dot{r})_{\text{NSA}}\right]_{\text{SA}} = 0, \tag{14b}$$

with corresponding Lagrangians

$$L_1 = e^{\gamma t}\tfrac{1}{2}\dot{r}^2 \tag{15a}$$

$$L_2 = \dot{r}\ln\dot{r} - \gamma r. \tag{15b}$$

Clearly, the symmetry under translation in time is not possessed by Lagrangian L_1 and the symmetry under translation in space is not possessed by Lagrangian L_2.

It should be stressed that this situation is due to the restriction on the symmetries to be manifest. If this restriction is lifted and arbitrary, generally nonmanifest symmetries are admitted, the equivalence between the class of symmetries of the equations of motion and that of a Lagrangian for its analytic representation can be recovered, as we shall indicate in Chart A.10.

Chart A.8 First Integrals and Conservation Laws

Symmetries are often applied in physics for the characterization of conservation laws. However, symmetries generally represent first integrals without any direct physical meaning. This is particularly true when the restriction to conservative systems is lifted and arbitrary Newtonian systems are admitted. To prevent misrepresentations of the problem of symmetries and conservation laws, it is therefore important to understand the difference between the mathematics and possible physical interpretation, beginning with the differentiation between first integrals and conservation laws.

Definition 1. A function $I(t, \mathbf{r}, \dot{\mathbf{r}})$ is a *first integral* of an (unconstrained) Newtonian system in Euclidean space,

$$m\ddot{r}_k - F_k(t, \mathbf{r}, \dot{\mathbf{r}}) = 0, \tag{1}$$

if it is a constant when computed along the solution of the system, that is

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + \frac{\partial I}{\partial r^k} \dot{r}^k + \frac{\partial I}{\partial \dot{r}^k} F_k/m \equiv 0. \tag{2}$$

A similar definition applies for a first integral $I(t, q, \dot{q})$ of a system in configuration space

$$A_{ki}(t, q, \dot{q})\ddot{q}^i + B_k(t, q, \dot{q}) = 0 \tag{3}$$

in which case we write

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + \frac{\partial I}{\partial q^k} \dot{q}^k + \frac{\partial I}{\partial \dot{q}^k} f^k \equiv 0 \tag{4a}$$

$$f^k = (\|A_{lm}\|^{-1})^{ki} B_i. \tag{4b}$$

Clearly, the notion of first integrals is of *mathematical* character, in the sense that a quantity $I(t, \mathbf{r}, \dot{\mathbf{r}})$ is, in general, an arbitrary function of local variables without any necessary direct physical meaning. This mathematical character is illustrated further by the following additional properties.

Definition 2. A set of m ($\leq 2n$) first integrals $I_i(t, q, \dot{q})$ of a system of n second-order ordinary differential equations are said to be *independent* when the rank of the $m \times 2n$ matrix,

$$\left(\frac{\partial I_i}{\partial a^\mu}\right), \qquad a = (q, \dot{q}), \tag{5}$$

$$i = 1, 2, \ldots, m; \qquad \mu = 1, 2, \ldots, 2n,$$

is m.

Without proof, we quote the following theorem.

Theorem 1 (Maximal Number of Independent First Integrals). *A system of n second-order ordinary differential equations can admit, at most, 2n independent first integrals in the neighborhood of a point of the local variables*:

$$I_\mu(t, q, \dot{q}) = C_\mu = \text{const.}, \qquad \mu = 1, 2, \ldots, 2n. \tag{6}$$

All possible additional first integrals can always be expressed as function of the I's in the neighborhood of the point considered.

The knowledge of $2n$ independent first integrals allows the *solution of the system by quadratures*. Under the condition that the rank of matrix (5) is $2n$, the Implicit Function Theorem I.1.1.1 can be applied to Equation (6) to yield coordinates and velocities as functions of time and the $2n$ arbitrary constants, c's,

$$q^k = q^k(t, c), \qquad \dot{q}^k = \dot{q}^k(t, c). \tag{7}$$

In this way, the solution of the system is provided by algebraic manipulations of the first integrals.

Needless to say, $2n$ independent first integrals are not known, in general, and their identification often turns out to be as difficult as the solution of the system itself (see, for instance, the historical case of the three-body classical problem, as presented in Hagihara (1970)). Nevertheless, the knowledge of as many independent first integrals as possible is important because they reduce the dimensionality of the systems and thus, the

difficulties in their solution. For this and related aspects, we refer the interested reader to the literature in ordinary differential equations, including Hagihara (*loc. cit.*).

We move now to a separate problem: whether a given first integral can be interpreted as a physical quantity, and law (2) or (4) as a physical conservation law.

Definition 3. A Newtonian system is called *conservative* when it is closed—that is, when it can be considered as isolated from the rest of the universe. A Newtonian system is called *nonconservative* when it is *open*—that is, when it cannot be isolated from its environment, which is then considered as external.

The Inverse Lagrangian Problem clearly permits the application of Noether's theorem and related techniques to conservative as well as nonconservative systems.

Definition 4. A quantity $I(t, \mathbf{r}, \dot{\mathbf{r}})$ is a *physical quantity* (such as energy, linear momentum, angular momentum, etc.) of an unconstrained non-self-adjoint Newtonian system,

$$\{[m\ddot{r}_k - f_k(t, \mathbf{r}, \dot{\mathbf{r}})]_{ESA} - F_k(t, \mathbf{r}, \dot{\mathbf{r}})\}_{NSA} = 0, \tag{8}$$

when it is a physical quantity of the maximal, essentially self-adjoint subsystem.

This definition is based on the idea that *physical quantities are unique* and do not depend (in their definition) on non-self-adjoint forces. These forces are merely responsible for the *behavior in time* of a given physical quantity. For instance, the *physical*[46] *linear momentum* is given by $\mathbf{p} = m\dot{\mathbf{r}}$, whether or not the forces are potential. Similarly, the *physical angular momentum* is uniquely given by $\mathbf{M} = \mathbf{r} \times m\dot{\mathbf{r}}$ and the possible presence of nonpotential forces may only affect its time evolution. More insidious is the case of the *physical energy* which, according to Defintion 4, is the (mechanical) energy of the maximal, essentially self-adjoint, subsystem; that is, the sum of the kinetic energy and the potential energy of all self-adjoint forces (Theorem $\mathscr{I}$.1 and Corollary $\mathscr{I}$.1a). The insidious character is that a Hamiltonian for the entire system (8) may exist and be interpreted as "energy." The point is that the notion of potential energy has no physical foundation for contact forces.

It is clear that the physical energy of system (8) is *not conserved*. This is the case for the systems of everyday life, such as spinning tops with drag

[46] The term "physical" is introduced here to stress the important differentiation from "canonical" momentum $\mathbf{p}_{can} = \partial L/\partial \dot{\mathbf{r}}$ (which can be an arbitrary function of the local variables, e.g., $\mathbf{p}_{can} = \alpha r e^{\beta r \cdot \dot{r}}$). The reader should recall the distinction between physical and canonical quantities whenever studying non-potential forces, from both a classical and quantum mechanical viewpoints. In fact, the differentiation is important to discourage the belief that mathematically consistent algorithms of the type $\Delta r \Delta p \geq \frac{1}{2}\hbar$ necessarily represent an indeterminacy occurring in nature (which is the case if and only if the operator "p" can be proved to represent the physical linear momentum and a number of other conditions are met; e.g., the consistency of the associative character of the operator enveloping algebra with the underlying dynamics). The interested reader may consult Santilli (1978d) for these quantum mechanical aspects.

torques, trajectory problems in atmosphere, damped oscillators, etc., which all admit action-at-a-distance potential forces and contact non-potential forces according to structure (8).

Inspection of physical reality therefore establishes the existence of *conservation laws* and *non-conservation laws*, depending on the physical case considered. The former can be first integrals (see below for counter-examples). The latter are not first integrals. We can therefore introduce the following definition.

Definition 5. A first integral is a *physical conservation law* when it represents directly a physical quantity of the maximal, essentially self-adjoint subsystem.

A few examples are in order. The Kepler problem in two-dimensions, Equation (9) of Chart A.7, admits the first integral

$$H = \tfrac{1}{2}(\dot{\mathbf{r}}^2 - \mathbf{r}^2/|r|^3) \tag{9}$$

which coincides with the total mechanical energy (the system is, in this case, essentially self-adjoint and conservative). The damped particle in one dimension, Equation (13) of Chart A.7, admits the first integral (a Hamiltonian of Example I.3.1)

$$H = \dot{r} \ln \dot{r} + \gamma r \tag{10}$$

which *does not* coincide with the energy of the system (the system is, in this case, non-essentially non-self-adjoint and nonconservative). The maximal, essentially self-adjoint subsystem is that of free motion. Physical energy is then given by the kinetic energy $T = \frac{1}{2}\dot{r}^2$, which is now non-conserved.

As a result, Hamiltonians (9) and (10) possess the mathematical property of being first integrals. However, only Hamiltonian (9) represents a physical law, while the "conservation" of Hamiltonian (10) has no direct physical meaning. Additional illustrations for energy as well as other physical quantities are given throughout our analysis.

We conclude that *a first integral does not represent, in general, a conservation law*, because Newtonian systems are generally non-conservative.

Intriguingly, *a conservation law does not constitute, in general, a first integral.* The Inverse Lagrangian Problem and underlying conditions of self-adjointness are again important for understanding this property.

The conventional notion of a conservative (closed) system is of a system of particles moving in vacuum at large mutual distances under internal action-at-a-distance, potential forces. Strictly speaking, the *only* example known at this time is given by the (Newtonian) treatment of our solar system, inclusive of satellite motion but excluding interior problem motions. The system is essentially self-adjoint and verifies the known ten Galilean conservation laws (of the total energy, linear momentum, angular momentum, and uniform motion of the center of mass). These conservation laws are indeed represented by first integrals of the equations of motion. We shall write the equations of motion and the first integrals in the self-explanatory notation:

$$[m_k \ddot{r}_{ka} - f_{ka}(\mathbf{r})]_{SA} = 0, \qquad k = 1, 2, 3, \qquad a = 1, 2, \ldots, N \tag{11a}$$

$$\dot{\varphi}_s(t, \mathbf{r}, \dot{\mathbf{r}}) = 0, \qquad s = 1, 2, \ldots, 10. \tag{11b}$$

As indicated earlier, no Newtonian system exists which is truly type (11) besides our solar system (unless very brief periods of time and major

approximations are admitted). This indicates that structures (11) are too simplistic and restrictive to represent physical reality.

A more general class of closed systems has been identified by Santilli (1978b,c and d), and can be presented as follows. Consider the Earth isolated from the rest of the universe (by ignoring planetary and other external interactions). The system is thus closed, that is, it verifies conservation laws (11b). Nevertheless, the internal forces are nonpotential, as established by the systems of our environment. Under these conditions, *the conservation laws are not first integrals but rather subsidiary constraints*, and we write in the self-explanatory notation

$$\begin{cases} \{[m_k \ddot{r}_{ka} - f_{ka}(t, \mathbf{r}, \dot{\mathbf{r}})]_{SA} - F_{ka}(t, \mathbf{r}, \dot{\mathbf{r}})\}_{NSA} = 0, \\ \qquad\qquad k = 1, 2, 3, \qquad a = 1, 2, \ldots, N, \qquad (12a) \\ \dot{\varphi}_s(t, \mathbf{r}, \dot{\mathbf{r}}) = 0, \qquad s = 1, 2, \ldots, 10. \qquad (12b) \end{cases}$$

In this way we reach the following important definition.

Definition 6. A *closed self-adjoint system* is one whose ten Galilean conservation laws are first integrals of the equations of motion or equivalently, one with self-adjoint internal forces. A *closed non-self-adjoint system* is one whose ten Galilean conservation laws are subsidiary constraints to the equations of motion or, equivalently, one with a combination of self-adjoint and non-self-adjoint internal forces.[47]

The study of systems with subsidiary constraints is quite involved technically and goes beyond the objectives of this text. Noether's theorem will therefore be studied for unconstrained systems, that is, for either closed self-adjoint systems or, more generally, for open non-self-adjoint systems.

Closed non-self-adjoint systems constitute a rather intriguing class of physical systems, particularly from the viewpoint of symmetries, conservation laws, and relativities. In fact, the systems considered disprove a number of beliefs lingering in contemporary theoretical physics.

It is generally believed that, when a system of particles verifies a given relativity, that same relativity applies to each constituent. To be more specific, when a Newtonian system of particles verifies the Galilean conservation laws of total quantities, it is generally believed that Galilei's relativity applies to the dynamic evolution of each constituent and, in particular, that the equations of motion are Galilei form-invariant. Similarly, at the relativistic level, it is generally believed that, when a composite system of particles on a Minkowski space verifies, as a whole, all the conditions of Einstein's special relativity, the relativity (and underlying physical laws) also apply to the characterization of the structure and dynamic evolution of each constituent.

[47] The notions of closed self-adjoint and non-self-adjoint systems were introduced primarily in an attempt to differentiate technically between closed systems with electromagnetic internal forces (e.g., atoms) and those with strong forces and other internal forces (e.g., hadrons). The main idea was to generalize the contemporary notion of interactions to unrestricted forces, while preserving conventional total conservation laws. The consistency of the model (e.g., existence of non-self-adjoint forces under which model (12) verifies the existence theorem of a solution) was proved via the existence theory of overdetermined systems. The model was subsequently studied in its statistical generalization by Tellez–Arenas *et al.* (1979), and it is currently studied in the yearly Workshops on Lie-Admissible Formulations (see the proceedings).

These beliefs are disproved by the considered class of systems in the following sense. No doubt, systems exist in nature where the whole systems and their constituents can be effectively treated via the same relativity. At the Newtonian level, this is clearly the case for our solar system. At the relativistic level, the possibility is established by the structure of atoms. An inspection of Equation (11) and Definition 6 shows that this is the case for all closed self-adjoint systems. In fact, as we shall see better via Noether's theorem, the ten conservation laws (11b) are a direct consequence of the symmetry of equations of motion (11a) under the ten-parameter Galilei group. The transition to relativistic, quantum mechanical, or quantum field theorerical settings implies increased technical complexity, although the fundamental physical ideas remain the same.

The important property expressed by Definition 6 is that systems exist in nature which violate this dual compatibility with the same relativity. At the Newtonian level, the property is established rather forcefully by our Earth. When seen by an outside observer, it is a *Galilean system*—that is a composite system whose center of mass dynamic evolution conforms to Galilei's relativity, as clearly expressed by the ten Galilean conservation laws of total quantities. However, when the dynamic evolution of each constituent is considered, a necessary condition for unrestricted forces, is that the equations of motion be, in general, not form-invariant under the Galilei group.[48] This is exactly the case of the more general closed non-self-adjoint systems. By recalling that Einstein's special relativity recovers Galilei's relativity for $v \ll c$, the occurrence is also expected, for consistency, at the relativistic level, and studies to this effect are in progress (see Santilli (1982d) and De Sabbata and Gasperini (1982)).

Admittedly, the situation being considered has not been experimentally established yet at the quantum mechanical level. Nevertheless, the problem of the relativity applicable to a particle under strong interactions is still open at this time on both experimental and theoretical levels. Thus the results outlined in this chart establish that *strict compliance with Einstein's special relativity for a composite particle under long-range interactions (e.g., a proton in a particle accelerator) does not necessarily imply that the same relativity is applicable to the characterization of the constituents.*[49] In fact, if the constituents of the protons have the same characteristics of all known strongly interacting particles (a wave packet with a charged radius on the order of the range of the strong interactions[12,14]), then each constituent moves within the medium constituted by the other particles,

[48] The imposition of any symmetry (whether Galilean or not) implies restrictions on the functional dependence of the forces.

[49] For the case of hadrons, we have the following two alternatives. If a hadron is a closed self-adjoint system, the center of mass dynamics conforms with the Lorentz transformations, and in addition the constituents are a suitable representation of the Lorentz group. In this case, we essentially have the same physical ideas and mathematical formulations as those for the atomic structure, only subjected to suitable additions (e.g., the addition of unitary internal symmetries). On the contrary, if a hadron is a closed non-self-adjoint system, the center of mass dynamics still conforms with the Lorentz transformations. However, the constituents *would not* be a representation of the Lorentz group but considerably more complex entities (e.g., two-sided representations of possible Lie-admissible generalizations of the Lorentz group). In this latter case, the generalization of the physical ideas and mathematical formulations in the transition from atoms to hadrons would perhaps be comparable to those in the transition from classical mechanics to quantum mechanics.

resulting in a structure which is substantially more complex than that of Equation (12). A few concluding remarks related to the connection of Definition 6 with the Inverse Problem are now in order.

Definition 7. A closed system is said to be *non-self-adjoint* when the equations of motion of the constituents generally *violate* the integrability conditions for the existence of a direct Lagrangian representation in the coordinate and time variables of the observer.

The reasons are evident. Imposing system (11a) to verify the conditions for an indirect Lagrangian representation (Theorem A.1.1) implies a substantial restriction on the forces, which may be contrary to nature. Therefore, *a closed non-self-adjoint system is truly achieved when the existence of a Lagrangian representation is prohibited and thus the applicability of Hamiltonian mechanics in the coordinate and time variables of the observer is also prohibited.*[50] Only when the researcher technically realizes these conditions can he see the dichotomy of relativities under consideration; that is, he can realize the applicability of conventional relativity for the dynamic evolution of a system as a whole (exterior problem) and the need of generalized formulations for the dynamic evolution of its constituents (interior problem).

The implications of these results are intriguing. For instance, a theory of the interior problem of gravitation is expected to permit a consistent description of simple interior motions (such as a satellite in the Earth's atmosphere) when the theory is not locally Lorentzian[51] in character and cannot be derived from a conventional (say, Riemannian) action principle.

In Chapter 6 we again study closed non-self-adjoint systems, this time in their equivalent first-order form. We will work out their Birkhoffian representation, symmetries, and conservation laws, and touch on the problem of the applicable relativity in more detail.

Chart A.9 Noether's Construction of First Integrals from Given Symmetries

In Charts A.6–A.8 we reviewed the notions of symmetries and first integrals in ways as disjoint as possible. In this chart we shall review the construction of first integrals from the given symmetries of a Lagrangian system, according to Noether's theorem. The topic is also called *Direct Noether's*

[50] As we indicate in Chart 6.1, the Birkhoffian generalization of Hamiltonian mechanics is applicable to the treatment of the system as a whole (exterior problem), while the Lie-admissible generalization of the Birkhoffian mechanics is applicable to each constituent (interior problem).

[51] One implication of the local Lorentz character of any gravitational theory for the interior problem is the locally exact character of the symmetry under the group of rotations, and this appears to be in irreconciliable disagreement with the local nonconservation of the angular momentum in interior motions (e.g., for a satellite in Earth's atmosphere). This can be expressed equivalently via the property that closed self-adjoint systems are based on the stability of the orbit (e.g., for planetary motions or atomic constituents), while closed non-self-adjoint systems are based on the *instability* of the orbits of the constituents, as apparently necessary under contact interactions

Problem to distinguish it from the *Inverse Noether's Problem,* which is the problem of constructing possible symmetries from known first integrals of a given Lagrangian. This latter topic is still open (and controversial), and we will not consider it at this time.

Consider a connected[52] r-dimensional Lie group,[53] G_r, of infinitesimal transformations of the first order

$$t' = t + \hat{\delta}t \stackrel{\text{def}}{=} t + \hat{\rho}_i \varepsilon^i; \quad \hat{\rho}_i = \left.\frac{\partial t'}{\partial \varepsilon^i}\right|_{\varepsilon=0}, \tag{1a}$$

$$q'^k = q^k + \hat{\delta}q^k \stackrel{\text{def}}{=} q^k + \hat{\eta}_i^k \varepsilon^i; \quad \hat{\eta}_i^k = \left.\frac{\partial q'^k}{\partial \varepsilon^i}\right|_{\varepsilon=0}, \tag{1b}$$

$$k = 1, 2, \ldots, n; \quad i = 1, 2, \ldots, r; \quad \varepsilon^i \approx 0.$$

According to Definition 3 of Chart A.6, a Lagrangian $L(t, q, \dot{q})$ possesses a symmetry under transformations (1) if a gauge function $\hat{\delta}G(t', q')$ exists such that

$$\begin{aligned} L(t, q, \dot{q})dt &= L'(t', q', \dot{q}')dt' \\ &= \left[L(t + \hat{\delta}t, q + \hat{\delta}q, \dot{q} + \hat{\delta}\dot{q}) + \frac{d}{dt'}\hat{\delta}G(t', q')\right]dt', \qquad (2) \\ \hat{\delta}G &\stackrel{\text{def}}{=} G_i \varepsilon^i. \end{aligned}$$

By using a Taylor expansion and rearranging the terms, expressions of type (12) of Chart A.7 can be reached.

Lemma 1. *A necessary and sufficient condition for a Lagrangian* $L(t, q, \dot{q})$ *to possess a symmetry under transformations* (1) *is that the following condition is identically verified*:

$$\hat{\delta}L + L\frac{d}{dt}(\hat{\delta}t) + \frac{d}{dt}\hat{\delta}G = 0, \tag{3a}$$

$$\hat{\delta}L = \frac{\partial L}{\partial t}\hat{\delta}t + \frac{\partial L}{\partial q^k}\hat{\delta}q^k + \frac{\partial L}{\partial \dot{q}^k}\hat{\delta}\dot{q}^k; \quad \frac{d}{dt'} \approx \frac{d}{dt}. \tag{3b}$$

A symmetry of a Lagrangian implies the following property of the action functional

$$\int_{t_1}^{t_2} dt\, L(t, q, \dot{q}) - \int_{t_1}^{t_2} dt'\, L(t', q', \dot{q}') = \int_{t_1}^{t_2} dt' \frac{d}{dt'} G_i(t', q')\varepsilon^i. \tag{4}$$

The left-hand side is the *first-order noncontemporaneous variation of the action*, Equation (I.1.3.39), i.e.,

$$\hat{\delta}A = -\int_{t_1}^{t_2} dt \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k}\right)\delta q^k + \int_{t_1}^{t_2} dt \frac{d}{dt} J, \tag{5}$$

$$J = \left[\frac{\partial L}{\partial \dot{q}^k}\hat{\eta}_i^k - \left(\frac{\partial L}{\partial \dot{q}^k}\dot{q}^k - L\right)\hat{\rho}_i\right]\varepsilon^i \stackrel{\text{def}}{=} J_i \varepsilon^i, \qquad \hat{\delta}q = \hat{\delta}q + \dot{q}\hat{\delta}t, \tag{5b}$$

[52] We exclude here discrete transformations (e.g., time and space inversions) because they cannot be incorporated in the context of Noether's theorem. Also, throughout this book the term "connected" always refers to the identity and not to possible additional, internal connectivity properties of a global group.

[53] The rudiments of Lie's theory are outlined in the charts at the end of Chapter 5.

where J is *the first-order end-point contribution*. In view of the independence of the parameters of G_r, a necessary condition for Equation (4) is that the coefficients of the parameters are equal. In this way we reach the following property.

Theorem 1 (First Noether's Theorem[54]). *If a Lagrangian $L(t, q, \dot{q})$ possesses a symmetry under an r-parameter connected Lie group G_r of infinitesimal transformations of the first order, then r distinct linear combinations of Lagrange's derivatives exist along an admissible path*[55] *which are exact differentials, i.e.,*

$$L_k(q)\eta_i^k = \frac{d}{dt} I_i, \qquad i = 1, 2, \ldots, r, \tag{6a}$$

$$I_i = J_i + G_i = I_i(t, q, \dot{q}), \tag{6b}$$

$$L_k(q) = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k}\right)(q), \tag{6c}$$

and which, when computed along an actual or possible path E_0, characterize r distinct first integrals, i.e.,

$$\frac{d}{dt} I_i(E_0) \equiv 0. \tag{7}$$

The following comments are in order.

1. *Theorem* 1 *clearly demands prior knowledge of a Lagrangian*. The Inverse Lagrangian Problem is therefore a prerequisite for its practical applicability. On formal grounds, the reader should keep in mind that, if Theorem A.1.1 is not applicable, the Theorem of Indirect Universality of the Hamiltonian of Chapter 6 holds and establishes a form of "universality" for the applicability of Noether's theorem to "all" Newtonian systems of the class admitted (regular and analytic). On practical grounds, the study of additional methods for constructing first integrals is advantageous, particularly when the computation of a Lagrangian is difficult, if not practically impossible.

2. *Theorem* 1 *also demands the prior knowledge of a symmetry*. This is clearly an additional limitation which confirms the advisability of studying alternative approaches for constructing first integrals. Lie's method has been reviewed in Chart A.7 and presented as one of the most effective methods available for the prior identification of symmetries. Again, on formal grounds, the method is expected to give all possible symmetries, but the practical situation may be different because of the known technical difficulties in solving the underlying quasilinear partial differential equations. Also, it is not known at this time, even formally, whether or not Lie's method can produce all symmetries needed for the solution of a system by quadratures.

[54] Noether (1918). The theorem is called "first" because of additional theorems proved by Noether in the same paper.

[55] According to the definition of Section I.1.3, an "admissible path" is an arbitrary path (not necessary a solution of the system) possessing the same continuity property of the solution. A "possible path" is a solution of the system without initial conditions and thus depends on arbitrary constants. The "actual path" is the solution of the system subject to all needed initial conditions.

3. *As originally formulated by Emmy Noether, Theorem* 1 *deals with a strictly mathematical property, the construction of first integrals.* The problem of whether or not first integrals represent physical conservation laws demands the explicit computation of the equations of motion, and the resolution of whether or not the first integrals are directly representative of physical quantities of the maximal, essentially self-adjoint subsystem (Chart A.8).

4. *Theorem* 1 *is applicable to all Lagrangians of the class admitted, including generalized Lagrangians of type* (*A*.1.19). *Thus the theorem applies in general for non-conservative systems, without excluding its conventional applicability to conservative systems as particular cases.* This confirms the expectation that first integrals originating from Theorem 1 *do not* represent, in general, physical laws.

5. *The quantities* I_i *verify Equations* (7) *by virtue of the equations of motion, as necessary for first integrals.* In fact, we can write

$$\frac{d}{dt} I_i(E_0) = L_k(E_0)\eta_i^k = 0. \tag{8}$$

6. *The r first integrals which can be constructed from an r-dimensional Lie symmetry via Theorem* 1 *are not necessarily independent among themselves.* This property can easily be proved by considering a free particle in Euclidean space. The Lagrangian $L = \frac{1}{2}m\dot{\mathbf{r}}^2$ is invariant (at least) under the ten-parameter Galilei group. Theorem 1 then yields ten different first integrals. These quantities, however, cannot all be independent among themselves because the maximal number of independent first integrals admitted by the system is six. A study of this situation is instructive (Problem A.9).

7. *Theorem* 1 *can be formulated in terms of contemporaneous transformations only.* In fact, only the quantities

$$\eta_i^k = \hat{\eta}_i^k - \dot{q}^k \hat{\rho}_i \tag{9}$$

enter into the formulation of Noether's identity (6a). We have chosen the formulation presented above mainly for pedagogical reasons. In fact, the symmetries of a Lagrangian are generally noncontemporaneous, particularly when originating from the use of Lie's method. Also, transformations involving time are particularly important in physics for a number of relativity aspects, and a formulation of Noether's theorem for generally noncontemporaneous symmetries appears advisable.

Celebrated examples of Noether's theorem are provided via the connected components of *Galilei's transformations.* By returning to the use of Cartesian coordinates in a two-dimensional (for simplicity) Euclidean space, connected Galilei's transformations in their (first-order) infinitesimal form can be written

$$t \to t' = t + \delta t_0, \qquad r^i \to r'^i = r^i + \delta\theta_0 R_j^i r^j + t\delta v_0^i + \delta r_0^i$$

$$(R_j^i) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad i, \qquad i = 1, 2 \tag{10}$$

where δt_0, δr_0, δv_0^i, and $\delta\theta_0$ are infinitesimal constants. For the case of translations in time,

$$t \to t' = t + \delta t_0, \tag{11}$$

the necessary and sufficient conditions (3) for the existence of symmetry become

$$\frac{\partial L}{\partial t} \equiv 0 \tag{12}$$

and first integral (6b) becomes the Hamiltonian

$$-I = \frac{\partial L}{\partial \dot{r}^i}\dot{r}^i - L = \text{const.} \tag{13}$$

For translations in the r^i coordinate (i fixed),

$$r^i \to r'^i = r^i + \delta r^i_0, \tag{14}$$

condition (3) becomes

$$\frac{\partial L}{\partial r^i} = 0, \tag{15}$$

and first integral (6b) is given by the ith component of the generalized momentum:

$$I = \frac{\partial L}{\partial \dot{r}^i} = \text{const.} \tag{16}$$

For the case of rotations

$$r^i \to r'^i = r^i + \delta\theta_0 R^i_j r^j \qquad i = 1, 2, \tag{17}$$

condition (3) becomes

$$R^i_j\left(\frac{\partial L}{\partial r^i}r^j + \frac{\partial L}{\partial \dot{r}^i}\dot{r}^j\right) = 0. \tag{18}$$

Assuming for simplicity a conventional structure for the Lagrangian,

$$L = \tfrac{1}{2}m\dot{\mathbf{r}}^2 - V(\mathbf{r}), \tag{19}$$

the term

$$R^i_j \frac{\partial L}{\partial \dot{r}^i}\dot{r}^j = R^i_j m\dot{r}_i\dot{r}^j \tag{20}$$

is identically null (because of the antisymmetry of R^i_j and the symmetry of $\dot{r}_i\dot{r}^j$). Condition (3) then reduces to

$$R^i_j \frac{\partial L}{\partial r^i}r^j = -R^i_j r^j F_i = 0, \tag{21}$$

that is, to the condition that the total torque is null. Integral (6b) then becomes the angular momentum

$$I = R^i_j \frac{\partial L}{\partial \dot{r}^i}r^j = r^1 p_2 - r^2 p_1 = \text{const.} \tag{22}$$

For the case of Galilei's boosts,

$$r^i \to r'^i = r^i + t\delta v^i_0, \tag{23}$$

condition (3) becomes

$$\delta v^i_0\left(t\frac{\partial L}{\partial r^i} + \frac{\partial L}{\partial \dot{r}^i}\right) + \frac{d}{dt}\delta G = 0. \tag{24}$$

For a free particle, the condition becomes

$$m(\delta \mathbf{v}_0 \cdot \dot{\mathbf{r}}) + \frac{d}{dt}\delta G = 0, \qquad \delta G = -m\delta \mathbf{v}_0 \cdot \mathbf{r}, \tag{25}$$

and first integrals (6b) are given by

$$\mathbf{I} = m\mathbf{r} - \mathbf{p}t = \text{const.} \tag{26}$$

The reader should keep in mind that these examples are mainly illustrative. For unrestricted Newtonian systems, conditions (15), (16), (21), and (24) are *generally violated*. Even when some of them are verified, the corresponding first integrals do not necessarily represent a physical quantity, as we shall see in Chart A.11.

Chart A.10 Isotopic Transformations, Symmetries, and First Integrals

As indicated in the text of this appendix, whenever the integrability conditions of Theorem A.1.1 are verified, a large variety of functionally different, yet equivalent, Lagrangians exists. They can be constructed via any of the following means and their combinations:

a) gauge transformations (Section A.2);
b) isotopic transformations (Section A.2); and
c) transformation theory of the local variables (Section A.3).

The degrees of freedom of a Lagrangian are clearly important for the identification of as many symmetries and first integrals as possible. In fact, symmetries which are not manifest for one given Lagrangian may be turned into manifest symmetries for an equivalent Lagrangian.

An example is provided by Lagrangians (15) of Chart A.7 for the particle with damping. In this case two independent first integrals are needed for the solution by quadratures. Suppose that these first integrals have to be identified via manifest symmetries and Noether's theorem. Knowledge of only the first Lagrangian, $L_1 = (\exp \gamma t)\frac{1}{2}\dot{r}^2$, is *insufficient* for this task, because its only manifest symmetry is that under space translation. The problem can be solved by including the second Lagrangian, $L_2 = \dot{r} \ln \dot{r} - \gamma r$, which is manifestly invariant under translations in time, thus yielding the needed second first integral.

In this chart we outline the studies by Santilli (1978c and 1979a) on the *enlargement of the problem of symmetries and first integrals through the inclusion of isotopic transformations of a Lagrangian within a fixed system of local variables*. The corresponding study of additional gauge transformations and transformations of local variables is left to the interested reader.

Consider rule (A.2.7) for the isotopically mapped Lagrangians, i.e.,

$$L_k^*(q) = h_k^i L_i(q), \tag{1}$$

and let (h_k^{-1i}) be the inverse of the matrix (h_k^i) of isotopic functions. The following property results:

$$\frac{d}{dt} I = L_k(q)\delta q^k = L_i(q) h_k^i h_j^{-1k} \delta q^j = L_k^*(q)\delta_* q^k,$$

$$\delta_* q^k = h_j^{-1k} \delta q^j. \tag{2}$$

From the validity of first integrals by virtue of the equations of motion, we can state the following lemma.

Lemma 1. *Functions $I_i(t, q, \dot{q})$, which are first integrals for one given Lagrangian $L(t, q, \dot{q})$, remain first integrals for all possible isotopically mapped Lagrangians $L^*(t, q, \dot{q})$.*

The considerable alteration of the structure of the Lagrangian under isotopic transformations on the one hand, and the preservation of the first integrals on the other hand suggest the possibility that the same first integral may be derived from fundamentally different symmetries. To investigate this possibility, Santilli (*loc. cit.*) introduced the following definition.

Definition 1. Two Lie symmetry groups, G_r and G_r^*, of (infinitesimal or finite) transformations of the same dimension r are called *isotopically related symmetry groups* when they constitute symmetries of two isotopically related Lagrangians, $L(t, q, \dot{q})$ and $L^*(t, q, \dot{q})$, respectively, which lead to the same (ordered) set of first integrals via Noether's theorem.

To illustrate this, let us consider the linear harmonic oscillator in three dimensions. The Lagrangian

$$L = \tfrac{1}{2}[(m\dot{x}^2 + m\dot{y}^2 + m\dot{z}^2) - (kx^2 + ky^2 + kz^2)] \tag{3}$$

possesses a symmetry under the group of rotations $G_3 = SO(3)$ which leads, via Noether's theorem, to the conservation of the physical angular momentum. One of the simplest possible isotopic images of L is given by

$$L^* = \tfrac{1}{2}[(m\dot{x}^2 - m\dot{y}^2 + m\dot{z}^2) - (kx^2 - ky^2 + kz^2)]. \tag{4}$$

This new Lagrangian is no longer invariant under G_3. Yet, the angular momentum conservation persists for L^*. We then expect the existence of a new symmetry, G_3^*, which leads to the conservation of the angular momentum via Noether's theorem. A study of the problem reveals that the symmetry G_3^* exists and is given by the Lorentz group in $(2 + 1)$ dimensions, $SO(2.1)$. This is the manifest symmetry of L^* which replaces $SO(3)$ for the characterization of the angular momentum conservation. Thus $SO(3)$ and $SO(2.1)$ are isotopically related symmetry groups with respect to the harmonic oscillator. The algebraic structure of the group isotopy is indicated in Chart 4.2 and turns out to be induced by a structural change in the Lie product.

As indicated earlier, the Inverse Noether's Problem (construction of symmetries from first integrals) is still quite controversial. The situation for isotopically mapped symmetries is different, however, because an original symmetry G_r leading to first integrals I_i is assumed to exist. The Inverse Noether's Problem is, in this case, restricted to the identification of the symmetry G_r^* leading to I_i. The integrability conditions for the existence of G_r^* result in being always verified under the existence of the original symmetry G_r.

Theorem 1 (Existence Theorem for Isotopically Mapped, First-Order Symmetries). *Suppose that an (analytic and regular) Lagrangian $L(t, q, \dot{q})$ admits an r-dimensional symmetry G_r of infinitesimal transformations of the first order with related first integrals I_i, $i = 1, 2, \ldots, r$. Then, an isotopically mapped Lagrangian $L^*(t, q, \dot{q})$ always admits a symmetry G_r^* of*

infinitesimal transformations also of the first order in the same parameters leading to the same first integrals.

This theorem is formulated specifically for the first-order case. The study of its extension to finite transformations is left to the interested reader. It is sufficient for our analysis to know that finite, isotopically mapped symmetry groups exist (this is the case for the symmetries $SO(3)$ and $SO(2.1)$ of the harmonic oscillator).

A simple inspection of the examples of isotopically related symmetries establishes the following property.

Lemma 2. *Two isotopically related symmetry groups are not necessarily isomorphic. In particular, the isotopic transformations do not necessarily preserve the Abelian or non-Abelian, compact or noncompact, and semi-simple or non-semi-simple character of the original symmetry.*

This property is sufficient to indicate that the isotopic transformations of a Lagrangian have a rather profound impact on symmetries, as expected. In fact, starting from Lagrangian (3) with the non-Abelian, compact, and semi-simple symmetry group $SO(3)$, its isotopic image can, in principle, be an arbitrary three-dimensional Lie group, including the Abelian case ($SO(2.1)$ is still non-Abelian and semi-simple but not compact; yet it is only one possibility).

Intriguingly, these changes in the structure of the symmetries have their origin in the integrability conditions for the existence of a Lagrangian. In this way we begin to see some of the implications of the Inverse Problem beyond those for the computation of a Lagrangian.

Chart A.11 Lack of a Unique Relationship between Space–Time Symmetries and Physical Laws

The existing literature sometimes explicitly states or implicitly assumes that conventional physical laws, such as the conservation of energy, linear momentum, and angular momentum (and charge), are uniquely characterized by known space–time symmetries, such as, translations in time, translations in space, and rotations (and field-theoretical gauge symmetries), respectively. This belief has been disproved by the Inverse Lagrangian Problem.

The best way to illustrate the situation is by explicit examples from Santilli (1978b and 1979a). The examples may also suggest caution in attributing physical meaning to a given space–time symmetry. The physical quantities considered are those defined in Chart A.8 and should not be confused with canonical quantities.

Occurrence 1. *When the total physical energy of a system is conserved, a Lagrangian (for its analytic representation) may violate the symmetry under translations in time.* The total physical energy of the harmonic oscillator

$$(\ddot{r} + r)_{SA} = 0, \qquad m = 1, \qquad k = 1 \tag{1}$$

is conserved. Nevertheless, this system admits a Lagrangian (Example A.1)

$$L = \tfrac{1}{6}\dot{r}^3 \cos t + \tfrac{1}{2}r\dot{r}^2 \sin t - r^2\dot{r} \cos t \tag{2}$$

which is explicitly dependent on time and, as such, violates the symmetry under translations in time.

Occurrence 2. *When the total physical energy of a system is not conserved, a Lagrangian may be invariant under translations in time.* The linearly damped oscillator

$$(\ddot{r} + \gamma\dot{r})_{\text{NSA}} = 0 \tag{3}$$

is nonconservative. Thus its total energy decays in time, according to experimental evidence. Nevertheless, the system admits the Lagrangian

$$L = \dot{r} \ln \dot{r} - \gamma r \tag{4}$$

which does not possess an explicit time dependence and is therefore manifestly invariant under translations in time.

Occurrence 3. *When the total physical linear momentum is conserved, a Lagrangian may violate the invariance under translations in space.* The total linear momentum of the system

$$\begin{pmatrix} \dot{x} - \dot{y} + 2x \\ \dot{x} + \dot{y} \end{pmatrix}_{\text{NSA}} = 0, \tag{5}$$

is conserved, as manifestly expressed by the second equation. Nevertheless, the system admits the Lagrangian

$$L = \tfrac{3}{2}\dot{x}^2 + \dot{x}\dot{y} + \tfrac{1}{2}\dot{y}^2 - x^2 \tag{6}$$

which violates the symmetry under translations in space (trivially, because it does not depend on the difference of the coordinates alone).

Occurrence 4. *When the total physical linear momentum is not conserved, a Lagrangian may be invariant under translations in space.* The physical linear momentum of the damped particle is manifestly not conserved. Nevertheless, the system admits the Lagrangian

$$L = e^{\gamma t}\tfrac{1}{2}\dot{r}^2 \tag{7}$$

which is independent of coordinate r and, as such, is manifestly invariant under translations in space.

Occurrence 5. *When the total physical angular momentum is conserved, a Lagrangian may violate the symmetry under rotations.* The physical angular momentum of the three-dimensional harmonic oscillator

$$(m\ddot{\mathbf{r}} + k\mathbf{r})_{\text{SA}} = 0 \tag{8}$$

is conserved. Nevertheless, the system admits the Lagrangian

$$L = \tfrac{1}{2}m(\dot{x}^2 - \dot{y}^2 + \dot{z}) - \tfrac{1}{2}k(x^2 - y^2 + z^2) \tag{9}$$

$$\mathbf{r} = (x, y, z)$$

which is manifestly noninvariant under rotations.

Occurrence 6. *When the total physical angular momentum is not conserved, a Lagrangian may be invariant under rotations.* Consider the motion of a particle in two dimensions in a dissipative medium (a liquid or a gas) by ignoring action-at-a-distance (conservative) forces:

$$\left[\begin{pmatrix} x \\ y \end{pmatrix}_{\text{SA}} + g\begin{pmatrix} \tfrac{1}{2}x\dot{x}^2 + y\dot{x}\dot{y} - \tfrac{1}{2}x\dot{y}^2 \\ \tfrac{1}{2}y\dot{y}^2 + x\dot{x}\dot{y} - \tfrac{1}{2}y\dot{x}^2 \end{pmatrix}\right]_{\text{NSA}} = 0, \qquad \mathbf{r} = (x, y). \tag{10}$$

Owing to the nonlinear dependence of the forces in the velocity, the system is in highly nonconservative motion. In particular, the physical angular

momentum $\mathbf{M}_{phy} = \mathbf{r} \times m\dot{\mathbf{r}}$ is nonconserved (as the reader is invited to verify). Nevertheless, the system admits the Lagrangian (Example A.3)

$$L = \exp\left(\frac{g}{2}(x^2 + y^2)\right)\tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) \tag{11}$$

which is manifestly invariant under rotations.

Occurrence 7. *The symmetry of a Lagrangian under Galilei's transformations does not necessarily imply the validity of the conventional conservation laws of total physical quantities (energy, linear momentum, etc.).* This is a consequence of Occurrences 2, 4, and 6. The illustration of the additional case of the lack of uniform motion of the center of mass, but invariance of a Lagrangian under Galilei's boosts, is left to the interested reader. The implications of the occurrence from a relativity viewpoint is self-evident. In fact, the mere existence of the symmetry under Galilei's group is not sufficient to establish the validity of Galilei's relativity as customarily intended in the physical literature, that is, as representative of physical laws. In turn, this illustrates a number of aspects presented in these volumes, such as the need of a return "ad originem." We are referring here to the assumption of Newton's equations of motion and their possible conservation laws as the physical foundations of the theory, and then the use of their Lagrangian representations, symmetries, and Noether's theorem as mathematical treatments. If the opposite approach is followed (assumption of symmetries and Lagrangian representations as fundamental), the physical content is not ensured. Indeed, all conventional space–time symmetries of examples 2, 4, and 6 produce first integrals via Noether's theorem. The point is that these first integrals have no physical meaning as conservation laws.

The examples above establish the following properties: (I) when conventional space–time symmetries hold, the conventionally expected physical laws do not necessarily hold; and (II) when conventional physical laws hold, the conventionally expected space–time symmetries do not necessarily exist. The use of the techniques of the Inverse Lagrangian Problem establishes that the lack of any unique relationship between space–time symmetries and physical laws is even deeper. In fact, we have the following property from Santilli (*loc. cit.*).

Theorem 1 (Lack of Unique Space, Time, and Space–Time Character of a Symmetry). *The conservation law of a physical quantity cannot necessarily be derived from a symmetry of unique space, time, or space–time character.*

This property is best illustrated by a specific example. Consider the harmonic oscillator (1). The conservation of the energy is often believed to be derivable only via symmetry of pure time character i.e., time translation (Definition 3 of Chart A.6). This belief is erroneous. Consider Lagrangian (2). Even though the symmetry under time translations is broken, an isotopic symmetry exists from Theorem 1 of Chart A.11, leading to the conservation of energy. Explicit calculations show that the symmetry is characterized by the following transformations (see Example A.7)

$$\hat{\delta}_* t = -\frac{H^* + p^*\dot{q}/\alpha}{L^*}\varepsilon, \qquad \hat{\delta}_* q = -\frac{\dot{q}}{\alpha}\varepsilon + \dot{q}\hat{\delta}_* t, \tag{12}$$

$$p^* = \partial L^*/\partial \dot{q}, \qquad H^* = p^*\dot{q} - L^*, \qquad \alpha = \dot{q}\cos t + q \sin t$$

in which we see a *combination of translations in space and time to produce the conservation of energy*. The interested reader should work out the isotopic image of the symmetries under translations in space and rotations of the preceding examples. He will see again the lack of unique space character for the characterization of the conservation of linear momentum and angular momentum. The lack of uniqueness of the field theoretical gauge symmetry for the conservation of the charge has been pointed out by Santilli (*loc. cit.*), together with the identification of the field theoretical extensions of the properties above.[56]

[56] Consider a charged scalar particle in interaction with an external electromagnetic field. The equations are given by

$$\left\{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\left[\begin{pmatrix}(\Box + m^2)\varphi\\ (\Box + m^2)\bar{\varphi}\end{pmatrix} + e\begin{pmatrix}eA_\alpha A^\alpha\varphi + 2iA_\alpha\varphi^{,\alpha}\\ eA_\alpha A^\alpha\bar{\varphi} - 2iA_\alpha\bar{\varphi}^{,\alpha}\end{pmatrix}\right]_{NSA}\right\}_{SA} = 0, \qquad \text{(a)}$$

$$\varphi^{,\alpha} = \partial\varphi/\partial x^\alpha, \qquad \alpha = 0, 1, 2, 3$$

and the conserved, physical, charge current is

$$j^\mu = ie(\bar{\varphi}\varphi^{,\mu} - \bar{\varphi}^{,\mu}\varphi) + 2eA^\mu\bar{\varphi}\varphi. \qquad \text{(b)}$$

This conservation law is customarily presented as one which can be derived from the invariance of the conventional Lagrangian

$$\mathscr{L} = (\bar{\varphi}_\mu^{,} + ieA_\mu)(\varphi^{,\mu} - ieA^\mu) - m^2\bar{\varphi}\varphi \qquad \text{(c)}$$

under the (Abelian, in this simple case) gauge transformations

$$\varphi \to \varphi' = e^{i\varepsilon e}\varphi, \qquad \bar{\varphi} \to \bar{\varphi}' = e^{-i\varepsilon e}\bar{\varphi}. \qquad \text{(d)}$$

To illustrate the *lack of uniqueness of the gauge symmetry for the characterization of the charge conservation,* Santilli (1978b and 1979a) constructed the following isotopically mapped Lagrangian

$$\mathscr{L}^* = \tfrac{1}{2}e^{2iex_\mu A^\mu}[\bar{\varphi}^{,\mu}\bar{\varphi}_\mu^{,} - (m^2 - e^2A_\mu A^\mu)] + \tfrac{1}{2}e^{-2iex_\mu A^\mu}[\varphi^{,\mu}\varphi_\mu^{,} - (m^2 - e^2A_\mu A^\mu)]. \qquad \text{(e)}$$

The physical charge current (b) is still conserved for $\mathscr{L}^*$. Nevertheless, the symmetry under gauge transformations is now manifestly broken. The lack of uniqueness of the gauge symmetry is of even deeper nature in that it extends to the internal character of the transformations (lack of participation of the Minkowski coordinates). In fact, the isotopic image of the gauge symmetry was constructed and can be given by the transformations

$$x^\mu \to x'^\mu = x^\mu + \hat{\delta}x^\mu, \qquad \hat{\delta}x^\mu = \frac{2eA^\mu\bar{\varphi}\varphi}{\mathscr{L}^*}\varepsilon$$

$$\begin{pmatrix}\varphi\\ \bar{\varphi}\end{pmatrix} \to \begin{pmatrix}\varphi'\\ \bar{\varphi}'\end{pmatrix} = \begin{pmatrix}\varphi\\ \bar{\varphi}\end{pmatrix} + \begin{pmatrix}\varphi_\mu^{,}\\ \bar{\varphi}_\mu^{,}\end{pmatrix}\hat{\delta}x^\mu + \begin{pmatrix}ie\bar{\varphi}\exp(2iex_\alpha A^\alpha)\\ -ie\varphi\exp(-2iex_\alpha A^\alpha)\end{pmatrix}\varepsilon. \qquad \text{(f)}$$

One can see that the conservation law of an internal quantity such as the charge can be derived via a symmetry which is of mixed space–time and internal character. Notice that these results extend to the minimal coupling rules. In fact, *Lagrangian (e) represents electromagnetic interactions with a manifest breaking of the minimal coupling rule.* Incidentally, the appearance of an explicit dependence of Lagrangian (e) in the Minkowski space–time coordinates fully conforms to physical laws. In fact, the electromagnetic field is *external* for system (a), which is therefore open and exhibits a rate of variation in time of the energy–momentum tensor. This space–time *non-conservation law* is fully reflected in the explicit dependence of $\mathscr{L}^*$ in the Minkowski coordinates. Therefore, the invariance of the conventional Lagrangian $\mathscr{L}$ under space–time translations is physically illusory. The nature of the symmetry breaking is identified in Chart 4.12.

In summary, Theorem 1 establishes that not only is the symmetry capable of representing a given physical law via Noether's theorem not unique, but the lack of uniqueness actually extends to the physical structure of the symmetry, whether of space, time, or space–time character.

Chart A.12 Classification of the Breakings of Space–Time Symmetries in Newtonian Mechanics

In the preceding charts we introduced the mathematical notion of *symmetry* through the form-invariance of the equations of motion or one of its Lagrangian representations. We then identified the methods for constructing *first integrals* from a known symmetry of a given system. Finally, we identified a possible physical meaning of the theory—that of representing *physical conservation laws*. Our primary emphasis was for conventional *space–time symmetries*, such as symmetries under translations in time, space, rotations, etc.

An unprejudiced inspection of our Newtonian reality reveals, quite firmly, that the breaking of conventional space–time symmetries is the rule and their preservation is the exception. In fact, unless the symmetries are "broken" in one way or another, we would have oscillators oscillating forever, spinning tops spinning forever, and satellites orbiting forever.

In the transition from these open nonconservative systems to their closed form, including their environment, total conservation laws are recovered. Yet the space–time symmetries remain generally broken, as established by the closed non-self-adjoint systems through the property that total conservation laws are not first integrals of the equations of motion (Chart A.8).

It follows that the study of "broken symmetries" at large and that of "broken space–time symmetries" in particular are of fundamental relevance, mathematically and physically. Some of the problems inherent in this study are (A) classification of the mechanisms of breaking a given symmetry; (B) study of the time rate of variation of given physical quantities, as a generalization of the particular case of conservation; and (C) search for generalized symmetries which (a) hold when the conventional symmetries are broken, (b) are representative of the rate of variation of physical quantities, and (c) when conventional conservations are recovered, recover the conventional ones.

The study of these (and related) problems was initiated by Santilli (1978c and e) in an attempt to generalize Galilei's relativity for application to Newtonian systems with unrestricted forces and dynamic conditions. The study has been continued in the Workshops on Lie-Admissible Formulations (see the proceedings of 1979 and 1981).

In this chart we review the studies related to Problem (A) only, that is, the classification of all possible ways in which a given symmetry can be broken, as presented in Santilli (*loc. cit.*; see also 1978b and 1979a).

Definition 1. A symmetry of a Lagrangian is said to be *physically exact*, or simply *exact*, not only when the Lagrangian is form-invariant according to the mathematical definition of Chart A.6, but also when the first integrals characterized by Noether's theorem are directly representative of physical conservation laws. The same symmetry under the same conditions is said to be *physically broken*, or *broken*, when either the original form-invariance of the Lagrangian or the original conservation laws or both are no longer valid.

The primary emphasis of the definition is on the historical motivation for the introduction of symmetries in physics—their representation of *physical laws*. Mathematical aspects are considered too, of course, but only subordinated to this physical objective. Notice that, according to the definition, a given space–time symmetry is broken when it is mathematically exact but the conventional physical conservation laws are lost. Also, the same symmetry is broken when the form-invariance is lost but the conventional physical conservation laws are preserved. Explicit illustrations of these intriguing occurrences follow.

I. *Isotopic breaking.* Consider a conservative Newtonian system in Euclidean space which possesses an exact space–time symmetry (ES) G; a (physical) conservation law $I(t, \mathbf{r}, \dot{\mathbf{r}})$; and a Lagrangian $L(\mathbf{r}, \dot{\mathbf{r}})$, i.e.,

$$G: [m\ddot{\mathbf{r}} - \mathbf{f}(\mathbf{r})]_{SA}^{ES} = 0 \tag{1a}$$

$$L: L = L_{free} + L_{int}, \qquad L_{free} = \tfrac{1}{2}m\dot{\mathbf{r}}^2, \qquad \mathbf{f} = \frac{\partial L_{int}}{\partial \mathbf{r}}, \tag{1b}$$

$$I: I = \frac{\partial I}{\partial t} + \frac{\partial I}{\partial \mathbf{r}} \cdot \dot{\mathbf{r}} + \frac{\partial I}{\partial \dot{\mathbf{r}}} \cdot \frac{\mathbf{f}}{m} \equiv 0, \tag{1c}$$

as is the case, say, for a Coulomb system in vacuum, where G is given by the symmetry under translations in time and I is the total energy.

The "weakest possible" (yet instructive) breaking is that for which the original symmetry is broken (BS) in such a way to leave the conservation law unchanged. This type of breaking is related to the rule of Lagrangian isotopy (Equations (A.2.7)) and occurs when the isotopic functions are not invariant under the original symmetry G. We shall then write

$$[L_k^*(\mathbf{r})]_{SA}^{SB} = \{h_k^i(t, \mathbf{r}, \dot{\mathbf{r}})[L_i(\mathbf{r})]_{SA}^{ES}\}_{SA}^{BS}. \tag{2}$$

The breaking is called of "isotopic type," in the sense of Chart A.10, to stress the fact that *no actual change of the physical system takes place (that is, the forces are the same), and the only variations occur in its mathematical treatment.*[57]

The breaking is instructive particularly for relativity profiles (Problem A.10). In fact, Galilei's group G (3.1) is now lost as the symmetry group of conventional physical laws and is replaced by a different, generally non-isomorphic (Chart 4.10) symmetry group G^* (3.1). This can easily be illustrated by considering a sufficiently complex isotopy of the Lagrangian for the free particle, and it shows that, even when Galilei's relativity is verified, its contemporary formulation is not unique.

II. *Self-adjoint breaking.* Suppose that system (1) enters a dissipative medium (say, our atmosphere) by acquiring new forces $\mathbf{F}(t, \mathbf{r}, \dot{\mathbf{r}})$. The originally conserved quantity (say, the energy) is now no longer constant in time, but acquires a (non-null) time rate of variation. This is a necessary consequence of the presence of contact-type forces created by the medium, and we can write the *nonconservation law*

$$\dot{I} = \frac{\partial I}{\partial t} + \frac{\partial I}{\partial \mathbf{r}} \cdot \dot{\mathbf{r}} + \frac{\partial I}{\partial \dot{\mathbf{r}}} \cdot \frac{\mathbf{f}}{m} + \frac{\partial I}{\partial \dot{\mathbf{r}}} \cdot \frac{\mathbf{F}}{m} \neq 0. \tag{3}$$

[57] A Newtonian example is provided by the time-dependent Lagrangian of the harmonic oscillator, Equation (2) of Chart A.11. One can see that the Lagrangian acquires an explicit time dependence, without affecting the conservation of the energy, because the isotopic function is explicitly dependent on time.

The loss of the original first integral is sufficient to establish the breaking of the original symmetry in the physical sense of Definition 1.

The method for the classical breaking of a symmetry, which is used rather generally in contemporary literature, is given by the addition of a symmetry-breaking potential in the Lagrangian (or Hamiltonian), i.e.

$$L^{ES} \to L^{BS} = L^{ES} + L^{BS}_{int}, \qquad \mathbf{F} = \frac{\partial L^{BS}_{int}}{\partial \mathbf{r}} - \frac{d}{dt}\frac{\partial L^{BS}_{int}}{\partial \dot{\mathbf{r}}}. \tag{4}$$

When reinspected within the context of the conditions of variational self-adjointness, this breaking is called "self-adjoint type" because it implies the addition of self-adjoint forces to equations (1), and we write

$$\{[m\ddot{\mathbf{r}} - \mathbf{f}(\mathbf{r})]^{ES}_{SA} - \mathbf{F}(t, \mathbf{r}, \dot{\mathbf{r}})\}^{BS}_{SA} = 0. \tag{5}$$

However, Newtonian forces are generally *not derivable* from a potential, particularly when they are of contact type. Conventional symmetry breaking (4) is therefore one of the simplest possible mechanisms of symmetry breaking, and additional, more realistic mechanisms exist.

III. *Semicanonical breaking.* The simplest nonpotential forces verify the integrability conditions of Theorem A.1.1 (non-essentially non-self-adjoint forces). In this case, mechanism (5) is generalized into the form

$$\{h^i_k[(m\ddot{r}_i - f_i)^{ES}_{SA} - F_i]^{ES}_{NSA}\}^{ES}_{SA} = 0. \tag{6}$$

The underlying Lagrangian is no longer of conventional type, but rather of generalized type (Section A.2)

$$L = L_{int,\,I}(t, \mathbf{r}, \dot{\mathbf{r}})L_{free}(\dot{\mathbf{r}}) + L_{int,\,II}(t, \mathbf{r}, \dot{\mathbf{r}}). \tag{7}$$

Often, the forces responsible for the time rate of variation (3) are still form-invariant under the symmetry G, and the same situation occurs for the genotopic functions h^i_k. Under these circumstances, generalized Lagrangian (7) is still invariant under G. Yet, the original physical meaning of the symmetry is lost. This type of breaking is called "semicanonical" in the sense that the canonical formalism of the symmetry can still be introduced. Yet, it loses the meaning of representing a physical law.

Clearly, semicanonical breakings are still restrictive. Nevertheless, they occur in a number of cases frequently considered in the literature, particularly when the symmetry breaking forces are approximated by linear velocity forces.

As an example, suppose that system (1) is a free particle, symmetry G is under space translation, the conservation law is one of linear momentum, $I = m\dot{r}$, and nonconservative extension (6) is given by

$$\{e^{\gamma t/m}[(m\ddot{r})^{ES}_{SA} + \gamma\dot{r}]^{ES}_{NSA}\}^{ES}_{SA} = 0, \qquad L = e^{\gamma t/m}\tfrac{1}{2}m\dot{r}^2. \tag{8}$$

In this case we have a semicanonical breaking of the space translation symmetry because L is still invariant under the original symmetry, while linear momentum $m\dot{r}$ is no longer conserved.[58]

[58] A field theoretical example of semicanonical breaking of the symmetry under space–time translations is given by the conventional Lagrangian for the charged scalar field under an external electromagnetic field, Equation (c) of footnote 56. The Lagrangian is indeed invariant under space–time translations, but this symmetry is not representative of the conservation of the energy–momentum tensor because the system is open. Notice that we call this type of breaking "semicanonical" rather than "self-adjoint" because the Lagrangian is of the simplest possible—yet generalized—type, with $\mathscr{L}_{int,\,I}$ being represented by a permutation, as expressed in Equation (a) of footnote 56. For more details on this point see Santilli (1977b).

An additional example is provided by the physical spinning top (the spinning top with drag torques responsible for the decay in time of the angular momentum). The drag torques are usually assumed to depend on angular velocity alone, therefore preserving the original symmetry under rotations. Nevertheless, the angular momentum is no longer conserved. This yields a semicanonical breaking of the symmetry under the group of rotations $SO(3)$ (see Example A.3).

IV. *Canonical breaking.* This class consists of systems whose non-potential forces still permit an indirect Lagrangian representation yet are no longer form-invariant under the original symmetry. We write

$$\{h_k^i[(m\ddot{r}_i - f_i)_{SA}^{ES} - F_i]_{NSA}^{BS}\}_{SA}^{BS} = 0. \tag{9}$$

This class of breaking is called "canonical" in the sense that the canonical formalism can still be introduced (because of the existence of a Hamiltonian). However, the canonical formalism of the symmetry G is now inapplicable. The increase in the complexity of the forces therefore ensures the loss not only of the original conservation law but also of the original space–time symmetry.[59]

This type of symmetry breaking is more realistic than the preceding ones (although still restrictive). In fact, an inspection of system (8) clearly indicates that a more realistic treatment of the contact forces calls for an explicit dependence on space (e.g., because of variations in the density of the medium)

$$\{h(t, r, \dot{r})[(m\ddot{r})_{SA}^{ES} + \gamma(r)\dot{r}]_{NSA}^{BS}\}_{SA}^{BS} = 0 \tag{10}$$

which implies a canonical breaking of the space translation symmetry.[60]

Similarly, a deeper inspection indicates that drag torques of spinning tops may depend on the angle of rotation. If the torques verify Theorem A.1.1, we have a canonical breaking of the symmetry under the group $SO(3)$, in the sense that tops can still be treated via the canonical formalism (e.g., via the Hamilton–Jacobi equations). Nevertheless, the formalism breaks down at the specifical level of the $SO(3)$ symmetry.

V. *Essentially non-self-adjoint breaking.* This is the most general mechanism of breaking conventional space–time symmetries which can be identified via the techniques of the Inverse Problem. In this case the forces not only break the original symmetry but actually violate the integrability

[59] Field theoretical examples of the canonical breaking of the symmetry under the Lorentz group are given by *generalized* Lagrangian models of the type

$$\mathscr{L} = m^2\varphi^2 + \varphi_{,\mu}^{i}\varphi^{i,\mu} + (\varphi_{,\mu}^{i}\varphi^{i,\mu})^2 + \cdots.$$

Despite superficial impressions that terms of the type $(\varphi_{,\mu}^{i}\varphi^{i,\mu})^2$ are invariant under the Lorentz group, the symmetry is broken because *the fields themselves, that is, the solutions of the field equations, do not transform covariantly under the Lorentz group*. Notice that the breaking in this case is canonical and *not* semicanonical. Notice also that models of this type are expected as field-theoretical images of systems ($\mathscr{I}.3$), namely, as field theoretical extensions of the contact effects of mutual wave penetration, with underlying nonlocal interactions approximated by power-series expansions of type ($\mathscr{I}.4$) (see footnotes 12 and 14).

[60] The breaking is canonical because a Lagrangian (and, therefore, a Hamiltonian) exists by virtue of Corollary A.1.1a.

conditions of Theorem A.1.1.[61] In the language of Definition 4.4.1, we therefore write

$$\{[m\ddot{\mathbf{r}} - \mathbf{f}(\mathbf{r})]^{ES}_{SA} - \mathbf{F}(t, \mathbf{r}, \dot{\mathbf{r}})\}^{BS}_{ENSA} = 0. \tag{11}$$

The methodological implications of this type of breakings are rather deep. Not only is the canonical formalism of the original symmetry broken, but in actuality, *the entire canonical formalism is inapplicable in the local variables of the experimenter*, as established by the lack of existence of a Hamiltonian. This implies, in particular, the impossibility of introducing a Lie algebra via the brackets of the time evolution in the local variables of the observer.

In this case the breaking of conventional space–time symmetries in general and Galilei's relativity in particular are brought to the level of inapplicability of the mathematical foundations, that is, Lie's theory. This becomes readily understandable and acceptable if one recognizes that contact forces in mechanics (whether discrete or continuous, classical or quantum mechanical, etc.) are not only of nonpotential type, but actually of nonlocal type.

This is a further indication that the problem of the relativity applicable in *Newtonian* mechanics (no relativistic, gravitational, and quantum mechanical extensions!) is still fundamentally open at this time.[62]

EXAMPLES

Example A.1

All the following Lagrangians and Hamiltonians

$$L = \tfrac{1}{2}(\dot{q}^2 - q^2), \tag{1a}$$

$$L_1^* = \tfrac{1}{6}\dot{q}^3 \cos t + \tfrac{1}{2}q\dot{q}^2 \sin t - q^2\dot{q} \cos t, \tag{1b}$$

$$L_2^* = 2\frac{\dot{q}}{q}\arctan\frac{\dot{q}}{q} - \ln(\dot{q}^2 + q^2), \tag{1c}$$

$$L_2^{*\dagger} = 2\frac{\dot{q}}{q}\arctan\frac{qc + \dot{q}q}{q^2 - \dot{q}c} - \ln(\dot{q}^2 + q^2), \quad c = \text{const.}, \tag{1d}$$

$$H = \tfrac{1}{2}(p^2 + q^2), \tag{1e}$$

$$H_2^* = 2 \ln |q \sec \tfrac{1}{2}qp^*|, \tag{1f}$$

[61] Examples of essentially self-adjoint breakings of the Lorentz symmetry in field theory are given by equations of the type

$$(\Box + m^2)\varphi + \varphi^i_\mu \varphi^{;\mu} + (\varphi^i_\mu \varphi^{;\mu})^2 + \cdots = 0,$$

for which the integrability conditions for the existence of an indirect Lagrangian representation (Santilli (1977a,b,c)) are inconsistent. In this case, the field equations are not form-invariant under the Lorentz group (because the solutions are not compatible with the representations of the group), and, in addition, the canonical formalism is not directly applicable.

[62] This problem is studied in Section 6.3 (see also footnote 19 of the same section).

$$H_2^{*\dagger} = 2 \ln \left| q \sec\left(\frac{1}{2} qp^{\dagger} - \arctan \frac{c}{q}\right)\right|, \tag{1g}$$

$$H_3^{*} = q(\tan t) \sec t |(e^{p'} \cot \tfrac{1}{2}t)^{\cos t} - \ln(e^{p'} \cot \tfrac{1}{2}t)^{\cos t} - 1|, \tag{1h}$$

$$p = \frac{\partial L}{\partial \dot{q}}, p^{*} = \frac{\partial L_2^{*}}{\partial \dot{q}}, p^{\dagger} = \frac{\partial L_2^{*\dagger}}{\partial \dot{q}}, p' = \frac{\partial L_3'}{\partial \dot{q}}$$

represent the one-dimensional harmonic oscillator

$$\ddot{q} + q = 0, \qquad m = 1, \qquad \omega^2 = 1, \tag{2}$$

that is, they *all* characterize the *same* implicit function $f = -q$. Thus the analytic equations in all functions (1) admit the *same solution*; any particular function of expressions (1) is admissible for the analytic representation of the system; and the selection of one representation versus another is therefore a matter of personal preference or practical convenience.

The example under consideration is intended to illustrate that the techniques of the Inverse Problem allow the construction, at least in principle, not only of a Lagrangian (or a Hamiltonian) when they exist but also of *all* possible Lagrangians (or Hamiltonians) for the analytic representation of equivalent self-adjoint forms of the system considered. These "degrees of freedom" are then used to study the possible identification of new first integrals, as well as for other applications (Charts A.6–A.12).

The methods for constructing representations (1) are the following. The trivial function (1a) is that producing the customary direct analytic representation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} \equiv (\ddot{q} + q)_{\mathrm{SA}}. \tag{3}$$

All other Lagrangians produce an indirect representation; that is, they verify the rule

$$\frac{d}{dt}\frac{\partial L^{*}}{\partial \dot{q}} - \frac{\partial L^{*}}{\partial q} \equiv [I(t, q, \dot{q})(\ddot{q} + q)_{\mathrm{SA}}]_{\mathrm{SA}} \tag{4}$$

where, in accordance with Corollary A.1.1a, the quantities $I(t, q, \dot{q})$ are first integrals. For instance, Lagrangians L_1^{*} and L_2^{*} are constructed via the first integrals

$$I_1 = \dot{q} \cos t + q \sin t, \tag{5a}$$

$$I_2 = (\dot{q}^2 + q^2)^{-1}, \tag{5b}$$

respectively.

Lagrangians L, L_1^{*}, and L_2^{*} are related by a chain of isotopies according to Equations (A.2.12). Lagrangians L_2^{*} and $L_2^{*\dagger}$ are related, instead, by a Newtonian gauge transformation, rule (A.2.1) or (A.2.3), because

$$L_2^{*\dagger} - L_2^{*} = \frac{2\dot{q}}{q} \arctan \frac{c}{q} = \frac{d}{dt} \int_{q_0}^{q} dz \frac{c}{z} \arctan \frac{c}{z} = \dot{C}(q). \tag{6}$$

All Hamiltonians of Equations (1) can be constructed via the Independent Inverse Hamiltonian Problem, that is, the methods for computing a Hamiltonian without any necessary prior knowledge of a Lagrangian (Sections I.3.8–I.3.12 and Chapter 4). Hamiltonians H, H_2^{*}, and $H_2^{*\dagger}$ are the Legendre transform of Lagrangians L, L_2^{*}, and $L_2^{*\dagger}$, respectively. The Hamiltonian corresponding to L_1^{*} and the Lagrangian corresponding to H_3^{*} have not been given in Equations (1). Notice that the coordinates of

all Hamiltonians of Equations (1) are the same. Nevertheless, their generalized momenta are different, even for the case of the gauge transformations. The relationship between these Hamiltonians is shown in Chapter 5.

The explicit computation of Lagrange's and Hamilton's equations with functions (1) is a truly instructive exercise for the interested student because it permits a knowledge of the structure of these fundamental equations which is often overlooked. We refer, for instance, to the proper use of the total time derivative appearing in Lagrange's equations according to the full structure ($\mathscr{I}.10$), or the crucial role of the theorem on implicit functions (Theorem I.1.1.1) for the construction of the second-order equations of motion from (first-order) Hamilton's equations (and vice versa). In turn, a sound knowledge of the structure of Lagrange's and Hamilton's equations at the Newtonian level is almost a necessity before passing to more complex branches of physics, such as field theory.[63]

In closing this example, I would like to express my gratitude to Eugene Saletan who taught me the possible varieties of admissible Lagrangian representations for a given system. In particular, the functions L_2^*, $L_2^{*\dagger}$, H_2^*, $H_2^{*\dagger}$, and H_3^* were computed for the first time by Currie and Saletan (1966) via conventional techniques.

Example A.2

The following two-dimensional conservative system (undamped and unforced oscillators with acceleration couplings)

$$\begin{cases} m_1\ddot{q}_1 + m_c\ddot{q}_2 + k_1q_1 + k_cq_2 = 0, \\ m_c\ddot{q}_1 + m_2\ddot{q}_2 + k_cq_1 + k_2q_2 = 0, \end{cases} \tag{1a}$$

$$\mathscr{H} = \det\begin{vmatrix} m_1 & m_c \\ m_c & m_2 \end{vmatrix} = m_1m_2 - m_c^2 \neq 0, \tag{1b}$$

$$k_1, k_2, k_c \neq 0$$

is self-adjoint, as the reader is encouraged to verify with the use of Equations (4.1.12).

The methods for the construction of a Lagrangian are trivial in this case, yielding the function

$$L = \tfrac{1}{2}(m_1\dot{q}_1^2 + m_c\dot{q}_1\dot{q}_2 + m_2\dot{q}_2^2) - \tfrac{1}{2}(k_1q_1^2 + k_cq_1q_2 + k_2q_2^2). \tag{2}$$

When system (1) is written in the equivalent kinematic form

$$\ddot{q}_k - f_k(q) = 0, \qquad k = 1, 2 \tag{3a}$$

$$(f_k) = \frac{1}{\mathscr{H}}\begin{pmatrix} (m_ck_c - m_2k_1)q_1 + (m_ck_2 - m_2k_c)q_2 \\ (m_ck_1 - m_1k_c)q_1 + (m_ck_c - m_1k_2)q_2 \end{pmatrix} \tag{3b}$$

$$m_ck_2 - m_2k_c \neq m_ck_1 - m_1k_c \tag{3c}$$

it becomes *non-self-adjoint*. Thus a Lagrangian for the direct representation of Equation (3a) via conventional Lagrange's equations does not exist. Nevertheless,

[63] The erroneous way of writing Lagrange's equations in field theory (pointed out in footnote 5 of the Introduction) is often due to lack of sufficient study at the Newtonian level.

system (3a) can still be directly represented by a variational principle. Indeed, the following genotopy of the conventional Hamilton's principle (Section A.2)

$$\delta^* \int_{t_1}^{t_2} dtL = -\int_{t_1}^{t_2} dt[\ddot{q}_k - f_k(q)]_{\text{NSA}}\, \delta q^k = 0 \tag{4a}$$

$$\delta^* q^k = h_i^k\, \delta q^i, \qquad \delta q^i = \varepsilon\eta(t),\ \varepsilon \approx 0 \tag{4b}$$

$$(h_i^j) = \frac{1}{\mathscr{H}} \begin{pmatrix} m_2 & -m_c \\ -m_c & m_1 \end{pmatrix} \tag{4c}$$

produces the representation desired.

The example is intended to illustrate the following aspects.

1. The acceleration couplings are often needed to reach a Lagrangian representation, in the sense that when they are elimitated via the theorem on implicit functions, the emerging equations are often non-self-adjoint. This property has been illustrated here with a *conservative* system. The reader should therefore expect an even greater occurrence of the property for non-conservative systems.

2. When the acceleration couplings are used, they necessarily result in a generalized structure of a Lagrangian even for *conservative* systems, with an understanding that such a generalized structure is necessary for Newtonian systems with non-self-adjoint forces. Indeed, Lagrangian (2) is of type (A.2.15) because the extra term $\frac{1}{2}m_c \dot{q}_1 \dot{q}_2$ does not represent free motion and cannot be incorporated into the additive term $L_{\text{int, II}}$, because of its nonlinearity.

3. The genotopically mapped Hamilton's principle (4a) does indeed allow the representation of equations of motion in their non-self-adjoint form, but in a purely *formal* treatment, in the sense that the Lagrangian, for the case considered, necessarily remains generalized.

Example A.3

The following equations describe the motion of a Newtonian particle in a Euclidean two-dimensional space with Cartesian coordinates

$$\begin{cases} m\ddot{x} + \gamma(x\dot{x}^2 + 2y\dot{x}\dot{y} - x\dot{y}^2) = 0, \\ m\ddot{y} + \gamma(y\dot{y}^2 + 2x\dot{x}\dot{y} - y\dot{x}^2) = 0. \end{cases} \tag{1}$$

Such a motion is highly nonconservative as a result of nonlinear velocity-dependent drag forces. The use of Equation ($\mathscr{A}$.12) confirms the expectation that the system is non-self-adjoint. The use of Theorem A.1.1 establishes that the system is non-essentially non-self-adjoint. A system of genotopic functions is given by

$$(h_i^j) = \exp\left(\frac{\gamma}{m}(x^2 + y^2)\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2}$$

The corresponding Lagrangian induced by Equation (A.1.10) is

$$L = \exp\left(\frac{\gamma}{m}\mathbf{r}^2\right)\tfrac{1}{2}m\dot{\mathbf{r}}^2, \qquad \mathbf{r} = (x, y). \tag{3}$$

Predictably, this Lagrangian is of the generalized type; that is, it exhibits an essential multiplicative term to that representing free motion, while the additive interaction term is, in this case, null (up to gauge as well as isotopic transformations). A methodologically significant aspect of this example is that the orbit of the particle is *unstable*,

because of nonlinear velocity-dependent drag forces (e.g., motion in a viscous medium). Thus the physical angular momentum *is not* conserved, i.e.,

$$\frac{d}{dt}\mathbf{M} = \frac{d}{dt}(\mathbf{r} \times m\dot{\mathbf{r}}) \neq 0. \tag{4}$$

Nevertheless, Lagrangian (3) is invariant under rotations. The example therefore illustrates semi-canonical breaking of the symmetry under the group of rotations, in the sense of Chart A.12.

Example A.4

The analytic, regular, nonlinear, and nonconservative system

$$\begin{aligned}
&(\tfrac{3}{2}\dot{q}_1^2 + \dot{q}_1\dot{q}_2)\ddot{q}_1 + (\tfrac{1}{2}\dot{q}_2^2 + \dot{q}_1\dot{q}_2)\ddot{q}_2 - \frac{\gamma}{3}\dot{q}_2\dot{q}_1^3 - \frac{\gamma}{3}\dot{q}_1\dot{q}_2^3 \\
&\quad + \gamma(\dot{q}_1 + \dot{q}_2 + 1)(\tfrac{1}{2}\dot{q}_2\dot{q}_1^2 + \tfrac{1}{2}\dot{q}_1\dot{q}_2^2 + \tfrac{1}{6}\dot{q}_1^3 + \tfrac{1}{6}\dot{q}_2^3) = 0, \\
&\tfrac{1}{2}\dot{q}_1^2\ddot{q}_1 + \tfrac{1}{2}\dot{q}_2^2\ddot{q}_2 \\
&\quad + \gamma(\dot{q}_1 + \dot{q}_2 + 1)(\tfrac{1}{2}\dot{q}_2\dot{q}_1^2 - \tfrac{1}{2}\dot{q}_1\dot{q}_2^2 + \tfrac{1}{6}\dot{q}_2^3 - \tfrac{1}{6}\dot{q}_1^3) = 0,
\end{aligned} \tag{1}$$

is non-self-adjoint, as the reader can verify. The only possibility (see Problem A.1) for the existence of a Lagrangian representation in the coordinate and time considered is that the system is of non-essentially non-self-adjoint type. In other words, Equation (1) admits consistent conditions of self-adjointness (A.1.9) in the genotopic functions h_k^i. A study of these equations confirms that this is indeed the case. *One* solution in the genotopic functions is given by

$$(h_i^j) = \tfrac{1}{2}e^{\gamma(q_1+q_2+t)}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{2}$$

As expected from the Cauchy–Kovalevski Theorem, this solution is analytic too. Theorem A.1.1 then applies and yields as *one* Lagrangian

$$L = e^{\gamma(q_1+q_2+t)}\tfrac{1}{6}(\dot{q}_2\dot{q}_1^3 + \dot{q}_1\dot{q}_2^3). \tag{3}$$

The study of the possible existence of different genotopic functions, and thus different Lagrangians, isotopically related to (3) is left to the interested reader.

Notice the need of five functions—four genotopic functions and one Lagrangian—for the construction of the analytic representation being studied.

Example A.5

Darboux's construction of a Lagrangian in its original formulation (Darboux, 1894) is illustrated here for the case of the one-dimensional, non-self-adjoint system

$$\ddot{q} - f(t, q, \dot{q}) = 0, \qquad f = 2\frac{(t\dot{q} - q)}{t^2}.$$

The problem is to identify a solution L of the second-order quasilinear partial differential equation

$$\frac{\partial^2 L}{\partial \dot{q}^2} f + \frac{\partial^2 L}{\partial \dot{q}\,\partial q}\dot{q} + \frac{\partial^2 L}{\partial \dot{q}\,\partial t} - \frac{\partial L}{\partial q} = 0,$$

which is consistent (because it verifies the condition of Theorem 2 of Chart A.3).

In this case the characteristic equations can be written

$$\frac{dt}{1} = \frac{dq}{\dot{q}} = \frac{d\dot{q}}{\dfrac{2(t\dot{q} - q)}{t^2}} = -\frac{dM}{2M/t}$$

$$M = \frac{\partial^2 L}{\partial \dot{q}^2}.$$

The first integrals are

$$\frac{t\dot{q} - q}{t^2} = u; \quad \frac{2\dot{q}}{t_1} - \dot{q} = v; \quad Mt^2 = m = F(u, v) \tag{4}$$

and the solution for M can be written

$$M = \frac{\partial^2 L}{\partial \dot{q}^2} = \frac{F(u, v)}{\sigma(t; u, v)} = \frac{1}{t^2} F\left[\left(\frac{t\dot{q} - q}{t^2}\right); \left(\frac{2q}{t} - \dot{q}\right)\right] \tag{5}$$

where F is an arbitrary function of its arguments.

The solution for L is then

$$L(t, q, \dot{q}) = \frac{1}{t^2} \int_0^q (z - \dot{q}) F\left[\left(\frac{tz - q}{t^2}\right); \left(\frac{2q}{t} - z\right)\right] dz$$

$$+ C(t, q) + D(t, q)\dot{q}, \tag{6}$$

and the consistency conditions for C and D become

$$\frac{\partial C}{\partial t} - \frac{\partial D}{\partial q} = 0. \tag{7}$$

Assume now, for simplicity, $F = 1$, $C = D = 0$. Then

$$L = -\frac{1}{2t^2}\dot{q}^2 + \frac{q\dot{q}}{t^3} - \frac{q^2}{2t^4} \tag{8a}$$

$$\frac{\partial^2 L}{\partial \dot{q}^2} = -\frac{1}{t^2}; \quad \frac{\partial^2 L}{\partial q\, \partial \dot{q}} = \frac{1}{t^3}; \quad \frac{\partial^2 L}{\partial t\, \partial \dot{q}} = \frac{2\dot{q}}{t^3} - \frac{3q}{t^4}; \quad \frac{\partial L}{\partial q} = \frac{\dot{q}}{t^3} - \frac{q}{t^4}. \tag{8b}$$

Thus

$$\frac{\partial^2 L}{\partial \dot{q}^2}\ddot{q} + \frac{\partial^2 L}{\partial \dot{q}\, \partial q}\dot{q} + \frac{\partial^2 L}{\partial \dot{q}\, \partial t} - \frac{\partial L}{\partial q}$$

$$= -\frac{1}{t^2}\ddot{q} + \frac{1}{t^3}\dot{q} + \frac{2\dot{q}}{t^3} - \frac{3q}{t^4} - \frac{\dot{q}}{t^3} + \frac{q}{t^4} \tag{9}$$

$$= -\left\{\frac{1}{t^2}\left[\ddot{q} - \frac{2(t\dot{q} - q)}{t^2}\right]_{\text{NSA}}\right\}_{\text{SA}}.$$

It should be stressed that Lagrangian (8a) *is not* unique.

A comparison of Darboux's approach with that used in these volumes is instructive. The problem, according to Equation (A.1.8), is to solve h of the condition of self-adjointness (A.1.11) for an equivalent self-adjoint form.

One solution is given by $h = 1/t^2$. This function is well defined on a star-shaped region centered at the origin because trivially, the star-shaped condition refers only to the q and $\dot{q}$ variables. Method (A.1.10) for the construction of a Lagrangian then applies, yielding Lagrangian (8a).

By recalling that Darboux's approach is equivalent to that presented here (Problem A.1), in the former we must solve a *second-order* partial differential equation in a Lagrangian, while in the latter we must solve a *first-order* equation in the genotopic function h and then compute a Lagrangian through integrals (A.1.10). As a result, when the use of any one of these methods presents technical difficulties, one can attempt to bypass them by using the other one.

Example A.6.

We shall illustrate here a property (Section A.3) according to which coordinate transformations can be self-adjoint genotopic, that is, capable of transforming a given non-self-adjoint system into an equivalent self-adjoint form.

The one-dimensional system of the general second-order form

$$(q\dot{q} + \dot{q}^2)_{\text{NSA}} = 0, \qquad q \neq 0, \tag{1}$$

is non-self-adjoint, yet the point transformation

$$q \to q' = \tfrac{1}{2}q^2 \tag{2}$$

turns system (1) into the equivalent self-adjoint form

$$(\ddot{q}')_{\text{SA}} = 0, \ddot{q}' = \frac{d}{dt}(q') = \frac{d}{dt}(q\dot{q}) = q\ddot{q} + \dot{q}^2. \tag{3}$$

At the Lagrangian level this case can be expressed as follows

$$\left(\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}'} - \frac{\partial L'}{\partial q'}\right)_{\text{SA}} = (\ddot{q}')_{\text{SA}}$$

$$= \left[\frac{\partial q}{\partial q'}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}\right)_{\text{SA}}\right]_{\text{NSA}} = \left[\frac{1}{q}(q^2\ddot{q} + q\dot{q})^{\text{A,R}}_{\text{SA}}\right]_{\text{NSA}} = 0 \tag{4}$$

$$L' = \tfrac{1}{2}\dot{q}'^2, \qquad L = \tfrac{1}{2}q^2\dot{q}^2$$

and illustrates Equations (A.3.20). Note the crucial role of the Jacobian in the transformation from a self-adjoint form to an equivalent non-self-adjoint one. Notice that it is also possible to obtain the Lagrangian representation in the original system by using Theorem A.1.1 with integrating factors. Thus the use of point transformations does not broaden the representational capabilities of Theorem A.1.1. Yet, these transformations are often useful in practical applications, e.g., by turning a system of partial differential equations (A.1.9) in one coordinate frame into a possibly more manageable form in a new coordinate frame.

Example A.7

We study here the relationship between the conservation of the (physical) total energy and the symmetry under translations in time for the simple one-dimensional harmonic oscillator

$$\ddot{q} + q = 0, \qquad m = 0, \qquad \omega = 0 \tag{1}$$

The transformation under considerations is

$$t \to t' = t + t_0 \tag{2}$$

where the constant t_0 is, at this point, finite. The corresponding transformations of the q-coordinate is given by

$$q(t) \to q(t) = q(t - t_0) \tag{3}$$

The equations of motion is trivially form-invariant under this transformation

$$\ddot{q}(t) + q(t) \equiv \ddot{q}'(t') + q'(t') \tag{4}$$

which, therefore, constitutes a symmetry of the equation of motion.

We now introduce the familiar Lagrangian

$$L = \tfrac{1}{2}(\dot{q}^2 - q^2) \tag{5}$$

Its form-invariance is expressed by

$$L(q, \dot{q}) \equiv L(q', \dot{q}'), \qquad \frac{\partial t'}{\partial t} = 1 \tag{6}$$

Thus, transformation (2) characterizes a finite, exact, noncontemporaneous symmetry of L.

To use Noether's Theorem for the identification of the underlying conservation law, we must restrict transformations (2) to be infinitesimal of the first order. This can be simply achieved by assuming $t_0 = w \in 0_\varepsilon$. The transformation is of pure time-type. Noether's theorem then yields the (first-order) conserved quantity

$$I = -\left(\dot{q}\frac{\partial L}{\partial \dot{q}} - L\right)w = -H(q, \dot{q})w \tag{7}$$

which represents the physical total energy of the system.

As established in Section A.2, Lagrangian (5) is highly non-unique. Among all possible isotopically mapped Lagrangians we consider the form (Example A.1)

$$L^* = \tfrac{1}{6}\dot{q}^3 \cos t + \tfrac{1}{2}q^2\dot{q}^2 \sin t - q^2\dot{q} \cos t \tag{8a}$$

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{q}} - \frac{\partial L^*}{\partial q} = \alpha(t)\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}\right), \quad \alpha = \dot{q} \cos t + q \sin t \tag{8b}$$

Since this Lagrangian is now explicitly dependent on time, the original symmetry is broken without affecting the conservation law of the total energy. This can be proved by ignoring all Lagrangian representations and verifying the following rule within the context of the equation of motion

$$\dot{H} = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial \dot{q}}f = q\dot{q} - \dot{q}q \equiv 0 \tag{9}$$

Theorem 1 of Chart A.10 holds and a *new* symmetry which leads to the same conservation law exists. This new symmetry can be identified by noting first that the *contemporaneous* variation associated with transformations (2) which, in this case is given by $\delta^1 q = -\dot{q}w$ and it is the variation appearing in the Noether's identity for the original Lagrangian.

We then construct the corresponding (still contemporaneous) variation $\delta^1_* q$ associated with L^* via the methods of Chart A.10, which is given by

$$\delta^1_* q = \frac{3}{\alpha(t)}\delta^1 q = -\frac{\dot{q}}{\alpha(t)}w \tag{10}$$

A solution the (first order) inverse Noether's problem is then given by

$$\begin{cases} \delta^1_* t = -\dfrac{H + \dfrac{\partial L^*}{\partial \dot{q}} \dfrac{\dot{q}}{\alpha(t)}}{L^*} \equiv \hat{\rho}_*(t, q, \dot{q})w & \text{(11a)} \\ \delta^1_* q = \left[\dfrac{\dot{q}}{\alpha(t)} - \dfrac{H\dot{q} + \dfrac{\partial L^*}{\partial \dot{q}} \dfrac{\dot{q}^2}{\alpha(t)}}{L^*} \right] w \equiv \hat{\eta}(t, q, \dot{q})w = & \text{(11b)} \\ \delta^1_* q \neq -\dot{q}\, \hat{\delta}^1_* t & \text{(11c)} \end{cases}$$

The symmetry above is of mixed space-time type. This first proves that the conservation of the total energy *is not* uniquely associated with the symmetry of the Lagrangian under translation in time and, second, illustrates Theorem 1 of Chart A.11, namely, that the isotopical image of the a pure time transformation can be of mixed space–time type. In fact, for a pure time translation we must have the condition

$$\hat{\delta}^1 q = -\dot{q}\, \hat{\delta}^1 t \tag{12}$$

which is violated by symmetry (11) as expressed by (11c).

Problems

A.1 Prove that the formulation of the Inverse Lagrangian Problem by Darboux (1891), Douglas (1941), and others,

$$\frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^i} f^i + \frac{\partial^2 L}{\partial \dot{q}^k \partial q^i} \dot{q}^i + \frac{\partial^2 L}{\partial \dot{q}^k \partial t} - \frac{\partial L}{\partial q^k} = 0,$$

is equivalent to that used in these volumes,

$$\frac{d}{dt} \frac{\partial L'}{\partial \dot{q}^k} - \frac{\partial L'}{\partial q^k} \equiv [h_{ki}(\ddot{q}^i - f^i)], \det(h) \neq 0,$$

that is, the existence of a solution $L(t, q, \dot{q})$ in one approach implies the existence of a solution $L\,(t, q, \dot{q})$ in the other approach and vice versa, up to equivalence transformations within fixed local variables, such as the gauge and isotopic transformations of Section A.2.

A.2 Prove Lemma A.2.1.

A.3 Prove that, if a Lagrangian L admits two different isotopic images according to the rules

$$\frac{d}{dt} \frac{\partial (L^{*1})^{*2}}{\partial \dot{q}^k} - \frac{\partial (L^{*1})^{*2}}{\partial q^k} \equiv \left[h^{(2)i}_k \left(\frac{d}{dt} \frac{\partial L^{*1}}{\partial \dot{q}^i} - \frac{\partial L^{*1}}{\partial q^i} \right)_{SA} \right]_{SA}$$

$$= \left\{ h^{(2)i}_k \left[h^{(i)j}_i \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^j} - \frac{\partial L}{\partial q^j} \right)_{SA} \right]_{SA} \right\}_{SA}$$

the following reformulations of Hamilton's principle hold

$$\delta \int_{t_1}^{t_2} dt(L^{*1})^{*2} = \delta^{*2} \int_{t_1}^{t_2} dt L^{*1} = (\delta^{*2})^{*1} \int_{t_1}^{t_2} dt L$$

$$\delta^{*2} q^k = h_i^{(2)k}\, \delta q^i, \qquad (\delta^{*2})^{*1} = h_j^{(1)i} h_i^{(2)k}\, \delta q^j.$$

A.4 Study the equations of variations of the generalized Lagrange's equations (15) of Chart 5.7. Identify the conditions under which the equations of variations reduce to an equivalence transformation of Jacobi's equations (Section I.3.3)

$$g_k^i(t)\left[\frac{d}{dt}\frac{\partial J}{\partial \dot{\eta}^i} - \frac{\partial J}{\partial \eta^i}\right] = 0,$$

$$J = \tfrac{1}{2}\left[\frac{\partial^2 L}{\partial \dot{q}^i\, \partial \dot{q}^j}\dot{\eta}^i\dot{\eta}^j + 2\frac{\partial^2 L}{\partial \dot{q}^i\, \partial q^j}\dot{\eta}^i\eta^j + \frac{\partial^2 L}{\partial q^i\, \partial q^j}\eta^i\eta^j\right];$$

finally, prove for the one-dimensional case that, when the function h is a first integral, the equations of variations coincide with their adjoint (i.e., are self-adjoint, according to the terminology of these volumes).

A.5 Construct a Lagrangian representation for the non-self-adjoint *Bessel equation* (see also Problem I.2.2)

$$[t^2\ddot{q}^2 + t\dot{q} + (t^2 - n^2)q]_{\mathrm{NSA}} = 0, \qquad t \neq 0.$$

A.6 Prove that the genotopic transformations

$$\{h_1(t)[Q(t)\ddot{q} + b(t)\dot{q} + c(t)q]_{\mathrm{NSA}}\}_{\mathrm{SA}} = 0$$

$$\{h_2(t)[a(t)\ddot{q} + b(t)\dot{q} + c(t)q - d(t)]_{\mathrm{NSA}}\}_{\mathrm{SA}} = 0$$

are verified by functions

$$h_1 = \frac{1}{a(t)}\exp\left\{\int_0^1 d\tau\, \frac{b(\tau)}{a(\tau)}\right\}$$

$$h_2 = \exp\left\{\int_0^1 d\tau\, \frac{b(\tau) - \dot{a}(\tau)}{a\tau}\right\}, \qquad \dot{a}(\tau) = \frac{da}{d\tau}.$$

A.7 Prove that the following systems are non-essentially non-self-adjoint, and compute a Lagrangian representation according to Theorem A.1.1:

$$\begin{cases} \ddot{q}_1 + \dot{q}_1 + 2\dfrac{\dot{q}_1^2}{q_1} + \dfrac{b_1}{k_1 q_1} + \dfrac{b_c q_2}{k_1 q_1^2} = 0, \\ \ddot{q}_2 + \dot{q}_2 + 2\dfrac{\dot{q}_2^2}{q_2} + \dfrac{b_2}{k_2 q_2} + \dfrac{b_c q_1}{k_2 q_2^2} = 0, \end{cases}$$

$$\begin{cases} \ddot{q}_1 + 2\gamma(\dot{q}_1 + \dot{q}_2)\dot{q}_1 + 2\gamma q_1 q_2 + q_2 = 0, \\ \ddot{q}_2 + 2\gamma(\dot{q}_1 + \dot{q}_2)\dot{q}_2 + 2\gamma q_1 q_2 + q_1 = 0, \end{cases}$$

$$\begin{cases} \ddot{q}_1 + \tfrac{1}{2}\gamma\dot{q}_1^2 + \gamma\dot{q}_1\dot{q}_3 = 0, \\ \ddot{q}_2 + \tfrac{1}{2}\dot{q}_2^2 + \gamma\dot{q}_2\dot{q}_3 = 0, \\ \ddot{q}_3 + \tfrac{1}{2}\gamma(\dot{q}_3 - \dot{q}_1^2 e^{\gamma q_1} - \dot{q}_2^2 e^{\gamma q_2}) = 0. \end{cases}$$

A.8 Extend the analysis of Chart A.4 to the case with an explicit dependence on time.

A.9 Consider a free particle in a three-dimensional Euclidean space and the ten first integrals which can be constructed through the use of the Galilei's symmetry (Chart A.9). Prove that only six first integrals are independent.

A.10 Consider a Lagrangian for the free motion in three-dimensions, $L = \frac{1}{2}m\dot{\mathbf{r}}^2$, its symmetry under the Galilei group $G(3.1)$, and the related ten conservation laws $\dot{I}_i(t, \mathbf{r}, \dot{\mathbf{r}}) = 0$. Construct an isotopic image L^* of L via the techniques of Section A.2, and then compute the infinitesimal symmetry $G^*(3.1)$ of L^* which is isotopically related to $G(3.1)$, that is (Chart A.10), which leads to the same conservation laws $\dot{I}_i = 0$. Identify the type of breaking of the original $G(3.1)$ symmetry along the lines of Chart A.12.

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