THE UNIVERSITY OF CHICAGO

DATE July 13, 1964

Zeman, J. Jay — Author
July 15, 1934 — Birth Date

The Graphical Logic of C. S. Peirce — Title of Dissertation

Philosophy — Department or School
Ph.D. — Degree
September, 1964 — Convocation

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THE GRAPHICAL LOGIC OF C. S. PEIRCE

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE HUMANITIES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF PHILOSOPHY

BY

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CHICAGO, ILLINOIS

SEPTEMBER, 1964

PREFACE

Virtually all of the material from the writings of Peirce upon which we have drawn in this paper is contained in The Collected Papers of C. S. Peirce.[1] In referring to these volumes we have adopted the method of reference in general use among Peirce scholars. Thus, "4.567" will mean paragraph 567 of Volume IV of the Collected Papers.

The principal aim of this paper is the study of Peirce's graphs as logics, employing in the study modern logical methods. The notations and abbreviations we employ are accepted as standard by contemporary logicians; we shall mention specifically that we employ "iff" as an abbreviation for "If and only if." This abbreviation, although of fairly recent vintage, may now be considered standard in logical and mathematical literature. Also, we often use single quotes in a technical manner, not to be confused with their use in indicating a quotation within a quotation. The complex consisting of a sign enclosed by single quotes in this usage is the name of the sign; note that all sentence punctuation, including periods and commas, will be placed *outside* the quotes in this case. This usage of single quotes is well established in the literature of logic. Another sign we shall mention is '▮'. This symbol will be used to

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[1] C. S. Peirce, The Collected Papers of C. S. Peirce, ed. Charles Hartshorne and Paul Weiss (Cambridge: Harvard, 1960), Vols. I-VI.

indicate the conclusion of the proof of a lemma, metatheorem, or corollary in this paper.

We have numbered in this paper the definitions, certain "rules of formation," and the lemmas, metatheorems, and corollaries. The numbering of definitions is consecutive throughout the paper; a 'D' followed immediately by an Arabic numeral indicates a definition. The numbering of rules of formation follows the numbering of sections throughout this paper; the number of a rule of formation will be the number of the section within which it occurs in Arabic numerals followed immediately by the number of the rule within that section in lower-case Roman numerals. Thus '1.21ii' would be the number of the second rule of formation in Section 1.21.

Lemmas, metatheorems, and corollaries are numbered consecutively in the chapter in which they occur, using a "decimal" notation. The number '2.09' would number the ninth lemma, metatheorem, or corollary of chapter ii. Whether a number indicates a metatheorem, a lemma, or a corollary is indicated by prefixing to it either an asterisk, '*', the word 'lemma', or the word 'corollary', respectively.

My very special thanks are due to Professor A. N. Prior who originally put me on the track of the graphs and steered me through my first months of research on them. To Professors Dudley Shapere, Sylvain Bromberger, Lars Svenonius, and Manley Thompson I am indebted; their criticism and suggestions were of great aid in the preparation of this paper. And finally, I thankfully acknowledge the heckling and help (with emphasis on the latter) of my wife, Bobbie; without the heckling my work would have been much less interesting, and without the help, perhaps impossible.

JJZ

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INTRODUCTION

A casual thumbing through Volumes III and IV of the Collected Papers of C. S. Peirce will turn up a fair number of kinds of diagrams, each of which has some claim to the title "Logical Diagram" or "Logical Graph." In this paper I shall examine in detail one family of these diagrams--or better, of systems of these diagrams--to which the name "Graphical Logic" may fairly be applied. These systems are the systems of "Existential Graphs," and the principal material on them is located in Volume IV of the Collected Papers, in paragraphs 372 to 584.

Peirce developed these systems about the turn of the century--Murray Murphey gives the year as 1896.[1] Peirce called the graphs his "*chef d'oeuvre*,"[2] but for the *chef d'oeuvre* of one of the great logicians, they have received scant attention till now. The present work aims at a thorough study of the existential graphs, using the methods of contemporary logic. I shall attempt to determine what the existential graphs are as systems, and how certain of these systems may be fruitfully extended in the light of modern formal logic.

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[1]Murray G. Murphey, *The Development of Peirce's Philosophy* (Cambridge: Harvard, 1961), p. 357.

[2]Peirce thus subtitles what is Book 2 of Volume IV of *The Collected Papers*. At this point we shall note again what we mentioned in the Preface, that citations from this collection shall generally be listed in the text of the paper without footnote, employing the decimal notation normally employed in Peirce scholarship; thus "4.327," for example, will mean paragraph 327 of Volume IV of the *Collected Papers*.

Although the primary aim of this paper is a study of the graphs as formal systems, I shall also devote some space in this introduction to a brief investigation of the graphs as part of Peirce's philosophy; I shall here present my views on why Peirce considered the existential graphs to be his most important work.

By 1885 Peirce had developed his "algebra of logic" into a fairly rich system or group of systems (3.359 ff.). He had definite ideas as to the <u>raison d'etre</u> of a system of symbolic logic; the purpose of such a system is "simply and solely the investigation of the theory of logic, and not at all the construction of a calculus to aid the drawing of inferences" (4.373). Some comment on Peirce's use of the word calculus is in order here. It is fairly clear that he was thinking of a calculus as a "computing aid" of some kind, possibly a system to be used as logarithms, for instance, frequently are. Peirce saw "calculi" as systems which would reduce to a minimum the number of steps in a deduction from premises to a conclusion. A logic, on the other hand, would be a system which would break down the steps in the deduction to the smallest possible units and thereby <u>exhibit</u> the deductive process involved. For Peirce, then, a calculus is a tool for turning out answers to specific problems, while a logic is a tool for investigating the deductive process itself. (We have taken time to distinguish between Peirce's use of these terms since "logic" is often taken to mean the same as "calculus" in contemporary literature on logic.)

Referring to the investigation of the deductive process, Peirce states, "This, then, is the purpose for which my logical algebras were designed" (4.429). But he adds, "In my opinion, they do not sufficiently fulfill" this purpose (4.429).

"This purpose," Peirce felt, was better fulfilled by the systems of existential graphs than by the algebra of logic. We shall examine in some detail the reasons why Peirce felt this way; First, however, we shall see what the existential graphs are. We shall familiarize ourselves with the symbols of these systems and with the rules that govern these systems; this will be a helpful--or even necessary--prologue both to our examination of the graphs as Peirce saw them and to our rigorous formal study of the graphs as logical systems, which will commence in chapter i.

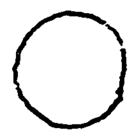
Symbols of the Existential Graphs

Before we consider the graphs as part of Peirce's philosophy or enter into a rigorous formal study of these systems, we shall introduce ourselves to the terminology and deductive method of the graphs, and we shall note some strands which connect the various systems.

First of all, Peirce defines "graph" as "the propositional expression in the System of Existential Graphs of any possible state of the universe" (4.359). The existential graphs are then intended by Peirce to be systems of "propositions" or "assertions."

Peirce presents the graphs as three general systems called, respectively, "alpha," "beta," and "gamma." This division corresponds fairly well to his division of the "algebra of logic" into "non-relative logic," "first-intentional logic of relations," and "second-intentional logic of relations" (3.359 ff.). The first and possibly most important sign of the graphical systems--one common to all three of them--is the "sheet of assertion" (for which we shall often use the abbreviation "SA"). SA is a surface upon which the graphs are to be "scribed" according to the "rules of transformation" of the systems. SA itself, even before any of these rules has been applied to it, is to be considered a graph (4.396). Also, according to Peirce, the SA is to be taken as "representing the universe of discourse, and as asserting whatever is taken for granted . . . to be true of that universe" (4.396). The SA in its "initial state" (before any of the rules we shall subsequently state has been applied to it), then, may be considered to represent a kind of "postulate set" to be operated upon by appropriate rules. The content of the "postulates" will depend upon what we wish to "reason about" with the graphs. We might imagine a sheet of assertion, for example, being set up initially to "deductively reason about" one of those cute little problems we find in the Lewis Carroll logic books. In this case we would consider the special "extralogical" premises or postualates needed for the problem at hand to be part of the initial SA. For the purpose of this paper, that of examining the graphs as formal systems, we shall find that very little in the way of such postulates or premises is needed. For alpha, as an example, all we will need to begin with is a completely blank SA.

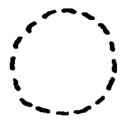
Among the signs which may, under appropriate circumstances, be scribed upon SA is one common to all three general graph systems. In some places (as in 4.399) Peirce calls this sign a "cut"; in others (as in 4.435) he refers to it as a "sep"--from



The Alpha Cut

the Latin *saepes*, "fence." The "alpha cut," which we will ordinarily refer to simply as the "cut," except when there is danger of confusion, is defined by Peirce as "a self-returning linear separation (naturally represented by a finely drawn or peculiarly colored line) which severs all that it encloses from the sheet of assertion on which is stands itself, or from any other area on which it stands itself" (4.399). The cut "cuts off" what it encloses from the area on which it stands. If a graph is an assertion, a sentence, then the enclosure of that graph by a cut is, in effect, an assertion that the enclosed graph is *not* asserted. The cut, then, may be considered a negation sign, inasmuch as it cuts something off from the asserted universe of discourse represented by SA. We shall find, indeed, that the alpha cut is completely analogous to the ordinary negation sign of the propositional calculus. And if two or more graphs occur in the *same* area, with *no* cuts between them, they may be considered to be *asserted simultaneously* in that area; such "unseparated occurrence" of graphs is analogous to PC conjunction.

Peirce mentions in passing other kinds of cuts than the one referred to above. The one of these other cuts to which he gives a significant amount of attention is one belonging to the gamma part of the graphs; it may be called the "broken cut." As



The Broken Cut

The "Scroll"--Two Versions

the name implies, it consists of a broken rather than a continuous line (4.516) and may be interpreted as asserting "possibly not" of its contents; a system containing the broken cut among its signs may therefore be considered a modal system.

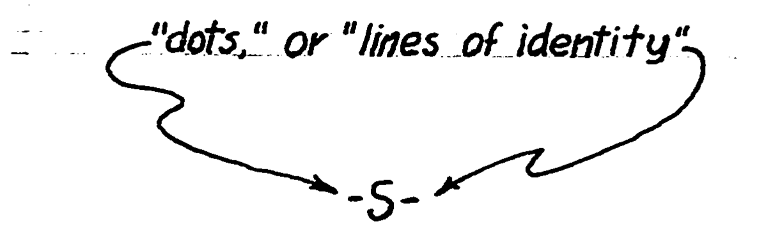
In 4.436 and 4.437 we find Peirce referring to what we might want to consider a superfluous sign, and referring to it in language which indicates that he thought of it as a primitive kind of sign. The sign is the "scroll," which consists of two cuts, one enclosing the other, whether they are connected by a node or not. Although the scroll is merely two cuts, and although we shall seldom refer to it by name, we shall find that the notion of the scroll--that is, of two cuts acting as a unit--does play a fairly important part at certain phases of our detailed discussion of alpha in chapter i.

Next we shall consider signs characteristic of each of the systems. The alpha system might be considered a set of graphical manipulations of "unanalysed" statements; that is, the "minimal unit" of alpha is a graph representing a complete, closed sentence. Alpha does not contain the apparatus for expressing or analysing the components of its minimal or atomic sentences; in this it is much like contemporary propositional calculi. We may, in fact, consider alpha to contain, among its signs, "propositional variables" directly analogous to those of the ordinary PC; Peirce himself employs letters of the Roman alphabet in his alpha graphs much as we use propositional variables in the PC.

We remarked earlier that comparisons could be drawn between Peirce's division of the existential graphs and division of the algebra of logic. In such a comparison, the beta graphs would be juxtaposed to the "first-intentional logic of relatives." In the one case as in the other, we enter the field of "analysed atomic sentences," sentences which contain a predicate and signs representing individuals for which the predicate holds iff the sentence is true. Peirce's favorite term for such a predicate is "rhema," or "rheme" (3.420 ff., 4.438 ff.). The "unanalysed expression" of a rheme in the systems of existential graphs--specifically, in beta--is called by Peirce a "spot" (4.403; 4.441). Some explication may be in order. A rheme is "a blank form of proposition produced by . . . erasures [the spaces left by which] can be filled, each with a proper name, to make a proposition again" (4.438). In other words, "____ is good," and "____ gives ____ to ____" are examples of rhemes, the first being monadic (or unary) and the other triadic (or ternary). We see from 4.438 that Peirce recognizes a 0-adic rheme, a rheme which, since it contains no blanks, is already a sentence; the unanalysed expression in the system of graphs of such a rheme might be considered to be one of the "atomic graphs" of alpha.

The "spot," then, as the unanalysed expression of a rheme with one or more blanks, would be a kind of predicate symbol, analogous to symbols with that function in the predicate calculus of today.

Peirce specifies that "on the periphery of every spot, a certain place shall be appropriated to each blank of the rheme, and such a place shall be called a 'hook' of the spot" (4.403). He wished the graphs to be graphic; the hooks are to be conceived of as connecting the predicate in question to the signs representing the individuals of which it is true. Strangely enough, however,



Binary Spot with "Dots" Attached to its Hooks

we shall not in practice see these hooks; in any graph properly so called, all hooks are already filled, connected to the appropriate signs for individuals. A spot with empty hooks would not even be analogous to an open sentence in the predicate calculus; it would rather be like the non-well-formed-formula which an n-ary predicate followed by fewer than n individual variables would be.

When we come to the method of representation of individuals in beta, we come to a point of marked difference between the notation of beta and that of ordinary predicate calculi. In an ordinary predicate calculus, individuals are referred to by individual variables, which variables are capable of standing as arguments of predicates of the system. In beta, however,

> A heavy dot attached at the hook of a spot shall be understood as filling the corresponding blank of the rheme of the spot with an indefinite sign of an individual, so that when there is a dot attached to every hook, the result shall be a proposition (4.404).

and further (we state paragraphs directly from CP):

> 4.405 Every heavily marked point, whether isolated, the extremity of a heavy line, or at the furcation of a heavy line, shall denote a single individual, without itself indicating what individual it is.

> 4.406 A heavily marked line without any sort of interruption . . . shall, under the name of a line of identity, be a graph, subject to all the conventions relating to graphs, and asserting precisely the identity of the individuals denoted by its extremities.

Instead of representing individuals by means of letters of the alphabet, beta uses "dots," and it asserts the identity of what is represented by one dot and what is represented by another by the simple device of allowing each to stand as the terminal of a heavy line drawn on the SA.

Unary Spots Connected by "Line of Identity"

In practice we shall find that much of Peirce's complex terminology for these signs may be avoided; instead of speaking of "dots," "ligatures," and like terminological complexities which Peirce seemed to delight in using, we shall find that we can get by with the notion of line of identity alone--for which see chapter ii.

Peirce also provides a subsidiary means of representing individuals in beta. I say "subsidiary," because--in 4.460, for example--he advocates it as an alternate notation whose purpose is to avoid complex tangles of lines of identity. This notation is the "selective." A selective is a sign, normally a letter of the Roman alphabet, which actually resembles in appearance and behavior the (bound) individual variable of the ordinary predicate calculus. But Peirce felt that the use of the selective can be avoided (as indeed it can--see 4.462), and should be avoided (4.473, 4.561n). As we shall see in the final section of this introduction, Peirce's preference for the line of identity to the exclusion of the selective and his reasons for this preference offer us important clues as to why he considered the existential graphs to be so important.

A Graph with a Line of Identity, and the same Graph with the Line of Identity Replaced by Selectives

In 4.406 Peirce informs us that "a point upon which three lines of identity abut is a graph expressing the relation of teridentity." As Peirce indicates in 4.445, this sign--the branching of a line of identity--gives us the apparatus for identifying any number of points on the sheet. One more bit of

A Branching

Peircean terminology may be mentioned at this point; a "network" of lines of identity, all of which are connected to each other and which may cross cuts and contain branchings is called by Peirce a "ligature" (4.407). The ligature, or network of lines of identity, ties together as representing the same individual all the points along its length, but is not necessarily itself a graph. We might note here that Peirce was loath to say flatly that a line of identity is capable of crossing a cut (4.401, 4.406), and so had to develop elaborate conventions to account for what happens when a line of identity appears to cross a cut. We shall find in chapter ii, however, that we may so set things up that we may think of lines of identity as crossing cuts and so avoid some of the terminology and conventions which Peirce feels it is necessary to use.

At this point we come to another distinctive feature of the beta graphs. Peirce formulated the system not only so that it contained signs for predicates and for individuals, but so that it would be able to <u>quantify</u> over the individuals represented as well. This is accomplished quite simply:

> Any line of identity whose outermost part is evenly enclosed [that is, by an even number of or by no cuts] refers to <u>something</u>, and any one whose outermost part is oddly enclosed [that is, by an odd number of cuts] refers to <u>anything</u> there might be (4.458).

The type of quantification applying to a given line of identity, then, is determined by examining the line and noting how many cuts enclose the least-enclosed part of the line. The very interesting feature here is that no explicit sign for quantification--that is, no quantifier--is required to "get quantification" in beta.

Although this by no means has been an exhaustive catalogue of the signs of the existential graphs, it will suffice as an introduction to the notation. Mention will be made of other signs at appropriate points in the text.

Transformations in the Existential Graphs

We are now in possession of a considerable number of signs of the graphs. The next step is to see how those signs work. As we remarked earlier, the starting point in alpha for our purposes is simply a blank SA. Casting about for an interpretation of a blank SA, we might take it as an exemplification of "He who says nothing does not lie"; or, since a blank SA is to be considered a graph, and so is to be taken as asserting something, as being a proposition, we might interpret it as asserting--in Fregean fashion--"the True" (or perhaps we might prefer to say "denoting" here rather than "asserting"--from this point of view, the blank SA would be a "pure denotation" of the True, with no connotation at all).

For the purposes of beta, one additional starting point or "axiomatic graph" is to be considered. Peirce states that

> since a Dot [the dot may here be considered a limiting case of the "freestanding" line of identity] merely asserts that some individual object exists . . . it may be inserted in any Area (4.567).

This amounts to an axiomatic assertion of the "dot"--or of the simple line of identity with no branchings and no connections, situated entirely in one area. The "assertion of the dot" amounts to a declaration of the non-emptyness of the beta universe of discourse. We thus see that the "axioms" required for alpha and beta as we shall study them are of the very simplest and most elementary kind.

For gamma as its signs have been presented, Peirce offers no additional axiomatic graphs. We shall find in our detailed examination of parts of gamma in chapter iii, however, that some interesting systems may be evolved in the notation of gamma through the use of certain such graphs.

We shall now list the "rules of transformation"[1] for the graphs essentially as we shall employ them in this paper; if the systems of graphs are considered systems of logic, these rules of transformation are, in effect, the rules of inference of the logics involved. These rules will enable us to produce "proofs" of "theorems" within these logics, taking the first step of any proof as the appropriate "axiomatic graph" mentioned above. The principal references in Peirce for these rules are 4.492, 4.505, and 4.516. Note that we shall employ a decimal system in numbering these rules, using 0 as the number to the left of the decimal

[1]At this point we will mention a "categorical rule of transformation" which is listed by Peirce, but which we will find no occasion to use in our work with the graphs. This is the rule which reads "Any graph well-understood to be true may be scribed unenclosed" (4.507). This rule may be considered a rule for the introduction of "extralogical" premises or hypotheses onto the sheet of assertion in order to "reason about" special problems.

point; the purpose of this is to make the numbering of these rules consistent with the numbering of the lemmas, metatheorems, and corollaries of the following chapters. Following are the rules for alpha:

0.01 In any area of the sheet enclosed by an <u>odd</u> number of cuts, any graph may be scribed. (This is the rule of "insertion in odd").

0.02 In any area of the sheet enclosed by an <u>even</u> number of, or by <u>no</u> cuts, any graph may be erased. (rule of "erasure in even").

0.03 If a graph X occurs in any area of the sheet, X may be <u>iterated</u> (that is, "written again") in that same area or in any area enclosed by <u>at least all</u> the cuts by which the original occurrence of X was enclosed. (rule of "iteration").

0.04 This rule is the exact converse of 0.03--it is the rule of "deiteration."

0.05 Any graph occurring in any area of the sheet may have scribed about it <u>two cuts</u>; there is to be <u>no</u> graph in the annular space between the cuts thus inserted. (rule of "biclosure").

0.06 This rule is the exact converse of 0.05--it is the rule of "negative biclosure." Note that two cuts removed by this rule are to have nothing in the annular space between them.

We shall now turn to the rules of transformation for the beta system. These are rules designed specifically to handle transformations involving lines of identity; it will be noted that they closely parallel the alpha rules given above, and might, in fact, be considered clauses extending the alpha rules to deal with lines of identity. We will here remark, by the way, that following our statement of the rules for beta and for gamma, we shall present some illustrations of the applications of these rules. These, then, are the rules which in addition to 0.01-0.06 above are needed for the beta system:

0.07 In any area enclosed by an odd number of cuts, two "loose ends" of lines of identity may be joined. (rule of "joining in odd").

0.08 In any area enclosed by an even number of, or by no cuts, any line of identity may be broken by erasing a portion of it. (rule of "breaking in even").

0.09 This is the rule of "beta iteration" and has three clauses:

(a) From any line of identity at any point a branch may be extended.

(b) Any loose end of a line of identity may be extended inwards through a "nest of cuts," crossing each cut just once.

(c) Any graph X may be iterated as in 0.03, with the added provision that if X includes a point on a line of identity, which point is outside all cuts in X, that point in the original occurrence of X may be connected by line of identity to the corresponding point in the new occurrence.[1]

0.10 This rule is the exact converse of 0.09; it has three clauses corresponding to those of 0.09, and it may be called the rule of "beta deiteration."

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[1]The rules of beta iteration and deiteration are somewhat difficult to state in a compact and perspicuous manner, as Peirce's own statements of them testify. Let us give this intuitive picture of how to do it. Given a graph Y containing a subgraph X which we wish to iterate, first copy the whole of Y on a separate sheet of paper. Now take a pair of scissors and cut the subgraph X out of the new copy of Y. In cutting X out of Y, we are not allowed to cut across any cuts, but we may cut through lines of identity which bind X to other parts of the whole graph Y. Also, the portion of Y thus cut out must be all one piece of paper. Take the piece of paper thus cut out and glue it onto the original graph Y, either in the same area within which the original subgraph X occurs, or in an area enclosed by at least all the cuts enclosing that original occurrence of X. Now if there are any points of lines of identity standing outside all cuts in X, any such point in the original occurrence of X may be connected to the corresponding point in the new occurrence of X by "geodesic line of identity" (this term will be defined in chapter ii). Or, if a "geodesic line of identity" connected to a point belonging to the original X but outside all cuts in the original X already goes from that point to within the area in which the new X has been "pasted," then the corresponding point in the new X may be connected directly to the already existing line of identity. But the line of identity connections are optional.

0.11 Any graph may be enclosed by two cuts as in 0.05, with the added provision that lines of identity may pass from entirely outside the outer to entirely inside the inner cut; all they are allowed is "free direct passage" through the annular space between the cuts. Aside from such lines of identity--which may not even branch within the annular space--no sign of any kind is permitted between the two cuts. This rule may be called "beta biclosure."

0.12 This rule is the exact converse of 0.11; it may be called "negative beta biclosure." It permits the removal of two cuts which might have been inserted by 0.11.

We shall state now the rules which Peirce offers for transformations involving the "broken cut" of gamma. We shall comment at length on these rules in chapter iii of this paper. These rules are drawn from 4.516:

0.13 In a broken cut already on SA any graph may be inserted.

0.14 A broken cut in an area enclosed by an <u>odd</u> number of cuts (which may be either alpha or broken cuts) may be transformed to an alpha cut (by "filling in" the breaks in it).

0.15 An alpha cut in an area enclosed by an even number of or by no cuts may be transformed to a broken cut (by erasing parts of it).

We shall now present some simple examples of transformations permitted by the above rules. Note that the "right arrow"-- → --indicates permission to transform "in one direction only." The "double arrow"-- ↔ --indicates "intertransformability." The examples then are:

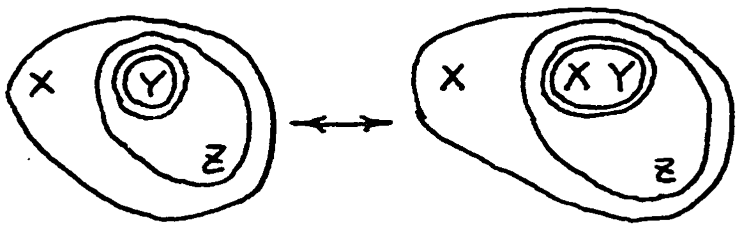
0.01 Insertion in odd:

0.02 Erasure in even:

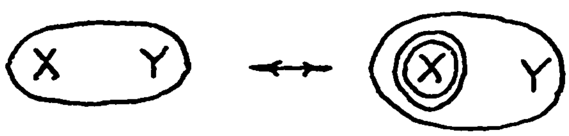
0.03 and 0.04 Iteration and deiteration:

In same area:

"Crossing cuts",

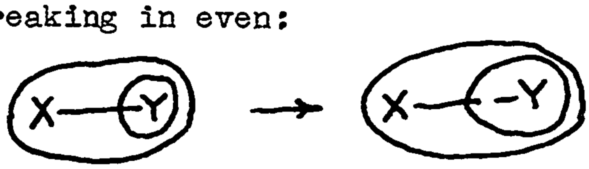


0.05 and 0.06 Biclosure:



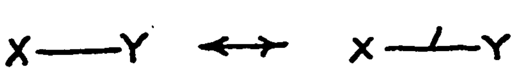
0.07 Joining in odd:

0.08 Breaking in even:

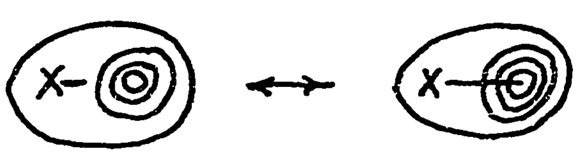


0.09 and 0.10 Beta iteration and deiteration:

(a) Extension of branch from line of identity:



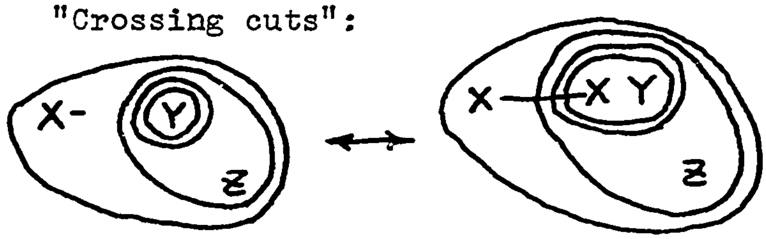
(b) Inward extension of line of identity end:



(c) Iteration or deiteration retaining connections:

In same area:

"Crossing cuts":



0.11 and 0.12 Beta biclosure:

0.14 Oddly enclosed broken cut to alpha cut:

0.15 Evenly enclosed alpha cut to broken cut:

I have considered it unnecessary to offer an example of 0.13. It should be clear by now that the alpha cut is very close in its function to the negation sign of the Classical Propositional Calculus--the rules of biclosure are the most obvious indication of this. Again, the simultaneous occurrence of a number of graphs may be interpreted as the conjunction of the sentences represented by those graphs. The spots of the beta system behave much like the predicates of the ordinary first-order calculus; the line of identity, however, is a sign with a very complex usage in beta, as should now be evident. It behaves sometimes like a variable, sometimes like a sentence, and sometimes like a quantifier. We shall investigate, in detail and formally, the alpha and beta systems in chapters i and ii respectively. In chapter iii we shall enter into a study of certain systems which may be classified as gamma systems. But before we do this, before we begin to make use of the "raw material" we have presented up to this point, we shall take a quick look at the existential graphs as they stand in the philosophy of Peirce.

The Continuity Interpretation

The systems of existential graphs are, as we shall see, systems of logic, and systems of logic in a very well-defined sense. Peirce had developed logics along more ordinary lines before he began his work on the graphs; nevertheless, he clearly preferred the graphs to his algebras of logic. As we quoted Peirce a bit earlier:

> [The development of a thorough understanding of mathematical reasoning] is the purpose for which my logical algebras were

> designed, but which, in my opinion, they do not sufficiently fulfill. The present system of existential graphs is far more perfect in that respect (4.429).

But why on earth did he believe that? What is there in the notation of the graphs that, for Peirce, makes them superior to the ordinary logical algebras? It should be understood, first of all, that "mathematical reasoning" covers considerable ground for Peirce. His definition of mathematics was "the science which draws necessary conclusions" (4.229). "Mathematical reasoning," then, in the broadest sense, is deductive reasoning, and Peirce's comments in 4.428 indicate to us that he did not wish to restrict it to a narrower notion in this case. And we find Peirce stating that "mathematics meddles with every other science without exception. There is no science whatever to which is not attached an application of mathematics" (1.245). For Peirce, this is virtually equivalent to saying that mathematical or deductive reasoning is an integral part of reality itself. The development of a thorough understanding of mathematical reasoning is a first and absolutely essential step towards the development of a thorough understanding of reality. The algebras of logic and the graphs were to foster such an understanding by presenting an analysis of the deductive process, "by breaking up inferences into the greatest possible number of steps, and exhibiting them under the most general categories possible" (4.373).

And for the analysis of the deductive process, Peirce preferred the systems of existential graphs to the algebras of logic. We can get a clue to the reasons for this preference by investigating a passage in which he discusses certain of the signs of the graphs. We will recall that we earlier mentioned a sign of the beta system called the "selective." The selective, as we noted, is used in many places where a line of identity might be used; it is, however, much like the ordinary bound variable of the algebras of logic--rather than being a line, it is simply a letter of the alphabet. Peirce criticises the selective as a sign of the graphical systems; his criticism of the selective may be applied directly to the variables of the algebras of logic to give a strong indication of why he preferred the graphs to those algebras. "The first respect," he states, "in which Selectives are not as analytical as they might be, and therefore ought to be, is in representing identity" (4.561n). Peirce remarks here that the way that two occurrences of a given selective (and the same would be true for two occurrences of a variable in an algebra) represent the same object is by "a special convention of interpretation." Peirce feels that given two occurrences of such a selective--or variable--although we know by convention that they are to be considered to represent the same object, "There is here no analysis of identity" (4.561n). Peirce wants a sign which will not merely be conventionally understood as signifying identity, but which will "wear its meaning on its sleeve," so to speak; which will offer in its very representation of identity an <u>analysis</u> of identity. And "the line of identity which may be substituted for the selectives very explicitly represents Identity to belong to the genus Continuity and to the species Linear Continuity" (4.561n). Identity is a continuity, and so too is an unbroken line. The self-identical individual is far better represented, Peirce felt, by a continuous line than by a batch of discrete occurrences of an individual variable, provided only a formalism can be found which gives the line the powers that the representative of an individual should have. In Peirce's opinion the beta formalism does the trick, and so the beta line of identity is far superior to the selective or the ordinary variable to represent the self-identical individual. And Peirce goes even further in telling us about this representing function of the graphs: "The continuity of [the sheet of assertion] being two dimensional . . . should represent an external continuity, and especially a continuity of experiential appearance" (4.561n). And the sheet of assertion,

> in representing the field of attention, represents the general object of that attention, the Universe of Discourse. This being the case, the continuity of the [sheet of assertion] in those places where, nothing being scribed, no <u>particular</u> attention is paid, is the most appropriate icon possible of the continuity of the Universe of Discourse where it only receives <u>general</u> attention as that Universe (4.561n).

We see from these remarks that Peirce felt that the graphs had a certain natural appropriateness about them for the task which he set them to do, and we see further that he felt that the <u>continuity</u> present in certain basic symbols of the graphs was the factor that made the graphs superior to the logical algebras for that task of representation and analysis.

Now, the alpha and beta sheets of assertion represent

> simply a universe of existent individuals, and the different parts of the sheet represent facts or true assertions made concerning that universe. At the cuts we pass into other areas, areas of conceived propositions which are not realized (4.512).

The cuts are <u>discontinuities</u> on the sheet of assertion, and they are meant by Peirce to correspond to "discontinuities of the universe of discourse." The non-existent, the unrealized, is in a definite sense discontinuous with the existent and realized insofar as it is not part of the universe of existent individuals. In general, a cut as a break in the continuity of any area indicates a certain break in continuity between what is inside and what is outside of it. The contents of a cut on the alpha or beta sheet of assertion represents that which is not part of the "continuity of experiential appearance."

In attempting to get across the idea of how the graphs represent, Peirce states:

> You may regard the ordinary blank sheet of assertion as a film upon which there is, as it were, an undeveloped photograph of the facts in the universe. I do not mean a literal picture, because its elements are propositions, and the meaning of a proposition is abstract and altogether of a different nature from a picture. But I ask you to imagine all the true propositions to have been formulated; and since facts blend into one another, it can only be in a continuum that we can conceive this to be done. . . . Of this continuum the blank sheet of assertion may be imagined to be a photograph (4.512).

Peirce then states, "So far I have called the sheet a photograph, so as not to overwhelm you with all the difficulties of the conception at once. But let us rather call it a map" (4.513). And just what is a map? Well, "a map of the simplest kind represents all the points of one surface by corresponding points of another surface in such a manner as to preserve the continuity unbroken, however great may be the distortion" (4.513). It is hardly necessary to add that if an existential graph is a map, it is not the "simplest kind" of map. Points in a given graph are intended to be correlated with what may loosely be called "features of the universe of discourse represented by the sheet of assertion upon which that graph is scribed," rather than with "points of another surface." If "____ is red" and "____ is round" are beta spots, they are to be correlated with the properties of being red and being round; it is quite clear that in the actual existing universe we do not have "redness" and "roundness" floating about as "things in themselves," without individuals that are red or round. This jibes with Peirce's refusal to call a spot whose hooks are empty a graph (4.439). But let the two spots mentioned be connected by a line of identity, and we do have a graph, the graph which states, "There is something both red and round." The continuity of the line of identity between spots expresses, for Peirce, the continuity which is the self-identity of the thing which is both red and round. It is thus that the graphs are to be considered "maps"; they are not pictures of facts--for who ever saw a fact that looked like an existential graph--but they are supposed to indicate continuity where there is continuity, and to represent discontinuity where there is discontinuity. A break in a line of identity is one such representation of discontinuity; the cut, as we have remarked, is another, of another kind.

All this emphasis on continuity is by no means accidental. The Peirce of the existential graphs, especially of the passages from which we have been quoting, is as well the Peirce of synechism.[1]

[1]Synechism is a metaphysical doctrine developed by Peirce in his later years and in which he set great store. An "instant characterization" of synechism might be that it is a doctrine asserting "the reality of continua and the continuity of reality." But this is, of course, rather oversimplified. For further reference on this topic, see Murphey, pp. 379 ff.

It is unquestionable that "continuity" is the key word of Peirce's synechism--all that is real is, insofar as it is real, continuous in some way or other. One passage, perhaps, will emphasize Peirce's rather firm commitment to the notion of continuity as a heuristic tool:

> Upon the assault of the enemy, when pressed for the explanation of any fact, I look myself up in my castle of impregnable logic and squirt out melted continuity upon the heads of my besiegers below.[1]

And closely connected to synechism is what Peirce sometimes called "synectics," but which is better known by the name of "topology." Murphey states, in fact, that

> the model upon which Peirce based his metaphysics quite obviously is the topology of Listing. And this is in fact what one would expect, for his work in mathematics had led him to the conclusion that topology is the mathematics of pure continua. If there is any formal system which ought to provide the key to the synechistic world, it is synectics or topology.[2]

There is no doubt whatsoever that Peirce in his later years was fascinated by topology, even though the only topology he knew was that of Listing, which was a rather low octane brand. It is also clear that he thought topology to be the mathematics of pure continua, and that continuity has a key place in his philosophy. The only point in doubt is just <u>how</u> Peirce planned to correlate the "synechistic world" with Listing's topology. For it is far from evident how we get from "continuity is the master key, and topology is the mathematics of pure continua" to "topology is the master key." Murphey does not feel that Peirce was

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[1] <u>Ibid.</u>, p. 406; Murphey here quotes from an unpublished paper of Peirce.

[2] <u>Ibid.</u>, p. 405.

able to make this particular jump successfully, and I would tend to agree with Murphey. But the interesting thing from our point of view here is how Peirce attempted that jump. I submit that Peirce tried it in a manner which, apparently, did not strike Murphey. It was through the existential graphs, I suggest, that Peirce tried to see the way from "the mathematics of pure continua" to "the synechistic world"--those same existential graphs that Murphey dismisses as merely one of the factors in Peirce's "lack of philosophical productivity" at the close of the nineteenth century.[1]

Peirce's "Algebra of Logic" owes its name and a large part of its form to what, in the later nineteenth century, became the algebra of real numbers. Although there are similarities between the algebra of logic and the algebra of real numbers, there is no doubt that they are different branches of mathematics. It seems to me that the relation between the existential graphs and Listing's topology is a very similar kind of thing. Peirce never indicates that the graphs <u>are</u> topology, even though it is very difficult to look at his presentations of them without thinking of topology. The graphs, in fact, rather than being a topological system, were to help us reason about topology and solve its problems, much as they were to help us solve the problems of mathematical reasoning in general (4.428). But much as the algebra of logic was fashioned after the image of the algebra of real numbers, I believe that the existential graphs were fashioned

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[1]*Ibid.*, p. 387.

after the image of Listing's topology. As what became the algebra of real numbers offered the algebra of logic a "pattern" into which its signs should fall and an indication of how its signs should behave, Listing's topology offered the existential graphs a pattern into which their signs should fall and an indication of the way their signs should behave. Where the systems of 3.359ff., for example, constitute an "algebra of logic," the existential graphs constitute a "topology of logic." And it is the features that they share with topology that render the graphs superior as a logical system, in the Peircean sense, to the algebra of logic.

Thus, while topology was, for Peirce, the mathematics of pure continua, the existential graphs were the apparatus for representing symbolically the world of continua, the world of reality. Listing's topology of itself offered little more help to Peirce's project of synechism than did real number algebra, of itself. It was necessary for Peirce to develop a logic of continua, a system which would permit a mapping of the continua of reality into itself. This, I feel, is the reason why Peirce in his later years put so much work into the graphs. And this too, I am sure, is the explanation for that puzzling "subtitle" which Peirce attached to the systems of graphs--"My chef d'oeuvre." The study of logical systems for their own sake as formal, mathematical systems was far from the chief interest of C. S. Peirce; hence it is unlikely that he would consider the graphs his major work merely because they are interesting formal systems. His fascination with and regard for the graphs goes beyond that; he saw for them, I am sure, a key place in his synechism. More than just a hint of this is given in a passage with which Peirce concludes a discussion of "An Improvement on the Gamma Graphs" (4.573ff):

> We here reach a point at which novel considerations about the constitution of knowledge and therefore of the constitution of nature burst in upon the mind with cataclysmal multitude and resistlessness. It is that synthesis of tychism and of pragmatism for which I long ago proposed the name, Synechism, to which one thus returns; but this time with stronger reasons than ever before (4.584).

But so far we have explicitly mentioned only the alpha and beta graphs in their "continuity interpretation." The alpha-beta sheet of assertion was to be considered as representing a very specific universe of discourse, the "universe of actual existent fact" (4.514). Synechism, however, was concerned with far more than this. Any "account of reality," in fact, which stopped with the actually existing universe and said nothing of the realm of possibility must, from the synechistic point of view, be radically incomplete. If we are correct about the place of the graphs in the thought of Peirce, then, we would expect him to have made provision for "the worlds of the possible" in his presentation of the graphs. He did just this; gamma was to be the system which did for reality as a whole what he felt that alpha and beta did for the "universe of actual existent fact." We have seen that the alpha and beta sheet of assertion, a two-dimensioned "space," a surface, was to represent this "universe of actual existent fact" although strictly speaking, at least three dimensions are required for a complete representation of beta, since there are instances when lines of identity are to be considered as crossing each other without joining;

this offers no difficulties for what follows, for we can replace "surface" with "three-dimensional space" and "three-dimensional space" with "four-dimensional space" with no difficulty at all. Peirce, taking the alpha or beta SA as a surface, remarks:

> . . . in order to represent to our minds the relation between the universe of possibilities and the universe of actual existent facts, if we are going to think of the latter as a surface, we must think of the former as three-dimensional space in which any surface would represent all the facts that might exist in one existential universe (4.514).

Gamma, then, is--quite literally--to add another dimension to the existential graphs. For gamma was to be the system which supplied the formalism necessary for the symbolic expression of the relationships between different universes of discourse. Alpha and beta are to offer an analysis of the "deductive (mathematical) reasoning" of the continua of the actually existing universe, but gamma was supposed to present an analysis of the deductive process existing between and relating the possible universes of discourse, including the actual existent universe of the alpha-beta sheet. The importance for Peirce of gamma must not be understated. The universe of "actual existent fact" does not exist in a vacuum, nor did it spring full-panoplied from the head of Zeus. What is today an "existent fact" may yesterday have been a "not-yet-realized fact," a possibility; today it is in the realm of alpha and beta, while yesterday it was not--what is the connection between that fact as of yesterday and as of today? The synechistic explanation must be that at both instances that fact must be part of a continuum which--unlike the continua represented in alpha and beta--transcends individual universes of discourses. But what is that continuum, and how are we to represent continua of that sort? This is the problem which gamma was to solve when it had been "brought to perfection." But how are we to go about the representations required of gamma? Peirce suggests that in order to handle the extra dimension required of gamma, we take as our gamma "sheet of assertion" not a single sheet, but "a book of separate sheets, tacked together at points, if not otherwise connected" (4.512). We are asked to think of the cut as being literally a cutting of the paper, which extends "down to one or another depth into the paper, so that the overturning of the piece cut out may expose one stratum or another, these being distinguished by their tints; the different tints representing different kinds of possibility" (4.578). The operation of "scribing" a cut now consists of two steps, cutting through the paper and turning over the portion we have cut out. That which is written on the reverse side of any of the sheets is to be considered denied in the universe of discourse which that sheet represents (4.574). When the area on the reverse of one of the sheets is "turned up" by the scribing of a cut, the denial of any graphs scribed on that area is asserted for the universe of discourse connected with the sheet to which the area belongs. We can picture, in this system, a nest of cuts with different cuts belonging to it penetrating to different strata of the book of sheets and expressing an extremely complex modal graph. All kinds of possible denial and possible assertion would be involved, and the situation could be extremely confusing.

It could be extremely confusing, that is, unless Peirce is at least able to provide us with a comprehensive set of rules of transformation for gamma which will enable us to know exactly which cuts we are allowed to make and what we may write in which area. Well, Peirce tells us quite frankly that so far as gamma goes, he was "able to gain mere glimpses, sufficient only to show me its reality, and to rouse my intense curiosity, without giving me any real insight into it" (4.576). The truth is that he was not able to discover such a comprehensive set of rules. We can see the general plan that Peirce had laid out for gamma. We can see just about where he intended to fit it into his synechism. We can see what he hoped to do with it, but we can see as well as he could that he was not able to realize those hopes. The final formulation of gamma was for Peirce an El Dorado, a golden city glistening just beneath the horizon.

Peirce set down a large number of signs which he considered within the scope of gamma. With most of these signs he did little more than to set them down. Since gamma was to be a logic of "second intentions," it is not strange at all that he set down signs of gamma which were to be a sort of metalanguage for the graphs. For gamma was to be able to reason about <u>ideas</u>, and if this was the case, some provision should be made in it for such signs. 4.524-529 offer examples of such symbols. An examination of these signs leads us to the conclusion that this project was presented by Peirce as one of the threads that his successors were going to have to weave into the warp of the completed gamma fabric. He gives us no specific rules for working with these signs, and tells us very little about how they are to be used.

There is also another suggestion, in 4.470, which is interesting but to which Peirce devotes very little time. Here he shows us a different kind of line of identity, one which expresses the identity of spots rather than of individuals. This is an intriguing move, since it strongly suggests at least the second order predicate calculus, with spots now acquiring quantifications. Peirce did very little with this idea, so far as I am able to determine, but it seems to me that there would not be too much of a problem in working it into a graphical system which would stand to the higher order calculi as beta stands to the first-order calculus. The continuity interpretation of the "spot line of identity" is fairly clear; it maps the continuity of a property or a relation. The redness of an apple is the same, in a sense, as the redness of my face if I am wrong; the continuity of the special line of identity introduced in 4.470 represents graphically this sameness. This sameness or continuity is not the same as the identity of individuals; although its representation is scribed upon the beta sheet of assertion, its "second intentional" nature seems to cause Peirce to classify it with the gamma signs. The same may be said of the "metalinguistic signs" mentioned above.

The metalinguistic and "higher-order" signs are, however, peripheral to the main thrust of gamma as envisioned by Peirce. As we noted, and as an examination of 4.510ff. and 4.573ff. will verify, gamma was for him, above all, the "three-dimensional," the multi-sheet system. As we also noted, he was unable to come even close to a complete formulation of rules for that system (the multi-sheet formulation is, I believe, doomed from the start by its very complexity if for no other reason). But he did try to make a start:

> In endeavoring to begin the construction of the gamma part of the system of existential graphs, what I had to do was to select, from the enormous mass of ideas thus suggested, a small number convenient to work with. It did not seem to be convenient to use more than one actual sheet at one time; but it seemed that various different kinds of cuts would be wanted (4.514).

He then begins telling us about one of these cuts, the "broken cut," which we mentioned earlier in this introduction. In chapter iii we shall study several interesting systems constructible with the use of the broken cut.

Before we get to chapter iii, however, we will have to move through chapters i and ii, in which we will view in detail the alpha and the beta systems respectively.

CHAPTER I

THE ALPHA SYSTEM

Our detailed study of the existential graphs as formal systems begins with alpha. This chapter will be devoted to a comparison of alpha with the classical propositional calculus (which we shall ordinarily abbreviate by "CPC" or simply "PC"). Alpha, as we shall see, is a logic; the rules of transformation for alpha presented in the Introduction are the rules of inference of that logic. As a logic, alpha has theorems, theorems provable through those rules of transformation; we shall see how those theorems are related to those of CPC.

"Project alpha," then, is an examination of relationships existing between alpha and the ordinary CPC. Obviously, since the notation of alpha is very different from that of garden-variety logics, the first thing we must do is to find a means of "translating" alpha graphs into wffs of ordinary PC's and vice-versa.

We describe a function, or set of instructions, which will enable us to translate any alpha graph into a unique formula of the PC; this function we call "f." In describing this function, we first show how to write "names" of alpha graphs, and then we designate a set of these names as "standard" names; the designation is such that each alpha graph has one and only one standard name. We then show how to transform standard names into unique CPC wffs. If X is any alpha graph, then the CPC wff formed by applying these instructions to X will be called f(X).

At this point we will note that in our work with alpha we will set out and use two different formulations of the CPC rather than just one. The first of these systems is one which has conjunction and negation as primitive operators; this formulation is called P_r (the "r" is for Rosser). The other system uses implication as a primitive operator and has a primitive constant false proposition; this formulation is called P_w (the "w" is for Wajsberg). P_r and P_w are both complete CPC's, and as such are "equivalent." The reason we use two systems rather than just one is one of convenience. P_r is a convenient system into which to translate alpha graphs in one-one fashion; for a given alpha graph X the formula f(X) will be a wff of P_r. But P_w is a convenient system from which to translate wffs into alpha graphs in one-one fashion, as we shall see.

We have mentioned that we shall describe a function f which "translates" alpha graphs into wffs of the system P_r. We shall also describe a one-one function, or set of instructions, which will enable us to translate any wff of the system P_w into a unique alpha graph; this function we shall call "g." Where A is any wff of P_w, the unique graph formed by applying this set of instructions to A will be called g(A).

We shall provide a precise definition of what it means to be a logic, and we shall show how alpha "fits" this definition;

that is, we shall show that alpha is a logic. In the process of doing this, we shall set down precise formulations of the alpha rules of transformation, which--as we have mentioned--are the rules of inference of the logic alpha.

Certain of the alpha graphs will be theorems of the logic alpha. Through a series of lemmas we shall establish that:

The graph X is a theorem of alpha iff the wff f(X) is a CPC theorem.

We shall also establish that:

The wff A is a CPC theorem iff the graph g(A) is a theorem of alpha.

We shall thus have established that the set of theorems of alpha maps one-one into the set of theorems of the CPC, and vice-versa. This means that given the natures of the functions f and g, alpha itself may be considered a complete classical CPC.

1.1 The Set of Alpha Graphs

We may first note that involved in the makeup of alpha as we have examined it is a set of objects, objects which are simply certain signs and combinations of signs that may be scribed, say, upon a sheet of paper or a blackboard. Consider a denumerably infinite set, M_a, of such signs, which we will denote individually by the letter 'b' followed by a sequence of primes, which sequence may be null. The members of the set M_a, then, are b, b', b'', ... It is understood, of course, that the correlation of the members of M_a to the signs 'b', 'b'', etc. is strictly one-one. No member of M_a is analysable into "smaller" signs, so we may call M_a the set of "minimal" or "atomic" graphs of alpha. The first member of this set, b, has a special status; it is the "null-graph" or "blank." Recall that the blank SA is to be considered a graph. The other members of M_a, b', b'', . . . , may be considered "graph-variables," with a function not unlike that of the propositional variables of the ordinary CPC. In the present treatment, we shall generally employ upper-case Roman letters 'X', 'Y', etc. as variables of the metalanguage ranging not only over the members of M_a, but over all objects which may come to be called graphs as well.

Consider now another sign of the metalanguage. Let S(X) be "the result of enclosing the graph X by the ordinary alpha cut," which as we learned in the Introduction is a "self returning linear separation" scribed upon the surface upon which we are working. Please note that if two "self returning linear separations" intersect each other, neither qualifies as a "cut." The cut itself is not a graph, but it plus the graph it encloses is a graph; that is, the scribing of a cut around a graph is a "graph-forming operation." Note that in practice a cut cannot be scribed without forming a graph, for it will always enclose at least b, the null-graph.

If a graph X is enclosed by a cut, and another graph Y lies outside that same cut, we shall say that X and Y are "separated"; otherwise they are "unseparated." This brings us to another sign of the metalanguage. Where X_1, . . . , X_n are n alpha graphs, each of which is either of the form S(Y) or is a member (other than b) of the set M_a, then $J(X_1 \ldots X_n)$ is "the result of scribing the n graphs $X_1, \ldots, X_n$ unseparated on SA." The operation of scribing graphs unseparated on SA is then also a "graph-forming" operation.

We may summarize the above by listing the following rules; these rules may be considered the rules characterizing the set of alpha graphs:

1.1i Each member of the set M_a is an alpha graph.

1.1ii Where X is an alpha graph, S(X) is an alpha graph.

1.1iii Where $X_1, \ldots, X_n$ are n alpha graphs other than b admitted by rules 1.1i or 1.1ii, then $J(X_1 \ldots X_n)$ is an alpha graph.

From what has gone before, we should already have a good idea of how to go about naming an alpha graph, as we have actually been using names of alpha graphs in our development. But we shall list some explicit rules similar to the above which will tell us formally how graphs are named.

1.1iv 'b' names the null-graph; 'b', followed by one prime or more, names a member of M_a; 'b' followed by n primes names the same graph as 'b' followed by m primes iff $n = m$.

1.1v The sequence of signs consisting of 'S(' followed by the name of the graph X followed by ')' names a graph; specifically, it names the graph formed by enclosing X by an alpha cut.

1.1vi Where the graphs $X_1, \ldots, X_n$ are n, $n \geq 2$, graphs admitted by rules 1.1i or 1.1ii, and none of them is the null-graph then the sequence of signs consisting of 'J(' followed by the n names of the $X_1, \ldots, X_n$ followed by ')' names a graph; specifically, it names the graph formed by scribing the $X_1, \ldots, X_n$ unseparated on SA.

Now note that since the set of alpha graphs is characterizable by the rules 1.li-iii, it is a recursive set. Since it is recursive, it may be ordered (every recursive set is recursively enumerable in increasing order without repetitions) in such a manner that alpha graphs X and Y are associated with the same natural number iff X and Y are identical, that is, are able to be "generated" by exactly the same applications of rules 1.li-iii. We could, if we wished, describe means for such an ordering. Such a description would, however, be long, tedious, and irrelevant. The important thing is to know that the set may be so ordered; the exact "how" of the ordering does not make too much difference. Let us then select some ordering of the alpha graphs such that each alpha graph is associated with a unique natural number, and vice-versa. If, then, the names of the n alpha graphs $X_1, \ldots, X_n$ are written in a sequence, we shall say that that sequence is "properly ordered" iff it is ordered according to our selected unique ordering of alpha graphs. Of course, for any n such graphs there is one and only one "proper ordering."

We shall now define what we shall call the "standard name" of an alpha graph:

1.lvii The names of graphs admitted by 1.liv are standard names.

1.lviii The sequence of signs consisting of 'S(' followed by the standard name of the graph X followed by ')' is a standard name.

1.lix Where $X_1, \ldots, X_n$ are n, n ≥ 2, alpha graphs admitted by rules 1.li or 1.lii, and none of them is the null-graph, then the sequence of signs consisting of 'J(' followed by the n standard names of the $X_1, \ldots, X_n$ followed by ')' is a standard name, provided the sequence of n names between the 'J(' and the ')' is properly ordered.

It should be evident that each alpha graph has one and only one standard name.

A question worth noting at this point is that of the cardinality of the set of alpha graphs. Although we have asserted that this set is recursive, and so at most denumerably infinite, the graphical nature of the system may well conjure up ghosts of the power of the continuum to haunt us. Let us simply remark here that our study involves only graphs containing a finite number of signs--an assumption well-grounded in the study, say, of the ordinary logical systems, which deal only in finitely long words or sentences. Since this is the case, it is impossible that the number of alpha graphs be larger than a denumerable infinity. We may seem in this to be somewhat less liberal than Peirce himself, who seems at one point to indicate that graphs containing infinitely many signs are permitted (4.494). We must comment, however, that there are marked differences between the presentation of a system permitting wffs of finite length only, and that of one permitting infinitely long wffs. What Peirce says in 4.494 is certainly a rather casual remark, the full implications of which did not occur to Peirce when he made it--and hardly could have, since the study of systems containing infinitely long sentences is of rather recent vintage. Indeed, so far as I have been able to determine, he does not mention the matter again in the writings on the graphs included in the Collected Papers. We shall thus stick by our assumption that an alpha graph may be of finite length only.

1.2 The Alpha Graphs and the WFFS of the Classical Propositional Calculus

We shall begin by setting out two systems of complete CPC; these systems will be called, respectively, P_r and P_w. The respective sets of wffs for these systems will be called (P_r) and (P_w). We shall employ the upper-case letters of the Roman alphabet as variables ranging over the CPC wffs. "$\vdash_r A$" shall be read, "A is a theorem of P_r," and "$\vdash_w A$" shall be read, "A is a theorem of P_w." The "primitive signs" of P_r include a denumerable infinity of signs, $p_0, p_1, p_2, \ldots$, called "propositional variables," left and right parentheses, '(' and ')', and the sign '-', called "the negation sign." The rules of formation for P_r are:

1.2i p_i is wf, $i \geq 0$.

1.2ii ⌜AB⌝ is wf where both A and B are wf.

1.2iii ⌜-(A)⌝ is wf where A is wf.

Definitions will be as usual in the CPC, with a definition considered as providing a notational abbreviation for a formula in primitive notation; where there is no danger of confusion, we shall employ the simple letters 'p', 'q', 'r', 's', . . . as propositional variables in place of the primitive p_1, p_2, p_3, ... As with all definitions, this is to be understood to be a space and time saving abbreviation. The variable p_0 will have a special use in this system, in the following definition:

$$1 \underset{\text{df}}{=} -(p_0-(p_0)).$$

As rules of inference, P_r shall have substitution for variables and detachment; the axioms shall be Rosser's set:

$$p \supset .p.p$$

$$p.q. \supset p$$

$$p \supset q. \supset .-(qr) \supset -(rp).$$

Note that we have presented the axioms in definitionally abbreviated form rather than in primitive notation.

The primitive signs of P_w are propositional variables, a sign '0', called the "constant false proposition," and a sign 'C', called the "sign of implication." The rules of formation are:

1.2iv p_i is wf, <u>where $i \geq 1$</u>.

1.2v 0 is wf.

1.2vi $\ulcorner C_{AB} \urcorner$ is wf where both A and B are wf.

Again, standard definitional abbreviations will be used, including the use of p, q, . . . as notational shorthand for p_1, p_2, . . . ; The rules of inference are substitution and detachment, and the axioms are:

CCpqCCqrCpr

CpCqp

CCCpqpp

C0p.

We are now prepared to set down a means of moving from the alpha graphs to the wffs of the CPC and vice-versa. We define a recursive word function f:

D1: The function f takes the set of alpha graphs as its domain, and finds its range in the set (P_r). Given any alpha graph X, write the <u>standard name</u> of X. Wherever in that standard name the "subname" $J(Y_1 \ldots Y_n)$ occurs, for all such "subnames" involving 'J', delete the 'J(' and the ')', leaving just the sequence $Y_1 \ldots Y_n$. Wherever 'S' occurs, replace it with the sign '-'. Wherever the simple 'b' occurs, replace it by '1', that is, by '$-(p_0-p_0)$'. Wherever 'b' occurs followed by i primes, $i \geq 1$, replace that occurence of 'b' and the primes following it by p_i. This completes the instruction for f; the result of the application of this instruction to an alpha graph X will be called f(X), and is a wff of P_r.

This last definition tells us how to transform an alpha graph into a wff of the CPC. As we remarked earlier, there is one and only one "standard name" for each alpha graph. It is quite clear that given any standard name, the latter part of the instructions in D1 will yield one and only one wff of P_r. The function f as defined above, then, is a one-one function;

f(X) will be the same formula as f(Y) iff X is the same graph as Y. The intuitively acceptable interpretations of the signs of the alpha graphs are preserved in the translation made possible by the function f; unseparated occurence of graphs in an area is correlated by f to PC conjunction, and enclosure by a cut is correlated to inclusion in the scope of a negation sign; the minimal graphs of alpha are correlated to the atomic formulas of P_r.

We now move on to the definition of the function which will enable us to translate wffs of the CPC into alpha graphs. First of all, we shall consider a subset of the set of alpha graphs, which we shall call $\mathcal{P}_\alpha$. Membership in $\mathcal{P}_\alpha$ is defined as follows:

1.2vii All members of the set M_a are members of $\mathcal{P}_\alpha$ with the exception of b.

1.2viii S(b) is a member of $\mathcal{P}_\alpha$.

1.2ix If X and Y are both members of $\mathcal{P}_\alpha$, then S(J(XS(Y))) is a member of $\mathcal{P}_\alpha$.

Let us introduce the following definition:

D2: $G(XY) \underset{df}{=} S(J(XS(Y)))$.

We might now restate the last rule thus: If X and Y both are members of $\mathcal{P}_\alpha$, then G(XY) is a member of $\mathcal{P}_\alpha$.

We may now move directly to a statement of the function g:

D3: The function g takes (P_w) as its domain, and $\mathcal{P}_\alpha$ (a subset of the set of alpha graphs) as its range. Given any wff of P_w, replace $\ulcorner C_{AB} \urcorner$ wherever it occurs by $\ulcorner G(AB) \urcorner$; wherever the variable p_i occurs, for each i and each such variable occurrence, replace it by 'b' followed by a string of i primes; wherever '0' (the constant false proposition) occurs, replace it by 'S(b)'. The result is the name of a graph. Where the wff with which we began was A, the graph named by the result of applying this instruction is g(A); draw the graph thus named. The instruction is complete.

It is clear that the function g is one-one and onto P_{α}.

1.3 Alpha as a Logic

Before entering into a detailed study of alpha as a logic, it will be well to say a few words about logics in general. The definitions we employ are in general adopted from Martin Davis; we retain, however, our numbering. First of all, let us say what we mean by "logic."[1]

D4: By a *logic* $\mathcal{L}$ we understand a recursive set A of words, called the *axioms of* $\mathcal{L}$, together with a finite set of recursive word predicates, none of which is singulary, called the *rules of inference of* $\mathcal{L}$.

It should be clear that a "recursive word predicate" is to a number-theoretic predicate as a recursive word function is to a (partial recursive) number-theoretic function.

D5: When $R(Y, X_1, \ldots, X_n)$ is a rule of inference of $\mathcal{L}$, we shall sometimes say that Y is a *consequence of* $X_1, \ldots, X_n$ *in* $\mathcal{L}$ *by* R.

D6: A finite sequence of words $X_1, X_2, \ldots, X_n$ is called

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[1]Martin Davis, *Computability and Unsolvability* (New York: McGraw Hill, 1958), p. 117ff.

a <u>proof in a logic $\mathcal{L}$</u> if, for each i, $1 \leq i \leq n$, either

(1) $X_i \varepsilon A$, or

(2) There exist $j_1, j_2, \ldots, j_k < i$ such that X_i is a consequence of $X_{j1}, X_{j2}, \ldots, X_{jk}$ in by one of the rules of inference of $\mathcal{L}$.

D7: We say that W is a <u>theorem of $\mathcal{L}$</u>, or that W <u>is provable in $\mathcal{L}$</u> and we write

$$\vdash_{\mathcal{L}} W$$

if there is a proof in $\mathcal{L}$ whose final step is W. This proof is then called a <u>proof of W in $\mathcal{L}$</u>.

1.31 "Theorem Generation" in Alpha

It will be our desire to show as quickly as possible that alpha is a logic. But before we state a metatheorem to this effect, let us examine alpha in the light of the requirements of Davis's definition. The first such requirement is for a recursive set of axioms; the usual criterion for determining what--in a given system--is an axiom is the word of the author of the system. Peirce was not so kind as to say explicitly, "Such-and-such a graph is an axiom of alpha," so we must look closely at the matter. An axiom is a theorem or "assertable word" of the system which need not be derived or proven within the system. Examining Peirce's presentation of alpha, we find one and only one graph which qualifies as such in what may be called the "unspecialized version" of alpha (that is, alpha with no "extralogical" premises). That graph is the blank sheet of assertion itself. We will recall that Peirce explicitly called SA a graph, and that we have notation in our metalanguage to speak about it. Any blank area on SA (and this includes the completely blank SA itself) is the null-graph, or b. Since SA is a presupposition of any graph whatsoever, we shall declare that the axiom set of alpha contains one and only one member, b.

It may seem curious to employ a <u>null</u> "something-or-other" as an axiom, since we very commonly think of the null-set as correlated in some way with the truth-value "false"; but let us recall that the "constant false proposition" employed in many systems of PC is often explicated as the proposition which says that "everything is true." This explication is exemplified in systems which have quantifiers ranging over propositional variables; in these systems the proposition '-(p)p' is a <u>theorem</u>. In the sense of "null" which is contrasted to the above sense of "universal," then, it is quite reasonable to take the "null-graph" as assertable.

The other requirement of Davis's definition of a logic is that it contain a finite set of recursive word predicates, the rules of inference of the system. It is clear that the rules of transformation for alpha qualify as such recursive word predicates. Employing 'R' with a subscript as notation for these predicates, we may list them as follows:

$R_{ins}(Y, X)$: Which is true iff Y is identical to X except for containing in a position enclosed by an odd number of cuts a graph Z which X does not contain at the corresponding position.

$R_{itr}(Y, X)$: Which is true iff Y is identical to X except for containing one more occurrence of a subgraph Z than does X (with X having at least one occurrence of Z); the extra occurrence of Z in Y is to be located in an area enclosed by at least all the cuts which enclose another occurrence of Z in Y, which last mentioned occurrence corresponds to an occurrence of Z in X.

$R_{bcl}(Y, X)$: Which is true iff Y is identical to X except for containing a graph S(S(Z)) where X contains simply Z.

The above predicates may be called the "positive rules of inference in alpha." In addition, we have the following "negative" rules:

$R_{ers}(Y, X)$: Which is true iff $R_{ins}(S(X), S(Y))$ is true.

$R_{dit}(Y, X)$: Which is true iff $R_{itr}(X, Y)$ is true.

$R_{nbc}(Y, X)$: Which is true iff $R_{bcl}(X, Y)$ is true.

The above six predicates are easily recognized respectively as Peirce's rules of insertion in odd, iteration, biclosure, erasure in even, deiteration, and negative biclosure. We may now state the following metatheorem:

*1.01 Alpha is a logic.

PROOF: By D4 and the above exposition of axiom and rules of inference in alpha. ∎

The axiom of alpha, then, gives us a starting point from which we may, by the stated rules of inference of alpha, generate the theorems of alpha. We now enter into an investigation of that set of theorems of alpha.

1.32 The Logic Alpha in the System P_r

In D1 we defined a function f which maps the set of alpha graphs into the set of wffs of P_r. Our project is now to show that whenever a given alpha graph X is a theorem of alpha, the corresponding wff, f(X), is a theorem of P_r, and then to show the converse, that whenever f(X) is a theorem of P_r, X is a theorem of alpha.

LEMMA 1.02 f(b) is a theorem of P_r.

PROOF: By D1, f(b) is '1', which is '$-(p_0.-p_0)$', which is immediately recognized as a theorem of P_r. ▮

The above lemma establishes that the axiom of alpha is correlated by f to a theorem of P_r. We now turn to the rules of inference in alpha, proving in the process two lemmas that are of some independent interest in the study of the CPC itself. We shall say that a given occurrence of wf subformula, B, of P_r is in "antecedental position," or "A-pos," in a P_r wff Q, and we shall write $\ulcorner Q^A(B)\urcorner$ for the whole formula iff the occurrence of the subformula B is within the scope of an odd number of negation signs in Q at the one occurrence of B in Q which is in question. We shall say that that occurrence of B is in "consequential position," or "C-pos," and shall write $\ulcorner Q^C(B)\urcorner$ for the whole formula otherwise, that is, when B is in the scope of an even number of, or of no negation signs in Q.

LEMMA 1.03 Where $\vdash_{\overline{r}} \ulcorner Q^A(B)\urcorner$ and $\vdash_{\overline{r}} \ulcorner D \supset B\urcorner$, then also $\vdash_{\overline{r}} \ulcorner Q^A(D)\urcorner$.

LEMMA 1.04 Where $\vdash_{\overline{r}} \ulcorner Q^C(B)\urcorner$ and $\vdash_{\overline{r}} \ulcorner B \supset D\urcorner$, then also $\vdash_{\overline{r}} \ulcorner Q^C(D)\urcorner$.

PROOF: These two lemmas shall be proven together. It is understood, of course, that in each lemma the D is to be considered as <u>replacing</u> the B.

We know that it is possible to reduce any wff of P_r to an equivalent conjunctive normal form. Reduce the formula Q(B) to such a form, making one exception to the usual procedure; that will be to treat B <u>in the one occurrence with which we are concerned</u> as if it were a propositional variable. All occurrences of propositional variables outside of that occurrence of B are to be treated as usual in the reduction, but the occurrence of B with which we are concerned will never be "broken up" in the reduction. The form we thus arrive at we may call the "quasi-normal-conjunctive" form of Q(B), or "Cnj(Q(B))." The PC laws needed for the transformation are (in parenthesis-free form):

1. EApKqrKApqApr
2. ENANpNqKpq
3. ENKpqANpNq
4. EpNNp.

These, together with the definitions of connectives in P_r, substitution for variables, and the substitutivity of the biconditional are all that is needed to perform the required transformation. An examination of the above laws will show that the property in a subformula occurrence in a wff A of being in A-pos or in C-pos in A is <u>hereditary</u> through applications to A of substitution instances of the above laws by substitutivity of the biconditional. But such applications are all that is needed, basically, to transform Q(B) to Cnj(Q(B)).

But this "hereditary nature" of A-pos and C-pos through the relevant transformations means that (with the aid of the law 'EpKpp') any $Cnj(Q^A(B))$ may be written in the general form:

5. $\ulcorner -B \vee F_1 . -B \vee F_2 . \ldots . -B \vee F_n . G_1 . G_2 . \ldots . G_m \urcorner$,

and any $Cnj(Q^C(B))$ may be written in the general form:

6. $\ulcorner G_1 . G_2 . \ldots . G_m . F_1 \vee B . F_2 \vee B . \ldots . F_n \vee B \urcorner$.

The n indicated occurrences of B in each of 5 and 6 above are the only occurrences of B in these formulas "derived from" the occurrence of B in question in $Q^A(B)$ and $Q^C(B)$ respectively; note that in 5 all such occurrences of B are in A-pos, and in 6 all such occurrences are in C-pos. Now, by laws 1-4 above, plus 'EpKpp', 5 may be transformed to a formula of form:

$$7. \quad \ulcorner -B \vee S \urcorner,$$

which, of course, is equivalent to

$$7'. \quad \ulcorner B \supset S \urcorner.$$

The formula 7' is, of course, equivalent to the original $Q^A(B)$, and the one occurrence of B as the antecedent of 7' is the only one in 7' which may be considered "ancestrally derived from" the occurrence of B in question in $Q^A(B)$. Similarly, 6 may be placed in the form

$$8. \quad \ulcorner R \vee .S.B \urcorner,$$

which is equivalent to a form

$$8'. \quad \ulcorner T \supset .S.B \urcorner.$$

The formula 8' is equivalent to $Q^C(B)$, and the one occurrence of B in the consequent of 8' is the only one in 8' which is "ancestrally derived from" the occurrence of B in question in $Q^C(B)$. The proof of the lemmas follows immediately. If

$\vdash_r \ulcorner D \supset B \urcorner$, then 7' is immediately transformable into $\ulcorner D \supset S \urcorner$; we can see immediately that this last formula is equivalent to $Q^A(D)$. Also, if $\vdash_r \ulcorner B \supset D \urcorner$, then 8' is immediately transformable into $\ulcorner T \supset .S.D \urcorner$; this formula is clearly equivalent to $Q^C(D)$. ∎

These two lemmas are of some interest in the study of the CPC as they provide us with a kind of analysis of the rule of substitutivity of the biconditional. Whereas the latter rule gives us a means for replacing a subformula B by a subformula D when $\ulcorner B \supset D.D \supset B \urcorner$, the lemmas 1.03 and 1.04 specify conditions when the replacement may be made when simply $\ulcorner D \supset B \urcorner$ or $\ulcorner B \supset D \urcorner$ respectively. These lemmas could not be used, of course, as they have been proven here, as parts of a proof of the rule of substitutivity of the biconditional, for that rule has been used in their proof. It is not difficult to visualize proofs of these lemmas, however, which would not make use of this rule.

We may now put these lemmas to use in the proof of the following:

LEMMA 1.05 Whenever, for alpha graphs X and Y, $R_{ins}(Y, X)$ is true, then if f(X) is a theorem of P_r, so too is f(Y).

LEMMA 1.06 Whenever, for alpha graphs X and Y, $R_{ers}(Y, X)$ is true, then if f(X) is a theorem of P_r, so too is f(Y).

PROOF: If $R_{ins}(Y, X)$ is true, then there is a subgraph Z oddly enclosed in Y which does not stand at the corresponding position in X; X and Y are otherwise identical. By D1 (the definition of f), then, f(X) and f(Y) are identical except that f(Y) contains the subformula which is f(Z) in an A-pos where f(X) does not contain it. In an A-pos, then, f(Y) contains the subformula ⌜A.f(Z).B⌝, where one but not both of A and B <u>may</u> be null; in the corresponding position, f(X) contains merely ⌜A.B⌝. But it is trivially the case that $\vdash_r$ ⌜A.f(Z).B.$\supset$.A.B⌝. But then, if f(X) is a theorem of P_r, by lemma 1.03 f(Y) must also be a theorem, and lemma 1.05 holds. The proof of lemma 1.06 is similar, employing, in its course, lemma 1.04. ∎

We now move to two more lemmas:

LEMMA 1.07 Wherever, for alpha graphs X and Y, R_{itr}(Y, X) is true, then if f(X) is a theorem of P_r, so too is f(Y).

LEMMA 1.08 Wherever, for alpha graphs X and Y, R_{dit}(Y, X) is true, then if f(X) is a theorem of P_r, so too is f(Y).

PROOF: It will be recalled that R_{itr}(Y, X) is true iff Y is identical to X except for containing one more occurrence of a subgraph Z than does X, with X having at least one occurrence of Z; the extra occurrence of Z in Y is to be located in an area enclosed by at least all the cuts which enclose another occurrence of Z in Y, which last-mentioned occurrence corresponds to an occurrence of Z in X. What we want at this point is a simple means of characterizing in P_r the differences between the wffs f(X) and f(Y). A good way of doing this will be to employ the notion of the "functor variable."[1] We shall not, however, employ

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[1]A. N. Prior, <u>Formal Logic</u> (2nd ed.; Oxford: Oxford, 1962), p. 64ff.

functor variables as signs of the object language, as Prior does in the passage cited above. The sign 'δ' will be considered a sign of the metalanguage; $\ulcorner \delta p \urcorner$ will be a schema which may represent any truth-function of p.

We may now characterize the difference between f(X) and f(Y). The wffs f(X) and f(Y) are identical except that where f(X) contains the subformula

$$\ulcorner f(Z).\delta(A.B)\urcorner,$$

f(Y) contains the subformula

$$\ulcorner f(Z).\delta(A.f(Z).B)\urcorner.$$

Note that one but not both of the wf subformulas A and B may be null. But it is easily proven (and intuitively acceptable) that

$$\vdash_{r} \ulcorner f(Z).\delta(A.B).\equiv.f(Z).\delta(A.f(Z).B)\urcorner.$$

In the presence of the substitutivity of the biconditional, this means that, given the definition of f and the problem as we have stated it,

$$\vdash_{r} \ulcorner f(X)\equiv f(Y)\urcorner.$$

This means that lemma 1.07 holds; and, since R_{dit} is the exact converse of R_{itr}, lemma 1.08 also holds. ∎

We now may move to the two final lemmas of this sequence:

LEMMA 1.09 Wherever, for alpha graphs X and Y, $R_{bcl}(Y, X)$ is true, then if f(X) is a theorem of P_r, so too is f(Y).

LEMMA 1.10 Wherever, for alpha graphs X and Y, $R_{nbc}(Y, X)$ is true, then if f(X) is a theorem of P_r, so too is f(Y).

PROOF: $R_{bcl}(Y, X)$ is true iff Y is like X except for containing a subgraph S(S(Z)) where X contains simply Z. But then f(Y) is like f(X) except for containing the subformula $\ulcorner$--f(Z)$\urcorner$ where f(X) contains just f(Z). But

$$\vdash_{r}\ p \equiv --p;$$

so by substitutivity of the biconditional,

$$\vdash_{r}\ \ulcorner f(X) \equiv f(Y) \urcorner .$$

Lemma 1.09 then holds, and since R_{nbc} is the exact converse of R_{bc1}, lemma 1.10 holds as well. ∎

We may now state the metatheorem to which these lemmas have been leading:

*1.11 If a graph X is a theorem of alpha, then f(X) is a theorem of P_r.

PROOF: Immediate from the preceding lemmas. For by lemma 1.02, f of the axiom of alpha is a theorem of P_r, and the lemmas 1.03-1.10 show that for each of the rules of inference of alpha there is a parallel derived rule of inference in P_r. If then there is a proof in alpha for a graph X, there is also a proof in P_r for f(X), and the metatheorem holds.

COROLLARY 1.12 Alpha is consistent.

PROOF: Suppose alpha to be inconsistent. Then any well-formed graph would be derivable by its rules of inference; specifically, the graph S(b) would be so derivable. But by *1.11, f(S(b)) would be a theorem of P_r. Now f(S(B)) is equivalent to '$p_0 \cdot -p_0$', which is obviously not a theorem of P_r. Neither, then, is S(b) a theorem of alpha, and alpha is consistent.

The next step we must undertake is to prove the converse of the metatheorem *1.11. Consider a function, which we shall call h. This function is to be many-one, and is to map the set (P_r) onto the set of alpha graphs. For any P_r wff A, form h(A) as follows: Wherever in A subformula -(B) occurs, replace it by B , and replace each P_r propositional variable by an appropriate minimal graph of alpha. The P_r variable 'p_0', we will recall, has a special use in the wff '$-(p_0 . -p_0)$', which is associated with the null graph of alpha. We may remark here that if any P_r wff A contains p_0 <u>outside</u> the context '$-(p_0 . -p_0)$', then A is clearly not in the range of the function f. In such cases, simply substitute for p_0 wherever it occurs in A a propositional variable entirely new to A. <u>Now</u> if '$-(p_0 . -p_0)$' occurs in a formula as it has been transformed to this point, simply erase it to form the null graph. This completes the instruction; the result of the application of this instruction to a wff A is the graph h(A). Note that where there were conjunctions of subformulas in A, there are unseparated occurrences of subgraphs in h(A), and where there were negation signs, there are cuts.

It should be evident that h is, as we claimed earlier, many-one and onto alpha. It is many-one (and not one-one) since the formulas '$p_1 . p_2$' and '$p_2 . p_1$', for example, would both translate through h as the same graph. It is onto alpha, since, quite clearly, from the definition of f,

$$h(f(X)) = X,$$

and for every alpha graph X there is a corresponding wff f(X).

We shall now establish the following lemma:

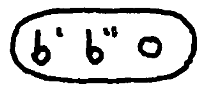
LEMMA 1.13 Where A is one of the axioms of P_r, h(A) is a theorem of alpha.

PROOF: We shall prove this lemma by deriving as theorems of alpha h of each of the three axioms of P_r which we listed earlier in this chapter.

Axiom, R_{bcl}

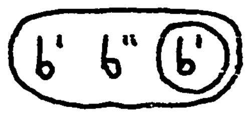
(1)

(1), R_{ins}



(2)

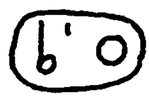
(2), R_{itr}



(3)

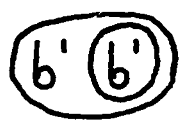
Note that (3) is h of the axiom 'p.q.⊃p'.

(1), R_{ins}



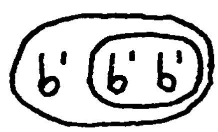
(4)

(4), R_{itr}



(5)

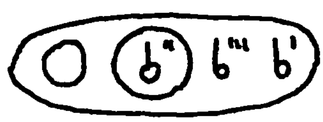
(5), R_{itr}



(6)

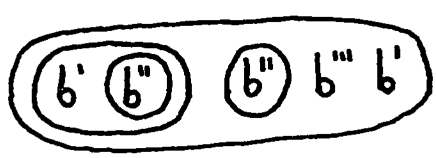
Note that (6) is h of the axiom 'p⊃.p.p'.

(1), R_{ins}



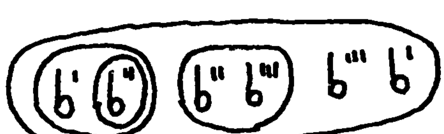
(7)

(7), R_{itr}



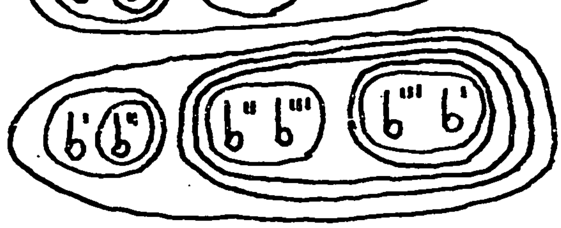
(8)

(8), R_{itr}



(9)

(9), R_{bcl}



(10)

Note that (10) is h of the axiom '$p \supset q . \supset . -(qr) \supset -(rp)$'; this concludes the proof.∎

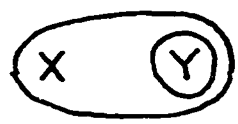
We shall now prove lemmas regarding "derived rules of inference" in alpha. Note that we shall make these lemmas do double duty, stating them both for h and the system P_r and g and the system P_w. This will save us the trouble of proving a parallel set of lemmas in the next section of this chapter.

LEMMA 1.14 Where $\ulcorner \delta p \urcorner$, as a truth-function of p, is a theorem of either P_r or P_w, and $\ulcorner \delta A \urcorner$ is the result of substituting the wff A for every occurrence of p in $\ulcorner \delta p \urcorner$, then if $\vdash_a$ h(δ p)--or g(δ p), depending upon the system of which $\ulcorner \delta p \urcorner$ is a theorem--then also $\vdash_a$ h(δ A)--or g(δ A), as the case may be. This lemma is stated only for the cases in which $\ulcorner \delta A \urcorner$ does not contain '$-(p_0 - p_0)$'.

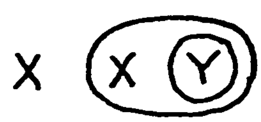
PROOF: This lemma is stated only for the "unspecialized" alpha system, that is, the system which has b as its sole axiom. If there is a proof in alpha for h(δ p)--or g(δ p)--I think it can be seen without elaborate argument that there will also be a proof for h(δ A)--or g(δ A). The minimal graph corresponding to p in h(δ p)--or g(δ p)--had originally to be introduced into the graph by an application of R_{ins}. But a graph corresponding to A could just as easily have been introduced at that point, and then treated in subsequent transformations just as the minimal graph corresponding to p was treated. If, then, h(δp)--or g(δ p)--is an alpha theorem, then h(δ A)--or g(δ A)--must also be an alpha theorem. This lemma states, in effect, the existence of a "derived rule of substitution" for minimal graphs in alpha; this derived rule may be called R_{sbs}, this being a binary recursive word predicate. ∎

LEMMA 1.15 Where $h(A \supset B)$ and $h(A)$--or $g(A \supset B)$ and $g(A)$, depending on whether A and B are wffs of P_r or P_w--are theorems of alpha, then $h(B)$--or $g(B)$--is also a theorem of alpha.

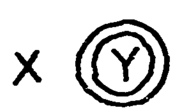
PROOF: Let $h(A)$--or $g(A)$--be the graph X; let $h(B)$--or $g(B)$--be the graph Y. The graph $h(A \supset B)$--or $g(A \supset B)$ is:



This is clear from the definitions of these functions. Now, by hypothesis, the following graph is a theorem of alpha:



By one application of R_{dit}, this becomes:



And by one application of R_{ers}, this becomes:

And finally, with an application of R_{nbc}, we get:

Y

But Y is h(B)--or g(B), as the case may be--and so the lemma holds. There is just one case in which there may be some question as to the truth of this lemma; this is with regard to the P_r theorem. For $h(-(p_0 . -p_0))$ is b, the null graph, and not a graph of form G(X.Y). But note that here the second hypothesis of the lemma--that h(A) be a theorem of alpha--can never be realized because of the consistency of alpha and the fact that h(A) will here be an alpha minimal graph other than b. Since that hypothesis can never be realized, the antecedent of the lemma is <u>false</u> in this special case, and so the lemma is true.

This lemma may be considered to assert the existence in alpha of a derived rule of detachment, which may be called R_{dtm}--a ternary recursive word predicate. ■

Now we may apply our lemmas thus:

*1.16 If A is a theorem of P_r, then h(A) is a theorem of alpha.

PROOF: We know from lemma 1.13 that when A is an axiom of P_r, h(A) is a theorem of alpha. We also know from lemmas 1.14 and 1.15 that analogs of P_r's rules of inference exist as derived rules of inference in alpha.

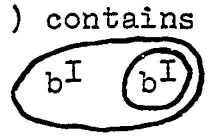
Let us now divide the theorems of P_r into two sets:

1. Those which do not contain the subformula '$-(p_0 . -p_0)$', and
2. Those which do contain occurrences of this subformula.

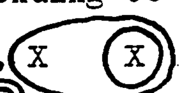
Clearly, if there is a proof in P_r for a wff A, and A belongs to the first of these sets, then there is a proof for h(A) in alpha, by lemmas 1.13, 1.14, and 1.15, and the definition of h.

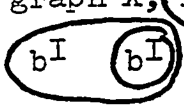
Now consider a wff A for which there is a proof in P_r, and which contains one or more occurrences of '$-(p_0 . -p_0)$'. But if there is a proof in P_r for A, then there is a proof for a wff A', which is exactly like A except for containing the subformula '$-(p_i . -p_i)$'--with p_i entirely new to the formula--wherever A contains '$-(p_0 . -p_0)$'. This we know because of the rule of substitutivity of the biconditional. A' is then a theorem, and belongs to the first of the sets of theorems mentioned above. By our remarks in the preceding paragraph, then, h(A') is a theorem of alpha.

The graph h(A') contains at certain places the subgraph



(with the I representing a string of i primes)

corresponding to the subformula '$-(p_i . -p_i)$' in A'. But for any graph X, 

is clearly a theorem of alpha. The subgraphs 

may then be deiterated from wherever they stand in h(A'), leaving blanks at those places. But by our definition of h, the result of such deiterations from the graph h(A') is precisely the graph h(A). In the case where A contains occurrences of '$-(p_0 . -p_0)$', then, and is a theorem of P_r, h(A) will be a theorem of alpha. Thus, in every case that A is a theorem of P_r, h(A) is a theorem of alpha. ∎

*1.17 If f(X) is a theorem of P_r, then X is a theorem of alpha.

PROOF: Suppose that f(X) is a theorem of P_r. Then, by *1.16, h(f(X)) is a theorem of alpha. But by the definitions of f and h, we know that h(f(X)) and X are the same graph; thus X is a theorem of alpha, and the metatheorem holds.

1.33 The System P_w in the Logic Alpha

We now move to a consideration of how a set of PC theorems maps into the set of alpha theorems. At first glance this may seem a strange project to enter into at this point, for we have already indicated that the function h maps the set of theorems of P_r onto the set of alpha theorems (although we have not formally stated this, it is nonetheless fairly evidently the case). But h is not a one-one function, and we wish to show that the set of PC theorems maps one-one into the set of alpha theorems. Here we shall make use of the one-one function g and the subset $\mathscr{P}_{\alpha}$ of alpha graphs.

We shall now introduce a sign of the metalanguage similar to one we used in the proof of lemmas 1.07 and 1.08, the sign 'δ'. This sign may be used along with the signs of alpha, and when it appears in a sign complex, that complex will be known as a "graph schema." The sign will be a Greek letter, usually 'δ', followed by a pair of curly braces within which will be scribed a graph (possibly the null-graph, in which case nothing will appear between the braces).

$$\delta\{X\}$$

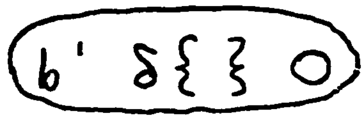
This sign, with a graph scribed between its braces, may stand in place of any alpha graph, and its relation to the alpha system is just the same as that of the 'δ' employed in the proof of lemmas 1.07 and 1.08 to the CPC. It represents, in other words, a "graph function" of the graph that appears between its braces.

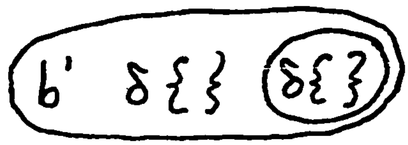
We shall now engage in some graphical proofs:

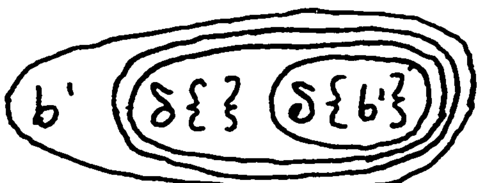
LEMMA 1.18 Let $\ulcorner\delta\{X\}\urcorner \in \mathcal{P}_\alpha$ whenever $X \in \mathcal{P}_\alpha$. Then the (5) graph is a theorem of alpha and a member of $\mathcal{P}_\alpha$.

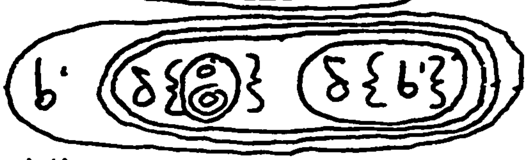
PROOF: The lemma is proven by the following graphical steps:

Axiom, R_{bcl}  (1)

(1), R_{ins}  (2)

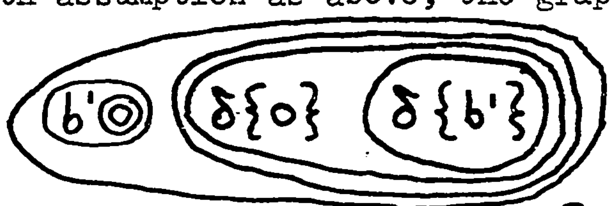
(2), R_{itr}  (3)

(3), R_{itr}, R_{bcl}  (4)

(4), R_{bcl}  (5)

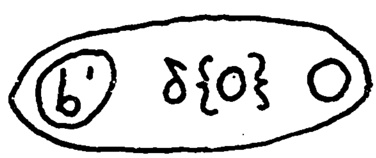
The graph at (5) is the one we wished to prove. That any graph belonging to this schema is a member of $\mathcal{P}_\alpha$ is a simple matter of inspection. ■

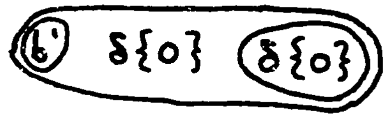
LEMMA 1.19 With assumption as above, the graph

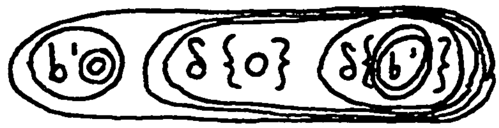


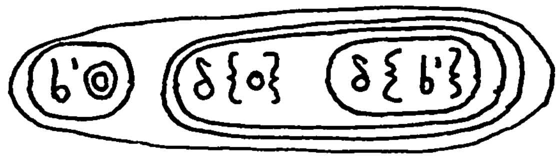
is a theorem and a member of $\mathcal{P}_\alpha$.

PROOF:

(1), R_{ins}  (6)

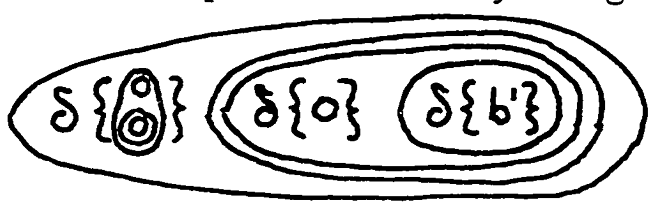
(6), R_{itr}  (7)

(7), R_{itr}, R_{bcl}  (8)

(8), R_{nbc}  (9)

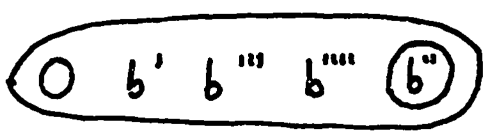
The graph at (9) is the one we wished to prove; again, determining that any graph belonging to this schema belongs to $\mathcal{P}_\alpha$ is a matter of inspection. ∎

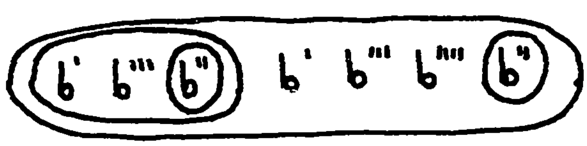
LEMMA 1.20 With assumptions as above, the graph

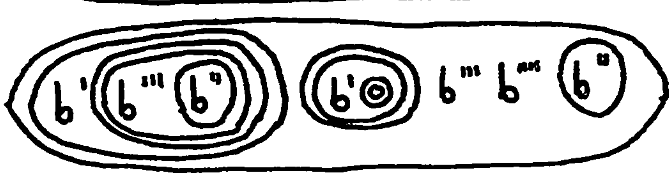


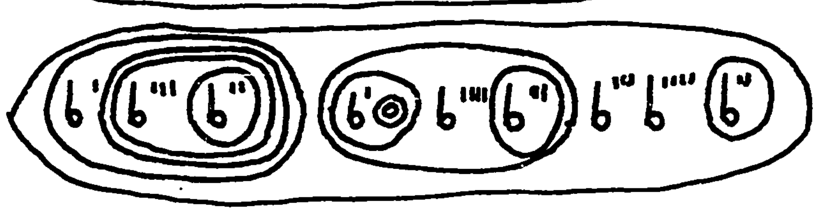
is a theorem of alpha and a member of $\mathcal{P}_\alpha$.

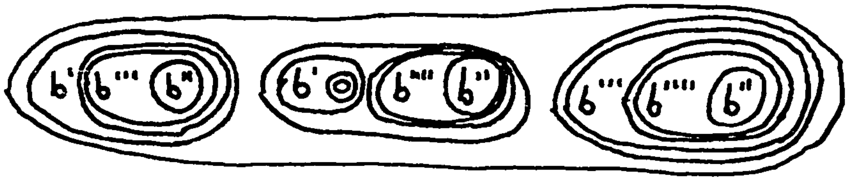
PROOF:

(1), R_{ins}  (10)

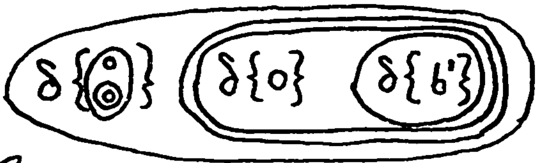
(10), R_{itr}  (11)

(11), R_{bcl}  (12)

(12), R_{itr}  (13)

(13), R_{bcl}  (14)

Now with (14), applying (14) first to the graph of lemma 1.18, and the result of that to the graph of lemma 1.19 by the derived rules R_{sbs} and R_{dtm} (taking first the graph of 1.18 and then that of 1.19 as "antecedent" in [14]), we get



(15)

Membership in $\mathcal{P}_\alpha$ by inspection. ∎

Let us now examine what we have. By the translation function g, the graph-theorem-schema of lemma 1.20 is correlated to the P_W theorem schema

$$\ulcorner C\delta 1 C\delta o \delta \bar{p} \urcorner.$$

Here '1' is defined as 'COO'. This will be recognized as being of the same form as the single axiom of Lukasiewicz' Classical PC with functor variables of 1951.[1] Although we are not using functor variables as primitive symbols, it should be evident that the formulas belonging to the above schema, when joined to the rules of inference of detachment and substitution for variables, form a sufficient basis for the complete CPC; the axioms of P_W are derivable in a system containing the above schema. The presence of the schema of lemma 1.20 in alpha emphasizes the extensional nature of alpha, and also illustrates the power of the rules of transformation of alpha. We may now state the metatheorem to which these lemmas have been leading:

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[1] Ibid., p. 306.

*1.21 A wff A is a theorem of P_W iff g(A) is a theorem of alpha.

PROOF: Lemmas 1.14 and 1.15, along with lemma 1.20, show us that whenever there is a proof for A in P_W, there is a proof for g(A) in alpha. Conversely, suppose that g(A) is a theorem of alpha, but A is not a theorem of P_r. P_W is strongly complete, and by the first part of this metatheorem, every theorem of P_W has a correlate theorem in ρ_α (the range--in alpha--of the function g). If in addition to these theorems we have g(A) as a theorem derivable by the alpha rules of transformation, while A is a non-theorem of P_W, it is easy to see that in the presence of the other theorems of alpha which are members of ρ_α and the derived rules of substitution and detachment in alpha that any member of ρ_α would then be provable as an alpha theorem. But S(b) is a member of ρ_α, and would in this case be an alpha theorem; this, however, would render alpha inconsistent. But, by corollary 1.12, alpha is consistent. Therefore, it must be the case that if g(A) is a theorem of alpha, A is a theorem of P_r, and the metatheorem holds in both directions. ∎

We shall now state another definition of Martin Davis.[1]

D8: Let $\mathcal{L}$ and $\mathcal{L}'$ be logics. Then we say that $\mathcal{L}$ is <u>translatable</u> into $\mathcal{L}'$ if there exists a recursive word function k(X) such that $\vdash_{\mathcal{L}}$ X iff $\vdash_{\mathcal{L}'}$ k(X), and moreover, if, whenever X = Y, we also have k(X) = k(Y). (That is, if k is one-one.)

---

[1]Davis, p. 119.

The following metatheorem applies this notion to alpha:

*1.22 Any complete CPC is translatable into alpha and vice-versa.

PROOF: Immediate from D8, *1.11, *1.17, and *1.21, and the definitions of the functions f and g. ■

Actually, considering the natures of the functions f and g, we are safe in saying that alpha is itself a complete classical propositional calculus. We may add the following concluding metatheorem:

*1.23 Alpha is complete in the strongest sense possible for a system lacking a primitive rule of direct substitution for variables.

PROOF: Immediate from *1.17, which asserts that if f(X) is a theorem of P_r, X is a theorem of alpha. ■

CHAPTER II

THE BETA SYSTEM

We now move to a close-up study of the beta part of the graphs. As we have seen, alpha may be considered to be a system of propositional calculus; to be sure, alpha's approach to truth-functional logic differs from approaches utilizing more ordinary notation, but it is relatively easy to grasp intuitively. We may even react quite favorably, in fact, to features of alpha like the symmetry--or irrelevancy of position--and associativity of conjunction implicit in the "simultaneous unseparated occurrence" of alpha graphs, this in spite of the complexities introduced by such features in, say, the translation of the graphs into ordinary notation and vice-versa. Beta, however, is a kettle of fish taken from somewhat deeper waters. Beta, we shall find, bears a relation to alpha similar to that borne by the ordinary first-order predicate calculus to the ordinary CPC. But as was indicated in the Introduction, the "individual variable" and the method of quantification in beta are, apparently, radically different from the corresponding elements in the ordinary predicate calculus. While intuition may give willing assent to the characteristic features of alpha, it is possible that it may be a bit more hesitant about beta. But a careful examination of beta should lead to a more willing acceptance of its features; such an examination will be the aim of this chapter.

The project of chapter i was a comparison of the alpha system with the ordinary CPC. As we might infer from the remarks we have just made, the project for beta will involve a comparison of beta with ordinary classical predicate calculus. We shall, in fact, compare beta with the complete classical first-order calculus with identity.

The notation of beta is--as we have already seen--considerably more complicated than that of alpha. We shall, then, first set down some remarks on that notation. Much of the terminology of the beta system will be explained, and a list of rules characterizing the beta graphs will be set down.

In our comparison of alpha with the CPC, we found it convenient to use two formulations of the CPC. Similarly, we shall here employ two formulations of the first-order calculus with identity. The first, F_r, will--like P_r of chapter i--have conjunction and negation as primitive operators; the other, F_w, will use implication and constant false proposition, as did P_w. Both systems will have the universal quantifier and the sign of identity as primitive. F_r and F_w are both full classical first-order calculi with identity, and are "equivalent" to each other in the sense that P_r and P_w are equivalent.

We shall provide a set of instructions, which shall be called the (one-one) function f', and which--when applied to any beta graph X as argument--will "translate" it into a unique closed wff--called f'(X)--of the logic F_r.

We shall also provide a set of instructions, to be called the (one-one) function g', which--when applied to any closed wff

A of the logic F_w--will translate it into a unique beta-graph--to be called, in this case, g'(A).

The description of the functions f' and g' will parallel in many ways the description of the functions f and g of chapter i.

Then will come an investigation of "theorem generation" in beta. This will involve an explicit statement of the rules of transformation, or inference, in beta. Here we shall show both that beta is a logic and that it is consistent.

The next step will be a proof--through a series of lemmas showing that the beta rules of transformation have analogs in F_r--that:

> If a beta graph X is a theorem of beta, then the closed wff f'(X) is a theorem of F_r.

The proof of the converse of this metatheorem, however, does not follow as easily as it did in alpha, and will be taken up later in this chapter.

Then we shall achieve a most remarkable result. We shall show that:

> A closed wff A is a theorem of the logic F_w iff the graph g'(A) is a beta theorem.

In the process we will give what amounts to a set of semantical rules for beta, that is, a means of assigning an <u>interpretation</u> to any beta graph. This will enable us to say what it means for a beta graph to be <u>valid</u>.

We may then enter the final phase of our beta project. This will be to prove that every valid beta graph is a theorem of beta. This is, in effect, to prove that beta is <u>complete</u>.

Once this is done, it will follow that:

If the closed wff $f'(X)$ is a theorem of F_r, then X is a theorem of beta.

We shall thus have established that the set of beta theorems maps one-one into the set of theorems of the first order calculus with identity, and more importantly, that the set of theorems of the first-order calculus with identity map into the set of beta theorems. Given the natures of the mapping functions, f' and g', we will then have shown beta itself to be a complete first order calculus with identity, and to have a recursively unsolvable decision problem.

2.1 The Set of Beta Graphs

In the Introduction we examined the beta system--at arm's length--and got some idea of what it is for a sign-complex to be a beta graph. In this section we shall look more closely at the set of beta graphs. Our purpose shall be two-fold; we wish:

(1) To recursively characterize the set of beta graphs, and in the process to say what it means for two beta graphs to be identical, and

(2) To lay the groundwork for the definition of the functions which will translate beta into an ordinary logic and vice-versa.

The characterization of the set of alpha graphs offered us no problem at all. Given the set of minimal alpha graphs and the "operations" $S(X)$ and $J(X_1, \ldots, X_n)$, we were easily able to characterize the set of alpha graphs.

As might be gathered from the Introduction, beta will be more difficult to handle than was alpha. It is fairly evident that beta, with its "spots," is a "calculus of predicates." And among the signs of beta is that protean "line of identity" (which we shall often abbreviate as "LI"); this sign is sometimes itself a graph--and then has a propositional interpretation--and sometimes, as we remarked in the Introduction, a kind of "bound variable." The presence of the LI in the vocabulary of beta considerably complicates both the characterization of the set of beta graphs and the definition of "translation functions" paralleling alpha's f and g.

Let us now define a term we shall find quite useful in our treatment of beta. We shall call a LI "geodesic" between two points on the SA iff it connects those points, crossing only the cuts which may be between them, and each of those cuts only once.

We shall now provide a set of rules which will characterize the set of beta graphs. These rules might also be considered rules for the "generation" of any beta graph.

2.1i The "null-graph," b, is a beta graph.

2.1ii '——'--a LI situated entirely in one area with no connections or branchings is a beta graph; we shall say this graph has two "loose ends," which are the end points of the LI.

2.1iii '>—'--the complex consisting of 3 LI's connected at one point, all in the same area, and with no other

other connections or branchings is a beta graph; this graph has 3 "loose ends."

2.liv ' $S \begin{smallmatrix}(1)\\(2)\\ \vdots \\(n)\end{smallmatrix}$ '--an n-adic spot with a LI attached to each of its n hooks, the entire complex in the same area, and none of the LI's having any other connections or branchings is a beta graph; this graph has n "loose ends."

2.lv Where X is a beta graph and has n loose ends, S(X) is a beta graph and has n loose ends.

2.lvi Where X_1, ..., X_n, n 2, are n beta graphs admitted by rules 2.lii-v above, and the total number of loose ends in these n graphs is m, then $J(X_1 \ldots X_n)$ is a beta graph and has m "loose ends."

2.lvii Where X is a beta graph having n loose ends, and X' is like X except for having two of those ends connected by a geodesic LI, then X' is also a beta graph, and has n-2 loose ends.

It should be fairly clear that these rules will admit all and only the sign complexes which Peirce would consider beta graphs. Two beta graphs are identical iff they have identical "histories of generation" by the above rules.

It is evident that for the generation of a given beta graph by these rules, the insertion by rules 2.lii-iv of certain signs involving LI's is <u>essential</u> to the generation of that graph. To put it another way, we may say that for the description of the "LI network" of a given graph, it is necessary that we know the locations in that graph of certain "critical points" in that network. These "critical points" are:

(1) The places where LI's connect to hooks of spots,

(2) Branchings in LI's,

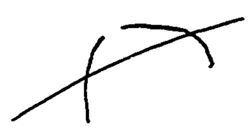
(3) The places where LI's come to "dead ends,"

(4) The places where LI's become "non-geodesic."

In the "generation" of a beta graph by rules 2.1i-vii, these are the points which must "get into" the graph through rules 2.1ii-iv. The remainder of the LI network of the graph may be "filled in" by rule 2.1vii.

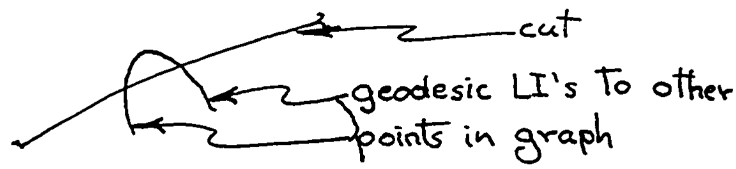
A comment on (4) above may be in order. If a LI is non-geodesic between two points, a little thought will tell us that there is at least one place along its length where it crosses a cut and then re-crosses the same cut thus:

Such LI formations will be called "loops." Consider any non-geodesic LI, snaking its way through a beta graph. If this LI were to be broken at all its loops, like this:



it is obvious that each of the LI's resulting from such breaking would be geodesic. If we wish to picture to ourselves the "generation" of a beta graph containing non-geodesic LI's (which is the same as saying, "Containing loops"), we may imagine that LI's with two "dead ends" are first inserted by rule 2.1ii at places in the graph where loops will eventually appear; the connections which will then "create" the loops may be made by 2.lvii. So, justified by 2.lii we have:

And then justified by 2.lvii:



The other three kinds of critical point in the LI network require, I think no further explanation. We shall see that these "critical points" play an important role in the translation of beta into an ordinary logical system in such a manner that X and Y translate as the same formula in ordinary logic iff the graphs X and Y are identical, that is, have the same "history of generation."

We will here note one special but unimportant kind of LI formation; if a LI "twists back on itself" and joins end to end, all in the same area, to form the graph:

(This is not a cut, but a LI) with no branchings and homeomorphic to a circle, that LI we shall call the "cyclic graph."

We may also add, as a bit of nomenclature, that we shall say that a LI "terminates" when and only when either it comes to a dead end, connecting to nothing, or it joins two other LI's at a branching.

2.2 The Beta Graphs and the WFFs of Classical Predicate Calculi

2.21 The Systems F_r and F_w

As was the case in Section 1.2 of our investigation of alpha, we shall set out two systems of classical logic; in this case these will be systems of first-order predicate calculus with identity. The systems themselves will be called F_r and F_w; we shall be interested in a particular subset of the set of wffs of each of these systems, the set of closed wffs (for closed wff we shall write "cwff"). The sets of cwffs of F_r and F_w respectively will be (F_r) and (F_w), and we shall specify what it is to be a cwff.

We shall employ the upper-case letters at the beginning of the Roman alphabet as metalinguistic variables for wffs and for predicates, and the lower-case letters at the beginning of the Roman alphabet as variables for individual variables. The letters 'x', 'y', . . . , will, as usual, be the individual variables themselves. An upper case letter will normally represent a predicate only when it is followed by a sequence of lower-case letters, which will represent or be the arguments of that predicate. "$\vdash_{\overline{R}}$ A" shall mean "the closure of A is a theorem of F_r," and "$\vdash_{\overline{W}}$ A" shall mean "the closure of A is a theorem of F_w." The notion of closure is as in the revised edition of Quine's Mathematical Logic.

The primitive signs of the system F_r are: A denumerable infinity of signs, w, x, y, z, w_1, x_1, . . . , called the "individual variables," the signs '(', ')', '-' as in P_r, the sign '=', a constant dyadic predicate called "the sign of identity."

The rest of the signs of F_r are "variable predicate symbols," which we need not show, and each of which has associated with it a natural number ≥ 1 called its "degree."

The primitive signs of F_w are: individual variables and variable predicate symbols as in F_r, the signs 'C' and 'O' as in P_w, the sign 'I', a constant dyadic predicate called "the sign of identity," and the sign '$\prod$', called the "sign of the universal quantifier."

Again, we shall assume all the standard definitions, including that of the existential quantifier. Note that the kind of notation used in practice is frequently a matter of convenience. It will be recalled that sometimes in our work with P_r in chapter i we used the Polish notation, even though the primitive symbols of that system are not in that notation. We did this because we feel that the Polish notation is superior to the "PM-type" notation for work in the PC. For extended work in quantificational systems, however, we prefer the "PM-type" notation; hence, we shall often, in this chapter, write out F_w formulas in that notation. It is always to be understood, however, that a formula so presented is a definitional "abbreviation" of a formula in primitive notation.

Now, the rules of formation for F_r are:

2.21i $\ulcorner a = b \urcorner$ is wf where a and b are individual variables.

2.21ii $\ulcorner Aa_1 \ldots a_n \urcorner$ is wf where A is a predicate symbol of degree n and $a_1, \ldots, a_n$ are exactly n individual variables.

2.21iii $\ulcorner -(A) \urcorner$ is wf where A is wf.

2.21if $\ulcorner AB \urcorner$ is wf where both A and B are wf.

2.21v $\ulcorner (a)(A) \urcorner$ is wf where a is an individual variable and A is wf.

An occurrence of a variable a is called "bound in a wff A" if it is in a wf part of A of form $\ulcorner (a)B \urcorner$. Otherwise it is called "free in A." A wff in which no variable has a free occurrence is called "closed." We write "cwff" for "closed well-formed-formula."

The rules of formation for F_W are:

2.21vi 0 is wf.

2.21vii $\ulcorner Iab \urcorner$ is wf where a and b are individual variables.

2.21viii This rule is the same as 2.21ii.

2.21ix $\ulcorner CAB \urcorner$ is wf where A and B are wf.

2.21x $\ulcorner \Pi aA \urcorner$ is wf where a is an individual variable and A is wf.

The remarks concerning "freedom," "binding," etc. for F_W apply here as well, <u>mutatis mutandis</u>.

As axioms for both systems we shall employ versions appropriate to the respective primitive notations of Quine's "axioms of quantification," *100, *101, *102, and *103. In addition we shall have the following axiom schemata for F_r:

$$\vdash_{\bar{R}} \ulcorner a = a \urcorner$$

$$\vdash_{\bar{R}} \ulcorner a = b \supset .Aa \supset Ab \urcorner.$$

In F_W the extra axiom schemata are:

$$\vdash_{\bar{W}} \ulcorner Iaa \urcorner$$

$$\vdash_{\bar{W}} \ulcorner CIabC_{BaBb} \urcorner.$$

The rule of inference for each system shall be an appropriate version of Quine's *104. All of Quine's metatheorems referred to here are, of course, from the revised edition of Mathematical Logic.

2.22 The Translation Function f'

In chapter i we showed how two functions, f and g, might be defined. The former was a function which translated alpha graphs into formulas of an ordinary PC, and the latter translated formulas of an ordinary PC into alpha graphs. We are now prepared to present the similar "translation functions" for the beta system. These functions will relate beta to the calculi F_r and F_w, and will be called f' and g' respectively.

We shall first consider the function f'. This function takes as arguments the members of the set of beta graphs, and finds its range in the set (F_r) of cwffs of F_r. For the purposes of translation, we shall add to the vocabulary of beta (temporarily) a "translation vocabulary" consisting of three spots:

(1) A unary spot, Q,

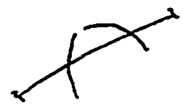
(2) A binary spot, L,

(3) A ternary spot, B.

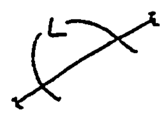
We now apply the function f' to an arbitrary beta graph X. Given X, attach an instance of the spot Q to each dead end of a LI occurring in X.

Next, we will recall from our earlier remarks what a loop is. Wherever a loop occurs in X,

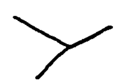
break it like this:



and then attach the two LI ends thus freed, one to each hook of the spot L:

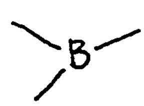


And then, there may be branchings in X. Each of them will look like this:



Erase the center of the branching thus:

and attach each of the loose ends of LI's thus formed to one of the hooks of the spot B, like this:



There may be "cyclic graphs" occurring in X. Wherever such a graph occurs, erase it and write the sign-complex '$(\exists w)(w = w)$'. Consider this sign-complex for the moment as if it were a graph.

It is now clear that all the LI's remaining in the graph X are, must be, geodesic. Effective means are certainly available for ordering the set of LI's remaining in X (we remark in passing that there are now, of course, no dead ends or branchings left in X). Order this set, and then associate with each such LI a distinct individual variable of the system F_r (excluding 'w'). If a given LI in the graph as it now stands is associated with the variable a, erase that LI and write the variable a at the points in the graph which were connected by the LI in question; that is, attach a to the hooks to which the LI had been connected. Do this for all LI's remaining in the graph. Where X was the original graph, call the version as so far transformed "X'."

X' is a graph free of LI's. Where X used LI's to identify arguments of spots of various kinds, X' uses individual variables. In this it is much closer to a formula of ordinary logic than was X. It is also much more similar to an alpha graph. In fact, we may now use an appropriately altered version of the function f of chapter i to convert X' to a unique wff of F_r. If the cyclic graph occurred in the original graph X, let the wff '$(\exists w)(w = w)$' be carried over into the formula we are forming unchanged. The spots Q, L, and B should be associated with constant predicates Q L, and B of F_r, the predicates being unary, binary, and ternary respectively. All individual variables should be carried over from X' as arguments of the predicates associated with the spots to which those individual variables were attached in X'; let the arguments of each of the predicates L and B be ordered alphabetically. Let each of the variable spots in X' be associated with an appropriate variable predicate symbol in the formula being formed from X'. Now assign quantifiers to the variables of this formula. Begin with the alphabetically lowest variable which appears free in the formula. Where this variable is a, write the quantifier $\ulcorner(\exists a)\urcorner$ in such a manner that it takes as its scope exactly the least subformula of the whole formula which contains all occurrences of a. Do this for each variable in turn, in alphabetical order.

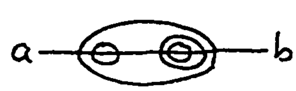
Now for each variable a, wherever $\ulcorner Qa\urcorner$ occurs, replace the $\ulcorner Qa\urcorner$ by $\ulcorner a = a\urcorner$. For each a and b, wherever $\ulcorner Lab\urcorner$ occurs, replace it by $\ulcorner a = b\urcorner$. For each a, b, and c, wherever $\ulcorner Babc\urcorner$ occurs, replace it by the formula $\ulcorner(\exists w)(a = w.b = w.c = w)\urcorner$.

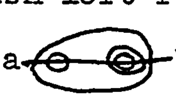
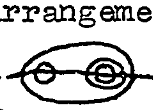
This completes the instruction for f'. Where the graph we started out with was X, the cwff of F_r we now have is f'(X). This formula is the unique translation by the function f' of the graph X into the logic F_r.

2.23 The Translation Function g'

The function g of chapter i translates the CPC P_W into the alpha system. The function g' will perform a similar function for the first-order calculus and the beta system. Consider an arbitrary member of the set (F_W), A. A is a cwff; its signs may consist of individual variables, predicate signs of various degrees, the sign of identity, 'I', the sign of implication, 'C', the constant false proposition, '0', and the sign of universal quantification, '$\prod$'. Now in the F_W wff A, let us for the time being consider the sign complexes ⌜$\prod$ a⌝ (the quantifiers) <u>as if</u> they were themselves wffs, and immediately before each '$\prod$' insert a 'C'; call the resulting formula A'--on the supposition that ⌜$\prod$ a⌝ is wf, A' is also wf. Now apply the function g (from chapter i) to A', with the following adjustments in instructions: g called for the replacement of P_w propositional variables by alpha minimal graphs, but here let the predicate signs of A' be replaced by appropriate beta spots: where a predicate sign of degree n is replaced by a spot (also of degree n, of course) let the argument variables of that original predicate sign be attached to appropriate hooks of the replacing spot. Let the subformulas ⌜$\prod$ a⌝ and ⌜Iab⌝ go into the graph unchanged; '0', the constant false proposition, will be replaced by the simple empty cut, S(b).

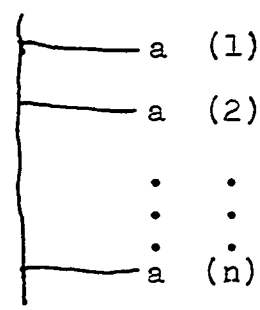
Now we have transformed A through A' into a graph, but one without lines of identity. Turn next to the subformulas ⌜Iab⌝ in that graph. Replace the formula Iab with the graph



for each such formula. The motivation behind this move is easy to understand. If we had taken the fairly obvious step of simply replacing Iab by the graph a——b , we would have no means of distinguishing the translation of ⌜Iab⌝ from that of ⌜Iba⌝--for a——b and b——a are, in the beta system, identical graphs; beta does not distinguish left from right or up from down. The arrangement of cuts in ab, however, distinguishes it from ba , and enables us to offer distinct translations for ⌜Iab⌝ and ⌜Iba⌝.

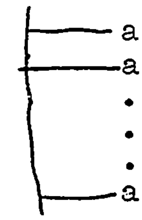
Now turn to the formulas $\ulcorner \prod a \urcorner$ in our graph; consider an arbitrary one of these formulas in that graph. In the original formula A there is a quantifier associated with the $\ulcorner \prod a \urcorner$ we are considering in our graph. In the scope of that quantifier in A are n occurrences of the variable a which n occurrences are bound by the quantifier in question. First of all, if n = 0, that is, if the quantifier in question is vacuous, replace the $\ulcorner \prod a \urcorner$ in our graph with ' — ', the simple "double-dead-end" LI situated entirely in the area in which the $\ulcorner \prod a \urcorner$ stood.

But if $n \neq 0$, replace the appropriate $\ulcorner \prod a \urcorner$ in our graph with the following sign-complex:



That is, replace it with a LI having n branchings, at the end of each of which is the variable a (the parenthesized numerals in the above diagram are not part of the sign-complex replacing $\ulcorner \prod a \urcorner$).

Now we are ready to complete our translation of the original formula A into a beta graph. In essence, what we are about to do is just the reverse of a step we took in our description of the function f'--there we replaced geodesic LI's with individual variables, and here we will replace individual variables with geodesic LI's. So go to one of the complexes



standing in our graph. This sign-complex corresponds to a quantifier in the original cwff A, which quantifier binds n occurrences of the variable a in A. There are, in our graph as it now stands, n points--each marked by the variable a--which points correspond to the n occurrences of a in A bound by the quantifier in question. Let each of the branchings in the above pictured complex be connected by a geodesic LI to one of these points; erase each of the a's as the LI's are drawn (these variables are no longer needed as place holders once the LI's are present). These connections should be made in some effectively determined order; that is, we should have a mechanical procedure for telling which branch of the pictured sign-complex is to be attached to which point in the graph--we will not describe such a procedure, but it should be clear that it would be quite easy (if irrelevantly tedious) to do so.

The above procedure of replacing variables by LI's should be continued until all variables in the graph have been replaced by LI's. What we have when we are finished is a beta graph with its variables replaced by LI's, and with distinctive signs involving LI's replacing quantifiers and identity formulas. Where A was the original F_w cwff, the graph we now have is g'(A).[1]

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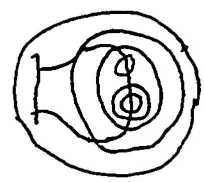
[1]Cf. Willard Van Orman Quine, Mathematical Logic (rev. ed.; Cambridge: Harvard, 1958), p. 70. Note the similarity between Quine's "curved line formulas" and the beta graphs which will constitute the range of g'.

We must draw attention here to the fact that g' as described above is not, strictly speaking, one-one, but many-one. This is so because if B is a mere alphabetic varient of the cwff A, then with the procedure we have outlined, g'(B) will be precisely the same graph as g'(A). As a simple example, the F_W cwffs '$\prod$xIxx' and '$\prod$yIyy' will both translate as the beta graph



One might comment that this is not a serious difficulty; from one point of view, you might even consider that if A and B are merely alphabetic variants of each other, then they are essentially the same formula--from this point of view, g' as it stands would already be one-one. But I submit that there are many possible ways of making g' take account of alphabetic variance, to yield distinct graphs as values for g' of cwffs A and B when A and B are distinct alphabetic variants of each other. Again, however, the description of such means has a definite tendency to be tedious--to be a kind of intellectual nit-picking. And it adds little to the main thrust of this paper. Hence, let us state simply that provisions may be built into g' to take care of alphabetic variance. Once this is accounted for and our explicit description of g' taken as above presented, we may assert that g' is a one-one recursive word function.

It will be recalled from chapter i that the function g mapped (P_W) onto a recursively characterizable subset of the set of alpha graphs, ρ_{α}. From the description we have given of g', it is clear that the beta graphs constituting the range of g' are members of a set in many ways similar to ρ_{α}. As we shall see later in this chapter, this set too is recursive, although its characterization is a bit more complex than that of ρ_{α}. The set of beta graphs which is the range of g' we will call ρ_{β}.

The mode of definition of the functions f' and g' tends to suggest, at least, that beta itself may be considered a complete notational basis for the first-order calculus with identity. This may seem somewhat weird, for it is quite apparent that beta has no notation specifically for explicit quantifiers. Weird it is if we restrict our concept of "quantification" to the quantification associated with the ordinary garden-variety quantifier we are accustomed to seeing. It will be recalled that in working with the functions f' and g' we make no attempt to correlate beta graphs with open formulas of predicate calculi; this is because every beta graph is, in effect, a cwff. One of the functions of the LI in beta is that of the individual variable. Quantification over these individual variables in beta is not accomplished by the insertion of a sign of quantification, but is implicit in a LI wherever it appears. The kind of quantification, the way the implicit quantification is to be interpreted, depends on the location in the beta graph of the LI being considered. There is nothing unusual about this; it makes the kind of quantification (that is, whether the quantification is universal or existential) depend upon "truth-functional" connectives in beta, specifically, upon the cuts in the graph by which the LI is enclosed. The ordinary classical quantification depends for distinction between universal and existential quantification on truth-functional features also. '(x)' is a universal quantifier, and '-(x)-' is an existential quantifier; the only difference between them is the presence of negation signs in the latter. Briefly, we may remark that a LI whose outermost point in a graph X is oddly enclosed in X bears an implicit universal quantification in X; otherwise it is existential. This will become easier to see in our discussion of the theorems of beta. It is sufficient for now to note that both the function f' and the function g' take account in their translations of this implicit quantification in beta.

2.3 Beta as a Logic

2.31 Theorem Generation in Beta

We now move from the study of the set of objects which qualify to be called beta graphs to the study of the subset of that set which is characteristic of beta as a logic--the theorems of beta. We have remarked that every beta graph is to be considered a closed quantificational formula. It follows rather trivially from this that every theorem of beta will be a closed formula. This fact is not as restrictive as it might seem at first; it is true that there are formulations of the predicate calculus which recognize some open formulas as theorems, but this is by no means necessary. The system of Quine's *Mathematical Logic*, for example, is one in which every theorem is a cwff; we have, in fact, based our systems F_r and F_w upon the "quantification theory" of Mathematical Logic.

Beta as a logic is built upon alpha; this becomes apparent in an examination of the set of rules for beta given in 4.505-508, for example. This means that the six rules of inference for alpha are to be considered to apply to beta graphs as well, insofar as the transformations wrought by these rules do not affect the LI's of a graph; in addition, the logic beta will contain what we may call extensions of these rules for cases involving transformations affecting LI's; we shall discuss this presently.

We will recall that the single axiom for alpha is the simple null-graph b. As we noted in the Introduction, Peirce indicates that another graph is to be what we would call an axiom for beta (4.567). For our purposes, we may take this graph to be the doubly terminating LI--the LI with two dead ends--standing alone on SA, with no branchings or connections. It is evident that given this graph as axiom and the rule R_{ers}, b may be derived as a theorem of beta. Let us then say that the double-dead-end LI standing on SA is the only axiom of beta.

In beta we have, as we have mentioned, the six alpha rules of transformation. In this sense beta is founded upon alpha just as quantification theory is founded upon the PC. There are also certain specific beta extensions of these rules which we shall now list (Cf. 4.505-508). Note that we use the same notation for these rules as we did for the alpha rules; the conditions we list here are to be considered clauses extending the alpha conditions for the respective rules of transformation.

Following, then, are the "extending clauses" for the rules of transformation as they shall be employed in beta:

$R_{ins}(Y, X)$: Is true provided Y is exactly like X except that in an <u>oddly enclosed</u> area where X has two LI dead end termini, Y has not those two termini but a continuous LI joining the points in Y corresponding to the points in X to which the terminating LI's were connected (this rule, in other words, permits the joining of two "loose ends" in an oddly enclosed area).

$R_{itr}(Y, X)$: Is true provided either:

1. Y is exactly like X except that at a point where X contains a continuous segment of LI, Y contains a <u>branching</u> with two of the branches connected respectively to the points connected by the continous LI in X and the third coming to a dead end with no cut-crossings, or
2. X contains at some point a "dead end" LI, and Y is identical to X except for having that dead end extended <u>geodesically</u> to any point in Y enclosed by <u>at least all</u> the cuts by which the original dead end in X was enclosed, or:
3. We shall call the "boundary" of a subgraph Z of a beta graph X an "imaginary self-returning

line" homeomorphic to a circle which might be drawn in X to enclose that subgraph; all, in fact, that is within that boundary will be considered the subgraph Z. The boundary may intersect LI's, but may not intersect cuts--the contents of the boundary is then a wf beta graph itself. Now, if a beta graph Y is exactly like a beta graph X except that while X contains somewhere a subgraph Z (that is, a boundary could be drawn somewhere in X, the entire contents of that boundary being Z), Y contains--at some point enclosed by at least all the cuts enclosing the original Z--another instance of the entire contents of the boundary of the subgraph Z--if all this is true, then $R_{itr}(Y, X)$ is true. Or,

4. X and Y are as described in clause 3 above and in addition, some point (or points) on an LI (or LI's) belonging to the second occurrence of Z in Y (that is, within the boundary of that occurrence of Z) but outside any cuts belonging to Z is connected (are connected) to a geodesic LI (LI's) running from the corresponding point (points) in the original instance of Z into the area in which the new instance of Z in Y has been

scribed. The connections are always to be made in that area, and the "geodesic LI" may have "branching points" along its length; that is, it may actually be composed of a number of LI's connecting at branching points.

$R_{bcl}(Y, X)$: Is true as provided in chapter i for the alpha system, and in addition, it is true if the two cuts of the biclosure present in Y but not in X intersect LI's in such a manner that the intersected segments of LI's pass from entirely outside the outer to entirely inside the inner cut--no branching, loops, dead ends, or any other kind of graph is permitted <u>in the annular space between the two cuts</u> of the biclosure. The only signs that may be there are the LI's enroute from outside to inside the biclosure.

$R_{ers}(Y, X)$: Is true iff $R_{ins}(S(X), S(Y))$ is true.

$R_{dit}(Y, X)$: Is true iff $R_{itr}(X, Y)$ is true.

$R_{nbc}(Y, X)$: Is true iff $R_{bcl}(X, Y)$ is true.

Thus do we state, for the purposes of this chapter, the beta versions of the rules of transformation, which in the Introduction were numbered 0.07 to 0,12. As is evident, these rules are not easy to express in English (or French, German, or Sanskrit, for that matter); but notwithstanding the complexity of their statement, they are intuitively not at all difficult to grasp. At this point it might be a good idea to review the examples of the applications of these rules which are presented in the Introduction.

We may now move to the statement of a metatheorem which parallels *1.01 in the alpha chapter:

*2.01 Beta is a Logic.

PROOF: Immediate from our definition of "logic" and the above statements regarding the axiom and rules of inference of beta. ∎

Alonzo Church[1] makes use of the notion of the "associated form of the propositional calculus"--abbreviated "afp"--in order to show that a first-order predicate calculus which he presents is consistent. We here introduce an analogous notion for the beta system; we shall speak of "the associated form of the alpha system" or "afa" of a given beta graph. It is clear that the spots of beta may be correlated one-one with the members of the set M_a (aside from b), the minimal graphs of alpha. Perform such a correlation, and for any beta graph X, replace each spot of X with its associated minimal graph of alpha; in the process, erase the <u>entire</u> LI network of X. The result of applying these instructions to a beta graph X is an alpha graph, and this graph will be called the afa of X.

*2.02 If X is a theorem of beta, then the afa of X is a theorem of alpha.

PROOF: The introduction of any spot with its connected LI's--the initial introduction--into a beta graph is exactly like the initial introduction of an alpha minimal graph in the proof of

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[1]Alonzo Church, <u>Introduction to Mathematical Logic</u> (Princeton: Princeton, 1956), I, pp. 180-81.

an alpha graph in that that introduction requires an application of R_{ins}. Furthermore, all the beta rules of inference as they apply to transformations involving spots and cuts are directly parallel to the alpha rules of transformation, prescinding from the effect these beta rules may have upon any LI's present. If, then, there is a proof for the beta graph X, then there is a proof in alpha directly paralleling the proof of X; where the proof of X performs some transformation involving a spot, the proof in alpha performs an analogous transformation involving an alpha minimal graph associated with the spot in question. The proof in alpha has no steps paralleling those in beta which affect only LI's, but this makes no difference, as these steps do not affect the location, etc., of the spots in X. But if such a proof exists in alpha, it is, by the definition of afa, a proof of the afa of X. Thus if X is a beta theorem, the afa of X is an alpha theorem. ▮

*2.03 Beta is consistent.

PROOF: It is clear that not every beta graph has an afa which is a theorem of alpha--an arbitrary spot with its LI's at its hooks, the whole complex standing alone on SA is an example of such a graph. But by *2.02 through modus tollens, then, there are beta graphs which are not theorems of beta, and beta is consistent. ▮

2.32 The Logic Beta in the System F_r

In section 2.22 we defined a function f' which maps the set of beta graphs into the set of cwffs of F_r. Our project now is to show that whenever X is a theorem of beta, f'(X) is a theorem of F_r. We state first the following lemma:

LEMMA 2.04 Where X is the axiom of beta, f'(X) is a theorem of F_r.

PROOF: The axiom of beta is a simple double-dead-end LI standing alone on SA. Then, by the definition of f', f'(X) is the cwff $\ulcorner(\exists a)(a = a.a = a)\urcorner$. This is trivially a theorem of the calculus F_r. ∎

This lemma shows that the axiom of beta is correlated by f' to a theorem of F_r. We now move to lemmas concerning the beta rules of inference; so far as the "alpha clauses" of these rules are concerned, we know by our work in chapter i that they "hold" in F_r, since F_r is, essentially, "based on" the CPC P_r.

LEMMA 2.05 Whenever, for beta graphs X and Y, $R_{ins}(Y, X)$ is true, then if $\vdash_{\bar{R}}$ f'(X), then $\vdash_{\bar{R}}$ f'(Y).

LEMMA 2.06 Whenever, for beta graphs X and Y, $R_{ers}(Y, X)$ is true, then if $\vdash_{\bar{R}}$ f'(X), then $\vdash_{\bar{R}}$ f'(Y).

PROOF: (As we will recall, "$\vdash_{\bar{R}}$ A" is to be read, "The closure of A is a theorem of F_r.") Based on our work in chapter i (lemmas 1.03 and 1.04) we may take it as fairly evident that the following hold:

If $\vdash_{\bar{R}} \ulcorner P^A(B)\urcorner$ and $\vdash_{\bar{R}} \ulcorner D \supset B\urcorner$, then $\vdash_{\bar{R}} \ulcorner P^A(D)\urcorner$, and (1)

If $\vdash_{\bar{R}} \ulcorner P^C(B)\urcorner$ and $\vdash_{\bar{R}} \ulcorner B \supset D\urcorner$, then $\vdash_{\bar{R}} \ulcorner P^C(D)\urcorner$. (2)

Now let us take as hypotheses:

For beta graphs X and Y, $R_{ins}(Y, X)$ is true (3)

$\vdash_{\bar{R}}$ f'(X) (4)

Now let us define as "Pren(A)" some specific prenex-normal form of the cwff A; we then have:

(4), Df. Pren(A) $\vdash_R$ Pren(f'(X)) (5)

By (3) and the definition of R_{ins}, there are, in an oddly enclosed area of the graph X two "loose ends" of LI's, which loose ends are connected in the graph Y. Hence, by the definition of the function f', Pren(f'(X)) may be represented by the schema $\ulcorner P^A(a = a.b = b)\urcorner$, where $\ulcorner a = a\urcorner$ and $\ulcorner b = b\urcorner$ are associated respectively through f' with the two loose ends in question. It should be clear that we have replaced f'(X) by one of its prenex normal forms to get any quantifiers which may have been introduced by f' out of the way. But now, since $\ulcorner P^A(a = a.b = b)\urcorner$ is the same formula as Pren(f'(X)), we have by (5) above:

$$\vdash_R \ulcorner P^A(a = a.b = b)\urcorner \quad (6).$$

But by the laws of F_r, it is trivially true that:

$$\vdash_R \ulcorner a = b \supset .a = a.b = b\urcorner \quad (7).$$

Then, by (1), (6), (7):

$$\vdash_R \ulcorner P^A(a = b)\urcorner \quad (8).$$

By Df. R_{ins}, the graph Y differs from the graph only in having the two "loose ends" in question connected; the "individuals" represented by these two "loose ends" in X are then identified in Y. Step (8) shows that the same identification may be accomplished in the system F_r. There should be no trouble in seeing, given the definition of f' and R_{ins} and the substitutivity of identity, that $\ulcorner P^A(a = b)\urcorner$ is a prenex normal form such that:

$$\vdash_R \ulcorner P^A(a = b) \equiv f'(Y)\urcorner \quad (9)$$

Then, by (8), (9) $\vdash_R$ f'(Y) (10)

This proves the first of the two lemmas involved. The relation of the second of these lemmas to the first is fairly clear, and once the first has been proven, the proof of the second should offer no difficulty. ∎

We now turn to two more lemmas:

LEMMA 2.07 Whenever, for beta graphs X and Y, $R_{itr}(Y, X)$ is true, then if $\vdash_{\bar{R}} f'(X)$, then $\vdash_{\bar{R}} f'(Y)$.

LEMMA 2.08 Whenever, for beta graphs X and Y, $R_{dit}(Y, X)$ is true, then if $\vdash_{\bar{R}} f'(X)$, then $\vdash_{\bar{R}} f'(Y)$.

PROOF: Recall that for R_{itr} (and R_{dit} is just the converse of R_{itr}) there are four basic cases to be considered:

1. When X is transformed into Y by the extension of a branch from a LI.
2. When X is transformed into Y by the extension inwards of the loose end of a LI.
3. When X is transformed into Y by the iteration of a subgraph Z (defined by the notion of the "boundary" of a subgraph) in the same area or inwards.
4. When X is transformed into Y by an iteration as in 3, with the maintenance of certain LI connections.

(For details, see the original statements of the definitions of R_{itr} and R_{dit} earlier in this chapter.)

For all of these cases, the following hypotheses will hold:

hyp. for beta graphs X and Y, $R_{itr}(Y, X)$ is true (1)

hyp. $\vdash_{\bar{R}} f'(X)$ (2).

Now let us consider the first case. Here Y differs from X only in having, at some point, a branch (with loose end) extending from an LI where X has no such branch. Let a be the variable in f'(X) associated with the LI involved (that from which the branching is to be extended). The cwff f'(X) will then contain at some point a subformula, a function of a containing all occurrences in f'(X) of a, Aa . The cwff f'(Y) contains a similar subformula in conjunction with a formula which by f' represents a branching with a loose end. By f', the conjunction occurring in f'(Y) in place of Aa will be of general form

$$\ulcorner(\exists b)(\exists c)(\exists w)(a = w.b = w.c = w.c = c.Ab)\urcorner \qquad (3),$$

where the three formulas involving 'w' represent the branching itself, and $\ulcorner c = c\urcorner$ represents the new "loose end" itself. But clearly, in F_r,

$$\vdash_R \ulcorner Aa \equiv (\exists b)(\exists c)(\exists w)(a = w.b = w.c = w.c = c.Ab)\urcorner \qquad (4).$$

Then, since f'(Y) differs from f'(X) only in having the equivalent of (3) where f'(X) has the simple Aa , by the substitutivity of the biconditional, (Z), and (4) we have

$$\vdash_R f'(Y), \text{ for case 1} \qquad (5).$$

We now consider the second case. Here Y differs from X only in having the loose end of a LI "extended geodesically" inwards from the original position it occupied in X. This means, by Df. f', that somewhere in f'(X) there will be a subformula $\ulcorner a = a\urcorner$ corresponding to the loose end in X and the scope of and bound by a quantifier $\ulcorner(\exists a)\urcorner$, and that in f'(Y) that formula $\ulcorner a = a\urcorner$ will not be where it was in f'(X), but will be elsewhere in the whole formula, though still in the scope of and bound by that same quantifier (we know from Df. f' that $\ulcorner a = a\urcorner$ in f'(Y) must be within the scope of at least all the negation signs within whose scope a = a is in f'(X))--this all is guaranteed by the fact that the LI in question is to be extended <u>geodesically inwards</u>. In all other respects f'(X) and f'(Y) are the same. But we have as an axiom of F_r that

$$\vdash_R \ulcorner a = a \urcorner \qquad (6).$$

It should be quite clear from the nature of F_r and (6), then, that whenever f'(X) and f'(Y) differ only as we have said they do, then

$$\vdash_R \ulcorner f'(X) \equiv f'(Y) \urcorner \qquad (7).$$

(2), (7) $\vdash_R$ f'(Y), for case 2 (8)

We turn now to the other cases of the rule R_{itr}. Although the statement of these cases looks very complicated, what they do in terms of the function f' is really quite simple. We may recall the function of the rule R_{itr} in the alpha system. This rule permits us to iterate a subgraph within the same area or "inwards." For the purposes of the beta version of this rule, we have defined subgraph through the notion of "boundary" in a graph. Let us ask, first of all, what we do when we iterate the contents of a "boundary" within a beta graph according to clause 4 of the definition of R_{itr}, supposing that each distinct LI outside all cuts in the iterated subgraph is connected by geodesic LI to the corresponding LI in the original instance of that subgraph. If there are no such LI's outside all cuts in this subgraph, then the iteration is simply an ordinary alpha iteration, justified by lemma 1.07. If there are such LI's, and all the connections are made as stated, the situation is not really too different. What such connections in Y assert is the identity of certain individual variables in f'(Y). Where Z is the iterated graph, and $a_1, \ldots, a_n$ are the n variables in f'(Z) which are associated with the n LI's connecting the original and the new instances of Z, we are assured by the requirement of R_{itr} (that such connected points be outside all cuts in Z) and by the definition of f' that all occurrences of the $a_1, \ldots, a_n$ in f'(Y) will be in the scope of the appropriate quantifiers, that no quantifications will be changed by the iteration. Indeed, the adjustment of the scope of the quantifiers as they occur in f'(X) to that as they occur in the formula with iteration, f'(Y), will require, at most, the movement to the right of a right parenthesis, the scope of each such quantifier in f'(Y) being included within the scopes of exactly the same negation signs in f'(Y) as was the scope of the correlate quantifier in f'(X). Other than the relatively minor adjustments in scopes of quantifiers in f'(Y), the justification of the transformation from f'(X) to f'(Y) by R_{itr} in this case is basically the same as the justification of the transformation from f(X) to f(Y) in lemma 1.07 of chapter i. Thus we see that there is no trouble justifying the iteration of the "contents of a boundary" in a graph X when each LI within that boundary and outside all cuts in that boundary is connected by geodesic LI to its correlate LI in the new instance of the "contents of that boundary."

Now note that any "significant" iteration is made into an evenly enclosed area (for <u>any</u> graph may be inserted in an oddly enclosed area by R_{ins}). Suppose that such an iteration has been made, with all LI's outside all cuts in Z, the iterated graph, being connected by geodesic LI to the correlate LI's in the original instance of Z. Since the iteration has been made into an evenly enclosed area, any or all of these geodesic LI's may be broken by R_{ers} (which has already been shown to have an analogous derived rule of inference in F_r) and withdrawn by clause 2 of R_{dit} (which also may be considered to have a correlate derived rule of inference in F_r, by our previous work in this proof). If <u>all</u> such LI's are broken and withdrawn, the result is the same as that allowed by clause 3 of R_{itr}. From the informal argument we have offered, then, I think that it is fairly easy to accept that when, for beta graphs X and Y, $R_{itr}(Y, X)$ is true in the cases 3 and 4 mentioned at the start of this proof, and at the same time $\vdash_{\bar{R}} f'(X)$, then also $\vdash_{\bar{R}} f'(Y)$.

We have dealt above primarily with R_{itr}. It is safe to say, I believe, that no elaborate arguments need be offered in support of lemma 2.08, which asserts the existence in F_r of a derived rule of inference analogous to R_{dit}. R_{dit} is the exact converse of R_{itr}, and it is clear that lemmas 2.07 and 2.08 stand or fall together. We may then say that, in effect, we have shown that these lemmas both hold for R_{itr} and R_{dit} as previously defined. ▮

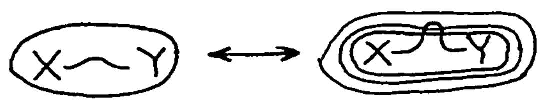
LEMMA 2.09 Whenever, for beta graphs X and Y, $R_{bcl}(Y, X)$ is true, then if $\vdash_{\bar{R}} f'(X)$, then $\vdash_{\bar{R}} f'(Y)$.

LEMMA 2.10 Whenever, for beta graphs X and Y, $R_{nbc}(Y, X)$ is true, then if $\vdash_{\bar{R}} f'(X)$, then $\vdash_{\bar{R}} f'(Y)$.

PROOF: We may say that there are two basic cases to deal with in this proof. The first of these is when the two cuts in the bi-closure which stands in Y but not in X intersect each LI involved as that LI stands in X no more than once (once, that is, for each of the cuts). In this case, these lemmas follow immediately from substitutivity of the biconditional and the theorem-schema

$$\vdash_{R} \ulcorner A \equiv --A \urcorner .$$

The other of these basic cases occurs in transformations of the following kind:



That is, it occurs when loops are introduced into the graph (or removed from it) by applications of R_{bcl} (or R_{nbc}). We shall argue very simply here. Note that the translation through f' of the graph to the left in the above diagram would be of the general form

$$\ulcorner -(\exists a)(Aa.Ba) \urcorner ,$$

and that of the graph to the right would be

$$\ulcorner -(\exists a)(\exists b)(a = b.--(Aa.Bb)) \urcorner .$$

Quite trivially in F_r we have

$$\vdash_{R} \ulcorner -(\exists a)(Aa.Ba) \equiv -(\exists a)(\exists b)(a = b.--(Aa.Bb) \urcorner .$$

This with the substitutivity of the biconditional shows that lemmas 2.09 and 2.10 hold in the above shown special case. There would be no trouble in extending the argument to more complex cases. ■

We may move immediately to the metatheorem to which these lemmas have been leading:

*2.11 If a graph X is a theorem of beta, then $\vdash_{R}$ f'(X).

PROOF: Immediate from the preceding lemmas. By 2.04 we know that there is a proof in F_r for f' of the axiom of beta, and by 2.05-2.10 we know that when there is a proof in beta leading from a theorem X of beta to a theorem Y, and f'(X) is a theorem of F_r, then there is a proof in F_r for f'(Y). Thus the metatheorem holds. ∎

Thus the function f' relates beta and F_r in one of the ways that f relates alpha and P_r. As we suggested earlier in this chapter, the proof of the converse of *2.11--that is, the completion of the proof that f' is a full-fledged mapping of the beta theorems into the F_r theorems--must await further developments in this chapter.

2.33 The System F_w in the Logic Beta

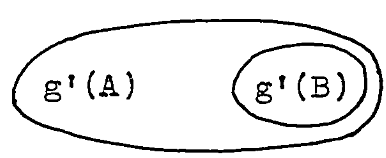
The conclusions of the preceding section are not startling. F_r is, after all, a powerful system--the complete first-order calculus with identity. What would be surprising, however, is to find that the set of theorems of such a calculus maps one-one into the set of beta theorems. It would be surprising from a formal point of view, because the notation of beta is so different from that of ordinary first-order calculi and because beta has no explicit quantifier, as well as because the beta-characteristic clauses of the rules of transformation seem so directly analogous to the purely alpha parts of the rules.

It would also be surprising historically, because although Peirce may be credited with being one of the midwives attending the birth of the modern quantifier, he seems--in his "algebra of logic"--never to have developed or even to have tried to develop a complete quantificational logic. If beta turns out to be, in effect, a complete first-order calculus with identity, it must be rated as a significant and remarkable accomplishment of a remarkable man, C. S. Peirce.

Again we shall enter into a series of lemmas leading up to the conclusions of this section. First of all, a lemma concerning the rule of inference in F_W.

LEMMA 2.12 Whenever, for F_W cwffs A and B, g'(A) and g'($A \supset B$) are both beta theorems, then so too is g'(B).

PROOF: Exactly as for the proof of the "derived rule of detachment" in alpha, lemma 1.15. This is quite clearly a "derived version" in beta of Quine's *104. The rule *104 reads, "If A and $\ulcorner A \supset B \urcorner$ are theorems, then B is a theorem." In the system of <u>Mathematical Logic</u>--and so in F_W--all theorems are cwffs; therefore, the statement of *104 guarantees that both A and B be cwffs, and so have, in effect, no variables in common. The graph g'($A \supset B$) will then be of form:



There will be no LI's connecting g'(A) and g'(B); thus as we mentioned, the proof follows exactly as in lemma 1.15. For beta, we may call this derived rule "R_{104}." ∎

This last lemma shows that Quine's "axiom of quantification" *104 "holds" in the beta system. We now move to a consideration of the other "axioms of quantification." First of all, we shall make some remarks concerning the set of "axioms" admitted by Quine's *100.

Quine's *100 (taken now for our system F_W) reads, "If A is tautologus, $\vdash_W$ A." It admits as axioms, then, the closures of all tautologous wffs of the system. Now, it will be recalled[1] that given the axiom schema

$$\ulcorner C\,\delta 1 C\,\delta 0\,\delta p \urcorner \quad (1),$$

where '0' and '1' are constant false and true propositions respectively and 'δ' is as we explained in chapter i, and given the rules of detachment and substitution for variables, we have a complete basis for the CPC; we have a system within which may be proven any CPC theorem, any member of the set of tautologous wffs of the system.

Note now the following schema:

$$\ulcorner \delta 1 \supset .\, \delta 0 \supset (a_n) \ldots (a_2)(a_1)\,\delta A \urcorner,\ n \geqq 0 \quad (2).$$

The schema of (2) is to be considered closed, to contain no free variables. The use of 'δ' is as it was in (1); where A is a wff of F_W, $\ulcorner \delta A \urcorner$ is a wff of F_W. In (2), A is any wff of F_W (subject to a restriction which will follow), '0' is the constant false proposition, and '1' is the definitional abbreviation of '$0 \supset 0$'. We specify that where $a_1, a_2, \ldots, a_m$ are the m distinct variables which are free in A, in the schema of (2), then each of these m variables is free at every occurrence in $\ulcorner \delta A \urcorner$. This precludes the "capturing" of any of A's free

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[1] Cf. Prior, p. 366.

variables by quantifiers which may be "lurking" in the context δ.

It should be evident that (2), when things are as we have stated them to be, is a theorem schema of F_W. But more interesting is the set of theorems which may be derived from (2) through the rule of inference *104. A little thought (in the light of the properties of the schema of (1)) will reveal that the set of consequences of (2) by *104 includes <u>the entire set of cwffs of F_W which are either themselves tautologous (this is when, in (2), n = 0) or are formed by prefixing a string of universal quantifiers to a "most general" tautologous wff of F_W</u>. A "most general" tautology is one which contains no unnecessary identifications of free variables. Thus, while '$Ax \supset By. \supset : By \supset Cz. \supset .Ax \supset Cz$' and '$Ax \supset Bx. \supset : Bx \supset Cx. \supset .Ax \supset Cx$' are both tautologies, and both belong to the same tautologous schema, the latter contains "unnecessary identifications" of free variables; the former, containing no such unnecessary identifications, is a "most general" tautology.

If a system, then, contains all the consequences of (2) by *104 and contains as well a theorem schema of the general form:

$$\ulcorner (a_n) \ldots (a_j) \ldots (a_1) \ldots (a_1)A \supset$$
$$(a_n) \ldots (a_i) \ldots (a_1)A' \urcorner \qquad (3),$$

where A' is like A except for containing free occurrences of a_i wherever A contains free occurrences of a_j, that system will contain as theorems <u>all</u> the cwffs which are either themselves tautologous or are formed by prefixing a string of universal quantifiers to any tautologous wff. For (3), which is clearly a theorem schema of F_W, permits us to move from the formulas involving "most general" tautologies to formulas involving less general tautologies by identifying variables free in the tautologies in question.

Note that by Quine's definition[1] the closure, properly so called, of a wff containing free variables is a particular one of the cwffs which may be formed by prefixing a string of universal quantifiers to the wff in question--specifically, the closure is that cwff which has the quantifiers in its prefix arranged alphabetically, from right to left, none of the quantifiers in question being vacuous. The set of closures of tautologous wffs, then, is a subset of the set of theorems provable in the system containing (2), (3), and *104.[2] We have already shown that *104 has an analog, R_{104}, in beta. If then we can show that there are theorem schemata in beta which correspond by g' to (2) and (3), we will have shown that the following holds:

LEMMA 2.13 When A is the closure of a tautologous wff of F_W, g'(A) is a theorem of beta.

PROOF: We shall first attack the schema (3) above. We shall prove it only for its simplest case, that is:

$$\vdash_W \ulcorner (b)(a)A \supset (a)A' \urcorner \qquad (4),$$

where A' is like A except for containing free occurrences of a

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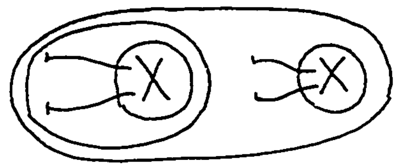
[1]Quine, p. 79.

[2]Note that since vacuous quantifiers are among those allowed to be introduced in the schema (2), Quine's *102 would also be easily provable in this system. We shall not, however, make use of this in the present development.

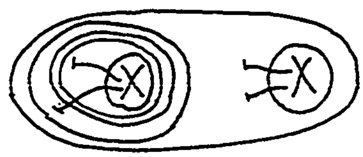
wherever A contains free occurrences of b. The extension to cases involving more quantifiers and free variables will, I think, be obvious.

By the alpha rules alone, we have:



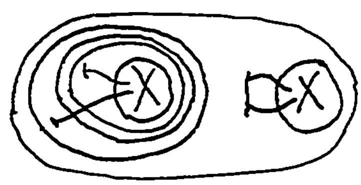
(5)

Applying R_{bcl}, beta biclosure, we get:



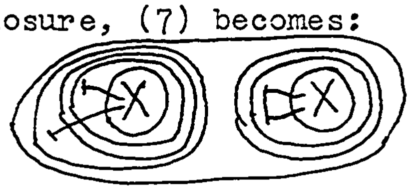
(6)

Applying R_{ins}, that is, "joining in odd":



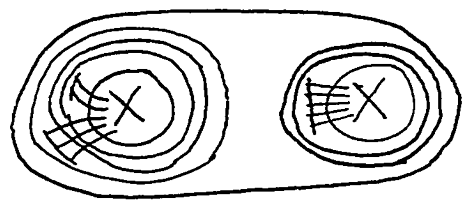
(7)

With a simple biclosure, (7) becomes:



(8)

By the function g', (8) may be considered a "graph schema" representing any formula admitted as a theorem of F_w by (4) above; if, then, B is such a theorem of F_w, then g'(B) is a theorem of beta. Note that, strictly speaking, if (8) is g' of (4) then A in (4) would contain only one free occurrence of a and one of b, and A' would contain two free occurrences of a. But let us consider that each of the two LI's pictured "running into" X in (8) may represent an indefinite finite number of LI's; (8) may then be considered to take care of all cases of (4), that is, to account for the cases when A contains occurrences of a and b in greater numbers. We will offer one example; suppose A to contain two occurrences of a and three of b. The schema (8) written out fully would then be

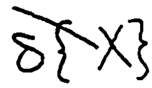


The proof of this last graph is almost exactly the same as that of (8) as explicitly shown. It may be said without hesitation that (8) represents g' of (4), then, regardless of the numbers of free occurrences of a and b in A .

As we mentioned before, it should be quite clear, now that we have (8), how one would go about proving g'(B) as a theorem of beta, where B is an F_W theorem which is a member of the more general schema (3). This would simply be a question of inserting biclosures at the proper times and places and of making the proper LI connection as was made at (7) in the above proof. We shall then state that whenever B is an F_W theorem of the general form of (3), g'(B) is a theorem of beta.

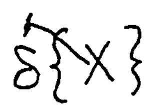
Now we turn our attention to the schema of (2), which is really the key schema of this lemma. In the series of graphs that follow, we shall execute the proof only for the case in which n = 1. Extension to cases involving more quantifiers and free variables will again, I think, be obvious. And the case where n = 0 is a matter entirely of proof by the alpha rules, for which see lemma 1.20.

The sign '$\delta\{\ \}$' will again be used, and in much the same manner as it was in chapter i. For beta purposes it will have a slight extension of usage. Used thus

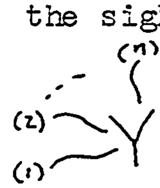


the sign will represent a graph containing occurrences of the subgraph X, with a <u>separate</u> geodesic LI running from entirely outside the graph

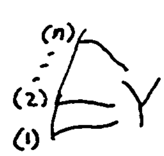
to each of the relevant hooks in the subgraph X. If there are n such hooks in the whole graph thus represented, then there will be n separate geodesic LI's represented by the LI shown. If the sign is scribed in this manner:



then it will indicate that all these LI's are connected by a "branching line" entirely outside the graph so represented. In the first case, then, the sign will represent a graph of form



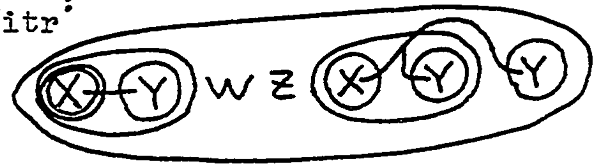
and in the second, one of form



By the alpha rules and beta R_{bcl}, then, we have

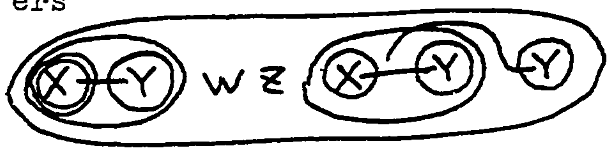
(9)

Applying beta R_{itr}:



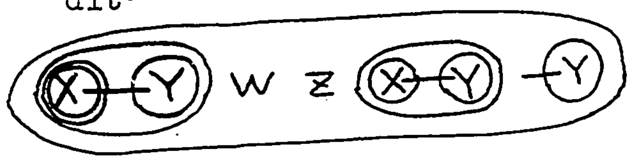
(10)

Applying beta R_{ers}, "breaking in even":



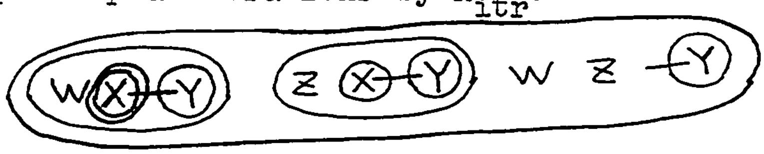
(11)

Applying beta R_{dit}, "withdrawal of LI":



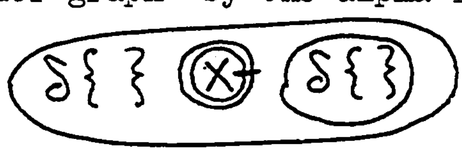
(12)

And simple alpha iterations by R_{itr}:



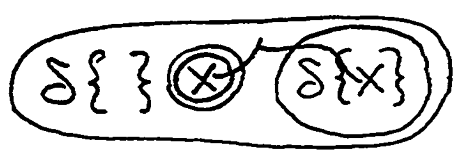
(13)

Now we prove another graph--by the alpha rules:



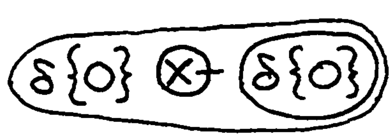
(14)

By beta R_{itr}, applied as often as necessary:



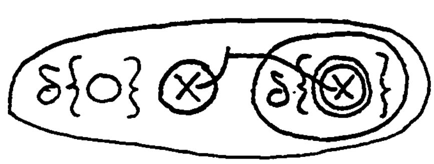
note also the application of R_{nbc}. (15)

Now another new graph--first, again, by the alpha rules:



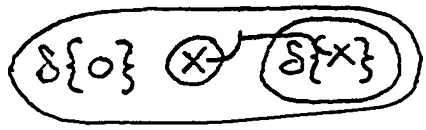
(16)

And by beta R_{itr} as at step (15):



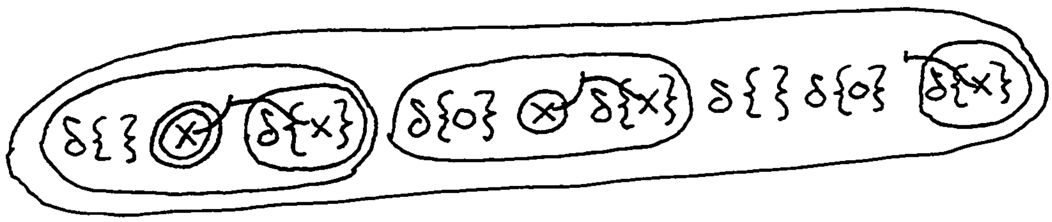
(17)

Now apply beta R_{nbc}, "negative biclosure," as often as needed:



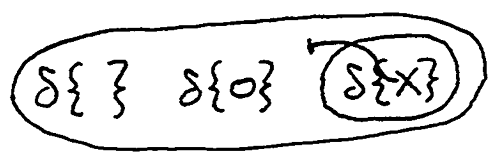
(18)

We may now write the following graph as a specific case of graph (13);



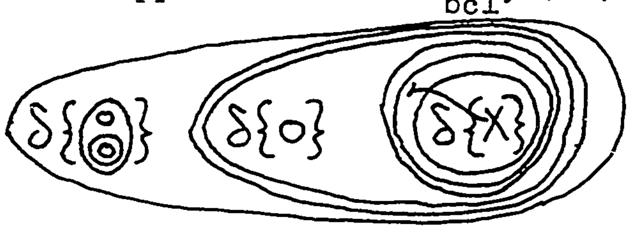
(19)

From the graph of (19) we may deiterate the graphs of (15) and (18) by R_{dit}:



(20)

But now with several applications of R_{bcl}, (20) becomes:



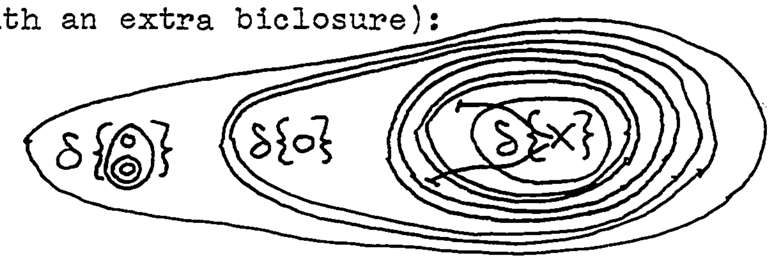
(21)

An examination of the definition of the function g' will show us that when B is a member of the schema

$$\ulcorner \delta 1 \supset . \delta 0 \supset (a) \delta A \urcorner \qquad (22),$$

where a is free in A (and B is therefore a theorem of F_r), the graph of (21) will do very nicely as g'(B). This means that when B is a member of the theorem schema (22), g'(B) is a theorem of beta. But (22) is the case of the more general schema (2) when n = 1.

As we mentioned earlier, the extension of the procedure to any finite n offers no problem at all. The entire above proof may be excecuted showing any number of "parallel" LI's explicitly attached to X instead of the one that we actually showed. If we had shown two such LI's, for example, step (21) would be (with an extra biclosure):



(23).

This, of course, would be g' of the schema

$$\ulcorner \delta 1 \supset . \delta 0 \supset (a_2)(a_1)\delta A \urcorner \qquad (24).$$

The cases we have been discussing have been cases involving non-vacuous quantifiers. To show that g' of the schema of (2) is a theorem schema of beta when some of the quantifiers shown are vacuous would be quite simple, but irrelevant to the purpose of the lemma we are proving. We have indicated sufficiently, I think, that when D is a member of the theorem schema (2) of F_w, and all the quantifiers shown are non-vacuous, then g'(D) is a theorem of beta. We had already shown that if B is a member of the theorem schema (3) of F_w, then g'(B) is a beta theorem. By lemma 2.12, we know that R_{104}, an analog in beta of the rule of detachment *104, exists as a derived rule of inference in beta. But by the remarks we made before we stated the present lemma, this means that if A is the closure of a tautologous wff of F_w, then g'(A) is a theorem of beta. And this is what we wanted to show. ∎

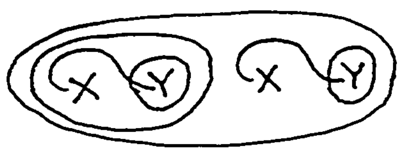
The last lemma shows that the "axioms of quantification" admitted to F_w by Quine's *100 map into the set of theorems of beta by the one-one function g'. The next lemma will show that the same is true of the axioms of quantification admitted by *101.

LEMMA 2.14 Whenever D is the closure of the F_W wff

$$\ulcorner (a)(A \supset B) \supset . (a)A \supset (a)B \urcorner,$$

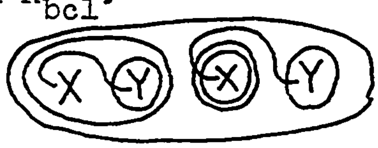
then g'(D) is a theorem of beta.

PROOF: We shall first prove this lemma for the case where D is precisely the formula $\ulcorner (a)(A \supset B) \supset . (a)A \supset (a)B \urcorner$. Then we shall show how it holds also in the cases where there are free variables in this latter formula. For the case when there are no variables free in the distribution formula: By the purely alpha clauses of the rules of transformation:



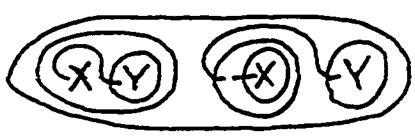
(1)

Applying beta R_{bcl}:



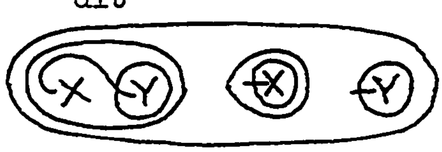
(2)

Applying beta R_{ers}, "erasure in even":



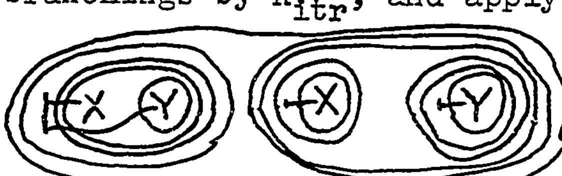
(3)

Applying beta R_{dit}, "withdrawal of LI":



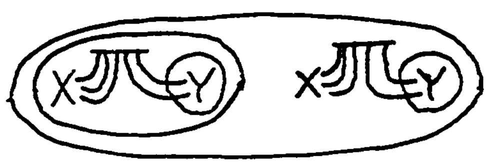
(4)

Extending branchings by R_{itr}, and applying R_{bcl} several times:



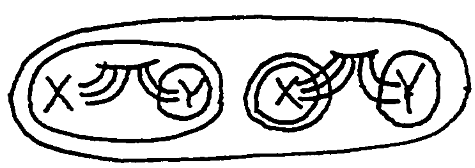
(5)

The graph at (5) is g'(D) when D is the "distribution of universal quantifier over implication" formula, with no free variables. Actually, however, (5) as it stands is, strictly speaking, g'(D) only for the case in which the subformulas A and B of D contain only one occurrence each of the distributed variable a. It would be well for us to indicate that g'(D) will be a theorem of beta regardless of the numbers of occurrences of the distributed variable in A and B. We shall indicate how the proof may be so extended by assuming that A contains, say, three occurrences of the distributed variable, and B two. Then in place of (1) above, we shall scribe, by the alpha rules alone:



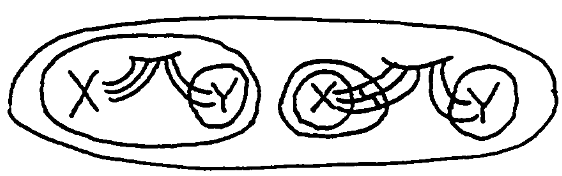
(6)

Applying beta R_{bcl} as before:



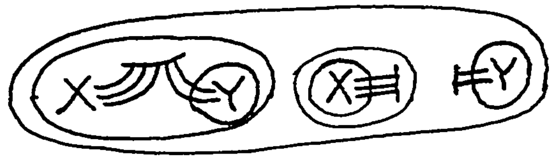
(7)

Now--the characteristic step for these situations--by beta R_{itr}:



(8)

Note what we have done in step (8). We have iterated segments of LI into the annular space between the two cuts of the biclosure by clause 4 of beta R_{itr}. We may now easily by beta R_{ers} and R_{dit} move to:



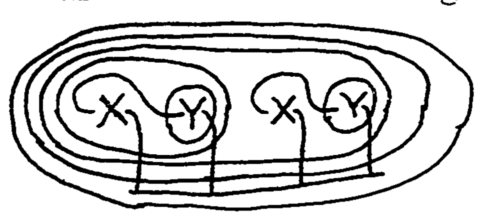
(9)

By applying R_{bcl} as in step (5) this last graph becomes g'(D), where D is as hypothesized, with three occurrences of the distributed variable in A and two in B. It should be evident that this procedure may be extended to cover any possible combination of numbers of occurrences of the distributed variable in A and B.

The above proofs were, as we said earlier, proofs for the case where the distribution formula contains no free variables, is its own closure. We shall offer one example of a case where the distribution formula contains a free variable; let D be the cwff

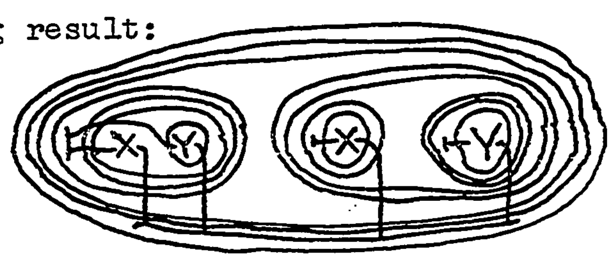
$$\ulcorner (b)((a)(A \supset B) \supset .(a)A \supset (a)B)\urcorner, \quad (10)$$

where b and only b is free in the distribution formula (with occurrences, say, in both A and B). The proof will be identical to the first we presented, except that we shall start out with a different first step. From our previous work (with lemma 2.13) it should be clear that the following is a beta theorem:



(11)

It should be clear that each step of the first proof may be carried out without difficulty, beginning with (10), to come to the following result:



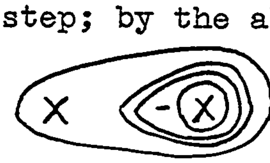
(12)

This last graph is g' of (10) above, assuming that there is one occurrence of a in each of A and B. For cases involving different numbers of occurrences of a in A and B, see steps (6) to (9) above and the remarks therewith. Note that the operations of our proof do not disturb the LI's representing the occurrences of b at all. It is clear that the procedure may be extended to any number of free variables in the distribution formula, and we may consider the lemma proven. ∎

We now move to a lemma which shows that the "axioms of quantification" admitted to F_W by *102 map into the set of beta theorems by g'.

LEMMA 2.15 Where D is the closure of the F_W wff $\ulcorner A \supset (a)A \urcorner$, and the variable a does not occur free in A, then g'(D) is a theorem of beta.

PROOF: We present the proof only for the case where there are no variables free in $\ulcorner A \supset (a)A \urcorner$; the extension to cases where there are variables free presents no problem and is quite similar to the analogous extension in the last lemma. Our explicit proof, then, is for the case where D <u>is</u> the formula $\ulcorner A \supset (a)A \urcorner$. The proof involves one step; by the alpha rules alone,

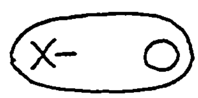


is a theorem of beta; this graph is g' of $\ulcorner A \supset (a)A \urcorner$ when a is not free in A, that is, when the quantifier $\ulcorner (a) \urcorner$ is vacuous. The double dead-end LI represents the vacuous quantifier according to g'. ∎

We now move to Quine's *103:

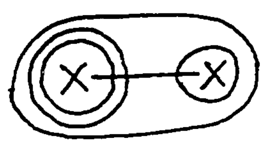
LEMMA 2.16 When D is the closure of the F_W wff ⌜(a)A ⊃ A'⌝, where A' differs from A only in having the (entirely new) variable b wherever A has a, then g'(D) is a theorem of beta.

PROOF: Once again, we shall show the proof only for the case where the only free variable in ⌜(a)A ⊃ A'⌝ is b, that is, where ⌜(a)A⌝ contains no variables free. The extension to cases where there are other free variables will be as in the preceding lemmas. By the alpha rules alone we have:



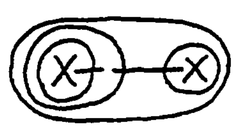
(1)

With beta R_{itr} and R_{bcl} we have:



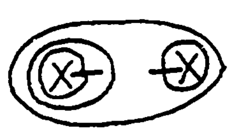
(2)

Applying beta R_{ers}, breaking in even:



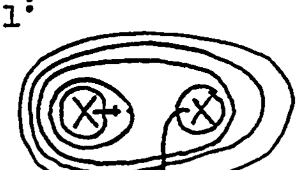
(3)

Then, by beta R_{dit}, withdrawal of LI:



(4)

And finally, R_{bcl}:



(5)

The graph of (5) is g'(D) where D is the cwff ⌜(b)((a)A ⊃ A')⌝.

The proof as sketched above is, of course, explicitly for the case when A contains only one occurrence of a free. The extension to cases where A contains more than one occurrence of a is similar to that of steps (6) to (9) of lemma 2.14. ■

We now have shown that all the axioms of quantification of F_W map into the set of beta theorems by the function g'. We shall now show that the same is true for the "axioms of identity" as well.

LEMMA 2.17 When B is one of the F_W cwffs

1. ⌜(a)(a = a)⌝, or the closure of
2. ⌜(b)(a)(a = b ⊃ .Aa ⊃ Ab)⌝,

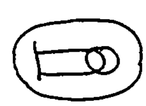
then g'(B) is a theorem of beta.

PROOF: The lemma will be proven by showing that the following proofs hold in beta:

By the alpha rules, we have:

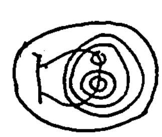
(1)

Then, by R_{itr}:



(2)

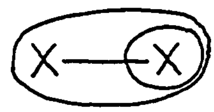
And applying R_{bcl} twice:



(3)

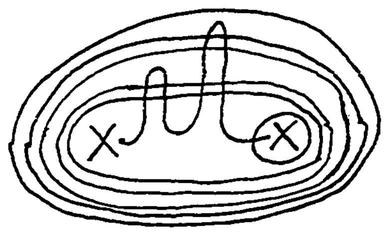
The graph of (3) is g'(B) when B is ⌜(a)(a = a)⌝.

We shall state the proof for the second schema of this lemma only for the case in which B itself is the formula ⌜(b)(a)(a = b ⊃ .Aa ⊃ Ab)⌝. The proofs for the cases in which A contains free variables other than a or b will be similar to those outlined in the previous lemmas. We have, then, by the alpha rules and R_{itr} of beta:



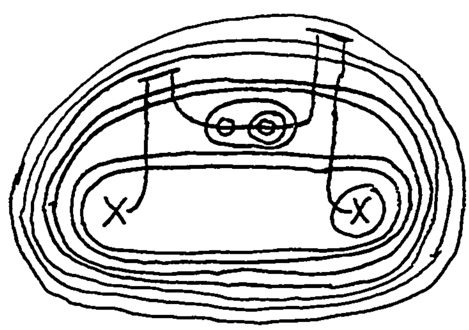
(4)

Applying R_{bcl} twice, we have:



(5)

Applying beta R_{itr} and, again, beta R_{bcl}:

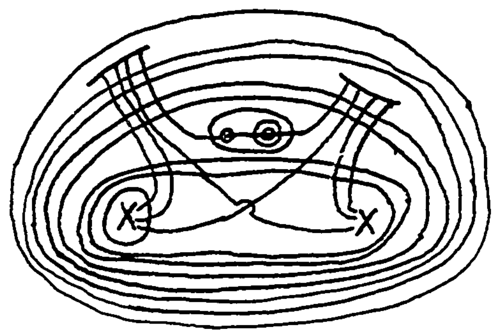


(6)

The graph of (6), strictly speaking, is g'(B) when B is the schema of substitutivity of identity with only one occurrence of a free in Aa and no variables free in B. We shall state simply that it is quite easy to extend the proof to any case of the substitutivity of identity. As an example, note a specific instance of this axiom schema:

$$\ulcorner(y)(x)(x = y \supset . A(xxy) \supset A(xyy))\urcorner.$$

The graph corresponding to this formula is



which is easily provable in beta. ∎

We have now established that when A is either one of the axioms of quantification or the axioms of identity of F_W, then g'(A) is a theorem of beta. But before we make use of the preceding lemmas in the proof of the central metatheorem of this section, let us examine briefly another issue which shall be of some importance in that proof.

Apart from the question of whether or not a given beta graph is a theorem of beta--which is the question of whether or not there is a proof in beta for that graph--we may raise the question of whether or not a given beta graph is valid. Although it is possible to speak of validity in a "purely syntactical" sense, it is undeniable that the notion--even that of syntactical validity--is intimately related to the interpretation of the system in question.

From what has gone before, we should have a good idea--even without the formal statement of semantical rules--of what the principal interpretation of beta will be. Beta graphs may be interpreted as statements, statements which are either true or false depending on "certain circumstances."

Speaking rather loosely, we might say that a given beta graph is valid iff it is true "regardless of what the circumstances are." It is incumbent upon us to be rather more specific, however, in setting down just what it means to say, "The beta graph X is valid."

The first step in this direction will be to provide--in definite terms--a "sound interpretation" of the logic beta. According to Church,

> we call an interpretation of a logistic system sound if, under it, all the axioms either denote truth or have always the value truth, and if further the same thing holds of the conclusion of any immediate inference if it holds of the premise.[1]

The system F_r which we set down earlier in this chapter is a complete first-order calculus with identity; there are, of course, sound interpretations of F_r. Might it not be possible to provide our interpretation of beta through F_r, and thus avoid the statement of very complex semantical rules for beta? Indeed it is possible, and the means to do so are already at hand:

> For a given sound interpretation of the system F_r, let the interpretation of the beta graph X be precisely the same as that of the F_r cwff f'(X).

Such an interpretation of beta is unquestionably sound, for our metatheorem *2.11 asserts that if a graph X is a theorem of beta, then $\vdash_R$ f'(X). This metatheorem guarantees that the conditions for soundness of an interpretation are met by our suggestion for an interpretation of the beta system, provided

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[1]Church, p. 55.

F_r has a sound interpretation, which it has.

Under this interpretation of beta, we may characterize validity in beta in this manner:

A beta graph X is valid iff the F_w cwff f'(X) is valid.

We may then say, in effect, that validity is "preserved" through the function f'^{-1}. In the light of this, we may state quickly the following lemma:

LEMMA 2.18 Every theorem of beta is valid.

PROOF: By our above remarks on validity in beta. ∎

Now we may move to the statement of the principal thesis of this section.

*2.19 The cwff A is a theorem of F_w iff g'(A) is a beta theorem.

PROOF: The "only if" part of this metatheorem follows immediately from lemma 2.12--which asserts the existence of detachment, or Quine's *104, as a derived rule in beta--lemmas 2.13, 2.14, 2.15, and 2.16--which assert that the axioms of quantification admitted to F_w by Quine's *100, *101, *102, and *103 respectively map into the set of beta theorems by g'--and lemma 2.17, which asserts that the axioms of identity of F_w map by g' into the theorem set of beta. All the above means that if there is a proof for a cwff A in F_w, then there is a proof for g'(A) in beta, and the "only if" part of the metatheorem holds.

Now, with the "only if" part holding, it is safe to say that we could, if we wished, assign a sound interpretation to F_w by means of beta, (through g') just as we assigned an interpretation to beta through F_r. We may say of the function g'^{-1}, just as we did of f'^{-1}, that "validity is preserved through it." If $g'(A)$ is valid, then, so too will be the cwff A. If the "if" part of this metatheorem does not hold, then there is some cwff A of F_W which is not a theorem of F_W while the graph $g'(A)$ is a beta theorem. But F_W is a complete system, which is to say that in it, every valid cwff is a theorem. So if A is not a theorem of F_W, then A is not valid. By modus tollens, then, $g'(A)$ must be invalid. But this is impossible, since our supposition was that $g'(A)$ is a beta theorem, and by lemma 2.18, every beta theorem is valid. Thus, if A is not a theorem of F_W, then $g'(A)$ cannot be a theorem of beta, and the metatheorem holds in both directions. ∎

2.34 The Completeness of Beta

As we stated in the rather trivial lemma 2.18, every beta theorem is valid. But more interesting than this is the question of whether or not every valid beta graph is a theorem; equivalently, this is the question of whether or not beta is complete. We shall now answer that question in the affirmative:

*2.20 Beta is complete.

PROOF: As we have remarked, beta is complete iff every valid beta graph is a theorem of beta; this we shall now prove. To avoid interrupting the general development with a long graphical proof, we shall, for the moment, assume the following, for which demonstrations shall subsequently be provided:

1. $\mathcal{P}_\beta$, the set of beta graphs which is the range of the function g', is recursively characterizable.
2. There is a (many-one) function k taking the set of beta

graphs as its domain and ρ_β as its range, and such that:

(a) X is a beta theorem iff k(X) is a beta theorem, and

(b) X is a valid graph iff k(X) is a valid graph.

Suppose that a beta graph X is valid (recall from our earlier remarks that a beta graph X is valid iff f'(X) is a valid cwff). Then, by 2-b above, k(X) is also valid. But k(X)ϵ ρ_β ; that is, there is a cwff A of F_w such that k(X) is the same graph as g'(A); g'(A), being precisely k(X), is valid. But then, as we remarked in the proof of *2.19, A itself must be valid, and so by the completeness of F_w, A must be a theorem of F_w. By *2.19, then, g'(A)--and so too k(X), which <u>is</u> g'(A)--must be a theorem of beta. But if k(X) is a theorem of beta, then--by 2-a above--so too must X be a theorem of beta. Therefore, if a beta graph X is valid, it must be a theorem of beta, and beta is complete. The completion of this proof awaits the demonstration of 1 and 2 above, which will follow an immediate corollary.∎

COROLLARY 2.21 If the cwff f'(X) is a theorem of F_r, then the graph X is a theorem of beta.

PROOF: Immediate: Every theorem of F_r is a valid cwff, so if f'(X) is a theorem, f'(X) is valid. But then the beta graph X is also valid, and by *2.20 is a beta theorem.∎

This last corollary is the long awaited converse of *2.11, and in conjunction with *2.11, it establishes that "A beta graph X is a theorem of beta iff f'(X) is a theorem of F_r."

We now move to assumptions 1 and 2 in the proof of *2.20. We shall first work with the first of these assumptions, that is, that the set $\mathcal{P}_\beta$ is recursively characterizable. We shall show that it is so characterizable by presenting a set of rules which characterizes it; these rules will be exactly parallel respectively to the rules characterizing the set of cwffs of the system F_w. The major problem here is that the cwffs of F_w are defined through the wffs in general, and there are wffs of F_w which are not closed; as we have indicated, however, there is no beta graph which is not a "closed formula."

We shall require a device in our characterization, then, to parallel the free variable of F_w. This device we shall call the "free LI terminal," or "flt."

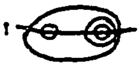
If a spot has attached to any of its hooks an LI which crosses no cuts and <u>comes to a dead end</u> without branchings or connections, that dead end will be called an flt. If either of the two LI ends <u>extending from</u> the graph '' comes to a dead end without cut-crossing, branching, or connections, it too will be called an flt. Only these termini will be called flts.

Note carefully that the flt is by no means to be considered to be related to the free variable of F_w by g'--it is merely an expedient we employ to help us define the set $\mathcal{P}_\beta$. We shall now define a set Q of beta graphs, of which $\mathcal{P}_\beta$ is a subset, just as the cwffs of F_w are a subset of the wffs. We shall in each case state a rule of formation of F_w, and with it, the analogous rule for Q.

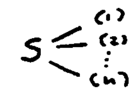
2.34i F_w rule: '0' (constant false proposition) is wf.

Q rule: The empty cut, ' ○ ', belongs to Q, and has <u>no</u> flts.

2.34ii F_w rule: Iab is wf where a and b are individual variables.

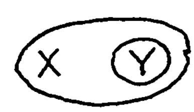
Q rule: The graph '' is a member of Q, and has 2 flts.

2.34iii F_w rule: Where A is a predicate symbol of degree n, and a_1, a_2, . . . , a_n are n individual variables, $Aa_1a_2 \ldots a_n$ is wf.

Q rule: Where s is an n-adic spot, the graph  is a member of Q, and has n flts. The n LI's attached to the n hooks of s are considered to terminate with no branchings or cut-crossings.

2.34iv F_w rule: Where A and B are both wf, C_{AB} is wf.

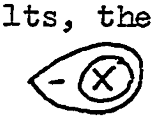
Q rule: Where X and Y are both members of Q, and X contains m flts and Y n flts, then the graph



is a member of Q and contains m + n flts.

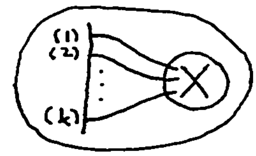
2.34v F_w rule: Where A is wf and a is an individual variable, $\ulcorner \prod aA \urcorner$ is wf.

Q rule: (1)--vacuous case--where X is a member of Q and has n flts, then the graph



is a member of Q and has n flts.

(2)--non-vacuous case--where X is a member of Q and has n flts, and Y is the graph



where both $k \leq n$ and $k \geq 1$ and each of the k LI's shown "branching off the main trunk" in the diagram is geodesic for all its length and connects to an appropriate one of the flts of X (that is, in a determined "proper order" from the first to the kth such LI) then Y is a member of Q and contains n - k flts.

2.34vi F_W rule: (Understanding the meaning of "free variable") Where A is wf and contains no free variables, A is a closed wff, or cwff.

Q rule: Where X is a member of Q, and X contains <u>no</u> flts, X is a member of $\mathcal{P}_\beta$.

This completes the definition of the set $\mathcal{P}_\beta$. The "proper ordering" referred to in 2.34v is meant to agree with the similar "ordering of connections" required in the definition of the function g' when that function correlates a non-vacuous quantifier with a "branching-complex" such as appears in the graph of 2.34v. In fact, if we look back at the definition of the function g', in Section 2.23, we shall easily be able to see that the characterization of $\mathcal{P}_\beta$ is also the characterization of the set of graphs which is the range of g'--the juxtaposition above of the rules of formation for the system F_W, whose cwffs constitute the domain of g', and those for the set Q will make this quite obvious. The set $\mathcal{P}_\beta$ above characterized is, then, the set of beta graphs constituting the range of the function g'.

The above recursive characterization takes care of assumption 1 in the proof of *2.20. We shall now move on to the second of these assumptions. This is that there is a function k with the set of beta graphs as its domain and the set $\mathcal{P}_\beta$ as its range such that:

(a) X is a beta theorem iff k(X) is a beta theorem, and

(b) X is a valid graph iff k(X) is a valid graph.

Because of the large variety of sign-complexes possible in beta, the demonstration that k exists will be largely a matter of busy-work, which we now enter into.

First of all, let us specify that if X is a member of $\mathcal{P}_\beta$, then k(X) = X. For graphs which are already members of $\mathcal{P}_\beta$, then, there is no problem. For them k is simply the identity function, and trivially possesses the required properties.

For graphs which are not members of the range of g', however, proving the existence of k is another problem. Our method will be to show that with applications of the beta rules of inference alone, any beta graph X as premise has as a consequence a member of $\mathcal{P}_\beta$, which we shall call k(X); and further, that k(X) as premise, by applications of the beta rules alone, has as a consequence the graph X. If this is the case, then obviously, X is a beta theorem iff k(X) is a beta theorem; this is one of the required properties of k.

We also wished k to be such that X is valid iff k(X) is valid. If the computation of k(X) from X consists solely in a deduction from X as premise by the beta rules of transformation, and if, with k(X) as premise X can be derived by the beta rules alone, then it is indeed the case that X is valid iff k(X) is valid. If R(Y, X) is one of the beta rules, then, as we have shown for each of these rules, if f'(X) is a theorem of F_r, then so too is f'(Y). Suppose X to be valid; then f'(X) must also be valid, by our definition of validity in beta.(since we have agreed that the interpretation of any beta graph is to be exactly the same as that of f' of that graph). By the completeness of F_r, then, f'(X) is a theorem of F_r. Then if for some Y, R(Y, X) is true, then also f'(Y) is a theorem of F_r. But then f'(Y) is trivially valid, and so too then is Y. We conclude then that no consequence of a valid beta graph by the beta rules of inference can be invalid, and so as we have suggested k should be, X is valid iff k(X) is valid.

We know, then, that if k(X) is computed by taking X as a premise and applying the beta rules of transformation, and if with k(X) as a premise X may be deduced by the beta rules alone, then both of the properties required of our function k in the proof of *2.20 are there, that is, X is a beta theorem iff k(X) is, and X is valid iff k(X) is.

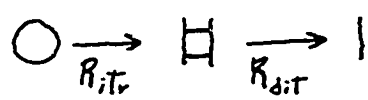
It is now a question of showing that any beta graph as premise may be transformed into a member of $\mathcal{P}_{\beta}$ by the beta rules alone, and that that member of $\mathcal{P}_{\beta}$ as premise may be transformed back into X by the beta rules of transformation alone.

We now take an arbitrary beta graph X which is not a member of $\mathcal{P}_{\beta}$. We may say this: if X is to be transformed into a member of $\mathcal{P}_{\beta}$, then there must be transformations available which will enable us, first, to get the LI network of X into a pattern characteristic of a member of $\mathcal{P}_{\beta}$, and secondly, to transform the cut-nest of X into a cut-nest characteristic of a member of $\mathcal{P}_{\beta}$.

We first approach the problem of the LI network. We shall do this in steps:

Step 0: If X is of form  , let it as it is; otherwise, transform it to  by R_{bcl}.

Step 1: The "cyclic graph" step. If there are any cyclic graphs in X, convert them to double dead end LI's by R_{itr} and R_{dit} thus:



Step 2: The "dead end" step. If a dead end occurs in a given area of the graph resulting from the previous steps, then for that area we have one of the fool-wing forms, where W is <u>the intire graph in that area apart from the dead end in question</u> (note that W could be as little as the "other end" of a double-dead-end LI, or even, in some cases, the null-graph), and--in (2) below--Z is the graph, whatever it might be, which is outside the area in question, and to which the dead end in question is connected.

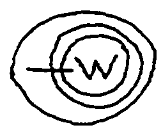
We thus have, for a given dead end in a given area, either

(1)

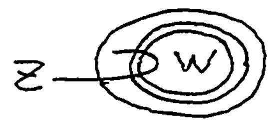
or (2)

These may be transformed by R_{bcl} respectively to

(1')



or (2')



Now, in some established order, carry out the applicable one of the transformations above for <u>each individual dead end</u> in the graph resulting from step 1.

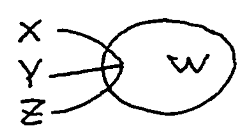
Step 3: The "branching" step. If a branching occurs in a given area of the graph resulting from the previous steps, then for that area we have one of the following forms, where W is as in step 2, and X,Y, and Z all have the function of Z in step 2.

(3)

or (4)

or (5)

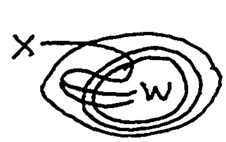
or (6)



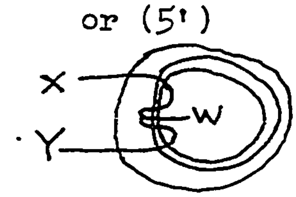
By R_{bcl}, these may be transformed respectively to

(3')

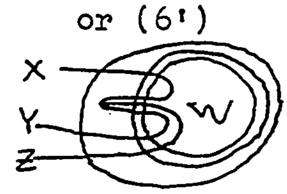
or (4')



or (5')



or (6')



Now, in some established order, carry out the applicable one of the above transformations for <u>every</u>

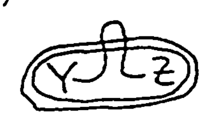
individual branching in the graph resulting from step 2.

Step 4: The "connecting LI" step. In the graph resulting from the previous steps, there may be places where two graphs in the same area, Y and Z, are connected by a LI which has its outermost portion in the same area in which Y and Z occur. This is case

(7)

By R_{bcl}, this may be transformed into

(7')



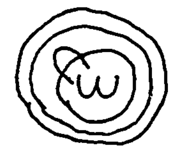
Now, in some established order, carry out the above transformation for every case in the graph resulting from the previous steps where two graphs are connected as in (7).

Step 5: The "in-pointing loop" step. In the graph resulting from the above steps, there may be cases where there are loops with "their ends pointed in"--if step 4 had to be applied, there will indeed be such loops. For a given area into which such a loop points, using W as before, we have

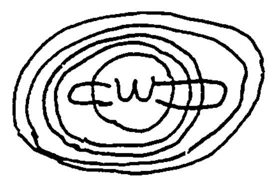
(8)

By R_{bcl}, this may be transformed to

(8')



For every such in-pointing loop in the graph resulting from step 4, in some established order, carry out the above transformation. Note that if at (8) there were two such loops, the final result of the transformation would look like

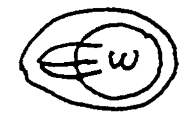


Step 6: The "quantifier-indicating" step. In the graph as so far transformed, dead-ends, "in-pointing" loops (those of step 5), and branchings <u>all occur</u>, respectively in subgraphs of the forms, now to be shown, where the dead end, loop, or branching is the only sign in the annular space between the two cuts shown with the possible exception of geodesic LI's which pass from completely outside the outer to completely inside the inner of the cuts. The operations to this point have been arranged, then, so that the dead end, in-pointing loop, or branching is the only graph standing in that annular space. Each dead end, in-loop, or branching of the graph, then, occurs, respectively in a subgraph of form

(9)--for dead end

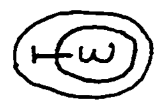
(10)--for in-pointing loop

(11)--for branching

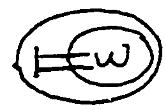


By R_{itr} these may be transformed into

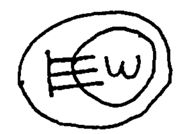
(9')



(10')



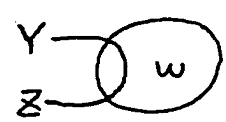
(11')



At this point a striking similarity to something we saw in rule 5 of the characterization of P_β shows up. Before we remark on that, however, we will take one further step.

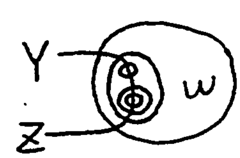
Step 7: The "out-pointing loop" step. All loops remaining in the graph as it now stands are "out-pointing"; where such a loop occurs in a given area, we have (using W,Y, and Z as before)

(12)

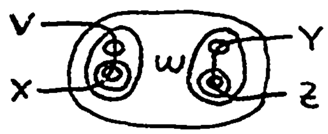


By R_{bcl} this is transformable to

(12')



Transform every loop remaining in the graph resulting from step 6 as above.



(12') when 2 loops stand in the same area as W

I contend that we have now transformed--using R_{bcl}, R_{itr}, and R_{dit} alone--the LI network of the original graph into an LI network characteristic of a member of $\mathcal{P}_\beta$. Note that all branchings and dead ends in the transformed graph occur in subgraphs of form (9'), (10'), or (11') above. This is precisely as is required for the branchings and dead ends of a member of $\mathcal{P}_\beta$ by rules 5 and 6 characterizing $\mathcal{P}_\beta$. Note too that <u>all</u> the <u>loops</u> in the graph we now have occur only in subgraphs '', by step 7. This also is as required by the characterization of $\mathcal{P}_\beta$. Any LI which crosses a cut--apart from those in the subgraphs ''--is <u>geodesic</u>, and moves inwards from one of the branching complexes '⊢', '⊨', or '⊫' as they occur in the forms (9'), (10'), and (11') to connect either to a hook of a spot or to one end of the graph ''. Thus I submit that we have transformed the LI network of the original graph into an LI network characteristic of a member of $\mathcal{P}_\beta$. What follows is easy. The steps required to make the graph as it now stands a member of $\mathcal{P}_\beta$ are almost the same as the steps by which any alpha graph may be transformed into a member of $\mathcal{P}_\alpha$. That is, by appropriate biclosures only, the graph may be transformed into the desired form. The problem in the general case, is that of getting the cut nest of the graph resulting from our seven steps into the form of a cut nest characteristic of a member of $\mathcal{P}_\beta$.

I shall go no further here than to offer the argument that just as any wff containing only the connectives 'K' and 'N' may be transformed into an equivalent wff containing only the connective 'C' and the constant false proposition '0' merely by application of the law 'EpNNp' through substitutivity of equivalence and the equivalences 'ENNKpNNqNCpNq' and then 'ENpCp0', so too may any graph resulting from the application of our seven steps be converted, by R_{bcl} alone, to a member of $\mathcal{P}_\beta$. Once the LI network is taken care of--which we did in the seven steps--the rest is easy.

Thus we will argue that the function k required for the proof of the completeness of beta exists, and has the properties required of it. We have shown explicitly that an arbitrary graph X may be converted by the beta rules R_{bcl}, R_{itr}, and R_{dit} alone into a member of $\mathcal{P}_\beta$ which we may call k(X). And since the above mentioned rules are all that are required to do the job, with K(X) as a premise, X may be derived, since the converse of each of the rules used is itself a rule of beta; the proof of X from k(X) is then just the "reverse" of that of k(X) from X. We have thus shown that with an arbitrary beta graph X as premise, k(X) such that k(X) is a member of $\mathcal{P}_\beta$ may be computed using the beta rules alone, and that if that k(X) is taken as a premise, X may be derived by the beta rules alone. Therefore, as required in the metatheorem on completeness, there is a k such that for any X, $k(X) \epsilon \mathcal{P}_\beta$, and both X is a theorem of beta iff k(X) is, and X is valid iff k(X) is.

2.35 The Intertranslatability of Beta and First Order Calculi with Identity

Recalling Martin Davis's definition of translatability, which we employed in chapter i, we may state the following metatheorem:

*2.22 Beta is translatable into any complete classical first-order calculus with identity, and any such calculus is translatable into beta.

PROOF: Immediate. the functions f' and g' are both one-one functions, as required in the definition of translatability, and we have proven that: X is a beta theorem iff f'(X) is a theorem of F_r, which means, by the definition, that beta is translatable into F_r. We have also proven that a cwff A is a theorem of F_w iff g'(A) is a theorem of beta, which means that F_w is translatable into beta. But F_w and F_r are translatable into each other, and into any full classical first-order calculus with identity, and any such calculus is translatable into either F_w or F_r. By the transitivity of translatability, then, the metatheorem holds. ▮

COROLLARY 2.23 The decision problem for beta is recursively unsolvable.

PROOF: By Martin Davis's theorem 8.1.3,[1] *2.22, and the recursive unsolvability of the decision problem for full first-order calculi. ▮

This chapter has been quite an effort--for reader as much as for author, I am sure--but I believe that it has been worth

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[1]Davis, p. 119.

it. It has established that the unlikely looking system beta is, in effect, a full first-order calculus with identity. And it has introduced us to and shown us the power of a novel concept in symbolic logic--that of implicit quantification as exemplified in the beta line of identity.

CHAPTER III

THE GAMMA SYSTEMS

In the Introduction, we showed that the "gamma part" of the Existential Graphs had a place near to the very heart of C. S. Peirce. Gamma as Peirce envisioned it was a system of many signs, of great complexity. Primarily, it was to be the system which, by using a book of sheets of assertion rather than just one, was to add an "extra dimension" to the logical analysis of reality; gamma was to help us extend this analysis to possible universes of discourse and to enable us to deal with problems beyond the scope of alpha and beta.

Hopes for the stereoscopic gamma went a-glimmering. Peirce was not able to pull the system together as he wanted to. But he did try to make a start; we restate here a passage we quoted in the Introduction:

> In endeavoring to begin the construction of the gamma part of the system of existential graphs, what I had to do was to select, from the enormous mass of ideas thus suggested, a small number convenient to work with. It did not seem to be convenient to use more than one actual sheet at one time; but it seemed that various different kinds of cuts would be wanted (4.514).

The "different cut" to which Peirce seems to have paid most attention is the "broken cut," which we will recall from the Introduction. This is, in fact, the only of the gamma cuts to which any real attention is paid in the writings of Peirce appearing in the Collected Papers. The bulk of this chapter will be devoted to a study of systems including the broken cut among their signs.

In 4.516, Peirce states two rules for the use of the broken cut; one of these rules is our 0.13 of the Introduction. The other rule has two clauses and is stated as our 0.14 and 0.15 in the Introduction. We may at this point repeat these rules as we stated them there:

0.13 In a broken cut already on SA any graph may be inserted.

0.14 A broken cut in an area enclosed by an odd number of cuts (which may be either alpha or broken cuts) may be transformed to an alpha cut (by "filling in" the breaks in it).

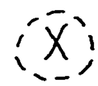
0.15 An alpha cut in an area enclosed by an even number of or by no cuts may be transformed to a broken cut (by erasing parts of it).

At this point we shall simply remark that these rules are open to amplification and interpretation, as we shall eventually see; and depending on how we amplify and interpret, they will yield us a variety of logical systems. In this they are like the rule of "strict implication introduction," for example, which is used by Anderson and Johnstone[1] to set up a system of natural deduction which is basically equivalent to the Lewis-modal S4. Although the rule as they present it enables us to derive the theorems of S4 within their system, there would be no trouble in so modifying the restrictions on the application of the rule as to enable us to derive the theorems of, say, the system S5.

---

[1]John M. Anderson and Henry W. Johnstone, Jr., Natural Deduction (Belmont, California: Wadsworth, 1962), p. 130.

We shall, then, construct several "broken cut" systems by slightly modifying certain rules for transformations involving the broken cut. As we will recall from the Introduction, the graph



may be interpreted as asserting, "It is possibly not the case that X." Systems containing the broken cut among their signs, then, may be considered systems of <u>modal logic</u>.

In chapters i and ii we compared the alpha and beta systems to ordinary logical calculi. The project for gamma will be similar. We shall compare the broken cut systems which we will define with ordinary systems of modal logic. We shall define four different broken cut systems, and shall compare them to four different standard modal systems. Three of the systems we shall thus use belong to the family of "Lewis-modal" systems, the "classical" modal systems of contemporary logic. The fourth system is a rather unusual modal system invented by Lukasiewicz.

The usual axiomatizations of modal systems ordinarily take either "necessity" (with 'Lp' read as "necessarily p") or "possibility" (with 'Mp' read as "possibly p") as primitive modal operators. Since the primitive modal operator in the gamma systems is the broken cut, which states "possibly not," it may be worthwhile to provide axiomatizations of our ordinary modal systems taking "possibly not" as primitive rather than the usual "possibly" or "necessarily." We thus shall state some rather neat axiomatizations of several Lewis-modal systems taking "possibly not" as primitive--these are, so far as I am aware, new bases for these systems.

We shall then present the axiomatic basis for that unusual system of Lukasiewicz, and then make some remarks about that calculus.

The next step will be to describe each of the four broken cut systems we have mentioned, and to compare each in turn to an appropriate standard modal system. We shall, in fact, show that our broken cut systems are equivalent to these standard systems in the same sense that alpha, say, is equivalent to the CPC.

3.1 Remarks on Some Standard Modal Systems

If the alpha cut is to be considered a negation sign, then the broken cut will be a sign that states "possibly not." The broken cut then represents a "weak negation." We shall presently be comparing the systems involving the broken cut to certain standard modal calculi. As we mentioned, these systems are ordinarily formulated using either 'M' (possibility) or 'L' (necessity) as a primitive modal operator. It is easily possible, however, to get nice axiomatizations of the standard modal systems with which we will be concerned using "possibly not" as the only primitive modal concept. We may read 'Rp' as "possibly not p." Then we have:

Definitions: $M \underset{df}{=} RN$; $L \underset{df}{=} NR$.

Rule of Inference: If α is a theorem, so too is $NR\alpha$. This rule may be called "RL." (This is the name of the rule, and not a sequence of operators.)

Stock of Axioms:

1. CNCRpRqRCqp

1'. CRCRpRqRCqp

1''. CRRCpqCRqRp

2. CNpRp

3. CRRpNRRNp.

The axioms and rule will be considered to be subjoined to a complete CPC base, including the rules of detachment and substitution of variables. The specific formulations of the systems are as follows:

For each of the systems below, any complete base for the CPC. Also for each of these systems, the rule "RL" mentioned above. In addition, the axioms for the respective systems from the above stock are:

For T: 1 and 2,

For S4: 1' and 2,

For S4.2: 1', 2, and 3,

For S5: 1'' and 2.

In what follows, we will allude to "well-known" axiomatizations of the above systems, in general, those developed by E. J. Lemmon.[1]

At this point I shall also remark that although we will develop no broken cut system analogous to the calculus T, we include its axiomatization in 'R' here as a matter of general interest.

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[1]An excellent quick reference to these well-known formulations may be found in Prior, pp. 312 ff.

The axioms are, in general, quite expressive of the nature of 'R' as a sign of "weak negation." Axiom 2 indicates that this weak negation of a statement follows from the ordinary negation; this axiom is equivalent in deductive power to the theses 'CLpp' and 'CpMp'. Axioms 1, 1', and 1'' are "laws of transposition" for the weak negation, analogous to ordinary PC laws of transposition like 'CCNpNqCqp'. By applications of appropriate CPC laws and the definition of 'L', axiom 1 becomes 'CLCpqCLpLq', and axiom 1' becomes 'CLCpqLCLpLq'. It is well-known that CPC, RL, and the two theses:

*1. CLCpqCLpLq

*2. CLpp

together are an axiomatic basis for the system T, while a replacement of *1 above by

*1'. CLCpqLCLpLq

yields the system S4. Our axioms 1 and 1', again, are easily shown to be theses of T and S4 respectively. We may state without hesitation, then, that the axioms in 'R' as we have given them form bases for T and S4. The system S4.2 is ordinarily formulated by subjoining the formula 'CMLpLMp' to a basis sufficient for S4; our axiom 3 is definitionally equivalent to this last formula, so we have also provided a sufficient base for S4.2.

The iterated modality 'RR' is equivalent to 'ML' ("possibly-not possibly-not" is the same as "possibly necessarily"). With this in mind, and transposing the consequent of 1'', 1'' becomes:

*1''. CMLCpqCLpLq.

This latter formula is, of course, also easily transformable back into 1''. If *1'' is a thesis of a system, and *2 is also a thesis, then *1 is easily provable. Now substitute 'Cpp' for 'p' in *1''; the result is:

4. CMLCCppqCLCppLq.

By PC and RL, the following strict equivalence holds:

5. LEqCCppq.

By PC and the substitutivity of strict equivalence, which holds in the system containing PC, RL, *1, and *2, all of which we have, and using the strict equivalence at 5, 4 becomes:

6. CLCppCMLqLq;

detaching now with the thesis 'LCpp', which holds in all Lewis-modal systems, we get:

7. CMLqLq.

This is a characteristic reduction formula for S5, the strongest of the Lewis-modal systems; CPC, RL, *1, *2, and 7, in fact, constitute a standard axiomatic base for S5. The formula *1'' is itself an S5 thesis, and easily shown to be such. It follows, then, that CPC, RL, *1'', and *2 form a sufficient base for S5; and then so too will CPC, RL, 1'', and 2--our axioms in 'R'-- as we stated earlier.

Of the systems for which we have provided "R-primitive" bases, we shall be interested in the relationship of S4, S4.2, and S5 to the broken cut systems. We shall also be concerned with one other modal system in this connection, and a rather unusual system at that. This is the "Ł-modal" system of

Łukasiewicz.[1] It should be pointed out that this is not the fairly well-known "three-valued" modal logic of Łukasiewicz, but a system based on his PC extended to include functor-variables, which we mentioned in chapter i. We will not, however, consider the system as we state it to include primitive functor variables; "formulas" like 'CpCδ1δp', for example, will be considered theorem-schemata rather than theorems. It is clear that for every theorem containing functor variables in the system as Łukasiewicz states it, there will be an identical-appearing theorem schema of the system as we state it. The Ł-modal system contains the rules of detachment and substitution for variables, and has as axioms:

8. ⌜CδpCδNpδq⌝

9. CLpp,

with standard definition "M $=_{df}$ NLN"; this is the system with 'L' primitive. It might also be stated with 'M' primitive, or even with 'R' primitive. Łukasiewicz also lists two "axiomatic rejections" for the system:

*CpLp

*NLp.

These will not be of too much interest for our purposes, for they do not actually enter into the generation of the set of theorems of the system.

The system as stated looks innocuous enough, but there is one catch. Functor signs, like 'δ' in 8, may stand in place

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[1]For details on this system, see Prior, pp. 208-209, and A. N. Prior, <u>Time and Modality</u> (Oxford: Oxford, 1957), pp. 1 ff.

of <u>any</u> context whatsoever, including modal contexts. This causes some unusual results; the formula

10. CLCpqCpLq,

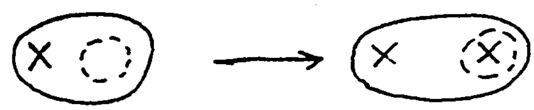
for example, is a thesis of the system. This formula could not be a thesis in any of the Lewis-modal systems, for they all contain as a law 'LCpp', which in the presence of 10 would yield as a theorem 'CpLp', and thereby destroy the modal nature of the system. Ł-modal, however, contains no thesis of the form ⌜Lα⌝, and so this problem does not arise there. Also among the strange laws of this system are formulas like

11. CMpMLp,

which are also incompatible with many characteristic Lewis-modal theses. This last is a reduction formula, but it differs from, say, the reduction theses of S5. Where the S5 reduction formulas "reduce" a string of modal operators to the <u>rightmost</u> member of the string, as indicated by 7, the Ł-modal reduction theses reduce such a string to the <u>leftmost</u> member, as indicated by 11. As an example, 'MLLMMMLMLp' is equivalent in S5 to the simple 'Lp', while in Ł-modal it is equivalent to 'Mp'.

3.2 <u>The Broken Cut Systems</u>

We now turn our attention back to the broken cut. Earlier in this chapter, we repeated from the Introduction the rules 0.13, 0.14, and 0.15 for the use of the broken cut. There is not too much doubt about how the latter two of these rules are to be interpreted and applied; 0.13, however, begs for study. The question raised by 0.13 is this: Just what is to be the over-all function of the alpha rules of transformation in a system containing the gamma cut? For example, the transformation



would involve the iteration of a graph "across a broken cut." Do we wish to permit such transformations at all? Do we wish to permit them with certain restrictions? Or do we wish to permit them in unlimited fashion? The rule 0.13 is in itself a very limited rule, as it is stated. But it does involve a certain "cross-breeding" of alpha and gamma concepts. Although it is really just a weak "gamma-version" of our alpha R_{ins}, it somewhat coercively turns our attention to the question of alpha and gamma "cross-breeding" in the rules in general.

Well, we now propose to open an experimental farm for the investigation of some of the possible hybrids, and for the comparison of them with some of the earlier mentioned standard breeds of modal logic. I think we shall find the alpha-gamma progeny, unlike most hybrids, relatively fertile.

3.21 Gamma-MR: Broken Cuts with Minimal Restrictions

The first broken cut system we shall develop and study is one which allows the most liberal possible interpretation within gamma of the alpha rules of inference; we shall call this system "gamma-MR." The basis for gamma-MR will be as follows:

The single axiom for gamma-MR will be b, the blank alpha SA.

The rules R_{ins} and R_{ers} will apply just as they do in alpha, and for purposes of their application, broken cuts will be counted as if they were alpha cuts to determine "oddness" or "eveness" of enclosure.

The rules R_{itr} and R_{dit} will apply just as they do in alpha; any graph may be iterated (or deiterated), across any kind of cut or combination or cuts, just as if they were alpha cuts.

The rules R_{bcl} and R_{nbc} may be applied just as they are in alpha, in any area at all. But it is understood that the only cuts that may be inserted or removed by these rules are alpha cuts.

In addition, there shall be two other rules which shall apply in this system; these rules correspond to our 0.14 and 0.15.

$R_{gam}(Y, X)$: Which is true iff X contains, in an area enclosed by an even number of or by no cuts of either kind or in any mixture, an alpha cut, and Y is like X except for having at that position a broken cut rather than an alpha cut.

$R_{ngm}(Y, X)$: Which is true iff $R_{gam}(S(X), S(Y))$ is true.

Before we go any further, let us state--without explicit proof, however--two theses which will be useful in what follows. These theses will be analogs for our modal systems of lemmas 1.03 and 1.04:

*3.01 When $\ulcorner Q^A(B) \urcorner$ and $\ulcorner D \supset B \urcorner$ are theorems of L-modal, so too is $\ulcorner Q^A(D) \urcorner$.

When $\ulcorner Q^A(B) \urcorner$ and $\ulcorner D \prec B \urcorner$ are theorems of S4, S4.2, or S5, then so too is $\ulcorner Q^A(D) \urcorner$ a theorem of the system in question.

*3.02 When $\ulcorner Q^C(B) \urcorner$ and $\ulcorner B \supset D \urcorner$ are theorems of Ł-modal, so too is $\ulcorner Q^C(D) \urcorner$.

When $\ulcorner Q^C(B) \urcorner$ and $\ulcorner B \prec D \urcorner$ are theorems of S4, S4.2, or S5, then so too is $\ulcorner Q^C(D) \urcorner$ a theorem of the system in question.

The formula '$p \prec q$', of course, is read, "p strictly implies q"; it is equivalent to the formula--in Polish notation--'LCpq'. The notation used above is used simply for the sake of consistency with that of lemmas 1.03 and 1.04. We may assume that the systems for which these theses are stated are "L-primitive" systems; and 'L' affects the A-pos or C-pos of subformulas in its scope in no way. The theses above are provable by an induction on the number of L's (belonging to the formula $Q^A(B)$ or $Q^C(B)$) within whose score B is located. But without going through the tedium of an explicit proof, I submit that the theses are intuitively quite acceptable. Recall the relationship of lemmas 1.03 and 1.04 to the rule of substitutivity of material equivalence in CPC. It may then help our intuition if we note that substitutivity of <u>material</u> equivalence holds in the Ł-modal system, while substitutivity of <u>strict</u> equivalence holds in all the Lewis-modal systems. That the above theses would hold in "M-primitive" or "R-primitive" systems is evident. Just note that since ERpNLp, an R will affect the A-pos or C-pos of formulas in its scope just as if it were an 'N'.

We may now begin our comparison of gamma-MR with an "ordinary" modal logic. The calculus used here will be the Ł-modal system. In our comparison of alpha with the CPC we found it convenient to use two formulations of that calculus, P_r and P_w, the former with "K-N" primitive, the latter with "C-O" primitive. We may similarly think of two formulations of Ł-modal, one with "K-N-R" primitive ('R', of course, being our possibly-not operator) and the other with "C-O-R" as primitive. in chapter i we defined functions f, h, and g; it should be evident that we may extend these functions so that they may be able to relate the broken cut systems to systems like Ł-modal. All that is required here is an instruction to correlate the broken cut to the modal operator 'R'. Call the functions thus extended f*, h*, and g* respectively.

In chapter i, we showed that f of the single axiom of alpha, the null-graph, is a theorem of P_r. The null-graph b is also the only axiom of gamma-MR; f*(b) is '$-(p_0.-p_0)$', just as was f(b). Clearly, then, f*(b) is a theorem of Ł-modal.

We also showed in chapter i that when R(Y, X) is one of the rules of inference of alpha, and is true for alpha graphs X and Y, then if f(X) is a CPC theorem, so too is f(Y).

The rules of inference of alpha hold unrestricted in gamma-MR. But *3.01 and *3.02 hold for Ł-modal, and the schema

$$\ulcorner C\delta pC\delta Np\delta q\urcorner$$

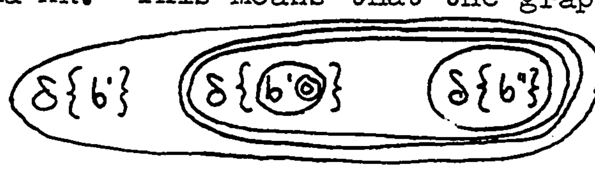
holds unrestricted in Ł-modal, as we have mentioned (it is, in fact, an axiom-schema of that system). There would then be no trouble in extending the proofs of chapter i to show that when R(Y, X) is one of the alpha rules as employed in gamma-MR, and is true for gamma-MR graphs X and Y, then if f*(X) is a theorem of Ł-modal, then so too is f*(Y).

Gamma-MR also includes two rules for the broken cut--R_{gam}, which permits an evenly enclosed alpha cut to be transformed to a broken cut, and R_{ngm}, which permits an oddly enclosed broken cut to be transformed to an alpha cut. Ł-modal, of course, contains the law 'CNpRp'; in the presence of *3.01 and *3.02, this means--since the broken cut is correlated by our functions to 'R' just as the alpha cut is to 'N'--that if either R_{gam}(Y, X) or R_{ngm}(Y, X) is true for gamma-MR graphs X and Y, then if f*(X) is an Ł-modal theorem, so too is f*(Y).

In summary, all of the above means that:

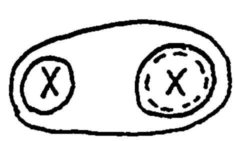
<u>If X is a theorem of gamma-MR, then f*(X) is a theorem of Ł-modal</u>.

Now turning to the "other direction" of the proof, recall again that the rules of insertion and erasure, iteration and deiteration, and positive and negative biclosure hold <u>unrestricted</u> in gamma-MR. This means that the graph



is provable in gamma-MR just as it is in alpha, even though here ' $\delta\{\}$ ' may stand for <u>any</u> graphical context, may include broken cuts. This graph corresponds by g* (or by h*) to the axiom-schema of Ł-modal, which is "unrestricted" in the same sense as is this "graph-schema."

The graph



is trivially provable in gamma-MR by R_{gam} and the alpha rules. This graph corresponds by g* (or by h*) to the Ł-modal axiom 'CNpRp' (which is, of course, equivalent to 'CLpp').

It is clear that the analogs of the rules of substitution and detachment hold in gamma-MR as "derived rules of inference" just as they do in alpha.

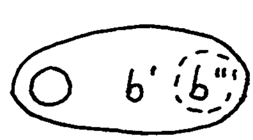
Given the above, we submit that the following hold, as analogs of *1.17 and *1.21 of chapter i:

If f*(X) is a theorem of Ł-modal, then X is a theorem of gamma-MR, and A is a theorem of Ł-modal iff g*(X) is a theorem of gamma-MR.

The three underscored statements in the above development assert equivalently that Ł-modal and gamma-MR are translatable into each other in our technical sense of "translatable." And they mean that gamma-MR and Ł-modal are equivalent to each other in the same sense that alpha is equivalent to the CPC. This result should not have been unexpected, since the unrestricted "cross-breeding" of alpha and gamma concepts in gamma-MR is strikingly like the unrestricted "cross-breeding" of truth-functional and modal concepts in the Ł-modal system.

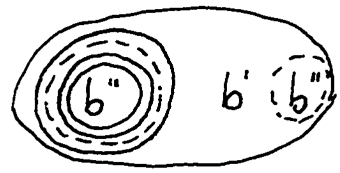
To drive the point home a little harder, we shall now engage in a few graphical derivations within gamma-MR. We shall prove within gamma-MR some typical graphs of the system; they shall be seen to correspond to characteristic theses of the Ł-modal system.

By alpha rules



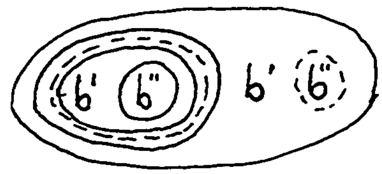
(1)

(1), R_{itr}, R_{bcl}



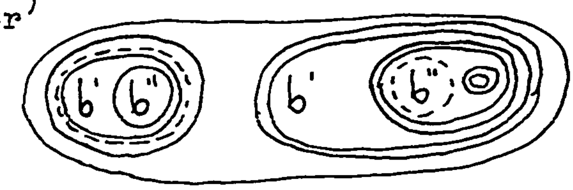
(2)

(2), R_{itr} (note the dependence on unrestricted R_{itr})



(3)

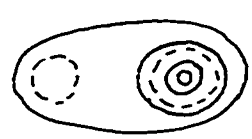
(3), R_{bcl}



(4)

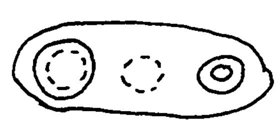
Note that given the definitions 'L' for 'NR' and 'Np' for 'CpO', (4) is equivalent to a characteristic Ł-modal thesis, 'CLCpqCpLq'. The critical step of this deduction is (3), where b' is iterated across a broken cut, as permitted by the rules of gamma-MR. We shall see that in the other broken cut systems we shall examine, this move, and so the graph of (4), is forbidden. We now go on to the proof of a graph which, while not peculiar to gamma-MR, may prove a bit surprising.

alpha rules



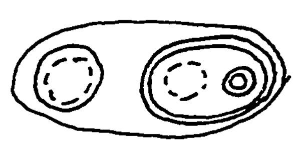
(5)

alpha rules

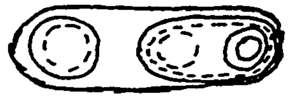


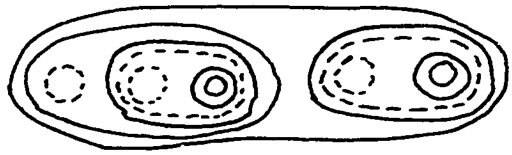
(6)

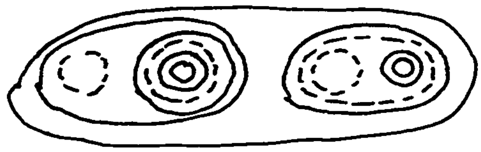
(6), R_{bcl}

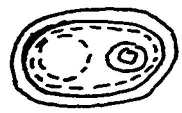


(7)

(7), R_{gam}  (8)

(8), R_{itr}  (9)

(9), R_{dit}  (10)

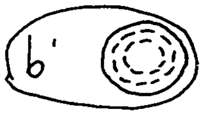
(10), (5), R_{dit}, R_{nbc}  (11)

(11), R_{nbc}  (12)

The graphs at (11) and (12) express, equivalently, the thesis 'RR1' or 'ML1'; they assert that the "true" is "possibly necessary." The thing one might find surprising is that the simple "broken cut biclosure" with a broken cut enclosed oddly is a thesis of gamma-MR, derivable only by the rules. It is not immediately apparent from an examination of R_{gam}, for example, that this is possible. Yet this theorem should not be too surprising considering that gamma-MR and Ł-modal are equivalent to each other, and 'ML1' is a law of Ł-modal.

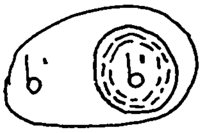
We set about the proof, now, of one more characteristic graph of gamma-MR, again, one that is equivalent to a characteristic thesis of Ł-modal.

(12), R_{bcl}, R_{ins}



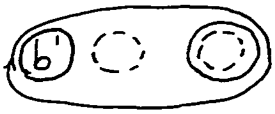
(13)

(13), R_{itr}



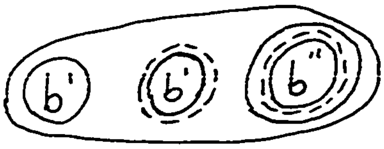
(14)

alpha rules



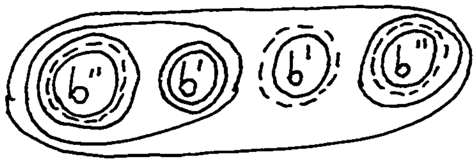
(15)

(15), R_{itr}, R_{ins}



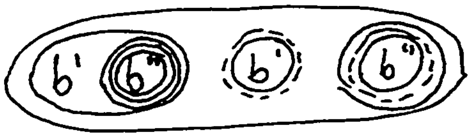
(16)

(16), R_{bcl}, R_{itr}



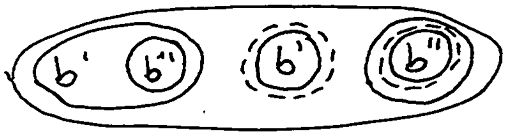
(17)

(17), R_{ngm}, R_{nbc}



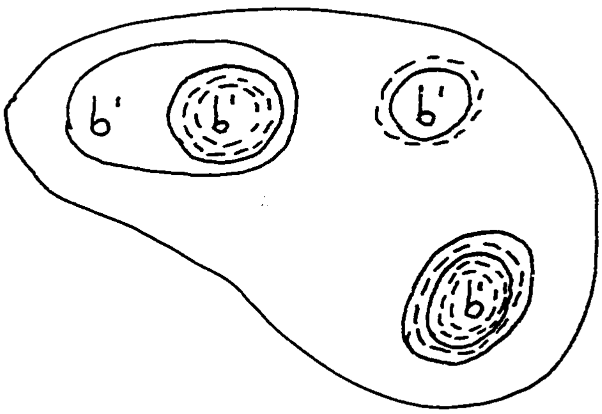
(18)

(18), R_{nbc}

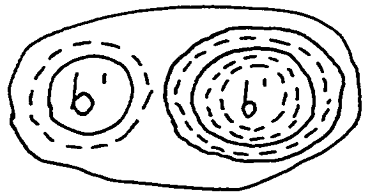


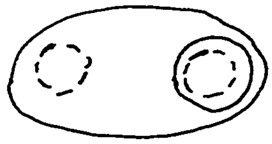
(19)

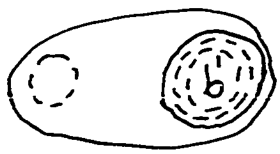
The graph at (19) is a typical gamma-MR graph, equivalent to the L-modal 'CCpqCMpMq'; now take b'' in (19) as  :

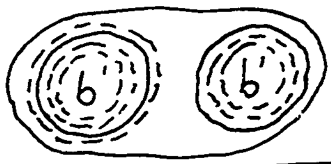


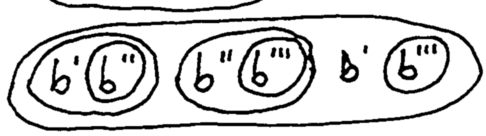
(20)

(14), (20), R_{dit}, R_{nbc}  (21)

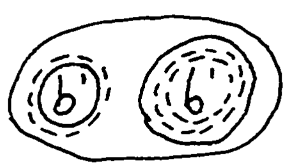
alpha rules  (22)

(22), R_{ins}  (23)

(23), R_{itr}  (24)

alpha rules  (25)

Now in (25) take b' as  , b'' as  , b''' as  . Then, with (25) (with these substitutions), (21), (24), R_{dit}, and R_{nbc}, we have:

 (26).

The graph of (26) says the same thing as the formula 'CMpMLp'; it is thus a "map" of a characteristic Ł-modal thesis. Note that step (16) in the above deduction, for example, again employs the unrestricted "iteration across broken cuts" characteristic of gamma-MR.

The system gamma-MR yields unusual modal graphs, just as

Ł-modal gives us unusual modal formulas. In both case the root of the strangeness is the "extensionalizing" of modal concepts. In Ł-modal this is accomplished by permitting unlimited or unrestricted use of modal contexts in schemata like C pC Np q '; this schema is often referred to as the "thesis of extensionality" (actually, it is but one of several schemata to which the name is appropriate). In the "classical" Lewis-modal systems, of course, such unrestricted treatment of modal operators is not permitted; the closest we can come to a thesis of extensionality in these systems is

$$CLCpqCLCqpC\delta p\delta q$$,[1]

which is a statement of the substitutivity of <u>strict</u> equivalence, a principle which, as a rule of inference, is characteristic of all the Lewis modal systems, and which is considerably weaker than the Ł-modal theses of extensionality.

In gamma-MR, the "extensionalizing" is accomplished by the unrestricted permission to use the rules R_{itr} and R_{dit}. The theses *3.01 and *3.02 show that the rules of insertion and erasure hold unrestricted even for gamma systems which might be called "Lewis-type," as we shall see; the rules of biclosure and for the broken cut also hold without restriction. The strange nature of gamma-MR may then be credited to its unrestricted rules of iteration and deiteration.

---

[1]This theorem-schema holds as stated in the systems S4, S4.2, and S5; at this point we will make no claims one way or the other for the other Lewis-modal systems. I am not aware of a general proof for this thesis in the literature, so I have added to this paper a short appendix which discusses the question of the deduction theorem in S4, S4.2, and S5, and shows that this schema holds in these systems.

Another remark is in order here: in 4.518 Peirce indicates that if a graph X is assertable at one "state of information," then  will be assertable at succeeding states of information. This is much like the move in some Lewis-modal systems justified by RL: "If α is a theorem, then $\ulcorner L\alpha \urcorner$ is a theorem." It is perhaps needless to say that this move cannot be permitted in gamma-MR. Such a permission, in the presence of the theorems of gamma-MR, would result in the collapsing of the modal structure of the system, just as the theoremhood of 'LCpp' would do for Ł-modal. The relationship of gamma-MR and Ł-modal, in fact, bars all graphs of form  from theoremhood in gamma-MR, since all formulas of form $\ulcorner L\alpha \urcorner$ are rejected in Ł-modal.

3.22 Gamma-4: A "Classical" Broken Cut System

There is little subtlety to the name we have chosen to give the broken cut system of the present section; it is a system which bears the same relation to the Lewis-modal S4 that gamma-MR bears to the Ł-modal system. As we have indicated, the statement of rules for gamma-4 will involve a restriction on the application of the rules R_{itr} and R_{dit}. It will also involve the axiomatic assertion of a certain extra graph. Although the graph  was derivable as a theorem of gamma-MR, the graph  was not derivable there, as indicated by the relationship between gamma-MR and Ł-modal. This latter graph, we might say, asserts "the necessity of the true." But in all the Lewis-modal systems, there is a sense in which the true <u>is</u> necessary;

even in the very weakest systems, like Lemmon's S0.5,[1] there are rules which permit us to move from the theoremhood of a formula α (belonging to a certain set of theorems, depending on the system) to the theoremhood of $\ulcorner L\alpha \urcorner$. In S0.5, this rule is simply, "If α is tautologous, $\ulcorner L\alpha \urcorner$ is a theorem"; but in the systems from T through S5, the presence of the unrestricted rule RL permits us to move from the assertion of any theorem to the assertion of the necessity of that theorem. In gamma-4 we shall discover that we can get the same results with the simple axiomatic assertion of the graph

which is what we might call "the necessity biclosure." Needless to say, the presence of this graph does not give us the right to use this pair of cuts as if it were a pair of alpha cuts in the application of the rules R_{bcl} and R_{nbc}. These rules will still apply just to the insertion or deletion of alpha biclosures. Note too that with R_{ngm} and R_{nbc} there is no trouble in getting from this graph as axiom to the null graph b as theorem.

We shall now state the axiom and the rules as they apply to gamma-4, just as we did for gamma-MR.

The sole axiom of gamma-4 is the graph  .

Rules R_{ins} and R_{ers} apply just as they did in gamma-MR, again counting alpha and broken cuts alike to determine oddness or eveness of enclosure.

---

[1]E. J. Lemmon, "New Foundations for the Lewis Modal Systems," Journal of Symbolic Logic, XXII (1956), 176-86.

The rules R_{itr} and R_{dit} will apply as before, <u>except for the following restriction</u>: The only graphs that may be iterated or deiterated across <u>broken cuts</u> are those of form  .

The rules R_{bcl} and R_{nbc} apply as in gamma-MR.

The rules R_{gam} and R_{ngm} (for the broken cut) apply as in gamma-MR.

The restriction on the rules of iteration and deiteration prevents an iteration in gamma-4 like that leading from step (2) to step (3) in the deductions of Section 3.21.

We turn our attention to the status of these rules in the system S4. First of all, the strict implication

LCNpRp

is a thesis of S4. By *3.01 and *3.02, this indicates that any subformula beginning with 'N' and standing in a C-pos in an S4 theorem--that is, a subformula $\ulcorner N\alpha \urcorner$ --may be replaced by the same formula with an 'R' in place of the 'N'--that is, by the subformula $\ulcorner R\alpha \urcorner$. And also, any subformula $\ulcorner R\alpha \urcorner$ standing in an A-pos in a law of S4 may be replaced by $\ulcorner N\alpha \urcorner$.

This indicates that whenever, for gamma-4 graphs X and Y, either $R_{gam}(Y, X)$ or $R_{ngm}(Y, X)$ is true, then if f*(X) is an S4 theorem, then so too is f*(Y).

The strict implication

LCKpqp

is also a law of S4; in the presence of *3.01 and *3.02 this means that the rules of insertion in odd and erasure in even have analogous derived rules of inference in S4, just as do the above mentioned rules for the broken cut.

A strict equivalence that holds in S4 is

$$LEpNNp.$$

Since the system S4 contains the rule of substitutivity of strict equivalence, this means that the rules of biclosure have analogous derived rules of inference in S4.

We move now to the rules of iteration and deiteration as stated for gamma-4. It will be recalled that the proof in chapter i that the CPC contains rules analogous to R_{itr} and R_{dit} pivots about the existence in CPC of a certain theorem schema which for our purposes here we may express as:

$$EKp\delta 1Kp\delta p \quad . \qquad (1)$$

If this schema is in a system, and the system contains the rule of substitutivity of material equivalence, then that system contains derived rules of inference analogous to R_{itr} and R_{dit}.

The schema at (1) quite definitely does not hold in the general case in S4; however, recalling the nature of the restriction on the rules of iteration and deiteration in gamma-4, it is possible to write a schema which does hold in S4 and whose presence makes possible proofs of the versions of iteration and deiteration appropriate to S4.

A little thought will tell us what this schema must be; we will recall that the restriction on iteration and deiteration for gamma-4 states that only graphs of form  may be iterated across broken cuts, that is, iterated "into modal contexts." This suggests that the schema we want in the general case in S4 is:

$$LEKLp\delta 1KLp\delta Lp \qquad (2).$$

Actually, the simple

$$EKLp\,\delta\,1KLp\,\delta\,Lp \quad (3)$$

will do, since we can always move from (3) to (2) by RL. The existence of (2) and the substitutivity of strict equivalence in S4 would permit us to say that in the general case, where 'δ' may be a modal context, analogs of the rules of iteration and deiteration of gamma-4 exist in S4.

It will be noted that when 'δ' is a non-modal context, that is, when the schema is to parallel iterations across alpha cuts only,

$$EKp\,\delta\,1Kp\,\delta\,p$$

does hold in S4 just as it does in CPC.

We shall now show that (3) is indeed a theorem schema in the general case in S4. The following schema holds in S4:[1]

$$CLCpqCLCqpC\,\delta\,q\,\delta\,p \quad (4),$$

where 'δ' may represent any S4 context at all.

By (4), q/1	$CLCp1CLC1pC\,\delta\,1\,\delta\,p$	(5)
(5), LCp1, PC	$CLC1pC\,\delta\,1\delta p$	(6)

The following is an easily provable S4 thesis:

	CLpLCqp	(7)
(6), (7), PC	$CLpC\,\delta 1\,\delta\,p$	(8)
(8), p/Lp	$CLLpC\,\delta\,1\,\delta Lp$	(9)

But 'LELLpLp' is a theorem of S4; by (9) and substitutivity of strict equivalence, then:

$$CLpC\,\delta 1\,\delta\,Lp \quad (10)$$

---

[1] As we noted earlier, see the Appendix for the proof of this.

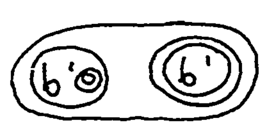
(10), PC CKLp δ 1KLp δ Lp (11)

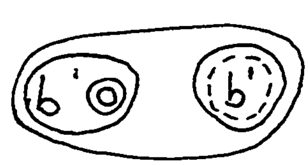
But (11) is one half of (3), which is the formula we wished to prove. The converse of (11) is provable along much the same lines that (11) was. We may then say that (3) holds in the general case in S4. But this means that when, for gamma-4 graphs X and Y, R_{itr}(Y, X) or R_{dit}(Y, X) is true with the gamma-4 restriction, then if f*(X) is an S4 theorem, so too is f*(Y). Iteration into non-modal contexts as in PC.

Where X is the sole axiom of gamma-4, then f*(X) is '$NRNKp_0Np_0$', which is the same as 'LCp_0p_0', and is clearly a theorem of S4. We may say, then, that:

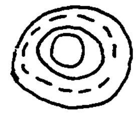
<u>If X is a theorem of gamma-4, f*(X) is a theorem of S4</u>.

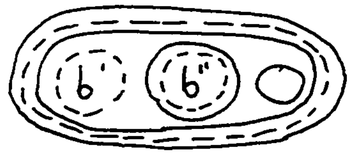
We shall now go on to show both that the converse of the above statement holds, and that the set of theorems of S4 maps into the set of theorems of gamma-4 by the function g* (f* and g*, we should recall, are the functions f and g of chapter i extended to account for the broken cut). Gamma-4 contains all that alpha contains and more. We shall have accomplished our purpose, then, if we can show that the axioms and rule of S4 beyond those of the CPC map appropriately into the set of theorems of gamma-4. First of all, we may easily prove the following:

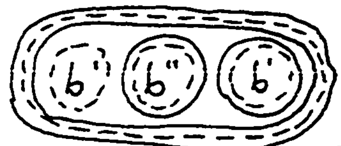
alpha rules  (12)

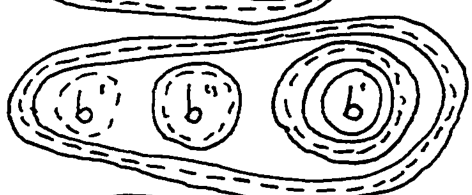
(12), R_{gam}  (13)

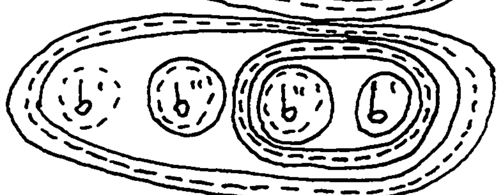
The graph at (13) is clearly a correlate by g* or h* of our axiom 2 for S4, 'CNpRp'.

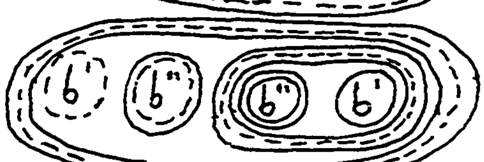
gamma-4 axiom, R_{bcl}  (14)

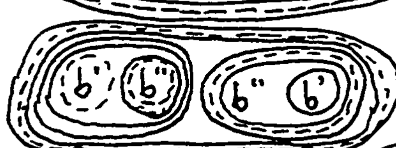
(14), R_{ins}  (15)

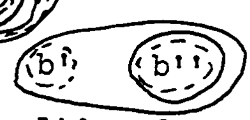
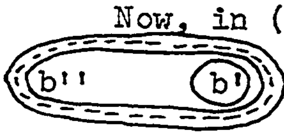
(15), R_{itr}  (16)

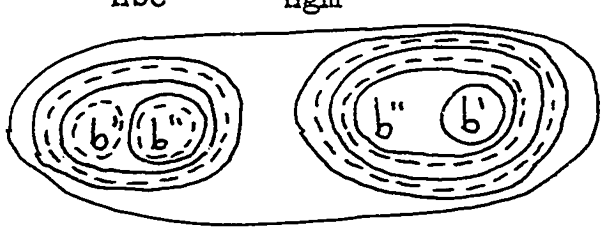
(16), R_{bcl}  (17)

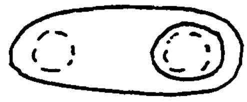
(17), R_{itr}  (18)

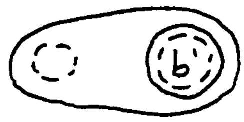
(18), R_{ngm}  (19)

(19), R_{bcl}, R_{nbc}  (20)

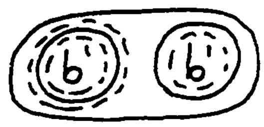
Now, in (20), take b' as  , and b'' as  ; the graph resulting from these substitutions contains a "replica" of (20) itself. Deiterate this "replica" with (20) and apply R_{nbc} and R_{ngm} to obtain

 (21)

by alpha rules  (22)

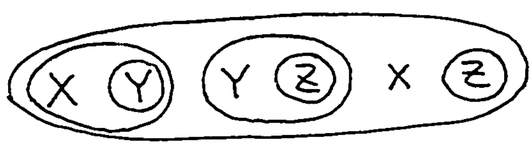
(22), R_{ins} 

(23)

(23), R_{itr} 

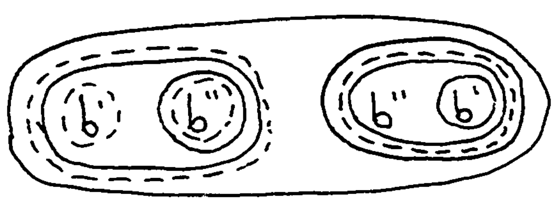
(24)

We may now move, by purely alpha methods, employing (21), (24), and the alpha graph



(25)

to



(26).

The graph at (26) will be found, upon examination, to correspond to our axiom 1' for S4, 'CRCRpRqCpq'. The graph of (24) corresponds to 'CRNRpRp', which becomes, on transposition of antecedent and consequent, 'CLpLLp', the characteristic reduction formula of S4.

Note that with  as an axiom of gamma-4, the equivalent of RL holds in gamma-4. This is so because any graph derivable as a theorem upon the blank SA alone will be derivable as a theorem within the "necessity biclosure" which is the gamma-4 axiom; this point is illustrated by the deduction leaeing to the graph of (20) above. It is now fairly clear that both:

A is a theorem of S4 iff g*(A) is a theorem of gamma-4, and

If f*(X) is a theorem of S4, then X is a theorem of gamma-4.

The latter by an extension of the argument leading to the proof of its analog in chapter i.

Gamma-4 is then a graphical version of the Lewis modal S4. One might suspect that there are other Lewis modal systems which may be formulated in the broken cut notation; this in fact is the case.

3.23 Gamma-4.2: Another "Lewis-Modal" Broken Cut System

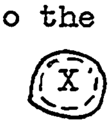
Situated between the Lewis-modal systems S4 and S5 are at least two other systems. They are situated between these systems in the sense that their sets of theorems include those of S4 and are included in those of S5. These systems may be formulated by subjoining to S4 certain additional axioms. One of these systems, S4.2, is formed by adding to S4 the axiom

CMLpLMp.

The other, S4.3, is formed by adding to S4 the axiom

ALCLpqLCLqp.

S4.2 is contained in S4.3. We shall here set down a broken cut system, gamma-4.2, which bears the same relation to S4.2 that gamma-4 bears to S4.

The system gamma-4.2 is based upon gamma-4. It is just the same as gamma-4 except for a slightly more liberal restriction attached to the rules R_{itr} and R_{dit}. Gamma-4 permitted graphs of form 

to be iterated and deiterated through

broken cuts; gamma-4.2 will permit <u>as well</u> graphs of form  to be so iterated. Otherwise the systems are the same.

All that we have said about gamma-4 and S4, then, applies to gamma-4.2 and S4.2; we have in addition, however, to prove the rules of iteration and deiteration with the new restriction in S4.2, and to prove a graph in gamma-4.2 corresponding to the special axiom of S4.2.

Recall how we went about the proof of the rules of iteration and deiteration in S4. Whether or not these rules held there was a function of whether or not the schema

$$EKLp\,\delta\,1KLp\,\delta\,Lp$$

held in S4. The above schema holds in S4.2, to be sure, as well as in S4. But we must show something more for S4.2; the restriction on the rules of iteration and deiteration has been relaxed in this system to permit graphs enclosed by two broken cuts to be iterated or deiterated through broken cuts. The pair of broken cuts is correlated by our translation functions to the modal prefix 'RR', which is equivalent to 'ML'. This suggests that the additional schema which we must show to exist in S4.2 is

$$EKMLp\,\delta\,1KMLp\,\delta\,MLp\ .$$

A little reflection will show us that this is indeed the case.

Let us turn to step (8) in the deductions of the preceding section, and instead of substituting Lp for p, substitute MLp (we begin the numbering anew in this section):

$$CLMLpC\,\delta\,1\,\delta\,MLp \qquad (1)$$

Now, in S4.2 the following thesis holds--it is <u>not</u> a law of S4:

CMLpIMLp (2).

(1), (2), PC CMLpCδ1δMLp (3)

(3), PC CKMLpδ1KMLpδMLp (4)

We noted that the <u>converse</u> of the analogous formula in Section 3.22 is proven much as is that formula itself; so too will the converse of (4) follow, and we may state that

EKMLpδ1KMLpδMLp (5).

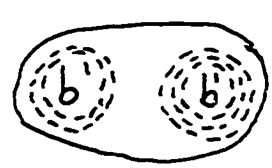
The formula at (5) is the one we wanted, and we are safe in saying that appropriate analogs of the rules of iteration and deiteration with their gamma-4.2 restrictions hold in S4.2. Since all else in gamma-4.2 is as in gamma-4, we may conclude that:

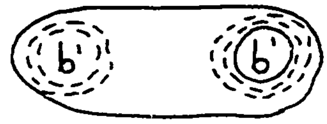
<u>A graph X is a theorem of gamma-4.2 only if f*(X) is a theorem of S4.2</u>.

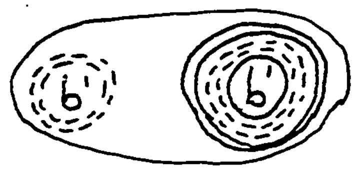
To show that the converse of the above holds, and also that the set of S4.2 theorems maps into the set of gamma-4.2 theorems by g*, we must show only that there is a theorem of gamma-4.2 which corresponds by g* and h* to the special axiom of S4.2, 'CMLpIMp'--or, in 'R', 'CRRpNRRNp'.

gamma-4 axiom  (6)

(6), R_{ins}  (7)

(7), R_{itr} (in gamma-4.2 version)  (8)

(8), R_{ngm}  (9)

(9), R_{bcl}  (10)

The graph of (10) states 'CRRpNRRNp', which is the same as 'CMLpIMp'. Note the characteristic gamma-4.2 operation at step (8). It is now safe for us to state, on the basis of the above, our work in Section 3.22, and the relationship between S4 and S4.2, that both:

<u>If f*(X) is an S4.2 theorem, then X is a theorem of gamma-4.2</u>,

and <u>A is a theorem of S4.2 iff g*(A) is a theorem of gamma-4.2</u>.

The systems S4.2 and gamma-4.2 are then translatable into each other, and also may be considered equivalent in the same sense that alpha is equivalent to the CPC.

3.24 Gamma-5: The Limiting "Lewis-Modal" Broken Cut System

In many respects, the system S5 may be considered a "limiting" system among the Lewis-modal calculi--to go into an extensive study of this, however, would really be outside the scope of this paper.[1]

The system gamma-5 is to be a "broken-cut" version of S5; this is why we have called it a "limiting" broken cut system. As might be suspected, gamma-5 differs from gamma-4 and gamma-4.2 only in having a different, more lenient restriction upon its

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[1]But cf., for example, C. I. Lewis and C. H. Langford, <u>Symbolic Logic</u> (New York: Dover, 1959), p. 501.

application of the rules of iteration and deiteration. The statement of this restriction is quite simple:

In gamma-5, the rules R_{itr} and R_{dit} apply as they do in alpha, except that a graph X may be iterated or deiterated across a broken cut only if <u>each</u> of X's minimal graphs is in the "scope" of a broken cut belonging to X, that is, provided each of the minimal graphs of X is part of a subgraph of X whose "outermost sign" is a broken cut.

As was the case in Sections 3.22 and 3.23, we shall state a formula whose theoremhood in S5 will guarantee the existence in S5 of derived rules of inference analogous to the rules of iteration and deiteration with the above restriction. The formula is:

$EK\alpha\delta 1K\alpha\delta\alpha$, where α is any formula of S5 having each of its propositional variables in the scope of a modal operator belonging to α. (1)

All the theorems of S4 and S4.2, of course, hold in S5; the schema at (1) is one which does not hold in general in S4 and S4.2, but which will be shown to hold in S5. As was the case in Section 3.23, we shall first turn to step (8) in the deductions of Section 3.22, and now, instead of substituting 'Lp' for 'p', we shall substitute α, where α is any wff having each of its propositional variables in the scope of a modal operator belonging to α. The result is:

$$CL\alpha C\delta 1\delta\alpha \qquad (2).$$

Now, there is a pair of rules of inference, due to Prior, which hold in S5 and which--when subjoined to the CPC--

actually form a basis sufficient for S5.[1] The rules are:

R1: If $\ulcorner C\alpha\beta \urcorner$ is a theorem, then $\ulcorner CL\alpha\beta \urcorner$ is a theorem, and

R2: If $\ulcorner C\alpha\beta \urcorner$ is a theorem, then $\ulcorner C\alpha L\beta \urcorner$ is a theorem, provided each of the variables of α is part of a subformula of α beginning with a modal operator.

We have, by PC alone:

$$C\alpha\alpha \quad (3)$$

But let α be of such a form that each of its propositional variables is part of a subformula of α beginning with a modal operator; this means both that the proviso of R2 is fulfilled and that α is the same as the α of the schema at (2). Then:

(3), R2	$C\alpha L\alpha$	(4)
(2), (4), PC	$C\alpha C\delta 1\delta\alpha$	(5)
(5), PC	$CK\alpha\delta 1K\alpha\delta\alpha$	(6).

Again, the converse of (6) will be easily provable, and we may state:

$$EK\alpha\delta 1K\alpha\delta\alpha \quad (7),$$

where α, of course, is as required in (1).

But (7) is the schema we needed, and so, in the light of all that has gone before, we may state:

<u>If X is a theorem of gamma-5, then f*(X) is a theorem of S5</u>.

To show that the converse of the above holds and that the set of S5 theorems maps into the set of gamma-5 theorems by g*, it would be sufficient to show that a graphical correlate of our axiom 1'' is a theorem of gamma-5; 1'' is, of course,

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[1]Prior, Formal Logic, p. 312.

'CRRCpqCRqRp'. But actually it will not be necessary to develop a proof for this particular graph. We know that if the formula 'CMLpLp' is added to a base sufficient for S4, the result is the system S5. Another formula which will do the job, and which, in fact, is just a transposed version of 'CMLpLp' is

CNLpLNLp.

Another way of writing this formula is

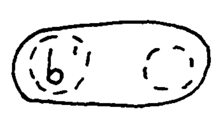
CRpNRRp.

Since gamma-5, with a more liberal restriction on its rules of iteration and deiteration than either of the other such systems we have discussed, will contain all the theorems of both of these systems, it is clear that if a graph corresponding to the last mentioned axiom can be proven in gamma-5, then graphs corresponding to all S5 theses will be derivable. The proof is as follows:

gamma-4 axiom

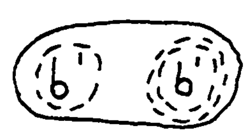
(8)

(8), R_{ins}



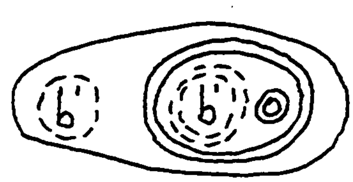
(9)

(9), R_{itr} (gamma-5 version)



(10)

(10), R_{bcl}



(11).

The graph of (11), a theorem of gamma-5, corresponds by g* or h* to 'CRpNRRp'. We are then safe in asserting that both:

If f*(X) is an S5 theorem, then X is a theorem of gamma-5, and

A is a theorem of S5 iff g*(A) is a theorem of gamma-5.

The systems S5 and gamma-5 are then translatable into each other, in our technical sense of translatable; and we may consider them equivalent in the same sense that alpha and the CPC are equivalent.

3.25 A Summary of the Broken Cut Systems Here Presented

In the preceding sections we have shown how systems equivalent to some contemporary modal systems may be formulated within the "broken cut" notation of Peirce's gamma graphs. The method of setting up these systems is quite simple. We add to the notation and rules of alpha the broken cut and the rules which permit us to change an evenly enclosed or unenclosed alpha cut into a broken cut, and an oddly enclosed broken cut into an alpha cut. We allow the rules of insertion in odd and erasure in even and of biclosure to hold just as they do in alpha, counting broken cuts as if they were alpha cuts to determine whether an enclosure is odd or even. If we add no new graphical axiom and allow the rules of iteration and deiteration to apply with no restrictions, the system obtained, gamma-MR, is equivalent to Ł-modal.

If we add as an axiom the graph  , and place certain restrictions on the rules of iteration and deiteration, we shall obtain several different systems, depending on what the restrictions are.

If we state that only graphs of form  may be iterated or reiterated through broken cuts, we get the system gamma-4, which is equivalent to the Lewis-modal S4.

If we permit also graphs of form  to be iterated or deiterated through broken cuts, the system yielded is gamma-4.2, which is equivalent to the Lewis-modal S4.2.

If we permit any graph all of whose minimal graphs are within the scope of broken cuts belonging to that graph to be so iterated or deiterated, we get gamma-5, which is equivalent to the system S5.

It is not by accident that the Lewis-modal broken cut systems are so formulable. In an article on these systems[1] I have shown that S4 and S4.2 may be formulated by the subjunction of Prior's rules R1 and R2 (which we saw in Section 3.25) to the CPC, changing the proviso of R2 to read, "Provided α is completely modalized." I distinguish between S4, S4.2, and S5 by stating for each of these systems what it is to be completely modalized. In S4 we find that a formula is completely modalized if it begins with an 'L' or is a conjunction of formulas each of which begins with an 'L'. In S4.2 we find that in addition formulas beginning with 'ML' will be completely modalized. For S5, complete modalization is as in Prior's version of the proviso.

It is interesting to note that these definitions of "complete modalization" correspond to the restrictions on iteration and deiteration for the respective systems gamma-4,

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[1]J. Jay Zeman, "Bases for S4 and S4.2 without Added Axioms," Notre Dame Journal of Formal Logic, IV (1963), 227-30.

gamma-4.2, and gamma-5. We might, then, state the restriction on these rules in another way, and specify that for all three of these systems iteration or deiteration through a broken cut is permitted only for a "completely modalized graph"--this would be a thread connecting these three gamma-systems. We would then identify each of these systems by a specification of what it is in that system for a graph to be completely modalized.

It is not at all impossible that other Lewis-modal systems might be formulated as broken cut systems--at least I know of no reason why it should be impossible. At the moment, however, we shall go no further in this direction. We will remark, however, that the formulability of these broken cut systems in really so simple a manner is another tribute to the power of the graphs and to the ingenuity of the man who first thought of them.

3.3 Cleaning Up

It is sad to note, perhaps, that from certain points of view the Existential Graphs, into which Peirce poured so much effort, are a failure. The final formulation of the gamma graphs as he envisaged it did not come off, and indeed, seems to have been doomed from the start. From this failure follows the failure of the graphs as the ultimate analytical instrument of deductive reasoning in the broadest sense. And this makes the graphs, from this point of view, just an unfinished wing in the incompleted structure that was the philosophy of C. S. Peirce.

But the graphs need not be viewed only from this point of view. As logical systems they are astoundingly successful. They seem somewhat awkward to work with, especially till you get the hang of them, but so too are many contemporary systems, when considered in their primitive notation. The graphs are, as we have seen, systems of considerable power, both in the theorems derivable within them, and in the insights they afford into logic in general.

Much remains to be done with the graphs. We have offered a sketch, in the Introduction, of how we feel the graphs fit into Peirce's philosophy as a whole; this sketch could be expanded into an interesting study of considerable length, I am sure. Much might also be said about the place of the graphs in Peirce's theory of signs. There are also intriguing little suggestions by Peirce regarding formal features of the graphs, such as the "state of information LI" which he mentions in 4.251--here we might find matter for further study and expansion, as we have studied and expanded the broken cut systems.

This paper has studied the more important parts of the graphs, those about which Peirce told us the most; it has studied them as logics. And from the point of view which this paper has adopted, we can only say that the graphs are not a failure at all, but a grand success.

APPENDIX

THE DEDUCTION THEOREM IN S4, S4.2, AND S5

The immediate purpose of our discussion of the deduction theorem in these modal systems is to show that

$$CLCpqCLCqpC\delta p\delta q$$

is a theorem-schema of S4, S4.2, and S5. But a statement of the deduction theorem for these systems is itself, I think, of considerable general interest.

Actually, there is no trick to merely stating the deduction theorem for any system. The problem is in getting an appropriate definition of "proof from hypotheses" for the system in question. Once we have such a definition, the statement and proof of the deduction theorem will ordinarily offer no problem.

The systems with which we will be concerned are considered to be formulated on a PC base. They will contain, first of all, any basis sufficient for the complete CPC, including the rules of substitution and detachment. Each of these systems will also contain the rule "RL": "If α is a theorem, so too is $L\alpha$." The additional axioms are, for S4:

1. CLCpqLCLpLq
2. CLpp.

For S4.2, axioms 1 and 2 and also:

3. CMLpLMp.

For S5, axioms 1 and 2 and also:

4. CNLpLNLp.

Since these systems are formulated on a PC base, we might suspect that a good part of the definition of "proof from hypotheses" will be exactly as for the CPC. This is the case; here we shall make use of Church's definition of "proof from hypotheses" for the CPC.[1] The clauses of the definition as he states them are quite easily extendable to our modal systems. We may thus state what will amount to most of the definition:

A finite sequence of wffs $B_1, B_2, \ldots, B_m$ is called a "proof from the hypotheses $A_1, A_2, \ldots A_n$" if for each i, $i \leq m$, either

1. B_i is one of the $A_1, A_2, \ldots, A_n$, or
2. B_i is a varient of an axiom, or
3. B_i is inferred by the rule of detachment from B_j and B_k, where $j, k < i$ and B_j is of form $\ulcorner B_k \supset B_i \urcorner$, or
4. B_i is inferred by the rule of substitution from B_j, where $j < i$, and the variable substituted for does not occur in the $A_1, A_2, \ldots, A_n$.

Note that Clause 2 above may be extended without difficulty to include the axioms of the modal systems with which we are concerned. We assume that the meaning of "varient" is understood.

But one thing is missing from the above definition, so far as S4, S4.2, and S5 are concerned. This is a consideration of the role of the rule RL in a "proof from hypotheses." It is obvious that this rule is analogous to the rule of "universal

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[1] Church, p. 87.

generalization" in predicate calculi; we might, then, expect to get a hint of how to account for this rule by an examination of the way that "universal generalization" is accounted for in statements of the deduction theorem for predicate calculi.

In the definition of "proof from hypotheses" in the predicate calculus, the following move is permitted[1] in the inference of a B_i from a B_j by universal generalization: B_i may be of form $(a)B_j$, where $j < i$ and the variable a does not occur free in any of the hypotheses $A_1, A_2, \ldots, A_n$.

The problem for us to find, for the systems S4, S4.2, and S5, an appropriate analog of the statement "The variable a does not occur free in any of the hypotheses $A_1, A_2, \ldots, A_n$."

Such an analog is available. Prior has shown that S5 is derivable,[2] and I have shown that S4 and S4.2 are derivable,[3] by subjoining to the CPC the following rules:

R1: If $\ulcorner C\alpha\beta \urcorner$ is a theorem, so too is $\ulcorner CL\alpha\beta \urcorner$.

R2: If $\ulcorner C\alpha\beta \urcorner$ is a theorem, so too is $\ulcorner C\alpha L\beta \urcorner$, provided α is completely modalized.

The definition of "completely modalized" varies among these systems, and is the factor which distinguishes them. In S4, a wff is completely modalized iff either:

1. It is a law of the system, every propositional variable

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[1]Cf. *ibid*., p. 196.

[2]Prior, *Formal Logic*, p. 312.

[3]Zeman, *Notre Dame Journal of Formal Logic*, Vol. IV (1963).

of which is in the scope of a modal operator belonging to α, or

2. It is of the form ⌜KLγKLδ . . . Lν⌝, with ⌜Lγ⌝ as a limiting case.

In S4.2 we have--in addition to the above--that α is completely modalized if:

3. It is of the form ⌜NLNLβ⌝.

In S5, a wff α is completely modalized provided every propositional variable of α is in the scope of a modal operator belonging to α.

Now, note that the complete quantification theory is formulable by subjoining to a complete CPC base the following:

RΠ1: If ⌜C$\alpha\beta$⌝ is a theorem, so too is ⌜CΠx$\alpha\beta$⌝.

RΠ2: If ⌜C$\alpha\beta$⌝ is a theorem, so too is ⌜C$\alpha\Pi$xβ⌝, provided x is not free in α.

The similarity of the above rules to R1 and R2 for 'L' is obvious. And this similarity tells us what the analog for S4, S4.2, and S5 for, "The variable a does not occur free in any of the hypotheses A_1, A_2, . . . , A_n" will be. Let us now move to a statement of the final clause in our definition of "proof from hypotheses" for S4, S4.2, and S5.

A finite sequence of wffs B_1, B_2, . . . , B_m is called a "proof from the hypotheses A_1, A_2, . . . , A_n" if for each i, i m, either one of the four previously mentioned clauses holds, or

5. B_i is infered from B_j by RL, where $j < i$ <u>and each of the hypotheses</u> A_1, A_2, . . . , A_n <u>is completely modalized</u> in the sense of the system in which we are working.

With these five clauses, then, defining "proof from hypotheses" in S4, S4.2, and S5, we shall write

$$A_1, A_2, \ldots, A_n \vdash B$$

for "there is a proof from the hypotheses $A_1, A_2, \ldots, A_n$ for the wff B."

The deduction theorem now will merely state that when

$$A_1, A_2, \ldots, A_n \vdash B,$$

then also

$$A_1, A_2, \ldots, A_{(n-1)} \vdash A_n \supset B.$$

The proof of this theorem for the first four clauses of the definition of "proof from hypotheses" will be just like Church's proof.[1]

The only extension of the proof needed is to cover our clause 5; this is easily accomplished. Let each of the $A_1, A_2, \ldots, A_n$ be completely modalized. And let B be B_i, such that if $k < i$, then

$$A_1, A_2, \ldots, A_{(n-1)} \vdash A_n \supset B_k,$$

that is, the deduction theorem holds for $A_1, A_2, \ldots, A_n \vdash B_k$. And also, B_i is inferred from B_j, $j < i$, by RL. This means that, by our definition of proof from hypotheses, and since B_i is B,

$$A_1, A_2, \ldots A_n \vdash B.$$

Now, since $j < i$, then also $j < k$, and we have

$$A_1, A_2, \ldots, A_{(n-1)} \vdash A_n \supset B_j.$$

But then, since each of the hypotheses is completely modalized, we have also, by RL and our definition of proof from hypotheses:

$$A_1, A_2, \ldots, A_{(n-1)} \vdash L(A_n \supset B_j).$$

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[1]Church, pp. 88-89.

It is easily provable as a theorem of S4, S4.2, or S5 that

$\ulcorner CLC\alpha pC\alpha Lp \urcorner$, where α is completely modalized.

But this means that we may move to

$$A_1, A_2, \ldots, A_{(n-1)} \vdash A_n \supset LB_j$$

since we originally assumed that A_n, along with the other hypotheses, is completely modalized. But B_i was infered from B_j by RL, and so is of the form 'LB_j'. And this means that the deduction theorem holds for clause 5 of our definition. Note the role in the above proof of our requirement that all of the hypotheses be completely modalized.

Now we may quickly prove that the schema

$$CLCpqCLCqpC\delta p\delta q$$

is indeed a theorem schema of S4, S4.2, and S5. With the <u>rule</u> of substitutivity of strict equivalence, the following holds for these systems:

$$LCpq, LCqp \vdash C\delta p\delta q.$$

Note that the hypotheses in this case are completely modalized in any of three systems in question. But by the deduction theorem for these systems, the schema we wished to prove is proven.

Note that we could <u>not</u> in the general case in these systems have stated:

$$Cpq, Cqp \vdash C\delta p\delta q,$$

even though--as a <u>rule</u>--the substitutivity of <u>material</u> equivalence holds in these systems. For in the general case there is no guarantee that the rule RL will not have to be applied to get the desired result, and by our definition of proof from hypotheses, its application would not be allowed in the last case, since the "hypotheses" there shown are <u>not</u> completely modalized in any of the three systems in question.

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