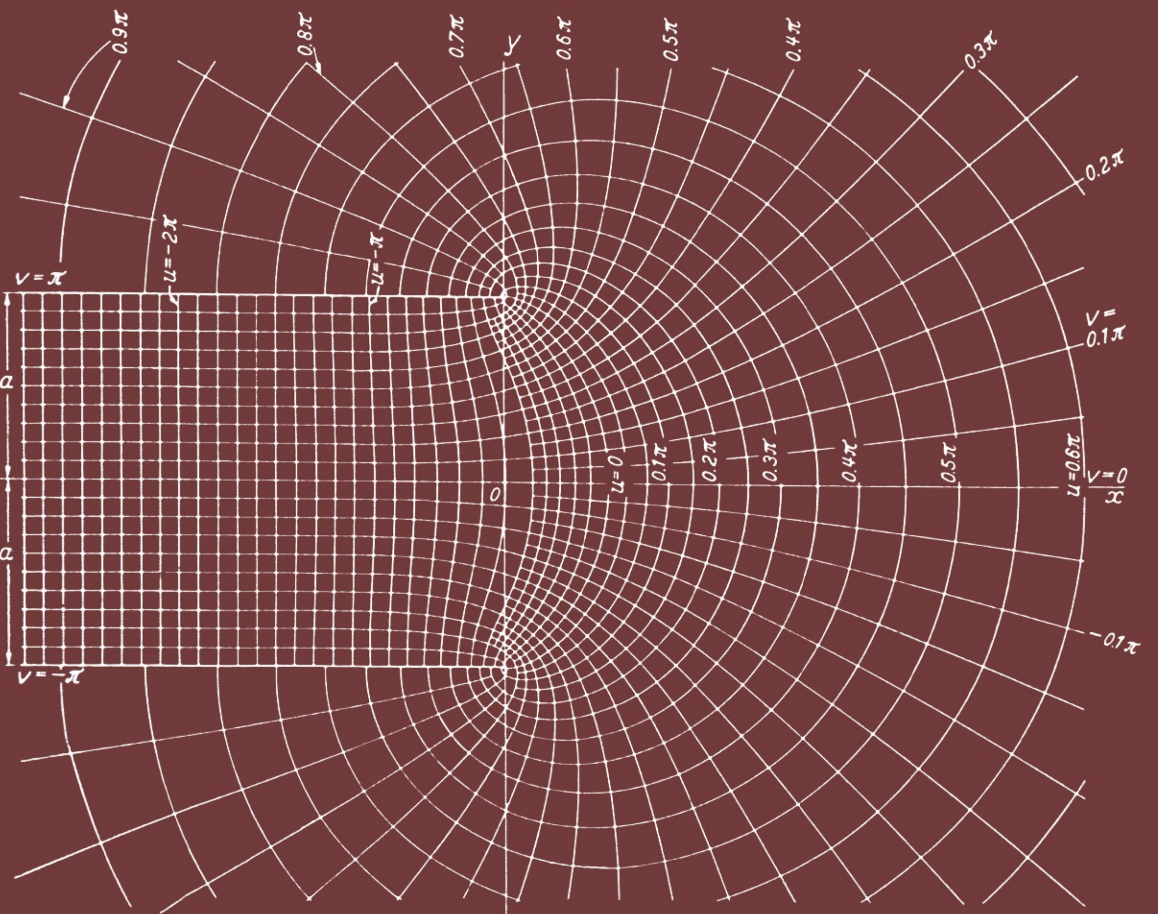
P. Moon and D. E. Spencer

Field Theory Handbook

Including Coordinate Systems Differential Equations and Their Solutions

2nd Edition



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Field Theory Handbook

Including Coordinate Systems, Differential Equations and Their Solutions

Second Edition

With 59 Figures

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PREFACE

Let us first state exactly what this book is and what it is not. It is a compendium of equations for the physicist and the engineer working with electrostatics, magnetostatics, electric currents, electromagnetic fields, heat flow, gravitation, diffusion, optics, or acoustics. It tabulates the properties of 40 coordinate systems, states the Laplace and Helmholtz equations in each coordinate system, and gives the separation equations and their solutions. But it is not a textbook and it does not cover relativistic and quantum phenomena.

The history of classical physics may be regarded as an interplay between two ideas, the concept of *action-at-a-distance* and the concept of a *field*. Newton's equation of universal gravitation, for instance, implies action-at-a-distance. The same form of equation was employed by COULOMB to express the force between charged particles. AMPÈRE and GAUSS extended this idea to the phenomenological action between currents. In 1867, LUDVIG LORENZ formulated electrodynamics as retarded action-at-a-distance. At almost the same time, MAXWELL presented the alternative formulation in terms of fields.

In most cases, the field approach has shown itself to be the more powerful. A partial differential equation is solved, and boundary conditions are fitted to give a unique solution of the problem. The partial differential equations of classical physics, considered in this book, are the Laplace equation, Poisson equation, diffusion equation, scalar wave equation, and vector wave equation. Several methods of handling these equations are possible, but *separation of variables* is generally the most valuable. The procedure is as follows:

(1) Transform the partial differential equation into the coordinate system that fits the geometry of the problem.

(2) Separate this equation into three ordinary differential equations.

(3) Obtain solutions of these ordinary differential equations.

(4) Build up the unique solution that fits the boundary conditions, using as building blocks the particular solutions obtained in (3).

The amount of labor involved in solving a practical problem by this method, particularly when the required coordinates are unfamiliar, is rather formidable. One may well hesitate about embarking on such a program; and this hesitancy is probably responsible for the dearth of engineering solutions of field problems.

Most of the labor, however, occurs in the first three steps; and *these parts of the solution can be completed, once for all, and the results tabulated.* This is the purpose of the Handbook—to remove the routine drudgery from field solutions so that the scientist can concentrate on (4), the unique and important part of the work. No such tables have been available previously.

The range of problems that can be handled by separation of variables depends to a marked extent on the number of available coordinate systems. Accordingly, we have not limited ourselves to the eleven systems of EISENHART but have chosen a number of others, bringing the total to 40. This by no means constitutes the totality of possible coordinates that allow simple separation or R-separation, though it includes most systems of reasonable simplicity and usefulness. Methods of obtaining further coordinates are explained in our book *Field Theory for Engineers* (D. Van Nostrand Co., Princeton, N. J., U.S.A., 1961).

For each of the 40 coordinate systems are given the relations to rectangular coordinates, the Stäckel matrix, metric coefficients, gradient, divergence, and curl, the Laplace and Helmholtz equations, the separation equations, and the solutions of these equations. Also listed are a tabulation of all the ordinary differential equations of field theory and their solutions, also a bibliography of works dealing with the mathematical functions involved.

Every equation has been checked independently by the two authors. But in a work that includes so many mathematical expressions, many of them given here for the first time, complete freedom from error would approach the miraculous. Any suggestions regarding errors or improvements will be greatly appreciated.

To a pure mathematician, our tabulations will seem ludicrously redundant. We have listed all special cases, even when they are obtainable from the basic form by a simple functional transformation. In this respect, the Handbook is similar to a table of integrals, whose practical value resides precisely in its redundancy.

We hope that the book will help the research worker in two ways:

(a) By providing new coordinate systems, thus extending the range of engineering problems that can be handled by separation of variables, and

(b) by freeing him from much of the annoyance and wasted effort usually associated with the routine part of the solution of partial differential equations.

The Authors

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Section I

ELEVEN COORDINATE SYSTEMS

The book is limited to orthogonal coordinate systems in euclidean 3-space. Skew coordinate systems do not allow separation of variables and will not be considered.

One method of obtaining new coordinates—a method employed particularly by LAMÉ [*1*]* and by DARBOUX [*2*]—is to study all the surfaces of a certain class and to determine what combinations of these surfaces will give orthogonal intersections. For surfaces of the *first* degree, we find only rectangular coordinates. Proceeding to equations of the *second* degree (and degenerate cases), we obtain the eleven coordinate systems of this section. EISENHART [*3*] has shown that all these systems allow simple separation of the Laplace and Helmholtz equations. Other coordinate systems, such as those built from fourth-degree surfaces [*4*], may have practical applications; but the eleven are undoubtedly the most important.

The use of esoteric coordinate systems may be of value, even in simple geometric considerations where field theory does not enter. With a spheroid, for instance, spheroidal coordinates eliminate the cumbersome mathematical expressions obtained with rectangular coordinates and allow the simple determination of areas and volumes.

But these unusual coordinate systems are particularly valuable in field theory. To express boundary conditions in a reasonably simple way, one must have coordinate surfaces that fit the physical boundaries of the problem [*5*]. In considering heat flow in a bar of elliptic cross section, for instance, one uses elliptic-cylinder coordinates; in calculating the effect of introducing a dielectric sphere into an electric field, one uses spherical coordinates; in obtaining the radiation from a slender spheroidal antenna, one uses prolate spheroid coordinates. Thus the range of field problems that can be handled effectively by an engineer or physicist will depend upon the number of coordinate systems with which he is familiar.

1.01 METRIC COEFFICIENTS

An orthogonal coordinate system (u^1, u^2, u^3) may be designated by the metric coefficients g_{11}, g_{22}, g_{33}. An infinitesimal distance is written [*5*]

$$(ds)^2 = g_{11}(du^1)^2 + g_{22}(du^2)^2 + g_{33}(du^3)^2, \qquad (1.01)$$

where

$$g_{ii} = \left(\frac{\partial x^1}{\partial u^i}\right)^2 + \left(\frac{\partial x^2}{\partial u^i}\right)^2 + \left(\frac{\partial x^3}{\partial u^i}\right)^2, \qquad (1.02)$$

and x^i are rectangular coordinates.

* See Bibliography at end of the book.

In parabolic coordinates, for instance, let $u^1=\mu$, $u^2=\nu$, $u^3=\psi$. Then

$$\begin{cases} x^1 = \mu\nu\cos\psi, \\ x^2 = \mu\nu\sin\psi, \\ x^3 = \frac{1}{2}(\mu^2-\nu^2). \end{cases}$$

The metric coefficients for parabolic coordinates are obtained by use of Eq. (1.02):

$$g_{11} = \left(\frac{\partial x^1}{\partial\mu}\right)^2 + \left(\frac{\partial x^2}{\partial\mu}\right)^2 + \left(\frac{\partial x^3}{\partial\mu}\right)^2 = (\nu\cos\psi)^2 + (\nu\sin\psi)^2 + \mu^2 = \mu^2+\nu^2.$$

Similarly,

$$g_{22} = \mu^2+\nu^2, \qquad g_{33} = \mu^2\nu^2.$$

Knowing the metric coefficients, one can easily write equations [*5*] for volume, gradient, curl. etc. Equation (1.01) shows that infinitesimal *distances* along the coordinate axes are $(g_{11})^{\frac{1}{2}}\,du^1$, $(g_{22})^{\frac{1}{2}}\,du^2$, $(g_{33})^{\frac{1}{2}}\,du^3$. Thus an element of *area* on the u^1u^2-surface is

$$d\mathscr{A} = [(g_{11})^{\frac{1}{2}}\,du^1]\,[(g_{22})^{\frac{1}{2}}\,du^2] = (g_{11}g_{22})^{\frac{1}{2}}\,du^1\,du^2. \tag{1.03}$$

Similarly, an element of *volume* is

$$d\mathscr{V} = (g_{11}g_{22}g_{33})^{\frac{1}{2}}\,du^1\,du^2\,du^3 = g^{\frac{1}{2}}\,du^1\,du^2\,du^3. \tag{1.04}$$

For example, what is the area of a paraboloid of revolution ($\mu=\mu_0$) from the vertex to a definite height designated by ν_0? Parabolic coordinates (μ, ν, ψ) are employed, with

$$g_{11} = g_{22} = \mu^2+\nu^2, \qquad g_{33} = \mu^2\nu^2.$$

Thus, from Eq. (1.03),

$$d\mathscr{A} = \mu_0\nu(\mu_0^2+\nu^2)^{\frac{1}{2}}\,d\nu\,d\psi.$$

The total area is

$$\mathscr{A} = \int_0^{2\pi}\int_0^{\nu_0} \mu_0\nu(\mu_0^2+\nu^2)^{\frac{1}{2}}\,d\nu\,d\psi = \frac{2\pi\mu_0}{3}\left[(\mu_0^2+\nu_0^2)^{\frac{3}{2}} - \mu_0^3\right].$$

Gradient in orthogonal curvilinear coordinates (u^1, u^2, u^3) is

$$\operatorname{grad}\varphi = \frac{\boldsymbol{a}_1}{(g_{11})^{\frac{1}{2}}}\frac{\partial\varphi}{\partial u^1} + \frac{\boldsymbol{a}_2}{(g_{22})^{\frac{1}{2}}}\frac{\partial\varphi}{\partial u^2} + \frac{\boldsymbol{a}_3}{(g_{33})^{\frac{1}{2}}}\frac{\partial\varphi}{\partial u^3}, \tag{1.05}$$

where $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ are unit vectors.

Divergence is expressed as

$$\operatorname{div}\boldsymbol{E} = g^{-\frac{1}{2}}\left\{\frac{\partial}{\partial u^1}[(g/g_{11})^{\frac{1}{2}}E_1] + \frac{\partial}{\partial u^2}[(g/g_{22})^{\frac{1}{2}}E_2] + \frac{\partial}{\partial u^3}[(g/g_{33})^{\frac{1}{2}}E_3]\right\}. \tag{1.06}$$

Curl is

$$\left.\begin{aligned} \operatorname{curl}\boldsymbol{E} = g^{-\frac{1}{2}}\Big\{&\boldsymbol{a}_1(g_{11})^{\frac{1}{2}}\left[\frac{\partial}{\partial u^2}[(g_{33})^{\frac{1}{2}}E_3] - \frac{\partial}{\partial u^3}[(g_{22})^{\frac{1}{2}}E_2]\right] \\ &+ \boldsymbol{a}_2(g_{22})^{\frac{1}{2}}\left[\frac{\partial}{\partial u^3}[(g_{11})^{\frac{1}{2}}E_1] - \frac{\partial}{\partial u^1}[(g_{33})^{\frac{1}{2}}E_3]\right] \\ &+ \boldsymbol{a}_3(g_{33})^{\frac{1}{2}}\left[\frac{\partial}{\partial u^1}[(g_{22})^{\frac{1}{2}}E_2] - \frac{\partial}{\partial u^2}[(g_{11})^{\frac{1}{2}}E_1]\right]\Big\}, \end{aligned}\right\} \tag{1.07}$$

or

$$\operatorname{curl} \boldsymbol{E} = \begin{vmatrix} \boldsymbol{a}_1 (g_{11}/g)^{\frac{1}{2}} & \boldsymbol{a}_2 (g_{22}/g)^{\frac{1}{2}} & \boldsymbol{a}_3 (g_{33}/g)^{\frac{1}{2}} \\ \frac{\partial}{\partial u^1} & \frac{\partial}{\partial u^2} & \frac{\partial}{\partial u^3} \\ (g_{11})^{\frac{1}{2}} E_1 & (g_{22})^{\frac{1}{2}} E_2 & (g_{33})^{\frac{1}{2}} E_3 \end{vmatrix}. \tag{1.07a}$$

The *scalar Laplacian* of φ is defined as

$$\nabla^2 \varphi \equiv \operatorname{div} \operatorname{grad} \varphi. \tag{1.08}$$

In orthogonal curvilinear coordinates,

$$\nabla^2 \varphi = g^{-\frac{1}{2}} \sum_{i=1}^{3} \frac{\partial}{\partial u^i} \left[\frac{g^{\frac{1}{2}}}{g_{ii}} \frac{\partial \varphi}{\partial u^i} \right]. \tag{1.09}$$

As an example, write Laplace's equation in parabolic coordinates. Since

$$g_{11} = g_{22} = \mu^2 + \nu^2, \qquad g_{33} = \mu^2 \nu^2, \qquad g^{\frac{1}{2}} = \mu\nu(\mu^2 + \nu^2),$$

Eq. (1.09) gives

$$\nabla^2 \varphi = \frac{1}{\mu\nu(\mu^2+\nu^2)} \left\{ \frac{\partial}{\partial \mu} \left[\mu\nu \frac{\partial \varphi}{\partial \mu} \right] + \frac{\partial}{\partial \nu} \left[\mu\nu \frac{\partial \varphi}{\partial \nu} \right] + \frac{\partial}{\partial \psi} \left[\frac{(\mu^2+\nu^2)}{\mu\nu} \frac{\partial \varphi}{\partial \psi} \right] \right\},$$

so Laplace's equation is

$$\frac{1}{\mu^2+\nu^2} \left[\frac{\partial^2 \varphi}{\partial \mu^2} + \frac{1}{\mu} \frac{\partial \varphi}{\partial \mu} + \frac{\partial^2 \varphi}{\partial \nu^2} + \frac{1}{\nu} \frac{\partial \varphi}{\partial \nu} \right] + \frac{1}{\mu^2 \nu^2} \frac{\partial^2 \varphi}{\partial \psi^2} = 0.$$

The *vector Laplacian* of $\boldsymbol{E}$ is defined as [6]

$$\mathbin{\hexstar} \boldsymbol{E} \equiv \operatorname{grad} \operatorname{div} \boldsymbol{E} - \operatorname{curl} \operatorname{curl} \boldsymbol{E}. \tag{1.10}$$

In orthogonal curvilinear coordinates (u^1, u^2, u^3), the general expression is [5]

$$\left.\begin{aligned} \mathbin{\hexstar} \boldsymbol{E} \equiv \boldsymbol{a}_1 &\left\{ \frac{1}{(g_{11})^{\frac{1}{2}}} \frac{\partial \Upsilon}{\partial u^1} + (g_{11}/g)^{\frac{1}{2}} \left[\frac{\partial \Gamma_3}{\partial u^2} - \frac{\partial \Gamma_2}{\partial u^3} \right] \right\} \\ + \boldsymbol{a}_2 &\left\{ \frac{1}{(g_{22})^{\frac{1}{2}}} \frac{\partial \Upsilon}{\partial u^2} + (g_{22}/g)^{\frac{1}{2}} \left[\frac{\partial \Gamma_1}{\partial u^3} - \frac{\partial \Gamma_3}{\partial u^1} \right] \right\} \\ + \boldsymbol{a}_3 &\left\{ \frac{1}{(g_{33})^{\frac{1}{2}}} \frac{\partial \Upsilon}{\partial u^3} + (g_{33}/g)^{\frac{1}{2}} \left[\frac{\partial \Gamma_2}{\partial u^1} - \frac{\partial \Gamma_1}{\partial u^2} \right] \right\}, \end{aligned}\right\} \tag{1.11}$$

where

$$\Upsilon = \operatorname{div} \boldsymbol{E},$$

$$\Gamma_1 = \frac{g_{11}}{g^{\frac{1}{2}}} \left\{ \frac{\partial}{\partial u^2} [(g_{33})^{\frac{1}{2}} E_3] - \frac{\partial}{\partial u^3} [(g_{22})^{\frac{1}{2}} E_2] \right\},$$

$$\Gamma_2 = \frac{g_{22}}{g^{\frac{1}{2}}} \left\{ \frac{\partial}{\partial u^3} [(g_{11})^{\frac{1}{2}} E_1] - \frac{\partial}{\partial u^1} [(g_{33})^{\frac{1}{2}} E_3] \right\},$$

$$\Gamma_3 = \frac{g_{33}}{g^{\frac{1}{2}}} \left\{ \frac{\partial}{\partial u^1} [(g_{22})^{\frac{1}{2}} E_2] - \frac{\partial}{\partial u^2} [(g_{11})^{\frac{1}{2}} E_1] \right\}.$$

1.02 DIFFERENTIAL EQUATIONS

The partial differential equations considered in this book are as follows:

(1) Laplace's equation, $\nabla^2 \varphi = 0$.

(2) Poissons's equation, $\nabla^2 \varphi = -K(u^1, u^2, u^3)$.

(3) The diffusion equation, $\nabla^2 \varphi = \frac{1}{h^2} \frac{\partial \varphi}{\partial t}$.

(4) The wave equation, $\nabla^2 \varphi = \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2}$.

(5) The damped wave equation, $\nabla^2 \varphi = \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} + R \frac{\partial \varphi}{\partial t}$.

(6) Transmission line equation, $\nabla^2 \varphi = \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} + R \frac{\partial \varphi}{\partial t} + S \varphi$.

(7) The vector wave equation, $✡ \boldsymbol{E} = \frac{1}{c^2} \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$.

In electrical problems, φ represents the electric potential (volts); in magnetic problems, φ is the magnetic scalar potential (ampere-turns); in thermal problems, φ is the temperature (deg. cent); in gravitation, φ is the gravitational potential (joule kg^{-1}); in vibration applications, φ is displacement (m); in hydrodynamics and acoustics, φ is the velocity potential ($\text{m}^2\,\text{sec}^{-1}$).

The solution of any of the scalar equations in the foregoing list may be reduced to a solution of the scalar *Helmholtz equation,*

$$\nabla^2 U + \varkappa^2 U = 0. \tag{1.12}$$

For the *diffusion equation* (3), let

$$U(u^i)\, T(t),$$

where U is a function of the space coordinates and T is a function of time only. Substitution into the diffusion equation allows the separation of the time part, giving

$$\left.\begin{aligned} &\nabla^2 U + \varkappa^2 U = 0, \\ &\frac{dT}{dt} + \varkappa^2 h^2 T = 0, \end{aligned}\right\} \tag{1.13}$$

where $\varkappa$ is the separation constant.

Similarly, separation of the *wave equation* (4) gives

$$\left.\begin{aligned} &\nabla^2 U + \varkappa^2 U = 0, \\ &\frac{d^2 T}{dt^2} + \varkappa^2 c^2 T = 0. \end{aligned}\right\} \tag{1.14}$$

For the damped wave equation (5),

$$\left.\begin{aligned} &\nabla^2 U + \varkappa^2 U = 0, \\ &\frac{d^2 T}{dt^2} + R c^2 \frac{dT}{dt} + \varkappa^2 c^2 T = 0. \end{aligned}\right\} \tag{1.15}$$

With (6),

$$\left.\begin{aligned} &\nabla^2 U + \varkappa^2 U = 0, \\ &\frac{d^2 T}{dt^2} + R c^2 \frac{dT}{dt} + (S + \varkappa^2)\, c^2 T = 0. \end{aligned}\right\} \tag{1.16}$$

The solution of the Helmholtz equation depends on the space variables and the boundary conditions, and will be different for each problem. The equation in time, however, is independent of the coordinate system. Thus the solution of the diffusion equation (3) is always

$$\varphi = U(u^1, u^2, u^3)\, e^{-\varkappa^2 h^2 t}, \tag{1.17}$$

and a particular solution of the wave equation (4) is

$$\varphi = U(u^1, u^2, u^3) \, {\sin \atop \cos} (\varkappa c t) . \tag{1.18}$$

For the damped wave equation (5),

$$\varphi = U(u^1, u^2, u^3) \, e^{-\alpha t \pm [\alpha^2 - \varkappa^2 c^2]^{\frac{1}{2}} t} , \tag{1.19}$$

where $\alpha = R c^2/2$. There are three cases:

(a) *Overdamped* ($\alpha > \varkappa c$). The solution is given by Eq. (1.19).

(b) *Oscillatory* ($\alpha < \varkappa c$). The solution may then be written

$$\varphi = U(u^1, u^2, u^3) \, [A \, e^{-\alpha t} \cos \omega t + B \, e^{-\alpha t} \sin \omega t] , \tag{1.20}$$

where

$$\omega = [\varkappa^2 c^2 - \alpha^2]^{\frac{1}{2}} .$$

(c) *Critically damped* ($\alpha = \varkappa c$), where

$$\varphi = U(u^1, u^2, u^3) \, [A \, e^{-\alpha t} + B \, t \, e^{-\alpha t}] . \tag{1.21}$$

Evidently (6) is the same as (5) except that

$$\omega = [(S + \varkappa^2) \, c^2 - \alpha^2]^{\frac{1}{2}} .$$

The Laplace equation (1) is, of course, merely a special case of the Helmholtz equation with $\varkappa = 0$. The Poisson equation (2) can be reduced to the Laplace equation by a change of variable. Let

$$\varphi = \Omega + f(u^i) ,$$

where $f(u^i)$ is so chosen that

$$\nabla^2 \Omega = 0 . \tag{1.22}$$

Thus *all the scalar equations mentioned in this section can be reduced to the Helmholtz equation* (or its special case, the Laplace equation). In this way, the scalar fields of electrostatics, electric conduction, magnetism, heat flow, and acoustics can be based on the solutions of the Helmholtz and Laplace equations given in this book. The vector wave equation requires a somewhat different treatment, which is given in Section V.

1.03 SIMPLE SEPARATION

The solution of any of the partial differential equations, (1) to (6) of Section 1.02, reduces to a solution of the Helmholtz equation or the Laplace equation [*7*]. First consider the separation of the Helmholtz equation in 3-space.

The formulation is facilitated by the introduction of the Stäckel matrix [*8*]. With each coordinate system (u^1, u^2, u^3) is associated a matrix:

$$[S] = \begin{bmatrix} \Phi_{11}(u^1) & \Phi_{12}(u^1) & \Phi_{13}(u^1) \\ \Phi_{21}(u^2) & \Phi_{22}(u^2) & \Phi_{23}(u^2) \\ \Phi_{31}(u^3) & \Phi_{32}(u^3) & \Phi_{33}(u^3) \end{bmatrix} , \tag{1.23}$$

whose principal characteristic is that each row contains functions of only one variable (or constants). The Stäckel determinant is the determinant of the above matrix:

$$S = \begin{vmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ \Phi_{21} & \Phi_{22} & \Phi_{23} \\ \Phi_{31} & \Phi_{32} & \Phi_{33} \end{vmatrix}. \tag{1.24}$$

The cofactors of the elements in the first column are

$$M_{11} = \begin{vmatrix} \Phi_{22} & \Phi_{23} \\ \Phi_{32} & \Phi_{33} \end{vmatrix}, \quad M_{21} = -\begin{vmatrix} \Phi_{12} & \Phi_{13} \\ \Phi_{32} & \Phi_{33} \end{vmatrix}, \quad M_{31} = \begin{vmatrix} \Phi_{12} & \Phi_{13} \\ \Phi_{22} & \Phi_{23} \end{vmatrix}.$$

The necessary and sufficient conditions for simple separation of the scalar Helmholtz equation [9] are

$$\left.\begin{aligned} g_{ii} &= S/M_{i1}, \\ g^{\frac{1}{2}}/S &= f_1(u^1)\cdot f_2(u^2)\cdot f_3(u^3). \end{aligned}\right\} \tag{1.25}$$

The first requirement introduces the restriction that it be possible to form a Stäckel determinant [10] that is related to the metric coefficients as in Eq. (1.25). The second requirement states that $g^{\frac{1}{2}}/S$ shall be a separable product. If these requirements are satisfied, the separation equations are

$$\frac{1}{f_i}\frac{d}{du^i}\left(f_i\frac{dU^i}{du^i}\right) + U^i\sum_{j=1}^{3}\alpha_j\,\Phi_{ij} = 0. \tag{1.26}$$

Here $i, j = 1, 2, 3$; Φ_{ij} are elements of the Stäckel matrix, and α_j are the separation constants with $\alpha_1 = \varkappa^2$.

In parabolic coordinates, for example, a few trials show that a possible form of Stäckel matrix is

$$[S] = \begin{bmatrix} \mu^2 & -1 & 1/\mu^2 \\ \nu^2 & 1 & 1/\nu^2 \\ 0 & 0 & 1 \end{bmatrix}.$$

The form is not unique: many equivalent forms would do just as well. From this matrix, we find

$$S = \mu^2 + \nu^2, \quad M_{11} = M_{21} = 1, \quad M_{31} = \frac{1}{\mu^2} + \frac{1}{\nu^2}.$$

Evidently, Eq. (1.25) is satisfied, and

$$f_1 = \mu, \quad f_2 = \nu, \quad f_3 = 1.$$

The separation equations are obtained from Eq. (1.26):

$$\left\{\begin{aligned} &\frac{1}{\mu}\frac{d}{d\mu}\left(\mu\frac{dM}{d\mu}\right) + M[\alpha_1\mu^2 - \alpha_2 + \alpha_3/\mu^2] = 0, \\ &\frac{1}{\nu}\frac{d}{d\nu}\left(\nu\frac{dN}{d\nu}\right) + N[\alpha_1\nu^2 + \alpha_2 + \alpha_3/\nu^2] = 0, \\ &\frac{d^2\Psi}{d\psi^2} + \Psi\alpha_3 = 0. \end{aligned}\right.$$

These equations are solved for M, N, Ψ; and the solution of the Helmholtz equation is

$$U(\mu,\nu,\psi) = M(\mu)\cdot N(\nu)\cdot\Psi(\psi).$$

By no means all orthogonal coordinate systems allow separation. In tangent-cylinder coordinates (μ, ν, ψ), for instance,

$$g_{11} = g_{22} = (\mu^2 + \nu^2)^{-2}, \quad g_{33} = 1.$$

A few attempts will convince one that no Stäckel determinant can be devised that will satisfy Eq. (1.25).

The necessary and sufficient conditions for simple separation of the Laplace equation [9] are

$$\left.\begin{aligned} \frac{g_{ii}}{g_{jj}} &= \frac{M_{j1}}{M_{i1}}, \\ \frac{g^{\frac{1}{2}}}{g_{ii}} &= f_1 f_2 f_3 M_{i1}. \end{aligned}\right\} \tag{1.27}$$

The separation equations are again given by Eq. (1.26), but with $\alpha_1 = 0$.

For any *cylindrical* coordinate system in which the Helmholtz equation separates, the Stäckel matrix may be written [7]

$$[S] = \begin{bmatrix} 0 & \Phi_{12} & \Phi_{13} \\ 0 & \Phi_{22} & \Phi_{23} \\ 1 & 0 & 1 \end{bmatrix}. \tag{1.28}$$

For any *rotational* system (coordinate surfaces symmetrical about the z-axis),

$$[S] = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ \Phi_{21} & \Phi_{22} & \Phi_{23} \\ 0 & 0 & 1 \end{bmatrix} \tag{1.29}$$

1.04 COORDINATE SYSTEMS

The eleven coordinate systems [3], formed from first and second degree surfaces, are as follows:

Cylindrical

1. Rectangular coordinates (x, y, z), Fig. 1.01.
2. Circular-cylinder coordinates (r, ψ, z), Fig. 1.02.
3. Elliptic-cylinder coordinates (η, ψ, z), Fig. 1.03.
4. Parabolic-cylinder coordinates (μ, ν, z), Fig. 1.04.

Rotational

5. Spherical coordinates (r, θ, ψ), Fig. 1.05.
6. Prolate spheroidal coordinates (η, θ, ψ), Fig. 1.06.
7. Oblate spheroidal coordinates (η, θ, ψ), Fig. 1.07.
8. Parabolic coordinates (μ, ν, ψ), Fig. 1.08.

General

9. Conical coordinates (r, θ, λ), Fig. 1.09.
10. Ellipsoidal coordinates (η, θ, λ), Fig. 1.10.
11. Paraboloidal coordinates (μ, ν, λ), Fig. 1.11.

In the cylindrical systems, the cylindrical axis is always taken as the z-direction. In the rotational systems, the axis of symmetry is always taken as the z-axis and the angle about this axis is called ψ.

The Helmholtz and Laplace equations are simply separable in all these coordinate systems. The separation equations and their solutions are listed in the following tables.

As an example [11] of the use of the table, take a uniform electric field,

$$\varphi = E_0 z .$$

A metal prolate spheroid $\eta = \eta_0$ at zero potential is now introduced into the field, the center of the spheroid being at the origin of coordinates. What is the resulting potential distribution ?

Since the field has symmetry about the z-axis, φ is independent of ψ and the table gives as particular solutions of Laplace's equation,

$$\varphi = P_p(\cosh\eta)\, P_p(\cos\theta),$$
$$\varphi = Q_p(\cosh\eta)\, P_p(\cos\theta).$$

The Q-functions of $\cos\theta$ cannot be used because they become infinite on the z-axis. Assume a solution,

$$\varphi = A\, P_p(\cosh\eta)\, P_p(\cos\theta) + B\, Q_p(\cosh\eta)\, P_p(\cos\theta). \tag{1.30}$$

If this assumed solution does not satisfy the boundary conditions, we introduce additional terms, using infinite series if necessary.

Boundary conditions are

$$\begin{cases} \eta \gg \eta_0, & \varphi = E_0 z = E_0 a \sinh\eta \sin\theta; \\ \eta = \eta_0, & \varphi = 0. \end{cases}$$

When $\eta \gg \eta_0$, $Q_p(\cosh\eta) \to 0$ and

$$\varphi = E_0 a \sinh\eta \sin\theta = A\, P_p(\cosh\eta)\, P_p(\cos\theta).$$

Thus, to satisfy the first boundary condition,

$$A = E_0 a, \quad p = 1.$$

For the second boundary condition,

$$\varphi = 0 = [E_0 a \cosh\eta_0 + B\, Q_0(\cosh\eta_0)] \cos\theta$$

or

$$B = -\frac{E_0 a \cosh\eta_0}{Q_0(\cosh\eta_0)}.$$

Therefore, the unique solution of the problem is

$$\varphi = E_0 a \left[\cosh\eta - \cosh\eta_0 \frac{Q_0(\cosh\eta)}{Q_0(\cosh\eta_0)}\right] \cos\theta. \tag{1.31}$$

The electric field strength is, according to the table,

$$\boldsymbol{E} = -\operatorname{grad}\varphi = -\frac{1}{a\,[\sinh^2\eta + \sin^2\theta]^{\frac{1}{2}}} \left[\boldsymbol{a}_\eta \frac{\partial\varphi}{\partial\eta} + \boldsymbol{a}_\theta \frac{\partial\varphi}{\partial\theta}\right],$$

which is evaluated by differentiating Eq. (1.31). The field is plotted in Fig. 1.12. A similar plot for a dielectric spheroid is shown in Fig. 1.13.

TABLE 1.01. RECTANGULAR COORDINATES (x, y, z)

$$u^1 = x, \quad -\infty < x < +\infty,$$
$$u^2 = y, \quad -\infty < y < +\infty,$$
$$u^3 = z, \quad -\infty < z < +\infty.$$

Surfaces of constant x, y, or z are mutually orthogonal planes (Fig. 1.01).

The Stäckel matrix may be written

$$[S] = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$S = 1,$$

$$M_{11} = M_{21} = M_{31} = 1.$$

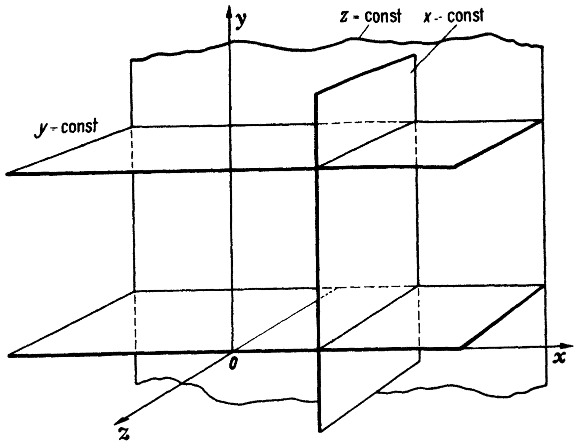

Fig. 1.01. Rectangular coordinates (x, y, z). Coordinate surfaces are the planes: $x = \text{const}$, $y = \text{const}$, $z = \text{const}$

Metric coefficients

$$g_{11} = g_{22} = g_{33} = 1, \quad g^{\frac{1}{2}} = 1.$$
$$f_1 = f_2 = f_3 = 1.$$

Important equations

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2.$$

$$\operatorname{grad} \varphi = \boldsymbol{a}_x \frac{\partial \varphi}{\partial x} + \boldsymbol{a}_y \frac{\partial \varphi}{\partial y} + \boldsymbol{a}_z \frac{\partial \varphi}{\partial z}.$$

$$\operatorname{div} \boldsymbol{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}.$$

$$\operatorname{curl} \boldsymbol{E} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}.$$

SEPARATION OF LAPLACE'S EQUATION, $\nabla^2 \varphi = 0$.

$$\frac{1}{f_i} \frac{d}{du^i}\left(f_i \frac{dU^i}{du^i}\right) + U^i \sum_{j=1}^{3} \Phi_{ij} \alpha_j = 0,$$

where $\alpha_1 = 0$ and $U^1 = X(x)$, $U^2 = Y(y)$, $U^3 = Z(z)$.

General case

$$\left\{\begin{aligned} &\frac{d^2 X}{dx^2} - (\alpha_2 + \alpha_3)\, X = 0, \\ &\frac{d^2 Y}{dy^2} + \alpha_2 Y = 0, \\ &\frac{d^2 Z}{dz^2} + \alpha_3 Z = 0. \end{aligned}\right.$$

If $\alpha_2 = p^2$ *and* $\alpha_3 = q^2$,

$$\frac{d^2X}{dx^2} - (p^2 + q^2)\,X = 0, \quad \{04\}^* \quad X = A\,e^{(p^2+q^2)^{\frac{1}{2}}x} + B\,e^{-(p^2+q^2)^{\frac{1}{2}}x}.$$

$$\frac{d^2Y}{dy^2} + p^2\,Y = 0, \quad \{04\} \quad Y = A\sin p\,y + B\cos p\,y.$$

$$\frac{d^2Z}{dz^2} + q^2 Z = 0, \quad \{04\} \quad Z = A\sin q\,z + B\cos q\,z.$$

If $\alpha_2 = p^2$ *and* $\alpha_3 = -\,q^2$,

$$\frac{d^2X}{dx^2} - (p^2 - q^2)\,X = 0, \quad \{04\} \quad X = A\,e^{(p^2-q^2)^{\frac{1}{2}}x} + B\,e^{-(p^2-q^2)^{\frac{1}{2}}x}.$$

$$\frac{d^2Y}{dy^2} + p^2\,Y = 0, \quad \{04\} \quad Y = A\sin p\,y + B\cos p\,y.$$

$$\frac{d^2Z}{dz^2} - q^2 Z = 0, \quad \{04\} \quad Z = A\,e^{qz} + B\,e^{-qz}.$$

For φ independent of z,

$$\begin{cases} \dfrac{d^2X}{dx^2} - \alpha_2 X = 0, \\[2mm] \dfrac{d^2Y}{dy^2} + \alpha_2 Y = 0. \end{cases}$$

If $\alpha_2 = p^2$,

$$\frac{d^2X}{dx^2} - p^2 X = 0, \quad \{04\} \quad X = A\,e^{px} + B\,e^{-px}.$$

$$\frac{d^2Y}{dy^2} + p^2\,Y = 0, \quad \{04\} \quad Y = A\sin p\,y + B\cos p\,y.$$

If $\alpha_2 = -\,p^2$,

$$\frac{d^2X}{dx^2} + p^2 X = 0, \quad \{04\} \quad X = A\sin p\,x + B\cos p\,x.$$

$$\frac{d^2Y}{dy^2} - p^2\,Y = 0, \quad \{04\} \quad Y = A\,e^{py} + B\,e^{-py}.$$

If $\alpha_2 = 0$,

$$\frac{d^2X}{dx^2} = \frac{d^2Y}{dy^2} = 0, \quad \{01\} \quad X = A + B\,x, \qquad Y = A + B\,y.$$

For φ independent of y and z,

$$\frac{d^2\varphi}{dx^2} = 0, \quad \{01\} \quad \varphi = A + B\,x.$$

SEPARATION OF HELMHOLTZ EQUATION, $\nabla^2\varphi + \varkappa^2\varphi = 0$.

$$\frac{1}{f_i}\,\frac{d}{du^i}\left(f_i\,\frac{dU^i}{du^i}\right) + U^i\sum_{j=1}^{3}\Phi_{ij}\,\alpha_j = 0,$$

where $\alpha_1 = \varkappa^2$ and $U^1 = X(x)$, $U^2 = Y(y)$, $U^3 = Z(z)$.

* The numbers in brackets designate the differential equation in terms of its singularities (see Section VI).

General case

$$\begin{cases} \dfrac{d^2X}{dx^2} - (\alpha_2 + \alpha_3)\, X = 0, \\ \dfrac{d^2Y}{dy^2} + \alpha_2 Y = 0, \\ \dfrac{d^2Z}{dz^2} + (\varkappa^2 + \alpha_3)\, Z = 0. \end{cases}$$

If $\alpha_2 = p^2$ *and* $\alpha_3 = q^2$,

$$\frac{d^2X}{dx^2} - (p^2 + q^2)\, X = 0, \quad \{04\} \quad X = A\, e^{(p^2+q^2)^{\frac{1}{2}} x} + B\, e^{-(p^2+q^2)^{\frac{1}{2}} x}.$$

$$\frac{d^2Y}{dy^2} + p^2 Y = 0, \quad \{04\} \quad Y = A \sin p y + B \cos p y.$$

$$\frac{d^2Z}{dz^2} + (\varkappa^2 + q^2)\, Z = 0, \quad \{04\} \quad Z = A \sin[(\varkappa^2 + q^2)^{\frac{1}{2}} z] + B \cos[(\varkappa^2 + q^2)^{\frac{1}{2}} z].$$

If $\alpha_2 = -p^2$ *and* $\alpha_3 = -q^2$,

$$\frac{d^2X}{dx^2} + (p^2 + q^2)\, X = 0, \quad \{04\} \quad X = A \sin[(p^2 + q^2)^{\frac{1}{2}} x] + B \cos[(p^2 + q^2)^{\frac{1}{2}} x].$$

$$\frac{d^2Y}{dy^2} - p^2 Y = 0, \quad \{04\} \quad Y = A\, e^{p y} + B\, e^{-p y}.$$

$$\frac{d^2Z}{dz^2} + (\varkappa^2 - q^2)\, Z = 0, \quad \{04\} \quad Z = A \sin[(\varkappa^2 - q^2)^{\frac{1}{2}} z] + B \cos[(\varkappa^2 - q^2)^{\frac{1}{2}} z].$$

If $\alpha_2 = \alpha_3 = 0$,

$$\frac{d^2X}{dx^2} = \frac{d^2Y}{dy^2} = 0, \quad \{01\} \quad X = A + Bx, \qquad Y = A + By.$$

$$\frac{d^2Z}{dz^2} + \varkappa^2 Z = 0, \quad \{04\} \quad Z = A \sin \varkappa z + B \cos \varkappa z.$$

For φ independent of z

$$\begin{cases} \dfrac{d^2X}{dx^2} - \alpha_2 X = 0, \\ \dfrac{d^2Y}{dy^2} + (\varkappa^2 + \alpha_2)\, Y = 0. \end{cases}$$

If $\alpha_2 = p^2$,

$$\frac{d^2X}{dx^2} - p^2 X = 0, \quad \{04\} \quad X = A\, e^{p x} + B\, e^{-p x}.$$

$$\frac{d^2Y}{dy^2} + (\varkappa^2 + p^2)\, Y = 0, \quad \{04\} \quad Y = A \sin[(\varkappa^2 + p^2)^{\frac{1}{2}} y] + B \cos[(\varkappa^2 + p^2)^{\frac{1}{2}} y].$$

If $\alpha_2 = -p^2$,

$$\frac{d^2X}{dx^2} + p^2 X = 0, \quad \{04\} \quad X = A \sin p x + B \cos p x.$$

$$\frac{d^2Y}{dy^2} + (\varkappa^2 - p^2)\, Y = 0, \quad \{04\} \quad Y = A \sin[(\varkappa^2 - p^2)^{\frac{1}{2}} y] + B \cos[(\varkappa^2 - p^2)^{\frac{1}{2}} y].$$

If $\alpha_2 = 0$,

$$\frac{d^2X}{dx^2} = 0, \quad \{01\} \quad X = A + Bx.$$

$$\frac{d^2Y}{dy^2} + \varkappa^2 Y = 0, \quad \{04\} \quad Y = A \sin \varkappa y + B \cos \varkappa y.$$

For φ independent of y and z,

$$\frac{d^2\varphi}{dx^2} + \varkappa^2 \varphi = 0, \quad \{04\} \quad \varphi = A \sin \varkappa x + B \cos \varkappa x.$$

TABLE 1.02. CIRCULAR-CYLINDER COORDINATES (r, ψ, z)

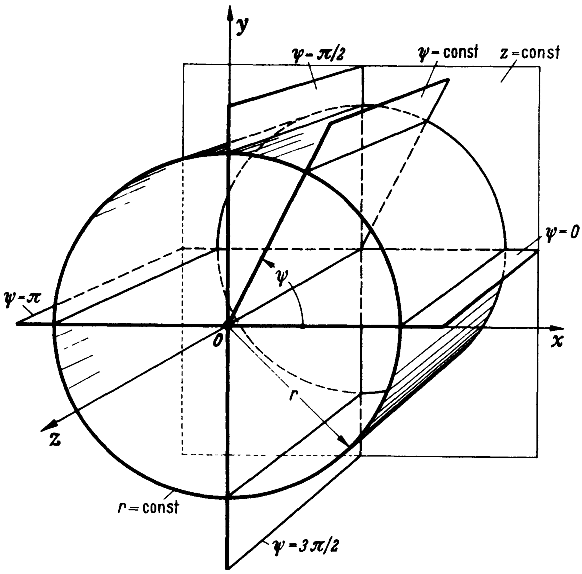

Fig. 1.02. Circular-cylinder coordinates (r, ψ, z). Coordinate surfaces are circular cylinders (r = const), half-planes (ψ = const) intersecting on the z-axis, and parallel planes (z = const)

$$u^1 = r, \qquad 0 \leqq r < \infty,$$
$$u^2 = \psi, \qquad 0 \leqq \psi < 2\pi,$$
$$u^3 = z. \qquad -\infty < z < +\infty.$$

$$\begin{cases} x = r\cos\psi, \\ y = r\sin\psi, \\ z = z. \end{cases}$$

Coordinate surfaces are

$x^2 + y^2 = r^2$ (circular cylinders, r = const),

$\tan\psi = y/x$ (half-planes, ψ = const),

z = const (planes).

The Stäckel matrix may be written

$$[S] = \begin{bmatrix} 0 & -1/r^2 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

$$S = 1, \quad M_{11} = 1, \quad M_{21} = 1/r^2, \quad M_{31} = 1.$$

Metric coefficients

$$g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = 1, \quad g^{\frac{1}{2}} = r.$$
$$f_1 = r, \quad f_2 = 1, \quad f_3 = 1.$$

Addendum see page 12a

Important equations

$$(ds)^2 = (dr)^2 + r^2(d\psi)^2 + (dz)^2.$$

$$\operatorname{grad}\varphi = \boldsymbol{a}_r \frac{\partial\varphi}{\partial r} + \frac{\boldsymbol{a}_\psi}{r}\frac{\partial\varphi}{\partial\psi} + \boldsymbol{a}_z \frac{\partial\varphi}{\partial z}.$$

$$\operatorname{div}\boldsymbol{E} = \frac{\partial E_r}{\partial r} + \frac{E_r}{r} + \frac{1}{r}\frac{\partial E_\psi}{\partial\psi} + \frac{\partial E_z}{\partial z}.$$

$$\operatorname{curl}\boldsymbol{E} = \frac{1}{r}\begin{vmatrix} \boldsymbol{a}_r & \boldsymbol{a}_\psi r & \boldsymbol{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\psi} & \frac{\partial}{\partial z} \\ E_r & E_\psi r & E_z \end{vmatrix}.$$

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\psi^2} + \frac{\partial^2\varphi}{\partial z^2}.$$

Addendum to page 12

Circular-Cylinder Coordinates

$$\Gamma_{ij}^{1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\Gamma_{ij}^{2} = \begin{bmatrix} 0 & 1/r & 0 \\ 1/r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\Gamma_{ij}^{3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

ALTERNATIVE CIRCULAR-CYLINDER SYSTEM (ξ, ψ, z)

Let $r = e^{\xi}$. Then

$$u^1 = \xi, \qquad -\infty < \xi < +\infty,$$
$$u^2 = \psi, \qquad 0 \leqq \psi < 2\pi,$$
$$u^3 = z. \qquad -\infty < z < +\infty.$$

$$\begin{cases} x = e^{\xi} \cos \psi, \\ y = e^{\xi} \sin \psi, \\ z = z. \end{cases}$$

The Stäckel matrix is

$$[S] = \begin{bmatrix} 0 & -1 & -e^{-2\xi} \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

$$S = e^{2\xi}, \quad M_{11} = M_{21} = 1, \quad M_{31} = e^{2\xi}.$$

Metric coefficients

$$g_{11} = g_{22} = e^{2\xi}, \quad g_{33} = 1, \quad g^{\frac{1}{2}} = e^{2\xi}.$$
$$f_1 = f_2 = f_3 = 1.$$

Important equations

$$(ds)^2 = e^{2\xi}[(d\xi)^2 + (d\psi)^2] + (dz)^2.$$

$$\operatorname{grad} \varphi = e^{-\xi}\left[\boldsymbol{a}_{\xi} \frac{\partial \varphi}{\partial \xi} + \boldsymbol{a}_{\psi} \frac{\partial \varphi}{\partial \psi}\right] + \boldsymbol{a}_z \frac{\partial \varphi}{\partial z}.$$

$$\operatorname{div} \boldsymbol{E} = e^{-\xi}\left[\frac{\partial E_{\xi}}{\partial \xi} + E_{\xi} + \frac{\partial E_{\psi}}{\partial \psi}\right] + \frac{\partial E_z}{\partial z}.$$

$$\operatorname{curl} \boldsymbol{E} = e^{-2\xi} \begin{vmatrix} \boldsymbol{a}_{\xi} e^{\xi} & \boldsymbol{a}_{\psi} e^{\xi} & \boldsymbol{a}_z \\ \dfrac{\partial}{\partial \xi} & \dfrac{\partial}{\partial \psi} & \dfrac{\partial}{\partial z} \\ E_{\xi} e^{\xi} & E_{\psi} e^{\xi} & E_z \end{vmatrix}.$$

$$\nabla^2 \varphi = e^{-2\xi}\left\{\frac{\partial^2 \varphi}{\partial \xi^2} + \frac{\partial^2 \varphi}{\partial \psi^2}\right\} + \frac{\partial^2 \varphi}{\partial z^2}.$$

SEPARATION OF LAPLACE'S EQUATION, $\nabla^2 \varphi = 0$.

$$\frac{1}{f_i} \frac{d}{du^i}\left(f_i \frac{dU^i}{du^i}\right) + U^i \sum_{j=1}^{3} \Phi_{ij} \alpha_j = 0,$$

where $\alpha_1 = 0$ and $U^1 = R(r)$, $U^2 = \Psi(\psi)$, $U^3 = Z(z)$.

General case

$$\begin{cases} \dfrac{d^2 R}{dr^2} + \dfrac{1}{r} \dfrac{dR}{dr} - \left(\dfrac{\alpha_2}{r^2} + \alpha_3\right) R = 0, \\ \dfrac{d^2 \Psi}{d\psi^2} + \alpha_2 \Psi = 0, \\ \dfrac{d^2 Z}{dz^2} + \alpha_3 Z = 0. \end{cases}$$

If $\alpha_2 = p^2$ *and* $\alpha_3 = q^2$,

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - (q^2 + p^2/r^2)\,R = 0, \quad \{24\} \quad R = A\mathcal{J}_p(iqr) + B\mathcal{J}_{-p}(iqr)$$

$$\text{or*} \quad R = A\mathcal{J}_n(iqr) + B\mathcal{Y}_n(iqr).$$

$$\frac{d^2\Psi}{d\psi^2} + p^2\Psi = 0, \quad \{04\} \quad \Psi = A\sin p\psi + B\cos p\psi.$$

$$\frac{d^2Z}{dz^2} + q^2Z = 0, \quad \{04\} \quad Z = A\sin qz + B\cos qz.$$

If $\alpha_2 = p^2$ *and* $\alpha_3 = -q^2$,

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + (q^2 - p^2/r^2)\,R = 0, \quad \{24\} \quad R = A\mathcal{J}_p(qr) + B\mathcal{J}_{-p}(qr)$$

$$\text{or} \quad R = A\mathcal{J}_n(qr) + B\mathcal{Y}_n(qr).$$

$$\frac{d^2\Psi}{d\psi^2} + p^2\Psi = 0, \quad \{04\} \quad \Psi = A\sin p\psi + B\cos p\psi.$$

$$\frac{d^2Z}{dz^2} - q^2Z = 0, \quad \{04\} \quad Z = A\,e^{qz} + B\,e^{-qz}.$$

If $\alpha_2 = 0$ *and* $\alpha_3 = q^2$,

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - q^2R = 0, \quad \{14\} \quad R = A\mathcal{J}_0(iqr) + B\mathcal{Y}_0(iqr).$$

$$\frac{d^2\Psi}{d\psi^2} = 0, \quad \{01\} \quad \Psi = A + B\psi.$$

$$\frac{d^2Z}{dz^2} + q^2Z = 0, \quad \{04\} \quad Z = A\sin qz + B\cos qz.$$

If $\alpha_2 = \alpha_3 = 0$,

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} = 0, \quad \{01\} \quad R = A + B\ln r.$$

$$\frac{d^2\Psi}{d\psi^2} = 0, \quad \{01\} \quad \Psi = A + B\psi.$$

$$\frac{d^2Z}{dz^2} = 0, \quad \{01\} \quad Z = A + Bz.$$

For φ independent of z,

$$\begin{cases} \dfrac{d^2R}{dr^2} + \dfrac{1}{r}\dfrac{dR}{dr} - \dfrac{\alpha_2}{r^2}R = 0, \\ \dfrac{d^2\Psi}{d\psi^2} + \alpha_2\Psi = 0. \end{cases}$$

If $\alpha_2 = p^2$,

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{p^2}{r^2}R = 0, \quad \{04\} \quad R = A\,r^p + B\,r^{-p}.$$

$$\frac{d^2\Psi}{d\psi^2} + p^2\Psi = 0, \quad \{04\} \quad \Psi = A\sin p\psi + B\cos p\psi.$$

If $\alpha_2 = 0$,

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} = 0, \quad \{04\} \quad R = A + B\ln r.$$

$$\frac{d^2\Psi}{d\psi^2} = 0, \quad \{01\} \quad \Psi = A + B\psi.$$

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* If p is an integer, the second form must be employed.

For φ independent of ψ,

$$\begin{cases} \dfrac{d^2R}{dr^2} + \dfrac{1}{r}\dfrac{dR}{dr} - \alpha_3 R = 0, \\ \dfrac{d^2Z}{dz^2} + \alpha_3 Z = 0. \end{cases}$$

If $\alpha_3 = q^2$,

$\dfrac{d^2R}{dr^2} + \dfrac{1}{r}\dfrac{dR}{dr} - q^2R = 0,$ {14} $R = A\mathscr{J}_0(iqr) + B\mathscr{Y}_0(iqr)$.

$\dfrac{d^2Z}{dz^2} + q^2Z = 0,$ {04} $Z = A\sin qz + B\cos qz$.

If $\alpha_3 = -q^2$,

$\dfrac{d^2R}{dr^2} + \dfrac{1}{r}\dfrac{dR}{dr} + q^2R = 0,$ {14} $R = A\mathscr{J}_0(qr) + B\mathscr{Y}_0(qr)$.

$\dfrac{d^2Z}{dz^2} - q^2Z = 0,$ {04} $Z = A\,e^{qz} + B\,e^{-qz}$.

If $\alpha_3 = 0$,

$\dfrac{d^2R}{dr^2} + \dfrac{1}{r}\dfrac{dR}{dr} = 0,$ {01} $R = A + B\ln r$.

$\dfrac{d^2Z}{dz^2} = 0,$ {01} $Z = A + Bz$.

For φ independent of r and z,

$\dfrac{d^2\varphi}{d\psi^2} = 0,$ {01} $\varphi = A + B\psi$.

For φ independent of ψ and z,

$\dfrac{d^2\varphi}{dr^2} + \dfrac{1}{r}\dfrac{d\varphi}{dr} = 0,$ {01} $\varphi = A + B\ln r$.

SEPARATION OF HELMHOLTZ EQUATION, $\nabla^2\varphi + \varkappa^2\varphi = 0$.

$$\frac{1}{f_i}\frac{d}{du^i}\left(f_i\frac{dU^i}{du^i}\right) + U^i\sum_{j=1}^{3}\Phi_{ij}\alpha_j = 0,$$

where $\alpha_1 = \varkappa^2$ and $U^1 = R(r)$, $U^2 = \Psi(\psi)$, $U^3 = Z(z)$.

General case

$$\begin{cases} \dfrac{d^2R}{dr^2} + \dfrac{1}{r}\dfrac{dR}{dr} - \left(\dfrac{\alpha_2}{r^2} + \alpha_3\right)R = 0, \\ \dfrac{d^2\Psi}{d\psi^2} + \alpha_2\Psi = 0, \\ \dfrac{d^2Z}{dz^2} + (\varkappa^2 + \alpha_3)Z = 0. \end{cases}$$

If $\alpha_2 = p^2$ *and* $\alpha_3 = q^2$,

$\dfrac{d^2R}{dr^2} + \dfrac{1}{r}\dfrac{dR}{dr} - (q^2 + p^2/r^2)R = 0,$ {24} $R = A\mathscr{J}_p(iqr) + B\mathscr{J}_{-p}(iqr)$

or $R = A\mathscr{J}_n(iqr) + B\mathscr{Y}_n(iqr)$.

$\dfrac{d^2\Psi}{d\psi^2} + p^2\Psi = 0,$ {04} $\Psi = A\sin p\psi + B\cos p\psi$.

$\dfrac{d^2Z}{dz^2} + (\varkappa^2 + q^2)Z = 0.$ {04} $Z = A\sin[(\varkappa^2 + q^2)^{\frac{1}{2}}z]$

$+ B\cos[(\varkappa^2 + q^2)^{\frac{1}{2}}z]$.

If $\alpha_2 = p^2$ *and* $\alpha_3 = -q^2$,

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + (q^2 - p^2/r^2)\,R = 0, \quad \{24\} \quad R = A\mathcal{J}_p(qr) + B\mathcal{J}_{-p}(qr)$$

$$\text{or} \quad R = A\mathcal{J}_n(qr) + B\mathcal{Y}_n(qr).$$

$$\frac{d^2\Psi}{d\psi^2} + p^2\Psi = 0, \quad \{04\} \quad \Psi = A\sin p\psi + B\cos p\psi.$$

$$\frac{d^2Z}{dz^2} + (\varkappa^2 - q^2)\,Z = 0, \quad \{04\} \quad Z = A\sin[(\varkappa^2 - q^2)^{\frac{1}{2}}z] + B\cos[(\varkappa^2 - q^2)^{\frac{1}{2}}z].$$

If $\alpha_2 = 0$ *and* $\alpha_3 = q^2$,

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - q^2R = 0, \quad \{14\} \quad R = A\mathcal{J}_0(iqr) + B\mathcal{Y}_0(iqr).$$

$$\frac{d^2\Psi}{d\psi^2} = 0, \quad \{01\} \quad \Psi = A + B\psi.$$

$$\frac{d^2Z}{dz^2} + (\varkappa^2 + q^2)\,Z = 0, \quad \{04\} \quad Z = A\sin[(\varkappa^2 + q^2)^{\frac{1}{2}}z] + B\cos[(\varkappa^2 + q^2)^{\frac{1}{2}}z].$$

If $\alpha_2 = \alpha_3 = 0$,

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} = 0, \quad \{01\} \quad R = A + B\ln r.$$

$$\frac{d^2\Psi}{d\psi^2} = 0, \quad \{01\} \quad \Psi = A + B\psi.$$

$$\frac{d^2Z}{dz^2} + \varkappa^2 Z = 0, \quad \{04\} \quad Z = A\sin\varkappa z + B\cos\varkappa z.$$

For φ independent of z,

$$\begin{cases} \dfrac{d^2R}{dr^2} + \dfrac{1}{r}\dfrac{dR}{dr} + (\varkappa^2 - \alpha_2/r^2)\,R = 0, \\ \dfrac{d^2\Psi}{d\psi^2} + \alpha_2\Psi = 0. \end{cases}$$

If $\alpha_2 = p^2$,

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + (\varkappa^2 - p^2/r^2)\,R = 0, \quad \{24\} \quad R = A\mathcal{J}_p(\varkappa r) + B\mathcal{J}_{-p}(\varkappa r)$$

$$\text{or} \quad R = A\mathcal{J}_n(\varkappa r) + B\mathcal{Y}_n(\varkappa r).$$

$$\frac{d^2\Psi}{d\psi^2} + p^2\Psi = 0, \quad \{04\} \quad \Psi = A\sin p\psi + B\cos p\psi.$$

If $\alpha_2 = 0$,

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \varkappa^2 R = 0, \quad \{14\} \quad R = A\mathcal{J}_0(\varkappa r) + B\mathcal{Y}_0(\varkappa r).$$

$$\frac{d^2\Psi}{d\psi^2} = 0, \quad \{01\} \quad \Psi = A + B\psi.$$

For φ independent of ψ,

$$\begin{cases} \dfrac{d^2R}{dr^2} + \dfrac{1}{r}\dfrac{dR}{dr} - \alpha_3 R = 0, \\ \dfrac{d^2Z}{dz^2} + (\varkappa^2 + \alpha_3)\,Z = 0. \end{cases}$$

If $\alpha_3 = q^2$,

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - q^2 R = 0, \quad \{14\} \quad R = A\mathcal{J}_0(iqr) + B\mathcal{Y}_0(iqr).$$

$$\frac{d^2Z}{dz^2} + (\varkappa^2 + q^2) Z = 0, \quad \{04\} \quad Z = A\sin[(\varkappa^2+q^2)^{\frac{1}{2}} z] + B\cos[(\varkappa^2+q^2)^{\frac{1}{2}} z].$$

If $\alpha_3 = -q^2$,

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + q^2 R = 0, \quad \{14\} \quad R = A\mathcal{J}_0(qr) + B\mathcal{Y}_0(qr).$$

$$\frac{d^2Z}{dz^2} + (\varkappa^2 - q^2) Z = 0, \quad \{04\} \quad Z = A\sin[(\varkappa^2-q^2)^{\frac{1}{2}} z] + B\cos[(\varkappa^2-q^2)^{\frac{1}{2}} z].$$

If $\alpha_3 = 0$,

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} = 0, \quad \{01\} \quad R = A + B\ln r.$$

$$\frac{d^2Z}{dz^2} + \varkappa^2 Z = 0, \quad \{04\} \quad Z = A\sin\varkappa z + B\cos\varkappa z.$$

For φ independent of ψ and z,

$$\frac{d^2\varphi}{dr^2} + \frac{1}{r}\frac{d\varphi}{dr} + \varkappa^2\varphi = 0, \quad \{14\} \quad \varphi = A\mathcal{J}_0(\varkappa r) + B\mathcal{Y}_0(\varkappa r).$$

TABLE 1.03. ELLIPTIC-CYLINDER COORDINATES (η, ψ, z)

$u^1 = \eta, \quad 0 \leqq \eta < \infty,$

$u^2 = \psi, \quad 0 \leqq \psi < 2\pi,$

$u^3 = z, \quad -\infty < z < +\infty.$

$$\begin{cases} x = a\cosh\eta\cos\psi, \\ y = a\sinh\eta\sin\psi, \\ z = z. \end{cases}$$

Coordinate surfaces are confocal cylinders and planes:

$\left(\frac{x}{a\cosh\eta}\right)^2 + \left(\frac{y}{a\sinh\eta}\right)^2 = 1$ (elliptic cylinders, $\eta = \text{const}$),

$\left(\frac{x}{a\cos\psi}\right)^2 - \left(\frac{y}{a\sin\psi}\right)^2 = 1$ (hyperbolic cylinders, $\psi = \text{const}$),

$z = \text{const}$ (planes).

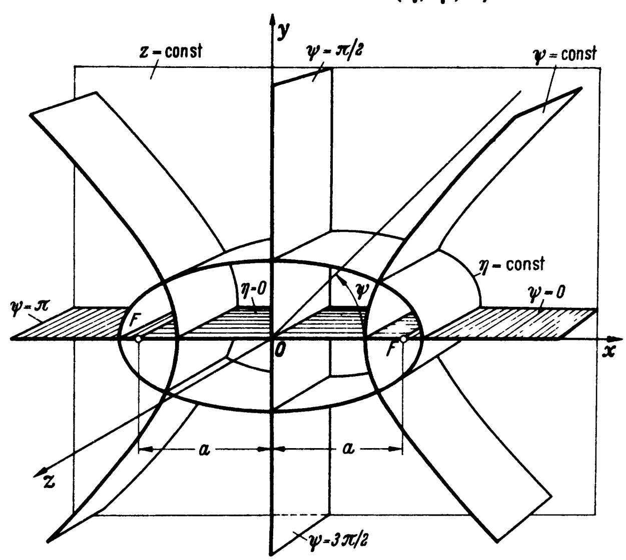

Fig. 1.03. Elliptic-cylinder coordinates (η, ψ, z). Coordinate surfaces are elliptic cylinders ($\eta = \text{const}$), hyperbolic cylinders ($\psi = \text{const}$), and planes ($z = \text{const}$)

The Stäckel matrix may be written

$$[S] = \begin{bmatrix} 0 & -1 & -a^2\cosh^2\eta \\ 0 & 1 & a^2\cos^2\psi \\ 1 & 0 & 1 \end{bmatrix}.$$

$$S = a^2(\cosh^2\eta - \cos^2\psi), \quad M_{11} = M_{21} = 1, \quad M_{31} = a^2(\cosh^2\eta - \cos^2\psi).$$

Metric coefficients

$$g_{11} = g_{22} = a^2(\cosh^2\eta - \cos^2\psi), \quad g_{33} = 1,$$
$$g^{\frac{1}{2}} = a^2(\cosh^2\eta - \cos^2\psi).$$
$$f_1 = f_2 = f_3 = 1.$$

Important equations

$$(ds)^2 = a^2(\cosh^2\eta - \cos^2\psi)\,[(d\eta)^2 + (d\psi)^2] + (dz)^2.$$

$$\operatorname{grad}\varphi = \frac{1}{a\,[\cosh^2\eta - \cos^2\psi]^{\frac{1}{2}}}\left[\boldsymbol{a}_\eta\frac{\partial\varphi}{\partial\eta} + \boldsymbol{a}_\psi\frac{\partial\varphi}{\partial\psi}\right] + \boldsymbol{a}_z\frac{\partial\varphi}{\partial z}.$$

$$\operatorname{div}\boldsymbol{E} = \frac{1}{a\,[\cosh^2\eta - \cos^2\psi]^{\frac{1}{2}}}\left\{\frac{\partial}{\partial\eta}\left[(\cosh^2\eta - \cos^2\psi)^{\frac{1}{2}}E_\eta\right]\right.$$
$$\left. + \frac{\partial}{\partial\psi}\left[(\cosh^2\eta - \cos^2\psi)^{\frac{1}{2}}E_\psi\right]\right\} + \frac{\partial E_z}{\partial z}.$$

$$\operatorname{curl}\boldsymbol{E} = \frac{1}{(\cosh^2\eta - \cos^2\psi)}$$
$$\times\begin{vmatrix} \boldsymbol{a}_\eta(\cosh^2\eta - \cos^2\psi)^{\frac{1}{2}} & \boldsymbol{a}_\psi(\cosh^2\eta - \cos^2\psi)^{\frac{1}{2}} & \boldsymbol{a}_z/a \\ \frac{\partial}{\partial\eta} & \frac{\partial}{\partial\psi} & \frac{\partial}{\partial z} \\ E_\eta(\cosh^2\eta - \cos^2\psi)^{\frac{1}{2}} & E_\psi(\cosh^2\eta - \cos^2\psi)^{\frac{1}{2}} & E_z/a \end{vmatrix}.$$

$$\nabla^2\varphi = \frac{1}{a^2[\cosh^2\eta - \cos^2\psi]}\left[\frac{\partial^2\varphi}{\partial\eta^2} + \frac{\partial^2\varphi}{\partial\psi^2}\right] + \frac{\partial^2\varphi}{\partial z^2}.$$

SEPARATION OF LAPLACE'S EQUATION, $\nabla^2\varphi = 0$.

$$\frac{1}{f_i}\frac{d}{du^i}\left(f_i\frac{dU^i}{du^i}\right) + U^i\sum_{j=1}^{3}\Phi_{ij}\alpha_j = 0,$$

where $\alpha_1 = 0$ and $U^1 = H(\eta)$, $U^2 = \Psi(\psi)$, $U^3 = Z(z)$.

General case

$$\begin{cases} \dfrac{d^2H}{d\eta^2} - (\alpha_2 + \alpha_3 a^2\cosh^2\eta)\,H = 0, \\ \dfrac{d^2\Psi}{d\psi^2} + (\alpha_2 + \alpha_3 a^2\cos^2\psi)\,\Psi = 0, \\ \dfrac{d^2Z}{dz^2} + \alpha_3 Z = 0. \end{cases}$$

If $q = \alpha_3 a^2/4$ *and* $\lambda = \alpha_2 + \alpha_3 a^2/2$,

$$\frac{d^2H}{d\eta^2} - (\lambda + 2q\cosh 2\eta)\,H = 0, \quad \{113\} \quad H = A\,\mathrm{ce}_m(i\eta, -q) + B\,\mathrm{fe}_m(i\eta, -q)$$
$$\text{or} \quad H = A\,\mathrm{se}_m(i\eta, -q) + B\,\mathrm{ge}_m(i\eta, -q).$$

$$\frac{d^2\Psi}{d\psi^2} + (\lambda + 2q\cos 2\psi)\,\Psi = 0, \quad \{113\} \quad \Psi = A\,\mathrm{ce}_m(\psi, -q) + B\,\mathrm{fe}_m(\psi, -q)$$
$$\text{or} \quad \Psi = A\,\mathrm{se}_m(\psi, -q) + B\,\mathrm{ge}_m(\psi, -q).$$

$$\frac{d^2Z}{dz^2} + \frac{4q}{a^2}Z = 0, \quad \{04\} \quad Z = A\sin(2q^{\frac{1}{2}}z/a) + B\cos(2q^{\frac{1}{2}}z/a),$$

where $m = 0, 1, 2, 3, \ldots$

If $q = -\alpha_3 a^2/4$ *and* $\lambda = \alpha_2 + \alpha_3 a^2/2$,

$$\frac{d^2H}{d\eta^2} - (\lambda - 2q\cosh 2\eta)\,H = 0, \quad \{1\,1\,3\} \quad H = A\,\mathrm{ce}_m(i\eta, q) + B\,\mathrm{fe}_m(i\eta, q)$$

$$\text{or} \quad H = A\,\mathrm{se}_m(i\eta, q) + B\,\mathrm{ge}_m(i\eta, q)\,.$$

$$\frac{d^2\Psi}{d\psi^2} + (\lambda - 2q\cos 2\psi)\,\Psi = 0, \quad \{1\,1\,3\} \quad \Psi = A\,\mathrm{ce}_m(\psi, q) + B\,\mathrm{fe}_m(\psi, q)$$

$$\text{or} \quad \Psi = A\,\mathrm{se}_m(\psi, q) + B\,\mathrm{ge}_m(\psi, q)\,.$$

$$\frac{d^2Z}{dz^2} - \frac{4q}{a^2}\,Z = 0, \quad \{0\,4\} \quad Z = A\,e^{2q^{\frac{1}{2}}z/a} + B\,e^{-2q^{\frac{1}{2}}z/a}.$$

For φ independent of z,

$$\begin{cases} \dfrac{d^2H}{d\eta^2} - \alpha_2 H = 0, \\ \dfrac{d^2\Psi}{d\psi^2} + \alpha_2 \Psi = 0. \end{cases}$$

If $\alpha_2 = p^2$,

$$\frac{d^2H}{d\eta^2} - p^2 H = 0, \quad \{0\,4\} \quad H = A\,e^{p\eta} + B\,e^{-p\eta}.$$

$$\frac{d^2\Psi}{d\psi^2} + p^2 \Psi = 0, \quad \{0\,4\} \quad \Psi = A\sin p\psi + B\cos p\psi.$$

For φ independent of η and z,

$$\frac{d^2\varphi}{d\psi^2} = 0, \quad \{0\,1\} \quad \varphi = A + B\psi.$$

For φ independent of ψ and z,

$$\frac{d^2\varphi}{d\eta^2} = 0, \quad \{0\,1\} \quad \varphi = A + B\eta.$$

SEPARATION OF HELMHOLTZ EQUATION, $\nabla^2\varphi + \varkappa^2\varphi = 0$.

$$\frac{1}{f_i}\frac{d}{du^i}\left(f_i \frac{dU^i}{du^i}\right) + U^i \sum_{j=1}^{3} \Phi_{ij}\alpha_j = 0,$$

where $\alpha_1 = \varkappa^2$ and $U^1 = H(\eta)$, $U^2 = \Psi(\psi)$, $U^3 = Z(z)$.

General case

$$\begin{cases} \dfrac{d^2H}{d\eta^2} - (\alpha_2 + \alpha_3 a^2\cosh^2\eta)\,H = 0, \\ \dfrac{d^2\Psi}{d\psi^2} + (\alpha_2 + \alpha_3 a^2\cos^2\psi)\,\Psi = 0, \\ \dfrac{d^2Z}{dz^2} + (\varkappa^2 + \alpha_3)\,Z = 0. \end{cases}$$

If $q = \alpha_3 a^2/4$ *and* $\lambda = \alpha_2 + \alpha_3 a^2/2$,

$$\frac{d^2H}{d\eta^2} - (\lambda + 2q\cosh 2\eta)\,H = 0, \quad \{113\} \quad H = A\,\mathrm{ce}_m(i\eta, -q) + B\,\mathrm{fe}_m(i\eta, -q)$$
$$\text{or} \quad H = A\,\mathrm{se}_m(i\eta, -q) + B\,\mathrm{ge}_m(i\eta, -q).$$

$$\frac{d^2\Psi}{d\psi^2} + (\lambda + 2q\cos 2\psi)\,\Psi = 0, \quad \{113\} \quad \Psi = A\,\mathrm{ce}_m(\psi, -q) + B\,\mathrm{fe}_m(\psi, -q)$$
$$\text{or} \quad \Psi = A\,\mathrm{se}_m(\psi, -q) + B\,\mathrm{ge}_m(\psi, -q).$$

$$\frac{d^2Z}{dz^2} + (\varkappa^2 + 4q/a^2)\,Z = 0, \quad \{04\} \quad Z = A\sin[(\varkappa^2 + 4q/a^2)^{\frac{1}{2}} z] + B\cos[(\varkappa^2 + 4q/a^2)^{\frac{1}{2}} z].$$

If $q = -\alpha_3 a^2/4$ *and* $\lambda = \alpha_2 + \alpha_3 a^2/2$,

$$\frac{d^2H}{d\eta^2} - (\lambda - 2q\cosh 2\eta)\,H = 0, \quad \{113\} \quad H = A\,\mathrm{ce}_m(i\eta, q) + B\,\mathrm{fe}_m(i\eta, q)$$
$$\text{or} \quad H = A\,\mathrm{se}_m(i\eta, q) + B\,\mathrm{ge}_m(i\eta, q).$$

$$\frac{d^2\Psi}{d\psi^2} + (\lambda - 2q\cos 2\psi)\,\Psi = 0, \quad \{113\} \quad \Psi = A\,\mathrm{ce}_m(\psi, q) + B\,\mathrm{fe}_m(\psi, q)$$
$$\text{or} \quad \Psi = A\,\mathrm{se}_m(\psi, q) + B\,\mathrm{ge}_m(\psi, q).$$

$$\frac{d^2Z}{dz^2} + (\varkappa^2 - 4q/a^2)\,Z = 0, \quad \{04\} \quad Z = A\sin[(\varkappa^2 - 4q/a^2)^{\frac{1}{2}} z] + B\cos[(\varkappa^2 - 4q/a^2)^{\frac{1}{2}} z].$$

If $\alpha_2 = \alpha_3 = 0$,

$$\frac{d^2H}{d\eta^2} = 0, \quad \{01\} \quad H = A + B\eta.$$

$$\frac{d^2\Psi}{d\psi^2} = 0, \quad \{01\} \quad \Psi = A + B\psi.$$

$$\frac{d^2Z}{dz^2} + \varkappa^2 Z = 0, \quad \{04\} \quad Z = A\sin\varkappa z + B\cos\varkappa z.$$

For φ independent of z,

$$\begin{cases} \dfrac{d^2H}{d\eta^2} - (\alpha_2 - \varkappa^2 a^2\cosh^2\eta)\,H = 0, \\ \dfrac{d^2\Psi}{d\psi^2} + (\alpha_2 - \varkappa^2 a^2\cos^2\psi)\,\Psi = 0. \end{cases}$$

If $q = \varkappa^2 a^2/4$ *and* $\lambda = \alpha_2 - \varkappa^2 a^2/2$,

$$\frac{d^2H}{d\eta^2} - (\lambda - 2q\cosh 2\eta)\,H = 0, \quad \{113\} \quad H = A\,\mathrm{ce}_m(i\eta, q) + B\,\mathrm{fe}_m(i\eta, q)$$
$$\text{or} \quad H = A\,\mathrm{se}_m(i\eta, q) + B\,\mathrm{ge}_m(i\eta, q).$$

$$\frac{d^2\Psi}{d\psi^2} + (\lambda - 2q\cos 2\psi)\,\Psi = 0, \quad \{113\} \quad \Psi = A\,\mathrm{ce}_m(\psi, q) + B\,\mathrm{fe}_m(\psi, q)$$
$$\text{or} \quad \Psi = A\,\mathrm{se}_m(\psi, q) + B\,\mathrm{ge}_m(\psi, q).$$

If $q = -\varkappa^2 a^2/4$ *and* $\lambda = \alpha_2 - \varkappa^2 a^2/2$,

$$\frac{d^2H}{d\eta^2} - (\lambda + 2q\cosh 2\eta)\,H = 0, \quad \{113\} \quad H = A\,\mathrm{ce}_m(i\eta, -q) + B\,\mathrm{fe}_m(i\eta, -q)$$
$$\text{or} \quad H = A\,\mathrm{se}_m(i\eta, -q) + B\,\mathrm{ge}_m(i\eta, -q).$$

$$\frac{d^2\Psi}{d\psi^2} + (\lambda + 2q\cos 2\psi)\,\Psi = 0, \quad \{113\} \quad \Psi = A\,\mathrm{ce}_m(\psi, -q) + B\,\mathrm{fe}_m(\psi, -q)$$
$$\text{or} \quad \Psi = A\,\mathrm{se}_m(\psi, -q) + B\,\mathrm{ge}_m(\psi, -q).$$

TABLE 1.04. PARABOLIC-CYLINDER COORDINATES (μ, ν, z)

$$u^1=\mu, \qquad 0\leqq\mu<+\infty,$$
$$u^2=\nu, \qquad -\infty<\nu<+\infty,$$
$$u^3=z. \qquad -\infty<z<+\infty.$$

$$\begin{cases} x=\frac{1}{2}(\mu^2-\nu^2), \\ y=\mu\nu, \\ z=z. \end{cases}$$

The coordinate surfaces are confocal parabolic cylinders and planes:

$$\begin{cases} y^2=\mu^2(\mu^2-2x) & \text{(parabolic cylinders, } \mu=\text{const)}, \\ y^2=\nu^2(\nu^2+2x) & \text{(parabolic cylinders, } \nu=\text{const)}, \\ z=\text{const} & \text{(planes)}. \end{cases}$$

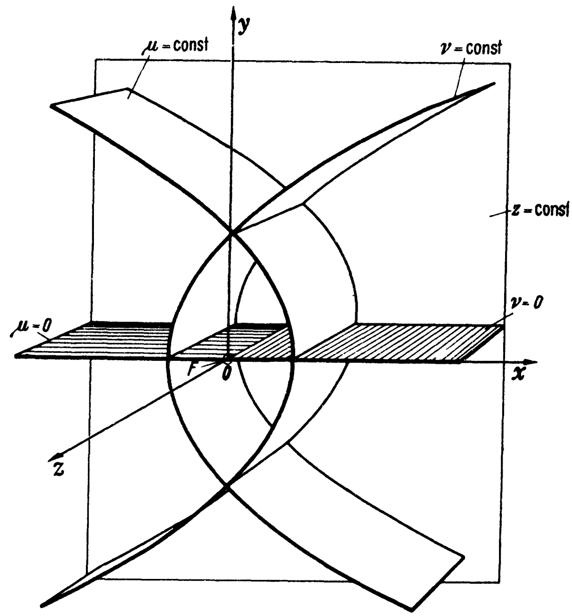

Fig. 1.04. Parabolic-cylinder coordinates (μ, ν, z). Coordinate surfaces are parabolic cylinders (μ = const, ν = const) and planes (z = const)

The Stäckel matrix may be written

$$[S]=\begin{bmatrix} 0 & -1 & -\mu^2 \\ 0 & 1 & -\nu^2 \\ 1 & 0 & 1 \end{bmatrix}.$$

$$S=\mu^2+\nu^2, \qquad M_{11}=M_{21}=1, \qquad M_{31}=\mu^2+\nu^2.$$

Metric coefficients

$$g_{11}=g_{22}=\mu^2+\nu^2, \qquad g_{33}=1, \qquad g^{\frac{1}{2}}=\mu^2+\nu^2.$$
$$f_1=f_2=f_3=1.$$

Important equations

$$(ds)^2=(\mu^2+\nu^2)\left[(d\mu)^2+(d\nu)^2\right]+(dz)^2.$$

$$\operatorname{grad}\varphi=(\mu^2+\nu^2)^{-\frac{1}{2}}\left[\boldsymbol{a}_\mu\frac{\partial\varphi}{\partial\mu}+\boldsymbol{a}_\nu\frac{\partial\varphi}{\partial\nu}\right]+\boldsymbol{a}_z\frac{\partial\varphi}{\partial z}.$$

$$\operatorname{div}\boldsymbol{E}=(\mu^2+\nu^2)^{-1}\left\{\frac{\partial}{\partial\mu}\left[(\mu^2+\nu^2)^{\frac{1}{2}}E_\mu\right]+\frac{\partial}{\partial\nu}\left[(\mu^2+\nu^2)^{\frac{1}{2}}E_\nu\right]\right\}+\frac{\partial E_z}{\partial z}.$$

$$\operatorname{curl}\boldsymbol{E}=(\mu^2+\nu^2)^{-1}\begin{vmatrix} \boldsymbol{a}_\mu(\mu^2+\nu^2)^{\frac{1}{2}} & \boldsymbol{a}_\nu(\mu^2+\nu^2)^{\frac{1}{2}} & \boldsymbol{a}_z \\ \dfrac{\partial}{\partial\mu} & \dfrac{\partial}{\partial\nu} & \dfrac{\partial}{\partial z} \\ E_\mu(\mu^2+\nu^2)^{\frac{1}{2}} & E_\nu(\mu^2+\nu^2)^{\frac{1}{2}} & E_z \end{vmatrix}.$$

$$\nabla^2\varphi=\frac{1}{\mu^2+\nu^2}\left[\frac{\partial^2\varphi}{\partial\mu^2}+\frac{\partial^2\varphi}{\partial\nu^2}\right]+\frac{\partial^2\varphi}{\partial z^2}.$$

SEPARATION OF LAPLACE'S EQUATION, $\nabla^2\varphi = 0$.

$$\frac{1}{f_i}\frac{d}{du^i}\left(f_i\frac{dU^i}{du^i}\right) + U^i\sum_{j=1}^{3}\Phi_{ij}\alpha_j = 0,$$

where $\alpha_1 = 0$ and $U^1 = M(\mu)$, $U^2 = N(\nu)$, $U^3 = Z(z)$.

General case

$$\begin{cases} \dfrac{d^2M}{d\mu^2} - (\alpha_2 + \alpha_3\mu^2)\,M = 0, \\ \dfrac{d^2N}{d\nu^2} + (\alpha_2 - \alpha_3\nu^2)\,N = 0, \\ \dfrac{d^2Z}{dz^2} + \alpha_3 Z = 0. \end{cases}$$

If $\alpha_2 = q^2(p+\frac{1}{2})$ *and* $\alpha_3 = q^4/4$,

$\dfrac{d^2M}{d\mu^2} - [q^2(p+\frac{1}{2}) + q^4\mu^2/4]\,M = 0,$ {06} $M = A\mathscr{W}_e(p, iq\mu) + B\mathscr{W}_o(p, iq\mu)$.

$\dfrac{d^2N}{d\nu^2} + [q^2(p+\frac{1}{2}) - q^4\nu^2/4]\,N = 0,$ {06} $N = A\mathscr{W}_e(p, q\nu) + B\mathscr{W}_o(p, q\nu)$.

$\dfrac{d^2Z}{dz^2} + \dfrac{q^4}{4}Z = 0,$ {04} $Z = A\sin(q^2z/2) + B\cos(q^2z/2)$.

If $\alpha_2 = -q^2(p+\frac{1}{2})$ *and* $\alpha_3 = q^4/4$,

$\dfrac{d^2M}{d\mu^2} + [q^2(p+\frac{1}{2}) - q^4\mu^2/4]\,M = 0,$ {06} $M = A\mathscr{W}_e(p, q\mu) + B\mathscr{W}_o(p, q\mu)$.

$\dfrac{d^2N}{d\nu^2} - [q^2(p+\frac{1}{2}) + q^4\nu^2/4]\,N = 0,$ {06} $N = A\mathscr{W}_e(p, iq\nu) + B\mathscr{W}_o(p, iq\nu)$.

$\dfrac{d^2Z}{dz^2} + \dfrac{q^4}{4}Z = 0,$ {04} $Z = A\sin(q^2z/2) + B\cos(q^2z/2)$.

If $\alpha_2 = 0$ *and* $\alpha_3 = -q^2$,

$\dfrac{d^2M}{d\mu^2} + q^2\mu^2M = 0,$ {06} $M = \mu^{\frac{1}{2}}[A\mathscr{J}_{\frac{1}{4}}(q\mu^2/2) + B\mathscr{J}_{-\frac{1}{4}}(q\mu^2/2)]$.

$\dfrac{d^2N}{d\nu^2} + q^2\nu^2N = 0,$ {06} $N = \nu^{\frac{1}{2}}[A\mathscr{J}_{\frac{1}{4}}(q\nu^2/2) + B\mathscr{J}_{-\frac{1}{4}}(q\nu^2/2)]$.

$\dfrac{d^2Z}{dz^2} - q^2Z = 0,$ {04} $Z = A\,e^{qz} + B\,e^{-qz}$.

For φ independent of z,

$$\begin{cases} \dfrac{d^2M}{d\mu^2} - \alpha_2 M = 0. \\ \dfrac{d^2N}{d\nu^2} + \alpha_2 N = 0. \end{cases}$$

If $\alpha_2 = p^2$,

$\dfrac{d^2M}{d\mu^2} - p^2M = 0,$ {04} $M = A\,e^{p\mu} + B\,e^{-p\mu}$.

$\dfrac{d^2N}{d\nu^2} + p^2N = 0,$ {04} $N = A\sin p\nu + B\cos p\nu$.

If $\alpha_2 = -p^2$,

$\dfrac{d^2M}{d\mu^2} + p^2M = 0,$ {04} $M = A\sin p\mu + B\cos p\mu$.

$\dfrac{d^2N}{d\nu^2} - p^2N = 0,$ {04} $N = A\,e^{p\nu} + B\,e^{-p\nu}$.

For φ independent of μ and z,

$$\frac{d^2\varphi}{d\nu^2}=0, \qquad \{01\} \qquad \varphi = A + B\nu.$$

For φ independent of ν and z,

$$\frac{d^2\varphi}{d\mu^2}=0, \qquad \{01\} \qquad \varphi = A + B\mu.$$

SEPARATION OF HELMHOLTZ EQUATION, $\nabla^2\varphi + \varkappa^2\varphi = 0$.

$$\frac{1}{f_i}\frac{d}{du^i}\left(f_i\frac{dU^i}{du^i}\right) + U^i\sum_{j=1}^{3}\Phi_{ij}\alpha_j = 0,$$

where $\alpha_1 = \varkappa^2$ and $U^1 = M(\mu)$, $U^2 = N(\nu)$, $U^3 = Z(z)$.

General case

$$\begin{cases} \dfrac{d^2M}{d\mu^2} - (\alpha_2 + \alpha_3\mu^2)\,M = 0, \\ \dfrac{d^2N}{d\nu^2} + (\alpha_2 - \alpha_3\nu^2)\,N = 0, \\ \dfrac{d^2Z}{dz^2} + (\varkappa^2 + \alpha_3)\,Z = 0. \end{cases}$$

If $\alpha_2 = q^2(p+\tfrac{1}{2})$ *and* $\alpha_3 = q^4/4$,

$$\frac{d^2M}{d\mu^2} - [q^2(p+\tfrac{1}{2}) + q^4\mu^2/4]\,M = 0, \qquad \{06\} \qquad M = A\mathscr{W}_e(p, iq\mu) + B\mathscr{W}_o(p, iq\mu).$$

$$\frac{d^2N}{d\nu^2} + [q^2(p+\tfrac{1}{2}) - q^4\nu^2/4]\,N = 0, \qquad \{06\} \qquad N = A\mathscr{W}_e(p, q\nu) + B\mathscr{W}_o(p, q\nu).$$

$$\frac{d^2Z}{dz^2} + (\varkappa^2 + q^4/4)\,Z = 0, \qquad \{04\} \qquad Z = A\sin[(\varkappa^2 + q^4/4)^{\frac{1}{2}}z] + B\cos[(\varkappa^2 + q^4/4)^{\frac{1}{2}}z].$$

If $\alpha_2 = -q^2(p+\tfrac{1}{2})$ *and* $\alpha_3 = q^4/4$,

$$\frac{d^2M}{d\mu^2} + [q^2(p+\tfrac{1}{2}) - q^4\mu^2/4]\,M = 0, \qquad \{06\} \qquad M = A\mathscr{W}_e(p, q\mu) + B\mathscr{W}_o(p, q\mu).$$

$$\frac{d^2N}{d\nu^2} - [q^2(p+\tfrac{1}{2}) + q^4\nu^2/4]\,N = 0, \qquad \{06\} \qquad N = A\mathscr{W}_e(p, iq\nu) + B\mathscr{W}_o(p, iq\nu).$$

$$\frac{d^2Z}{dz^2} + (\varkappa^2 + q^4/4)\,Z = 0, \qquad \{04\} \qquad Z = A\sin[(\varkappa^2 + q^4/4)^{\frac{1}{2}}z] + B\cos[(\varkappa^2 + q^4/4)^{\frac{1}{2}}z].$$

If $\alpha_2 = 0$ *and* $\alpha_3 = -q^2$,

$$\frac{d^2M}{d\mu^2} + q^2\mu^2M = 0, \qquad \{06\} \qquad M = \mu^{\frac{1}{2}}[A\mathscr{J}_{\frac{1}{4}}(q\mu^2/2) + B\mathscr{J}_{-\frac{1}{4}}(q\mu^2/2)].$$

$$\frac{d^2N}{d\nu^2} + q^2\nu^2N = 0, \qquad \{06\} \qquad N = \nu^{\frac{1}{2}}[A\mathscr{J}_{\frac{1}{4}}(q\nu^2/2) + B\mathscr{J}_{-\frac{1}{4}}(q\nu^2/2)].$$

$$\frac{d^2Z}{dz^2} + (\varkappa^2 - q^2)\,Z = 0, \qquad \{04\} \qquad Z = A\sin[(\varkappa^2 - q^2)^{\frac{1}{2}}z] + B\cos[(\varkappa^2 - q^2)^{\frac{1}{2}}z].$$

For φ independent of z,

$$\begin{cases} \dfrac{d^2M}{d\mu^2} - (\alpha_2 - \varkappa^2\mu^2)\,M = 0, \\ \dfrac{d^2N}{d\nu^2} + (\alpha_2 + \varkappa^2\nu^2)\,N = 0. \end{cases}$$

If $\alpha_2 = q^2$,

$$\frac{d^2M}{d\mu^2} - (q^2 - \varkappa^2\mu^2)\,M = 0, \quad \{06\} \quad M = \sqrt{i\mu}\,[A\mathscr{S}_{\frac{1}{2}}(\varkappa, q, i\mu) + B\mathscr{S}_{-\frac{1}{2}}(\varkappa, q, i\mu)].$$

$$\frac{d^2N}{d\nu^2} + (q^2 + \varkappa^2\nu^2)\,N = 0, \quad \{06\} \quad N = \nu^{\frac{1}{2}}[A\mathscr{S}_{\frac{1}{2}}(\varkappa, q, \nu) + B\mathscr{S}_{-\frac{1}{2}}(\varkappa, q, \nu)].$$

If $\alpha_2 = -q^2$,

$$\frac{d^2M}{d\mu^2} + (q^2 + \varkappa^2\mu^2)\,M = 0, \quad \{06\} \quad M = \mu^{\frac{1}{2}}[A\mathscr{S}_{\frac{1}{2}}(\varkappa, q, \mu) + B\mathscr{S}_{-\frac{1}{2}}(\varkappa, q, \mu)].$$

$$\frac{d^2N}{d\nu^2} - (q^2 - \varkappa^2\nu^2)\,N = 0, \quad \{06\} \quad N = \sqrt{i\nu}\,[A\mathscr{S}_{\frac{1}{2}}(\varkappa, q, i\nu) + B\mathscr{S}_{-\frac{1}{2}}(\varkappa, q, i\nu)].$$

If $\alpha_2 = 0$,

$$\frac{d^2M}{d\mu^2} + \varkappa^2\mu^2 M = 0, \quad \{06\} \quad M = \mu^{\frac{1}{2}}[A\mathscr{S}_{\frac{1}{4}}(\varkappa\mu^2/2) + B\mathscr{S}_{-\frac{1}{4}}(\varkappa\mu^2/2)].$$

$$\frac{d^2N}{d\nu^2} + \varkappa^2\nu^2 N = 0, \quad \{06\} \quad N = \nu^{\frac{1}{2}}[A\mathscr{S}_{\frac{1}{4}}(\varkappa\nu^2/2) + B\mathscr{S}_{-\frac{1}{4}}(\varkappa\nu^2/2)].$$

TABLE 1.05. SPHERICAL COORDINATES (r, θ, ψ)

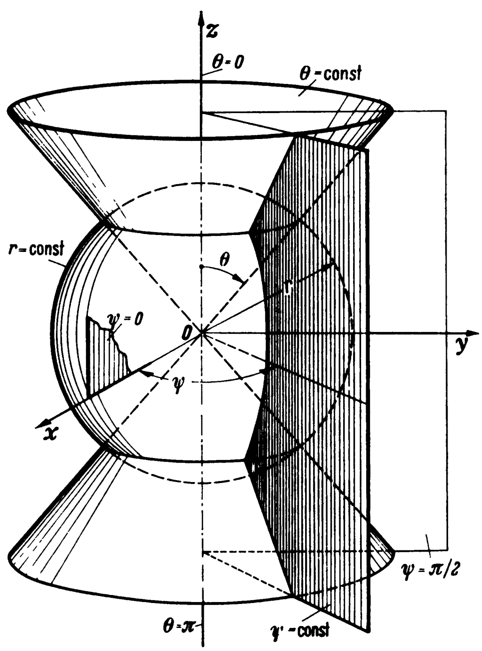

Fig. 1.05. Spherical coordinates (r, θ, ψ). Coordinate surfaces are spheres (r = const), circular cones (θ = const), and half planes (ψ = const)

$$u^1 = r, \quad 0 \leqq r < \infty,$$
$$u^2 = \theta, \quad 0 \leqq \theta \leqq \pi,$$
$$u^3 = \psi, \quad 0 \leqq \psi < 2\pi.$$

$$\begin{cases} x = r\sin\theta\cos\psi, \\ y = r\sin\theta\sin\psi, \\ z = r\cos\theta. \end{cases}$$

Coordinate surfaces are

$$\begin{cases} x^2 + y^2 + z^2 = r^2 \\ \qquad \text{(spheres, } r = \text{const)}, \\ \tan\theta = (x^2 + y^2)^{\frac{1}{2}}/z \\ \qquad \text{(circular cones, } \theta = \text{const)}, \\ \tan\psi = y/x \\ \qquad \text{(half planes, } \psi = \text{const)}. \end{cases}$$

The Stäckel matrix may be written

$$[S] = \begin{bmatrix} 1 & -1/r^2 & 0 \\ 0 & 1 & -1/\sin^2\theta \\ 0 & 0 & 1 \end{bmatrix}.$$

$$S = 1, \quad M_{11} = 1,$$
$$M_{21} = 1/r^2, \quad M_{31} = 1/(r^2\sin^2\theta).$$

Metric coefficients

$$g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = r^2\sin^2\theta, \quad g^{\frac{1}{2}} = r^2\sin\theta.$$
$$f_1 = r^2, \quad f_2 = \sin\theta, \quad f_3 = 1.$$

Addendum to page 25

Spherical Coordinates

$$\Gamma_{ij}^{1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & -r\sin^2\theta \end{bmatrix},$$

$$\Gamma_{ij}^{2} = \begin{bmatrix} 0 & 1/r & 0 \\ 1/r & 0 & 0 \\ 0 & 0 & -\sin\theta\cos\theta \end{bmatrix},$$

$$\Gamma_{ij}^{3} = \begin{bmatrix} 0 & 0 & 1/r \\ 0 & 0 & \operatorname{ctn}\theta \\ 1/r & \operatorname{ctn}\theta & 0 \end{bmatrix}.$$

SEPARATION OF LAPLACE'S EQUATION, $\nabla^2 \varphi = 0$.

$$\frac{1}{f_i}\frac{d}{du^i}\left(f_i\frac{dU^i}{du^i}\right) + U^i \sum_{j=1}^{3} \Phi_{ij}\alpha_j = 0,$$

where $\alpha_1 = 0$ and $U^1 = R(r)$, $U^2 = \Theta(\theta)$, $U^3 = \Psi(\psi)$.

General case

$$\begin{cases} \dfrac{d^2R}{dr^2} + \dfrac{2}{r}\dfrac{dR}{dr} - \dfrac{\alpha_2}{r^2}R = 0, \\ \dfrac{d^2\Theta}{d\theta^2} + \cot\theta\dfrac{d\Theta}{d\theta} + \left(\alpha_2 - \dfrac{\alpha_3}{\sin^2\theta}\right)\Theta = 0, \\ \dfrac{d^2\Psi}{d\psi^2} + \alpha_3\Psi = 0. \end{cases}$$

If $\alpha_2 = p(p+1)$ *and* $\alpha_3 = q^2$,

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{p(p+1)}{r^2}R = 0, \qquad \{22\} \quad R = A\,r^p + B\,r^{-(p+1)}.$$

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta\frac{d\Theta}{d\theta} + \left[p(p+1) - \frac{q^2}{\sin^2\theta}\right]\Theta = 0, \quad \{222\} \quad \Theta = A\mathscr{P}_p^q(\cos\theta) + B\mathscr{Q}_p^q(\cos\theta).$$

$$\frac{d^2\Psi}{d\psi^2} + q^2\Psi = 0, \qquad \{04\} \quad \Psi = A\sin q\psi + B\cos q\psi.$$

For φ independent of ψ,

$$\begin{cases} \dfrac{d^2R}{dr^2} + \dfrac{2}{r}\dfrac{dR}{dr} - \dfrac{\alpha_2}{r^2}R = 0, \\ \dfrac{d^2\Theta}{d\theta^2} + \cot\theta\dfrac{d\Theta}{d\theta} + \alpha_2\Theta = 0. \end{cases}$$

If $\alpha_2 = p(p+1)$,

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{p(p+1)}{r^2}R = 0, \qquad \{22\} \quad R = A\,r^p + B\,r^{-(p+1)}.$$

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta\frac{d\Theta}{d\theta} + p(p+1)\Theta = 0, \quad \{112\} \quad \Theta = A\mathscr{P}_p(\cos\theta) + B\mathscr{Q}_p(\cos\theta).$$

For φ independent of r,

$$\begin{cases} \dfrac{d^2\Theta}{d\theta^2} + \cos\theta\dfrac{d\Theta}{d\theta} - \dfrac{\alpha_3}{\sin^2\theta}\Theta = 0, \\ \dfrac{d^2\Psi}{d\psi^2} + \alpha_3\Psi = 0. \end{cases}$$

If $\alpha_3 = q^2$,

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta\frac{d\Theta}{d\theta} - \frac{q^2}{\sin^2\theta}\Theta = 0, \qquad \{220\} \quad \Theta = A\mathscr{P}_0^q(\cos\theta) + B\mathscr{Q}_0^q(\cos\theta).$$

$$\frac{d^2\Psi}{d\psi^2} + q^2\Psi = 0, \qquad \{04\} \quad \Psi = A\sin q\psi + B\cos q\psi.$$

For φ independent of θ and ψ,

$$\frac{d^2\varphi}{dr^2} + \frac{2}{r}\frac{d\varphi}{dr} = 0, \qquad \{01\} \quad \varphi = A + B r^{-1}.$$

For φ independent of r and ψ,

$$\frac{d^2\varphi}{d\theta^2} + \cot\theta\frac{d\varphi}{d\theta} = 0, \qquad \{01\} \quad \varphi = A + B\ln\cot(\theta/2).$$

SEPARATION OF HELMHOLTZ EQUATION, $\nabla^2\varphi + \varkappa^2\varphi = 0$.

$$\frac{1}{f_i}\frac{d}{du^i}\left(f_i\frac{dU^i}{du^i}\right) + U^i\sum_{j=1}^{3}\Phi_{ij}\alpha_j = 0,$$

where $\alpha_1 = \varkappa^2$ and $U^1 = R(r)$, $U^2 = \Theta(\theta)$, $U^3 = \Psi(\psi)$.

General case

$$\begin{cases} \dfrac{d^2R}{dr^2} + \dfrac{2}{r}\dfrac{dR}{dr} + (\varkappa^2 - \alpha_2/r^2)R = 0, \\ \dfrac{d^2\Theta}{d\theta^2} + \cot\theta\dfrac{d\Theta}{d\theta} + \left(\alpha_2 - \dfrac{\alpha_3}{\sin^2\theta}\right)\Theta = 0, \\ \dfrac{d^2\Psi}{d\psi^2} + \alpha_3\Psi = 0. \end{cases}$$

If $\alpha_2 = p(p+1)$ and $\alpha_3 = q^2$,

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left[\varkappa^2 - \frac{p(p+1)}{r^2}\right]R = 0,$$

$$\{24\} \quad R = r^{-\frac{1}{2}}\left[A\mathscr{J}_{p+\frac{1}{2}}(\varkappa r) + B\mathscr{J}_{-(p+\frac{1}{2})}(\varkappa r)\right].$$

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta\frac{d\Theta}{d\theta} + \left[p(p+1) - \frac{q^2}{\sin^2\theta}\right]\Theta = 0,$$

$$\{222\} \quad \Theta = A\mathscr{P}_p^q(\cos\theta) + B\mathscr{Q}_p^q(\cos\theta).$$

$$\frac{d^2\Psi}{d\psi^2} + q^2\Psi = 0, \qquad \{04\} \quad \Psi = A\sin q\psi + B\cos q\psi.$$

For φ independent of ψ,

$$\begin{cases} \dfrac{d^2R}{dr^2} + \dfrac{2}{r}\dfrac{dR}{dr} + (\varkappa^2 - \alpha_2/r^2)R = 0, \\ \dfrac{d^2\Theta}{d\theta^2} + \cot\theta\dfrac{d\Theta}{d\theta} + \alpha_2\Theta = 0. \end{cases}$$

If $\alpha_2 = p(p+1)$,

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left[\varkappa^2 - \frac{p(p+1)}{r^2}\right]R = 0,$$

$$\{24\} \quad R = r^{-\frac{1}{2}}\left[A\mathscr{J}_{p+\frac{1}{2}}(\varkappa r) + B\mathscr{J}_{-(p+\frac{1}{2})}(\varkappa r)\right].$$

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta\frac{d\Theta}{d\theta} + p(p+1)\Theta = 0,$$

$$\{112\} \quad \Theta = A\mathscr{P}_p(\cos\theta) + B\mathscr{Q}_p(\cos\theta).$$

For φ independent of θ and ψ,

$$\frac{d^2\varphi}{dr^2} + \frac{2}{r}\frac{d\varphi}{dr} + \varkappa^2\varphi = 0, \qquad \{04\} \quad \varphi = \frac{1}{r}\left[A\sin\varkappa r + B\cos\varkappa r\right].$$

TABLE 1.06. PROLATE SPHEROIDAL COORDINATES (η, θ, ψ)

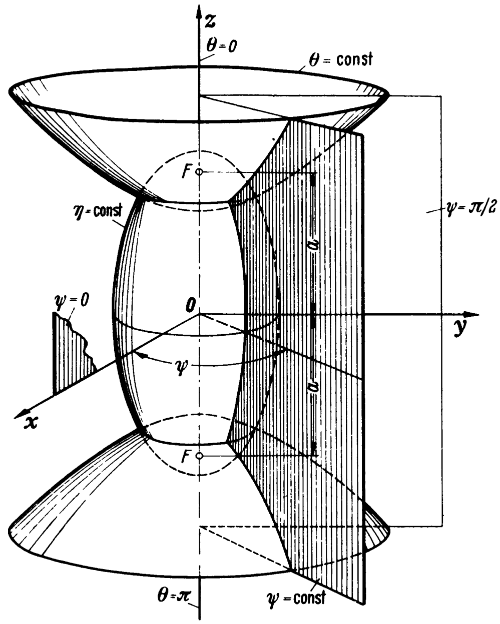

Fig. 1.06. Prolate spheroidal coordinates (η, θ, ψ). Coordinate surfaces are prolate spheroids ($\eta = \text{const}$), hyperboloids of revolution ($\theta = \text{const}$), and half-planes ($\psi = \text{const}$)

$$u^1 = \eta, \quad 0 \leqq \eta < \infty,$$
$$u^2 = \theta, \quad 0 \leqq \theta \leqq \pi,$$
$$u^3 = \psi, \quad 0 \leqq \psi < 2\pi.$$

$$\begin{cases} x = a \sinh\eta \sin\theta \cos\psi, \\ y = a \sinh\eta \sin\theta \sin\psi, \\ z = a \cosh\eta \cos\theta. \end{cases}$$

Coordinate surfaces are

$$\frac{x^2}{a^2 \sinh^2\eta} + \frac{y^2}{a^2 \sinh^2\eta} + \frac{z^2}{a^2 \cosh^2\eta} = 1$$

(prolate spheroids, $\eta = \text{const}$),

$$\frac{x^2}{a^2 \sin^2\theta} - \frac{y^2}{a^2 \sin^2\theta} + \frac{z^2}{a^2 \cos^2\theta} = 1$$

(hyperboloids of two sheets, $\theta = \text{const}$),

$$\tan\psi = y/x$$

(half planes, $\psi = \text{const}$).

The Stäckel matrix may be written

$$[S] = \begin{bmatrix} a^2 \sinh^2\eta & -1 & -1/\sinh^2\eta \\ a^2 \sin^2\theta & 1 & -1/\sin^2\theta \\ 0 & 0 & 1 \end{bmatrix}$$

$$S = a^2 (\sinh^2\eta + \sin^2\theta) = a^2 (\cosh^2\eta - \cos^2\theta),$$

$$M_{11} = M_{21} = 1, \quad M_{31} = \frac{\sinh^2\eta + \sin^2\theta}{\sinh^2\eta \sin^2\theta}.$$

Metric coefficients

$$g_{11} = g_{22} = a^2 (\sinh^2\eta + \sin^2\theta), \quad g_{33} = a^2 \sinh^2\eta \sin^2\theta,$$

$$g^{\frac{1}{2}} = a^3 (\sinh^2\eta + \sin^2\theta) \sinh\eta \sin\theta.$$

$$f_1 = \sinh\eta, \quad f_2 = \sin\theta, \quad f_3 = a.$$

Important equations

$$(ds)^2 = a^2 (\sinh^2\eta + \sin^2\theta) [(d\eta)^2 + (d\theta)^2] + a^2 \sinh^2\eta \sin^2\theta (d\psi)^2.$$

$$\operatorname{grad}\varphi = \frac{1}{a (\sinh^2\eta + \sin^2\theta)^{\frac{1}{2}}} \left[\boldsymbol{a}_\eta \frac{\partial\varphi}{\partial\eta} + \boldsymbol{a}_\theta \frac{\partial\varphi}{\partial\theta} \right] + \frac{\boldsymbol{a}_\psi}{a \sinh\eta \sin\theta} \frac{\partial\varphi}{\partial\psi}.$$

$$\operatorname{div} \boldsymbol{E} = \frac{1}{a(\sinh^2\eta + \sin^2\theta)} \left\{ \frac{1}{\sinh\eta} \frac{\partial}{\partial\eta} [(\sinh^2\eta + \sin^2\theta)^{\frac{1}{2}} \sinh\eta\, E_\eta] + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} [(\sinh^2\eta + \sin^2\theta)^{\frac{1}{2}} \sin\theta\, E_\theta] \right\} + \frac{1}{a \sinh\eta \sin\theta} \frac{\partial E_\psi}{\partial\psi}.$$

$$\operatorname{curl} \boldsymbol{E} = \frac{1}{a(\sinh^2\eta + \sin^2\theta)\sinh\eta\sin\theta} \times \begin{vmatrix} \boldsymbol{a}_\eta (\sinh^2\eta + \sin^2\theta)^{\frac{1}{2}} & \boldsymbol{a}_\theta (\sinh^2\eta + \sin^2\theta)^{\frac{1}{2}} & \boldsymbol{a}_\psi \sinh\eta \sin\theta \\ \dfrac{\partial}{\partial\eta} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\psi} \\ E_\eta (\sinh^2\eta + \sin^2\theta)^{\frac{1}{2}} & E_\theta (\sinh^2\eta + \sin^2\theta)^{\frac{1}{2}} & E_\psi \sinh\eta\sin\theta \end{vmatrix}.$$

$$\nabla^2\varphi = \frac{1}{a^2(\sinh^2\eta + \sin^2\theta)} \left\{ \frac{\partial^2\varphi}{\partial\eta^2} + \coth\eta \frac{\partial\varphi}{\partial\eta} + \frac{\partial^2\varphi}{\partial\theta^2} + \cot\theta \frac{\partial\varphi}{\partial\theta} \right\} + \frac{1}{a^2\sinh^2\eta\sin^2\theta} \frac{\partial^2\varphi}{\partial\psi^2}.$$

SEPARATION OF LAPLACE'S EQUATION, $\nabla^2\varphi = 0$.

$$\frac{1}{f_i} \frac{d}{du^i}\left(f_i \frac{dU^i}{du^i}\right) + U^i \sum_{j=1}^{3} \Phi_{ij}\alpha_j = 0,$$

where $\alpha_1 = 0$ and $U^1 = H(\eta)$, $U^2 = \Theta(\theta)$, $U^3 = \Psi(\psi)$.

General case

$$\begin{cases} \dfrac{d^2H}{d\eta^2} + \coth\eta \dfrac{dH}{d\eta} - \left(\alpha_2 + \dfrac{\alpha_3}{\sinh^2\eta}\right) H = 0, \\ \dfrac{d^2\Theta}{d\theta^2} + \cot\theta \dfrac{d\Theta}{d\theta} + \left(\alpha_2 - \dfrac{\alpha_3}{\sin^2\theta}\right)\Theta = 0, \\ \dfrac{d^2\Psi}{d\psi^2} + \alpha_3\Psi = 0. \end{cases}$$

If $\alpha_2 = p(p+1)$ *and* $\alpha_3 = q^2$,

$$\frac{d^2H}{d\eta^2} + \coth\eta\frac{dH}{d\eta} - \left[p(p+1) + \frac{q^2}{\sinh^2\eta}\right] H = 0,$$

$\{222\}$ $H = A\mathscr{P}_p^q(\cosh\eta) + B\mathscr{Q}_p^q(\cosh\eta)$.

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta\frac{d\Theta}{d\theta} + \left[p(p+1) - \frac{q^2}{\sin^2\theta}\right]\Theta = 0,$$

$\{222\}$ $\Theta = A\mathscr{P}_p^q(\cos\theta) + B\mathscr{Q}_p^q(\cos\theta)$.

$\dfrac{d^2\Psi}{d\psi^2} + q^2\Psi = 0,$ $\{04\}$ $\Psi = A\sin q\psi + B\cos q\psi$.

For φ independent of ψ,

$$\begin{cases} \dfrac{d^2H}{d\eta^2} + \coth\eta\dfrac{dH}{d\eta} - \alpha_2 H = 0, \\ \dfrac{d^2\Theta}{d\theta^2} + \cot\theta\dfrac{d\Theta}{d\theta} + \alpha_2\Theta = 0. \end{cases}$$

If $\alpha_2 = p(p+1)$,

$\dfrac{d^2H}{d\eta^2} + \coth\eta\dfrac{dH}{d\eta} - p(p+1)H = 0,$ $\{112\}$ $H = A\mathscr{P}_p(\cosh\eta) + B\mathscr{Q}_p(\cosh\eta)$.

$\dfrac{d^2\Theta}{d\theta^2} + \cot\theta\dfrac{d\Theta}{d\theta} + p(p+1)\Theta = 0,$ $\{112\}$ $\Theta = A\mathscr{P}_p(\cos\theta) + B\mathscr{Q}_p(\cos\theta)$.

For φ independent of θ and ψ,

$$\frac{d^2\varphi}{d\eta^2} + \coth\eta \frac{d\varphi}{d\eta} = 0, \qquad \{01\} \quad \varphi = A + B \ln\coth(\eta/2) = A + C \ln\tanh(\eta/2).$$

For φ independent of η and ψ,

$$\frac{d^2\varphi}{d\theta^2} + \cot\theta \frac{d\varphi}{d\theta} = 0, \qquad \{01\} \quad \varphi = A + B \ln\cot(\theta/2) = A + C \ln\tan(\theta/2).$$

SEPARATION OF HELMHOLTZ EQUATION, $\nabla^2\varphi + \varkappa^2\varphi = 0$.

$$\frac{1}{f_i}\frac{d}{du^i}\left(f_i \frac{dU^i}{du^i}\right) + U^i \sum_{j=1}^{3} \Phi_{ij}\alpha_j = 0,$$

where $\alpha_1 = \varkappa^2$ and $U^1 = H(\eta)$, $U^2 = \Theta(\theta)$, $U^3 = \Psi(\psi)$.

General case

$$\begin{cases} \dfrac{d^2H}{d\eta^2} + \coth\eta \dfrac{dH}{d\eta} + \left(\varkappa^2 a^2 \sinh^2\eta - \alpha_2 - \dfrac{\alpha_3}{\sinh^2\eta}\right) H = 0, \\ \dfrac{d^2\Theta}{d\theta^2} + \cot\theta \dfrac{d\Theta}{d\theta} + \left(\varkappa^2 a^2 \sin^2\theta + \alpha_2 - \dfrac{\alpha_3}{\sin^2\theta}\right)\Theta = 0, \\ \dfrac{d^2\Psi}{d\psi^2} + \alpha_3 \Psi = 0. \end{cases}$$

If $\alpha_2 = p(p+1)$ *and* $\alpha_3 = q^2$,

$$\frac{d^2H}{d\eta^2} + \coth\eta \frac{dH}{d\eta} + \left[\varkappa^2 a^2 \sinh^2\eta - p(p+1) - \frac{q^2}{\sinh^2\eta}\right] H = 0,$$

$$\{224\} \quad H = A\mathscr{P}_p^q(\varkappa a, \cosh\eta) + B\mathscr{Q}_p^q(\varkappa a, \cosh\eta),$$

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta \frac{d\Theta}{d\theta} + \left[\varkappa^2 a^2 \sin^2\theta + p(p+1) - \frac{q^2}{\sin^2\theta}\right]\Theta = 0,$$

$$\{224\} \quad \Theta = A\mathscr{P}_p^q(\varkappa a, \cos\theta) + B\mathscr{Q}_p^q(\varkappa a, \cos\theta).$$

$$\frac{d^2\Psi}{d\psi^2} + q^2\Psi = 0, \qquad \{04\} \quad \Psi = A \sin q\psi + B \cos q\psi.$$

For φ independent of ψ,

$$\begin{cases} \dfrac{d^2H}{d\eta^2} + \coth\eta \dfrac{dH}{d\eta} + (\varkappa^2 a^2 \sinh^2\eta - \alpha_2) H = 0, \\ \dfrac{d^2\Theta}{d\theta^2} - \cot\theta \dfrac{d\Theta}{d\theta} + (\varkappa^2 a^2 \sin^2\theta + \alpha_2)\Theta = 0. \end{cases}$$

If $\alpha_2 = p(p+1)$,

$$\frac{d^2H}{d\eta^2} + \coth\eta \frac{dH}{d\eta} + [\varkappa^2 a^2 \sinh^2\eta - p(p+1)] H = 0,$$

$$\{114\} \quad H = A\mathscr{P}_p(\varkappa a, \cosh\eta) + B\mathscr{Q}_p(\varkappa a, \cosh\eta).$$

$$\frac{d^2\Theta}{d\theta^2} - \cot\theta \frac{d\Theta}{d\theta} + [\varkappa^2 a^2 \sin^2\theta + p(p+1)]\Theta = 0,$$

$$\{114\} \quad \Theta = A\mathscr{P}_p(\varkappa a, \cos\theta) + B\mathscr{Q}_p(\varkappa a, \cos\theta).$$

TABLE 1.07. OBLATE SPHEROIDAL COORDINATES (η, θ, ψ)

$$u^1 = \eta, \quad 0 \leqq \eta < \infty,$$
$$u^2 = \theta, \quad 0 \leqq \theta \leqq \pi,$$
$$u^3 = \psi, \quad 0 \leqq \psi < 2\pi.$$

$$\begin{cases} x = a\cosh\eta\sin\theta\cos\psi, \\ y = a\cosh\eta\sin\theta\sin\psi, \\ z = a\sinh\eta\cos\theta. \end{cases}$$

Coordinate surfaces are

$$\begin{cases} \dfrac{x^2}{a^2\cosh^2\eta} + \dfrac{y^2}{a^2\cosh^2\eta} + \dfrac{z^2}{a^2\sinh^2\eta} = 1 \\ \quad \text{(oblate spheroids, } \eta = \text{const)}, \\ \dfrac{x^2}{a^2\sin^2\theta} + \dfrac{y^2}{a^2\sin^2\theta} - \dfrac{z^2}{a^2\cos^2\theta} = 1 \\ \quad \text{(hyperboloids of one sheet, } \theta = \text{const)}, \\ \tan\psi = y/x \\ \quad \text{(half planes, } \psi = \text{const)}. \end{cases}$$

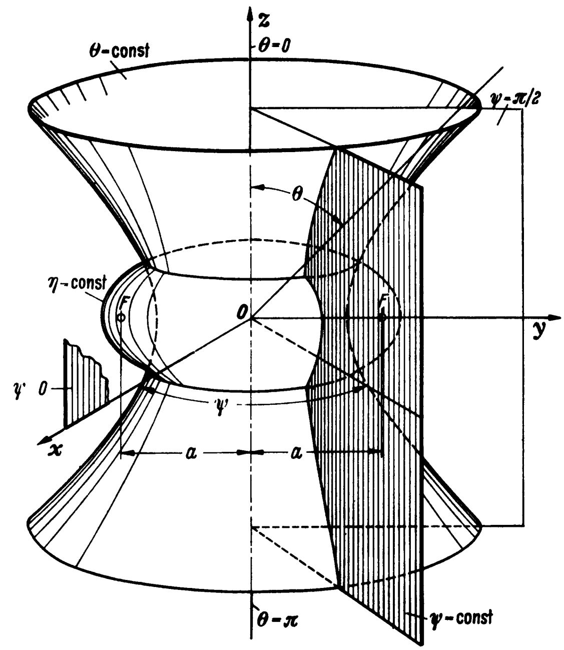

Fig. 1.07. Oblate spheroidal coordinates (η, θ, ψ). Coordinate surfaces are oblate spheroids (η = const), hyperboloids of revolution (θ = const), and half-planes (ψ = const)

The Stäckel matrix may be written

$$[S] = \begin{bmatrix} a^2\cosh^2\eta & -1 & 1/\cosh^2\eta \\ -a^2\sin^2\theta & 1 & -1/\sin^2\theta \\ 0 & 0 & 1 \end{bmatrix}.$$

$$S = a^2(\cosh^2\eta - \sin^2\theta) = a^2(\sinh^2\eta + \cos^2\theta),$$

$$M_{11} = M_{21} = 1, \quad M_{31} = \frac{\cosh^2\eta - \sin^2\theta}{\cosh^2\eta\sin^2\theta}.$$

Metric coefficients

$$g_{11} = g_{22} = a^2(\cosh^2\eta - \sin^2\theta), \quad g_{33} = a^2\cosh^2\eta\sin^2\theta,$$

$$g^{\frac{1}{2}} = a^3(\cosh^2\eta - \sin^2\theta)\cosh\eta\sin\theta.$$

$$f_1 = \cosh\eta, \quad f_2 = \sin\theta, \quad f_3 = a.$$

Important equations

$$(ds)^2 = a^2(\cosh^2\eta - \sin^2\theta)\,[(d\eta)^2 + (d\theta)^2] + a^2\cosh^2\eta\sin^2\theta\,(d\psi)^2.$$

$$\operatorname{grad}\varphi = \frac{1}{a(\cosh^2\eta - \sin^2\theta)^{\frac{1}{2}}}\left[\boldsymbol{a}_\eta\frac{\partial\varphi}{\partial\eta} + \boldsymbol{a}_\theta\frac{\partial\varphi}{\partial\theta}\right] + \boldsymbol{a}_\psi\frac{1}{a\cosh\eta\sin\theta}\frac{\partial\varphi}{\partial\psi}.$$

$$\operatorname{div} \boldsymbol{E} = \frac{1}{a(\cosh^2\eta - \sin^2\theta)} \left\{ \frac{1}{\cosh\eta} \frac{\partial}{\partial\eta} [(\cosh^2\eta - \sin^2\theta)^{\frac{1}{2}} \cosh\eta\, E_\eta] + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} [(\cosh^2\eta - \sin^2\theta)^{\frac{1}{2}} \sin\theta\, E_\theta] \right\} + \frac{1}{a\cosh\eta\sin\theta} \frac{\partial E_\psi}{\partial\psi}.$$

$$\operatorname{curl} \boldsymbol{E} = \frac{1}{a(\cosh^2\eta - \sin^2\theta)\cosh\eta\sin\theta} \times \begin{vmatrix} \boldsymbol{a}_\eta (\cosh^2\eta - \sin^2\theta)^{\frac{1}{2}} & \boldsymbol{a}_\theta (\cosh^2\eta - \sin^2\theta)^{\frac{1}{2}} & \boldsymbol{a}_\psi \cosh\eta\sin\theta \\ \dfrac{\partial}{\partial\eta} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\psi} \\ E_\eta (\cosh^2\eta - \sin^2\theta)^{\frac{1}{2}} & E_\theta (\cosh^2\eta - \sin^2\theta)^{\frac{1}{2}} & E_\psi \cosh\eta\sin\theta \end{vmatrix}.$$

$$\nabla^2\varphi = \frac{1}{a^2(\cosh^2\eta - \sin^2\theta)} \left\{ \frac{\partial^2\varphi}{\partial\eta^2} + \tanh\eta \frac{\partial\varphi}{\partial\eta} + \frac{\partial^2\varphi}{\partial\theta^2} + \cot\theta \frac{\partial\varphi}{\partial\theta} \right\} + \frac{1}{a^2\cosh^2\eta\sin^2\theta} \frac{\partial^2\varphi}{\partial\psi^2}.$$

SEPARATION OF LAPLACE'S EQUATION, $\nabla^2\varphi = 0$.

$$\frac{1}{f_i} \frac{d}{du^i} \left(f_i \frac{dU^i}{du^i} \right) + U^i \sum_{j=1}^{3} \Phi_{ij} \alpha_j = 0,$$

where $\alpha_1 = 0$ and $U^1 = H(\eta)$, $U^2 = \Theta(\theta)$, $U^3 = \Psi(\psi)$.

General case

$$\begin{cases} \dfrac{d^2H}{d\eta^2} + \tanh\eta \dfrac{dH}{d\eta} + \left(-\alpha_2 + \dfrac{\alpha_3}{\cosh^2\eta} \right) H = 0, \\ \dfrac{d^2\Theta}{d\theta^2} + \cot\theta \dfrac{d\Theta}{d\theta} + \left(\alpha_2 - \dfrac{\alpha_3}{\sin^2\theta} \right) \Theta = 0, \\ \dfrac{d^2\Psi}{d\psi^2} + \alpha_3 \Psi = 0. \end{cases}$$

If $\alpha_2 = p(p+1)$ and $\alpha_3 = q^2$,

$$\frac{d^2H}{d\eta^2} + \tanh\eta \frac{dH}{d\eta} + \left[-p(p+1) + \frac{q^2}{\cosh^2\eta} \right] H = 0,$$

$\{222\}$ $H = A\mathscr{P}_p^q(i\sinh\eta) + B\mathscr{Q}_p^q(i\sinh\eta)$.

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta \frac{d\Theta}{d\theta} + \left[p(p+1) - \frac{q^2}{\sin^2\theta} \right] \Theta = 0,$$

$\{222\}$ $\Theta = A\mathscr{P}_p^q(\cos\theta) + B\mathscr{Q}_p^q(\cos\theta)$.

$$\frac{d^2\Psi}{d\psi^2} + q^2\Psi = 0,$$

$\{04\}$ $\Psi = A\sin q\psi + B\cos q\psi$.

For φ independent of ψ,

$$\begin{cases} \dfrac{d^2H}{d\eta^2} + \tanh\eta \dfrac{dH}{d\eta} - \alpha_2 H = 0, \\ \dfrac{d^2\Theta}{d\theta^2} + \cot\theta \dfrac{d\Theta}{d\theta} + \alpha_2 \Theta = 0. \end{cases}$$

If $\alpha_2 = p(p+1)$,

$$\frac{d^2H}{d\eta^2} + \tanh\eta\frac{dH}{d\eta} - p(p+1)H = 0, \quad \{112\} \quad H = A\mathscr{P}_p(i\sinh\eta) + B\mathscr{Q}_p(i\sinh\eta).$$

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta\frac{d\Theta}{d\theta} + p(p+1)\Theta = 0, \quad \{112\} \quad \Theta = A\mathscr{P}_p(\cos\theta) + B\mathscr{Q}_p(\cos\theta).$$

For φ independent of θ and ψ,

$$\frac{d^2\varphi}{d\eta^2} + \tanh\eta\frac{d\varphi}{d\eta} = 0, \quad \{01\} \quad \varphi = A + B\cot^{-1}(\sinh\eta)$$

$$\text{or} \quad \varphi = C + D\tan^{-1}(\sinh\eta).$$

For φ independent of η and ψ,

$$\frac{d^2\varphi}{d\theta^2} + \cot\theta\frac{d\varphi}{d\theta} = 0, \quad \{01\} \quad \varphi = A + B\ln\cot(\theta/2).$$

SEPARATION OF HELMHOLTZ EQUATION, $\nabla^2\varphi + \varkappa^2\varphi = 0$.

$$\frac{1}{f_i}\frac{d}{du^i}\left(f_i\frac{dU^i}{du^i}\right) + U^i\sum_{j=1}^{3}\Phi_{ij}\alpha_j = 0,$$

where $\alpha_1 = \varkappa^2$ and $U^1 = H(\eta)$, $U^2 = \Theta(\theta)$, $U^3 = \Psi(\psi)$.

General case

$$\begin{cases} \dfrac{d^2H}{d\eta^2} + \tanh\eta\dfrac{dH}{d\eta} + \left(\varkappa^2a^2\cosh^2\eta - \alpha_2 + \dfrac{\alpha_3}{\cosh^2\eta}\right)H = 0, \\ \dfrac{d^2\Theta}{d\theta^2} + \cot\theta\dfrac{d\Theta}{d\theta} + \left(-\varkappa^2a^2\sin^2\theta + \alpha_2 - \dfrac{\alpha_3}{\sin^2\theta}\right)\Theta = 0, \\ \dfrac{d^2\Psi}{d\psi^2} + \alpha_3\Psi = 0. \end{cases}$$

If $\alpha_2 = p(p+1)$ *and* $\alpha_3 = q^2$,

$$\frac{d^2H}{d\eta^2} + \tanh\eta\frac{dH}{d\eta} + \left[\varkappa^2a^2\cosh^2\eta - p(p+1) + \frac{q^2}{\cosh^2\eta}\right]H = 0,$$

$$\{224\} \quad H = A\mathscr{P}_p^q(i\varkappa a, i\sinh\eta) + B\mathscr{Q}_p^q(i\varkappa a, i\sinh\eta).$$

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta\frac{d\Theta}{d\theta} + \left[-\varkappa^2a^2\sin^2\theta + p(p+1) - \frac{q^2}{\sin^2\theta}\right]\Theta = 0,$$

$$\{224\} \quad \Theta = A\mathscr{P}_p^q(i\varkappa a, \cos\theta) + B\mathscr{Q}_p^q(i\varkappa a, \cos\theta).$$

$$\frac{d^2\Psi}{d\psi^2} + q^2\Psi = 0, \quad \{04\} \quad \Psi = A\sin q\psi + B\cos q\psi.$$

For φ independent of ψ,

$$\begin{cases} \dfrac{d^2H}{d\eta^2} + \tanh\eta\dfrac{dH}{d\eta} + (\varkappa^2a^2\cosh^2\eta - \alpha_2)H = 0, \\ \dfrac{d^2\Theta}{d\theta^2} + \cot\theta\dfrac{d\Theta}{d\theta} + (-\varkappa^2a^2\sin^2\theta + \alpha_2)\Theta = 0. \end{cases}$$

$$If\ \alpha_2 = p(p+1),$$

$$\frac{d^2H}{d\eta^2} + \tanh\eta \frac{dH}{d\eta} + [\varkappa^2 a^2 \cosh^2\eta - p(p+1)]\, H = 0,$$

$$\{1\,1\,4\} \quad H = A\mathscr{P}_p(i\varkappa a, i\sinh\eta) + B\mathscr{Q}_p(i\varkappa a, i\sinh\eta).$$

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta \frac{d\Theta}{d\theta} + [-\varkappa^2 a^2 \sin^2\theta + p(p+1)]\,\Theta = 0,$$

$$\{1\,1\,4\} \quad \Theta = A\mathscr{P}_p(i\varkappa a, \cos\theta) + B\mathscr{Q}_p(i\varkappa a, \cos\theta).$$

TABLE 1.08. PARABOLIC COORDINATES (μ, ν, ψ)

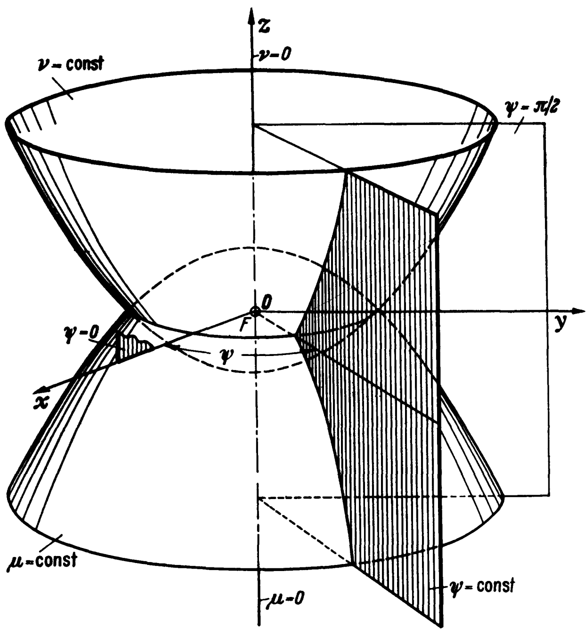

Fig. 1.08. Parabolic coordinates (μ, ν, ψ). Coordinate surfaces are paraboloids of revolution (μ = const, ν = const) and half-planes (ψ = const)

$$u^1 = \mu, \quad 0 \leqq \mu < \infty,$$
$$u^2 = \nu, \quad 0 \leqq \nu < \infty,$$
$$u^3 = \psi, \quad 0 \leqq \psi < 2\pi.$$

$$\begin{cases} x = \mu\nu\cos\psi, \\ y = \mu\nu\sin\psi, \\ z = \tfrac{1}{2}(\mu^2 - \nu^2). \end{cases}$$

The coordinate surfaces are

$x^2 + y^2 = \mu^2(\mu^2 - 2z)$ (paraboloids of revolution, μ = const),

$x^2 + y^2 = \nu^2(\nu^2 + 2z)$ (paraboloids of revolution, ν = const),

$\tan\psi = y/x$ (half planes, ψ = const).

The Stäckel matrix may be written

$$[S] = \begin{bmatrix} \mu^2 & -1 & -1/\mu^2 \\ \nu^2 & 1 & -1/\nu^2 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$S = \mu^2 + \nu^2, \quad M_{11} = M_{21} = 1, \quad M_{31} = \frac{\mu^2 + \nu^2}{\mu^2\nu^2}.$$

Metric coefficients

$$g_{11} = g_{22} = \mu^2 + \nu^2, \quad g_{33} = \mu^2\nu^2, \quad g^{\frac{1}{2}} = \mu\nu(\mu^2 + \nu^2).$$

$$f_1 = \mu, \quad f_2 = \nu, \quad f_3 = 1.$$

Important equations

$$(ds)^2 = (\mu^2 + \nu^2)\,[(d\mu)^2 + (d\nu)^2] + \mu^2\nu^2(d\psi)^2.$$

$$\operatorname{grad}\varphi = \frac{1}{(\mu^2+\nu^2)^{\frac{1}{2}}}\left[\boldsymbol{a}_\mu \frac{\partial\varphi}{\partial\mu} + \boldsymbol{a}_\nu \frac{\partial\varphi}{\partial\nu}\right] + \frac{\boldsymbol{a}_\psi}{\mu\nu}\frac{\partial\varphi}{\partial\psi}.$$

$$\operatorname{div}\boldsymbol{E} = \frac{1}{\mu^2+\nu^2}\left\{\frac{1}{\mu}\frac{\partial}{\partial\mu}\left[\mu(\mu^2+\nu^2)^{\frac{1}{2}}E_\mu\right] + \frac{1}{\nu}\frac{\partial}{\partial\nu}\left[\nu(\mu^2+\nu^2)^{\frac{1}{2}}E_\nu\right]\right\} + \frac{1}{\mu\nu}\frac{\partial E_\psi}{\partial\psi}.$$

$$\operatorname{curl}\boldsymbol{E} = \frac{1}{\mu\nu(\mu^2+\nu^2)}\times\begin{vmatrix} \boldsymbol{a}_\mu(\mu^2+\nu^2)^{\frac{1}{2}} & \boldsymbol{a}_\nu(\mu^2+\nu^2)^{\frac{1}{2}} & \boldsymbol{a}_\psi\mu\nu \\ \dfrac{\partial}{\partial\mu} & \dfrac{\partial}{\partial\nu} & \dfrac{\partial}{\partial\psi} \\ E_\mu(\mu^2+\nu^2)^{\frac{1}{2}} & E_\nu(\mu^2+\nu^2)^{\frac{1}{2}} & E_\psi\mu\nu \end{vmatrix}.$$

$$\nabla^2\varphi = \frac{1}{\mu^2+\nu^2}\left[\frac{\partial^2\varphi}{\partial\mu^2} + \frac{1}{\mu}\frac{\partial\varphi}{\partial\mu} + \frac{\partial^2\varphi}{\partial\nu^2} + \frac{1}{\nu}\frac{\partial\varphi}{\partial\nu}\right] + \frac{1}{\mu^2\nu^2}\frac{\partial^2\varphi}{\partial\psi^2}.$$

SEPARATION OF LAPLACE'S EQUATION, $\nabla^2\varphi = 0$.

$$\frac{1}{f_i}\frac{d}{du^i}\left(f_i\frac{dU^i}{du^i}\right) + U^i\sum_{j=1}^{3}\Phi_{ij}\alpha_j = 0,$$

where $\alpha_1 = 0$ and $U^1 = M(\mu)$, $U^2 = N(\nu)$, $U^3 = \Psi(\psi)$.

General case

$$\begin{cases} \dfrac{d^2M}{d\mu^2} + \dfrac{1}{\mu}\dfrac{dM}{d\mu} - \left(\alpha_2 + \dfrac{\alpha_3}{\mu^2}\right)M = 0, \\ \dfrac{d^2N}{d\nu^2} + \dfrac{1}{\nu}\dfrac{dN}{d\nu} + \left(\alpha_2 - \dfrac{\alpha_3}{\nu^2}\right)N = 0, \\ \dfrac{d^2\Psi}{d\psi^2} + \alpha_3\Psi = 0. \end{cases}$$

If $\alpha_2 = q^2$ and $\alpha_3 = p^2$,

$$\frac{d^2M}{d\mu^2} + \frac{1}{\mu}\frac{dM}{d\mu} - (q^2 + p^2/\mu^2)M = 0, \quad \{24\} \quad M = A\mathscr{J}_p(iq\mu) + B\mathscr{J}_{-p}(iq\mu)$$
$$\text{or} \quad M = A\mathscr{J}_n(iq\mu) + B\mathscr{Y}_n(iq\mu).$$

$$\frac{d^2N}{d\nu^2} + \frac{1}{\nu}\frac{dN}{d\nu} + (q^2 - p^2/\nu^2)N = 0, \quad \{24\} \quad N = A\mathscr{J}_p(q\nu) + B\mathscr{J}_{-p}(q\nu)$$
$$\text{or} \quad N = A\mathscr{J}_n(q\nu) + B\mathscr{Y}_n(q\nu).$$

$$\frac{d^2\Psi}{d\psi^2} + p^2\Psi = 0, \quad \{04\} \quad \Psi = A\sin p\psi + B\cos p\psi.$$

For φ independent of ψ,

$$\begin{cases} \dfrac{d^2M}{d\mu^2} + \dfrac{1}{\mu}\dfrac{dM}{d\mu} - \alpha_2 M = 0, \\ \dfrac{d^2N}{d\nu^2} + \dfrac{1}{\nu}\dfrac{dN}{d\nu} + \alpha_2 N = 0. \end{cases}$$

If $\alpha_2 = q^2$,

$$\frac{d^2M}{d\mu^2} + \frac{1}{\mu}\frac{dM}{d\mu} - q^2 M = 0, \qquad \{14\} \quad M = A\mathscr{I}_0(iq\mu) + B\mathscr{Y}_0(iq\mu).$$

$$\frac{d^2N}{d\nu^2} + \frac{1}{\nu}\frac{dN}{d\nu} + q^2 N = 0, \qquad \{14\} \quad N = A\mathscr{I}_0(q\nu) + B\mathscr{Y}_0(q\nu).$$

For φ independent of ν and ψ,

$$\frac{d^2\varphi}{d\mu^2} + \frac{1}{\mu}\frac{d\varphi}{d\mu} = 0, \qquad \{01\} \quad \varphi = A + B\ln\mu.$$

For φ independent of μ and ψ,

$$\frac{d^2\varphi}{d\nu^2} + \frac{1}{\nu}\frac{d\varphi}{d\nu} = 0, \qquad \{01\} \quad \varphi = A + B\ln\nu.$$

SEPARATION OF HELMHOLTZ EQUATION, $\nabla^2\varphi + \varkappa^2\varphi = 0$.

$$\frac{1}{f_i}\frac{d}{du^i}\left(f_i\frac{dU^i}{du^i}\right) + U^i\sum_{j=1}^{3}\Phi_{ij}\alpha_j = 0,$$

where $\alpha_1 = \varkappa^2$ and $U^1 = M(\mu)$, $U^2 = N(\nu)$, $U^3 = \Psi(\psi)$.

General case

$$\begin{cases} \dfrac{d^2M}{d\mu^2} + \dfrac{1}{\mu}\dfrac{dM}{d\mu} + (\varkappa^2\mu^2 - \alpha_2 - \alpha_3/\mu^2)M = 0, \\ \dfrac{d^2N}{d\nu^2} + \dfrac{1}{\nu}\dfrac{dN}{d\nu} + (\varkappa^2\nu^2 + \alpha_2 - \alpha_3/\nu^2)N = 0, \\ \dfrac{d^2\Psi}{d\psi^2} + \alpha_3\Psi = 0. \end{cases}$$

If $\alpha_2 = q^2$ *and* $\alpha_3 = p^2$,

$$\frac{d^2M}{d\mu^2} + \frac{1}{\mu}\frac{dM}{d\mu} + (\varkappa^2\mu^2 - q^2 - p^2/\mu^2)M = 0,$$

$$\{26\} \quad M = A\mathscr{I}_p(\varkappa, iq\mu) + B\mathscr{I}_{-p}(\varkappa, iq\mu)$$

$$\text{or} \quad M = A\mathscr{I}_n(\varkappa, iq\mu) + B\mathscr{Y}_n(\varkappa, iq\mu).$$

$$\frac{d^2N}{d\nu^2} + \frac{1}{\nu}\frac{dN}{d\nu} + (\varkappa^2\nu^2 + q^2 - p^2/\nu^2)N = 0,$$

$$(26\} \quad N = A\mathscr{I}_p(\varkappa, q\nu) + B\mathscr{I}_{-p}(\varkappa, q\nu)$$

$$\text{or} \quad N = A\mathscr{I}_n(\varkappa, q\nu) + B\mathscr{Y}_n(\varkappa, q\nu).$$

$$\frac{d^2\Psi}{d\psi^2} + p^2\Psi = 0, \qquad \{04\} \quad \Psi = A\sin p\psi + B\cos p\psi.$$

For φ independent of ψ,

$$\begin{cases} \dfrac{d^2M}{d\mu^2} + \dfrac{1}{\mu}\dfrac{dM}{d\mu} + (\varkappa^2\mu^2 - \alpha_2)M = 0, \\ \dfrac{d^2N}{d\nu^2} + \dfrac{1}{\nu}\dfrac{dN}{d\nu} + (\varkappa^2\nu^2 + \alpha_2)N = 0. \end{cases}$$

If $\alpha_2 = q^2$,

$$\frac{d^2M}{d\mu^2} + \frac{1}{\mu}\frac{dM}{d\mu} + (\varkappa^2\mu^2 - q^2)M = 0, \quad \{16\} \quad M = A\mathscr{I}_0(\varkappa, iq\mu) + B\mathscr{Y}_0(\varkappa, iq\mu).$$

$$\frac{d^2N}{d\nu^2} + \frac{1}{\nu}\frac{dN}{d\nu} + (\varkappa^2\nu^2 + q^2)N = 0, \quad \{16\} \quad N = A\mathscr{I}_0(\varkappa, q\nu) + B\mathscr{Y}_0(\varkappa, q\nu).$$

TABLE 1.09. CONICAL COORDINATES (r, θ, λ)

$$u^1 = r, \quad 0 \leqq r < \infty,$$
$$u^2 = \theta, \quad b^2 < \theta^2 < c^2,$$
$$u^3 = \lambda, \quad 0 < \lambda^2 < b^2.$$

$$\begin{cases} (x)^2 = \left(\frac{r\theta\lambda}{bc}\right)^2, \\ (y)^2 = \frac{r^2(\theta^2 - b^2)(b^2 - \lambda^2)}{b^2(c^2 - b^2)}, \\ (z)^2 = \frac{r^2(c^2 - \theta^2)(c^2 - \lambda^2)}{c^2(c^2 - b^2)}, \end{cases}$$

with $c^2 > \theta^2 > b^2 > \lambda^2 > 0$.

Coordinate surfaces are

$$\begin{cases} x^2 + y^2 + z^2 = r^2 \\ \quad \text{(spheres, } r = \text{const)}, \\ \frac{x^2}{\theta^2} + \frac{y^2}{\theta^2 - b^2} - \frac{z^2}{c^2 - \theta^2} = 0 \\ \quad \text{(elliptic cones, } \theta = \text{const)}, \\ \frac{x^2}{\lambda^2} - \frac{y^2}{b^2 - \lambda^2} - \frac{z^2}{c^2 - \lambda^2} = 0 \\ \quad \text{(elliptic cones, } \lambda = \text{const)}. \end{cases}$$

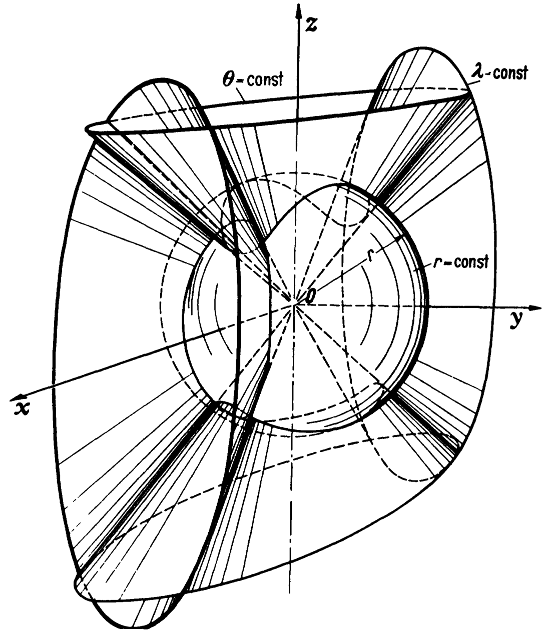

Fig. 1.09. Conical coordinates (r, θ, λ). Coordinate surfaces are spheres (r = const), and elliptic cones (θ = const, λ = const)

The Stäckel matrix may be written

$$[S] = \begin{bmatrix} 1 & -1/r^2 & 0 \\ 0 & \frac{\theta^2}{(\theta^2 - b^2)(c^2 - \theta^2)} & \frac{-1}{(\theta^2 - b^2)(c^2 - \theta^2)} \\ 0 & \frac{-\lambda^2}{(b^2 - \lambda^2)(c^2 - \lambda^2)} & \frac{1}{(b^2 - \lambda^2)(c^2 - \lambda^2)} \end{bmatrix}.$$

$$S = \frac{(\theta^2 - \lambda^2)}{(\theta^2 - b^2)(c^2 - \theta^2)(b^2 - \lambda^2)(c^2 - \lambda^2)},$$

$$M_{11} = S,$$

$$M_{21} = \frac{1}{r^2(b^2 - \lambda^2)(c^2 - \lambda^2)},$$

$$M_{31} = \frac{1}{r^2(\theta^2 - b^2)(c^2 - \theta^2)}.$$

Metric coefficients

$$g_{11} = 1,$$

$$g_{22} = \frac{r^2(\theta^2 - \lambda^2)}{(\theta^2 - b^2)(c^2 - \theta^2)},$$

$$g_{33} = \frac{r^2(\theta^2 - \lambda^2)}{(b^2 - \lambda^2)(c^2 - \lambda^2)},$$

$$g^{\frac{1}{2}} = \frac{r^2(\theta^2 - \lambda^2)}{[(\theta^2 - b^2)(c^2 - \theta^2)(b^2 - \lambda^2)(c^2 - \lambda^2)]^{\frac{1}{2}}}.$$

$$f_1 = r^2,$$

$$f_2 = [(\theta^2 - b^2)(c^2 - \theta^2)]^{\frac{1}{2}},$$

$$f_3 = [(b^2 - \lambda^2)(c^2 - \lambda^2)]^{\frac{1}{2}}.$$

Important equations

$$(ds)^2 = (dr)^2 + r^2(\theta^2 - \lambda^2)\left[\frac{(d\theta)^2}{(\theta^2 - b^2)(c^2 - \theta^2)} + \frac{(d\lambda)^2}{(b^2 - \lambda^2)(c^2 - \lambda^2)}\right].$$

$$\operatorname{grad}\varphi = \boldsymbol{a}_r \frac{\partial\varphi}{\partial r} + \frac{1}{r(\theta^2 - \lambda^2)^{\frac{1}{2}}}$$
$$\times\left\{\boldsymbol{a}_\theta(\theta^2 - b^2)^{\frac{1}{2}}(c^2 - \theta^2)^{\frac{1}{2}}\frac{\partial\varphi}{\partial\theta} + \boldsymbol{a}_\lambda(b^2 - \lambda^2)^{\frac{1}{2}}(c^2 - \lambda^2)^{\frac{1}{2}}\frac{\partial\varphi}{\partial\lambda}\right\}.$$

$$\operatorname{div}\boldsymbol{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) + \frac{1}{r(\theta^2 - \lambda^2)}\left\{(\theta^2 - b^2)^{\frac{1}{2}}(c^2 - \theta^2)^{\frac{1}{2}}\right.$$
$$\left.\times\frac{\partial}{\partial\theta}\left[(\theta^2 - \lambda^2)^{\frac{1}{2}}E_\theta\right] + (b^2 - \lambda^2)^{\frac{1}{2}}(c^2 - \lambda^2)^{\frac{1}{2}}\frac{\partial}{\partial\lambda}\left[(\theta^2 - \lambda^2)^{\frac{1}{2}}E_\lambda\right]\right\}.$$

$$\operatorname{curl}\boldsymbol{E} = \frac{\left[(\theta^2 - b^2)(c^2 - \theta^2)(b^2 - \lambda^2)(c^2 - \lambda^2)\right]^{\frac{1}{2}}}{(\theta^2 - \lambda^2)}$$
$$\times\begin{vmatrix} \dfrac{\boldsymbol{a}_r}{r} & \boldsymbol{a}_\theta\dfrac{(\theta^2 - \lambda^2)^{\frac{1}{2}}}{(\theta^2 - b^2)^{\frac{1}{2}}(c^2 - \theta^2)^{\frac{1}{2}}} & \boldsymbol{a}_\lambda\dfrac{(\theta^2 - \lambda^2)^{\frac{1}{2}}}{(b^2 - \lambda^2)^{\frac{1}{2}}(c^2 - \lambda^2)^{\frac{1}{2}}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\lambda} \\ \dfrac{E_r}{r} & E_\theta\dfrac{(\theta^2 - \lambda^2)^{\frac{1}{2}}}{(\theta^2 - b^2)^{\frac{1}{2}}(c^2 - \theta^2)^{\frac{1}{2}}} & E_\lambda\dfrac{(\theta^2 - \lambda^2)^{\frac{1}{2}}}{(b^2 - \lambda^2)^{\frac{1}{2}}(c^2 - \lambda^2)^{\frac{1}{2}}} \end{vmatrix}.$$

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial r^2} + \frac{2}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2(\theta^2 - \lambda^2)}\left\{(\theta^2 - b^2)(c^2 - \theta^2)\frac{\partial^2\varphi}{\partial\theta^2}\right.$$
$$- \theta\left[2\theta^2 - (b^2 + c^2)\right]\frac{\partial\varphi}{\partial\theta} + (b^2 - \lambda^2)(c^2 - \lambda^2)\frac{\partial^2\varphi}{\partial\lambda^2}$$
$$\left. + \lambda\left[2\lambda^2 - (b^2 + c^2)\right]\frac{\partial\varphi}{\partial\lambda}\right\}.$$

SEPARATION OF LAPLACE'S EQUATION, $\nabla^2\varphi = 0$.

$$\frac{1}{f_i}\frac{d}{du^i}\left(f_i\frac{dU^i}{du^i}\right) + U^i\sum_{j=1}^{3}\Phi_{ij}\alpha_j = 0,$$

where $\alpha_1 = 0$ and $U^1 = R(r)$, $U^2 = \Theta(\theta)$, $U^3 = \Lambda(\lambda)$.

General case

$$\begin{cases} \dfrac{d^2R}{dr^2} + \dfrac{2}{r}\dfrac{dR}{dr} - \dfrac{\alpha_2}{r^2}R = 0, \\ (\theta^2 - b^2)(c^2 - \theta^2)\dfrac{d^2\Theta}{d\theta^2} - \theta\left[2\theta^2 - (b^2 + c^2)\right]\dfrac{d\Theta}{d\theta} + \left[\alpha_2\theta^2 - \alpha_3\right]\Theta = 0, \\ (b^2 - \lambda^2)(c^2 - \lambda^2)\dfrac{d^2\Lambda}{d\lambda^2} + \lambda\left[2\lambda^2 - (b^2 + c^2)\right]\dfrac{d\Lambda}{d\lambda} - \left[\alpha_2\lambda^2 - \alpha_3\right]\Lambda = 0. \end{cases}$$

If $\alpha_2 = p(p+1)$ and $\alpha_3 = (b^2 + c^2)\,q$,

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{p(p+1)}{r^2}R = 0, \qquad \{22\}\quad R = A\,r^p + B\,r^{-(p+1)}.$$

$$(\theta^2 - b^2)(\theta^2 - c^2)\frac{d^2\Theta}{d\theta^2} + \theta\left[2\theta^2 - (b^2 + c^2)\right]\frac{d\Theta}{d\theta} + \left[(b^2 + c^2)\,q - p(p+1)\,\theta^2\right]\Theta = 0,$$
$$\{1112\}\quad \Theta = A\mathscr{E}_p^q(\theta) + B\mathscr{F}_p^q(\theta).$$

$$(\lambda^2-b^2)(\lambda^2-c^2)\frac{d^2\Lambda}{d\lambda^2}+\lambda[2\lambda^2-(b^2+c^2)]\frac{d\Lambda}{d\lambda}+[(b^2+c^2)q-p(p+1)\lambda^2]\Lambda=0,$$

$$\{1112\}\quad \Lambda=A\mathscr{E}_p^q(\lambda)+B\mathscr{F}_p^q(\lambda).$$

If $\alpha_2=\alpha_3=0$,

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}=0,\qquad \{01\}\quad R=A+B/r.$$

$$\frac{d}{d\theta}\left[(\theta^2-b^2)^{\frac12}(c^2-\theta^2)^{\frac12}\frac{d\Theta}{d\theta}\right]=0,\qquad \{01\}\quad \Theta=A+B\,\mathrm{sn}^{-1}\left(\sqrt{\frac{c^2-\theta^2}{c^2-b^2}},\sqrt{\frac{c^2-b^2}{c^2}}\right).$$

$$\frac{d}{d\lambda}\left[(b^2-\lambda^2)^{\frac12}(c^2-\lambda^2)^{\frac12}\frac{d\Lambda}{d\lambda}\right]=0,\qquad \{01\}\quad \Lambda=A+B\,\mathrm{sn}^{-1}\left(\frac{\lambda}{b},\frac{b}{c}\right).$$

For φ independent of r,

$$\begin{cases}(\theta^2-b^2)(\theta^2-c^2)\dfrac{d^2\Theta}{d\theta^2}+\theta[2\theta^2-(b^2+c^2)]\dfrac{d\Theta}{d\theta}+\alpha_3\Theta=0,\\[2mm](\lambda^2-b^2)(\lambda^2-c^2)\dfrac{d^2\Lambda}{d\lambda^2}+\lambda[2\lambda^2-(b^2+c^2)]\dfrac{d\Lambda}{d\lambda}+\alpha_3\Lambda=0.\end{cases}$$

If $\alpha_3=q(b^2+c^2)$,

$$(\theta^2-b^2)(\theta^2-c^2)\frac{d^2\Theta}{d\theta^2}+\theta[2\theta^2-(b^2+c^2)]\frac{d\Theta}{d\theta}+q(b^2+c^2)\Theta=0,$$

$$\{1111\}\quad \Theta=A\mathscr{E}_0^q(\theta)+B\mathscr{F}_0^q(\theta).$$

$$(\lambda^2-b^2)(\lambda^2-c^2)\frac{d^2\Lambda}{d\lambda^2}+\lambda[2\lambda^2-(b^2+c^2)]\frac{d\Lambda}{d\lambda}+q(b^2+c^2)\Lambda=0,$$

$$\{1111\}\quad \Lambda=A\mathscr{E}_0^q(\lambda)+B\mathscr{F}_0^q(\lambda).$$

For φ independent of r and λ,

$$\frac{d}{d\theta}\left[(\theta^2-b^2)^{\frac12}(c^2-\theta^2)^{\frac12}\frac{d\varphi}{d\theta}\right]=0,\qquad \{01\}\quad \varphi=A+B\,\mathrm{sn}^{-1}\left(\sqrt{\frac{c^2-\theta^2}{c^2-b^2}},\sqrt{\frac{c^2-b^2}{c^2}}\right).$$

For φ independent of r and θ,

$$\frac{d}{d\lambda}\left[(b^2-\lambda^2)^{\frac12}(c^2-\lambda^2)^{\frac12}\frac{d\varphi}{d\lambda}\right]=0,\qquad \{01\}\quad \varphi=A+B\,\mathrm{sn}^{-1}\left(\frac{\lambda}{b},\frac{b}{c}\right).$$

SEPARATION OF HELMHOLTZ EQUATION, $\nabla^2\varphi+\varkappa^2\varphi=0$.

$$\frac{1}{f_i}\frac{d}{du^i}\left(f_i\frac{dU^i}{du^i}\right)+U^i\sum_{j=1}^{3}\Phi_{ij}\alpha_j=0,$$

where $\alpha_1=\varkappa^2$ and $U^1=R(r)$, $U^2=\Theta(\theta)$, $U^3=\Lambda(\lambda)$.

General case

$$\begin{cases}\dfrac{d^2R}{dr^2}+\dfrac{2}{r}\dfrac{dR}{dr}+(\varkappa^2-\alpha_2/r^2)R=0,\\[2mm](\theta^2-b^2)(c^2-\theta^2)\dfrac{d^2\Theta}{d\theta^2}-\theta[2\theta^2-(b^2+c^2)]\dfrac{d\Theta}{d\theta}+[\alpha_2\theta^2-\alpha_3]\Theta=0,\\[2mm](b^2-\lambda^2)(c^2-\lambda^2)\dfrac{d^2\Lambda}{d\lambda^2}+\lambda[2\lambda^2-(b^2+c^2)]\dfrac{d\Lambda}{d\lambda}-[\alpha_2\lambda^2-\alpha_3]\Lambda=0.\end{cases}$$

If $\alpha_2 = p(p+1)$ *and* $\alpha_3 = q(b^2 + c^2)$,

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left(\varkappa^2 - p(p+1)/r^2\right)R = 0,$$

$$\{24\} \quad R = r^{-\frac{1}{2}}\left[A\mathscr{I}_{p+\frac{1}{2}}(\varkappa r) + B\mathscr{I}_{-(p+\frac{1}{2})}(\varkappa r)\right].$$

$$(\theta^2 - b^2)(\theta^2 - c^2)\frac{d^2\Theta}{dt^2} + \theta\left[2\theta^2 - (b^2 + c^2)\right]\frac{d\Theta}{d\theta} - \left[p(p+1)\theta^2 - q(b^2 + c^2)\right]\Theta = 0,$$

$$\{1\,1\,1\,2\} \quad \Theta = A\mathscr{E}_p^q(\theta) + B\mathscr{F}_p^q(\theta).$$

$$(\lambda^2 - b^2)(\lambda^2 - c^2)\frac{d^2\Lambda}{d\lambda^2} + \lambda\left[2\lambda^2 - (b^2 + c^2)\right]\frac{d\Lambda}{d\lambda} - \left[p(p+1)\lambda^2 - q(b^2 + c^2)\right]\Lambda = 0,$$

$$\{1\,1\,1\,2\} \quad \Lambda = A\mathscr{E}_p^q(\lambda) + B\mathscr{F}_p^q(\lambda).$$

If $\alpha_2 = \alpha_3 = 0$,

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \varkappa^2 R = 0, \qquad \{04\} \quad R = \frac{1}{r}\left[A\sin\varkappa r + B\cos\varkappa r\right].$$

$$\frac{d}{d\theta}\left[(\theta^2 - b^2)^{\frac{1}{2}}(c^2 - \theta^2)^{\frac{1}{2}}\frac{d\Theta}{d\theta}\right] = 0, \qquad \{01\} \quad \Theta = A + B\,\mathrm{sn}^{-1}\left(\sqrt{\frac{c^2 - \theta^2}{c^2 - b^2}}, \sqrt{\frac{c^2 - b^2}{c^2}}\right).$$

$$\frac{d}{d\lambda}\left[(b^2 - \lambda^2)^{\frac{1}{2}}(c^2 - \lambda^2)^{\frac{1}{2}}\frac{d\Lambda}{d\lambda}\right] = 0, \qquad \{01\} \quad \Lambda = A + B\,\mathrm{sn}^{-1}\left(\frac{\lambda}{b}, \frac{b}{c}\right).$$

For φ independent of θ and λ,

$$\frac{d^2\varphi}{dr^2} + \frac{2}{r}\frac{d\varphi}{dr} + \varkappa^2\varphi = 0, \qquad \{04\} \quad R = \frac{1}{r}\left[A\sin\varkappa r + B\cos\varkappa r\right].$$

TABLE 1.10. ELLIPSOIDAL COORDINATES (η, θ, λ)

$$u^1 = \eta, \quad c^2 < \eta^2 < \infty^2,$$
$$u^2 = \theta, \quad b^2 < \theta^2 < c^2,$$
$$u^3 = \lambda, \quad 0 \leqq \lambda^2 < b^2.$$

$$\begin{cases} (x)^2 = \left(\dfrac{\eta\theta\lambda}{bc}\right)^2, \\ (y^2) = \dfrac{(\eta^2 - b^2)(\theta^2 - b^2)(b^2 - \lambda^2)}{b^2(c^2 - b^2)}, \\ (z)^2 = \dfrac{(\eta^2 - c^2)(c^2 - \theta^2)(c^2 - \lambda^2)}{c^2(c^2 - b^2)}, \end{cases}$$

with $\eta^2 > c^2 > \theta^2 > b^2 > \lambda^2 > 0$.

Coordinate surfaces are

$$\begin{cases} \dfrac{x^2}{\eta^2} + \dfrac{y^2}{\eta^2 - b^2} + \dfrac{z^2}{\eta^2 - c^2} = 1 & \text{(ellipsoids, } \eta = \text{const),} \\ \dfrac{x^2}{\theta^2} + \dfrac{y^2}{\theta^2 - b^2} - \dfrac{z^2}{c^2 - \theta^2} = 1 & \text{(hyperboloids of one sheet, } \theta = \text{const),} \\ \dfrac{x^2}{\lambda^2} - \dfrac{y^2}{b^2 - \lambda^2} - \dfrac{z^2}{c^2 - \lambda^2} = 1 & \text{(hyperboloids of two sheets, } \lambda = \text{const).} \end{cases}$$

The Stäckel matrix may be written

$$[S] = \begin{bmatrix} \dfrac{\eta^4}{(\eta^2-b^2)(\eta^2-c^2)} & \dfrac{1}{(\eta^2-b^2)(\eta^2-c^2)} & \dfrac{\eta^2}{(\eta^2-b^2)(\eta^2-c^2)} \\ \dfrac{-\theta^4}{(\theta^2-b^2)(c^2-\theta^2)} & \dfrac{-1}{(\theta^2-b^2)(c^2-\theta^2)} & \dfrac{-\theta^2}{(\theta^2-b^2)(c^2-\theta^2)} \\ \dfrac{\lambda^4}{(b^2-\lambda^2)(c^2-\lambda^2)} & \dfrac{1}{(b^2-\lambda^2)(c^2-\lambda^2)} & \dfrac{\lambda^2}{(b^2-\lambda^2)(c^2-\lambda^2)} \end{bmatrix}.$$

$$S = \frac{(\eta^2-\theta^2)(\eta^2-\lambda^2)(\theta^2-\lambda^2)}{(\eta^2-b^2)(\eta^2-c^2)(\theta^2-b^2)(c^2-\theta^2)(b^2-\lambda^2)(c^2-\lambda^2)},$$

$$M_{11} = \frac{(\theta^2-\lambda^2)}{(\theta^2-b^2)(c^2-\theta^2)(b^2-\lambda^2)(c^2-\lambda^2)},$$

$$M_{21} = \frac{(\eta^2-\lambda^2)}{(\eta^2-b^2)(\eta^2-c^2)(b^2-\lambda^2)(c^2-\lambda^2)},$$

$$M_{31} = \frac{(\eta^2-\theta^2)}{(\eta^2-b^2)(\eta^2-c^2)(\theta^2-b^2)(c^2-\theta^2)}.$$

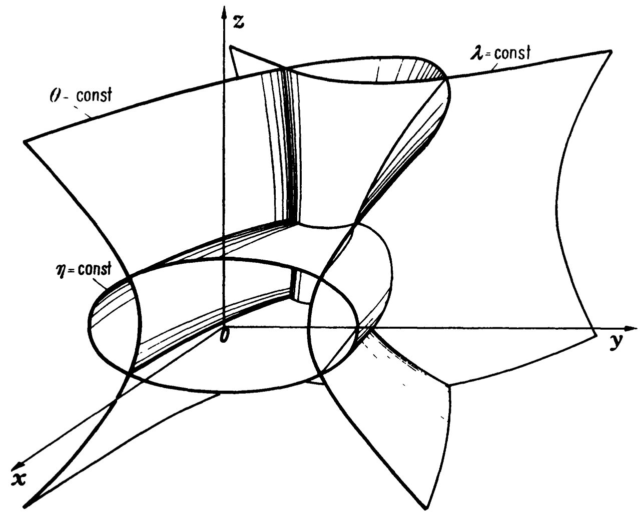

Fig. 1.10. Ellipsoidal coordinates (η, θ, λ). Coordinate surfaces are ellipsoids (η = const), and hyperboloids (θ = const, λ = const)

Metric coefficients

$$g_{11} = \frac{(\eta^2-\theta^2)(\eta^2-\lambda^2)}{(\eta^2-b^2)(\eta^2-c^2)},$$

$$g_{22} = \frac{(\theta^2-\lambda^2)(\eta^2-\theta^2)}{(\theta^2-b^2)(c^2-\theta^2)},$$

$$g_{33} = \frac{(\eta^2-\lambda^2)(\theta^2-\lambda^2)}{(b^2-\lambda^2)(c^2-\lambda^2)},$$

$$g^{\frac{1}{2}} = \frac{(\eta^2-\theta^2)(\eta^2-\lambda^2)(\theta^2-\lambda^2)}{[(\eta^2-b^2)(\eta^2-c^2)(\theta^2-b^2)(c^2-\theta^2)(b^2-\lambda^2)(c^2-\lambda^2)]^{\frac{1}{2}}}.$$

$$f_1 = [(\eta^2-b^2)(\eta^2-c^2)]^{\frac{1}{2}},$$

$$f_2 = [(\theta^2-b^2)(c^2-\theta^2)]^{\frac{1}{2}},$$

$$f_3 = [(b^2-\lambda^2)(c^2-\lambda^2)]^{\frac{1}{2}}.$$

Important equations

$$(ds)^2 = \left[\frac{(\eta^2-\theta^2)(\eta^2-\lambda^2)}{(\eta^2-b^2)(\eta^2-c^2)}\right](d\eta)^2 + \left[\frac{(\theta^2-\lambda^2)(\eta^2-\theta^2)}{(\theta^2-b^2)(c^2-\theta^2)}\right](d\theta)^2 + \left[\frac{(\eta^2-\lambda^2)(\theta^2-\lambda^2)}{(b^2-\lambda^2)(c^2-\lambda^2)}\right](d\lambda)^2.$$

$$\operatorname{grad}\varphi = \boldsymbol{a}_\eta\left[\frac{(\eta^2-b^2)(\eta^2-c^2)}{(\eta^2-\theta^2)(\eta^2-\lambda^2)}\right]^{\frac{1}{2}}\frac{\partial\varphi}{\partial\eta} + \boldsymbol{a}_\theta\left[\frac{(\theta^2-b^2)(c^2-\theta^2)}{(\theta^2-\lambda^2)(\eta^2-\theta^2)}\right]^{\frac{1}{2}}\frac{\partial\varphi}{\partial\theta} + \boldsymbol{a}_\lambda\left[\frac{(b^2-\lambda^2)(c^2-\lambda^2)}{(\eta^2-\lambda^2)(\theta^2-\lambda^2)}\right]^{\frac{1}{2}}\frac{\partial\varphi}{\partial\lambda}.$$

$$\operatorname{div}\boldsymbol{E} = \frac{(\eta^2-b^2)^{\frac{1}{2}}(\eta^2-c^2)^{\frac{1}{2}}}{(\eta^2-\theta^2)(\eta^2-\lambda^2)}\frac{\partial}{\partial\eta}\left[(\eta^2-\theta^2)^{\frac{1}{2}}(\eta^2-\lambda^2)^{\frac{1}{2}}E_\eta\right] + \frac{(\theta^2-b^2)^{\frac{1}{2}}(c^2-\theta^2)^{\frac{1}{2}}}{(\eta^2-\theta^2)(\theta^2-\lambda^2)}\frac{\partial}{\partial\theta}\left[(\eta^2-\theta^2)^{\frac{1}{2}}(\theta^2-\lambda^2)^{\frac{1}{2}}E_\theta\right] + \frac{(b^2-\lambda^2)^{\frac{1}{2}}(c^2-\lambda^2)^{\frac{1}{2}}}{(\eta^2-\lambda^2)(\theta^2-\lambda^2)}\frac{\partial}{\partial\lambda}\left[(\eta^2-\lambda^2)^{\frac{1}{2}}(\theta^2-\lambda^2)^{\frac{1}{2}}E_\lambda\right].$$

$$\operatorname{curl}\boldsymbol{E} = \frac{[(\eta^2-b^2)(\eta^2-c^2)(\theta^2-b^2)(c^2-\theta^2)(b^2-\lambda^2)(c^2-\lambda^2)]^{\frac{1}{2}}}{(\eta^2-\theta^2)(\eta^2-\lambda^2)(\theta^2-\lambda^2)} \times \begin{vmatrix} \boldsymbol{a}_\eta\left[\frac{(\eta^2-\theta^2)(\eta^2-\lambda^2)}{(\eta^2-b^2)(\eta^2-c^2)}\right]^{\frac{1}{2}} & \boldsymbol{a}_\theta\left[\frac{(\theta^2-\lambda^2)(\eta^2-\theta^2)}{(\theta^2-b^2)(c^2-\theta^2)}\right]^{\frac{1}{2}} & \boldsymbol{a}_\lambda\left[\frac{(\eta^2-\lambda^2)(\theta^2-\lambda^2)}{(b^2-\lambda^2)(c^2-\lambda^2)}\right]^{\frac{1}{2}} \\ \frac{\partial}{\partial\eta} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\lambda} \\ E_\eta\left[\frac{(\eta^2-\theta^2)(\eta^2-\lambda^2)}{(\eta^2-b^2)(\eta^2-c^2)}\right]^{\frac{1}{2}} & E_\theta\left[\frac{(\theta^2-\lambda^2)(\eta^2-\theta^2)}{(\theta^2-b^2)(c^2-\theta^2)}\right]^{\frac{1}{2}} & E_\lambda\left[\frac{(\eta^2-\lambda^2)(\theta^2-\lambda^2)}{(b^2-\lambda^2)(c^2-\lambda^2)}\right]^{\frac{1}{2}} \end{vmatrix}.$$

$$\nabla^2\varphi = \frac{(\eta^2-b^2)^{\frac{1}{2}}(\eta^2-c^2)^{\frac{1}{2}}}{(\eta^2-\theta^2)(\eta^2-\lambda^2)}\frac{\partial}{\partial\eta}\left[(\eta^2-b^2)^{\frac{1}{2}}(\eta^2-c^2)^{\frac{1}{2}}\frac{\partial\varphi}{\partial\eta}\right] + \frac{(\theta^2-b^2)^{\frac{1}{2}}(c^2-\theta^2)^{\frac{1}{2}}}{(\eta^2-\theta^2)(\theta^2-\lambda^2)}\frac{\partial}{\partial\theta}\left[(\theta^2-b^2)^{\frac{1}{2}}(c^2-\theta^2)^{\frac{1}{2}}\frac{\partial\varphi}{\partial\theta}\right] + \frac{(b^2-\lambda^2)^{\frac{1}{2}}(c^2-\lambda^2)^{\frac{1}{2}}}{(\eta^2-\lambda^2)(\theta^2-\lambda^2)}\frac{\partial}{\partial\lambda}\left[(b^2-\lambda^2)^{\frac{1}{2}}(c^2-\lambda^2)^{\frac{1}{2}}\frac{\partial\varphi}{\partial\lambda}\right].$$

SEPARATION OF LAPLACE'S EQUATION, $\nabla^2\varphi = 0$.

$$\frac{1}{f_i}\frac{d}{du^i}\left(f_i\frac{dU^i}{du^i}\right) + U^i\sum_{j=1}^{3}\Phi_{ij}\alpha_j = 0,$$

where $\alpha_1 = 0$ and $U^1 = H(\eta)$, $U^2 = \Theta(\theta)$, $U^3 = \Lambda(\lambda)$.

General case

$$\begin{cases} (\eta^2-b^2)(\eta^2-c^2)\dfrac{d^2H}{d\eta^2} + \eta[2\eta^2-(b^2+c^2)]\dfrac{dH}{d\eta} + (\alpha_2+\alpha_3\eta^2)H = 0, \\ (\theta^2-b^2)(c^2-\theta^2)\dfrac{d^2\Theta}{d\theta^2} - \theta[2\theta^2-(b^2+c^2)]\dfrac{d\Theta}{d\theta} - (\alpha_2+\alpha_3\theta^2)\Theta = 0, \\ (b^2-\lambda^2)(c^2-\lambda^2)\dfrac{d^2\Lambda}{d\lambda^2} + \lambda[2\lambda^2-(b^2+c^2)]\dfrac{d\Lambda}{d\lambda} + (\alpha_2+\alpha_3\lambda^2)\Lambda = 0. \end{cases}$$

If $\alpha_2 = (b^2 + c^2)\, q$ *and* $\alpha_3 = -p(p+1)$,

$$(\eta^2 - b^2)(\eta^2 - c^2)\frac{d^2H}{d\eta^2} + \eta[2\eta^2 - (b^2 + c^2)]\frac{dH}{d\eta} + [(b^2 + c^2)q - p(p+1)\eta^2]H = 0,$$

$$\{1112\} \quad H = A\mathscr{E}_p^q(\eta) + B\mathscr{F}_p^q(\eta).$$

$$(\theta^2 - b^2)(\theta^2 - c^2)\frac{d^2\Theta}{d\theta^2} + \theta[2\theta^2 - (b^2 + c^2)]\frac{d\Theta}{d\theta} + [(b^2 + c^2)q - p(p+1)\theta^2]\Theta = 0,$$

$$\{1112\} \quad \Theta = A\mathscr{E}_p^q(\theta) + B\mathscr{F}_p^q(\theta).$$

$$(\lambda^2 - b^2)(\lambda^2 - c^2)\frac{d^2\Lambda}{d\lambda^2} + \lambda[2\lambda^2 - (b^2 + c^2)]\frac{d\Lambda}{d\lambda} + [(b^2 + c^2)q - p(p+1)\lambda^2]\Lambda = 0,$$

$$\{1112\} \quad \Lambda = A\mathscr{E}_p^q(\lambda) + B\mathscr{F}_p^q(\lambda).$$

If $\alpha_2 = \alpha_3 = 0$,

$$\frac{d}{d\eta}\left[(\eta^2 - b^2)^{\frac{1}{2}}(\eta^2 - c^2)^{\frac{1}{2}}\frac{dH}{d\eta}\right] = 0, \qquad \{01\} \quad H = A + B\,\mathrm{sn}^{-1}\left(\frac{c}{\eta}, \frac{b}{c}\right).$$

$$\frac{d}{d\theta}\left[(\theta^2 - b^2)^{\frac{1}{2}}(c^2 - \theta^2)^{\frac{1}{2}}\frac{d\Theta}{d\theta}\right] = 0, \qquad \{01\} \quad \Theta = A + B\,\mathrm{sn}^{-1}\left(\sqrt{\frac{c^2 - \theta^2}{c^2 - b^2}}, \sqrt{\frac{c^2 - b^2}{c^2}}\right).$$

$$\frac{d}{d\lambda}\left[(b^2 - \lambda^2)^{\frac{1}{2}}(c^2 - \lambda^2)^{\frac{1}{2}}\frac{d\Lambda}{d\lambda}\right] = 0, \qquad \{01\} \quad \Lambda = A + B\,\mathrm{sn}^{-1}\left(\frac{\lambda}{b}, \frac{b}{c}\right).$$

For φ independent of θ und λ,

$$\frac{d}{d\eta}\left[(\eta^2 - b^2)^{\frac{1}{2}}(\eta^2 - c^2)^{\frac{1}{2}}\frac{d\varphi}{d\eta}\right] = 0, \qquad \{01\} \quad \varphi = A + B\,\mathrm{sn}^{-1}\left(\frac{c}{\eta}, \frac{b}{c}\right)$$

For φ independent of η and λ,

$$\frac{d}{d\theta}\left[(\theta^2 - b^2)^{\frac{1}{2}}(c^2 - \theta^2)^{\frac{1}{2}}\frac{d\varphi}{d\theta}\right] = 0, \qquad \{01\} \quad \varphi = A + B\,\mathrm{sn}^{-1}\left(\sqrt{\frac{c^2 - \theta^2}{c^2 - b^2}}, \sqrt{\frac{c^2 - b^2}{c^2}}\right).$$

For φ independent of η and θ,

$$\frac{d}{d\lambda}\left[(b^2 - \lambda^2)^{\frac{1}{2}}(c^2 - \lambda^2)^{\frac{1}{2}}\frac{d\varphi}{d\lambda}\right] = 0, \qquad \{01\} \quad \varphi = A + B\,\mathrm{sn}^{-1}\left(\frac{\lambda}{b}, \frac{b}{c}\right).$$

SEPARATION OF THE HELMHOLTZ EQUATION, $\nabla^2\varphi + \varkappa^2\varphi = 0$.

$$\frac{1}{f_i}\frac{d}{du^i}\left(f_i\frac{dU^i}{du^i}\right) + U^i\sum_{j=1}^{3}\Phi_{ij}\alpha_j = 0,$$

where $\alpha_1 = \varkappa^2$ and $U^1 = H(\eta)$, $U^2 = \Theta(\theta)$, $U^3 = \Lambda(\lambda)$.

General case

$$\begin{cases} (\eta^2 - b^2)(\eta^2 - c^2)\dfrac{d^2H}{d\eta^2} + \eta[2\eta^2 - (b^2 + c^2)]\dfrac{dH}{d\eta} + [\varkappa^2\eta^4 + \alpha_3\eta^2 + \alpha_2]H = 0, \\ (\theta^2 - b^2)(c^2 - \theta^2)\dfrac{d^2\Theta}{d\theta^2} - \theta[2\theta^2 - (b^2 + c^2)]\dfrac{d\Theta}{d\theta} - [\varkappa^2\theta^4 + \alpha_3\theta^2 + \alpha_2]\Theta = 0, \\ (b^2 - \lambda^2)(c^2 - \lambda^2)\dfrac{d^2\Lambda}{d\lambda^2} + \lambda[2\lambda^2 - (b^2 + c^2)]\dfrac{d\Lambda}{d\lambda} + [\varkappa^2\lambda^4 + \alpha_3\lambda^2 + \alpha_2]\Lambda = 0. \end{cases}$$

If $\alpha_2 = q(b^2 + c^2)$ *and* $\alpha_3 = -p(p+1)$,

$$(\eta^2 - b^2)(\eta^2 - c^2)\frac{d^2H}{d\eta^2} + \eta[2\eta^2 - (b^2+c^2)]\frac{dH}{d\eta}$$
$$+ [\varkappa^2\eta^4 - p(p+1)\eta^2 + q(b^2+c^2)]H = 0, \quad \{1113\} \quad H = A\mathscr{E}_p^q(\varkappa,\eta) + B\mathscr{F}_p^q(\varkappa,\eta).$$

$$(\theta^2 - b^2)(\theta^2 - c^2)\frac{d^2\Theta}{d\theta^2} + \theta[2\theta^2 - (b^2+c^2)]\frac{d\Theta}{d\theta}$$
$$+ [\varkappa^2\theta^4 - p(p+1)\theta^2 + q(b^2+c^2)]\Theta = 0, \quad \{1113\} \quad \Theta = A\mathscr{E}_p^q(\varkappa,\theta) + B\mathscr{F}_p^q(\varkappa,\theta).$$

$$(\lambda^2 - b^2)(\lambda^2 - c^2)\frac{d^2\Lambda}{d\lambda^2} + \lambda[2\lambda^2 - (b^2+c^2)]\frac{d\Lambda}{d\lambda}$$
$$+ [\varkappa^2\lambda^4 - p(p+1)\lambda^2 + q(b^2+c^2)]\Lambda = 0, \quad \{1113\} \quad \Lambda = A\mathscr{E}_p^q(\varkappa,\lambda) + B\mathscr{F}_p^q(\varkappa,\lambda).$$

TABLE 1.11. PARABOLOIDAL COORDINATES (μ, ν, λ)

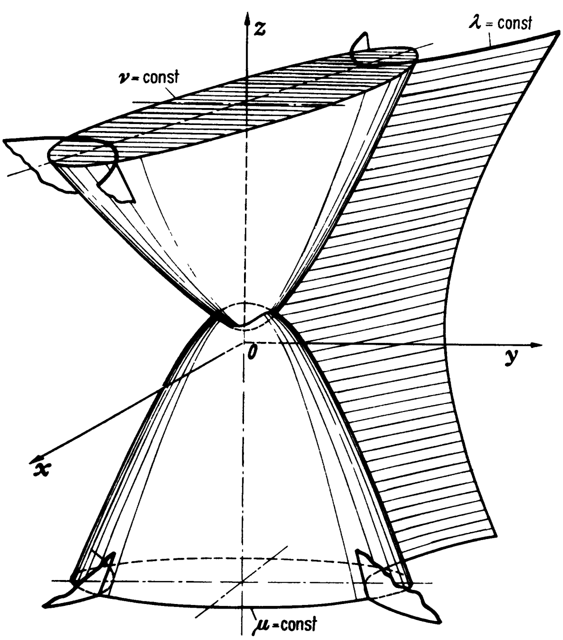

Fig. 1.11. Paraboloidal coordinates (μ, ν, λ). Coordinate surfaces are elliptic paraboloids (μ = const, ν = const), and hyperbolic paraboloids (λ = const)

$$u^1 = \mu, \quad b < \mu < \infty,$$
$$u^2 = \nu, \quad 0 < \nu < c,$$
$$u^3 = \lambda, \quad c < \lambda < b.$$

$$\begin{cases} (x)^2 = \dfrac{4}{(b-c)}(\mu - b) \times (b-\nu)(b-\lambda), \\ (y)^2 = \dfrac{4}{(b-c)}(\mu - c) \times (c-\nu)(\lambda - c), \\ z = \mu + \nu + \lambda - b - c, \end{cases}$$

where $\mu > b > \lambda > c > \nu > 0$.

Coordinate surfaces are

$\dfrac{x^2}{\mu - b} + \dfrac{y^2}{\mu - c} = -4(z - \mu)$ (elliptic paraboloid opening downward, μ = const),

$\dfrac{x^2}{b - \nu} + \dfrac{y^2}{c - \nu} = 4(z - \nu)$ (elliptic paraboloid opening upward, ν = const),

$\dfrac{x^2}{b - \lambda} - \dfrac{y^2}{\lambda - c} = 4(z - \lambda)$ (hyperbolic paraboloid, λ = const).

The Stäckel matrix may be written

$$[S] = \begin{bmatrix} \dfrac{\mu^2}{(\mu-b)(\mu-c)} & \dfrac{-1}{(\mu-b)(\mu-c)} & \dfrac{\mu}{(\mu-b)(\mu-c)} \\ \dfrac{\nu^2}{(b-\nu)(c-\nu)} & \dfrac{-1}{(b-\nu)(c-\nu)} & \dfrac{\nu}{(b-\nu)(c-\nu)} \\ \dfrac{-\lambda^2}{(b-\lambda)(\lambda-c)} & \dfrac{1}{(b-\lambda)(\lambda-c)} & \dfrac{-\lambda}{(b-\lambda)(\lambda-c)} \end{bmatrix}.$$

$$S = \frac{(\mu-\nu)(\mu-\lambda)(\lambda-\nu)}{(\mu-b)(\mu-c)(b-\nu)(c-\nu)(b-\lambda)(\lambda-c)},$$

$$M_{11} = \frac{(\lambda - \nu)}{(b-\nu)(c-\nu)(b-\lambda)(\lambda - c)},$$
$$M_{21} = \frac{(\mu - \lambda)}{(\mu - b)(\mu - c)(b-\lambda)(\lambda - c)},$$
$$M_{31} = \frac{(\mu - \nu)}{(\mu - b)(\mu - c)(b-\nu)(c-\nu)}.$$

Metric coefficients

$$g_{11} = \frac{(\mu-\nu)(\mu-\lambda)}{(\mu-b)(\mu-c)},$$
$$g_{22} = \frac{(\mu-\nu)(\lambda-\nu)}{(b-\nu)(c-\nu)},$$
$$g_{33} = \frac{(\lambda-\nu)(\mu-\lambda)}{(b-\lambda)(\lambda-c)},$$
$$g^{\frac{1}{2}} = \frac{(\mu-\nu)(\mu-\lambda)(\lambda-\nu)}{[(\mu-b)(\mu-c)(b-\nu)(c-\nu)(b-\lambda)(\lambda-c)]^{\frac{1}{2}}}.$$
$$f_1 = (\mu-b)^{\frac{1}{2}}(\mu-c)^{\frac{1}{2}},$$
$$f_2 = (b-\nu)^{\frac{1}{2}}(c-\nu)^{\frac{1}{2}},$$
$$f_3 = (b-\lambda)^{\frac{1}{2}}(\lambda-c)^{\frac{1}{2}}.$$

Important equations

$$(ds)^2 = \left[\frac{(\mu-\nu)(\mu-\lambda)}{(\mu-b)(\mu-c)}\right](d\mu)^2 + \left[\frac{(\mu-\nu)(\lambda-\nu)}{(b-\nu)(c-\nu)}\right](d\nu)^2 + \left[\frac{(\lambda-\nu)(\mu-\lambda)}{(b-\lambda)(\lambda-c)}\right](d\lambda)^2.$$

$$\operatorname{grad}\varphi = \boldsymbol{a}_\mu\left[\frac{(\mu-b)(\mu-c)}{(\mu-\nu)(\mu-\lambda)}\right]^{\frac{1}{2}}\frac{\partial\varphi}{\partial\mu} + \boldsymbol{a}_\nu\left[\frac{(b-\nu)(c-\nu)}{(\mu-\nu)(\lambda-\nu)}\right]^{\frac{1}{2}}\frac{\partial\varphi}{\partial\nu} + \boldsymbol{a}_\lambda\left[\frac{(b-\lambda)(\lambda-c)}{(\lambda-\nu)(\mu-\lambda)}\right]^{\frac{1}{2}}\frac{\partial\varphi}{\partial\lambda}.$$

$$\operatorname{div}\boldsymbol{E} = \frac{(\mu-b)^{\frac{1}{2}}(\mu-c)^{\frac{1}{2}}}{(\mu-\nu)(\mu-\lambda)}\frac{\partial}{\partial\mu}[(\mu-\nu)^{\frac{1}{2}}(\mu-\lambda)^{\frac{1}{2}}E_\mu] + \frac{(b-\nu)^{\frac{1}{2}}(c-\nu)^{\frac{1}{2}}}{(\mu-\nu)(\lambda-\nu)}\frac{\partial}{\partial\nu}[(\mu-\nu)^{\frac{1}{2}}(\lambda-\nu)^{\frac{1}{2}}E_\nu] + \frac{(b-\lambda)^{\frac{1}{2}}(\lambda-c)^{\frac{1}{2}}}{(\mu-\lambda)(\lambda-\nu)}\frac{\partial}{\partial\lambda}[(\mu-\lambda)^{\frac{1}{2}}(\lambda-\nu)^{\frac{1}{2}}E_\lambda].$$

$$\operatorname{curl}\boldsymbol{E} = \frac{[(\mu-b)(\mu-c)(b-\nu)(c-\nu)(b-\lambda)(\lambda-c)]^{\frac{1}{2}}}{(\mu-\nu)(\mu-\lambda)(\lambda-\nu)} \times \begin{vmatrix} \boldsymbol{a}_\mu\left[\frac{(\mu-\nu)(\mu-\lambda)}{(\mu-b)(\mu-c)}\right]^{\frac{1}{2}} & \boldsymbol{a}_\nu\left[\frac{(\mu-\nu)(\lambda-\nu)}{(b-\nu)(c-\nu)}\right]^{\frac{1}{2}} & \boldsymbol{a}_\lambda\left[\frac{(\lambda-\nu)(\mu-\lambda)}{(b-\lambda)(\lambda-c)}\right]^{\frac{1}{2}} \\ \frac{\partial}{\partial\mu} & \frac{\partial}{\partial\nu} & \frac{\partial}{\partial\lambda} \\ E_\mu\left[\frac{(\mu-\nu)(\mu-\lambda)}{(\mu-b)(\mu-c)}\right]^{\frac{1}{2}} & E_\nu\left[\frac{(\mu-\nu)(\lambda-\nu)}{(b-\nu)(c-\nu)}\right]^{\frac{1}{2}} & E_\lambda\left[\frac{(\lambda-\nu)(\mu-\lambda)}{(b-\lambda)(\lambda-c)}\right]^{\frac{1}{2}} \end{vmatrix}.$$

$$\nabla^2\varphi = \left[\frac{(\mu-b)(\mu-c)}{(\mu-\nu)(\mu-\lambda)}\right]^{\frac{1}{2}}\frac{\partial}{\partial\mu}\left[(\mu-b)^{\frac{1}{2}}(\mu-c)^{\frac{1}{2}}\frac{\partial\varphi}{\partial\mu}\right] + \left[\frac{(b-\nu)(c-\nu)}{(\mu-\nu)(\lambda-\nu)}\right]^{\frac{1}{2}}\frac{\partial}{\partial\nu}\left[(b-\nu)^{\frac{1}{2}}(c-\nu)^{\frac{1}{2}}\frac{\partial\varphi}{\partial\nu}\right] + \left[\frac{(b-\lambda)(\lambda-c)}{(\mu-\lambda)(\lambda-\nu)}\right]^{\frac{1}{2}}\frac{\partial}{\partial\lambda}\left[(b-\lambda)^{\frac{1}{2}}(\lambda-c)^{\frac{1}{2}}\frac{\partial\varphi}{\partial\lambda}\right].$$

SEPARATION OF LAPLACE'S EQUATION, $\nabla^2 \varphi = 0$.

$$\frac{1}{f_i}\frac{d}{du^i}\left(f_i \frac{dU^i}{du^i}\right) + U^i \sum_{j=1}^{3} \Phi_{ij}\alpha_j = 0,$$

where $\alpha_1 = 0$ and $U^1 = M(\mu)$, $U^2 = N(\nu)$, $U^3 = \Lambda(\lambda)$.

General case

$$\begin{cases} (\mu - b)(\mu - c)\dfrac{d^2M}{d\mu^2} + \dfrac{1}{2}[2\mu - (b+c)]\dfrac{dM}{d\mu} - [\alpha_2 - \alpha_3\mu]M = 0, \\ (b - \nu)(c - \nu)\dfrac{d^2N}{d\nu^2} + \dfrac{1}{2}[2\nu - (b+c)]\dfrac{dN}{d\nu} - [\alpha_2 - \alpha_3\nu]N = 0, \\ (b - \lambda)(\lambda - c)\dfrac{d^2\Lambda}{d\lambda^2} - \dfrac{1}{2}[2\lambda - (b+c)]\dfrac{d\Lambda}{d\lambda} + [\alpha_2 - \alpha_3\lambda]\Lambda = 0. \end{cases}$$

If $\alpha_2 = (b+c)q$ *and* $\alpha_3 = -p(p+1)$,

$$(\mu - b)(\mu - c)\frac{d^2M}{d\mu^2} + \frac{1}{2}[2\mu - (b+c)]\frac{dM}{d\mu} - [p(p+1)\mu + q(b+c)]M = 0,$$

$$\{113\} \quad M = A\mathscr{B}_p^q(\mu) + B\mathscr{C}_p^q(\mu).$$

$$(\nu - b)(\nu - c)\frac{d^2N}{d\nu^2} + \frac{1}{2}[2\nu - (b+c)]\frac{dN}{d\nu} - [p(p+1)\nu + q(b+c)]N = 0,$$

$$\{113\} \quad N = A\mathscr{B}_p^q(\nu) + B\mathscr{C}_p^q(\nu).$$

$$(\lambda - b)(\lambda - c)\frac{d^2\Lambda}{d\lambda^2} + \frac{1}{2}[2\lambda - (b+c)]\frac{d\Lambda}{d\lambda} - [p(p+1)\lambda + q(b+c)]\Lambda = 0,$$

$$\{113\} \quad \Lambda = A\mathscr{B}_p^q(\lambda) + B\mathscr{C}_p^q(\lambda).$$

If $\alpha_2 = \alpha_3 = 0$,

$$\frac{d}{d\mu}\left[(\mu - b)^{\frac{1}{2}}(\mu - c)^{\frac{1}{2}}\frac{dM}{d\mu}\right] = 0, \qquad \{01\} \quad M = A + B\ln[2\mu - b - c + 2\sqrt{\mu - b}\,\sqrt{\mu - c}$$

$$\frac{d}{d\nu}\left[(b - \nu)^{\frac{1}{2}}(c - \nu)^{\frac{1}{2}}\frac{dN}{d\nu}\right] = 0, \qquad \{01\} \quad N = A + B\ln[b + c - 2\nu - 2\sqrt{b - \nu}\,\sqrt{c - \nu}\,]$$

$$\frac{d}{d\lambda}\left[(b - \lambda)^{\frac{1}{2}}(\lambda - c)^{\frac{1}{2}}\frac{d\Lambda}{d\lambda}\right] = 0, \qquad \{01\} \quad \Lambda = A + B\sin^{-1}\left[\frac{2\lambda - (b+c)}{b - c}\right].$$

For φ independent of ν and λ,

$$\frac{d}{d\mu}\left[(\mu - b)^{\frac{1}{2}}(\mu - c)^{\frac{1}{2}}\frac{d\varphi}{d\mu}\right] = 0, \qquad \{01\} \quad \varphi = A + B\ln[2\mu - b - c + 2\sqrt{\mu - b}\,\sqrt{\mu - c}$$

For φ independent of μ and λ,

$$\frac{d}{d\nu}\left[(b - \nu)^{\frac{1}{2}}(c - \nu)^{\frac{1}{2}}\frac{d\varphi}{d\nu}\right] = 0, \qquad \{01\} \quad \varphi = A + B\ln[b + c - 2\nu - 2\sqrt{b - \nu}\,\sqrt{c - \nu}\,]$$

For φ independent of μ and ν,

$$\frac{d}{d\lambda}\left[(b - \lambda)^{\frac{1}{2}}(\lambda - c)^{\frac{1}{2}}\frac{d\varphi}{d\lambda}\right] = 0, \qquad \{01\} \quad \varphi = A + B\sin^{-1}\left[\frac{2\lambda - (b+c)}{b - c}\right].$$

SEPARATION OF THE HELMHOLTZ EQUATION, $\nabla^2 \varphi + \varkappa^2 \varphi = 0$.

$$\frac{1}{f_i} \frac{d}{du^i}\left(f_i \frac{dU^i}{du^i}\right) + U^i \sum_{j=1}^{3} \Phi_{ij} \alpha_j = 0,$$

where $\alpha_1 = \varkappa^2$ and $U^1 = M(\mu)$, $U^2 = N(\nu)$, $U^3 = \Lambda(\lambda)$.

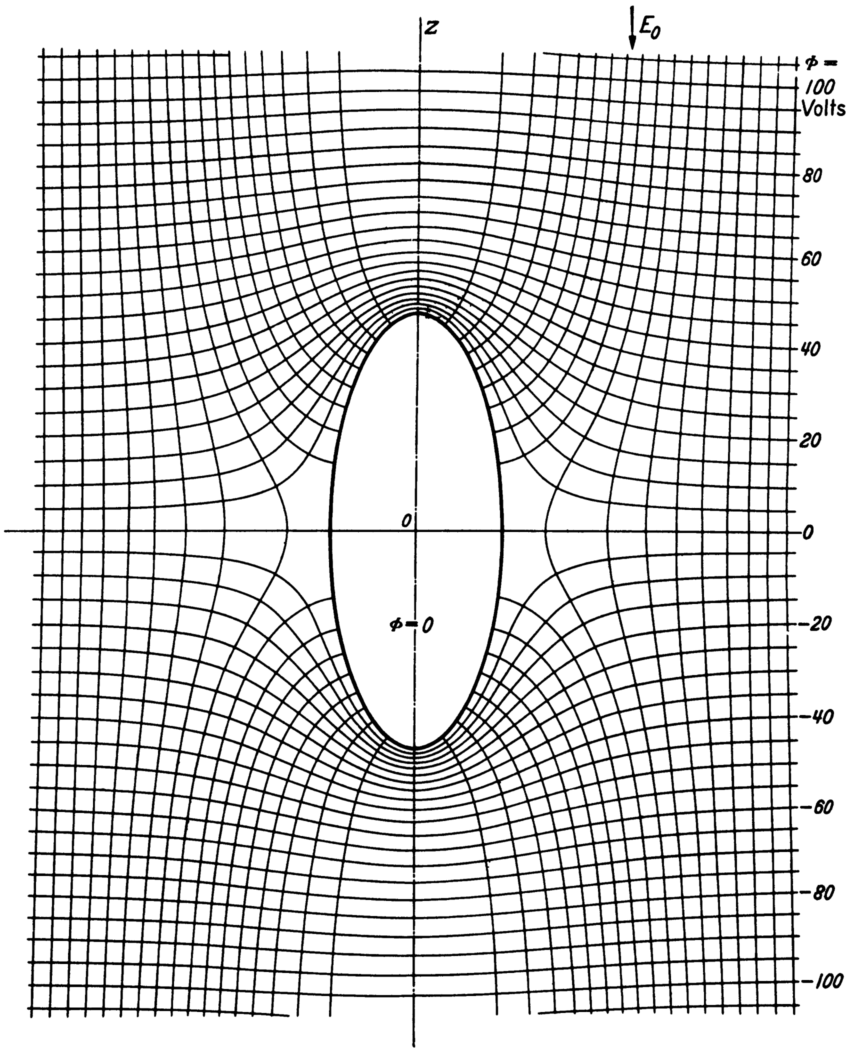

Fig. 1.12. The distorted electric field produced by the introduction of a metal spheroid ($\eta_0 = 0.444$) into a uniform field

General case

$$\begin{cases} (\mu - b)(\mu - c) \dfrac{d^2M}{d\mu^2} + \dfrac{1}{2}[2\mu - (b + c)] \dfrac{dM}{d\mu} + [\varkappa^2 \mu^2 + \alpha_3 \mu - \alpha_2] M = 0, \\ (b - \nu)(c - \nu) \dfrac{d^2N}{d\nu^2} + \dfrac{1}{2}[2\nu - (b + c)] \dfrac{dN}{d\nu} + [\varkappa^2 \nu^2 + \alpha_3 \nu - \alpha_2] N = 0, \\ (b - \lambda)(\lambda - c) \dfrac{d^2\Lambda}{d\lambda^2} - \dfrac{1}{2}[2\lambda - (b + c)] \dfrac{d\Lambda}{d\lambda} - [\varkappa^2 \lambda^2 + \alpha_3 \lambda - \alpha_2] \Lambda = 0. \end{cases}$$

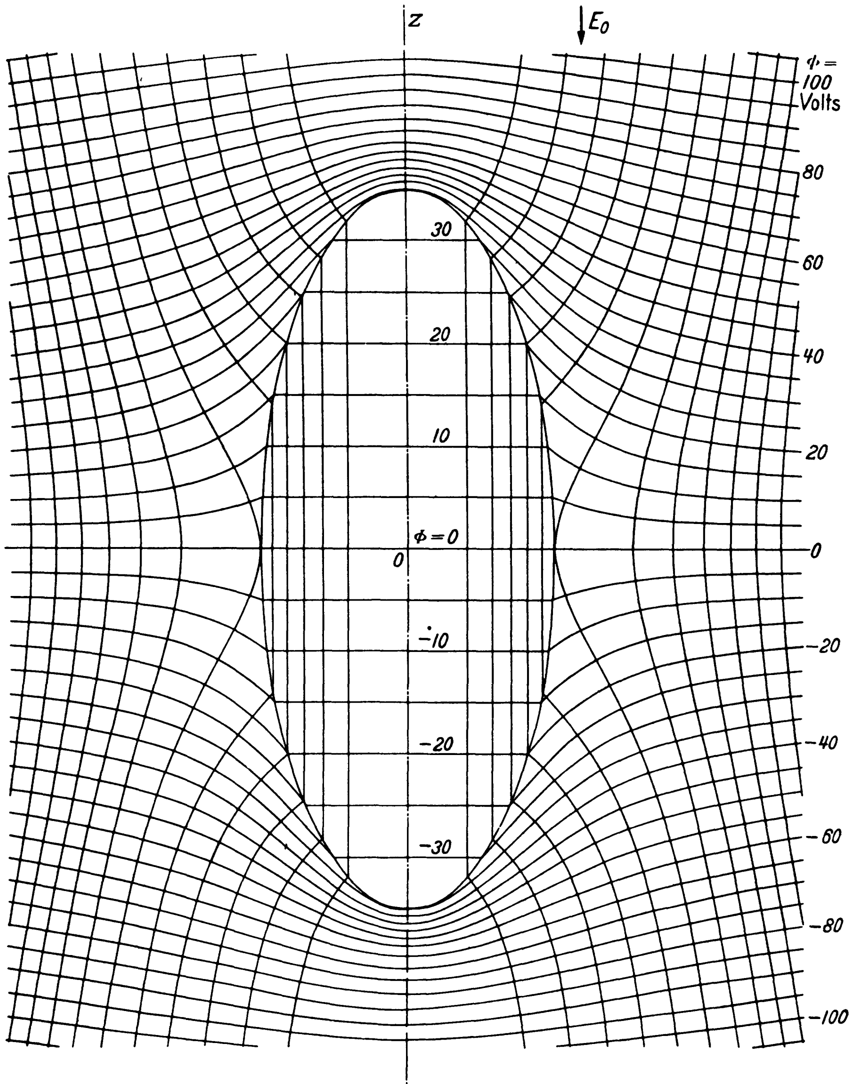

Fig. 1.13. A dielectric spheroid introduced into a uniform electric field ($\varepsilon_2/\varepsilon_1 = 10$, $\eta_0 = 0.444$)

If $\alpha_2 = (b+c)q$ *and* $\alpha_3 = -p(p+1)$,

$$(\mu-b)(\mu-c)\frac{d^2M}{d\mu^2} + \frac{1}{2}[2\mu-(b+c)]\frac{dM}{d\mu} + [\varkappa^2\mu^2 - p(p+1)\mu - q(b+c)]M = 0,$$

$$\{114\} \quad M = A\mathscr{B}_p^q(\varkappa,\mu) + B\mathscr{C}_p^q(\varkappa,\mu).$$

$$(\nu-b)(\nu-c)\frac{d^2N}{d\nu^2} + \frac{1}{2}[2\nu-(b+c)]\frac{dN}{d\nu} + [\varkappa^2\nu^2 - p(p+1)\nu - q(b+c)]N = 0,$$

$$\{114\} \quad N = A\mathscr{B}_p^q(\varkappa,\nu) + B\mathscr{C}_p^q(\varkappa,\nu).$$

$$(\lambda-b)(\lambda-c)\frac{d^2\Lambda}{d\lambda^2} + \frac{1}{2}[2\lambda-(b+c)]\frac{d\Lambda}{d\lambda} + [\varkappa^2\lambda^2 - p(p+1)\lambda - q(b+c)]\Lambda = 0,$$

$$\{114\} \quad \Lambda = A\mathscr{B}_p^q(\varkappa,\lambda) + B\mathscr{C}_p^q(\varkappa,\lambda).$$

Section II

TRANSFORMATIONS IN THE COMPLEX PLANE

The most promising way of extending the engineering applications of field theory is to develop new coordinate systems. Section I listed the eleven systems whose coordinate surfaces are of the first or second degree. KLEIN [*12*] and BÔCHER [*4*] extended this list to include a class of fourth-degree surfaces known as cyclides [*13*]. All possible systems of this class are treated by BÔCHER. They include the eleven coordinates of Section I, as well as more complicated coordinates; but all are either simply separable or R-separable (Section IV).

Another procedure for obtaining new coordinate systems is *inversion* [*14*]. The coordinate surfaces are reflected in a sphere, the new surfaces forming an orthogonal coordinate system which is generally more complicated than the original. The mathematics are simple but will not be considered here.

A third procedure is to form new coordinate systems [*15*] by complex-plane transformations. A rectangular map in the w-plane is transformed into a curvilinear but orthogonal map in the z-plane. The map is then translated to form a new *cylindrical coordinate system* in 3-space, or it is twirled about an axis of symmetry to form a *rotational coordinate system*. Infinitely many systems can be obtained in this way. There is no guarantee, however, that the Helmholtz equation or the Laplace equation will separate in the new coordinates; and this question must be investigated by the methods of Sections I and IV.

The complex-plane transformation is a very fruitful way of extending the totality of coordinate systems. Of course it does not produce the asymmetric systems such as Nos. 9, 10, and 11 of Section I or the general cyclidal systems of BÔCHER [*4*]. But asymmetric systems are of little practical value, so nothing of importance is lost by restriction to the symmetric case.

2.01 CONFORMAL TRANSFORMATIONS [*16*]

Take an arbitrary relation between the w-plane and the z-plane:

$$z = \mathscr{F}(w). \tag{2.01}$$

The function $\mathscr{F}$ may be chosen at random or may be found by the Schwarz-Christoffel method [*16*]. The Cauchy-Riemann equations apply:

$$\frac{\partial x}{\partial u} = \frac{\partial y}{\partial v}, \qquad \frac{\partial x}{\partial v} = -\frac{\partial y}{\partial u}, \tag{2.02}$$

where $w = u + i\,v$, $z = x + i\,y$. Thus angles are preserved by the transformation, and squares in the w-plane always map into curvilinear squares in the z-plane. Separation of Eq. (2.02) into real and imaginary parts gives the two equations,

$$\left.\begin{aligned} x &= \xi_1(u, v), \\ y &= \xi_2(u, v). \end{aligned}\right\} \tag{2.03}$$

These equations can be used in plotting the two families of curves, $u = \text{const}$ and $v = \text{const}$, which form the orthogonal map in the z-plane. Twenty-one transformations of this kind are listed in Tables 2.01 and 2.02, and accurate maps are included. Note that all intersections are at right angles and all subdivisions are curvilinear squares.

We now generate coordinate systems in 3-space. *Cylindrical coordinates* are formed by translating the z-plane map perpendicular to itself, thus forming families of cylinders. The resulting coordinate system (u^1, u^2, u^3) is specified by

$$\left.\begin{aligned} x &= \xi_1(u^1, u^2), \\ y &= \xi_2(u^1, u^2), \\ z &= u^3, \end{aligned}\right\} \tag{2.04}$$

where ξ_1 and ξ_2 are the same as in the plane case, Eq. (2.03). The z-axis is always taken parallel to the generators of the cylinders. Metric coefficients are

$$g_{11} = g_{22} = \left(\frac{\partial \xi_1}{\partial u^1}\right)^2 + \left(\frac{\partial \xi_2}{\partial u^1}\right)^2, \quad g_{33} = 1. \tag{2.05}$$

Data on cylindrical systems are listed in Section III.

If the plane map is rotated about what was originally the y-axis, the *rotational system* (u^1, u^2, ψ) is specified by

$$\left.\begin{aligned} x &= \xi_1(u^1, u^2) \cdot \cos\psi, \\ y &= \xi_1(u^1, u^2) \cdot \sin\psi, \\ z &= \xi_2(u^1, u^2). \end{aligned}\right\} \tag{2.06}$$

Metric coefficients are

$$g_{11} = g_{22} = \left(\frac{\partial \xi_1}{\partial u^1}\right)^2 + \left(\frac{\partial \xi_2}{\partial u^1}\right)^2, \quad g_{33} = [\xi_1(u^1, u^2)]^2. \tag{2.07}$$

If the plane map is rotated about what was originally the x-axis,

$$\left.\begin{aligned} x &= \xi_2(u^1, u^2) \cdot \cos\psi, \\ y &= \xi_2(u^1, u^2) \cdot \sin\psi, \\ z &= \xi_1(u^1, u^2), \end{aligned}\right\} \tag{2.08}$$

and the metric coefficients are

$$g_{11} = g_{22} = \left(\frac{\partial \xi_1}{\partial u^1}\right)^2 + \left(\frac{\partial \xi_2}{\partial u^1}\right)^2, \quad g_{33} = [\xi_2(u^1, u^2)]^2. \tag{2.09}$$

In all rotational coordinate systems, the axis of rotation is called the z-axis and the angle about this axis is called ψ. Data on a number of rotational systems are given in Section IV.

Conformal transformations have been widely used in *two-dimensional* field problems [*17*], particularly in electrostatics. Practical applications have stimulated the study of a great number of such transformations [*18*], and these transformations can be employed in the development of new coordinate systems. Note, however, that the previous applications have been valid only for two-dimensional fields and usually only where the potential is constant on the lines $v = \text{const}$. In the present treatment, on the other hand, we are using the transformation as a foundation for coordinate systems in 3-space; and this method allows a wide variety of applications with arbitrary potential distributions on the boundaries.

TABLE 2.01. TRANSFORMATIONS

in the complex plane,

$$z = \mathscr{F}(w),$$

where $z = x + i\,y,\ w = u + i\,v;\ \bar{z} = x - i\,y,\ \bar{w} = u - i\,v.$

No.	Equation	Designation	Fig. No.
	Power Functions		
P 1	$\bar{z} = 1/w$	Tangent circles	2.01
P 2	$z = \frac{1}{2} w^2$	Parabolas	2.02
P 3	$\bar{z} = \frac{1}{2} w^{-2}$	Cardioids	2.03
P 4	$z = \sqrt{2}\, w^{\frac{1}{2}}$	Hyperbolas	2.04
P 5	$\bar{z} = \sqrt{2}\, w^{-\frac{1}{2}}$	4-leaf Roses	2.05
	Exponential Functions		
E 1	$z = e^w$	Circles	2.06
E 2	$z = a(e^w + 1)^{\frac{1}{2}}$	Cassinian ovals	2.07
E 3	$\bar{z} = a(e^w + 1)^{-\frac{1}{2}}$	Inverse Cassinian ovals	2.08
E 4	$\bar{z} = \dfrac{a(e^w + 1)}{e^w - 1}$	Bipolar circles	2.09
E 5	$z = \dfrac{a}{\pi}(w + 1 + e^w)$	Maxwell curves	2.10
	Logarithmic Functions		
L 1	$z = \dfrac{2a}{\pi} \ln w$	Logarithmic curves	2.11
L 2	$z = \dfrac{2a}{\pi} \ln \tan w - i\,a$	ln tan	2.12
L 3	$z = \dfrac{2a}{\pi} \ln \cosh w$	ln cosh	2.13
	Hyperbolic Functions		
H 1	$z = a \cosh w$	Ellipses	2.14
H 2	$z = a\,\mathrm{sech}\, w$	Inverse ellipses	2.15
	Elliptic Functions		
J 1	$z = a\,\mathrm{sn}\, w$	sn	2.16
J 2	$\bar{z} = a\,\mathrm{cn}\, w$	cn	2.17
J 3	$\bar{z} = \dfrac{k^{\frac{1}{2}}}{i\,2a}\left(\dfrac{1 + i k^{\frac{1}{2}}\,\mathrm{sn}\, w}{1 - i k^{\frac{1}{2}}\,\mathrm{sn}\, w}\right)$	Inverse sn	2.18
J 4	$\bar{z} = \dfrac{a}{\pi} \ln\left(\dfrac{1}{k\,\mathrm{sn}^2 w}\right)$	ln sn	2.19
J 5	$\bar{z} = \dfrac{2a}{\pi} \ln \mathrm{cn}\, w$	ln cn	2.20
J 6	$\bar{z} = \dfrac{2Ka}{\pi} Z(w + i\,K') + i\,a$	Zeta function	2.21

TABLE 2.02. TRANSFORMATIONS

No.	Transformation	Fig. No.	Inversion of	x	y	g_{11}
			Power Functions			
P 1	$\bar{z} = 1/w$	2.01	Rect-ang. coords.	$\frac{u}{u^2+v^2}$	$\frac{v}{u^2+v^2}$	$\frac{1}{(u^2+v^2)^2}$
P 2	$z = \frac{1}{2} w^2$	2.02	P 3	$\frac{1}{2}(u^2 - v^2)$	$u\,v$	$u^2 + v^2$
P 3	$\bar{z} = \frac{1}{2} w^{-2}$	2.03	P 2	$\frac{1}{2}\frac{u^2-v^2}{(u^2+v^2)^2}$	$\frac{u\,v}{(u^2+v^2)^2}$	$\frac{1}{(u^2+v^2)^3}$
P 4	$z = \sqrt{2}\, w^{\frac{1}{2}}$	2.04	P 5	$(\varrho + u)^{\frac{1}{2}}$ where $\varrho = +(u^2+v^2)^{\frac{1}{2}}$	$(\varrho - u)^{\frac{1}{2}}$	$\frac{1}{2}(u^2+v^2)^{-\frac{1}{2}}$
P 5	$\bar{z} = \sqrt{2}\, w^{-\frac{1}{2}}$	2.05	P 4	$\frac{1}{\varrho}(\varrho + u)^{\frac{1}{2}}$ where $\varrho = +(u^2+v^2)^{\frac{1}{2}}$	$\frac{1}{\varrho}(\varrho - u)^{\frac{1}{2}}$	$\frac{1}{2}(u^2+v^2)^{-\frac{3}{2}}$
			Exponentials			
E 1	$z = e^w$	2.06	E 1	$e^u \cos v$	$e^u \sin v$	e^{2u}
E 2	$z = a(e^w + 1)^{\frac{1}{2}}$	2.07	E 3	$\frac{a}{\sqrt{2}}[\varrho_1 + (e^u \cos v + 1)]^{\frac{1}{2}}$	$\frac{a}{\sqrt{2}}[\varrho_1 - (e^u \cos v + 1)]^{\frac{1}{2}}$	$\frac{a^2 e^{2u}}{4\varrho_1}$

E3	$\bar{z} = a(e^w + 1)^{-\frac{1}{2}}$	2.08	E2	$\frac{a}{\sqrt{2}\varrho_1}[\varrho_1 + (e^u \cos v + 1)]^{\frac{1}{2}}$ where $\varrho_1 = +(e^{2u} + 2e^u \cos v + 1)^{\frac{1}{2}}$	$\frac{a}{\sqrt{2}\varrho_1}[\varrho_1 - (e^u \cos v + 1)]^{\frac{1}{2}}$	$\frac{a^2 e^{2u}}{4\varrho_1^3}$
E4	$\bar{z} = \frac{a(e^w + 1)}{e^w - 1}$	2.09	E1	$\frac{a \sinh u}{\cosh u - \cos v}$	$\frac{a \sin v}{\cosh u - \cos v}$	$\frac{a^2}{(\cosh u - \cos v)^2}$
E5	$z = \frac{a}{\pi}(w + 1 + e^w)$	2.10	—	$\frac{a}{\pi}(u + 1 + e^u \cos v)$	$\frac{a}{\pi}(v + e^u \sin v)$	$\left(\frac{a}{\pi}\right)^2 (e^{2u} + 2e^u \cos v + 1)$
				Logarithmic Functions		
L1	$z = \frac{2a}{\pi} \ln w$	2.11	—	$\frac{a}{\pi} \ln(u^2 + v^2)$	$\frac{2a}{\pi} \tan^{-1}(v/u)$	$\left(\frac{2a}{\pi}\right)^2 \frac{1}{u^2 + v^2}$
L2	$z = \frac{2a}{\pi} \ln \tan w - ia$	2.12	—	$\frac{a}{\pi} \ln\left[\frac{\sin^2 u + \sinh^2 v}{\cos^2 u + \sinh^2 v}\right]$	$\frac{2a}{\pi} \tan^{-1}\left(\frac{\sinh 2v}{\sin 2u}\right)$	$\left(\frac{4a}{\pi}\right)^2 \frac{1}{\sin^2 2u + \sinh^2 2v}$
L3	$z = \frac{2a}{\pi} \ln \cosh w$	2.13	—	$\frac{a}{\pi} \ln(\cosh^2 u - \sin^2 v)$	$\frac{2a}{\pi} \tan^{-1}(\tanh u \tan v)$	$\left(\frac{2a}{\pi}\right)^2 \left\{\frac{\cosh^2 u \sinh^2 u + [\sinh u \cosh u + \sin v \cos v]^2}{(\cosh^2 u - \sin^2 v)^2}\right\}$
				Hyperbolic Functions		
H1	$z = a \cosh w$	2.14	H2	$a \cosh u \cos v$	$a \sinh u \sin v$	$a^2(\cosh^2 u - \cos^2 v)$
H2	$\bar{z} = a\,\mathrm{sech}\, w$	2.15	H1	$\frac{a \cosh u \cos v}{\cosh^2 u - \sin^2 v}$	$\frac{a \sinh u \sin v}{\cosh^2 u - \sin^2 v}$	$\frac{a^2(\cosh^2 u - \cos^2 v)}{(\cosh^2 u - \sin^2 v)^2}$

Table 2.02. Continuation

No.	Transformation	Fig. No.	Inversion of	x	y	g_{11}
Elliptic Functions						
J 1	$z = a \operatorname{sn} w$	2.16	J 3	$\frac{a}{\Lambda} \operatorname{sn} u \operatorname{dn} v$ where $\Lambda = 1 - \operatorname{dn}^2 u \operatorname{sn}^2 v$	$\frac{a}{\Lambda} \operatorname{cn} u \operatorname{dn} u \operatorname{sn} v \operatorname{cn} v$	$\frac{a^2 \Omega^2}{\Lambda^2}$ where $\Omega^2 = (1 - \operatorname{sn}^2 u \operatorname{dn}^2 v) \times (\operatorname{dn}^2 v - k^2 \operatorname{sn}^2 u)$
J 2	$\bar{z} = a \operatorname{cn} w$	2.17	J 2	$\frac{a}{\Lambda} \operatorname{cn} u \operatorname{cn} v$	$\frac{a}{\Lambda} \operatorname{sn} u \operatorname{dn} u \operatorname{sn} v \operatorname{dn} v$	$\frac{a^2 \Gamma^2}{\Lambda^2}$ where $\Gamma^2 = (\operatorname{sn}^2 v + \operatorname{sn}^2 u \operatorname{cn}^2 v) \times (\operatorname{dn}^2 v - k^2 \operatorname{sn}^2 u)$
J 3	$\bar{z} = \frac{k^{\frac{1}{2}}}{i\,2a} \left(\frac{1 + i k^{\frac{1}{2}} \operatorname{sn} w}{1 - i k^{\frac{1}{2}} \operatorname{sn} w} \right)$	2.18	J 1	$\frac{\Lambda}{a \Upsilon} \operatorname{sn} u \operatorname{dn} v$ where $\Upsilon = \operatorname{sn}^2 u \operatorname{dn}^2 v + [(\Lambda/\sqrt{k}) + \operatorname{cn} u \operatorname{dn} u \operatorname{sn} v \operatorname{cn} v]^2$	$\frac{k^{\frac{1}{2}} \Pi}{2 a \Upsilon}$ where $\Pi = (\Lambda^2/k) - (\operatorname{sn}^2 u \operatorname{dn}^2 u + \operatorname{cn}^2 u \times \operatorname{dn}^2 u \operatorname{sn}^2 v \operatorname{cn}^2 v)$	$\left(\frac{\Lambda \Omega}{a \Upsilon} \right)^2$
J 4	$\bar{z} = \frac{a}{\pi} \ln \left(\frac{1}{k \operatorname{sn}^2 w} \right)$	2.19	—	$\frac{a}{\pi} \ln \left(\frac{\Lambda^2}{k \Sigma} \right)$	$\frac{2a}{\pi} \times \tan^{-1} \left(\frac{\operatorname{cn} u \operatorname{dn} u \operatorname{sn} v \operatorname{cn} v}{\operatorname{sn} u \operatorname{dn} v} \right)$	$\left(\frac{2 a \Theta}{\pi \Sigma} \right)^2$ where $\Sigma = \operatorname{sn}^2 u \operatorname{dn}^2 v + \operatorname{cn}^2 u \operatorname{dn}^2 u \operatorname{sn}^2 v \operatorname{cn}^2 v$; $\Theta = \operatorname{sn}^2 u \operatorname{cn}^2 u \operatorname{dn}^2 u (\operatorname{cn}^2 v - \operatorname{sn}^2 v \operatorname{dn}^2 v)^2 + \operatorname{sn}^2 v \operatorname{cn}^2 v \times \operatorname{dn}^2 v (\operatorname{cn}^2 u + \operatorname{sn}^2 u \operatorname{dn}^2 u)^2$

J 5	$\bar{z} = \frac{2a}{\pi} \ln \operatorname{cn} w$	2.20	—	$\frac{a}{\pi} (\Xi / \Lambda^2)$	$\frac{2a}{\pi} \times \tan^{-1}\left(\frac{\operatorname{sn} u \operatorname{dn} u \operatorname{sn} v \operatorname{dn} v}{\operatorname{cn} u \operatorname{cn} v}\right)$	$\left(\frac{2a}{\pi \Xi}\right)^2 [\operatorname{sn}^2 u \operatorname{cn}^2 u \operatorname{dn}^2 u (\operatorname{dn}^2 v - k'^2 \operatorname{sn}^2 v \operatorname{cn}^2 v)^2 + \operatorname{sn}^2 v \times \operatorname{cn}^2 v \operatorname{dn}^2 v (\operatorname{dn}^2 u - k^2 \operatorname{sn}^2 u \operatorname{cn}^2 u)^2]$ where $\Xi = \operatorname{cn}^2 u \operatorname{cn}^2 v + \operatorname{sn}^2 u \operatorname{dn}^2 u \operatorname{sn}^2 v \operatorname{dn}^2 v$
J 6	$\bar{z} = \frac{2Ka}{\pi} Z(w + iK') + ia$	2.21	—	$\frac{2Ka}{\pi} \left\{Z(u) + \frac{k^2}{\Lambda'} \times \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u \operatorname{sn}^2 v'\right\}$	$\frac{2Ka}{\pi} \left\{Z(v') + \frac{\pi v}{2KK'} - \frac{1}{\Lambda'} \times \operatorname{dn}^2 u \operatorname{sn} v' \operatorname{cn} v' \operatorname{dn} v'\right\}$	$\left(\frac{2Ka}{\pi}\right)^2 \left\{\frac{4k^4}{\Lambda'^4} \operatorname{sn}^2 u \operatorname{cn}^2 u \times \operatorname{dn}^2 u \operatorname{sn}^2 v' \operatorname{cn}^2 v' \operatorname{dn}^2 v' + \left[\operatorname{dn}^2 u - \frac{E}{K} + \frac{k^2 \operatorname{sn}^2 v'}{\Lambda'^2} (\operatorname{dn}^2 u \times \operatorname{cn}^2 v' (\operatorname{cn}^2 u - \operatorname{sn}^2 u \operatorname{dn}^2 u) - k^2 \operatorname{sn}^2 u \cdot (1 - k^2 \operatorname{sn}^4 u))\right]^2\right\}$ where $\Lambda' = 1 - \operatorname{dn}^2 u \operatorname{sn}^2 v'$, $v' = v + K'$, $K = \int_0^{\pi/2} \frac{d\varphi}{(1 - k^2 \sin^2 \varphi)^{\frac{1}{2}}}$, $K' = \int_0^{\pi/2} \frac{d\varphi}{(1 - k'^2 \sin^2 \varphi)^{\frac{1}{2}}}$, $E = \int_0^{\pi/2} (1 - k^2 \sin^2 \varphi)^{\frac{1}{2}} d\varphi$

With all elliptic functions, the modulus k is associated with the variable u and k' is associated with v. For example, $\operatorname{sn} v \equiv \operatorname{sn}(v, k')$, $\operatorname{dn} u \equiv \operatorname{dn}(u, k)$.

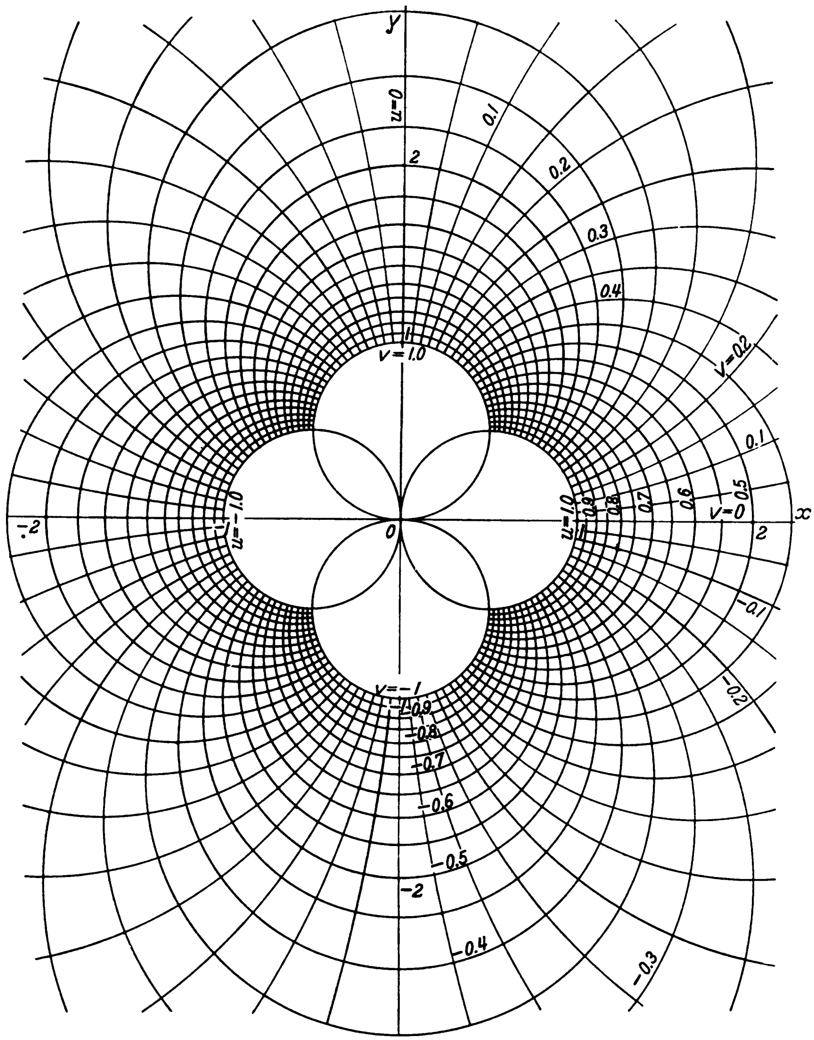

Fig. 2.01. Tangent circles obtained by the transformation $\bar{z} = 1/w$ (Transformation P1)

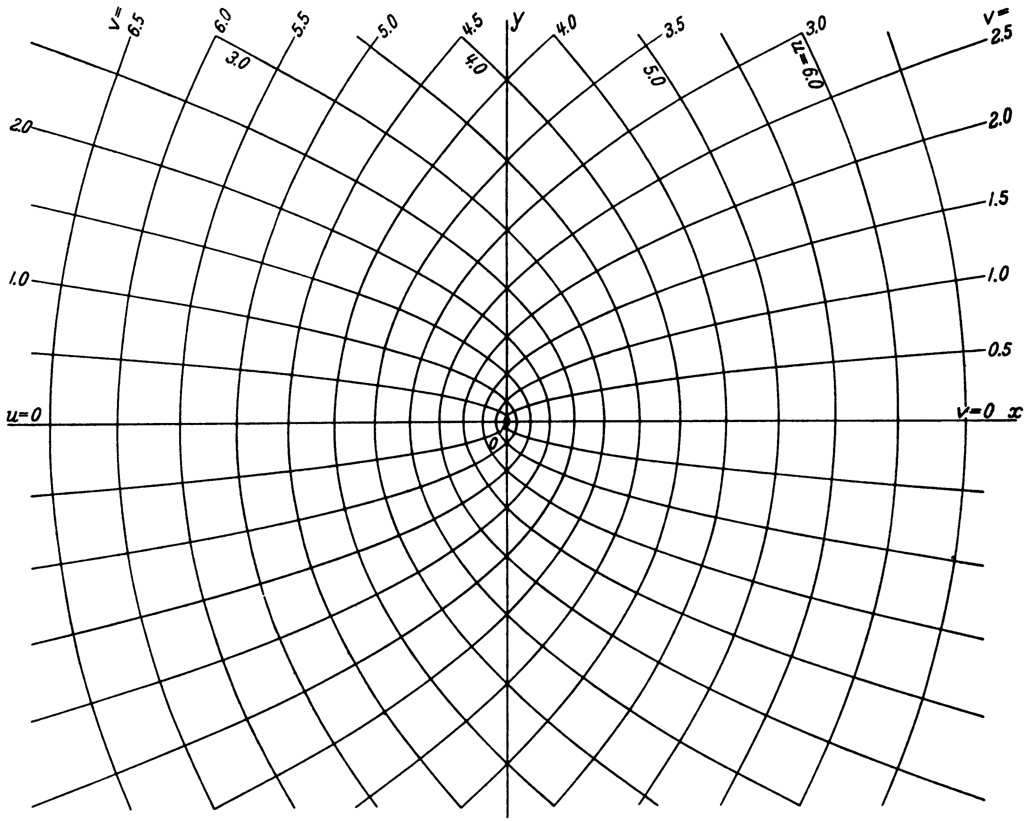

Fig. 2.02. Parabolas, $z = \frac{1}{2} w^2$ (P 2)

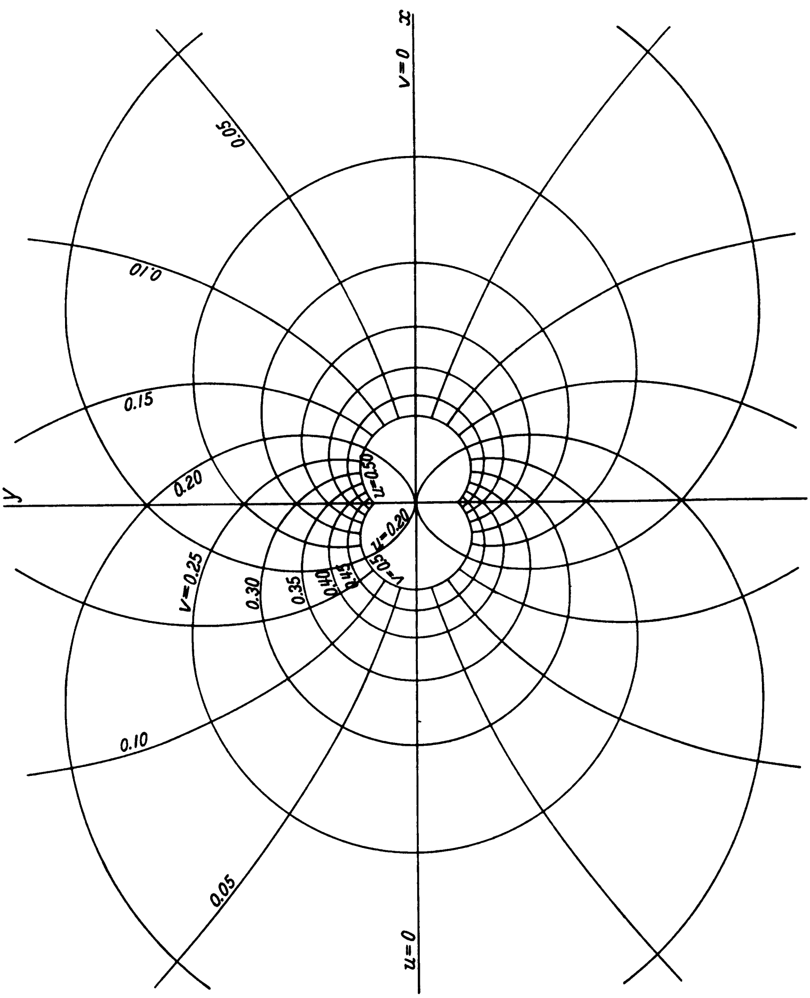

Fig. 2.03. Cardioids, $\bar{z} = \frac{1}{2} w^{-2}$ (P 3)

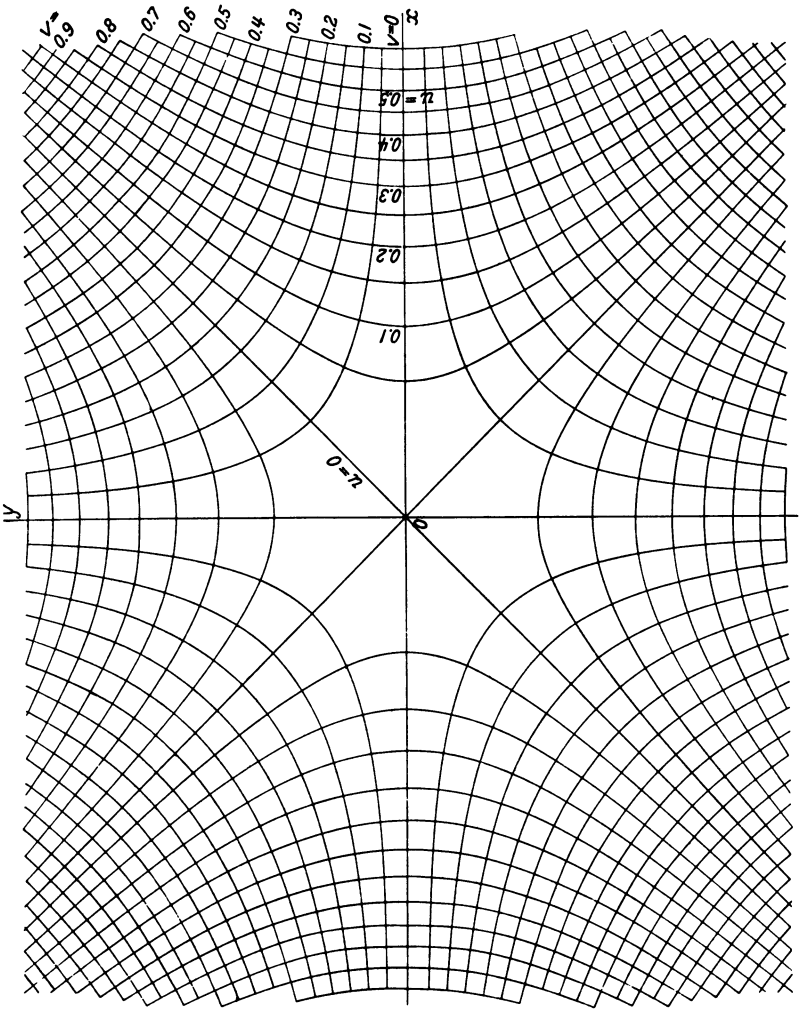

Fig. 2.04. Hyperbolas, $z = \sqrt{2}\, w^{\frac{1}{2}}$ (P 4)

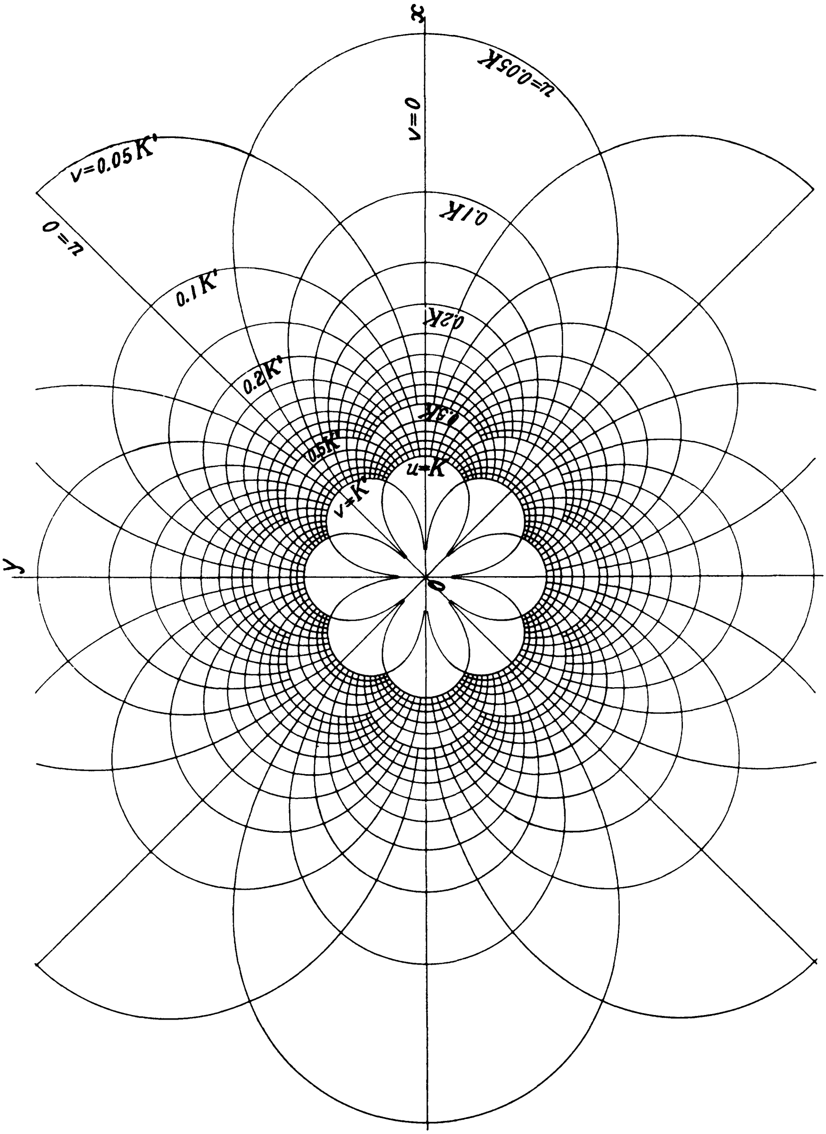

Fig. 2.05. 4-leaf roses, $\bar{z} = \sqrt{2}\, w^{-\frac{1}{2}}$ (P 5)

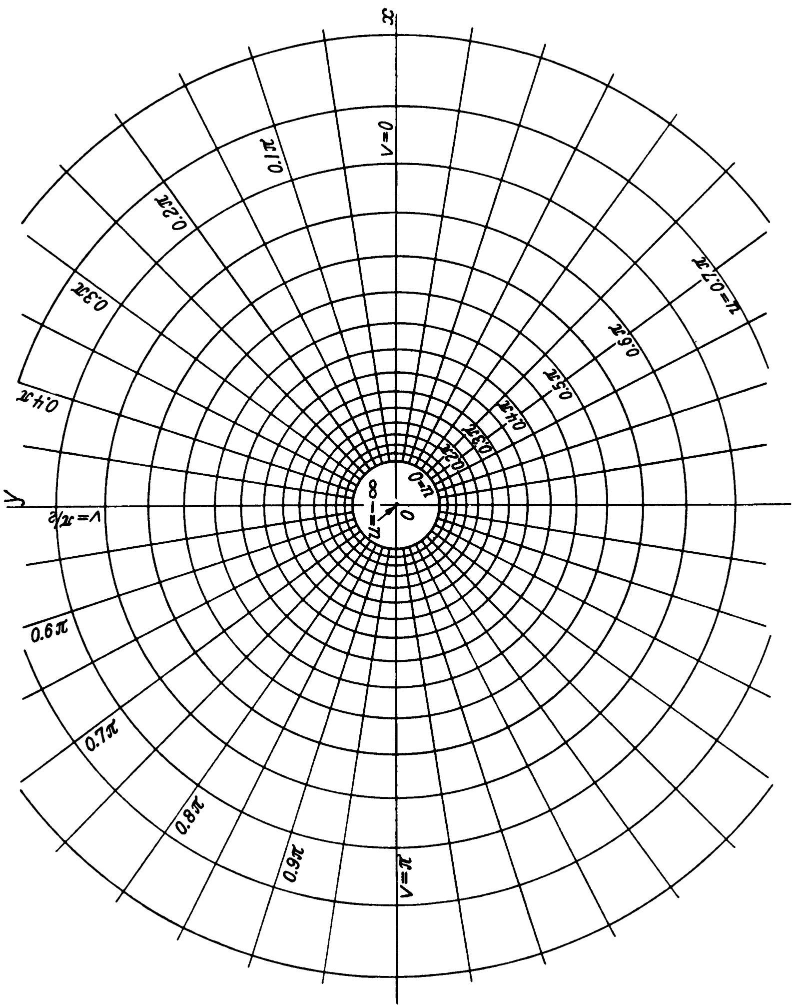

Fig. 2.06. Circles, $z = e^w$ (E 1)

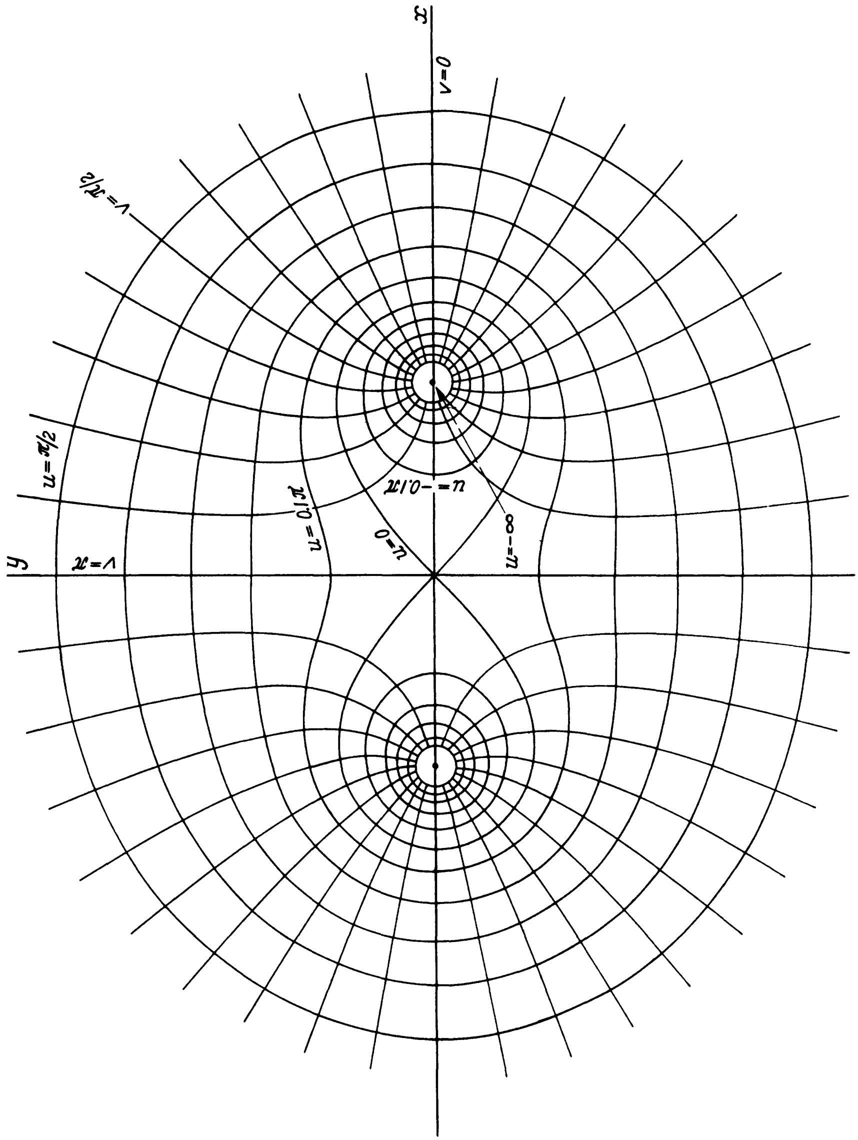

Fig. 2.07. Cassinian ovals, $z = a(e^{w}+1)^{\frac{1}{2}}$ (E 2)

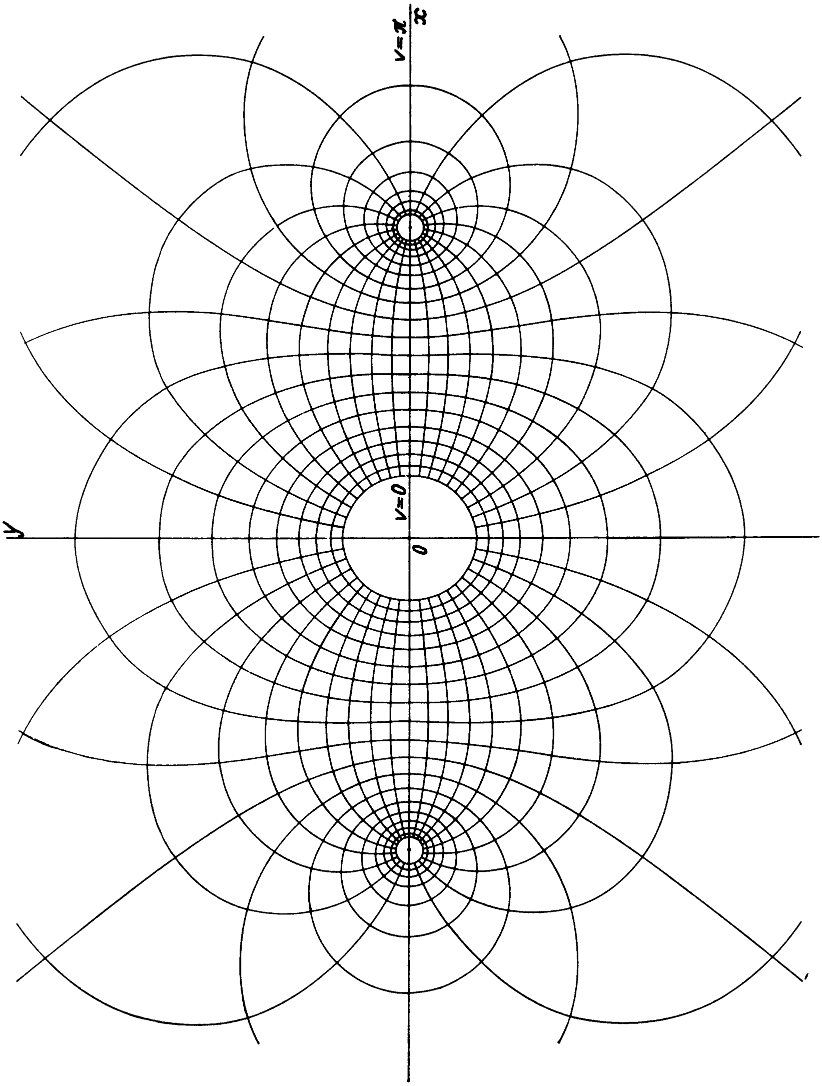

Fig. 2.08. Inverse Cassinian ovals, $\bar{z} = a(e^{w} + 1)^{-\frac{1}{2}}$ (E 3)

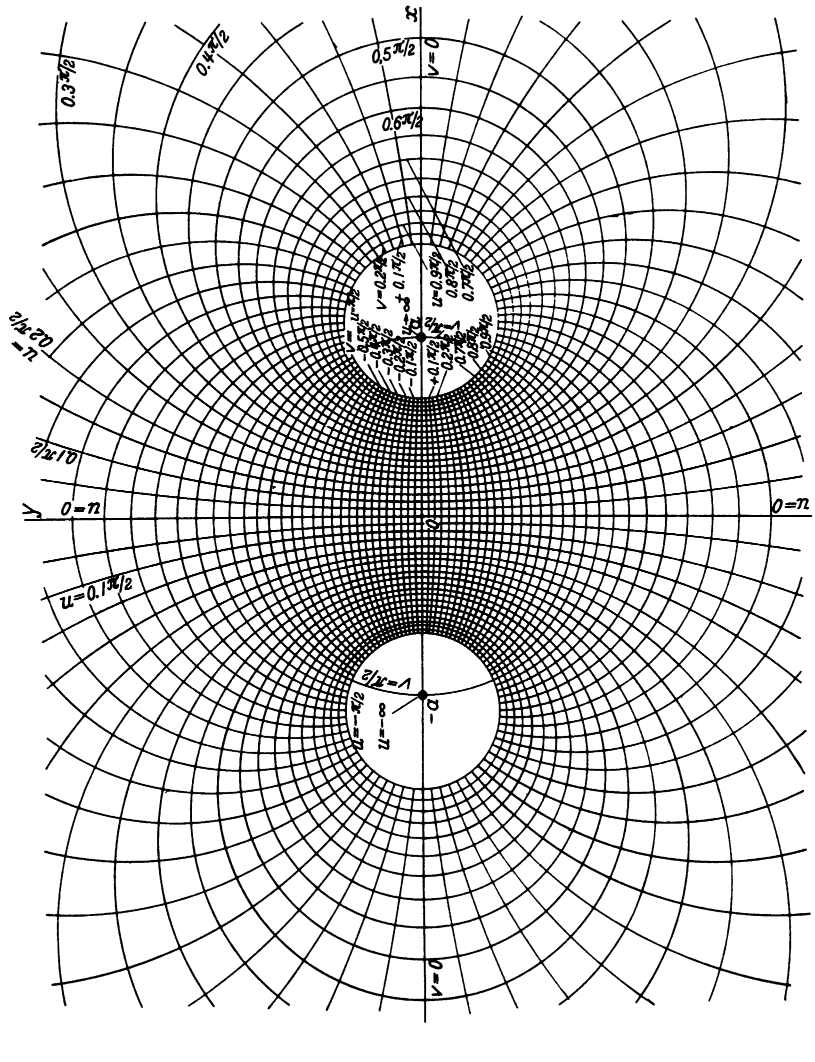

Fig. 2.09. Bipolar circles, $\bar{z} = \dfrac{a(e^{w}+1)}{e^{w}-1}$ (E 4)

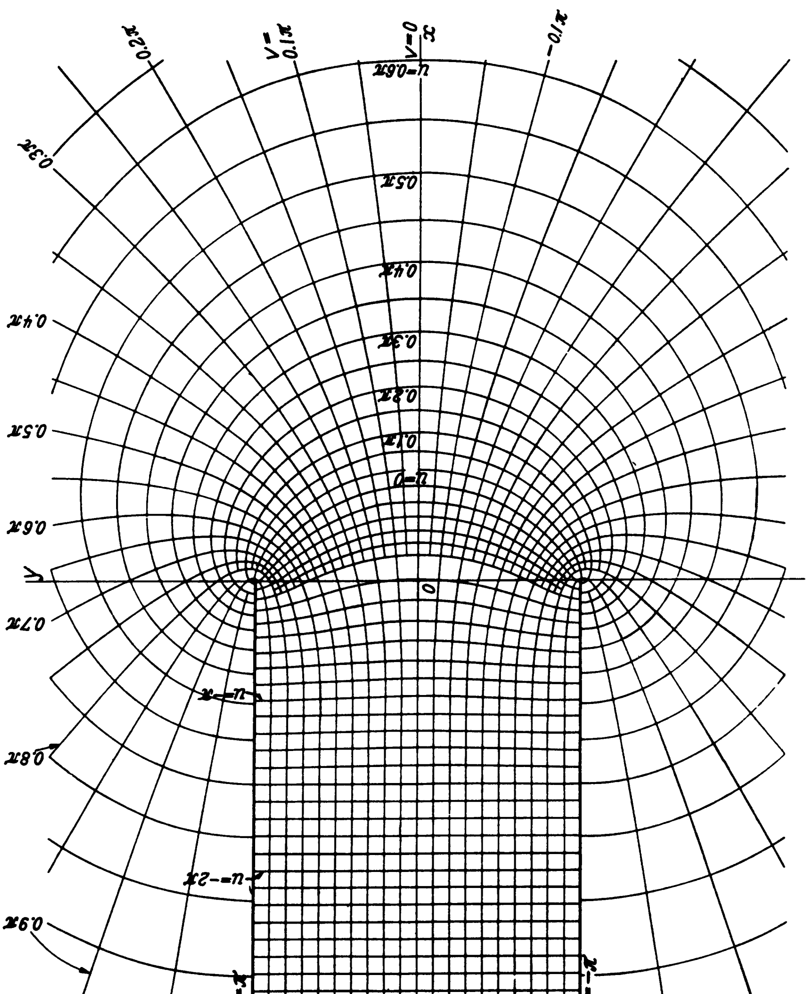

Fig. 2.10. Maxwell curves, $z = \frac{a}{\pi}(w + 1 + e^w)$ (E 5)

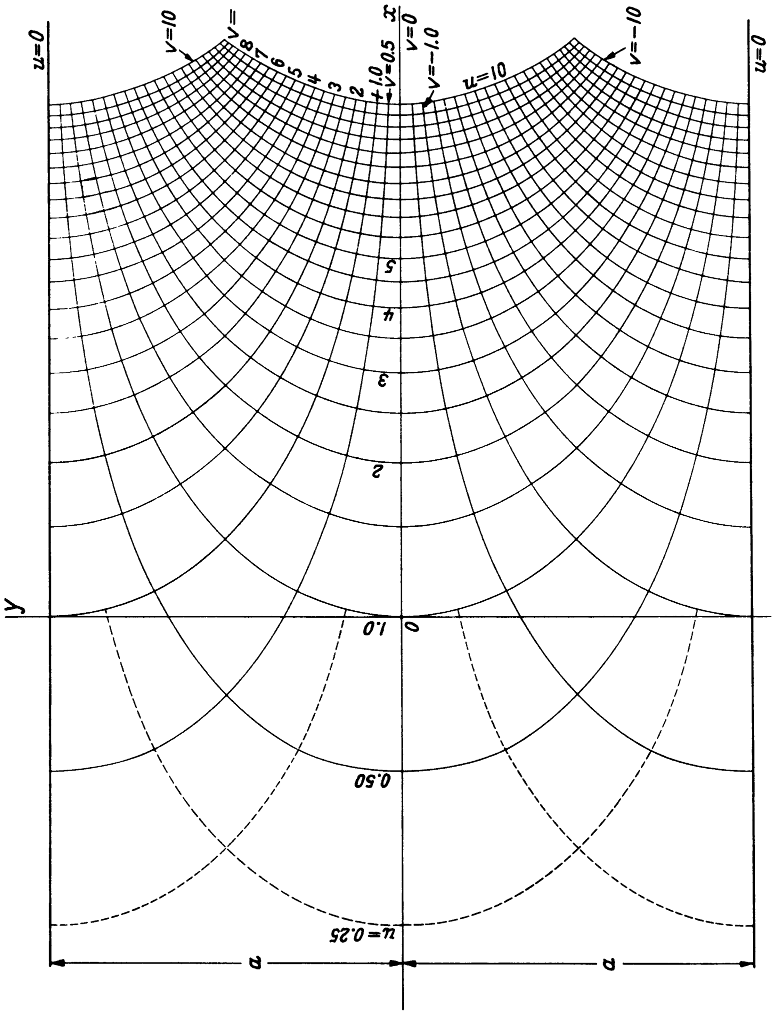

Fig. 2.11. Logarithmic curves, $z = \frac{2a}{\pi} \ln w$ (L 1)

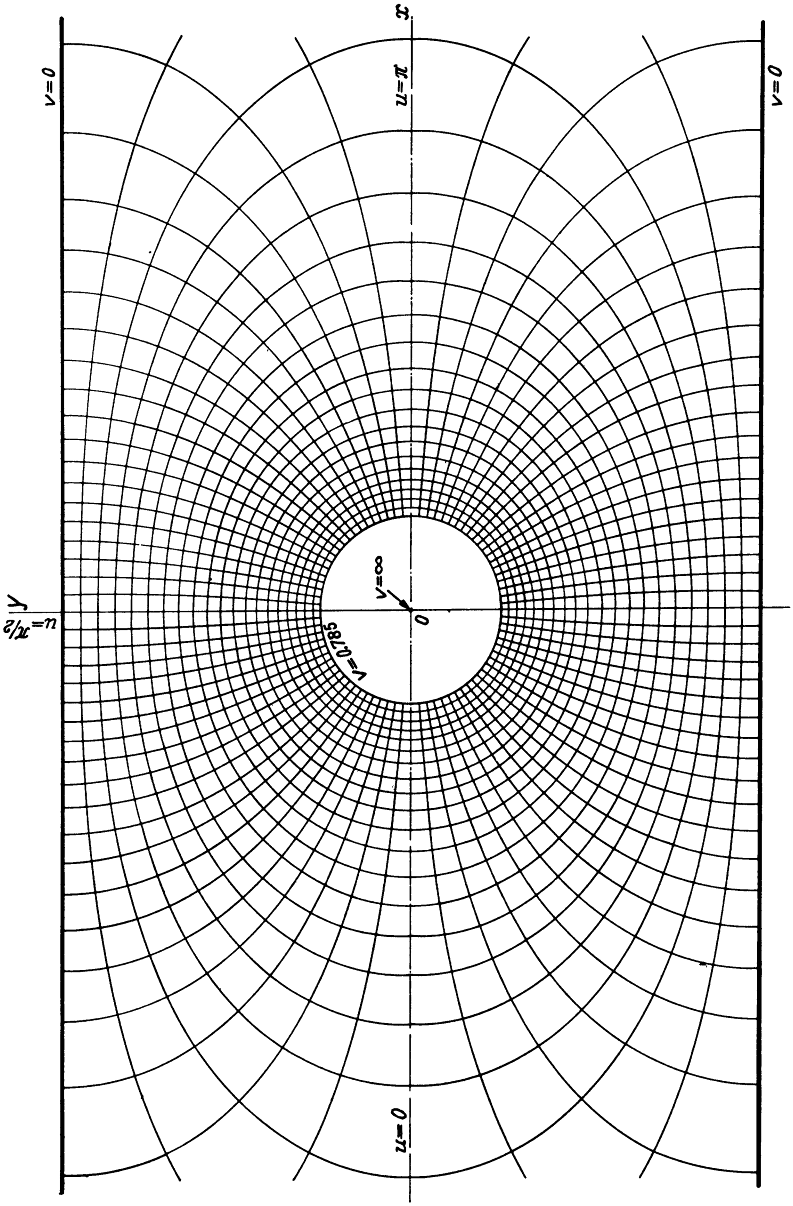

Fig. 2.12. ln tan curves, $z = \frac{2a}{\pi} \ln \tan w - ia$ (L 2)

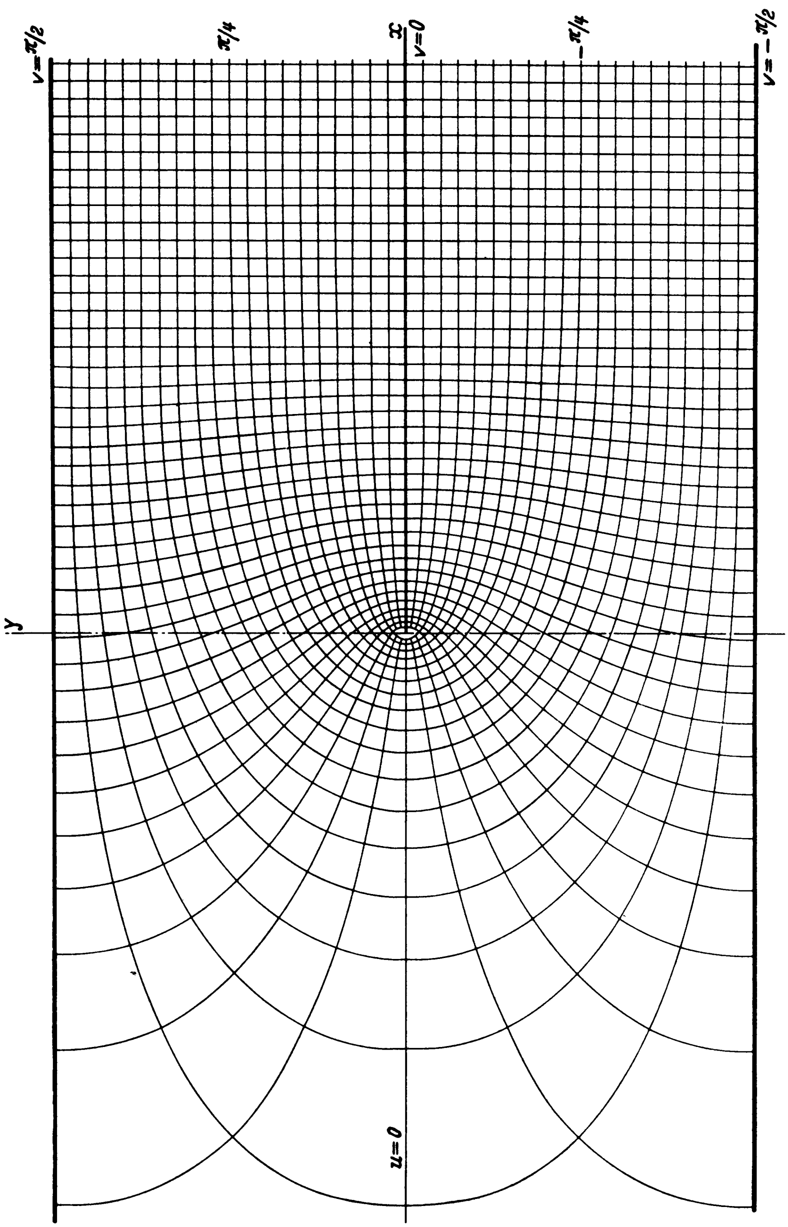

Fig. 2.13. ln cosh curves, $z = \frac{2a}{\pi} \ln \cosh w$ (I. 3)

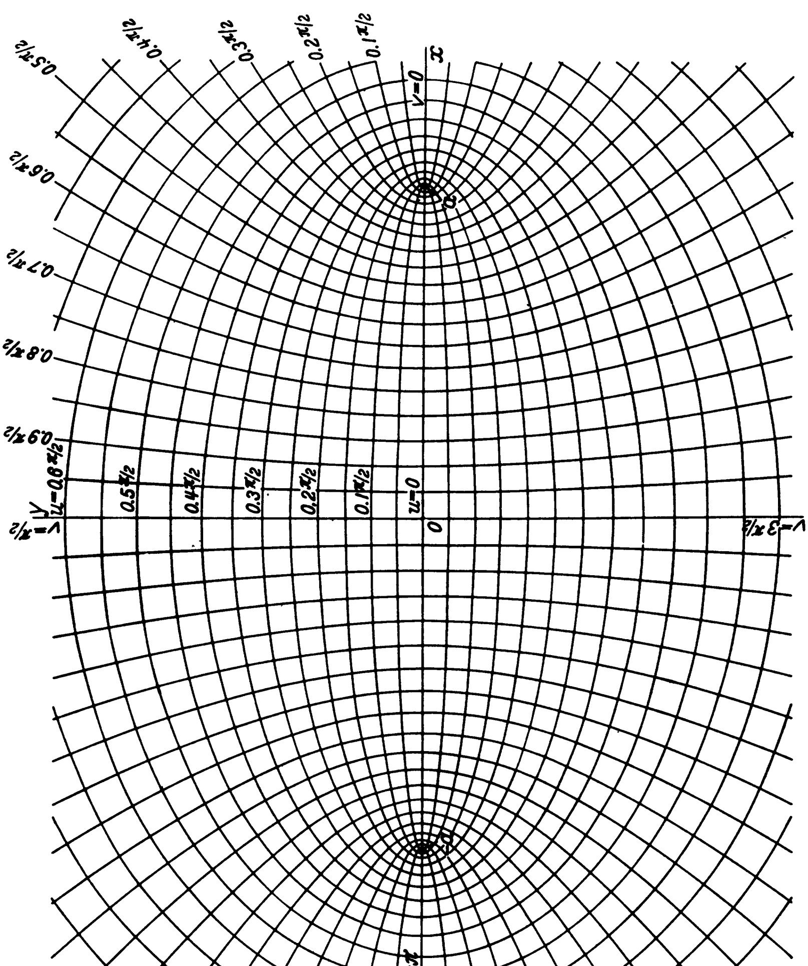

Fig. 2.14. Ellipses, $z = a \cosh w$ (H 1)

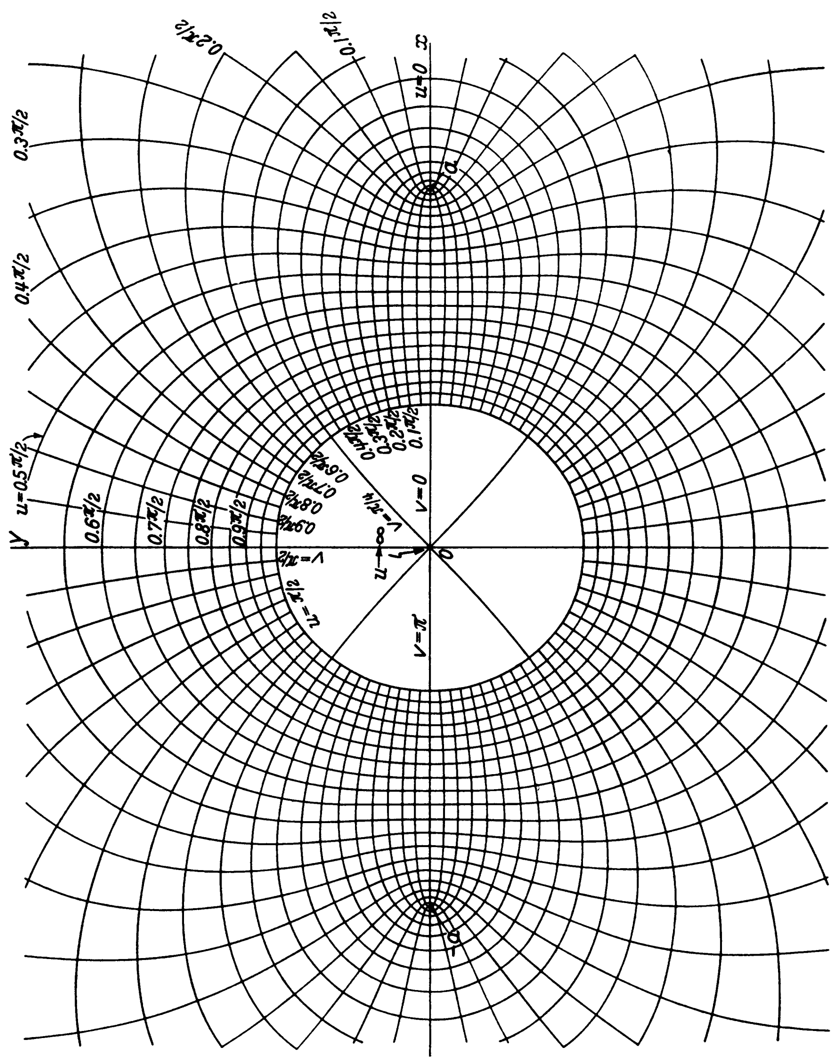

Fig. 2.15. Inverse ellipses, $\bar{z} = a \operatorname{sech} w$ (H 2)

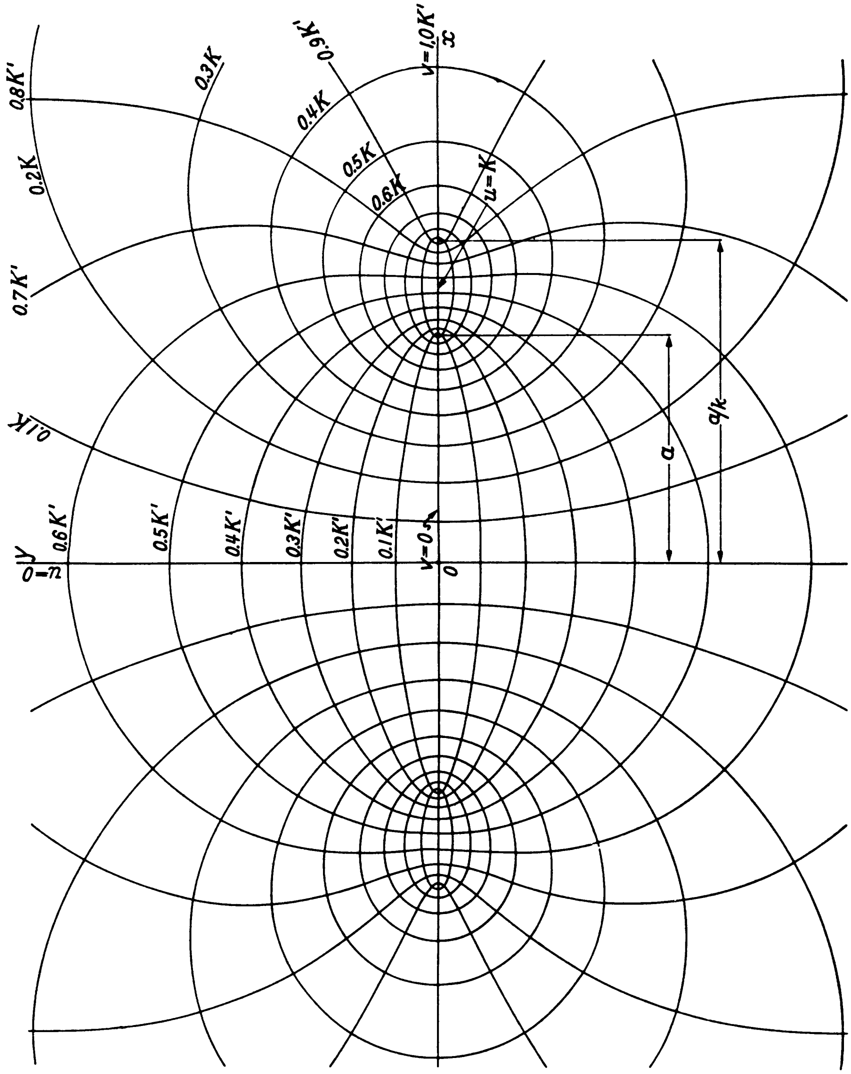

Fig. 2.16. sn curves, $z = a\,\mathrm{sn}\,w$ (J 1)

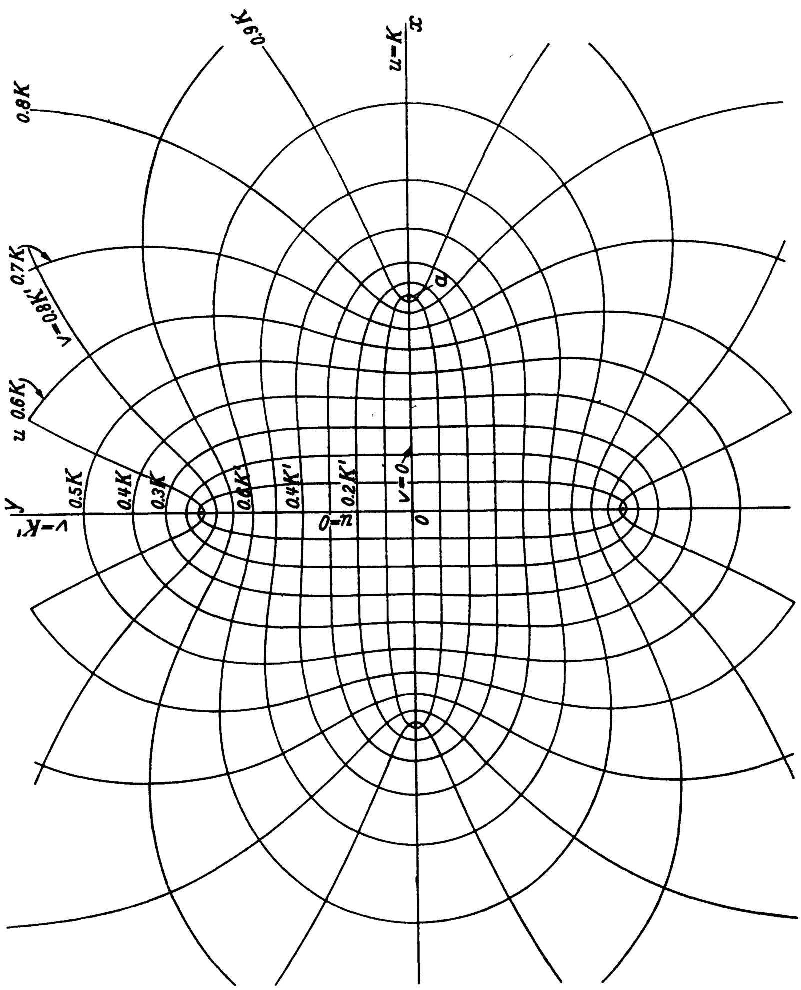

Fig. 2.17. cn curves, $\bar{z} = a\,\mathrm{cn}\,\bar{w}$ (J 2)

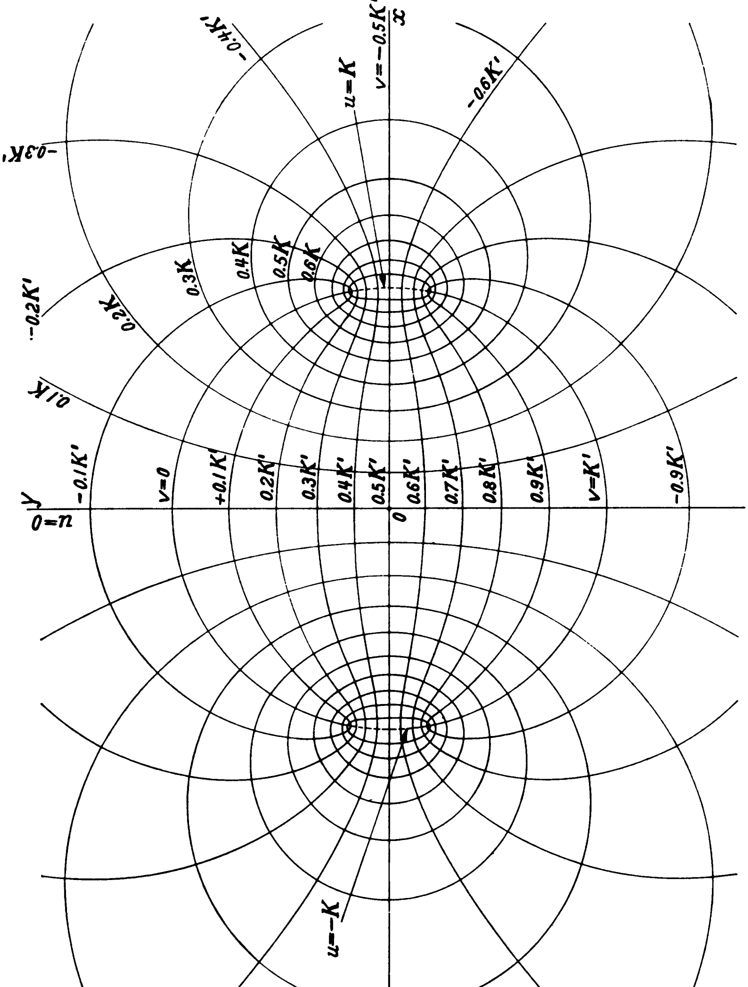

Fig. 2.18. Inverse sn, $\bar{z} = \frac{k^{1/2}}{i2a}\left(\frac{1 + ik^{1/2}\operatorname{sn} w}{1 - ik^{1/2}\operatorname{sn} w}\right)$ (J 3)

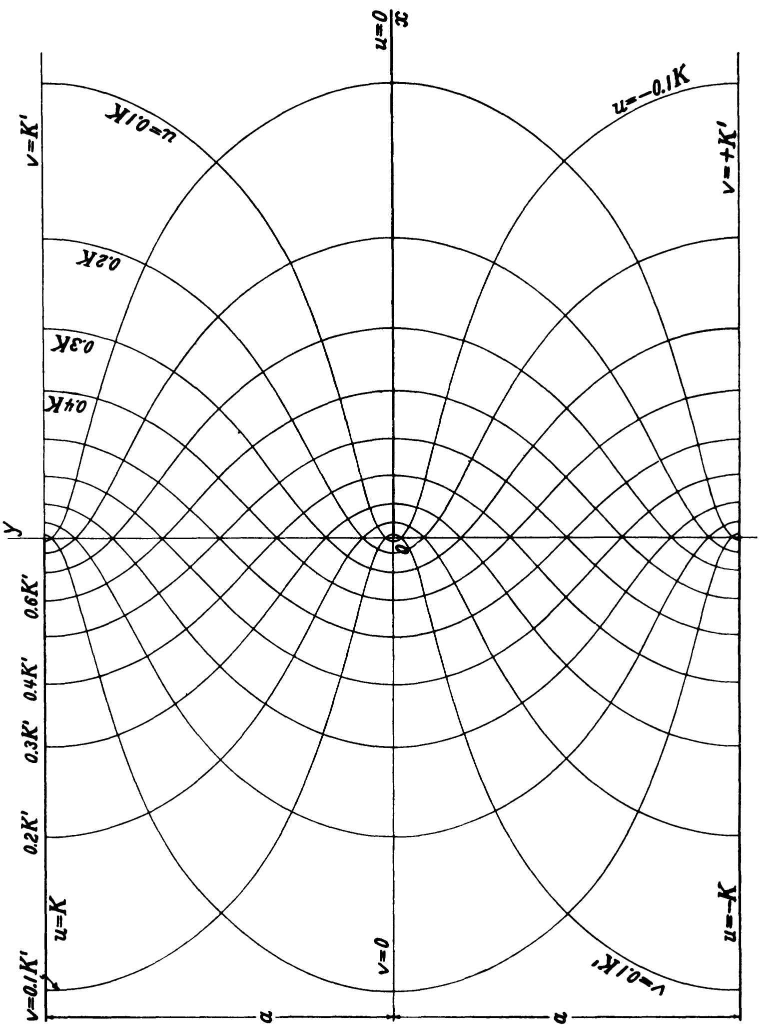

Fig. 2.19. ln sn curves, $\bar{z} = \frac{a}{\pi}\ln\left(\frac{1}{k\,\mathrm{sn}^2 w}\right)$ (J 4)

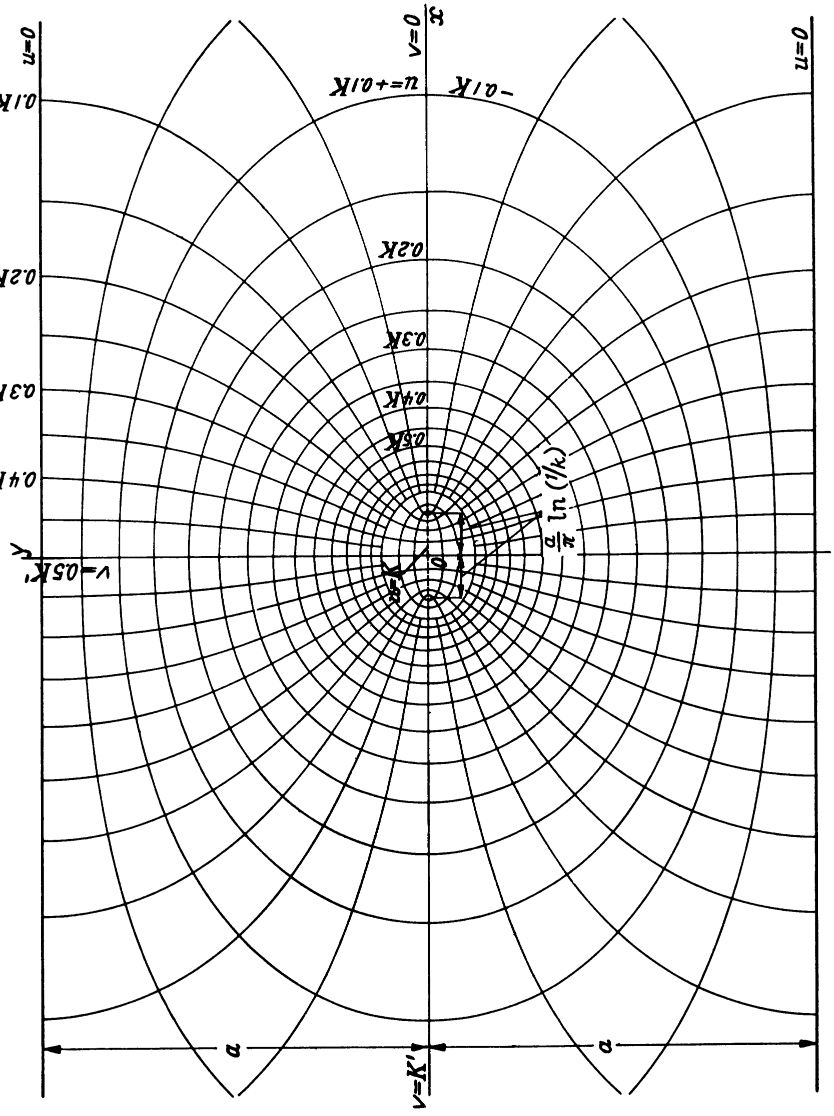

Fig. 2.20. ln cn curves, $\bar{z} = \frac{2a}{\pi} \ln \operatorname{cn} w$ (J 5)

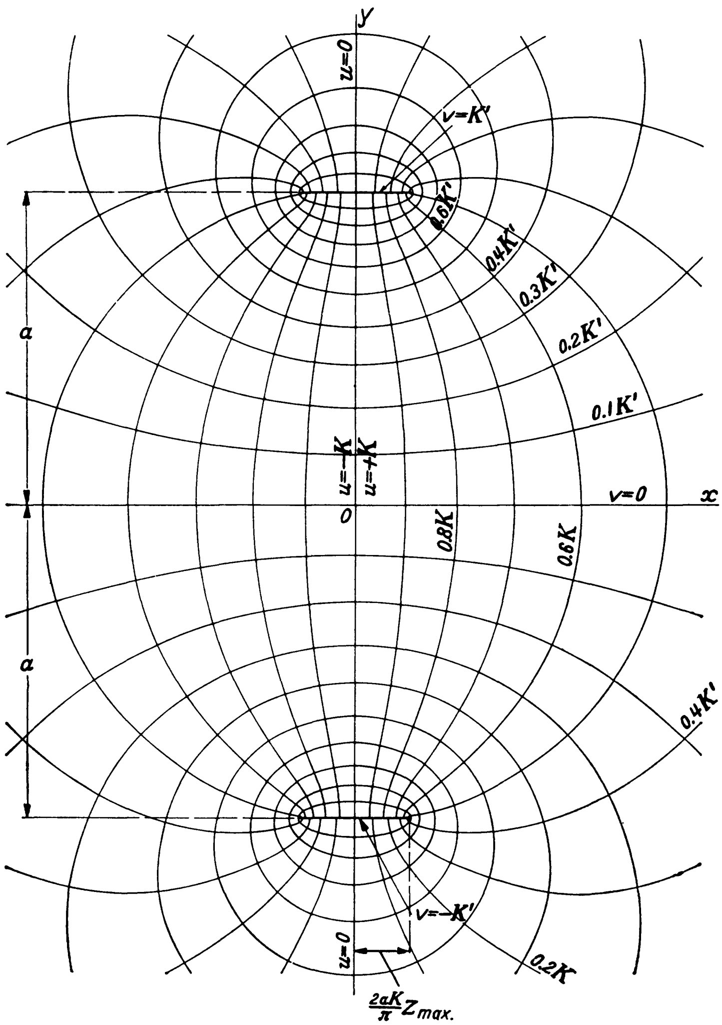

Fig. 2.21. ζ-function curves, $\bar{z} = \frac{2Ka}{\pi} Z(w + iK) + ia$ (J 6)

Section III

CYLINDRICAL SYSTEMS

The cylindrical coordinate systems are obtained by translating each of the maps, Figs. 2.01 to 2.21, in a direction perpendicular to the graphs, thus forming two orthogonal families of cylinders. The third family of coordinate surfaces consists of parallel planes, $z = \text{const}$. The coordinate axis that is parallel to the generators of the cylinders is called the z-axis in all cases.

The following coordinate systems are considered in this section:

No.	Name	Fig. No.
P 1 C	Tangent-cylinder coordinates	2.01
P 2 C	Parabolic-cylinder coordinates	2.02, 1.04
P 3 C	Cardioid-cylinder coordinates	2.03
P 4 C	Hyperbolic-cylinder coordinates	2.04
P 5 C	Rose coordinates	2.05
E 1 C	Circular-cylinder coordinates	2.06, 1.02
E 2 C	Cassinian-oval coordinates	2.07
E 3 C	Inverse Cassinian-oval coordinates	2.08
E 4 C	Bi-cylindrical coordinates	2.09
E 5 C	Maxwell-cylinder coordinates	2.10
L 1 C	Logarithmic-cylinder coordinates	2.11
L 2 C	ln tan cylinder coordinates	2.12
L 3 C	ln cosh cylinder coordinates	2.13
H 1 C	Elliptic-cylinder coordinates	2.14, 1.03
H 2 C	Inverse elliptic-cylinder coordinates	2.15
J 1 C	sn-cylinder coordinates	2.16
J 2 C	cn-cylinder coordinates	2.17
J 3 C	Inverse sn-cylinder coordinates	2.18
J 4 C	ln sn cylinder coordinates	2.19
J 5 C	ln cn cylinder coordinates	2.20
J 6 C	Zeta coordinates	2.21

This list can be extended indefinitely. We have included here merely a few coordinate systems that seemed promising with regard to engineering applications. Table 3.01 lists the cylindrical coordinate systems obtained from the transformations of Section II. For each system, the metric coefficients are given and the separability data are evaluated. Table 3.02 presents the gradient, divergence, curl, and Laplacian for each of these coordinate systems. The appearance of the coordinate surfaces can be visualized by reference to Figs. 2.01 to 2.21.

3.01 CHARACTERISTICS

The equations of Table 3.02 are obtained from the general expressions of Section I, which are considerably simplified in this special case. For cylindrical systems with $g_{11} = g_{22}$,

$$(ds)^2 = g_{11}[(du^1)^2 + (du^2)^2] + (dz)^2, \tag{3.01}$$

$$\operatorname{grad}\varphi = \frac{1}{(g_{11})^{\frac{1}{2}}}\left[\boldsymbol{a}_1 \frac{\partial\varphi}{\partial u^1} + \boldsymbol{a}_2 \frac{\partial\varphi}{\partial u^2}\right] + \boldsymbol{a}_z \frac{\partial\varphi}{\partial z}, \tag{3.02}$$

$$\operatorname{div}\boldsymbol{E} = \frac{1}{g_{11}}\left\{\frac{\partial}{\partial u^1}[(g_{11})^{\frac{1}{2}}E_1] + \frac{\partial}{\partial u^2}[(g_{11})^{\frac{1}{2}}E_2]\right\} + \frac{\partial E_z}{\partial z}, \tag{3.03}$$

$$\left.\begin{aligned}\operatorname{curl}\boldsymbol{E} &= \boldsymbol{a}_1\left[\frac{1}{(g_{11})^{\frac{1}{2}}}\frac{\partial E_z}{\partial u^2} - \frac{\partial E_2}{\partial z}\right] + \boldsymbol{a}_2\left[\frac{\partial E_1}{\partial z} - \frac{1}{(g_{11})^{\frac{1}{2}}}\frac{\partial E_z}{\partial u^1}\right]\\ &+ \frac{\boldsymbol{a}_z}{g_{11}}\left[\frac{\partial}{\partial u^1}[(g_{11})^{\frac{1}{2}}E_2] - \frac{\partial}{\partial u^2}[(g_{11})^{\frac{1}{2}}E_1]\right],\end{aligned}\right\} \tag{3.04}$$

$$\nabla^2\varphi = \frac{1}{g_{11}}\left[\frac{\partial^2\varphi}{(\partial u^1)^2} + \frac{\partial^2\varphi}{(\partial u^2)^2}\right] + \frac{\partial^2\varphi}{\partial z^2}. \tag{3.05}$$

No cylindrical coordinate system allows R-separation [5]. Also, except for the four cylindrical systems of Section I, no cylindrical system is known that allows separation of the Helmholtz equation [9] or that allows separation of the Laplace equation where $\varphi = \varphi(u^1, u^2, u^3)$. Therefore, *the only separation that can be considered* in Table 3.02 *is simple separation of the Laplace equation with* $\varphi = \varphi(u^1, u^2)$.

But if φ is independent of z, Eq. (3.05) gives for Laplace's equation,

$$\frac{\partial^2\varphi}{(\partial u^1)^2} + \frac{\partial^2\varphi}{(\partial u^2)^2} = 0, \tag{3.06}$$

which is the same as the equation obtained in rectangular coordinates. Particular solutions are

$$\varphi = e^{\pm p u^1}\begin{pmatrix}\sin\\ \cos\end{pmatrix} p u^2$$

or

$$\varphi = e^{\pm p u^2}\begin{pmatrix}\sin\\ \cos\end{pmatrix} p u^1. \tag{3.07}$$

These solutions apply to all the coordinate systems [18] of Table 3.02.

A one-dimensional solution is also possible for Laplace's equation (but not for the Helmholtz equation) in cylindrical coordinates. If φ is independent of z and u^2, Eq. (3.05) gives for Laplace's equation,

$$\frac{d^2\varphi}{(du^1)^2} = 0, \tag{3.08}$$

whose general solution is

$$\varphi = A + B u^1. \tag{3.09}$$

Thus there is no need of listing solutions under each coordinate system, as was done in Section I: Eqs. (3.07) and (3.09) apply to all the coordinates of Section III.

The one-dimensional solution, Eq. (3.09), is equivalent to the conventional method of field mapping by conformal transformation [17]. Here boundary conditions fix the potentials of a set of surfaces such as $u^1 =$ const. But this approach is inadequate for the general case, where arbitrary potential distributions exist on the boundaries. In this case, the usual conformal-mapping technique fails; but its extension, by the development of new cylindrical coordinate systems, allows the treatment of two-dimensional field problems with arbitrary boundary conditions.

TABLE 3.01. SOME CYLINDRICAL COORDINATE SYSTEMS

No.	Transformation	Fig. No.	Equations	Metric Coefficients	Simple Separability: Helmholtz $\varphi(u^1,u^2,u^3)$	Helmholtz $\varphi(u^1,u^2)$	Laplace $\varphi(u^1,u^2,u^3)$	Laplace $\varphi(u^1,u^2)$	One-dimensional Solution: Helm-holtz	Laplace
P1C	$\bar{z} = 1/w$ Tangent cylinder	2.01	$x = \frac{\mu}{\mu^2+\nu^2}$ $y = \frac{\nu}{\mu^2+\nu^2}$ $z = z$	$g_{11} = g_{22} = (\mu^2+\nu^2)^{-2}$ $g_{33} = 1$	×	×	×	●	×	●
P2C	$z = \frac{1}{2} w^2$ Parabolic cylinder	2.02	$x = \frac{1}{2}(\mu^2 - \nu^2)$ $y = \mu\nu$ $z = z$	$g_{11} = g_{22} = \mu^2+\nu^2$ $g_{33} = 1$	●	●	●	●	×	●
P3C	$\bar{z} = \frac{1}{2} w^{-2}$ Cardioid cylinder	2.03	$x = \frac{1}{2} \frac{\mu^2-\nu^2}{(\mu^2+\nu^2)^2}$ $y = \frac{\mu\nu}{(\mu^2+\nu^2)^2}$ $z = z$	$g_{11} = g_{22} = (\mu^2+\nu^2)^{-3}$ $g_{33} = 1$	×	×	×	●	×	●
P4C	$z = \sqrt{2}\, w^{\frac{1}{2}}$ Hyperbolic-cylinder coordinates	2.04	$x = (\varrho+\mu)^{\frac{1}{2}}$ $y = (\varrho-\mu)^{\frac{1}{2}}$ $z = z$ where $\varrho = +(\mu^2+\nu^2)^{\frac{1}{2}}$	$g_{11} = g_{22} = \frac{1}{2}(\mu^2+\nu^2)^{-\frac{1}{2}}$ $g_{33} = 1$	×	×	×	●	×	●

Table 3.01. Continuation

No.	Transformation	Fig. No.	Equations	Metric Coefficients	Simple Separability: HELMHOLTZ $\varphi(u^1,u^2,u^3)$	Simple Separability: HELMHOLTZ $\varphi(u^1,u^2)$	Simple Separability: LAPLACE $\varphi(u^1,u^2,u^3)$	Simple Separability: LAPLACE $\varphi(u^1,u^2)$	One-dimensional Solution: HELM-HOLTZ	One-dimensional Solution: LAPLACE
P 5 C	$\bar{z}=\sqrt{2}\,w^{-\frac{1}{2}}$ Rose coordinates	2.05	$x=\frac{1}{\varrho}(\varrho+\mu)^{\frac{1}{2}}$ $y=\frac{1}{\varrho}(\varrho-\mu)^{\frac{1}{2}}$ $z=z$ where $\varrho=+(\mu^2+\nu^2)^{\frac{1}{2}}$	$g_{11}=g_{22}=\frac{1}{2(\mu^2+\nu^2)^{\frac{3}{2}}}$ $g_{33}=1$	×	×	×	●	×	●
E 1 C	$z=e^w$ Circular-cylinder coordinates	2.06	$x=e^\eta\cos\psi$ $y=e^\eta\sin\psi$ $z=z$	$g_{11}=g_{22}=e^{2\eta}$ $g_{33}=1$	●	●	●	●	●	●
E 2 C	$z=a(e^w+1)^{\frac{1}{2}}$ Cassinian-oval coordinates	2.07	$x=\frac{a}{\sqrt{2}}[\varrho_1+(e^\eta\cos\psi+1)]^{\frac{1}{2}}$ $y=\frac{a}{\sqrt{2}}[\varrho_1-(e^\eta\cos\psi+1)]^{\frac{1}{2}}$ $z=z$ where $\varrho_1=+(e^{2\eta}+2e^\eta\cos\psi+1)^{\frac{1}{2}}$	$g_{11}=g_{22}=\frac{a^2e^{2\eta}}{4\varrho_1}$ $g_{33}=1$	×	×	×	●	×	●

E3C	$\bar{z} = \dfrac{a}{(e^w + 1)^{\frac{1}{2}}}$	2.08	$x = \dfrac{a}{\sqrt{2}\,\varrho}\,[\varrho + (e^\eta \cos\psi + 1)]^{\frac{1}{2}}$ $y = \dfrac{a}{\sqrt{2}\,\varrho}\,[\varrho - (e^\eta \cos\psi + 1)]^{\frac{1}{2}}$ $z = z$ where $\varrho = +\,(e^{2\eta} + 2e^\eta \cos\psi + 1)^{\frac{1}{2}}$	$g_{11} = g_{22} = \dfrac{a^2 e^{2\eta}}{4\varrho^3}$ $g_{33} = 1$	×	×	×	●	×	●
E4C	$\bar{z} = \dfrac{a(e^w + 1)}{e^w - 1}$ Bi-cylindrical coordinates	2.09	$x = \dfrac{a \sinh\eta}{\cosh\eta - \cos\psi}$ $y = \dfrac{a \sin\psi}{\cosh\eta - \cos\psi}$ $z = z$	$g_{11} = g_{22} = \dfrac{a^2}{(\cosh\eta - \cos\psi)^2}$ $g_{33} = 1$	×	×	×	●	×	●
E5C	$z = \dfrac{a}{\pi}(w + 1 + e^w)$ Maxwell-cylinder coordinates	2.10	$x = \dfrac{a}{\pi}(\eta + 1 + e^\eta \cos\psi)$ $y = \dfrac{a}{\pi}(\psi + e^\eta \sin\psi)$ $z = z$	$g_{11} = g_{22} = \left(\dfrac{a}{\pi}\right)^2$ $\times (1 + 2e^\eta \cos\psi + e^{2\eta})$ $g_{33} = 1$	×	×	×	●	×	●
L1C	$z = \dfrac{2a}{\pi} \ln w$	2.11	$x = \dfrac{a}{\pi} \ln(\mu^2 + \nu^2)$ $y = \dfrac{2a}{\pi} \tan^{-1}(\nu/\mu)$ $z = z$	$g_{11} = g_{22} = \left(\dfrac{2a}{\pi}\right)^2 \dfrac{1}{\mu^2 + \nu^2}$ $g_{33} = 1$	×	×	×	●	×	●

Table 3.01. Continuation

No.	Transformation	Fig. No.	Equations	Metric Coefficients	Simple Separability: HELMHOLTZ $\varphi(u^1,u^2,u^3)$	HELMHOLTZ $\varphi(u^1,u^2)$	LAPLACE $\varphi(u^1,u^2,u^3)$	LAPLACE $\varphi(u^1,u^2)$	One-dimensional Solution: HELM-HOLTZ	LAPLACE
L 2 C	$z=\frac{2a}{\pi}\ln\tan w$	2.12	$x=\frac{a}{\pi}\ln\left[\frac{\sinh^2\eta+\sin^2\psi}{\sinh^2\eta+\cos^2\psi}\right]$ $y=\frac{2a}{\pi}\tan^{-1}\left(\frac{\sinh 2\eta}{\sin 2\psi}\right)$ $z=z$	$g_{11}=g_{22}=\left(\frac{4a}{\pi}\right)^2 \times \frac{1}{\sinh^2 2\eta+\sin^2 2\psi}$ $g_{33}=1$	×	×	×	●	×	●
L 3 C	$z=\frac{2a}{\pi}\ln\cosh w$	2.13	$x=\frac{a}{\pi}\ln(\cosh^2\eta-\sin^2\psi)$ $y=\frac{2a}{\pi}\tan^{-1}(\tanh\eta\tan\psi)$ $z=z$	$g_{11}=g_{22}$ $=\left(\frac{2a}{\pi}\right)^2\left\{\frac{\cosh^2\eta\sinh^2\eta}{(\cosh^2\eta+[\sinh\eta\cosh\eta+\sin\psi\cos\psi]^2-\sin^2\psi)^2}\right\}$ $g_{33}=1$	×	×	×	●	×	●
H 1 C	$z=a\cosh w$ Elliptic-cylinder coordinates	2.14	$x=a\cosh\eta\cos\psi$ $y=a\sinh\eta\sin\psi$ $z=z$	$g_{11}=g_{22}$ $=a^2(\cosh^2\eta-\cos^2\psi)$ $g_{33}=1$	●	●	●	●	×	●
H 2 C	$\bar{z}=a\operatorname{sech} w$	2.15	$x=\frac{a\cosh\eta\cos\psi}{\cosh^2\eta-\sin^2\psi}$ $y=\frac{a\sinh\eta\sin\psi}{\cosh^2\eta-\sin^2\psi}$ $z=z$	$g_{11}=g_{22}=\frac{a^2(\cosh^2\eta-\cos^2\psi)}{(\cosh^2\eta-\sin^2\psi)^2}$ $g_{33}=1$	×	×	×	●	×	●

J1C	$z = a \operatorname{sn} w$	2.16	$x = \frac{a}{\Lambda} \operatorname{sn}\mu \operatorname{dn}\nu$ $y = \frac{a}{\Lambda} \operatorname{cn}\mu \operatorname{dn}\mu \operatorname{sn}\nu \operatorname{cn}\nu$ $z = z$ where $\Lambda = 1 - \operatorname{dn}^2\mu \operatorname{sn}^2\nu$	$g_{11} = g_{22} = \frac{a^2 \Omega^2}{\Lambda^2}$ $g_{33} = 1$ where $\Omega^2 = (1 - \operatorname{sn}^2\mu \operatorname{dn}^2\nu)$ $\times (\operatorname{dn}^2\nu - k^2 \operatorname{sn}^2\mu)$	×	×	×	●	×	●
J2C	$\bar{z} = a \operatorname{cn} w$	2.17	$x = \frac{a}{\Lambda} \operatorname{cn}\mu \operatorname{cn}\nu$ $y = \frac{a}{\Lambda} \operatorname{sn}\mu \operatorname{dn}\mu \operatorname{sn}\nu \operatorname{dn}\nu$ $z = z$	$g_{11} = g_{22} = \frac{a^2 \Gamma^2}{\Lambda^2}$ $g_{33} = 1$ where $\Gamma^2 = (\operatorname{sn}^2\nu + \operatorname{sn}^2\mu \operatorname{cn}^2\nu)$ $\times (\operatorname{dn}^2\nu - k^2 \operatorname{sn}^2\mu)$	×	×	×	●	×	●
J3C	$\bar{z} = \frac{k^{\frac{1}{2}}}{i 2a} \times \left(\frac{1 + i k^{\frac{1}{2}} \operatorname{sn} w}{1 - i k^{\frac{1}{2}} \operatorname{sn} w}\right)$	2.18	$x = \frac{\Lambda}{a \Upsilon} \operatorname{sn}\mu \operatorname{dn}\nu$ $y = \frac{k^{\frac{1}{2}} \Pi}{2 a \Upsilon}$ $z = z$ where $\Upsilon = \operatorname{sn}^2\mu \operatorname{dn}^2\nu + [(\Lambda/\sqrt{k})$ $+ \operatorname{cn}\mu \operatorname{dn}\mu \operatorname{sn}\nu \operatorname{cn}\nu]^2$ $\Pi = (\Lambda^2/k) - (\operatorname{sn}^2\mu \operatorname{dn}^2\nu$ $+ \operatorname{cn}^2\mu \operatorname{dn}^2\mu \operatorname{sn}^2\nu \operatorname{cn}^2\nu)$	$g_{11} = g_{22} = \frac{\Lambda^2 \Omega^2}{a^2 \Upsilon^2}$ $g_{33} = 1$	×	×	×	●	×	●

Table 3.01. Continuation

No.	Transformation	Fig. No.	Equations	Metric Coefficients	Simple Separability: Helmholtz $\varphi(u^1,u^2,u^3)$	Simple Separability: Helmholtz $\varphi(u^1,u^2)$	Simple Separability: Laplace $\varphi(u^1,u^2,u^3)$	Simple Separability: Laplace $\varphi(u^1,u^2)$	One-dimensional Solution: Helmholtz	One-dimensional Solution: Laplace
J 4 C	$\bar{z}=\frac{a}{\pi}\ln\left(\frac{1}{k\,\mathrm{sn}^2 w}\right)$	2.19	$x=\frac{a}{\pi}\ln\left(\frac{\Lambda^2}{k\Sigma}\right)$ $y=\frac{2a}{\pi}\times\tan^{-1}\left(\frac{\mathrm{cn}\,\mu\,\mathrm{dn}\,\mu\,\mathrm{sn}\,\nu\,\mathrm{cn}\,\nu}{\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu}\right)$ $z=z$	$g_{11}=g_{22}=\left(\frac{2a\Theta}{\pi\Sigma}\right)^2$ $g_{33}=1$ where $\Sigma=\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu+\mathrm{cn}^2\mu\,\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu\,\mathrm{cn}^2\nu$ $\Theta^2=\mathrm{sn}^2\mu\,\mathrm{cn}^2\mu\,\mathrm{dn}^2\mu\times(\mathrm{cn}^2\nu-\mathrm{sn}^2\nu\,\mathrm{dn}^2\nu)^2+\mathrm{sn}^2\nu\,\mathrm{cn}^2\nu\,\mathrm{dn}^2\nu\times(\mathrm{cn}^2\mu+\mathrm{sn}^2\mu\,\mathrm{dn}^2\mu)^2$	×	×	×	●	×	●
J 5 C	$\bar{z}=\frac{2a}{\pi}\ln\mathrm{cn}\,w$	2.20	$x=\frac{a}{\pi}\ln(\Xi/\Lambda^2)$ $y=\frac{2a}{\pi}\times\tan^{-1}\left(\frac{\mathrm{sn}\,\mu\,\mathrm{dn}\,\mu\,\mathrm{sn}\,\nu\,\mathrm{dn}\,\nu}{\mathrm{cn}\,\mu\,\mathrm{cn}\,\nu}\right)$ $z=z$	$g_{11}=g_{22}=\left(\frac{2a\Theta'}{\pi\Xi}\right)^2$ $g_{33}=1$ where $\Xi=\mathrm{cn}^2\mu\,\mathrm{cn}^2\nu+\mathrm{sn}^2\mu\,\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu\,\mathrm{dn}^2\nu$ $\Theta'^2=\mathrm{sn}^2\mu\,\mathrm{cn}^2\mu\,\mathrm{dn}^2\mu\times(\mathrm{dn}^2\nu-k'^2\mathrm{sn}^2\nu\,\mathrm{cn}^2\nu)^2+\mathrm{sn}^2\nu\,\mathrm{cn}^2\nu\,\mathrm{dn}^2\nu\times(\mathrm{dn}^2\mu-k^2\mathrm{sn}^2\mu\,\mathrm{cn}^2\mu)^2$	×	×	×	●	×	●

J6C	$\bar{z} = \frac{2Ka}{\pi} \times Z(w + iK') + ia.$	2.21	$x = \frac{2Ka}{\pi}\Big\{Z(\mu) + \frac{k^2}{\Lambda'} \times \operatorname{sn}\mu \operatorname{cn}\mu \operatorname{dn}\mu \operatorname{sn}^2 v'\Big\}$ $y = \frac{2Ka}{\pi}\Big\{Z(v') + \frac{\pi v}{2KK'} - \frac{1}{\Lambda'} \times \operatorname{dn}^2\mu \operatorname{sn} v' \operatorname{cn} v' \operatorname{dn} v'\Big\}$ $z = z$ where $\Lambda' = 1 - \operatorname{dn}^2\mu \operatorname{sn}^2 v'$ $v' = v + K'$ K, K', and E are complete elliptic integrals	$g_{11} = g_{22} = \left(\frac{2Ka}{\pi}\right)^2 \Big\{\frac{4k^4}{\Lambda'^4} \times \operatorname{sn}^2\mu \operatorname{cn}^2\mu \operatorname{dn}^2\mu \operatorname{sn}^2 v' \times \operatorname{cn}^2 v' \operatorname{dn}^2 v' + \Big[\operatorname{dn}^2\mu - \frac{E}{K} + \frac{k^2 \operatorname{sn}^2 v'}{\Lambda'^2} (\operatorname{dn}^2\mu \operatorname{cn}^2 v' \times (\operatorname{cn}^2\mu - \operatorname{sn}^2\mu \operatorname{dn}^2\mu) - k^2 \operatorname{sn}^2\mu (1 - k^2 \operatorname{sn}^4\mu))\Big]^2\Big\}$ $g_{33} = 1$	×	×	×	●	×	●

Crosses indicate non-separability, dots indicate separability. The modulus k is associated with the variable μ, and k' is associated with v: e.g., $\operatorname{sn}\mu \equiv \operatorname{sn}(\mu, k)$, $\operatorname{sn} v \equiv \operatorname{sn}(v, k')$.

TABLE 3.02. IMPORTANT EQUATIONS FOR CYLINDRICAL SYSTEMS

P1C. TANGENT-CYLINDER COORDINATES (μ, ν, z), Fig. 2.01.

$$\begin{cases} x = \dfrac{\mu}{\mu^2+\nu^2}, \\ y = \dfrac{\nu}{\mu^2+\nu^2}, \\ z = z. \end{cases}$$

$$g_{11} = g_{22} = g^{\frac{1}{2}} = (\mu^2+\nu^2)^{-2}, \qquad g_{33} = 1.$$

$$(ds)^2 = \frac{1}{(\mu^2+\nu^2)^2}\left[(d\mu)^2 + (d\nu)^2\right] + (dz)^2.$$

$$\operatorname{grad} \varphi = (\mu^2+\nu^2)\left[\boldsymbol{a}_\mu \frac{\partial \varphi}{\partial \mu} + \boldsymbol{a}_\nu \frac{\partial \varphi}{\partial \nu}\right] + \boldsymbol{a}_z \frac{\partial \varphi}{\partial z}.$$

$$\operatorname{div} \boldsymbol{E} = (\mu^2+\nu^2)^2\left[\frac{\partial}{\partial \mu}\left(\frac{E_\mu}{\mu^2+\nu^2}\right) + \frac{\partial}{\partial \nu}\left(\frac{E_\nu}{\mu^2+\nu^2}\right)\right] + \frac{\partial E_z}{\partial z}.$$

$$\operatorname{curl} \boldsymbol{E} = \boldsymbol{a}_\mu\left[(\mu^2+\nu^2)\frac{\partial E_z}{\partial \nu} - \frac{\partial E_\nu}{\partial z}\right] + \boldsymbol{a}_\nu\left[\frac{\partial E_\mu}{\partial z} - (\mu^2+\nu^2)\frac{\partial E_z}{\partial \mu}\right]$$
$$+ \boldsymbol{a}_z(\mu^2+\nu^2)^2\left[\frac{\partial}{\partial \mu}\left(\frac{E_\nu}{\mu^2+\nu^2}\right) - \frac{\partial}{\partial \nu}\left(\frac{E_\mu}{\mu^2+\nu^2}\right)\right].$$

$$\nabla^2 \varphi = (\mu^2+\nu^2)^2\left[\frac{\partial^2 \varphi}{\partial \mu^2} + \frac{\partial^2 \varphi}{\partial \nu^2}\right] + \frac{\partial^2 \varphi}{\partial z^2}.$$

P2C. PARABOLIC-CYLINDER COORDINATES (see Section I).

P3C. CARDIOID-CYLINDER-COORDINATES (μ, ν, z), Fig. 2.03.

$$\begin{cases} x = \dfrac{1}{2}\,\dfrac{\mu^2-\nu^2}{(\mu^2+\nu^2)^2}, \\ y = \dfrac{\mu\nu}{(\mu^2+\nu^2)^2}, \\ z = z. \end{cases}$$

$$g_{11} = g_{22} = g^{\frac{1}{2}} = (\mu^2+\nu^2)^{-3}, \qquad g_{33} = 1.$$

$$(ds)^2 = \frac{1}{(\mu^2+\nu^2)^3}\left[(d\mu)^2 + (d\nu)^2\right] + (dz)^2.$$

$$\operatorname{grad} \varphi = (\mu^2+\nu^2)^{\frac{3}{2}}\left[\boldsymbol{a}_\mu \frac{\partial \varphi}{\partial \mu} + \boldsymbol{a}_\nu \frac{\partial \varphi}{\partial \nu}\right] + \boldsymbol{a}_z \frac{\partial \varphi}{\partial z}.$$

$$\operatorname{div} \boldsymbol{E} = (\mu^2+\nu^2)^3\left\{\frac{\partial}{\partial \mu}\left[(\mu^2+\nu^2)^{-\frac{3}{2}} E_\mu\right] + \frac{\partial}{\partial \nu}\left[(\mu^2+\nu^2)^{-\frac{3}{2}} E_\nu\right]\right\} + \frac{\partial E_z}{\partial z}.$$

$$\operatorname{curl} \boldsymbol{E} = \boldsymbol{a}_\mu\left[(\mu^2+\nu^2)^{\frac{3}{2}}\frac{\partial E_z}{\partial \nu} - \frac{\partial E_\nu}{\partial z}\right] + \boldsymbol{a}_\nu\left[\frac{\partial E_\mu}{\partial z} - (\mu^2+\nu^2)^{\frac{3}{2}}\frac{\partial E_z}{\partial \mu}\right]$$
$$+ \boldsymbol{a}_z(\mu^2+\nu^2)^3\left[\frac{\partial}{\partial \mu}\left((\mu^2+\nu^2)^{-\frac{3}{2}} E_\nu\right) - \frac{\partial}{\partial \nu}\left((\mu^2+\nu^2)^{-\frac{3}{2}} E_\mu\right)\right].$$

$$\nabla^2 \varphi = (\mu^2+\nu^2)^3\left[\frac{\partial^2 \varphi}{\partial \mu^2} + \frac{\partial^2 \varphi}{\partial \nu^2}\right] + \frac{\partial^2 \varphi}{\partial z^2}.$$

P4C. HYPERBOLIC-CYLINDER COORDINATES (μ, ν, z), Fig. 2.04.

$$\begin{cases} x = (\varrho + \mu)^{\frac{1}{2}}, \\ y = (\varrho - \mu)^{\frac{1}{2}}, \\ z = z, \quad \text{where} \quad \varrho = +(\mu^2 + \nu^2)^{\frac{1}{2}}. \end{cases}$$

$$g_{11} = g_{22} = g^{\frac{1}{2}} = \tfrac{1}{2}(\mu^2+\nu^2)^{-\frac{1}{2}}, \qquad g_{33} = 1.$$

$$(ds)^2 = \frac{1}{2(\mu^2+\nu^2)^{\frac{1}{2}}}\left[(d\mu)^2 + (d\nu)^2\right] + (dz)^2.$$

$$\operatorname{grad}\varphi = \sqrt{2}\,(\mu^2+\nu^2)^{\frac{1}{4}}\left[\boldsymbol{a}_\mu \frac{\partial\varphi}{\partial\mu} + \boldsymbol{a}_\nu \frac{\partial\varphi}{\partial\nu}\right] + \boldsymbol{a}_z \frac{\partial\varphi}{\partial z}.$$

$$\operatorname{div}\boldsymbol{E} = \sqrt{2}\,(\mu^2+\nu^2)^{\frac{1}{2}}\left\{\frac{\partial}{\partial\mu}\left[(\mu^2+\nu^2)^{-\frac{1}{4}}E_\mu\right] + \frac{\partial}{\partial\nu}\left[(\mu^2+\nu^2)^{-\frac{1}{4}}E_\nu\right]\right\} + \frac{\partial E_z}{\partial z}.$$

$$\operatorname{curl}\boldsymbol{E} = \boldsymbol{a}_\mu\left[\sqrt{2}\,(\mu^2+\nu^2)^{\frac{1}{4}}\frac{\partial E_z}{\partial\nu} - \frac{\partial E_\nu}{\partial z}\right] + \boldsymbol{a}_\nu\left[\frac{\partial E_\mu}{\partial z} - \sqrt{2}\,(\mu^2+\nu^2)^{\frac{1}{4}}\frac{\partial E_z}{\partial\mu}\right]$$
$$+ \boldsymbol{a}_z\sqrt{2}\,(\mu^2+\nu^2)^{\frac{1}{2}}\left[\frac{\partial}{\partial\mu}\left((\mu^2+\nu^2)^{-\frac{1}{4}}E_\nu\right) - \frac{\partial}{\partial\nu}\left((\mu^2+\nu^2)^{-\frac{1}{4}}E_\mu\right)\right].$$

$$\nabla^2\varphi = 2\,(\mu^2+\nu^2)^{\frac{1}{2}}\left[\frac{\partial^2\varphi}{\partial\mu^2} + \frac{\partial^2\varphi}{\partial\nu^2}\right] + \frac{\partial^2\varphi}{\partial z^2}.$$

P5C. ROSE-CYLINDER COORDINATES (μ, ν, z), Fig. 2.05.

$$\begin{cases} x = \dfrac{1}{\varrho}(\varrho + \mu)^{\frac{1}{2}}, \\ y = \dfrac{1}{\varrho}(\varrho - \mu)^{\frac{1}{2}}, \\ z = z, \quad \text{where} \quad \varrho = +(\mu^2 + \nu^2)^{\frac{1}{2}}. \end{cases}$$

$$g_{11} = g_{22} = g^{\frac{1}{2}} = \frac{1}{2(\mu^2+\nu^2)^{\frac{3}{2}}}, \qquad g_{33} = 1.$$

$$(ds)^2 = \frac{1}{2(\mu^2+\nu^2)^{\frac{3}{2}}}\left[(d\mu)^2 + (d\nu)^2\right] + (dz)^2.$$

$$\operatorname{grad}\varphi = \sqrt{2}\,(\mu^2+\nu^2)^{\frac{3}{4}}\left[\boldsymbol{a}_\mu \frac{\partial\varphi}{\partial\mu} + \boldsymbol{a}_\nu \frac{\partial\varphi}{\partial\nu}\right] + \boldsymbol{a}_z \frac{\partial\varphi}{\partial z}.$$

$$\operatorname{div}\boldsymbol{E} = \sqrt{2}\,(\mu^2+\nu^2)^{\frac{3}{2}}\left\{\frac{\partial}{\partial\mu}\left[(\mu^2+\nu^2)^{-\frac{3}{4}}E_\mu\right] + \frac{\partial}{\partial\nu}\left[(\mu^2+\nu^2)^{-\frac{3}{4}}E_\nu\right]\right\} + \frac{\partial E_z}{\partial z}.$$

$$\operatorname{curl}\boldsymbol{E} = \boldsymbol{a}_\mu\left[\sqrt{2}\,(\mu^2+\nu^2)^{\frac{3}{4}}\frac{\partial E_z}{\partial\nu} - \frac{\partial E_\nu}{\partial z}\right] + \boldsymbol{a}_\nu\left[\frac{\partial E_\mu}{\partial z} - \sqrt{2}\,(\mu^2+\nu^2)^{\frac{3}{4}}\frac{\partial E_z}{\partial\mu}\right]$$
$$+ \boldsymbol{a}_z\sqrt{2}\,(\mu^2+\nu^2)^{\frac{3}{2}}\left[\frac{\partial}{\partial\mu}\left((\mu^2+\nu^2)^{-\frac{3}{4}}E_\nu\right) - \frac{\partial}{\partial\nu}\left((\mu^2+\nu^2)^{-\frac{3}{4}}E_\mu\right)\right].$$

$$\nabla^2\varphi = 2\,(\mu^2+\nu^2)^{\frac{3}{2}}\left[\frac{\partial^2\varphi}{\partial\mu^2} + \frac{\partial^2\varphi}{\partial\nu^2}\right] + \frac{\partial^2\varphi}{\partial z^2}.$$

E 1C. CIRCULAR-CYLINDER COORDINATES (see Section I).

E 2C. CASSINIAN-OVAL COORDINATES (η, ψ, z), Fig. 2.07.

$$\begin{cases} x = \dfrac{a}{\sqrt{2}}\,[\varrho + (e^{\eta}\cos\psi + 1)]^{\frac{1}{2}}, \\ y = \dfrac{a}{\sqrt{2}}\,[\varrho - (e^{\eta}\cos\psi + 1)]^{\frac{1}{2}}, \\ z = z, \quad \text{where} \quad \varrho = +\,(e^{2\eta} + 2e^{\eta}\cos\psi + 1)^{\frac{1}{2}}. \end{cases}$$

$$g_{11} = g_{22} = g^{\frac{1}{2}} = \frac{a^2 e^{2\eta}}{4\varrho}, \qquad g_{33} = 1.$$

$$(ds)^2 = \frac{a^2 e^{2\eta}}{4\varrho}\,[(d\eta)^2 + (d\psi)^2] + (dz)^2.$$

$$\operatorname{grad}\varphi = \frac{2\varrho^{\frac{1}{2}}}{a e^{\eta}}\left[\boldsymbol{a}_\eta \frac{\partial\varphi}{\partial\eta} + \boldsymbol{a}_\psi \frac{\partial\varphi}{\partial\psi}\right] + \boldsymbol{a}_z \frac{\partial\varphi}{\partial z}.$$

$$\operatorname{div}\boldsymbol{E} = \frac{2\varrho}{a e^{2\eta}}\left\{\frac{\partial}{\partial\eta}\left[\frac{e^{\eta}}{\sqrt{\varrho}}E_\eta\right] + \frac{\partial}{\partial\psi}\left[\frac{e^{\eta}}{\sqrt{\varrho}}E_\psi\right]\right\} + \frac{\partial E_z}{\partial z}.$$

$$\operatorname{curl}\boldsymbol{E} = \boldsymbol{a}_\eta\left[\frac{2\varrho^{\frac{1}{2}}}{a e^{\eta}}\frac{\partial E_z}{\partial\psi} - \frac{\partial E_\psi}{\partial z}\right] + \boldsymbol{a}_\psi\left[\frac{\partial E_\eta}{\partial z} - \frac{2\varrho^{\frac{1}{2}}}{a e^{\eta}}\frac{\partial E_z}{\partial\eta}\right]$$
$$+\,\boldsymbol{a}_z \frac{2\varrho}{a e^{2\eta}}\left[\frac{\partial}{\partial\eta}\left(\frac{e^{\eta}}{\sqrt{\varrho}}E_\psi\right) - \frac{\partial}{\partial\psi}\left(\frac{e^{\eta}}{\sqrt{\varrho}}E_\eta\right)\right].$$

$$\nabla^2\varphi = \frac{4\varrho}{a^2 e^{2\eta}}\left[\frac{\partial^2\varphi}{\partial\eta^2} + \frac{\partial^2\varphi}{\partial\psi^2}\right] + \frac{\partial^2\varphi}{\partial z^2}.$$

E 3C. INVERSE CASSINIAN-OVAL COORDINATES (η, ψ, z), Fig. 2.08.

$$\begin{cases} x = \dfrac{a}{\sqrt{2}\,\varrho}\,[\varrho + (e^{\eta}\cos\psi + 1)]^{\frac{1}{2}}, \\ y = \dfrac{a}{\sqrt{2}\,\varrho}\,[\varrho - (e^{\eta}\cos\psi + 1)]^{\frac{1}{2}}, \\ z = z, \quad \text{where} \quad \varrho = +\,(e^{2\eta} + 2e^{\eta}\cos\psi + 1)^{\frac{1}{2}}. \end{cases}$$

$$g_{11} = g_{22} = g^{\frac{1}{2}} = \frac{a^2 e^{2\eta}}{4\varrho^3}, \qquad g_{33} = 1.$$

$$(ds)^2 = \frac{a^2 e^{2\eta}}{4\varrho^3}\,[(d\eta)^2 + (d\psi)^2] + (dz)^2.$$

$$\operatorname{grad}\varphi = \frac{2\varrho^{\frac{3}{2}}}{a e^{\eta}}\left[\boldsymbol{a}_\eta \frac{\partial\varphi}{\partial\eta} + \boldsymbol{a}_\psi \frac{\partial\varphi}{\partial\psi}\right] + \boldsymbol{a}_z \frac{\partial\varphi}{\partial z}.$$

$$\operatorname{div}\boldsymbol{E} = \frac{2\varrho^3}{a e^{2\eta}}\left\{\frac{\partial}{\partial\eta}\left[\frac{e^{\eta}}{\varrho^{\frac{3}{2}}}E_\eta\right] + \frac{\partial}{\partial\psi}\left[\frac{e^{\eta}}{\varrho^{\frac{3}{2}}}E_\psi\right]\right\} + \frac{\partial E_z}{\partial z}.$$

$$\operatorname{curl}\boldsymbol{E} = \boldsymbol{a}_\eta\left[\frac{2\varrho^{\frac{3}{2}}}{a e^{\eta}}\frac{\partial E_z}{\partial\psi} - \frac{\partial E_\psi}{\partial z}\right] + \boldsymbol{a}_\psi\left[\frac{\partial E_\eta}{\partial z} - \frac{2\varrho^{\frac{3}{2}}}{a e^{\eta}}\frac{\partial E_z}{\partial\eta}\right]$$
$$+\,\boldsymbol{a}_z \frac{2\varrho^3}{a e^{2\eta}}\left[\frac{\partial}{\partial\eta}\left(\frac{e^{\eta}}{\varrho^{\frac{3}{2}}}E_\psi\right) - \frac{\partial}{\partial\psi}\left(\frac{e^{\eta}}{\varrho^{\frac{3}{2}}}E_\eta\right)\right].$$

$$\nabla^2\varphi = \frac{4\varrho^3}{a^2 e^{2\eta}}\left[\frac{\partial^2\varphi}{\partial\eta^2} + \frac{\partial^2\varphi}{\partial\psi^2}\right] + \frac{\partial^2\varphi}{\partial z^2}.$$

E 4C. BI-CYLINDER COORDINATES (η, ψ, z), Fig. 2.09.

$$\begin{cases} x = \dfrac{a \sinh \eta}{\cosh \eta - \cos \psi}, \\ y = \dfrac{a \sin \psi}{\cosh \eta - \cos \psi}, \\ z = z. \end{cases}$$

$$g_{11} = g_{22} = g^{\frac{1}{2}} = \frac{a^2}{(\cosh \eta - \cos \psi)^2}, \qquad g_{33} = 1.$$

$$(ds)^2 = \frac{a^2}{(\cosh \eta - \cos \psi)^2} [(d\eta)^2 + (d\psi)^2] + (dz)^2.$$

$$\operatorname{grad} \varphi = \frac{1}{a} (\cosh \eta - \cos \psi) \left[\boldsymbol{a}_\eta \frac{\partial \varphi}{\partial \eta} + \boldsymbol{a}_\psi \frac{\partial \varphi}{\partial \psi}\right] + \boldsymbol{a}_z \frac{\partial \varphi}{\partial z}.$$

$$\operatorname{div} \boldsymbol{E} = \frac{1}{a} (\cosh \eta - \cos \psi)^2 \times \left\{\frac{\partial}{\partial \eta} [(\cosh \eta - \cos \psi)^{-1} E_\eta] + \frac{\partial}{\partial \psi} [(\cosh \eta - \cos \psi)^{-1} E_\psi]\right\} + \frac{\partial E_z}{\partial z}.$$

$$\operatorname{curl} \boldsymbol{E} = \boldsymbol{a}_\eta \left[\frac{1}{a} (\cosh \eta - \cos \psi) \frac{\partial E_z}{\partial \psi} - \frac{\partial E_\psi}{\partial z}\right] + \boldsymbol{a}_\psi \left[\frac{\partial E_\eta}{\partial z} - \frac{1}{a} (\cosh \eta - \cos \psi) \frac{\partial E_z}{\partial \eta}\right] + \frac{\boldsymbol{a}_z}{a} (\cosh \eta - \cos \psi)^2 \left[\frac{\partial}{\partial \eta} \left(\frac{E_\psi}{\cosh \eta - \cos \psi}\right) - \frac{\partial}{\partial \psi} \left(\frac{E_\eta}{\cosh \eta - \cos \psi}\right)\right].$$

$$\nabla^2 \varphi = \frac{1}{a^2} (\cosh \eta - \cos \psi)^2 \left[\frac{\partial^2 \varphi}{\partial \eta^2} + \frac{\partial^2 \varphi}{\partial \psi^2}\right] + \frac{\partial^2 \varphi}{\partial z^2}.$$

E 5C. MAXWELL-CYLINDER COORDINATES (η, ψ, z), Fig. 2.10.

$$\begin{cases} x = \dfrac{a}{\pi} (\eta + 1 + e^\eta \cos \psi), \\ y = \dfrac{a}{\pi} (\psi + e^\eta \sin \psi), \\ z = z. \end{cases}$$

$$g_{11} = g_{22} = g^{\frac{1}{2}} = \left(\frac{a}{\pi}\right)^2 (1 + 2e^\eta \cos \psi + e^{2\eta}), \qquad g_{33} = 1.$$

$$(ds)^2 = \left(\frac{a}{\pi}\right)^2 (1 + 2e^\eta \cos \psi + e^{2\eta}) [(d\eta)^2 + (d\psi)^2] + (dz)^2.$$

$$\operatorname{grad} \varphi = \frac{\pi}{a} (1 + 2e^\eta \cos \psi + e^{2\eta})^{-\frac{1}{2}} \left[\boldsymbol{a}_\eta \frac{\partial \varphi}{\partial \eta} + \boldsymbol{a}_\psi \frac{\partial \varphi}{\partial \psi}\right] + \boldsymbol{a}_z \frac{\partial \varphi}{\partial z}.$$

$$\operatorname{div} \boldsymbol{E} = \frac{\pi}{a} (1 + 2e^\eta \cos \psi + e^{2\eta})^{-1} \times \left\{\frac{\partial}{\partial \eta} [(1 + 2e^\eta \cos \psi + e^{2\eta})^{\frac{1}{2}} E_\eta] + \frac{\partial}{\partial \psi} [(1 + 2e^\eta \cos \psi + e^{2\eta})^{\frac{1}{2}} E_\psi]\right\} + \frac{\partial E_z}{\partial z}.$$

$$\operatorname{curl} E = \boldsymbol{a}_\eta \left[\frac{\pi}{a} (1 + 2e^\eta \cos \psi + e^{2\eta})^{-\frac{1}{2}} \frac{\partial E_z}{\partial \psi} - \frac{\partial E_\psi}{\partial z}\right] + \boldsymbol{a}_\psi \left[\frac{\partial E_\eta}{\partial z} - \frac{\pi}{a} (1 + 2e^\eta \cos \psi + e^{2\eta})^{-\frac{1}{2}} \frac{\partial E_z}{\partial \eta}\right] + \boldsymbol{a}_z \frac{\pi}{a} (1 + 2e^\eta \cos \psi + e^{2\eta})^{-1} \times \left[\frac{\partial}{\partial \eta} ((1 + 2e^\eta \cos \psi + e^{2\eta})^{\frac{1}{2}} E_\psi) - \frac{\partial}{\partial \psi} ((1 + 2e^\eta \cos \psi + e^{2\eta})^{\frac{1}{2}} E_\eta)\right].$$

$$\nabla^2 \varphi = \left(\frac{\pi}{a}\right)^2 (1 + 2e^\eta \cos \psi + e^{2\eta})^{-1} \left[\frac{\partial^2 \varphi}{\partial \eta^2} + \frac{\partial^2 \varphi}{\partial \psi^2}\right] + \frac{\partial^2 \varphi}{\partial z^2}.$$

L1C. LOGARITHMIC-CYLINDER COORDINATES (μ, ν, z), Fig. 2.11.

$$\begin{cases} x = \dfrac{a}{\pi} \ln(\mu^2 + \nu^2), \\ y = \dfrac{2a}{\pi} \tan^{-1}(\nu/\mu), \\ z = z. \end{cases}$$

$$g_{11} = g_{22} = g^{\frac{1}{2}} = \left(\frac{2a}{\pi}\right)^2 \frac{1}{\mu^2 + \nu^2}, \qquad g_{33} = 1.$$

$$(ds)^2 = \left(\frac{2a}{\pi}\right)^2 \frac{1}{\mu^2 + \nu^2} [(d\mu)^2 + (d\nu)^2] + (dz)^2.$$

$$\operatorname{grad} \varphi = \frac{\pi}{2a} (\mu^2 + \nu^2)^{\frac{1}{2}} \left[\boldsymbol{a}_\mu \frac{\partial \varphi}{\partial \mu} + \boldsymbol{a}_\nu \frac{\partial \varphi}{\partial \nu}\right] + \boldsymbol{a}_z \frac{\partial \varphi}{\partial z}.$$

$$\operatorname{div} \boldsymbol{E} = \frac{\pi}{2a} (\mu^2 + \nu^2) \left\{\frac{\partial}{\partial \mu} [(\mu^2 + \nu^2)^{-\frac{1}{2}} E_\mu] + \frac{\partial}{\partial \nu} [(\mu^2 + \nu^2)^{-\frac{1}{2}} E_\nu]\right\} + \frac{\partial E_z}{\partial z}.$$

$$\operatorname{curl} \boldsymbol{E} = \boldsymbol{a}_\mu \left[\frac{\pi}{2a} (\mu^2 + \nu^2)^{\frac{1}{2}} \frac{\partial E_z}{\partial \nu} - \frac{\partial E_\nu}{\partial z}\right] + \boldsymbol{a}_\nu \left[\frac{\partial E_\mu}{\partial z} - \frac{\pi}{2a} (\mu^2 + \nu^2)^{\frac{1}{2}} \frac{\partial E_z}{\partial \mu}\right]$$
$$+ \boldsymbol{a}_z \frac{\pi}{2a} (\mu^2 + \nu^2) \left[\frac{\partial}{\partial \mu}\left(\frac{E_\nu}{(\mu^2 + \nu^2)^{\frac{1}{2}}}\right) - \frac{\partial}{\partial \nu}\left(\frac{E_\mu}{(\mu^2 + \nu^2)^{\frac{1}{2}}}\right)\right].$$

$$\nabla^2 \varphi = \left(\frac{\pi}{2a}\right)^2 (\mu^2 + \nu^2) \left[\frac{\partial^2 \varphi}{\partial \mu^2} + \frac{\partial^2 \varphi}{\partial \nu^2}\right] + \frac{\partial^2 \varphi}{\partial z^2}.$$

L2C. LN TAN-CYLINDER COORDINATES (η, ψ, z), Fig. 2.12.

$$\begin{cases} x = \dfrac{a}{\pi} \ln\left[\dfrac{\sinh^2 \eta + \sin^2 \psi}{\sinh^2 \eta + \cos^2 \psi}\right], \\ y = \dfrac{2a}{\pi} \tan^{-1}\left(\dfrac{\sinh 2\eta}{\sin 2\psi}\right), \\ z = z. \end{cases}$$

$$g_{11} = g_{22} = g^{\frac{1}{2}} = \left(\frac{4a}{\pi}\right)^2 \frac{1}{\sinh^2 2\eta + \sin^2 2\psi}, \qquad g_{33} = 1.$$

$$(ds)^2 = \left(\frac{4a}{\pi}\right)^2 \frac{1}{\sinh^2 2\eta + \sin^2 2\psi} [(d\eta)^2 + (d\psi)^2] + (dz)^2.$$

$$\operatorname{grad} \varphi = \frac{\pi}{4a} [\sinh^2 2\eta + \sin^2 2\psi]^{\frac{1}{2}} \left[\boldsymbol{a}_\eta \frac{\partial \varphi}{\partial \eta} + \boldsymbol{a}_\psi \frac{\partial \varphi}{\partial \psi}\right] + \boldsymbol{a}_z \frac{\partial \varphi}{\partial z}.$$

$$\operatorname{div} \boldsymbol{E} = \frac{\pi}{4a} [\sinh^2 2\eta + \sin^2 2\psi] \left\{\frac{\partial}{\partial \eta} [(\sinh^2 2\eta + \sin^2 2\psi)^{-\frac{1}{2}} E_\eta]\right.$$
$$\left. + \frac{\partial}{\partial \psi} [(\sinh^2 2\eta + \sin^2 2\psi)^{-\frac{1}{2}} E_\psi]\right\} + \frac{\partial E_z}{\partial z}.$$

$$\operatorname{curl} \boldsymbol{E} = \boldsymbol{a}_\eta \left[\frac{\pi}{4a} (\sinh^2 2\eta + \sin^2 2\psi)^{\frac{1}{2}} \frac{\partial E_z}{\partial \psi} - \frac{\partial E_\psi}{\partial z}\right]$$
$$+ \boldsymbol{a}_\psi \left[\frac{\partial E_\eta}{\partial z} - \frac{\pi}{4a} (\sinh^2 2\eta + \sin^2 2\psi)^{\frac{1}{2}} \frac{\partial E_z}{\partial \eta}\right] + \boldsymbol{a}_z \frac{\pi}{4a} (\sinh^2 2\eta + \sin^2 2\psi)$$
$$\times \left[\frac{\partial}{\partial \eta} \left((\sinh^2 2\eta + \sin^2 2\psi)^{-\frac{1}{2}} E_\psi\right) - \frac{\partial}{\partial \psi} \left((\sinh^2 2\eta + \sin^2 2\psi)^{-\frac{1}{2}} E_\eta\right)\right].$$

$$\nabla^2 \varphi = \left(\frac{\pi}{4a}\right)^2 (\sinh^2 2\eta + \sin^2 2\psi) \left[\frac{\partial^2 \varphi}{\partial \eta^2} + \frac{\partial^2 \varphi}{\partial \psi^2}\right] + \frac{\partial^2 \varphi}{\partial z^2}.$$

L3C. LN COSH-CYLINDER COORDINATES (η, ψ, z), Fig. 2.13.

$$\begin{cases} x = \frac{a}{\pi}\ln(\cosh^2\eta - \sin^2\psi), \\ y = \frac{2a}{\pi}\tan^{-1}(\tanh\eta\tan\psi), \\ z = z. \end{cases}$$

$$g_{11} = g_{22} = g^{\frac{1}{2}} = \left(\frac{2a\Pi}{\pi}\right)^2, \qquad g_{33} = 1.$$

where

$$\Pi^2 = \frac{\cosh^2\eta\sinh^2\eta + (\sinh\eta\cosh\eta + \sin\psi\cos\psi)^2}{(\cosh^2\eta - \sin^2\psi)^2}.$$

$$(ds)^2 = \left(\frac{2a\Pi}{\pi}\right)^2[(d\eta)^2 + (d\psi)^2] + (dz)^2.$$

$$\operatorname{grad}\varphi = \frac{\pi}{2a\Pi}\left[\boldsymbol{a}_\eta\frac{\partial\varphi}{\partial\eta} + \boldsymbol{a}_\psi\frac{\partial\varphi}{\partial\psi}\right] + \boldsymbol{a}_z\frac{\partial\varphi}{\partial z}.$$

$$\operatorname{div}\boldsymbol{E} = \frac{\pi}{2a\Pi^2}\left\{\frac{\partial}{\partial\eta}[\Pi E_\eta] + \frac{\partial}{\partial\psi}[\Pi E_\psi]\right\} + \frac{\partial E_z}{\partial z}.$$

$$\operatorname{curl}\boldsymbol{E} = \boldsymbol{a}_\eta\left[\frac{\pi}{2a\Pi}\frac{\partial E_z}{\partial\psi} - \frac{\partial E_\psi}{\partial z}\right] + \boldsymbol{a}_\psi\left[\frac{\partial E_\eta}{\partial z} - \frac{\pi}{2a\Pi}\frac{\partial E_z}{\partial\eta}\right]$$
$$+ \boldsymbol{a}_z\frac{\pi}{2a\Pi^2}\left[\frac{\partial}{\partial\eta}[\Pi E_\psi] - \frac{\partial}{\partial\psi}[\Pi E_\psi]\right].$$

$$\nabla^2\varphi = \left(\frac{\pi}{2a\Pi}\right)^2\left[\frac{\partial^2\varphi}{\partial\eta^2} + \frac{\partial^2\varphi}{\partial\psi^2}\right] + \frac{\partial^2\varphi}{\partial z^2}.$$

E1C. ELLIPTIC-CYLINDER COORDINATES (see Section I).

H2C. INVERSE ELLIPTIC-CYLINDER COORDINATES (η, ψ, z), Fig. 2.15.

$$\begin{cases} x = \frac{a\cosh\eta\cos\psi}{\cosh^2\eta - \sin^2\psi}, \\ y = \frac{a\sinh\eta\sin\psi}{\cosh^2\eta - \sin^2\psi}, \\ z = z. \end{cases}$$

$$g_{11} = g_{22} = g^{\frac{1}{2}} = \frac{a^2(\cosh^2\eta - \cos^2\psi)}{(\cosh^2\eta - \sin^2\psi)^2}, \qquad g_{33} = 1.$$

$$(ds)^2 = \frac{a^2(\cosh^2\eta - \cos^2\psi)}{(\cosh^2\eta - \sin^2\psi)^2}[(d\eta)^2 + (d\psi)^2] + (dz)^2.$$

$$\operatorname{grad}\varphi = \frac{\cosh^2\eta - \sin^2\psi}{a(\cosh^2\eta - \cos^2\psi)^{\frac{1}{2}}}\left[\boldsymbol{a}_\eta\frac{\partial\varphi}{\partial\eta} + \boldsymbol{a}_\psi\frac{\partial\varphi}{\partial\psi}\right] + \boldsymbol{a}_z\frac{\partial\varphi}{\partial z}.$$

$$\operatorname{div}\boldsymbol{E} = \frac{(\cosh^2\eta - \sin^2\psi)^2}{a(\cosh^2\eta - \cos^2\psi)}$$
$$\times\left\{\frac{\partial}{\partial\eta}\left[\frac{(\cosh^2\eta - \cos^2\psi)^{\frac{1}{2}}}{\cosh^2\eta - \sin^2\psi}E_\eta\right] + \frac{\partial}{\partial\psi}\left[\frac{(\cosh^2\eta - \cos^2\psi)^{\frac{1}{2}}}{\cosh^2\eta - \sin^2\psi}E_\psi\right]\right\} + \frac{\partial E_z}{\partial z}.$$

$$\operatorname{curl} \boldsymbol{E} = \boldsymbol{a}_\eta \left\{ \frac{\cosh^2\eta - \sin^2\psi}{a(\cosh^2\eta - \cos^2\psi)^{\frac{1}{2}}} \frac{\partial E_z}{\partial \psi} - \frac{\partial E_\psi}{\partial z} \right\}$$
$$+ \boldsymbol{a}_\psi \left\{ \frac{\partial E_\eta}{\partial z} - \frac{\cosh^2\eta - \sin^2\psi}{a(\cosh^2\eta - \cos^2\psi)^{\frac{1}{2}}} \frac{\partial E_z}{\partial \eta} \right\} + \boldsymbol{a}_z \frac{(\cosh^2\eta - \sin^2\psi)^2}{a(\cosh^2\eta - \cos^2\psi)}$$
$$\times \left\{ \frac{\partial}{\partial \eta} \left[\frac{(\cosh^2\eta - \cos^2\psi)^{\frac{1}{2}}}{\cosh^2\eta - \sin^2\psi} E_\psi \right] - \frac{\partial}{\partial \psi} \left[\frac{(\cosh^2\eta - \cos^2\psi)^{\frac{1}{2}}}{\cosh^2\eta - \sin^2\psi} E_\eta \right] \right\}.$$

$$\nabla^2 \varphi = \frac{(\cosh^2\eta - \sin^2\psi)^2}{a^2(\cosh^2\eta - \cos^2\psi)} \left[\frac{\partial^2\varphi}{\partial \eta^2} + \frac{\partial^2\varphi}{\partial \psi^2} \right] + \frac{\partial^2\varphi}{\partial z^2}.$$

J 1C. SN-CYLINDER COORDINATES (μ, ν, z), Fig. 2.16.

$$0 \leqq \mu \leqq K, \quad 0 \leqq \nu \leqq K', \quad -\infty < z < +\infty.$$

$$\begin{cases} x = \dfrac{a}{\Lambda} \operatorname{sn}\mu \operatorname{dn}\nu, \\ y = \dfrac{a}{\Lambda} \operatorname{cn}\mu \operatorname{dn}\mu \operatorname{sn}\nu \operatorname{cn}\nu, \\ z = z, \quad \text{where} \quad \Lambda = 1 - \operatorname{dn}^2\mu \operatorname{sn}^2\nu. \end{cases}$$

$$g_{11} = g_{22} = g^{\frac{1}{2}} = \frac{a^2 \Omega^2}{\Lambda^2}, \quad g_{33} = 1,$$

where

$$\Omega^2 = (1 - \operatorname{sn}^2\mu \operatorname{dn}^2\nu)(\operatorname{dn}^2\nu - k^2 \operatorname{sn}^2\mu).$$

$$(ds)^2 = \frac{a^2\Omega^2}{\Lambda^2} [(d\mu)^2 + (d\nu)^2] + (dz)^2.$$

$$\operatorname{grad} \varphi = \frac{\Lambda}{a\Omega} \left[\boldsymbol{a}_\mu \frac{\partial \varphi}{\partial \mu} + \boldsymbol{a}_\nu \frac{\partial \varphi}{\partial \nu} \right] + \boldsymbol{a}_z \frac{\partial \varphi}{\partial z}.$$

$$\operatorname{div} \boldsymbol{E} = \frac{\Lambda^2}{a\Omega^2} \left\{ \frac{\partial}{\partial \mu} \left[\frac{\Omega}{\Lambda} E_\mu \right] + \frac{\partial}{\partial \nu} \left[\frac{\Omega}{\Lambda} E_\nu \right] \right\} + \frac{\partial E_z}{\partial z}.$$

$$\operatorname{curl} \boldsymbol{E} = \boldsymbol{a}_\mu \left[\frac{\Lambda}{a\Omega} \frac{\partial E_z}{\partial \nu} - \frac{\partial E_\nu}{\partial z} \right] + \boldsymbol{a}_\nu \left[\frac{\partial E_\mu}{\partial z} - \frac{\Lambda}{a\Omega} \frac{\partial E_z}{\partial \mu} \right]$$
$$+ \boldsymbol{a}_z \frac{\Lambda^2}{a\Omega^2} \left\{ \frac{\partial}{\partial \mu} \left[\frac{\Omega}{\Lambda} E_\nu \right] - \frac{\partial}{\partial \nu} \left[\frac{\Omega}{\Lambda} E_\mu \right] \right\}.$$

$$\nabla^2 \varphi = \frac{\Lambda^2}{a^2\Omega^2} \left\{ \frac{\partial^2\varphi}{\partial \mu^2} + \frac{\partial^2\varphi}{\partial \nu^2} \right\} + \frac{\partial^2\varphi}{\partial z^2}.$$

J 2C. CN-CYLINDER COORDINATES (μ, ν, z), Fig. 2.17.

$$0 \leqq \mu \leqq K, \quad 0 \leqq \nu \leqq K', \quad -\infty < z < +\infty.$$

$$\begin{cases} x = \dfrac{a}{\Lambda} \operatorname{cn}\mu \operatorname{cn}\nu, \\ y = \dfrac{a}{\Lambda} \operatorname{sn}\mu \operatorname{dn}\mu \operatorname{sn}\nu \operatorname{dn}\nu, \\ z = z, \quad \text{where} \quad \Lambda = 1 - \operatorname{dn}^2\mu \operatorname{sn}^2\nu. \end{cases}$$

$$g_{11} = g_{22} = g^{\frac{1}{2}} = \frac{a^2 \Gamma^2}{\Lambda^2}, \quad g_{33} = 1,$$

where

$$\Gamma^2 = (\operatorname{sn}^2\nu + \operatorname{sn}^2\mu \operatorname{cn}^2\nu)(\operatorname{dn}^2\nu - k^2 \operatorname{sn}^2\mu).$$

$$(ds)^2 = \frac{a^2\Gamma^2}{\Lambda^2}[(d\mu)^2 + (d\nu)^2] + (dz)^2.$$

$$\operatorname{grad}\varphi = \frac{\Lambda}{a\Gamma}\left[\boldsymbol{a}_\mu \frac{\partial\varphi}{\partial\mu} + \boldsymbol{a}_\nu \frac{\partial\varphi}{\partial\nu}\right] + \boldsymbol{a}_z \frac{\partial\varphi}{\partial z}.$$

$$\operatorname{div}\boldsymbol{E} = \frac{\Lambda^2}{a\Gamma^2}\left\{\frac{\partial}{\partial\mu}\left[\frac{\Gamma}{\Lambda}E_\mu\right] + \frac{\partial}{\partial\nu}\left[\frac{\Gamma}{\Lambda}E_\nu\right]\right\} + \frac{\partial E_z}{\partial z}.$$

$$\operatorname{curl}E = \boldsymbol{a}_\mu\left[\frac{\Lambda}{a\Gamma}\frac{\partial E_z}{\partial\nu} - \frac{\partial E_\nu}{\partial z}\right] + \boldsymbol{a}_\nu\left[\frac{\partial E_\mu}{\partial z} - \frac{\Lambda}{a\Gamma}\frac{\partial E_z}{\partial\mu}\right]$$

$$+ \boldsymbol{a}_z \frac{\Lambda^2}{a\Gamma^2}\left\{\frac{\partial}{\partial\mu}\left[\frac{\Gamma}{\Lambda}E_\nu\right] - \frac{\partial}{\partial\nu}\left[\frac{\Gamma}{\Lambda}E_\mu\right]\right\}.$$

$$\nabla^2\varphi = \frac{\Lambda^2}{a^2\Gamma^2}\left\{\frac{\partial^2\varphi}{\partial\mu^2} + \frac{\partial^2\varphi}{\partial\nu^2}\right\} + \frac{\partial^2\varphi}{\partial z^2}.$$

J 3C. INVERSE SN-CYLINDER COORDINATES (μ, ν, z), Fig. 2.18.

$$0 \leqq \mu \leqq K, \quad 0 \leqq \nu \leqq K', \quad -\infty < z < +\infty.$$

$$\begin{cases} x = \dfrac{\Lambda}{a\Upsilon}\operatorname{sn}\mu\,\operatorname{dn}\nu, \\ y = \dfrac{k^{\frac{1}{2}}\Pi}{2a\Upsilon}, \\ z = z, \quad \text{where } \Lambda = 1 - \operatorname{dn}^2\mu\,\operatorname{sn}^2\nu, \end{cases}$$

$$\Omega = \operatorname{sn}^2\mu\,\operatorname{dn}^2\nu + \operatorname{cn}^2\mu\,\operatorname{dn}^2\mu\,\operatorname{sn}^2\nu\,\operatorname{cn}^2\nu,$$

$$\Upsilon = \operatorname{sn}^2\mu\,\operatorname{dn}^2\nu + [(\Lambda/\sqrt{k}) + \operatorname{cn}\mu\,\operatorname{dn}\mu\,\operatorname{sn}\nu\,\operatorname{cn}\nu],$$

$$\Pi = (\Lambda^2/k) - (\operatorname{sn}^2\mu\,\operatorname{dn}^2\nu + \operatorname{cn}^2\mu\,\operatorname{dn}^2\mu\,\operatorname{sn}^2\nu\,\operatorname{cn}^2\nu).$$

$$g_{11} = g_{22} = g^{\frac{1}{2}} = \left(\frac{\Lambda\Omega}{a\Upsilon}\right)^2, \quad g_{33} = 1.$$

$$(ds)^2 = \left(\frac{\Lambda\Omega}{a\Upsilon}\right)^2[(d\mu)^2 + (d\nu)^2] + (dz)^2.$$

$$\operatorname{grad}\varphi = \frac{a\Upsilon}{\Lambda\Omega}\left[\boldsymbol{a}_\mu \frac{\partial\varphi}{\partial\mu} + \boldsymbol{a}_\nu \frac{\partial\varphi}{\partial\nu}\right] + \boldsymbol{a}_z \frac{\partial\varphi}{\partial z}.$$

$$\operatorname{div}\boldsymbol{E} = \frac{a\Upsilon^2}{\Lambda^2\Omega^2}\left\{\frac{\partial}{\partial\mu}\left[\frac{\Lambda\Omega}{\Upsilon}E_\mu\right] + \frac{\partial}{\partial\nu}\left[\frac{\Lambda\Omega}{\Upsilon}E_\nu\right]\right\} + \frac{\partial E_z}{\partial z}.$$

$$\operatorname{curl}\boldsymbol{E} = \boldsymbol{a}_\mu\left[\frac{a\Upsilon}{\Lambda\Omega}\frac{\partial E_z}{\partial\nu} - \frac{\partial E_\nu}{\partial z}\right] + \boldsymbol{a}_\nu\left[\frac{\partial E_\mu}{\partial z} - \frac{a\Upsilon}{\Lambda\Omega}\frac{\partial E_z}{\partial\mu}\right]$$

$$+ \boldsymbol{a}_z \frac{a\Upsilon^2}{\Lambda^2\Omega^2}\left\{\frac{\partial}{\partial\mu}\left[\frac{\Lambda\Omega}{\Upsilon}E_\nu\right] - \frac{\partial}{\partial\nu}\left[\frac{\Lambda\Omega}{\Upsilon}E_\mu\right]\right\}.$$

$$\nabla^2\varphi = \left(\frac{a\Upsilon}{\Lambda\Omega}\right)^2\left\{\frac{\partial^2\varphi}{\partial\mu^2} + \frac{\partial^2\varphi}{\partial\nu^2}\right\} + \frac{\partial^2\varphi}{\partial z^2}.$$

J 4C. LN SN-CYLINDER COORDINATES (μ, ν, z), Fig. 2.19.

$$0 \leqq \mu \leqq K, \quad 0 \leqq \nu \leqq K', \quad -\infty < z < +\infty$$

$$\begin{cases} x = \frac{a}{\pi} \ln\left(\frac{\Lambda^2}{k\Sigma}\right), \\ y = \frac{2a}{\pi} \tan^{-1}\left(\frac{\operatorname{cn}\mu \operatorname{dn}\mu \operatorname{sn}\nu \operatorname{cn}\nu}{\operatorname{sn}\mu \operatorname{dn}\nu}\right), \\ z = z, \quad \text{where} \quad \Lambda = 1 - \operatorname{dn}^2\mu \operatorname{sn}^2\nu, \end{cases}$$

$$\Sigma = \operatorname{sn}^2\mu \operatorname{dn}^2\nu + \operatorname{cn}^2\mu \operatorname{dn}^2\mu \operatorname{sn}^2\nu \operatorname{cn}^2\nu.$$

$$g_{11} = g_{22} = g^{\frac{1}{2}} = \left(\frac{2a\Theta}{\pi\Sigma}\right)^2, \quad g_{33} = 1,$$

where

$$\Theta^2 = \operatorname{sn}^2\mu \operatorname{cn}^2\mu \operatorname{dn}^2\mu (\operatorname{cn}^2\nu - \operatorname{sn}^2\nu \operatorname{dn}^2\nu)^2 + \operatorname{sn}^2\nu \operatorname{cn}^2\nu \operatorname{dn}^2\nu (\operatorname{cn}^2\mu + \operatorname{sn}^2\mu \operatorname{dn}^2\mu)^2.$$

$$(ds)^2 = \left(\frac{2a\Theta}{\pi\Sigma}\right)^2 [(d\mu)^2 + (d\nu)^2] + (dz)^2.$$

$$\operatorname{grad}\varphi = \frac{\pi\Sigma}{2a\Theta}\left[\boldsymbol{a}_\mu \frac{\partial\varphi}{\partial\mu} + \boldsymbol{a}_\nu \frac{\partial\varphi}{\partial\nu}\right] + \boldsymbol{a}_z \frac{\partial\varphi}{\partial z}.$$

$$\operatorname{div}\boldsymbol{E} = \frac{\pi\Sigma^2}{2a\Theta^2}\left\{\frac{\partial}{\partial\mu}\left[\frac{\Theta}{\Sigma}E_\mu\right] + \frac{\partial}{\partial\nu}\left[\frac{\Theta}{\Sigma}E_\nu\right]\right\} + \frac{\partial E_z}{\partial z}.$$

$$\operatorname{curl}\boldsymbol{E} = \boldsymbol{a}_\mu\left[\frac{\pi\Sigma}{2a\Theta}\frac{\partial E_z}{\partial\nu} - \frac{\partial E_\nu}{\partial z}\right] + \boldsymbol{a}_\nu\left[\frac{\partial E_\mu}{\partial z} - \frac{\pi\Sigma}{2a\Theta}\frac{\partial E_z}{\partial\mu}\right] + \boldsymbol{a}_z \frac{\pi\Sigma^2}{2a\Theta^2}\left\{\frac{\partial}{\partial\mu}\left[\frac{\Theta}{\Sigma}E_\nu\right] - \frac{\partial}{\partial\nu}\left[\frac{\Theta}{\Sigma}E_\mu\right]\right\}.$$

$$\nabla^2\varphi = \left(\frac{\pi\Sigma}{2a\Theta}\right)^2\left\{\frac{\partial^2\varphi}{\partial\mu^2} + \frac{\partial^2\varphi}{\partial\nu^2}\right\} + \frac{\partial^2\varphi}{\partial z^2}.$$

J 5C. LN CN-CYLINDER COORDINATES (μ, ν, z), Fig. 2.20.

$$0 \leqq \mu \leqq K, \quad 0 \leqq \nu \leqq K', \quad -\infty < z < +\infty.$$

$$\begin{cases} x = \frac{a}{\pi} \ln(\Xi/\Lambda^2), \\ y = \frac{2a}{\pi} \tan^{-1}\left(\frac{\operatorname{sn}\mu \operatorname{dn}\mu \operatorname{sn}\nu \operatorname{dn}\nu}{\operatorname{cn}\mu \operatorname{cn}\nu}\right), \\ z = z, \quad \text{where} \quad \Lambda = 1 - \operatorname{dn}^2\mu \operatorname{sn}^2\nu, \end{cases}$$

$$\Xi = \operatorname{cn}^2\mu \operatorname{cn}^2\nu + \operatorname{sn}^2\mu \operatorname{dn}^2\mu \operatorname{sn}^2\nu \operatorname{dn}^2\nu.$$

$$g_{11} = g_{22} = g^{\frac{1}{2}} = \left(\frac{2a\Theta'}{\pi\Xi}\right)^2, \quad g_{33} = 1,$$

where

$$\Theta'^2 = \operatorname{sn}^2\mu \operatorname{cn}^2\mu \operatorname{dn}^2\mu (\operatorname{dn}^2\nu - k'^2 \operatorname{sn}^2\nu \operatorname{cn}^2\nu)^2 + \operatorname{sn}^2\nu \operatorname{cn}^2\nu \operatorname{dn}^2\nu (\operatorname{dn}^2\mu - k^2 \operatorname{sn}^2\mu \operatorname{cn}^2\mu)^2.$$

$$(ds)^2 = \left(\frac{2a\Theta'}{\pi\Xi}\right)^2 [(d\mu)^2 + (d\nu)^2] + (dz)^2.$$

$$\operatorname{grad}\varphi = \frac{\pi\Xi}{2a\Theta'}\left[\boldsymbol{a}_\mu \frac{\partial\varphi}{\partial\mu} + \boldsymbol{a}_\nu \frac{\partial\varphi}{\partial\nu}\right] + \boldsymbol{a}_z \frac{\partial\varphi}{\partial z}.$$

$$\operatorname{div}\boldsymbol{E} = \frac{\pi\Xi^2}{2a\Theta'^2}\left\{\frac{\partial}{\partial\mu}\left[\frac{\Theta'}{\Xi}E_\mu\right] + \frac{\partial}{\partial\nu}\left[\frac{\Theta'}{\Xi}E_\nu\right]\right\} + \frac{\partial E_z}{\partial z}.$$

$$\operatorname{curl}\boldsymbol{E} = \boldsymbol{a}_\mu\left[\frac{\pi\Xi}{2a\Theta'}\frac{\partial E_z}{\partial\nu} - \frac{\partial E_\nu}{\partial z}\right] + \boldsymbol{a}_\nu\left[\frac{\partial E_\mu}{\partial z} - \frac{\pi\Xi}{2a\Theta'}\frac{\partial E_z}{\partial\mu}\right]$$
$$+ \boldsymbol{a}_z \frac{\pi\Xi^2}{2a\Theta'^2}\left\{\frac{\partial}{\partial\mu}\left[\frac{\Theta'}{\Xi}E_\nu\right] - \frac{\partial}{\partial\nu}\left[\frac{\Theta'}{\Xi}E_\mu\right]\right\}.$$

$$\nabla^2\varphi = \left(\frac{\pi\Xi}{2a\Theta'}\right)^2\left\{\frac{\partial^2\varphi}{\partial\mu^2} + \frac{\partial^2\varphi}{\partial\nu^2}\right\} + \frac{\partial^2\varphi}{\partial z^2}.$$

J 6C. ZETA-CYLINDER COORDINATES (μ, ν, z), Fig. 2.21.

$$0 \leqq \mu \leqq K, \quad 0 \leqq \nu \leqq K', \quad -\infty < z < +\infty.$$

$$\begin{cases} x = \dfrac{2Ka}{\pi}\left\{Z(\mu) + \dfrac{k^2}{\Lambda'}\operatorname{sn}\mu\operatorname{cn}\mu\operatorname{dn}\mu\operatorname{sn}^2\nu'\right\}, \\ y = \dfrac{2Ka}{\pi}\left\{Z(\nu') + \dfrac{\pi\nu}{2KK'} - \dfrac{1}{\Lambda'}\operatorname{dn}^2\mu\operatorname{sn}\nu'\operatorname{cn}\nu'\operatorname{dn}\nu'\right\}, \\ z = z, \quad \text{where } \Lambda' = 1 - \operatorname{dn}^2\mu\operatorname{sn}^2\nu', \end{cases}$$

$$\nu' = \nu + K',$$

$$K = \int_0^{\pi/2} \frac{d\varphi}{(1 - k^2\sin^2\varphi)^{\frac{1}{2}}},$$

$$K' = \int_0^{\pi/2} \frac{d\varphi}{(1 - k'^2\sin^2\varphi)^{\frac{1}{2}}},$$

$$E = \int_0^{\pi/2} (1 - k^2\sin^2\varphi)^{\frac{1}{2}}\,d\varphi.$$

$$g_{11} = g_{22} = g^{\frac{1}{2}} = \left(\frac{2Ka}{\pi}\right)^2\left\{\frac{4k^4}{\Lambda'^4}\operatorname{sn}^2\mu\operatorname{cn}^2\mu\operatorname{dn}^2\mu\operatorname{sn}^2\nu'\operatorname{cn}^2\nu'\operatorname{dn}^2\nu'\right.$$
$$+ \left[\operatorname{dn}^2\mu - \frac{E}{K} + \frac{k^2\operatorname{sn}^2\nu'}{\Lambda'^2}\left(\operatorname{dn}^2\mu\operatorname{cn}^2\nu'[\operatorname{cn}^2\mu - \operatorname{sn}^2\mu\operatorname{dn}^2\mu]\right.\right.$$
$$\left.\left.\left. - k^2\operatorname{sn}^2\mu[1 - k^2\operatorname{sn}^4\mu]\right)\right]^2\right\}.$$

Other equations can be written with the help of the general relations of § 3.01.

Section IV

ROTATIONAL SYSTEMS

Each of the 21 transformations of Section II yields one or two rotational coordinate systems, obtained by twirling the plane map about an axis. A tabulation of the most interesting of these systems is given in this section.

4.01 R-SEPARATION

In most cases, the rotational coordinate systems do not allow simple separation of the Laplace and Helmholtz equations, though they may allow R-separation of the Laplace equation. This subject is treated in previous publications [*7*], [*9*].

Definition I. If the assumption

$$\varphi = U^1(u^1) \cdot U^2(u^2) \cdot U^3(u^3)$$

permits the separation of the partial differential equation into three ordinary differential equations, the equation is said to be *simply separable.*

Definition II. If the assumption

$$\varphi = \frac{U^1(u^1) \cdot U^2(u^2) \cdot U^3(u^3)}{R(u^1, u^2, u^3)}$$

permits the separation of the partial differential equation into three ordinary differential equations, and if $R \neq \text{const}$, the equation is said to be R-*separable.*

R-separation introduces two new quantities, $R(u^1, u^2, u^3)$ and $Q(u^1, u^2, u^3)$, which greatly improve the possibility of effecting separation. These quantities are defined by the relations,

$$\begin{cases} g_{ii} = \dfrac{S}{M_{i1}} Q, & (4.01) \\ \dfrac{g^{\frac{1}{2}}}{S} = f_1 f_2 f_3 (R)^2 Q. & (4.02) \end{cases}$$

Also, α_1 is not zero as it was in simple separation of the Laplace equation, but

$$\alpha_1 = -\frac{Q}{R} \sum_{i=1}^{3} \frac{1}{f_i g_{ii}} \frac{\partial}{\partial u^i}\left(f_i \frac{\partial R}{\partial u^i}\right) = \text{const}. \quad (4.03)$$

Equation (4.03) constitutes the necessary and sufficient condition for separability of the Laplace equation in euclidean 3-space [*7*]. A comparison of the equations for simple separation and R-separation is given in Table 4.01.

No case is known in which the Helmholtz equation is R-separable, so the question that arises in this section is merely whether the Laplace equation is

R-separable in a given rotational coordinate system. For example, in toroidal coordinates,

$$g_{11} = g_{22} = \frac{a^2}{(\cosh \eta - \cos \theta)^2},$$

$$g_{33} = \frac{a^2 \sinh^2 \eta}{(\cosh \eta - \cos \theta)^2}.$$

The complicated form of the denominator precludes the possibility of obtaining a Stäckel matrix for simple separation. But for R-separation, we let

$$Q = \frac{a^2}{(\cosh \eta - \cos \theta)^2}.$$

Then

$$\frac{M_{11}}{S} = \frac{M_{21}}{S} = 1, \quad \frac{M_{31}}{S} = \frac{1}{\sinh^2 \eta},$$

and the Stäckel determinant is very simple:

$$S = \begin{vmatrix} 1 & -1 & -\frac{1}{\sinh^2 \eta} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

Also, $\alpha_1 = +\frac{1}{4}$ and $R^2 = (\cosh \eta - \cos \theta)^{-1}$. Requirements for R-separation of the Laplace equation are satisfied.

4.02 TABLES

Metric coefficients and separability data are listed in Table 4.02. Some of the rotational systems obtained from Figs. 2.01 to 2.21 do not allow separation of any kind. After discarding the non-separable cases, we find 4 systems in which Laplace's equation is simply separable and 10 that are R-separable. The simply separable coordinates have already been listed in Section I, and the R-separable cases are tabulated in this section. One additional system in introduced—the inversion of rectangular coordinates—making a total of 11 coordinate systems treated in Section 4:

No.	Name	Fig. No.
P1R	Tangent-sphere coordinates	4.01
P3R	Cardioid coordinates	4.02
E4Rx	Bispherical coordinates	4.03
E4Ry	Toroidal coordinates	4.04
H2Rx	Inverse prolate spheroidal coordinates	4.05
H2Ry	Inverse oblate spheroidal coordinates	4.06
— —	6-sphere coordinates	4.07
J1Rx	Bi-cyclide coordinates	4.08
J1Ry	Flat-ring cyclide coordinates	4.09
J2R	Disk-cyclide coordinates	4.10
J3R	Cap-cyclide coordinates	4.11

Details of these eleven coordinate systems are given in Table 4.03 and Figs. 4.01 to 4.11. Table 4.04 summarizes four rotational systems obtained from elliptic functions. It should be emphasized that these coordinates by no means constitute the totality of all R-separable systems.

TABLE 4.01. SEPARABILITY

for $\varphi = \varphi(u^1, u^2, u^3)$.

Simple separability of Helmholtz equation	R-separability of Laplace equation
$\nabla^2 \varphi + \varkappa^2 \varphi = 0$	$\nabla^2 \varphi = 0$
$\varphi = U^1 \cdot U^2 \cdot U^3$	$\varphi = \dfrac{U^1 \cdot U^2 \cdot U^3}{R}$
$g_{ii} = \dfrac{S}{M_{i1}}$	$g_{ii} = \dfrac{S}{M_{i1}} Q$
$\dfrac{g^{\frac{1}{2}}}{S} = f_1 f_2 f_3$	$\dfrac{g^{\frac{1}{2}}}{S} = f_1 f_2 f_3 (R)^2 Q$
$\alpha_1 = \varkappa^2$	$\alpha_1 = -\dfrac{Q}{R} \sum_{i=1}^{3} \dfrac{1}{f_i g_{ii}} \dfrac{\partial}{\partial u^i}\left(f_i \dfrac{\partial R}{\partial u^i}\right)$

Separated equations

$$\frac{1}{f_i} \frac{d}{du^i}\left(f_i \frac{dU^i}{du^i}\right) + U^i \sum_{j=1}^{3} \Phi_{ij} \alpha_j = 0.$$

Stäckel matrix

$$[S] = \begin{bmatrix} \Phi_{11}(u^1) & \Phi_{12}(u^1) & \Phi_{13}(u^1) \\ \Phi_{21}(u^2) & \Phi_{22}(u^2) & \Phi_{23}(u^2) \\ \Phi_{31}(u^3) & \Phi_{32}(u^3) & \Phi_{33}(u^3) \end{bmatrix}.$$

TABLE 4.02. SOME ROTATIONAL COORDINATE SYSTEMS

No.	Name of Coordinate System	Fig. No.	Equations	Metric Coefficients	Separability: Helmholtz $\varphi(u^1,u^2,u^3)$	Separability: Helmholtz $\varphi(u^1,u^2)$	Separability: Laplace $\varphi(u^1,u^2,u^3)$	Separability: Laplace $\varphi(u^1,u^2)$	One-dimensional Solution: Helm-holtz	One-dimensional Solution: Laplace	α_1
P1R	Tangent-sphere	4.01 Rot. about y axis	$x = \frac{\mu}{\mu^2+\nu^2}\cos\psi$ $y = \frac{\mu}{\mu^2+\nu^2}\sin\psi$ $z = \frac{\nu}{\mu^2+\nu^2}$	$g_{11} = g_{22} = (\mu^2+\nu^2)^{-2}$ $g_{33} = \frac{\mu^2}{(\mu^2+\nu^2)^2}$	×	×	R	R	×	×	0
P2R	Parabolic	1.08 Rot. about x axis	$x = \mu\nu\cos\psi$ $y = \mu\nu\sin\psi$ $z = \frac{1}{2}(\mu^2-\nu^2)$	$g_{11} = g_{22} = \mu^2+\nu^2$ $g_{33} = \mu^2\nu^2$	S	S	S	S	×	●	0
P3R	Cardioid	4.02 Rot. about x axis	$x = \frac{\mu\nu}{(\mu^2+\nu^2)^2}\cos\psi$ $y = \frac{\mu\nu}{(\mu^2+\nu^2)^2}\sin\psi$ $z = \frac{1}{2}\frac{\mu^2-\nu^2}{(\mu^2+\nu^2)^2}$	$g_{11} = g_{22} = (\mu^2+\nu^2)^{-3}$ $g_{33} = \frac{\mu^2\nu^2}{(\mu^2+\nu^2)^4}$	×	×	R	R	×	×	0

Table 4.02. Continuation

No.	Name of Coordinate System	Fig. No.	Equations	Metric Coefficients	Separability: Helmholtz $\varphi(u^1,u^2,u^3)$	Separability: Helmholtz $\varphi(u^1,u^2)$	Separability: Laplace $\varphi(u^1,u^2,u^3)$	Separability: Laplace $\varphi(u^1,u^2)$	One-dimensional Solution: Helmholtz	One-dimensional Solution: Laplace	α_1
P 4 R	Hyperbolic	2.04 Rot. about y axis	$x=(\varrho+\mu)^{\frac{1}{2}}\cos\psi$ $y=(\varrho+\mu)^{\frac{1}{2}}\sin\psi$ $z=(\varrho-\mu)^{\frac{1}{2}}$ where $\varrho=+(\mu^2+\nu^2)^{\frac{1}{2}}$	$g_{11}=g_{22}=\frac{1}{2}(\mu^2+\nu^2)^{-\frac{1}{2}}$ $g_{33}=\varrho+\mu$	×	×	×	×	×	×	—
E 1 R	Spherical	1.05 Rot. about x axis	$x=e^{\eta}\sin\theta\cos\psi$ $y=e^{\eta}\sin\theta\sin\psi$ $z=e^{\eta}\cos\theta$	$g_{11}=g_{22}=e^{2\eta}$ $g_{33}=e^{2\eta}\sin^2\theta$	S	S	S	S	●	●	0
E 4 Rx	Bispherical	4.03 Rot. about x axis	$x=\frac{a\sin\theta\cos\psi}{\cosh\eta-\cos\theta}$ $y=\frac{a\sin\theta\sin\psi}{\cosh\eta-\cos\theta}$ $z=\frac{a\sinh\eta}{\cosh\eta-\cos\theta}$	$g_{11}=g_{22}$ $=\frac{a^2}{(\cosh\eta-\cos\theta)^2}$ $g_{33}=\frac{a^2\sin^2\theta}{(\cosh\eta-\cos\theta)^2}$	×	×	R	R	×	×	$-\frac{1}{4}$

E 4 Ry	Toroidal	4.04 Rot. about y axis	$x = \dfrac{a \sinh \eta \cos \psi}{\cosh \eta - \cos \theta}$ $y = \dfrac{a \sinh \eta \sin \psi}{\cosh \eta - \cos \theta}$ $z = \dfrac{a \sin \theta}{\cosh \eta - \cos \theta}$	$g_{11} = g_{22} = \dfrac{a^2}{(\cosh \eta - \cos \theta)^2}$ $g_{33} = \dfrac{a^2 \sinh^2 \eta}{(\cosh \eta - \cos \theta)^2}$	×	×	R	R	×	×	$\frac{1}{4}$
H 1 Rx	Prolate spheroidal	1.06 Rot. about x axis	$x = a \sinh \eta \sin \theta \cos \psi$ $y = a \sinh \eta \sin \theta \sin \psi$ $z = a \cosh \eta \cos \theta$	$g_{11} = g_{22} = a^2(\cosh^2 \eta - \cos^2 \theta) = a^2(\sinh^2 \eta + \sin^2 \theta)$ $g_{33} = a^2 \sinh^2 \eta \sin^2 \theta$	S	S	S	S	×	●	0
H 1 Ry	Oblate spheroidal	1.07 Rot. about y axis*	$x = a \cosh \eta \sin \theta \cos \psi$ $y = a \cosh \eta \sin \theta \sin \psi$ $z = a \sinh \eta \cos \theta$	$g_{11} = g_{22} = a^2(\cosh^2 \eta - \sin^2 \theta)$ $g_{33} = a^2 \cosh^2 \eta \sin^2 \theta$	S	S	S	S	×	●	0
H 2 Rx	Inversion of prolate spheroidal	4.05 Rot. about x axis	$x = \dfrac{a \sinh \eta \sin \theta \cos \psi}{\cosh^2 \eta - \sin^2 \theta}$ $y = \dfrac{a \sinh \eta \sin \theta \sin \psi}{\cosh^2 \eta - \sin^2 \theta}$ $z = \dfrac{a \cosh \eta \cos \theta}{\cosh^2 \eta - \sin^2 \theta}$	$g_{11} = g_{22} = \dfrac{a^2(\sinh^2 \eta + \sin^2 \theta)}{(\cosh^2 \eta - \sin^2 \theta)^2}$ $g_{33} = \dfrac{a^2 \sinh^2 \eta \sin^2 \theta}{(\cosh^2 \eta - \sin^2 \theta)^2}$	×	×	R	R	×	×	0
H 2 Ry	Inversion of oblate spheroidal	4.06 Rot. about y axis*	$x = \dfrac{a \cosh \eta \sin \theta \cos \psi}{\cosh^2 \eta - \cos^2 \theta}$ $y = \dfrac{a \cosh \eta \sin \theta \sin \psi}{\cosh^2 \eta - \cos^2 \theta}$ $z = \dfrac{a \sinh \eta \cos \theta}{\cosh^2 \eta - \cos^2 \theta}$	$g_{11} = g_{22} = \dfrac{a^2(\cosh^2 \eta - \sin^2 \theta)}{(\cosh^2 \eta - \cos^2 \theta)^2}$ $g_{33} = \dfrac{a^2 \cosh^2 \eta \sin^2 \theta}{(\cosh^2 \eta - \cos^2 \theta)^2}$	×	×	R	R	×	×	0

* $\theta = v - \pi/2$ in this case, to make the equations agree with usual oblate nomenclature.

Table 4.02. Continuation

No.	Name of Coordinate System	Fig. No.	Equations	Metric Coefficients	Separability: Helmholtz $\varphi(u^1,u^2,u^3)$	Separability: Helmholtz $\varphi(u^1,u^2)$	Separability: Laplace $\varphi(u^1,u^2,u^3)$	Separability: Laplace $\varphi(u^1,u^2)$	One-dimensional Solution: Helm-holtz	One-dimensional Solution: Laplace	α_1
J 1 Rx	$z = a \operatorname{sn} w$	4.08 Rot. about x axis	$x = \frac{a}{\Lambda} \operatorname{cn}\mu \operatorname{dn}\mu \times \operatorname{sn}\nu \operatorname{cn}\nu \cos\psi$ $y = \frac{a}{\Lambda} \operatorname{cn}\mu \operatorname{dn}\mu \times \operatorname{sn}\nu \operatorname{cn}\nu \sin\psi$ $z = \frac{a}{\Lambda} \operatorname{sn}\mu \operatorname{dn}\nu$ where $\Lambda = 1 - \operatorname{dn}^2\mu \operatorname{sn}^2\nu$	$g_{11} = g_{22} = \frac{a^2 \Omega^2}{\Lambda^2}$ $g_{33} = \frac{a^2}{\Lambda^2} \operatorname{cn}^2\mu \operatorname{dn}^2\mu \times \operatorname{sn}^2\nu \operatorname{cn}^2\nu$ where $\Omega^2 = (1 - \operatorname{sn}^2\mu \operatorname{dn}^2\nu) \times (\operatorname{dn}^2\nu - k^2 \operatorname{sn}^2\mu)$	×	×	R	R	×	×	− 2
J 1 Ry	$z = a \operatorname{sn} w$	4.09 Rot. about y axis	$x = \frac{a}{\Lambda} \operatorname{sn}\mu \operatorname{dn}\nu \cos\psi$ $y = \frac{a}{\Lambda} \operatorname{sn}\mu \operatorname{dn}\nu \sin\psi$ $z = \frac{a}{\Lambda} \operatorname{cn}\mu \operatorname{dn}\mu \times \operatorname{sn}\nu \operatorname{cn}\nu$	$g_{11} = g_{22} = \frac{a^2 \Omega^2}{\Lambda^2}$ $g_{33} = \frac{a^2}{\Lambda^2} \operatorname{sn}^2\mu \operatorname{dn}^2\nu$	×	×	R	R	×	×	− 1

J2R	$\bar{z} = a \operatorname{cn} w$	4.10 Rot. about y axis	$x = \frac{a}{\Lambda} \operatorname{cn}\mu \operatorname{cn}\nu \cos\psi$ $y = \frac{a}{\Lambda} \operatorname{cn}\mu \operatorname{cn}\nu \sin\psi$ $z = \frac{a}{\Lambda} \operatorname{sn}\mu \operatorname{dn}\mu$ $\times \operatorname{sn}\nu \operatorname{dn}\nu$	$g_{11} = g_{22} = \frac{a^2 \Gamma^2}{\Lambda^2}$ $g_{33} = \frac{a^2}{\Lambda^2} \operatorname{cn}^2\mu \operatorname{cn}^2\nu$ where $\Gamma^2 = (\operatorname{sn}^2\nu + \operatorname{sn}^2\mu \operatorname{cn}^2\nu)$ $\times (\operatorname{dn}^2\nu - k^2 \operatorname{sn}^2\mu)$	×	×	R	R	×	×	−1
J3R	$\bar{z} = \frac{k^{\frac{1}{2}}}{i2a} \times \left(\frac{1 + ik^{\frac{1}{2}} \operatorname{sn} w}{1 - ik^{\frac{1}{2}} \operatorname{sn} w}\right)$	4.11 Rot. about y axis	$x = \frac{\Lambda}{a\Upsilon} \operatorname{sn}\mu \operatorname{dn}\nu \cos\psi$ $y = \frac{\Lambda}{a\Upsilon} \operatorname{sn}\mu \operatorname{dn}\nu \sin\psi$ $z = \frac{k^{\frac{1}{2}} \Pi}{2a\Upsilon}$ where $\Upsilon = \operatorname{sn}^2\mu \operatorname{dn}^2\nu$ $+ [(\Lambda/\sqrt{k}) + \operatorname{cn}\mu$ $\times \operatorname{dn}\mu \operatorname{sn}\nu \operatorname{cn}\nu]^2$ $\Pi = (\Lambda^2/k) - (\operatorname{sn}^2\mu$ $\times \operatorname{dn}^2\nu + \operatorname{cn}^2\mu$ $\times \operatorname{dn}^2\mu \operatorname{sn}^2\nu \operatorname{cn}^2\nu)$	$g_{11} = g_{22} = \frac{\Lambda^2 \Omega^2}{a^2 \Upsilon^2}$ $g_{33} = \frac{\Lambda^2}{a^2 \Upsilon^2} \operatorname{sn}^2\mu \operatorname{dn}^2\nu$	×	×	R	R	×	×	−1

Crosses indicate non-separability, *S* indicates simple separability, *R* indicates *R*-separability. The modulus k is associated with μ, k' is associated with ν: $\operatorname{sn}\mu \equiv \operatorname{sn}(\mu, k)$, $\operatorname{sn}\nu \equiv \operatorname{sn}(\nu, k')$.

TABLE 4.03

P1R. TANGENT-SPHERE COORDINATES (μ, ν, ψ), Fig. 4.01.

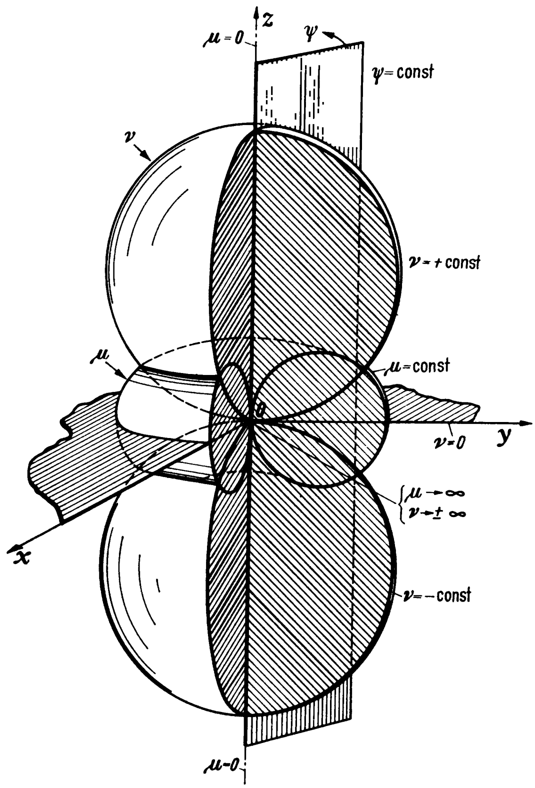

Fig. 4.01. Tangent-sphere coordinates (μ, ν, ψ). This and the following figures are drawn as if the coordinate surfaces were filled with jelly and then one quarter were cut out (from $\psi = 0$ to $\psi = \pi/2$). The coordinate surfaces are tangent spheres ($\nu = \pm$ const), toroids without center opening ($\mu =$ const), and half-planes ($\psi =$ const). The solid arrow heads indicate the direction associated with an increase in the parameter

$$0 < \mu < +\infty,$$
$$-\infty < \nu < +\infty,$$
$$0 \leqq \psi < 2\pi.$$

$$\begin{cases} x = \dfrac{\mu \cos\psi}{\mu^2 + \nu^2}, \\ y = \dfrac{\mu \sin\psi}{\mu^2 + \nu^2}, \\ z = \dfrac{\nu}{\mu^2 + \nu^2}. \end{cases}$$

Coordinate surfaces

$x^2 + y^2 + z^2 = \frac{1}{\mu}(x^2 + y^2)^{\frac{1}{2}}$ (toroids without center opening, $\mu =$ const),

$x^2 + y^2 + \left(z - \frac{1}{2\nu}\right)^2 = \frac{1}{4\nu^2}$ (spheres tangent to xy-plane at origin, $\nu =$ const),

$\tan\psi = y/x$ (half planes, $\psi =$ const).

Stäckel matrix

$$[S] = \begin{bmatrix} 1 & -1 & -1/\mu^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$S = 1,$$

$$M_{11} = 1, \quad M_{21} = 1, \quad M_{31} = 1/\mu^2.$$

Metric coefficients

$$g_{11} = g_{22} = 1/(\mu^2 + \nu^2)^2, \quad g_{33} = \mu^2/(\mu^2 + \nu^2)^2, \quad g^{\frac{1}{2}} = \mu/(\mu^2 + \nu^2)^3.$$

$$R = (\mu^2 + \nu^2)^{-\frac{1}{2}}, \quad Q = (\mu^2 + \nu^2)^{-2},$$

$$f_1 = \mu, \quad f_2 = 1, \quad f_3 = 1, \quad \alpha_1 = 0.$$

Important equations

$$(ds)^2 = \frac{(d\mu)^2 + (d\nu)^2 + \mu^2 (d\psi)^2}{(\mu^2 + \nu^2)^2}.$$

$$\operatorname{grad}\varphi = (\mu^2 + \nu^2)\left[\boldsymbol{a}_\mu \frac{\partial\varphi}{\partial\mu} + \boldsymbol{a}_\nu \frac{\partial\varphi}{\partial\nu} + \frac{\boldsymbol{a}_\psi}{\mu}\frac{\partial\varphi}{\partial\psi}\right].$$

$$\operatorname{div}\boldsymbol{E} = \frac{(\mu^2 + \nu^2)^3}{\mu}\left[\frac{\partial}{\partial\mu}\left(\frac{\mu}{(\mu^2 + \nu^2)^2} E_\mu\right) + \mu \frac{\partial}{\partial\nu}\left(\frac{1}{(\mu^2 + \nu^2)^2} E_\nu\right)\right] + \frac{1}{\mu}(\mu^2 + \nu^2)\frac{\partial E_\psi}{\partial\psi}.$$

$$\operatorname{curl} \boldsymbol{E} = \frac{(\mu^2+\nu^2)^2}{\mu} \begin{vmatrix} \boldsymbol{a}_\mu & \boldsymbol{a}_\nu & \boldsymbol{a}_\psi \mu \\ \dfrac{\partial}{\partial \mu} & \dfrac{\partial}{\partial \nu} & \dfrac{\partial}{\partial \psi} \\ \dfrac{E_\mu}{\mu^2+\nu^2} & \dfrac{E_\nu}{\mu^2+\nu^2} & \dfrac{E_\psi \mu}{\mu^2+\nu^2} \end{vmatrix}.$$

$$\nabla^2 \varphi = \frac{(\mu^2+\nu^2)^3}{\mu} \left\{ \frac{\partial}{\partial \mu}\left(\frac{\mu}{\mu^2+\nu^2} \frac{\partial \varphi}{\partial \mu}\right) + \mu \frac{\partial}{\partial \nu}\left(\frac{1}{\mu^2+\nu^2} \frac{\partial \varphi}{\partial \nu}\right)\right\} + \frac{(\mu^2+\nu^2)^2}{\mu^2} \frac{\partial^2 \varphi}{\partial \psi^2}.$$

SEPARATION OF LAPLACE'S EQUATION, $\nabla^2 \varphi = 0$.

$$\frac{1}{f_i} \frac{d}{du^i}\left(f_i \frac{dU^i}{du^i}\right) + U^i \sum_{j=1}^{3} \Phi_{ij} \alpha_j = 0,$$

where $\alpha_1 = 0$ and $U^1 = M(\mu)$, $U^2 = N(\nu)$, $U^3 = \Psi(\psi)$,

$$\varphi = (\mu^2+\nu^2)^{\frac{1}{2}} M \cdot N \cdot \Psi.$$

General case

$$\begin{cases} \dfrac{d^2 M}{d\mu^2} + \dfrac{1}{\mu} \dfrac{dM}{d\mu} - \left(\alpha_2 + \dfrac{\alpha_3}{\mu^2}\right) M = 0, \\ \dfrac{d^2 N}{d\nu^2} + \alpha_2 N = 0, \\ \dfrac{d^2 \Psi}{d\psi^2} + \alpha_3 \Psi = 0. \end{cases}$$

If $\alpha_2 = -q^2$ *and* $\alpha_3 = p^2$,

$$\frac{d^2 M}{d\mu^2} + \frac{1}{\mu} \frac{dM}{d\mu} + (q^2 - p^2/\mu^2) M = 0, \quad \{24\}^* \quad M = A \mathscr{J}_p(q\mu) + B \mathscr{J}_{-p}(q\mu)$$

$$\text{or}^{**} \quad M = A \mathscr{J}_n(q\mu) + B \mathscr{Y}_n(q\mu).$$

$$\frac{d^2 N}{d\nu^2} - q^2 N = 0, \quad \{04\} \quad N = A e^{q\nu} + B e^{-q\nu}.$$

$$\frac{d^2 \Psi}{d\psi^2} + p^2 \Psi = 0, \quad \{04\} \quad \Psi = A \sin p\psi + B \cos p\psi.$$

If $\alpha_2 = +q^2$ *and* $\alpha_3 = p^2$,

$$\frac{d^2 M}{d\mu^2} + \frac{1}{\mu} \frac{dM}{d\mu} - (q^2 + p^2/\mu^2) M = 0, \quad \{24\} \quad M = A \mathscr{J}_p(iq\mu) + B \mathscr{J}_{-p}(iq\mu)$$

$$\text{or}^{**} \quad M = A \mathscr{J}_n(iq\mu) + B \mathscr{Y}_n(iq\mu).$$

$$\frac{d^2 N}{d\nu^2} + q^2 N = 0, \quad \{04\} \quad N = A \sin q\nu + B \cos q\nu.$$

$$\frac{d^2 \Psi}{d\psi^2} + p^2 \Psi = 0, \quad \{04\} \quad \Psi = A \sin p\psi + B \cos p\psi.$$

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* Designation in terms of singularities. See Section VI.

** If $p = n$, an integer, use the second solution.

If $\alpha_2 = 0$ *and* $\alpha_3 = p^2$,

$$\frac{d^2M}{d\mu^2} + \frac{1}{\mu}\frac{dM}{d\mu} - \frac{p^2}{\mu^2}M = 0, \quad \{04\} \quad M = A\mu^p + B\mu^{-p}.$$

$$\frac{d^2N}{d\nu^2} = 0, \qquad \{01\} \quad N = A + B\nu.$$

$$\frac{d^2\Psi}{d\psi^2} + p^2\Psi = 0, \qquad \{04\} \quad \Psi = A\sin p\psi + B\cos p\psi.$$

If $\alpha_2 = \alpha_3 = 0$,

$$\frac{d^2M}{d\mu^2} + \frac{1}{\mu}\frac{dM}{d\mu} = 0, \qquad \{01\} \quad M = A + B\ln\mu.$$

$$\frac{d^2N}{d\nu^2} = 0, \qquad \{01\} \quad N = A + B\nu.$$

$$\frac{d^2\Psi}{d\psi^2} = 0, \qquad \{01\} \quad \Psi = A + B\psi.$$

For φ independent of ψ,

$$\begin{cases} \dfrac{d^2M}{d\mu^2} + \dfrac{1}{\mu}\dfrac{dM}{d\mu} - \alpha_2 M = 0, \\[2ex] \dfrac{d^2N}{d\nu^2} + \alpha_2 N = 0. \end{cases}$$

If $\alpha_2 = -q^2$,

$$\frac{d^2M}{d\mu^2} + \frac{1}{\mu}\frac{dM}{d\mu} + q^2M = 0, \quad \{14\} \quad M = A\mathscr{J}_0(q\mu) + B\mathscr{Y}_0(q\mu).$$

$$\frac{d^2N}{d\nu^2} - q^2N = 0, \qquad \{04\} \quad N = Ae^{q\nu} + Be^{-q\nu}.$$

If $\alpha_2 = +q^2$,

$$\frac{d^2M}{d\mu^2} + \frac{1}{\mu}\frac{dM}{d\mu} - q^2M = 0, \quad \{14\} \quad M = A\mathscr{J}_0(iq\mu) + B\mathscr{Y}_0(iq\mu).$$

$$\frac{d^2N}{d\nu^2} + q^2N = 0, \qquad \{04\} \quad N = A\sin q\nu + B\cos q\nu.$$

If $\alpha_2 = 0$,

$$\frac{d^2M}{d\mu^2} + \frac{1}{\mu}\frac{dM}{d\mu} = 0, \qquad \{01\} \quad M = A + B\ln\mu.$$

$$\frac{d^2N}{d\nu^2} = 0, \qquad \{01\} \quad N = A + B\nu.$$

P3R. CARDIOID COORDINATES (μ, ν, ψ), Fig. 4.02.

$$0 \leqq \mu < \infty,$$
$$0 \leqq \nu < \infty,$$
$$0 \leqq \psi < 2\pi.$$

$$\begin{cases} x = \dfrac{\mu\nu}{(\mu^2+\nu^2)^2}\cos\psi, \\ y = \dfrac{\mu\nu}{(\mu^2+\nu^2)^2}\sin\psi, \\ z = \dfrac{1}{2}\,\dfrac{(\mu^2-\nu^2)}{(\mu^2+\nu^2)^2}. \end{cases}$$

Coordinate surfaces

$$x^2+y^2+z^2 = \frac{1}{4\mu^2}\left[(x^2+y^2+z^2)^{\frac{1}{2}}+z\right]$$
(cardioids of revolution intersecting positive half of z-axis, $\mu = \text{const}$),

$$x^2+y^2+z^2 = \frac{1}{4\nu^2}\left[(x^2+y^2+z^2)^{\frac{1}{2}}-z\right]$$
(cardioids of revolution intersecting negative half of z-axis, $\nu = \text{const}$),

$$\tan\psi = y/x$$
(half planes, $\psi = \text{const}$).

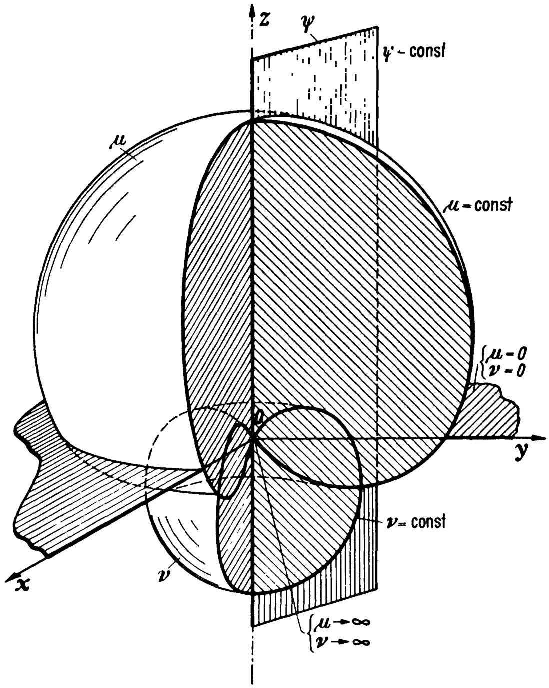

Fig. 4.02. Cardioid coordinates (μ, ν, ψ). The coordinate surfaces are cardioids of revolution ($\mu = \text{const}$ and $\nu = \text{const}$), and half-planes ($\psi = \text{const}$)

Stäckel matrix

$$[S] = \begin{bmatrix} \mu^2 & -1 & -1/\mu^2 \\ \nu^2 & 1 & -1/\nu^2 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$S = \mu^2+\nu^2, \quad M_{11} = 1, \quad M_{21} = 1, \quad M_{31} = \frac{\mu^2+\nu^2}{\mu^2\nu^2}.$$

Metric coefficients

$$g_{11} = g_{22} = 1/(\mu^2+\nu^2)^3, \quad g_{33} = \mu^2\nu^2/(\mu^2+\nu^2)^4, \quad g^{\frac{1}{2}} = \mu\nu/(\mu^2+\nu^2)^5.$$

$$R = 1/(\mu^2+\nu^2), \quad Q = 1/(\mu^2+\nu^2)^4,$$

$$f_1 = \mu, \quad f_2 = \nu, \quad f_3 = 1, \quad \alpha_1 = 0.$$

Important equations

$$(ds)^2 = \frac{1}{(\mu^2+\nu^2)^3}\left[(d\mu)^2+(d\nu)^2\right] + \frac{\mu^2\nu^2}{(\mu^2+\nu^2)^4}(d\psi)^2.$$

$$\operatorname{grad}\varphi = (\mu^2+\nu^2)^{\frac{3}{2}}\left[\boldsymbol{a}_\mu\frac{\partial\varphi}{\partial\mu} + \boldsymbol{a}_\nu\frac{\partial\varphi}{\partial\nu}\right] + \boldsymbol{a}_\psi\frac{(\mu^2+\nu^2)}{\mu\nu}\,\frac{\partial\varphi}{\partial\psi}.$$

$$\operatorname{div}\boldsymbol{E} = \frac{(\mu^2+\nu^2)^5}{\mu\nu}\left[\nu\frac{\partial}{\partial\mu}\left(\frac{\mu}{(\mu^2+\nu^2)^{\frac{7}{2}}}E_\mu\right) + \mu\frac{\partial}{\partial\nu}\left(\frac{\nu}{(\mu^2+\nu^2)^{\frac{7}{2}}}E_\nu\right)\right] + \frac{(\mu^2+\nu^2)^2}{\mu\nu}\,\frac{\partial E_\psi}{\partial\psi}.$$

$$\operatorname{curl} \boldsymbol{E} = \frac{(\mu^2+\nu^2)^{\frac{7}{2}}}{\mu\nu} \begin{vmatrix} \boldsymbol{a}_\mu & \boldsymbol{a}_\nu & \dfrac{\boldsymbol{a}_\psi \mu\nu}{(\mu^2+\nu^2)^{\frac{1}{2}}} \\ \dfrac{\partial}{\partial\mu} & \dfrac{\partial}{\partial\nu} & \dfrac{\partial}{\partial\psi} \\ \dfrac{E_\mu}{(\mu^2+\nu^2)^{\frac{3}{2}}} & \dfrac{E_\nu}{(\mu^2+\nu^2)^{\frac{3}{2}}} & \dfrac{E_\psi \mu\nu}{(\mu^2+\nu^2)^2} \end{vmatrix}.$$

$$\nabla^2 \varphi = \frac{(\mu^2+\nu^2)^5}{\mu\nu}$$
$$\times \left\{ \nu \frac{\partial}{\partial\mu}\left(\frac{\mu}{(\mu^2+\nu^2)^2}\frac{\partial\varphi}{\partial\mu}\right) + \mu\frac{\partial}{\partial\nu}\left(\frac{\nu}{(\mu^2+\nu^2)^2}\frac{\partial\varphi}{\partial\nu}\right)\right\} + \frac{(\mu^2+\nu^2)^4}{\mu^2\nu^2}\frac{\partial^2\varphi}{\partial\psi^2}.$$

SEPARATION OF LAPLACE'S EQUATION, $\nabla^2\varphi = 0$.

$$\frac{1}{f_i}\frac{d}{du^i}\left(f_i\frac{dU^i}{du^i}\right) + U^i\sum_{j=1}^{3}\Phi_{ij}\alpha_j = 0,$$

where $\alpha_1 = 0$ and $U^1 = M(\mu)$, $U^2 = N(\nu)$, $U^3 = \Psi(\psi)$,

$$\varphi = (\mu^2+\nu^2)\, M\cdot N\cdot\Psi.$$

General case

$$\begin{cases} \dfrac{d^2M}{d\mu^2} + \dfrac{1}{\mu}\dfrac{dM}{d\mu} - \left(\alpha_2 + \dfrac{\alpha_3}{\mu^2}\right)M = 0, \\ \dfrac{d^2N}{d\nu^2} + \dfrac{1}{\nu}\dfrac{dN}{d\nu} + \left(\alpha_2 - \dfrac{\alpha_3}{\nu^2}\right)N = 0, \\ \dfrac{d^2\Psi}{d\psi^2} + \alpha_3\Psi = 0. \end{cases}$$

If $\alpha_2 = q^2$ and $\alpha_3 = p^2$,

$$\frac{d^2M}{d\mu^2} + \frac{1}{\mu}\frac{dM}{d\mu} - (q^2 + p^2/\mu^2)M = 0, \quad \{24\} \quad M = A\mathcal{J}_p(iq\mu) + B\mathcal{J}_{-p}(iq\mu)$$
or* $M = A\mathcal{J}_n(iq\mu) + B\mathcal{Y}_n(iq\mu)$.

$$\frac{d^2N}{d\nu^2} + \frac{1}{\nu}\frac{dN}{d\nu} + (q^2 - p^2/\nu^2)N = 0, \quad \{24\} \quad N = A\mathcal{J}_p(q\nu) + B\mathcal{J}_{-p}(q\nu)$$
or* $N = A\mathcal{J}_n(q\nu) + B\mathcal{Y}_n(q\nu)$.

$$\frac{d^2\Psi}{d\psi^2} + p^2\Psi = 0, \quad \{04\} \quad \Psi = A\sin p\psi + B\cos p\psi.$$

If $\alpha_2 = -q^2$ and $\alpha_3 = p^2$,

$$\frac{d^2M}{d\mu^2} + \frac{1}{\mu}\frac{dM}{d\mu} + (q^2 - p^2/\mu^2)M = 0, \quad \{24\} \quad M = A\mathcal{J}_p(q\mu) + B\mathcal{J}_{-p}(q\mu)$$
or* $M = A\mathcal{J}_n(q\mu) + B\mathcal{Y}_n(q\mu)$.

$$\frac{d^2N}{d\nu^2} + \frac{1}{\nu}\frac{dN}{d\nu} - (q^2 + p^2/\nu^2)N = 0, \quad \{24\} \quad N = A\mathcal{J}_p(iq\nu) + B\mathcal{J}_{-p}(iq\nu)$$
or* $N = A\mathcal{J}_n(iq\nu) + B\mathcal{Y}_n(iq\nu)$.

$$\frac{d^2\Psi}{d\psi^2} + p^2\Psi = 0, \quad \{04\} \quad \Psi = A\sin p\psi + B\cos p\psi.$$

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* If $p = n$, an integer, use the second solution.

If $\alpha_2 = 0$ *and* $\alpha_3 = p^2$,

$$\frac{d^2M}{d\mu^2} + \frac{1}{\mu}\frac{dM}{d\mu} - \frac{p^2}{\mu^2}M = 0, \qquad \{04\} \quad M = A\mu^p + B\mu^{-p}.$$

$$\frac{d^2N}{d\nu^2} + \frac{1}{\nu}\frac{dN}{d\nu} - \frac{p^2}{\nu^2}N = 0, \qquad \{04\} \quad N = A\nu^p + B\nu^{-p}.$$

$$\frac{d^2\Psi}{d\psi^2} + p^2\Psi = 0, \qquad \{04\} \quad \Psi = A\sin p\psi + B\cos p\psi.$$

If $\alpha_2 = \alpha_3 = 0$,

$$\frac{d^2M}{d\mu^2} + \frac{1}{\mu}\frac{dM}{d\mu} = 0, \qquad \{01\} \quad M = A + B\ln\mu.$$

$$\frac{d^2N}{d\nu^2} + \frac{1}{\nu}\frac{dN}{d\nu} = 0, \qquad \{01\} \quad N = A + B\ln\nu.$$

$$\frac{d^2\Psi}{d\psi^2} = 0, \qquad \{01\} \quad \Psi = A + B\psi.$$

For φ independent of ψ,

$$\begin{cases} \dfrac{d^2M}{d\mu^2} + \dfrac{1}{\mu}\dfrac{dM}{d\mu} - \alpha_2 M = 0, \\[2ex] \dfrac{d^2N}{d\nu^2} + \dfrac{1}{\nu}\dfrac{dN}{d\nu} + \alpha_2 N = 0. \end{cases}$$

If $\alpha_2 = q^2$,

$$\frac{d^2M}{d\mu^2} + \frac{1}{\mu}\frac{dM}{d\mu} - q^2 M = 0, \qquad \{14\} \quad M = A\mathscr{I}_0(iq\mu) + B\mathscr{Y}_0(iq\mu).$$

$$\frac{d^2N}{d\nu^2} + \frac{1}{\nu}\frac{dN}{d\nu} + q^2 N = 0, \qquad \{14\} \quad N = A\mathscr{I}_0(q\nu) + B\mathscr{Y}_0(q\nu).$$

If $\alpha_2 = -q^2$,

$$\frac{d^2M}{d\mu^2} + \frac{1}{\mu}\frac{dM}{d\mu} + q^2 M = 0, \qquad \{14\} \quad M = A\mathscr{I}_0(q\mu) + B\mathscr{Y}_0(q\mu).$$

$$\frac{d^2N}{d\nu^2} + \frac{1}{\nu}\frac{dN}{d\nu} - q^2 N = 0, \qquad \{14\} \quad N = A\mathscr{I}_0(iq\nu) + B\mathscr{Y}_0(iq\nu).$$

If $\alpha_2 = 0$,

$$\frac{d^2M}{d\mu^2} + \frac{1}{\mu}\frac{dM}{d\mu} = 0, \qquad \{01\} \quad M = A + B\ln\mu.$$

$$\frac{d^2N}{d\nu^2} + \frac{1}{\nu}\frac{dN}{d\nu} = 0, \qquad \{01\} \quad N = A + B\ln\nu.$$

E4Rx. BISPHERICAL COORDINATES (η, θ, ψ), Fig. 4.03.

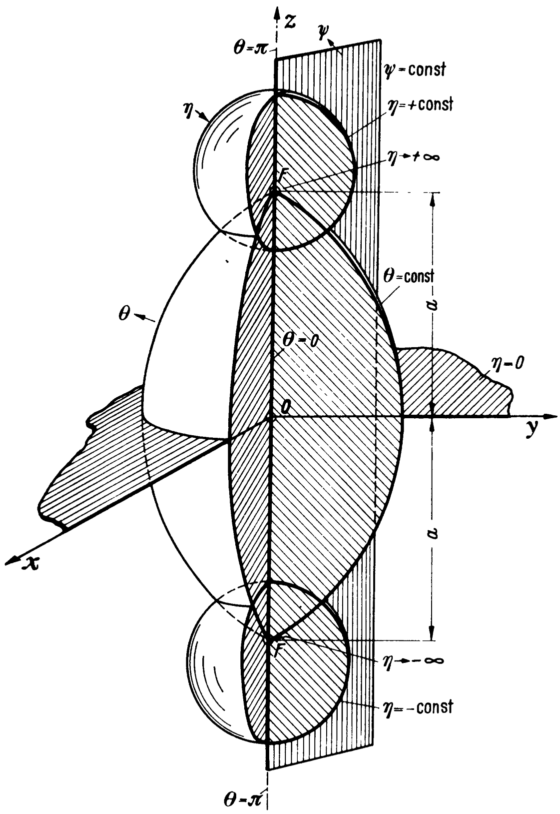


Fig. 4.03. Bispherical coordinates (η, θ, ψ). The spheres are designated by $\eta = \text{const}$, half-planes by $\psi = \text{const}$. The surfaces $\theta = \text{const}$ are spindles if $\theta < \pi/2$, a sphere at $\theta = \pi/2$, and apple-shaped surfaces if $\theta > \pi/2$

$$-\infty < \eta < +\infty,$$
$$0 \leqq \theta < \pi,$$
$$0 \leqq \psi < 2\pi.$$

$$\left\{\begin{aligned} x &= \frac{a \sin\theta \cos\psi}{\cosh\eta - \cos\theta}, \\ y &= \frac{a \sin\theta \sin\psi}{\cosh\eta - \cos\theta}, \\ z &= \frac{a \sinh\eta}{\cosh\eta - \cos\theta}. \end{aligned}\right.$$

Coordinate surfaces

$x^2 + y^2 + (z - a\coth\eta)^2 = \dfrac{a^2}{\sinh^2\eta}$

(spheres, $\eta = \text{const}$),

$x^2 + y^2 + z^2 - 2a(x^2 + y^2)^{\frac{1}{2}} \cot\theta = a^2$

(apple-shaped surfaces for $\theta < \pi/2$, spindle-shaped surfaces for $\theta > \pi/2$. $\theta = \text{const}$),

$\tan\psi = y/x$

(half-planes, $\psi = \text{const}$).

Stäckel matrix

$$[S] = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1/\sin^2\theta \\ 0 & 0 & 1 \end{bmatrix}.$$

$$S = 1, \quad M_{11} = 1, \quad M_{21} = 1, \quad M_{31} = 1/\sin^2\theta.$$

Metric coefficients

$$g_{11} = g_{22} = \frac{a^2}{(\cosh\eta - \cos\theta)^2},$$

$$g_{33} = \frac{a^2 \sin^2\theta}{(\cosh\eta - \cos\theta)^2},$$

$$g^{\frac{1}{2}} = \frac{a^3 \sin\theta}{(\cosh\eta - \cos\theta)^3}.$$

$$R = [\cosh\eta - \cos\theta]^{-\frac{1}{2}}, \qquad Q = a^2/[\cosh\eta - \cos\theta]^2,$$

$$f_1 = 1, \quad f_2 = \sin\theta, \quad f_3 = a, \quad \alpha_1 = -\tfrac{1}{4}.$$

Important equations

$$(ds)^2 = \frac{a^2}{(\cosh\eta - \cos\theta)^2}\left[(d\eta)^2 + (d\theta)^2 + \sin^2\theta\,(d\psi)^2\right].$$

$$\operatorname{grad}\varphi = \frac{1}{a}(\cosh\eta - \cos\theta)\left[\boldsymbol{a}_\eta \frac{\partial\varphi}{\partial\eta} + \boldsymbol{a}_\theta \frac{\partial\varphi}{\partial\theta} + \frac{\boldsymbol{a}_\psi}{\sin\theta}\frac{\partial\varphi}{\partial\psi}\right].$$

$$\operatorname{div} \boldsymbol{E} = \frac{(\cosh\eta - \cos\theta)^3}{a\sin\theta}\left[\sin\theta \frac{\partial}{\partial\eta}\left(\frac{E_\eta}{(\cosh\eta-\cos\theta)^2}\right) + \frac{\partial}{\partial\theta}\left(\frac{\sin\theta\, E_\eta}{(\cosh\eta-\cos\theta)^2}\right)\right] + \frac{(\cosh\eta-\cos\theta)}{a\sin\theta}\frac{\partial E_\psi}{\partial\psi}.$$

$$\operatorname{curl} \boldsymbol{E} = \frac{(\cosh\eta-\cos\theta)^2}{a\sin\theta}\begin{vmatrix} \boldsymbol{a}_\eta & \boldsymbol{a}_\theta & \boldsymbol{a}_\psi \sin\theta \\ \frac{\partial}{\partial\eta} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\psi} \\ \frac{E_\eta}{\cosh\eta-\cos\theta} & \frac{E_\theta}{\cosh\eta-\cos\theta} & \frac{E_\psi\sin\theta}{\cosh\eta-\cos\theta}\end{vmatrix}.$$

$$\nabla^2\varphi = \frac{(\cosh\eta-\cos\theta)^3}{a^2\sin\theta}\left\{\sin\theta\frac{\partial}{\partial\eta}\left(\frac{1}{\cosh\eta-\cos\theta}\frac{\partial\varphi}{\partial\eta}\right) + \frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{\cosh\eta-\cos\theta}\frac{\partial\varphi}{\partial\theta}\right)\right\} + \frac{(\cosh\eta-\cos\theta)^2}{a^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\psi^2}.$$

SEPARATION OF LAPLACE'S EQUATION, $\nabla^2\varphi = 0$.

$$\frac{1}{f_i}\frac{d}{du^i}\left(f_i\frac{dU^i}{du^i}\right) + U^i\sum_{j=1}^{3}\Phi_{ij}\alpha_j = 0,$$

where $\alpha_1 = -\frac{1}{4}$ and $U^1 = H(\eta)$, $U^2 = \Theta(\theta)$, $U^3 = \Psi(\psi)$,

$$\varphi = (\cosh\eta - \cos\theta)^{\frac{1}{2}} H\cdot\Theta\cdot\Psi.$$

General case

$$\begin{cases} \frac{d^2H}{d\eta^2} - \left(\frac{1}{4}+\alpha_2\right)H = 0, \\ \frac{d^2\Theta}{d\theta^2} + \cot\theta\frac{d\Theta}{d\theta} + \left(\alpha_2 - \frac{\alpha_3}{\sin^2\theta}\right)\Theta = 0, \\ \frac{d^2\Psi}{d\psi^2} + \alpha_3\Psi = 0. \end{cases}$$

If $\alpha_2 = p(p+1)$ and $\alpha_3 = q^2$,

$$\frac{d^2H}{d\eta^2} - \left(p+\tfrac{1}{2}\right)^2 H = 0, \qquad \{04\}\quad H = A\, e^{(p+\frac{1}{2})\eta} + B\, e^{-(p+\frac{1}{2})\eta}.$$

$$(1-\xi^2)\frac{d^2\Theta}{d\xi^2} - 2\xi\frac{d\Theta}{d\xi} + \left[p(p+1) - \frac{q^2}{1-\xi^2}\right]\Theta = 0,$$

$\{222\}$ where $\xi = \cos\theta$,

$$\Theta = A\mathscr{P}_p^q(\cos\theta) + B\mathscr{Q}_p^q(\cos\theta).$$

$$\frac{d^2\Psi}{d\psi^2} + q^2\Psi = 0, \qquad \{04\}\quad \Psi = A\sin q\psi + B\cos q\psi.$$

If $\alpha_2 = p(p+1)$ and $\alpha_3 = 0$,

$$\frac{d^2H}{d\eta^2} - \left(p+\tfrac{1}{2}\right)^2 H = 0, \qquad \{04\}\quad H = A\, e^{(p+\frac{1}{2})\eta} + B\, e^{-(p+\frac{1}{2})\eta}.$$

$$(1-\xi^2)\frac{d^2\Theta}{d\xi^2} - 2\xi\frac{d\Theta}{d\xi} + p(p+1)\Theta = 0, \quad \{112\}\quad \Theta = A\mathscr{P}_p(\cos\theta) + B\mathscr{Q}_p(\cos\theta).$$

$$\frac{d^2\Psi}{d\psi^2} = 0, \qquad \{01\}\quad \Psi = A + B\psi.$$

If $\alpha_2 = \alpha_3 = 0$,

$$\frac{d^2H}{d\eta^2} - \tfrac{1}{4}H = 0, \qquad \{04\} \quad H = A\,e^{\eta/2} + B\,e^{-\eta/2}.$$

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta\,\frac{d\Theta}{d\theta} = 0, \qquad \{01\} \quad \Theta = A + B\ln\cot(\theta/2).$$

$$\frac{d^2\Psi}{d\psi^2} = 0, \qquad \{01\} \quad \Psi = A + B\psi.$$

For φ independent of ψ,

$$\begin{cases} \dfrac{d^2H}{d\eta^2} - (\tfrac{1}{4} + \alpha_2)\,H = 0, \\ \dfrac{d^2\Theta}{d\theta^2} + \cot\theta\,\dfrac{d\Theta}{d\theta} + \alpha_2\,\Theta = 0. \end{cases}$$

If $\alpha_2 = p(p+1)$,

$$\frac{d^2H}{d\eta^2} - (p + \tfrac{1}{2})^2 H = 0, \qquad \{04\} \quad H = A\,e^{(p+\frac{1}{2})\eta} + B\,e^{-(p+\frac{1}{2})\eta}.$$

$$(1-\xi^2)\frac{d^2\Theta}{d\xi^2} - 2\xi\,\frac{d\Theta}{d\xi} + p(p+1)\,\Theta = 0, \quad \{112\} \quad \Theta = A\mathscr{P}_p(\cos\theta) + B\mathscr{Q}_p(\cos\theta).$$

If $\alpha_2 = 0$,

$$\frac{d^2H}{d\eta^2} - \tfrac{1}{4}H = 0, \qquad \{04\} \quad H = A\,e^{\eta/2} + B\,e^{-\eta/2}.$$

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta\,\frac{d\Theta}{d\theta} = 0, \qquad \{01\} \quad \Theta = A + B\ln\cot(\theta/2).$$

E4Ry. TOROIDAL COORDINATES (η, θ, ψ), Fig. 4.04.

$$0 \leqq \eta < +\infty, \qquad -\pi < \theta \leqq +\pi, \qquad 0 \leqq \psi < 2\pi.$$

$$\begin{cases} x = \dfrac{a\sinh\eta\cos\psi}{\cosh\eta - \cos\theta}, \\ y = \dfrac{a\sinh\eta\sin\psi}{\cosh\eta - \cos\theta}, \\ z = \dfrac{a\sin\theta}{\cosh\eta - \cos\theta}. \end{cases}$$

Coordinate surfaces

$$\begin{cases} x^2 + y^2 + z^2 + a^2 = 2a(x^2+y^2)^{\frac{1}{2}}\coth\eta & \text{(toroids, } \eta = \text{const)}, \\ x^2 + y^2 + (z - a\cot\theta)^2 = \dfrac{a^2}{\sin^2\theta} & \text{(spherical bowls, } \theta = \text{const)}, \\ \tan\psi = y/x & \text{(half-planes, } \psi = \text{const)}. \end{cases}$$

Stäckel matrix,

$$[S] = \begin{bmatrix} 1 & -1 & -1/\sinh^2\eta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$S = 1, \qquad M_{11} = 1, \qquad M_{21} = 1, \qquad M_{31} = 1/\sinh^2\eta.$$

Metric coefficients

$$g_{11} = g_{22} = \frac{a^2}{(\cosh\eta - \cos\theta)^2},$$

$$g_{33} = \frac{a^2 \sinh^2\eta}{(\cosh\eta - \cos\theta)^2},$$

$$g^{\frac{1}{2}} = \frac{a^3 \sinh\eta}{(\cosh\eta - \cos\theta)^3}.$$

$$R = [\cosh\eta - \cos\theta]^{-\frac{1}{2}}, \qquad Q = a^2/(\cosh\eta - \cos\theta)^2,$$

$$f_1 = \sinh\eta, \quad f_2 = 1, \quad f_3 = a, \quad \alpha_1 = +\tfrac{1}{4}.$$

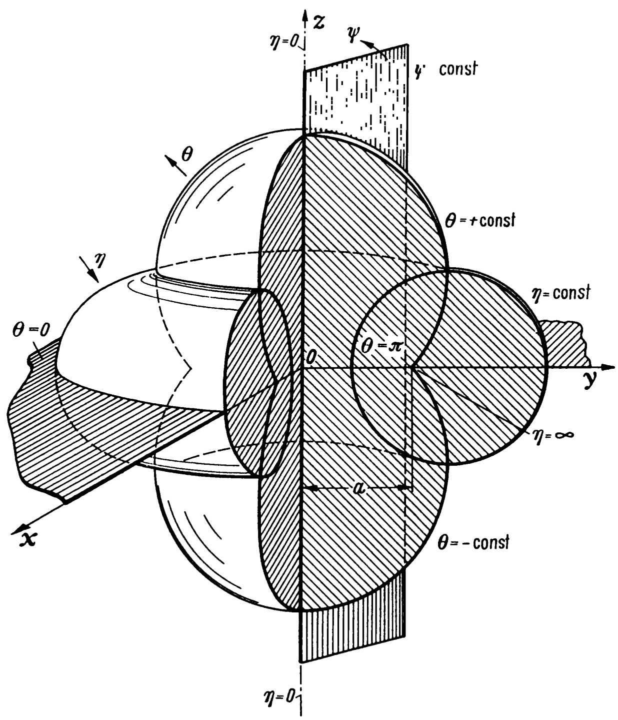


Fig. 4.04. Toroidal coordinates (η, θ, ψ). Coordinate surfaces are toroids ($\eta = \text{const}$), spherical bowls ($\theta = \text{const}$), and half-planes ($\psi = \text{const}$)

Important equations,

$$(ds)^2 = \frac{a^2}{(\cosh\eta - \cos\theta)^2}\left[(d\eta)^2 + (d\theta)^2 + \sinh^2\eta\,(d\psi)^2\right].$$

$$\operatorname{grad}\varphi = \frac{1}{a}(\cosh\eta - \cos\theta)\left[\boldsymbol{a}_\eta \frac{\partial\varphi}{\partial\eta} + \boldsymbol{a}_\theta \frac{\partial\varphi}{\partial\theta} + \frac{\boldsymbol{a}_\psi}{\sinh\eta}\frac{\partial\varphi}{\partial\psi}\right].$$

$$\operatorname{div}\boldsymbol{E} = \frac{(\cosh\eta - \cos\theta)^3}{a\sinh\eta}\left[\frac{\partial}{\partial\eta}\left(\frac{\sinh\eta}{(\cosh\eta - \cos\theta)^2}E_\eta\right)\right.$$

$$\left. + \sinh\eta\frac{\partial}{\partial\theta}\left(\frac{E_\theta}{(\cosh\eta - \cos\theta)^2}\right)\right] + \frac{\cosh\eta - \cos\theta}{a\sinh\eta}\frac{\partial E_\psi}{\partial\psi}.$$

$$\operatorname{curl} \boldsymbol{E} = \frac{(\cosh\eta - \cos\theta)^2}{a \sinh\eta} \begin{vmatrix} \boldsymbol{a}_\eta & \boldsymbol{a}_\theta & \boldsymbol{a}_\psi \sinh\eta \\ \frac{\partial}{\partial\eta} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\psi} \\ \frac{E_\eta}{\cosh\eta - \cos\theta} & \frac{E_\theta}{\cosh\eta - \cos\theta} & \frac{E_\psi \sinh\eta}{\cosh\eta - \cos\theta} \end{vmatrix}.$$

$$\nabla^2 \varphi = \frac{(\cosh\eta - \cos\theta)^3}{a^2 \sinh\eta} \left\{ \frac{\partial}{\partial\eta} \left(\frac{\sinh\eta}{\cosh\eta - \cos\theta} \frac{\partial\varphi}{\partial\eta} \right) + \sinh\eta \frac{\partial}{\partial\theta} \left(\frac{1}{\cosh\eta - \cos\theta} \frac{\partial\varphi}{\partial\theta} \right) \right\} + \frac{(\cosh\eta - \cos\theta)^2}{a^2 \sinh^2\eta} \frac{\partial^2\varphi}{\partial\psi^2}.$$

SEPARATION OF LAPLACE'S EQUATION, $\nabla^2 \varphi = 0$.

$$\frac{1}{f_i} \frac{d}{du^i} \left(f_i \frac{dU^i}{du^i} \right) + U^i \sum_{j=1}^{3} \Phi_{ij} \alpha_j = 0,$$

where $\alpha_1 = \frac{1}{4}$ and $U^1 = H(\eta)$, $U^2 = \Theta(\theta)$, $U^3 = \Psi(\psi)$,

$$\varphi = (\cosh\eta - \cos\theta)^{\frac{1}{2}} H \cdot \Theta \cdot \Psi.$$

General case

$$\begin{cases} \dfrac{d^2H}{d\eta^2} + \coth\eta \dfrac{dH}{d\eta} + \left(\frac{1}{4} - \alpha_2 - \dfrac{\alpha_3}{\sinh^2\eta} \right) H = 0, \\ \dfrac{d^2\Theta}{d\theta^2} + \alpha_2 \Theta = 0, \\ \dfrac{d^2\Psi}{d\psi^2} + \alpha_3 \Psi = 0. \end{cases}$$

If $\alpha_2 = p^2$ and $\alpha_3 = q^2$,

$$(\xi^2 - 1) \frac{d^2H}{d\xi^2} + 2\xi \frac{dH}{d\xi} - \left[\frac{q^2}{\xi^2 - 1} + (p^2 - \tfrac{1}{4}) \right] H = 0,$$

{222} where $\xi = \cosh\eta$,

$$H = A \mathscr{P}_{p-\frac{1}{2}}^{q}(\cosh\eta) + B \mathscr{Q}_{p-\frac{1}{2}}^{q}(\cosh\eta).$$

$$\frac{d^2\Theta}{d\theta^2} + p^2 \Theta = 0,$$ {04} $\Theta = A \sin p\,\theta + B \cos p\,\theta$.

$$\frac{d^2\Psi}{d\psi^2} + q^2 \Psi = 0,$$ {04} $\Psi = A \sin p\,\psi + B \cos p\,\psi$.

If $\alpha_2 = p^2$ and $\alpha_3 = 0$,

$$(\xi^2 - 1) \frac{d^2H}{d\xi^2} + 2\xi \frac{dH}{d\xi} - (p^2 - \tfrac{1}{4}) H = 0,$$

{112} $H = A \mathscr{P}_{p-\frac{1}{2}}(\cosh\eta) + B \mathscr{Q}_{p-\frac{1}{2}}(\cosh\eta)$.

$$\frac{d^2\Theta}{d\theta^2} + p^2 \Theta = 0,$$ {04} $\Theta = A \sin p\,\theta + B \cos p\,\theta$.

$$\frac{d^2\Psi}{d\psi^2} = 0,$$ {01} $\Psi = A + B\psi$.

If $\alpha_2 = \alpha_3 = 0$,

$$(\xi^2 - 1)\frac{d^2H}{d\xi^2} + 2\xi\frac{dH}{d\xi} + \tfrac{1}{4}H = 0, \quad \{112\} \quad H = A\mathscr{P}_{-\frac{1}{2}}(\cosh\eta) + B\mathscr{Q}_{-\frac{1}{2}}(\cosh\eta).$$

$$\frac{d^2\Theta}{d\theta^2} = 0, \qquad \{01\} \quad \Theta = A + B\theta.$$

$$\frac{d^2\Psi}{d\psi^2} = 0, \qquad \{01\} \quad \Psi = A + B\psi.$$

For φ independent of ψ,

$$\begin{cases} \dfrac{d^2H}{d\eta^2} + \coth\eta\,\dfrac{dH}{d\eta} + (\tfrac{1}{4} - \alpha_2)H = 0, \\ \dfrac{d^2\Theta}{d\theta^2} + \alpha_2\Theta = 0. \end{cases}$$

If $\alpha_2 = p^2$,

$$(\xi^2 - 1)\frac{d^2H}{d\xi^2} + 2\xi\frac{dH}{d\xi} - (p^2 - \tfrac{1}{4})H = 0,$$

$$\{112\} \quad H = A\mathscr{P}_{p-\frac{1}{2}}(\cosh\eta) + B\mathscr{Q}_{p-\frac{1}{2}}(\cosh\eta).$$

$$\frac{d^2\Theta}{d\theta^2} + p^2\Theta = 0, \qquad \{04\} \quad \Theta = A\sin p\theta + B\cos p\theta.$$

If $\alpha_2 = 0$,

$$\frac{d^2H}{d\eta^2} + \coth\eta\,\frac{dH}{d\eta} + \tfrac{1}{4}H = 0,$$

$$\{112\} \quad H = A\mathscr{P}_{-\frac{1}{2}}(\cosh\eta) + B\mathscr{Q}_{-\frac{1}{2}}(\cosh\eta).$$

$$\frac{d^2\Theta}{d\theta^2} = 0, \qquad \{01\} \quad \Theta = A + B\theta.$$

H2Rx. INVERSE PROLATE SPHEROIDAL COORDINATES (η, θ, ψ), Fig. 4.05.

$$0 \leqq \eta < +\infty, \quad 0 \leqq \theta \leqq \pi, \quad 0 \leqq \psi < 2\pi.$$

$$\begin{cases} x = \dfrac{a\sinh\eta\sin\theta\cos\psi}{\cosh^2\eta - \sin^2\theta}, \\ y = \dfrac{a\sinh\eta\sin\theta\sin\psi}{\cosh^2\eta - \sin^2\theta}, \\ z = \dfrac{a\cosh\eta\cos\theta}{\cosh^2\eta - \sin^2\theta}. \end{cases}$$

Coordinate surfaces

$$\begin{cases} x^2 + y^2 + z^2 = a\left[\dfrac{x^2+y^2}{\sinh^2\eta} + \dfrac{z^2}{\cosh^2\eta}\right]^{\frac{1}{2}} & \text{(rotation cyclides, } \eta = \text{const)}, \\ x^2 + y^2 + z^2 = a\left[-\dfrac{x^2+y^2}{\sin^2\theta} + \dfrac{z^2}{\cos^2\theta}\right]^{\frac{1}{2}} & \text{(rotation cyclides, } \theta = \text{const)}, \\ \tan\psi = y/x & \text{(half-planes, } \psi = \text{const)}. \end{cases}$$

Stäckel matrix

$$[S] = \begin{bmatrix} a^2 \sinh^2 \eta & -1 & -1/\sinh^2 \eta \\ a^2 \sin^2 \theta & 1 & -1/\sin^2 \theta \\ 0 & 0 & 1 \end{bmatrix}.$$

$$S = a^2(\sinh^2 \eta + \sin^2 \theta), \quad M_{11} = 1, \quad M_{21} = 1, \quad M_{31} = \frac{\sinh^2 \eta + \sin^2 \theta}{\sinh^2 \eta \sin^2 \theta}.$$

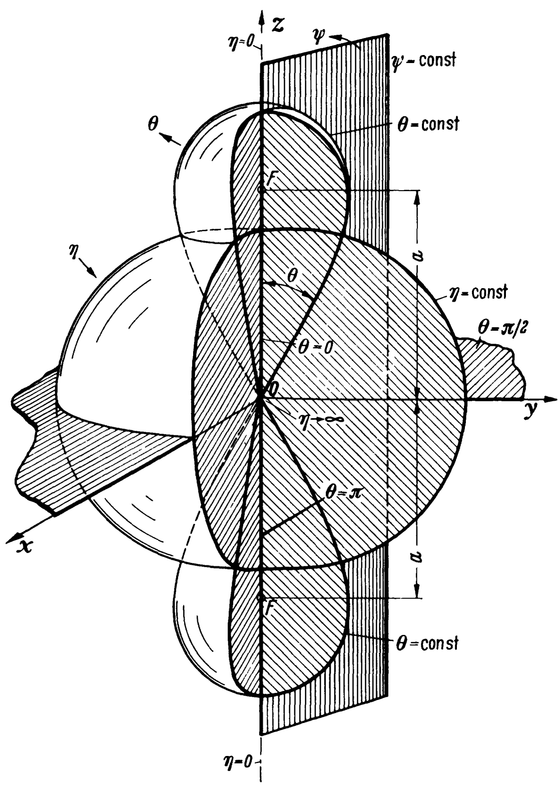


Fig. 4.05. Inverse prolate spheroidal coordinates (η, θ, ψ). The surfaces $\eta = \text{const}$ are inversions of prolate spheroids; the surfaces $\theta = \text{const}$ are inversions of hyperboloids of two sheets

Metric coefficients

$$g_{11} = g_{22} = \frac{a^2(\sinh^2 \eta + \sin^2 \theta)}{(\cosh^2 \eta - \sin^2 \theta)^2},$$

$$g_{33} = \frac{a^2 \sinh^2 \eta \sin^2 \theta}{(\cosh^2 \eta - \sin^2 \theta)^2},$$

$$g^{\frac{1}{2}} = \frac{a^3 \sinh \eta \sin \theta (\sinh^2 \eta + \sin^2 \theta)}{(\cosh^2 \eta - \sin^2 \theta)^3},$$

$$R = [\cosh^2 \eta - \sin^2 \theta]^{-\frac{1}{2}}, \qquad Q = [\cosh^2 \eta - \sin^2 \theta]^{-2},$$

$$f_1 = \sinh \eta, \quad f_2 = \sin \theta, \quad f_3 = a, \quad \alpha_1 = 0.$$

Important equations

$$(ds)^2 = \frac{a^2}{\Omega^2}\left[\Pi[(d\eta)^2 + (d\theta)^2] + \sinh^2\eta \sin^2\theta (d\psi)^2\right],$$

where

$$\Pi = \sinh^2\eta + \sin^2\theta,$$

$$\Omega = \cosh^2\eta - \sin^2\theta.$$

$$\operatorname{grad}\varphi = \frac{\Omega}{a}\left[\Pi^{-\frac{1}{2}}\left(\boldsymbol{a}_\eta \frac{\partial\varphi}{\partial\eta} + \boldsymbol{a}_\theta \frac{\partial\varphi}{\partial\theta}\right) + \frac{\boldsymbol{a}_\psi}{\sinh\eta\sin\theta}\frac{\partial\varphi}{\partial\psi}\right].$$

$$\operatorname{div}\boldsymbol{E} = \frac{\Omega^3}{a\Pi\sinh\eta\sin\theta}\left[\sin\theta\frac{\partial}{\partial\eta}\left(\frac{\Pi^{\frac{1}{2}}\sinh\eta}{\Omega^2}E_\eta\right) + \sinh\eta\frac{\partial}{\partial\theta}\left(\frac{\Pi^{\frac{1}{2}}\sin\theta}{\Omega^2}E_\theta\right)\right]$$
$$+ \frac{\Omega}{a\sinh\eta\sin\theta}\frac{\partial E_\psi}{\partial\psi}.$$

$$\operatorname{curl}\boldsymbol{E} = \frac{\Omega^2}{a\Pi^{\frac{1}{2}}\sinh\eta\sin\theta}\begin{vmatrix} \boldsymbol{a}_\eta & \boldsymbol{a}_\theta & \boldsymbol{a}_\psi \Pi^{-\frac{1}{2}}\sinh\eta\sin\theta \\ \dfrac{\partial}{\partial\eta} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\psi} \\ E_\eta\dfrac{\Pi^{\frac{1}{2}}}{\Omega} & E_\theta\dfrac{\Pi^{\frac{1}{2}}}{\Omega} & E_\psi\dfrac{\sinh\eta\sin\theta}{\Omega} \end{vmatrix}.$$

$$\nabla^2\varphi = \frac{\Omega^3}{a^2\Pi\sinh\eta\sin\theta}\left\{\sin\theta\frac{\partial}{\partial\eta}\left(\frac{\sinh\eta}{\Omega}\frac{\partial\varphi}{\partial\eta}\right) + \sinh\eta\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{\Omega}\frac{\partial\varphi}{\partial\theta}\right)\right\}$$
$$+ \frac{\Omega^2}{a^2\sinh^2\eta\sin^2\theta}\frac{\partial^2\varphi}{\partial\psi^2}.$$

SEPARATION OF LAPLACE'S EQUATION, $\nabla^2\varphi = 0$.

$$\frac{1}{f_i}\frac{d}{du^i}\left(f_i\frac{dU^i}{du^i}\right) + U^i\sum_{j=1}^{3}\Phi_{ij}\alpha_j = 0,$$

where $\alpha_1 = 0$ and $U^1 = H(\eta)$, $U^2 = \Theta(\theta)$, $U^3 = \Psi(\psi)$,

$$\varphi = (\cosh^2\eta - \sin^2\theta)^{\frac{1}{2}} H\cdot\Theta\cdot\Psi.$$

General case

$$\begin{cases} \dfrac{d^2H}{d\eta^2} + \coth\eta\dfrac{dH}{d\eta} - \left(\alpha_2 + \dfrac{\alpha_3}{\sinh^2\eta}\right)H = 0, \\ \dfrac{d^2\Theta}{d\theta^2} + \cot\theta\dfrac{d\Theta}{d\theta} + \left(\alpha_2 - \dfrac{\alpha_3}{\sin^2\theta}\right)\Theta = 0, \\ \dfrac{d^2\Psi}{d\psi^2} + \alpha_3\Psi = 0. \end{cases}$$

If $\alpha_2 = p(p+1)$ *and* $\alpha_3 = q^2$,

$$(\xi^2 - 1)\frac{d^2H}{d\xi^2} + 2\xi\frac{dH}{d\xi} - \left[p(p+1) + \frac{q^2}{\xi^2-1}\right]H = 0,$$

{222} where $\xi = \cosh\eta$,

$$H = A\mathscr{P}_p^q(\cosh\eta) + B\mathscr{Q}_p^q(\cosh\eta).$$

$$(1-\zeta^2)\frac{d^2\Theta}{d\zeta^2} - 2\zeta\frac{d\Theta}{d\zeta} + \left[p(p+1) - \frac{q^2}{\zeta^2-1}\right]\Theta = 0,$$

$\{222\}$ where $\zeta = \cos\theta$,
$\Theta = A\mathscr{P}_p^q(\cos\theta) + B\mathscr{Q}_p^q(\cos\theta)$.

$$\frac{d^2\Psi}{d\psi^2} + q^2\Psi = 0,$$ $\{04\}$ $\Psi = A\sin q\psi + B\cos q\psi$.

If $\alpha_2 = p(p+1)$ *and* $\alpha_3 = 0$,

$$(\xi^2-1)\frac{d^2H}{d\xi^2} + 2\xi\frac{dH}{d\xi} - p(p+1)H = 0,$$

$\{112\}$ $H = A\mathscr{P}_p(\cosh\eta) + B\mathscr{Q}_p(\cosh\eta)$.

$$(1-\zeta^2)\frac{d^2\Theta}{d\zeta^2} - 2\zeta\frac{d\Theta}{d\zeta} + p(p+1)\Theta = 0,$$

$\{112\}$ $\Theta = A\mathscr{P}_p(\cos\theta) + B\mathscr{Q}_p(\cos\theta)$.

$$\frac{d^2\Psi}{d\psi^2} = 0,$$ $\{01\}$ $\Psi = A + B\psi$.

If $\alpha_2 = \alpha_3 = 0$,

$$\frac{d^2H}{d\eta^2} + \coth\eta\frac{dH}{d\eta} = 0,$$ $\{01\}$ $H = A + B\ln\coth(\eta/2)$.

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta\frac{d\Theta}{d\theta} = 0,$$ $\{01\}$ $\Theta = A + B\ln\cot(\theta/2)$.

$$\frac{d^2\Psi}{d\psi^2} = 0,$$ $\{01\}$ $\Psi = A + B\psi$.

For $\boldsymbol{\varphi}$ independent of $\boldsymbol{\psi}$,

$$\begin{cases} \dfrac{d^2H}{d\eta^2} + \coth\eta\dfrac{dH}{d\eta} - \alpha_2 H = 0, \\ \dfrac{d^2\Theta}{d\theta^2} + \cot\theta\dfrac{d\Theta}{d\theta} + \alpha_2\Theta = 0. \end{cases}$$

If $\alpha_2 = p(p+1)$,

$$(\xi^2-1)\frac{d^2H}{d\xi^2} + 2\xi\frac{dH}{d\xi} - p(p+1)H = 0,$$

$\{112\}$ where $\xi = \cosh\eta$,
$H = A\mathscr{P}_p(\cosh\eta) + B\mathscr{Q}_p(\cosh\eta)$.

$$(1-\zeta^2)\frac{d^2\Theta}{d\zeta^2} - 2\zeta\frac{d\Theta}{d\zeta} + p(p+1)\Theta = 0,$$

$\{112\}$ where $\zeta = \cos\theta$,
$\Theta = A\mathscr{P}_p(\cos\theta) + B\mathscr{Q}_p(\cos\theta)$.

If $\alpha_2 = 0$,

$$\frac{d^2H}{d\eta^2} + \coth\eta\frac{dH}{d\eta} = 0,$$ $\{01\}$ $H = A + B\ln\coth(\eta/2)$.

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta\frac{d\Theta}{d\theta} = 0,$$ $\{01\}$ $\Theta = A + B\ln\cot(\theta/2)$.

H2Ry. INVERSE OBLATE SPHEROIDAL COORDINATES (η, θ, ψ), Fig. 4.06.

$$0 \leqq \eta < \infty,$$
$$0 \leqq \theta \leqq \pi,$$
$$0 \leqq \psi < 2\pi.$$

$$\begin{cases} x = \dfrac{a \cosh \eta \sin \theta \cos \psi}{\cosh^2 \eta - \cos^2 \theta}, \\ y = \dfrac{a \cosh \eta \sin \theta \sin \psi}{\cosh^2 \eta - \cos^2 \theta}, \\ z = \dfrac{a \sinh \eta \cos \theta}{\cosh^2 \eta - \cos^2 \theta}. \end{cases}$$

Coordinate surfaces

$$\begin{cases} x^2 + y^2 + z^2 \\ \quad = a\left[\dfrac{x^2+y^2}{\cosh^2 \eta} + \dfrac{z^2}{\sinh^2 \eta}\right]^{\frac{1}{2}} \\ \quad \text{(rotation cyclides, } \eta = \text{const)}, \\ x^2 + y^2 + z^2 \\ \quad = a\left[\dfrac{x^2+y^2}{\sin^2 \theta} - \dfrac{z^2}{\cos^2 \theta}\right]^{\frac{1}{2}} \\ \quad \text{(rotation cyclides, } \theta = \text{const)}, \\ \tan \psi = y/x \\ \quad \text{(half-planes, } \psi = \text{const)}. \end{cases}$$

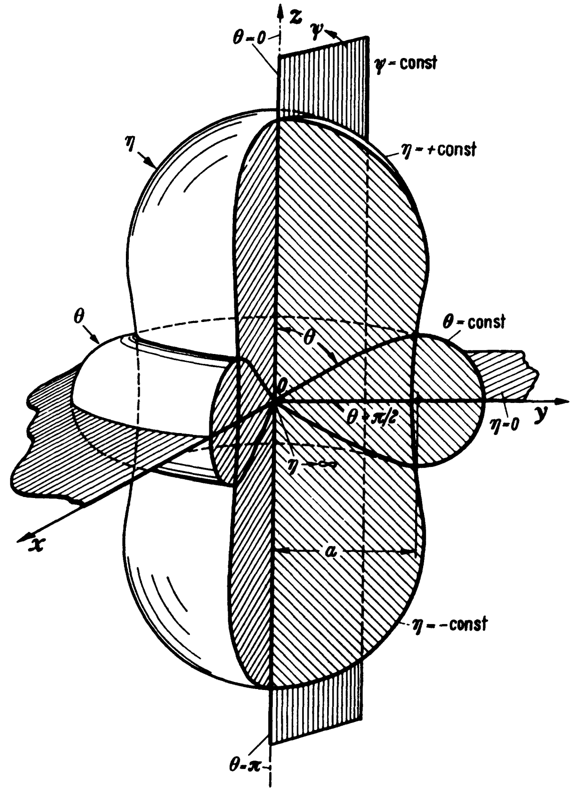


Fig. 4.06. Inverse oblate spheroidal coordinates (η, θ, ψ). The surfaces $\eta = \text{const}$ are inversions of oblate spheroids; the surfaces $\theta = \text{const}$ are inversions of hyperboloids of one sheet

Stäckel matrix

$$[S] = \begin{bmatrix} a^2 \cosh^2 \eta & -1 & 1/\cosh^2 \eta \\ -a^2 \sin^2 \theta & 1 & -1/\sin^2 \theta \\ 0 & 0 & 1 \end{bmatrix}.$$

$$S = a^2(\cosh^2 \eta - \sin^2 \theta), \quad M_{11} = 1, \quad M_{21} = 1, \quad M_{31} = \frac{\cosh^2 \eta - \sin^2 \theta}{\cosh^2 \eta \sin^2 \theta}.$$

Metric coefficients

$$g_{11} = g_{22} = \frac{a^2(\cosh^2 \eta - \sin^2 \theta)}{(\cosh^2 \eta - \cos^2 \theta)^2},$$

$$g_{33} = \frac{a^2 \cosh^2 \eta \sin^2 \theta}{(\cosh^2 \eta - \cos^2 \theta)^2},$$

$$g^{\frac{1}{2}} = \frac{a^3 \cosh \eta \sin \theta (\cosh^2 \eta - \sin^2 \theta)}{(\cosh^2 \eta - \cos^2 \theta)^3}.$$

$$R = [\cosh^2 \eta - \cos^2 \theta]^{-\frac{1}{2}}, \qquad Q = 1/(\cosh^2 \eta - \cos^2 \theta)^2,$$

$$f_1 = \cosh \eta, \quad f_2 = \sin \theta, \quad f_3 = a, \quad \alpha_1 = 0.$$

Important equations

$$(ds)^2 = \frac{a^2}{\Sigma^2}\{\Omega[(d\eta)^2 + (d\theta)^2] + \cosh^2\eta \sin^2\theta (d\psi)^2\},$$

where

$$\Omega = \cosh^2\eta - \sin^2\theta,$$

$$\Sigma = \cosh^2\eta - \cos^2\theta.$$

$$\operatorname{grad}\varphi = \frac{\Sigma}{a}\left\{\Omega^{-\frac{1}{2}}\left[\boldsymbol{a}_\eta \frac{\partial\varphi}{\partial\eta} + \boldsymbol{a}_\theta \frac{\partial\varphi}{\partial\theta}\right] + \frac{\boldsymbol{a}_\psi}{\cosh\eta\sin\theta}\frac{\partial\varphi}{\partial\psi}\right\}.$$

$$\operatorname{div}\boldsymbol{E} = \frac{\Sigma^3}{a\Omega\cosh\eta\sin\theta}\left[\sin\theta\frac{\partial}{\partial\eta}\left(\frac{\Omega^{\frac{1}{2}}\cosh\eta}{\Sigma^2}E_\eta\right) + \cosh\eta\frac{\partial}{\partial\theta}\left(\frac{\Omega^{\frac{1}{2}}\sin\theta}{\Sigma^2}E_\theta\right)\right] + \frac{\Sigma}{a\cosh\eta\sin\theta}E_\psi.$$

$$\operatorname{curl}\boldsymbol{E} = \frac{\Sigma^2}{a\Omega^{\frac{1}{2}}\cosh\eta\sin\theta}\begin{vmatrix} \boldsymbol{a}_\eta & \boldsymbol{a}_\theta & \boldsymbol{a}_\psi\Omega^{-\frac{1}{2}}\cosh\eta\sin\theta \\ \frac{\partial}{\partial\eta} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\psi} \\ E_\eta\frac{\Omega^{\frac{1}{2}}}{\Sigma} & E_\theta\frac{\Omega^{\frac{1}{2}}}{\Sigma} & E_\psi\frac{\cosh\eta\sin\theta}{\Sigma}\end{vmatrix}.$$

$$\nabla^2\varphi = \frac{\Sigma^3}{a^2\Omega\cosh\eta\sin\theta}\left\{\sin\theta\frac{\partial}{\partial\eta}\left(\frac{\cosh\eta}{\Sigma}\frac{\partial\varphi}{\partial\eta}\right) + \cosh\eta\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{\Sigma}\frac{\partial\varphi}{\partial\theta}\right)\right\} + \frac{\Sigma^2}{a^2\cosh^2\eta\sin^2\theta}\frac{\partial^2\varphi}{\partial\psi^2}.$$

SEPARATION OF LAPLACE'S EQUATION, $\nabla^2\varphi = 0$.

$$\frac{1}{f_i}\frac{d}{du^i}\left(f_i\frac{dU^i}{du^i}\right) + U^i\sum_{j=1}^{3}\Phi_{ij}\alpha_j = 0,$$

where $\alpha_1 = 0$ and $U^1 = H(\eta)$, $U^2 = \Theta(\theta)$, $U^3 = \Psi(\psi)$,

$$\varphi = (\cosh^2\eta - \cos^2\theta)^{\frac{1}{2}} H\cdot\Theta\cdot\Psi.$$

General case

$$\begin{cases} \dfrac{d^2H}{d\eta^2} + \tanh\eta\dfrac{dH}{d\eta} + \left(-\alpha_2 + \dfrac{\alpha_3}{\cosh^2\eta}\right)H = 0, \\ \dfrac{d^2\Theta}{d\theta^2} + \cot\theta\dfrac{d\Theta}{d\theta} + \left(\alpha_2 - \dfrac{\alpha_3}{\sin^2\theta}\right)\Theta = 0, \\ \dfrac{d^2\Psi}{d\psi^2} + \alpha_3\Psi = 0. \end{cases}$$

If $\alpha_2 = p(p+1)$ *and* $\alpha_3 = q^2$,

$$(\xi^2 - 1)\frac{d^2H}{d\xi^2} + 2\xi\frac{dH}{d\xi} - \left[p(p+1) - \frac{q^2}{\xi^2 - 1}\right]H = 0,$$

{222} where $\xi = i\sinh\eta$,

$$H = A\mathscr{P}_p^q(i\sinh\eta) + B\mathscr{Q}_p^q(i\sinh\eta).$$

$$(1-\zeta^2)\frac{d^2\Theta}{d\zeta^2}-2\zeta\frac{d\Theta}{d\zeta}+\left[p(p+1)-\frac{q^2}{\zeta^2-1}\right]\Theta=0,$$

$\{222\}$ where $\zeta=\cos\theta$,

$$\Theta=A\mathscr{P}_p^q(\cos\theta)+B\mathscr{Q}_p^q(\cos\theta).$$

$$\frac{d^2\Psi}{d\psi^2}+q^2\Psi=0, \qquad \{04\} \quad \Psi=A\sin q\psi+B\cos q\psi.$$

If $\alpha_2=p(p+1)$ *and* $\alpha_3=0$,

$$(\xi^2-1)\frac{d^2H}{d\xi^2}+2\xi\frac{dH}{d\xi}-p(p+1)H=0,$$

$$\{112\} \quad H=A\mathscr{P}_p(i\sinh\eta)+B\mathscr{Q}_p(i\sinh\eta).$$

$$(1-\zeta^2)\frac{d^2\Theta}{d\zeta^2}-2\zeta\frac{d\Theta}{d\zeta}+p(p+1)\Theta=0,$$

$$\{112\} \quad \Theta=A\mathscr{P}_p(\cos\theta)+B\mathscr{Q}_p(\cos\theta).$$

$$\frac{d^2\Psi}{d\psi^2}=0, \qquad \{01\} \quad \Psi=A+B\psi.$$

If $\alpha_2=\alpha_3=0$,

$$\frac{d^2H}{d\eta^2}+\tanh\eta\frac{dH}{d\eta}=0, \qquad \{01\} \quad H=A+B\cot^{-1}(\sinh\eta).$$

$$\frac{d^2\Theta}{d\theta^2}+\cot\theta\frac{d\Theta}{d\theta}=0, \qquad \{01\} \quad \Theta=A+B\ln\cot(\theta/2).$$

$$\frac{d^2\Psi}{d\psi^2}=0, \qquad \{01\} \quad \Psi=A+B\psi.$$

For φ independent of ψ,

$$\begin{cases} \dfrac{d^2H}{d\eta^2}+\tanh\eta\dfrac{dH}{d\eta}-\alpha_2 H=0, \\ \dfrac{d^2\Theta}{d\theta^2}+\cot\theta\dfrac{d\Theta}{d\theta}+\alpha_2\Theta=0. \end{cases}$$

If $\alpha_2=p(p+1)$,

$$(\xi^2-1)\frac{d^2H}{d\xi^2}+2\xi\frac{dH}{d\xi}-p(p+1)H=0,$$

$$\{112\} \quad H=A\mathscr{P}_p(i\sinh\eta)+B\mathscr{Q}_p(i\sinh\eta).$$

$$(1-\zeta^2)\frac{d^2\Theta}{d\zeta^2}-2\zeta\frac{d\Theta}{d\zeta}+p(p+1)\Theta=0,$$

$$\{112\} \quad \Theta=A\mathscr{P}_p(\cos\theta)+B\mathscr{Q}_p(\cos\theta).$$

If $\alpha_2=0$,

$$\frac{d^2H}{d\eta^2}+\tanh\eta\frac{dH}{d\eta}=0, \qquad \{01\} \quad H=A+B\cot^{-1}(\sinh\eta).$$

$$\frac{d^2\Theta}{d\theta^2}+\cot\theta\frac{d\Theta}{d\theta}=0, \qquad \{01\} \quad \Theta=A+B\ln\cot(\theta/2).$$

6-SPHERE COORDINATES (u, v, w), Fig. 4.07.

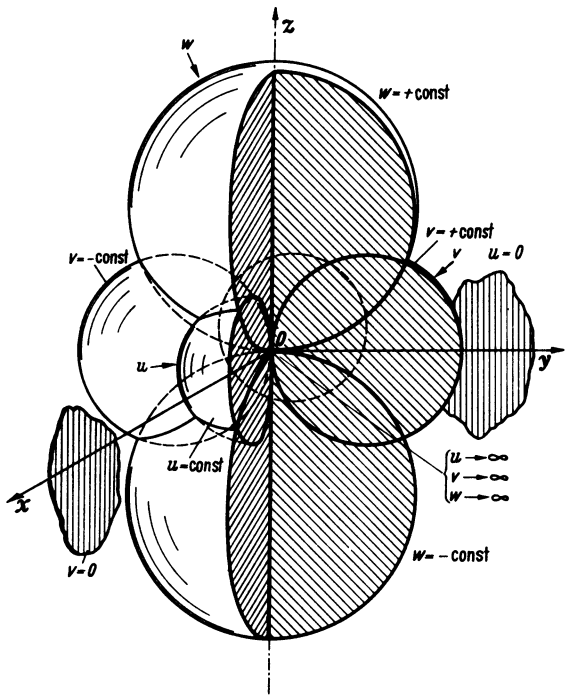


Fig. 4.07. 6-sphere coordinates (u, v, w). The inversion of rectangular coordinates. All coordinate surfaces are spheres

$$-\infty < u < +\infty,$$
$$-\infty < v < +\infty,$$
$$-\infty < w < +\infty.$$

$$\begin{cases} x = \dfrac{u}{u^2+v^2+w^2}, \\ y = \dfrac{v}{u^2+v^2+w^2}, \\ z = \dfrac{w}{u^2+v^2+w^2}. \end{cases}$$

Coordinate surfaces

$$\left(x - \frac{1}{2u}\right)^2 + y^2 + z^2 = \frac{1}{4u^2}$$

(spheres tangent to yz-plane at origin, $u = \text{const}$),

$$x^2 + \left(y - \frac{1}{2v}\right)^2 + z^2 = \frac{1}{4v^2}$$

(spheres tangent to xz-plane at origin, $v = \text{const}$),

$$x^2 + y^2 + \left(z - \frac{1}{2w}\right)^2 = \frac{1}{4w^2}$$

(spheres tangent to xy-plane at origin, $w = \text{const}$).

Stäckel matrix

$$[S] = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

$$S = 1, \quad M_{11} = M_{21} = M_{31} = 1.$$

Metric coefficients,

$$g_{11} = g_{22} = g_{33} = 1/[u^2+v^2+w^2]^2, \quad g^{\frac{1}{2}} = 1/[u^2+v^2+w^2]^3.$$
$$R = [u^2+v^2+w^2]^{-\frac{1}{2}}, \quad Q = 1/[u^2+v^2+w^2]^2,$$
$$f_1 = f_2 = f_3 = 1, \quad \alpha_1 = 0.$$

Important equations,

$$(ds)^2 = \frac{(du)^2 + (dv)^2 + (dw)^2}{[u^2+v^2+w^2]^2}.$$

$$\operatorname{grad} \varphi = (u^2+v^2+w^2)\left[\boldsymbol{a}_u \frac{\partial \varphi}{\partial u} + \boldsymbol{a}_v \frac{\partial \varphi}{\partial v} + \boldsymbol{a}_w \frac{\partial \varphi}{\partial w}\right].$$

$$\operatorname{div} \boldsymbol{E} = (u^2+v^2+w^2)^3 \left[\frac{\partial}{\partial u}\left(\frac{E_u}{(u^2+v^2+w^2)^2}\right) + \frac{\partial}{\partial v}\left(\frac{E_v}{(u^2+v^2+w^2)^2}\right) + \frac{\partial}{\partial w}\left(\frac{E_w}{(u^2+v^2+w^2)^2}\right)\right].$$

$$\operatorname{curl} \boldsymbol{E} = (u^2+v^2+w^2)^2 \begin{vmatrix} \boldsymbol{a}_u & \boldsymbol{a}_v & \boldsymbol{a}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ \frac{E_u}{u^2+v^2+w^2} & \frac{E_v}{u^2+v^2+w^2} & \frac{E_w}{u^2+v^2+w^2} \end{vmatrix}.$$

$$\nabla^2 \varphi = (u^2+v^2+w^2)^3 \left[\frac{\partial}{\partial u}\left(\frac{1}{u^2+v^2+w^2}\frac{\partial \varphi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{u^2+v^2+w^2}\frac{\partial \varphi}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{1}{u^2+v^2+w^2}\frac{\partial \varphi}{\partial w}\right)\right].$$

SEPARATION OF LAPLACE'S EQUATION, $\nabla^2 \varphi = 0$.

$$\frac{1}{f_i}\frac{d}{du^i}\left(f_i \frac{dU^i}{du^i}\right) + U^i \sum_{j=1}^{3} \Phi_{ij}\alpha_j = 0,$$

where $\alpha_1 = 0$ and $U^1 = U(u)$, $U^2 = V(v)$, $U^3 = W(w)$,

$$\varphi = (u^2+v^2+w^2)^{\frac{1}{2}}\, U \cdot V \cdot W.$$

General case

$$\begin{cases} \frac{d^2U}{du^2} - (\alpha_2+\alpha_3)\,U = 0, \\ \frac{d^2V}{dv^2} + \alpha_2\, V = 0, \\ \frac{d^2W}{dw^2} + \alpha_3\, W = 0. \end{cases}$$

If $\alpha_2 = p^2$ *and* $\alpha_3 = q^2$,

$\frac{d^2X}{dx^2} - (p^2+q^2)\,X = 0,$ {04} $X = A\,e^{(p^2+q^2)^{\frac{1}{2}}x} + B\,e^{-(p^2+q^2)^{\frac{1}{2}}}$

$\frac{d^2Y}{dy^2} + p^2\,Y = 0,$ {04} $Y = A\sin p\,y + B\cos p\,y.$

$\frac{d^2Z}{dz^2} + q^2\,Z = 0,$ {04} $Z = A\sin q\,z + B\cos q\,z.$

If $\alpha_2 = -p^2$ *and* $\alpha_3 = -q^2$,

$\frac{d^2X}{dx^2} + (p^2+q^2)\,X = 0,$ {04} $X = A\sin(p^2+q^2)^{\frac{1}{2}}\,x + B\cos(p^2+q^2)^{\frac{1}{2}}\,x.$

$\frac{d^2Y}{dy^2} - p^2\,Y = 0,$ {04} $Y = A\,e^{py} + B\,e^{-py}.$

$\frac{d^2Z}{dz^2} - q^2\,Z = 0,$ {04} $Z = A\,e^{qz} + B\,e^{-qz}.$

If $\alpha_2 = p^2$ *and* $\alpha_3 = 0$,

$\frac{d^2X}{dx^2} - p^2X = 0,$ {04} $X = A\,e^{px} + B\,e^{-px}.$

$\frac{d^2Y}{dy^2} + p^2\,Y = 0,$ {04} $Y = A\sin p\,y + B\cos p\,y.$

$\frac{d^2Z}{dz^2} = 0,$ {01} $Z = A + B\,z.$

If $\alpha_2 = \alpha_3 = 0$,

$\frac{d^2X}{dx^2} = \frac{d^2Y}{dy^2} = \frac{d^2Z}{dz^2} = 0,$ $X = A + B\,x,\ Y = A + B\,y,\ Z = A + B\,z.$

J 1Rx. BI-CYCLIDE COORDINATES (μ, ν, ψ), Fig. 4.08.

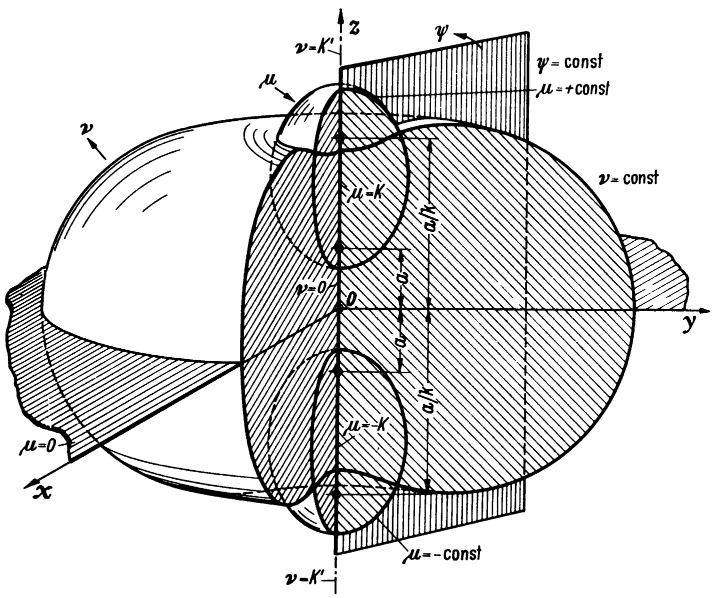


Fig. 4.08. Bi-cyclide coordinates (μ, ν, ψ). Similar to bispherical coordinates (Fig. 4.03), but with 4th-degree surfaces instead of 2nd-degree for $\mu = \text{const}$. Corresponding to the two spheres of bispherical coordinates are two cyclides $\mu = \pm\,\text{const}$. Corresponding to the spindles or apple-shaped surfaces are the cyclides $\nu = \text{const}$

$$0 \leqq \mu \leqq K, \quad 0 \leqq \nu \leqq K', \quad 0 \leqq \psi < 2\pi.$$

$$\begin{cases} x = \dfrac{a}{\Lambda} \operatorname{cn}\mu \operatorname{dn}\mu \operatorname{sn}\nu \operatorname{cn}\nu \cos\psi, \\ y = \dfrac{a}{\Lambda} \operatorname{cn}\mu \operatorname{dn}\mu \operatorname{sn}\nu \operatorname{cn}\nu \sin\psi, \\ z = \dfrac{a}{\Lambda} \operatorname{sn}\mu \operatorname{dn}\nu, \end{cases}$$

$$\Lambda = 1 - \operatorname{dn}^2\mu \operatorname{sn}^2\nu.$$

Coordinate surfaces

$$\begin{cases} (x^2+y^2+z^2)^2 + \dfrac{a^2}{k^4} \dfrac{[(1-k^2)^2 - 2(1-k^2)\operatorname{dn}^2\mu + (1+k^2)\operatorname{dn}^4\mu]}{\operatorname{dn}^2\mu \operatorname{cn}^2\mu} (x^2+y^2) \\ \qquad - a^2\left(\operatorname{sn}^2\mu + \dfrac{1}{k^2\operatorname{sn}^2\mu}\right) z^2 + \dfrac{a^4}{k^2} = 0 \qquad \text{(bi-cyclides, } \mu = \text{const)}, \\ \left[\dfrac{\operatorname{cn}^2\nu}{a^2\operatorname{sn}^2\nu}(x^2+y^2) + \dfrac{\operatorname{dn}^2\nu}{a^2} z^2\right]^2 - \dfrac{2\operatorname{cn}^2\nu}{a^2\operatorname{sn}^2\nu}(x^2+y^2) - \dfrac{2\operatorname{dn}^2\nu}{a^2} z^2 + 1 = 0 \\ \qquad \text{(rotation-cyclides, } \nu = \text{const)}, \\ \tan\psi = y/x \qquad \text{(half planes, } \psi = \text{const)}. \end{cases}$$

Stäckel matrix

$$[S] = \begin{bmatrix} -k^2\,\mathrm{sn}^2\mu & -1 & -\dfrac{k'^4\,\mathrm{sn}^2\mu}{\mathrm{cn}^2\mu\,\mathrm{dn}^2\mu} \\ \mathrm{dn}^2\nu & 1 & -\dfrac{\mathrm{dn}^2\nu}{\mathrm{sn}^2\nu\,\mathrm{cn}^2\nu} \\ 0 & 0 & 1 \end{bmatrix}.$$

$$S = \mathrm{dn}^2\nu - k^2\,\mathrm{sn}^2\mu = \mathrm{dn}^2\mu - k'^2\,\mathrm{sn}^2\nu,$$

$$M_{11} = M_{21} = 1, \quad M_{31} = \frac{\Omega^2}{\mathrm{cn}^2\mu\,\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu\,\mathrm{cn}^2\nu}.$$

Metric coefficients

$$g_{11} = g_{22} = \frac{a^2\Omega^2}{\Lambda^2},$$

$$g_{33} = \frac{a^2}{\Lambda^2}\,\mathrm{cn}^2\mu\,\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu\,\mathrm{cn}^2\nu,$$

$$g^{\frac{1}{2}} = \frac{a^3\Omega^2}{\Lambda^3}\,\mathrm{cn}\,\mu\,\mathrm{dn}\,\mu\,\mathrm{sn}\,\nu\,\mathrm{cn}\,\nu,$$

where

$$\Omega^2 = (1 - \mathrm{sn}^2\mu\,\mathrm{dn}^2\nu)(\mathrm{dn}^2\nu - k^2\,\mathrm{sn}^2\mu).$$

$$R = \Lambda^{-\frac{1}{2}}, \quad Q = \frac{a^2}{\Lambda^2}(1 - \mathrm{sn}^2\mu\,\mathrm{dn}^2\nu),$$

$$f_1 = \mathrm{cn}\,\mu\,\mathrm{dn}\,\mu, \quad f_2 = \mathrm{sn}\,\nu\,\mathrm{cn}\,\nu, \quad f_3 = a, \quad \alpha_1 = -2.$$

Important equations

$$(ds)^2 = \frac{a^2\Omega^2}{\Lambda^2}\left[(d\mu)^2 + (d\nu)^2\right] + \frac{a^2}{\Lambda^2}\,\mathrm{cn}^2\mu\,\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu\,\mathrm{cn}^2\nu\,(d\psi)^2.$$

$$\mathrm{grad}\,\varphi = \frac{\Lambda}{a\Omega}\left[\boldsymbol{a}_\mu \frac{\partial\varphi}{\partial\mu} + \boldsymbol{a}_\nu \frac{\partial\varphi}{\partial\nu}\right] + \boldsymbol{a}_\psi \frac{\Lambda}{a\,\mathrm{cn}\,\mu\,\mathrm{dn}\,\mu\,\mathrm{sn}\,\nu\,\mathrm{cn}\,\nu}\frac{\partial\varphi}{\partial\psi}.$$

$$\mathrm{div}\,\boldsymbol{E} = \frac{\Lambda^3}{a\Omega^2\,\mathrm{cn}\,\mu\,\mathrm{dn}\,\mu\,\mathrm{sn}\,\nu\,\mathrm{cn}\,\nu}\left\{\mathrm{sn}\,\nu\,\mathrm{cn}\,\nu\frac{\partial}{\partial\mu}\left[\frac{\Omega\,\mathrm{cn}\,\mu\,\mathrm{dn}\,\mu}{\Lambda^2}E_\mu\right]\right.$$
$$\left. + \mathrm{cn}\,\mu\,\mathrm{dn}\,\mu\frac{\partial}{\partial\nu}\left[\frac{\Omega\,\mathrm{sn}\,\nu\,\mathrm{cn}\,\nu}{\Lambda^2}E_\nu\right]\right\} + \frac{\Lambda}{a\,\mathrm{cn}\,\mu\,\mathrm{dn}\,\mu\,\mathrm{sn}\,\nu\,\mathrm{cn}\,\nu}\frac{\partial E_\psi}{\partial\psi}.$$

$$\mathrm{curl}\,\boldsymbol{E} = \frac{\Lambda^2}{a\Omega^2\,\mathrm{cn}\,\mu\,\mathrm{dn}\,\mu\,\mathrm{sn}\,\nu\,\mathrm{cn}\,\nu}\begin{vmatrix} \boldsymbol{a}_\mu\Omega & \boldsymbol{a}_\nu\Omega & \boldsymbol{a}_\psi\,\mathrm{cn}\,\mu\,\mathrm{dn}\,\mu\,\mathrm{sn}\,\nu\,\mathrm{dn}\,\nu \\ \dfrac{\partial}{\partial\mu} & \dfrac{\partial}{\partial\nu} & \dfrac{\partial}{\partial\psi} \\ E_\mu\Omega/\Lambda & E_\nu\Omega/\Lambda & E_\psi\,\mathrm{cn}\,\mu\,\mathrm{dn}\,\mu\,\mathrm{sn}\,\nu\,\mathrm{cn}\,\nu/\Lambda \end{vmatrix}.$$

$$\nabla^2\varphi = \frac{\Lambda^3}{a^2\Omega^2\,\mathrm{cn}\,\mu\,\mathrm{dn}\,\mu\,\mathrm{sn}\,\nu\,\mathrm{cn}\,\nu}\left\{\mathrm{sn}\,\nu\,\mathrm{cn}\,\nu\frac{\partial}{\partial\mu}\left[\frac{\mathrm{cn}\,\mu\,\mathrm{dn}\,\mu}{\Lambda}\frac{\partial\varphi}{\partial\mu}\right]\right.$$
$$\left. + \mathrm{cn}\,\mu\,\mathrm{dn}\,\mu\frac{\partial}{\partial\nu}\left[\frac{\mathrm{sn}\,\nu\,\mathrm{cn}\,\nu}{\Lambda}\frac{\partial\varphi}{\partial\nu}\right]\right\} + \frac{\Lambda^2}{a^2\,\mathrm{cn}^2\mu\,\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu\,\mathrm{cn}^2\nu}\frac{\partial^2\varphi}{\partial\psi^2}.$$

SEPARATION OF LAPLACE'S EQUATION, $\nabla^2\varphi = 0$.

$$\frac{1}{f_i}\frac{d}{du^i}\left(f_i\frac{dU^i}{du^i}\right) + U^i\sum_{j=1}^{3}\Phi_{ij}\alpha_j = 0,$$

where $\alpha_1 = -2$ and $U^1 = M(\mu)$, $U^2 = N(\nu)$, $U^3 = \Psi(\psi)$,

$$\varphi = \Lambda^{\frac{1}{2}} M N \Psi.$$

General case

$$\begin{cases} \dfrac{d^2M}{d\mu^2} - \dfrac{\operatorname{sn}\mu(\operatorname{dn}^2\mu + k^2\operatorname{cn}^2\mu)}{\operatorname{cn}\mu\operatorname{dn}\mu}\dfrac{dM}{d\mu} + \left[2k^2\operatorname{sn}^2\mu - \alpha_2 - \alpha_3\dfrac{k'^4\operatorname{sn}^2\mu}{\operatorname{cn}^2\mu\operatorname{dn}^2\mu}\right]M = 0, \\ \dfrac{d^2N}{d\nu^2} + \dfrac{\operatorname{dn}\nu(\operatorname{cn}^2\nu - \operatorname{sn}^2\nu)}{\operatorname{sn}\nu\operatorname{cn}\nu}\dfrac{dN}{d\nu} + \left[-2\operatorname{dn}^2\nu + \alpha_2 - \alpha_3\dfrac{\operatorname{dn}^2\nu}{\operatorname{sn}^2\nu\operatorname{cn}^2\nu}\right]N = 0, \\ \dfrac{d^2\Psi}{d\psi^2} + \alpha_3\Psi = 0. \end{cases}$$

Substitution of $\operatorname{sn}^2\mu = z$ in the first equation and $\operatorname{dn}^2\nu = z$ in the second, reduces both to the canonical form

$$\frac{d^2Z}{dz^2} + \frac{1}{2}\left[\frac{1}{z-a_1} + \frac{2}{z-a_2} + \frac{2}{z-a_3}\right]\frac{dZ}{dz} + \frac{1}{4}\left[\frac{\bar{A}_0 + \bar{A}_1 z + \bar{A}_2 z^2 + \bar{A}_3 z^3}{(z-a_1)(z-a_2)^2(z-a_3)^2}\right]Z = 0,$$

where, for M with $\alpha_2 = p^2$ and $\alpha_3 = q^2$,

$$a_1 = 0, \quad a_2 = b = 1, \quad a_3 = c = 1/k^2,$$
$$\bar{A}_0 = -p^2 c, \quad \bar{A}_1 = p^2 c(1+c) - q^2(1-c)^2 + 2c,$$
$$\bar{A}_2 = -[p^2 c + 2(1+c)], \quad \bar{A}_3 = 2.$$

For N with $\alpha_2 = p'^2$ and $\alpha_3 = q^2$,

$$a_1 = 0, \quad a_2 = b = 1, \quad a_3 = c = k^2,$$
$$\bar{A}_0 = -p'^2 c, \quad \bar{A}_1 = p'^2(1+c) - q^2(1-c)^2 + 2c,$$
$$\bar{A}_2 = -[p'^2 + 2(1+c)], \quad \bar{A}_3 = 2.$$

Evidently $p'^2 = p^2 c$. This is a Bôcher equation with designation $\{1222\}$, and the general solution is

$$Z = A\,\mathcal{U}_p^q(k, z) + B\,\mathcal{V}_p^q(k, z).$$

General case, $\varphi = \varphi(u^1, u^2, u^3)$

$$\begin{cases} M = A\,\mathcal{U}_p^q(k, \operatorname{sn}^2\mu) + B\,\mathcal{V}_p^q(k, \operatorname{sn}^2\mu), \\ N = A\,\mathcal{U}_{p'}^q(1/k, \operatorname{dn}^2\nu) + B\,\mathcal{V}_{p'}^q(1/k, \operatorname{dn}^2\nu), \\ \Psi = A\sin q\psi + B\cos q\psi, \quad \text{where} \quad p'^2 = p^2/k^2. \end{cases}$$

For φ independent of ψ,

$$\begin{cases} M = A\,\mathcal{U}_p(k, \operatorname{sn}^2\mu) + B\,\mathcal{V}_p(k, \operatorname{sn}^2\mu), \\ N = A\,\mathcal{U}_{p'}(1/k, \operatorname{dn}^2\nu) + B\,\mathcal{V}_{p'}(1/k, \operatorname{dn}^2\nu). \end{cases}$$

J 1Ry. FLAT-RING CYCLIDE COORDINATES (μ, ν, ψ), Fig. 4.09.

$$0 \leqq \mu \leqq K, \quad 0 \leqq \nu \leqq K', \quad 0 \leqq \psi < 2\pi.$$

$$\begin{cases} x = \dfrac{a}{\Lambda}\operatorname{sn}\mu\operatorname{dn}\nu\cos\psi, \\ y = \dfrac{a}{\Lambda}\operatorname{sn}\mu\operatorname{dn}\nu\sin\psi, \\ z = \dfrac{a}{\Lambda}\operatorname{cn}\mu\operatorname{dn}\mu\operatorname{sn}\nu\operatorname{cn}\nu, \end{cases}$$

where

$$\Lambda = 1 - \operatorname{dn}^2\mu\operatorname{sn}^2\nu.$$

Coordinate surfaces

$$\left\{\begin{aligned}
&(x^2+y^2+z^2)^2+\frac{a^2}{k^4}\,\frac{[(1-k^2)^2-2(1-k^2)\,\mathrm{dn}^2\mu+(1+k^2)\,\mathrm{dn}^4\mu]}{\mathrm{dn}^2\mu\,\mathrm{cn}^2\mu}\,z^2\\
&\qquad -a^2\left(\mathrm{sn}^2\mu+\frac{1}{k^2\,\mathrm{sn}^2\mu}\right)(x^2+y^2)+\frac{a^4}{k^2}=0 \quad \text{(flat-ring cyclides, } \mu=\text{const)},\\
&\left[\frac{\mathrm{dn}^2\nu}{a^2}(x^2+y^2)+\frac{\mathrm{cn}^2\nu}{a^2\,\mathrm{sn}^2\nu}\,z^2\right]^2-\frac{2\,\mathrm{cn}^2\nu}{a^2\,\mathrm{sn}^2\nu}\,z^2-\frac{2\,\mathrm{dn}^2\nu}{a^2}(x^2+y^2)+1=0\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(rotation-cyclides, } \nu=\text{const)},\\
&\tan\psi=y/x \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{(half-planes, } \psi=\text{const)}.
\end{aligned}\right.$$

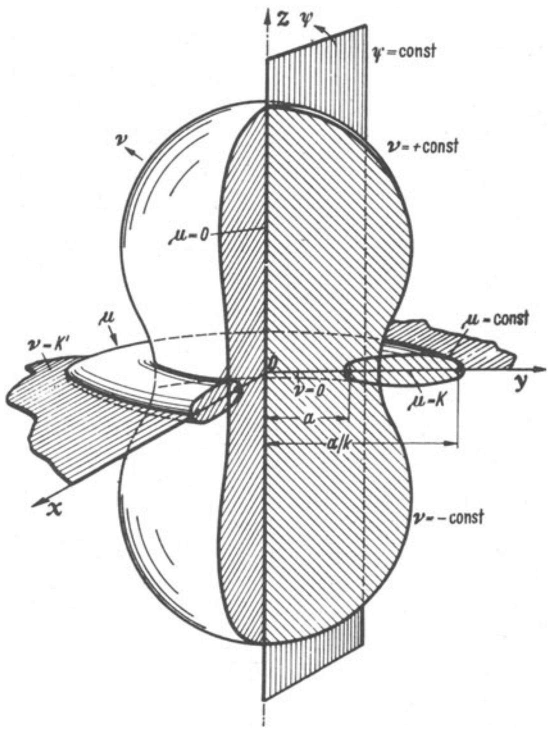


Fig. 4.09. Flat-ring cyclide coordinates (μ, ν, ψ). Similar to toroidal coordinates (Fig. 4.04) but with 4th-degree surfaces instead of 2nd-degree for ν = const. The toroids of circular cross-section are replaced by flattened rings (μ = const), and the spherical bowls of toroidal coordinates are replaced by rotation cyclides (ν = const)

Stäckel matrix

$$[S]=\begin{bmatrix} -k^2\,\mathrm{sn}^2\mu & -1 & -\left(k^2\,\mathrm{sn}^2\mu+\dfrac{1}{\mathrm{sn}^2\mu}\right)\\ \mathrm{dn}^2\nu & 1 & \left(\mathrm{dn}^2\nu+\dfrac{k^2}{\mathrm{dn}^2\nu}\right)\\ 0 & 0 & 1\end{bmatrix}.$$

$$S=\mathrm{dn}^2\nu-k^2\,\mathrm{sn}^2\mu=\mathrm{dn}^2\mu-k'^2\,\mathrm{sn}^2\nu,$$

$$M_{11}=M_{21}=1\,,\qquad M_{31}=\frac{\Omega^2}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}\,.$$

Metric coefficients

$$g_{11} = g_{22} = a^2\Omega^2/\Lambda^2,$$

$$g_{33} = \frac{a^2}{\Lambda^2} \operatorname{sn}^2\mu \operatorname{dn}^2\nu,$$

$$g^{\frac{1}{2}} = \frac{a^3\Omega^2}{\Lambda^3} \operatorname{sn}\mu \operatorname{dn}\nu,$$

where

$$\Omega^2 = (1 - \operatorname{sn}^2\mu \operatorname{dn}^2\nu)(\operatorname{dn}^2\nu - k^2 \operatorname{sn}^2\mu).$$

$$R = \Lambda^{-\frac{1}{2}}, \qquad Q = \frac{a^2}{\Lambda^2}(1 - \operatorname{sn}^2\mu \operatorname{dn}^2\nu),$$

$$f_1 = \operatorname{sn}\mu, \quad f_2 = \operatorname{dn}\nu, \quad f_3 = a, \quad \alpha_1 = -1.$$

Important equations

$$(ds)^2 = \frac{a^2\Omega^2}{\Lambda^2}[(d\mu)^2 + (d\nu)^2] + \frac{a^2}{\Lambda^2} \operatorname{sn}^2\mu \operatorname{dn}^2\nu (d\psi)^2.$$

$$\operatorname{grad}\varphi = \frac{\Lambda}{a\Omega}\left[\boldsymbol{a}_\mu \frac{\partial\varphi}{\partial\mu} + \boldsymbol{a}_\nu \frac{\partial\varphi}{\partial\nu}\right] + \frac{\boldsymbol{a}_\psi \Lambda}{a \operatorname{sn}\mu \operatorname{dn}\nu} \frac{\partial\varphi}{\partial\psi}.$$

$$\operatorname{div}\boldsymbol{E} = \frac{\Lambda^3}{a\Omega^2 \operatorname{sn}\mu \operatorname{dn}\nu} \times \left\{\operatorname{dn}\nu \frac{\partial}{\partial\mu}\left[\frac{\Omega \operatorname{sn}\mu}{\Lambda^2} E_\mu\right] + \operatorname{sn}\mu \frac{\partial}{\partial\nu}\left[\frac{\Omega \operatorname{dn}\nu}{\Lambda^2} E_\nu\right]\right\} + \frac{\Lambda}{a \operatorname{sn}\mu \operatorname{dn}\nu} \frac{\partial E_\psi}{\partial\psi}.$$

$$\operatorname{curl}\boldsymbol{E} = \frac{\Lambda^2}{a\Omega^2 \operatorname{sn}\mu \operatorname{dn}\nu}\begin{vmatrix} \boldsymbol{a}_\mu \Omega & \boldsymbol{a}_\nu \Omega & \boldsymbol{a}_\psi \operatorname{sn}\mu \operatorname{dn}\nu \\ \dfrac{\partial}{\partial\mu} & \dfrac{\partial}{\partial\nu} & \dfrac{\partial}{\partial\psi} \\ E_\mu \Omega/\Lambda & E_\nu \Omega/\Lambda & E_\psi \operatorname{sn}\mu \operatorname{dn}\nu/\Lambda \end{vmatrix}.$$

$$\nabla^2\varphi = \frac{\Lambda^3}{a^2\Omega^2 \operatorname{sn}\mu \operatorname{dn}\nu} \times \left\{\operatorname{dn}\nu \frac{\partial}{\partial\mu}\left[\frac{\operatorname{sn}\mu}{\Lambda} \frac{\partial\varphi}{\partial\mu}\right] + \operatorname{sn}\mu \frac{\partial}{\partial\nu}\left[\frac{\operatorname{dn}\nu}{\Lambda} \frac{\partial\varphi}{\partial\nu}\right]\right\} + \frac{\Lambda^2}{a^2 \operatorname{sn}^2\mu \operatorname{dn}^2\nu} \frac{\partial^2\varphi}{\partial\psi^2}.$$

SEPARATION OF LAPLACE'S EQUATION, $\nabla^2\varphi = 0$.

$$\frac{1}{f_i} \frac{d}{du^i}\left(f_i \frac{dU^i}{du^i}\right) + U^i \sum_{j=1}^{3} \Phi_{ij}\alpha_j = 0,$$

where $\alpha_1 = -1$ and $U^1 = M(\mu)$, $U^2 = N(\nu)$, $U^3 = \Psi(\psi)$,

$$\varphi = \Lambda^{\frac{1}{2}} M N \Psi.$$

General case

$$\begin{cases} \dfrac{d^2M}{d\mu^2} + \dfrac{\operatorname{cn}\mu \operatorname{dn}\mu}{\operatorname{sn}\mu} \dfrac{dM}{d\mu} + \left[k^2 \operatorname{sn}^2\mu - \alpha_2 - \alpha_3\left(k^2 \operatorname{sn}^2\mu + \dfrac{1}{\operatorname{sn}^2\mu}\right)\right] M = 0, \\ \dfrac{d^2N}{d\nu^2} - \dfrac{k'^2 \operatorname{sn}\nu \operatorname{cn}\nu}{\operatorname{dn}\nu} \dfrac{dN}{d\nu} + \left[-\operatorname{dn}^2\nu + \alpha_2 + \alpha_3\left(\operatorname{dn}^2\nu + \dfrac{k^2}{\operatorname{dn}^2\nu}\right)\right] N = 0, \\ \dfrac{d^2\Psi}{d\psi^2} + \alpha_3 \Psi = 0. \end{cases}$$

Substitution of $\mathrm{sn}^2\mu = z$ in the first equation and $\mathrm{dn}^2\nu = z$ in the second, reduces both to the canonical form

$$\frac{d^2Z}{dz^2} + \frac{1}{2}\left[\frac{1}{z-a_1} + \frac{1}{z-a_2} + \frac{2}{z-a_3}\right]\frac{dZ}{dz} + \frac{1}{4}\left[\frac{\bar{A}_0 + \bar{A}_1 z + \bar{A}_2 z^2}{(z-a_1)(z-a_2)(z-a_3)^2}\right]Z = 0,$$

where, for M with $\alpha_2 = p^2$ and $\alpha_3 = q^2$,

$$a_1 = b = 1, \quad a_2 = c = 1/k^2, \quad a_3 = 0,$$
$$\bar{A}_0 = -q^2 c, \quad \bar{A}_1 = -p^2 c, \quad \bar{A}_2 = 1 - q^2.$$

For N, with $\alpha_2 = p'^2$ and $\alpha_3 = q^2$,

$$a_1 = b = 1, \quad a_2 = c = k^2, \quad a_3 = 0,$$
$$\bar{A}_0 = -q^2 c, \quad \bar{A}_1 = p'^2, \quad \bar{A}_2 = 1 - q^2.$$

These are Bôcher equations with designation $\{1122\}$, and their solutions are Wangerin functions:

$$Z = A\mathscr{S}_p^q(k, z) + B\mathscr{T}_p^q(k, z).$$

General case, $\varphi = \varphi(u^1, u^2, u^3)$,

$$\begin{cases} M = A\mathscr{S}_p^q(k, \mathrm{sn}^2\mu) + B\mathscr{T}_p^q(k, \mathrm{sn}^2\mu), \\ N = A\mathscr{S}_{p'}^q(1/k, \mathrm{dn}^2\nu) + B\mathscr{T}_{p'}^q(1/k, \mathrm{dn}^2\nu), \\ \Psi = A\sin q\psi + B\cos q\psi, \quad \text{where} \quad p'^2 = p^2/k^2. \end{cases}$$

For φ independent of ψ,

$$\begin{cases} M = A\mathscr{S}_p(k, \mathrm{sn}^2\mu) + B\mathscr{T}_p(k, \mathrm{sn}^2\mu), \\ N = A\mathscr{S}_{p'}(1/k, \mathrm{dn}^2\nu) + B\mathscr{T}_{p'}(1/k, \mathrm{dn}^2\nu). \end{cases}$$

J2R. DISK-CYCLIDE COORDINATES (μ, ν, ψ), Fig. 4.10.

$$0 \leqq \mu \leqq K, \quad 0 \leqq \nu \leqq K', \quad 0 \leqq \psi < 2\pi.$$

$$\begin{cases} x = \dfrac{a}{\Lambda}\,\mathrm{cn}\,\mu\,\mathrm{cn}\,\nu\cos\psi, \\ y = \dfrac{a}{\Lambda}\,\mathrm{cn}\,\mu\,\mathrm{cn}\,\nu\sin\psi, \\ z = \dfrac{a}{\Lambda}\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\mu\,\mathrm{sn}\,\nu\,\mathrm{dn}\,\nu, \end{cases}$$

$$\Lambda = 1 - \mathrm{dn}^2\mu\,\mathrm{sn}^2\nu.$$

Coordinate surfaces

$$\left(\frac{x^2+y^2}{a^2\mathrm{cn}^2\mu} + \frac{k^2\mathrm{sn}^2\mu}{a^2\mathrm{dn}^2\mu}z^2\right)^2 - \frac{2}{a^2\mathrm{cn}^2\mu}(x^2+y^2) - \frac{2k^2\mathrm{sn}^2\mu}{a^2\mathrm{dn}^2\mu}z^2 + 1 = 0$$

(rotation-cyclides, $\mu = \text{const}$),

$$\left(\frac{\mathrm{cn}^2\nu}{a^2}(x^2+y^2) + \frac{k'^2\mathrm{sn}^2\nu}{a^2\mathrm{dn}^2\nu}z^2\right)^2 - \frac{2\,\mathrm{cn}^2\nu}{a^2}(x^2+y^2) - \frac{2k'^2\mathrm{sn}^2\nu}{a^2\mathrm{dn}^2\nu}z^2 + 1 = 0$$

(disk-cyclides, $\nu = \text{const}$),

$\tan\psi = y/x$ (half-planes, $\psi = \text{const}$).

Stäckel matrix

$$[S] = \begin{bmatrix} -k^2 \operatorname{sn}^2 \mu & -1 & \left(k^2 \operatorname{cn}^2 \mu - \dfrac{k'^2}{\operatorname{cn}^2 \mu}\right) \\ \operatorname{dn}^2 \nu & 1 & \left(k'^2 \operatorname{cn}^2 \nu - \dfrac{k^2}{\operatorname{cn}^2 \nu}\right) \\ 0 & 0 & 1 \end{bmatrix}.$$

$$S = \operatorname{dn}^2 \nu - k^2 \operatorname{sn}^2 \mu = \operatorname{dn}^2 \mu - k'^2 \operatorname{sn}^2 \nu,$$

$$M_{11} = M_{21} = 1, \quad M_{31} = \frac{\Gamma^2}{\operatorname{cn}^2 \mu \operatorname{cn}^2 \nu}.$$

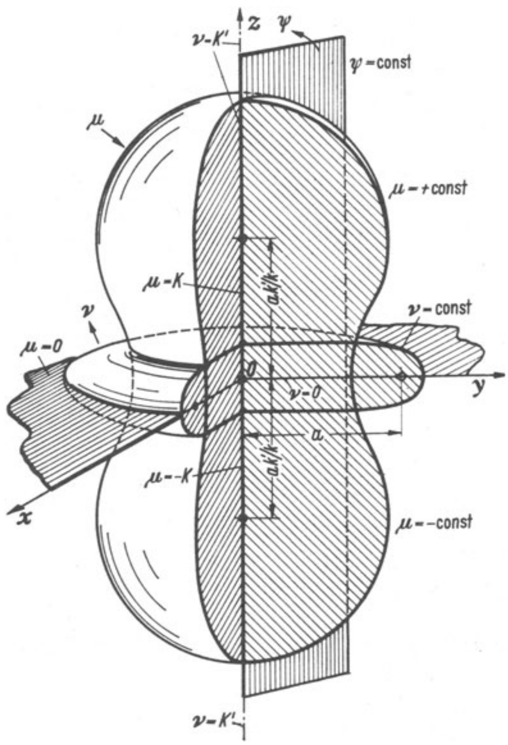


Fig. 4.10. Disk-cyclide coordinates (μ, ν, ψ). Coordinate surfaces are cyclides (μ = const and ν = const) and half-planes (ψ = const)

Metric coefficients

$$g_{11} = g_{22} = a^2 \Gamma^2 / \Lambda^2,$$

$$g_{33} = \frac{a^2}{\Lambda^2} \operatorname{cn}^2 \mu \operatorname{cn}^2 \nu,$$

$$g^{\frac{1}{2}} = \frac{a^3 \Gamma^2}{\Lambda^3} \operatorname{cn} \mu \operatorname{cn} \nu,$$

where

$$\Gamma^2 = (\operatorname{sn}^2 \nu + \operatorname{sn}^2 \mu \operatorname{cn}^2 \nu)(\operatorname{dn}^2 \nu - k^2 \operatorname{sn}^2 \mu).$$

$$R = \Lambda^{-\frac{1}{2}}, \quad Q = \frac{a^2}{\Lambda^2} (\operatorname{sn}^2 \nu + \operatorname{sn}^2 \mu \operatorname{cn}^2 \nu),$$

$$f_1 = \operatorname{cn} \mu, \quad f_2 = \operatorname{cn} \nu, \quad f_3 = a, \quad \alpha_1 = -1.$$

Important equations

$$(ds)^2 = \frac{a^2 \Gamma^2}{\Lambda^2}\left[(d\mu)^2 + (d\nu)^2\right] + \frac{a^2}{\Lambda^2}\operatorname{cn}^2\mu \operatorname{cn}^2\nu\,(d\psi)^2.$$

$$\operatorname{grad}\varphi = \frac{\Lambda}{a\Gamma}\left[\boldsymbol{a}_\mu \frac{\partial\varphi}{\partial\mu} + \boldsymbol{a}_\nu \frac{\partial\varphi}{\partial\nu}\right] + \boldsymbol{a}_\psi \frac{\Lambda}{a \operatorname{cn}\mu \operatorname{cn}\nu}\frac{\partial\varphi}{\partial\psi}.$$

$$\operatorname{div}\boldsymbol{E} = \frac{\Lambda^3}{a\Gamma^2 \operatorname{cn}\mu \operatorname{cn}\nu} \times\left\{\operatorname{cn}\nu \frac{\partial}{\partial\mu}\left[\frac{\Gamma \operatorname{cn}\mu}{\Lambda^2}E_\mu\right] + \operatorname{cn}\mu\frac{\partial}{\partial\nu}\left[\frac{\Gamma\operatorname{cn}\nu}{\Lambda^2}E_\nu\right]\right\} + \frac{\Lambda}{a\operatorname{cn}\mu\operatorname{cn}\nu}\frac{\partial E_\psi}{\partial\psi}.$$

$$\operatorname{curl}\boldsymbol{E} = \frac{\Lambda^2}{a\Gamma^2\operatorname{cn}\mu\operatorname{cn}\nu}\begin{vmatrix} \boldsymbol{a}_\mu \Gamma & \boldsymbol{a}_\nu \Gamma & \boldsymbol{a}_\psi \operatorname{cn}\mu\operatorname{cn}\nu \\ \dfrac{\partial}{\partial\mu} & \dfrac{\partial}{\partial\nu} & \dfrac{\partial}{\partial\psi} \\ E_\mu \Gamma/\Lambda & E_\nu\Gamma/\Lambda & E_\psi \operatorname{cn}\mu\operatorname{cn}\nu/\Lambda \end{vmatrix}.$$

$$\nabla^2\varphi = \frac{\Lambda^3}{a^2\Gamma^2\operatorname{cn}\mu\operatorname{cn}\nu} \times\left\{\operatorname{cn}\nu\frac{\partial}{\partial\mu}\left[\frac{\operatorname{cn}\mu}{\Lambda}\frac{\partial\varphi}{\partial\mu}\right] + \operatorname{cn}\mu\frac{\partial}{\partial\nu}\left[\frac{\operatorname{cn}\nu}{\Lambda}\frac{\partial\varphi}{\partial\nu}\right]\right\} + \frac{\Lambda^2}{a^2\operatorname{cn}^2\mu\operatorname{cn}^2\nu}\frac{\partial^2\varphi}{\partial\psi^2}.$$

SEPARATION OF LAPLACE'S EQUATION, $\nabla^2\varphi = 0$.

$$\frac{1}{f_i}\frac{d}{du^i}\left(f_i\frac{dU^i}{du^i}\right) + U^i\sum_{j=1}^{3}\Phi_{ij}\alpha_j = 0,$$

where $\alpha_1 = -1$ and $U^1 = M(\mu)$, $U^2 = N(\nu)$, $U^3 = \Psi(\psi)$,

$$\varphi = \Lambda^{\frac{1}{2}} M N \Psi.$$

General case

$$\begin{cases} \dfrac{d^2M}{d\mu^2} - \dfrac{\operatorname{sn}\mu\operatorname{dn}\mu}{\operatorname{cn}\mu}\dfrac{dM}{d\mu} + \left[k^2\operatorname{sn}^2\mu - \alpha_2 + \alpha_3\left(k^2\operatorname{cn}^2\mu - \dfrac{k'^2}{\operatorname{cn}^2\mu}\right)\right]M = 0, \\ \dfrac{d^2N}{d\nu^2} - \dfrac{\operatorname{sn}\nu\operatorname{dn}\nu}{\operatorname{cn}\nu}\dfrac{dN}{d\nu} + \left[-\operatorname{dn}^2\nu + \alpha_2 + \alpha_3\left(k'^2\operatorname{cn}^2\nu - \dfrac{k^2}{\operatorname{cn}^2\nu}\right)\right]N = 0, \\ \dfrac{d^2\Psi}{d\psi^2} + \alpha_3\Psi = 0. \end{cases}$$

Substitution of $\operatorname{cn}^2\mu = z$ and $\operatorname{cn}^2\nu = z$ in the first two equations reduces them to the canonical form

$$\frac{d^2Z}{dz^2} + \frac{1}{2}\left[\frac{1}{z-a_1} + \frac{1}{z-a_2} + \frac{2}{z-a_3}\right]\frac{dZ}{dz} + \frac{1}{4}\left[\frac{\bar{A}_0 + \bar{A}_1 z + \bar{A}_2 z^2}{(z-a_1)(z-a_2)(z-a_3)}\right]Z = 0,$$

where, for M with $\alpha_2 = p^2$ and $\alpha_3 = q^2$,

$$a_1 = b = 1, \quad a_2 = c = -(k'/k)^2, \quad a_3 = 0,$$

$$\bar{A}_0 = -q^2c, \quad \bar{A}_1 = p^2(1-c) - 1, \quad \bar{A}_2 = 1 - q^2.$$

For N, with $\alpha_2 = p'^2$ and $\alpha_3 = q^2$,

$$a_1 = b = 1, \quad a_2 = c = -(k/k')^2, \quad a_3 = 0,$$

$$\bar{A}_0 = -q^2c, \quad \bar{A}_1 = -[p'^2(1-c) + c], \quad \bar{A}_2 = 1 - q^2.$$

Evidently $p'^2 = 1 - p^2$. These are Bôcher equations with designation $\{1\,1\,2\,2\}$, and their solutions are the Wangerin functions,

$$Z = A\,\mathscr{S}_p^q(k, z) + B\,\mathscr{T}_p^q(k, z).$$

For $\varphi = \varphi(u^1, u^2, u^3)$,

$$\begin{cases} M = A\,\mathscr{S}_p^q(i\,k/k', \mathrm{cn}^2\mu) + B\,\mathscr{T}_p^q(i\,k/k', \mathrm{cn}^2\mu), \\ N = A\,\mathscr{S}_{p'}^q(i\,k'/k, \mathrm{cn}^2\nu) + B\,\mathscr{T}_{p'}^q(i\,k'/k, \mathrm{cn}^2\nu), \\ \Psi = A\sin q\psi + B\cos q\psi, \quad \text{where} \quad p'^2 = 1 - p^2. \end{cases}$$

For φ independent of ψ,

$$\begin{cases} M = A\,\mathscr{S}_p(i\,k/k', \mathrm{cn}^2\mu) + B\,\mathscr{T}_p(i\,k/k', \mathrm{cn}^2\mu), \\ N = A\,\mathscr{S}_{p'}(i\,k'/_k, \mathrm{cn}^2\nu) + B\,\mathscr{T}_{p'}(i\,k'/k, \mathrm{cn}^2\nu). \end{cases}$$

J 3R. CAP-CYCLIDE COORDINATES (μ, ν, ψ), Fig. 4.11.

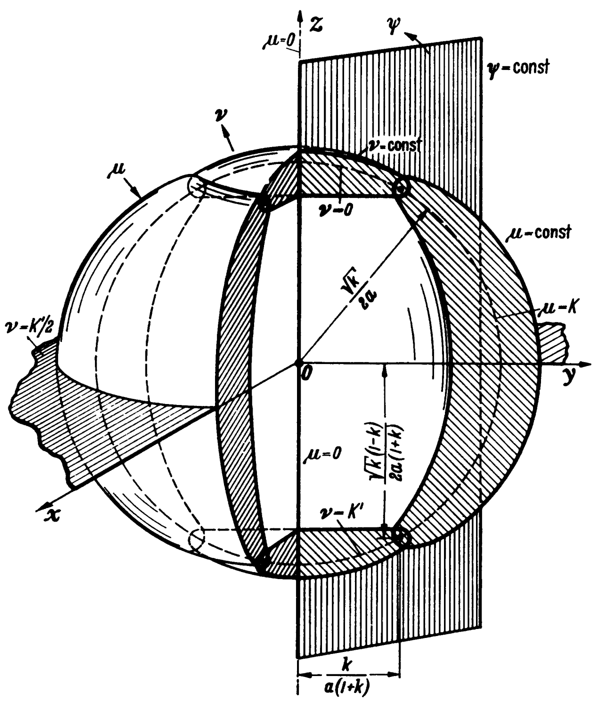


Fig. 4.11. Cap-cyclide coordinates (μ, ν, ψ). An inversion of bi-cyclide coordinates (Fig. 4.08). Coordinate surfaces are cap cyclides (ν = const), ring cyclides (μ = const), and half-planes (ψ = const)

$$0 \leqq \mu \leqq K, \quad 0 \leqq \nu \leqq K', \quad 0 \leqq \psi < 2\pi.$$

$$\begin{cases} x = \dfrac{A}{a\Upsilon}\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu\cos\psi, \\ y = \dfrac{A}{a\Upsilon}\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu\sin\psi, \\ z = \dfrac{k^{\frac{1}{2}}\Pi}{2a\Upsilon}, \end{cases}$$

where
$$\Lambda = 1 - \mathrm{dn}^2\mu\,\mathrm{sn}^2\nu,$$
$$\Upsilon = \mathrm{sn}^2\mu\,\mathrm{dn}^2\nu + \left[(\Lambda/\sqrt{k}) + \mathrm{cn}\,\mu\,\mathrm{dn}\,\mu\,\mathrm{sn}\,\nu\,\mathrm{cn}\,\nu\right]^2,$$
$$\Pi = (\Lambda^2/k) - (\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu + \mathrm{cn}^2\mu\,\mathrm{dn}^2\mu\,\mathrm{sn}^2\nu\,\mathrm{cn}^2\nu).$$

Coordinate surfaces

$$(x^2+y^2+z^2)^2 + A(x^2+y^2) + Bz^2 + C = 0,$$
where
$$A = -(R_1+R_2),$$
$$B = \frac{4(R_1+R_2)\,\mathrm{sn}^2\mu - \left(4a^2R_1R_2 + \dfrac{k^2}{4a^2}\right)(1/k + \mathrm{sn}^2\mu)^2}{k(1/k - \mathrm{sn}^2\mu)^2}.$$
$$C = R_1R_2,$$
$$R_1 = \frac{k(1-\mathrm{dn}^2\mu/(1+k))^2\,\mathrm{sn}^2\mu}{a^2\left[k\,\mathrm{sn}^2\mu + \dfrac{1}{\sqrt{k}}\left(1 - \dfrac{\mathrm{dn}^2\mu}{1+k}\right) + (\mathrm{cn}\,\mu\,\mathrm{dn}\,\mu)\dfrac{\sqrt{k}}{1+k}\right]^2}.$$
$$R_2 = \frac{k\left(1 - \dfrac{\mathrm{dn}^2\mu}{1+k}\right)^2\mathrm{sn}^2\mu}{a^2\left[k\,\mathrm{sn}^2\mu + \dfrac{1}{\sqrt{k}}\left(1 - \dfrac{\mathrm{dn}^2\mu}{1+k}\right) - \dfrac{\sqrt{k}}{1+k}\,\mathrm{cn}\,\mu\,\mathrm{dn}\,\mu\right]^2}$$
(ring-cyclides, $\mu = \mathrm{const}$).

$$(x^2+y^2+z^2)^2 + A(x^2+y^2) + Bz^2 + C = 0,$$
where
$$A = \frac{k^2}{8a^2}\left[\frac{(1+k')(\mathrm{cn}^4\nu + 6k\,\mathrm{sn}^2\nu\,\mathrm{cn}^2\nu + k^2\mathrm{sn}^4\nu)\left(\dfrac{(1-k'\mathrm{sn}^2\nu)}{k} - \dfrac{(1+k'\mathrm{sn}^2\nu)}{1+k'}\right)^2}{\mathrm{dn}^2\nu(\mathrm{cn}^2\nu - k^2\mathrm{sn}^2\nu)^2} - 1\right],$$
$$B = -\frac{k}{2a^2}\,\frac{(\mathrm{cn}^4\nu + 6k\,\mathrm{sn}^2\nu\,\mathrm{cn}^2\nu + k^2\mathrm{sn}^4\nu)}{(\mathrm{cn}^2\nu - k^2\mathrm{sn}^2\nu)^2},$$
$$C = \frac{k^2}{16a^4}$$
(cap-cyclides, $\nu = \mathrm{const}$).

$\tan\psi = y/x$ (half-planes, $\psi = \mathrm{const}$).

Stäckel matrix

$$[S] = \begin{bmatrix} -k^2\mathrm{sn}^2\mu & -1 & -\left(k^2\mathrm{sn}^2\mu + \dfrac{1}{\mathrm{sn}^2\mu}\right) \\ \mathrm{dn}^2\nu & 1 & \left(\mathrm{dn}^2\nu + \dfrac{k^2}{\mathrm{dn}^2\nu}\right) \\ 0 & 0 & 1 \end{bmatrix}.$$
$$S = \mathrm{dn}^2\nu - k^2\mathrm{sn}^2\mu = \mathrm{dn}^2\mu - k'^2\mathrm{sn}^2\nu,$$
$$M_{11} = M_{21} = 1, \qquad M_{31} = \frac{\Omega^2}{\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}.$$

Metric coefficients

$$g_{11} = g_{22} = \left(\frac{\Lambda}{a\Upsilon}\right)^2\Omega^2,$$
$$g_{33} = \left(\frac{\Lambda}{a\Upsilon}\right)^2\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu,$$
$$g^{\frac{1}{2}} = \left(\frac{\Lambda}{a\Upsilon}\right)^3\Omega^2\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu,$$

where

$$\Omega^2 = (1 - \mathrm{sn}^2\mu\,\mathrm{dn}^2\nu)\,(\mathrm{dn}^2\nu - k^2\,\mathrm{sn}^2\mu),$$

$$\Upsilon = \mathrm{sn}^2\mu\,\mathrm{dn}^2\nu + [\Lambda/\sqrt{k} + \mathrm{cn}\,\mu\,\mathrm{dn}\,\mu\,\mathrm{sn}\,\nu\,\mathrm{cn}\,\nu]^2.$$

$$R = \Lambda^{\frac{1}{2}}\Upsilon^{-\frac{1}{2}}, \qquad Q = \left(\frac{\Lambda}{a\Upsilon}\right)^2 (1 - \mathrm{sn}^2\mu\,\mathrm{dn}^2\nu),$$

$$f_1 = \mathrm{sn}\,\mu, \qquad f_2 = \mathrm{dn}\,\nu, \qquad f_3 = 1/a, \qquad \alpha_1 = -1.$$

Important equations

$$(ds)^2 = \left(\frac{\Lambda}{a\Upsilon}\right)^2 \Omega^2 [(d\mu)^2 + (d\nu)^2] + \left(\frac{\Lambda}{a\Upsilon}\right)^2 \mathrm{sn}^2\mu\,\mathrm{dn}^2\nu\,(d\psi)^2.$$

$$\operatorname{grad}\varphi = \frac{a\Upsilon}{\Lambda\Omega}\left[\boldsymbol{a}_\mu \frac{\partial\varphi}{\partial\mu} + \boldsymbol{a}_\nu \frac{\partial\varphi}{\partial\nu}\right] + \boldsymbol{a}_\psi \frac{a\Upsilon}{\Lambda\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu}\frac{\partial\varphi}{\partial\psi}.$$

$$\operatorname{div}\boldsymbol{E} = \frac{a\Upsilon^3}{\Lambda^3\Omega^2\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu} \times \left\{\mathrm{dn}\,\nu \frac{\partial}{\partial\mu}\left[\frac{\Lambda^2\Omega\,\mathrm{sn}\,\mu}{\Upsilon^2}E_\mu\right] + \mathrm{sn}\,\mu \frac{\partial}{\partial\nu}\left[\frac{\Lambda^2\Omega\,\mathrm{dn}\,\nu}{\Upsilon^2}E_\nu\right]\right\} + \frac{a\Upsilon}{\Lambda\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu}\frac{\partial E_\psi}{\partial\psi}.$$

$$\operatorname{curl}\boldsymbol{E} = \frac{a\Upsilon^2}{\Lambda^2\Omega^2\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu}\begin{vmatrix} \boldsymbol{a}_\mu\Omega & \boldsymbol{a}_\nu\Omega & \boldsymbol{a}_\psi\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu \\ \dfrac{\partial}{\partial\mu} & \dfrac{\partial}{\partial\nu} & \dfrac{\partial}{\partial\psi} \\ E_\mu\Lambda\Omega/\Upsilon & E_\nu\Lambda\Omega/\Upsilon & E_\psi\Lambda\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu/\Upsilon \end{vmatrix}.$$

$$\nabla^2\varphi = \frac{a^2\Upsilon^3}{\Lambda^3\Omega^2\,\mathrm{sn}\,\mu\,\mathrm{dn}\,\nu} \times \left\{\mathrm{dn}\,\nu \frac{\partial}{\partial\mu}\left[\frac{\Lambda\,\mathrm{sn}\,\mu}{\Upsilon}\frac{\partial\varphi}{\partial\mu}\right] + \mathrm{sn}\,\mu \frac{\partial}{\partial\nu}\left[\frac{\Lambda\,\mathrm{dn}\,\nu}{\Upsilon}\frac{\partial\varphi}{\partial\nu}\right]\right\} + \frac{a^2\Upsilon^2}{\Lambda^2\,\mathrm{sn}^2\mu\,\mathrm{dn}^2\nu}\frac{\partial^2\varphi}{\partial\psi^2}.$$

SEPARATION OF LAPLACE'S EQUATION, $\nabla^2\varphi = 0$.

$$\frac{1}{f_i}\frac{d}{du^i}\left(f_i \frac{dU^i}{du^i}\right) + U^i \sum_{j=1}^{3} \Phi_{ij}\alpha_j = 0,$$

where $\alpha_1 = -1$ and $U^1 = M(\mu)$, $U^2 = N(\nu)$, $U^3 = \Psi(\psi)$,

$$\varphi = \Lambda^{-\frac{1}{2}}\Upsilon^{\frac{1}{2}} M N \Psi.$$

General case

$$\begin{cases} \dfrac{d^2M}{d\mu^2} + \dfrac{\mathrm{cn}\,\mu\,\mathrm{dn}\,\mu}{\mathrm{sn}\,\mu}\dfrac{dM}{d\mu} + \left[k^2\,\mathrm{sn}^2\mu - \alpha_2 - \alpha_3\left(k^2\,\mathrm{sn}^2\mu + \dfrac{1}{\mathrm{sn}^2\mu}\right)\right]M = 0, \\ \dfrac{d^2N}{d\nu^2} - \dfrac{k'^2\,\mathrm{sn}\,\nu\,\mathrm{cn}\,\nu}{\mathrm{dn}\,\nu}\dfrac{dN}{d\nu} + \left[-\mathrm{dn}^2\nu + \alpha_2 + \alpha_3\left(\mathrm{dn}^2\nu + \dfrac{k^2}{\mathrm{sn}^2\nu}\right)\right]N = 0, \\ \dfrac{d^2\Psi}{d\psi^2} + \alpha_3\Psi = 0. \end{cases}$$

Substitution of $\mathrm{sn}^2\mu = z$ in the first equation, and $\mathrm{dn}^2\nu = z$ in the second equation, reduces both to the canonical form

$$\frac{d^2Z}{dz^2} + \frac{1}{2}\left[\frac{1}{z-a_1} + \frac{1}{z-a_2} + \frac{2}{z-a_3}\right]\frac{dZ}{dz} + \frac{1}{4}\left[\frac{\bar{A}_0 + \bar{A}_1 z + \bar{A}_2 z^2}{(z-a_1)(z-a_2)(z-a_3)^2}\right]Z = 0,$$

where, for M with $\alpha_2 = p^2$ and $\alpha_3 = q^2$,

$$a_1 = b_1 = 1, \quad a_2 = c = 1/k^2, \quad a_3 = 0,$$
$$\bar{A}_0 = -q^2 c, \quad \bar{A}_1 = -p^2 c, \quad \bar{A}_2 = 1 - q^2.$$

For N, with $\alpha_2 = p'^2$ and $\alpha_3 = q^2$,

$$a_1 = b = 1, \quad a_2 = c = k^2, \quad a_3 = 0,$$
$$\bar{A}_0 = -q^2 c, \quad \bar{A}_1 = -p'^2, \quad \bar{A}_2 = 1 - q^2.$$

Here $p'^2 = p^2/k^2$. These are Bôcher equations with designation $\{1122\}$, and their solutions are Wangerin functions,

$$Z = A\mathscr{S}_p^q(k, z) + B\mathscr{T}_p^q(k, z).$$

For $\varphi = \varphi(u^1, u^2, u^3)$,

$$\begin{cases} M = A\mathscr{S}_p^q(k, \mathrm{sn}^2\mu) + B\mathscr{T}_p^q(k, \mathrm{sn}^2\mu), \\ N = A\mathscr{S}_{p'}^q(i/k, \mathrm{dn}^2\nu) + B\mathscr{T}_{p'}^q(i/k, \mathrm{dn}^2\nu), \\ \Psi = A\sin q\psi + B\cos q\psi, \quad \text{where} \quad p'^2 = p^2/k^2. \end{cases}$$

For φ independent of ψ,

$$\begin{cases} M = A\mathscr{S}_p(k, \mathrm{sn}^2\mu) + B\mathscr{T}_p(k, \mathrm{sn}^2\mu), \\ N = A\mathscr{S}_{p'}(i/k, \mathrm{dn}^2\nu) + B\mathscr{T}_{p'}(i/k, \mathrm{dn}^2\nu). \end{cases}$$

A summary of the coordinates obtained from elliptic functions is given in Table 4.04. Further information on the differential equations and their solutions may be found in Section VII.

TABLE 4.04.
SEPARATION EQUATIONS OBTAINED IN FOUR COORDINATE SYSTEMS

		a_1	a_2	a_3	$\bar{A}_0$	$\bar{A}_1$	$\bar{A}_2$	$\bar{A}_3$
J1Rx $\{1222\}$ $p'^2=p^2/k^2$	M	0	1	$1/k^2=c$	$-p^2c^2$	$[p^2c(1+c) - q^2(1-c)^2+2c]$	$-[p^2c+2(1+c)]$	2
	N	0	1	$k^2=c$	$-p'^2c$	$[p'^2(1+c) - q^2(1-c)^2+2c]$	$-[p'^2+2(1+c)]$	2
J1Ry, J3R $\{1122\}$ $p'^2=p^2/k^2$	M	1	$1/k^2=c$	0	$-q^2c$	$-p^2c$	$1-q^2$	0
	N	1	$k^2=c$	0	$-q^2c$	$-p'^2$	$1-q^2$	0
J2R $\{1122\}$ $p'^2=1-p^2$	M	1	$-(k'/k)^2=c$	0	$-q^2c$	$[p^2(1-c)-1]$	$1-q^2$	0
	N	1	$-(k/k')^2=c$	0	$-q^2c$	$-[p'^2(1-c)+c]$	$1-q^2$	0

Section V

THE VECTOR HELMHOLTZ EQUATION

The vector Helmholtz equation, which occurs particularly in electromagnetic theory [*19*], is more complicated than the scalar Helmholtz equation and its separation presents new problems.

5.01 CYLINDRICAL SYSTEMS

The vector Laplacian in general orthogonal coordinates is expressed by Eq. (1.11). For a *cylindrical* coordinate system, these results can be simplified because g_{33} is always unity [*9*]. If $g_{11}=g_{22}$ also,

$$\Upsilon=\frac{1}{g_{11}}\left\{\frac{\partial}{\partial u^1}\left[(g_{11})^{\frac{1}{2}}E_1\right]+\frac{\partial}{\partial u^2}\left[(g_{11})^{\frac{1}{2}}E_2\right]\right\}+\frac{\partial E_z}{\partial z},$$

$$\Gamma_1=\frac{\partial E_3}{\partial u^2}-(g_{11})^{\frac{1}{2}}\frac{\partial E_2}{\partial z},$$

$$\Gamma_2=(g_{11})^{\frac{1}{2}}\frac{\partial E_1}{\partial z}-\frac{\partial E_z}{\partial u^1},$$

$$\Gamma_3=\frac{1}{g_{11}}\left\{\frac{\partial}{\partial u^1}\left[(g_{11})^{\frac{1}{2}}E_2\right]-\frac{\partial}{\partial u^2}\left[(g_{11})^{\frac{1}{2}}E_1\right]\right\}.$$

Substitution into Eq. (1.11) gives the vector Laplacian in any *cylindrical* system with $g_{11}=g_{22}$:

$$\left.\begin{aligned}(\text{✡}\,\boldsymbol{E})_1=&\frac{1}{g_{11}}\left\{\frac{\partial^2E_1}{(\partial u^1)^2}+\frac{\partial^2E_1}{(\partial u^2)^2}\right\}+\frac{\partial^2E_1}{\partial z^2}+\frac{E_1}{2(g_{11})^2}\left[\frac{\partial^2g_{11}}{(\partial u^1)^2}+\frac{\partial^2g_{11}}{(\partial u^2)^2}\right]\\&-\frac{3E_1}{4(g_{11})^3}\left[\left(\frac{\partial g_{11}}{\partial u^1}\right)^2+\left(\frac{\partial g_{11}}{\partial u^2}\right)^2\right]+\frac{1}{(g_{11})^2}\left[\frac{\partial g_{11}}{\partial u^2}\frac{\partial E_2}{\partial u^1}-\frac{\partial g_{11}}{\partial u^1}\frac{\partial E_2}{\partial u^2}\right],\end{aligned}\right\}\qquad(5.01)$$

$$\left.\begin{aligned}(\text{✡}\,\boldsymbol{E})_2=&\frac{1}{g_{11}}\left\{\frac{\partial^2E_2}{(\partial u^1)^2}+\frac{\partial^2E_2}{(\partial u^2)^2}\right\}+\frac{\partial^2E_2}{\partial z^2}+\frac{E_2}{2(g_{11})^2}\left[\frac{\partial^2g_{11}}{(\partial u^1)^2}+\frac{\partial^2g_{11}}{(\partial u^2)^2}\right]\\&-\frac{3E_2}{4(g_{11})^3}\left[\left(\frac{\partial g_{11}}{\partial u^1}\right)^2+\left(\frac{\partial g_{11}}{\partial u^2}\right)^2\right]+\frac{1}{(g_{11})^2}\left[\frac{\partial g_{11}}{\partial u^1}\frac{\partial E_1}{\partial u^2}-\frac{\partial g_{11}}{\partial u^2}\frac{\partial E_1}{\partial u^1}\right],\end{aligned}\right\}\qquad(5.02)$$

$$(\text{✡}\,\boldsymbol{E})_z=\frac{1}{g_{11}}\left\{\frac{\partial^2E_z}{(\partial u^1)^2}+\frac{\partial^2E_z}{(\partial u^2)^2}\right\}+\frac{\partial^2E_z}{\partial z^2}.\qquad(5.03)$$

Equation (5.03) shows that the z-component of the vector Laplacian has exactly the form of the scalar Laplacian:

$$(\text{✡}\,\boldsymbol{E})_z=\nabla^2E_z.\qquad(5.04)$$

But the other components of $\text{✡}\,\boldsymbol{E}$ contain additional terms [*20*].

Details for the separation of the vector Helmholtz equation in cylindrical systems are given in Table 5.01. For rectangular coordinates, the three components of the vector Helmholtz equation are identical with the scalar Helmholtz equation. Thus separation presents no problem and the solutions are those given in Section I. Circular-cylinder coordinates, though less satisfactory than rectangular coordinates, still allow solutions in a number of cases. For the other cylindrical systems, separation appears to be possible only for the z-component and to occur only in the four cylindrical systems in which the scalar wave equation is known to separate. Solutions for E_z are identical with those previously given for φ.

5.02 ROTATIONAL SYSTEMS

A simplification of Eq. (1.11) is also effected in coordinate systems having axial symmetry. Here the metric coefficients do not contain the third independent variable $u^3 = \psi$. If we also stipulate that $g_{11} = g_{22}$, then

$$\begin{aligned}
\Upsilon &= \frac{1}{(g_{11})^{\frac12}}\left[\frac{\partial E_1}{\partial u^1} + \frac{\partial E_2}{\partial u^2}\right] + \frac{1}{(g_{33})^{\frac12}}\frac{\partial E_\psi}{\partial \psi} \\
&\quad + \frac{E_1}{2(g_{11})^{\frac12}}\left[\frac{1}{g_{11}}\frac{\partial g_{11}}{\partial u^1} + \frac{1}{g_{33}}\frac{\partial g_{33}}{\partial u^1}\right] + \frac{E_2}{2(g_{11})^{\frac12}}\left[\frac{1}{g_{11}}\frac{\partial g_{11}}{\partial u^2} + \frac{1}{g_{33}}\frac{\partial g_{33}}{\partial u^2}\right], \\
\Gamma_1 &= \frac{\partial E_\psi}{\partial u^2} - \left(\frac{g_{11}}{g_{33}}\right)^{\frac12}\frac{\partial E_2}{\partial \psi} + \frac{E_\psi}{2g_{33}}\frac{\partial g_{33}}{\partial u^2}, \\
\Gamma_2 &= \left(\frac{g_{11}}{g_{33}}\right)^{\frac12}\frac{\partial E_1}{\partial \psi} - \frac{\partial E_\psi}{\partial u^1} - \frac{E_\psi}{2g_{33}}\frac{\partial g_{33}}{\partial u^1}, \\
\Gamma_3 &= \left(\frac{g_{33}}{g_{11}}\right)^{\frac12}\left[\frac{\partial E_2}{\partial u^1} - \frac{\partial E_1}{\partial u^2} + \frac{1}{2g_{11}}\left(\frac{\partial g_{11}}{\partial u^1}E_2 - \frac{\partial g_{11}}{\partial u^2}E_1\right)\right],
\end{aligned}$$

Therefore,

$$\begin{aligned}
(\text{✡} \boldsymbol{E})_1 &= \frac{1}{g_{11}}\left[\frac{\partial^2 E_1}{(\partial u^1)^2} + \frac{\partial^2 E_1}{(\partial u^2)^2}\right] + \frac{1}{g_{33}}\frac{\partial^2 E_1}{\partial \psi^2} + \frac{1}{2g_{11}g_{33}}\left[\frac{\partial g_{33}}{\partial u^1}\frac{\partial E_1}{\partial u^1} + \frac{\partial g_{33}}{\partial u^2}\frac{\partial E_2}{\partial u^2}\right] \\
&\quad + \frac{E_1}{2g_{11}}\left\{\frac{1}{g_{11}}\left[\frac{\partial^2 g_{11}}{(\partial u^1)^2} + \frac{\partial^2 g_{11}}{(\partial u^2)^2}\right] - \frac{3}{2(g_{11})^2}\left[\left(\frac{\partial g_{11}}{\partial u^1}\right)^2 + \left(\frac{\partial g_{11}}{\partial u^2}\right)^2\right]\right. \\
&\quad \left. - \frac{1}{2g_{11}g_{33}}\left[\frac{\partial g_{11}}{\partial u^1}\frac{\partial g_{33}}{\partial u^1} - \frac{\partial g_{11}}{\partial u^2}\frac{\partial g_{33}}{\partial u^2}\right] + \frac{1}{g_{33}}\left[\frac{\partial^2 g_{33}}{(\partial u^1)^2} - \frac{1}{g_{33}}\left(\frac{\partial g_{33}}{\partial u^1}\right)^2\right]\right\} \\
&\quad + \frac{1}{(g_{11})^2}\left[\frac{\partial g_{11}}{\partial u^2}\frac{\partial E_2}{\partial u^1} - \frac{\partial g_{11}}{\partial u^1}\frac{\partial E_2}{\partial u^2}\right] \\
&\quad + \frac{E_2}{2g_{11}g_{22}}\left[\frac{\partial^2 g_{33}}{\partial u^1 \partial u^2} - \frac{\partial g_{33}}{\partial u^2}\left(\frac{1}{g_{11}}\frac{\partial g_{11}}{\partial u^1} + \frac{1}{g_{33}}\frac{\partial g_{33}}{\partial u^1}\right)\right] \\
&\quad - \frac{1}{(g_{11})^{\frac12}(g_{33})^{\frac32}}\frac{\partial g_{33}}{\partial u^1}\frac{\partial E_\psi}{\partial \psi}.
\end{aligned}$$

$$\begin{aligned}
(\text{✡} \boldsymbol{E})_2 &= \frac{1}{g_{11}}\left[\frac{\partial^2 E_2}{(\partial u^1)^2} + \frac{\partial^2 E_2}{(\partial u^2)^2}\right] + \frac{1}{g_{33}}\frac{\partial^2 E_2}{\partial \psi^2} + \frac{1}{2g_{11}g_{33}}\left[\frac{\partial g_{33}}{\partial u^1}\frac{\partial E_2}{\partial u^1} + \frac{\partial g_{33}}{\partial u^2}\frac{\partial E_2}{\partial u^2}\right] \\
&\quad + \frac{E_2}{2g_{11}}\left\{\frac{1}{g_{11}}\left[\frac{\partial^2 g_{11}}{(\partial u^1)^2} + \frac{\partial^2 g_{11}}{(\partial u^2)^2}\right] - \frac{3}{2(g_{11})^2}\left[\left(\frac{\partial g_{11}}{\partial u^1}\right)^2 + \left(\frac{\partial g_{11}}{\partial u^2}\right)^2\right]\right. \\
&\quad \left. - \frac{1}{2g_{11}g_{33}}\left[\frac{\partial g_{11}}{\partial u^2}\frac{\partial g_{33}}{\partial u^2} - \frac{\partial g_{11}}{\partial u^1}\frac{\partial g_{33}}{\partial u^1}\right] + \frac{1}{g_{33}}\left[\frac{\partial^2 g_{33}}{(\partial u^2)^2} - \frac{1}{g_{33}}\left(\frac{\partial g_{33}}{\partial u^2}\right)^2\right]\right\} \\
&\quad + \frac{1}{(g_{11})^2}\left[\frac{\partial g_{11}}{\partial u^1}\frac{\partial E_1}{\partial u^2} - \frac{\partial g_{11}}{\partial u^2}\frac{\partial E_1}{\partial u^1}\right] \\
&\quad + \frac{E_1}{2g_{11}g_{33}}\left[\frac{\partial^2 g_{33}}{\partial u^1 \partial u^2} - \frac{\partial g_{33}}{\partial u^1}\left(\frac{1}{g_{11}}\frac{\partial g_{11}}{\partial u^2} + \frac{1}{g_{33}}\frac{\partial g_{33}}{\partial u^2}\right)\right] \\
&\quad - \frac{1}{(g_{11})^{\frac12}(g_{33})^{\frac32}}\frac{\partial g_{33}}{\partial u^2}\frac{\partial E_\psi}{\partial \psi},
\end{aligned}$$

$$(\text{✡}\,\boldsymbol{E})_\psi = \frac{1}{g_{11}}\left[\frac{\partial^2 E_\psi}{(\partial u^1)^2} + \frac{\partial^2 E_\psi}{(\partial u^2)^2}\right] + \frac{1}{g_{33}}\frac{\partial^2 E_\psi}{\partial \psi^2} + \frac{1}{2 g_{11} g_{33}}\left[\frac{\partial g_{33}}{\partial u^1}\frac{\partial E_\psi}{\partial u^1} + \frac{\partial g_{33}}{\partial u^2}\frac{\partial E_\psi}{\partial u^2}\right]$$
$$+ \frac{E_\psi}{2 g_{11} g_{33}}\left\{\left[\frac{\partial^2 g_{33}}{(\partial u^1)^2} + \frac{\partial^2 g_{33}}{(\partial u^2)^2}\right] - \frac{1}{g_{33}}\left[\left(\frac{\partial g_{33}}{\partial u^1}\right)^2 + \left(\frac{\partial g_{33}}{\partial u^2}\right)^2\right]\right\}$$
$$+ \frac{1}{(g_{11})^{\frac{1}{2}}(g_{33})^{\frac{3}{2}}}\left[\frac{\partial g_{33}}{\partial u^1}\frac{\partial E_1}{\partial \psi} + \frac{\partial g_{33}}{\partial u^2}\frac{\partial E_2}{\partial \psi}\right].$$

If the field is axially symmetric, the ψ-component of the vector Helmholtz equation reduces to

$$\frac{1}{g_{11}}\left[\frac{\partial^2 E_\psi}{(\partial u^1)^2} + \frac{\partial^2 E_\psi}{(\partial u^2)^2}\right] + \frac{1}{2 g_{11} g_{33}}\left[\frac{\partial g_{33}}{\partial u^1}\frac{\partial E_\psi}{\partial u^1} + \frac{\partial g_{33}}{\partial u^2}\frac{\partial E_\psi}{\partial u^2}\right]$$
$$+ \left[\varkappa^2 + \frac{1}{2 g_{11} g_{33}}\left\{\left[\frac{\partial^2 g_{33}}{(\partial u^1)^2} + \frac{\partial^2 g_{33}}{(\partial u^2)^2}\right] - \frac{1}{g_{33}}\left[\left(\frac{\partial g_{33}}{\partial u^1}\right)^2 + \left(\frac{\partial g_{33}}{\partial u^2}\right)^2\right]\right\}\right] E_\psi = 0.$$

This equation is simply separable in the rotational coordinate systems in which the scalar Helmholtz equation separates. Details are given in Table 5.02.

TABLE 5.01.
THE VECTOR HELMHOLTZ EQUATION IN CYLINDRICAL SYSTEMS

FIG. 1.01. RECTANGULAR COORDINATES (x, y, z)

$$g_{11} = g_{22} = g_{33} = g^{\frac{1}{2}} = 1.$$
$$\Upsilon = \frac{\partial E_1}{\partial u^1} + \frac{\partial E_2}{\partial u^2} + \frac{\partial E_3}{\partial u_3},$$
$$\Gamma_1 = \frac{\partial E_3}{\partial u^2} - \frac{\partial E_2}{\partial u^3},$$
$$\Gamma_2 = \frac{\partial E_1}{\partial u^3} - \frac{\partial E_3}{\partial u^1},$$
$$\Gamma_3 = \frac{\partial E_2}{\partial u^1} - \frac{\partial E_1}{\partial u^2}.$$
$$\text{✡}\,\boldsymbol{E} = \boldsymbol{a}_x \nabla^2 E_x + \boldsymbol{a}_y \nabla^2 E_y + \boldsymbol{a}_z \nabla^2 E_z.$$

Thus the vector Helmholtz equation in rectangular coordinates reduces to three ordinary scalar Helmholtz equations:

$$\begin{cases} \nabla^2 E_x + \varkappa^2 E_x = 0, \\ \nabla^2 E_y + \varkappa^2 E_y = 0, \\ \nabla^2 E_z + \varkappa^2 E_z = 0. \end{cases}$$

Solutions are given in Section I.

FIG. 1.02. CIRCULAR-CYLINDER COORDINATES (r, ψ, z)

$$g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = 1, \quad g^{\frac{1}{2}} = r.$$
$$\Upsilon = \frac{\partial E_r}{\partial r} + \frac{E_r}{r} + \frac{1}{r}\frac{\partial E_\psi}{\partial \psi} + \frac{\partial E_z}{\partial z},$$
$$\Gamma_1 = \frac{1}{r}\frac{\partial E_z}{\partial \psi} - \frac{\partial E_\psi}{\partial z},$$
$$\Gamma_2 = r\left[\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}\right],$$
$$\Gamma_3 = \frac{\partial E_\psi}{\partial r} + \frac{E_\psi}{r} - \frac{1}{r}\frac{\partial E_r}{\partial \psi}.$$

From Eq. (1.11), the vector Helmholtz equation is

$$\frac{\partial^2 E_r}{\partial r^2} + \frac{1}{r}\frac{\partial E_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E_r}{\partial \psi^2} + \frac{\partial^2 E_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial E_\psi}{\partial \psi} + (\varkappa^2 - 1/r^2)\, E_r = 0,$$

$$\frac{\partial^2 E_\psi}{\partial r^2} + \frac{1}{r}\frac{\partial E_\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E_\psi}{\partial \psi^2} + \frac{\partial^2 E_\psi}{\partial z^2} + \frac{2}{r^2}\frac{\partial E_r}{\partial \psi} + (\varkappa^2 - 1/r^2)\, E_\psi = 0,$$

$$\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r}\frac{\partial E_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \psi^2} + \frac{\partial^2 E_z}{\partial z^2} + \varkappa^2 E_z = 0.$$

These equations can be written

$$\begin{cases} \nabla^2 E_r - \dfrac{2}{r^2}\dfrac{\partial E_\psi}{\partial \psi} + (\varkappa^2 - 1/r^2)\, E_r = 0, \\ \nabla^2 E_\psi + \dfrac{2}{r^2}\dfrac{\partial E_r}{\partial \psi} + (\varkappa^2 - 1/r^2)\, E_\psi = 0, \\ \nabla^2 E_z + \varkappa^2 E_z = 0. \end{cases}$$

Solutions

For $\boldsymbol{E} = \boldsymbol{a}_r E_r$,

$$\begin{cases} \dfrac{\partial^2 E_r}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial E_r}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 E_r}{\partial \psi^2} + \dfrac{\partial^2 E_r}{\partial z^2} + (\varkappa^2 - 1/r^2)\, E_r = 0, \\ \dfrac{\partial E_r}{\partial \psi} = 0. \end{cases}$$

Because of the second equation, E_r must be independent of ψ and the vector Helmholtz equation is

$$\frac{\partial^2 E_r}{\partial r^2} + \frac{1}{r}\frac{\partial E_r}{\partial r} + \frac{\partial^2 E_r}{\partial z^2} + (\varkappa^2 - 1/r^2)\, E_r = 0.$$

Thus, for $\boldsymbol{E} = \boldsymbol{a}_r E_r(r, z)$, separation equations and solutions are

$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + (q^2 - 1/r^2)\, R = 0, \quad \{24\} \quad R = A\mathcal{J}_1(q\,r) + B\mathcal{Y}_1(q\,r).$$

$$\frac{d^2 Z}{dz^2} + (\varkappa^2 - q^2)\, Z = 0, \qquad \{04\} \quad Z = A\sin(\varkappa^2 - q^2)^{\frac{1}{2}} z + B\cos(\varkappa^2 - q^2)^{\frac{1}{2}} z.$$

For $\boldsymbol{E} = \boldsymbol{a}_r E_r(r)$,

$$\frac{d^2 E_r}{dr^2} + \frac{1}{r}\frac{dE_r}{dr} + (\varkappa^2 - 1/r^2)\, E_r = 0, \quad \{24\} \quad E_r = A\mathcal{J}_1(\varkappa\, r) + B\mathcal{Y}_1(\varkappa\, r).$$

For $\boldsymbol{E} = \boldsymbol{a}_\psi E_\psi$,

$$\begin{cases} \dfrac{\partial^2 E_\psi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial E_\psi}{\partial r} + \dfrac{\partial^2 E_\psi}{\partial z^2} + \dfrac{1}{r^2}\dfrac{\partial^2 E_\psi}{\partial \psi^2} + (\varkappa^2 - 1/r^2)\, E_\psi = 0, \\ \dfrac{\partial E_\psi}{\partial \psi} = 0. \end{cases}$$

Because of the second equation, E_ψ is independent of ψ and the Helmholtz equation is

$$\frac{\partial^2 E_\psi}{\partial r^2} + \frac{1}{r}\frac{\partial E_\psi}{\partial r} + \frac{\partial^2 E_\psi}{\partial z^2} + (\varkappa^2 - 1/r^2)\, E_\psi = 0.$$

Separation equations and solutions, for $\boldsymbol{E} = \boldsymbol{a}_\psi E_\psi(r, z)$, are

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + (q^2 - 1/r^2) R = 0, \quad \{24\} \quad R = A\mathscr{J}_1(qr) + B\mathscr{Y}_1(qr).$$

$$\frac{d^2Z}{dz^2} + (\varkappa^2 - q^2) Z = 0, \quad \{04\} \quad Z = A\sin(\varkappa^2 - q^2)^{\frac{1}{2}} z + B\cos(\varkappa^2 - q^2)^{\frac{1}{2}} z.$$

For $\boldsymbol{E} = \boldsymbol{a}_\psi E_\psi(r)$,

$$\frac{d^2E_\psi}{dr^2} + \frac{1}{r}\frac{dE_\psi}{dr} + (\varkappa^2 - 1/r^2) E_\psi = 0, \quad \{24\} \quad E_\psi = A\mathscr{J}_1(\varkappa r) + B\mathscr{Y}_1(\varkappa r).$$

The differential equation in E_z can always be separated to give solutions identical with those for the scalar Helmholtz equation, Table 1.02. Evidently these results apply, irrespective of other components that $\boldsymbol{E}$ may have.

FIG. 1.03. ELLIPTIC-CYLINDER COORDINATES (η, ψ, z)

$$g_{11} = g_{22} = g^{\frac{1}{2}} = a^2(\cosh^2\eta - \cos^2\psi), \qquad g_{33} = 1.$$

The vector Helmholtz equation is obtained from Eqs. (5.01), (5.02), (5.03):

$$\frac{1}{a^2(\cosh^2\eta - \cos^2\psi)}\left\{\frac{\partial^2 E_\eta}{\partial\eta^2} + \frac{\partial^2 E_\eta}{\partial\psi^2}\right\} + \frac{\partial^2 E_\eta}{\partial z^2}$$
$$+ \frac{2}{a^2(\cosh^2\eta - \cos^2\psi)^2}\left\{\cos\psi\sin\psi\frac{\partial E_\psi}{\partial\eta} - \cosh\eta\sinh\eta\frac{\partial E_\psi}{\partial\psi}\right\}$$
$$+ \left\{\varkappa^2 - \frac{5(\cosh^2\eta - \sin^2\psi)}{4a^2(\cosh^2\eta - \cos^2\psi)^2}\right\} E_\eta = 0,$$

$$\frac{1}{a^2(\cosh^2\eta - \cos^2\psi)}\left\{\frac{\partial^2 E_\psi}{\partial\eta^2} + \frac{\partial^2 E_\psi}{\partial\psi^2}\right\} + \frac{\partial^2 E_\psi}{\partial z^2}$$
$$+ \frac{2}{a^2(\cosh^2\eta - \cos^2\psi)^2}\left\{\cosh\eta\sinh\eta\frac{\partial E_\eta}{\partial\psi} - \cos\psi\sin\psi\frac{\partial E_\eta}{\partial\eta}\right\}$$
$$+ \left\{\varkappa^2 - \frac{5(\cosh^2\eta - \sin^2\psi)}{4a^2(\cosh^2\eta - \cos^2\psi)^2}\right\} E_\psi = 0,$$

$$\frac{1}{a^2(\cosh^2\eta - \cos^2\psi)}\left\{\frac{\partial^2 E_z}{\partial\eta^2} + \frac{\partial^2 E_z}{\partial\psi^2}\right\} + \frac{\partial^2 E_z}{\partial z^2} + \varkappa^2 E_z = 0.$$

Since the form of the equation for E_z is exactly the same as for the scalar Helmholtz equation in elliptic-cylinder coordinates, the solutions are given in Table 1.03.

FIG. 1.04. PARABOLIC-CYLINDER COORDINATES (μ, ν, z)

$$g_{11} = g_{22} = g^{\frac{1}{2}} = \mu^2 + \nu^2, \qquad g_{33} = 1.$$

The vector Helmholtz equation is obtained from Eqs. (5.01), (5.02), (5.03):

$$\frac{1}{\mu^2 + \nu^2}\left\{\frac{\partial^2 E_\mu}{\partial\mu^2} + \frac{\partial^2 E_\mu}{\partial\nu^2}\right\} + \frac{\partial^2 E_\mu}{\partial z^2}$$
$$+ \frac{2}{(\mu^2 + \nu^2)^2}\left\{\nu\frac{\partial E_\nu}{\partial\mu} - \mu\frac{\partial E_\nu}{\partial\nu}\right\} + \left\{\varkappa^2 - \frac{1}{(\mu^2 + \nu^2)^2}\right\} E_\mu = 0,$$

$$\frac{1}{\mu^2 + \nu^2}\left\{\frac{\partial^2 E_\nu}{\partial\mu^2} + \frac{\partial^2 E_\nu}{\partial\nu^2}\right\} + \frac{\partial^2 E_\nu}{\partial z^2}$$
$$+ \frac{2}{(\mu^2 + \nu^2)^2}\left\{\mu\frac{\partial E_\mu}{\partial\nu} - \nu\frac{\partial E_\mu}{\partial\mu}\right\} + \left\{\varkappa^2 - \frac{1}{(\mu^2 + \nu^2)^2}\right\} E_\nu = 0,$$

$$\frac{1}{\mu^2 + \nu^2}\left\{\frac{\partial^2 E_z}{\partial\mu^2} + \frac{\partial^2 E_z}{\partial\nu^2}\right\} + \frac{\partial^2 E_z}{\partial z^2} + \varkappa^2 E_z = 0.$$

Solutions for E_z are given in Table 1.04.

TABLE 5.02. ROTATIONAL COORDINATES

FIG. 1.05. SPHERICAL COORDINATES (r, θ, ψ)

$$g_{11}=1, \quad g_{22}=r^2, \quad g_{33}=r^2\sin^2\theta, \quad g^{\frac{1}{2}}=r^2\sin\theta.$$

The vector Helmholtz equation has the three components:

$$\frac{\partial^2 E_r}{\partial r^2}+\frac{2}{r}\frac{\partial E_r}{\partial r}+\frac{1}{r^2}\frac{\partial^2 E_r}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial E_r}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 E_r}{\partial\psi^2}$$
$$-\frac{2}{r^2}\frac{\partial E_\theta}{\partial\theta}-\frac{2\cot\theta}{r^2}E_\theta-\frac{2}{r^2\sin\theta}\frac{\partial E_\psi}{\partial\psi}+(\varkappa^2-2/r^2)\,E_r=0,$$

$$\frac{\partial^2 E_\theta}{\partial r^2}+\frac{2}{r}\frac{\partial E_\theta}{\partial r}+\frac{1}{r^2}\frac{\partial^2 E_\theta}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial E_\theta}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 E_\theta}{\partial\psi^2}$$
$$+\frac{2}{r^2}\frac{\partial E_r}{\partial\theta}-\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial E_\psi}{\partial\psi}+\left(\varkappa^2-\frac{1}{r^2\sin^2\theta}\right)E_\theta=0,$$

$$\frac{\partial^2 E_\psi}{\partial r^2}+\frac{2}{r}\frac{\partial E_\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 E_\psi}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial E_\psi}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 E_\psi}{\partial\psi^2}$$
$$+\frac{2}{r^2\sin\theta}\frac{\partial E_r}{\partial\psi}+\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial E_\theta}{\partial\psi}+\left(\varkappa^2-\frac{1}{r^2\sin^2\theta}\right)E_\psi=0.$$

These equations can be written

$$\begin{cases} \nabla^2 E_r-\dfrac{2}{r^2}\dfrac{\partial E_\theta}{\partial\theta}-\dfrac{2\cot\theta}{r^2}E_\theta-\dfrac{2}{r^2\sin\theta}\dfrac{\partial E_\psi}{\partial\psi}+(\varkappa^2-2/r^2)\,E_r=0,\\ \nabla^2 E_\theta+\dfrac{2}{r^2}\dfrac{\partial E_r}{\partial\theta}-\dfrac{2\cot\theta}{r^2\sin\theta}\dfrac{\partial E_\psi}{\partial\psi}+\left(\varkappa^2-\dfrac{1}{r^2\sin^2\theta}\right)E_\theta=0,\\ \nabla^2 E_\psi+\dfrac{2}{r^2\sin\theta}\dfrac{\partial E_r}{\partial\psi}+\dfrac{2\cot\theta}{r^2\sin\theta}\dfrac{\partial E_\theta}{\partial\psi}+\left(\varkappa^2-\dfrac{1}{r^2\sin^2\theta}\right)E_\psi=0.\end{cases}$$

Solutions

For $\boldsymbol{E}=\boldsymbol{a}_r E_r$,

$$\begin{cases} \nabla^2 E_r+(\varkappa^2-2/r^2)\,E_r=0,\\ \dfrac{\partial E_r}{\partial\theta}=0, \quad \dfrac{\partial E_r}{\partial\psi}=0.\end{cases}$$

Thus, if $\boldsymbol{E}$ is in the radial direction, it must be a function of r alone, and

$$\frac{d^2E_r}{dr^2}+\frac{2}{r}\frac{dE_r}{dr}+[\varkappa^2-2/r^2]\,E_r=0, \quad \{24\} \quad E_r=r^{-\frac{1}{2}}[A\mathscr{I}_{\frac{3}{2}}(\varkappa z)+B\mathscr{I}_{-\frac{3}{2}}(\varkappa z)].$$

For $\boldsymbol{E}=\boldsymbol{a}_\theta E_\theta$,

$$\begin{cases} \nabla^2 E_\theta+\left(\varkappa^2-\dfrac{1}{r^2\sin^2\theta}\right)E_\theta=0,\\ \dfrac{\partial E_\theta}{\partial\theta}+\cot\theta\,E_\theta=0, \quad \dfrac{\partial E_\theta}{\partial\psi}=0.\end{cases}$$

Thus if $\boldsymbol{E}$ is in the θ-direction, it must depend on both r and θ. Separation equations and solutions are

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\varkappa^2R=0, \qquad \{04\} \quad R=\frac{1}{r}[A\sin\varkappa r+B\cos\varkappa r].$$

$$\frac{d\Theta}{d\theta}+(\cot\theta)\,\Theta=0, \quad \Theta=\frac{A}{\sin\theta}.$$

For $\boldsymbol{E} = \boldsymbol{a}_\psi E_\psi$,

$$\begin{cases} \nabla^2 E_\psi + \left(\varkappa^2 - \dfrac{1}{r^2 \sin^2\theta}\right) E_\psi = 0, \\ \dfrac{\partial E_\psi}{\partial \psi} = 0. \end{cases}$$

Thus, if $\boldsymbol{E}$ is in the ψ-direction, the field must be axially symmetric, $\boldsymbol{E} = \boldsymbol{a}_\psi E_\psi(r, \theta)$. If $E_\psi = R(r)\,\Theta(\theta)$, separation equations and solutions are

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left[\varkappa^2 - \frac{p(p+1)}{r^2}\right] R = 0,$$

$$\{24\} \quad R = r^{-\frac{1}{2}}\left[A \mathscr{I}_{p+\frac{1}{2}}(\varkappa r) + B \mathscr{I}_{-(p+\frac{1}{2})}(\varkappa r)\right].$$

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta \frac{d\Theta}{d\theta} + \left[p(p+1) - \frac{1}{\sin^2\theta}\right]\Theta = 0,$$

$$\{222\} \quad \Theta = A \mathscr{P}_p^1(\cos\theta) + B \mathscr{Q}_p^1(\cos\theta).$$

FIG. 1.06. PROLATE SPHEROIDAL COORDINATES (η, θ, ψ)

$$g_{11} = g_{22} = a^2(\sinh^2\eta + \sin^2\theta), \qquad g_{33} = a^2 \sinh^2\eta \sin^2\theta,$$

$$g^{\frac{1}{2}} = a^2(\sinh^2\eta + \sin^2\theta)\sinh\eta\sin\theta.$$

For $\boldsymbol{E}$ independent of ψ, the ψ-component of the vector Helmholtz equations is

$$\frac{\partial^2 E_\psi}{\partial\eta^2} + \coth\eta\frac{\partial E_\psi}{\partial\eta} + \frac{\partial^2 E_\psi}{\partial\theta^2} + \cot\theta\frac{\partial E_\psi}{\partial\theta} + \left[\varkappa^2 a^2(\sinh^2\eta + \sin^2\theta) - \left(\frac{1}{\sinh^2\eta} + \frac{1}{\sin^2\theta}\right)\right] E_\psi = 0.$$

For $E_\psi = H(\eta)\cdot\Theta(\theta)$, separation equations are

$$\begin{cases} \dfrac{d^2H}{d\eta^2} + \coth\eta\dfrac{dH}{d\eta} + \left[\varkappa^2 a^2 \sinh^2\eta - p(p+1) - \dfrac{1}{\sinh^2\eta}\right] H = 0, \\ \dfrac{d^2\Theta}{d\theta^2} + \cot\theta\dfrac{d\Theta}{d\theta} + \left[\varkappa^2 a^2 \sin^2\theta + p(p+1) - \dfrac{1}{\sin^2\theta}\right]\Theta = 0. \end{cases}$$

Solutions are

$$H = A \mathscr{P}_p^1(\varkappa a, \cosh\eta) + B \mathscr{Q}_p^1(\varkappa a, \cosh\eta),$$

$$\Theta = A \mathscr{P}_p^1(\varkappa a, \cos\theta) + B \mathscr{Q}_p^1(\varkappa a, \cos\theta).$$

FIG. 1.07. OBLATE SPHEROIDAL COORDINATES (η, θ, ψ)

$$g_{11} = g_{22} = a^2(\cosh^2\eta - \sin^2\theta), \qquad g_{33} = a^2\cosh^2\eta\sin^2\theta,$$

$$g^{\frac{1}{2}} = a^3(\cosh^2\eta - \sin^2\theta)\cosh\eta\sin\theta.$$

For $\boldsymbol{E}$ independent of ψ, the ψ-component of the vector Helmholtz equation is

$$\frac{\partial^2 E_\psi}{\partial\eta^2} + \tanh\eta\frac{\partial E_\psi}{\partial\eta} + \frac{\partial^2 E_\psi}{\partial\theta^2} + \cot\theta\frac{\partial E_\psi}{\partial\theta} + \left[\varkappa^2 a^2(\cosh^2\eta - \sin^2\theta) + \left(\frac{1}{\cosh^2\eta} - \frac{1}{\sin^2\theta}\right)\right] E_\psi = 0.$$

For $E_\psi = H(\eta)\cdot\Theta(\theta)$, separation equations are

$$\begin{cases} \dfrac{d^2H}{d\eta^2} + \tanh\eta\,\dfrac{dH}{d\eta} + \left[\varkappa^2 a^2\cosh^2\eta - p(p+1) + \dfrac{1}{\cosh^2\eta}\right]H = 0, \\ \dfrac{d^2\Theta}{d\theta^2} + \cot\theta\,\dfrac{d\Theta}{d\theta} + \left[-\varkappa^2 a^2\sin^2\theta + p(p+1) - \dfrac{1}{\sin^2\theta}\right]\Theta = 0. \end{cases}$$

Solutions are

$$H = A\mathscr{P}_p^1(i\varkappa a, i\sinh\eta) + B\mathscr{Q}_p^1(i\varkappa a, i\sinh\eta),$$

$$\Theta = A\mathscr{P}_p^1(i\varkappa a, \cos\theta) + B\mathscr{Q}_p^1(i\varkappa a, \cos\theta).$$

FIG. 1.08. PARABOLIC COORDINATES (μ, ν, ψ)

$$g_{11} = g_{22} = \mu^2 + \nu^2, \qquad g_{33} = \mu^2\nu^2, \qquad g^{\frac{1}{2}} = \mu\nu(\mu^2+\nu^2).$$

For $\boldsymbol{E}$ independent of ψ, the ψ-component of the vector Helmholtz equation is

$$\frac{\partial^2 E_\psi}{\partial\mu^2} + \frac{1}{\mu}\frac{\partial E_\psi}{\partial\mu} + \frac{\partial^2 E_\psi}{\partial\nu^2} + \frac{1}{\nu}\frac{\partial E_\nu}{\partial\nu} + \left[\varkappa^2(\mu^2+\nu^2) - \left(\frac{1}{\mu^2} + \frac{1}{\nu^2}\right)\right]E_\psi = 0.$$

For $E_\psi = M(\mu)\cdot N(\nu)$, separation equations are

$$\begin{cases} \dfrac{d^2M}{d\mu^2} + \dfrac{1}{\mu}\dfrac{dM}{d\mu} + [\varkappa^2\mu^2 - q^2 - 1/\mu^2]\,M = 0, \\ \dfrac{d^2N}{d\nu^2} + \dfrac{1}{\nu}\dfrac{dN}{d\nu} + [\varkappa^2\nu^2 + q^2 - 1/\nu^2]\,N = 0. \end{cases}$$

Solutions are

$$M = A\mathscr{I}_1(\varkappa, q, i\mu) + B\mathscr{Y}_1(\varkappa, q, i\mu),$$

$$N = A\mathscr{I}_1(\varkappa, q, \nu) + B\mathscr{Y}_1(\varkappa, q, \nu).$$

Section VI

DIFFERENTIAL EQUATIONS

Previous sections have treated the separation of the Laplace and Helmholtz equations in 40 coordinate systems. In this section, all the separation equations are tabulated in a systematic manner. Each equation is designated in terms of its singularities in the complex plane, and the general solutions of the differential equations are listed.

A comprehensive view of the separation equations of mathematical physics seems to have been first attempted by FELIX KLEIN [*21*] and MAXIME BÔCHER [*22*] in 1894. Their method was to consider the various separation equations as degenerate cases of a "generalized Lamé equation" with 5 singularities. Unfortunately, this scheme does not seem to cover all the necessary equations. Their work was extended by INCE [*23*], who developed a specification in terms of singularities.

To be satisfactory, any such specification must provide

(a) Different specifications for all equations whose solutions involve different functions,

(b) The same specification for all equations whose solutions are the same.

Since the Klein-Bôcher-Ince method does not satisfy these criteria [*24*], we shall employ a different method of specification.

6.01 BÔCHER EQUATIONS

It is convenient to consider a fairly general form of ordinary differential equation of the second order, called the *Bôcher equation*. All the foregoing separation equations can be written as Bôcher equations or can be obtained from Bôcher equations by transformation of independent or dependent variable. The Bôcher equation is

$$\frac{d^2Z}{dz^2} + P(z)\frac{dZ}{dz} + Q(z)Z = 0, \tag{6.01}$$

where

$$P(z) = \frac{1}{2}\left[\frac{m_1}{z-a_1} + \frac{m_2}{z-a_2} + \cdots + \frac{m_{n-1}}{z-a_{n-1}}\right],$$

$$Q(z) = \frac{1}{4}\left[\frac{\bar{A}_0 + \bar{A}_1 z + \cdots + \bar{A}_l z^l}{(z-a_1)^{m_1}(z-a_2)^{m_2}\ldots(z-a_{n-1})^{m_{n-1}}}\right],$$

and where m_i, n, and l are non-negative integers.

Bôcher equations can be classified in terms of the singularities of $P(z)$ and $Q(z)$. Evidently, Eq. (6.01) has no essential singularities, though both $P(z)$ and

$Q(z)$ have poles at $z = a_1, a_2, \dots a_{n-1}$. The *order of a pole* of the differential equation may be defined as the larger of the two integers representing the order of the pole of P and the order of the pole of Q at $z = a_i$. Usually the poles of Q are of higher (or equal) order compared with those of P. The integers m_i are then taken as the orders of the poles of the equation.

A special case occurs where $(z - a_i)$ is a factor of the numerator of Q. This particular pole is then reduced in order for Q, though it remains unchanged for P. Another degenerate case occurs when the numerator of Q is zero. The poles of the equation are then fixed by P. Usually, however, the orders of the poles of the differential equation are $m_1, m_2, \dots m_{n-1}$.

The above discussion applies to the singularities in the finite z-plane. Generally there is also a pole at infinity. Its order is obtained by taking the larger of the two integers representing the orders of

$$[2z - z^2 P(z)] \quad \text{and} \quad z^4 Q(z)$$

as $z \to \infty$.

6.02 SPECIFICATION

A Bôcher equation may be specified by writing a sequence of integers representing the order of the poles of the differential equation:

$$\{m_1, m_2, m_3, \dots m_n\}.$$

The final integer m_n in this sequence holds a privileged position, since it refers to the pole at $z \to \infty$. The other integers indicate the orders of the poles in the finite z-plane, and they may be arranged in any convenient order [*25*].

The most complicated separation equation found in the preceding sections has four singularities ($n = 4$) and is written

$$\frac{d^2Z}{dz^2} + \frac{1}{2}\left[\frac{1}{z-a_1} + \frac{2}{z-a_2} + \frac{2}{z-a_3}\right]\frac{dZ}{dz} + \frac{1}{4}\left[\frac{\bar{A}_0 + \bar{A}_1 z + \bar{A}_2 z^2 + \bar{A}_3 z^3}{(z-a_1)(z-a_2)^2(z-a_3)^2}\right]Z = 0. \quad (6.02)$$

Evidently, there are three singularities in the finite plane: one first-order pole and two second-order poles. There is also a second-order pole at infinity. Thue the specification is $\{1\,2\,2\,2\}$.

A Bôcher equation with three singularities is

$$\frac{d^2Z}{dz^2} + \frac{1}{2}\left[\frac{2}{z-a_1} + \frac{2}{z-a_2}\right]\frac{dZ}{dz} + \frac{1}{4}\left[\frac{\bar{A}_0 + \bar{A}_2 z^2}{(z-a_1)^2(z-a_2)^2}\right]Z = 0. \quad (6.03)$$

If $a_1 = 1$, $a_2 = -1$, $\bar{A}_0 = 4[p(p+1) - q^2]$, and $\bar{A}_2 = -4p(p+1)$, then

$$\frac{d^2Z}{dz^2} + \frac{1}{2}\left[\frac{2}{z-1} + \frac{2}{z+1}\right]\frac{dZ}{dz} + \left[\frac{p(p+1) - q^2 - p(p+1)z^2}{(z-1)^2(z+1)^2}\right]Z = 0.$$

Second-order poles occur at $z = \pm 1$. At infinity,

$$[2z - z^2 P(z)] \quad \text{is of order } 0,$$

$$z^4 Q(z) \quad \text{is of order } 2,$$

so the pole at ∞ is of order 2 and the equation is designated as $\{222\}$. It is usually written

$$(z^2-1)\frac{d^2Z}{dz^2}+2z\frac{dZ}{dz}-\left[p(p+1)+\frac{q^2}{z^2-1}\right]Z=0 \qquad (6.04)$$

which is *Legendre's equation.*

An example of a differential equation with two singularities is

$$\frac{d^2Z}{dz^2}+\frac{1}{2}\left[\frac{2}{z-a_1}\right]\frac{dZ}{dz}+\frac{1}{4}\left[\frac{\bar{A}_0+\bar{A}_2 z^2}{(z-a_1)^2}\right]Z=0. \qquad (6.05)$$

Evidently the specification is $\{24\}$. If $\bar{A}_0=-4p^2$ and $\bar{A}_2=4q^2$,

$$\frac{d^2Z}{dz^2}+\frac{1}{z-a_1}\frac{dZ}{dz}+\left[\frac{q^2z^2-p^2}{(z-a_1)^2}\right]Z=0 \qquad (6.05\,\text{a})$$

and if $a_1=0$, we have the familiar *Bessel equation,*

$$\frac{d^2Z}{dz^2}+\frac{1}{z}\frac{dZ}{dz}+(q^2-p^2/z^2)Z=0. \qquad (6.05\,\text{b})$$

Equations (6.05), (6.05 a), and (6.05 b) are all designated as $\{24\}$. But if $p=0$ and $a_1=0$ in Eq. (6.05 a), the z's cancel in Q, leaving

$$\frac{d^2Z}{dz^2}+\frac{1}{z}\frac{dZ}{dz}+q^2Z=0. \qquad (6.06)$$

If the cancellation is not performed, the equation apparently remains a $\{24\}$; but as written in (6.06), it is $\{14\}$. The ordinary Bôcher $\{14\}$ is

$$\frac{d^2Z}{dz^2}+\frac{1}{2}\left[\frac{1}{z-a_1}\right]\frac{dZ}{dz}+\frac{1}{4}\left[\frac{\bar{A}_0+\bar{A}_2 z}{z-a_1}\right]Z=0,$$

which differs from Eq. (6.06) by the $\frac{1}{2}$ in the second term. To prevent ambiguity, we designate Eq. (6.06) as $\{14\text{D}\}$, indicating that it is not the ordinary Bôcher $\{14\}$ but is a degenerate form of a higher type of Bôcher equation.

For an equation with one singularity, the specification must still contain two integers to distinguish between a pole at infinity and a pole in the finite z-plane. For instance, the simple differential equation

$$\frac{d^2Z}{dz^2}=0 \qquad (6.07)$$

has no singularity in the finite plane and is specified as $\{01\}$. But the transformation $z=\zeta^{-1}$ gives the differential equation

$$\frac{d^2Z}{d\zeta^2}+\frac{2}{\zeta}\frac{dZ}{d\zeta}=0,$$

which may be designated as $\{10\}$. Lists of differential equations and their specifications are given in Tables 6.01 and 6.02.

6.03 TRANSFORMATIONS

Some of the separation equations of mathematical physics do not appear in Bôcher form; but all of them can be changed into Bôcher equations by suitable transformations of the independent (or the dependent) variable. For $n \geqq 2$, each separation equation has a unique specification, obtained by examination of its Bôcher form. As an example, a separation equation in spherical coordinates is

$$\frac{d^2Z}{d\zeta^2} + \cot\zeta \frac{dZ}{d\zeta} + p(p+1)Z = 0,$$

which is not in Bôcher form and which appears to have an infinite number of singularities. But the transformation $z = \cos\zeta$ gives

$$(z^2-1)\frac{d^2Z}{dz^2} + 2z\frac{dZ}{dz} - p(p+1)Z = 0.$$

Evidently both equations have the designation $\{1\,1\,2\}$.

The only cases where our method of designation leads to ambiguity are in the simplest examples [*25*], such as $\{0\,1\}$ and $\{0\,4\}$. For instance, the separation equation

$$\frac{d^2Z}{d\zeta^2} + \frac{1}{\zeta}\frac{dZ}{d\zeta} - \frac{p^2Z}{\zeta} = 0$$

is a Bôcher equation and might be designated as $\{2\,2\}$. But the transformation $z = -i\ln\zeta$ gives the elementary Bôcher equation

$$\frac{d^2Z}{dz^2} + p^2Z = 0,$$

whose designation is $\{0\,4\}$. In these simple equations, it seems preferable to arbitrarily employ the same designation for all forms that can be obtained by functional transformations (see Table 6.02).

6.04 TABLES

Table 6.01 shows that, of all the differential equations obtained by separation of the Laplace and Helmholtz equations in 40 coordinate systems, there are only 9 distinct types, with some degenerate cases [*26*]. Table 6.02 gives canonical forms of the Bôcher equation for the separation equations needed in this work. Table 6.03 lists all the separation equations and shows how they are obtained from Table 6.02 by functional transformations. Also given are the general solutions of the differential equations. Details of the functions are treated in Section VII.

In only two cases is there an apparent violation of our uniqueness criterion:

(a) Solutions of $\{0\,6\}$ are listed as

$$Z = z^{\frac{1}{2}}[A\mathscr{I}_{\frac{1}{4}}(qz^2/2) + B\mathscr{I}_{-\frac{1}{4}}(qz^2/2)],$$

$$Z = A\mathscr{W}_e(p, qz) + B\mathscr{W}_o(p, qz),$$

$$Z = z^{\frac{1}{2}}[A\mathscr{I}_{\frac{1}{3}}(\varkappa, qz) + B\mathscr{I}_{-\frac{1}{3}}(\varkappa, qz)];$$

(b) Solutions of $\{1\,1\,3\}$ are listed as Baer functions and as Mathieu functions.

It might seem that the specification of the differential equation should be different when it leads to a Bessel function $\mathscr{I}_{\frac{1}{4}}$ and when it leads to a Weber function $\mathscr{W}$. It can be shown, however, that the above Bessel functions are special cases of the Weber functions. Similarly, the Mathieu functions are obtainable from the Baer functions when the singularities of the latter are moved to $a_1 = 0$, $a_2 = 1$. Thus our specifications $\{0\,6\}$ and $\{1\,1\,3\}$ are unambiguous.

TABLE 6.01. CLASSIFICATION OF BÔCHER EQUATIONS

obtained by separation of Laplace and Helmholtz equations in 40 coordinate systems.

No. Singularities n	Name	Original	Degenerate cases
1	Elementary Weber equation	{04} {06}	{01}
2	Elementary Bessel wave equation	{42} {26}	{22D} {14D}, {16D}, {24}
3	Baer wave equation Legendre wave equation	{114} {224}	{113} {112D}, {114D}, {220}, {222}
4	Lamé wave equation Wangerin equation Heine equation	{1113} {1122} {1222}	{1111}, {1112}

D indicates a degenerate form in which a cancellation of z occurs in numerator and denominator of $Q(z)$.

TABLE 6.02. CANONICAL EQUATIONS

occurring in field theory.

Basic form	Degenerate form	Equation
		One singularity
{04}		$\frac{d^2Z}{dz^2}+\frac{1}{2}\left[\frac{m_1}{z-a_1}\right]\frac{dZ}{dz}+\frac{1}{4}\left[\frac{\bar{A}_0}{(z-a_1)^{m_1}}\right]Z=0,$ $m_1=0,\ \bar{A}_0=4p^2$; or $\frac{d^2Z}{dz^2}+p^2Z=0.$
	{01}	$\frac{d^2Z}{dz^2}=0,$ {04} with $p=0$.
{06}		$\frac{d^2Z}{dz^2}+\frac{1}{2}\left[\frac{m_1}{z-a_1}\right]\frac{dZ}{dz}+\frac{1}{4}\left[\frac{\bar{A}_0+\bar{A}_2z^2}{(z-a_1)^{m_1}}\right]Z=0,$ with $m_1=0$. If $\bar{A}_0=4M,\ \bar{A}_2=4N$, the foregoing equation becomes $\frac{d^2Z}{dz^2}+[M+Nz^2]Z=0.$

Table 6.02. Continuation

Basic form	Degenerate form	Equation
{06}		If $M = 0$ and $N = q^2$, $\frac{d^2Z}{dz^2} + q^2 z^2 Z = 0.$
{06}		If $M = q^2(p + \frac{1}{2})$ and $N = -q^4/4$, $\frac{d^2Z}{dz^2} + [q^2(p + \frac{1}{2}) - q^4 z^2/4] Z = 0.$ (Weber equation)
{06}		If $M = q^2$ and $N = \varkappa^2$, $\frac{d^2Z}{dz^2} + [q^2 + \varkappa^2 z^2] Z = 0.$
		Two singularities
{42}		$\frac{d^2Z}{dz^2} + \frac{1}{2}\left[\frac{4}{z - a_1}\right]\frac{dZ}{dz} + \frac{1}{4}\left[\frac{\bar{A}_2 z^2}{(z - a_1)^4}\right] Z = 0.$ If $a_1 = 0$ and $\bar{A}_2 = -4p(p+1)$, the above equation reduces to
	{22D}	$\frac{d^2Z}{dz^2} + \frac{2}{z}\frac{dZ}{dz} - \frac{p(p+1)}{z^2} Z = 0.$
{26}		$\frac{d^2Z}{dz^2} + \frac{1}{2}\left[\frac{2}{z - a_1}\right]\frac{dZ}{dz} + \frac{1}{4}\left[\frac{\bar{A}_0 + \bar{A}_2 z^2 + \bar{A}_4 z^4}{(z - a_1)^2}\right] Z = 0.$ If $a_1 = 0$, $\bar{A}_0 = -4p^2$, $\bar{A}_2 = 4q^2$, $\bar{A}_4 = 4\varkappa^2$, $\frac{d^2Z}{dz^2} + \frac{1}{z}\frac{dZ}{dz} + (\varkappa^2 z^2 + q^2 - p^2/z^2) Z = 0.$ (Bessel wave equation)
	{24}	$\frac{d^2Z}{dz^2} + \frac{1}{z}\frac{dZ}{dz} + (q^2 - p^2/z^2) Z = 0,$ {26} with $\varkappa = 0$. (Bessel equation)
	{16D}	$\frac{d^2Z}{dz^2} + \frac{1}{z}\frac{dZ}{dz} + (\varkappa^2 z^2 + q^2) Z = 0.$ {26} with $p = 0$.
	{14D}	$\frac{d^2Z}{dz^2} + \frac{1}{z}\frac{dZ}{dz} + q^2 Z = 0,$ {26} with $p = 0$ and $\varkappa = 0$.

Table 6.02. Continuation

Basic form	Degenerate form	Equation
		Three singularities
$\{114\}$		$$\frac{d^2Z}{dz^2}+\frac{1}{2}\left[\frac{1}{z-a_1}+\frac{1}{z-a_2}\right]\frac{dZ}{dz}+\frac{1}{4}\left[\frac{\bar{A}_0+\bar{A}_1 z+\bar{A}_2 z^2}{(z-a_1)(z-a_2)}\right]Z=0.$$ If $\bar{A}_0=-4q^2$, $\bar{A}_1=-4p^2$, $\bar{A}_2=4\varkappa^2$, $$(z-a_1)(z-a_2)\frac{d^2Z}{dz^2}+\frac{1}{2}\left[2z-(a_1+a_2)\right]\frac{dZ}{dz}+\left[\varkappa^2 z^2-p^2 z-q^2\right]Z=0.$$ (Baer wave equation)
	$\{113\}$	If $\varkappa=0$, $$(z-a_1)(z-a_2)\frac{d^2Z}{dz^2}+\frac{1}{2}\left[2z-(a_1+a_2)\right]\frac{dZ}{dz}-\left[p^2 z+q^2\right]Z=0.$$ (Baer equation)
	$\{113\}$	In general, Bôcher $\{113\}$ is $$\frac{d^2Z}{dz^2}+\frac{1}{2}\left[\frac{1}{z-a_1}+\frac{1}{z-a_2}\right]\frac{dZ}{dz}+\frac{1}{4}\left[\frac{\bar{A}_0+\bar{A}_1 z}{(z-a_1)(z-a_2)}\right]Z=0.$$ If $a_1=0$, $a_2=+1$, $\bar{A}_0=-(2q+\lambda)$, $\bar{A}_1=4q$, then $$\frac{d^2Z}{dz^2}+\frac{1}{2}\left[\frac{1}{z}+\frac{1}{z-1}\right]\frac{dZ}{dz}+\frac{1}{4}\left[\frac{4qz-(2q+\lambda)}{z(z-1)}\right]Z=0.$$
	$\{113\}$	Substitution of $z=\cos^2\zeta$ gives $$\frac{d^2Z}{d\zeta^2}+(\lambda-2q\cos 2\zeta)Z=0.$$ (Mathieu equation)
$\{224\}$		$$\frac{d^2Z}{dz^2}+\frac{1}{2}\left[\frac{2}{z-a_1}+\frac{2}{z-a_2}\right]\frac{dZ}{dz}+\frac{1}{4}\left[\frac{\bar{A}_0+\bar{A}_1 z+\bar{A}_2 z^2+\bar{A}_3 z^3+\bar{A}_4 z^4}{(z-a_1)^2(z-a_2)^2}\right]Z=0.$$

Table 6.02. Continuation

Basic form	Degenerate form	Equation
		If $a_1 = 1,\ a_2 = -1,\ \bar{A}_0 = 4[\varkappa^2 a^2 + p(p+1) - q^2]$, $\bar{A}_2 = -4[2\varkappa^2 a^2 + p(p+1)]$, $\bar{A}_1 = \bar{A}_3 = 0$, $\bar{A}_4 = 4\varkappa^2 a^2$, $$(z^2-1)\frac{d^2Z}{dz^2} + 2z\frac{dZ}{dz} + \left[\varkappa^2 a^2 (z^2-1) - p(p+1) - \frac{q^2}{z^2-1}\right] Z = 0.$$ (Legendre wave equation)
	{222}	$$(z^2-1)\frac{d^2Z}{dz^2} + 2z\frac{dZ}{dz} - \left[p(p+1) + \frac{q^2}{z^2-1}\right] Z = 0,$$ Legendre wave equation {224} with $\varkappa = 0$. (Legendre equation)
	{220}	$$(z^2-1)\frac{d^2Z}{dz^2} + 2z\frac{dZ}{dz} - \frac{q^2}{z^2-1} Z = 0,$$ {224} with $\varkappa = 0$ and $p = 0$.
	{114D}	$$(z^2-1)\frac{d^2Z}{dz^2} + 2z\frac{dZ}{dz} + [\varkappa^2 a^2 (z^2-1) - p(p+1)] Z = 0,$$ {224} with $q = 0$.
	{112D}	$$(z^2-1)\frac{d^2Z}{dz^2} + 2z\frac{dZ}{dz} - p(p+1) Z = 0,$$ {224} with $\varkappa = 0$ and $q = 0$.

Four singularities

{1113}		$$\frac{d^2Z}{dz^2} + \frac{1}{2}\left[\frac{1}{z-a_1} + \frac{1}{z-a_2} + \frac{1}{z-a_3}\right]\frac{dZ}{dz} + \frac{1}{4}\left[\frac{\bar{A}_0 + \bar{A}_1 z + \bar{A}_2 z^2}{(z-a_1)(z-a_2)(z-a_3)}\right] Z = 0.$$ If $a_1 = 0,\ \bar{A}_0 = (a_2^2 + a_3^2)\, q,\ \bar{A}_1 = -p(p+1),\ \bar{A}_2 = \varkappa^2$, $$\frac{d^2Z}{dz^2} + \frac{1}{2}\left[\frac{1}{z} + \frac{1}{z-a_2} + \frac{1}{z-a_3}\right]\frac{dZ}{dz} + \frac{1}{4}\left[\frac{(a_2^2 + a_3^2)\, q - p(p+1) z + \varkappa^2 z^2}{z(z-a_2)(z-a_3)}\right] Z = 0.$$ (Lamé wave equation)

Table 6.02. Continuation

Basic form	Degenerate form	Equation
	$\{1111\}$	$\frac{d^2Z}{dz^2}+\frac{1}{2}\left[\frac{1}{z-a_1}+\frac{1}{z-a_2}+\frac{1}{z-a_3}\right]\frac{dZ}{dz}+\frac{1}{4}\left[\frac{\bar{A}_0}{(z-a_1)(z-a_2)(z-a_3)}\right]Z=0.$
	$\{1112\}$	$\frac{d^2Z}{dz^2}+\frac{1}{2}\left[\frac{1}{z-a_1}+\frac{1}{z-a_2}+\frac{1}{z-a_3}\right]\frac{dZ}{dz}+\frac{1}{4}\left[\frac{\bar{A}_0+\bar{A}_1 z}{(z-a_1)(z-a_2)(z-a_3)}\right]Z=0.$ (Lamé equation)
$\{1122\}$		$\frac{d^2Z}{dz^2}+\frac{1}{2}\left[\frac{1}{z-a_1}+\frac{1}{z-a_2}+\frac{2}{z-a_3}\right]\frac{dZ}{dz}+\frac{1}{4}\left[\frac{\bar{A}_0+\bar{A}_1 z+\bar{A}_2 z^2}{(z-a_1)(z-a_2)(z-a_3)^2}\right]Z=0.$ (Wangerin equation)
$\{1222\}$		$\frac{d^2Z}{dz^2}+\frac{1}{2}\left[\frac{1}{z-a_1}+\frac{2}{z-a_2}+\frac{2}{z-a_3}\right]\frac{dZ}{dz}+\frac{1}{4}\left[\frac{\bar{A}_0+\bar{A}_1 z+\bar{A}_2 z^2+\bar{A}_3 z^3}{(z-a_1)(z-a_2)^2(z-a_3)^2}\right]Z=0.$ (Heine equation)

TABLE 6.03. THE SEPARATION EQUATIONS

of field theory, for Laplace and Helmholtz equations in 40 coordinate systems.

Standard designation	Transformation	Differential equation and solution
		One singularity
		Canonical form
$\{01\}$		$\frac{d^2Z}{dz^2}=0,\quad Z=A+Bz.$
		Other forms
	$z=1/\zeta$	$\frac{d^2Z}{d\zeta^2}+\frac{2}{\zeta}\frac{dZ}{d\zeta}=0,\quad Z=A+B/\zeta.$
	$z=\ln\zeta$	$\frac{d^2Z}{d\zeta^2}+\frac{1}{\zeta}\frac{dZ}{d\zeta}=0,\quad Z=A+B\ln\zeta.$

Table 6.03. Continuation

Standard designation	Transformation	Differential equation and solution
For corrected version of the upper part of Table 6.03 see page 153a	$z = \ln\cot(\zeta/2)$	$\frac{d^2Z}{d\zeta^2} + \cot\zeta\,\frac{dZ}{d\zeta} = 0,$ $Z = A + B\ln\cot(\zeta/2).$
	$z = \ln\coth(\zeta/2)$	$\frac{d^2Z}{d\zeta^2} + \coth\zeta\,\frac{dZ}{d\zeta} = 0,$ $Z = A + B\ln\coth(\zeta/2).$
	$z = \cot^{-1}(\sinh\zeta)$	$\frac{d^2Z}{d\zeta^2} + \tanh\zeta\,\frac{dZ}{d\zeta} = 0,$ $Z = A + B\cot^{-1}(\sinh\zeta).$
	$z = \mathrm{sn}^{-1}\left(\frac{\zeta}{b}, \frac{b}{c}\right)$	$\frac{d}{d\zeta}\left[(b^2-\zeta^2)^{\frac{1}{2}}(c^2-\zeta^2)^{\frac{1}{2}}\frac{dZ}{d\zeta}\right] = 0$ or $\frac{d}{d\zeta}\left[(\zeta^2-b^2)^{\frac{1}{2}}(\zeta^2-c^2)^{\frac{1}{2}}\frac{dZ}{d\zeta}\right] = 0$ or $\frac{d}{d\zeta}\left[(b^2-\zeta^2)^{\frac{1}{2}}(\zeta^2-c^2)^{\frac{1}{2}}\frac{dZ}{d\zeta}\right] = 0,$ $Z = A + B\,\mathrm{sn}^{-1}\left(\frac{\zeta}{b}, \frac{b}{c}\right).$
	$z = \sin^{-1}\left[\frac{2\zeta-(b+c)}{b-c}\right]$	$\frac{d}{d\zeta}\left[(b-\zeta)^{\frac{1}{2}}(c-\zeta)^{\frac{1}{2}}\frac{dZ}{d\zeta}\right] = 0$ or $\frac{d}{d\zeta}\left[(\zeta-b)^{\frac{1}{2}}(\zeta-c)^{\frac{1}{2}}\frac{dZ}{d\zeta}\right] = 0$ or $\frac{d}{d\zeta}\left[(b-\zeta)^{\frac{1}{2}}(\zeta-c)^{\frac{1}{2}}\frac{dZ}{d\zeta}\right] = 0,$ $Z = A + B\sin^{-1}\left[\frac{2\zeta-(b+c)}{b-c}\right].$
{04}		$\boldsymbol{\frac{d^2Z}{dz^2} + p^2Z = 0,}$ $Z = A\sin pz + B\cos pz.$
	$z = i\zeta$	$\frac{d^2Z}{d\zeta^2} - p^2Z = 0,$ $Z = Ae^{p\zeta} + Be^{-p\zeta}.$
	$z = -i\ln\zeta$	$\frac{d^2Z}{d\zeta^2} + \frac{1}{\zeta}\frac{dZ}{d\zeta} - \frac{p^2Z}{\zeta^2} = 0,$ $Z = A\zeta^p + B\zeta^{-p}.$
	$X = Z/z$	$\frac{d^2X}{dz^2} + \frac{2}{z}\frac{dX}{dz} + p^2X = 0,$ $X = \frac{1}{z}[A\sin pz + B\cos pz].$

Table 6.03. Continuation

Standard designation	Transformation	Differential equation and solution
{06}		$\dfrac{d^2Z}{dz^2}+[M+Nz^2]\,Z=0.$
		If $M=0$ and $N=q^2$,
		$\dfrac{d^2Z}{dz^2}+q^2z^2Z=0,$
		$Z=z^{\frac{1}{2}}[A\mathscr{J}_{\frac{1}{4}}(qz^2/2)+B\mathscr{J}_{-\frac{1}{4}}(qz^2/2)].$
{06}		If $M=q^2(p+\frac{1}{2})$ and $N=-q^4/4$,
		$\dfrac{d^2Z}{dz^2}+[q^2(p+\frac{1}{2})-q^4z^2/4]\,Z=0,$
		$Z=A\mathscr{W}_e(p,qz)+B\mathscr{W}_o(p,qz).$
		(Weber equation)
	$z=i\zeta$	$\dfrac{d^2Z}{d\zeta^2}-[q^2(p+\frac{1}{2})+q^4\zeta^2/4]\,Z=0,$
		$Z=A\mathscr{W}_e(p,iq\zeta)+B\mathscr{W}_o(p,iq\zeta).$
{06}		If $M=q^2$ and $N=\varkappa^2$,
		$\dfrac{d^2Z}{dz^2}+[q^2+\varkappa^2z^2]\,Z=0,$
		$Z=z^{\frac{1}{2}}[A\mathscr{J}_{\frac{1}{2}}(\varkappa,qz)+B\mathscr{J}_{-\frac{1}{2}}(\varkappa,qz)].$
	$z=i\zeta$	$\dfrac{d^2Z}{d\zeta^2}-[q^2-\varkappa^2\zeta^2]\,Z=0,$
		$Z=\sqrt{i\zeta}\,[A\mathscr{J}_{\frac{1}{2}}(\varkappa,iq\zeta)+B\mathscr{J}_{-\frac{1}{2}}(\varkappa,iq\zeta)].$

Two singularities

{22D}		$\dfrac{d^2Z}{dz^2}+\dfrac{2}{z}\dfrac{dZ}{dz}-\dfrac{p(p+1)}{z^2}Z=0,$
		$Z=Az^p+Bz^{-(p+1)}.$
{14D}		$\dfrac{d^2Z}{dz^2}+\dfrac{1}{z}\dfrac{dZ}{dz}+q^2Z=0,$
		$Z=A\mathscr{J}_0(qz)+B\mathscr{Y}_0(qz).$
	$z=i\zeta$	$\dfrac{d^2Z}{d\zeta^2}+\dfrac{1}{\zeta}\dfrac{dZ}{d\zeta}-q^2Z=0,$
		$Z=A\mathscr{J}_0(iq\zeta)+B\mathscr{Y}_0(iq\zeta).$
{16D}		$\dfrac{d^2Z}{dz^2}+\dfrac{1}{z}\dfrac{dZ}{dz}+(\varkappa^2z^2+q^2)\,Z=0,$
		$Z=A\mathscr{J}_0(\varkappa,qz)+B\mathscr{Y}_0(\varkappa,qz).$

Table 6.03. Continuation

Standard designation	Transformation	Differential equation and solution
	$z = i\zeta$	$\frac{d^2Z}{d\zeta^2} + \frac{1}{\zeta}\frac{dZ}{d\zeta} + (\varkappa^2\zeta^2 - q^2)Z = 0,$ $Z = A\mathscr{J}_0(\varkappa, iq\zeta) + B\mathscr{Y}_0(\varkappa, iq\zeta).$
{24}		$\boldsymbol{\frac{d^2Z}{dz^2} + \frac{1}{z}\frac{dZ}{dz} + (q^2 - p^2/z^2)Z = 0},$ $Z = A\mathscr{J}_p(qz) + B\mathscr{J}_{-p}(qz)$ or $Z = A\mathscr{J}_n(qz) + B\mathscr{Y}_n(qz).$ (Bessel equation)
	$z = i\zeta$	$\frac{d^2Z}{d\zeta^2} + \frac{1}{\zeta}\frac{dZ}{d\zeta} - (q^2 + p^2/\zeta^2)Z = 0,$ $Z = A\mathscr{J}_p(iq\zeta) + B\mathscr{J}_{-p}(iq\zeta)$ or $Z = A\mathscr{J}_n(iq\zeta) + B\mathscr{Y}_n(iq\zeta).$
	$Z = Xz^{\frac{1}{2}},$ $s = p - \frac{1}{2}$	$\frac{d^2X}{dz^2} + \frac{2}{z}\frac{dX}{dz} + \left[q^2 - \frac{s(s+1)}{z^2}\right]X = 0,$ $X = z^{-\frac{1}{2}}[A\mathscr{J}_{s+\frac{1}{2}}(qz) + B\mathscr{J}_{-(s+\frac{1}{2})}(qz)].$
{26}		$\boldsymbol{\frac{d^2Z}{dz^2} + \frac{1}{z}\frac{dZ}{dz}}$ $\boldsymbol{+ (\varkappa^2z^2 + q^2 - p^2/z^2)Z = 0},$ $Z = A\mathscr{J}_p(\varkappa, q, z) + B\mathscr{J}_{-p}(\varkappa, q, z)$ or $Z = A\mathscr{J}_n(\varkappa, q, z) + B\mathscr{Y}_n(\varkappa, q, z).$ (Bessel wave equation)
	$z = i\zeta$	$\frac{d^2Z}{d\zeta^2} + \frac{1}{\zeta}\frac{dZ}{d\zeta}$ $+ (\varkappa^2\zeta^2 - q^2 - p^2/\zeta^2)Z = 0,$ $Z = A\mathscr{J}_p(\varkappa, q, i\zeta) + B\mathscr{J}_{-p}(\varkappa, q, i\zeta)$ or $Z = A\mathscr{J}_n(\varkappa, q, i\zeta) + B\mathscr{Y}_n(\varkappa, q, i\zeta).$
		Three singularities
{112D}		$\boldsymbol{(z^2 - 1)\frac{d^2Z}{dz^2} + 2z\frac{dZ}{dz} - p(p+1)Z = 0},$ $Z = A\mathscr{P}_p(z) + B\mathscr{Q}_p(z).$
	$z = \cos\zeta$	$\frac{d^2Z}{d\zeta^2} + \cot\zeta\frac{dZ}{d\zeta} + p(p+1)Z = 0,$ $Z = A\mathscr{P}_p(\cos\zeta) + B\mathscr{Q}_p(\cos\zeta).$

Table 6.03. Continuation

Standard designation	Transformation	Differential equation and solution
	$z = \cosh\zeta$	$\frac{d^2Z}{d\zeta^2} + \coth\zeta\,\frac{dZ}{d\zeta} - p(p+1)Z = 0,$ $Z = A\mathscr{P}_p(\cosh\zeta) + B\mathscr{Q}_p(\cosh\zeta).$ $\frac{d^2Z}{d\zeta^2} + \coth\zeta\,\frac{dZ}{d\zeta} - (p^2 - \frac{1}{4})Z = 0,$ $Z = A\mathscr{P}_{p-\frac{1}{2}}(\cosh\zeta) + B\mathscr{Q}_{p-\frac{1}{2}}(\cosh\zeta).$ $\frac{d^2Z}{d\zeta^2} + \coth\zeta\,\frac{dZ}{d\zeta} + Z/4 = 0,$ $Z = A\mathscr{P}_{-\frac{1}{2}}(\cosh\zeta) + B\mathscr{Q}_{-\frac{1}{2}}(\cosh\zeta).$
	$z = i\sinh\zeta$	$\frac{d^2Z}{d\zeta^2} + \tanh\zeta\,\frac{dZ}{d\zeta} - p(p+1)Z = 0,$ $Z = A\mathscr{P}_p(i\sinh\zeta) + B\mathscr{Q}_p(i\sinh\zeta).$
{113}		$\frac{d^2Z}{dz^2} + \frac{1}{2}\left[\frac{1}{z} + \frac{1}{z-1}\right]\frac{dZ}{dz} + \frac{1}{4}\left[\frac{4qz - (2q+\lambda)}{z(z-1)}\right]Z = 0.$
	$z = \cos^2\zeta$	$\frac{d^2Z}{d\zeta^2} + (\lambda - 2q\cos 2\zeta)Z = 0,$ $Z = A\,\mathrm{ce}_m(\zeta, q) + B\,\mathrm{fe}_m(\zeta, q)$ or $Z = A\,\mathrm{se}_m(\zeta, q) + B\,\mathrm{ge}_m(\zeta, q).$ $\frac{d^2Z}{d\zeta^2} + (\lambda + 2q\cos 2\zeta)Z = 0,$ $Z = A\,\mathrm{ce}_m(\zeta, -q) + B\,\mathrm{fe}_m(\zeta, -q)$ or $Z = A\,\mathrm{se}_m(\zeta, -q) + B\,\mathrm{ge}_m(\zeta, -q).$
	$z = \cosh^2\zeta$	$\frac{d^2Z}{d\zeta^2} - (\lambda - 2q\cosh 2\zeta)Z = 0,$ $Z = A\,\mathrm{ce}_m(i\zeta, q) + B\,\mathrm{fe}_m(i\zeta, q)$ or $Z = A\,\mathrm{se}_m(i\zeta, q) + B\,\mathrm{ge}_m(i\zeta, q).$ $\frac{d^2Z}{d\zeta^2} - (\lambda + 2q\cosh 2\zeta)Z = 0,$ $Z = A\,\mathrm{ce}_m(i\zeta, -q) + B\,\mathrm{fe}_m(i\zeta, -q)$ or $Z = A\,\mathrm{se}_m(i\zeta, -q) + B\,\mathrm{ge}_m(i\zeta, -q).$

Table 6.03. Continuation

Standard designation	Transformation	Differential equation and solution
{113}		$(z-a_1)(z-a_2)\frac{d^2Z}{dz^2} + \frac{1}{2}[2z-(a_1+a_2)]\frac{dZ}{dz} - [p(p+1)z+q(a_1+a_2)]Z=0,$ $Z = A\mathscr{B}_p^q(z) + B\mathscr{C}_p^q(z).$ (Baer equation)
{114}		$(z-a_1)(z-a_2)\frac{d^2Z}{dz^2} + \frac{1}{2}[2z-(a_1+a_2)]\frac{dZ}{dz} + [\varkappa^2z^2-p(p+1)z - q(a_1+a_2)]Z=0,$ $Z = A\mathscr{B}_p^q(\varkappa, z) + B\mathscr{C}_p^q(\varkappa, z).$ (Baer wave equation)
{114D}		$(z^2-1)\frac{d^2Z}{dz^2} + 2z\frac{dZ}{dz} + [\varkappa^2a^2(z^2-1)-p(p+1)]Z=0,$ $Z = A\mathscr{P}_p(\varkappa a, z) + B\mathscr{Q}_p(\varkappa a, z).$
		$(z^2-1)\frac{d^2Z}{dz^2} + 2z\frac{dZ}{dz} - [\varkappa^2a^2(z^2-1)+p(p+1)]Z=0,$ $Z = A\mathscr{P}_p(i\varkappa a, z) + B\mathscr{Q}_p(i\varkappa a, z).$
	$z=\cos\zeta$	$\frac{d^2Z}{d\zeta^2} + \cot\zeta\frac{dZ}{d\zeta} + [\varkappa^2a^2\sin^2\zeta + p(p+1)]Z=0,$ $Z = A\mathscr{P}_p(\varkappa a, \cos\zeta) + B\mathscr{Q}_p(\varkappa a, \cos\zeta).$
	$z=\cos\zeta$	$\frac{d^2Z}{d\zeta^2} + \cot\zeta\frac{dZ}{d\zeta} + [-\varkappa^2a^2\sin^2\zeta + p(p+1)]Z=0,$ $Z = A\mathscr{P}_p(i\varkappa a, \cos\zeta) + B\mathscr{Q}_p(i\varkappa a, \cos\zeta).$
	$z=\cosh\zeta$	$\frac{d^2Z}{d\zeta^2} + \coth\zeta\frac{dZ}{d\zeta} + [\varkappa^2a^2\sinh^2\zeta - p(p+1)]Z=0,$ $Z = A\mathscr{P}_p(\varkappa a, \cosh\zeta) + B\mathscr{Q}_p(\varkappa a, \cosh\zeta).$

Table 6.03. Continuation

Standard designation	Transformation	Differential equation and solution
	$z = i\sinh\zeta$	$\frac{d^2Z}{d\zeta^2} + \tanh\zeta\,\frac{dZ}{d\zeta} + [\varkappa^2 a^2\cosh^2\zeta - p(p+1)]\,Z = 0,$ $Z = A\mathscr{P}_p(i\varkappa a, i\sinh\zeta) + B\mathscr{Q}_p(i\varkappa a, i\sinh\zeta).$
{220}		$(z^2-1)\frac{d^2Z}{dz^2} + 2z\frac{dZ}{dz} - \left(\frac{q^2}{z^2-1}\right)Z = 0,$ $Z = A\mathscr{P}_0^q(z) + B\mathscr{Q}_0^q(z).$
	$z = \cos\zeta$	$\frac{d^2Z}{dz^2} + \cot\zeta\,\frac{dZ}{d\zeta} - \left(\frac{q^2}{\sin^2\zeta}\right)Z = 0,$ $Z = A\mathscr{P}_0^q(\cos\zeta) + B\mathscr{Q}_0^q(\cos\zeta).$
{222}		$(z^2-1)\frac{d^2Z}{dz^2} + 2z\frac{dZ}{dz} - \left[p(p+1) + \frac{q^2}{z^2-1}\right]Z = 0,$ $Z = A\mathscr{P}_p^q(z) + B\mathscr{Q}_p^q(z).$ (Legendre equation)
	$z = \cos\zeta$	$\frac{d^2Z}{d\zeta^2} + \cot\zeta\,\frac{dZ}{dz} + \left[p(p+1) - \frac{q^2}{\sin^2\zeta}\right]Z = 0,$ $Z = A\mathscr{P}_p^q(\cos\zeta) + B\mathscr{Q}_p^q(\cos\zeta).$
	$z = \cosh\zeta$	$\frac{d^2Z}{d\zeta^2} + \coth\zeta\,\frac{dZ}{d\zeta} - \left[p(p+1) + \frac{q^2}{\sinh^2\zeta}\right]Z = 0,$ $Z = A\mathscr{P}_p^q(\cosh\zeta) + B\mathscr{Q}_p^q(\cosh\zeta).$ $\frac{d^2Z}{d\zeta^2} + \coth\zeta\,\frac{dZ}{d\zeta} - \left[(p^2 - \tfrac{1}{4}) + \frac{q^2}{\sinh^2\zeta}\right]Z = 0,$ $Z = A\mathscr{P}_{p-\frac{1}{2}}^q(\cosh\zeta) + B\mathscr{Q}_{p-\frac{1}{2}}^q(\cosh\zeta).$
	$z = i\sinh\zeta$	$\frac{d^2Z}{d\zeta^2} + \tanh\zeta\,\frac{dZ}{d\zeta} + \left[-p(p+1) + \frac{q^2}{\cosh^2\zeta}\right]Z = 0,$ $Z = A\mathscr{P}_p^q(i\sinh\zeta) + B\mathscr{Q}_p^q(i\sinh\zeta).$

Table 6.03. Continuation

Standard designation	Transformation	Differential equation and solution
{224}		$(z^2-1)\frac{d^2Z}{dz^2}+2z\frac{dZ}{dz}+\left[\varkappa^2a^2(z^2-1)-p(p+1)-\frac{q^2}{z^2-1}\right]Z=0.$ $Z=A\mathscr{P}_p^q(\varkappa a,z)+B\mathscr{Q}_p^q(\varkappa a,z).$ (Legendre wave equation)
		$(z^2-1)\frac{d^2Z}{dz^2}+2z\frac{dZ}{dz}-\left[\varkappa^2a^2(z^2-1)+p(p+1)+\frac{q^2}{z^2-1}\right]Z=0,$ $Z=A\mathscr{P}_p^q(i\varkappa a,z)+B\mathscr{Q}_p^q(i\varkappa a,z).$
	$z=\cos\zeta$	$\frac{d^2Z}{d\zeta^2}+\cot\zeta\frac{dZ}{d\zeta}+\left[\varkappa^2a^2\sin^2\zeta+p(p+1)-\frac{q^2}{\sin^2\zeta}\right]Z=0,$ $Z=A\mathscr{P}_p^q(\varkappa a,\cos\zeta)+B\mathscr{Q}_p^q(\varkappa a,\cos\zeta).$
	$z=\cos\zeta$	$\frac{d^2Z}{d\zeta^2}+\cot\zeta\frac{dZ}{d\zeta}+\left[-\varkappa^2a^2\sin^2\zeta+p(p+1)-\frac{q^2}{\sin^2\zeta}\right]Z=0,$ $Z=A\mathscr{P}_p^q(i\varkappa a,\cos\zeta)+B\mathscr{Q}_p^q(i\varkappa a,\cos\zeta).$
	$z=\cosh\zeta$	$\frac{d^2Z}{d\zeta^2}+\coth\zeta\frac{dZ}{d\zeta}+\left[\varkappa^2a^2\sinh^2\zeta-p(p+1)-\frac{q^2}{\sinh^2\zeta}\right]Z=0,$ $Z=A\mathscr{P}_p^q(\varkappa a,\cosh\zeta)+B\mathscr{Q}_p^q(\varkappa a,\cosh\zeta).$
	$z=i\sinh\zeta$	$\frac{d^2Z}{d\zeta^2}+\tanh\zeta\frac{dZ}{d\zeta}+\left[\varkappa^2a^2\cosh^2\zeta-p(p+1)+\frac{q^2}{\cosh^2\zeta}\right]Z=0,$ $Z=A\mathscr{P}_p^q(i\varkappa a,i\sinh\zeta)+B\mathscr{Q}_p^q(i\varkappa a,i\sinh\zeta).$
Four singularities		
{1111}		$\frac{d^2Z}{dz^2}+\frac{1}{2}\left[\frac{1}{z-a_1}+\frac{1}{z-a_2}+\frac{1}{z-a_3}\right]\frac{dZ}{dz}+\frac{1}{4}\left[\frac{\bar{A}_0}{(z-a_1)(z-a_2)(z-a_3)}\right]Z=0.$ If $a_1=0$ and $\bar{A}_0=(a_2^2+a_3^2)\,q$, $Z=A\mathscr{E}_0^q(z^{\frac{1}{2}})+B\mathscr{F}_0^q(z^{\frac{1}{2}}).$

Table 6.03. Continuation

Standard designation	Transformation	Differential equation and solution
	$z=\zeta^2$	$(\zeta^2-a_2^2)(\zeta^2-a_3^2)\frac{d^2Z}{d\zeta^2}+\zeta[2\zeta^2-(a_2^2+a_3^2)]\times\frac{dZ}{d\zeta}+q^2(a_2^2+a_3^2)Z=0,$ $Z=A\mathscr{E}_0^q(\zeta)+B\mathscr{F}_0^q(\zeta).$
{1112}		$\frac{d^2Z}{dz^2}+\frac{1}{2}\left[\frac{1}{z-a_1}+\frac{1}{z-a_2}+\frac{1}{z-a_3}\right]\frac{dZ}{dz}+\frac{1}{4}\left[\frac{\bar{A}_0+\bar{A}_1 z}{(z-a_1)(z-a_2)(z-a_3)}\right]Z=0.$ If $a_1=0$, $\bar{A}_0=(a_2^2+a_3^2)q$, and $\bar{A}_1=-p(p+1)$, $Z=A\mathscr{E}_p^q(z^{\frac{1}{2}})+B\mathscr{F}_p^q(z^{\frac{1}{2}}).$
	$z=\zeta^2$	$(\zeta^2-a_2^2)(\zeta^2-a_3^2)\frac{d^2Z}{d\zeta^2}+\zeta[2\zeta^2-(a_2^2+a_3^2)]\frac{dZ}{d\zeta}+[(a_2^2+a_3^2)q-p(p+1)\zeta^2]Z=0,$ $Z=A\mathscr{E}_p^q(\zeta)+B\mathscr{F}_p^q(\zeta).$ (Lamé equation)
{1113}		$\frac{d^2Z}{dz^2}+\frac{1}{2}\left[\frac{1}{z-a_1}+\frac{1}{z-a_2}+\frac{1}{z-a_3}\right]\frac{dZ}{dz}+\frac{1}{4}\left[\frac{\bar{A}_0+\bar{A}_1 z+\bar{A}_2 z^2}{(z-a_1)(z-a_2)(z-a_3)}\right]Z=0.$ If $a_1=0$, $\bar{A}_0=(a_2^2+a_3^2)q$, $\bar{A}_1=-p(p+1)$, and $\bar{A}_2=\varkappa^2$, $Z=A\mathscr{E}_p^q(\varkappa,z^{\frac{1}{2}})+B\mathscr{F}_p^q(\varkappa,z^{\frac{1}{2}}).$ (Lamé wave equation)
	$z=\zeta^2$	$(\zeta^2-a_2^2)(\zeta^2-a_3^2)\frac{d^2Z}{d\zeta^2}+\zeta[2\zeta^2-(a_2^2+a_3^2)]\frac{dZ}{d\zeta}+[\varkappa^2\zeta^4-p(p+1)\zeta^2+(a_2^2+a_3^2)q]Z=0,$ $Z=A\mathscr{E}_p^q(\varkappa,\zeta)+B\mathscr{F}_p^q(\varkappa,\zeta).$

Table 6.03. Continuation

Standard designation	Transformation	Differential equation and solution
{1122}		$$\frac{d^2Z}{dz^2}+\frac{1}{2}\left[\frac{1}{z-a_1}+\frac{1}{z-a_2}+\frac{2}{z-a_3}\right]\frac{dZ}{dz}+\frac{1}{4}\left[\frac{\bar{A}_0+\bar{A}_1 z+\bar{A}_2 z^2}{(z-a_1)(z-a_2)(z-a_3)^2}\right]Z=0.$$ If $a_1=1,\ a_2=1/k^2,\ a_3=0,\ \bar{A}_0=-\alpha_3/k^2,\ \bar{A}_1=-\alpha_2/k^2,\ \bar{A}_2=1-\alpha_3,$ $$\frac{d^2Z}{dz^2}+\frac{1}{2}\left[\frac{1}{z-1}+\frac{1}{z-1/k^2}+\frac{2}{z}\right]\frac{dZ}{dz}+\frac{1}{4}\left[\frac{-\alpha_3/k^2-\alpha_2 z/k^2+(1-\alpha_3)z^2}{z^2(z-1)(z-1/k^2)}\right]\times Z=0,$$ $$Z=A\mathscr{S}_p^q(k,z)+B\mathscr{T}_p^q(k,z).$$ (Wangerin equation)
	$z=\mathrm{sn}^2\zeta$	$$\frac{d^2Z}{d\zeta^2}+\frac{\mathrm{cn}\,\zeta\,\mathrm{dn}\,\zeta}{\mathrm{sn}\,\zeta}\frac{dZ}{d\zeta}+\left[k^2\,\mathrm{sn}^2\zeta-\alpha_2-\alpha_3\left(k^2\,\mathrm{sn}^2\zeta+\frac{1}{\mathrm{sn}^2\zeta}\right)\right]Z=0.$$ $$Z=A\mathscr{S}_p^q(k,\mathrm{sn}^2\zeta)+B\mathscr{T}_p^q(k,\mathrm{sn}^2\zeta).$$
	$z=\mathrm{cn}^2\zeta$	$$\frac{d^2Z}{d\zeta^2}-\frac{\mathrm{sn}\,\zeta\,\mathrm{dn}\,\zeta}{\mathrm{cn}\,\zeta}\frac{dZ}{d\zeta}+\left[k^2\,\mathrm{sn}^2\zeta-\alpha_2+\alpha_3\left(k^2\,\mathrm{cn}^2\zeta-\frac{k'^2}{\mathrm{cn}^2\zeta}\right)\right]Z=0,$$ If $a_1=1,\ a_2=-(k'/k)^2,\ a_3=0,\ \bar{A}_0=(k'/k)^2,\ \bar{A}_1=(\alpha_2-k^2)/k^2,\ \bar{A}_2=\alpha_3-1,$ $$Z=A\mathscr{S}_p^q(k,\mathrm{cn}^2\zeta)+B\mathscr{T}_p^q(k,\mathrm{cn}^2\zeta).$$
	$z=\mathrm{cn}^2\zeta$	$$\frac{d^2Z}{d\zeta^2}-\frac{\mathrm{sn}\,\zeta\,\mathrm{dn}\,\zeta}{\mathrm{cn}\,\zeta}\frac{dZ}{d\zeta}+\left[-\mathrm{dn}^2\zeta+\alpha_2+\alpha_3\left(k'^2\,\mathrm{cn}^2\zeta-\frac{k^2}{\mathrm{cn}^2\zeta}\right)\right]Z=0,$$ If $a_1=1,\ a_2=-(k'/k)^2,\ a_3=0,\ \bar{A}_0=\alpha_3,\ \bar{A}_1=(k'^2-\alpha_2)/k^2,\ \bar{A}_2=1-\alpha_3(k'/k)^2,$ $$Z=A\mathscr{S}_p^q(k,\mathrm{cn}^2\zeta)+B\mathscr{T}_p^q(k,\mathrm{cn}^2\zeta).$$

Table 6.03. Continuation

Standard designation	Transformation	Differential equation and solution
	$z=\mathrm{dn}^2\zeta$	$$\frac{d^2Z}{d\zeta^2}-\frac{k'^2\,\mathrm{sn}\,\zeta\,\mathrm{cn}\,\zeta}{\mathrm{dn}\,\zeta}\frac{dZ}{d\zeta}+\Big[-\mathrm{dn}^2\zeta+\alpha_2+\alpha_3\Big(\mathrm{dn}^2\zeta+\frac{k^2}{\mathrm{dn}^2\zeta}\Big)\Big]Z=0,$$ If $a_1=1$, $a_2=k^2$, $a_3=0$, $\bar{A}_0=-\alpha_3k^2$, $\bar{A}_1=-\alpha_2$, $\bar{A}_3=1-\alpha_3$ in the original {1122} equation $$Z=A\mathcal{S}_p^q(k,\mathrm{dn}^2\zeta)+B\mathcal{T}_p^q(k,\mathrm{dn}^2\zeta).$$
{1222}		$$\frac{d^2Z}{dz^2}+\frac{1}{2}\Big[\frac{1}{z-a_1}+\frac{2}{z-a_2}+\frac{2}{z-a_3}\Big]\frac{dZ}{dz}+\frac{1}{4}\Big[\frac{\bar{A}_0+\bar{A}_1z+\bar{A}_2z^2+\bar{A}_3z^3}{(z-a_1)(z-a_2)^2(z-a_3)^2}\Big]\times Z=0,$$ $$Z=A\mathcal{U}_p^q(k,z)+B\mathcal{V}_p^q(k,z).$$ (Heine equation) Let $a_1=0$, $a_2=1$, $a_3=1/k^2$, $\bar{A}_0=-\alpha_2/k^2$, $\bar{A}_1=(\alpha_2+2)+\alpha_2/k^2-\alpha_3k'^4/k^2$, $\bar{A}_2=(\alpha_2+2)+2k^2$, $\bar{A}_3=2k^4$.
	$z=\mathrm{sn}^2\zeta$	$$\frac{d^2Z}{d\zeta^2}-\frac{\mathrm{sn}\,\zeta(\mathrm{dn}^2\zeta+k^2\,\mathrm{cn}^2\zeta)}{\mathrm{cn}\,\zeta\,\mathrm{dn}\,\zeta}\frac{dZ}{d\zeta}+\Big[2k^2\,\mathrm{sn}^2\zeta-\alpha_2-\alpha_3\frac{k'^4\,\mathrm{sn}^2\zeta}{\mathrm{cn}^2\zeta\,\mathrm{dn}^2\zeta}\Big]\times Z=0,$$ $$Z=A\mathcal{U}_p^q(k,\mathrm{sn}^2\zeta)+B\mathcal{V}_p^q(k,\mathrm{sn}^2\zeta).$$ Let $a_1=0$, $a_2=1$, $a_3=k^2$, $\bar{A}_0=-\alpha_2k^2$, $\bar{A}_1=(\alpha_2-\alpha_3)+k^2(\alpha_2+2)$, $\bar{A}_2=-(\alpha_2+2)-2k^2$, $\bar{A}_3=2$.
	$z=\mathrm{dn}^2\zeta$	$$\frac{d^2Z}{d\zeta^2}+\frac{\mathrm{dn}\,\zeta(\mathrm{cn}^2\zeta-\mathrm{sn}^2\zeta)}{\mathrm{sn}\,\zeta\,\mathrm{cn}\,\zeta}\frac{dZ}{d\zeta}+\Big[-2\,\mathrm{dn}^2\zeta+\alpha_2+\alpha_3\frac{\mathrm{dn}^2\zeta}{\mathrm{sn}^2\zeta\,\mathrm{cn}^2\zeta}\Big]\times Z=0,$$ $$Z=A\mathcal{U}_p^q(k,\mathrm{dn}^2\zeta)+B\mathcal{V}_p^q(k,\mathrm{dn}^2\zeta).$$

Section VII

FUNCTIONS

The purpose of this section is to provide a summary of the mathematical functions obtained as solutions of the differential equations of field theory. The summary is necessarily incomplete, since many of the functions have never been thoroughly investigated. In particular, almost nothing is known about the properties of the various *wave functions*. Recent tabulation of spheroidal wave functions is a beginning in this direction, but much remains to be done.

Even with the functions that have been studied, the question of *notation* is a troublesome one. With Mathieu functions, for instance, there are almost as many notations as there are investigators. We have tried in such cases to adopt a notation that is modern and logical.

In particular, there seems to be no advantage in employing different symbols for the same function of real and imaginary arguments. Bessel functions of the first kind, for example, are here denoted by $\mathscr{I}_p(z)$ for the complex argument z, and are written $\mathscr{I}_p(iy)$, not $I_p(y)$, for imaginary arguments. Similarly, Mathieu functions of the first kind are written $\mathrm{ce}_m(q, iy)$, $\mathrm{se}_m(q, iy)$ rather than $\mathrm{Ce}_m(q, y)$, $\mathrm{Se}_m(q, y)$. Where an additional parameter enters, as in transition from Laplace to Helmholtz equations, it seems best to retain the basic letter for the function, as when $\mathscr{I}_p(z)$ changes to $\mathscr{I}_p(\varkappa, z)$.

7.01 FUNCTIONS

Many of the differential equations of Section VI are satisfied by elementary functions. The properties of these functions are well-known and will not be considered here. But other differential equations of field theory require Bessel, Legendre, and other functions for their solution. We shall attempt to tabulate useful information on these functions. For additional data, see the references listed in the Bibliography, Section VIII. The Bibliography also refers to numerical tables of the various functions, in so far as such tables are known to the authors.

The functions considered are as follows:

Differential equation	Functions	Differential equation	Functions
	Weber functions		**Bessel functions**
{06}	$\mathscr{W}_e(p, qz)$, $\mathscr{W}_e(p, iqz)$,	{06}	$\mathscr{I}_{\frac{1}{2}}(\varkappa, q, z)$, $\mathscr{I}_{\frac{1}{2}}(\varkappa, q, iz)$,
	$\mathscr{W}_o(p, qz)$, $\mathscr{W}_o(p, iqz)$.		$\mathscr{I}_{-\frac{1}{2}}(\varkappa, q, z)$, $\mathscr{I}_{-\frac{1}{2}}(\varkappa, q, iz)$.

Differential equation	Functions	
{06}	$\mathscr{J}_{\frac{1}{2}}(qz)$,	$\mathscr{J}_{-\frac{1}{2}}(qz)$.
{14}	$\mathscr{J}_0(qz)$,	$\mathscr{J}_0(iqz)$,
	$\mathscr{Y}_0(qz)$,	$\mathscr{Y}_0(iqz)$.
{16}	$\mathscr{J}_0(\varkappa, q, z)$,	$\mathscr{J}_0(\varkappa, q, iz)$,
	$\mathscr{Y}_0(\varkappa, q, z)$,	$\mathscr{Y}_0(\varkappa, q, iz)$.
{24}	$\mathscr{J}_p(qz)$,	$\mathscr{J}_p(iqz)$,
	$\mathscr{J}_{-p}(qz)$,	$\mathscr{J}_{-p}(iqz)$.
	$\mathscr{Y}_n(qz)$,	$\mathscr{Y}_n(iqz)$.
{24}	$\mathscr{J}_{s+\frac{1}{2}}(qz)$,	$\mathscr{J}_{-(s+\frac{1}{2})}(qz)$.
{26}	$\mathscr{J}_p(\varkappa, q, z)$,	$\mathscr{J}_p(\varkappa, q, iz)$,
	$\mathscr{J}_{-p}(\varkappa, q, z)$,	$\mathscr{J}_{-p}(\varkappa, q, iz)$,
	$\mathscr{Y}_n(\varkappa, q, z)$,	$\mathscr{Y}_n(\varkappa, q, iz)$.

Baer functions

{113}	$\mathscr{B}_p^q(z)$,	
	$\mathscr{C}_p^q(z)$,	
{114}	$\mathscr{B}_p^q(\varkappa, z)$,	
	$\mathscr{C}_p^q(\varkappa, z)$.	

Mathieu functions

{113}	$\mathrm{ce}_m(q, z)$,	$\mathrm{ce}_m(q, iz)$,
	$\mathrm{fe}_m(q, z)$,	$\mathrm{fe}_m(q, iz)$,
	$\mathrm{se}_m(q, z)$,	$\mathrm{se}_m(q, iz)$,
	$\mathrm{ge}_m(q, z)$,	$\mathrm{ge}_m(q, iz)$.

Legendre functions

{112}	$\mathscr{P}_p(z)$,	$P_n(z)$, (Polynomials)
	$\mathscr{Q}_p(z)$,	$Q_n(z)$,
	$\mathscr{P}_{p-\frac{1}{2}}(z)$,	
	$\mathscr{Q}_{p-\frac{1}{2}}(z)$,	
	$\mathscr{P}_{-\frac{1}{2}}(z)$,	
	$\mathscr{Q}_{-\frac{1}{2}}(z)$.	

Differential equation	Functions	
{114}	$\mathscr{P}_p(\varkappa a, z)$,	$\mathscr{P}_p(i\varkappa a, z)$,
	$\mathscr{Q}_p(\varkappa a, z)$,	$\mathscr{Q}_p(i\varkappa a, z)$,
		$\mathscr{P}_p(i\varkappa a, iz)$,
		$\mathscr{Q}_p(i\varkappa a, iz)$.
{220}	$\mathscr{P}_0^q(z)$,	
	$\mathscr{Q}_0^q(z)$.	
{222}	$\mathscr{P}_p^q(z)$,	$P_n^m(z)$,
	$\mathscr{Q}_p^q(z)$,	$Q_n^m(z)$,
	$\mathscr{P}_{p-\frac{1}{2}}^q(z)$,	
	$\mathscr{Q}_{p-\frac{1}{2}}^q(z)$.	
{224}	$\mathscr{P}_p^q(\varkappa a, z)$,	$\mathscr{P}_p^q(i\varkappa a, z)$,
	$\mathscr{Q}_p^q(\varkappa a, z)$,	$\mathscr{Q}_p^q(i\varkappa a, z)$,
		$\mathscr{P}_p^q(i\varkappa a, iz)$,
		$\mathscr{Q}_p^q(i\varkappa a, iz)$.

Lamé functions

{1111}	$\mathscr{E}_0^q(z)$,	$E_0^q(z)$,
	$\mathscr{F}_0^q(z)$,	(Polynomials. Applicable in only a few special cases.)
{1112}	$\mathscr{E}_p^q(z)$,	$E_n^q(z)$,
	$\mathscr{F}_p^q(z)$,	(Polynomials)
{1113}	$\mathscr{E}_p^q(\varkappa, z)$,	
	$\mathscr{F}_p^q(\varkappa, z)$.	

Wangerin functions

{1122}	$\mathscr{S}_p^q(k, z)$,	
	$\mathscr{T}_p^q(k, z)$.	

Heine functions

{1222}	$\mathscr{U}_p^q(k, z)$,	
	$\mathscr{V}_p^q(k, z)$	

7.02 SERIES SOLUTIONS

We are interested in the solution of equations of the Bôcher type:

$$\frac{d^2Z}{dz^2} + P(z)\frac{dZ}{dz} + Q(z)Z = 0, \tag{7.01}$$

where $P(z)$ and $Q(z)$ are given in §6.01. The simple equations are usually satisfied by elementary functions; the more complicated equations are handled by use of infinite series.

Suppose that the solution is to be obtained about an *ordinary point*, $z = z_0$, where $P(z)$ and $Q(z)$ are analytic. Assume a solution of the form

$$Z = \sum_{n=0}^{\infty} C_n (z - z_0)^n. \tag{7.02}$$

This series is substituted into the differential equation, and the coefficients C_n are evaluated. If $P(z)$ and $Q(z)$ are polynomials in $(z - z_0)$, they fit nicely into the assumed power series. If P and Q are other functions, they are analytic at z_0 and can be expanded in power series in $(z - z_0)$. In either case, the solution of Eq. (7.01) is obtained by equating the coefficient of each power of $(z - z_0)$ to zero. The series solution converges within the circle whose center is at z_0 and whose radius extends to the nearest singularity of the differential equation.

If the equation has poles in the finite z-plane, one may find it convenient to expand about a *singular point* z_0. Assume a solution

$$Z = (z - z_0)^\beta \sum_{j=0}^{\infty} C_j (z - z_0)^j. \tag{7.03}$$

Equation (7.01) may be written

$$(z - z_0)^2 \frac{d^2Z}{dz^2} + (z - z_0)\,[(z - z_0)\,P(z)]\frac{dZ}{dz} + [(z - z_0)^2\,Q(z)]\,Z = 0. \tag{7.01 a}$$

If $[(z - z_0)\,P(z)]$ and $[(z - z_0)^2\,Q(z)]$ are analytic at z_0, this singularity is said to be a *regular singularity*. Then

$$\left.\begin{aligned} [(z - z_0)\,P(z)] &= \sum_{i=0}^{\infty} A_i (z - z_0)^i, \\ [(z - z_0)^2\,Q(z)] &= \sum_{i=0}^{\infty} B_i (z - z_0)^i. \end{aligned}\right\} \tag{7.04}$$

Substitution of Eqs. (7.03) and (7.04) into (7.01 a) gives

$$\left.\begin{aligned} &C_0 f_0(\beta) = 0, \\ &C_1 f_0(\beta+1) + C_0 f_1(\beta) = 0, \\ &C_2 f_0(\beta+2) + C_1 f_1(\beta+1) + C_0 f_2(\beta) = 0, \\ &\dots\dots\dots\dots\dots\dots \\ &C_j f_0(\beta+j) + C_{j-1} f_1(\beta+j-1) + C_{j-2} f_2(\beta+j-2) + \cdots \\ &\qquad + C_2 f_{j-2}(\beta+2) + C_1 f_{j-1}(\beta+1) + C_0 f_j(\beta) = 0, \\ &\dots\dots\dots\dots\dots\dots \end{aligned}\right\} \tag{7.05}$$

where

$$\left.\begin{aligned} f_0(\beta) &= \beta^2 + (A_0 - 1)\,\beta + B_0, \\ f_j(\beta) &= \beta A_j + B_j, \quad j = 1, 2, 3, \dots \end{aligned}\right\} \tag{7.06}$$

For a non-trivial solution, we have the *indicial equation*,

$$\beta^2 + (A_0 - 1)\beta + B_0 = 0. \tag{7.07}$$

Take the roots of the indicial equation as b_1 and b_2, with $b_1 > b_2$. Then

$$f_0(\beta) = (\beta - b_1)(\beta - b_2). \tag{7.08}$$

The case of equal roots will be considered later (Section 7.03).

A solution of Eq. (7.01) requires first the evaluation of the roots b_1 and b_2. The f's are then obtained from Eq. (7.06), and the coefficients C_n are found from Eq. (7.05). If the roots are distinct and do not differ by an integer, this procedure yields the two series required for the general solution of the second-order differential equation. Eq. (7.05) expresses each coefficient in terms of the preceding ones. Or any coefficient may be obtained directly by use of the determinant $\Delta_j(\beta)$:

$$C_j(\beta) = \frac{(-1)^j C_0 \Delta_j(\beta)}{f_0(\beta+1)\cdot f_0(\beta+2)\ldots f_0(\beta+j)}, \tag{7.09}$$

where $\Delta_0 = 1$ and, for $j \geqq 1$,

$$\Delta_j(\beta) = \begin{vmatrix} f_1(\beta) & f_0(\beta+1) & 0 & 0 & 0\ldots 0 & 0 \\ f_2(\beta) & f_1(\beta+1) & f_0(\beta+2) & 0 & 0\ldots 0 & 0 \\ f_3(\beta) & f_2(\beta+1) & f_1(\beta+2) & f_0(\beta+3) & 0\ldots 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ f_{j-1}(\beta) & f_{j-2}(\beta+1) & f_{j-3}(\beta+2) & \cdots & f_1(\beta+j-2) & f_0(\beta+j-1) \\ f_j(\beta) & \cdots & \cdots & \cdots & f_2(\beta+j-2) & f_1(\beta+j-1) \end{vmatrix}$$

According to Eq. (7.08), the denominator of Eq. (7.09) may be written

$$[(\beta - b_1 + 1)(\beta - b_2 + 1)][(\beta - b_1 + 2)(\beta - b_2 + 2)]\ldots[(\beta - b_1 + j)(\beta - b_2 + j)]$$

$$= \prod_{l=1}^{j}(\beta - b_1 + l)\cdot\prod_{l=1}^{j}(\beta - b_2 + l) = \frac{\Gamma[(\beta - b_1) + j + 1]\cdot\Gamma[(\beta - b_2) + j + 1]}{\Gamma[(\beta - b_1) + 1]\cdot\Gamma[(\beta - b_2) + 1]}.$$

Evidently,

$$\Delta_0 = 1,$$

$$\Delta_1(\beta) = f_1(\beta),$$

$$\Delta_2(\beta) = \begin{vmatrix} f_1(\beta) & f_0(\beta+1) \\ f_2(\beta) & f_1(\beta+1) \end{vmatrix},$$

$$\Delta_3(\beta) = \begin{vmatrix} f_1(\beta) & f_0(\beta+1) & 0 \\ f_2(\beta) & f_1(\beta+1) & f_0(\beta+2) \\ f_3(\beta) & f_2(\beta+1) & f_1(\beta+2) \end{vmatrix}, \quad \text{etc.}$$

If $b_1 - b_2 = k$ is not an integer, the denominator is

$$\frac{j!\,\Gamma(j+1+k)}{\Gamma(1+k)} \text{ if } \beta = b_1 \quad \text{and} \quad \frac{j!\,\Gamma(j+1-k)}{\Gamma(1-k)} \text{ if } \beta = b_2,$$

and

$$\left.\begin{aligned} C_j(b_1) &= \frac{(-1)^j C_0 \Gamma(1+k)\Delta_j(b_1)}{j!\,\Gamma(j+1+k)}, \\ C_j(b_2) &= \frac{(-1)^j C_0 \Gamma(1-k)\Delta_j(b_2)}{j!\,\Gamma(j+1-k)}. \end{aligned}\right\} \tag{7.10}$$

If the roots *differ by an integer*, however, C_j may become infinite at $\beta \to b_2$, since the denominator is then

$$[(1-k)(2-k)(3-k)\dots(k-k)\dots(j-k)]\,j!$$

Thus, when the roots differ by an integer,

$$\left.\begin{aligned} C_j(b_2) &= \frac{(k-j-1)!\,C_0\,\Delta_j(b_2)}{(k-1)!\,j!}, & j<k;\\ C_j(b_2) &= \frac{(-1)^{j-k+1}C_0}{(k-1)!\,(j-k)!\,j!}\left[\frac{\Delta_j(\beta)}{\beta-b_2}\right]_{\beta\to b_2}, & j\geqq k. \end{aligned}\right\} \quad (7.10\text{a})$$

The determinant may possibly contain the factor $(\beta - b_2)$; in which case, $C_j(b_2)$ remains finite. Generally, however, a new quantity is introduced:

$$D_j(\beta) = (\beta-b_2)\cdot C_j(\beta) = \frac{(-1)^j C_0\,\Delta_j(\beta)}{\prod\limits_{l=1}^{k-1}(\beta-b_1+l)\prod\limits_{l=k+1}^{j}(\beta-b_1+l)\prod\limits_{l=1}^{j}(\beta-b_2+l)}. \quad (7.11)$$

When $\beta \to b_2$, Eq. (7.11) reduces to

$$D_j(b_2) = \frac{(-1)^{j-k+1}C_0\,\Delta_j(b_2)}{(k-1)!\,(j-k)!\,j!}. \quad (7.12)$$

Froms Eqs. (7.05) and (7.10),

$$\frac{C_{k+j}(b_2)}{C_k(b_2)} = \frac{C_j(b_1)}{C_0} = \frac{(-1)^j k!\,\Delta_j(b_1)}{j!\,(k+j)!}.$$

But, according to Eq. (7.12),

$$\frac{C_{k+j}(b_2)}{C_k(b_2)} = \frac{(-1)^j k!\,\Delta_{k+j}(b_2)}{j!\,(k+j)!\,\Delta_k(b_2)}.$$

Thus,

$$\Delta_{k+j}(b_2) = \Delta_k(b_2)\cdot\Delta_j(b_1) \quad \text{if } k \text{ is an integer.} \quad (7.13)$$

Also needed are derivatives with respect to β. According to Eq. (7.09),

$$\left.\begin{aligned} \frac{\partial C_j}{\partial\beta} &= \frac{(-1)^j C_0}{\prod\limits_{l=1}^{j}(\beta-b_1+l)\cdot\prod\limits_{l=1}^{j}(\beta-b_2+l)}\\ &\quad\times\left\{\frac{\partial\Delta_j}{\partial\beta} - \Delta_j(\beta)\left[\sum_{l=1}^{j}\frac{1}{\beta-b_1+l} + \sum_{l=1}^{j}\frac{1}{\beta-b_2+l}\right]\right\}. \end{aligned}\right\} \quad (7.14)$$

If $\beta = b_1$,

$$\left[\frac{\partial C_j}{\partial\beta}\right]_{\beta=b_1} = \frac{(-1)^j C_0\,\Gamma(k+1)}{j!\,\Gamma(k+j+1)}\left\{\left[\frac{\partial\Delta_j}{\partial\beta}\right]_{\beta=b_1} - \Delta_j(b_1)\left[\sum_{l=1}^{j}\frac{1}{l} + \sum_{l=1}^{j}\frac{1}{k+l}\right]\right\}. \quad (7.15)$$

Differentiation of Eq. (7.11) gives, when k is an integer,

$$\left.\begin{aligned} \frac{\partial D_j}{\partial\beta} &= \frac{(-1)^j C_0}{\prod\limits_{l=1}^{k-1}(\beta-b_1+l)\prod\limits_{l=k+1}^{j}(\beta-b_1+l)\prod\limits_{l=1}^{j}(\beta-b_2+l)}\\ &\quad\times\left\{\frac{\partial\Delta_j}{\partial\beta} - \Delta_j(\beta)\left[\sum_{l=1}^{k-1}\frac{1}{\beta-b_1+l} + \sum_{l=k+1}^{j}\frac{1}{\beta-b_1+l} + \sum_{l=1}^{j}\frac{1}{\beta-b_2+l}\right]\right\}. \end{aligned}\right\} \quad (7.16)$$

If $\beta \to b_2$ and $j \geqq k$, Eq. (7.16) becomes

$$\left.\begin{aligned}\left[\frac{\partial D_j}{\partial \beta}\right]_{\beta\to b_2} &= \frac{(-1)^{j-k+1}C_0}{(k-1)!\,(j-k)!\,j!}\\ &\times\left\{\left[\frac{\partial \Delta_j}{\partial \beta}\right]_{\beta\to b_2} - \Delta_j(b_2)\left[-\sum_{l=1}^{k-1}\frac{1}{l} + \sum_{l=1}^{j-k}\frac{1}{l} + \sum_{l=1}\frac{1}{l}\right]\right\}.\end{aligned}\right\} \quad (7.17)$$

7.03 THE FROBENIUS METHOD [27]

A solution of a Bôcher equation is to be obtained as a series expansion about a *regular singularity*. Evidently there are three possibilities [28] with respect to the roots of the indicial equation:

I. The roots are distinct and do not differ by an integer.

II. The roots are identical.

III. The roots differ by an integer.

Case I. In I, the procedure of § 7.02 yields two distinct series:

$$\left.\begin{aligned}Z_1 &= C_0\Gamma(1+k)\cdot(z-z_0)^{b_1}\sum_{j=0}^{\infty}\frac{(-1)^j\Delta_j(b_1)}{j!\,\Gamma(j+1+k)}(z-z_0)^j,\\ Z_2 &= C_0\Gamma(1-k)\cdot(z-z_0)^{b_2}\sum_{j=0}^{\infty}\frac{(-1)^j\Delta_j(b_2)}{j!\,\Gamma(j+1-k)}(z-z_0)^j.\end{aligned}\right\} \quad (7.18)$$

The general solution of Eq. (7.01) is therefore

$$Z = A\,Z_1(b_1, z) + B\,Z_2(b_2, z).$$

But in II and III, only the first solution is obtained in this way. A new second solution, independent of Z_1 is found by differentiation [28].

Case II. Consider the case of equal roots. The first solution is given by Eq. (7.18) with $k = 0$:

$$Z_1 = C_0(z-z_0)^{b_1}\sum_{j=0}^{\infty}\frac{(-1)^j\Delta_j(b_1)}{(j!)^2}(z-z_0)^j. \quad (7.18\text{a})$$

For the second solution, assume

$$Z(\beta, z) = C_0(z-z_0)^{\beta} + \sum_{j=1}^{\infty} C_j(\beta)\cdot(z-z_0)^{\beta+j}, \quad (7.19)$$

where β is considered to be a variable and where C_0 is the constant of § 7.02. Thus $Z(\beta, z)$ satisfies the recursion formula, Eq. (7.05), but does not satisfy the indicial equation (7.07). Introduce the operator L:

$$L \equiv (z-z_0)^2\frac{\partial^2}{\partial z^2} + (z-z_0)\left[(z-z_0)\,P(z)\right]\frac{\partial}{\partial z} + \left[(z-z_0)^2\,Q(z)\right].$$

If $\beta \neq b_1$, Eq. (7.08) gives (for equal roots)

$$f_0(\beta) = (\beta - b_1)^2.$$

Thus,

$$L[Z(\beta, z)] = C_0(\beta - b_1)^2 (z - z_0)^\beta \tag{7.20}$$

and

$$\frac{\partial}{\partial\beta} L(Z) = 2C_0(\beta - b_1)(z - z_0)^\beta + C_0(\beta - b_1)^2 (z - z_0)^\beta \ln(z - z_0).$$

In general, $\partial Z/\partial\beta$ does not constitute a solution of the differential equation. But if $\beta \to b_1$, the second solution is

$$Z_2 = \left[\frac{\partial Z}{\partial\beta}\right]_{\beta\to b_1}. \tag{7.21}$$

Differentiation of Eq. (7.19) gives

$$\frac{\partial Z}{\partial\beta} = C_0(z - z_0)^\beta \ln(z - z_0) + \sum_{j=1}^{\infty} \frac{\partial C_j}{\partial\beta} \cdot (z - z_0)^{\beta+j} + \sum_{j=1}^{\infty} C_j(\beta) \cdot (z - z_0)^{\beta+j} \ln(z - z_0).$$

Thus, according to Eq. (7.15),

$$\left[\frac{\partial Z}{\partial\beta}\right]_{\beta\to b_1} = \left[\sum_{j=0}^{\infty} C_j(b_1) \cdot (z - z_0)^{b_1+j}\right] \ln(z - z_0) + C_0 \sum_{j=1}^{\infty} \frac{(-1)^j (z - z_0)^{b_1+j}}{(j!)^2} \left\{\left[\frac{\partial \Delta_j}{\partial\beta}\right]_{\beta\to b_1} - 2\Delta_j(b_1) \sum_{l=1}^{j} \frac{1}{l}\right\}.$$

But the first bracket is Z_1, by Eq. (7.03), so the second solution is

$$\left.\begin{aligned} Z_2 &= Z_1 \ln(z - z_0) \\ &\quad + C_0(z - z_0)^{b_1} \sum_{j=1}^{\infty} \frac{(-1)^j (z - z_0)^j}{(j!)^2} \left\{\left[\frac{\partial \Delta_j}{\partial\beta}\right]_{\beta\to b_1} - 2\Delta_j(b_1) \sum_{l=1}^{j} \frac{1}{l}\right\}. \end{aligned}\right\} \tag{7.22}$$

Case III. For III, the roots of the indicial equation are distinct but differ by an integer. The first solution is

$$Z_1 = C_0 k!(z - z_0)^{b_1} \sum_{j=0}^{\infty} \frac{(-1)^j \Delta_j(b_1)}{j!(k+j)!} (z - z_0)^j. \tag{7.18b}$$

To obtain an independent second solution, let

$$\bar{Z}(\beta, z) = (\beta - b_2)\left[C_0(z - z_0)^\beta + \sum_{j=1}^{k-1} C_j(\beta) \cdot (z - z_0)^{\beta+j}\right] + \sum_{j=k}^{\infty} D_j(\beta) \cdot (z - z_0)^{\beta+j}, \tag{7.23}$$

where $D_j(\beta) = (\beta - b_2) \cdot C_j(\beta)$ as in § 7.02. A relation similar to Eq. (7.20) is

$$L[\bar{Z}(\beta, z)] = C_0(\beta - b_2)^2 (\beta - b_1)(z - z_0)^\beta. \tag{7.24}$$

Also,

$$L\left(\frac{\partial\bar{Z}}{\partial\beta}\right) = \frac{\partial}{\partial\beta}[C_0(\beta - b_2)^2(\beta - b_1)(z - z_0)^\beta].$$

But

$$L\left(\frac{\partial\bar{Z}}{\partial\beta}\right) = 2C_0(\beta - b_2)(\beta - b_1)(z - z_0)^\beta + C_0(\beta - b_2)^2(z - z_0)^\beta + C_0(\beta - b_2)^2(\beta - b_1)(z - z_0)^\beta \ln(z - z_0).$$

The entire expression vanishes when $\beta = b_2$. Thus a second solution is

$$Z_2 = \left[\frac{\partial\bar{Z}}{\partial\beta}\right]_{\beta\to b_2}. \tag{7.25}$$

By differentiation of Eq. (7.23), we obtain

$$\left.\begin{aligned}\frac{\partial \bar{Z}}{\partial \beta} &= \sum_{j=0}^{k-1} C_j(\beta)\cdot(z-z_0)^{\beta+j} + (\beta-b_2)\Bigg[C_0(z-z_0)^{\beta}\cdot\ln(z-z_0)\\ &\quad+\sum_{j=1}^{k-1}\left\{\frac{\partial C_j}{\partial\beta}\cdot(z-z_0)^{\beta+j}+C_j(\beta)\cdot(z-z_0)^{\beta+j}\cdot\ln(z-z_0)\right\}\Bigg]\\ &\quad+\sum_{j=k}^{\infty}\left\{\frac{\partial D_j}{\partial\beta}\cdot(z-z_0)^{\beta+j}+D_j(\beta)\cdot(z-z_0)^{\beta+j}\cdot\ln(z-z_0)\right\}.\end{aligned}\right\} \quad (7.26)$$

As $\beta \to b_2$, the first term becomes

$$\sum_{j=0}^{k-1} C_j(b_2)\cdot(z-z_0)^{b_2+j}.$$

All terms in the square bracket are finite, so $(\beta - b_2)$ reduces this part of Eq. (7.26) to zero. There remain the summations containing $D_j(\beta)$ and its derivative [29]. Substitution of Eqs. (7.10a), (7.12), and (7.17) gives the second solution,

$$\begin{aligned}Z_2 &= \frac{C_0(z-z_0)^{b_2}}{(k-1)!}\sum_{j=0}^{k-1}\frac{(k-j-1)!\,\Delta_j(b_2)}{j!}(z-z_0)^j\\ &\quad+\frac{C_0(z-z_0)^{b_2}}{(k-1)!}\ln(z-z_0)\sum_{j=k}^{\infty}\frac{(-1)^{j-k+1}\Delta_j(b_2)\cdot(z-z_0)^j}{(j-k)!\,j!}\\ &\quad+\frac{C_0(z-z_0)^{b_2}}{(k-1)!}\sum_{j=k}^{\infty}\frac{(-1)^{j-k+1}(z-z_0)^j}{(j-k)!\,j!}\\ &\quad\times\left\{\left[\frac{\partial\Delta_j}{\partial\beta}\right]_{\beta\to b_2}-\Delta_j(b_2)\left[\sum_{l=1}^{j-k}\frac{1}{l}+\sum_{l=1}^{j}\frac{1}{l}-\sum_{l=1}^{k-1}\frac{1}{l}\right]\right\}.\end{aligned}$$

The second summation may be written

$$\frac{C_0(z-z_0)^{b_2+k}}{(k-1)!}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\Delta_{n+k}(b_2)\cdot(z-z_0)^n}{n!\,(n+k)!},$$

or, by Eq. (7.13),

$$\frac{C_0(z-z_0)^{b_1}\Delta_k(b_2)}{(k-1!)}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\Delta_n(b_1)\cdot(z-z_0)^n}{n!\,(n+k)!}.$$

Comparison with Eq. (7.18) shows that

$$\left.\begin{aligned}Z_2 &= -\frac{\Delta_k(b_2)}{k!\,(k-1)!}Z_1\ln(z-z_0)\\ &\quad+\frac{C_0(z-z_0)^{b_2}}{(k-1)!}\sum_{j=0}^{k-1}\frac{(k-j-1)!\,\Delta_j(b_2)}{j!}(z-z_0)^j\\ &\quad+\frac{C_0(z-z_0)^{b_2}}{(k-1)!}\sum_{j=k}^{\infty}\frac{(-1)^{j-k+1}(z-z_0)^j}{(j-k)!\,j!}\\ &\quad\times\left\{\left[\frac{\partial\Delta_j}{\partial\beta}\right]_{\beta\to b_2}-\Delta_j(b_2)\left[\sum_{l=1}^{j-k}\frac{1}{l}+\sum_{l=1}^{j}\frac{1}{l}-\sum_{l=1}^{k-1}\frac{1}{l}\right]\right\}.\end{aligned}\right\} \quad (7.27)$$

7.04 AN EXAMPLE, CASE I

As an example of the methods of obtaining a series solution, consider the *Bessel wave equation* {26}:

$$\frac{d^2Z}{dz^2} + \frac{1}{z}\frac{dZ}{dz} + (\varkappa^2 z^2 + q^2 - p^2/z^2)\,Z = 0. \tag{7.28}$$

The equation has a second-order pole at the origin ($z = 0$) and a sixth-order pole at infinity. An expansion will be obtained about the origin, which is a regular singularity.

As in § 7.02, let

$$Z = z^\beta \sum_{j=0}^{\infty} C_j z^j.$$

Then

$$[z\,P(z)] = 1, \qquad A_0 = 1, \qquad A_i = 0 \quad \text{for } i > 0;$$

$$[z^2 Q(z)] = \varkappa^2 z^4 + q^2 z^2 - p^2,$$

$$B_0 = -p^2, \qquad B_2 = q^2, \qquad B_4 = \varkappa^2,$$

with all other B's equal to zero. The indicial equation is

$$\beta^2 - p^2 = 0,$$

so $b_1 = +p$, $b_2 = -p$, and

$$f_0(\beta) = (\beta - p)(\beta + p).$$

From Eq. (7.06), $f_2 = q^2$, $f_4 = \varkappa^2$, and all other f's are zero. Thus,

$$C_j(\beta) = \frac{(-1)^j C_0 \Delta_j(\beta)}{[1]\,[2]\ldots[j]},$$

where

$$\Delta_j(\beta) = \begin{vmatrix} 0 & [1] & 0 & 0 & \cdots & & 0 \\ q^2 & 0 & [2] & 0 & \cdots & & 0 \\ 0 & q^2 & 0 & [3] & \cdots & & 0 \\ \varkappa^2 & 0 & q^2 & 0 & \cdots & & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & [j-1] \\ 0 & 0 & 0 & 0 & \cdots\ \varkappa^2 & 0\ \ q^2 & 0 \end{vmatrix}, \tag{7.29}$$

and

$$[j] = [(\beta + j)^2 - p^2].$$

This determinant vanishes for odd values of j. For even values, the determinant may be simplified by expanding in terms of the elements [1], [2], etc. Then

$$C_{2m}(\beta) = \frac{(-1)^m C_0 \Delta'_m(\beta)}{[2]\,[4]\ldots[2m]},$$

where

$$\Delta'_m(\beta) = \begin{vmatrix} q^2 & [2] & 0 & 0 & \cdots & & 0 \\ \varkappa^2 & q^2 & [4] & 0 & \cdots & & 0 \\ 0 & \varkappa^2 & q^2 & [6] & \cdots & & 0 \\ 0 & 0 & \varkappa^2 & q^2 & \cdots & & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots\ \varkappa^2 & q^2 & [2(m-1)] \\ 0 & 0 & \cdots & & 0 & \varkappa^2 & q^2 \end{vmatrix}, \tag{7.30}$$

with

$$m = j/2.$$

The principal diagonal consists of elements q^2, the adjacent diagonals contain $\varkappa^2$ and functions of β, and all other elements are zero.

For the first root, $\beta = b_1 = +p$, and

$$[2] = [(\beta + 2)^2 - p^2] = 4(p+1),$$
$$[4] = [(\beta + 4)^2 - p^2] = 8(p+2),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$[2m] = [(\beta + 2m)^2 - p^2] = 4m(p+m),$$

$$C_{2m}(p) = \frac{(-1)^m C_0 \Gamma(p+1)\, \Delta'_m(p)}{2^{2m} m!\, \Gamma(p+m+1)}.$$

Therefore,

$$C_2 = -\frac{C_0 (q/2)^2}{1!(p+1)},$$

$$C_4 = \frac{C_0 (q/2)^4}{2!(p+1)(p+2)} [1 - 4(p+1)\varkappa^2/q^4],$$

$$C_6 = -\frac{C_0 (q/2)^6}{3!(p+1)(p+2)(p+3)} [1 - 4(3p+5)\varkappa^2/q^4], \ldots.$$

For convenience, let

$$C_0 = \frac{(q/2)^p}{\Gamma(p+1)}.$$

The first solution of Eq. (7.28) is then defined as a *Bessel wave function* $\mathscr{I}_p(\varkappa, q, z)$:

$$\left.\begin{aligned} Z_1 = \mathscr{I}_p(\varkappa, q, z) &= \frac{(qz/2)^p}{\Gamma(p+1)} \left\{1 - \frac{(qz/2)^2}{1!(p+1)} + \frac{(qz/2)^4 [1 - 4(p+1)\varkappa^2/q^4]}{2!(p+1)(p+2)} \right. \\ &\quad \left. - \frac{(qz/2)^6 [1 - 4(3p+5)\varkappa^2/q^4]}{3!(p+1)(p+2)(p+3)} + \cdots\right\} \\ &= (qz/2)^p \sum_{m=0}^{\infty} \frac{(-1)^m \Delta'_m(p) \cdot (z/2)^{2m}}{m!\, \Gamma(m+p+1)}, \end{aligned}\right\} \quad (7.31)$$

with $\Delta_0 = 1$, $\Delta_1 = q^2$. Similarly, for the other root of the indicial equation, $\beta = -p$ and

$$\mathscr{I}_{-p}(\varkappa, q, z) = (qz/2)^{-p} \sum_{m=0}^{\infty} \frac{(-1)^m \Delta'_m(-p) \cdot (z/2)^{2m}}{m!\, \Gamma(m-p+1)}. \quad (7.32)$$

The complete solution of Eq. (7.28), for non-integral values of p, is

$$Z = A\mathscr{I}_p(\varkappa, q, z) + B\mathscr{I}_{-p}(\varkappa, q, z). \quad (7.33)$$

If $\varkappa = 0$, Eq. (7.31) reverts to the ordinary Bessel function of the first kind,

$$\left.\begin{aligned} \mathscr{I}_p(qz) &= \frac{(qz/2)^p}{\Gamma(p+1)} \left\{1 - \frac{(qz/2)^2}{1!(p+1)} \right. \\ &\quad \left. + \frac{(qz/2)^4}{2!(p+1)(p+2)} - \frac{(qz/2)^6}{3!(p+1)(p+2)(p+3)} + \cdots\right\} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m (qz/2)^{p+2m}}{m!\, \Gamma(m+p+1)}. \end{aligned}\right\} \quad (7.34)$$

7.05 EXAMPLE, CASE II

With Bessel wave functions, the only case of equal roots occurs when $b_1 = b_2 = 0$. Then $C_0 = 1$ and the first solution is

$$\left.\begin{aligned} Z_1 = \mathscr{I}_0(\varkappa, q, z) &= 1 - \frac{(qz/2)^2}{(1!)^2} + \frac{(qz/2)^4}{(2!)^2}[1 - 4\varkappa^2/q^4] \\ &\qquad - \frac{(qz/2)^6}{(3!)^2}[1 - 20\varkappa^2/q^4] + \cdots \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m \Delta'_m(0) \cdot (z/2)^{2m}}{(m!)^2}. \end{aligned}\right\} \quad (7.31\,\mathrm{a})$$

Since all odd determinants are zero, Eq. (7.22) becomes

$$Z_2 = Z_1 \ln z + \sum_{m=1}^{\infty} \frac{z^{2m}}{(2m)!}\left\{\left[\frac{\partial \Delta_{2m}}{\partial \beta}\right]_{\beta\to 0} - 2\Delta_m(0)\sum_{l=1}^{2m}\frac{1}{l}\right\}. \quad (7.22\,\mathrm{a})$$

But

$$\Delta_{2m}(\beta) = (-1)^m [1][3]\ldots[2m-1]\,\Delta'_m(\beta),$$

$$\frac{\partial \Delta_{2m}}{\partial \beta} = (-1)^m [1][3]\ldots[2m-1]\left\{\frac{\partial \Delta'_m}{\partial \beta} + 2\Delta'_m(\beta)\sum_{l=1}^{m}\frac{1}{\beta + 2l - 1}\right\},$$

$$\begin{aligned} \frac{\partial \Delta'_m}{\partial \beta} &= -\left\{\frac{\partial[2]}{\partial \beta} M_{12}(\beta) + \frac{\partial[4]}{\partial \beta} M_{23}(\beta) + \cdots + \frac{\partial[2(m-1)]}{\partial \beta} M_{(m-1)m}(\beta)\right\} \\ &= -2\left\{(\beta+2) M_{12}(\beta) + (\beta+4) M_{23}(\beta) + \cdots + [\beta + 2(m-1)] M_{(m-1)m}(\beta)\right\}. \end{aligned}$$

Thus, the second solution may be written

$$Z_2 = Z_1 \ln z - \sum_{m=1}^{\infty} \frac{(-1)^m (z/2)^{2m}}{(m!)^2}\left\{4\sum_{l=1}^{m-1} l M_{l(l+1)}(0) + \Delta'_m(0)\sum_{l=1}^{m}\frac{1}{l}\right\}, \quad (7.35)$$

where $M_{l(l+1)}$ is the minor of the corresponding element in the determinant Δ'_m.

Bessel functions of the second kind, however, are usually written in a slightly different form from the above. Let

$$\mathscr{Y}_0(\varkappa, q, z) = \frac{2}{\pi}\left[Z_2 + [\gamma + \ln(q/2)]\,\mathscr{I}_0(\varkappa, q, z)\right]. \quad (7.36)$$

Substitution of Eq. (7.35) gives

$$\left.\begin{aligned} \mathscr{Y}_0(\varkappa, q, z) = \frac{2}{\pi}\Bigg\{&[\gamma + \ln(qz/2)]\,\mathscr{I}_0(\varkappa, q, z) \\ &- \sum_{m=1}^{\infty} \frac{(-1)^m (z/2)^{2m}}{(m!)^2}\left[4\sum_{l=1}^{m-1} l M_{l(l+1)}(0) + \Delta'_m(0)\sum_{l=1}^{m}\frac{1}{l}\right]\Bigg\}. \end{aligned}\right\} \quad (7.37)$$

For the special case of $\varkappa = 0$, Eq. (7.37) reduces to

$$\mathscr{Y}_0(qz) = \frac{2}{\pi}\left\{[\gamma + \ln(qz/2)]\,\mathscr{I}_0(qz) - \sum_{m=1}^{\infty} \frac{(-1)^m (qz/2)^{2m}}{(m!)^2}\sum_{l=1}^{m}\frac{1}{l}\right\}, \quad (7.37\,\mathrm{a})$$

in accordance with the usual formulation of NIELSEN [*30*].

7.06 EXAMPLE, CASE III

If $p = n$, an integer not equal to zero, the second solution of the Bessel wave equation is given by Eq. (7.27) with $b_1 = n$, $b_2 = -n$, $k = 2n$, $j = 2m$:

$$\left.\begin{aligned} Z_2 = &-\frac{\Delta_{2n}(-n)}{(2n)!\,(2n-1)!} Z_1 \ln z \\ &+ \frac{C_0 z^{-n}}{(2n-1)!} \sum_{m=0}^{n-1} \frac{(2n-2m-1)!\,\Delta_{2m}(-n)}{(2m)!} z^{2m} \\ &+ \frac{C_0 z^{-n}}{(2n-1)!} \sum_{m=n}^{\infty} \frac{(-1)^{2m-2n-1} z^{2m}}{(2m-2n)!\,(2m)!} \\ &\times \left\{ \left[\frac{\partial \Delta_{2m}}{\partial \beta}\right]_{\beta \to -n} - \Delta_{2m}(-n) \left[\sum_{l=1}^{2m-2n} \frac{1}{l} + \sum_{l=1}^{2m} \frac{1}{l} - \sum_{l=1}^{2n-1} \frac{1}{l} \right] \right\}. \end{aligned}\right\} \quad (7.27\text{a})$$

But

$$\frac{\Delta_{2n}(-n)}{(2n)!\,(2n-1)!} = \frac{\Delta'_n(-n)}{2^{2n-1} n!\,(n-1)!}. \qquad (7.38)$$

For $m < n$,

$$\frac{(2n-2m-1)!\,\Delta_{2m}(-n)}{(2m)!} = \frac{(2n)!\,(n-m-1)!\,\Delta'_m(-n)}{2^{2m+1} m!\,n!}. \qquad (7.39)$$

For $m > n$,

$$\Delta_{2m}(-n) = (-1)^{m+n} \frac{(2m)!\,(2n)!\,(2m-2n)!}{2^{2m} m!\,n!\,(m-n)!} \Delta'_m(-n). \qquad (7.40)$$

Also,

$$\frac{\partial \Delta_{2m}}{\partial \beta} = (-1)^{m+n} \frac{(2m)!\,(2n)!\,(2m-2n)!}{2^{2m} m!\,n!\,(m-n)!}$$

$$\times \left\{ \frac{\partial \Delta'_m}{\partial \beta} + 2\Delta'_m(\beta) \left[\frac{\beta+1}{[1]} + \frac{\beta+3}{[3]} + \cdots + \frac{\beta+2m-1}{[2m-1]} \right] \right\},$$

$$\left[\frac{\partial \Delta'_m}{\partial \beta}\right]_{\beta \to -n} = 2 \sum_{l=1}^{m-1} (n-2l) M_{l(l+1)},$$

where $M_{l(l+1)}$ is the minor of the corresponding term in the determinant Δ'_m.

Substitution of these relations into Eq. (7.27a) gives

$$\begin{aligned} Z_2 = &-\frac{\Delta'_n(-n)}{2^{2n-1} n!\,(n-1)!} Z_1 \ln z + \frac{C_0 z^{-n}}{(n-1)!} \sum_{m=0}^{n-1} \frac{(n-m-1)!\,\Delta'_m(-n) \cdot (z/2)^{2m}}{m!} \\ &+ \frac{C_0 z^{-n}}{(n-1)!} \sum_{m=n}^{\infty} \frac{(-1)^{m-n-1} (z/2)^{2m}}{m!\,(m-n)!} \\ &\times \left\{ 4 \sum_{l=1}^{m-1} (n-2l) M_{l(l+1)} - \Delta'_m(-n) \left[\sum_{l=1}^{m-n} \frac{1}{l} + \sum_{l=1}^{m} \frac{1}{l} - \sum_{l=1}^{n-1} \frac{1}{l} \right] \right\}. \end{aligned}$$

A change of the second summation so that it begins at $m = 0$, and use of the relation,

$$\Delta'_{m+n}(b_2) = \Delta'_m(b_2) \cdot \Delta'_n(b_1),$$

gives

$$\left.\begin{aligned}
Z_2 = &- \frac{1}{2^{2n-1} n! (n-1)!} \\
&\times \left\{ \left[\ln(qz/2) - \ln(q/2) + 2^{2n-1} \sum_{l=1}^{n-1} \frac{1}{l} \right] \Delta_n'(-n) \cdot \mathscr{I}_n(\varkappa, q, z) \right. \\
&- \tfrac{1}{2} (q/2)^n 2^{2n} z^{-n} \sum_{m=0}^{n-1} \frac{(n-m-1)!\, \Delta_m'(-n) \cdot (z/2)^{2m}}{m!} \\
&+ \tfrac{1}{2} (qz/2)^n \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m}}{m!(m+n)!} \\
&\times \left. \left\{ 4 \sum_{l=1}^{m+n-1} (n-2l) M_{l(l+1)} - \Delta_m'(n) \Delta_n'(-n) \left[\sum_{l=1}^{m} \frac{1}{l} + \sum_{l=1}^{m+n} \frac{1}{l} \right] \right\} \right\}.
\end{aligned}\right\} \quad (7.41)$$

To make this solution more nearly like the familiar Bessel function, we introduce the *Bessel wave function of the second kind*, defined by

$$\mathscr{Y}_n(\varkappa, q, z) = - \frac{2^{2n} n! (n-1)!}{\pi q^{2n}} Z_2 + \frac{2 \mathscr{I}_n(\varkappa, q, z)}{\pi q^{2n}} \left[\gamma + \ln(q/2) - 2^{2n-1} \sum_{l=1}^{n-1} \frac{1}{l} \right].$$

Substitution of Eq. (7.41) results in the second solution,

$$\left.\begin{aligned}
\mathscr{Y}_n(\varkappa, q, z) = &\frac{2}{\pi q^{2n}} \left\{ [\gamma + \ln(qz/2)] \Delta_n'(-n) \mathscr{I}_n(\varkappa, q, z) \right. \\
&- \tfrac{1}{2} (q/2)^n 2^{2n} z^{-n} \sum_{m=0}^{n-1} \frac{(n-m-1)!\, \Delta_m'(-n) \cdot (z/2)^{2m}}{m!} \\
&+ \tfrac{1}{2} (qz/2)^n \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m}}{m!(m+n)!} \\
&\times \left. \left\{ 4 \sum_{l=1}^{m+n-1} (n-2l) M_{l(l+1)} - \Delta_m'(n) \Delta_n'(-n) \left[\sum_{l=1}^{m} \frac{1}{l} + \sum_{l=1}^{m+n} \frac{1}{l} \right] \right\} \right\}.
\end{aligned}\right\} \quad (7.42)$$

If $\varkappa = 0$, Eq. (7.42) reduces to the Bessel function as written by NIELSEN [*30*]:

$$\left.\begin{aligned}
\mathscr{Y}_n(qz) = &\frac{2}{\pi} \left\{ [\gamma + \ln(qz/2)] \mathscr{I}_n(qz) - \tfrac{1}{2} (qz/2)^{-n} \sum_{m=0}^{n-1} \frac{(n-m-1)!\, (qz/2)^{2m}}{m!} \right. \\
&- \left. \tfrac{1}{2} (qz/2)^n \sum_{m=0}^{\infty} \frac{(-1)^m (qz/2)^{2m}}{m!(m+n)!} \left[\sum_{l=1}^{m} \frac{1}{l} + \sum_{l=1}^{m+n} \frac{1}{l} \right] \right\}.
\end{aligned}\right\} \quad (7.42a)$$

7.07 ORTHOGONALITY

Solution of a Bôcher equation of field theory yields two functions $Z_1(\varkappa, p, q, z)$ and $Z_2(\varkappa, p, q, z)$ where p and q are separation constants. A series of these functions is then employed to fit the boundary conditions and thus to obtain a unique solution of a field problem. The evaluation of the coefficients in the series is greatly facilitated if the functions are *orthogonal* on the given interval.

The question of orthogonality could be investigated separately for each function. But it is advantageous to consider the general subject of orthogonality for all the functions obtained in field theory. The STURM-LIOUVILLE [*31*] treatment is presented here in slightly modified form.

The Bôcher equation is

$$\frac{d^2Z}{dz^2} + P(z)\frac{dZ}{dz} + Q(z)Z = 0, \tag{7.43}$$

with general boundary conditions

$$\left.\begin{aligned} z = a, \quad & k_1 Z_i'(a) + k_2 Z_i(a) = 0; \\ z = b, \quad & k_3 Z_i'(b) + k_4 Z_i(b) = 0; \end{aligned}\right\} \tag{7.44}$$

where $Z_i'(a) = (dZ/dz)_{z=a}$, $i = 1, 2$, and the k's are constants. These general conditions include DIRICHLET, NEUMANN and mixed conditions.

Introduce a new quantity u:

$$u(z) = K e^{\int P(z)\,dz}$$

where K is an arbitrary constant, or

$$\frac{1}{u}\frac{du}{dz} = P(z). \tag{7.45}$$

Also let

$$Q(z) = \frac{1}{u(z)}\left[v(z) + f(\lambda)\,w(z)\right]. \tag{7.46}$$

Here λ is an *eigenvalue*, and $w(z)$ is a *weighting function*. If the differential equation contains only one separation constant, say p, then λ is a function of p only, and the various eigenvalues λ_n are expressed in terms of the allowable values p_n that fit the boundary conditions. In this case, the quantities u, v, and w do not contain p. If there are two separation constants, p and q, boundary conditions determine which is to be associated with the eigenvalues and which is to be regarded as a constant. The latter may appear in v and w. Substitution of Eqs. (7.45) and (7.46) into Eq. (7.43) gives

$$\frac{d^2Z}{dz^2} + \left(\frac{1}{u}\frac{du}{dz}\right)\frac{dZ}{dz} + \frac{1}{u}\left[v(z) + f(\lambda)\,w(z)\right]Z = 0$$

or

$$\frac{d}{dz}\left[u\frac{dZ}{dz}\right] + \left[v + f(\lambda)\,w\right]Z = 0. \tag{7.47}$$

Eqs. (7.47) and (7.44) are said to constitute a *Sturm-Liouville system* [*31*].

It is easily proved that the solutions of a Sturm-Liouville system are *orthogonal on an arbitrary interval* (a, b). Take Z_m and Z_n as any two distinct solutions of Eq. (7.43), corresponding to the distinct eigenvalues λ_m and λ_n. Then

$$\left\{\begin{aligned} &\frac{d}{dz}\left[u\frac{dZ_m}{dz}\right] + \left[v + f(\lambda_m)\,w\right]Z_m = 0, \\ &\frac{d}{dz}\left[u\frac{dZ_n}{dz}\right] + \left[v + f(\lambda_n)\,w\right]Z_n = 0. \end{aligned}\right.$$

Multiplication by Z_n and Z_m:

$$\begin{cases} Z_n \dfrac{d}{dz}\left[u\dfrac{dZ_m}{dz}\right] + [v + f(\lambda_m)\,w]\,Z_m Z_n = 0, \\ Z_m \dfrac{d}{dz}\left[u\dfrac{dZ_n}{dz}\right] + [v + f(\lambda_n)\,w]\,Z_m Z_n = 0, \end{cases}$$

and subtraction gives

$$[f(\lambda_m) - f(\lambda_n)]\,wZ_m Z_n = \frac{d}{dz}\,[uZ'_n Z_m - uZ'_m Z_n]. \tag{7.48}$$

Integration over the arbitrary interval (a, b) yields

$$[f(\lambda_m) - f(\lambda_n)]\int_a^b wZ_m Z_n\,dz = u(b)\,[Z'_n(b)\,Z_m(b) - Z'_m(b)\,Z_n(b)] \\ - u(a)\,[Z'_n(a)\,Z_m(a) - Z'_m(a)\,Z_n(a)].$$

Boundary conditions, Eq. (7.44), are now substituted, reducing the bracketed quantities to zero. Thus,

$$\int_a^b wZ_m Z_n\,dz = 0, \tag{7.49}$$

which is the *condition for orthogonality on the interval* (a, b) *with respect to the weighting function* $w(z)$. As indicated by Table 7.01, all the ordinary differential equations of this book, with general boundary conditions of field theory, constitute Sturm-Liouville systems. Consequently, the solutions of these equations can be used to build up orthogonal series of eigenfunctions.

Given a Sturm-Liouville system and an arbitrary function $f(z)$ on the interval (a, b). This function can be expressed in terms of the eigenfunctions Z_n obtained as solutions of the differential equation:

$$f(z) = \sum_{n=0}^{\infty} A_n Z_n(z).$$

Because of orthogonality, with respect to weighting function $w(z)$, on interval (a, b),

$$A_n = \frac{1}{N_n}\int_a^b w(z)\,f(z)\,Z_n(z)\,dz,$$

where the norm is

$$N_n = \int_a^b w(z)\,[Z_n(z)]^2\,dz.$$

There are, of course, restrictions on the functions $f(z)$ that can be handled in this way. But these restrictions are so light that they can be ignored in practically all physical applications. Accordingly, no further attention will be paid to such restrictions. The fitting of boundary values by this method has had a great many practical applications [*32*] in physics and engineering; and the additional coordinate systems now available should allow a still wider field of usefulness.

TABLE 7.01. ORTHOGONALITY

Equation	u	v	w	$f(\lambda)$	λ
{04} $\frac{d^2Z}{dz^2}+p^2Z=0$. Eigenfunctions: $\sin pz$, $\cos pz$.	1	0	1	λ^2	p
{06} Weber $\frac{d^2Z}{dz^2}+[q^2(p+\frac{1}{2})-q^4z^2/4]Z=0$, $\mathscr{W}_e(p,qz)$, $\mathscr{W}_o(p,qz)$.	$1/q^2$	$-q^2z^2/4$	1	$\lambda+\frac{1}{2}$	p
{24} Bessel $\frac{d^2Z}{dz^2}+\frac{1}{z}\frac{dZ}{dz}+(q^2-p^2/z^2)Z=0$, $\mathscr{I}_p(qz)$, $\mathscr{I}_{-p}(qz)$.	z z	$-p^2/z$ q^2	z $1/z$	λ^2 $-\lambda^2$	q p
{26} Bessel wave $\frac{d^2Z}{dz^2}+\frac{1}{z}\frac{dZ}{dz}+(\varkappa^2z^2+q^2-p^2/z^2)Z=0$, $\mathscr{I}_p(\varkappa,q,z)$, $\mathscr{I}_{-p}(\varkappa,q,z)$.	z z	$(\varkappa^2z^3-p^2/z)$ $(\varkappa^2z^2+q^2)z$	z $1/z$	λ^2 $-\lambda^2$	q p

{113} Baer $\frac{d^2Z}{dz^2}+\frac{1}{2}\left[\frac{1}{z-b}+\frac{1}{z-c}\right]\frac{dZ}{dz}$ $-\left[\frac{p(p+1)z+q(b+c)}{(z-b)(z-c)}\right]Z=0,$ $\mathscr{B}_p^q(z), \quad \mathscr{C}_p^q(z).$	$[(z-b)(z-c)]^{\frac{1}{2}}$ $[(z-b)(z-c)]^{\frac{1}{2}}$	$-q(b+c)/u$ $-p(p+1)z/u$	z/u $(b+c)/u$	$-\lambda(\lambda+1)$ $-\lambda$	p q
{114} Baer wave $\frac{d^2Z}{dz^2}+\frac{1}{2}\left[\frac{1}{z-b}+\frac{1}{z-c}\right]\frac{dZ}{dz}$ $+\left[\frac{\varkappa^2z^2-p(p+1)z-q(b+c)}{(z-b)(z-c)}\right]Z=0,$ $\mathscr{B}_p^q(\varkappa,z), \quad \mathscr{C}_p^q(\varkappa,z).$	$[(z-b)(z-c)]^{\frac{1}{2}}$ $[(z-b)(z-c)]^{\frac{1}{2}}$	$\frac{1}{u}[\varkappa^2z^2-q(b+c)]$ $\frac{1}{u}[\varkappa^2z^2-p(p+1)z]$	z/u $(b+c)/u$	$-\lambda(\lambda+1)$ $-\lambda$	p q
{222} Legendre $\frac{d^2Z}{dz^2}+\left(\frac{2z}{z^2-1}\right)\frac{dZ}{dz}-\left[\frac{p(p+1)(z^2-1)+q^2}{(z^2-1)^2}\right]Z=0,$ $\mathscr{P}_p^q(z), \quad \mathscr{Q}_p^q(z).$	z^2-1 z^2-1	$-q^2/u$ $-p(p+1)$	1 $1/u$	$-\lambda(\lambda+1)$ $-\lambda^2$	p q

Table 7.01. Continuation

Equation	u	v	w	$f(\lambda)$	λ
$\{224\}$ Legendre wave $\dfrac{d^2Z}{dz^2}+\left(\dfrac{2z}{z^2-1}\right)\dfrac{dZ}{dz}$ $+\left[\dfrac{\varkappa^2a^2(z^2-1)^2-p(p+1)(z^2-1)-q^2}{(z^2-1)^2}\right]Z=0,$ $\mathscr{P}_p^q(\varkappa a,z),\quad \mathscr{Q}_p^q(\varkappa a,z).$	z^2-1	$\frac{1}{u}[\varkappa^2a^2u^2-q^2]$	1	$-\lambda(\lambda+1)$	p
	z^2-1	$[\varkappa^2a^2u-p(p+1)]$	$1/u$	$-\lambda^2$	q
$\{1112\}$ Lamé $\dfrac{d^2Z}{dz^2}+\dfrac{1}{2}\left[\dfrac{1}{z}+\dfrac{1}{z-b}+\dfrac{1}{z-c}\right]\dfrac{dZ}{dz}$ $+\left[\dfrac{(b^2+c^2)q-p(p+1)z}{z(z-b)(z-c)}\right]Z=0,$ $\mathscr{E}_p^q(z^{\frac{1}{2}}),\quad \mathscr{F}_p^q(z^{\frac{1}{2}}).$	$[z(z-b)(z-c)]^{\frac{1}{2}}$	$(b^2+c^2)q/u$	z/u	$-\lambda(\lambda+1)$	p
	$[z(z-b)(z-c)]^{\frac{1}{2}}$	$-p(p+1)z/u$	$(b^2+c^2)/u$	λ	q
$\{1113\}$ Lamé wave $\dfrac{d^2Z}{dz^2}+\dfrac{1}{2}\left[\dfrac{1}{z}+\dfrac{1}{z-b}+\dfrac{1}{z-c}\right]\dfrac{dZ}{dz}$ $+\left[\dfrac{\varkappa^2z^2-p(p+1)z+(b^2+c^2)q}{z(z-b)(z-c)}\right]Z=0,$ $\mathscr{E}_p^q(\varkappa,z^{\frac{1}{2}}),\quad \mathscr{F}_p^q(\varkappa,z^{\frac{1}{2}}).$	$[z(z-b)(z-c)]^{\frac{1}{2}}$	$\frac{1}{u}[\varkappa^2z^2+(b^2+c^2)q]$	z/u	$-\lambda(\lambda+1)$	p
	$[z(z-b)(z-c)]^{\frac{1}{2}}$	$\frac{1}{u}[\varkappa^2z^2-p(p+1)z]$	$(b^2+c^2)/u$	λ	q

{1122} Wangerin $\frac{d^2Z}{dz^2}+\frac{1}{2}\left[\frac{1}{z-1}+\frac{1}{z-c}+\frac{2}{z}\right]\frac{dZ}{dz}$ $+\frac{1}{4}\left[\frac{-q^2c-p^2cz+(1-q^2)z^2}{z^2(z-1)(z-c)}\right]Z=0,$ $\mathcal{S}_p^q(k,z), \quad \mathcal{T}_p^q(k,z).$ $u=z[(z-1)(z-c)]^{\frac{1}{2}}, \quad f(\lambda)=-\lambda^2, \quad \lambda=p.$ $w(z)=\frac{cz}{4u}, \quad v(z)=\frac{1}{4u}[-q^2c+(1-q^2)z^2].$ Or $f(\lambda)=-\lambda^2, \quad \lambda=q.$ $w(z)=\frac{1}{4u}[z^2+c], \quad v=\frac{z}{4u}[z-p^2c].$					
{1222} Heine $\frac{d^2Z}{dz^2}+\frac{1}{2}\left[\frac{1}{z}+\frac{2}{z-1}+\frac{2}{z-c}\right]\frac{dZ}{dz}$ $+\frac{1}{4}\left[\frac{-p^2c^2+[p^2c(1+c)-q^2(1-c)^2+2c]z}{z(z-1)^2}\right.$ $\left.\frac{-[p^2c+2(1+c)]z^2+2z^3}{\times(z-c)^2}\right]Z=0,$ $\mathcal{U}_p^q(k,z), \quad \mathcal{V}_p^q(k,z).$ $u=z^{\frac{1}{2}}(z-1)(z-c), \quad f(\lambda)=-\lambda^2, \quad \lambda=p.$ $w(z)=\frac{c}{4u}[c-(1+c)z+z^2],$ $v(z)=\frac{z}{4u}[-q^2(1-c)^2+2c-2(1+c)z+2z^2].$ Or $f(\lambda)=-\lambda^2, \quad \lambda=q. \quad w(z)=\frac{z}{4u}(1-c)^2,$ $v(z)=\frac{1}{4u}[-p^2c^2+[p^2c(1+c)+2c]z$ $-[p^2c+2(1+c)]z^2+2z^3].$					

7.08 WEBER FUNCTIONS

As the basic form of the {06} equation, we have, according to Table 6.02,

$$\frac{d^2Z}{dz^2} + [M + Nz^2]\, Z = 0. \tag{7.50}$$

If $M = q^2(p + \frac{1}{2})$ and $N = -q^4/4$, Eq. (7.50) becomes

$$\frac{d^2Z}{dz^2} + [q^2(p + \tfrac{1}{2}) - q^4 z^2/4]\, Z = 0, \tag{7.51}$$

which is *Weber's equation* [*33*].

Expansion in an infinite series about the origin gives

$$\left.\begin{aligned}\mathscr{W}_e(p, qz) &= e^{-(qz)^2/4}\Big\{1 - \frac{p}{2!}(qz)^2 + \frac{p}{4!}(p-2)(qz)^4 \\ &\qquad - \frac{p}{6!}(p-2)(p-4)(qz)^6 + \cdots\Big\} \\ &= e^{-(qz)^2/4}\Big\{1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n!)}p(p-2)(p-4)\ldots(p-2[n-1])(qz)^{2n}\Big\},\end{aligned}\right\} \tag{7.52}$$

$$\left.\begin{aligned}\mathscr{W}_o(p, qz) &= e^{-(qz)^2/4}(qz)\Big\{1 - \frac{1}{3!}(p-1)(qz)^2 + \frac{1}{5!}(p-1)(p-3)(qz)^4 \\ &\qquad - \frac{1}{7!}(p-1)(p-3)(p-5)(qz)^6 + \cdots\Big\} \\ &= e^{-(qz)^2/4}(qz)\Big\{1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}(p-1)(p-3)\ldots \\ &\qquad (p-[2n-1])(qz)^{2n}\Big\}.\end{aligned}\right\} \tag{7.53}$$

Both series are absolutely convergent for the finite z-plane. The first function is even, the second odd. They may be called *Weber functions*, and the general solution of Eq. (7.51) may be written

$$Z = A\mathscr{W}_e(p, qz) + B\mathscr{W}_o(p, qz).$$

For the special case of p an *even integer*,

$$\mathscr{W}_e(p, qz) = (-2)^{p/2}\frac{(p/2)!}{p!}\, e^{-(qz)^2/4} H_p(qz). \tag{7.54}$$

If p is an *odd integer*,

$$\mathscr{W}_o(p, qz) = (-2)^{(p-1)/2}\frac{[(p-1)/2]!}{p!}\, e^{-(qz)^2/4} H_p(qz), \tag{7.55}$$

where $H_p(qz)$ are the *Hermite polynomials* [*34*]:

$$\begin{aligned} H_0 &= 1, \\ H_1 &= qz, \\ H_2 &= (qz)^2 - 1, \\ H_3 &= (qz)^3 - 3(qz), \\ H_4 &= (qz)^4 - 6(qz)^2 + 3, \\ H_5 &= (qz)^5 - 10(qz)^3 + 15(qz), \\ H_6 &= (qz)^6 - 15(qz)^4 + 45(qz)^2 - 15, \ldots. \end{aligned}$$

Except for these special cases, the series does not terminate and solutions are given by the infinite series of Eqs. (7.52) and (7.53).

For the alternative form of the Weber equation,

$$\frac{d^2Z}{dz^2} - [q^2(p+\tfrac{1}{2}) + q^4 z^2/4]\,Z = 0,$$

the solution is

$$Z = A\mathscr{W}_e(p, iqz) + B\mathscr{W}_o(p, iqz).$$

Here

$$\left.\begin{aligned}\mathscr{W}_e(p, iqz) &= e^{(qz)^2/4}\Big\{1 + \frac{p}{2!}(qz)^2 + \frac{p}{4!}(p-2)(qz)^4 \\ &\qquad + \frac{p}{6!}(p-2)(p-4)(qz)^6 + \cdots\Big\} \\ &= e^{(qz)^2/4}\Big\{1 + \sum_{n=1}^{\infty}\frac{1}{(2n)!}p(p-2)(p-4)\ldots(p-2[n-1])(qz)^{2n}\Big\},\end{aligned}\right\} \quad (7.56)$$

$$\left.\begin{aligned}\mathscr{W}_o(p, iqz) &= e^{(qz)^2/4}(iqz)\Big\{1 + \frac{1}{3!}(p-1)(qz)^2 + \frac{1}{5!}(p-1)(p-3)(qz)^4 \\ &\qquad + \frac{1}{7!}(p-1)(p-3)(p-5)(qz)^6 + \cdots\Big\} \\ &= e^{(qz)^2/4}(iqz)\Big\{1 + \sum_{n=1}^{\infty}\frac{1}{(2n+1)!}(p-1)(p-3)\ldots \\ &\qquad (p-[2n-1])(qz)^{2n}\Big\}.\end{aligned}\right\} \quad (7.57)$$

WHITTAKER [35] expresses the solution of the Weber equation in degenerate forms of the confluent hypergeometric function, which he denotes [36] by D:

$$Z = A\,D_p(qz) + B\,D_{-(p+1)}(iqz).$$

There are some advantages, however, in the use of the simple even and odd functions of Eqs. (7.52) and (7.53).

Special values

$$\mathscr{W}_e(p, 0) = 1, \qquad \mathscr{W}_o(p, 0) = 0;$$

$$\mathscr{W}_e(p, \pm\infty) = 0, \quad \mathscr{W}_o(p, \pm\infty) = 0.$$

Asymptotic expansions

$$\mathscr{W}_e(p, qz) \sim (-2)^{p/2}\frac{\Gamma(p/2+1)}{\Gamma(p+1)}e^{-(qz)^2/4}(qz)^p\Big[1 - \frac{p(p-1)}{2(qz)^2} + \frac{p(p-1)(p-2)(p-3)}{2\cdot 4(qz)^4} - \frac{p(p-1)(p-2)(p-3)(p-4)(p-5)}{2\cdot 4\cdot 6(qz)^6} + \cdots\Big],$$

$$\mathscr{W}_o(p, qz) \sim (-2)^{(p-1)/2}\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma(p+1)}e^{-(qz)^2/4}(qz)^p\Big[1 - \frac{p(p-1)}{2(qz)^2} + \frac{p(p-1)(p-2)(p-3)}{2\cdot 4(qz)^4} - \frac{p(p-1)(p-2)(p-3)(p-4)(p-5)}{2\cdot 4\cdot 6(qz)^6} + \cdots\Big].$$

Contour integrals

$$\mathscr{W}_e(p, qz) = \frac{2^{p/2}\, e^{-(qz)^2/4}}{i \sin p\pi\, \Gamma(-p/2)} \int_{\infty}^{0+} \cos qzt\, e^{-t^2/2} (-t)^{-p-1}\, dt,$$

$$\mathscr{W}_o(p, qz) = \frac{2^{(p-1)/2}\, e^{-(qz)^2/4}}{i \sin p\pi\, \Gamma\left(\frac{1-p}{2}\right)} \int_{\infty}^{0+} \sin qzt\, e^{-t^2/2} (-t)^{-p-1}\, dt.$$

Recurrence formulas. For either the even or the odd functions,

$$\mathscr{W}(p+1, qz) = (qz)\, \mathscr{W}(p, qz) - p\, \mathscr{W}(p-1, qz),$$

$$\frac{d\mathscr{W}(p, qz)}{dz} = p\, \mathscr{W}(p-1, qz) - \frac{qz}{2}\, \mathscr{W}(p, qz).$$

Orthogonality. Weber functions are orthogonal on an arbitrary interval (a, b) if boundary conditions are of the form

$$\begin{cases} k_1 \mathscr{W}(p, qa) + k_2 \mathscr{W}'(p, qa) = 0, \\ k_3 \mathscr{W}(p, qb) + k_4 \mathscr{W}'(p, qb) = 0. \end{cases}$$

We have merely a special case of a Sturm-Liouville system (§ 7.07), with $w = 1$ and $\lambda = p$. Therefore,

$$\int_a^b \mathscr{W}(p_m, qz)\, \mathscr{W}(p_n, qz)\, dz = 0, \qquad m \neq n,$$

for either the odd or even functions.

An arbitrary function $f(z)$ can be expanded in terms of Weber functions on any interval (a, b):

$$f(z) = \sum_{m=0}^{\infty} A_m \mathscr{W}(p_m, qz),$$

where

$$A_m = \frac{1}{N_m} \int_a^b f(z)\, \mathscr{W}(p_m, qz)\, dz,$$

$$N_m = \int_a^b [\mathscr{W}(p_m, qz)]^2\, dz.$$

Here all conditions apply to $\mathscr{W}_e$-functions, or $\mathscr{W}_o$-functions, or linear combinations of them.

On the infinite interval, since $\mathscr{W}(p, \pm\infty) = 0$,

$$\int_{-\infty}^{\infty} \mathscr{W}(p_m, qz)\, \mathscr{W}(p_n, qz)\, dz = 0, \qquad m \neq n.$$

Thus

$$f(z) = \sum_{m=0}^{\infty} A_m \mathscr{W}(p_m, qz),$$

$$A_m = \frac{1}{N_m} \int_{-\infty}^{\infty} f(z)\, \mathscr{W}(p_m, qz)\, dz,$$

$$N_m = \int_{-\infty}^{\infty} [\mathscr{W}(p_m, qz)]^2\, dz.$$

In particular if $p = m$,

$$N_m = (2\pi)^{\frac{1}{2}}\, m!$$

7.09 BESSEL FUNCTIONS

The wave equation. The Bessel wave equation {26} is

$$\frac{d^2Z}{dz^2} + \frac{1}{z}\frac{dZ}{dz} + (\varkappa^2 z^2 + q^2 - p^2/z^2) Z = 0. \tag{7.28}$$

The series solution was developed in § 7.04. The general solution of Eq. (7.28) may be written, for $p \neq$ integer,

$$Z = A\mathscr{I}_p(\varkappa, q, z) + B\mathscr{I}_{-p}(\varkappa, q, z), \tag{7.33}$$

where the Bessel wave functions are

$$\mathscr{I}_p(\varkappa, q, z) = (qz/2)^p \sum_{m=0}^{\infty} \frac{(-1)^m \Delta'_m(p) \cdot (z/2)^{2m}}{m!\,\Gamma(m+p+1)}, \tag{7.31}$$

$$\mathscr{I}_{-p}(\varkappa, q, z) = (qz/2)^{-p} \sum_{m=0}^{\infty} \frac{(-1)^m \Delta'_m(-p) \cdot (z/2)^{2m}}{m!\,\Gamma(m-p+1)}, \tag{7.32}$$

and where the determinant Δ'_m is given by Eq. (7.30). These series are valid everywhere in the finite complex plane.

If the differential equation is written

$$\frac{d^2Z}{dz^2} + \frac{1}{z}\frac{dZ}{dz} + (\varkappa^2 z^2 - q^2 - p^2/z^2) Z = 0,$$

the general solution (for $p \neq$ integer) is

$$Z = A\mathscr{I}_p(\varkappa, q, iz) + B\mathscr{I}_{-p}(\varkappa, q, iz).$$

If $p = n$, an integer, $\mathscr{I}_{-p}$ is no longer independent of $\mathscr{I}_p$ and the general solution of Eq. (7.28) is

$$Z = A\mathscr{I}_n(\varkappa, q, z) + B\mathscr{Y}_n(\varkappa, q, z). \tag{7.33a}$$

The Bessel wave function of the second kind may be written

$$\left.\begin{aligned} \mathscr{Y}_n(\varkappa, q, z) = \frac{2}{\pi q^{2n}} \Bigg\{ & [\gamma + \ln(qz/2)]\, \Delta'_n(-n)\, \mathscr{I}_n(\varkappa, q, z) \\ & - \tfrac{1}{2}(q/2)^n 2^{2n} z^{-n} \sum_{m=0}^{n-1} \frac{(n-m-1)!\, \Delta'_m(-n) \cdot (z/2)^{2m}}{m!} \\ & + \tfrac{1}{2}(qz/2)^n \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m}}{m!\,(m+n)!} \\ & \times \left\{ 4 \sum_{l=1}^{m+n-1} (n-2l) M_{l(l+1)} - \Delta'_m(n)\, \Delta'_n(-n) \left[\sum_{l=1}^{m} \frac{1}{l} + \sum_{l=1}^{m+n} \frac{1}{l} \right] \right\} \Bigg\}. \end{aligned}\right\} \tag{7.42}$$

For $n = 0$,

$$\left.\begin{aligned} \mathscr{Y}_0(\varkappa, q, z) = \frac{2}{\pi} \Bigg\{ & [\gamma + \ln(qz/2)]\, \mathscr{I}_0(\varkappa, q, z) \\ & - \sum_{m=1}^{\infty} \frac{(-1)^m (z/2)^{2m}}{(m!)^2} \left[4 \sum_{l=1}^{m-1} l M_{l(l+1)}(0) + \Delta'_m(0) \sum_{l=1}^{m} \frac{1}{l} \right] \Bigg\}. \end{aligned}\right\} \tag{7.37}$$

Degenerate cases. The other Bessel functions [37] listed in § 7.01 are all degenerate forms of the foregoing. All series converge within a circle of infinite radius in the z-plane.

$$\mathcal{J}_p(qz) = \sum_{m=0}^{\infty} \frac{(-1)^m (qz/2)^{p+2m}}{m!\,\Gamma(m+p+1)}. \tag{7.34}$$

$$\mathcal{J}_{-p}(qz) = \sum_{m=0}^{\infty} \frac{(-1)^m (qz/2)^{-p+2m}}{m!\,\Gamma(m-p+1)}. \tag{7.34a}$$

These functions are shown in Fig. 7.01.

$$\left.\begin{aligned}\mathcal{Y}_n(qz) = \frac{2}{\pi}\Bigg\{&[\gamma + \ln(qz/2)]\,\mathcal{J}_n(qz) - \tfrac{1}{2}(qz/2)^{-n}\sum_{m=0}^{n-1}\frac{(n-m-1)!\,(qz/2)^{2m}}{m!}\\ &- \tfrac{1}{2}(qz/2)^{n}\sum_{m=0}^{\infty}\frac{(-1)^m (qz/2)^{2m}}{m!\,(m+n)!}\left[\sum_{l=1}^{m}\frac{1}{l} + \sum_{l=1}^{m+n}\frac{1}{l}\right]\Bigg\}.\end{aligned}\right\} \tag{7.42a}$$

$$\mathcal{J}_p(iqz) = \sum_{m=0}^{\infty} \frac{(-1)^m (iqz/2)^{p+2m}}{m!\,\Gamma(m+p+1)}. \tag{7.34b}$$

$$\mathcal{J}_{-p}(iqz) = \sum_{m=0}^{\infty} \frac{(-1)^m (iqz/2)^{-p+2m}}{m!\,\Gamma(m-p+1)}. \tag{7.34c}$$

$$\left.\begin{aligned}\mathcal{Y}_n(iqz) = \frac{2}{\pi}\Bigg\{&[\gamma + \ln(iqz/2)]\,\mathcal{J}_n(iqz)\\ &- \tfrac{1}{2}(iqz/2)^{-n}\sum_{m=0}^{n-1}\frac{(n-m-1)!\,(iqz/2)^{2m}}{m!}\\ &- \tfrac{1}{2}(iqz/2)^{n}\sum_{m=0}^{\infty}\frac{(-1)^m (iqz/2)^{2m}}{m!\,(m+n)!}\left[\sum_{l=1}^{m}\frac{1}{l} + \sum_{l=1}^{m+n}\frac{1}{l}\right]\Bigg\}.\end{aligned}\right| \tag{7.42b}$$

$$\mathcal{J}_0(qz) = 1 - \frac{(qz/2)^2}{(1!)^2} + \frac{(qz/2)^4}{(2!)^2} - \frac{(qz/2)^6}{(3!)^2} + \cdots + (-1)^n \frac{(qz/2)^{2n}}{(n!)^2} + \cdots.$$

A plot of this function is given in Fig. 7.02.

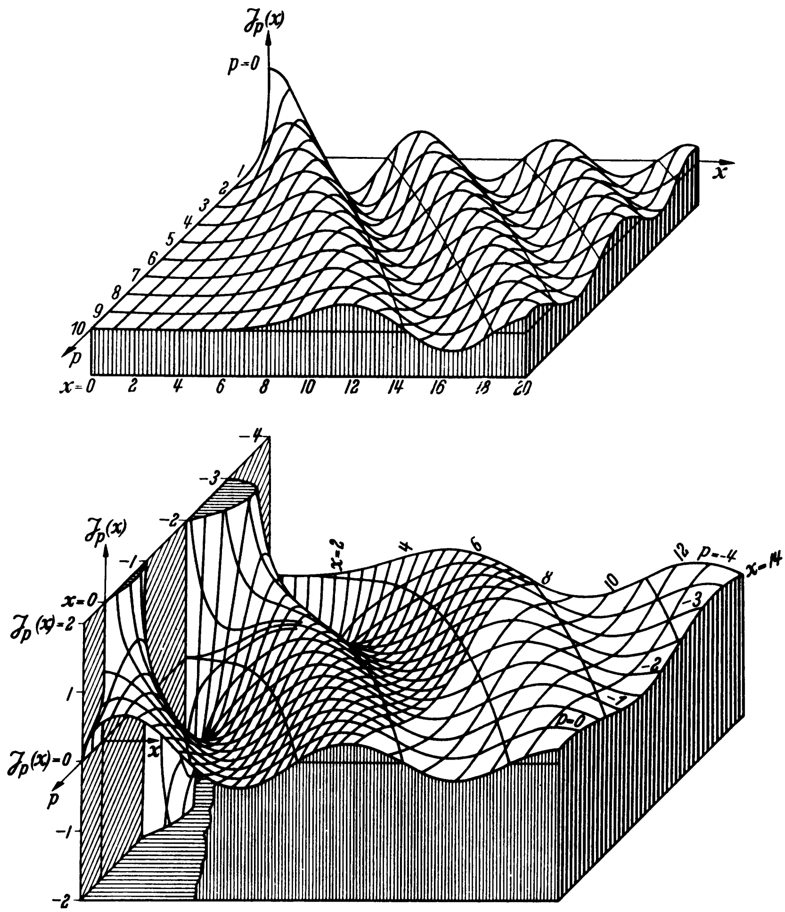
$$\begin{aligned}\mathcal{Y}_0(qz) &= \frac{2}{\pi}[\gamma + \ln(qz/2)]\,\mathcal{J}_0(qz)\\ &+ \frac{2}{\pi}\left\{\frac{(qz/2)^2}{(1!)^2} - \frac{(qz/2)^4}{(2!)^2}(1+\tfrac{1}{2}) + \frac{(qz/2)^6}{(3!)^2}(1+\tfrac{1}{2}+\tfrac{1}{3}) - \cdots\right\}.\end{aligned}$$

See Fig. 7.03 for the $\mathcal{Y}_0$-function and Fig. 7.04 for $\mathcal{J}_{\frac{1}{2}}$ and $\mathcal{J}_{\frac{3}{2}}$.

$$\mathcal{J}_0(iqz) = 1 + \frac{(qz/2)^2}{(1!)^2} + \frac{(qz/2)^4}{(2!)^2} + \frac{(qz/2)^6}{(3!)^2} + \cdots.$$

$$\begin{aligned}\mathcal{Y}_0(iqz) &= \frac{2}{\pi}\left[\gamma + \ln(qz/2) + i\frac{\pi}{2}\right]\mathcal{J}_0(iqz)\\ &- \frac{2}{\pi}\left\{\frac{(qz/2)^2}{(1!)^2} + \frac{(qz/2)^2}{(2!)^2}(1+\tfrac{1}{2}) + \frac{(qz/2)^6}{(3!)^2}(1+\tfrac{1}{2}+\tfrac{1}{3}) + \cdots\right\}.\end{aligned}$$

$$\mathcal{J}_0(\varkappa, q, z) = 1 - \frac{(qz/2)^2}{(1!)^2} + \frac{(qz/2)^4}{(2!)^2}[1 - 4\varkappa^2/q^4] - \frac{(qz/2)^6}{(3!)^2}[1 - 20\varkappa^2/q^4] + \cdots.$$



b

Fig. 7.01 a and b. Bessel functions of the first kind, as functions of x and p. Note that for positive p, the Besse functions are bounded; but for negative p, the functions become infinite at $x=0$. (From JAHNKE and EMDE, Funktionentafeln, by courtesy of the publishers, B. G. Teubner, Leipzig, and Dover Publications, New York City)

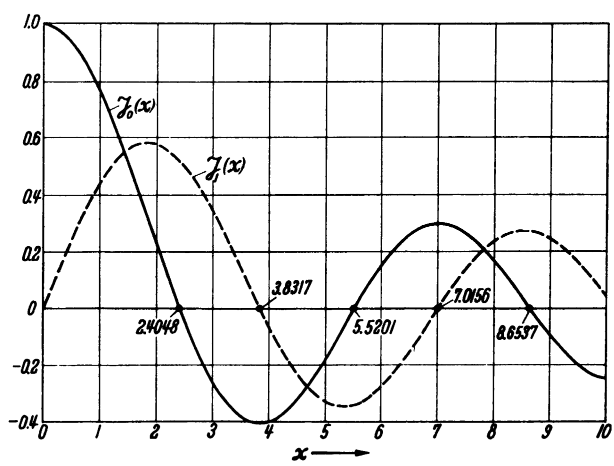


Fig. 7.02. Bessel functions $\mathcal{J}_0(x)$ and $\mathcal{J}_1(x)$, showing the first few zeros

$$\mathscr{Y}_0(\varkappa, q, z) = \frac{2}{\pi}\Big\{[\gamma + \ln(qz/2)]\,\mathscr{J}_0(\varkappa, q, z) + \frac{(qz/2)^2}{(1!)^2} - \frac{(qz/2)^4}{(2!)^2}[4\varkappa^2/q^4 + (1 - 4\varkappa^2/q^4)(1+\tfrac{1}{2})] + \frac{(qz/2)^6}{(3!)^2}[12\varkappa^2/q^4 + (1 - 20\varkappa^2/q^4)(1 + \tfrac{1}{2} + \tfrac{1}{3})] - \cdots\Big\}.$$

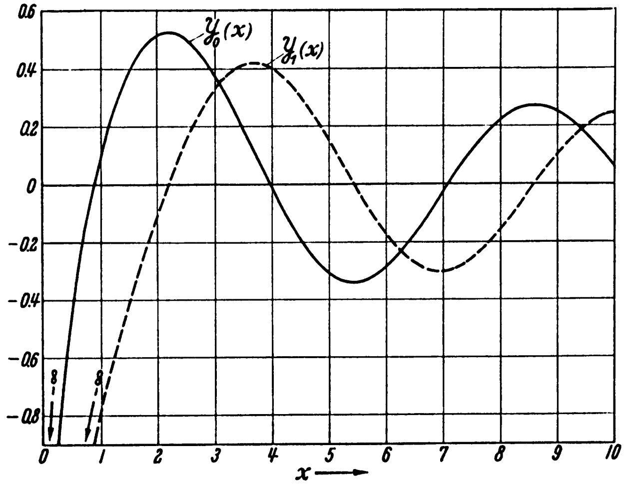


Fig. 7.03. Bessel functions $\mathscr{Y}_0(x)$ and $\mathscr{Y}_1(x)$

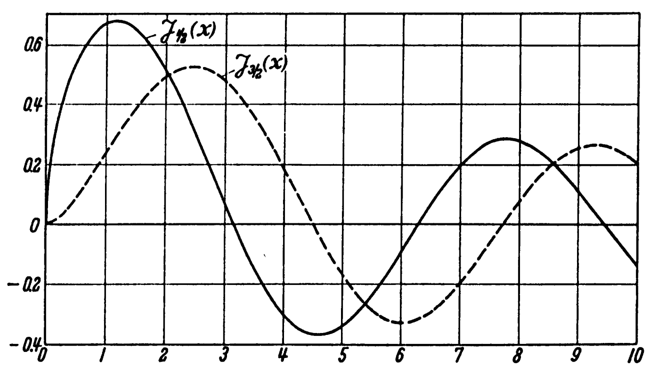


Fig. 7.04. Bessel functions $\mathscr{J}_{\frac{1}{2}}(x)$ and $\mathscr{J}_{\frac{3}{2}}(x)$

$$\mathscr{J}_0(\varkappa, q, iz) = 1 + \frac{(qz/2)^2}{(1!)^2} + \frac{(qz/2)^4}{(2!)^2}[1 - 4\varkappa^2/q^4] + \frac{(qz/2)^6}{(3!)^2}[1 - 20\varkappa^2/q^4] + \cdots.$$

$$\mathscr{Y}_0(\varkappa, q, iz) = \frac{2}{\pi}\Big\{[\gamma + \ln(iqz/2)]\,\mathscr{J}_0(\varkappa, q, iz) - \frac{(qz/2)^2}{(1!)^2} - \frac{(qz/2)^4}{(2!)^2}[4\varkappa^2/q^4 + (1 - 4\varkappa^2/q^4)(1+\tfrac{1}{2})] - \frac{(qz/2)^6}{(3!)^2}[12\varkappa^2/q^4 + (1 - 20\varkappa^2/q^4)(1 + \tfrac{1}{2} + \tfrac{1}{3})] - \cdots\Big\}.$$

$$\mathscr{J}_p(\varkappa, q, iz) = \frac{(iqz/2)^p}{\Gamma(p+1)}\Big\{1 + \frac{(qz/2)^2}{1!(p+1)} + \frac{(qz/2)^4[1 - 4(p+1)\varkappa^2/q^4]}{2!(p+1)(p+2)} + \frac{(qz/2)^6[1 - 4(3p+5)\varkappa^2/q^4]}{3!(p+1)(p+2)(p+3)} + \cdots\Big\}.$$

$$\mathscr{I}_{-p}(\varkappa, q, iz) = \frac{(iqz/2)^{-p}}{\Gamma(-p+1)}\left\{1 - \frac{(qz/2)^2}{1!(p-1)} + \frac{(qz/2)^4[1+4(p-1)\varkappa^2/q^4]}{2!(p-1)(p-2)} - \frac{(qz/2)^6[1+4(3p-5)\varkappa^2/q^4]}{3!(p-1)(p-2)(p-3)} + \cdots\right\}.$$

$$\mathscr{Y}_n(\varkappa, q, iz) = \frac{2}{\pi q^{2n}}\Big\{[\gamma + \ln(iqz/2)]\, \Delta_n'(-n)\, \mathscr{I}_n(\varkappa, q, iz) - \tfrac{1}{2}(q/2)^n 2^{2n}(iz)^{-n}\sum_{m=0}^{n-1}\frac{(-1)^m(n-m-1)!\,\Delta_m'(-n)(z/2)^{2m}}{m!} + \tfrac{1}{2}(iqz/2)^n\sum_{m=0}^{\infty}\frac{(z/2)^{2m}}{m!(m+n)!} \times\left\{4\sum_{l=1}^{m+n-1}(n-2l)M_{l(l+1)} - \Delta_m'(n)\,\Delta_n'(-n)\left[\sum_{l=1}^{m}\frac{1}{l} + \sum_{l=1}^{m+n}\frac{1}{l}\right]\right\}\Big\}.$$

$$\mathscr{I}_{\frac{1}{2}}(\varkappa, q, z) = \frac{(qz/2)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} \times\left\{1 - \frac{(qz/2)^2}{1!(\frac{3}{2})} + \frac{(qz/2)^4[1-3\varkappa^2/q^4]}{2!(\frac{3}{2})(\frac{5}{2})} - \frac{(qz/2)^6[1-13\varkappa^2/q^4]}{3!(\frac{3}{2})(\frac{5}{2})(\frac{7}{2})} + \cdots\right\}.$$

$$\mathscr{I}_{-\frac{1}{2}}(\varkappa, q, z) = \frac{(qz/2)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} \times\left\{1 - \frac{(qz/2)^2}{1!(\frac{1}{2})} + \frac{(qz/2)^4[1-\varkappa^2/q^4]}{2!(\frac{1}{2})(\frac{3}{2})} - \frac{(qz/2)^6[1-7\varkappa^2/q^4]}{3!(\frac{1}{2})(\frac{3}{2})(\frac{5}{2})} + \cdots\right\}.$$

$$\mathscr{I}_{\frac{1}{2}}(\varkappa, q, iz) = \frac{(iqz/2)^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} \times\left\{1 + \frac{(qz/2)^2}{1!(\frac{3}{2})} + \frac{(qz/2)^4[1-3\varkappa^2/q^4]}{2!(\frac{3}{2})(\frac{5}{2})} + \frac{(qz/2)^6[1-13\varkappa^2/q^4]}{3!(\frac{3}{2})(\frac{5}{2})(\frac{7}{2})} + \cdots\right\}.$$

$$\mathscr{I}_{-\frac{1}{2}}(\varkappa, q, iz) = \frac{(iqz/2)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} \times\left\{1 + \frac{(qz/2)^2}{1!(\frac{1}{2})} + \frac{(qz/2)^4[1-\varkappa^2/q^4]}{2!(\frac{1}{2})(\frac{3}{2})} + \frac{(qz/2)^6[1-7\varkappa^2/q^4]}{3!(\frac{1}{2})(\frac{3}{2})(\frac{5}{2})} + \cdots\right\}.$$

$$\mathscr{I}_{\frac{1}{4}}(qz) = \frac{(qz/2)^{\frac{1}{4}}}{\Gamma(\frac{5}{4})} \times\left\{1 - \frac{(qz/2)^2}{1!(\frac{5}{4})} + \frac{(qz/2)^4}{2!(\frac{5}{4})(\frac{9}{4})} - \frac{(qz/2)^6}{3!(\frac{5}{4})(\frac{9}{4})(\frac{13}{4})} + \cdots\right\}.$$

$$\mathscr{I}_{-\frac{1}{4}}(qz) = \frac{(qz/2)^{-\frac{1}{4}}}{\Gamma(\frac{3}{4})} \times\left\{1 - \frac{(qz/2)^2}{1!(\frac{3}{4})} + \frac{(qz/2)^4}{2!(\frac{3}{4})(\frac{7}{4})} - \frac{(qz/2)^6}{3!(\frac{3}{4})(\frac{7}{4})(\frac{11}{4})} + \cdots\right\}.$$

$$\mathscr{I}_{s+\frac{1}{2}}(qz) = \frac{(qz/2)^{s+\frac{1}{2}}}{\Gamma(s+\frac{3}{2})}\left\{1 - \frac{(qz/2)^2}{1!(s+\frac{3}{2})} + \frac{(qz/2)^4}{2!(s+\frac{3}{2})(s+\frac{5}{2})} - \frac{(qz/2)^6}{3!(s+\frac{3}{2})(s+\frac{5}{2})(s+\frac{7}{2})} + \cdots\right\}.$$

$$\mathscr{I}_{-(s+\frac{1}{2})}(qz) = \frac{(qz/2)^{-(s+\frac{1}{2})}}{\Gamma(-s+\frac{1}{2})}\left\{1 + \frac{(qz/2)^2}{1!(s-\frac{1}{2})} + \frac{(qz/2)^4}{2!(s-\frac{1}{2})(s-\frac{3}{2})} + \frac{(qz/2)^6}{3!(s-\frac{1}{2})(s-\frac{3}{2})(s-\frac{5}{2})} + \cdots\right\}.$$

Hankel functions (Bessel functions of the third kind) are linear combinations of Bessel functions of the first and second kinds:

$$\left.\begin{aligned} \mathscr{H}_p^{(1)}(qz) &= \mathscr{I}_p(qz) + i\mathscr{Y}_p(qz), \\ \mathscr{H}_p^{(2)}(qz) &= \mathscr{I}_p(qz) - i\mathscr{Y}_p(qz). \end{aligned}\right\} \tag{7.58}$$

The general solution of Eq. (7.28) may be written

$$Z = A\mathscr{H}_p^{(1)}(qz) + B\mathscr{H}_p^{(2)}(qz). \tag{7.59}$$

Other relations among Bessel functions may be written

$$\mathscr{I}_p(qz) = \tfrac{1}{2}[\mathscr{H}_p^{(1)}(qz) + \mathscr{H}_p^{(2)}(qz)],$$

$$\mathscr{I}_{-p}(qz) = \tfrac{1}{2}[e^{ip\pi}\mathscr{H}_p^{(1)}(qz) + e^{-ip\pi}\mathscr{H}_p^{(2)}(qz)],$$

$$\begin{aligned} \mathscr{Y}_p(qz) &= \frac{1}{\sin p\pi}[\mathscr{I}_p(qz)\cos p\pi - \mathscr{I}_{-p}(qz)] \\ &= \frac{1}{2i}[\mathscr{H}_p^{(1)}(qz) - \mathscr{H}_p^{(2)}(qz)], \end{aligned}$$

$$\begin{aligned} \mathscr{Y}_{-p}(qz) &= \frac{1}{\sin p\pi}[\mathscr{I}_p(qz) - \mathscr{I}_{-p}(qz)\cos p\pi] \\ &= \frac{1}{2i}[e^{ip\pi}\mathscr{H}_p^{(1)}(qz) - e^{-ip\pi}\mathscr{H}_p^{(2)}(qz)], \end{aligned}$$

$$\mathscr{H}_p^{(1)}(qz) = \frac{1}{i\sin p\pi}[\mathscr{I}_{-p}(qz) - e^{-ip\pi}\mathscr{I}_p(qz)],$$

$$\mathscr{H}_p^{(2)}(qz) = \frac{1}{i\sin p\pi}[e^{ip\pi}\mathscr{I}_p(qz) - \mathscr{I}_{-p}(qz)].$$

Bessel functions as definite integrals

$$\mathscr{I}_p(qz) = \frac{(qz/2)^p}{\Gamma(p+\frac{1}{2})\,\Gamma(\frac{1}{2})}\int_0^\pi \cos(qz\cos u)\sin^{2p}u\,du,$$

$$\mathscr{I}_p(qz) = \frac{2(qz/2)^p}{\Gamma(p+\frac{1}{2})\,\Gamma(\frac{1}{2})}\int_0^1 (1-t^2)^{p-\frac{1}{2}}\cos(qzt)\,dt,$$

$$\mathscr{I}_p(qz) = \frac{(qz/2)^p}{\Gamma(p+\frac{1}{2})\,\Gamma(\frac{1}{2})}\int_{-1}^1 (1-t^2)^{p-\frac{1}{2}}\cos(qzt)\,dt,$$

$$\mathscr{I}_p(qz) = \frac{(qz/2)^p}{\Gamma(p+\frac{1}{2})\,\Gamma(\frac{1}{2})}\int_{-1}^1 (1-t^2)^{p-\frac{1}{2}}e^{izt}\,dt,$$

$$\mathscr{I}_p(qz) = \frac{(qz/2)^p}{\Gamma(p+\frac{1}{2})\,\Gamma(\frac{1}{2})}\int_0^\pi e^{iqz\cos\theta}\sin^{2p}\theta\,d\theta,$$

$$\mathscr{I}_{n+\frac{1}{2}}(qz) = (-i)^n\left(\frac{qz}{2\pi}\right)^{\frac{1}{2}}\int_0^\pi e^{iqz\cos\theta}P_n(\cos\theta)\sin\theta\,d\theta.$$

Asymptotic expansions

$$\mathcal{J}_p(qz) \cong \left(\frac{2}{\pi q z}\right)^{\frac{1}{2}} \left\{ \cos(qz - p\pi/2 - \pi/4) \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(p + 2m + \frac{1}{2})}{(2qz)^{2m}\, m!\, \Gamma(p - 2m + \frac{1}{2})} \right.$$

$$\left. - \sin(qz - p\pi/2 - \pi/4) \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(p + 2m + \frac{3}{2})}{(2qz)^{2m+1}\, (m+1)!\, \Gamma(p - 2m - \frac{1}{2})} \right\},$$

$$\mathcal{Y}_p(qz) \cong \left(\frac{2}{\pi q z}\right)^{\frac{1}{2}} \left\{ \sin(qz - p\pi/2 - \pi/4) \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(p + 2m + \frac{1}{2})}{(2qz)^{2m}\, m!\, \Gamma(p - 2m + \frac{1}{2})} \right.$$

$$\left. + \cos(qz - p\pi/2 - \pi/4) \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(p + 2m + \frac{3}{2})}{(2qz)^{2m+1}\, (m+1)!\, \Gamma(p - 2m - \frac{1}{2})} \right\},$$

$$\mathcal{H}_p^{(1)}(qz) \cong \left(\frac{2}{\pi q z}\right)^{\frac{1}{2}} e^{i(qz - p\pi/2 - \pi/4)} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(p + m + \frac{1}{2})}{(2iqz)^m\, m!\, \Gamma(p - m + \frac{1}{2})},$$

$$\mathcal{H}_p^{(2)}(qz) \cong \left(\frac{2}{\pi q z}\right)^{\frac{1}{2}} e^{-i(qz - p\pi/2 - \pi/4)} \sum_{m=0}^{\infty} \frac{\Gamma(p + m + \frac{1}{2})}{(2iqz)^m\, m!\, \Gamma(p - m + \frac{1}{2})}.$$

Recursion formulas. The same recursion formulas apply to all Bessel functions [*38*]. The same is true for formulas of differentiation and integration. Let $\mathcal{Z}$ be a Bessel function of first, second, or third kind. Then

$$\mathcal{Z}_{p-1}(qz) + \mathcal{Z}_{p+1}(qz) = \frac{2p}{qz} \mathcal{Z}_p(qz),$$

$$\mathcal{Z}_{p-1}(qz) - \mathcal{Z}_{p+1}(qz) = \frac{2}{q} \frac{d}{dz} \mathcal{Z}_p(qz),$$

$$z \frac{d}{dz} \mathcal{Z}_p(qz) + p\, \mathcal{Z}_p(qz) = qz\, \mathcal{Z}_{p-1}(qz),$$

$$z \frac{d}{dz} \mathcal{Z}_p(qz) - p\, \mathcal{Z}_p(qz) = -\, qz\, \mathcal{Z}_{p+1}(qz),$$

$$\mathcal{Z}_p(qz)\, \mathcal{Z}_{1-p}(qz) + \mathcal{Z}_{-p}(qz)\, \mathcal{Z}_{p-1}(qz) = \frac{2 \sin p\pi}{\pi q z}.$$

Differentiation. Let $\mathcal{Z}_p(qz)$ be a Bessel function of first, second, or third kind, or any linear combination of these functions. Then

$$\frac{d}{dz}[\mathcal{Z}_p(qz)] = -\frac{p}{z} \mathcal{Z}_p(qz) + q\, \mathcal{Z}_{p-1}(qz)$$

$$= \frac{p}{z} \mathcal{Z}_p(qz) - q\, \mathcal{Z}_{p+1}(qz)$$

$$= \frac{q}{2} [\mathcal{Z}_{p-1}(qz) - \mathcal{Z}_{p+1}(qz)].$$

$$\frac{d}{dz}[z^p\, \mathcal{Z}_p(qz)] = q z^p\, \mathcal{Z}_{p-1}(qz).$$

$$\frac{d}{dz}[z^{-p}\, \mathcal{Z}_p(qz)] = -\, q z^{-p}\, \mathcal{Z}_{p+1}(qz).$$

$$\frac{d}{dz}[z^{p/2}\, \mathcal{Z}_p[(qz)^{\frac{1}{2}}]] = \frac{q^{\frac{1}{2}}}{2} z^{(p-1)/2}\, \mathcal{Z}_{p-1}[(qz)^{\frac{1}{2}}].$$

$$\frac{d}{dz}\left[z^{-p/2}\mathscr{Z}_p[(qz)^{\frac{1}{2}}]\right] = -\frac{q^{\frac{1}{2}}}{2} z^{-(p+1)/2}\mathscr{Z}_{p+1}[(qz)^{\frac{1}{2}}].$$

$$\frac{d^2}{dz^2}[\mathscr{Z}_p(qz)] = \left[\frac{p(p-1)}{z^2} - q^2\right]\mathscr{Z}_p(qz) + \frac{q}{z}\mathscr{Z}_{p+1}(qz).$$

$$\left(\frac{1}{qz}\frac{d}{dz}\right)^m [z^p\mathscr{Z}_p(qz)] = (qz)^{p-m}\mathscr{Z}_{p-m}(qz).$$

$$\left(\frac{1}{qz}\frac{d}{dz}\right)^m [z^{-p}\mathscr{Z}_p(qz)] = (-1)^m (qz)^{-(p+m)}\mathscr{Z}_{p+m}(qz).$$

$$\frac{d}{dz}[\mathscr{Z}_0(qz)] = -q\mathscr{Z}_1(qz).$$

$$\frac{d}{dz}[\mathscr{Z}_1(qz)] = q\mathscr{Z}_0(qz) - \frac{1}{z}\mathscr{Z}_1(qz).$$

$$\frac{d^2}{dz^2}[\mathscr{Z}_0(qz)] = -q^2\mathscr{Z}_0(qz) + \frac{q}{z}\mathscr{Z}_1(qz).$$

$$\frac{d^2}{dz^2}[\mathscr{Z}_1(qz)] = -q^2\mathscr{Z}_1(qz) + \frac{q}{z}\mathscr{Z}_2(qz).$$

Integration

$$\int z^{p+1}\mathscr{Z}_p(qz)\,dz = \frac{z^{p+1}}{q}\mathscr{Z}_{p+1}(qz).$$

$$\int z^{-p+1}\mathscr{Z}_p(qz)\,dz = -\frac{z^{-p+1}}{q}\mathscr{Z}_{p-1}(qz).$$

$$\int z[\mathscr{Z}_p(qz)]^2 dz = \frac{z^2}{2}\{[\mathscr{Z}_p(qz)]^2 - \mathscr{Z}_{p-1}(qz)\,\mathscr{Z}_{p+1}(qz)\}.$$

$$\int z[\mathscr{Z}_0(qz)]^2 dz = \frac{z^2}{2}\{[\mathscr{Z}_0(qz)]^2 + [\mathscr{Z}_1(qz)]^2\}.$$

$$\int \mathscr{Z}_1(qz)\,dz = -\frac{1}{q}\mathscr{Z}_0(qz).$$

$$\int z\mathscr{Z}_0(qz)\,dz = \frac{z}{q}\mathscr{Z}_1(qz).$$

$$\int z\mathscr{Z}_p(\alpha z)\,\mathscr{Z}_p(\beta z)\,dz = \frac{z}{\alpha^2-\beta^2}[\beta\mathscr{Z}_p(\alpha z)\,\mathscr{Z}_{p-1}(\beta z) - \alpha\mathscr{Z}_{p-1}(\alpha z)\,\mathscr{Z}_p(\beta z)].$$

$$\int z\mathscr{Z}_0(\alpha z)\,\mathscr{Z}_0(\beta z)\,dz = \frac{z}{\alpha^2-\beta^2}[\beta\mathscr{Z}_0(\alpha z)\,\mathscr{Z}_{-1}(\beta z) - \alpha\mathscr{Z}_{-1}(\alpha z)\,\mathscr{Z}_0(\beta z)].$$

$$\int z\mathscr{Z}_1(\alpha z)\,\mathscr{Z}_1(\beta z)\,dz = \frac{z}{\alpha^2-\beta^2}[\beta\mathscr{Z}_1(\alpha z)\,\mathscr{Z}_0(\beta z) - \alpha\mathscr{Z}_0(\alpha z)\,\mathscr{Z}_1(\beta z)].$$

Orthogonality. Bessel wave functions are orthogonal on the interval (a, b) with respect to the weighting function z:

$$\int_a^b z\mathscr{Z}_p(\varkappa, q_m, z)\cdot\mathscr{Z}_p(\varkappa, q_n, z)\,dz = 0, \qquad m \neq n, \tag{7.60}$$

where boundary conditions of the form

$$k_1\mathscr{Z}_p(\varkappa, q_m, z) + k_2\mathscr{Z}_p'(\varkappa, q_m, z) = 0$$

apply at $z = a$ and at $z = b$ (Sturm-Liouville conditions, § 7.07).

An arbitrary function $f(z)$ can be expanded in a series of Bessel wave functions:

$$f(z) = \sum_{m=0}^{\infty} A_m \mathscr{Z}_p(\varkappa, q_m, z), \tag{7.61}$$

where

$$A_m = \frac{1}{N_m} \int_a^b z f(z) \mathscr{Z}_p(\varkappa, q_m, z)\, dz \tag{7.62}$$

and

$$N_m = \int_a^b z [\mathscr{Z}_p(\varkappa, q_m, z)]^2\, dz.$$

If $\varkappa = 0$,

$$\left.\begin{aligned} N_m = \frac{b^2}{2}\{[\mathscr{Z}_p(q_m b)]^2 - \mathscr{Z}_{p-1}(q_m b)\cdot \mathscr{Z}_{p+1}(q_m b)\} \\ - \frac{a^2}{2}\{[\mathscr{Z}_p(q_m a)]^2 - \mathscr{Z}_{p-1}(q_m a)\cdot \mathscr{Z}_{p+1}(q_m a)\}. \end{aligned}\right\} \tag{7.63}$$

For $p = 0$, the coefficients of the expansion, Eq. (7.61), are

$$A_m = \frac{1}{N_m} \int_a^b z f(z) \cdot \mathscr{Z}_0(\varkappa, q_m, z)\, dz$$

where

$$N_m = \int_a^b z [\mathscr{Z}_0(\varkappa, q_m, z)]^2\, dz.$$

If $\varkappa = 0$,

$$\left.\begin{aligned} N_m = \frac{b^2}{2}\{[\mathscr{Z}_0(q_m b)]^2 + [\mathscr{Z}_1(q_m b)]^2\} \\ - \frac{a^2}{2}\{[\mathscr{Z}_0(q_m a)]^2 + [\mathscr{Z}_1(q_m a)]^2\}. \end{aligned}\right\} \tag{7.63 a}$$

For boundary conditions $\mathscr{Z}_0(q_m a) = 0$ and $\mathscr{Z}_0(q_m b) = 0$, $\mathscr{Z}_0$ may be taken as a linear combination of $\mathscr{I}_0$ and $\mathscr{Y}_0$, so adjusted that its zeros are at a and b. Then Eq. (7.63 a) becomes

$$N_m = \frac{b^2}{2}[\mathscr{Z}_1(q_m b)]^2 - \frac{a^2}{2}[\mathscr{Z}_1(q_m a)]^2. \tag{7.63 b}$$

In the special case of $a = 0$, Bessel wave functions of the second kind are not applicable, and the expansion is in terms of $\mathscr{I}_p(\varkappa, q_m, z)$. Then, for $p = 0$,

$$f(z) = \sum_{m=0}^{\infty} A_m \mathscr{I}_0(\varkappa, q_m, z),$$

$$A_m = \frac{1}{N_m} \int_a^b z f(z)\, \mathscr{I}_0(\varkappa, q_m, z)\, dz,$$

where

$$N_m = \int_0^b z [\mathscr{I}_0(\varkappa, q_m, z)]^2\, dz.$$

If $\varkappa = 0$,

$$N_m = \frac{b^2}{2}[\mathscr{I}_1(q_m b)]^2,$$

where q_m are determined by the relation,

$$\mathscr{I}_0(q_m b) = 0.$$

7.10 BAER FUNCTIONS

The wave functions. The Baer wave equation {114} may be written [*39*]

$$\frac{d^2Z}{dz^2} + \frac{1}{2}\left[\frac{1}{z-b} + \frac{1}{z-c}\right]\frac{dZ}{dz} + \left[\frac{\varkappa^2 z^2 - p(p+1)\,z - q(b+c)}{(z-b)\,(z-c)}\right] Z = 0. \quad (7.64)$$

For a solution about the origin, take

$$Z = z^\beta \sum_{j=0}^{\infty} C_j z^j.$$

Applying the method of § 7.03, we obtain

$$A_0 = 0,$$

$$A_1 = -\frac{b+c}{2bc},$$

$$A_2 = -\frac{(b^2+c^2)}{2(bc)^2},$$

$$A_j = \frac{1}{bc}\left[A_{j-1}(b+c) - A_{j-2}\right] \quad \text{for } j > 2;$$

$$B_0 = B_1 = 0,$$

$$B_2 = -\frac{q(b+c)}{bc},$$

$$B_3 = -\frac{1}{bc}\left[p(p+1) + q\frac{(b+c)^2}{bc}\right],$$

$$B_4 = \frac{1}{bc}\left[\varkappa^2 - p(p+1)\left(\frac{b+c}{bc}\right) + q\left(\frac{b+c}{bc}\right)\left[1 - \frac{(b+c)^2}{bc}\right]\right],$$

$$B_j = \frac{1}{bc}\left[B_{j-1}(b+c) - B_{j-2}\right] \quad \text{for } j > 4.$$

The indicial equation is

$$\beta(\beta-1) = 0,$$

so $b_1 = 1$, $b_2 = 0$. For the first root, the determinant is

$$\Delta_j(1) = \begin{vmatrix} A_1 & 2\cdot 1 & 0 & 0 & \dots & 0 \\ A_2 + B_2 & 2A_1 & 3\cdot 2 & 0 & \dots & 0 \\ A_3 + B_3 & 2A_2 + B_2 & 3A_1 & 4\cdot 3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & j(j-1) \\ A_j + B_j & \dots & \dots & \dots & \dots & jA_1 \end{vmatrix}. \quad (7.65)$$

Thus,

$$C_j(1) = \frac{(-1)^j C_0 \Delta_j(1)}{[1\cdot 2]\,[2\cdot 3]\,[3\cdot 4]\dots[j(j+1)]}$$

and the first solution is

$$Z_1 = C_0 z \sum_{j=0}^{\infty} \frac{(-1)^j \Delta_j(1)\, z^j}{j!\,(j+1)!}. \quad (7.66)$$

For $\beta = b_2 = 0$,

$$\Delta_j(0) = 0 \text{ since } f_0(1) = 0$$

and

$$C_j(0) = 0$$

or

$$Z_2 = C_0\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m \Delta'_m(0) z^m}{m!(m-1)!}\right], \tag{7.67}$$

where

$$\frac{\Delta'_m(0)}{(m-1)!} = \begin{vmatrix} A_1 & 1 & 0 & 0 & 0 & 0 \dots & 0 \\ B_2 & A_1 & 1 & 0 & 0 & 0 \dots & 0 \\ B_3 & A_2 + B_2 & A_1 & 2 & 0 & 0 \dots & 0 \\ B_4 & A_3 + B_3 & A_2 + \frac{B_2}{2} & A_1 & 3 & 0 \dots & 0 \\ B_5 & A_4 + B_4 & A_3 + \frac{B_3}{2} & A_2 + \frac{B_2}{3} & A_1 & 4 \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ B_j & A_{j-1} + B_{j-1} & \dots & \dots & \dots & \dots & A_1 \end{vmatrix} \tag{7.68}$$

If $C_0 = 1$, the general solution of Eq. (7.64) may be written

$$Z = A\mathscr{B}_p^q(\varkappa, z) + B\mathscr{C}_p^q(\varkappa, z), \tag{7.69}$$

where

$$\left.\begin{aligned} \mathscr{B}_p^q(\varkappa, z) &= z\sum_{m=0}^{\infty} \frac{(-1)^m \Delta'_m(1) z^m}{m!(m+1)!}, \\ \mathscr{C}_p^q(\varkappa, z) &= 1 + \sum_{m=1}^{\infty} \frac{(-1)^m \Delta'_m(0) z^m}{m!(m-1)!}. \end{aligned}\right\} \tag{7.70}$$

These expressions converge within a circle about the origin with radius b ($b < c$).

Expansion about another singularity. For an expansion about the point $z_0 = b$,

$$Z = (z-b)^\beta \sum_{j=0}^{\infty} C_j (z-b)^j,$$

and

$$\begin{aligned} A_0 &= \tfrac{1}{2}, \\ A_1 &= \frac{1}{2(b-c)}, \\ A_j &= -\frac{A_j}{b-c} \quad \text{for } j > 1; \\ B_0 &= 0, \\ B_1 &= \frac{1}{b-c}\left[b^2\varkappa^2 - p(p+1)b - q(b+c)\right], \\ B_2 &= \frac{1}{(b-c)^2}\left[b\varkappa^2 - 2bc\varkappa^2 + p(p+1)c + q(b+c)\right], \\ B_3 &= \frac{1}{(b-c)^3}\left[c\varkappa^2 - p(p+1)c - q(b+c)\right], \\ B_j &= -\frac{B_{j-1}}{b-c} \quad \text{for } j > 3. \end{aligned}$$

The indicial equation is

$$\beta(\beta - \tfrac{1}{2}) = 0,$$

so $b_1 = \frac{1}{2}$, $b_2 = 0$. For $\beta = b_1$,

$$\Delta_j(\tfrac{1}{2}) = \begin{vmatrix} A_1/2 + B_1 & \frac{3}{2} & 0 & 0 \ldots & 0 \\ A_2/2 + B_2 & \frac{3}{2}A_1 + B_1 & 5 & 0 \ldots & 0 \\ A_3/2 + B_3 & \frac{3}{2}A_2 + B_2 & \frac{5}{2}A_1 + B_1 & \frac{21}{2} \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & (j - \frac{1}{2})(j-1) \\ A_j/2 + B_j & \ldots & \ldots & \ldots & (j - \frac{1}{2}) A_1 + B_1 \end{vmatrix}. \quad (7.71)$$

For $\beta = b_2 = 0$,

$$\Delta_j(0) = \begin{vmatrix} B_1 & \frac{1}{2} & 0 & 0 \ldots & 0 \\ B_2 & A_1 + B_1 & 3 & 0 \ldots & 0 \\ B_3 & A_2 + B_2 & 2A_1 + B_1 & \frac{15}{2} \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & (j-1)(j-\frac{3}{2}) \\ B_j & \ldots & \ldots & \ldots & (j-1) A_1 + B_1 \end{vmatrix}. \quad (7.72)$$

Thus the solutions are

$$\left.\begin{aligned} Z_1 &= C_0 (z-b) \sum_{j=0}^{\infty} \frac{(-1)^j 2^{2j} \Delta_j(\frac{1}{2}) \cdot (z-b)^j}{(2j+1)!}, \\ Z_2 &= C_0 \sum_{j=0}^{\infty} \frac{(-1)^j 2^{2j} \Delta_j(0)(z-b)^j}{(2j)!}. \end{aligned}\right\} \quad (7.73)$$

These series represent the same functions as Eq. (7.70) but over a different region in the z-plane. Accordingly, Eq. (7.73) with $C_0 = 1$ represents the Baer functions $\mathscr{B}_p^q(\varkappa, z)$ and $\mathscr{C}_p^q(\varkappa, z)$ about the point $z = b$ and extending out to the nearest singularity.

Orthogonality. According to § 7.07, a set of Baer functions with appropriate boundary conditions form an orthogonal set (with weighting function) on the arbitrary interval (a, b). Thus, from Table 7.01,

$$\int_a^b \frac{z}{u} \mathscr{B}_{p_m}^q(\varkappa, z)\, \mathscr{B}_{p_n}^q(\varkappa, z)\, dz = 0, \qquad m \neq n,$$

where $u = [(z-b)(z-c)]^{\frac{1}{2}}$. The same relation holds for $\mathscr{C}_p^q(\varkappa, z)$; and a similar relation holds if q is varied instead of p (Table 7.01).

An arbitrary function $f(z)$ can be expanded in a series of Baer functions. For instance,

$$f(z) = \sum_{m=0}^{\infty} A_m \mathscr{B}_{p_m}^q(\varkappa, z),$$

where the coefficients are

$$A_m = \frac{1}{N_m} \int_a^b \frac{z}{u} f(z)\, \mathscr{B}_{p_m}^q(\varkappa, z)\, dz,$$

$$N_m = \int_a^b \frac{z}{u} [\mathscr{B}_{p_m}^q(\varkappa, z)]^2\, dz.$$

7.11 MATHIEU FUNCTIONS

The Baer equation {1 1 3} may be written

$$\frac{d^2Z}{d\zeta^2} + \frac{1}{2}\left[\frac{1}{\zeta - b} + \frac{1}{\zeta - c}\right]\frac{dZ}{d\zeta} + \frac{1}{4}\left[\frac{\bar{A}_0 + \bar{A}_1\zeta}{(\zeta - b)(\zeta - c)}\right]Z = 0.$$

A special case of the Baer equation is the *Mathieu equation* [*40*]. Let $b = 0$, $c = 1$, $\bar{A}_0 = -(2q + \lambda)$, $\bar{A}_1 = 4q$. Then, introducing the new variable defined by $\zeta = \cos^2 z$, we obtain the Mathieu equation,

$$\frac{d^2Z}{dz^2} + (\lambda - 2q\cos 2z)\,Z = 0, \tag{7.74}$$

where λ and q are arbitrary parameters [*41*].

Periodic solutions. For the special case of $q = 0$ and $\lambda = p^2$, Eq. (7.74) reduces to the familiar {0 4}:

$$\frac{d^2Z}{dz^2} + p^2 Z = 0,$$

whose solutions are

$$Z = \begin{pmatrix}\cos\\ \sin\end{pmatrix} p z.$$

Periodic solutions are obtainable also for other values of q, but only when certain relations hold among λ, p, and q. These periodic solutions are called *Mathieu functions of the first kind* and are denoted by

$$\mathrm{ce}_p(q, z), \qquad \mathrm{se}_p(q, z),$$

corresponding to $\cos pz$ and $\sin pz$.

Define the function $\mathrm{ce}_p(q, z)$ as a periodic function that reduces to $\cos pz$ when $q = 0$. Let [*42*]

$$\begin{cases} Z = \mathrm{ce}_p(q, z) = \cos pz + \sum_{j=1}^{\infty} q^j C_j(z), \\ \lambda = p^2 + \sum_{j=1}^{\infty} \alpha_j q^j. \end{cases}$$

Substitution into Eq. (7.74) yields the series

$$\left.\begin{aligned} \mathrm{ce}_p(q, z) = \cos pz &- \frac{q}{4}\left[\frac{\cos(p+2)z}{p+1} - \frac{\cos(p-2)z}{p-1}\right] \\ &+ \frac{q^2}{32}\left[\frac{\cos(p+4)z}{(p+1)(p+2)} + \frac{\cos(p-4)z}{(p-1)(p-2)}\right] \\ &- \frac{q^3}{128}\left[\frac{(p^2+4p+7)\cos(p+2)z}{(p-1)(p+1)^3(p+2)} - \frac{(p^2-4p+7)\cos(p-2)z}{(p+1)(p-1)^3(p-2)}\right. \\ &+ \left.\frac{\cos(p+6)z}{3(p+1)(p+2)(p+3)} - \frac{\cos(p-6)z}{3(p-1)(p-2)(p-3)}\right] + \cdots, \end{aligned}\right\} \tag{7.75}$$

and

$$\left.\begin{aligned} \lambda = p^2 &+ \frac{q^2}{2(p^2-1)} \\ &+ \frac{(5p^2+7)q^4}{32(p^2-1)^3(p^2-4)} + \frac{(9p^4+58p^2+29)q^6}{64(p^2-1)^5(p^2-4)(p^2-9)} + \cdots. \end{aligned}\right\} \tag{7.76}$$

Similarly,

$$\left.\begin{aligned} \mathrm{se}_p(q,z) &= \sin pz - \frac{q}{4}\left[\frac{\sin(p+2)z}{p+1} - \frac{\sin(p-2)z}{p-1}\right] \\ &+ \frac{q^2}{32}\left[\frac{\sin(p+4)z}{(p+1)(p+2)} + \frac{\sin(p-4)z}{(p-1)(p-2)}\right] - \cdots, \end{aligned}\right\} \tag{7.77}$$

where λ is again given by Eq. (7.76). The series of Eqs. (7.75), (7.76), and (7.77) fail if p is an integer other than zero.

For $p \neq$ integer, $\mathrm{ce}_p(q,z)$ and $\mathrm{se}_p(q,z)$ are independent functions, and the general solution of Eq. (7.74) is

$$Z = A\,\mathrm{ce}_p(q,z) + B\,\mathrm{se}_p(q,z), \tag{7.78}$$

with p related to λ and q as in Eq. (7.76). For the more familiar case of p an integer, only one solution can be periodic and Eq. (7.78) cannot be used as the general solution [*43*].

Integer p. Now consider the special case where $p = m$, an integer [*44*]. If $p = 0$, Eqs. (7.75) and (7.76) reduce to

$$\begin{aligned} \mathrm{ce}_0(q,z) &= 1 - \frac{q}{2}\cos 2z + \frac{q^2}{32}\cos 4z \\ &- \frac{q^3}{128}\left(\frac{1}{9}\cos 6z - 7\cos 2z\right) + \frac{q^4}{73728}(\cos 8z - 320\cos 4z) + \cdots \end{aligned}$$

with

$$\lambda = -\frac{q^2}{2} + \frac{7q^4}{128} - \frac{29q^6}{2304} + \cdots.$$

Improved methods of computation have been developed by INCE and by GOLDSTEIN [*41*]. For $m = 0$, a periodic solution requires the above relation between λ and q. However, the second solution obtained from Eq. (7.77) vanishes identically, so a non-periodic second solution must be found.

For $p = 1$, neither Eq. (7.75) nor Eq. (7.77) yields a convergent solution. However, somewhat similar periodic solutions, using an entirely different expression for λ can be found. These are Mathieu functions for integer values.

$$\begin{aligned} \mathrm{ce}_1(q,z) &= \cos z - \frac{q}{8}\cos 3z + \frac{q^2}{64}\left(\frac{1}{3}\cos 5z - \cos 3z\right) \\ &- \frac{q^3}{512}\left(\frac{1}{18}\cos 7z - \frac{4}{9}\cos 5z + \frac{1}{3}\cos 3z\right) \\ &+ \frac{q^4}{4096}\left(\frac{1}{180}\cos 9z - \frac{1}{12}\cos 7z + \frac{1}{6}\cos 5z + \frac{11}{9}\cos 3z\right) - \cdots \end{aligned}$$

with

$$\lambda = 1 + q - \frac{q^2}{8} - \frac{q^3}{64} - \frac{q^4}{1536} + \frac{11q^5}{36864} + \cdots.$$

The interesting way in which a cosine changes into other Mathieu functions is indicated in Fig. 7.05, where ce_1 is plotted as a function of q and z.

$$\begin{aligned} \mathrm{ce}_2(q,z) &= \cos 2z - \frac{q}{8}\left(\frac{2}{3}\cos 4z - 2\right) + \frac{q^2}{384}\cos 6z \\ &- \frac{q^3}{512}\left(\frac{1}{45}\cos 8z + \frac{43}{27}\cos 4z + \frac{40}{3}\right) \\ &+ \frac{q^4}{4096}\left(\frac{1}{540}\cos 10z + \frac{293}{540}\cos 6z\right) - \cdots \end{aligned}$$

with

$$\lambda = 4 + \frac{5}{12} q^2 - \frac{763}{13824} q^4 + \frac{1002401}{79626240} q^6 - \cdots.$$

The λ's for the ce-functions are indicated by the curves marked $a_0, a_1, \ldots$ in Fig. 7.06. Corresponding curves marked $b_1, b_2, b_3, \ldots$ refer to the values of λ

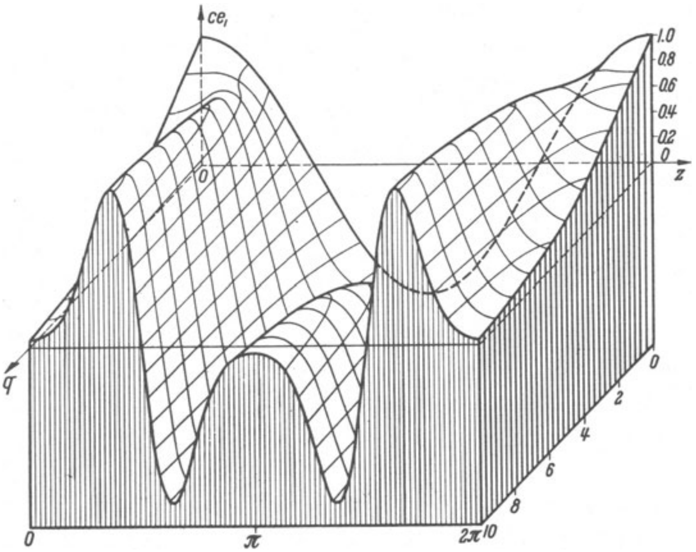


Fig. 7.05. The Mathieu function $ce_1(qz)$. For $q = 0$, the function is an ordinary cosine, but as q increases, the cosine curve is progressively distorted

for the se-functions:

$$\mathrm{se}_1(q, z) = \sin z - \frac{q}{8} \sin 3z + \frac{q^2}{64}\left(\frac{1}{3} \sin 5z + \sin 3z\right) - \frac{q^3}{512}\left(\frac{1}{18} \sin 7z + \frac{4}{9} \sin 5z + \frac{1}{3} \sin 3z\right) + \frac{q^4}{4096}\left(\frac{1}{180} \sin 9z + \frac{1}{12} \sin 7z + \frac{1}{6} \sin 5z - \frac{11}{9} \sin 3z\right) - \cdots$$

with

$$\lambda = 1 - q - \frac{q^2}{8} + \frac{q^3}{64} - \frac{q^4}{1536} - \frac{11 q^5}{36864} + \cdots.$$

$$\mathrm{se}_2(q, z) = \sin 2z - \frac{q}{12} \sin 4z + \frac{q^2}{384} \sin 6z - \frac{q^3}{512}\left(\frac{1}{45} \sin 8z - \frac{5}{27} \sin 4z\right) + \frac{q^4}{4096}\left(\frac{1}{540} \sin 10z - \frac{37}{540} \sin 6z\right) - \cdots$$

with

$$\lambda = 4 - \frac{q^2}{12} + \frac{5}{13824} q^4 - \cdots.$$

Since ce_m and se_m do not correspond to the same value of λ, their linear combination, Eq. (7.78), does not constitute the general solution of the Mathieu equation. Either ce_m or se_m may be used if a series of such terms fits the boundary conditions. If neither will satisfy the boundary conditions, one must introduce Mathieu functions of the second kind which are non-periodic.

We have seen that periodic solutions of the Mathieu equation occur only for certain relations between λ, m and q, as shown by the heavy curves of Fig. 7.06. For all other conditions, the solutions are non-periodic. The shaded areas indicate non-periodic solutions which tend to finite values as $z \to \infty$; the unshaded areas indicate non-periodic solutions which approach ∞ as $z \to \infty$.

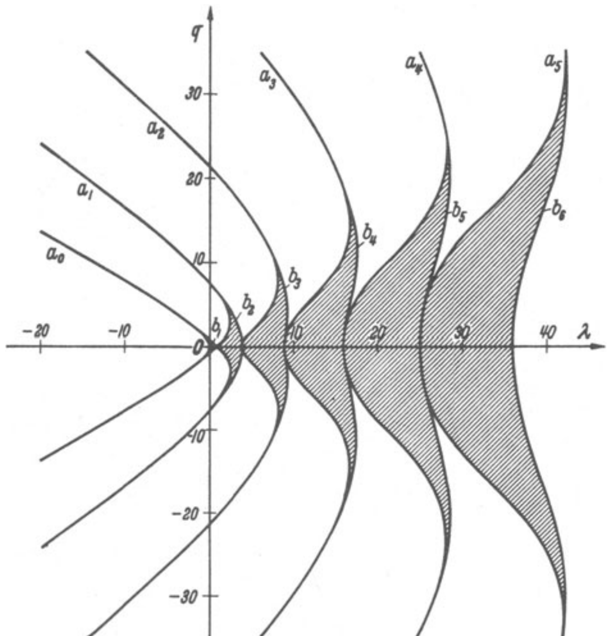


Fig. 7.06. Relation between λ and q for Mathieu functions. Periodic solutions are obtained only for points located on the curves. All other solutions are non-periodic, the shaded areas representing stable solutions and the unshaded areas unstable solutions

Functions of the second kind. Corresponding to Mathieu functions of the first kind, we have non-periodic functions of the second kind:

$$\left.\begin{aligned} \mathrm{fe}_m(q,z) &= \mathrm{ce}_m(q,z)\int_0^z \frac{dz}{[\mathrm{ce}_m(q,z)]^2}, \\ \mathrm{ge}_m(q,z) &= \mathrm{se}_m(q,z)\int_0^z \frac{dz}{[\mathrm{se}_m(q,z)]^2}. \end{aligned}\right\} \tag{7.79}$$

Orthogonality. If m and n are positive integers,

$$\int_0^{2\pi} \mathrm{ce}_m(q,z)\,\mathrm{ce}_n(q,z)\,dz = 0, \qquad m \neq n;$$

$$\int_0^{2\pi} \mathrm{se}_m(q,z)\,\mathrm{se}_n(q,z)\,dz = 0, \qquad m \neq n;$$

$$\int_0^{2\pi} \mathrm{ce}_m(q,z)\,\mathrm{se}_n(q,z)\,dz = 0,$$

$$\int_0^{2\pi} \mathrm{ce}_m^2(q,z)\,dz = \int_0^{2\pi} \mathrm{se}_m^2(q,z)\,dz = \pi, \qquad m > 0.$$

Thus an arbitrary function $f(z)$ can be expanded in a series of Mathieu functions:

$$f(z) = \sum_{m=0}^{\infty} A_m \operatorname{ce}_m(q, z),$$

where

$$A_m = \frac{1}{\pi} \int_0^{2\pi} f(z) \operatorname{ce}_m(q, z)\, dz.$$

Or

$$f(z) = \sum_{m=1}^{\infty} A_m \operatorname{se}_m(q, z),$$

$$A_m = \frac{1}{\pi} \int_0^{2\pi} f(z) \operatorname{se}_m(q, z)\, dz.$$

7.12 LEGENDRE FUNCTIONS

The wave equation. The Legendre wave equation {224} is

$$(z^2 - 1)\frac{d^2 Z}{dz^2} + 2z\frac{dZ}{dz} + \left[\varkappa^2 a^2 (z^2 - 1) - p(p+1) - \frac{q^2}{z^2 - 1}\right] Z = 0. \quad (7.80)$$

Expanding about the origin by the methods of § 7.03, we obtain

$$A_2 = A_4 = A_6 = \cdots = -2,$$
$$B_2 = \varkappa^2 a^2 + p(p+1) - q^2,$$
$$B_j = p(p+1) - jq^2/2$$

for j even and greater than 2. All other coefficients are zero. The indicial equation is

$$\beta(\beta - 1) = 0$$

with the roots $b_1 = 1$, $b_2 = 0$. The determinants for these two cases are

$$\Delta'_m(1) = \left|\begin{array}{cccccc} B_2 - 2 & 2\cdot 3 & 0 & 0 & \ldots & 0 \\ B_4 - 2 & B_2 - 6 & 4\cdot 5 & 0 & \ldots & 0 \\ B_6 - 2 & B_4 - 6 & B_2 - 10 & 6\cdot 7 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ B_{2m-2} - 2 & B_{2m-4} - 6 & \ldots & & \ldots & (2m-2)(2m-1) \\ B_{2m} - 2 & B_{2m-2} - 6 & \ldots & & \ldots & B_2 - 4m + 2 \end{array}\right|, \quad (7.81)$$

$$\Delta'_m(0) = \left|\begin{array}{cccccc} B_2 & 1\cdot 2 & 0 & 0 & \ldots & 0 \\ B_4 & B_2 - 4 & 3\cdot 4 & 0 & \ldots & 0 \\ B_6 & B_4 - 4 & B_2 - 8 & 5\cdot 6 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ B_{2m-2} & B_{2m-4} - 4 & B_{2m-6} - 8 & \ldots & \ldots & (2m-3)(2m-2) \\ B_{2m} & B_{2m-2} - 4 & B_{2m-4} - 8 & \ldots & \ldots & B_2 - 4(m-1) \end{array}\right|, \quad (7.82)$$

where

$$B_2 = \varkappa^2 a^2 + p(p+1) - q^2,$$
$$B_{2l} = p(p+1) - lq^2, \quad l > 1.$$

The general solution of Eq. (7.80) is

$$Z = AZ_1 + BZ_2,$$

where

$$Z_1 = C_0 z \sum_{m=0}^{\infty} \frac{(-1)^m \Delta'_m(1)\, z^{2m}}{(2m+1)!}, \tag{7.83}$$

$$Z_2 = C_0 \sum_{m=0}^{\infty} \frac{(-1)^m \Delta'_m(0)\, z^{2m}}{(2m)!}. \tag{7.84}$$

This solution satisfies the differential equation for all values (integer or non-integer) of $\varkappa$, p, and q. It applies within a unit circle about the origin, $|z| < 1$. The same method may be employed to obtain a series solution for the region outside the unit circle.

The constant C_0 is arbitrary. For simplicity, let $C_0 = 1$ and define the *Legendre wave functions* by the above series:

$$\mathscr{P}_p^q(\varkappa a, z) = z \sum_{m=0}^{\infty} \frac{(-1)^m \Delta'_m(1)\, z^{2m}}{(2m+1)!}, \tag{7.83a}$$

$$\mathscr{Q}_p^q(\varkappa a, z) = \sum_{m=0}^{\infty} \frac{(-1)^m \Delta'_m(0)\, z^{2m}}{(2m)!}. \tag{7.84a}$$

The function $\mathscr{P}_p^q$ is always an odd function, the function $\mathscr{Q}_p^q$ is always even.

Degenerate cases. As indicated in § 7.01, the general Legendre wave functions of Eqs. (7.83a) and (7.84a) are solutions of {224}. These solutions may be written

$$\begin{aligned}\mathscr{P}_p^q(\varkappa a, z) = z\Big\{1 &- \frac{z^2}{3!}\,[\varkappa^2 a^2 + p(p+1) - q^2 - 2]\\ &+ \frac{z^4}{5!}\,[\varkappa^4 a^4 + 2\varkappa^2 a^2[p(p+1) - q^2 - 4]\\ &+ p(p+1)\,[p(p+1) - 2q^2 - 14] + q^4 + 20q^2 + 24] - \cdots\Big\},\end{aligned}$$

$$\begin{aligned}\mathscr{Q}_p^q(\varkappa a, z) = 1 &- \frac{z^2}{2!}\,[\varkappa^2 a^2 + p(p+1) - q^2]\\ &+ \frac{z^4}{4!}\,[\varkappa^4 a^4 + 2\varkappa^2 a^2[p(p+1) - q^2 - 4]\\ &+ p(p+1)\,[p(p+1) - 2q^2 - 6] + q^4 + 8q^2] - \cdots.\end{aligned}$$

As degenerate cases, we have {222}, {220}, {114}, and {112}.

I. Equation {222}, $\varkappa = 0$. Then

$$\begin{aligned}\mathscr{P}_p^q(z) = z\Big\{1 &- \frac{z^2}{3!}\,[p(p+1) - q^2 - 2]\\ &+ \frac{z^4}{5!}\,[p(p+1)\,[p(p+1) - 2q^2 - 14] + q^4 + 20q^2 + 24] - \cdots\Big\},\end{aligned}$$

$$\begin{aligned}\mathscr{Q}_p^q(z) = 1 &- \frac{z^2}{2!}\,[p(p+1) - q^2]\\ &+ \frac{z^4}{4!}\,[p(p+1)\,[p(p+1) - 2q^2 - 6] + q^4 + 8q^2] - \cdots\end{aligned}$$

II. Equation {220}, $p = 0$.

$$\mathcal{P}_0^q(\varkappa a, z) = z\left\{1 - \frac{z^2}{3!}[\varkappa^2 a^2 - q^2 - 2] + \frac{z^4}{5!}[\varkappa^4 a^4 - 2\varkappa^2 a^2(q^2 + 4) + q^4 + 20q^2 + 24] - \cdots\right\},$$

$$\mathcal{Q}_0^q(\varkappa a, z) = 1 - \frac{z^2}{2!}[\varkappa^2 a^2 - q^2] + \frac{z^4}{4!}[\varkappa^4 a^4 - 2\varkappa^2 a^2[q^2 + 4] + q^4 + 8q^2] - \cdots,$$

III. Equation {114}, $q = 0$.

$$\mathcal{P}_p(\varkappa a, z) = z\left\{1 - \frac{z^2}{3!}[\varkappa^2 a^2 + p(p+1) - 2] + \frac{z^4}{5!}[\varkappa^4 a^4 + 2\varkappa^2 a^2[p(p+1) - 4] + p(p+1)[p(p+1) - 14] + 24] - \cdots\right\},$$

$$\mathcal{Q}_p(\varkappa a, z) = 1 - \frac{z^2}{2!}[\varkappa^2 a^2 + p(p+1)] + \frac{z^4}{4!}[\varkappa^4 a^4 + 2\varkappa^2 a^2[p(p+1) - 4] + p(p+1)[p(p+1) - 6]] - \cdots.$$

IV. Equation {112}, $\varkappa = 0$ and $q = 0$.

$$\mathcal{P}_p(z) = z\left\{1 - \frac{z^2}{3!}[p(p+1) - 1\cdot 2] + \frac{z^4}{5!}[p(p+1) - 1\cdot 2][p(p+1) - 3\cdot 4] - \frac{z^6}{7!}[p(p+1) - 1\cdot 2][p(p+1) - 3\cdot 4][p(p+1) - 5\cdot 6] + \cdots\right\},$$

$$\mathcal{Q}_p(z) = 1 - \frac{z^2}{2!}p(p+1) + \frac{z^4}{4!}p(p+1)[p(p+1) - 2\cdot 3] - \frac{z^6}{6!}p(p+1)[p(p+1) - 2\cdot 3][p(p+1) - 4\cdot 5] + \cdots.$$

$$\mathcal{P}_{p-\frac{1}{2}}(z) = z\left\{1 - \frac{z^2}{3!}[(p^2 - \tfrac{1}{4}) - 1\cdot 2] + \frac{z^4}{5!}[(p^2 - \tfrac{1}{4}) - 1\cdot 2][(p^2 - \tfrac{1}{4}) - 3\cdot 4] - \frac{z^6}{7!}[(p^2 - \tfrac{1}{4}) - 1\cdot 2][(p^2 - \tfrac{1}{4}) - 3\cdot 4][(p^2 - \tfrac{1}{4}) - 5\cdot 6] + \cdots\right\},$$

$$\mathcal{Q}_{p-\frac{1}{2}}(z) = 1 - \frac{z^2}{2!}(p^2 - \tfrac{1}{4}) + \frac{z^4}{4!}(p^2 - \tfrac{1}{4})[(p^2 - \tfrac{1}{4}) - 2\cdot 3] - \frac{z^6}{6!}(p^2 - \tfrac{1}{4})[(p^2 - \tfrac{1}{4}) - 2\cdot 3][(p^2 - \tfrac{1}{4}) - 4\cdot 5] + \cdots.$$

$$\mathcal{P}_{-\frac{1}{2}}(z) = z\left\{1 + \frac{z^2}{3!}(1\cdot 2 + \tfrac{1}{4}) + \frac{z^4}{5!}(1\cdot 2 + \tfrac{1}{4})(3\cdot 4 + \tfrac{1}{4}) + \frac{z^6}{7!}(1\cdot 2 + \tfrac{1}{4})(3\cdot 4 + \tfrac{1}{4})(5\cdot 6 + \tfrac{1}{4}) + \cdots\right\},$$

$$\mathcal{Q}_{-\frac{1}{2}}(z) = 1 + \frac{z^2}{2!\,4} + \frac{z^4}{4!\,4}[2\cdot 3 + \tfrac{1}{4}] + \frac{z^6}{6!\,4}[2\cdot 3 + \tfrac{1}{4}][4\cdot 5 + \tfrac{1}{4}] + \cdots.$$

Ordinary Legendre functions. Note that the functions $\mathcal{P}_p^q$ and $\mathcal{Q}_p^q$ do not reduce directly to the familiar Legendre [*45*] functions $P_n(z)$ and $Q_n(z)$. In obtaining the latter, one introduces other constants and makes other changes, as will be shown.

If $\varkappa = 0$, $q = 0$, and $p = n$, an integer, a simplification occurs because the infinite series become polynomials in certain cases. Evidently all B's are equal,

$$B_{2l} = p(p+1),$$

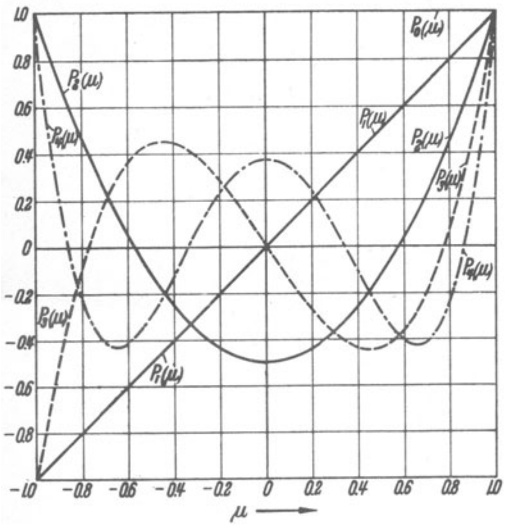


Fig. 7.07. Ordinary Legendre polynomials $P_n(\mu)$ for the range $-1 \leqq \mu \leqq +1$. The functions are alternately even and odd and all go through ± 1 at the ends of the interval

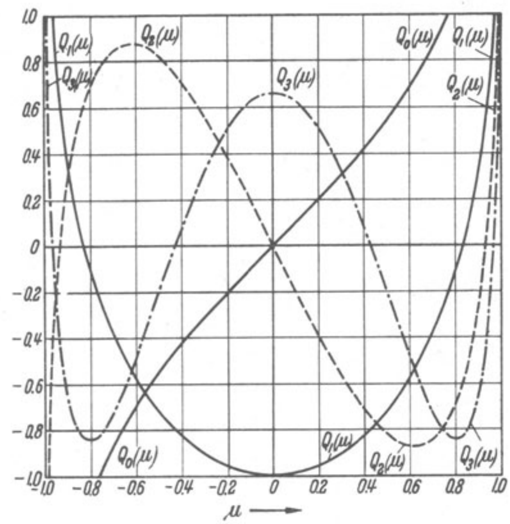


Fig. 7.08. Ordinary Legendre functions $Q_n(\mu)$. All functions become infinite at $\mu = \pm 1$

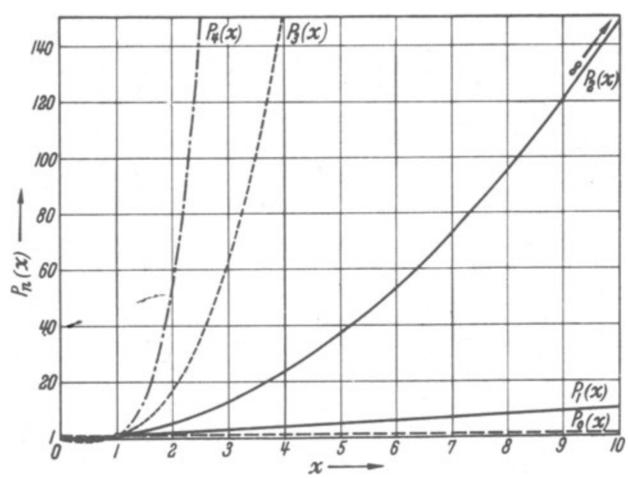


Fig. 7.09. Ordinary Legendre functions $P_n(x)$ over a wider range. Except for $P_0(x)$, all $P_n(x) \to \infty$ as $\mu \to \infty$

and the determinant $\Delta'_m(1)$ is zero if the last two rows are identical. Since the B's are equal, the two rows are identical if

$$(2m-2)(2m-1) = p(p+1) - 4m + 2$$

or

$$p(p+1) = 2m(2m-1).$$

Therefore, $p = 2m - 1$, an *odd integer*. Similarly, $\Delta'_m(0)$ becomes zero only if $p = 2m - 2$, an *even integer*.

Thus, for $\varkappa = 0$ and $p = n$, an integer, it is convenient to remove the polynomials from both Z_1 and Z_2 and to classify them as P_n. The remaining functions are called Q_n, and C_0 is so chosen that

$$\left.\begin{aligned} P_n(z) &= \frac{(-1)^{n/2}\, n!}{2^n[(n/2)!]^2} \sum_{m=0}^{\infty} \frac{(-1)^m \Delta'_m(0)\, z^{2m}}{(2m)!}, && n \text{ even}, \\ P_n(z) &= \frac{(-1)^{(n+1)/2}\, n!\, z}{2^{n-1}\left[\left(\frac{n-1}{2}\right)!\right]^2} \sum_{m=0}^{\infty} \frac{(-1)^m \Delta'_m(1)\, z^{2m}}{(2m+1)!}, && n \text{ odd}. \end{aligned}\right\} \quad (7.85)$$

$$\left.\begin{aligned} Q_n(z) &= \frac{(-1)^{n/2}\, 2^n[(n/2)!]^2\, z}{n!} \sum_{m=0}^{\infty} \frac{(-1)^m \Delta'_m(1)\, z^{2m}}{(2m+1)!}, && n \text{ even}, \\ Q_n(z) &= \frac{(-1)^{(n+1)/2}\, 2^{n-1}}{n!} \left[\left(\frac{n-1}{2}\right)!\right]^2 \sum_{m=0}^{\infty} \frac{(-1)^m \Delta'_m(0)\, z^{2m}}{(2m)!}, && n \text{ odd}. \end{aligned}\right\} \quad (7.86)$$

Fig. 7.10. Ordinary Legendre functions $Q_n(x)$. All are infinite at $x = \pm 1$; all approach zero as $x \to \infty$

$P_n(z)$ and $Q_n(z)$ for the interval $(-1, +1)$ are shown in Figs. 7.07 and 7.08. These are the *ordinary Legendre functions*, which will be employed only for $\varkappa = 0$ and $p = n$, an integer. They must be clearly distinguished from the general functions $\mathscr{P}_p^q(\varkappa a, z)$ and $\mathscr{Q}_p^q(\varkappa a, z)$, which do not reduce to Eqs. (7.85) and (7.86) but which are required except in the above special case.

The first six functions [*46*] obtained from Eq. (7.85) are

$$\begin{aligned} P_0(z) &= 1, \\ P_1(z) &= z, \\ P_2(z) &= \tfrac{1}{2}(3z^2 - 1), \\ P_3(z) &= \tfrac{1}{2}(5z^3 - 3z), \\ P_4(z) &= \tfrac{1}{8}(35z^4 - 30z^2 + 3), \\ P_5(z) &= \tfrac{1}{8}(63z^5 - 70z^3 + 15z). \end{aligned}$$

Graphs are given in Figs. 7.07 and 7.09 for real values of z.

Legendre functions of the second kind may be obtained from Eq. (7.86):

$$Q_0(z) = \tanh^{-1} z = \tfrac{1}{2} \ln\left(\frac{1+z}{1-z}\right), \qquad |z| < 1;$$

$$Q_0(z) = \coth^{-1} z = \tfrac{1}{2} \ln\left(\frac{z+1}{z-1}\right), \qquad |z| > 1;$$

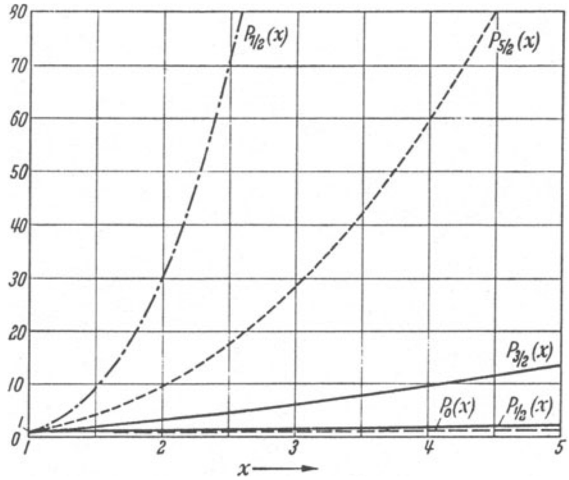


Fig. 7.11. Ordinary Legendre functions $P_{\frac{1}{2}}(x)$, $P_{\frac{3}{2}}(x)$, $P_{\frac{5}{2}}(x)$, $P_{\frac{7}{2}}(x)$

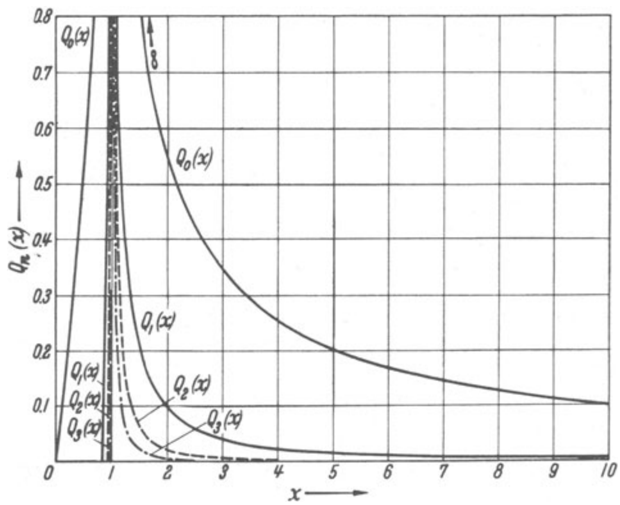


Fig. 7.12. Ordinary Legendre functions $Q_{\frac{1}{2}}(x)$, $Q_{\frac{3}{2}}(x)$, $Q_{\frac{5}{2}}(x)$, $Q_{\frac{7}{2}}(x)$

$$Q_1(z) = z\,Q_0(z) - 1,$$

$$Q_2(z) = P_2(z)\,Q_0(z) - \tfrac{3}{2} z,$$

$$Q_3(z) = P_3(z)\,Q_0(z) - \tfrac{5}{2} z^2 + \tfrac{2}{3},$$

$$Q_4(z) = P_4(z)\,Q_0(z) - \tfrac{35}{8} z^3 + \tfrac{55}{24} z,$$

$$Q_5(z) = P_5(z)\,Q_0(z) - \tfrac{63}{8} z^4 + \tfrac{49}{8} z^2 - \tfrac{8}{15}, \ldots.$$

These functions are indicated in Figs. 7.08 and 7.10. Other graphs are given in Figs. 7.11 to 7.14. The general solution of Legendre's equation {112D}, with $p = n$, is

$$Z = A\,P_n(z) + B\,Q_n(z).$$

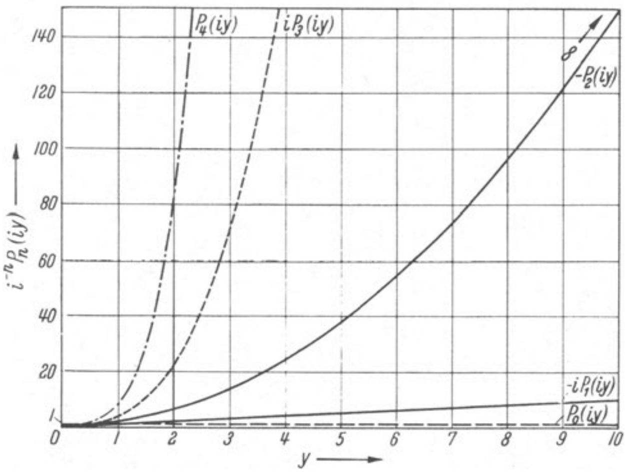


Fig. 7.13. Ordinary Legendre functions of the first kind for imaginary arguments

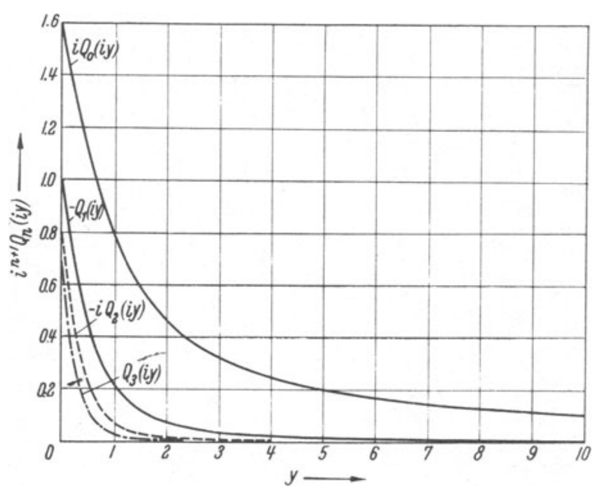


Fig. 7.14. Ordinary Legendre functions of the second kind for imaginary arguments

Legendre associated functions [45]. The general form of the Legendre equation {222} is

$$(1 - z^2)\frac{d^2Z}{dz^2} - 2z\frac{dZ}{dz} + \left[p(p+1) - \frac{q^2}{1 - z^2}\right]Z = 0. \qquad (7.87)$$

If the constants are integers, the general solution is

$$Z = A\,P_n^m(z) + B\,Q_n^m(z),$$

where P_n^m and Q_n^m are the Legendre associated functions. For $|z| < 1$,

$$\left.\begin{aligned} P_n^m(z) &= (1-z^2)^{m/2}\frac{d^m P_n(z)}{dz^m}, \\ Q_n^m(z) &= (1-z^2)^{m/2}\frac{d^m Q_n(z)}{dz^m}. \end{aligned}\right\} \tag{7.88}$$

For $|z| > 1$,

$$\left.\begin{aligned} P_n^m(z) &= (z^2-1)^{m/2}\frac{d^m P_n(z)}{dz^m}, \\ Q_n^m(z) &= (z^2-1)^{m/2}\frac{d^m Q_n(z)}{dz^m}. \end{aligned}\right\} \tag{7.88a}$$

Definite integrals

$$P_n(z) = \frac{1}{\pi}\int_0^\pi [z \pm (z^2-1)^{\frac{1}{2}}\cos u]^n\, du.$$

$$P_n(z) = \frac{1}{\pi}\int_0^\pi \frac{du}{[z \mp (z^2-1)^{\frac{1}{2}}\cos u]^{n+1}}.$$

$$P_n(z) = \frac{2}{\pi}\int_0^{\cos^{-1} z} \frac{\cos(n+\frac{1}{2})\,u\,du}{[2(\cos u - z)]^{\frac{1}{2}}}.$$

$$P_n(z) = \frac{2}{\pi}\int_{\cos^{-1} z}^{\pi} \frac{\sin(n+\frac{1}{2})\,u\,du}{[2(z - \cos u)]^{\frac{1}{2}}}.$$

$$Q_n(z) = \frac{1}{2}\int_{-1}^{1} \frac{P_n(u)}{z-u}\, du.$$

$$Q_n(z) = \int_0^\infty \frac{du}{[z + (z^2-1)^{\frac{1}{2}}\cosh u]^{n+1}}.$$

$$P_n^m(z) = \frac{2^m\, m!\,(n+m)!}{(2m)!\,(n-m)!}(z^2-1)^{m/2}\int_0^\pi \frac{\sin^{2m} u\, du}{[z + (z^2-1)^{\frac{1}{2}}\cos u]^{n+m+1}}.$$

$$Q_n^m(z) = (-1)^m \frac{2^m (n+m)!\, m!}{(n-m)!\,(2m)!}(z^2-1)^{m/2}\int_0^\infty \frac{\sinh^{2m} u\, du}{[z + (z^2-1)^{\frac{1}{2}}\cosh u]^{n+m+1}},$$

$(n-m) \geqq 0$ and z not on real axis between ± 1.

$$Q_n^m(z) = (-1)^m \frac{n!}{(n-m)!}\int_0^\infty \frac{\cosh mu\, du}{[z + (z^2-1)^{\frac{1}{2}}\cosh u]^{n+}}$$

$n \geqq m$ and z not on real axis between ± 1.

Relations

$$n P_n - (2n-1)\, z P_{n-1} + (n-1)\, P_{n-2} = 0.$$

$$(z^2-1)\frac{dP_n}{dz} = -(n+1)(z P_n - P_{n+1}).$$

$$(n+1)(P_{n+1} - z P_n) - n(z P_n - P_{n-1}) = 0.$$

$$n P_n = z\frac{dP_n}{dz} - \frac{dP_{n-1}}{dz}.$$

$$(n+1)\, P_n = -z\frac{dP_n}{dz} + \frac{dP_{n+1}}{dz}.$$

$$(2n+1)(z^2-1)\frac{dP_n}{dz} = n(n+1)(P_{n+1}-P_{n-1}).$$

$$n\,Q_n-(2n-1)\,z\,Q_{n-1}+(n-1)\,Q_{n-2}=0.$$

$$(2n+1)\,Q_n=\frac{dQ_{n+1}}{dz}-\frac{dQ_{n-1}}{dz}.$$

$$P_n\,Q_{n-1}-P_{n-1}\,Q_n=1/n.$$

$$P_n\,Q_{n-2}-P_{n-2}\,Q_n=\frac{(2n-1)\,z}{(n-1)\,n}.$$

$$P_n\frac{dQ_n}{dz}-Q_n\frac{dP_n}{dz}=(1-z^2)^{-1}.$$

$$P_n^{m+2}+2(m+1)\frac{z}{(z^2-1)^{\frac{1}{2}}}P_n^{m+1}-(n-m)(n+m+1)\,P_n^m=0.$$

$$(2n+1)\,z\,P_n^m-(n-m+1)\,P_{n+1}^m-(n+m)\,P_{n-1}^m=0.$$

$$(n-m+2)\,P_{n+2}^m-(2n+3)\,z\,P_{n+1}^m+(n+m+1)\,P_n^m=0.$$

Orthogonality. Solutions of the Legendre wave equation {224} are

$$\mathscr{P}_p^q(\varkappa a, z) \quad \text{and} \quad \mathscr{Q}_p^q(\varkappa a, z).$$

Either of these functions, with proper boundary conditions, forms an orthogonal set on an arbitrary interval (a, b). According to Table 7.01,

$$\int_a^b \mathscr{P}_{p_m}^q(\varkappa a, z)\,\mathscr{P}_{p_n}^q(\varkappa a, z)\,dz=0,$$

and

$$\int_a^b \mathscr{Q}_{p_m}^q(\varkappa a, z)\,\mathscr{Q}_{p_n}^q(\varkappa a, z)\,dz=0, \qquad m\neq n.$$

Therefore, an arbitrary function $f(z)$ may be expressed as a series of either $\mathscr{P}$ or $\mathscr{Q}$ functions,

$$f(z)=\sum_{m=0}^{\infty}A_m\,\mathscr{P}_{p_m}^q(\varkappa a, z),$$

where

$$A_m=\frac{1}{N_m}\int_a^b f(z)\,\mathscr{P}_{p_m}^q(\varkappa a, z)\,dz,$$

$$N_m=\int_a^b[\mathscr{P}_{p_m}^q(\varkappa a, z)]^2\,dz.$$

For the ordinary Legendre polynomials,

$$\int_{-1}^{1} z^k P_n(z)\,dz=0, \qquad k=0,1,2,\ldots(n-1).$$

$$\int_0^1 z^k P_n(z)\,dz=\frac{k(k-1)(k-2)\ldots(k-n+2)}{(k+n+1)(k+n-1)\ldots(k-n+3)}.$$

$$\int_{-1}^{1}P_m(z)\,P_n(z)\,dz=0, \qquad m\neq n.$$

$$\int_{-1}^{1}[P_m(z)]^2\,dz=\frac{2}{2m+1}.$$

An arbitrary function $f(z)$ may be expressed as

$$f(z) = \sum_{m=0}^{\infty} A_m P_m(z),$$

where

$$A_m = \frac{2m+1}{2} \int_{-1}^{1} f(z) P_m(z)\, dz.$$

7.13 LAMÉ FUNCTIONS

The wave functions. In the Lamé wave equation {1113}, a_1 may be set equal to zero without loss of generality. Then

$$\left.\begin{aligned} &\frac{d^2Z}{dz^2} + \frac{1}{2}\left[\frac{1}{z} + \frac{1}{z-b} + \frac{1}{z-c}\right]\frac{dZ}{dz} \\ &\quad + \frac{1}{4}\left[\frac{(b^2+c^2)\,q - p(p+1)\,z + \varkappa^2 z^2}{z(z-b)(z-c)}\right] Z = 0. \end{aligned}\right\} \tag{7.89}$$

To obtain a series solution about the origin, let

$$Z = z^{\beta} \sum_{j=0}^{\infty} C_j z^j.$$

Then, according to § 7.03,

$$A_0 = \tfrac{1}{2},$$

$$A_1 = -\frac{b+c}{2bc},$$

$$A_2 = -\frac{1}{2(bc)^2}\left[b^2 + c^2\right],$$

$$A_j = \frac{1}{bc}\left[A_{j-1}(b+c) - A_{j-2}\right] \quad \text{for } j > 2.$$

Also,

$$B_0 = 0,$$

$$B_1 = \frac{q(b^2+c^2)}{4bc},$$

$$B_2 = \frac{1}{4(bc)^2}\left[q(b+c)(b^2+c^2) - p(p+1)\,bc\right],$$

$$B_3 = \frac{1}{4(bc)^3}\left[q(b^2+c^2)(b^2+bc+c^2) - p(p+1)\,bc(b+c) + \varkappa^2(bc)^2\right],$$

$$B_j = \frac{1}{bc}\left[B_{j-1}(b+c) - B_{j-2}\right] \quad \text{for } j > 3.$$

The indicial equation is

$$\beta(\beta - \tfrac{1}{2}) = 0,$$

$b_1 = \frac{1}{2}$, $b_2 = 0$. Therefore

$$A_j(\tfrac{1}{2}) = \begin{vmatrix} \frac{1}{2}A_1 + B_1 & \frac{3}{2} & 0 & 0 & \cdots & 0 \\ \frac{1}{2}A_2 + B_2 & \frac{3}{2}A_1 + B_1 & 5 & 0 & \cdots & 0 \\ \frac{1}{2}A_3 + B_3 & \frac{3}{2}A_2 + B_2 & \frac{5}{2}A_1 + B_1 & \frac{21}{2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & (j-\frac{1}{2})(j-1) \\ \frac{1}{2}A_j + B_j & \cdots & \cdots & \cdots & \cdots & (j-\frac{1}{2})A_1 + B_1 \end{vmatrix}, \tag{7.90}$$

and

$$\Delta_j(0) = \begin{vmatrix} B_1 & \frac{1}{2} & 0 & 0 \; . \; . \; . & 0 \\ B_2 & A_1 + B_1 & 3 & 0 \; . \; . \; . & 0 \\ B_3 & A_2 + B_2 & 2A_1 + B_1 & \frac{15}{2} \; . \; . \; . & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & (j-1)(j-\frac{3}{2}) \\ B_j & A_{j-1} + B_{j-1} & \cdots & \cdots & (j-1)A_1 + B_1 \end{vmatrix}. \tag{7.91}$$

The solutions of the Lamé wave equation are

$$\left.\begin{aligned} Z_1 &= C_0 z^{\frac{1}{2}} \sum_{j=0}^{\infty} \frac{(-1)^j 2^{2j} \Delta_j(\frac{1}{2}) z^j}{(2j+1)!}, \\ Z_2 &= C_0 \sum_{j=0}^{\infty} \frac{(-1)^j 2^{2j} \Delta_j(0) z^j}{(2j)!}. \end{aligned}\right\} \tag{7.92}$$

It is customary to express Lamé functions with a different variable. If $z = \zeta^2$ and $C_0 = 1$, we have the *Lamé wave functions*

$$\left.\begin{aligned} \mathscr{E}_p^q(\varkappa, \zeta) &= \zeta \sum_{j=0}^{\infty} \frac{(-1)^j 2^{2j} \Delta_j(\frac{1}{2}) \zeta^{2j}}{(2j+1)!}, \\ \mathscr{F}_p^q(\varkappa, \zeta) &= \sum_{j=0}^{\infty} \frac{(-1)^j 2^{2j} \Delta_j(0) \zeta^{2j}}{(2j)!}. \end{aligned}\right\} \tag{7.92a}$$

The general solution of Eq. (7.89) is then

$$Z = A\mathscr{E}_p^q(\varkappa, \zeta) + B\mathscr{F}_p^q(\varkappa, \zeta). \tag{7.93}$$

The series are convergent within a circle about the origin and extending out to the nearest singularity. The same procedure, however, may be employed in expanding about any other point.

Polynomial solutions. The solutions (7.92a) and (7.93) apply also to the ordinary Lamé equation $\{1\,1\,1\,2\}$ and to the degenerate case $\{1\,1\,1\,1\}$ where $p = 0$. There are a few very special cases in which the infinite series reduce to polynomials [*47*], much as with Legendre polynomials. These are the ordinary Lamé functions $E_p^q(z)$ treated in books on the subject. Not only does $\varkappa$ have to be zero and p an integer, but q must be related to p in a definite way to allow a polynomial.

Case I. $C_0 = 1$, $\beta = 0$, $\varkappa = p = q = 0$, $\mathscr{F}_0^0(0, z) = 1$.

Case II. $C_0 = 1$, $\beta = 1$, $\varkappa = 0$, $p = 2$, $q = 1$, $\mathscr{E}_1^2(0, z) = z$.

Case III. $C_0 = 1$, $\beta = 0$, $\varkappa = 0$, $p = 2$,

$$q_1 = 2b^2c^2\left[1 + \left(1 - \frac{3b^2c^2}{(b^2+c^2)^2}\right)^{\frac{1}{2}}\right],$$

$$q_2 = 2b^2c^2\left[1 - \left(1 - \frac{3b^2c^2}{(b^2+c^2)^2}\right)^{\frac{1}{2}}\right],$$

$$\mathscr{F}_2^{q_1}(0, z) = 1 - \frac{3z^2}{(b^2+c^2) + [(b^2+c^2)^2 - 3b^2c^2]^{\frac{1}{2}}},$$

$$\mathscr{F}_2^{q_1}(0, z) = 1 - \frac{3z^2}{(b^2+c^2) - [(b^2+c^2)^2 - 3b^2c^2]^{\frac{1}{2}}}.$$

Case IV. $C_0 = 1,\ \beta = 1,\ \varkappa = 0,\ p = 3,$

$$q_1 = 1 - \frac{6b^2c^2}{5}\left[2 + \left(4 - \frac{15b^2c^2}{(b^2+c^2)^2}\right)^{\frac{1}{2}}\right],$$

$$q_2 = 1 - \frac{6b^2c^2}{5}\left[2 - \left(4 - \frac{15b^2c^2}{(b^2+c^2)^2}\right)^{\frac{1}{2}}\right],$$

$$\mathscr{E}_3^{q_1}(0, z) = 1 - \frac{5z^3}{2(b^2+c^2) + [4(b^2+c^2)^2 - 15b^2c^2]^{\frac{1}{2}}},$$

$$\mathscr{E}_3^{q_2}(0, z) = 1 - \frac{5z^3}{2(b^2+c^2) - [4(b^2+c^2)^2 - 15b^2c^2]^{\frac{1}{2}}}.$$

Polynomial solutions of higher degree also occur for discrete values of p and q.

Other interesting special cases can be expressed in closed form:

$$\mathscr{F}_1^q(0, z) = \frac{1}{ib}(z^2 - b^2)^{\frac{1}{2}} \quad \text{if} \quad q = \frac{c^2}{b^2+c^2},$$

$$\mathscr{F}_1^q(0, z) = \frac{1}{ic}(z^2 - c^2)^{\frac{1}{2}} \quad \text{if} \quad q = \frac{b^2}{b^2+c^2},$$

$$\mathscr{F}_2^1(0, z) = \frac{1}{bc}[(z^2 - b^2)(z^2 - c^2)]^{\frac{1}{2}}.$$

Related to these solutions are further solutions which reduce to polynomials times square roots, for suitable values of p and q.

Orthogonality. Solutions of the Lamé wave equation {1113} are $\mathscr{E}_p^q(\varkappa, \zeta)$ and $\mathscr{F}_p^q(\varkappa, \zeta)$. These functions may be employed in expressing an arbitrary function $f(\zeta)$ over an arbitrary interval (a, b). According to Table 7.01, orthogonality is obtained with a weighting function, assuming Sturm-Liouville boundary conditions. Thus, for example,

$$\int_a^b \frac{z}{u}\, \mathscr{E}_{p_m}^q(\varkappa, \zeta)\, \mathscr{E}_{p_n}^q(\varkappa, \zeta)\, d\zeta = 0, \qquad m \neq n.$$

Thus,

$$f(\zeta) = \sum_{m=0}^{\infty} A_m\, \mathscr{E}_{p_m}^q(\varkappa, \zeta),$$

where

$$A_m = \frac{1}{N_m}\int_a^b \frac{\zeta^2}{u} f(\zeta)\, \mathscr{E}_{p_m}^q(\varkappa, \zeta)\, d\zeta,$$

$$N_m = \int_a^b \frac{\zeta^2}{u}[\mathscr{E}_{p_m}^q(\varkappa, \zeta)]^2\, d\zeta.$$

A similar expansion can be made on q_m, or the $\mathscr{F}$-functions can be employed instead of the $\mathscr{E}$-functions.

7.14 WANGERIN FUNCTIONS

Equation {1122} is written [48]

$$\frac{d^2Z}{dz^2} + \frac{1}{2}\left[\frac{1}{z-b} + \frac{1}{z-c} + \frac{2}{z}\right]\frac{dZ}{dz} + \frac{1}{4}\left[\frac{\bar{A}_0 + \bar{A}_1 z + \bar{A}_2 z^2}{z^2(z-b)(z-c)}\right]Z = 0, \quad (7.94)$$

if $a_3 = 0$. For a solution about the origin,

$$Z = z^{\beta}\sum_{j=0}^{\infty} C_j z^j.$$

The method of § 7.02 then gives

$$A_0 = 1,$$
$$A_1 = -\frac{b+c}{2bc},$$
$$A_2 = -\frac{1}{2(bc)^2}[b^2+c^2],$$
$$A_j = \frac{1}{bc}[A_{j-1}(b+c) - A_{j-2}], \qquad j > 2.$$

Also,

$$B_0 = \frac{\bar{A}_0}{4bc},$$
$$B_1 = \frac{1}{4(bc)^2}[\bar{A}_0(b+c) + \bar{A}_1 bc],$$
$$B_2 = \frac{1}{4(bc)^3}[\bar{A}_0(b^2+bc+c^2) + \bar{A}_1 bc(b+c) + \bar{A}_2(bc)^2],$$
$$B_j = \frac{1}{bc}[B_{j-1}(b+c) - B_{j-2}], \qquad j > 2.$$

Therefore, the determinant is

$$\Delta_j(\beta) = \begin{vmatrix} \beta A_1 + B_1 & (\beta+1)^2 + B_0 & 0 & 0 \ldots & 0 \\ \beta A_2 + B_2 & (\beta+1)A_1 + B_1 & (\beta+2)^2 + B_0 & 0 \ldots & 0 \\ \beta A_3 + B_3 & (\beta+1)A_2 + B_2 & (\beta+2)A_1 + B_1 & \ldots\ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & (\beta+j-1)^2 + B_0 \\ \beta A_j + B_j & (\beta+1)A_{j-1} + B_{j-1} & \cdots & \cdots & (\beta+j-1)A_1 + B_1 \end{vmatrix}. \quad (7.95)$$

The indicial equation gives

$$b_1 = +\left(-\frac{\bar{A}_0}{4bc}\right)^{\frac{1}{2}}, \quad b_2 = -\left(-\frac{\bar{A}_0}{4bc}\right)^{\frac{1}{2}},$$

and the solutions are

$$\left.\begin{aligned} Z_1 &= C_0 z^{b_1} \sum_{j=0}^{\infty} \frac{(-1)^j \Delta_j(b_1) z^j}{\prod_{l=1}^{j} [(b_1+l)^2 + B_0]}, \\ Z_2 &= C_0 \sum_{j=0}^{\infty} \frac{(-1)^j \Delta_j(b_2) z^j}{\prod_{l=1}^{j} [(b_2+l)^2 + B_0]}, \end{aligned}\right\} \quad (7.96)$$

if b_1 and b_2 are distinct and do not differ by an integer.

The *Wangerin functions* may be defined for $C_0 = 1$, $b = 1$, $c = 1/k^2$, $\bar{A}_0 = -q^2 c$, $\bar{A}_1 = -p^2 c$, $\bar{A}_2 = 1 - q^2$. Then Eq. (7.96) becomes

$$\left.\begin{aligned} \mathscr{S}_p^q(k,z) &= z^{q/2} \sum_{m=0}^{\infty} \frac{(-1)^m \Delta_m(q/2) z^m}{\prod_{l=1}^{m} [(l+q/2)^2 - q^2/4]}, \\ \mathscr{T}_p^q(k,z) &= \sum_{m=0}^{\infty} \frac{(-1)^m \Delta_m(-q/2) z^m}{\prod_{l=1}^{m} [(l-q/2)^2 - q^2/4]}, \end{aligned}\right\} \quad (7.96\text{a})$$

where

$$A_0 = 1,$$
$$A_1 = -\left(\frac{1+c}{2c}\right),$$
$$A_2 = -\left(\frac{(1+c^2)}{2c^2}\right),$$
$$A_j = \frac{1}{c}\,[A_{j-1}(1+c) - A_{j-2}] \quad \text{for } j > 2:$$
$$B_0 = -q^2/4,$$
$$B_1 = -\frac{1}{4c}\,[q^2(1+c) + p^2],$$
$$B_2 = -\frac{1}{4c^2}\,[q^2(1+c)^2 + p^2(1+c) - c],$$
$$B_j = \frac{1}{c}\,[B_{j-1}(1+c) - B_{j-2}].$$

Equation (7.96a) converges for a circular region with center at the origin and extending out to the next singularity. The same procedure allows the $\mathscr{S}$ and $\mathscr{T}$ functions to be expanded about any other point.

Orthogonality. The Wangerin functions $\mathscr{S}_p^q(k, z)$ and $\mathscr{T}_p^q(k, z)$ form orthogonal sets, with weighting function, on an arbitrary interval (a, b). With Sturm-Liouville boundary conditions and with weighting function $w = z/u$,

$$\int_a^b \frac{z}{u}\,\mathscr{S}_{p_m}^q(k, z)\,\mathscr{S}_{p_n}^q(k, z)\,dz = 0, \qquad m \neq n.$$

An arbitrary function $f(z)$ may be expressed therefore as

$$f(z) = \sum_{m=0}^{\infty} A_m\,\mathscr{S}_{p_m}^q(k, z)\,dz,$$

where

$$A_m = \frac{1}{N_m}\int_a^b \frac{z}{u}\,f(z)\,\mathscr{S}_{p_m}^q(k, z)\,dz,$$

$$N_m = \int_a^b \frac{z}{u}\,[\mathscr{S}_{p_m}^q(k, z)]^2\,dz.$$

This result applies equally well to the $\mathscr{T}$-function.

7.15 HEINE FUNCTIONS

The Heine equation {1222} is

$$\frac{d^2Z}{dz^2} + \frac{1}{2}\left[\frac{1}{z} + \frac{2}{z-b} + \frac{2}{z-c}\right]\frac{dZ}{dz} + \frac{1}{4}\left[\frac{\bar{A}_0 + \bar{A}_1 z + \bar{A}_2 z^2 + \bar{A}_3 z^3}{z(z-b)^2(z-c)^2}\right]Z = 0, \quad (7.97)$$

if $a_1 = 0$. By the method of § 7.03, expanding about $z = 0$,

$$A_0 = \tfrac{1}{2},$$
$$A_1 = -\frac{(b+c)}{bc},$$
$$A_2 = -\frac{(b^2+c^2)}{(bc)^2},$$
$$A_j = \frac{1}{bc}\,[A_{j-1}(b+c) - A_{j-2}] \quad \text{for } j > 2.$$

Also,

$$B_0 = 0,$$

$$B_1 = \frac{\bar{A}_0}{4(bc)^2},$$

$$B_2 = \frac{1}{4(bc)^3}\,[2(b+c)\,\bar{A}_0 + bc\,\bar{A}_1],$$

$$B_3 = \frac{1}{4(bc)^4}\,[(3b^2 + 4bc + 3c^2)\,\bar{A}_0 + 2bc(b+c)\,\bar{A}_1 + (bc)^2\,\bar{A}_2],$$

$$B_4 = \frac{1}{4(bc)^5}\,[2(b+c)(2b^2 + bc + 2c^2)\,\bar{A}_0$$

$$+ bc(3b^2 + bc + 3c^2)\,\bar{A}_1 + 2(bc)^2(b+c)\,\bar{A}_2 + (bc)^3\,\bar{A}_3],$$

$$B_j = \frac{1}{(bc)^2}\,[2B_{j-1}\,bc(b+c) - B_{j-2}(b^2 + 4bc + c^2)$$

$$+ 2B_{j-3}(b+c) - B_{j-4}] \quad \text{for } j > 4.$$

The indicial equation is

$$\beta(\beta - \tfrac{1}{2}) = 0,$$

so $b_1 = \frac{1}{2}$, $b_2 = 0$. For $\beta = b_1 = \frac{1}{2}$, the determinant is

$$\Delta_j(\tfrac{1}{2}) = \left|\begin{array}{llllll} \tfrac{1}{2}A_1 + B_1 & \tfrac{3}{2} & 0 & 0 & \cdots & 0 \\ \tfrac{1}{2}A_2 + B_2 & \tfrac{3}{2}A_1 + B_1 & \tfrac{10}{2} & 0 & \cdots & 0 \\ \tfrac{1}{2}A_3 + B_3 & \tfrac{3}{2}A_2 + B_2 & \tfrac{5}{2}A_1 + B_1 & \tfrac{21}{2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & (j-\tfrac{1}{2})(j-1) \\ \tfrac{1}{2}A_j + B_j & \cdots & \cdots & \cdots & \cdots & (j-\tfrac{1}{2})A_1 + B_1 \end{array}\right|.$$

For $\beta = b_2 = 0$,

$$\Delta_j(0) = \left|\begin{array}{llllll} B_1 & \tfrac{1}{2} & 0 & 0 & \cdots & 0 \\ B_2 & A_1 + B_1 & 3 & 0 & \cdots & 0 \\ B_3 & A_2 + B_2 & 2A_1 + B_1 & \tfrac{15}{2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & (j-1)(j-1-\tfrac{1}{2}) \\ B_j & A_{j-1} + B_{j-1} & \cdots & \cdots & \cdots & (j-1)A_1 + B_1 \end{array}\right|.$$

Thus the solutions of Eq. (7.97) are

$$\left.\begin{aligned} Z_1 &= C_0 z^{\frac{1}{2}} \sum_{j=0}^{\infty} \frac{(-1)^j\, 2^{2j}\, \Delta_j(\frac{1}{2})\, z^j}{(2j+1)!}, \\ Z_2 &= C_0 \sum_{j=0}^{\infty} \frac{(-1)^j\, 2^{2j}\, \Delta_j(0)\, z^j}{(2j)!}. \end{aligned}\right\} \tag{7.98}$$

These series are convergent within a circular region whose center is at the origin and whose perimeter passes through the nearest singularity ($z = b$).

The form of Eq. (7.97) needed in field theory has $b=1$, $\bar{A}_0=-p^2c^2$, $\bar{A}_1=p^2c(1+c)-q^2(1-c)^2+2c$, $\bar{A}_2=-[p^2c+2(1+c)]$, $\bar{A}_3=2$. Setting $C_0=1$ and $c=1/k^2$, we obtain the *Heine functions:*

$$\left.\begin{aligned} \mathcal{U}_p^q(k,z) &= z^{\frac{1}{2}}\sum_{j=0}^{\infty}\frac{(-1)^j 2^{2j}\Delta_j(\frac{1}{2})\,z^j}{(2j+1)!}, \\ \mathcal{V}_p^q(k,z) &= \sum_{j=0}^{\infty}\frac{(-1)^j 2^{2j}\Delta_j(0)\,z^j}{(2j)!}, \end{aligned}\right\} \tag{7.99}$$

where

$$\begin{aligned} A_0 &= \tfrac{1}{2}, \\ A_1 &= -(1+c)/c, \\ A_2 &= -(1+c^2)/c^2, \\ A_j &= \frac{1}{c}[A_{j-1}(1+c)-A_{j-2}] \qquad \text{for } j>2; \\ B_0 &= 0, \\ B_1 &= -\frac{p^2}{4c}, \\ B_2 &= -\frac{1}{4c^2}[p^2(1+c)+q^2-2c], \\ B_3 &= -\frac{1}{4c^3}[p^2(1+c+c^2)+2q^2(1+c)-2c(1+c)], \ldots. \end{aligned}$$

Orthogonality. Like the other functions of this book, the Heine functions form orthogonal sets on the arbitrary interval (a, b) with respect to a weighting function $w(z)$. With Sturm-Liouville boundary conditions,

$$\int_a^b w(z)\,\mathcal{U}_{p_m}^q(k,z)\,\mathcal{U}_{p_n}^q(k,z)\,dz=0, \qquad m\neq n.$$

Therefore, an arbitrary function of z may be expressed as

$$f(z)=\sum_{m=0}^{\infty}A_m\,\mathcal{U}_{p_m}^q(k,z),$$

where

$$A_m=\frac{1}{N_m}\int_a^b w(z)\,f(z)\,\mathcal{U}_{p_m}^q(k,z)\,dz,$$

$$N_m=\int_a^b w(z)\,[\mathcal{U}_{p_m}^q(k,z)]^2\,dz.$$

These relations hold also for the functions $\mathcal{V}_p^q(k,z)$, the weighting functions being taken from Table 7.01.

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APPENDIX

SYMBOLS USED IN THE TEXT

A, B, C, D = constants.

A_i = coefficients in series expansion of $[(z - z_0)\, P(z)]$, Section VII.

$\bar{A}_0, \bar{A}_1, \ldots$ = constants in Q-term of a Bôcher equation.

$\mathscr{A}$ = area.

a = distance to focus.

$a, b, c,$ a_1, a_2, a_3 = constants.

$\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ $\boldsymbol{a}_x, \boldsymbol{a}_y, \boldsymbol{a}_z$ = unit vectors.

B_i = coefficients in series expansion of $[(z - z_0)^2\, Q(z)]$, Section VII.

$\mathscr{B}_p^q$ = Baer function.

b_1, b_2 = roots of indicial equation, § 7.02.

C_n = coefficients in series expansion of Z.

$\mathscr{C}_p^q$ = Baer function.

c = a constant.

ce_m = a Mathieu function.

$D_j(\beta) = (\beta - b_2) \cdot C_j(\beta)$, § 7.01.

$\boldsymbol{E}$ = a vector.

E_1, E_2, E_3 = components of the vector $\boldsymbol{E}$.

E_n^q = ordinary Lamé polynomial.

$\mathscr{E}_p^q$ = generalized Lamé function.

$e = 2.71828\ldots$

F_n^q = ordinary Lamé function of second kind.

$\mathscr{F}_p^q$ = generalized Lamé function of the second kind.

f_i = a function of u^i.

fe_m = a Mathieu function.

g_{11}, g_{22}, g_{33} = metric coefficients, § 1.01.

$g = g_{11}\, g_{22}\, g_{33}$.

ge_m = a Mathieu function.

h = a constant.

$i = \sqrt{-1}$.

$\mathscr{I}_p$ = Bessel function of first kind.

$$K = \int_0^{\pi/2} \frac{d\varphi}{(1 - k^2 \operatorname{sn}^2 \varphi)^{\frac{1}{2}}},\ \text{complete elliptic integral.}$$

$$K' = \int_0^{\pi/2} \frac{d\varphi}{(1 - k'^2 \operatorname{sn}^2 \varphi)^{\frac{1}{2}}},\ \text{complete elliptic integral.}$$

$k = b_1 - b_2$, § 7.02.

k = modulus of elliptic function.
$k' = (1 - k^2)^{\frac{1}{2}}$.
L = an operator, Section VII.
M_{ij} = cofactor of element Φ_{ij} in the Stäckel matrix.
m, n = integers.
N_n = nth norm.
$P_n(z)$ = ordinary Legendre polynomial.
$P_n^m(z)$ = Legendre associated function.
$P(z)$ = coefficient in Bôcher equation, Section VI.
$\mathscr{P}_p^q$ = generalized Legendre functions of first kind.
p, q = constants, not necessarily integers.
$Q_n(z)$ = ordinary Legendre function of second kind.
$Q_n^m(z)$ = Legendre associated function.
Q = a function of u^1, u^2, u^3 used in R-separation (Section IV).
$Q(z)$ = coefficient in Bôcher equation.
$\mathscr{Q}_p^q$ = generalized Legendre function of second kind.
$R(r)$ = a function of r.
R = a function of u^1, u^2, u^3 used in R-separation (Section IV).
r, θ, λ = conical coordinates (Table 1.09).
r, θ, ψ = spherical coordinates (Table 1.05).
r, ψ, z = circular-cylinder coordinates (Table 1.02).
S = Stäckel determinant.
$[S]$ = Stäckel matrix.
$\mathscr{S}_p^q$ = Wangerin function.
s = distance.
se_m = a Mathieu function.
T = a function of time.
$\mathscr{T}_p^q$ = Wangerin function.
t = time.
U^i = a function of u^i.
$\mathscr{U}_p^q$ = Heine function.
u^1, u^2, u^3 = general coordinates.
$u(z), v(z), w(z)$ = parameters in the Sturm-Liouville equation (§ 7.07).
$\mathscr{V}$ = volume.
$\mathscr{V}_p^q$ = Heine function.
$\mathscr{W}_e, \mathscr{W}_o$ = Weber functions.
$w = u + iv$, a complex number (Section II).
X, Y, Z = functions of x, y, z, respectively.
x, y, z = rectangular coordinates.
$\mathscr{Y}_n$ = Bessel function of the second kind.
Z = a function of z.
Z_1, Z_2 = independent solutions of a Bôcher equation.
$\mathscr{Z}_p$ = generic designation of Bessel functions, including Bessel functions of first, second, and third kinds and their linear combinations.
z_0 = singular point.
$z = x + iy$, a complex number.
$\bar{z}$ = complex conjugate of z (Section II).

α_j = separation constant.
β = a variable (Section VII).
Γ = gamma function (Section VII).
γ = a constant.
$\gamma = 0.5772157\ldots$, Euler's constant (Section VII).
Δ_j = a determinant.
ζ = independent variable.
$H(\eta)$ = a function of η.
η = a coordinate in elliptic-cylinder coordinates (and in other systems).
η, θ, λ = ellipsoidal coordinates (Table 1.10).
η, ψ, z = elliptic-cylinder coordinates (Table 1.03), and other cylindrical systems (Section III).
Θ = a function of θ.
θ = angle from z-axis.
$\varkappa$ = constant in Helmholtz equation.
Λ = a function of λ.
$M(\mu)$ = a function of μ.
μ, ν = coordinates.
μ, ν, z = parabolic-cylinder coordinates (Table 1.04), and other cylindrical systems (Section III).
μ, ν, ψ = parabolic coordinates (Table 1.08), and other rotational systems (Section IV).
μ, ν, λ = paraboloidal coordinates (Table 1.11).
$N(\nu)$ = a function of ν.
ξ = a coordinate in alternative circular-cylinder coordinates and in alternative spherical coordinates.
$\pi = 3.14159\ldots$
Φ_{ij} = element in Stäckel matrix.
φ = scalar potential.
Ψ = a function of ψ.
ψ = angle about the z-axis.
Ω = a solution of Laplace's equation (Section I).
∇^2 = scalar Laplacian.
✡ = vector Laplacian.

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