## NICOLE ORESME

## De proportionibus proportionum <br> AND

Ad pauca respicientes

EDITED WITH
INTRODUCTIONS, ENGLISH TRANSLATIONS, AND CRITICAL NOTES BY

EDWARD GRANT

THE UNIVERSITY OF WISCONSIN PRESS MADISON, MILWAUKEE, AND LONDON, 1966

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## Foreword

THE publication of Professor Grant's edition of two of Nicole Oresme's scientific works is the first step in a project that has been in the planning stage for some time, namely the publication in this series of all of the scientific works of this remarkable French schoolman of the fourteenth century. No scientific figure in the Middle Ages combines in his works such originality with the more traditional views of natural philosophy as does Oresme. It is evident that Oresme presents to us many faces: that of the traditional scholastic preparing questiones on standard authors, that of an original author preparing a tractatus as an original excursus into a specialized topic with the object of establishing doctrina or disciplina rather than mere exercises (exercitatus), that of a determined critic of astrology and magic, and finally that of a rather humanistic translator into French of the works of Aristotle. And so it is clear that the projected publication of Oresme's works will not only reveal to us the daring and interesting speculations of a creative mind at work but will at the same time cast significant light on some of the most important trends and problems in four-teenth-century mathematics and natural philosophy. They should show to what extent that philosophy, while still formed within the Aristotelian framework, was subversive of it.

The two works that Professor Grant has edited, translated, and analyzed in this volume are essentially original treatises. In the De proportionibus proportionum, Oresme, starting from Thomas Bradwardine's fundamental exponential relationship,

$$
\frac{F_{2}}{R_{2}}=\left(\frac{F_{1}}{R_{1}}\right)^{V_{2} / V_{1}}
$$

with $F_{\mathrm{I}} F_{2}$ forces, $R_{\mathrm{I}} R_{2}$ resistances, and $V_{1} V_{2}$ velocities, gave an extraordinary elaboration of the whole problem of relating ratios exponentially. It became essentially a treatment of fractional exponents conceived as
the "ratios of ratios." In this treatment Oresme makes a new and apparently original distinction between irrational ratios whose fractional exponents are rational, e.g., $(2 / \mathrm{I})^{1 / 2}$, and those whose exponents are themselves irrational, apparently of the form $(2 / \mathrm{I})^{1 / \sqrt{2}}$. In the course of making this distinction Oresme introduces new significations for the terms pars, partes, commensurabilis, and incommensurabilis. Thus pars is used to stand for the exponential part that one ratio is of another. For example, starting with the ratio $(2 /)^{1 / 2}$, Oresme would say, in terms of his exponential calculus, that this irrational ratio is "one half part" of the ratio $2 / \mathrm{I}$, meaning, of course, that if one took the original ratio twice and composed a ratio therefrom, $2 / \mathrm{I}$ would result. Or one would say that the ratio $2 / \mathrm{I}$ can be divided into two "parts" exponentially, each part being $(2 / \mathrm{I})^{1 / 2}$, or more succinctly in modern representation: $2 / \mathrm{I}=\left[(2 / \mathrm{I})^{1 / 2}\right]^{2}$. Furthermore, Oresme would say a ratio like $(3 / \mathrm{I})^{2 / 3}$ is "two third parts" of $3 / \mathrm{I}$, meaning that if we exponentially divided $3 / \mathrm{I}$ into $(3 / \mathrm{I})^{1 / 3}$. $(3 / 1)^{1 / 3} \cdot(3 / 1)^{1 / 3}$, then $(3 / 1)^{2 / 3}$ is two of the three "parts" by which we compose the ratio $3 / \mathrm{I}$, again representable in modern symbols as $(3 / 1)=(3 / \mathrm{I})^{2 / 3}$. $3 / \mathrm{I})^{1 / 3}$. This new signification of pars and partes also led to a new exponential treatment of commensurability, as Professor Grant has shown neatly in his lengthy summary of the main conclusions of the tract. The editor also focuses on an interesting conclusion about these exponential ratios from which the title of the book takes its name, the "ratio of ratios." Oresme claims, without any real proof to be sure, that as we take a larger and larger group of whole number ratios greater than one and relate them exponentially two at a time, the number of irrational "ratios of ratios" (i.e., irrational fractional exponents relating the pairs of whole number ratios) rises in relation to the number of rational "ratios of ratios." From such a mathematical conclusion, Oresme then jumps to a central theme whose implications reappear in a number of his works: it is probable that the ratio of any two unknown ratios, each of which expresses a celestial motion, time, or distance, will be an irrational ratio. This renders astrology, whose predictions are based on the precise determinations of conjunctions and oppositions, fallacious at the very beginning of its operations. While the idea of the possible incommensurability of celestial motions was certainly not new with Oresme, it seems that the extensive kinematic elaboration of this view as found first in the Ad pauca respicientes (presented here as the second of the two scientific treatises edited by Professor Grant) was indeed original with Oresme. It is clear that Oresme in this tract made a start toward forming a disciplina of the kinematics of circular motion, a start that he himself was later to expand in his De commensurabilitate vel
incommensurabilitate motuum celi, a text which Mr. Grant hopes later to publish in this series.

Beyond the three works whose editing was undertaken by Mr. Grant, it should also be noted that the following editions of Oresme or pseudoOresme treatises are in process: De configurationibus qualitatum et motuum (Marshall Clagett), a new edition of the Livre du ciel (Albert D. Menut), the Questiones in libros de caelo (Claudia Kren), the Questiones in librum de spera (Garrett Droppers), De latitudinibus formarum attributed to Oresme, but probably by Jacobus de Sancto Martino (Thomas Smith), De proportionibus velocitatum in motibus, attributed to Oresme, but by Symon de Castello (James F. McCue). All of these editions will include, in addition to the Latin text, English translations and critical commentary. It is planned to assign still other scientific works of Oresme for editing as the aforementioned texts near completion.

Marshall Clagett
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Princeton, New Jersey
1964

## Preface

THE significant place that Nicole Oresme holds in the history of medieval science is now, at last, properly recognized. But an exact and intimate knowledge of his thought and an evaluation of its originality must await the publication of additional treatises from the rather extensive corpus of works of this unusual and gifted fourteenth-century Frenchman. Hopefully, the two treatises published in this volume will contribute toward this worthy end. Although the general subject matter of these two works is different, they reveal to us Oresme's intense interest in a fundamental theme, that of commensurability and incommensurability. The De proportionibus proportionum shows Oresme concerned primarily with the purely mathematical aspects of these concepts, but by no means neglecting relationships involving physical magnitudes. In the $A d$ pauca respicientes-the exact title is unknown and $I$ use as a substitute the first three words of the treatise-he concentrates solely on the physical relations between points, or, in some propositions, celestial bodies moving in circular paths. For a given set of initial conditions, Oresme frequently investigates the different consequences deriving from an assumption that the velocities of the points or bodies are commensurable and then incommensurable.
Complete translations are provided for both treatises and in addition each is preceded by an introduction and followed by critical notes keyed to the order of the chapters or parts and the line numbers of the Latin texts. The De proportionibus is annotated in considerably greater detail than the $A d$ pauca respicientes, which is explained by the fact that the propositions of the latter treatise have been expounded sequentially and systematically in the Introduction, thus requiring little further elucidation. As to the De proportionibus, only the major concepts have been emphasized in the Introduction, all other materials being relegated to the Critical Notes. To find all discussion relevant to any passage in the Latin text or translation, the reader
is urged to locate in the Critical Notes the chapter and terminal, or inclusive, line numbers that embrace that particular line or passage and to use the copious cross-references that have been provided throughout.
It is now my pleasant duty to acknowledge a great indebtedness to Marshall Clagett, Professor of the History of Science and Director of the Institute for Research in the Humanities at the University of Wisconsin. As teacher, friend, and editor of this series, he has contributed greatly to whatever merit this book may possess. My first acquaintance with Nicole Oresme's De proportionibus proportionum came some years ago as a student in his seminar in the history of medieval science. His profound historical insight and critical judgment have helped shape the very structure and substance of this volume. A number of errors were averted and numerous improvements suggested as a consequence of his careful reading of the manuscript.
I must also express my grateful appreciation for the many kindnesses rendered by my friend, Professor John E. Murdoch of the Committee on the History of Science at Harvard University. Not only did he read the manuscript, pruning errors and offering valuable suggestions and criticisms, but he has on many occasions generously brought to my attention manuscripts, articles, and books which made possible a sounder final version. All errors and inadequacies of interpretation are, of course, my sole responsibility.
On the institutional side, my greatest obligation is to the Social Science Division of the National Science Foundation whose financial support enabled me to work through four consecutive summers and to acquire microfilms and other materials essential to the total research effort. My sincere thanks also to the American Council of Learned Societies for awarding me travel and expense funds during the summer of 1961. For generously contributing research funds to defray the heavy cost of typing this manuscript, I should like to express my gratitude to the Graduate School of Indiana University.

Finally, it is a pleasure to thank the following libraries for their kind permission to reproduce pages from manuscript codices in their possession: Biblioteca Capitular Colombina, Seville; Biblioteca Nazionale Marciana, Venice; Biblioteca Vaticana; Bibliothèque Nationale, Paris; Wissenschaftliche Bibliothek der Stadt Erfurt; Magdalene College, Cambridge.
E. G.

Indiana University
August I, 1963

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# Note on Texts, Translations and Abbreviations 

IN establishing the two texts presented in this volume, the primary concern has been to produce editions which are intelligible and faithfully reflect the substantive thought of their author, Nicole Oresme. Every textual decision has been arrived at with this in mind. It soon became apparent that no single manuscript for either treatise was sufficiently reliable to supply consistently readings in difficult or unclear passages. The familial, or genetic, connections between the manuscripts have not been explored or traced, since in every manuscript instances can be cited where at one or more places in the text a sequence of words is omitted which is substantially intact in the remaining manuscripts. Furthermore, the difficulties besetting such an undertaking are so formidable and the results of such dubious value that it was disregarded.
The Latin texts have been punctuated wherever it was deemed essential. Since early Latin printed editions were usually punctuated in chaotic fashion, it has been necessary to supply punctuation in numerous quotations as well as correct a great many typographical errors.
In order to indicate unambiguously textual corrections and additions, the following procedures have been adopted. Where all the manuscripts have an erroneous reading, the correction is entered into the text without resorting to brackets of any kind. The erroneous manuscript readings are then cited in the variants at the bottom of the page preceded always by the correct reading and the abbreviation corr ex. Additions to the text which are thought to have been in the original version are enclosed within angle brackets $\rangle$. The translation of such passages or words will be free of brackets. Square brackets [ ] are employed in all instances where a word or
phrase has been included that was almost certainly lacking in the pristine text. In the translation such additions will also be contained within square brackets, as will be the many editorial expansions and elaborations that have been incorporated into the translation for the sake of clarity and intelligibility.

Every effort has been made to render the translation in an accurate and readable manner. However, a reasonable fidelity to the Latin text and to medieval modes of expression has at times required that technical descriptions of varying lengths be translated somewhat literally (see, for example, De proportionibus, Chapter III, lines 25-30 and the next paragraph, for an explanation of this reference). In virtue of the usual difficulties of translation, and in order to resolve the many obscurities and perplexities lurking in these medieval scientific texts, a rather full introduction and a summary have been supplied for each text, supplemented by critical notes systematically following the chapter and line numbers of the respective texts.

The necessity for a host of references and cross-references has made it advisable to introduce suitable abbreviations for our texts. All references to any of the four chapters of the De proportionibus proportionum are by chapter and line number. Thus when the reader sees a reference to IV.350-5 2, he should turn to Chapter IV, lines $350-52$, of the text of the De proportionibus. The two parts of Chapter II are distinguished by the Arabic numerals i or 2 immediately following the Roman numeral designating the chapter number. For example, II.2.110-12 signifies lines ino-12 of the second part of the second chapter. Line references to the Ad pauca respicientes are always preceded by APr or $\mathrm{AP}_{2}$, which specify the first or second parts of the treatise respectively. For example, AP2.108-12 signifies that reference is being made to lines 108-12 of the second part of the Ad pauca respicientes.
Finally, for convenience, the few articles and books listed below, which are cited with considerable frequency, have been assigned abbreviated titles. Works listed in the Bibliography are cited in shortened form in the notes. All other works are cited in full upon first occurrence.

Crosby, Brad.

Euc.-Campanus
H. Lamar Crosby, Jr. (ed. and tr.), Thomas of Bradwardine His Tractatus de Proportionibus (Madison, Wis., 1955).
Euclidis Megarensis mathematici clarissimi Elementorum geometricorum libri XV. Cum expositione Theonis in priores XIII a Bartholo-

Grant, "Oresme: Prop."

Grant, "Oresme: Comm."
maeo Veneto Latinitate donata, Campani in omnes, et Hypsicles Alexandrini in duos postremos...(Basle, 1546).
E. Grant, "Nicole Oresme and His De proportionibus proportionum," Isis, Vol. sI (1960), 293-314.
E. Grant, "Nicole Oresme and the Commensurability or Incommensurability of the Celestial Motions," Arcbive for History of Exact Sciences, Vol. I (1961), 420-58.

## Latin Terms and Abbreviations Used in Variant Readings

| add (addidit) | has added |
| :--- | :--- |
| ante | before |
| corr ex (correxi ex) | I have corrected from |
| bab (babet) | has |
| mg hab (in margine babet) | has in the margin |
| obs (obscuravit) | has obscured (usually by ink spots) |
| om (omisit) | has omitted |
| post | after, following |
| rep (repetivit) | has repeated |
| scr et del (scripsit et delevit) | has written and deleted |
| sed | but |
| $t r$ (transposuit) | has transposed |

If only one manuscript diverges from the text for any word or series of words, only the divergent manuscript is recorded. For example:

120 habetur: patet $H$
signifies that at line 120 in that particular chapter or part MS $H$ has patet in place of the chosen textual reading babetur. The remaining manuscripts agree with the textual reading-i.e., all have babetur.
If, however, more than one manuscript has a variant for a specific textual reading, all the manuscripts with the chosen reading are listed immediately after the textual reading followed by the variants. Thus

```
60 assignarentur \(E H R V\); om \(E d\) assignentur \(C\)
signarentur \(P S\)
```

denotes that at line 60 manuscripts $E, H, R$, and $V$ have the textual reading while $E d$ omits it and $C, P, S$ have substituted other readings.
Punctuation has been held to a minimum by omitting periods after all abbreviations. When, only one manuscript varies, a colon is used to separate the textual reading from the variant. Should more than one manuscript diverge, no punctuation is employed to separate the textual reading from the variants except where an omission or transposition immediately follows the manuscripts having the textual reading, in which event a semicolon will separate them. Where there are variants for more than one word or phrase on a given line, these will be separated by slant lines.

## Biographical Sketch of Nicole Oresme ${ }^{\text {² }}$

THUS far the earliest reliable date associated with the life of Nicole Oresme ${ }^{2}$ is 1348 , in which year he apparently entered the College of Na varre at the University of Paris where his name was recorded among the theological students. ${ }^{3}$ His date of birth is conjecturally given by most
${ }^{1}$ There is at present no full and adequate biography of Oresme and much research remains to be done towards this end. Only a very brief sketch will be attempted here. The soundest biographical portrait now available is that by Albert D. Menut in the introduction to his edition of Maistre Nicole Oresme: Le Livre de Etbiques d' Aristote. Briefer, but also helpful, is the introduction in Maistre Nicole Oresme: Le Livre du ciel et du monde, Text and Commentary, edited by Albert D. Menut and Alexander J. Denomy, in Mediaeval Studies, Vol. 5 (1943), 23945.

The only work solely devoted to a biography of Oresme is that by Francis Meunier, Essai sur la vie et les ouvrages de Nicole Oresme. Meunier also compiled a list of works attributed to Oresme, including those of certain and uncertain authorship in both Latin and French. He cited all the Parisian manuscripts of which he had knowledge.
Later research by various scholars challenged numerous assertions by Meunier; in 1906 Emile Bridrey published his La Théorie de la monnaie, au xive siècle, Nicole

Oresme étude d'histoire des doctrines et des faits économiques, pronouncing Meunier's work as outdated and containing numerous statements contradicted by later research. "Nos recherches particulières nous ont permis de réunir nous-mêmes sur la vie d'Oresme un assez grand nombre de pièces originales, que nous ne pouvions songer à utiliser ici, et avec lesquelles nous pourrons peut-être donner un jour une biographie rectifiée de l'auteur'" (pp. 1-2, n. r). Unfortunately Bridrey never published this biography.
${ }_{2}$ Meunier, Essai sur Oresme, p. 3, gives the numerous divergent spellings of O resme's name: "On l'a nommé en latin Orem, Oresmus, Oresmius, Oremius; en français Oresmius, Orème, Oresmes, d'Oresme, d'Oresmieux, Orem, Orême. Son véritable prénom était Nicole et son nom s'écrivait Oresmius en latin, Oresme en français." To these may be added "Horen," which appears in the title of the Venice edition ( 1505 ), fol. 17 r, c.i. For full title entry of the edition, see below, p. I31.
${ }_{3}$ Launoy, Regii Navarrae Gymnasii Parisiensis Historia, p. 92. Menut cites as his
scholars as some time between 1320 and 1340, ${ }^{4}$ but possibly 1320-1325 is a more judicious guess. ${ }^{5}$
It is certain that Oresme came originally from Normandy, perhaps from the village of Allemagne ${ }^{6}$ in the vicinity of the city of Caen. "Practically nothing is known concerning Oresme's family. The fact that Nicole attended the royally sponsored and subsidized College of Navarre, where students were admitted only upon proof that they were too poor to pay their expenses while studying at the University of Paris, makes it seem probable that he came from one of those sturdy peasant families from which spring so many famous men in the annals of French scholarship." ${ }^{7}$

We also find, in a document of November 29, 1348, Oresme's name included in a list of students of the "Natio Normannorum" seeking to have certain unspecified examinations and reading requirements waived. 8 The only other bit of information relevant to his career as a theological student is a later reference by Oresme to "quedam alias dixi in tertio sententiarum." 9
source C. E. Du Boulay, Historia Universitatis Parisiensis, Vol. 4, 977, but the earliest date given by Du Boulay is that Oresme "factus Collegij Navarraci Archdidascalus" in 1355.-Oresme: Le Livre de Ethiques, ed. Menut, p. 11, and n. 2.
4 Oresme: Le Livre de Ethiques, ed. Menut, p. ir. E. Amman gives a date of ca. 1325 . See "Oresme," Dictionnaire de théologie catholique, Vol. II, Pt. 2, c. 1405.
${ }^{5}$ It can reasonably be assumed that O resme would have been no younger than twenty-three years of age if he became a bachelor in theology in 1348 ; hence the latest plausible birth date of 1325 . On the assumption of his birth occurring between 1320 and 1325, Oresme, who probably received his doctorate in 1356 (see below, p. 5), would have been thirty-one to thirtysix years of age when he became a doctor of theology. The statutes of Robert Curzon in 1219 required a doctor to have reached the age of thirty-five. Further, the requirements for obtaining the doctorate became more complicated and the period of bachelorhood was extended, culminating in 1366 with a formal requirement of sixteen years study for the doctorate in theology, though the faculty frequently shortened this period by various dispensations. See Rashdall, Universities of Europe in the Middle Ages,

Vol. I, 471-72. Borchert assumes that Oresme had some requirements waived (see below, p. 5, n. IO) to account for only an eight-year course in theology ( 1348 -56 ). Thus, allowing for exemptions, the conjectures given above seem plausible, even for such obviously vague evidence.
${ }^{6}$ Oresme: Le Liure de Ethiques, ed. Menut, p. I i. Féret cites passages from Huet's Les Origines de la ville de Caen (Rouen, 1706), p. 33 I , in which Huet states that the name "'Oresme" appears in the fourteenth century in the Caen region and that in Huet's own day several families possessing the name still lived in the area.-Féret, Faculté de théologie de Paris, Vol. 3, 289, n. 1.
${ }^{7}$ Oresme: Le Livre de Ethiques, ed. Menut, p. II.
${ }^{8}$ Denifle and Chatelain, Chartularium, Vol. 2, 638. The name of a 'Magistro Henrico Oresme, Baiocen. dioc." also appears on this list. In 1352 we find a Guillelmus Oresme listed as a grammaticus, and in 1353 amongst the theological students.-Launoy, Regii Navarrae Gymnasii Parisiensis Historia, p. 93. For further details, see below, p. ir, n. i.

9 Der Einfluss des Nominalismus auf die Cbristologie der Spätscholastik nach dem Traktat De communicatione Idiomatum des Nicolaus Oresme, ed. Borchert, in Beiträge zur Ge-

Biographical Sketch of Nicole Oresme
Borchert, hypothetically reconstructing Oresme's theological career on the basis of Oresme's having entered the College of Navarre in 1348, ${ }^{10}$ dates these lectures on the Sentences 1355 and assumes that Oresme received his doctorate in 1356 , the year he became grand master of his college, since the doctorate was prerequisite to the grand mastership.

Before becoming grand master, Oresme probably wrote the Latin version of his Tractatus de origine, natura, jure et mutationibus monetarum. Bridrey dates a first version at the end of 1355 and a second towards the end of 1357 or very early in $1358 .{ }^{11}$ Indeed, Meunier believed that Oresme composed all his Latin astrological and physical works while at the College of Navarre, i.e., I348-ca.1362. ${ }^{12}$ Menut and Denomy, on the other hand, hold "that the scientific writings of Oresme were accomplished in large part between 1360 and $1370 .{ }^{\prime \prime}{ }^{13}$ From a recent argument by Clagett, it now appears likely that Meunier's estimate is more nearly correct and that Oresme wrote many of his significant Latin treatises prior to 1360 . Although only one Latin treatise bears a definite date, ${ }^{14}$ Clagett has shown ${ }^{15}$ that if Oresme's French Livre de divinacions was written prior to his French translation of Ptolemy's Quadripartitum (ca. 1356-1360)-and there is good
schichte der Pbilosophie und Theologie des Mit telalters, Vol. 35, Bk. 4/5, Pt. 2, 5 .
${ }^{10}$ Borchert, taking the dates $1348-56$ to comprise Oresme's entire theological career, supposes that after five years (1353) Oresme became a "bacalarius biblicus" and after another year or two ( $1354 / 55$ ) rose to "bacalarius sententiarius." During the next year ( $1355 / 56$ ) Oresme explained the four Sentence books, delivering a "principium," or opening lecture, for each book. Upon completion of that year he became a licentiate and after costly formal preparations he became a doctor of theology in 1356. ''So hat Oresme wahrscheinlich den kürzest möglichen Aufsteig zum Magister der Theologie durchgemacht, ebenso hat er wohl nicht das Jahr Sentenzenstudium mitgemacht, ... und auch als bacalarius biblicushat er wahrscheinlich nur die kürzeste vorgesehene Frist, nämlich ein Jahr lang." -Ibid., p. 14. Oresme became grand master of Navarre on October 4, 1356.-Meunier, Essai sur Oresme, p. 8.
${ }^{11}$ Bridrey, Théorie de la monnaie, p. $s 4$. For a list of the manuscripts and editions, see pp. 23-33.
${ }^{12}$ Meunier, Essai sur Oresme, p. Io. Meunier gives Oresme's date of departure from the College of Navarre as 1361, which is incorrect (see below, p. 7, n. 25). Since it was compulsory to speak and write Latin in the College of Navarre (p. 8), Meunier assumed that Oresme would have written his Latin works while at the college. Obviously, Oresme could have written some of them after he left Navarre; and if he translated Ptolemy's Quadripartitum from Latin into French, he would then have written in French sometime between 1356 and 1360 while at Navarre. See below, P. II, n. I; and Oresme: Le Livre du ciel. eds. Menut and Denomy, in Mediaeval Studies, Vol. s, 241.
${ }_{13}$ Oresme: Le Livre du ciel, eds. Menut and Denomy, in Mediaeval Studies, Vol. s, 245.

14 The Contra divinatores boroscopios is dated in 1370.-Jourdain, "Oresme et les astrologues," Revue des questions historiques, Vol. 18, 144.
${ }_{15}$ Clagett, Science of Mechanics, p. 338, n. II.
reason for accepting this if Nicole Oresme, and not a certain G. Oresme, was the actual translator (see p. ir) -then a chain-like sequence of citations to other of his treatises can be established showing unmistakably that many of his most thoughtful Latin treatises antedate 1360. This gains some indirect support from the probability that it was an already-established scholarly reputation which attracted the attention of the royal family to Oresme and brought him into intimate contact with the future Charles $V$ not later than $1359,{ }^{16}$ and probably before.

Whether Oresme's first association with the royal family was as précepteur or instructeur to the dauphin Charles is not known, but a few writers have argued the point, interpreting the terms broadly or narrowly in accordance with their points of view. ${ }^{17}$ Royal reliance on Oresme's capabilities is evidenced in 1360 when the grand master of Navarre was sent by the dauphin ${ }^{18}$ to seek a loan from the municipal authorities of Rouen. ${ }^{19}$

In 136I Oresme, while grand master of Navarre, was appointed arch-
${ }_{16}$ Bridrey, Théorie de la monnaie, p. 449, cites a document of the Chambres des Comptes, now lost, from Abraham Tessereau, Histoire cbronologique de la Grande Cbancelerie de France (Paris, 1710 ), Vol. I, 22, which was signed by Nicole Oresme as "secretaire du roi" on November 2, 1359.
${ }^{17}$ Meunier disbelieved that Oresme was précepteur or instructeur and says: "...qu'il faut descendre jusqu'à du Haillan, c'est-àdire jusqu'en 1576, et jusqu'à La Croix du Maine, c'est-à-dire jusqu'en 1s84, pour trouver enfin Oresme appelé, chez l'un instructeur, chez l'autte précepteur de Charles V."-Essai sur Oresme, p. 24. Charles Jourdain says it is quite uncertain whether Oresme was précepteur in the proper sense of this word: "....mais il avait contribué, du moins, à lui enseigner la philosophie et la religion; il avait été 'son instructeur en ces sciences,' comme dit un historien du temps de Charles VII, dans un passage que M. Meunier n'a pas connu...."-"'Oresme et les astrologues," Revue des questions bistoriques, Vol. 18, $156-57$. Jourdain's source was a fifteenth-century manuscript (MS Paris, BN fr. 1223, fol. 116r-v).-Ibid., p. 157, nn. 1, 2. Thus he finds support for a teacher-student relationship as early as the fifteenth century, rather than in the sixteenth century as Meunier thought. Bri-
drey, Théorie de la monnaie, p. 446, seems in general agreement with Jourdain and calls Oresme a kind of "directeur d'études" for King Charles. Menut (Oresme: Le Livre de Etbiques, p. 13) believes the question unimportant, only emphasizing that once Charles "ascended the throne he consulted Oresme intimately, made good use of his services and rewarded him amply for his loyalty." However, Menut makes an unwarranted interpretation of Jourdain's quotation of the BN manuscript by saying (p. 12): "Probably it was his reputation as a mathematician that brought him to the attention of the king, John II, who may have engaged him as a tutor to the dauphin, even before he received the doctorate in theology and became, on October 4, 1356, grand master of his college. This seems to be the purport of a statement in a manuscript dating from the reign of Charles VII, cited by Charles Jourdain." Jourdain includes only five words of the manuscript, in which nothing is said about mathemat${ }_{1}{ }_{18}$
${ }^{18}$ At this time King John II was still being held for ransom in England by Edward III, the consequence of being captured at the battle of Poitiers in 1356.
${ }^{19}$ Bridrey, Théorie de la monnaie, p. 449.
deacon of Bayeux, ${ }^{20}$ probably with the support of Charles. This appointment was challenged by Symon Freron, an eminent master of theology, who took the case before the Parlement of Paris where Oresme was compelled to surrender one of the two posts, at his own discretion. Oresme appealed, but a second adverse decision was handed down on December 4, 1361, with Oresme choosing to remain grand master. On November 23, 1362,21 Oresme was appointed canon of the Cathedral of Rouen and on February 10, 1363 , was made a canon at La Sainte Chapelle and given a semiprebend. ${ }^{22}$ On March 18, 1364, Oresme was elevated to the post of dean of the Cathedral of Rouen. ${ }^{23}$ Prior to his selection as dean, Oresme journeyed to Avignon to deliver a Christmas Eve sermon in 1363 before the papal court of Urban V. ${ }^{24}$ At precisely what point Oresme left his post as grand master is unknown, but it is doubtful if he was grand master beyond the time of his election as dean of Rouen and perhaps he left after his appointment as canon. ${ }^{25}$
Whether Oresme spent much of his time in Paris during his tenure in
${ }^{20}$ Oresme: Le Livre de Ethiques, ed. Menut, p. 13, and n. 14.
${ }_{21}$ Denifle and Chatelain, Chartularium, Vol. 3, 78-79. At the time of his appointment to the canonship of Rouen, Oresme was still teaching regularly at the University of Paris. The rotulus on which Oresme's name appears is exclusively of "regent masters," that is, "masters actually engaged in teaching in the schools."-Rashdall, Universities of Europe in the Middle Ages, Vol. I, 409.
${ }_{22}$ "...an. I363, February 10, can. Paris., ubi semipraeb. obtinuit (Suppl., Urb. V, an. I, p. 3, fol. 77)."-Denifle and Chatelain, Chartularium, Vol. 2, 64I, n. 3. Féret corrects an earlier view that Oresme was treasurer of La Sainte-Chapelle by listing the names of two men holding this office during the years $1352-76$, thus leaving no period in which Oresme could have held this post.- Faculté de théologie de Paris, Vol. 3, 295, and n. 2.
${ }^{23}$ "....an. 1364 (non 1361, ut hactenus semper assertum est), Martii 88 , fit decanus eccl. Rotomag. (Supplic. Urbani V, an. 2, p. 2, fol. $49^{\text {b }}$ )."-Denifle and Chatelain, Chartularium, Vol. 2, 641, n.3. "In these frequent changes of position it is not unlikely that the royal hand of John II was
impelled by the suggestions of the dauphin. On January 3, 1364, John set sail on what was destined to be his last journey to London...This act left Charles once more in the position of regent and there can be little doubt that he availed himself of this situation to secure the appointment of Oresme as dean of the cathedral of Rouen." -Oresme: Le Livre de Etbiques, ed. Menut, pp. 13-14.
${ }_{24}$ Oresme: Le Livre de Ethiques, ed. Menut, p. 14, and n. 19; and cf. ibid, p. 31.
${ }^{25}$ Meunier held that Oresme had to yield his position as grand master upon election as dean of the church of Rouen, which Meunier erroneously believed occurred in 136 I , since "elle était en opposition formelle avec le testament de la fondatrice de la maison."-Essai sur Oresme, p. 10. May we infer from this that Oresme surrendered his grand mastership when appointed canon of Rouen in 1362 , since it would have conflicted with the rules of the College of Navarre? That Oresme left his post at Navarre upon appointment as canon of Rouen is accepted without question by Menut and Denomy. See their edition of Oresme: Le Livre du ciel, in Mediaeval Studies, Vol. $\varsigma, 242$.
the successive posts at the cathedral of Rouen ( $1364-1377$ ) is a moot point. Documents in the Cbartularium show he was in Paris in the month of November, 1364 , serving on a committee of theological masters that was drawing up a document of revocation against a bachelor of theology, Dionysius Foullechat. ${ }^{26}$ On February 26, 1371, ${ }^{27}$ he was present at the confirmation of a new chancellor of Paris (Johannes de Calore) ${ }^{28}$ and on March 17, 1372, ${ }^{29}$ was selected as sole representative of the Norman nation to sit with single representative masters of theology from the French and Picard nations to hear complaints brought by the English nation. This same committee is mentioned again on April 3, 1372, and gave its decision May 3, $1372 .{ }^{30}$ Finally, in December, $1375,{ }^{31}$ Oresme took part in an investigation of the faculty of theology to determine if any of the masters had translated from Latin into French the banned Defensor pacis of Marsilius of Padua and John of Jandun. Three official investigators were chosen, who, before proceeding, were made to give sworn answers to three specific questions. Oresme was one of three masters who put the questions to the three official investigators, after which Oresme himself was asked the same three questions. In all, thirty-two masters were questioned. ${ }^{32}$
During the period ${ }_{1364-1375}$ we see that Oresme was in Paris on at least four separate occasions, 1364, 1371, 1372, and 1375, from which Amman concludes that despite his ecclesiastical post at Rouen Oresme continued to reside at Paris and to be a "professeur de théologie en exercice." 33 Both assertions seem unwarranted and are certainly unsupported by the available evidence. It seems more probable that, except for occasional trips to
${ }^{26}$ Denifle and Chatelain, Chartularium, Vol. 3, II $5-23$. See also Le Clerc and Renan, Histoire littéraire, Vol. I, 376-77, where he is called Denis Soulechat. Foullechat was accused of holding the error of the Fraticelli on the interdiction of all property.
${ }_{27}$ Denifle and Chatelain, Chartularium, Vol. 3, 193.
${ }^{28}$ In 1348 we find two students with the name Johannes de Calore entering the College of Navarre, one as an arts student, the other as a theologian whose name appears on the same list with Oresme. For the list of entrants to Navarte, see Launoy, Regii Navarrae Gymnasii Parisiensis Historia, p. 92. On Johannes de Calore, see Denifle and Chatelain, Chartularium, Vol. 2, 637, n. 9 .

29 Denifle and Chatelain, Cbartularium, Vol. 3, 204-5.
${ }^{30}$ Ibid., pp. 205-6.
${ }^{31}$ II Ibid., pp. 223-25.
${ }^{32}$ Menut, in his edition of Oresme: Le Livre de Ethiques, p. 18, says, "A rumor, probably unfounded, was spread about that Oresme was actually the translator." There is certainly no evidence that Oresme was under more suspicion than any other of the masters who swore under oath. Indeed, he was honored with a place on the committee that swore in the official investigators. If there were any rumors, they failed to influence the officials seeking the guilty party.
${ }^{33}$ Amman, "Oresme," Dictionnaire de théologie catholique, Vol. II, Pt. 2, c. 1406.

Paris, Oresme resided at Rouen tending to his official duties during the period $1364-1369 / 70$. With the commencement of his prolonged translating activities at the request of Charles $V$, Oresme did indeed reside almost continuously at Paris, as is shown by official documents dated 1372.34 His residency at Paris seems fairly continuous and probably extended to 1380.

Though Oresme was active in the affairs of the University of Paris on the dates cited from the Cbartularium, we cannot infer that he was also teaching, which seems to be implied by Amman's "professeur de théologie en exercice." On the occasions he served the University Oresme may have been requested or summoned to appear in the capacity of a non-regent master to contribute his experience and knowledge to these special committees. After 1370 his full-time translating activity would almost preclude any other time-consuming duties such as regular teaching.

Oresme probably began work on his translation of Aristotle's Ethics in 1369, completing it in 1370. The Politics, with possible later revisions, and the Economics seem to have been completed between 1372 and 1374, and the De caelo et mundo in $1377 .{ }^{35}$ As early as 1371 Oresme appears to have been the recipient of a pension from the royal treasury as reward for his labors. ${ }^{36}$

On August 3, 1377, Oresme, with support of the king, was named bishop
${ }^{34}$ Letters, dated November 11, 1372, from Charles $V$ to Rouen show that Oresme was residing away from Rouen, probably in Paris, during the early years of translation.-Reg. cap. de Notre Dame de Rouen, Archives Seine-Inférieure, G. 2118. An even earlier letter of August 28, 1372, grants Oresme permission to enjoy the fruits of his post at Rouen while completing the translation of the Politics: "Fuit concessum domine decano, tam ad suam requestam, quam ad requestam regis, quod percipiat fructus et distributiones suas, usque ad perfectionem libri Politicorum, quod scribit pro praedicto domino rege, nonobstante quod idem dominus decanus non veniat in capitulo, aliquin nec ad eccle-siam."-Reg. cap. de 1365-1373, Archives Seine-Inférieure, G. 2115 . These references appear in Oresme: Le Livre de Etbiques, ed. Menut, p. 17, and n. 30.

One gets the impression that only for so important and special a task would permission have been granted Oresme to reside away from his official post. Though we cannot argue from silence, it is noteworthy that Oresme's name appears in no university documents in the Chartularium in the period 1364-1371, whereas during his stay in Paris for the translations he took part in some official university functions.
${ }^{35}$ For a detailed discussion of the translations, see Oresme: Le Livre de Ethiques, ed. Menut, p. 5, n. 11, and pp. 15-17. See also Oresme: Le Livre du ciel, eds. Menut and Denomy, in Mediaeval Studies, Vol. 5, 23945 passim; and for the translation of De caelo et mundo in particular, pp. 254-57.
${ }^{36}$ Oresme: Le Livre de Ethiques, ed. Menut, p. 15.
of Lisieux, ${ }^{37}$ though he seems not to have taken up residence at Lisieux until September $1380 .{ }^{38} \mathrm{He}$ died in Lisieux on July II, 1382, and was buried in the cathedral church. ${ }^{39}$

## The De proportionibus <br> proportionum

## Date of Composition

THE De proportionibus bears no date of composition, but may have been written sometime between 1351 and 1360. If Nicole, and not G. (Guillaume?), Oresme translated Ptolemy's Quadripartitum ${ }^{\text {I }}$ from Latin into
${ }^{1}$ In a prologue to the translation, the translator refers to himself as " $G$. Oresme": "Et quant a present et son commendement, par moy, G. Oresme, sera translaté a l'aide de Dieu de latin en françois le Quadriperti de Ptholomee avecques le comment de Haly afin que li tres noble science ne perisse mais soit manifeste a l'honneur de Dieuetau prouffit publique."-"LeQuadripartit Ptholomee, Edited from the Text of MS Français 1348 of the Bibliothèque Na tionale in Paris," by Jay W. Gossner, pp. 22, 56. Gossner writes (p. 23): "The disturbing note of this passage is the reference to G . Oresme as the translator, for little knowledge of him exists in our own day." Meunier reports that a Guillaume Oresme attended the College of Navarre from 1348 to 1356 (Essai sur Oresme, p. 10), and Menut, in his edition of Oresme: Le Livre de Etbiques (p. 11, n. 8), notes that a Guillaume Oresme was canon of Bayeux in 1376. In what seems a reasonable assessment, Gossner concludes (p. 24): "The lack of information about Guillaume Oresme, and
the knowledge that Nicole Oresme was entrusted by Charles $V$ with translations of Aristotle's works and that he was the king's advisor, have led some scholars to attribute the translation to Nicole. There is not as yet sufficient evidence to warrant denying its authorship to Guillaume."
We must also mention another Oresme -Henricus. In a brief report entitled "Three Notes," Isis, Vol. 48, I82, Marshall Clagett cites this brief passage from MS Paris, BN lat. 7380, fol. 83 v : "Iste liber est henrici de fontanis qui eum habuit ex dono venerabilis viri magistri henrici oresme iunioris condam nepotis excellentissimi doctoris magistri nicolai oresme olim episcopilexoniensis..."Thus Henricus Oresme, the Younger, was Nicole's nephew and may even be the same Henricus Oresme who, with Nicole, was mentioned as a member of the Norman nation in 1348 (Denifle and Chatelain, Chartularium, Vol. 2, 638); or is, perhaps, the son of that Henricus (Clagett, "Three Notes," Isis, Vol. 48, 183, n. 10). There is yet another

French, then a sound argument in favor of 1360 as a terminus ante quem is given by M. Clagett as follows:
In the course of his translation of Plato of Tivoli's Latin version of the Quadripartitum with an attendant commentary of 'Alī ibn Ridwān, Oresme mentions Charles as "hoir de France, a present gouverneur du royalme." This, according to Menut and Denomy (Mediaeval Studies, Vol. 4 [1942], 241), "must signify that he was writing during the period of King John's absence in England, between 1356-1360, when the dauphin Charles was acting as regent." Now in Oresme's French work on astrology, Livre de divinacions, he tells us in the proemium (edition of G. W. Coopland [Cambridge, Mass., 1952], p. 50), "...et supplie que on me ait pour excuse de la rude maniere de parler, car je n’ay pas aprins de (estre) acoustume de riens baillier ou escripre en françois." From which statement that "I have never learned or been used to set forth or write anything in French," one might deduce that this is his first French effort. If so, then the work precedes the translation of the Quadripartitum and thus was written before the end of the period 1356-60.... The De divinacions...also cites by title his De commensurabilitate motuum celestium which puts this work also in the fifties. Furthermore, the $D e$ commensurabilitate (BN lat. 7281, fol. 267r) cites by title the brilliant De proportionibus proportionum. ${ }^{2}$

Justification for 1351 as a terminus post quem comes from the preface of Oresme's Algorismus proportionum, in which he requests Phillipe de Vitry, bishop of Meaux (he is referred to as "Reverende Presul Meldensis Phillipe"), to correct the work and thereby guarantee its soundness against would-be critics. ${ }^{3}$ Phillipe de Vitry, a renowned scholar and friend of Pe-
occurrence of the name Henricus Oresme, the Younger, this time as the owner of a codex (Avranches, Bibl. Municipale, MS 223) containing Nicole Oresme's Le Livre de Politiques, Le Livre de Yconomique and glosses from Le Livre de Ethiques. On fol. 348 cd , we read: "Liber iste Politicorum est Henrici Oresme, junioris canonici Baiocensis." Menut, who furnishes this quote and a description of the codex in his edition of Maistre Nicole Oresme: Le Livre de Yconomique d'Aristote, in Transactions of the American Pbilosopbical Society, New Series, Vol. 47(1957), 801, c.2, mentions Leopold Delisle's suggestion that this codex belonged to Nicole and was given by him "to his nephew Henri Oresme, who was canon of Bayeux in 1385 ." The Oresme family seems to have established a tradition for
church offices in Bayeux, with Nicole holding a brief appointment as archdeacon during 1361-62, Guillaume as canon in 1376, and Henricus as canon in 1385. See also above, p. 4, n. 8.
${ }^{2}$ Clagett, Science of Mechanics, p. 338, n. II. The passage in the De commensurabilitate citing the De proportionibus by title is quoted below on p. 61, n. 81. The same note contains other explicit references to the De proportionibus.
${ }^{3}$ The text of Oresme's brief prologue follows: "Algorismum proportionum Reverende Presul Meldensis Phillipe, quem Pictagoram dicerem si fas esset credere sententie ipsius de reditu animarum, vestre excellentie, si placeat, offero corrigendum ipsum quem audacter proferam in medium. Si tanti viri auctoritate probatum et exa-
trarch, was bishop of Meaux (Melden) from January 3, 1351, to June 9, 1361. ${ }^{4}$ Since Phillipe held no ecclesiastical post at Meaux prior to 1351 , his designation by Oresme as "Meldensis" indicates that the prologue, and very likely the Algorismus itself, was written during the period of Phillipe's bishopric at Meaux, i.e., in or after 135 I but no later than 1361 .

However, the 1351 terminus post quem for the Algorismus is relevant in dating the De proportionibus only if the former were composed prior to the latter. Fortunately, reasonable evidence is available to suggest that the Algorismus is the earlier treatise. Mention of the Algorismus in Oresme's De configurationibus qualitatum 5 places the former earlier than the latter. Now in the De configurationibus Oresme makes the following important reference to irrational ratios: "And yet between such irrational ratios there is a great difference in this regard, [namely] that some are more irrational than others, as is evident from the tenth book of Euclid; and some are even unknowable and unnameable, as is clear in a comment on the fifth book of Euclid." ${ }^{6}$ Although the first distinction-some irrationals are more irra-
mine fuerit emendatum nam omne quod fuerit lima vestre correctionis politum. Et si detractor latrare potuit ubi tamen dentem laniandi figeret, non invenit."-Cited from my text of the first part of the Algorismus in "Mathematical Theory of Oresme," p. 33 1. Since this prologue was lacking in the single manuscript utilized by Maximilian Curtze for his edition of the Algorismus, it is not to be found in the Curtze edition. The reader will find the prologue in any of the following manuscripts: (1) MS Paris, Bibliothèque de l'Arsenal 522 , fol. 12Ir; (2) MS Oxford, Bodleian Library, St. John's College 188, fol. 104r; (3) MS Florence, Biblioteca Medicea Laurenziana, Ashburnham 210, fol. 172r; (4) MS Utrecht, Bibliotheek der Rijksuniversiteit te Utrecht, 725 , fol. 165 r .
${ }^{4}$ Conrad Eubel, Hierarchia Catbolica Medii Aevi (4 vols.; Regensberg, 1898 1935), Vol. $I$, 349. For a brief biograph ical sketch of Phillipe de Vitry, see A. Coville, "Philippe de Vitri Notes Biographiques," Romania, Vol. s9 (1933), $520-$ 47. Coville quotes the incipit of the prologue of the Algorismus, but was unaware of Oresme's authorship (p. 543, n. 3). The reader will observe that I have followed
the manuscript spelling of "Phillipe," whereas Coville uses "Philippe." We differ also in our spelling of his place name.
5 "Qualiter autem unaquaque proportio addatur alteri vel ab altera subtrahatur ego dicam in quodam tractatu quod voc[e]m algorismus proportionum." Quoted from Wieleitner, "Über den Funktionsbegriff und die graphische Darstellung bei Oresme," Bibliotheca Mathematica, Vol. 14, 229. Wieleitner used MS Paris, BN lat. 7371, fols. 214r-266r. See also Clagett, Science of Mechanics, p. 338, n. ri.

6 "Et adhuc inter huiusmodi proportiones irrationales est magna differentia quo ad istud secundum hoc quod alique sunt magis irrationales quam alie, ut patet decimo Euclidis; et alique etiam inscibiles et innominabiles ut patet in commento quint Euclidis." This passage appears in Part II, Ch. 17, and was sent to me by Professor Marshall Clagett who is currently editing the De configurationibus. The claim that some irrationals are more irrational than others is probably based upon Book X, Def. 4, in Euc.-Campanus, p. 243 (its counterpart in the modern editon of the Elements is Def. 3), where Euclid distinguishes between lines incommensurable in length only (these
tional than others-is not considered in the De proportionibus (see p. 13, n. 6), the second "unknowable and unnameable" kind is treated at some length along with the relevant commentary to Book V, Definition i6, of Euc.-Campanus (see I.297-308; see also below on pp. 37-38, 329-31). Had the De proportionibus already been written, Oresme would probably have cited it rather than have merely referred to the comment on the fifth book of Euclid. From this it is a plausible inference that the De proportionibus was composed after the De configurationibus, and a fortiori after the Algorismus. Furthermore, since the De proportionibus treats exponential relationships in a much more sophisticated and complex manner than the Algoris$m u s$, it seems reasonable to assume that the latter treatise was composed earlier (see below, pp. 65, 68).
In sum, if Oresme translated the Quadripartitum, we are justified in assuming that the De proportionibus was written sometime between 1351 and 1360 ; if he is not the translator we may only say that it was probably written after 135 I .

## The Significance of Thomas Bradwardine's Tractatus de proportionibus

In contrast to so many works in the history of medieval science, Nicole Oresme's De proportionibus proportionum ${ }^{7}$ falls within a class of scientific literature that originated with a particular treatise written barely thirty years before the De proportionibus itself. Any estimation and evaluation of Oresme's work must begin with a consideration of Thomas Bradwardine's Tractatus proportionum seu de proportionibus velocitatum in motibus. ${ }^{8}$ Written in 1328 , it introduced innovations into medieval physics that pro-
would be commensurable in square) and lines that are incommensurable in both length and square. Apparently, Oresme thought of all ratios between lines in the second category as more irrational than those in the first category. This interpretation plays no role in the De proportionibus where Oresme makes a quite different distinction between two types of irrational ratio. In our treatise, as will be seen, the basic distinction is between irrational ratios with rational exponents, and those with irrational exponents (see below, pp. 33-34).
${ }^{7}$ Chapters I-III (there are four in all) of the De proportionibus were summarized in my earlier article-Grant, "Oresme: Prop." Although my summary and analysis of the first three chapters will be partially based on this article, some corrections and many additions and expansions have been made. To avoid tedious and repetitious references, my article will be cited only where interpretations have been altered and errors corrected.
${ }^{8}$ Crosby, Brad.
foundly influenced not only the mathematical representation of motions arising from ratios of force to resistance, but also the larger context of mathematico-physical discussion.

The departure initiated by Bradwardine came to be designated in the fourteenth century by the expression proportio proportionum ${ }^{9}$ and in recent years has been called "Bradwardine's function." ${ }^{10}$ Before interpreting Bradwardine's function, it is essential to describe, briefly, the context from which the new departure emerged.

Medieval discussions of ratios of motion were founded upon a series of scattered remarks in Aristotle's Physics and De caelo. In these treatises we find the Stagirite relating motive forces, mobiles (or things moved), distances, and times, as well as discussing the motions of bodies through different media. ${ }^{\text {II }}$ Although in Pbysics VII, Ch. 5, Aristotle furnishes a few

9 The expression appears neither in the text nor variant readings of Crosby, Brad., although Crosby in at least two places leaves the impression that Bradwardine used it. In one place (p. 31 ) Crosby says, "So long as velocities are taken (as they are by Bradwardine) to vary according to the 'proportion of proportions'...," and then (pp. 112, 113) translates "proportiones potentiarum moventium ad potentias resistivas, ..." as "the proportion of the proportions of motive to resistive powers...." We find the expression used illustratively in a non-mathematical discussion in the Questions on the Physics ascribed to Oresme. Here Oresme considers whether every accident is the subject of another accident so that we would have an "accident of an accident" (accidens accidentis), just as we say there is a "likeness of likenesses" (similitudo similitudinum) and a "ratio of ratios" (proportio proportionum). ("Tertia difficultas est utrum unum accidens sit subiectum alterius ita quod sit accidens accidentis sicut diceremus(?) quod esse quantum habet esse quale. Et forte quod sic est et ideo dicimus quod accidentia habent proprietates et quod est similitudo similitudinum et proportio proportionum, et sic de aliis. Et forte non est inconveniens quod sit processus in infinitum, sicut dictum est."MS Seville, Biblioteca Colombina, $7-6-30$, fol. sv, c.i. I am indebted to Professor

John Murdoch of Harvard University for bringing this to my attention and for supplying the Latin quotation.) In the De proportionibus proportionum its importance and centrality is revealed in the very title and by its usage throughout the first four chapters. However, the expression was not orig inal with Oresme, since it appears in earlier works by John Dumbleton and Jean Buridan. Latin passages containing the expression proportio proportionum are quoted from both Dumbleton and Buridan by Marshall Clagett in The Science of Mechanics, pp. 44I, n. 39, and 442, n. 40 . Clagett observes: "It became conventional, at least from the time of Dumbleton and Buridan..., to express the Bradwardine formula by saying that velocity follows 'the proportion of proportions'" (p. 441, n. 39). For the meaning of the expression proportio proportionum, see below, p. 49. This important technical term will be translated in this volume as "ratio of ratios" rather than "proportion of proportions" (see below, p. 16, n. 14).

10 Maier, Vorläufer Galileis, p. 94.
${ }^{11}$ Pbysics IV.8.215a. 24-215b.1o; IV.8. 216a. II-I6; VIII. 5.249 b. 30-250a. 24. In the De caelo, see I.6.273b. 30-274a. 2; III.2.301b. 4-5, 11-13; IV.I.308a. 29-33; IV.2.309b. 12-1 5 . These passages may be consulted in The Works of Aristotle, translated into English under the editorship of W. D. Ross, Vol. 2. For an excellent sum-
special rules or examples, he nowhere provides a general formulation. In the Middle Ages a number of attempts were made to supply the general rule which Aristotle ostensibly held but failed to make explicit. It is this quest for the general rule that prompted Bradwardine, and others before him, to propose a "correct" interpretation of Aristotle's remarks (see pp. 308-9).
One of the most important general rules that emerged in medieval physics prior to the composition of Bradwardine's famous treatise is that which modern scholars represent as $F / R \propto V$, where $F$ is force or motive power, $R$ is the resistance of a medium or mobile, and $V$ is velocity or speed. ${ }^{12}$ Whenever motion is produced, it is understood that $F$ must be greater than $R .{ }^{13}$ Usually either $F$ is varied and $R$ held constant, or vice versa. Of four "erroneous" opinions on ratios ${ }^{14}$ of motion refuted by Bradwardine
mary account of Aristotelian dynamics of motion, see Clagett, Science of Mechanics, pp. 425-32.
${ }_{12}$ Although most scholars would consider this the most appropriate representation of Aristotle's scattered remarks, Stephen Toulmin asserts: "So far from formulating any mathematical function or equation relating force and velocity, Aristotle does not even employ any word for 'speed' or 'velocity,' and his word 'dynamis' can only very dubiously be rendered as 'force,'..""-"Criticism in the History of Science," Pbilosophical Review, Vol. 68 (1959), I, n.I. Whatever the merits of this claim, there is little question that force and velocity were functionally related in the Middle Ages and the above formulation was only one of the interpretations in which this was done (see below, pp. 17, 308-9).

13 Throughout the medieval period it was usually made explicit that no motion arises from a ratio of equality ( $\mathrm{F}=\mathrm{R}$ ) or one of lesser inequality ( $\mathrm{F}<\mathrm{R}$ ). Indeed, Bradwardine devotes a special proposition to demonstrate this (Theorem VIII, in Crosby, Brad., p. IIs). This principle is not, however, supported by any explicit statement from Aristotle, but is perhaps derived from Physics VII.s.2soa. I $5-20$, which is quoted below on p. 369 . In that passage Aristotle does not exclude the possibility that as the force is successively
halved, motion might arise when the force becomes equal to, or less than, the resistance (contrary to my statement in Grant, "Oresme: Prop.," pp. 294, 295). We are only entitled to say that the successive halvings of the force may reduce the force to a point where it is incapable of moving an object that it has previously moved. See Morris R. Cohen and Israel E. Drabkin, A Source Book in Greek Science (2nd ed.; Cambridge, Mass., 1958), p. 203, n. 3.

14 Note that the term proportio, which is equivalent to contemporary use of the term "ratio," will be translated as "ratio" in this volume. Furthermore, the term "proportion" will be employed, wherever possible, in accordance with present usage, to signify equality of ratios. In the Middle Ages the term proportionalitas (i.e., "proportionality") was used to express equality of ratios. Thus, in his widely used edition of Euclid's Elements, Campanus of Novara says, "Proportionality is a similitude of ratios" ("proportionalitas est similitudo proportionum"). In explaining this, he writes: "Ut dicamus quod quae est proportio $A$ ad $B$, ea est etiam $C$ ad $D$. Proportio quae est inter $A$ et $B$ similis est illi quae est inter $C$ et $D$. Haec autem similitudo quae ex istis proportionibus resultat, dicitur proportionalitas."-Euc.-Campanus p. 104.

## De proportionibus proportionum

this was the third ${ }^{15}$ and, apparently, most formidable, since Bradwardine devotes the greatest amount of space to its refutation, and it persisted as a rival theory until both passed from the historical scene.
Bradwardine musters two arguments ${ }^{16}$ against this opinion, but only the second, and most important, requires elucidation here. Those who believe that $F \mid R \propto V$ are committed to the absurd position that any given force can move any resistance whatever, and, consequently, any force is potentially of infinite capacity. ${ }^{17}$ Thus, if $F / R \propto V$, and $\mathrm{F}>R$ as required, then by continually doubling $R$ we can make $R>F$. This is shown symbolically by $(F / n R) \propto V / n$, where $n=2,4,8,16,32, \ldots$ It is evident that when $n R>F$ we still attain some velocity $V / n$ that, for all scholastics, violates the assumption that no motion can arise when a resistance exceeds its mover. Similarly, we can represent by $(F: n) \mid R \propto V / n$ the case in which $R$ is held constant and $F$ repeatedly halved so that when $R>(F: n)$ some velocity $V / n$ is still produced.
To avoid this absurdity, Bradwardine proposed his own solution, which may be anachronistically, but adequately, expressed as follows: $F_{2} / R_{2}=$ $\left(F_{\mathrm{I}} / R_{\mathrm{I}}\right)^{n}$, where $n=V_{2} / V_{\mathrm{I}}$, and $F, R$, and $V$ represent the same physical quantities as before. ${ }^{18}$ We note first that two ratios of force and resist-
${ }^{15}$ Bradwardine takes the whole of Chap ter II to refute the four erroneous opinions. See Crosby, Brad., pp. 87-111 and $32-38$; the third opinion is found on pp. 95-105.
${ }^{16}$ The first argument asserts that this false opinion is only capable of handling ratios of velocities where "either the mover or the mobile are constant. Concerning motions in which the moving forces, as well as the mobilia, are varied, it tells us almost nothing."-Crosby, Brad., p. 99. This limitation is built into Bradwardine's very formulation of the third erroneous opinion: "There follows the third erroneous theory, which claims that: (with the moving power remaining constant) the proportion of the speeds of motions varies in accordance with the proportion of resistances, and (with the resistance remaining constant) that it varies in accordance with the proportions of moving powers."-Ibid., p. 95 . Symbolically, $V_{2} / V_{\mathrm{I}}=R_{\mathrm{I}} / R_{2}$ when $F_{2}=$ $\dot{F}_{1}$; and $V_{2} / V_{1}=F_{2} / F_{1}$ when $R_{2}=R_{1}$. Bradwardine's solution, an exponential one, permitted $F$ and $R$ to vary conjointly,
since the whole ratio varies when altered exponentially.
${ }_{17}$ The second argument reads: "The theory is, on the other hand, to be refuted on grounds of falsity, for the reason that a given motive power can move a given mobile with a given degree of slowness and can also cause a motion of twice that slowness. According to this theory, therefore, it can move double the mobile. And, since it can move with four times the slowness, it can move four times the mobile, and so on ad infinitum. Therefore, any motive power would be of infinite capacity.
"A similar argument may be made from the standpoint of the mobile. For any mobile may be moved with a given degree of slowness, with twice that degree, four times, and so on without end; and, therefore, by the given mover, and by half of it, one fourth of it, and so on, without end. Any mobile could, therefore, be moved by any mover."-Crosby, Brad., p. 99.

18 The importance of this formulation is justification for presenting both the Latin
ance are related geometrically or exponentially. But the exponent ${ }^{19}$ itself is a ratio of velocities expressing an arithmetic relation between the velocities arising from the two ratios of force and resistance. Thus for Bradwardine, any geometric progression (beginning with unity) expressed as successive ratios of force and resistance can be utilized, and for any two ratios selected in the progression, the ratio of velocities is given from the corresponding terms in the arithmetic progression comprised of the natural numbers successively numbering the terms in the geometric series. ${ }^{20}$ For
text and English translation. I offer my own translation because of objections to Crosby's unwarranted introduction of the expression "proportion of the proportions" (see above, p. is, n. 9). "Proportio velocitatum in motibus sequitur proportionem potentiarum moventium ad potentias resistivas, et etiam econtrario. Vel sic sub alis verbis, eadem sententia remanente: Proportiones potentiarum moventium ad potentias resistivas, et velocitates in motibus, eodem ordine proportionales existunt, et similiter econtrario. Et hoc de geometrica proportionalitate intelligas."Crosby, Brad., p. i12. My translation follows: "A ratio of velocities in motions depends on ratios of motive powers to resistive powers, and conversely. Or, expressing the same thing in other words: ratios of motive powers to resistive powers and the velocities [which they produce] are proportional when taken in the same order, and conversely. And you should understand that [only] geometric proportionality is involved." My rendering of the term velocitas as velocity must not be construed as conferring upon that term any vectorial connotations. Throughout this volume velocity is scalar in the sense of speed.
${ }_{19}$ No special term, or terms, is used by Bradwardine to express an equivalent for the exponent. Oresme uses the expression proportio proportionum as an exclusive designation for any exponent relating two ratios(see below, p.49). In Grant, "Oresme: Prop.," p. 295, n. 12, it was erroneously asserted that Oresme used no special term to designate the exponent.
${ }^{20}$ Thus the $m$ th and $n$th ratios of force and resistance in a geometric progression yield a ratio of velocities $m / n$. Functions similar to Bradwardine's but relating different physical quantities were put forth by Sadi Carnot and Robert Malthus. Carnot, in his Réflexions sur la Puissance Motrice du Feu et sur les Machines Propres à Développer cette Puissance (Paris, 1824), enunciated a law of specific heat as follows:
"When a gas varies in volume without change of temperature, the quantities of beat absorbed or liberated by this gas are in arithmetical progression, if the increments or the decrements of volume are found to be in geometrical progression.
"When a liter of air maintained at a temperature of ten degrees is compressed, and when it is reduced to one half a liter, a certain quantity of heat is set free. This quantity will be found always the same if the volume is further reduced from a half liter to a quarter liter, from a quarter liter to an eighth, and so on.
"If, instead of compressing the air, we carry it successively to two liters, four liters, eight liters, etc., it will be necessary to supply to it always equal quantities of heat in order to maintain a constant tempera-ture."-Reflections on the Motive Power of Fire, trans. and ed. Thurston, p. 27; cf. also pp. 29, 3 I.
Malthus, in his law of population, holds that "population, when unchecked, increases in a geometrical ratio. Subsistence increases only in an arithmetical ratio. A slight acquaintance with numbers will shew the immensity of the first power in comparison of the second."-First Essay on Population, 1798 , p. 14.

Bradwardine, to double a velocity arising from some ratio $F / R$ it is necessary to square the ratio; to triple a velocity one must cube $F \mid R$; and generally to obtain $n$ times any velocity the ratio $F \mid R$ must be raised to the $n$th power, so that $V_{2}=n V_{\mathrm{I}}$ when $F_{2} / R_{2}=\left(F_{\mathrm{I}} / R_{\mathrm{I}}\right)^{n}$ (but see below, pp. 20-2 I). Conversely, to halve a velocity we take the square root of $F \mid R$; to find $1 / 3$ of a velocity arising from a given ratio $F / R$ its cube root must be taken; and generally, to find the $n$th part of any initial velocity one takes the $n$th root of $F / R$, so that $V_{2}=V_{\mathrm{I}} / n$ when $F_{2} / R_{2}=\left(F_{\mathrm{I}} / R_{\mathrm{I}}\right)^{1 / n}$.
It can now be seen why Bradwardine's function avoids and remedies the defects inherent in the repudiated theory. Given initially that $F>\mathrm{R}$, it can never happen that $R$ becomes equal to or greater than $F$. This is obvious, because to repeatedly halve a velocity arising from a given ratio $F \mid R$ is simply to take $(F / R)^{1 / n}$, where $n=2,4,8,16,32, \ldots$. In this way Bradwardine remains faithful to contemporary Aristotelian physics by still maintaining that some relation of force and resistance (where $F>R$ ) determines a velocity, but avoids the mathematical difficulties. ${ }^{21}$
${ }^{21}$ Aristotle, who actually formulated and subscribed to the third erroneous theory, was quite aware of the difficulties raised by Bradwardine and qualified his own rules of motion in a passage (Pbysics VII.5.250a. 15-20) which I have called the shiphauler example or argument (the passage is quoted and discussed below, p. 369). The fact is, however, that Bradwardine nowhere mentions or alludes to the shiphauler example. That he was acquainted with it must, it seems, be granted. He makes frequent references to Physics VII, no doubt intending Chapter 5 , which contains the only section in that book treating motion by explicit rules of proportion. We may conjecture, then, that being cognizant of the shiphauler passage he found in it no obstacles to a continuing belief or conviction that Aristotle, far from being the originator of the third erroneous opinion, was very likely adumbrating the correct theory, which Bradwardine himself was to formalize explicitly. In truth, Bradwardine could have turned the shiphaulers to his own advantage. The shiphaulers represent a specific example of the general statement: "If $E$ move $F$ a distance $C$ in a time $D$, it does not necessarily follow that $E$ can
move twice $F$ half the distance $C$ in the same time. If, then, $A$ move $B$ a distance $C$ in a time $D$, it does not follow that $E$, being half of $A$, will in the time $D$ or in any fraction of it cause $B$ to traverse a par of $C$ the ratio between which and the whole of $C$ is proportionate to that between $E$ and $A$ [the text says $A$ and $E$, but see Heath, Mathematics in Aristotle, p. 143 (whatever fraction of $A E$ may be): in fact it might well be that it will cause no motion at all." The translation is from Works of Aristotle, ed. Ross, Vol. 2, Pbysics VII.s. 250 . $10-15$; the bracketed material is mine.

We see that half of a given force, namely $A / 2=E$, may not necessarily move a given mobile, $B$, half the distance, $C / 2$, in time $D$, even though $A$ moved $B$ distance $C$ in time $D$. In Theorem VI of Ch. III in Bradwardine's Tractatus de proportionibus, we find a situation which seems illustrative of the passage just quoted from Aristotle. "If the proportion of the power of the mover to that of its mobile is less than two to one, when the power moving this mobile is doubled it will increase the speed to more than twice what it was." - Crosby, Brad., p. I13. That is, if $F_{1} / R_{1}<2 / 1$ and $F_{2}=2 F_{1}, R_{2}=R_{\mathrm{I}}$, then $F_{2} / R_{2}>\left(F_{1} / R_{\mathrm{I}}\right)^{2 / 4}$,

It is, however, important to realize that Bradwardine applied his "function" to propositions involving only the simplest cases in which $V_{2}=\mathbf{2} V_{\mathrm{I}}$ and $V_{2}=V_{\mathrm{I}} / 2$. This may be partially explicable on the grounds that Bradwardine was arguing against adherents of the third erroneous opinion, who, like Aristotle, considered instances of doubling and halving velocities. Hence Bradwardine was anxious to show that almost all instances of doubling or halving velocities by the erroneous theory would not yield, for identical data, the same results according to his "function."
where the exponent ${ }^{2} / 1=V_{2} / V_{1}$. Hence doubling the force produces more than twice the velocity. Conversely-and here is where Bradwardine could proclaim his fidelity to the shiphaulers- $F_{1}$ moves $R_{\mathrm{I}}$ with less than half the velocity with which $F_{2}$ moves $R_{2}$, despite the fact that $F_{1}=$ $1 / 2\left(F_{2}\right)$. Thus Bradwardine's function could produce the very sort of case that Aristotle said could occur. Half of a given force might not move the same mobile with half the velocity and, consequently, the mobile will not traverse half the initial distance in the same time. In refuting Theory III, Bradwardine introduces a "sense experience" (experimentum sensibile) which, in effect, illustrates Theorem VI. "We see, indeed, that if, to a single man who is moving some weight which he can scarcely manage with a very slow motion, a second man joins himself, the two together can move it much more than twice as fast."-Crosby, Brad., p. 99. Obviously the single man will move the weight with less than half the speed with which it is moved by the two men.

It appears, then, that we may have here a plausible justification for Bradwardine's conviction that Aristotle had also opposed the third erroneous theory. The latter theory could not produce the very results that Aristotle said might arise.
But how, it might be asked, could Bradwardine interpret as favorable to his theory the rules of proportion that Aristotle enunciates in the first part of Physics VII.s (i.e., 249b. 30-250a.9). Those rules (they are admirably summarized by Heath in Mathematics in Aristotle, p. 144) seem clearly reducible to $F \mid R \propto V$, which represents the
third erroneous theory. Would not those rules produce results directly at variance with his own function? Not, however, if they were construed as particular cases of that function, for then they would yield identical results. Two of the rules differentiated by Aristotle are summarized by Heath as follows:
"If $A$ moves $B$ a distance $C$ in the time $D$, then
(1) $A$ will move $1 / 2 B$ the distance $2 C$ in the time $D(250 a 1-3)$
(2) $A$ will move $1 / 2 B$ the distance $C$ in the time $1 / 2 D\left(a_{3}\right) \ldots$ "
Now if $A \mid B=F_{\mathrm{I}} / R_{\mathrm{I}}=4 / 2$ and $B / 2=$ $R_{1} / 2=R_{2}$ with $A=F_{1}=F_{2}$, then $F_{2} / R_{2}$ $=4 / \mathrm{so}$ that $F_{2} / R_{2}=\left(F_{1} / R_{1}\right)^{2 / 1}$, or $F_{2} / R_{2}$ $=2 / 1\left(F_{1} / R_{\mathrm{I}}\right)$. Thus if an initial ratio $F / R$ is made proportional to $2 / \mathrm{t}$ it is possible to make Aristotle a precursor of Bradwardine. Later scholastics who accepted Bradwardine's function realized that this was the only way in which Aristotle could be "saved." This is exactly the line followed by Oresme in IV.165-72, although Oresme, along with most Parisian scholastics, departed radically from Bradwardine and came to believe that Aristotle did actually subscribe to the false theory. Oresme also fails to mention the physical restrictions placed by Aristotle on his own rules of motion, but with far less justification than Bradwardine since Oresme actually accuses Aristotle of formulating false rules and then fails to credit Aristotle with being fully aware of the limitations to those rules. See below on pp. 368-70 for Oresme's attitude and for a further general discussion of this problem.

We see this by considering the three cases $V_{2}>2 V_{\mathrm{I}}, V_{2}=2 V_{1}$, and $V_{2}<{ }_{2} V_{1}$. If $F_{2}={ }_{2} F_{1}$ and $R_{2}=R_{1}$, the figure below reveals that the rival theories agree in one instance, case (a), and disagree in the others, (b) and (c): ${ }^{22}$

## Third Erroneous Theory

(a) $F_{2} / R_{2}={ }_{2} F_{1} / R_{1}$
so that $V_{2}={ }_{2} V_{1}$
When $F_{\mathrm{I}} / R_{\mathrm{I}}=2 / \mathrm{I}$
Bradwardine's Function
(a) $F_{2} / R_{2}=\left(F_{\mathrm{I}} / R_{\mathrm{I}}\right)^{2}$ so that $V_{2}=2 V_{1}$
(b) $F_{2} / R_{2}=2 F_{1} / R_{\text {I }}$ so that $V_{2}={ }_{2} V_{1}$
(b) $F_{2} / R_{2}<\left(F_{1} / R_{\mathrm{I}}\right)^{2}$ so that $V_{2}<{ }_{2} V_{\text {I }}$
When $F_{\mathrm{I}} / R_{\mathrm{I}}>{ }^{2 / \mathrm{I}}$

When $F_{\mathrm{I}} / R_{\mathrm{I}}<2 / \mathrm{I}$
(c) $F_{2} / R_{2}={ }_{2} F_{1} / R_{1}$ so that $V_{2}=2 V_{1}$
(c) $F_{2} / R_{2}>\left(F_{1} / R_{\mathrm{I}}\right)^{2}$ so that $V_{2}>2 V_{I}$

A similar arrangement could be made for the three cases $V_{2}>V_{1} / 2, V_{2}=$ $V_{\mathrm{I}} / 2$, and $V_{2}<V_{\mathrm{I}} / \mathbf{2} .{ }^{23}$

The mathematical basis of Bradwardine's function is the application of geometric proportionality to ratios of force and resistance producing motion. To achieve this, Bradwardine utilized two propositions from the $D e$ proportionibus of Jordanus de Nemore. ${ }^{24}$ The first of these as presented by
${ }^{22}$ In the figure, the three cases (a), (b), and (c) under Bradwardine's Function correspond respectively to Theorems-or Propositions-II, IV, and VI of Ch. III of Bradwardine's Tractatus de proportionibus (see Crosby, Brad., p. inz). These three propositions have been wholly misrepresented by Crosby because of a strange error in which he represents Bradwardine's propositions by the very formulation that Bradwardine had repudiated. The source of the difficulty is found in Crosby's representation (p. 38) of Theorem II, where he sets $F \mid R=V$, which is, of course, the third erroneous theory (see above, pp. 1617). Curiously, in Theorem I, Crosby (p. 38) quite properly says that $V=\log _{n}$ $(F \mid R)$. Here is the proof of Theorem II:
"Let $F=2 R$ and $F^{\prime}=2 F$, then
$F^{\prime} \mid R=(F \mid R)^{2}$ (Theorem I, Chapter I)
Then, if $F / R=V$,
$F^{\prime} \mid R($ or $2 F \mid R)=2 V$."

The result is correct in this instance only because it corresponds to case (a) in the figure, where the results for the Third Erroneous Theory and Bradwardine's Function are identical. But for Theorem IV, Crosby gives the following (p. 39): "If $F \mid R>2 /$, then $2 F / R<2 V$." This is obviously false for if $V=F \mid R$, as Crosby says in Theorem II, then $2 F \mid R=2 V$ and Crosby has attributed to Bradwardine case (b) under the Third Erroneous Theory rather than case (b) under Bradwardine's Function, as given in the figure. This crucial error invalidates his representation of Theorems II-VII.
${ }^{23}$ These three cases correspond, in order, to Theorems III, V, and VII of Ch. III of Bradwardine's treatise (see Crosby, Brad., p. 112).
${ }^{24}$ Bradwardine gives the title but not the author of the De proportionibus (see Crosby, Brad., pp. 76, 77). Crosby (p. 28)

Bradwardine asserts:"Given two extreme terms, and interposing an intermediate term possessing a given proportion to each, the proportion of the first to the third will be the product of the proportions of the first to the second and the second to the third." 25 That is, if a mean term $B$ is assigned between extreme terms $A$ and $C$, it follows that $A / C=A / B \cdot B / C$. In giving the second of the propositions from Jordanus, Bradwardine says: "Given two or more intermediate terms placed between two extremes, the proportion of the first to the last will be the product of the proportions of the first to the second, the second to the third, the third to the fourth, and so on, to the last term." ${ }^{26}$ Here, the previous supposition is extended to $n$ means, where $n$ is any integer. Although the propositions on ratios of motions are based exclusively on geometric proportionality, ${ }^{27}$ Bradwardine relied on Jordanus for two suppositions whichembrace geometric and non-geometric proportionality. Any examples involving continuous proportionality of a
lists this treatise as "unknown at present," but the text quoted by Bradwardine agrees with the corresponding passages in Jordanus' De proportionibus (see n. 25 , below).
${ }_{25}$ This is the second supposition of Ch . I, Part 3 (see Crosby, Brad., pp. 76-77). The Latin text from Jordanus' De proportionibus reads: 'Quocumque duobus interposito medio cuius ad utrumque aliqua proportio erit, proportio primi ad tertium composita ex proportione primi ad secundum et proportione secundi ad tertium." -MS Florence, Biblioteca Nazionale Centrale, Conv. Soppr. J.V.30, fol. 8r. In Euc.Campanus (p. ı04), Bk.V, Def. io is restricted to geometric proportionality involving one mean term where the ratio of the first to the third term is said to be equal to the ratio of the first to the second "duplicated" (duplicata), i.e., squared. This definition appears as Bk. V, Def. 9, in the modern edition of the Elements translated by Thomas L. Heath, The Thirteen Books of Euclid's Elements, Vol. 2, i14. Bradwardine actually demonstrates Def. io of Bk. V of Euc.Campanus as a theorem (see below, p. 23).
${ }^{26}$ Crosby, Brad., pp. 76, 77. The corresponding Latin text from Jordanus' $D e$ proportionibus says: "Duobus vel quotcumque mediis inter duo extrema positis, proportio primi ad extremum producitur ex


#### Abstract

prima ad secundum, secundi ad tertium, tertii ad quartum, sicque deinceps usque ad extremum proportionibus."-MS Florence, Biblioteca Nazionale Centrale, Conv. Soppr. J.V.30, fol. 8v. Crosby (Brad., p. 28) remarks that the second and third suppositions quoted above from Bradwardine's treatise "correspond rather closely to Campanus' Definitions io and in, but have a generality not possessed by Euclid's version. Campanus, himself, remarking on this lack of generality, in his comment on Definition 1 , explains Euclid's failure to generalize the axiom beyond four terms by the fact that three dimensional solids, which represent a natural limit to geometric abstraction, are denominated by no more than four terms. In his commentary, Campanus indicates what the projection to ' $n$ ' terms would yield, ..." Actually, Defs. io and in of Bk. V do not correspond "rather closely" to the suppositions in Bradwardine's treatise because the Euclidean definitions are narrower in scope, being confined exclusively to geometric proportionality, whereas the Bradwardinian suppositions apply as well to continuous proportionality of a non-geometric kind. ${ }^{27}$ Crosby, Brad., pp. 112, 113.


non-geometric kind would, however, conflict with Bradwardine's function, based as it is on geometric proportionality. Definitions io and II of Book V of Euc.-Campanus plus the commentary on the latter definition would have fitted Bradwardine's requirements perfectly. The choice of basic suppositions compelled Bradwardine to demonstrate as Theorem I, Chapter I, Part 3 :"If a proportion of greater inequality between a first and a second term is the same as that between the second and a third, the proportion of the first to the third will be exactly the square of the proportions between the first and the second, and the second and the third." ${ }^{28}$ In Theorem II, Chapter I, Part 3, he demonstrates: "If four terms are continuously proportional, the proportion of the first to the last will be the cube of the proportion between any of them to the one succeeding it. If there are five terms, it will be to the fourth power, and so forth, ad infinitum, in such a way that the denomination of the proportion is always one less than the number of terms." ${ }^{29}$ Thus Bradwardine chose to postulate as suppositions the broadest concept of composition of ratios, and then, by demonstrating two theorems, narrowed his operations to composition of ratios involving only geometric proportionality. All this might have been simplified by postulating as suppositions the Euclidean definitions mentioned above. But such a move may have held little appeal for Bradwardine who, it appears, wished to derive his fundamental theorems from the widest notion of continuous proportionality in anticipation of their specific application to ratios of motion.
Indeed, the direct application of Theorems I and II of Chapter I, Patt 3, form the paradigm case for Bradwardine in coping with ratios of motion. Thus in Theorem II, Chapter III, ${ }^{30}$ he sets $F / R=2 / 1$ and with $R$ constant asserts that ${ }_{2} F$ will move $R$ twice as quickly as $F$ moves $R$. By forming a geometric proportionality from these quantities we obtain $2 F / R=2 F / F$. $F \mid R$ so that $2 F / R=(F \mid R)^{2 / /}$ where $2 / \mathrm{I}$, the exponent, is a ratio of velocities. Then, as we have already seen (p. 21), Bradwardine goes on to consider cases where $F \mid R \leqslant 2 / \mathrm{r}$.

Bradwardine had made a momentous contribution to medieval physics by mathematizing rules of motion in a reasonably consistent manner and bequeathing to his successors a function which was to achieve the widest dissemination and popularity during the next two centuries. But, like so many pioneers, Bradwardine had raised more problems than he was aware of or could cope with, and for the next great contribution-and apparently

[^0]the last significant development of this function-we must turn to Nicole Oresme's De proportionibus proportionum.

## Summary and Analysis of Oresme's De proportionibus proportionum

In the form in which Oresme's De proportionibus has been preserved, it is safe to say that it contained at least four chapters, while the last two, the fifth and sixth, present vexing problems in their relationships to the first four. ${ }^{31}$ The initial three chapters (see p. 14, n. 7) furnish both a mathematical foundation for Bradwardine's function and an extension of its concepts into areas of mathematics not even hinted at by Bradwardine. Chapter IV is, for the most part, a straightforward application of previous mathematical propositions and concepts to local motion with some attention to celestial motion in the closing section of the chapter, heralding, so it seems, what was to be a more complete discussion of circular motion and celestial motion in the final chapter or chapters.

By the time Oresme wrote his treatise, Bradwardine's ideas were so well known and influential that it was possible to begin in medias res. Where Bradwardine had elaborately refuted four "erroneous" theories on ratios of velocities, Oresme considers only the principal opposition theory ${ }^{32}$ and mentions only one other without discussion (I.2-7; see pp. 308-9). He also accepts many definitions and explanations from other authors, which were discussed in considerable detail by Bradwardine (see p. 3 10). For Oresme the study of "ratios of ratios" (see p. 49) was valuable not only as a means for more consistently representing ratios of motion, but was a potent tool for understanding the difficulties and secrets of philosophy on a cosmic scale (I.14-17). That this assertion was not mere rhetoric is borne out by the widespread application of the mathematical conclusions derived in the first three chapters to the physical problems in the latter part of the treatise.
The first three chapters constitute a fairly well integrated unit and will be summarized and evaluated without regard to sequence of propositions. The first chapter sets out many operational procedures, definitions, pos-

[^1]tulates, suppositions, and distinctions of considerable importance, while the second and third chapters present a series of propositions.
The content and approach of Oresme's De proportionibus differs markedly from Bradwardine's treatise because of the special interpretation given to the term pars, which in turn determined not only the way he was to use that term, but also the terms commensurabilis and multiplex. These terms are used almost exclusively within the context of geometric proportionality and applied almost exclusively to ratios of greater inequality, i.e., $A / B$ where $A>B .{ }^{33}$ The term pars is taken in two ways-properly and improperly (I.227-32). A "proper" part is an aliquot or "multiplicative" part to which the whole is multiple, as in $\mathrm{I} / q$ where $q$ is any integer. The term partes is simply an aliquot part taken more than once as in $p / q$, where $p$ and $q$ are integers prime to each other and $q>p>1 .{ }^{34}$ An "improper" part is non-aliquot, or "aggregative," and however many times it is taken will not precisely constitute the whole. ${ }^{35}$ Oresme says he will use the term pars in its "proper" signification.

Thus far we seem to have a mere repetition of customary distinctions and definitions, but it soon becomes obvious that such is not the case, and that Oresme is using the term pars in a manner lying outside the main Euclidean tradition. For Oresme pars signifies any unit ratio which is the common measure, or base, of all greater ratios related to it in the same
${ }^{33}$ Apart from an extended discussion showing how ratios of lesser inequalityi.e., $B \mid A$, where $A>B$-correspond reciprocally to ratios of greater inequality (I. 90-207), Oresme expressly eliminates further discussion of ratios of lesser inequality (I. 207-10). One obvious reason for omitting ratios of lesser inequality was simply that ratios of motions were confined exclusively to ratios of greater inequality. But even more important than this was the fact that manipulating ratios of lesser inequality posed many perplexing problems to the medieval mathematician and most authors would have happily forsaken them. See below, pp. 321-22.
${ }_{34}$ In his commentary on Euclid V, Def. I, Campanus says that "sometimes a part is taken properly," and proceeds to give essentially the same account as Oresme, remarking that such a part is called "multiplicative." "Pars, quandoque sumitur pro-
prie et haec est quae aliquoties sumpta, suum totum praecise constituit sine diminutione vel augmento. Et dicitur suum totum numerare per illum numerum secundum quem sumitur ad ipsius totius constitutionem talem autem partem, quam multiplicativam dicimus, hic diffinit." Euc.-Campanus, p. 103. Campanus does not consider the term partes.
${ }_{35}$ Continuing his comment on Euclid V, Def. I (see n. 34, above), Campanus calls the equivalent of Oresme's "improper" part, a part taken "commonly" (communiter), which is called "aggregative." "Quandoque sumitur [i.e., pars] communiter, et haec est quelibet quantitas minor, que quotienscunque sumpta, suo toto minus aut maius constituit, quam aggregativam dicimus eo quod cum alia quantitate diversa totum suum constituat, per se autem quotienscunque sumpta fuerit, non producat." The brackets are mine.
geometric series. It will be convenient to summarize the way Oresme uses the term pars, and the related terms multiplex and commensurabilis:
If $A$ and $B$ are two ratios and $m$ and $n$ are integers, then if $A>B$
(1) ratio $B$ is a part of ratio $A$ when $B=(A)^{1 / n}$
and (2) $A$ is commensurable and multiple to $B$, since $A=B^{n}$;
(3) $B$ is parts of $A$ when $B=(A)^{m / n}$ where $n>m>\mathrm{I}$ and $m$ and $n$ are in their lowest terms
so that (4) $A$ is commensurable to $B$, since they have a common measure in the unit ratio $(A)^{\mathrm{I} / n}$. However, $A$ is not multiple to $B$ since $A \neq(B)^{n} .{ }^{36}$
Examples are plentiful to show that Oresme is plainly employing the term pars in an exponential, and not arithmetic, sense, applying it to both rational and irrational ratios. This can be seen in I.38I-4I 3, where Oresme imagines seven possible ways of dividing a rational ratio. For the fifth way (I.400-404), he says that any rational ratio can be divided into unequal irrational ratios each of which is a part or parts of the whole. He divides $4 / \mathrm{x}$ into $(4 / \mathrm{I})^{1 /+}$ (pars) and (4/I) ${ }^{3 / 4}$ (partes). A key proposition is Proposition II of Chapter II, where Oresme shows: "If there should be no mean proportional number or numbers between the prime numbers of some rational ratio, such a ratio cannot be divided into several equal rational ratios and, consequently, no rational ratio is an aliquot part of it (II.r.53-56)." Without a geometric mean between its extreme terms, a given rational ratio $A$ cannot be divided into two equal rational ratios $B$ and $C$ where $B=C=(A)^{1 / n}$ and $n=\mathbf{2}$. Indeed $A$ could not be divided into any number of equal rational ratios. Hence no rational ratio such as $B$ or $C$ is an "aliquot part" (i.e., exponential part) of $A$.

In Chapter II, Proposition III, Oresme says: "If any quantity were divided into two unequal parts and one of them is a part or parts of that quantity, those two parts are related as two numbers in their least terms" (II.r. $87-88$ ). To facilitate the understanding of this proposition, Oresme enunciates three suppositions. The first (II.I.99-104) asserts that every quantity which is a part, or parts, of another may be assigned two numbers, a numerator and denominator, which are prime to each other. Now this ratio of numbers must be interpreted in an exponential sense. For example, if we have some quantity $A$ and take three of its five equal parts, then $(A)^{3 / 5}$ is what Oresme intends, where 3 is the numerator and 5 the denom-
${ }^{36}$ In II.r. $289-97$, Oresme distinguishes two senses of the term multiplex. See below, p. 29 .
inator of the exponent. The second supposition (II.r.ios-9) says that if we have some quantity $A$ and take away some part or parts of it, the remainder will also be a part or parts of the original quantity. Thus, if we take away $(A)^{3 / 5}$ we have left $(A)^{2 / 5}$, since $A=(A)^{3 / 5} \cdot(A)^{2 / 5}$. The third supposition (II.I.IIo-13) holds that any quantity divided in two, where one is a part or parts of the whole, will have the two parts related as their numerators, just as $(A)^{2 / 5}$ and $(A)^{3 / 5}$ are related as a ratio of numbers $2 / 3$, i.e., as the numerators, or conversely as $3 / 2$. Obviously, Oresme does not mean that $(A)^{2 / 5} \cdot(A)^{3 / 5}=2 / 3$, or that $\left(A^{2 / 5}\right) /\left(A^{3 / 5}\right)=2 / 3$, but rather that $(A)^{2 / 5}$ is related to $(A)^{3 / 5}$ as two exponential parts to three exponential parts, where each part is $(A)^{1 / 5}$; or simply that $\left(A^{2 / 5}\right)=\left[(A)^{3 / 5}\right]^{2 / 3}$, and conversely that $(A)^{3 / 5}=\left[(A)^{2 / 5}\right]^{3 / 2}$. Although Oresme has spoken only of "quantities" in these three suppositions, one is justified in substituting "ratios" for quantities by the fifth supposition in I.26I-62, where it is stated that any ratio is divisible just like a continuous quantity (see also p. 339).

These suppositions reveal that Oresme is here dealing with purely numerical relations between the parts, or unit ratios of the quantities, and explains his citation of Euclid VII, the first of the arithmetic books. That is, he is relating the numerical exponents themselves where it is understood that each exponent represents one or more unit ratios. But when he refers to Euclid V and X he appears to be thinking only of relations between ratios considered as quantities or magnitudes in the widest sense, embracing both rational and irrational quantities or ratios. Paradoxically, while Oresme seems to preserve a traditional distinction between Books V and VII, he was also helping break down the artificial barriers between number and magnitude which plagued mathematics ever since Greek antiquity. This paradoxical approach was necessary since two ratios such as $(A)^{2 / 5}$ and $(A)^{3 / 5}$ could be irrational while their exponential relations are represented by a ratio of numbers or, as Oresme says, "The ratio of these ratios will be as the ratio of those numbers. You can discover the ratio of the numbers by arithmetic" (II.r.430-32). Hence in every such case, Books V or X, on the one hand, and one of the arithmetic books (VII, VIII, IX), on the other, would be simultaneously involved (see pp. 336-39). ${ }^{37}$
${ }_{37}$ The modest efforts of Oresme in cially from Euclid, was too narrow. This bringing number and magnitude together number concept included natural numbers were far transcended by Simon Stevin who, as well as rational and certain irrational according to Dirk Struik, was "quite con vinced that the traditional number concept, as it had come down from the Greek through the early Renaissance and espe-
ones, the latter usually conceived as
icals. Stevin now draws the conclusion icals. Stevin now draws the conclusion
that number is a continuous quantity 'as continuous water corresponds to a contin-

In Chapter II, Propositions V-VII, Oresme provides the criteria for commensurability and multiplicity. In Proposition V we are told: "If there is no mean proportional number or numbers between the prime numbers of some ratio, that ratio will be incommensurable to any smaller rational, and to any greater rational ratio that is not multiple to it" (II.I.226-29). Thus if ratio $A$ has no mean proportional term between its prime numbers, we do not have at least three terms in geometric series and no smaller rational ratio of greater inequality, say $B$, formed by any two successive terms can be part of $A$ (by Chapter II, Proposition II), i.e., $B \neq(A)^{1 / n}$ or parts of $A$, i.e. $B \neq(A)^{m / n}$, where $n>m>\mathrm{I}$ and $m$ and $n$ are integers. Under these conditions $A$ and $B$ are incommensurable.

It is of importance to realize that in this proposition Oresme stresses the arithmetic character of the exponents rather than the geometric character of the relationship between the ratios. This is seen when (II.r.231-34) he introduces Euclid X.s, to say that ratios $A$ and $B$ are related as two numbers-i.e., by an exponent-and then moves directly to Euclid VII. 4, asserting that the lesser number representing ratio $B$, would have to be a part or parts of the greater number, representing $A . .^{38}$
The second part of Chapter II, Proposition V, shows that if $C>A$ these two ratios will be incommensurable unless $C=(A)^{n}$ thereby necessitating that $A$ be a unit ratio with respect to $C$, i.e., $A=(C)^{1 / n}$.

Thus far we know only that if a ratio $A$ has no mean proportional terms between its extremes, it follows, necessarily, that $A$ cannot be commensurable to a smaller ratio $B$. In Proposition VI, Oresme considers the exponential relationship between $A$ and $B$ when $A$ does have mean proportionals between its extreme terms, and the two ratios are commensurable. If $A$ is multiple to $B$, i.e., if $A=(B)^{n}$, "there will be between the extremes of the greater ratio one less mean proportional than [the number of times] the greater ratio contains the lesser" (II.r.258-60). Thus if

[^2]$A=(B)^{n}$, there will be $(n-1)$ mean proportionals and $A$ contains $B n$ times. In Proposition VII, Oresme not only introduces another factor in his concept of commensurability, but also distinguishes between two senses of the term multiplex (II.1.289-97). A ratio is said to be "absolutely multiple" (multiplex absolute) when it is of the form $n / \mathrm{I}$, where $n$ is any integer. This is the customary use of the term. However, when we wish to compare or relate ratios exponentially, the term "comparatively multiple" (multiplex comparatione seu relatione) is introduced for all relationships of the form $A=(B)^{n}$, where $A$ and $B$ may be rational or irrational ratios, $n>\mathrm{I}$ and $n$ is an integer. ${ }^{39}$

For the remainder of Proposition VII, Oresme shows that ratios can be commensurable exponentially where the greater is not "comparatively multiple" to the lesser. Thus if $A>B$ and $B$ is parts of $A$, where $B=$ $(A)^{m i n}$ with $n>m>\mathrm{I}$ and $m$ and $n$ are integers in their least terms, he shows that $A$ and $B$ are commensurable but $A$ is not multiple to $B$. Ratios $A$ and $B$ are commensurable because they have a common measure in $(A)^{1 / n}$, which is their unit ratio or common base. The concept of unit ratio only hinted at earlier in the treatise in connection with the concept of part, is for the first time expressed in the seventh proposition when Oresme says: "If a greater ratio is commensurable to a lesser ratio but not multiple to it, it is necessary that the prime numbers of one unite in the means with the prime numbers of the other" (II.I.298-300).

The reader is urged to examine the enunciations of Propositions VII and VIII ${ }^{40}$ of Chapter II (II.r.283-88 and 327-30 respectively) to see the complicated and prolix mode of expression forced upon Oresme by lack of an adequate terminology or symbolism.

Having examined the ways in which Oresme uses the terms pars, partes, commensurabilis, and multiplex, we can plainly see that he has departed not only from Bradwardine but from Euclid as well, though whenever he cites Euclid in connection with the terms under consideration he does so
$\begin{array}{ll}39 \text { Although Oresme first distinguishes } & \text { both have mean proportionals which form } \\ \text { a }\end{array}$ two senses of the term multiplex in Prop. VII, and even says that he has previously utilized the absolute sense of the term (II.r. ${ }^{296-97)}$ and henceforth will use the comparative sense, instances of the comparative, or exponential, sense are found in Props. V and VI (II.r. 228, 257, 262-63).
${ }_{40}$ Prop. VIII, the converse of VI and VII, says that if $A$ has mean proportionals which form ratio $B$ (i.e., $\left.A=(B)^{n}\right)$; or
a common unit ratio (i.e., $A=(B)^{m / n}$,
where $(A)^{l / n}$ is the unit ratio), then $A$ and $B$ will be commensurable. See below, pp. 344-45. Prop. IX of Ch. II ("' How ] to find whether two given ratios are commensura-ble"-II. $1.360-61$ ) presents a final summation of commensurability criteria by citing specific examples based on Props. V-VIII.
without qualification or comment, thereby implying customary usage.41
By citing the key definitions and propositions from Euclid and seeing how Oresme utilized them, we shall better understand Oresme's approach. The definition of commensurability given in Euclid X, Definition I, reads, "Quantities are said to be commensurable which have a common quantity numbering them." ${ }^{42}$ When Oresme employs this definition we must understand him to substitute implicitly the phrase "ratios of quantities" for "quantities" (or "ratio of quantities" for "quantity"). It would then read, "Ratios of quantities [not simply 'quantities'] are said to be commensurable which have a common ratio of quantities [instead of 'quantity'] numbering them." This same substitution must also be made implicitly in Euclid X. 5, which reads, "Any two commensurable quantities have to one another the ratio which a number has to a number." ${ }^{43}$ Again substituting the phrase "ratio of quantities" for the term "quantities" this would read, "Any two commensurable ratios of quantities have to one another the ratio which a number has to a number." ${ }^{44}$ Indeed, this difference can be illustrated when both versions of Euclid X. 5 are symbolized as follows:

Euclid: Given two commensurable quantities, $A$ and $B$,
then $A / B=m / n$, where $m$ and $n$ are integers.
Oresme: Given two commensurable ratios of quantities, $C / D$ and $E / F$, then $C / D=(E / F)^{m / n}$, where $m$ and $n$ are integers.
Oresme has turned X.s into a "ratio of ratios" (proportio proportionum). Euclid X.6, the converse of X.5, can be treated similarly. The term commensurabilis, as used in the De proportionibus proportionum, applies only to ratios or terms in a geometric series so that any given ratio, rational or irrational, is commensurable to all ratios in that same series, but incommensurable to all those not in the same series. This concept of commensurability leads Oresme to seemingly paradoxical expressions such as "an irrational ratio which is commensurable to a rational ratio." 45

Consistently applying the crucial terms described above, Oresme distin-
${ }^{41}$ It cannot be determined to what extent, if at all, Oresme was aware that he was using Euclid in a "non-Euclidean" way. He also cites, without comment, Jordanus de Nemore's De numeris datis in ways completely foreign to anything intended by Jordanus. See below, pp. 366-67.
42 "Quantitates quibus fuerit una quantitas communis eas numerans, dicentur communicantes."-Euc.-Campanus, p. 243.

43 "Omnium duarum quantitatum communicantium est proportio tanquam numeri ad numerum."-Ibid., p. 247 .
${ }^{44}$ The analogy between the treatment of a ratio of quantities and a ratio of ratios was beautifully summarized by George Lokert in the sixteenth century. See p. 70, n. 9I, below, for the text of this passage.
${ }_{45}$ For example, see I.28r-85.

De proportionibus proportionum
guishes three types of ratios (III.336-41): (1) rational ratios; (2) irrational ratios which have mediate denominations and are therefore commensurable to rational ratios; and (3) irrational ratios which have no denominations, and are consequently incommensurable to at least one and possibly more, and even, perhaps, incommensurable to all rational ratios. Since this classification seems original with Oresme and central to the treatise we must examine what he understood by each one of these categories.
(1) Rational ratios are always immediately denominated by some number or numbers. If, for example, we have a ratio of commensurable quantities, $A / B$, then $A / B=n$ where $n$ is some integer or ratio of integers (I.278-80). ${ }^{46}$
(2) Irrational ratios which, though incapable of immediate denomination by any number or numbers, can be mediately denominated by some number. This is possible in all cases where the irrational ratio is a part or parts of some rational ratio, and is equivalent to an assertion of commensurability between the irrational and rational ratios ( $\mathrm{I} .28 \mathrm{I}-85$ ).

Given an irrational ratio $(A / B)^{p l q}$, where $p$ and $q$ are integers in their least terms with $p<q$ and $(A / B)$ is a rational ratio, what can Oresme mean in $\mathrm{I} .28 \mathrm{I}-8$; by saying: "An irrational ratio is said to be mediately denominated by some number when it is an aliquot part or parts of some rational ratio, or when it is commensurable to some rational ratio, which is the same thing..."? The meager and vague remarks made by Oresme in this connection are insufficient to provide a clear-cut answer, but the following interpretation seems to best fit the available evidence.

The rational ratio $A / B$ immediately denominates the entire irrational ratio $(A / B)^{p / q}$ since Oresme says: "It is understood that every irrational ratio whose denomination is known is denominated by a rational ratio. Therefore, it is either denominated by a greater or lesser rational ratio" (IV.42325). Thus when $(A / B)^{p l q}<(A / B)$, Oresme would say that the irrational ratio $(A / B)^{p i q}$ is immediately denominated by a greater rational ratio $A / B$; when, however, $(A / B)^{p / q}>(A / B)$, the irrational ratio is denominated by a smaller rational ratio. An instance of the former is $(2 / \mathrm{I})^{1 / 2}$ and of the latter $(2 / \mathrm{I})^{3 / 2}$.

But what is the mediate numerical denomination of this irrational ratio? It would seem that the best candidate is the exponent itself, namely $p / q$. After describing what he means by the denomination of an irrational ratio by a greater rational ratio (quoted in the preceding paragraph), Oresme goes on to say: "If it is denominated by a greater rational then that irra-

[^3]tional is said to be a part of that rational ratio, as a second, a third, or fourth part, etc.; or it is parts of it, as two-thirds, or three-fourths, etc. And one number is the numerator, the other the denominator of these parts or this part" (IV.427-30). Thus we might have $(A / B)^{1 / 2}$, or $(A / B)^{1 / 3}$, or $(A / B)^{2 / 3}$, etc., where the mediate denominations consist of the numerators and denominators which are the indicators of what part or parts the irrational is of the rational. A similar situation obtains when the irrational is denominated by a smaller rational ratio, in which event "The irrational will contain the rational by which it is denominated one or more times and some part or parts of it; and of this part or parts one number will be the denominator, and the other the numerator" (IV.454-57). Here again we may conjecture that Oresme would understand as mediate numerical denominations the numbers representing the numerators and denominators.
In Chapter III, Proposition VII, Oresme indicates again that the numbers of the exponent are mediate denominations of the irrational ratios. He shows (III. 187-244) that all the terms in a series such as $(3 / 2)^{p / q}$, where $p / q=2 / 1,4 / 1,6 / 1,8 / 1,10 / 1$, etc., are square numbers. For example, $(3 / 2)^{2}$ is $9 / 4 ;(3 / 2)^{4}$ is $81 / 16$, etc., where the square root of each term is a ratio of integers. But if both terms of each ratio are greater than one but not square numbers and, in addition, are prime to each other, such ratios can never be the square, or fourth power, or sixth power, etc., of any rational ratio; and, conversely, no rational ratio can be the square root, or fourth root, etc., of such a ratio. For example, $(27 / 8)$ which is $(3 / 2)^{3}$ has cube numbers but no square numbers, so that $27 / 8$ is not the square of any rational ratio and, conversely, no rational ratio is its square root, or fourth root, etc. Indeed, says Oresme, "Any such part of it denominated by an even number will be an irrational ratio" (III.243-44). Thus, in my example $(27 / 8)^{q / p}$, where $q / p=1 / 2,1 / 4,1 / 6,1 / 8,1 / 10$, etc., constitutes a series of irrational ratios. The importance of all this is that with respect to the exponent $q \mid p$, Oresme speaks of denominating $p$ by an even number, thereby indicating that the mediate numerical denomination is the ratio of numbers denominating the exponential part, $q / p$, which signifies the part that the irrational ratio is of the rational ratio. 47
In this interpretation of mediate numerical denomination of irrational ratios the rational ratio $(A / B)$ is the base which immediately denominates
${ }^{47}$ The three suppositions in Ch. II, numerical denomination. See above, pp. Prop. III (II.I.98-113) seem also to point to an exponential interpretation of mediate
the irrational ratio, while the exponent enables one to distinguish between all irrational ratios with the same base..$^{48}$

In all cases where the exponent $p / q$ is itself irrational, $(A / B)$ and $p / q$ cannot be the immediate and mediate denominations, respectively, of the irrational ratio $(A / B)^{p l q}$, for Oresme says: "The denominations of some [irrationals] are not knowable because they have no denominations at all, since every denomination, whether mediate or immediate, is denominated by some number" (I.329-32).
(3) Irrational ratios which have no numerical denomination. Such a ratio is not an aliquot part or parts of the rational ratio which is supposed to
${ }_{48}$ Further evidence for this interpretation is found in Oresme's Algorismus proportionum, where he says: "Omnis proportio irrationalis, de qua nunc est intentio, denominatur a proportione rationali taliter quod dicitur pars eius aut partes sicut dicendo medietas duple, aut tertia pars quadruple, vel due tertie quadruple. Unde patet quod in denominatione talis proportionis sunt tria, videlicet numerator, et denominator, et proportio rationalis a qua ipsa denominatur, scilicet cuius illa irrationalis dicitur pars vel partes, sicut cum dicitur una medietas duple unitas est numerator vel in loco numeratoris, duo est denominator, et proportio dupla est illa a qua ipsa denominatur." This passage is from my edition of the first part of the Algorismus appearing in "Mathematical Theory of Oresme," pp. 331-32.

Thus Oresme distinguishes three elements in denominating an irrational ratio. The rational ratio, or base, denominates the irrational ratio, while the exponent, consisting of numerator and denominator, determines what part or parts the irrational ratio is of the rational. In the Algorismus Oresme does not distinguish between immediate and mediate numerical denominations, nor does he mention irrational ratios that have no denomination whatever
Previously, I had offered another interpretation (Grant, "Oresme: Prop.," p. 301), maintaining that if $(A \mid B)^{p / q}$ is irrational, then $(A \mid B)$, which is rational, immediately denominates the entire irrational ratio $(A \mid B)^{p / q}$. But then, mistakenly, I in-
sisted that since every rational ratio is, in turn, immediately denominated by some number $n$, it followed that $A \mid B=n$ and that " $n$ " may be said to mediately denominate the given irrational proportion by way of the rational proportion $A / B$. That is, $A / B$ mediates between the irrational proportion, $(A / B)^{p / q}$, on the one hand, and the number $n$, on the other. It is evident from all this that every irrational proportion of the form $(A / B)^{p / q}$ is immediately denominated by the same rational proportion, namely $A \mid B$, and therefore by the same mediate number. In such cases, the only way to distinguish one irrational proportion from another is by specifying the proportion of numbers, or the exponent, $p / q$, which denotes what part or parts the irrational proportion is of the rational." (Note that in my article the term proportio was rendered as "proportion" rather than "ratio.") In truth, this interpretation may be tenable for Bradwardine but not, it seems, for Oresme. Bradwardine says that irrational ratios "are not immediately but only mediately denominated by a given number, for they are immediately denominated by a given proportion, which is, in turn, immediately denominated by a num-ber."-Crosby, Brad., p. 67. According to my earlier interpretation, if $A / B$ immediately denominates the irrational ratio, then $A / B$ is "in turn immediately denominated by a number" $n$, so that $A \mid B=n$. Although Bradwardine does not indicate how one is to distinguish different irrational ratios with the same base $A \mid B$, he may
denominate it immediately. Thus if $A / B$ is a double ratio, namely $2 / 1$, then an irrational ratio of this type cannot be expressed in the form $(2 / \mathrm{I}) p l q$, where $p$ and $q$ are integers. In other words, $p / q$ is itself irrational and, consequently, not a number or ratio of numbers. Hence $p / q$, the exponent, cannot mediately denominate the irrational ratio. Oresme has, therefore, distinguished two types of irrational ratios, those where the exponent is rational and those where it is irrational. The former kind can be mediately denominated by a number, the latter is incapable of such denomination.
The crucial passage which underlies this interpretation is found in I.286303. Here, Oresme arrives at the concept of an irrational exponent by
have understood that the exponent served this function without considering it as a mediate denomination.
Crosby, in his explanation of mediate numerical denomination, opts for the exponential interpretation, but offers a peculiar literal account of the passage quoted above from Bradwardine. Centering his discussion around Bradwardine's example, Crosby says (p. 20): "Bradwardine uses the example of 'half a double proportion' (medietas duplae proportionis) which, he says, is the proportion of the diagonal of a square to its side. This is not expressible as a single, simple proportion of integers but may be expressed by two such immediately denominated integral proportions, the one being denominated by the other-i.e., $(2 / 5)^{1 / 2}$." By "two such immediately denominated integral proportions," Crosby must mean ( $2 / \mathrm{I}$ ) and the exponent $\mathrm{I} / 2$, rather than the entire irrational ratio, which cannot be immediately denominated by two integral ratios-or proportions, as Crosby calls them. But if we have $2 / 1$ and $1 / 2$, with the "one being denominated by the other," then $1 / 2$ must denominate $2 / 1$ and $1 / 2=2 / 1$. Thus Bradwardine's peculiar description involves two successive immediate denominations, which led Crosby to say, in effect, that $1 / 2$ denominates $2 / 1$, where the exponent, $1 / 2$, is the mediate numerical denomination of the irrational ratio. Lacking further clarification, Bradwardine's statement is hopelessly obscure and Crosby's explanation fails to illuminate the obscurities.

Although Oresme never explains precisely what $h e$ means by mediate denomination, it is significant that he says nothing of two successive immediate denominations. That is, he nowhere states-as does Bradwardine--that by a mediate numerical denomination he means the immediate numerical denomination of the rational ratio that, in turn, immediately denominates the irrational ratio. For Oresme, therefore, we are justified in dissociating mediate from immediate denomination, understanding by this that the mediate numerical denomination does not itself immediately denominate the rational ratio that immediately denominates the irrational ratio. They are two separate designations, applied as follows: the irrational ratio $(A / B)^{p / q}$ is immediately denominated by the rational ratio or base $A \mid B$, and mediately by the exponent, or ratio of numbers, $p / q$, where it is understood that $p / q$ does not immediately denominate $A / B$.
It is instructive to compare the definitions of mediate numerical denomination as given by Bradwardine, Albert of Saxony, and Oresme.
(i) Bradwardine: "...irrationalis vocatur, quae non immediate denominatur ab aliquo numero, sed mediate tantum (quia immediate denominatur ab aliqua proportione, quae immediate denominatur a numero: sicut medietas duplae proportionis, quae est proportio diametri ad costam, et medietas sesquioctavae proportionis, quae toni medietatem constituit."-Crosby, Brad., p. 66.
pushing to the limit his use of the terms pars and commensurabilis. Since all ratios of the form $(2 / \mathrm{I})^{p l q}$, where $p$ and $q$ are integers with $q>p>\mathrm{I}$, are commensurable to the rational ratio $2 / 1$, Oresme asks whether there may be some irrational ratio which is not any part of $2 / \mathrm{I}$. The ratio $(2 / \mathrm{I})^{1 / \sqrt{q}}$ seems to represent what Oresme had in mind, since raising it to the $q$ th power will never raise it to ${ }^{2} / \mathrm{I}$. Hence it is not a part or parts of $2 / \mathrm{I}$, and the two ratios cannot be related as a number to a number. Oresme, of course, was incapable of expressing this with the available mathematical tools and language, and was limited to negative statements about this category of ratios (see p. 371). This becomes apparent when it is realized that in utilizing the concept of "part," Oresme was able to make statements about, and even to manipulate, irrational ratios which could be designated as exponential parts of some rational ratio. The next step was to ask whether there might be other irrational ratios which are not parts of rational ratios. It will be seen in the next paragraph that Oresme answered this in the affirmative, but having done so he came to a dead end since the concept
(2) Albert: "...proportio irrationalis est que non potest immediate denominari ab aliquo numero sed immediate denominatur ab aliqua proportione que immediate denominatur ab aliquo numero sicut proportio que medietas duple nominatur qualis est proportio dyametri quadrati ad costam eiusdem."-Tractatus proportionum Alberti de saxonia, sig. Aiir, c.I. (No date of publication is given, but see below, p. 131, n. 22.)
(3) Oresme: "...proportio, vero, irrationalis dicitur mediate denominari ab aliquo numero quando ipsa est pars aliquota aut partes alicuius proportionis rationalis, aut quando est commensurabilis alicui rationali, quod est idem, sicut proportio dyametri ad costam est medietas duple propor-tionis."-I. 28 I-85.
Albert's definition is almost identical with Bradwardine's and equally unclear. Oresme's definition has been based on the terms pars and commensurabilis, which are clear concepts discussed at considerable length in this introduction. By introducing the notions of part and commensurability, Oresme was able to take the next step and conceive of an irrational ratio that has no denomination at all-mediate or immediate
(I.329-32; III.338-4I; see above, p. 34). That is, he conceives of an irrational ratio that is neither a part or parts of some rational ratio and is therefore incommensurable to it. Since such irrationals have irrational exponents, our attention once again is directed to the exponent as the entity denoted by "mediate numerical denomination." Oresme's definition is wholly different from the other two because Oresme has provided an elaborate mathematical context for this concept, making it dependent on his exponential interpretation of the terms pars and commensurabilis.
Bradwardine and Albert seem to lack any genuine comprehension of their definitions. Indeed Bradwardine's definition, which plays no role whatever in his treatise, may even have been derived from some presently unknown source which he failed to understand. As a cloak for his ignorance, he perhaps reformulated or merely repeated the passage quoted above. If the concept were original with Bradwardine, we would expect to find a more detailed discussion of such an innovation. Albert's definition may ultimately derive from Bradwardine.
of part was no longer applicable and some new concept, not forthcoming, was necessary to make further progress. Lack of adequate terminology, symbols, and rules prevented effective handling of irrational exponents. Where such ratios were involved Oresme held (IV.365-73) that it was sufficient if one could approximate to such unknowable ratios by determining, sufficiently closely, one greater and one lesser knowable ratio, thus locating the unknowable ratio.

Although Oresme seems to have arrived at the concept of an irrational exponent from a consistent-and persistent-logical application of the terms pars and commensurabilis, what justification did he have for believing that any such ratios might acutally exist? Here Oresme draws upon an earlier supposition which we may call his "principle of mathematical plenitude." In the fifth supposition of Chapter I, Oresme says: "... any ratio is as a continuous quantity in the sense that it is divisible into infinity just like a continuous quantity. [It is divisible] into two equal parts, or three, or four, etc.; into unequal parts in any way; into commensurable parts and into parts incommensurable to it, etc.; and, indeed, in any other way" (I.26I-6s). Indeed, Oresme cites this very supposition as the basis for his belief in the existence of irrational ratios which lack rational denominations (I.286-89). His pattern of thought seems to be as follows: any ratio is like a continuous quantity and therefore divisible in any conceivable mathematically logical manner; now one conceivable way is to divide a rational ratio into smaller ratios one of which, at least, is irrational and no part, or parts, of the given rational, and hence incommensurable to it; therefore, such a conceivable ratio, logically possible, must correspond to some real category of ratios even though we cannot express a single instance of it.

Then by extending the cases, one can conceive of irrational ratios incommensurable to every rational ratio. A deductive proof of this was out of the question for Oresme because of the inductive form of the argument. It was only possible to show, on the basis of his "principle of mathematical plenitude," that there is some irrational ratio which is no part whatever of some larger given rational ratio. But from that point on, the argument is inductive, since the fifth supposition applies to the division of any one particular ratio, but not to all. In order to extend the conclusion to two or more rational ratios we find Oresme resorting to the expression pari ratione (I.294), which he recognized as having only reasonable persuasive-ness-not logical force. For if such an irrational ratio is incommensurable to $2 / \mathrm{I}$ and, consequently, to every rational ratio of the form $(2 / \mathrm{I})^{n}$, where
$n$ is any integer or improper fraction, then pari ratione it is reasonable to suppose that there exists a ratio incommensurable to $(2 / \mathrm{I})^{n}$ and $(3 / \mathrm{I})^{n}$, and so on inductively. Hence it is only reasonable to assume that one or more such irrational ratios are incommensurable to every rational ratio-but this "does not follow from the form of the argument" (I.300).49

A ware of the inconclusiveness of this argument, Oresme appeals next to the authority of Campanus of Novara. After arguing that there might be irrational ratios incommensurable to one or more rational ratios, Oresme says, with reference to Campanus: "This is also apparent in the comment on the last definition of the fifth book of Euclid, where it is said that there are infinite irrational ratios whose denominations are not knowable. If the passage from this authority is valid, it follows that not every irrational ratio is commensurable to some rational ratio, or capable of denomination by some rational,..." (I.303-8). Following this Oresme provides a proof (1.309-20) to show the propriety of Campanus' remarks. Thus, from the completely general statement by Campanus "that there are infinite irrational ratios whose denominations are not knowable," Oresme justifies the particular conclusion that there must be irrational ratios which are incommensurable to rational ratios, or, in other words, irrational ratios with irrational exponents-i.e., whose denominations are unknowable. Did Oresme wish to imply that Campanus, in his comment on the last definition of Book V , was aware of the existence of the third category of ratios? Probably not, although he subtly trades upon the generality of Campanus' assertion. In any event, an examination Campanus' comment on Definition 16 of Book V reveals nothing of the kind. In the course of discussing what Euclid meant by equality of ratios, $5^{50}$ Campanus states that Euclid's crite-

49 For a detailed discussion, see below on pp. 326-31.
so Oresme's reference to Campanus pertains to the following part of Bk. V, Def. 16: "Et si esset omnis proportio scita sive rationalis, tunc facile esset intellectu cognoscere quae proportiones essent una et quae diversae. Quae enim haberent unam denominationem essent una; quae autem diversas, diversae. Haec autem facilitas manifesta est ex arithmetica, quoniam omnium numerorum proportio scita et rationalis est. Unde Jordanus in secundo Arithmeticae suae deffiniens quae proportiones sunt eadem et quae diversae, dicit easdem esse quae eandem denominationem reci-
piunt; maiorem, vero, quae maiorem, et minorem quae minorem. Sed infinitae sunt proportiones irrationales quarum denominatio scibilis non est, quare cum Euclides consideret in hoc libro suo proportionalia communiter non contrahendo ad rationales vel irrationales quoniam considerat proportionem repertam in continuis que communis est ad istas non potuit diffinire identitatem proportionum per identitatem denominationum, sicut arithmeticus, eo quod multarum proportionum (ut dictum est) sunt denominationes simpliciter ignotae, diffinitionem autem oportet fieri ex notis unde malitia proportionum irrationalium coegit Euclidem tales diffinitiones ponere.
rion was not that of equality of denominations of ratios since this would have been applicable only to rational ratios. All rational ratios have numerical denominations so that any two rational ratios are equal when they can be assigned equal numerical denominations. But this cannot be extended to irrational ratios all of which lack numerical denominations. Since there are an infinite number of irrational ratios whose denominations are unknowable and unknown, Euclid, in the fifth book, did not define the identity, or equality, of ratios by the equality of denominations but resorted to the concept of equimultiples, which embraces both rational and irrational ratios. ${ }^{51}$ This is the context of Campanus' discussion and is obviously unrelated to the specific distinctions formulated by Oresme.
We have now seen the three types of ratios which Oresme would distinguish when considering ratios independently. But Oresme is interested in "ratios of ratios" (proportiones proportionum) 52 and he must, therefore, investigate the different ways in which any two ratios can be related exponentially. Two such ratios may be drawn from any one of the three types already enumerated (III. 336-4II), or one ratio from each of two different types (III.412-15). Oresme's attention is devoted almost exclusively to ratios of ratios of the former kind.
All ratios of ratios are either rational or irrational. A ratio of ratios is said to be rational when the two ratios are commensurable, i.e., related by a rational exponent; or, as Oresme would also express it, when the smaller ratio is a part or parts of the greater. But a ratio of ratios is irrational when the two ratios are incommensurable, i.e., related by an irrational exponent.
In the class of rational ratios of ratios there are three major subdivisions:
(r) A rational ratio that is commensurable to another rational ratio. An example is $8 / \mathrm{I}$ and $2 / \mathrm{I}$, which are related as a number to a number, namely $3 / \mathrm{I}$, since $8 / \mathrm{I}=(2 / \mathrm{I})^{3 / \mathrm{I}}$ (II.1.389-92, 398-400, 435-39). It is obvious that Quia, ergo, non potuit (ut patet ex prae- ros-6) gives a confused definition of equimissis) diffinire proportionalitatem sive identitatem proportionum per identitatem habitudinum, sive denominationum, ipsorum terminorum propter irrationalitatem habitudinum et inconvenientiam terminorum, coactus est refugere ad terminorum multiplicia."-Euc.-Campanus, p. III. The italics are mine. (Def. i6 of Euc.-Campanus is Def. 17 in the modern edition of Euclid's Elements, translated by Heath, Vol. 2, IIs, 136.)
${ }_{51}$ Bk. V, Def. 6, of Euc.-Campanus (pp.
multiples. In the modern edition, the correct counterpart appears as Def. $s$ of Bk. V. See Euclid's Elements, trans. Heath, Vol. 2, 114, 120-29. Campanus, and medieval mathematicians generally, failed to understand the famous Euclidean fifth definition concerning equality of ratios. On this point see John E. Murdoch, "Medieval Language of Proportions," in Scientific Change, ed. Crombie, pp. 25 I-6r.
${ }^{52}$ See below on p. 49 for the meaning of this expression.
when Oresme relates two rational ratios he is not concerned with their immediate denominations, but rather in what he calls their "meeting or participating in means" (see II.I.321-22 and especially III. $25-30$ ), or, as we would say, in their common base.
(2) An irrational ratio denominated by, and therefore commensurable to, some rational ratio, which is commensurable to another irrational ratio also denominated by, and commensurable to, some rational ratio. In other words, Oresme is relating two ratios from the second category so that each ratio is irrational with a rational exponent. An example in which $(4 / 1)^{1 / 3}$ and $(2 / 1)^{1 / 2}$ are related as a sesquitertian ratio, namely as $4 / 3$, is found in I. $368-80$. That is, $(4 / 1)^{1 / 3}=\left[(2 /)^{1 / 1 / 2}\right]^{4 / 3}$.
(3) An irrational ratio having no rational denomination is commensurable to some other irrational ratio which also lacks any rational denomination. Here Oresme is relating two ratios from the third category where each irrational ratio has an irrational exponent. Unable to express such ratios properly (see pp. 33-36), Oresme merely alludes to them in another context saying: "....if it should happen that the proposed ratios belonged to the third type of ratio, if there are any such ratios with no denominations and it is probable there are, but if not they can be imagined ${ }^{53}$-the same applies to them as to the others with respect to this, namely that among the ratios of these ratios, rationals are fewer than irrationals..." (III.406ro). The following example represents the third kind of rational ratio of ratios: $(8 / \mathrm{I}) \sqrt{2}^{\sqrt{2}}$ and $\left.(2 / \mathrm{I})\right)^{\sqrt{2}}$ are related as $3 / \mathrm{I}$ since $\left.(8 / \mathrm{I})\right)^{\sqrt{2}}=\left[(2 / \mathrm{I})^{\sqrt{2}}\right]^{3 / \mathrm{I}}$.
Each of the above types of rational ratios of ratios has a counterpart in the class of irrational ratios of ratios:
(I) A rational ratio which is incommensurable to another rational ratio. For example (II.r. $393-97$ ), $9 / \mathrm{I}$ and $2 / \mathrm{I}$ since $9 / \mathrm{I} \neq(2 / \mathrm{I})^{p l q}$, where $p / q$ is a ratio of integers with $p>q$. The same holds for any two rational ratios where $A \mid B \neq(C \mid D)^{p / q}$ and $A|B, C| D, p \mid q$ are rational ratios (see also II. I. 380-83, 401-4, and III, Props. I-V).
(2) An irrational ratio denominated by, and therefore commensurable to, some rational ratio that is incommensurable to another irrational ratio also denominated by, and commensurable to, some rational ratio. Oresme is here relating two ratios from the second category of ratios described on pp. 31-33. An example (I.357-6I) is $(3 / 1)^{1 / 4}$ and $(2 / 1)^{1 / 2}$, where $(3 / 1)^{1 / 4} \neq\left[(2 / 1)^{1 / 2}\right]^{p / 9}$

[^4]and $p / q$, the exponent, is a ratio of integers, that is, a rational ratio.
(3) An irrational ratio having no rational denomination that is incommensurable to some other irrational ratio also lacking any rational denomination. These two ratios are drawn from the third category of ratios described on $\mathrm{pp} .33-36$. The same passage cited above on p. 39-i.e., III. 406-10-in support of the third subdivision of rational ratios of ratios is also applicable here since Oresme says: "Among the ratios of these ratios, rationals are fewer than irrationals." Once again, Oresme can offer no examples, but one fitting the specifications would be $(8 / \mathrm{I}) \sqrt{2}^{\sqrt{2}}$ and $(5 / \mathrm{I}) \sqrt{2}^{2}$ which are unrelatable as a number to a number, that is they are unrelatable by any rational exponent $p / q$ since $(8 / \mathrm{I}) \sqrt{2} \neq\left[(5 / \mathrm{I})^{\sqrt{2}}\right]^{p / q}$.

The reader of the De proportionibus proportionum upon arriving almost at the end of Chapter III, the last of the strictly mathematical chapters, is met with an unexpected burst of enthusiasm which serves as introduction to the tenth proposition. Oresme informs his readers (III.329-32) that the more deeply they reflect upon this proposition and its consequences, the more they will come to admire it.

In Chapter III, Proposition X, Oresme moves on to a consideration of probability relations involving-for the first time in the treatise-unknown ratios. He shows the high degree of probability that any two proposed unknown ratios would be incommensurable because if many unknown ratios were selected it would be very probable that any one of them, taken at random, would be incommensurable to any other of them also taken at random (III.333-35). In other words, Oresme is saying that any two such ratios would probably form an irrational, rather than rational, ratio of ratios. The basic cases which he considers are identical to the three categories of rational and irrational ratios of ratios already discussed (see III.342-90 for the first class; 391-40s for the second; and 406-1I for the third) but are also intended to apply to ratios of ratios involving one ratio from any two of the three categories mentioned on p. 3 I (see III.41 2-15). Thus Proposition X is made completely general, applying to all categories of ratios and all possible combinations of ratios of ratios that can be formed from the basic categories.

The detailed argument is made for ratios of ratios involving only rational ratios. The demonstration of Proposition $X$ consists of two parts-an antecedent and a consequent (III.342-48). The antecedent asserts that when any sequence of rational ratios is taken there can be formed a greater number of irrational than rational ratios of ratios. Based on probability, the
consequent states that any two proposed unknown ratios are probably incommensurable.

The truth of the antecedent is shown in terms of a specific case (III. 349-5 8). Oresme takes 100 rational ratios from $2 / \mathrm{I}$ to ${ }^{101} / \mathrm{I}$, and by relating them two at a time shows that we can have 4,950 possible ratios of ratios of which only 25 are rational and the rest irrational. In a practica conclusio (III.440-98), Proposition XI, Oresme reveals how he arrives at these figures. There he shows "[how] to find the number of ratios between any proposed number of unequal terms by relating every one of them to every other of them" (III.440-42). In terms of the particular example, 100 is multiplied by 99 resulting in 9900 , the total number of possible ratios of ratios when 100 ratios are taken two at a time. Since only ratios of ratios of greater inequality are under consideration (III.447-55), half of 9900 is taken, so that 4950 ratios of ratios of greater inequality are possible. Of this total, only 25 are rational ratios of ratios constituted from the following geometric series: $(2 / 1)^{n}$, where $n=1,2,3,4,5,6$, from which is possible rational ratios of ratios can be produced; $(3 / \mathrm{I})^{n}$, where $n=1,2$, 3, 4, yielding 6 rational ratios of ratios; and one rational ratio of ratios from each of the following four pairs of ratios: $(5 / \mathrm{I})^{n},(6 / \mathrm{I})^{n},(7 / 1)^{n}$, and $(10 / \mathrm{I})^{n}$ where in each case $n=1,2$. The remaining 4,925 ratios of ratios are irrational, so that the resultant ratio of irrational to rational ratios of ratios is 197 to 1 . As more and more rational ratios are taken, say 200 or 300 , and so on, the ratio of irrational to rational ratios of ratios becomes greater and greater (III. $355-58$ ). ${ }^{54}$
${ }^{54} \mathrm{An}$ important question arises as to whether Oresme conceived this disparity of ratio to increase as one takes ever great-er-but always finite-numbers of ratios, or whether he meant to extend it to any number of ratios taken to infinity. If the number of ratios is always restricted to a finite group-and the evidence inclines in this direction (see below)-then Oresme seems correct, provided that the terms of the sequence of ratios are properly chosen. Obviously it would not do to select as the terms of the sequence a particular geomet ric series, since the members of such a series would form only rational ratios of ratios. Every sequence must be chosen such that the members of the subsets forming geometric series thin out as the entire sequence is extended to embrace ever more
terms. But if Oresme meant to extend the sequence of ratios to infinity, it appears that his claim would be invalid since a one-to-one correspondence could be established yielding as many rational as irrational ratios of ratios. Thus if he meant to extend the series $2 / 1,3 / 1,4 / 1,5 / 1$, etc., to infinity, then for every irrational ratio of ratios formed from the sequence, there can be a corresponding rational ratio of ratios between any two ratios formed from the members of any particular geometric series that is a subset of the infinite sequence of ratios-e.g., $(2 /)^{n}$, where $n$ is the sequence of natural numbers. Since Oresme shows no awareness of the possibility of such a one-to-one correspondence, we may be confident that if he avoided this pitfall it must have been for other reasons.

The consequent follows directly from this, for if one were asked whether or not any two unknown rational ratios formed an irrational ratio of ratios, the answer, based on probability, ought to be in the affirmative(III.386-90). Oresme illustrates the probability argument by analogy with cube and perfect numbers (III.370-80). As more numbers are taken, the ratio of non-cube numbers to cube numbers becomes greater. Should one be asked whether an unknown number is a cube or not, it is more prudent to reply in the negative since this is more probable. The same argument applies to rational versus irrational ratios of ratios from whatever category such ratios may be formed.
In Chapter IV Oresme applies to local and celestial motion some of the earlier propositons concerning ratios of ratios. The chapter consists of nine suppositions (IV.r-75; see pp. 363-67) and seven propositions, the first of which treats some of the same material covered by Bradwardine in Chapter II, Part 3, and Chapter III of his Tractatus de proportionibus. The first prop-

One-to-one correspondences between an infinite set and its subsets were known to the Stoics. Sambursky, Pbysics of the Stoics, p. 97, observes, "The main characteristic of the infinite set-the fact that it contains subsets which are equivalent to the whole-was known to the Stoics and formulated as follows: 'Man does not consist of more parts than his finger, nor the cosmos of more parts than man. For the division of bodies goes on infinitely, and among the infinities there is no greater and smaller nor generally any quantity which exceeds the other, nor cease the parts of the remainder to split up and to supply quantity out of themselves.' Here the most important sentence is the first one. It seems to be a literal quotation from the writings of Chrysippos. The infinite sets 'man' and 'cosmos' are compared with their respective subsets 'finger' and 'man,' and it is clearly stated that the subset is equivalent to its set in the sense defined by the modern theory of sets. This property of the infinite set was rediscovered after the Stoics by Galileo who shows the equivalence of the denumerable set of natural numbers and its subset of square numbers." It should be noted that Galileo's significance was not to 'rediscover' this property of an infinite set as applied to physical things-this was
done in the Middle Ages (see Crombie, Medieval and Early Modern Science, Vol. 2, 42; and The Opus Majus of Roger Bacon, trans. Burke, Vol. 2, 455)-but that he applied it to numbers in a strictly mathematical context.
The problem of determining whether Oresme meant to restrict his probability theorem to finite sequences of ratios, or extend it to infinite sequences, is made difficult by the vague language employed in the crucial passages. For example, what does Oresme mean in the statement "quibuscumque et quotlibet proportionibus rationalibus secundum unum ordinem denominationum" (III.343-45)? Are we to understand some definite number of ratios that may be increased by some finite amount, though not indefinitely; or, does Oresme mean a series of ratios taken to infinity? Evidence in favor of the first alternative derives from the fact that Oresme does not use the expressions ad infinitum, et cetera, or et sic ultra, though these and equivalent expressions are used frequently in Ch. I, where sequences of ratios are clearly intended to be taken to infinity. It is further significant that in these unambiguous instances where infinity is meant, Oresme does not employ the terms "quibuscumque" or "quotlibet."
osition (IV.76-172) is independent of the last six which constitute an integral and related whole.

In Proposition I the extent to which Oresme has compressed and somewhat altered the material treated by Bradwardine in the sections cited above reflects the fact that Bradwardine's arguments were well known at Paris and any extensive repetition was unnecessary. The purpose of Proposition $I$ is to refute the very opinion represented by the third erroneous theory of Bradwardine's treatise (see pp. 16-21), and like Bradwardine, Oresme divides the erroneous view into two parts, which he calls the "false" rules.

The two false rules are as follows (IV.76-79): "If a power moves a mobile with a certain velocity, double the power will move the same mobile twice as quickly. And this [rule]: If a power moves a mobile, the same power can move half the mobile twice as quickly." These false rules represent to Oresme the rules of motion formulated by Aristotle in Pbysics VII.5. 249b.27-250a.20, although, strictly speaking, Aristotle does not specifically discuss the case where double a given motive power moves the some resistance (i.e., the first false rule). It is, however, clearly implied.

The first false rule ${ }^{55}$ asserts that if $F / R \propto V$ then $(2 F / R) \propto 2 V$ and is actually a particular case of the second part of Theory III in Bradwardine's treatise which we represented as $V_{2} / V_{\mathrm{I}}=F_{2} / F_{\mathrm{I}}$ with $R_{2}=R_{\mathrm{I}} .{ }^{56}$ Oresme refutes this rule by showing that, with the exception of the particular case where $F_{\mathrm{I}} / R_{\mathrm{I}}=2 / \mathrm{I}$, the rule fails to yield the same results as the "true" law. Thus (IV.98-105) if $F_{2}=8, F_{\mathrm{I}}=4$, and $R_{2}=R_{\mathrm{I}}=2$, then $F_{2} / R_{2}$ $=2\left(F_{\mathrm{I}} / R_{\mathrm{I}}\right)$ and produces the same result as $F_{2} / R_{2}=\left(F_{\mathrm{I}} / R_{\mathrm{I}}\right)^{2 / \mathrm{I}} 57$ since $8 / 2=2(4 / 2)$ and $8 / 2=(4 / 2)^{2 / 1}$. But should $R_{2}=R_{1}=3$, the true law shows that $8 / 3>(4 / 3)^{2 / 5}$ so that $V_{2}>2 V_{1}$, while the false rule shows that $8 / 3=2(4 / 3)$ and $V_{2}=2 V_{\mathrm{I}}$. A similar difference arises when $R_{2}=R_{\mathrm{I}}=\mathrm{I}$. In this case $8 / 1<(4 / 1)^{2 / 1}$ and $V_{2}<2 V_{1}$. The false rule, however, would yield $8 / 1=2(4 / 1)$ so that $V_{2}=2 V_{1} .{ }^{58}$
${ }_{55}$ The discussion of the two false rules and Oresme's criticisms have been summarized in my article, "Aristotle's Restriction on His Law of Motion," in Mélanges Alexandre Koyré: L'Aventure de la science, pp. 173-97. I am here following substantially what I have written in my article.
${ }^{56}$ See above, p. 17, n. 16.
${ }^{57}$ In this singular case Oresme, referring
to III.93-roi, says that the agreement
arises because "a ratio of a quadruple to a double ratio is just like a ratio of [their] denominations, but this is not found in other ratios" (IV.I s 2-s4).
${ }^{58}$ Bradwardine gives the same sequence of relationships in Theorems II, IV, and VI of Ch. III of his Tractatus de proportionibus (see Crosby, Brad., p. II2). See also above, p. 21 , and n. 22.

Oresme's refutation is hardly convincing, since it depends upon the fact that Brad-

The second false rule, which holds that if $F / R \propto V$ then $F /(R: 2) \propto 2 V$, is a special case of the first part of Bradwardine's third erroneous theory represented earlier as $V_{2} / V_{1}=R_{1} / R_{2}$ with $F_{2}=F_{1} .{ }^{59}$ Oresme musters two arguments against the second false rule. The first (IV.112-17) is similar to the refutation of the first rule and trades upon the fact that the second false rule produces results in conflict with the true function. ${ }^{60}$ It is in the second argument that Oresme formulates his major attack against the second rule. Oresme says (IV.ir8-20): "In the second place, I argue against the second rule thus: if it were true, it follows that any power, however feeble, can move any mobile, whatever its resistance." Thus, like Bradwardine, Oresme draws the same mathematical consequence from the false rule. ${ }^{61}$ Indeed, as we shall see, Oresme actually offers a demonstration to justify the consequence. But where Bradwardine, without further consideration, repudiated the consequence as a self-evident physical absurdity, Oresme avoids all appeal to physical experience and produces instead a strictly mathematical counterinstance to demonstrate that although the consequence quoted above is necessarily entailed by the second false rule it does not in fact follow mathematically.

The following is a summary of Oresme's argument in IV.121-64. He shows first that if $A$ is a power which can move some mobile $C$, then it follows that $A$ can also move mobiles $D=2 C, E=2 D, F=2 E, G=$ $2 F$, and so on. He must first demonstrate that if $A$ moves $C$, it can move $D$. In order to prove this he assumes that $B$ is another motive power or force capable of moving mobile $D$ with half the velocity with which $A$ moves $C$. Now since the second false rule says that "if a power moves a mobile, the same power can move half the mobile twice as quickly," it follows that $B$, which moves $D$ with a certain velocity, can also move $C=D / 2$ with twice the velocity with which it moves $D$. Therefore $B / C$ $=2(B / D)$ and, consequently, $V_{B / C}=2 V_{B / D}$ which represent the velocities arising from ratios $B / C$ and $2(B / D)$ respectively. But by assumption
wardine's function produces results that differ from those arrived at by the false rule. No argument has been advanced to convince the reader that the true function has virtues lacking in the false rule, nor has it been shown that the false rule produces absurd consequences.
${ }^{59}$ See above, p. 17, n. 16.
${ }^{60}$ Oresme does not trouble to provide examples for this argument, since the form of the refutation is almost identical with

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$B / D=1 / 2(A / C)$ or $A / C=2(B / D)$, so that $V_{A / C}=2 V_{B / D}$. Thus we see that $2 V_{B / D}=V_{B / C}=V_{A / C}$, so that $A / C=B / C$ and $A=B$ (by Euclid V.9), from which it follows that $A$ can move $D$, since by assumption $B$ can move $D .{ }^{62}$

By the same procedure Oresme shows (IV.i38-42) that if $A$ can move $D$ it can also move $E$, which equals $2 D$. Here again a third motive power,
${ }^{62}$ A possible source for this portion of Oresme's attack against the second false rule may have been Jean Buridan, who offers substantially the same argument against the same rule (Oresme's second false rule which Buridan calls the "first rule" [prima regula]) in his Questions on the Eight Books of the Physics of Aristotle.
Buridan discusses first a rule that is a direct consequence of the false rule. This consequential rule says, "If a power moves a mobile through a certain distance in a certain time, the same, or an equal, power would move double the mobile through half the distance in an equal time." ("... s aliqua virtus movet aliquod mobile per aliquod spacium in aliquo tempore, eadem, vel equalis, virtus movet duplex mobile per dimidium spacium in equali tempore." -Acutissimi philosophi reverendi Magister Jobannis Buridani subtilissime questiones super octo phisicorum libros Aristotelis, fol. 107 v , c.r.) This rule, says Buridan, is manifestly false, since the mobile or resistance may be sufficiently large so that the force might be wholly incapable of moving double the resistance, since a velocity can be produced only when the force exceeds the resistance. ("Quod autem hec ultima regula sit falsa manifestum est per se quia tantum mobile vel tanta resistentia mota a tua virtute posset signari quod tu non posses movere duplam per aliquod spacium nec in aliquo tempore. Et causa huius est quia virtus motoris debet excedere virtutem resistentie, et forte non excederet si resistentia duplaretur..." [fol. 107v, c.1].) See also above, p. 16, ก. 13 .
But Buridan says that this obviously false rule would be true if another previously discussed rule were true. This rule, called by Buridan the prima regula, is essen-
tially identical with Oresme's second false rule and says, "If a force moves a mobile through a certain distance in a certain time, the same or equal force will move half this mobile through twice the distance in an equal time." ("Si aliqua virtus movet aliquod mobile per aliquod spacium in aliquo tempore, eadem vel equalis virtus movebit medietatem illius mobilis per duplex spacium in equali tempore" [fol. 107r, c.2].) Buridan shows why the former rule would be true if the prima regula were true Assume that (1) $A$ moves $B$ one league in one hour, and (2) $C$ is a mobile double $B$. Buridan concludes from this that $A$ will move $C$ one half-a-league in one hour. For if $A$ is unable to move $C(=2 B)$ half-aleague in an hour, we can postulate some force or power, $D$, that can accomplish this. Now if the first rule (prima regula) is true and $D$ can move $C$ half-a-league in one hour, it follows that $D$ can move $B$ ( $=C / 2$ ) a distance of one league in one hour. Therefore $A=D$ since both powers can move $B$ equal distances in the same time. We may now properly conclude that $A$, like $D$, can move $C(=2 B)$ one half-aleague in one hour; and this is what was to be demonstrated. ("Sed tunc ego probo consequentiam argumenti principalis, scilicet, quod si prima regula sit vera alia etiam, que dicta fuit, erit vera quia pono quod $A$ movet $B$ per unam leucam una hora, et pono etiam quod $C$ est duplum ipsi $B$, tunc ego concludam quod $A$ movebit $C$ per dimidiam leucam in una hora. Quia si non potest tunc ego sumam aliam virtutem maiorem que hoc potest, et sit illa virtus $D$. Cum ergo $D$ moveat $C$ per dimidiam leucam in una hora sequitur, si prima regula erat vera, quod etiam $D$ movebat medietatem ipsius $C$, que quidem medietas est $B$,
say $P$, is introduced and assumed capable of moving $E$ with half the velocity with which $A$ moves $D$. Following the same sequence of steps, Oresme demonstrates that $A=P$ and, therefore, $A$ can move $E$ with half the velocity with which it moves $D$. Since this can be carried on ad infinitum as the succession of mobiles is doubled, it follows that "any power, however feeble, can move any mobile, whatever its resistance" (IV.iri-20).

Oresme says next (IV.143-§I) that, on the basis of what has already been shown, if $A / C=4 / \mathrm{I}$ and $B / D=4 / 2$, it follows that $B$ can move $C$, where $C=D / 2$, with the same speed as $A$ moves $C$, from which it may be concluded that $A=B$. Therefore, $A / D=B / D$ and $A$ can move $D$ with a double ratio, namely $4 / 2$.

But this is true only because in this particular case the false rule and the true function yield the same results, since $A / C=2(B / D)$ and $A / C=$ $(B / D)^{2 / 1}$. But in other cases, the results would be dissimilar (IV.1 ${ }^{2-54}$ ). Thus, according to the second false rule, if $A$ moves $D$ in a double ratio, i.e., $A / D=2 / 1$, and $E=2 D$, then should another force $B$ move $E$ with half the velocity with which $A$ moves $D$, it follows that $B / D=A / D$. But by Supposition I (IV.3-8), which is Bradwardine's function, this is false, for according to that function to say that $B$ moves $E$ with half the velocity with which $A$ moves $D$ is to say that $B / E=(A / D)^{1 / 2}$ rather than $B / E=1 / 2(A / D)$, as required by the false rule. Now $E / D=2 / \mathrm{I}$ and $A / D$ $={ }^{2} / \mathrm{I}$. Consequently, $B / E=(A / D)^{1 / 2}=(2 / \mathrm{I})^{1 / 2}$ and $B / D=B / E \cdot E / D=$ $(2 / \mathrm{I})^{1 / 2} \cdot(2 / \mathrm{I})$, so that $B / D=(2 / \mathrm{I})^{3 / 2}$. It is now evident that $B / D>A / D$, since $(2 / \mathrm{I})^{3 / 2}>(2 / \mathrm{I})$. Therefore, $B>A$ and $B / E>A / E$ (where $E=2 D$ ). Thus from the fact that $A<B$, it does not follow that because $A$ can move $D$ it can move $E=2 D$, even though $B$ can move $E$ and a fortiori can move $D$, which is half of $E$. Indeed, $A$ may be incapable of producing any motion whatever in $E$. But the second false rule requires that $B / D=$

[^5]$A / D$ making $B=A$, in which event $A$ must necessarily be capable of moving $D$ and $E$.

The success of Oresme's demonstration depends entirely on setting $B / E$ $=(A / D)^{1 / 2}$ rather than $B / E=1 / 2(A / D) .{ }^{63}$ This move enabled him to show that $B / D>A / D$ instead of $B / D=A / D$. The entire demonstration is improper since Oresme has substituted his own function for the false rule and merely demonstrated that one obtains quite different results from the rival theories. He has produced no good reasons for repudiating the false rule, but has only shown that in terms of Bradwardine's function the consequence of the false rule-namely, that any power, however weak, can move any resistance, however great-is contradicted. In general, the sum total of Oresme's efforts against the false rules reduces to the revelation that in all but one case, they produce results which disagree with those arising from Bradwardine's function. But this was already known and furnished no proper grounds for rejecting the false rules.

Propositions II-VII of Chapter IV are not found in Bradwardine's treatise, nor, to my knowledge, in any other similar work in the fourteenth century which considers ratios of velocities involving forces and resistances. These propositions indicate the degree to which Oresme extended the application of ratios of ratios to problems of motion.

The above-numbered propositions are largely concerned with showing what one may learn about unknown ratios of force and resistance when certain limited data are furnished or already known. Oresme would have liked to present the reader with precise rules for determining the exact terms of any unknown ratio of force and resistance, but he acknowledges, in a significant passage in Proposition VI (IV. $369-67$ ), that this is unattainable for "if there were some velocity that arises from such a ratio whose denomination is not knowable, it is impossible to make its ratio known." In terms of the subject matter of the De proportionibus, what Oresme means, as will be seen, is that there are certain ratios of the form
${ }^{63}$ A supporter of the false rule would certainly have objected to the substitution of $B \mid E=(A \mid D)^{1 / 2}$ for $B \mid E=1 / 2(A \mid D)$. How might Oresme justify this crucial move? It is possible that he would have pointed out that since $A / D=2 / \mathrm{It}$ follows that $B / E=1 / 2(A / D)=1 / 2(2 / \mathrm{s})=2 / 2$. Thus, when the velocity with which $A$ moves $D$ is halved, the false rule produces a ratio of equality where force, $B$, equals resistance, $E$; and no motion is possible from such a
ratio of equality. This can be avoided by invoking Bradwardine's function, or Supposition I of Ch . IV, since the square root of $A / D=2 / \mathrm{y}$ is a ratio of greater inequality and force, $B$, remains greater than resistance, $E$, and motion is produced. An opponent, however, might have insisted that if application of the false rule made $B=E$, at that point motion ceases and the mathematical rules of proportion are no longer operative. On this point, see below, p. 369
$(A / B)^{n}$, where $A / B$ is rational but $n$, the exponent, is irrational. ${ }^{64}$ Should some force move a resistance and produce a velocity which is expressible only by such a ratio, it would be impossible to determine $F / R$. With such ratios it is only possible to "investigate whether any ratio given, or to be given to us, is greater or smaller than such an irrational, unknowable, and unnameable ratio. Finally, in this way we can find two ratios sufficiently close so that such an unknown ratio will be greater than the lesser and smaller than the greater. And this ought to suffice" (IV.367-73). ${ }^{65}$

But even if a velocity should arise from a ratio whose denomination is knowable, Oresme says he "can discover no general rule for determining it in every case" (IV.374-76). However, Propositions I-V are helpful in providing such information (IV.376-77). ${ }^{66}$

In Proposition II (IV.173-224) we see that in some cases it is possible to determine whether an unknown ratio, which produces a velocity, is equal to, greater than, or less than some proposed ratio (see p. 370). Thus if $B$ is an unknown ratio and $A$ is known, Oresme shows how to determine whether $B \gtreqless A$. He assumes that $C=B \cdot A$ and then transforms them into ratios involving forces and resistances where $C=D / F, B=D / E$, and $A=E / F$ where $D$ is a force and $E, F$, are mobiles or resistances with $E>F$. Since $C=B \cdot A$, it follows that $D / F=D / E \cdot E / F$. Now if $D$ moves $F$ with a velocity twice that with which $D$ moves $E$, we have $D / F=(D / E)^{2 / \mathrm{I}}$ where ${ }^{2} / \mathrm{I}=V_{F} / V_{E}$ and $V_{F}$ and $V_{E}$ represent the velocities of $F$ and $E$ respectively. Thus $C=B^{2}$, since the ratio of velocities depends on the ratios of force to resistance (by the first supposition, IV.3-7), which means that we can reason from the ratio of velocities, or the exponent, to the relations between the ratios (see IV.334-37). But if $C=B^{2}$, then $B=A$ by Supposition V (IV.16-18). The unknown ratio $B$ is now known to be equal to the known ratio $A$. Oresme goes on to show (IV.198-207) that if $(D / F)>(D / E)^{2 / \mathrm{r}}$, then $C>B^{2}$ and $B<\mathrm{A}$ by the second part of Supposition V (IV.17); on the other hand, if $D / F<$ $(D / E)^{2 / 1}$, then $C<B^{2}$ and $B>A$ by the third part of Supposition $V$ (IV.17-18). ${ }^{67}$
${ }^{64}$ This is the third category of ratios mentioned above on pp. 33-37. See also 1.286-332, III. 333-69, and below on pp. 327-31 for discussion of the existence of such a class of ratios.
${ }^{65}$ This is taken up in Prop. II of Ch. IV (IV.173-224).
${ }_{66}$ Prop. I is not actually mentioned in
IV.376-403, where Oresme summarizes the contribution of the preceding propositions. However, Prop.I, which establishes Bradwardine's function, underlies the entire fourth chapter.
${ }^{67}$ For an example, see IV.208-14, and below on p. 370 .

The problem in Proposition III is to determine the ratio of a force to two different mobiles or resistances when the ratio of velocities and the ratio of the mobiles are given. Let $A$ be a force, and $B, C$, mobiles with $A>B>C$, and assume that $A / C=D, A / B=E, B / C=F$, so that $D=E \cdot F$. Now $D=E^{g}$ where $g$, the exponent, is a ratio of numbers, or-as Oresme would say-a ratio of ratios (see below) equal to the ratio of the velocities $V_{C} / V_{B}$. There are, in fact, four ratios $D, E, F$, and $g$, of which $g, F$ are known and $D, E$ are to be determined. Relying on Supposition VIII (IV. 37-48), Oresme shows that if the ratio of whole to part is known-namely $g$, which relates $D$ and $E$ since $D=E^{g}$ - the ratios, or exponents, relating $E$ with $F$, and $D$ with $F$ can also be known. This is clear because if $g=n / m$ where $m<n$ then, according to Supposition VIII, $E=D^{m / n}, F=D^{p / n}$ (with $m+p=n$ ), so that $E=F^{m / p}$ and $D=F^{n / p}$. Now $F$ is known by hypothesis and, therefore, ratio $E$ is easily determined from the relationship $E=F^{m / p}$ as is $D$ from the relationship $D=F^{n} \mid p$ or $F=D^{p / n}$. Since $D$ and $E$ have now been found, we know the two ratios of force and resistance, namely $A / C$ and $A / B$, which equal $D$ and $E$ respectively. ${ }^{68}$

In terms of the more familiar $V$ for velocity, $F$ for force, and $R$ for resistance, the details and objectives of the preceding paragraph may be easily summarized. The following relationships are involved: $F_{2} / R_{2}=$ $F_{\mathrm{I}} / R_{\mathrm{I}} \cdot R_{\mathrm{I}} / R_{2}$ where $F_{2}=F_{\mathrm{I}}, R_{\mathrm{I}}>R_{2}$, and $F_{2} / R_{2}=\left(F_{\mathrm{I}} / R_{\mathrm{I}}\right)^{V_{2} / V_{\mathrm{I}}}$. The ratio of velocities $V_{2} / V_{\mathrm{I}}$ and the ratio of resistances $R_{\mathrm{I}} / R_{2}$ are known, from which one can easily determine the ratios $F_{2} / R_{2}$ and $F_{\mathrm{I}} / R_{\mathrm{I}}$.

Should there be only one mobile and two powers or forces (IV.262-70), one could also find the two unknown ratios of force and resistance. Thus if $F_{2} / R_{2}=F_{2} / F_{\mathrm{I}} \cdot F_{\mathrm{I}} / R_{\mathrm{I}}$ where $R_{2}=R_{\mathrm{I}}, \quad F_{2}>F_{\mathrm{I}}$, and $F_{2} / R_{2}=$ $\left(F_{\mathrm{I}} / R_{\mathrm{I}}\right)^{V_{2} / V_{\mathrm{I}}}$, then should $V_{2} / V_{\mathrm{I}}$ and $F_{2} / F_{\mathrm{I}}$ be known the two ratios of force and resistance can be found.

In this proposition Oresme uses the important expression "ratio of ratios" (proportio proportionum) with greater clarity than anywhere else in the treatise. An inspection of the three occurrences of the expression (IV. 233-34, 235, and 250) makes it immediately apparent that a ratio of ratios applies exclusively to the exponent. Oresme calls $g$ a ratio of ratios, since $D=E^{g} .{ }^{69}$ The expression is most appropriate, since it tells us that a ratio
${ }^{68}$ A specific example is given in IV. to understand by the phrase proportio pro-248-58 and summarized below on p. 37,
${ }^{69}$ In my article, "Oresme: Prop.," pp. 295-96, n. 13, I remarked,"Oresme appears
to understand by the phrase proportio pro$(F \mid R)^{n}$, where $n$ is a proportion of numbers, or an exponent, relating the two pro-
of numbers, or an exponent, relates two other ratios and is, therefore, a ratio of ratios or, as Oresme might have said, a "ratio which relates two other ratios." It must be added, however, that a ratio of ratios can also be irrational (see pp. 33-36), in which event it is no longer a ratio of numbers but of quantities, i.e., magnitudes.
In Proposition IV there is a twofold problem. The first aspect is to find whether some unknown ratio is commensurable to a given ratio; if it is commensurable, the second objective is to make known the unknown ratio.
Here $C=A \cdot B$ where $A$ is a known ratio, $B$ unknown. Let $D$ be the power or force, and $E, F$ mobiles with $E>F$; finally, we have $C=D / F$, $B=D / E$, and $A=E / F$, the ratio of mobiles. Now $D / F$ and $D / E$ give rise to velocities which may be designated as $V_{F}$ and $V_{E}$, respectively. Oresme asks whether these two velocities are commensurable (IV.278-80). If they are-i.e., if $V_{F} / V_{E}=m / n$, where $m$ and $n$ are integers in their lowest terms-then $D / F$ and $D / E$, or $C$ and $B$, are commensurable, since
 are commensurable they are related as whole to part, since $C=A \cdot B$; and by Supposition VI (IV. 19-26) it follows that $B$ will also be commensurable to $A$, the remainder. That is, $A=B p l q$ where $p$ and $q$ are mutually prime integers.
If, on the other hand, $V_{F}$ and $V_{E}$ are incommensurable, then $C$ and $B$ are incommensurable and $C \neq B^{m / n}$ where $m / n$ is a rational ratio. Therefore, $C$ and $B$ are not related as whole to part or parts, and consequently $B$ and $A$ are not so related, which makes them incommensurable (IV. 285-92).
Assuming next that an unknown ratio $B$ is commensurable to a given and known ratio $A$, Oresme shows how to find $B$ when the ratio of velocities, $V_{F} / V_{E}$, is known (IV.293-303). Since $V_{F} / V_{E}$ is known, we have the exponential relationship $D / F=(D / E)^{V} F^{\prime V} E$ and also $C=B^{m / n}$
portions. But proportion $n$, or the exponent $n$, is also a proportion of velocities $V_{\mathrm{I}} / V$ which varies as, or depends on, the 'proportion of proportions." This interpretation is clearly erroneous. In the first place, the entire expression in the quotation is not a proportion of proportions-or ratio of ratios as we have expressed it in this volume-since, as we have seen, only the exponent may bear that designation. Furthermore, Oresme distinguishes be-
tween a proportion of proportions, or ratio of ratios, and a proportion of velocities, or ratio of velocities. The former is strictly applicable only to the exponent in a mathematical relationship between two non-physical-i.e., strictly mathematical-ratios. When we deal with two ratios of force and resistance, however, they are related by a ratio of velocities which may be like -but is not to be identified with-a ratio of ratios. See below, pp. $\mathrm{si}^{1-\varsigma 2 .}$
where $m>n$ and $m / n=V_{F} / V_{E}$. Now $B$ is a part or parts of $C$ and we may express this as $B=(C)^{n / m}$ where $n<m$ and they are in their lowest terms. It then follows by Supposition VIII that $A=(C)^{p / m}$ (where, of course, $n+p=m$ ) and $B=(A)^{n / p}$. But the eighth supposition tells us that $n / p$ will be known and by hypothesis $A$ is known, so that $B$ can easily be determined by the ninth supposition. $7^{0}$

At the conclusion of Proposition IV, Oresme says: "The process of arriving at a ratio of ratios from a ratio of velocities is a posteriori. When, however, a ratio of velocities is derived from a ratio of ratios, the procedure is by way of the cause and is a priori" (IV.334-37). Thus if $F_{2} / R_{2}=$ $\left(F_{\mathrm{I}} / R_{\mathrm{I}}\right)^{V_{2} / V_{\mathrm{I}}}$ where $V$ is velocity, $F$ force, and $R$ resistance, we have a case in which two ratios of force and resistance give rise to a ratio of velocities. Now if $F_{2} / R_{2}=C, F_{\mathrm{I}} / R_{\mathrm{I}}=B$, and $V_{2} / V_{\mathrm{I}}=m / n$, then $C=(B)^{m / n}$ is equivalent to the formulation involving forces, resistances, and velocities. We have, consequently, moved from a ratio of velocities, $V_{2} / V_{1}$, to a ratio of ratios, $m / n$ (see p. 49). But why should Oresme call this a posteriori? In all likelihood because he conceived it as a process from the physical to the mathematical where the physical relations are dependent on, or follow, the mathematical.

For Oresme the link between mathematical and "physical" ratios is given by Supposition I of Chapter IV (IV.3-8) which served as justification for the straightforward application of previously established mathematical principles and propositions to ratios of motion. This is made quite explicit when he says: "...by means of the first supposition of this chapter and the propositions given in the third chapter, one who understands can demonstrate many propositions about velocities" (IV.522-24). This seems to account for his remark that when a ratio of velocities is derived from a ratios of ratios, we go from cause to effect and the process is a priori. Thus if $C$ and $B$ are ratios and $C=B^{m / n}$ where $m$ and $n$ are integers in their lowest terms, then if, as before, $C=F_{2} / R_{2}, B=F_{1} / R_{\mathrm{I}}$ and $\mathrm{m} / \mathrm{n}$ $=V_{2} / V_{1}$, the transition is made directly from the mathematical to the physical.
The attitudes embodied in the passage cited above from IV.334-37 seem to underscore the difference between Aristotle and Oresme. Where for Aristotle, actual physical conditions and experience set limits on the applicability of mathematics to problems of motion (see shiphauler discussion on p. 369), Oresme, in sharp contrast, places the physical in direct dependence on the mathematical possibilities, almost as if he believed in a

[^6]direct correspondence between mathematics and physics, with the former revealing the possibilities and actualities of the latter. ${ }^{71}$
When one of two ratios is unknown, Proposition V shows that the unknown ratio can be found if the ratio of velocities and the other ratio are known. Let $A$ represent the known ratio and $B$ the unknown. The relationships between these elements is $B=A^{m / n}$ where $m / n$ is a ratio, or exponent, of integers. Now $m / n$, which is a ratio of ratios (see p. 49), is as a ratio of velocities, and for this reason we can link $B$ and $A$ by a ratio of velocities, so that $B=(A)^{V_{B} / V} A$ where $V_{B}$ and $V_{A}$ are the velocities produced by ratios $B$ and $A$ respectively. Since $A$ and $V_{B} / V_{A}$ are known, one can find $B .{ }^{72}$

The purpose of Propositions II-V is to enable us to "come to know the ratio producing a certain velocity so that we could say 'such a velocity arises from such a ratio,' as, for example, when we say that the velocity with which a certain mobile traverses one mile in an hour arises from a double ratio" (IV.391-94). ${ }^{73}$ To show how these propositions are helpful in this connection, Oresme very briefly summarizes their function in Proposition VI (IV.365-403), but prior to any actual discussion of the proposition. The summary is sufficiently lucid to require no further exposition. Of special interest, however, is an attempt by Oresme to extend the range of his mathematical physics to embrace celestial and circular as well as terrestrial and rectilinear motion (IV.399-416). Oresme says that if some body, presumably a planet, which we may call $A$, moves with a circular motion, and the ratio which produces its velocity is known, namely $F_{\mathrm{I}} / A$, then we could theoretically determine a ratio of velocities between $A$ 's motion and that of some planetary orb, say $B$. How is such a ratio of velocities
${ }^{71}$ A different attitude seems to characterize Oresme's De configurationibus qualitatum. There he insists, in Clagett's words (Science of Mechanics, p. 340), that his geometrical "representations are entirely 'ymaginationes,'...They are concerned with a figurative presentation of hypothetical quality variations and thus are totally unrelated to any empirical investigations of actual quality variations." Of course, Oresme made no empirical investigations appropos of his claims about ratios of velocities, but he did believe in the physical truth of the mathematical rules, or laws, of motion that he employed in the $D e$
proportionibus. They were not merely hypothetical or imaginative formulations, as in the De configurationibus.
${ }^{72}$ The example in IV. $351-56$ is outlined below on pp. 372-73.
${ }^{73}$ These propositions are valid, Oresme emphasizes, only if the medium is homo-geneous-or, as he puts it, not defective (IV.395-96)-the respective velocities uniform (IV. $387-88$ ), the power or force is inseparable from the mobile or resistance and is applied to only one mobile at a time, and every mobile is moved by only one power (IV.401-3).

De proportionibus proportionum
obtained? In astronomy these velocities would be arrived at "from the ratio of the quantities of the motions or circles described, and from the ratio of the times in which they revolve" (IV.407-9; see also IV.332-34). Thus if we know $S_{B} / S_{A}$ when $T_{B}=T_{A}$ (where $S$, the distance, is measured either in terms of angles swept out by radius vectors, or by the respective number of complete circulations), ${ }^{74}$ it follows that $S_{B} / S_{A}=V_{B} / V_{A}$; or, when $T_{B} / T_{A}$ is known with $S_{B}=S_{A}$, it follows that $T_{B} / T_{A}=V_{A} / V_{B}$.

Assuming the determination of such a ratio of velocities $V_{B} / V_{A}$, and knowing ratio $F_{1} / A$, we can by Proposition $V$ of Chapter IV obtain the ratio of force and resistance which moves body $B$ and which we shall represent by $F_{2} / B$. The formulation is $F_{2} / B=\left(F_{1} / A\right)^{V_{B} / V} A$ where only $F_{2} / B$ is unknown and is, therefore, readily found.

In this way, ratios of force and resistance may be found even in celestial bodies where, however, the forces would be called moving intelligences and the resistances, planetary spheres. But Oresme is quick to add (IV.41 2r6) that it is improper to speak of ratios of celestial force and resistance except by way of analogy or similitude because, strictly speaking, the immaterial intelligences move the celestial spheres effortlessly through a medium which offers no resistance. 75
${ }^{74}$ For example, if $S_{B} / S_{A}=5 / 2$ when tor, just as when a man makes a clock $T_{B}=T_{A}$, this is understood to signify that in the time in which $B$ completed ; revolutions on its circle, $A$ will have completed 2. Or more generally, in equal times $B$ moves $2^{1} / 2$ times the angular distance traversed by $A$. See API. 5 $5-83$, and below on pp. 88-90. See also Grant, "Oresme: Comm.," p. 42 I.
${ }^{75}$ It appears that Oresme later entertained the possibility that forces and resistances might be operative in the celestial region. In his Le Livre du ciel et du monde, Bk. II, Ch. 2, he speculates that when God created the intelligences, He may also have placed motive forces within them, just as He gave weight and heaviness to terrestrial things. Into these intelligences He may also have put resistances to the motive forces. Furthermore God may have coordinated the forces and resistances in such a manner that the celestial bodies move without violence. Having established the proper proportions, the heavens are moved without further interference or aid from the Crea-
which moves by itself when properly made and prepared.

Here is the relevant passage: "Et selon verité, nulle intelligence n'est simplement immobile et ne convient pas que chascune soit par tout le ciel que elle meut ne en chascune partie de tel ciel, posé que les cielz soient meüz par intelligences, car par aventure, quant Dieu les crea, Il mist en eulz qualitéz et vertus motivez aussi comme Il mist pesanteur es choses terrestes, et mist en eulz resistances contre ces vertus motivez. Et sont ces vertus et ces resistances d'autre nature et d'autre ma[t]iere que quelcunque chose sensible ou qualité qui soit ici-bas. Et sont ces vertus contre ces resistances telement moderees, attrempees et accordees que les mouvemens sont faiz sanz violence; et excepté la violence, c'est aucunement semblable quant un honme a fait un horloge et il le lesse aler et estre meü par soy. Ainsi lessa Dieu les cielz estre meüz continuelment selon les proporcions que les vertus motivez ont aus resistances

In Proposition VI Oresme discusses methods of denomination for the various kinds of ratios of force and resistance which can arise. For ratios that are rational and known it is a simple matter to find their prime numbers (IV.420-22). ${ }^{76}$ For example, should $F / R$ be a quadruple ratio its prime numbers would be $4 / \mathrm{I}$.

The remainder of the proposition (IV.422-95) is devoted to the more complicated problem of denominating irrational ratios of force and resistance. Since irrational ratios cannot be immediately denominated by ratios of numbers, and Oresme is seeking some means of direct denomination, he resorts to representing them by lines. It is apparent that Oresme is dealing solely with irrational ratios that are commensurable to rational ratios, namely irrational ratios of the form $(A / B)^{m / n}$ where $m / n$, the exponent, is a ratio of integers in its lowest terms and $A / B$ is rational. There are two possible cases (IV.425-26): (1) where $m<n$ in which case the irrational ratio is a part or parts of the rational ratio $A / B$ (see I.281-8s and p. 3 I ) and obviously $(A / B)^{m / n}<(A / B)$; and (2) where $m>n$ and $(A / B)^{m / n}>(A / B)$ (see p. 31).

In the first case (IV.427-50) where $(A / B)^{m / n}<(A / B)$, Oresme assigns two lines representing the ratio $A$ to $B$ and then divides $A / B$ into $n$ equal parts by assigning $n$ - I mean proportional lines. Since $m$ is the numerator, we relate any one of the mean proportional lines to the $m$ th line before or after it as required by the position of the lines.
et selon l'ordenance establie."-Oresme: Le Livre du ciel, eds. Menut and Denomy, in Mediaeval Studies, Vol. 4, 170.
Pierre Duhem, who cited this passage to show how Oresme disagreed with Aristotle and sided with Buridan in considering the possibility of celestial forces and resistances, believed the sentiments expressed in this section reveal Oresme's conception of the study of motion as one in which the principles of mechanics were like those of a clock. However, in the very passage he quoted, Duhem failed to take cognizance of Oresme's quite explicit assertion that such celestial forces and resistances are different in nature from their terrestrial counterparts. If Duhem is right, we must restrict his remarks to the principles of celestial mechanics, for it is obvious on the basis of this passage that in Oresme's mind one could never relate celestial with terrestrial
mechanics. Duhem's evaluation is as follows: "Mais, en même temps et de cette même bouche, nous entendons la déclaration que voici: L'étude des mouvements de l'Univers est un problème qui dépend tout entier des mêmes principes de Mécanique, de ceux qui règleraient une horloge immense et compliquée. En effet, un jour, dans son traité De horologio oscillatorio, Huygens posera les théorèmes qui permettront à Newton d'analyser le mécanisme de l'Univers."-Système du monde, Vol. 8,345. It is certainly a fact that the clock, or watch, analogy was to become extremely popular in the seventeenth and eighteenth centuries.
${ }^{76}$ Oresme cites the first chapter as the place where he discussed how to find the prime numbers of a rational ratio. Actually this was done in II.2.3-56.

In the second of two examples (IV.444-50) the given irrational ratio is $(4 / \mathrm{I})^{2 / 3}$ (due tertie quadruple). Let $A / B=4 / \mathrm{I}$ and then assign two lines related as $A / B$. Next assign mean proportional lines $C$ and $D$ so that $A, C, D, B$ are four continuously proportional lines. Forming successive ratios we have $(A / C) \cdot(C / D) \cdot(D / B)$ which are related as $\left[4 /(4)^{2 / 3}\right] \cdot\left[(4)^{2 / 3} /(4)^{1 / 3}\right] \cdot$ $\left[(4)^{1 / 3} / \mathrm{I}\right]$ where each ratio is equal to $(4 / \mathrm{I})^{1 / 3}$. Since 2 is the numerator of the exponent in the given irrational ratio, either of the mean proportional lines $C$ or $D$ can be related to any line once removed from it. Thus line $D$ can be related only to line $A$, while line $C$ can be associated only with $B$. Linking lines $D$ and $A$, we see that $A$ is greater than $D$ and must represent the power or force, while $D$ represents the mobile or resistance. The ratio of force to resistance represented by $A / D$ is as $4 /(4)^{1 / 3}=(4)^{2 / 3} / \mathrm{I}$. The same can be done for $C$ and $B$ where $C>B$ and $C / B$ is also a ratio of $(4)^{2 / 3} / \mathrm{x}$.

The second of the two cases (IV.45I-9I) distinguished above is preceded by a brief commentary on the relations between greater irrational ratios and smaller rationals (IV. 45 1-57). Oresme, referring to Chapter II, Proposition I, observes that no irrational ratio can be multiple to any smaller rational ratio. With respect to the present discussion, this means that no irrational ratio is relatable to a smaller rational by an integral exponent which has the form of a multiple ratio $n / \mathrm{r}$ and, consequently, the rational is not a part of the greater irrational. However, should the exponent of the greater irrational ratio be any one of the four remaining types of ratios (superparticular, superpartient, multiple superparticular, and multiple superpartient; see II.2.6-20), then it will contain the smaller rational ratio one or more times plus an exponential part or parts of the rational. For example, if the irrational is $(A / B)^{m / n}$ where $m / n=3 / 2$, a superparticular ratio, the irrational contains the rational once plus $1 / 2$ part of the ra-tional-understood in an exponential sense-since $3 / 2=1$ I/2.

As one example (IV.464-82), Oresme selects as his ratio of force and resistance the irrational ratio $(2 / \mathrm{I})^{\mathrm{I}^{2 / 3}}$. He then outlines the steps for determining a ratio of two lines which will serve to denominate this ratio. First he assigns two lines, $A$ and $B$, related as $2 / 1$, or as we shall represent it $4 / 2$. Now the denominator of the exponent, namely 3 , indicates that the irrational ratio contains three equal exponential parts of the rational ratio $4 / 2$ and, consequently, Oresme assigns mean proportional lines $C$ and $D$ between $A$ and $B$, and there are now four lines, $A, C, D$, and $B$, in geometric proportionality forming the sequence of ratios $(2)^{2} /(2)^{\mathrm{r}^{2 / 3}} \cdot(2)^{\mathrm{r}^{2 / 3} /(2)^{1^{1 / 3}}}$. (2) $)^{1^{1 / 3}} /(2)^{1}$ where each ratio or part equals $(2 /)^{1 / 3}$. But the numerator of the exponent, namely 2 , indicates that there are yet two further exponential
parts each equal to the previously derived parts. Thus two additional mean proportional lines must be assigned, which may be either greater or smaller than the four other lines. Let them be smaller, says Oresme, and call them $E$ and $F$. The six geometrically proportional lines, $A, C, D, B, E$, and $F$, now form the series of ratios $(2)^{2} /(2)^{1^{2} / 3} \cdot(2)^{1 / 3 / 3} /(2)^{1^{1 / 3}} \cdot(2)^{1^{1 / 3} /(2)^{1}} \cdot(2)^{1 /(2)^{2 / 3}}$, $(2)^{2 / 3} /(2)^{1 / 3}$, or $A / C \cdot C / D \cdot D / B \cdot B / E \cdot E / F$. Ratio $A / F$ is, therefore, a ratio of lines where $A$, the longest line, represents the motive force, and $F$, the shortest line, is surrogate for the resistance. That is, $A / F=$ $(2)^{2} /(2)^{1 / 3}=(2 / \mathrm{I})^{5 / 3}=(2 / 1)^{12 / 3} .77$

Proposition VII consists largely of a direct application of propositions from the first part of Chapter II to ratios of velocities. Once again Supposition I is invoked to justify the move (IV.500-501; see also p. 5 I).

The first series of relationships (IV.500-s 14) depends on Proposition V

77 We can see now why it was desirable to represent the double ratio as $4 / 2$ rather than ${ }^{2} /{ }_{1}$. Had $2 / 1$ been used, the ultimate sequence of ratios would have involved negative and zero exponents, since the following series would result:
$\frac{2}{(2)^{2 / 3}} \cdot \frac{(2)^{2 / 3}}{(2)^{1 / 3}} \cdot \frac{(2)^{1 / 3}}{(2)^{0}} \cdot \frac{(2)^{0}}{(2)^{-1 / 3}} \cdot \frac{(2)^{-1 / 3}}{(2)^{-2 / 3}}$
It is extremely improbable that Oresme knew of, or could have represented, such concepts. Indeed, in all this Oresme provides no operational details and it is evident that he could have achieved his objective without any knowledge of how to find the actual mean proportional, since he simply assumes that any number of mean lines can be found between two given lines (IV.49293 ). In the present example he needed to know only that five exponential parts were required, which was obvious from the fact that the given ratio is $(2 / \mathrm{J})^{5 / 3}$. From the denominator of the exponent he knew that each of the five parts had to be $\left(2_{1}\right)^{1 / 3}$. He then simply assumes that the first two mean proportionals are appropriate, so that $A$, $C, D$, and $B$ are in geometric proportionality, from which it follows that $A / C \cdot C / D$. $D \mid B$ constitute three of the five required exponential parts. Obviously two more extreme terms, or lines, are needed and it is indifferent whether they are greater or smaller than $A, C, D, B$. Oresme chose to
operate with smaller extreme terms, $E$ and $F$, and to assert that they are in the same series with the preceding four terms. Hence the two additional ratios, or parts, are now available. Upon completion of the example Oresme says that "ratio $A$ to $F$ contains a double ratio and two-thirds of a double and, consequently, $A$ to $F$ is the given ratio" (IV.480-82). By this he means that $A|F=A| B \cdot B \mid F=2 / \mathrm{I} \cdot(2 / \mathrm{I})^{2 / 2}$.
To cope with these irrational ratios, Oresme needed only his concept of part and parts. Since the given ratio is $(2 / 1)^{5 / 3}$, it is clear that four geometric means will divide $4 / 2$ into five parts, where each part must be a "one-third part"-namely $(2 / 1 /)^{1 / 3}$.
Alvarus Thomas, in his Liber de triplici motu, criticized Oresme for assuming that he could find four continuously proportional lines starting with two lines related as a double ratio and then assigning two mean proportional lines. In Proposition (or Conclusion) 8 of the second part of his treatise (sig. e.I verso, c.2; the folios are unnumbered), Alvarus discusses how to find a mean proportional line between two given lines and cites Euclid VI.I 3 , where this is demonstrated (see Euclid's Elements, trans. Heath, Vol. 2, ir6). His criticism of Oresme appears in the tenth proposition of Part II (sig. e.2r, c.1-c.2). Alvarus wonders how Oresme would proceed in finding two mean proportional lines between two given
of Chapter II (IV.903-4) and is concerned with velocities produced by ratios smaller than some given rational ratio between whose mutually prime numbers no mean proportional number is assignable. For example, let $V_{B}$ represent a velocity produced by a double ratio ${ }^{2 /}$ and assume that $V_{C}$ is any velocity produced by a ratio less than $2 / \mathrm{I}$. Now if $V /{ }_{B} V_{C}$, the ratio of velocities, is rational, then, says Oresme, the ratio giving rise to $V_{C}$ must be irrational. This is so because $(2 / \mathrm{I})=\left[(2 / \mathrm{I})^{m / n}\right]^{n / m}$ where $m$ and $n$ are integers in their lowest terms with $m<n$, with the consequence that $(2 / \mathrm{I})^{m / n}$ must be irrational. And since a ratio of ratios is as a ratio of velocities (see p. 51), it follows that $n / m=V_{B} / V_{C}$ and the velocities are commensurable. But not all irrational ratios smaller than $2 / \mathrm{I}$ produce velocities commensurable to $V_{B}$ as, for example, all irrational ratios $(2 / \mathrm{I})^{m / n}$ where the exponent $m / n$ is itself irrational.
lines, say $A$ and $B$, related as a double in mathematicis scientiis videtur michi ratio. Now Euclid VI.13, demonstrates quod per artem medie rei inventionis non how to find one such line between $A$ and $B$. If this line were $C$, then $A \mid B=A / C$. $C / B$, where $A / C=C / B$. The second mean proportional line, say $D$, must now be found between $A$ and $C$ or $C$ and $B$. Let us assume it is found between $A$ and $C$ so that we have $A|B=A| D \cdot D / C \cdot C \mid B$, where $A|D=D / C \neq C| B$, or $(2 / 1)^{1 / 4}=$ $(2 / 1)^{1 / 4} \neq(2 / 5)^{1 / 2}$. Hence the lines are not in geometric proportionality. The same reasoning applies if $D$ is found between $C$ and $B$.
Here is the text of Alvarus' argument: "Decima conclusio. Quamvis facile sit cuilibet proportioni invenire subduplam, subquadruplam, suboctuplam, subsexdecuplam, et sic in infinitum ascendendo per numeros pariter pares. Difficile tamen est subtriplam, subquintuplam, subsextuplam, et sic in infinitum per numeros impares vel impariter pares ascendendo invenire. Prima pars patet ex priori conclusione; et secunda est michi experimento comperta, quamvis Nicholas Horen [i.e., Oresme] in suo tractatu proportionum capite quarto velit dare modum per artem medie rei inventionis ad inveniendam proportionem et subduplam et subtriplam et subsexquialteram. Sed salvo meliori iudicio [the text has "indicio"] et auctoritate tam circumspecti [the text has "circuaspecti"] viri signanter
possunt inveniri quatuor linee continuo proportionabiliter se habentes. Quod sic ostendo: quia captis duabus lineis se habentibus in proportione dupla ad inveniendam quatuor lineas continuo proportionabiles oportet inter illas duas invenire alias duas continuo proportionabiles inter se et cum extremis ut ipsemet fatetur. Sed hoc non potest fieri per medie rei inventionem igitur minor probatur quia vel prima illarum duarum linearum que invenitur inter illas duas invenitur per illam artem vel non. Si non habeo propositum quod oportet dare aliam artem; si sic, tunc manifestum est quod illa erit medio loco proportionabilis inter lineas se habentes in proportione dupla et per consequens maioris linee ad ipsam et etiam ipsius ad minimum erit proportio que est medietas duple. Et tunc quero de inventione secunde linee intermedie quia vel ille invenietur per artem medie rei inventionis vel non. Si non, habeo propositum; si sic, quero vel illa debet inveniri per illam artem inter illam mediam lineam et ultimam, vel inter primam et illam mediam. Sed neutrum istorum est dicendum igitur probatur minor quoniam si inveniatur inter mediam et ultimam iam ille quatuor linee non erunt continuo proportionabiles quoniam prime ad secundam erit medietas duple, et secunde ad tertiam

However, all smaller rational ratios produce velocities incommensurable to $V_{B}$. By way of illustration, let us assume that $3 / 2$ produces a velocity $V_{C}$. But $2 / \mathrm{I} \neq(3 / 2)^{n / m}$-i.e., no rational exponent $n / m$ can raise $3 / 2$ to $2 / 1$ so that $n / m$ must be irrational, and since $n / m=V_{B} / V_{C}$ the ratio of velocities is incommensurable.

The next set of relationships deals with velocities arising from ratios greater than some given rational ratio (IV.515-22). Once again, let $V_{B}$ be the velocity produced by a given rational ratio and $V_{C}$ a velocity derived from a rational ratio greater than that producing $V_{B}$. Now if $V_{C} / V_{B}=$ $n / \mathrm{r}$, where $n$ is any integer, then the ratios producing $V_{C}$ and $V_{B}$ are commensurable. For example, if $V_{C} / V_{B}=3 / \pm$ and the ratios $8 / I$ and $2 / \mathrm{I}$ give rise to $V_{C}$ and $V_{B}$, respectively, then $8 / \mathrm{I}=(2 / \mathrm{I})^{3 / 1}$ and the ratios creating the velocities are commensurable because the velocities them-
et etiam tertie ad quartam erit subquadrupla duple quia erit medietas medietatis duple ut patet ex nona conclusione huius. Si vero inveniatur inter primam et mediam idem sequitur. Ex quo sequitur Horen [i.e., Oresme] non tradidisse doctrinam ad inveniendam proportionem compositam ex duabus tertiis proportionis duple puta subsexquialteram ad duplam. Probatur quia ut sonant verba eius videtur innuere illas lineas inveniendas essentie(?) per artem medie rei inventionis quod stare non potest ut probatum est. Et si hec non fuit intentio et mens venerabilis magistri Nicholai Horen detur imbecillitati et parvitati ingenioli mei venia. Eligat igitur unusquisque quod vult et me magis studiosum quam malivolum probet."
The criticism by Alvarus would be justified if Oresme could not have found two mean proportional lines between given lines $A$ and $B$. But Oresme refers to the very same Euclidean proposition cited by Alvarus (Oresme invokes Euclid VI.9 in Campanus' edition [Euc.-Campanus, pp. 145 -46], which corresponds to VI.I3 in the edition of Zamberti used by Alvarus [Euc.Campanus, pp. 147-48] and in the modern text as translated by Heath [Euclid's Elements, Vol. 2, II6]), and states that while Euclid shows how to find only one mean proportional line between two given lines, he believes (IV.494-95) that Johannes de

Muris (d. ca. 1350) has shown how to find any number of them. I have been unable to locate such a discussion, but a likely candidate would be de Muris' very lengthy Quadripartitum numerorum (for manuscripts of this treatise, see Clagett and Murdoch, "Medieval Mathematics, Physics and Philosophy," Manuscripta, Vol. 3, 24). In $A r$ chimedes, Vol. I: The Arabo-Latin Tradition, Appendix V, Marshall Clagett presents the text of a proof showing how to find two mean proportionals between two given quantities. This proof appears in Jordanus de Nemore's Liber de triangulis and, along with two different proofs, in Leonardo Pisano's Practica geometrie. Whether Oresme knew of one, or more, of these proofs, or had any direct knowledge of Johannes de Muris' alleged proof, is irrelevant to his purpose. For, as we have seen, he simply assumed that any number of mean proportionals could be found and then formulated his discussion in terms of exponential parts. Alvarus, however, chose to challenge Oresme's assumption, convinced as he was that Oresme could not even demonstrate his case for two mean proportionals-to say nothing of any number of them. But a careful reading of IV.492-95-Alvarus' objections were levelled at this very sixth proposition of Ch. IV of the De propor-tionibus-might have persuaded Alvarus that his criticism was beside the point.
selves are commensurable. But if $V_{C} / V_{B} \neq n / \mathrm{x}$, all rational ratios greater than $B$-where $B$ is a rational ratio between whose prime numbers there are no assignable mean proportional numbers-are incommensurable to $B$. Thus, if $B$ is $2 / \mathrm{I}$ and $V_{C}$ arises from a ratio of $6 / \mathrm{I}$, it is clear that $n / m$, where $n>m>1$, must be irrational, since $6 / \mathrm{I} \neq(2 / \mathrm{I})^{n!m}$ if $n / m$ is rational. Consequently, $V_{C} / V_{B}$ is also irrational and the velocities are incommensurable.

Indeed, any greater ratio which produces a velocity commensurable, but not multiple, to a velocity produced by a smaller rational ratio must be an irrational ratio (IV.515-I8). In an example expressed initially in terms of distances, Oresme assumes that $S_{C} / S_{B}=3 / 2$ (IV.SI8-22), where $S_{C}$ and $S_{B}$ are the distances traversed in equal times by $V_{C}$ and $V_{B}$, respectively. Since $S_{C} / S_{B}=V_{C} / V_{B}$ when $T_{C}=T_{B}$ (IV. 332-34; see also pp. 371-72), it follows that $V_{C} / V_{B}=3 / 2$. Without furnishing any additional information, Oresme states categorically that the ratio which produces $V_{C}$ must be irrational. However, this is true because $V_{B}$ is generated by ${ }^{2} / \mathrm{I}$, the ratio representing force and resistance, so that $V_{C}$, which is $3 / 2\left(V_{B}\right)$, would result from $(8 / \mathrm{I})^{1 / 2}$, an irrational ratio, since $(8 / \mathrm{I})^{1 / 2}=(2 / \mathrm{I})^{3 / 2}$.
In a paragraph concluding this portion of Proposition VII (IV.525-32), Oresme draws upon other propositions in the third chapter. By Chapter III, Proposition III, all velocities resulting from multiple ratios $n / \mathrm{I}$, where $n$ is any integer greater than I , are incommensurable to all velocities arising from ratios of the form $p / q$, where $p>q>\mathrm{I}$ and $p / q$ is in its lowest terms. This is obvious because $n / \mathrm{I} \neq(p / q)^{m / r}$ where $m / r$ is a ratio of integers. Clearly, in all such instances $m / r$ must be irrational and since $m / r=V_{C} / V_{B}$ the ratio of velocities is also irrational (IV.525-27). Not even all multiple ratios yield mutually commensurable velocities unless they are in the same geometric series. Thus all ratios in the sequence $(2 / \mathrm{I})^{n}$, where $n=1,2,3$, $4, \ldots$, would produce commensurable velocities; but such velocities would be incommensurable to those generated in the series $(3 / \mathrm{I})^{n}$.

Finally, by Proposition V of Chapter III, all superparticular ratios

$$
\frac{n+\mathrm{x}}{n}
$$

produce velocities incommensurable to one another (IV.529-32) since rational exponents cannot relate two such ratios. That is,

$$
\frac{n+\mathrm{I}}{n} \neq\left(\frac{m+\mathrm{I}}{m}\right)^{p / q}
$$

where $p / q$ is a ratio of integers.
One might continue in this manner and show that "one or more [additional] propositions about velocities can be demonstrated from any prop-
osition of the third chapter" (IV. $533-34$ ), ${ }^{78}$ but Oresme decides instead to concentrate on the application of Chapter III, Proposition X-the climactic proposition of the first three mathematical chapters-to ratios of velocities. Citing Supposition I of ChapterIV and applying what was previously said about ratios of ratios (III. $426-34$ ), he says it is probable that any two velocities proposed are probably incommensurable and constitute an irrational ratio. The greater the number of velocities proposed, the greater the probability that any two of them are incommensurable and form an irrational ratio. This follows because a ratio of velocities is as a ratio of ratios by Chapter IV, Supposition I (see p. 363), and because it was already shown in Proposition X of Chapter III (see pp. 40-42), that any two unknown ratios are probably incommensurable and form an irrational ratio of ratios. The more ratios taken, the greater the probability that they are related by an irrational exponent. Thus if two ratios $A / B$ and $C / D$ are related as $C / D=(A / B)^{m / n}$, it is probable that $m / n$ is irrational and the ratios incommensurable. Should $F_{2} / R_{2}=C / D$ and $F_{\mathrm{I}} / R_{\mathrm{r}}=A / B$, then $F_{2} / R_{2}=$ $\left(F_{\mathrm{r}} / R_{\mathrm{T}}\right)^{V_{2} / V_{\mathrm{r}}}$ and $V_{2} / V_{\mathrm{I}}$, the ratio of velocities, is irrational since a ratio of ratios is related as a ratio of velocities, namely $m / n=V_{2} / V_{1}$.
The probability argument is applied not only to ratios of velocities but also to times and distances and, indeed, to any quantities relatable as times and distances. In a general assertion covering all these cases, Oresme says: "When there have been proposed any two things whatever acquirable [or traversable] by a continuous motion and whose ratio is unknown, it is probable that they are incommensurable. And if more are proposed, it is more probable that any [one of them] is incommensurable to any [other]. The same thing can be said of two times and of any continuous quantities whatever" (IV.596-6I). By way of example, Oresme says (IV.562-69) that if there are two unequal motions and $T_{2}=T_{1}$, where $T$ is time, then it is probable that the unequal distances traversed would be incommensurable -i.e., $S_{2} / S_{1}$ is irrational. Obviously, the ratio of velocities would also be irrational, since $S_{2} / S_{\mathrm{I}}=V_{2} / V_{\mathrm{I}}$ (IV.332-34; see also pp. 371-72).
In a second example (IV. $566-72$ ), if a ratio of times, $T_{2} / T_{1}$, is unknown when $S_{2}=S_{1}$, the times are probably incommensurable and the ratio irrational. Were more times to be taken, the incommensurability of any two would be even more likely. As an illustration of this, Oresme says that if $T_{2} / T_{\mathrm{I}}$ were the lengths of the day and solar year, respectively, it is
${ }^{78}$ It is hardly surprising that Oresme applies ratios of ratios to problems of motion since Bradwardine, the originator of
this special mathematical treatment, formulated it in response to a mathematicalphysical problem.
probable that they would be incommensurable, in which event it would be impossible to find the true length of the year. 79
From the general claim made earlier (IV.5s6-6I) that any two things acquirable by a continuous motion are probably incommensurable-a claim based in turn on Chapter III, Proposition X-Oresme draws a special conclusion pertaining to celestial motions. He says: "When two motions of celestial bodies have been proposed, it is probable that they would be incommensurable, and most probable that any celestial motion would be incommensurable to the motion of any other [celestial] sphere..." (IV. $573-76) .{ }^{80}$ Following this enunciation, Oresme presents a series of propositions, without proofs or elaboration, which can be shown to follow from the general proposition that any two celestial motions are probably incommensurable. Since some form of these propositions is found in the Ad pauca respicientes, references to them will be made in the discussion of that treatise.
It must be noted, however, that the special conclusion (IV. 573 -76) in particular, and sometimes the general conclusion (IV.556-6I), served as important weapons in Oresme's attacks against astrological prediction. Derived ultimately, as we have seen, from the mathematical arguments advanced in Proposition X of Chapter III, where it was shown that there are more irrational than rational ratios of ratios, one or the other or both are cited by Oresme in a number of works, ${ }^{81}$ some of which were attacks

79 Oresme makes essentially the same assertion in AP2.192-97 and in his De commensurabilitate vel incommensurabilitate motuum celi. In the latter he writes: "Adhuc, autem, ex predicta incommensurabilitate contingeret quod annus solaris medius contineret aliquos dies et portionem diei incommensurabilem suo toti. Que posito, impossibile est precisam anni quantitatem deprehendere, aut perpetuum almanac condere, seu verum kalendarium invenire."MS Vat. lat. 4082, fol. iost, c.2. Quoted from Grant, "Oresme: Comm.," p. 456, n. 80 .
${ }^{80}$ In APr.125-34, Oresme defines the various ways in which two mobiles can be incommensurable when moving with circular motion (see below, pp. 432-33).
${ }^{81}$ In his De commensurabilitate Oresme expressly cites (MS Vat. lat. 4082, fol. IO4r, c.1) the De proportionibus proportionum as
follows: "Cum ergo medietas sesquitertie et medietas sesquialtere sint proportiones incommensurabiles ut patet ex libro $D e$ proportionibus proportionum." Many passages in the De proportionibus are consonant with this reference and it is sufficient to list only Prop. VII of II. I. Of particular relevance to the present discussion is an apparent second reference to the De proportionibus embedded in an argument favoring the assumption that the celestial motions are incommensurable. Both the general and special conclusions are cited following a statement that the difficulties stemming from the arguments of those who believe they can foretell the future are avoided if it can be shown that the heavenly motions are incommensurable. "It seems better, therefore, to assume the incommensurability of the celestial motions, since these difficulties do not follow from that [sup-
against those astrologers who claimed it was possible to make precise astrological predictions from exact celestial data (see APı.I-4). If astrology depends on precise predictions of conjunctions, oppositions, quadratures, entries of planets into the different signs of the zodiac, and so forth, then Oresme believes he has the means of destroying the very basis for such predictions. In his attack two main arguments are discernible-one physical, the other mathematical. First, Oresme assumes that we can never know ratios of any quantities pertaining to celestial motions, since, in fact, our senses are so deficient that we are not even capable of knowing ratios of quantities lying immediately by us (this is the substance of the third supposition of API.45-90; see also AP2.239-44). Though we are, by the very nature of things, doomed to ignorance concerning actual ratios of velocities, we might consider whether the actual, but un-
position]. Indeed, incommensurability is shown in yet another way, for, as demonstrated elsewhere, when any two unknown magnitudes have been designated, it is more probable that they are incommensurable rather than commensurable, just as it is more probable that any unknown [number] proposed from a multitude of numbers would be non-perfect rather than perfect. Consequently, with regard to any two motions whose ratio is unknown to us, it is more probable that that ratio is irrational than rational." ("Magis igiturponenda est incommensuratio motuum celestium ex qua hec inconvenientia non sequitur. Que quidem incommensurabilitas adhuc aliter ostenditur quoniam sicut alibi probatum est 〈quibuslibet〉 ignotis magnitudinibus demonstratis verisimilius est istas esse incommensurabiles quam commensurabiles sicut quantumque ignota multitudine proposita magis verisimile est quod sit non perfectus numerus quam perfectus. Igitur de proportione quorumlibet duorum motuum nobis ignota verisimilius et probabilius est ipsam esse irrationalem quam rationalem" [MS Vat. lat. 4082, fol. 108v].)
It seems reasonable to assume that we have here a reference to the De proportionibus proportionum, since the phrase "sicut alibi probatum est" explicitly refers to ( I ) the demonstrations concerning the probability that any given unknown magnitudes
are incommensurable, and (2) the example concerning perfect and non-perfect numbers. The assertion of ( 1 ) is made in IV. 556-6I, but is based on Ch. III, Prop. X; the citation of (2) is an example from $D e$ proportionibus III.370-74.
The De commensurabilitate, in turn, was cited by Oresme in his Le Livre du ciel et du monde, in support of the contention that any motions of the heavens are incommensurable. Oresme writes: "Et que aucuns des mouvemens du ciel soient incommensurables, ce est plus vraysemblable que n'est l'opposite, si comme je monstray jadys par plusseurs persuasions en un traitie intitulé De Commensurabilitate vel incommensurabilitate motuum celi."-Oresme: Le Livre du ciel, eds. Menut and Denomy, in Mediaeval Studies, Vol. 3, 252.
The De commensurabilitate is explicitly cited again in Oresme's Livre de divinacions, a treatise attacking astrology. Oresme says: "La premiere partie d'astrologie est speculative et mathematique, tres noble et tres excellente science, et baillie es livres moult soubtilment et la puet on suffisament savoir, mais ce ne puet estre precisement et a point, si comme jay declaire en mon traictie de la Mesure des Mouvemens du Ciel et l'ay prouve par raison fondee sur demoustracion mathematique."-Oresme and the Astrologers, ed. and trans. Coopland, p. 54 . Since Oresme says that the "speculative
known, ratios of motion are commensurable or not. Chapter III, Proposition X, supplies the mathematical answer for Oresme. Even were it possible to know ratios of celestial motion, the probability is great that any such ratios would be irrational. Any two celestial velocities would probably be incommensurable, as would two times or distances (IV.552-77). It follows, of course, that the ratios which give rise to these celestial velocities, times, or distances are also incommensurable. Indeed, as more and more velocities, times, or distances are taken there are correspondingly more irrational than rational ratios of ratios. Astrologers, forced to rely on irrational ratios for the most basic physical quantities, could not possibly predict precise celestial positions (see AP2.185-97). The very best astronomical data could not remedy this inherent mathematical indeterminacy. ${ }^{82}$
and mathematical" part of astrology ("astronomy" not astrology is meant; the terms were used interchangeably) "can be adequately known but it cannot be known precisely and with punctual exactness" (the translation is Coopland's), and cites the De commensurabilitate as support, it is likely that his reference is to the statements already quoted on the incommensurability of the celestial motions. If this is correct, then it is even possible that his closing remark that he has proved it "by reason founded on mathematical demonstration" is a reference to his De proportionibus, for only in that treatise does Oresme offer mathematical proofs for the probable incommensurability of any two magnitudes or distances traversable by continuous motions. Since the probable incommensurability of the celestial motions is a special case of this general proposition, it seems we have another reference-albeit vague-to the De proportionibus.

The general and special conclusions in the De proportionibus may underlie a statement by Oresme in his Contra judiciarios astronomos concerning astrological prediction. He says: "...tamen hoc astrologi nequeunt prescire, tum quia proporciones sunt inscibiles, ut alibi demonstravi, tum quia..."-Studien $z^{u}$ Langenstein, ed. Pruckner, p. 235.

The De proportionibus is cited by name in

Oresme's Questiones de sphera, where the general conclusion is given as a supposition: "Hoc posito pono istam suppositionem quod ista tempora, scilicet $A$ et $B$, sint incommensurabilia et hoc est verisimile, et hoc probatur quia quibuscumque temporibus vel quantitatis duabus demonstratis verisimile est illa esse incommensurabilia, et quod eorum proportio sit irrationalis sicut in libro De proportionibus."MS Florence, Bibl. Riccardiana II7, fol. 134v.
Finally, the general conclusion is also found in the Ad pauca respicientes, where it is given as the second supposition of Part I (API. $36-38$ ).
${ }_{82}$ The many inferences and demonstrations that Oresme based on his derived but fundamental proposition that any two celestial motions are probably incommensurable were attacked by John de Fundis in a treatise composed at Bologna in 145 I . John's work constitutes a defense of astrology against Oresme's onslaughts, and bears the rubric Tractatus reprobationis eorum que scripsit Nicolaus orrem in suo libello intitulato de proportionalitate motuum celestium contra astrologos et sacram astrorum scientiam, compilatus per Iobannem Lauratium de Fundis. This information is given by Thorndike, Magic and Experimental Science, Vol. 4, 235. Thorndike conjectures (p. 236, n. 12) that John's attack is more specifically against

Why was Oresme so determined an opponent of astrology? He no doubt opposed it on general religious grounds, but his dislike for it must have been intensified immeasurably because of the strong fascination it exerted on the king of France, Charles V, friend and benefactor of Oresme. The latter had been in close association with Charles long before his formal ascension to the French throne in 1364, and had ample opportunity to note the considerable influence of court astrologers. ${ }^{83}$ His vigorous attacks may have been prompted by a fear that important decisions of state might be influenced by vain astrological predictions. ${ }^{84}$ The mathematical argument of Chapter III, Proposition X, as applied to celestial motions, may have been, in Oresme's mind, the proper antidote to the insidious poison of astrological delusion. Oresme had formulated an intellectual mathematical argument in the hope of appealing to a king well known for his intellectual interests, so that with one mighty blow the king might be brought to reason and made to realize that he had been duped into believing in the possibility of accurate astrological prediction which was now shown to be utterly impossible. Unfortunately, for all his prodigious and praiseworthy effort, Oresme, judging from Charles's unswerving allegiance to his astrologers, failed dismally in his reform attempt.

It should not be thought, however, that Oresme turned away from the science of astronomy because he believed it inexact. In his Livre de divinacions

Oresme's De commensurabilitate vel incommensurabilitate motuum celi. But as this book went to press, I received a microfilm copy of John's treatise in MS Paris, Bibliothéque Nationale, fonds latin, 10271 , fols. 63r-75r (although the attack on Oresme terminates on fol. 75 r , the complete work as cited by Thorndike occupies fols. $63 \mathrm{r}^{-}$ 153 V ) and discovered that it is, in part, a commentary on Oresme's Ad pauca respicientes. That part of this treatise relevant to the $A d$ pauca will be edited as an appendix to a future edition of Oresme's De commensurabilitate. Even a cursory perusal, however, reveals that John made no attempt to cope with the technical and difficult propositions of the $A d$ pauca, but, for the most part, remained content to defend astrology and argue against Oresme's assumption that the celestial motions are probably incommensurable. For example, after commenting upon the introductory material and Prop. I of Part 2, John, seemingly with
a sigh of relief, leaps to Props. XIX and XX which, strictly speaking, are not propositions at all, but climactic pronouncements repudiating astrological prediction on the basis of the technical propositions wholly omitted in John's commentary.
${ }^{83}$ On this point, see the following: Jourdain, "Oresme et les astrologues," Revue des questions bistoriques, Vol. I8; Thorndike, Magic and Experimental Science, Vol. 3, 585-89; Oresme: Le Livre du ciel, eds. Menut and Denomy, in Mediaeval Studies, Vol. s, 240-4I.
84 Though Oresme does not expressly mention Charles V in, his Livre de divinacions, he probably had Charles in mind when, after denouncing efforts to foretell the future by astrology and other occult arts, he remarks, "Such things are most dangerous to those of high estate, such as princes and lords to whom appertains the government of the commonwealth."-Oresme and the Astrologers, ed. and trans. Coopland, p. s I.

## De proportionibus proportionum

he divides astrology into six parts, of which the first is essentially what we would call astronomy. He describes it as "speculative and mathematical, a very noble and excellent science and set forth in the books very subtly, and this part can be adequately known but it cannot be known precisely and with punctual exactness...."85 And one might add that Oresme, like so many medieval thinkers, believed firmly in an overall physical influence of the celestial bodies on human activities. ${ }^{86}$ But he was wholly skeptical that human beings could foretell future events from celestial motions which were, very likely, mathematically incommensurable.

## Brief Remarks on <br> Oresme's Algorismus proportionum

Oresme's interest in manipulating ratios and formulating rules about their combination and reduction is brought out in his Algorismus proportionum, a treatise written, in all likelihood, before the De proportionibus (see p. 14). There is little doubt that in the Algorismus Oresme had already arrived at the concept of "part" in an exponential sense (see above, p. 33, n. 48), although it was only to receive its full and advanced expression in the $D e$ proportionibus. Of the central idea of the De proportionibus-commensurability involving exponential parts-there is no hint in the Algorismus, nor is there any allusion to irrational exponents.

Let us now examine Oresme's discussion of "part" as found in the third and fourth rules of the first part of the Algorismus. ${ }^{87}$ In the third
${ }^{85}$ Ibid., p. 55 . The French text of this passage is given above on p. 62, n. 8 I . See also AP 2.263 -64.
${ }^{86}$ In the Livre de divinacions, Oresme says: "And of the three subdivisions of the third part of astrology the first, which is concerned with the great events of the world, can be and is sufficiently well known but only in general terms. Especially we cannot know in what country, in what month, through what persons or under what conditions, such things will happen, or other particular circumstances. Secondly, as regards change in the weather, this part by its nature permits of knowledge being acquired therein but it is very difficult and is not now, nor has it ever been to any one who has studied it, more than
worthless, for the rules of the second part are mostly false as I have said, and are assumed in this branch....In the third place, so far as medicine is concerned, we can know a certain amount as regards the effects which ensue from the course of the sun and moon but beyond this little or nothing. All this third part of astrology has to do chiefly with physical effects;..."Oresme and the Astrologers, ed. and trans. Coopland, pp. 55, 57. See also Thorndike, Magic and Experimental Science, Vol. 3 , 417.
${ }^{87}$ For a summary of Part I of the Algorismus, see Grant, "Mathematical Theory of Oresme," pp. 288-303. Part I contains nine rules and one "general rule," but only the third and fourth rules and the material
rule 88 he explains that when some irrational ratio is "parts" of some rational ratio, it is possible to express that irrational ratio as a part of some other rational ratio. Assume that $B$, an irrational ratio, is parts of $A$, a rational ratio. That is, $B=(A)^{m / n}$, where $m / n$ expresses the number of parts which $B$ is of $A$. Without altering the denominator, the relationship of $B$ to $A$ can be changed from that of parts to that of part by expanding $(A)^{m}$ to $(D)^{1 / n}$ where $D=(A)^{m}$. Indeed, Oresme insists that an irrational ratio is in its most proper form when expressed as a part rather than parts. He illustrates this in the following example: if we are given an irrational ratio $(4 / \mathrm{I})^{2 / 3}$, we obtain $(16 / \mathrm{I})^{1 / 3}$ after expanding $(4 / \mathrm{I})^{2}$. In the De proportionibus Oresme does not concern himself with such transformations, although the same notion of part and parts is found there.

In the fourth rule ${ }^{89}$ rational ratios are given a twofold classification.
summarized below on Pp. 314-15 have a genuine relevance for the De proportionibus.

88 "Tertia regula. Si proportio irrationalis fuerit partes alicuius rationalis ipsam possibile est partem vocare et hoc alterius rationalis licet non eiusdem, unde competentius nominatur pars quam partes.
"Sit itaque $B$, irrationalis, partes $A$ rationalis, igitur ipsum $B$ habebit numeratorem et denominatorem. Dico, ergo, quod non mutato denominatore, ipsum $B$ erit pars alicuius proportionis multiplicis ad $A$ secundum numeratorem et quia omni proportioni rationali continget dare quomodolibet multiplicem, erit quelibet irrationalis, de qua est intentio, pars alicuius rationalis.
"Verbi gratia, proponatur proportio que sit due tertie quadruple et quia 2 est numerator ipsa erit una tertia quadruple duplicate, scilicet sedecuple, et sic de aliis."Grant, "Mathematical Theory of Oresme," p. 333.
${ }_{89}$ "Quarta regula. Denominationem proportionis irrationalis proprissime assignare. Pro isto est sciendum quod proportio rationalis dicitur primaria que non potest dividi in proportiones rationales equales et est illa inter cuius numeros minimos nullus est numerus medius proportionaliter seu numeri medio loco proportionales sicut est dupla aut tripla aut sexquialtera. Sed illa vocatur secundaria que potest sic dividi
et inter cuius numeros est numerus vel numeri medii proportionaliter in medio loco proportionales sicut sunt quadrupla que dividitur in duas duplas, et octupla in tres duplas, similiter nonupla in duas triplas et sic de aliis. Proposita itaque proportione irrationali quomodolibet si denominetur partes tunc per regulam precedentem fiat quod vocetur pars.
"Quo posito videatur si proportio rationalis a qua denominatur sit primaria et si sit tunc standum est quia proportio irrationalis, de qua est sermo, est competentissime nominata, sicut dicendo unam tertiam sextuple vel duple, et sic de aliis.
"Si, vero, proportio rationalis a qua denominatur sit secundaria videatur quot habet proportiones rationales primarias que sunt eius partes equales et si numerus quotiens istarum partium et denominator proportionis irrationalis proposite sint incommunicantes standum est in tali denominatione sicut si dicatur una medietas octuple talis est propria quia octupla habet tres partes equales rationales, scilicet tres duplas, et 2 est denominator proportionis irrationalis proposite modo 3 et 2 sunt numeri incommunicantes. Ideo medietas octuple non est pars alicuius proportionis rationalis minoris quam octupla, quamvis bene sit partes quia medietas octuple est tres quarte quadruple sed talis denominatio non esset propria.

One type of rational ratio is called "primary" (primaria) and embraces all ratios which can have no mean proportional numbers assigned between their terms and are, consequently, not divisible into equal rational ratios. Instances of primary rational ratios are $2 / 1,3 / 1,3 / 2$, and so on. The second type of rational ratio is designated as "secondary" (secundaria) and includes all those ratios which can be divided into equal rational ratios by assigning mean proportional numbers. For example, $4 / \mathrm{I}=4 / 2 \cdot 2 / \mathrm{I}, 8 / \mathrm{I}=8 / 4 \cdot 4 / 2 \cdot 2 / \mathrm{I}$, $9 / 1=9 / 3 \cdot 3 / 1$, and so on.
Oresme then shows how these types of rational ratios may serve as bases for irrational ratios. If a primary rational ratio denominates some irrational ratio-i.e. serves as base-it must remain in that form since the irrational ratio is then "most properly denominated" (est competentissime nominata). Thus, $(2 / 5)^{1 / 3}$ and $(6 / 5)^{1 / 3}$ are in final form because no mean proportional numbers are assignable between the terms of these two rational bases.

But if the base is a secondary rational ratio, Oresme outlines certain steps which must be followed in order to determine the most proper denomina-tion-i.e., the most proper rational ratio that will serve as base. If we are given an irrational ratio such as $(B)^{1 / q}$, where $B$ is a secondary ratio, it is necessary to decompose $B$ into its constituent primary ratios. Let us represent the primary ratios by $A$ so that $B=(A)^{m}$, where $m$ is an integer, and $(B)^{1 / q}=\left(A^{m}\right)^{1 / q}$. Should $m$ and $q$ be mutually prime numbers greater than I , we must revert to the original form $(B)^{1 / q}$, which expresses the irrational ratio as a part of rational ratio $B$. For example, $(8 / \mathrm{I})^{1 / 2}$ is most
"Si autem, numerus minimarum, id est una quarta proportionis $64^{\text {p }}$. Sed quia $64^{p}$ primariarum, partium talis proportionis componitur ex duplis et 6 , qui est numerus rationalis secundarie a qua denominatur proportio irrationalis et denominator illius proportionis irrationalis que est pars ipsius sint numeri communicantes tunc accipiatur maximus numerus in quo communicant et per ipsum dividendus est uterque illorum. Et dividendo numerum partium proportionis secundarie provenit numerus proportionum partialium ex quibus componitur proportio rationalis a qua denominatur proprissime proportio proposita. Dividendo, vero, dominatorem propositum per eundem maximum numerum prius habitum venit denominator proportionis irrationalis proprissimus et quesitus.
"Verbi gratia, proponatur proportio que vocetur tres quarte quadruple tunc agendo per tertiam regulam patet quod ipsa est
partium primararium istius $64^{\mathrm{ple}}$, et 4 , qui
est denominator proportionis proposite, sunt communicantes in 2 igitur dividendo 6 per 2 exit 3, ergo proportio proposita est pars trium duplarum, scilicet pars $8^{\text {ple }}$. Similiter dividendo 4 per 2 venit 2 , igitur proportio proposita est una medietas. Pa tet, ergo, ex hac regula quod proportio proposita est una medietas octuple et scribitur sic $1 / 28^{p}$ et ista est eius denominatio competentior. Eodem modo una duodecima quatuor triplarum, scilicet $8 \mathrm{I}^{\mathrm{ple}}$, est $\mathrm{I} / 3$ $3^{\text {ple }}$ et similiter una quarta sex triplarum est una medietas trium triplarum, scilicet $27^{\text {ple }}$, et cetera."-Grant, "Mathematical Theory of Oresme," pp. 333-35. A few abbreviated forms, such as $8^{\text {ple }}$ and $3 / 44^{\text {p }}$, have been written out in full.
properly denominated because the alternative form is $(2 /)^{3 / 2}$ and does not express the relationship in terms of a single part. Another alternative form, $(4 / \mathrm{I})^{3 / 4}$, is rejected for the same reason.
If, however, $m$ and $q$ are not mutually prime, we must make the appropriate divisions which would reduce them to their lowest terms. If after reduction $m / q=\mathrm{I} / n$, where $n$ is an integer, the final form will be $(A)^{1 / n}$. But should $m / q$ reduce to $p / n$, where $p$ and $n$ are integers greater than x , the proper form will not be $(A)^{p / n}$ but rather $(D)^{1 / n}$, where $D=(A)^{p}$.

Two examples will serve to illustrate the two cases distinguished in the preceding paragraph. If $\left(8_{1} /\right)^{1 / 12}$ is a given irrational ratio, we can reduce this to $\left[(3 / \mathrm{I})^{4}\right]^{1 / 2}$, where $4 / 12$ reduces to $1 / 3$, yielding the final form $(3 / \mathrm{I})^{1 / 3}$, which most properly denominates $\left({ }^{81} /\right)^{1 / 1 / 2}$. As an instance of the second case, Oresme offers $(4 / \mathrm{I})^{3 / 4}$. By the third rule it is necessary to expand $(4 / \mathrm{I})^{3}$ to $(64 / \mathrm{I})$. Now $(64 / \mathrm{I})$ contains six primary double ratios, i.e., $(64 / \mathrm{I})=(2 / \mathrm{I})^{6}$, so that $(64 /)^{1 / 4}=(2 / \mathrm{I})^{6 / 4}$, which reduces to $(2 / \mathrm{I})^{3 / 2}$. But this is unacceptable because $(2 / \mathrm{I})^{3 / 2}$ is not expressed as a part of a rational ratio even though the base, $2 / 1$, is a primary ratio. Since the most fundamental criterion in such reductions is that the irrational be expressed ultimately as a part of some rational ratio, Oresme expands $(2 / 1)^{3 / 2}$ to $(8 / \mathrm{I})^{1 / 2}$, which under the circumstances is the most proper ultimate form for $(4 / \mathrm{I})^{3 / 4}$.
In the De proportionibus, the extent of Oresme's advance beyond the level of the Algorismus is immediately apparent in the sevenfold division of rational ratios (I.381-4I3 and II.I.I- 5 2), of which six are possible and the seventh impossible. Although the terms primaria and secundaria do not occur in the De proportionibus, the two concepts together constitute the first of the seven subdivisions, where it is stated that rational ratios that can have geometric means assigned are divisible into equal rational ratios-this corresponds to secondary ratios in the Algorismus-and those that cannot have such means assigned are not so divisible and may be equated with the primary ratios of the Algorismus. The other six ways play no role in the Algorismus, which is, perhaps, another indication that the Algorismus is an earlier treatise (see p. 14). Nevertheless, the terms pars and partes are used exponentially in the Algorismus, and this represents an important step toward the later developments in the De proportionibus. The essential conceptual ingredients of the De proportionibus were fully fashioned when Oresme added the notion of commensurability and incommensurability to the notion of part and parts found in the Algorismus.

## The Origin and Influence of the Central Theme in the De proportionibus

In the De proportionibus the major concern was with rational and irrational ratios of ratios based ultimately on the notion of the commensurability of exponential parts. This division into rational and irrational ratios of ratios -probably original with Oresme-arose ultimately from Bradwardine's function, which, soon after its formulation, had been quickly adopted at the University of Paris where, in all likelihood, Oresme first became acquainted with it. His subsequent contributions have already been described (see above, pp. 24-6s) and constituted a unique departure from his predecessors and contemporaries.

The influence of Oresme's special treatment of ratios of ratios and its application to ratios of motion is surprisingly limited. To be sure, his name is mentioned often enough as one of the partisans of Bradwardine's function, ${ }^{00}$ but few concerned themselves with specific propositions or concepts
${ }^{90}$ Alessandro Achillini, for example, insisting that a ratio of ratios varies as a ratio of denominations (but see III.93-10I, and below on p. 357), says that almost all the "moderns" he has read would disagree with his view. The "moderns" mentioned are Bradwardine, Swineshead, and Oresme. "Hae regulae modernis fere omnibus quos legerim de proportione proportionum loquentibus contrariae sunt. Ideo advertendae, ut Thomae Braduardino et consequenter Suiset Calculatori, Nicolao Orem, etc." He goes on to say that Aristotle and Averroes agree with him and the "ancients" generally.-De proportione motuum, in Achillini...opera omnia in unum collecta, fol. $18 \rho \mathrm{v}$, c. 1 .

Oresme's De proportionibus is mentioned with disapproval in a treatise, Volumniz Rodulphi Spoletani De proportione proportionum disputatio, published in Rome in 1516. The author, Rodulphus of Spoleto, is unknown to me. Rodulphus, who rejects the ways of the "moderns," says that ratios can be formed legitimately only from quantities. He refers to Campanus' commentary on Euclid Bk.V, Def. I (see De proportionibus I.272-77), to the effect that two quantities are related as a line, three as a surface,
and four as a solid, and then accuses Oresme of violating this by making ratios indiscriminately and dividing quantities in any manner beyond four terms. Furthermore, Oresme divides ratios as if they were quantities (an apparent reference to I.26171). This seems to be the sense of a rather murky passage. "Campanus, qui in expositione xi diffinitionis quinti, voluit proportionem in duabus quantitatibus esse simplex intervallum et habere naturam lineae; proportionem vero extremorum in tribus terminis habere naturam superficiei; et in terminis quattuor retinere naturam solidi. Hoc idem et adversarii usurpant cum Nicolaus horem in suo libello De proportionibus id supponat proportionesque ipsas passim ac si mere essent quantitates dividat, ac regulis quibus et alias quantitates subiiciat et obnoxias esse velit." The treatise is unnumbered, but this passage appears in the third part. A copy of the work is in the Rare Book Library of the University of Wisconsin.

Giovanni Marliani, an opponent of Bradwardine's function, mentions Oresme's De proportionibus in his Questio de proportione motuum in velocitate, written in 1464. See Clagett, Giovanni Marliani, p. 139.
from the De proportionibus. There are but two exceptions known to me. These are George Lokert and Alvarus Thomas. The former, in his Tractatus proportionum, uses the crucial concepts of rational and irrational ratios of ratios ${ }^{91}$ and the notion of a unit ratio as the common measure of two commensurable ratios. ${ }^{92}$ Although Lokert does not mention Oresme, it is likely that his source was Oresme or Alvarus Thomas, who was probably a contemporary of Lokert's at the University of Paris in the early sixteenth century. If Alvarus Thomas was his immediate source, then Lokert would

It is of interest to realize that these three opponents are Italian. In the Italian schools of the fifteenth and sixteenth centuries there was a strong tendency to interpret Aristotle as faithfully as possible. A departure of the kind represented by Bradwardine's function was not too warmly received, since its acceptance would have required an interpretation of Aristotle's law of motion grossly at variance with the Áristotelian texts.
${ }^{91}$ "Sequitur secundus articulus. Consequenter est dicendum de proportione proportionum et proportionalitate. Unde est advertendum quod differentia est inter proportionem proportionum et proportionalitatem. Nam proportio proportionum est unius proportionis ad alteram certa habitudo ut proportionis duple ad sesquialteram. Et potest dividi in proportionem proportionum equalitatis et in proportione proportionum inequalitatis; et ita subdividatur proportio proportionum inequalitatis in proportionem proportionum rationalem et in proportionem proportionum irrationalem. Et proportio proportionum rationalis inequalitatis in proportionem proportionum rationalem maioris inequalitatis et in proportionem proportionum rationalem minoris inequalitatis, et sic consequenter sicut diximus de proportione quantitatum. Et proportionabiliter dentur diffinitiones membrorum illo dempto quod in proposito sit comparatio proportionis ad proportionem sicut in precedentibus fiebat comparatio quantitatis ad quantitatem."-Lokert, Tractatus proportionum, sigs. Aaaiiiir, c.2-Aaaiiiiv, c.I. Lo-
kert emphasizes the analogy between relating quantities and relating a ratio to a ratio. This was, of course, basic to Oresme's entire treatise.
${ }_{92}$ Lokert expresses the unit ratio concept in discussing commensurability between ratios. He says that $(2 / \mathrm{I})^{1 / 2}$ is the common measure of all ratios $(2 /)^{n}$, where $n$ is any integer. Furthermore, he says that a rational ratio and an irrational ratio can be commensurable (indeed the example just given is such an instance). Finally, he notes that it does not follow that one rational ratio must be commensurable to another rational ratio, and it does not follow that an irrational ratio must be incommensurable to another irrational ratio. All of this is ultimately from Oresme's De proportionibus. Here is the Latin text: "Ad secundum partem dubitationis de commensurabilitate proportionum dicitur. Proportiones ille sunt commensurabiles que eadem proportione commensurari possunt, ut dupla, quadrupla, octupla, [sedecupla], etc. Omnes tales mensurantur medietate proportionis duple. Et opposito modo dicuntur incommensurabiles que non possunt eadem proportione commensurari, ut dupla, tripla, sesquialtera, etc. Sequitur correlarie proportionem rationalem proportioni irrationali esse commensurabilem. Et ex consequenti, rationalis ad irrationalem esse proportionem rationalem; non tamen oportet quod quarumcumque rationalium adinvicem sit proportio rationalis, nec quod quarumcumque irrationalium adinvicem sit proportio irrationalis."-Lokert, Tractatus proportionum, sig. Aaaiiiiv, c.I.
still be indebted ultimately to Oresme since Alvarus relies heavily on Oresme, citing the De proportionibus in a number of places.

Alvarus is the only author known to me who shows an extensive acquaintance with, and understanding of, Oresme's treatise. Although Alvarus was lavish with praise for Oresme, ${ }^{93}$ he did not hesitate to criticize and reformulate propositions. ${ }^{94} \mathrm{He}$ adopted the basic notion of rational and irrational ratios of ratios as evidenced by a chapter which is concerned with "a ratio of ratios and their commensurability and incommensurability." ${ }_{95}$ Many propositions from Chapters II, III, and IV of the De proportionibus are substantially repeated with many additions, elaborations, and further illustrations. In a section consisting of four propositions dealing with ratios of velocities and methods for determining the ratios of force and resistance producing them, Alvarus makes it clear that all four are taken from Chapter IV of Oresme's De proportionibus and adds that he wishes this to be known because he refuses to profit improperly from the labor of others. ${ }^{96}$ Furthermore, Alvarus is the only author known to me who so much as attempts
${ }_{93} \mathrm{He}$ calls Oresme "the most learned investigator of proportions." ("Respondeo ponendo quandam propositionem quam ponit doctissimus proportionum indagator magister Nicholaus Horen."-Alvarus Thomas, Liber de triplici motu, sig. q.6v, c.2.)
${ }^{94}$ See ibid., sig. d.sv, c.2; and above, pp. $56-58$, n. 77 .
${ }_{95}$ "Capitulum sextum in quo agitur de proportionum proportione commensurabilitate earundem et incommensurabilitate." This is PartII, Ch. 6 of ibid., sig. d. 4 r , c.I.
${ }^{96}$ "Et hec quatuor conclusiones (ne alienis spoliis triumphare videamur) ex officina et perspicaci minerva doctissimi magistri Nicolai Horen deprompte sunt et excerpte quas in suo tractatu proportionum quarto capite suis fulcimentis et probationibus mathematicis reperies munitas." -Alvarus Thomas, Liber de triplici motu, sig. p. 4v, c.2. The four propositions correspond to three propositions in Ch.IV of the De proportionibus. Alvarus' first proposition (sigs. p. $3 \mathrm{v}, \mathrm{c} .2-\mathrm{p} .4 \mathrm{r}, \mathrm{c}$. $)$ corresponds to Prop. II; the second proposition (sig. p. 4r, c.1.) to Prop. III (IV.262-70); the third proposition (sig. p. $4 \mathrm{r}, \mathrm{c} .1-\mathrm{c} .2$ ) to

Prop. III (IV.22s-61); the fourth proposition (sigs. p. 4r, c. 2-p. 4v, c.2) to Prop.IV. Earlier, Alvarus gave the substance of part of Prop. VII of Ch.IV (IV.500-507; 525-29) without mentioning Oresme. He says that if $F_{2} / R_{2}=6 / 2$ and $F_{\mathrm{I}} / R_{\mathrm{I}}=3 / 2$, where $R_{2}=R_{1}$, then the distances traversed will be incommensurable because the ratio, or exponent, relating $6 / 2$, a triple ratio, and $3 / 2$, a sesquialterate ratio, is irrational. " Et si aliqua virtus moveat aliquam resistentiam a proportione sexquialtera et alia movet eandem resistentiam in proportione tripla, tunc virtus movens a proportione tripla velocius movet virtute movens proportione sexquialtera in ea proportione qua tripla sexquialteram exuperat. Et quia talis proportio que est inter triplam et sexquialteram est irrationalis, ut ex sexto et septimo capitibus secunde partis facile monstratur, ideo nec spacium pertransitum a proportione tripla excedit spacium pertransitum a proportione sexquialtera in proportione aliqua multiplici, nec superparticulari, nec suprapartiente, nec multiplici superparticulari, nec multiplici suprapartiente, quod postea magis elucidabitur."-Ibid., sig. f. $3 \mathrm{r}, \mathrm{c} .2$.
to repeat Oresme's proposition that among many ratios an unknown ratio of ratios will probably be irrational. Attributing the proposition to Oresme, Alvarus says that "wherever a multiplicity of ratios occurs between which a ratio is not easily found, it must be understood that many of them are irrational." 97 This is a pale reflection of Proposition X of Chapter III and would have undoubtedly confused anyone unfamiliar with Oresme's treatise. It fails to mention that any such unknown ratio of ratios would very probably be irrational. Instead, Alvarus says only that many such ratios will be irrational. Indeed, nothing more is said of this anywhere else in the Liber de triplici motu. Thus the one person who is known to have largely understood and utilized Oresme's De proportionibus failed to grasp the culminating proposition of the entire work. It is not surprising, therefore, that Oresme's treatise was, for the most part, unread and little discussed.

## The Missing Fifth and Sixth Chapters and the Ad pauca respicientes

That the De proportionibus proportionum was meant to comprise more than four chapters seems incontestable from the fact that in all the manuscripts it is expressly divided into six chapters (I.24-33; but see below for the peculiar status of the description of the fifth chapter as given in I.31). Despite this unanimity, however, not a single one of the manuscripts numbers more than four chapters. Indeed, it is only in the two early printed editions (see pp. 130-32) that we find a total of six chapters, where the fifth and sixth chapters are none other than the first and second parts, respectively, of the $A d$ pauca respicientes. It would appear that here is genuine evidence that the two editions were made from manuscripts containing the fifth and sixth chapters of the De proportionibus and that, somehow, the final two chapters were detached and circulated as an independent treatise. A closer inspection, however, reveals the implausibility of this interpretation.

Of all the known manuscripts of the De proportionibus only two are followed immediately by the treatise which, for convenience, we have called the Ad pauca respicientes, after its opening words. In one of these manuscripts, $H$ (see pp. 125-26), we find on fol. mor the following con-

[^7]
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cluding statement after Chapter IV: "Explicit tractatus de proportionibus datus a magistro Nicolao Oresme." On fol. I Iov the Ad pauca respicientes follows as a separate anonymous treatise ${ }^{98}$ bearing the title De astrologia aliqua specialia (see p. 126).

It is the second of these manuscripts, however, that provides the basic clue explaining the divergence of the editions from the manuscripts. In MS $V$ (see p. 128) the first four chapters, which are properly numbered, are followed immediately by the $A d$ pauca respicientes without any indication of a break between the two treatises. Since the Ad pauca is untitled, one cannot discern that a new treatise has begun. It is quite likely that both editions, which are very similar, were based on MS $V$ itself or on some unknown manuscript sharing certain peculiarities with $V$. This is indicated by the fact that of the five $A d$ pauca manuscripts only $V$ omits the opening lines of the treatise (APı.1-13) and Proposition VIII of the second part (AP2.93-105). Since both editions also omit the very same lines, there is good reason for suspecting that both editions were probably made from MS $V$ or some other very closely related manuscript.

But if $V$ served as the basis for the two editions, how are we to explain the fact that the editions-but not $V$-designate Parts One and Two of the $A d$ pauca respicientes as Chapters V and VI, respectively, of the $D e$ proportionibus? Apparently, the editor or editors, reading in I.24-33 that there were six chapters in the De proportionibus and finding no break between the end of Chapter IV of the De proportionibus and the beginning of the $A d$ pauca, assumed that Part One of the latter was Chapter V of the former and that Part Two was Chapter VI. Such a move would have received further support from the fact that I.32-33 describes the proposed subject matter of the sixth chapter of the De proportionibus as treating of the incommensurability of the celestial motions-precisely the topic treated in both parts of the Ad pauca respicientes. ${ }^{99}$ Finally, at the end of the $A d$
${ }^{98}$ The Ad pauca is definitely ascribed to Oresme only in MS Paris, BN lat. 7378A, fols. ${ }^{4} \mathrm{~V}-17 \mathrm{~V}$ (see variant readings $\mathrm{AP}_{2}$. 270), MS London, British Museum, Sloane 2542, fols. $5 \int \mathrm{~V}-59 \mathrm{r}$ (see below, p. 38 I ), and the two editions. But there is no reason to doubt its attribution to Oresme.
${ }^{99}$ How the editors reconciled Part I of the Ad pauca respicientes with the description of Ch. V in I .3 I is unclear. But they may have interpreted "velocities of motions," the subject matter of the pro-
posed fifth chapter of the De proportionibus, in a kinematic sense, which could easily be taken to embrace the material of Part I of the Ad pauca. This would have contrasted with the dynamic approach of Ch. IV, where "ratios of motions" (I.30-3I)-i.e., ratios involving forces and resistanceswere treated. Unfortunately, the only certainty in all this is the unquestioned carelessness of the editors who supervised the enterprise.
pauca, MS $V$ concludes with "Deo gracias. Explicit tractatus," signifying the end of the work. Recalling that in MS $V$ there were no concluding remarks at the end of the fourth chapter of the De proportionibus, we see that the editor was led to believe that he had now truly reached the end of the $D e$ proportionibus and not the end of a quite separate treatise. In this manner, the two treatises were united to comprise the De proportionibus with its full complement of six chapters. Although the incorporation of the Ad pauca respicientes into the De proportionibus was a mistake, there is indeed a genuine connection between these two treatises-a connection which offers the best solution to the fate of the missing chapters.
Having established in a preliminary way the distinctness of the two treatises edited in this volume, we must now conjecture as to the connections between them and then argue for our conclusions. In the first place, the $A d$ pauca respicientes was probably written before the De proportionibus itself. When, however, Oresme came to write the De proportionibus he realized that it provided a solid foundation for the earlier $A d$ pauca and viewed the latter as a fitting kinematic complement to Chapter IV of the De proportionibus, which had dealt with ratios of motions dynamically. ${ }^{100}$ Oresme, therefore, decided to attach the Ad pauca respicientes as the last chapter or chapters (see below, pp. $76-78$ ) of the De proportionibus and so asserted in the Proemium (I.24-33). But before this plan materialized, Oresme became dissatisfied with the Ad pauca respicientes and proceded on a major revision of that treatise which culminated in a completely new and much expanded work that he called De commensurabilitate vel incommensurabilitate motuum celi. As a consequence of this, the Ad pauca respicientes was completely superseded and abandoned, and no attempt was made to append the De commensurabilitate as the concluding portion of the De proportionibus. Thus, the De proportionibus never received the final sections announced in its own Proemium and circulated instead as a four-chapter treatise.

Evidence in support of the contention that the $A d$ pauca respicientes was written as a separate treatise is overwhelming. The $A d$ pauca is not divided into chapters but rather into two parts. Thus the opening line of Part Two reads: "The second part of this work [now] begins" (AP2.I ; see also AP2. 50, 161-62, 166-67, 181, 208-9, 212), signifying the second part of an independent treatise and not the sixth chapter of the De proportionibus. In the second part references are made to the first part (see AP2.92, 190-91,
${ }^{100}$ Bradwardine's Tractatus de proportioni- cally and kinematically. This became quite bus also treated ratios of motion dynami- commonplace.

2I2) but never to a fifth chapter. Indeed, there is not a single reference from the $A d$ pauca to Chapters I-IV of the De proportionibus. If the $A d$ pauca were truly the fifth and sixth chapters of the De proportionibus, Oresme would almost certainly have made references from the former to the latter. Thus, in Supposition II of the Ad pauca respicientes (APr.36-38) Oresme asserts the probability that "if many quantities are proposed and their ratios are unknown, it is possible, doubtful, and probable that any [one of them] would be incommensurable to any other." It should be noted (see pp. $6 \mathrm{I}-63,88$ ) that the mathematical basis for this supposition is undoubtedly De proportionibus, Chapter III, Proposition X, and the general proposition enunciated in IV.556-6I. If the first part of the Ad pauca were Chapter V of the De proportionibus, it is hardly conceivable that Oresme, who furnished abundant and usually appropriate cross-references throughout Chapters I-IV, would have failed to mention so crucial a discussion in an earlier chapter. The same two references-Chapter III, Proposition X, and IV. 556-6I-should have been cited to account for the statement in APr.I3234 that any two motions are probably incommensurable. Instead, Oresme offers the weak and peculiar reason that incommensurable ratios have many causes thereby making it more likely that any ratio of circular motions would be incommensurable.

Another significant instance can be cited from APr.28-32 where Oresme says that the number of stars is probably not a cube number since there are fewer cube numbers than other kinds of numbers. Now in III.370-80 Oresme asserts essentially the same opinion, and yet no reference is made from APr to Chapter III, or vice versa. Further evidence is found in the fact that the Great Year is mentioned as an error in the last paragraph of Chapter IV (IV.606-9) and ridiculed anew in the introductory lines of the first part of the $A d$ pauca (API.7-13) without any reference from the latter to the former. Indeed, the similarity between IV.606-9 and API. 7-13 may explain why the scribe who copied MS $V$-the only manuscript where an attempt is made to fuse the De proportionibus and Ad paucaeliminated APr.i-13. It may be guessed that whoever placed the $A d$ pauca respicientes immediately after Chapter IV recognized the repetitious character of API.7-13 with respect to IV.606-9 (they deal with the Great Year and are separated by only a few lines) and eliminated the former lines by simply omitting API.I-I3. If this were so, it would reinforce our contention that we have two separate works which could be given the appearance of one overall treatise only by scribal or editorial surgery of the kind just mentioned. Whatever the merits of this conjecture, it is a fact that the
other five manuscripts of the $A d$ pauca respicientes circulated as quite independent works, thereby solidly buttressing the strong evidence adduced above for the view that the Ad pauca respicientes was composed as a separate treatise.

We must now produce evidence to show that the Adpauca respicientes was written prior to the De proportionibus and that Oresme had genuinely intended to append it as the concluding chapter or chapters of the De proportionibus. In IV.579-82 he says: "Now that I have declared that any celestial motion might be incommensurable to any other celestial motion, many very beautiful propositions that I arranged at another time follow, and I intend to demonstrate them more perfectly later, in the last chapter, among which will be these." This is followed by the enunciation of four propositions(IV.583-600) offered as samplings of the "many" propositions slated for the "last chapter." Thus, the treatise referred to by Oresme is one that he seems to have written prior to the composition of the De proportionibus and that he deemed suitable for addition to the latter treatise after some needed revision. Furthermore, this unnamed treatise should contain, it seems, not only the four illustrative propositions in some recognizable form, but a reasonable number of other propositions involving the incommensurability of celestial or circular motions. Of all the treatises ascribed to Oresme, only two qualify as genuine candidates for the above reference-namely, the Ad pauca respicientes and the De commensurabilitate vel incommensurabilitate motuum celi. But the De commensurabilitate cites the De proportionibus (see pp. 6I-62, n. 81) and was almost certainly composed after it. Only the $\operatorname{Ad}$ pauca respicientes properly qualifies since it contains not only the four illustrative propositions mentioned in IV.583-600 (see p. 375), but many others involving incommensurable motions. Furthermore, the $A d$ pauca seems to have been written earlier than the De proportionibus, since, as already noted, it fails to cite the latter work in support of Supposition II (API.36-38) for the likely reason that Oresme had not yet formulated Proposition X of Chapter III. Indeed, the very fact that the De proportionibus is not even mentioned in the $A d$ pauca-despite the obvious relevance of the former for the latter-is reasonable evidence for assuming that the $D e$ proportionibus had not yet been written.
The treatise to which Oresme alludes in IV.579-82 is without doubt the same work referred to in I.32-33 where we are told that in the sixth chapter of the De proportionibus he will "speak about the incommensurability of celestial motions by correcting certain things which, on another occasion, I
had treated briefly in [the course of] reflecting on a few matters." Thus, IV. 579-82 and I.32-33 refer to a treatise that was scheduled to become the last chapter of the De proportionibus. On the specific information provided in IV.579-82 this work is most plausibly identified as the Ad pauca respicientes. ${ }^{101}$ It is even possible, though perhaps farfetched, that in I.32-33
${ }^{101}$ V. P. Zoubov, in his article, "Quelques observations," in Isis, Vol. so, 13034, suggests that I.32-33 may refer to an anonymous work Questio de proportione dyametri quadrati ad costam ejusdem, which Heinrich Suter published as "Die Quaestio 'De proportione dyametri quadrati ad costam ejusdem' des Albertus de Saxonia," in Zeitscbrift für Mathematik und Pbysik, Vol. 32, 41-56. Suter ascribed the treatise to Albert of Saxony, but Zoubov (see his article in Isis, Vol. 50, 131) believes it is by Oresme because of a brief section in the work devoted to the incommensurability of the celestial motions-a topic identified with Oresme but not Albert. With reference to I.32-33, and citing the Venice edition of IsOs, Zoubov writes (p. 134): "La Quaestio ne serait-elle alors cet écrit d'Oresme, dont il parle dans son traité $D e$ proportionibus proportionum: 'corrigendo quedam que alias ad pauca respiciens breviter pertransivi.'" This suggestion is untenable if, as we have already argued, I. 32-33 and IV. $579-82$ refer to the same treatise. One of the propositions that we expect to find in the unnamed treatise mentioned in IV. 579-82 involves the simultaneous motion of three planets (IV. 591-94) which having once conjuncted will never conjunct again. No counterpart to such a proposition appears in the anonymous Quaestio, where only two mobiles are mentioned in the few brief propositions devoted to the incommensurability of the celestial motions. Furthermore, no proposition in the Quaestio resembles the illustrative proposition in IV.595-600. Indeed, the discussion pertinent to the incommensurability of celestial motions is so meager in the anonymous Quaestio that it could not be the treatise in question, since the treatise referred to by Oresme was to include
"many other no less beautiful propositions" (see IV.604-6).
However, Zoubov's attribution of the Quaestio to Oresme seems reasonable not only because it incorporates a theme dear to Oresme's heart, but also because of the striking resemblance between it and portions of Oresme's Quaestiones super geometriam Euclidis. All of the propositions in the Quaestio concerned with the incommensutability of celestial motions, or points moving on circles, find counterparts in Questions 7 and 9 of the Quaestiones super geometriam Euclidis. These latter two questions are both concerned with the single problem "whether the diagonal of a square is commensurable to its side" ("Utrum dyameter quadrati sit commensurabilis coste"). Although the anonymous Quaestio has a fuller treatment, in some instances almost identical passages occur in the two treatises. For example, the Quaestio has the following section:
"Ulterius sequitur, supposito quod tempus in quo sol facit unam revolutionem annalem sit incommensurabile diei, sicut est verisimile et etiam ignotum est an ita sit, tunc impossibile est, verum kalendarium invenire. Ulterius sequitur, supposito quod tempus in quo sol facit unam revolutionem et tempus quo luna facit unam sint incommensurabilia, et posito cum hoc, quod omnes revolutiones solis sint equales et similiter lunae, dico quod si luna oriatur punctualiter in aliquo instanti alicujus horae, ut forte in instanti medio horae tertiae alicujus diei, si mundus duraret in eternum, nunquam in instanti consimili consimilis horae oriretur; declaratio istorum patet ex declaratione secundi corrollarii illati ex conclusione $3^{a}$ et $4^{\text {a }}$. Ex quibus sequitur quod judicia astrologorum sunt aliquando valde incerta."-Suter, "Die Quaestio,"

Oresme is actually citing the Ad pauca respicientes by name．If so，ourtransla－ tion would have to be altered as follows：＂And in the sixth chapter I shall speak about the incommensurability of celestial motions by correcting certain things which，on another occasion，I had treated briefly in the $A d$ pauca aspicientes．＂The title Ad pauca aspicientes could be a variant of $A d$ pauca respicientes，or Ad pauca respiciens（see variant readings for the term aspicientes in I .33 and for respicientes in APr．1）．One is reluctant to pursue this argument since Oresme would，presumably，have assigned a specific title ${ }^{102}$ rather than refer to it by its opening words，as we have chosen to do as a matter of convenience．

Up to this point，we have argued that the Ad pauca respicientes was com－

Zeitschrift für Mathematik und Pbysik，Vol． 32，p． 50.
The same two propositions，in reverse order，are found in Oresme＇s Quaestiones super geometriam Euclidis．Despite the re－ versed order of these two propositions in the two treatises，it will be evident imme－ diately that the wording is sufficiently close to eliminate the element of coincidence．
＂Quintum correlarium et unum suppo－ nendo et est istud，quod dies solis，puta tempus in quo sol facit revolucionem，et dies lune sunt tempora incommensurabilia， sicut est verisimile et puto cum hoc，quod omnes revoluciones solis，essent equales et similiter omnes lune，quod si non sit ve－ rum，adhuc illud，quod sequitur，est magis verisimile quam［instead of＇et＇］est istud， quod si luna oriatur punctualiter in aliquo instanti alicuius hore，ut puta in instanti ［instead of＇〈in instanti medio〉＇］ $3^{e}$ hore〈alicuius diei〉，dico quod，si mundus dura－ ret in eternum，numquam alias oriretur in consimili instanti consimilis hore，quia si detur［instead of＇dico＇］oppositum， statim probatur oppositum antecedentis， sicut patet ex una alia questione［instead of iconclusione＇］；et ex hoc sequitur，quod ＇udicia astrologorum sunt valde incerta．
＇Sextum correlarium est，quod supposi－ to，quod tempus，in quo sol facit unam revolucionem annualem［instead of＇anna－ lem＇］，sit incommensurabile diei，sicut est verisimile et est eciam ignotum＜an ita sit＞，dico，quod impossibile est，verum ca－
lendarium invenire et hoc potest ex prece－ dentibus declarari quia［instead of＇sim－ iliter＇］impossibile est［instead of＇（est＞＇］ veram quantitatem anni invenire．＂ Oresme：Quaestionessuper geometriam Euclidis， ed．Busard，Fasc．I，24－25．（In a long review of Busard＇s edition in Scripta Matbe－ matica，Vol．27，67－91，John Murdoch has supplied numerous essential emendations and corrections to Busard＇s text．Where required，these have been incorporated into the passages just quoted．The emenda－ tions appear on p． 82 of Murdoch＇s review； the discarded readings from Busard＇s text appear in square brackets．）

Other similarities unrelated to the in－ commensurability of the celestial motions could also be noted．There is obviously a connection between these two treatises， and，even more to the point，since Oresme is thus far the only author known to have discussed in detail the incommensurability of the celestial motions－and this in a num－ ber of different treatises－there is ample reason for tentatively assigning the anony－ mous Quaestio to him．But，neither of these two treatises is a proper candidate for the references in I．32－33 and IV．579－82．
102 Various titles do appear in MSS $F$ （see below，p．380），$B$（see p．380），$H$（see p．126），and British Museum，Sloane 2542 （see p．380），but since no two agree it is impossible to determine whether any one of them originated with Oresme．
posed as a separate treatise prior to the writing of the De proportionibus， and that Oresme intended to append it as the final chapter of the latter work．That he appears not to have done this is borne out by the fact that the $A d$ pauca was never integrated with the first four chapters of the $D e$ proportionibus．Our previous discussion pointing out the lack of cross－ref－ erences where these would normally be expected bears this out．But the question now arises as to why Oresme did not attach the $A d$ pauca to the De proportionibus．According to his own assertions in I．32－33 and IV．579－ 82，his intention was to correct and perfect the treatise we have called $A d$ pauca respicientes and then add it to the De proportionibus as the final chapter． It seems that in the process of revision he became dissatisfied with its overall organization，as well as with some of its propositions and con－ cepts，so that by the time he had thoroughly reworked it，a new and much expanded version emerged，which he called De commensurabilitate vel in－ commensurabilitate motuum celi．This revised treatise completely superseded the earlier $A d$ pauca．

Possible evidence that the revision was in process while Oresme was still at work on the first four chapters of the De proportionibus is found in IV． $577-78$ ，where Oresme justifies the probable incommensurability of any two celestial motions by remarking，＂．．．this seems especially true since， as I shall declare afterward，harmony comes from incommensurable mo－ tions．＂No mention is made of this notion in the Ad pauca respicientes，but it is discussed in the De commensurabilitate（see p．375）．Perhaps this was one of the numerous additions Oresme had in mind for the revised $A d$ pauca．

The evidence is quite plausible for supposing that the De commensurabi－ litate is an enlarged and revised version of the Ad pauca．The latter is a poorly organized and skeletal treatise．Where the $A d$ pauca indiscriminately intermingles propositions dealing with both commensurable and incom－ mensurable motions，the De commensurabilitate gathers into Part I all prop－ ositions involving commensurable motions，and into Part II all those con－ cerned with incommensurable motions．${ }^{103}$ Most of the propositions retained from the＇Ad pauca have been improved！and expanded in the De commen－ surabilitate．For example，the content of Proposition II of the first part of the $A d$ pauca is distributed over Propositions VI，VII，and VIII of Part I of the De commensurabilitate．The unfruitful concept of properly and im－ properly similar dispositions（see pp． $9^{8-100}$ ）is dropped in the De commen－ surabilitate，as are the strange tenth and eleventh propositions of the second
${ }^{103}$ See Grant，＂Oresme：Comm．＂
part of the $\operatorname{Adpauca}\left(\mathrm{AP}_{2}\right.$.127-38). A significant piece of evidence indicating that Oresme looked upon the mature De commensurabilitate as superseding the $A d$ pauca respicientes emerges from the fact that although the former treatise makes reference to two places in the first four chapters of the De proportionibus (see pp. 6x-62, n. 81), it never mentions any proposition in the $\operatorname{Ad}$ pauca. This is astonishing since the subject matter of the two treatises is identical. It is readily explicable, however, if we see the $A d$ pauca as an early full-scale attempt by Oresme to cope with the problem of commensurability and incommensurability of circular motions. But the mature De commensurabilitate so completely replaced the $A d$ pauca that Oresme could find no good reason for citing an earlier treatise, the sound parts of which were now incorporated in the De commensurabilitate and whose unsound features and details were better forgotten. A brief remark in the De commensurabilitate seems to substantiate this interpretation. In what is almost certainly an allusion to the $A d$ pauca respicientes, Oresme explains, "In this book [i.e., the De commensurabilitate] I have set forth some assumptions from other mathematical treatises [and] from them have inferred propositions, a few of which I discovered after I had written elsewhere." ${ }^{104}$

One more difficulty must now be considered. Was the De proportionibus originally envisioned as a five- or as a six-chapter treatise? According to all the manuscripts, the answer is unequivocally in favor of six chapters. But the description of Chapter V (I.3r) does not correspond to any one of the four chapters of the De proportionibus nor to anything in the Ad pauca respicientes. Ratios of motion were to be dealt with in Chapter IV (I.30-3I) and the description of Chapter VI (I.32-33) must apply to the entire $A d$ pauca respicientes, since both parts of it deal equally with the incommensurability of the celestial motions. This interpretation is corroborated by the fact that of the four illustrative propositions (IV. $583-600$ ) which Oresme says are to appear in the last chapter (in ultimo capitull ; IV. 579-82), three are found in the first part of the $A d$ pauca (see p. 375). This clearly indicates that Oresme looked upon the Ad pauca respicientes as a single, concluding chapter to the De proportionibus. Now the fifth chapter was to

[^8]deal with "velocities of motions" but Chapter IV deals not only with ratios of force and resistance, but also with ratios of velocities. Thus, on the one hand, material which is ostensibly to be taken up in Chapter V is actually considered in Chapter IV of the De proportionibus, and, on the other hand, the description of Chapter $V$ does not correspond with anything in the $A d$ pauca, the content of which agrees with the description of Chapter VI in I. 32-33.

Is the description of Chapter V an error or an interpolation? Was the De proportionibus envisioned as a five-chapter work, where the fifth and final chapter was to embrace the entire Ad pauca respicientes, which, however, was numbered as chapter six because, somehow, the description of an imaginary fifth chapter was interpolated in I. 31 and perpetuated in all the manuscripts? Or, indeed, did Oresme actually write a fifth chapter dealing with "velocities of motions," 105 which, like the $A d$ pauca respicientes, was never attached to the work?

## Capsule Summary of De proportionibus

Having completed a detailed summary analysis of the De proportionibus, we shall now present a capsule review of the substance of Oresme's achievements. In essence, his major contribution was to distinguish two types of irrational ratios, namely ( 1 ) those with rational exponents and (2) those with irrational exponents. This distinction derives from Oresme's special interpretation of the Euclidean terms "part" (pars) and "commensurable" (commensurabilis). Applying these terms exclusively to geometric proportionality, Oresme would hold that if $(A / B) p / q$ is irrational, where $(A / B)$ is a rational ratio and $p / q$, the exponent, is a ratio of mutually prime numbers, then when $p=\mathrm{r},(A \mid B)^{p / q}$ is a part of $(A \mid B)$ and when $q>p>\mathrm{I}$, $(A \mid B)^{p l q}$ is parts of $(A \mid B)$. In each case the irrational ratio $(A \mid B)^{p l q}$ is said to be commensurable to $A / B$. However, should $p / q$ be itself irrational, then $(A / B)^{p / q}$ cannot be a part or parts of $(A / B)$.
This important distinction reveals that Oresme had a clear grasp of the concept of an irrational exponent. This is, perhaps, the most significant

[^9]contribution emerging from the De proportionibus and may have been original with Oresme. This twofold division of irrational ratios was subsequently applied to probability considerations and problems of terrestrial and celestial motion.

Toward the conclusion of his thitd chapter, Oresme shows by way of example that for a given set of rational ratios not in geometrical progression, it is probable that any two of them will be relatable only by an irrational exponent. This follows from his example in which it is shown that rational ratios such as $3 / 1$ and $4 / 1$, etc., cannot be made equal by rational exponents, but are relatable only by irrational exponents. That is, $3 / 1 \neq$ $(4 / \mathrm{I})^{p l q}$ if $p / q$ is rational. Now because Oresme accepts Bradwardine's function as true, he can immediately apply his probability results to hypothetical physical relationships described in terms of Bradwardine's function. Thus, it is probable that velocities produced by $F_{2} / R_{2}$ and $F_{\mathrm{I}} / R_{\mathrm{I}}$, respectively, are incommensurable so that $F_{2} / R_{2}=\left(F_{\mathrm{I}} / R_{\mathrm{I}}\right)^{V_{2} / V_{\mathrm{I}}}$, where $V_{2} / V_{\mathrm{I}}$ would very likely be irrational. Similarly, Oresme shows that any two celestial motions are probably incommensurable. In fact, he extends the application of this special case of mathematical probability to all continuous magnitudes which are relatable as ratios.

By extending mathematical probability to all ratios of continuous quantities Oresme has, in effect, made inexactitude and imprecision an essential aspect of mathematical physics and astronomy. The exponents representing the relationships between any two ratios of quantities are probably irrational, and, consequently, inexpressible. This is the end product of Oresme's mathematics of geometric proportionality and his special interpretation of the terms pars and commensurabilis. Actually, Oresme chose to invoke and dwell upon mathematical imprecision only with respect to celestial motion, and this largely because it served him well in his attacks upon astrology and its practitioners. This aspect is elaborated in the $A d$ pauca respicientes and the later De commensurabilitate.
Although largely uninfluential, the' significance of Oresme's De proportionibus lies in the mathematical foundation it furnished for Bradwardine's function, or law of motion, and the manner in which it logically extended the basic mathematical concepts outlined above into the realm of mathematical probability, culminating in the application of mathematical probability to physical and astronomical phenomena.

## III

# The $\operatorname{Ad}$ pauca respicientes 

Summary and Analysis

THE relationship between the earlier Ad pauca respicientes and the De proportionibus proportionum has already been considered in the preceding section. Here it is only necessary to summarize and evaluate the $A d$ pauca respicientes as well as note certain other connections between it and the De proportionibus.

We have already seen how important was the principle that between any two celestial motions there is probably an incommensurable relationship (IV.573-76). It was a key weapon in Oresme's battle against astrology. But apart from such direct utility, Oresme seems to have been genuinely and intellectually interested in investigating the consequences that arise from the motions of two or more bodies moving with uniform circular motion and having velocities assumed now commensurable and now incommensurable. How often, if at all, would they be in conjunction, or opposition, and when and where would these events occur? Answers to such questions were radically different depending on the initial assumption of mutually commensurable or incommensurable velocities. It is with problems of this kind that Oresme is concerned in the Ad pauca respicientes. Indeed, in some cases, when consequences or conclusions are sufficiently interesting and important, Oresme transforms abstract bodies moving on circles into celestial bodies carried on their spherical orbs.
The sharp contrast in approach between Chapter IV of the De proportionibus, on the one hand, and the Ad pauca respicientes, on the other, is noteworthy. In both places there is a common concern with ratios of veloc-
ity. But in Chapter IV the basic interest was dynamic in the sense that the propositions related ratios of velocities and the ratios of force and resistance which produced those velocities. Even for celestial motion Oresme noted the dynamic analogy between moving intelligences and terrestrial forces on the one hand, and resistance of celestial spheres with terrestrial resistances or mobiles on the other. But in the Ad pauca respicientes the interest is exclusively kinematic. It is by means of distance and time-not force and resistance-that velocities are compared.

The main elements that concern Oresme in Chapter IV and the $A d$ pauca respicientes may be outlined as follows: (a) Two ratios $A$ and $B$ are related as a ratio of ratios $m / n$ when $A=(B)^{m / n}$. (b) If $F_{2} / R_{2}=A$ and $F_{1} / R_{\mathrm{I}}=B$, then $F_{2} / R_{2}=\left(F_{1} / R_{\mathrm{I}}\right)^{V_{2} / V_{\mathrm{I}}}$, where $F, R$, and $V$ are force, resistance, and velocity, in that order, and $V_{2} / V_{I}=m / n$ since a ratio of velocities varies as a ratio of ratios. (c) $S_{2} / S_{I}=V_{2} / V_{\mathrm{I}}$ when $T_{2}=T_{1}$, where $S$ is distance and $T$ time, which signifies that a ratios of distances varies as a ratio of velocities. (d) And finally $T_{\mathrm{I}} / T_{2}=V_{2} / V_{\mathrm{I}}$ when $S_{2}=S_{\mathrm{I}}$ and thus a ratio of times varies (inversely) with a ratios of velocities. In Chapter IV of the De proportionibus Oresme moves from (a) $\rightarrow$ (b) $\rightarrow$ (c) $\rightarrow$ (d) as well as from $(\mathrm{d}) \rightarrow$ (c) $\rightarrow(\mathrm{b}) \rightarrow(\mathrm{a})$, concentrating, however, on relations between (a) and (b). But throughout the $A d$ pauca respicientes the problems are couched exclusively in terms of elements (c) and (d). Thus Oresme has followed a traditional division between dynamics and kinematics. ${ }^{\text {I }}$

At the outset of the $A d$ pauca (API.1-4), Oresme reveals his disagreement with the opinions of those astrologers who believe they can know the exact punctual positions of conjunctions, oppositions, and any other celestial aspect. He intends to show that this is overwhelmingly improbable. His skepticism on this point must not, however, be construed as disgust with the imprecision of astronomy, for he says in the last paragraph of the work, "...it is sufficient for a good astronomer to judge motions and aspects near a point, and that his senses do not observe and judge the opposite. But one who wishes to seek more, or believes he knows, labors in vain..." (AP2.263-65). ${ }^{2}$ But if astronomers must settle for approxima-

[^10]tions, Oresme, who is not an astronomer, is not constrained by such limitations. Indeed, it is just such a quest for punctual exactness that serves as the basic motivation of the Ad pauca respicientes. ${ }^{3}$ Where conditions of commensurability or incommensurability permit, it is precise positions and times of occurrence of various celestial aspects that Oresme seeks.
Of the four suppositions in the first part of the $A d$ pauca, the second (APr.36-38) and third (APr.45-50) are of special significance. The second supposition is in substance practically identical to the general proposition enunciated in De proportionibus IV.556-61 (see p. 60) and asserts: "If many quantities are proposed and their ratios are unknown, it is possible, doubtful, and probable that any [one of them] would be incommensurable to any other" (APr.36-38). But why "possible, doubtful, and probable"? For an explanation we must turn to APr.14-32 where Oresme discusses the different usages of the term "possible."

Oresme distinguishes two general uses of the term. (r) In one sense possible applies to statements that are contingent or necessary. That is, the statements to which we apply the term possible are themselves either contingent or necessary propositions. If a statement is contingent, it is obviously possible. But, under certain conditions, even a necessary proposition may claim no higher status than that of mere possibility (see under
strong arguments in behalf of concepts regarded as absurd from the standpoint of traditional physics and astronomy. In such instances his objective was to show that seemingly ridiculous physical concepts could not be conclusively repudiated and, indeed, could even be made to appear as plausible as the traditionally accepted views. Scientific truths about physical nature usually were unattainable. As a theologian, Oresme was anxious to convince others that if the human mind was largely incapable of arriving at physical truth, how much the less could it philosophize about and demonstrate truths of the Christian faith. See Grant, "Late Medieval Thought," Journal of the History of Ideas, Vol. 23, $210-11$.
${ }^{3}$ This motive is unexpressed in the $A d$ pauca respicientes, but in the De commensurabilitate, where Oresme deals with much the same subject matter, he is quite explicit: "Intentio in hoc libello est loqui de precisis et punctualibus aspectibus mobilium circu-
lariter, et non de aspectibus prope punctum de quibus communiter intendunt astronomi qui non curant nisi quod non sit sensibilis defectus, quamvis modicus error imperceptibilis multiplicatus per tempus notabilem efficat." -MS Vat. lat. 4082, fol. 98r, c.I, quoted in Grant, "Oresme: Comm.," p. 42I, n. 5 . For a translation, see below on p. 440 . Many of the quotations from the De commensurabilitate will be drawn from Grant, "Oresme: Comm." Since the Latin passages quoted there are based on a number of manuscripts, the references to MS Vat. lat. 4082 are given for convenience only and it should be understood that the Vatican manuscript will sometimes vary from the textual material in Grant, "Oresme: Comm." Where no reference is made to Grant, "Oresme: Comm.," the Latin passage quoted from the De commensurabilitate will be based solely on MS Vat. lat. 4082 unless otherwise specified.

2b). (2) The second general use of the term possible is linked with doubt, and this is twofold. (2a) In one way (APr.15-17) we have a pair of contradictory contingent statements where each statement is initially possible in virtue of its contingency. But we do not know which statement is true and, consequently, each is possible because of this doubt. (2b) In the second sense of possible used in the context of doubt (AP1.17-19) we have two contradictory statements where one is necessary and the other impossible. Assuming now that we are ignorant of which statement is necessary and which impossible, then, in face of such doubt, a threefold sub-division is necessary: (2b-r) Each contradictory is equally possible. (2b-2) One of the contradictories is improbable. (2b-3) The other is probable.

Oresme offers examples for each of the cases under (2b). Thus for (2b-r) he offers, "The number of stars is even; the number of stars is odd" (APr. 20-21). ${ }^{4}$ Assuming the stars to be finite in number, Oresme insists that one
${ }^{4}$ Cicero, in his Academica, raises the same question during a discussion on uncertainty and probability: "For if a question be put to him [i.e., to a wise man] about duty or about a number of other matters in which practice has made him an expert, he would not reply in the same way as he would if questioned as to whether the number of stars is even or odd, and say that he did not know; for in things uncertain there is nothing probable, but in things where there is probability the wise man will not be at a loss either what to do or what to answer." - Academica, II. (xxxiv) in De Natura Deorum; Academica, with an English translation by Rackham, p. 609. The brackets are mine.
In commenting upon De caelo I. 12.28 Ib . 7-14, Oresme considers the meaning of the terms possible and impossible in his Le Livre du ciel et du monde (edition of Menut and Denomy in Mediaeval Studies, Vol. 3, $258-59$ ), and, in the course of his discussion of the term possible, includes the example of the stars and whether they are odd or even.
The term possible, says Oresme, is used in three ways. In one sense, it applies to every thing that is contingent or necessary. This is identical with ( I ) above, in API 14-I5. In the second way, possible is applic-
able to anything imaginable even though that which is imagined could not be generated naturally. For example, it is possible that there might be another world, or that a place could exist in a vacuum, or that the heavens might rest, or the earth move, and so on. This second aspect has no counterpart in the Ad pauca respicientes. But the third subdivision links possible with doubt and is identical with (2), in APr.15-19, although it lacks the more elaborate distinctions of the Ad pauca. Something doubtful is neither true nor false, necessary nor impossible. For example, continues Oresme, we say that it is possible that the number of stars is odd-numbered. Aristotle, on the other hand, would argue that it is either necessary that the stars be odd-numbered, or it is impossible that they be odd-numbered. Oresme accuses Aristotle-who, let it be noted, makes no mention of doubt in the present context-of failing to realize that in cases of doubt we must use possible and not insist that a decision be made in terms of necessity or impossibility. In cases where knowledge is impossible by the very nature of things, we must, perforce, settle for both alternatives being possible. By denying this distinction and following Aristotle, one could demonstrate that the stars are both odd- and even-numbered,
of these statements is necessarily true and the other impossible. But since we cannot determine whether the stars are even- or odd-numbered, it is obvious that each is equally possible.
The examples for ( $2 \mathrm{~b}-2$ ) and ( $2 \mathrm{~b}-3$ ) are concerned with the following pair of contradictories: "The number of stars is a cube [number]" (APr.28) and "The number of stars is not a cube [number]" (AP1.3r). The first statement is possible, but improbable and unlikely (APr.28-30), since Oresme holds there are fewer cube numbers than other kinds of numbers, ${ }^{5}$
that the motions of the heavens are both commensurable and incommensurable, and so on.

Oresme's discussion in Le Livre du ciel et du monde is more extensive than in the Ad pauca and includes one additional usage of the term possible-namely, that which can be imagined without contradiction. But the distinctions made in the Ad pauca under doubt-(2a) and (2b), with the latter further subdivided into three parts-are not repeated in Le Livre du ciel et du monde. I now give the French text of the arguments summarized above: "Je di aprés que possible est dit quant a propos en .iii. manieres. Premièrement, generalment de tout ce qui est possible en quelconque(s) maniere, soit necessaire ou contingent, comme que soit. Secondement, possible est dit de ce qui pourroit estre selonc ymagynacion sanz contradiction, combien que ce ne puisse estre mis en estre naturelment. Et ainsi est possible que un autre monde soit et que un lieu soit du tout vieu ou que le ciel repouse ou que la terre soit meüe et mise hors de son lieu, ou que elle soit [perciee] et que l'en voie de l'autre part et telles choses. Tiercement, possible est dit de ce qui est doubteus ou en doubte et n'appert pass[e] c'est vray ou faulz, necessaire ou impossible, et ainsi disons nous estre possible que le nombre des estoilles est nomper, et toutevoies selonc Aristote, ou c'est necessaire ou c'est impossible selonc ce que dit est. Et semblablement diroit l'en que c'est possible que aucuns des mouvemens du ciel sont inconmensurables, et toutesvoies, selonc la philosophie d'Aristote, c'est necessaire ou impossible-a prendre impossible selonc aucune autre des significacions de-
vant mises. Et par ygnorance de ceste dis$\mathrm{t}[$ inct $]$ ion, aucuns ont cuidé faire nouvelles demonstracions a prouver que Diex est par sophisme telz, si comme qui voudroit prover que les estoilles sont nomper en ceste maniere. Il est possible, que les estoilles soient nomperet ne enclot quelconque(s) contradicion, et donques se ce est mis en estre, il ne s'ensuit quelconque(s) impossible. Or pousons donques que aucune foys elles soient nomper, et d'autre partie, selonc Aristote, c'est impossible que ou ciel soit faite addicion ou substraction d'aucune estoille. Et donques est ce chose perpetuelment necessaire que elles soient nomper et ne puet estre autrement. Mais l'en puet veoir clerement que cest argüement est sophistique, quar, par semblable l'en prouveroit que elles sont per, et semblablement de la conmensurableté ou inconmensurableté des mouvemens du ciel. Et la deffaute de tel argüement est quar quant l'en dit que c'est possible, se l'en prent possible selonc la premiere ou la seconde significacion, ce ne seroit pas a octroier selonc Aristote, mais est a doubter; et donques la conclusion seroit doubteuse. Et se possible est prins en la tierce maniere, premisses et conclusion,--tot demoure en doubte."-Oresme: Le Livre du ciel, eds. Menut and Denomy, in Mediaeval Studies, Vol. 3, 259.
5 This is asserted in the later De proportionibus III. 370-80, where it is stated that there are fewer cube numbers than many other kinds of numbers. In terms of a one-to-one correspondence there are, of course, as many cube numbers as square or natural numbers.
say the natural numbers, or square numbers, etc. Here, then, we have case (2b-2), where one of the contradictories is improbable and unlikely, though still possible. The other contradictory, "The number of stars is not a cube [number]," is not only possible but also probable and likely (APr.31-32), and this is an illustration of $(2 b-3)$.

We are now in a position to understand why Oresme says of Supposition II that it is "possible, doubtful, and probable." That any two magnitudes may be incommensurable is obviously "possible" because Supposition II is either contingent or necessary in the sense of (r) on p. 85 . But neither Supposition II nor its contradictory is demonstrably true and consequently each is "possible." From this it follows that the truth of Supposition II is "doubtful" in the sense of (2b). But Oresme says that Supposition II is also "probable" (verisimile) as in (2b-3), but offers no reasons for this statement, perhaps because in a suppositio or assumption it was unnecessary to do so. But had the De proportionibus already been written it is very likely that both IV. 556-6I and Proposition X of Chapter III would have been cited as grounds for believing in the probability that the ratio between any two unknown quantities would be incommensurable.
Supposition III (APr.45-50) makes it evident why Oresme labels Supposition II as "possible, doubtful, and probable." Man is simply unable to acquire exact knowledge about ratios of quantities-e.g., ratios of circles or distances traversed-pertaining to celestial motions. Indeed, we cannot even know ratios of quantities involving things very close to us. For this reason consideration of such celestial magnitudes must, of necessity, be couched in terms of "possible, doubtful, and probable." Thus Supposition III, while concentrating on celestial motions, serves to underscore the more general second supposition.

Part One of the Ad pauca respicientes comprises nine propositions. In Proposition I Oresme gives the necessary conditions for two mobiles to enter into conjunction (for definition of conjunction, see APr.52-54) in a point in which they must previously have conjuncted, and in which they will conjunct in the future. He then shows how to determine the time interval between successive conjunctions in a given point (APı.72-76).
Two mobiles can conjunct repeatedly in the same point if they travel on unequal circles or equal circles. If they travel on unequal circles, three essential conditions must obtain (API.55-57). The circles, or circumferences, must be commensurable and, secondly, the mobiles must in equal times traverse distances which are commensurable. The third condition is negative (API.57) and says that circle $A /$ circle $B \neq S_{A} / S_{B}$, where circle
$A$ is the circle on which mobile $A$ travels and circle $B$ that on which mobile $B$ travels, and $S_{A}$ and $S_{B}$ are the angular distances traversed by mobiles $A$ and $B$ respectively. This is necessary, as Oresme explains (APı. 60-64), because if circle $A /$ circle $B=S_{A} / S_{B}$, then in equal times mobiles $A$ and $B$ would traverse equal angles and the relative positions of $A$ and $B$ would remain constant. Hence if they were once in conjunction, they would forever remain in conjunction; if not initially in conjunction, they could never conjunct but would remain equidistant.
If the circles are equal, then it is necessary that the mobiles be moved with unequal but commensurable velocities.
In the first part of the proof (API. $65-71$ ), the circles are assumed unequal and the velocities of mobiles $A$ and $B$ are commensurable. Now if $A$ and $B$ are in conjunction in point $c$, they will conjunct again in $c$ when circle $A$ has been traversed $n$ times and circle $B m$ times so that circle $A \cdot n=$ circle $B \cdot m$, where $m$ and $n$ are integers. ${ }^{6}$ Given the same conditions, conjunctions in $c$ must have occurred previously and will occur regularly in the future. The same situation obtains if the circles are equal and the distances traversed are unequal but commensurable.
In finding the time interval between any two successive conjunctions in point $c$, Oresme takes up first the case where circles are unequal and then when they are equal. ${ }^{7}$

The first case is presented in two parts ( $\mathrm{AP}_{\mathrm{I} .72-76}$ ), in each of which the circles are unequal and the arcal, or curvilinear, velocities are equal. ${ }^{8}$ In the
${ }_{6}$ This part of Prop. I has its counterpart in Part I, Prop. IV, of Oresme's De commensurabilitate. The enunciation of the proposition reads, "Si duo mobilia nunc sint coniuncta necesse est ut alias in puncto eodem coniungantur."-MS Vat. lat. 4082, fol. 98v, c.2. See Grant, "Oresme: Comm.," p. 425.
7 Essentially the same proposition appears in the De commensurabilitate, Part I, Prop. V. "Tempus invenire quando primitus coniungentur in puncto in quo nunc sunt."-MS Vat. lat. 4082, fol. 99r, c.i. See Grant, "Oresme: Comm.," p. 425.
${ }^{8}$ In his De configurationibus qualitatum, Oresme clearly distinguishes between curvilinear velocity and angular velocity. In Part II, Ch. IV, of that treatise, Oresme has a section "On different kinds of velocity," in which he says: "It should not be over-
looked that the same movement or flux is called by many terms connoting different things. Hence velocity denominated in different ways is attended or measured accordingly as its quantity of gradual intension is assigned in different ways....For example, in the first way, in circular motion a body is said 'to be moved' (moveri), and it is [also] said 'to revolve' (circuire). Now the intension of a velocity of motion (i.e., rectilinear or curvilinear motion) is measured by the linear space which will be traversed at that degree [of speed]. But the intension of a degree of a rotary velocity (velocitas circuitionis) is measured by the angles described about the center. Hence it happens that one body moved circularly in comparison to another is moved more quickly but revolves less rapidly. Thus perhaps Mars is moved more rapidly in its proper
first part (APr.72-75), if circle $A$ is greater than circle $B$, let them be related as circle $A /$ circle $B=m / n$, where $m$ and $n$ are integers in their lowest terms and $m$ is multiple to $n$. Should $m$ and $n$ be the number of days in which mobiles $A$ and $B$ complete one circulation of their respective circles, we can divide circle $A$ into $m$ parts and circle $B$ into $n$ parts. Therefore, $A$ traverses I $/ m$ part of its circle every day and $B$ moves over $\mathrm{I} / n$ part of its circle daily. Since $m / n$ is in its lowest terms and $m$ is multiple to $n$, the number of days between conjunctions in $c$ is given by $m$ (APr.74-76; the specific example in APr.77-80 is given on pp. 431-32).

But if-and this is the second part (APr.75-76)-m/n is in its lowest terms and $m>n>\mathrm{I}$, it is obvious that $m$ is not multiple to $n$. Since we are told that on any given day $A$ traverses $\mathrm{I} / m$ part of its circle and $B$ I $/ n$ part of its circle, we must multiply $m$ and $n$ and the product yields the number of days between successive conjunctions in $c$ (see API. 8 I - 83 and p. 432 for a specific example).

In the second case (APr.84-85) the circles are equal and the curvilinear velocities are unequal. Since, for the same data, mobiles moving under these conditions would produce the same results as in the two parts of the first case, Oresme simply asserts that the same steps should be followed as when the circles are unequal.
Proposition II subsumes a number of propositions that were later formulated as separate propositions in the De commensurabilitate. If mobiles are moving under conditions laid down in Proposition I, then either the mobiles will conjunct in places other than $c$, the only point of conjunction discussed in Proposition I, or they will conjunct elsewhere in the interim between successive conjunctions in $c$ (APr.89-92). If they do not conjunct elsewhere in the interim, then $c$ is their only point of conjunction. Oresme assumes they do conjunct in more than one point and describes how to find the total number of fixed places of conjunction (API.95-100). 9 If the
motion than the sun because of the magnitude of the circle described, and yet the sun makes a quicker circuit [in terms of angular velocity] and revolves more swiftly around the center...."-Translation by Clagett, Science of Mechanics, pp. 355-56 (the Latin text appears on p .376 ). This distinction is never made in the Ad pauca respicientes.
Although Oresme sometimes speaks of circles moving with unequal speeds and presumably carrying the mobiles with
them (APr.72-73), at other times he talks of mobiles moving on circles as if the latter were stationary. Since these are equivalent, I shall, for the sake of consistency, attribute the motion to the mobiles, thus assuming that the circles are stationary.
${ }^{9}$ In the De commensurabilitate, this is done in Part I, Prop. VII. "Datis duobus motibus duorum mobilium, numerum coniunctionum totius revolutionis invenire." -MS Vat. lat. 4082 , fol. 99V, c.I. A simpler procedure is given in Part I, Prop.

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speeds of mobiles $A$ and $B$ are represented by $V_{A}$ and $V_{B}$ respectively, where $V_{B}>V_{A}$, then Oresme prescribes the following steps:
(I) Divide the difference of the velocities into the number of parts into which the circle is divided. Thus if $q$ is the number of parts into which the circle is divided, then the quotient of $q /\left(V_{B}-V_{A}\right)$ yields the time of the first conjunction of $A$ and $B$ after departure from their previous point of conjunction. ${ }^{10}$ In an example (APr.104-6) where $q$ is given as 12 equal parts, mobile $A$ moves 4 parts per day and $B 9$ parts so that $q /\left(V_{B}-V_{A}\right)$ $=12 /(9-4)=22 / 5$. Thus $22 / 5$ days elapse between the departure of $A$ and $B$ from conjunction to the very next conjunction in a different point.
(2) Next divide the time of one revolution of $A$ and $B$-i.e., the time between any two successive conjunctions of $A$ and $B$ in the same pointwhich can be found by Proposition I, by the time of the first conjunction. To continue with Oresme's example (APr.ro6-11), $A$ moves $4 / 12=1 / 3$ of its circle per day and $B$ moves $9 / 12=3 / 4$ of its circle every day. By Proposition I they will conjunct in the same point every 12 days (multiply the denominators as specified in APr.75-76) and this is their period of revolution. Then dividing 12 , the period of revolution, by $2 / 5$, the time elapsed before the next conjunction in another point, we get

$$
\frac{12}{2^{2} / 5}=5
$$

the total number of fixed points of conjunction. Thus, there will be $s$ conjunctions of $A$ and $B$ during every period of revolution of 12 days. The successive conjunctions occur every $22 / 5$ days for a total of $s$ over the 12-day period of revolution.
After determining the time between successive conjunctions, Oresme shows how to find all the places of conjunction of $A$ and $B$ which occur during the course of a period of revolution (APr.iol-3). ${ }^{\text {II }}$ This is done by first multiplying either $V_{A}$ or $V_{B}$-actually $S_{A}$ or $S_{B}$, since $V \propto S$ where $S$ is distance-with the time between two successive conjunctions
XI. See Grant, "Oresme: Comm.," pp. 426, 429-30.
10 Part I, Prop. VI, of the De commensurabilitate tells " $[\mathrm{how}$ ] to find the time of the first conjunction following when the velocities of two mobiles now in conjunction have been given" ("Datis velocitatibus duorum mobilium nunc coniunctorum, tempus prime coniunctionis sequen-
tis reperire").-MS Vat. lat. 4082, fol. 99v, c.I. See Grant, "Oresme: Comm.," p. 425 .
${ }^{11}$ In the De commensurabilitate this is shown in Part I, Prop. VII, where the enunciation reads as follows: "Datis duobus motibus duorum mobilium, numerum conjunctionum totius revolutionis invenire." -MS Vat. lat. fol. 99v, c.I. See Grant, "Oresme: Comm.," p. 426.
occurring during the period of revolution. Thus, in the previous example, $2^{2} / 5$ days elapse between two successive conjunctions, and $A$ travels $1 / 3$ of its circle every day, so that $22 / 5 \cdot 1 / 3=4 / 5$. Then subtract $4 / 5$ from the next highest integer representing the whole circle (APr.102-3), i.e., $\mathrm{I}-$ $4 / 5=1 / 5$. Therefore, $A$ and $B$ will conjunct in $s$ points of the circle with each point separated from its neighbors by $1 / 5$ of a circle. The same results are obtained if we multiply $V_{B}$ (i.e., $S_{B}$ ) by $22 / 5$, namely $22 / 5 \cdot 3 / 4=$ I $4 / 5$. Subtracting I $4 / 5$ from the next highest integer, namely 2 , we get $2-\mathrm{I} 4 / 5=1 / 5$.
Now if we wish to find the first place of conjunction following upon the departure of $A$ and $B$ from conjunction in point $c$, which is the initial point of conjunction commencing a period of revolution, we multiply either $V_{A}$ or $V_{B}$ (i.e., $S_{A}$ or $S_{B}$ ) with the time between successive conjunctions in the course of a single revolution of $A$ and $B .{ }^{12}$ Thus $A$ travels $1 / 3$ of its circle daily and $1 / 3 \cdot 22 / 5=4 / 5$, signifying that after departure from $c, A$ and $B$ will conjunct first in a point $4 / 5$ of the way round the circle from $c$. The same result is achieved by using the distance traversed by $B$ since $3 / 4 \cdot 2^{2} / 5=14 / 5$, where only the fraction is relevant since $B$ will have completed one more circuit around its circle than $\operatorname{did} A$ on its own circle, but will have overtaken $A$ at a point $4 / 5$ of the circle distant from the immediately preceding place of conjunction, $c$.

Proposition III asserts that mobiles moving commensurably as described in Proposition II will have passed through identical dispositions an infinite number of times, and will do so an infinite number of times in the future (APr.112-13). In Proposition II there were five points or places of conjunction, and Proposition III tells us that mobiles $A$ and $B$ will have conjuncted in these points an infinite number of times in the past and will do so an infinite number of times in the future.
Points of conjunction, opposition, quadrature, and other aspects are fixed in number for any given set of conditions of commensurable motion specified in Proposition II. But for any particular point of conjunction or opposition there will be only one disposition or arrangement of mobiles. Then without further elaboration, Oresme says, "...the places of other dispositions, however, are twice as many [in number]" (APr.iri6). ${ }^{13}$ But in the De commensurabilitate he says: "However, we must distinguish
${ }_{12}$ This is demonstrated in Part I, Prop. -MS Vat. lat. 4082, fol., 99v, c.r. See VIII, of the De commensurabilitate. "Datis Grant, "Oresme: Comm.," pp. 426-27. duobus mobilibus nunc coniunctis, locum prime coniunctionis sequentis assignare."
${ }^{13}$ See AP2.5-18, and below on pp. 9798.
the trinal aspect before a conjunction from the trinal aspect following the same conjunction, and this applies to any aspect with the exception of conjunction and opposition since every other aspect is twofold, namely before conjunction and after, in one way from the right and the other from the left." ${ }^{14}$

A trinal aspect is one where the mobiles are separated by four signs or an angle of $120^{\circ}$. Prior to any conjunction, the slower mobile precedes the faster, in the sense that for conjunction to occur the faster-moving mobile must overtake the slower. Applied to a trinal aspect, before conjunction the slower mobile will precede the faster by $120^{\circ}$. After conjunction, however, the two mobiles reverse their relative positions, and it is now the faster mobile which precedes the slower by $120^{\circ}$. The same reasoning may be applied to other aspects such as sextilis and quartilis, which are separated by two and three signs of the zodiac respectively. ${ }^{15}$
Inferring from Proposition III to planetary motions, Oresme says that if the sun and moon were moved commensurably they would bein conjunction and opposition in only a finite number of places. Consequently, there would be an infinite number of places in which they would never conjunct (APr.117-19). Thus, if Mars and the sun traversed their orbits in exactly two and one years respectively, they would conjunct in only one place.
In Proposition IV, Oresme considers the case where two mobiles are moving unequally and incommensurably with respect to the common center of their respective circles (APr.122-24). ${ }^{16}$ As a consequence of such motion the mobiles will eternally describe incommensurable angles in equal times. Now there are a number of different ways in which two mobiles can move with respect to each other and traverse incommensurable angles in equal times. Oresme outlines four different ways (APi.127-34; see also pp. 432-33), selecting one of them for the purpose of demonstration.
In his reductio ad absurdum proof, Oresme assumes that mobiles $A$ and
${ }^{14}$ "Verumtamen, distinguendus est aspectus trinus ante coniunctionem ab aspectu trino ipsam coniunctionem sequenti et sic de quolibet aspectu seu modo se habendi exceptis coniunctione et oppositione, quoniam omnis alter aspectus est dupliciter, scilicet ante coniunctionem et post, una vice a dextris et alia a sinistris."-MS Vat. lat. 4082 , fol. 102 It , c. i. See Grant, "Oresme: Comm.," p. 438, n. 39, where I said mistakenly that aspectus trinus "refers to the three kinds of aspects, namely sextile,
quartile, and trinal"; it applies only to the trinal aspect.
${ }^{15}$ Grant, "Oresme: Comm.," p. 437.
${ }^{16}$ In the De commensurabilitate, Oresme rigorously separates propositions where the motions are commensurable from those where they are incommensurable. The former are found in Part I, the latter in Part II. See Grant, "Oresme: Comm." In the Ad pauca respicientes, Oresme mixes them indiscriminately.
$B$ are moving with equal curvilinear velocities on circumferences which are unequal and incommensurable (as described in APr.127-29), so that if they were in conjunction in point $c$ they would traverse unequal and incommensurable angles in equal times. He assumes that $A$ and $B$ conjunct again in $c$ and then shows that for this to happen, circumferences $A$ and $B$ must be commensurable by APr.39-44, which is contrary to the supposition that they are incommensurable. ${ }^{17}$ Although Oresme fails to make explicit the obvious consequence that $A$ and $B$ will never conjunct twice in the same point, he makes use of this in the next proposition.
Two mobiles moving under the conditions described in Proposition IV will have been in conjunction in an infinite number of different points, and will conjunct in an infinite number of yet different points in the future. This is the message of Proposition V based on the obvious consequence from Proposition IV (mentioned at the end of the previous paragraph) that $A$ and $B$ will never conjunct twice in the same point, and the assumption of motion in an infinite past and an eternal future. ${ }^{18}$ As one example (APi.rso-53), Oresme says that should two mobiles conjunct at the intersection, or nodal point, of two circles they would never again conjunct there (see pp. 433-34 for further discussion).
Oresme applies the results of Propositions I and II, where only two mobiles were considered, to Proposition VI where three mobiles are in motion with commensurable velocities. Applying Proposition I first, he states that three mobiles now in conjunction "will be and have been in conjunction an infinite number of times through an eternal motion" (APr. 161-62). Proposition II is relevant when Oresme says that the three mobiles can conjunct in only a finite number of places. Proposition VI includes little else than the bare enunciation, and no procedural details are offered for treating three mobiles (but see Proposition VII of Part I and pp. 95-96). However, in Proposition XIV of Part I of the De commensurabilitate, ${ }^{19}$ which

17 An almost identical proof is given in Part II, Prop. I, of the De commensurabilitate. The enunciation reads as follows: "Si duo talia mobilia incommensurabiliter mota, nunc sint coniuncta numquam alias in puncto eodem coniungentur."-MS Vat. lat. 4082 , fols. $102 \mathrm{~V}, \mathrm{c} .2-103 \mathrm{r}, \mathrm{c} . \mathrm{I}$. See Grant, "Oresme: Comm.," p. 443.

18 When Oresme assumes eternal motion he is "speaking naturally" (naturaliter loquendo; see IV. 58 -86)-i.e., in terms of natural philosophy and physics. Obvious-
ly, as a Christian and theologian, Oresme believed that the world had a beginning in time and would terminate in the future.
19 MS Vat. lat. 4082, fol. IoIr, c.i. See Grant, "Oresme: Comm.," p. 434. The second part of Prop. VI, which asserts that the three mobiles can conjunct in only a finite number of places, can be derived from Part I, Props. XVII and XIX, of the De commensurabilitate (the enunciations are given on p. 434 of the article cited above).

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is similar to the first part of Proposition VI above, Oresme assumes that three mobiles start from a point of conjunction and then by taking them two at a time demonstrates that they must conjunct again in the same point. As in Proposition VI, he draws on an identical earlier proposition which demonstrated the same thing for two mobiles.
Under certain specified conditions, Proposition VII demonstrates that three mobiles moving with commensurable velocities may never conjunct. ${ }^{20}$ In this proposition, unlike the previous ones, the calculations do not begin with the mobiles in conjunction since the object of the proposition is to demonstrate that they cannot conjunct. Hence only two of the three mobiles are assumed to be in conjunction at the outset, with the third removed from them by a certain angular distance.
Let the mobiles be $A, B$, and $C$. Now if it can be shown that none of the points of conjunction of $A$ and $C$ are also points of conjunction of $A$ and $B$, the proposition will be demonstrated. The remaining combination of $B$ and $C$ is irrelevant, for if $A, C$ and $A, B$ share no points of conjunction, then wherever $B$ and $C$ may conjunct, $A$ cannot simultaneously join them, since this signifies that $A, C$ and $A, B$ would share a point of conjunction, although the opposite was assumed to have been shown.
In the proof Oresme assumes that

$$
\frac{\text { circle } A}{\text { circle } C}=\frac{6}{3}=\frac{2}{1} \text { and } \frac{\text { circle } A}{\text { circle } B}=\frac{6}{4}=\frac{3}{2} .
$$

Finally, mobile $B$ precedes by $1 / 6$ of its circle mobiles $A$ and $C$ which are in conjunction in point $d$ (see p. 4or, Figure 5). The mobiles move with equal curvilinear velocities but the unequal sizes of the circles produce unequal, though commensurable, velocities. If they commence moving clockwise toward $f$, it is evident by Proposition I that $A$ and $C$ can conjunct only in $d$. On the other hand, $A$ and $B$ are assumed to conjunct in point $e$, which is opposite $d$. Thereafter, $A$ and $B$ will conjunct only in $e$. Oresme offers no details, but if $B$ precedes $A$ by $1 / 6$ of its circle it must move only $2 / 6$ of a circle to arrive at $e$, half way round the circle from $d$. During the same time interval, $A$ will have moved $3 / 6$, or $1 / 2$, of its circle and conjunct with $B$ in $e$. Thus every sixth day $A$ and $B$ will conjunct in
${ }^{20}$ Prop. XII of Part I of the De commensurabilitate demonstrates the very same thing. "Si fuerint mobilia plura duobus possibile est quod numquam coniungentur simul plura quam duo."-MS Vat. lat. 4082, fol. roov, c.i. See Grant, "Oresme:

Comm.," pp. 430-31. Since in Part I only commensurable velocities are considered, it is clear that this proposition is relevant to commensurable velocities, and therefore corresponds to Part 1, Prop. VII, of the Ad pauca respicientes.
$e$ and nowhere else. Since $A, C$ conjunct only in $d$, while $A, B$ conjunct only in $e$, the three mobiles will never conjunct simultaneously.

The enunciation of Proposition VIII does not agree with the data furnished in the demonstration itself. Although the objective of the proposition is to demonstrate that three mobiles moving with incommensurable angular velocities will never conjunct, two of the three mobiles are made to travel with commensurable angular velocities. The three mobiles, $A, B$, and $C$, move with equal curvilinear velocities, but

$$
\frac{\text { circle } B}{\text { circle } C}=\frac{\mathbf{2}}{\mathrm{I}} \text { and } \frac{\text { circle } A}{\text { circle } B}=\frac{\sqrt{\mathbf{2}}}{\mathbf{I}}
$$

so that $V_{B}$ and $V_{C}$ are commensurable angular velocities and $V_{A}, V_{B}$ incommensurable.

If $B$ and $C$ are in conjunction in point $d$, then by Proposition I and the $2 / \mathrm{I}$ ratio of angular velocities, it follows that they will conjunct only in $d$. Now if it is assumed that $A$ and $B$ were once in conjunction in $d$ when $C$ was elsewhere (APr.189-90), it follows by Proposition IV that $A$ and $B$, moving with incommensurable angular velocities, will never again conjunct in $d$. It is then obvious that $A, B$, and $C$ could never conjunct in $d$.
The conditions laid down for Proposition IX are identical with those in Proposition VIII except that initially mobiles $A, B$, and $C$ are assumed to be in conjunction in point $d$. This move enables Oresme to modify the previous proposition to the extent that it can now be said of $A, B$, and $C$ that they will have conjuncted in $d$ once through all eternity.
Whereas Part I of the Ad pauca respicientes was concerned largely with conjunctions and oppositions, Part 2 makes the other aspects of central importance. Oresme divides dispositions into the three categories (AP2. 2-5) of (I) conjunction, (2) opposition, and (3) all other dispositions such as, for example, quartile, sextile, and trinal (see pp. 92-93). Conjunctions and oppositions have two properties in common. They form no central angles, by which Oresme seems to mean that a single straight line connects all mobiles in conjunction or opposition. Thus any opposition or conjunction can occur in only one way (AP2.5-6), namely, without a central angle where the mobiles involved are joined by a straight line. However, in oppositions the mobiles can be opposed in various ways depending on the number involved ( $\mathrm{AP}_{2.23-27}$ ). Where only two mobiles are in motion they can oppose each other in only one way, with one mobile at each end of a line drawn through the two concentric circles on which the mobiles
are assumed to move. But if three mobiles are involved, they can be combined to oppose each other two at a time in three ways; four mobiles can be arranged two at a time in six ways; and generally $p$ mobiles can be in opposition in

$$
\frac{p(p-1)}{2} \text { ways. }
$$

All other dispositions that fall into the third category do form central angles, and Oresme refers to them as "angular dispositions" (dispositiones angulares; $\mathrm{AP}_{2}$.8). Common to all these dispositions is the fact that the mobiles involved are related differently before conjunction or opposition than they are after conjunction or opposition (AP2.6-7). Oresme says no more than this. Let us arbitrarily take three mobiles $A, B$, and $C$ moving clockwise on their respective circles with angular velocities such that $V_{A}>V_{B}>$ $V_{C}$. Now before any conjunction in some point $b$ it is necessary that the mobiles be so ordered that C is closest to $b, B$ next, and $A$ farthest from $b$ (see figure below). Thus immediately before conjunction the order of mobiles is from slowest to fastest. This arrangement is necessary if all the

mobiles are to reach $b$ simultaneously for conjunction. For if A were nearer $b$ than $B$ it would be impossible for $B$ to conjunct with $A$ since the latter, moving more quickly, would pull away from $B$ as it approached $b$.

Now immediately after any conjunction of the three mobiles in $b$ their order is reversed and the fastest mobile, $A$, is first, followed by $B$, the next fastest, with $C$, the slowest, last and closest to $h$. Thus the relative
positions of the mobiles before and after conjunction are a function of their respective speeds. ${ }^{21}$
The distinction made between dispositions of mobiles before and after conjunction or opposition is crucial to the next step when Oresme divides all angular dispositions into what he calls "properly similar" (proprie similes) and "improperly similar" (improprie similes) angular dispositions (AP2. 8-12). Angular dispositions are properly similar when they occur after conjunctions or oppositions (AP2.8-9). In Figure 1 (p. 97) the mobiles, after conjunction in point $b$, are arranged as $C, B, A$, with $C$ the slowest and nearest to $h, B$ next and in the middle, and $A$ fastest and farthest from $b$. Let us call this disposition $D_{\mathrm{I}}$. Now immediately after the next conjunction of all three mobiles-wherever it may occur-the same arrangement of mobiles must occur as in $D_{\mathrm{I}}$. Thus the two dispositions are identical and constitute a properly similar disposition. This will be true for all dispositions after conjunction, for any given number of mobiles.
Now it is obvious that before any two conjunctions of mobiles $A, B$, and $C$, the mobiles will also be arranged in the same relative positions and will, therefore, be properly similar. Curiously, Oresme fails to mention this case in his brief description in AP2.8-9, but declares later, "...there are no 'properly similar' dispositions unless one [angular disposition] moves to one conjunction and another [angular disposition] to another conjunction, or both come from two [conjunctions]" (AP2.38-40). Thus if the mobiles in two angular dispositions have identical relative positions moving toward different points of conjunction or the same point of conjunction, they are properly similar. In Figure I (p. 97) the mobiles approaching points of conjunction $b$ and $e$ have identical relative positions and, consequently, are properly similar angular dispositions. ${ }^{22}$ Finally, the arrangement of a set of mobiles immediately after any two conjunctions will also be properly similar.
In the most general sense improperly similar angular dispositions obtain between a given number of mobiles immediately before conjunction and
${ }^{21}$ In APi.ir6, Oresme mentioned that dispositions other than conjunction and opposition were twofold. See above, p. 92. What has been said for dispositions after conjunction applies also to oppositions, but the situation is more complicated since the greater the number of mobiles the greater the number of possible ways in which they can be opposed (AP2.19-27).

This will not be discussed here except to say that in any given arrangement of mobiles the relative positions of the mobiles before that particular opposition will always be the same.
${ }_{22}$ Here, as in post-conjunctive dispositions, there are also properly similar angular dispositions before oppositions.
immediately after conjunction, or immediately before opposition and immediately after opposition. In Figure I (p. 97) the three mobiles immediately prior to conjunction in point $b$ are arranged clockwise from the fastest to slowest-i.e., from $A$ to $C$. But immediately after conjunction, their clockwise positions are reversed, arranged now from slowest to fast-est-i.e., from $C$ to $A$. The relative positions of any set of mobiles involved in conjunction or opposition are reversed with respect to before and after, and for this reason the expression "improperly similar" is used in contrast to "properly similar" where there is an exact-not reverse-identity of relative positions.
Oresme's division of angular dispositions into proper and improper is built upon the fact that conjunctions and oppositions form no central angles. There is always a change of disposition when mobiles pass through conjunctions and oppositions. Whether the new disposition is proper or improper depends upon the comparisons that are drawn. Thus properly similar angular dispositions obtain when the following comparisons are made for all the mobiles of any group that conjunct as often as they oppose:
(1) All dispositions immediately before any conjunction.
(2) All dispositions immediately before any opposition.
(3) All dispositions immediately after any conjunction.
(4) All dispositions immediately after any opposition.

Improperly similar angular dispositions obtain when the following comparisons are made: ${ }^{23}$
${ }_{23}$ A serious question arises with respect to improperly similar dispositions involving three or more mobiles. It is possible that mobiles $C, B$, and $A$, immediately after conjunction in point $b$ (see Figure I , p. 97) will not undergo opposition-i.e., where two mobiles are in conjunction and the third opposes them-before reversing their relative positions to assume the clockwise order $A B C$ prior to another conjunction in $b$ or some other point. That is, it does not follow that an opposition will intervene between two successive conjunctions. And yet the relative positions of the mobiles immediately after conjunction would be the reverse of their order immediately before the next conjunction even without the occurrence of an opposition, i.e., without a zero angle intervening. The
problem is whether Oresme would consider such a comparison to constitute an improperly similar angular disposition. This difficulty is never mentioned, but had Oresme been aware of it, he would perhaps have rejected such a comparison, since in Part 2, Prop. XIII (AP2.156-59) he concludes that where conjunctions and oppositions are absent so also are properly and improperly similar dispositions lacking.
The distinction between "properly similar" and "improperly similar" dispositions is omitted from the De commensurabilitate. However, the distinction between angular dispositions before and after conjunction is made explicitly in Part I, Prop. XXI, and is utilized earlier in Part I, Prop. XIII. The relevant passage from Prop. XXI has been quoted above on page 93, n.14. For
(1) All dispositions immediately before conjunction with all dispositions immediately after conjunction.
(2) All dispositions immediately before opposition with those after opposition.

In Proposition I Oresme shows that to pass successively from one similar disposition, whether properly or improperly similar, to another similar disposition two mobiles must form no angle-i.e., they must be either in conjunction or opposition. Two mobiles which conjunct at all will repeatedly conjunct, and Oresme notes that an opposition must intervene or mediate before those two successive conjunctions can occur. That is, the mobiles must pass through an opposition before the faster mobile can once again overtake the slower. On the other hand, between the occurrence of any other two dispositions-say trinal, quartile, or sextile (see p. 93)a conjunction or opposition must occur.
Moving specifically to improper and proper angular dispositions in Proposition II, which also deals with only two mobiles (AP2.43-44), Oresme asserts, as the first part of the proposition, that a conjunction or opposition must take place between any two successive dispositions which are improperly similar. Now Proposition I states that similar dispositions are only altered when two mobiles pass through a phase where they form no angle-i.e., through opposition or conjunction. But when the two mobiles pass through any conjunction or opposition they will reverse their relative positions, thereby constituting an improperly similar disposition. Thus the first part of Proposition II is a special case of Proposition I.
But in the second part of Proposition II Oresme moves a step beyond and explains why it is necessary that between any two successive properly similar dispositions there must be one conjunction and one opposition. With regard to conjunctions, ${ }^{24}$ two dispositions can be properly similar in either of the following ways (AP2.38-40): ( I ) when the mobiles are immediately before two conjunctions either in the same point or in different points; (2) immediately after two separate conjunctions in the same point or immediately after conjunction in two separate points.

Now let us see why it is necessary that between two successive properly similar dispositions one conjunction and one opposition must occur. Since Oresme is dealing with only two mobiles, let us assume that two mobiles are approaching conjunction in some point $b$ (see Figure 1, p. 97). After
Prop. XIII, see Grant, "Oresme: Comm.," ${ }^{24}$ The same applies to oppositions. pp. 432-33.

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conjunction in $b$ the two mobiles must pass through at least one opposition before they again approach conjunction in $b$ or any other point. This is obvious since the mobiles have unequal velocities and after conjunction the quicker mobile will pull away from the slower until it opposes it. Eventually the quicker will overtake the slower for another conjunction. Hence there will be one conjunction and one opposition before the next properly similar disposition when the two mobiles approach conjunction.
From all this Oresme infers ( $\mathrm{AP}_{2}$.43-45) that a particular properly similar disposition can occur in as many places as there are points of conjunction. If there are two points of conjunction and $A$ and $B$ are the mobiles, with $V_{A}>V_{B}$, then $B$ will precede $A$ before conjunction in each of the two points. There will also be two other properly similar dispositions after conjunction in each of the two points. In that situation, $A$, the faster mobile, will precede and move away from $B$, the slower mobile. The two dispositions which are properly similar before conjunction differ from the two which are properly similar but occur after conjunction. However, by comparing one disposition before conjunction with one after conjunction, we form an improperly similar disposition.
Shifting explicitly to a consideration of three mobiles, Oresme, in Proposition III, merely applies Proposition I to three mobiles moving commensurably or incommensurably. It is again a special case of Proposition I, in the sense that a change occurs from an unspecified disposition to an improperly similar disposition with an intervening conjunction or opposition. The intervening conjunction can occur in the transition from before conjunction to after conjunction, or after opposition to after conjunction. An intervening opposition would result when the three mobiles go from a post-conjunction disposition to a post-opposition disposition, or when they go from before opposition to after opposition. In any event, the crucial factor is the lack of angle separating two successive dispositions.

Since the velocities of the mobiles in Proposition III could be either commensurable or incommensurable, it is likely, in the absence of any statement to the contrary, that Proposition IV is also applicable to each alternative. This proposition asserts that if three mobiles have been in two successive dispositions which are properly similar, it follows that they must have undergone one conjunction and one opposition. Let mobiles $A$, $B$, and $C$ be related, as in Figure I (p.97), immediately before conjunction in $h$. Now immediately prior to the next conjunction-wherever that may
be ${ }^{25}-A, B$, and $C$ will be related in the same way as before conjunction in $h$, and we shall then have two successively proper dispositions. But before this can happen, the three mobiles must have undergone conjunction and opposition.
They will have passed through conjunction in $b$. Immediately after emerging from conjunction in $h$, there will be an improperly similar disposition because $A, B$, and $C$ have now reversed the relative positions which they held immediately before conjunction in $b\left(\mathrm{AP}_{2} .53-55\right)$. But according to Proposition IV, they must also pass through one opposition. In support of this Oresme cites the second part of Proposition II (AP2.36-37). But in the absence of any specific information, it cannot be determined whether Oresme holds that there must be an opposition of the three mobiles before the next conjunction (see p. 99, n.23). He has not in any way demonstrated this. The situation is even further complicated by the fact that the motions may be incommensurable as well as commensurable. Thus what was valid for two mobiles in Proposition II may be invalid for three mobiles in Propositions III and IV.
In Proposition $V$ we are told that there will be an infinite number of properly and improperly similar dispositions when three mobiles are moved with commensurable motions. Let the mobiles be $A, B$, and $C$. Now if $A$ and $B$ are in a certain disposition and their motions are commensurable, it follows by Proposition III of Part I (API.112-21) that they have been and will be in this identical relationship an infinite number of times. Every time this particular relationship occurs in the very same places $A$ will have made a certain number of revolutions on its circle.

When $A$ and $B$ enter into the unspecified relationship just mentioned, $A$ will bear simultaneously a certain relationship to $C$, and this will have happened and will continue to happen an infinite number of times, since $A$ and $C$ have commensurable velocities. In order for $A$ to enter into this particular relationship with $C$, it must have made a certain number of revolutions.

The two simultaneous relationships which $A$ has with $B$ and $C$, respectively, constitute one overall disposition, which will be repeated an infinite number of times. If the number of revolutions, say $n$, required to bring $A$ into the same relationship with $B$ is equal to the number of revolutions required to produce the other simultaneous relationship with $C$, then after

[^11]every $n$ revolutions of $A$ the three mobiles will again be in the same disposition. But if the number of revolutions required for $A$ and $B$ is $n$, and that required for $A$ and $C$ is $m$, then after $m n$ revolutions of $A$ the three mobiles will again enter into the same disposition.
This infinitely repetitive and identical disposition would appear to constitute a sequence of properly similar dispositons. There would also be an infinite sequence of improperly similar dispositions. For example, should the unspecified properly similar disposition just discussed occur before a particular conjunction or opposition, then immediately after that conjunction or opposition we would have an improperly similar disposition.

Since Proposition V has shown that similar dispositions of three mobiles will repeat infinitely, and Proposition III of Part 2 demonstrated that any two successive similar dispositions are separated by a zero angle-i.e., by opposition or conjunction-Oresme infers in Proposition VI that at least one opposition or conjunction will be repeated an infinite number of times when the motions are commensurable. Even if no conjunction could occur -and this was shown to be possible in Proposition VII of Part x -an opposition would take place and this would be repeated an infinite number of times. Although it is true that in the particular example used in Proposition VII of Part a there would be an opposition despite the impossibility of conjunction of the three mobiles, Oresme has never demonstrated that with three mobiles an opposition must necessarily occur whether or not a conjunction takes place (see p. 99, n.23).

Applying the conditions of Part 1, Proposition IX, ${ }^{26}$ to Part 2, Proposition VII, Oresme assumes that $B$ and $C$ are moved commensurably but that $A$ and $B$ have incommensurable velocities. The further assumption is made that $A, B$, and $C$ are in conjunction in some point, say $d$. From this it follows that $B$ and $C$ will conjunct repeatedly in $d$, while $B$ and $A$ have never before, and will never again, conjunct in $d$. What applies to conjunctions applies also to any other disposition formed by $A, B$, and $C$. That is, they will never form the same disposition more than once in a given place or places.

Adopting, once again, the same relationships between $A, B$, and $C$ as in the preceding proposition and in Part r, Proposition IX, Oresme demonstrates in Part 2, Proposition VIII, that if the three mobiles are once in conjunction-as they were in Part 1 , Proposition IX-they can never be in opposition.

As stated in the preceding proposition, mobiles $B$ and $C$, whose circles
${ }_{26}$ These relationships, in turn, are drawn directly from Part 1, Prop. VIII.
are related as $2 / 1$, can conjunct only in point $d$. All three mobiles can conjunct only once in point $d$, and assuming that this has already occurred Oresme wishes to show that no opposition is possible thereafter.
Under the conditions adopted in Part 2, Proposition VIII, opposition can happen in only one place and in only one way, namely when $B$ and $C$ are in conjunction in $d$ with $A$ simultaneously in point $e$, opposite to $d$. Every time $B$ and $C$ conjunct in $d$ they have traversed commensurable distances expressed in terms of the number of times each has traversed its circle. That is, $S_{B} / S_{C}=m / n$ where $S$ is distance, and $m, n$ are integers with $m$ representing the number of times $B$ has traversed its circle, and $n$ the number of revolutions made by $C$ in the same time. During the same time interval, however, $A$ has traversed a total distance which is incommensurable to the distances traversed by $B$ and $C$ respectively. For this reason $A$ cannot have arrived in $e$ when $B$ and $C$ conjunct in $d$ since the distance from $d$ to $e$, a half circle, is commensurable to the respective distances traversed by $B$ and $C$, but incommensurable to the distance covered by $A$. Hence $A$ cannot have reached $e$ when $B$ and $C$ conjunct in $d$, and no opposition is possible which involves all three mobiles.
Oresme shows next ( $\mathrm{AP}_{2}$.1oI- $\varsigma$ ) that if $A, B$, and $C$ have been in conjunction in point $d$, it is impossible thereafter for $A$ and $B$ to conjunct when $C$ is simultaneously in opposition. The argument is very straightforward. Calculating all motions from $d$, it is evident that when $A$ and $B$ conjunct anywhere on the circle (not, however, in point $d$ ) after departing from $d$, they will have traversed distances which are mutually incommensurable since their velocities are incommensurable. But when $B$ and $C$ are in opposition after leaving $d$ they will have traversed mutually commensurable distances in equal times. Now if it happened that $A$ and $B$ conjuncted and $C$ was simultaneously in opposition, then certain consequences follow which are contrary to the assumptions of the proposition. Obviously, if $B$ and $C$ are in opposition and traversed commensurable distances, then the distance traversed by $A$ is also commensurable to that travelled by $C$, since $A$ and $B$ are in conjunction and both are opposed to $C$. But if the distances traversed by $A$ and $B$ are respectively commensurable to that traversed by $C$, then the distances travelled by $A$ and $B$ must be mutually commensurable and also their velocities, since in equal times velocity is proportional to distance. But this is contrary to the assumption that the velocities of $A$ and $B$ are incommensurable.
Since Oresme has not provided a specific detailed proof, another demonstration would be as follows. The velocities of $B$ and $C$ are related as

2 to I so that subsequent to departure from $d$ they could enter into opposition only in points $d$ and $e$. But $A$ and $B$ also started from conjunction in $d$ moving with incommensurable velocities. Therefore, by Part I, Proposition IV, they can never again conjunct in $d$, nor can they conjunct in $e$ for then, in equal times, they would have traversed commensurable distances since $e$ is distant from $d$ by a semi-circle. It follows, then, that $C$ cannot oppose $B$ when $A$ and $B$ are in conjunction.
Once again, in Part 2, Proposition IX, Oresme assumes that $A, B$, and $C$ are in conjunction in $d ; B$ and $C$ move with commensurable angular velocities and $A$ and $B$ with incommensurable angular velocities. He then demonstrates that in equal times they can never traverse mutually commensurable angles which are also commensurable to a right angle.
The argument proceeds as follows (see Figure 6, p. 415): (1) $\angle C B$ is commensurable to a right angle only when $\angle B d$ is commensurable to a right angle. (2) But $\angle A B$ cannot be commensurable to a right angle when $\angle B d$ is commensurable to a right angle since $A$ and $B$ are moved incommensurably. (3) Therefore, $\angle A B$ and $\angle C B$ will never be simultaneously commensurable to a right angle.
Oresme explains the formal argument (AP2.114-18) by noting that in the major premise, namely ( I ) above, $\angle B d$ and $\angle C d$ will always be commensurable since $B$ and $C$ have commensurable, angular velocities. And because $\angle B d-\angle C d=\angle C B$ it follows that $\angle C B$ is commensurable to both $\angle B d$ and $\angle C d$. Therefore, if $\angle B d$ is commensurable to a right angle so is $\angle C B$.
The minor premise, i.e., (2) above, is explained in a similar manner. Here $\angle A d$ and $\angle B d$ will be incommensurable, since mobiles $A$ and $B$ are moved with incommensurable angular velocities. Since $\angle A d-\angle B d$ $=\angle A B$, and $\angle A d$ and $\angle B d$ are incommensurable, it follows that $\angle A B$ is incommensurable to both $\angle A d$ and $\angle B d .{ }^{27}$ Thus when $\angle B d$ is commensurable to a right angle, $\angle A B$ must be incommensurable to a right angle.

27 In AP2.119-20, Oresme is actually citing Campanus' edition of Euclid Bk.X. 8, which says, "Si fuerint duae quantitates uni quantitati communicantes, ipsas quoque invicem commensurabiles necesse est."-Euc.-Campanus, p. 25 I. However Campanus' additio to X. 9 (Euc.-Campanus, p. 25 2) seems more appropriate. Campanus explains that if a whole constituted of two
parts is incommensurable to one of the parts, it will also be incommensurable to the remaining part; and the parts will also be incommensurable. As applied to O resme's problem, it follows that $\angle A B$ is incommensubable to $\angle A d$ and $\angle B d$, since $\angle A d=\angle B d+\angle A B$, and $\angle A d$ is incommensurable to $\angle B d$.

Proposition X is indeed curious. Having shown in the preceding proposition that when $\angle B d$ is commensurable to a right angle, $\angle C B$ will also be commensurable to a right angle, but $\angle A B$ incommensurable, in Proposition X Oresme assumes that $A$ and $C$ are in conjunction so that $\angle C B=\angle A B$. On the basis of Part 2, Proposition IX, he concludes that despite the equality of these two angles they cannot both be commensurable to a right angle, apparently because $A$ and $C$ move with incommensurable angular velocities. Only $\angle C B$ is commensurable to a right angle since $C$ and $B$ move with commensurable velocities, but not $\angle A B$ where $A$ and $B$ have incommensurable angular velocities. The same paradoxical conclusions are drawn with respect to a conjunction of $A$ and $B$, whose velocities are also incommensurable, thus making $\angle A C=\angle B C$.
It is not difficult to understand why Oresme abandoned this proposition in the later De commensurabilitate. Although the paradox depends on the distinction between commensurable and incommensurable, it has, in fact, obliterated that distinction.
In Proposition XI Oresme pushes further the paradox of the preceding tenth proposition. He now says that even if angles $A B$ and $C B$ were simultaneously commensurable to a right angle-and this was denied in Part 2, Proposition X-mobiles $A, B$, and $C$ never were and never will be in conjunction. The reason for this appears to be the paramount fact that $A$ is moved incommensurably with respect to both $B$ and $C$.
Thus if we assume that $A$ and $C$ are in conjunction as in the previous proposition, then should $B$ be a commensurable distance from the point of conjunction of $A$ and $C$-say in quadrature to $A$ and $C\left(\mathrm{AP}_{2.135-36)-}\right.$ the three mobiles have never been, nor ever will be in conjunction. ${ }^{28}$ This, in fact, repeats the circumstances of Part 2, Proposition X, where $\angle A B=\angle C B$.
Indeed, the same may be said if $A, B$, and $C$ should form two right angles or three mutually commensurable angles. Generally, then, Oresme holds that even in cases where angles $A B$ and $C B$ are commensurable (recall that $A$ and $B$ move with incommensurable angular velocities and $B, C$ with commensurable velocities), one must nonetheless conclude that mobiles $A, B$, and $C$ have never conjuncted and never will conjunct as long as $A$ 's motion is incommensurable to the velocities of both $B$ and $C$.
Proposition XII offers a proof for one of the special cases mentioned in Part 2, Proposition XI, but is extended to cover a number of other situations. Oresme takes the case in Proposition XI (AP2.135-36) where two

[^12]Ad pauca respicientes
mobiles, $A$ and $B$, are in conjunction while $C$ is in quadrature to them, and demonstrates that the three mobiles could never have been in conjunction. ${ }^{29}$ In addition, he shows that they were not in opposition.
As in the preceding propositions, $B$ and $C$ have commensurable angular velocities, while $A$ moves incommensurably with respect to $B$ and $C$. Initially $A$ and $B$ are in conjunction in point $d$, while, at the same time, $C$ is a quadrant away-i.e., in quadrature. Oresme shows that whenever $B$ and $C$ form an angle commensurable to a right angle or when they are in conjunction or opposition, they will have traversed commensurable distances and their respective distances from point $d$ will always be commensurable. In other words, $\angle B d$ and $\angle C d$ will always be commensurable or equal whether they are in conjunction or opposition or form an angle commensurable to a right angle.
But when $A$ and $C$, which move incommensurably, form an angle commensurable to a right angle or are in conjunction or opposition, they will have traversed incommensurable distances and $\angle A d$ and $\angle C d$ will be incommensurable.
Thus the points of conjunction and opposition measured from $d$ are such that those for $B$ and $C$ will differ from $A$ and $C$. This is evident from the fact that when $B$ and $C$ are in conjunction or opposition their respective distances from $d$ are commensurable, whereas for $A$ and $C$ the distances will be incommensurable. Consequently, there will be no common points of conjunction or opposition, 30 and the three mobiles can never be in conjunction or opposition simultaneously.

The same arguments apply if $B$ and $C$ conjuncted in $d$ and $A$ was in quadrature, or if $C$ and $A$ were in conjunction and $B$ in quadrature. Under these conditions three mobiles might never be in conjunction or opposition through all eternity.
Drawing on Part 1, Proposition VIII, and Part 2, Proposition XII, Oresme, in Proposition XIII, observes that three mobiles may never be in conjunction or opposition, in which event there can be no properly or improperly similar dispositions. It will be recalled from Propositions III and IV of Part 2 that properly and improperly similar dispositions occur only if there are intervening conjunctions or oppositions. Should no con-
${ }^{29}$ Oresme claims to have demonstrated this in Part 2, Prop. XI, and says that he is proving it again, along with additional claims ( $\mathrm{AP}_{2.142}$ ). It was asserted as a particular case of Prop. XI.
${ }^{30}$ Oresme drew this inference earlier in

Part 2, Prop. VI, when he said (AP2.8486): "If the places of conjunction of one [mobile] with each of the others separately are incommunicant, it is impossible for this mobile to have any common places of conjunction with the other two mobiles."
junctions and oppositions occur, there will be no similar dispositions at all, and it follows that no disposition of the three mobiles will ever be repeated.

In Proposition XIV, Oresme next applies the concept of "similar dispositions" to Part I, Proposition IX, where it was shown that three mobiles might conjunct only once through all eternity. In this situation the single conjunction serves to separate two improperly similar dispositions of infinite duration. For example ( $\mathrm{AP}_{2} .173-75$ ), one day after this unique conjunction the mobiles will be in reverse order to the relative positions they held one day before conjunction; two days after conjunction will find their order the reverse of that held two days before conjunction, and so on ad infinitum.

With such an arrangement of mobiles there could never be any properly similar dispositions, and this is shown in Proposition XV. One conjunction and one opposition are required before two successive properly similar dispositions are possible (but see p. 102). But in Part 2, Proposition VIII, it was demonstrated that no opposition can occur when three mobiles are moving as described in Part I, Proposition IX. Therefore, the three mobiles have never been and will never be in a properly similar disposition.

In a significant proposition, XVI, the consequences deducible from incommensurable motions are described. Precise knowledge of astronomical aspects and dispositions could be obtained if the motions of the celestial bodies were commensurable. But if these motions are incommensurable, then such knowledge is impossible. The precise length of the solar year would be indeterminable because the length of the day and the time in which the sun traverses its annual orbit would be incommensurable. Any exact calendar would be impossible (cf. IV. $568-72$; see also pp. 60-6I). The same would apply to the lunar year.
Although in Proposition XVI Oresme does not decide which of the two kinds of motions are operative in the heavens, it is clear from Part I , Supposition II, and the next proposition that he believes it likely that celestial motions are incommensurable.
In Proposition XVII, Oresme invokes Part x, Supposition II, to show that in any instant of time, it is probable that a unique disposition of mobiles is formed which never appeared before and will never appear again. The argument takes this simple form: (1) There are many circles, latitudes, distances, eccentricities, and other kinds of celestial magnitudes. (2) And therefore, by Supposition II of Part r, we can infer the probability that any two such like quantities are incommensurable.

After asserting the probability that any two unknown celestial quanti-
ties are incommensurable, Oresme simply cites Propositions VII and XIII of Part 2 where the motions were assumed incommensurable and it was shown that aspects and dispositions never repeat. But now we see that the abstract and hypothetical incommensurability of earlier propositions is transformed by Supposition II of Part I into probable relationships in nature. At this point Parts I and 2 cease to be mere imaginative exercises.

In a number of propositions thus far Oresme has stressed the conditions producing unique or non-repetitive dispositions, or failing entirely to yield a certain disposition. For example, there may be no conjunction whatever (Propositions VII, VIII of Part I ; Propositions XI, XII, XIII of Part 2); or only one conjunction through all eternity (Proposition IX of Part r ; Proposition XIV of Part 2); or two properly similar dispositions may never occur (Proposition XV of Part 2); or there may never be two similar dispositions whatever (Proposition XVII of Part 2).

Oresme continues this tendency in Proposition XVIII and, as in Proposition XVII, applies previous results to planetary motions. Thus three or more planets may conjunct only once through all eternity as was demonstrated for three mobiles under the conditions specified in Part 1 , Proposition IX. Indeed, perhaps four or more planets may only conjunct once in an eternal time. ${ }^{31}$ But is it not perhaps possible, asks Oresme (AP2. 214-16), that such a unique conjunction may have produced or caused some unique effect which will last through eternity? ${ }^{32}$ Or, perhaps it was the cause of a unique but temporary effect-as the Biblical flood-which did not occur prior to that single conjunction and will never again appear through an infinite future (AP2.217-19).
But it seems astonishing that such a conjunction should be directly connected with such a unique effect, and that in the infinite past it must have been true that in the eternal future such a conjunction and unique effect would occur of necessity. Furthermore, why did it happen in this particular instant rather than another? It is futile to seek rational explanations for
${ }^{31}$ In IV.595-600, Oresme described a similar situation.
${ }^{32}$ A very similar notion appears in the De commensurabilitate: "Et si constellationes sint cause inferiorum effectuum continue erit talis dispositio quod numquam erit similis in hoc mundo. Cum que notabiles aspectus respiciant totam unam speciem, non videtur inopinabile, loquendo naturaliter, quod una magna coniunctio
planetarum cui numquam fuit similis producat aliquod individuum cui non fuerit simile in specie... Et forte possibile est quod talis species incepta numquam desineret si mundus perpetuaretur, aut quod aliquando desineret virtute alterius con-stellationis."-MS Vat. lat. 4082, fols. 104v, c.2-105r, c.r. See Grant, "Oresme: Comm.," p. 453, n.71.
such events, and one must simply realize that God, as a free agent, can arrange things an infinite time before they take place. ${ }^{33}$

The last two propositions are appeals to the reader to understand and recognize, on the basis of what has already been propounded, that attempts to predict future events are futile and that astrology is an empty discipline.

In Proposition XIX Oresme concedes, for the sake of argument, some of the fundamental assumptions of astrology. He grants (AP2.228-32) the domination of the earth by the heavens, the uniformity of the heavens, determinism, an eternal world and eternal motion. But even with all this granted, the future could in no way be predicted because such prediction depends on the past. But by Propositions XVI and XVII the probability of the incommensurability of the celestial motions would preclude the prediction of future events since no two configurations are repeatable. Each disposition is unique. No backlog of properly similar dispositions can be collected since there are none. Without such data astrology cannot predict the future from the past and is the victim of an endless string of unique and unpredictable events.

But even if we assume that the celestial motions are commensurable (AP2.239-44), we might not come to know any ratios involving such motions since by Supposition III of Part I (APr.45-50) all ratios involving celestial motions are unknown. Let us even discount Supposition III of Part I , and assume that such celestial ratios are theoretically knowable. In this circumstance, how could it be known when the calculations had been carried sufficiently far, since fractions would surely be involved? Indeed, having found such a ratio, we might be wholly unaware of the discovery because our senses are incapable of verifying calculations with any precision.

Finally (AP2.245-46), we cannot truly know whether any two or more planets move commensurably or incommensurably. It is probable-but only probable-that such motions would be incommensurable. But the element of uncertainty makes us, in effect, "ignorant of the antecedent"-
${ }_{33}$ The very same puzzle outlined in this paragraph is elaborated in the De commensurabilitate, Part II, Prop. XI. The eternal necessity for the occurrence of a particular configuration is expressed as follows: "Supposita namque incommensurabilitate motuum et eternitate pulcrum est considerare qualiter talis constellatio sicut esset coniunctio punctualis eveniet semel solum
in toto tempore infinito, et quomodo ab eterno futura erat necessario pro hoc instanti nulla simili precedente aut sequente ...Nec est querenda ratio quare magis eveniret tunc quam alias, nisi quia tales sunt velocitates motuum et immutabiles voluntates moventium."-MS Vat. lat. 4082, fol. $104 \mathrm{v}, \mathrm{c} .2$. See Grant, "Oresme: Comm.," p. 453, nn.69, 70.
i.e., ignorant of whether the motions are commensurable or not-and "who is ignorant of the antecedent is necessarily ignorant of the consequent."

Despite a whole series of concessions to astrology, Oresme has shown that even under the most ideal of possible conditions, astrology cannot fulfill its claims and must be written off as a delusive enterprise. The objectives of astrology "lie hidden behind Him who numbers the multitude of stars, and who governs the world by reason everlasting" (AP2.247-49). The future can be known only by revelation ( $\mathrm{AP}_{2} .23 \mathrm{I}-32$ ) and the astronomer would be wise to realize that he cannot attain punctual exactness but must remain content with a reasonable approximation (AP2.263-64). ${ }^{34}$

## The Origin and Influence <br> of Oresme's Concept of the Incommensurability of the Celestial Motions

The Ad pauca respicientes was concerned with a series of propositions dealing, in large measure, with the incommensurability of the celestial motions. Oresme was certainly not the first to announce the possibility that the celestial motions might be incommensurable. The initiator of this concept is simply unknown ${ }^{35}$ and will quite likely remain unknown. However, a preliminary and germinal discussion can be traced to Greek origins in a work by Theodosius of Bithynia (born ca.180 в.c.). In the five concluding propositions of his On Days and Nights (De diebus et noctibus), ${ }^{36}$ Theodosius considers commensurability and incommensurability with respect to the sun's motion.

In Proposition is Theodosius shows that if the year consisted of a rational number of days and nights, the corresponding days and nights of
${ }^{34}$ See above, pp. 64-65.
35 "Nous ne saurions nommer le premier mathématicien qui ait émis cette supposition: Le rapport des durées de deux révolutions célestes peut être un nombre irrationnel, un nombre sourd."-Duhem, $S y$ steme du monde, Vol. 8, 444. Duhem devotes all of Ch. 8, pp. 443-501, to a consideration of the problem of the incommensurability of the celestial motions. No attempt is made to investigate the origin of the problem; Duhem begins his discussion
with Abraham ben Ezra (ca. 1089-1 167).
${ }^{36}$ This Greek treatise has been edited and translated into Latin by Rudolf Fecht under the title Theodosii De babitationibus, De diebus et noctibus, in Abbandlungen der Gesellschaft der Wissenschaften zu Göttingen, New Series, Vol. 19, 4. Biographical and bibliographical information is furnished by Fecht on pp. 1-12. Fecht's edition and the pertinence of Theodosius' propositions were brought to my attention by John Murdoch.
each year would be equal in length, the solstices would occur at the same points of the ecliptic, and the risings and settings of the sun take place at the same points of the horizon circle. Finally, the sun would arrive at the tropics and equinoxes at the very same hour in every year. ${ }^{37}$
Assuming, in Proposition 16, that the year equals some whole number of diurnal solar revolutions plus a fractional part of a solar revolution-i.e., the year equals an integral number of days plus a fractional part-Theodosius demonstrates that the days and nights of any given year will not equal in length the corresponding days and nights of the following year, nor, indeed, will the solstices occur at the same points of the ecliptic, or the risings and settings take place at the same points of the horizon. Finally, the sun will not reach the tropics or equinoctial points in the same hour in any two successive years. ${ }^{38}$
In the next proposition, the seventeenth, we are told that if the year consisted of an integral number of diurnal solar revolutions-i.e., an integral number of days-everything would occur in exactly the same way. ${ }^{39}$
${ }^{37}$ The opening paragraph of Prop. Is summarizes these points. I give Fecht's Latin translation: "Si annus totis conversionibus solis, h.e. numero rationali dierum noctiumque constat, etiam dies et noctes singulorum annorum magnitudine et numero aequales erunt, et ad eadem puncta horizontis et circuli solaris solstitia, ortus, occasus erunt, praeterea autem etiam eadem hora sol ad tropicos et aequinoctialem per-veniet."-Ibid., p. 145. A detailed description and discussion of these propositions will be given in my future edition of Oresme's De commensurabilitate.
${ }_{38}$ " Sin annus totis solis conversionibus non constat, sed ad totas conversiones etiam pars quaedam accedit, dies et noctes primi anni diebus et noctibus anni sequentis magnitudine inaequales erunt, neque solstitia neque ortus neque occasus ad eadem puncta horizontis et circuli solaris erunt, neque sol eadem hora ad tropicos neque ad aequinoctialem perveniet." Ibid., p. 149.
${ }^{39}$ Theodosius fails to specify the sense in which all things will repeat. Does he mean that all celestial and terrestrial events will repeat, or only those events already mentioned, such as risings and settings, sol-
stices and equinoxes, and so on? In Prop. 19 (see below, p. 113, and n. 43), he specifies the conditions under which solar events would never repeat. In the absence of more definite evidence, it seems plausible to confine the repetition of events, or lack thereof, to solar phenomena. On a number of occasions Oresme applies propositions of a similar nature to both celestial and terrestrial events.
That part of Fecht's translation of Prop. 17 which will be described here is as follows (ibid., pp. 149, is 1): "Si supponimus solis conversiones aequali inter se tempore fieri, quod ex sensu videtur ita esse, et totus annus totis solis conversionibus constat, singulis annis omnia eodem modo evenient, quo supra exposuimus. Sin annus totis conversionibus non constat, sed etiam conversionis pars accedit, si quidem pars accedens cum tota conversione commensurabilis est, annis proxime sequentibus eadem non evenient, sicut dictum est, sed post nonnullos annos omnia eodem modo evenient; quot autem post annos hoc fiat, hoc modo indicabitur: Duobus numeris inter se primis in eadem proportione sumptis, quam habet conversio ad partem accedentem, post tot annos, quot maioris nu-

But if the year embraced an integral number of solar revolutions plus a part of a revolution commensurable to a whole revolution (i.e., an integral number of days plus a fraction of a day commensurable to a whole day), then in immediately successive years the same events would not occur. The repetition of such events would occur only after the lapse of a certain number of years, depending on what fractional part of a diurnal solar revolution is taken. ${ }^{40}$ However, should the additional fractional part be incommensurable to one diurnal solar revolution, no events would occur in exactly the same way. ${ }^{4}$

Proposition 18 is a further elaboration of the preceding proposition where instead of a part of a day, Theodosius takes parts of a day. Thus he shows that if the year is $3655 / 19$ days, the same events will repeat every nineteen years. ${ }^{42}$

In the final proposition of the treatise-Proposition 19-Theodosius asserts that if the part added to an integral number of solar revolutions is incommensurable to a single diurnal solar revolution, the sun will never return to the same place in equal time intervals from year to year. ${ }^{43}$

Did Theodosius' De diebus et noctibus exert any influence? It was translated into Arabic by Costa ben Luca ${ }^{44}$ in the ninth century, but there is no evidence of a medieval Latin translation from either Greek or Arabic. 45
meri quantitas indicat, omnia rursus eodem modo evenient.
"Sin pars accedens cum tota conversione incommensurabilis est,", numquam omnia eodem modo evenient."
${ }^{40}$ Prop. XXII of Part I of Oresme's De commensurabilitate is similar to this proposition. See Grant, "Oresme: Comm.," pp. 438-39.
${ }_{41}$ A proposition containing this idea in much more advanced and elaborate form is offered by Oresme in Part II, Prop. XII, of the De commensurabilitate (Grant, "Oresme:Comm.," p -454). See also Prop.XVI of the Ad pauca, below on pp. $42 \mathrm{I}, 423$.
${ }^{42}$ "Rursus igitur secundum Metonem et Euctemonem, quoniam annus iis diebus trecentis sexaginta quinque et praeterea quinque undevicesimis partibus conversionis constare videtur, post undeviginti annos omnia eodem modo evenient."-Theodosii De babitationibus; De diebus et noctibus, ed. and trans. Fecht, in Abbandlungen der Gesellschaft der Wi.ssenschaften zu Göttingen,
${ }_{43}$ Although omitted by Theodosius, the qualification "in equal time intervals" must be added, since the sun always retraces its path over the ecliptic. "Sin pars accedens cum tota conversione commensurabilis non sit, nunquam eadem eventura esse, h.e. solem nunquam ad idem reversurum esse hoc modo demonstrabimus: ..."-Ibid., p. is5. For references to similar propositions in Oresme's treatises, see above, n . 4 I .
${ }^{44}$ This is reported by Moritz Steinschneider in his Die arabischen Ubersetzungen aus dem Griechischen, p. 22 1.

45 Steinschneider mentions that both the De sphaera and the De habitationibus were translated from Arabic into Latin, but is silent about the De diebus et noctibus. F. Carmody, in an article on Autolycus of Pitane published in Catalogus Translationum et Commentariorum, edited by P. Kristeller (Vol. I, 170, c.2-171, c.1), reports that certain Greek codices in the Vatican library

Its influence, if any, in the Latin West is simply unknown, and we find no link connecting the few propositions offered by Theodosius and the more mature, sophisticated, and enormously more detailed works of Oresme. Indeed, Theodosius considers only quite elementary propositions concerned exclusively with the sun's motion, and for this reason cannot be the initiator of discussions on the incommensurability of the celestial motions. In sharp contrast, Oresme deals with simultaneous motions of two and three bodies. For the most part, Oresme's possible indebtedness to Theodosius, direct or indirect, could not extend beyond a few propositions and statements including, perhaps, the former's assertion, "...if the time in which the sun traverses its circle were incommensurable to a day, so that the solar year would last through a certain number of days and a part of a day incommensurable to a whole day, the length of the year was, is, and will be perpetually unknown." ${ }^{6}$ Unlike Oresme, Theodosius does not discuss whether or not it is more probable that the length of the year is expressible by a rational or irrational number, nor does he link the discussion of incommensurability with astrological prediction. In general, the discussion by Theodosius is so meager and far removed from Oresme's full-blown treatises that we must seek a series of later substantial developments prior to Oresme or concede that his work was, for all practical purposes, an original contribution dependent ultimately, perhaps, on Theodosius only for the idea that the time it takes to complete a given celestial motion might be conceived as commensurable or incommensurable to some other temporal unit.
If, at present, Theodosius' De diebus et noctibus represents the first treatise known to apply notions of commensurability or incommensurability to at least one celestial motion, namely the sun, certain problems raised by Aristotle near the end of the De generatione et corruptione also seem germane. Aristotle says ( $337 \mathrm{a} .35-337 \mathrm{~b} .3$ ):47
contain the writings of the minor Greek mathematicians and astronomers, including the De diebus et noctibus, De sphaera, and De babitationibus of Theodosius. He believes that a similar codex served as the basis for a translation of these treatises into Arabic. Furthermore, he observes that Latin codex Bibliothèque Nationale 9335 includes four treatises from these lesser Greek writers, so that it is very likely they were translated into Latin from an Arabic codex quite similar in content to one of the origi-
nal Greek codices. Of the four treatises two are the De babitationibus and De spbaera of Theodosius. The De diebus et noctibus is missing, and while one may not properly infer that it was untranslated, the fact remains that no such medieval Latin translation is as yet known.

46 See below, pp. $42 \mathrm{I}, 423$ (AP2.19295); and above, p. 113, nn. 40, 4i.

47 Works of Aristotle, ed. Ross, Vol. 2, De generatione et corruptione.

Wherever there is continuity in any process (coming-to-be or "alteration" or any kind of change whatever) we observe "consecutiveness", i.e., this coming-to-be after that without any interval. Hence we must investigate whether, amongst consecutive members, there is any whose future being is necessary, or whether, on the contrary, every one of them may fail to come-to-be.

After some discussion, Aristotle decides that "the coming-to-be of anything, if it is absolutely necessary, must be cyclical-i.e., must return upon itself" (338a.4-5), for
it is in circular movement,... and in cyclical coming-to-be that the "absolutely necessary" is to be found. In other words, if the coming-to-be of any things is cyclical, it is "necessary" that each of them is coming-to-be and has come-to-be: and if the coming-to-be of any things is "necessary," their coming-to-be is cyclical. The result we have reached is logically concordant with the eternity of circular motion, i.e., the eternity of the revolution of the heavens (a fact which approved itself on other and independent evidence), since precisely those movements which belong to, and depend upon, this eternal revolution "come-to-be" of necessity, and of necessity "will be". For since the revolving body is always setting something else in motion, the movement of the things it moves must also be circular. Thus, from the being of the "upper revolution" it follows that the sun revolves. in this determinate manner; and since the sun revolves thus, the seasons in consequence come-to-be in a cycle, i.e., return upon themselves; and since they come to be cyclically, so in their turn do the things whose coming-to-be the seasons. initiate (Aristotle, $338 \mathrm{a} .15-338 \mathrm{~b} .6$ ).

Since the heavens are imperishable, their eternal revolution produces a sequence of numerically identical events in contradistinction to other sequences which are the same only in species, ${ }^{48}$ as, for example, the coming-to-be of individual men or animals where there is a never-ending sequence of men and animals but no two individuals are identical.

To astrologers Aristotle's position is obviously favorable, since it is theoretically possible to predict future events which will have happened in the past and will repeat in an identical manner in the future. Partisans of a Great Year (see pp. 429-31) may have found Aristotle's arguments useful.

Averroes, in his Epitome on Aristotle's De generatione et corruptione, agrees with Aristotle that individual men or animals cannot return and in the course of his discussion considers, all too briefly, the subject which is of concern here. He notes that Alexander of Aphrodisias
${ }^{48}$ Ibid., 338b.12-14.
believes that the state and disposition of the spheres at any given time never revert individually. He maintains that if we assume all of the stars to be at a particular point in the sphere of the constellations, for example, in Ram, and then all of them, both the fast and the slow ones, begin to move, they need not necessarily all of them revert to the exact same point from which they began their movement, but the revolutions of some will be proportionate to those of others, so that, for example, when the sun completes one revolution the moon will have completed twelve. And there will be a similar relationship between the revolution of the sun and each one of the stars. Then it should be possible for all of them to return to any one place, to any place you may postulate. But we find the exact opposite to take place. For the sun traverses its sphere in $3651 / 4$ days and the moon traverses its sphere in $271 / 2$ days. When $271 / 2$ days are multiplied (by twelve), they do not yield $3651 / 4$ days. Since this is so, and the efficient cause does not return upon itself numerically, and neither can the material cause do so, it becomes evident that it is impossible in any way whatsoever for the individual to recur. Now that is what we set out to prove.
We might add to what we have already said that even though the revolution of the moon is not commensurable with that of the sun in days, it does not follow that they are not commensurable with one another at all. For it is possible that their common unit of measurement is a shorter time. But if that were so, the common measure would have to be one-quarter of a day. To ascertain whether these revolutions of the stars are commensurable or not is most difficult or well nigh impossible, for that would have to be based upon a knowledge of the time of a single revolution in the case of each star as it is in truth. That is impossible because of the limited and approximate nature of our observation of these things. What we can ascertain in this matter is that they are approximately commensurate to one another, as the astronomers believe. Whatever the case may be, it is impossible for the individual to recur. ${ }^{49}$

Averroes, arguing initially against Alexander's position, says that the celestial bodies will not return to particular places if their motions are incommensurable. Using the sun and moon he says that the accepted values for the month and year are incommensurable when the unit of common measure is $27 / 2$ days. But, in the most important passage, he concedes that a smaller unit measure would make them commensurable and readily admits that the commensurability or incommensurability of the celestial motions is indeterminable ${ }^{50}$ since it depends upon exact knowledge of the period of revolution of each celestial body. Since astronomers know they must
${ }^{49}$ Averroes on Aristotle's De generatione, translated from the original Arabic and the Hebrew and Latin versions, with notes and introduction by Samuel Kurland,
pp. 137-38. ${ }^{50}$ In an absolute sense, Oresme also ad-
mits this (AP2.245-46), but believes it probable that they are incommensurable.

Ad pauca respicientes
rely on approximate observations, they operate on the assumption that the heavenly motions are approximately commensurate.
On the basis of this extremely important passage appearing in a work of one of the most widely read and studied authors in the Latin West, one might reasonably conjecture that Averroes transmitted the problem to the medieval scholastics, and eventually to Oresme. But such is not the case. Neither in the Ad pauca respicientes nor in the De commensurabilitate, does Oresme make any reference to Aristotle's De generatione et corruptione or to the commentaries of Averroes on that treatise. Indeed, in his own Questions on the De generatione et corruptione ${ }^{51}$, Oresme does not even mention the problem when discussing the very passages quoted above from Aristotle. ${ }^{52}$

Are there other likely sources for the transmission of the problem? The most plausible candidates are the astrological and anti-astrological traditions. In the Tetrabiblos, or Quadripartitum as it was called in the Middle Ages, Ptolemy says:

The ancient configurations of the planets, upon the basis of which we attach to similar aspects of our own day the effects observed by the ancients in theirs, can
${ }^{51}$ MS Florence, Biblioteca Nazionale Centrale, Conv. Soppr. H 9 1628, containing 39 unnumbered folios, on the last of which is the following attribution to Oresme: "Explicit liber de generatione et corruptione Nicolai Orem."
${ }_{52}$ It is probable that discussions of incommensurability in the Latin West were not incorporated into commentaries and questions on the De generatione, because it appears that Averroes' Epitome became available in Latin too late to exert any influence. Kurland says (Averroes on Aristotle's De generatione, p. xiv): "Our Epitome was translated into Hebrew by Moses ibn Tibbon (fl. ca. 1240-1283) in the year 1250 It appears in the Juntine edition of Venice 1550 in the Latin version made from the Hebrew by Vital Nissus." In a note Kurland remarks that nothing is known of the translator. The fact that no medieval Latin manuscript tradition is mentioned indicates a late translation. It was Averroes' Middle Commentary on the De generatione, translated from Arabic to Latin in the thirteenth century by Michael Scot, which served as
the popular version, but it lacks any section corresponding to the discussion in the Epitome cited above. There was, of course, nothing to prevent Oresme and others from introducing such discussions independently of Averroes-but this seems not to have occurred.
Quite apart from any possible connection between the passages quoted from Aristotle's De generatione et corruptione (see above, p. ifs) and the incommensurability of the celestial motions, it is obvious that Oresme is in fundamental disagreement with Aristotle. The "cyclical coming-to-be" of which Aristotle speaks is dependent on the never-ending repetition of celestial dispositions and configurations. Oresme, on the other hand, believing that the celestial motions are probably incommensurable, argues that celestial dispositions may never repeat thus resulting in unique generations and corruptions never to be repeated through all eternity (see above, p. 109). This was un-Aristotelian, but well-suited to the unique events in the Christian drama.
be more or less similar to the modern aspects, and that, too, at long intervals, but not identical, since the exact return of all the heavenly bodies and the earth to the same positions, unless one holds vain opinions of his ability to comprehend and know the incomprehensible, either takes place not at all or at least not in the period of time that falls within the experience of man; so that for this reason predictions sometimes fail, because of the disparity of the examples on which they are based. ${ }^{53}$
In this ambiguous passage there is justification both for belief in a "Great Year" and for its repudiation. But Ptolemy's authority in support of the opinion that the planets do not exactly return to the same positions may have spurred someone hostile to astrological prediction and pretention to seek a sound reason for such irregularity. Belief in the incommensurability of the celestial motions may have emerged independently from such considerations, perhaps even before Theodosius. But Theodosius and Ptolemy notwithstanding, specific utterances concerning the incommensurability of two or more celestial motions have not yet been found earlier than the twelfth century.

Indeed, even in the Latin West between the twelfth century and the midfourteenth when Oresme was actively writing, there are only a few mentions of the topic. Duhem cites only Henri Bate de Malines 54 and Duns Scotus ${ }^{55}$ while, as we have already seen, Thorndike cites an anonymous author ${ }^{56}$ of the fourteenth century. Duns Scotus raises the problem in connection with his rejection of the Great Year. The concept of a Great Year can be refuted, says Duns, if it could be shown that the celestial motions are incommensurable. They would be incommensurable if with equal curvilinear velocities and in equal times, the distances traversed by any two celestial motions were incommensurable. But such a thing would be difficult to determine and with this Duns leaves off.
With such a meager background, Oresme's treatment of this difficult subject is truly astonishing. His is a full-blown mathematical treatment seemingly without benefit of any discernible previous tradition. Upon learning that some people believed the celestial motions were incommensurable, he may have set about investigating the mathematical consequen-
${ }^{53}$ Ptolemy: Tetrabiblos, ed. and trans. Robbins, pp. 15-17.
${ }_{54}$ Duhem, Systeme du monde, Vol. 8, 44447. If the passage Duhem quotes is truly by Henri Bate, then the latter mentions the incommensurability of the celestial motions only for the purpose of repudiating it.
${ }^{55}$ Ibid., pp. 447-48.
${ }^{56}$ See Thorndike, Magic and Experimental Science, Vol. 3, 582. If the anonymous Questio de proportione dyametri quadrati ad costam ejusdem (see above, pp. 77-78, n. 10I) is not by Oresme, then it should be added to this meager list.
ces of such a doctrine under a variety of conditions and circumstances. Given the paucity of literature on this difficult topic and Oresme's demonstrated ability in many areas of medieval science and mathematics, it is no strain on our historical instinct to believe that Oresme fashioned his treatises from his own creative mind. This reasonable conjecture gains more credence when it is realized that even after Oresme produced fullblown treatises on the subject of the incommensurability of the celestial motions, we can, as yet, point to no subsequent treatise devoted wholly or partially to this topic. ${ }^{57}$ Given Oresme's reputation and both the intrinsic and general interest of the subject matter, we may plausibly expect that other authors subsequently considered this topic and that such discussions may come to light in the future. Indeed, a few eminent contemporary and later writers mention with praise, or at least allude to, Oresme's treatises on the subject. Henry of Hesse, ${ }^{58}$ Jean Gerson, ${ }^{59}$ and Pico della Mirandola ${ }^{60}$ mention Oresme by name, while it is quite likely that both
${ }_{57}$ This assertion is justified despite the commentary on the Ad pauca respicientes by John de Fundis, who simply ignored the difficult and technical material while concentrating solely on defending astrology against the implications of Oresme's assumption that the celestial motions are probably incommensurable (see above, p. 63, n. 82 ).
${ }_{58}$ Henry of Hesse (Heinrich von Langenstein), Oresme's contemporary at the University of Paris, mentions, in his Tractatus de reductione effectuum specialium, that Oresme has shown the impossibility of determining whether the motions and speeds of all the planets are mutually commensurable. See Duhem, Système du monde, Vol. 8, 483. This could be a general ref erence to the De commensurabilitate and $A d$ pauca respicientes. Arguing against astrological prediction in his Tractatus contra as trologos coniunctionistas, Henry, likely with Oresme in mind, asserts that the foundations of astrology cannot be based on identically recurring astronomical experiences, since astronomical events are not of this type--"propter motuum superiorum va-type-"propter motuum superiorum va-
rietatem et incommensurabilitatem." The rietatem et incommensurabilitatem." The
passage containing this statement is in Studien $\chi^{u}$ Langenstein, ed. Pruckner, p. 159 .

59 Duhem, Système du monde, Vol. 8, $454^{-}$

59 , translates the following passage from Gerson's Trilogium Astrologiae theologizatae, sent to the future Charles VII in 1419. Gerson asserts only that it cannot be determined whether or not the heavens are commensurable and does not invoke Oresme's further claim that they are probably incommensurable. Thus Gerson's familiarity with the details of Oresme's treatises was probably slender. "Proposition IX. Il est absolument incertain si les mouvements des signes du Ciel sont commensurables ou incommensurables; il l'est également si une planète bien déterminée domine sur telle ou telle nation. Commentaire. Ils sont tombés dans l'erreur, comme l'a montré l'expérience, ceux qui ont voulu apporter la certitude, là où une probabilité rhétorique pouvait seule être atteinte; Maitre Nicole Oresme l'a demontré et, après lui, Monseigneur Pierre, cardinal de Cambrai, qui en a tiré une des causes des difficultés que présentent les jugements astrologiques. Peut-être est-ce pour cette raison que la grandeur précise de l'année solaire ne parait pas avoir été trouvée jusqu'ici; sinon de quelle raison cela provient-il?" Note the familiar argument about the inability to find the exact length of the year.

60 Pico della Mirandola ( $1463-96$ ) says Oresme showed in his De proportionibus

Pierre d'Ailly ${ }^{61}$ and Nicholas of Cusa ${ }^{62}$ were familiar with Oresme's work in this area. But none of these individuals used, or even plagiarized, any of Oresme's propositions-to say nothing of adding to them. Thus whereas his ideas in the De proportionibus on rational and irrational ratios of
proportionum that celestial bodies can never return to the positions they held previously. This vague reference could be to Oresme's arguments against the Great Year (IV. 606-9) or to special propositions enunciating the impossibility of the repetition of any particular celestial disposition (IV. 583-600). Possibly Pico had a version of the De proportionibus that included the $A d$ pauca respicientes. If so, his remarks could apply to quite a number of propositions. "Nos vero istam insaniam coarguere pluribus non est necesse, quoniam Haly ipse, in expositione primi libri Apostelesmatum ex sententia ipsius Ptolemaei idem dicentis, illam indignanter etiam obiurgat nec esse aliquem ait peritum arithmeticae, cui non iste error evidens fiat. Tum post eum Nicolaus Orem philosophus acutissimus et diligens mathematicus, in tractatu de proportionibus proportionum, mathematicis rationibus opinionem illam falsam et impossibilem demonstravit, putantem scilicet posse eandem unquam caeli et siderum redire positionem quae alias fuerit."-Pico della Mirandola, Disputationes adversus astrologiam divinatricem, ed. Garin, Vol. 2, 12-14. Oresme is also mentioned in Vol. i, 58 , and Vol. 2, 420, 530 . See also Thorndike, Magic and Experimental Science, Vol. 3, 423.
${ }^{61}$ Coopland, in his edition of Oresme and the Astrologers, p. 46, quotes a brief statement from Pierre d'Ailly's Tractatus contra astronomos, in which the latter says: "'Quod proposita quaestio de commensurabilitate motuum coelestium est problema neutrum de quo naturaliter haberi non potest evidens certitudo'; for even in terrestrial matters and those close at hand, 'nequeat saepe punctualis praecisio deprehendi, sed minor pars quam millesima aequalitatem tollat et proportionem ad irrationalem commutet.'" D'Ailly's claim that a part smaller than a thousandth could destroy an equality and change a ratio from rational to irra-
tional is almost a verbatim repetition of a statement made by Oresme in his De commensurabilitate. In Part III of that treatise, which takes the form of a debate presided over by Apollo, Oresme has the latter say: "...nequit deprehendi precisio punctualis. Si enim excessus imperceptibilis, ymo minor pars quam eius millesima [Vat. lat. 4082 has the numeral 1,000 ] equalitatem tollit et proportionem mutat ['et' is repeated twice at this point; both have been eliminated] de rationali ad irrationalem quomodo motuum aut magnitudinum celestium punctualem proportionem poteris agnoscere."-MS Vat. lat. 4082, fol. rosv, c.I-c.2. ("...exact precision is undetectable. For if an imperceptible excess-even a part smaller than a thousandth-could destroy an equality and alter a ratio from rational to irrational, how will you be able to know a punctual or [exact] ratio of motions or celestial magnitudes?") Like Gerson (see above, p. 119, n. 59), however, Pierre d'Ailly emphasizes that no "natural" evidence can decide whether the heavenly motions are commensurable. He calls it a problema neutrum, which is a technical term used in the Middle Ages to denote a situation in which two alternatives are either equally probable or equally incapable of demonstration (see Maier, Vorläufer Galileis, p. 199). If Oresme adopted the first alternative, he would not, like D'Ailly, have characterized this question as a problema neutrum, since he thought it more probable that the celestial motions were incommensurable. But Oresme recognized that one could never demonstrate scientifically whether the celestial motions were commensurable or incommensurable. Therefore, had he emphasized, or confined himself to, the second alternative, the question would have constituted a problema neutrum. Finally, if one interprets a problema neutrum as embracing all problems where

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ratios were utilized by Alvarus Thomas and George Lokert, we have yet to discover any authors who genuinely continued Oresme's interest in the commensurability or incommensurability of the celestial motions.

## Capsule Summary of $A d$ pauca respicientes

Leaving aside the details and intricacies of the many propositions, the basic aims and motives of the Ad pauca respicientes are fairly straightforward. It is a treatise devoted exclusively to the kinematics of circular motioni.e., to distance and time factors as applied to circular motion. Oresme's purpose was to determine the consequences deriving from the motions of points or bodies moving with different uniform circular velocities. These bodies, usually two or three, are, in some propositions, assumed to move
the alternatives are indemonstrable and each is equally probable, then, of course, the question at issue could not be a problema neutrum for Oresme.
D'Ailly also repeats the now familiar claim about the inability of astronomy to fix the true and exact length of the year (see Oresme and the Astrologers, ed. and trans. Coopland, pp. 193-94, n.94).
On the basis of d'Ailly's unacknowledged quotation from Oresme, it seems plausible to accept Coopland's conjecture (p. 11) thatd'Ailly probably had some acquaintance with Oresme's work-and perhaps with Oresme himself-since he was a student in the College of Navarre in 1368 at the age of eighteen, which, some years earlier, had also been Oresme's college. Indeed, Gerson's statement (see above, p. 119, n. 59) linking Oresme and d'Ailly helped convince Duhem of Oresme's influence on d'Ailly, so that he attributed the former's De commensurabilitate, which he knew in an anonymous manuscript, to the latter. See Duhem, Système du monde, Vol. 8,455, and Grant, "Oresme: Comm.," pp. 457-58. Thorndike, Magic and Experimental Science, Vol. 3, 423, maintains that d'Ailly knew Oresme's arguments against astrology, but mentions them in order to reject them. No evidence is presented to substantiate this claim. Duhem, however, tells us that if ever d'Ailly was hostile to astrology
it was only as a young man, for he later succumbed to the influence of Roger Bacon and embraced astrology wholeheartedly.
${ }^{62}$ Although Nicholas of Cusa did not explicitly mention Oresme's treatises on the incommensurability of celestial motion, Duhem argues that Cusa was influenced by the anonymous treatise that he attributed to Pierre d'Ailly (see previous note), but which was actually Oresme's De commensurabilitate. In his Reparatio calendarii, published in 1436, Cusa says (the translation is Duhem's): "La durée de l'année demeure douteuse,... Par là, certains astronomes ont été réduits à déclarer que tout mouvement de corps céleste est incommensurable à la raison humaine, qu'il dépend d'un rapport irrationnel, qu'il admet une racine sourd et qu'on ne peut dénommer; étant donnéeune mesurehumaine, qui mesure approximativement un certain mouvement, on en peut toujours donner une autre qui soit plus approchée." Duhem's translation is from the collected works entitled D. Nicolai de Cusa Cardinalis, utriusque juris doctoris, in omnique philosophia incomparabilis viri Opera... ex officina Henricpetrina (Basle, is6s), Vol. 3, II57.
Once again, the length of the year is the raison d'etre for mentioning the incommensurability of the celestial motions. Cusa's interest in the problem seems confined to this particular passage.
with mutually commensurable velocities and in others with incommensurable velocities.
In contrast to astronomers, and especially in Part 1 , Oresme is generally interested in precise punctual relations between the moving bodies. That is, he wishes to specify the exact point or points at which the bodies will meet or oppose one another, and if the bodies move commensurably, to determine the precise time intervals between such occurences. On the assumption of incommensurable speeds, such precision is impossible.
In Part 2 Oresme shifts his point of emphasis and concerns himself with general dispositions and relative positions of two or more mobiles before and after they enter into actual conjunction or opposition. The particular points in which these events occur are now of little significance.

Of great interest is Supposition II of Part I (p. 385), where Oresme assumes that any two quantities are probably incommensurable. Although this supposition is not utilized until the very end of the treatise, its application is the highlight of the $A d$ pauca respicientes. Having formulated numerous abstract propositions involving points or bodies moving with commensurable or incommensurable speeds, Oresme, in Proposition XVII of Part 2 (p. 423), invokes Supposition II and applies it to celestial motions in order to demonstrate that no configuration or relationship of celestial bodies can ever repeat. This constitutes the basis of his repudiation of astrological prediction in Proposition XIX (pp. 425, 427). If every celestial configuration is unique-as it must be on the assumption of incommensurable speeds, which by Supposition II are probable-astrological prediction is hopeless for it must rely on cumulative observations of precisely recurring events. This culminating attack against astrology may have initially provided Oresme with sufficient incentive to develop what was probably his first major effort to cope with the problem of the incommensurability of the celestial motions. Together with his later De commensurabilitate vel incommensurabilitate motuum celi, Oresme may have left us the only two treatises devoted wholly to an investigation of this interesting topic.


Plate I: Opening page of the De proportionibus. MS Paris, Bibl. Nationale, fonds latin, 7371, fol. 269 r.


Plate 2: Figure in Ch. III of the De proportionibus (see p. 234). MS Paris, Bibl. Nationale, fonds latin, 1662 I , fol 103 r




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Plate 3: End of Ch. II and beginning of Ch. III of the De proportionibus. MS Erfurt, Wissenschaftliche Bibl., Amplonius Q. 389 , fol. 77r.























[^13] Bibl. Pepysiana 2329 (actually in Magdalene College), fol. 93 v .


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6: End of Ch. I and beginning of Ch. II of the De proportionibus. MS Seville, Bibl. Colombina, 7-7-13, fol. ifsv.




















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page of the Adpauca respicientes. MS Paris, Bibl. Nationale, fonds atin, 7378 A , fol. 14 V


Plate 10: Page from Part I of the $A d$ pauca respicientes including all but the opening line of Prop. IV and extending through the first few words of Prop. VIII. MS Paris, Bibl. Nationale, fonds latin, 1662 1, fol. itiv.


Plate II: Page showing the beginning of the $A d$ pauca respicientes in the second column. MS Vat. Palatine lat. 1354, fol. 233 v .

## De proportionibus proportionum

# Manuscripts and Editions 

## Manuscripts Used in Establishing Text

I. $P=$ Paris, Bibliothèque Nationale, fonds latin, 7371 , fols. $269 \mathrm{r}-278 \mathrm{v}$. This manuscript is written in a single column in a fairly legible hand. The folios are numbered on successive recto sides. There are inkspots at the bottom of some folios obscuring a few words, although in one or two instances the underlying text can be read. The manuscript is incomplete containing only the first two chapters with numerous omissions, often of two or more consecutive lines. This manuscript has been collated fully only through Chapter I, line 131, and thereafter utilized occasionally in a few special instances.
Charles Thurot, who examined the manuscript, dated it in the fourteenth century ${ }^{\mathrm{I}}$ and Curtze, who had not seen it, placed it in the fifteenth century. ${ }^{2}$
2. $H=$ Paris, Bibliothèque Nationale, fonds latin, 16621 , fols. $94 \mathrm{r}-\mathrm{I}$ ior. The text is written in a single column and has numbered folios. On fol. 93 v there appear three lines on an otherwise blank page, which read as follows: "Sequitur tractatus proportionum Orem cum quodam tractatu astrologico ad pauca aspicientes habito ab illo de Muris item exepta

[^14]de Dumbleton(?) de proportione motuum in velocitate pene et similia tractata(?)." ${ }^{3}$ The "tractatus proportionum" occupies folios $94 \mathrm{r}-\mathrm{r}$ ror. The "astrological tract" is the Ad pauca respicientes, which is referred to here as "ad pauca aspicientes" and called De astrologia aliqua specialia at the top of fol. inov, the first page of the $A d$ pauca respicientes. Did the scribe who added these lines wish to claim Oresme as the author of the astrological treatise and to inform the reader that Oresme took his material from (Johannes?) de Muris ("habito ab illo de Muris")? Or are we to understand that de Muris is the author? The introduction of the name of Johannes de Muris would seem unwarranted, since to my knowledge he has not been associated in any manner with the subject matter of the Ad pauca respicientes. There is certainly no reason to doubt Oresme's authorship (see p. 73, n. 98).
Pierre Duhem described codex 16621 as a poorly organized collection of notebooks assembled toward the end of the fourteenth century by a Parisian student. ${ }^{4}$ The different works deal with various doctrines in vogue at Oxford in the fourteenth century
There is no title on fol. 94 r but "Oresme" is written at the top of the page. As we have already seen the treatise is called "tractatus proportionum" on fol. 93 v , but in the explicit on fol. nor it is called "tractatus de proportionibus." Since the $A d$ pauca respicientes is in the same codex on fols. roov-ri4r, the letter $H$ will serve also as its siglum. However, any reference to $H$ will specify the treatise. MS $H$ of the De proportionibus has been collated through the entire length of its four chapters.
3. $E=$ Erfurt, Wissenschaftliche Bibliothek, Amplonius Q.385, fols. 67r82v.
The codex has a total of thirty-two works totaling 222 leaves. ${ }^{5}$ The
${ }^{3}$ Lynn Thorndike furnishes a full description of BN i662I in an article, "Some Medieval and Renaissance Manuscripts on Physics," Procedings of the American Pbilosophical Society, Vol. 104, 188-201. Our codex is discussed on pp. 188, c.2-191, c.r. Thorndike's transcription of these three lines differs somewhat from my own. Where I have "pauca" he has "peluca (pauca)"; for "aspicientes" he has "aspiciens"; for "Dumbleton(?)" he has "efmion (Exafrenon?)"; and, finally, for "et similia tractata(?)" he offers "consimilia." However, Thorndike's transcription permitted me to
correct a few readings in my earlier version of this brief passage.
4 Duhem, Etudes sur Léonard de Vinci, Vol. 3, 41 II . Charles Thurot, in a review of Carl Prantl's Geschichte der Logik im Abendlande, also dates 16621 in the fourteenth century.-Revue critique d'bistoire et de littérature, Vol. 6, Pt. I (1872), 143.
${ }_{5}$ A complete description of the codex is given in Schum, Amplonianischen Hand-schriften-Sammlung, pp. 641-44. My information is entirely derived from Schum. The De proportionibus is specifically described on p. 642.

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De proportionibus is the ninth work. Schum characterizes the hand as a small, forceful cursive one at the beginning of the work, becoming deformed as the work progresses. The manuscript contains almost three full chapters lacking lines III.s 14-28. It has been completely collated. A partial table of contents, by a fifteenth-century hand, appears on the front inside cover where the author and title of the work in question are given as, "Nicolai Orem tractatus de proportionibus proportionum."

This codex does not appear in the catalogue left by Amplonius Ratinck in 1412. It was acquired sometime in the fifteenth century and assigned a signature number of 95 in the mathematics section of the original Amplonius collection. Schum dates the codex at the beginning of the second half of the fourteenth century which, if true, would make this a very early manuscript. Although Schum offers no reason for this estimation, it is possible that he based it on a notarial document written in German and pasted on the inside cover of the codex. The document is dated November or December 1364. This would, of course, be very slender evidence at best since the notarial document could have been added long after the codex was bound, or the codex may have been bound long after 1364, the date of the document.

In another codex of the Amplonian catalogue of 1412-number seven under the division of logic-we find the title Orem de proporcione proporcionum subtilis. Unfortunately, this codex is among those of the original Amplonian catalogue which are unlocated and presumed lost. ${ }^{6}$
4. $C=$ Cambridge, Peterhouse 277, Bibliotheca Pepysiana 2329, fols. $93 \mathrm{v}-$ irov (actually in possession of Magdalene College, Cambridge).

This codex was "given to Peterhouse in $\mathbf{1 4 7 2}$... borrowed in 1556 by Dr. John Dee, and was in Pepys' possession in 1697 when the 'Catalogi Manuscriptorum Angliae et Hiberniae' were published. The contents appear in that work (Vol. II., p. 208, 9) catalogued as separate items under nos. 6767-76, 6778, 6780-84." ${ }^{7}$

Our manuscript-the fifth work in the codex-is written in double columns in a fine, clear hand. The first treatise in the codex is Jordanus' Arithmetica, which was copied at Paris in 1407 by Servatius Tomlinger who also copied the De proportionibus, ${ }^{8}$ and whose name appears in the

[^15]colophon. Thus it is reasonable to assume that the De proportionibus was copied ca.1407.
All four chapters of the De proportionibus are included and have been collated throughout in editing the text.
5. $V=$ Venice, Bibliotheca Marciana, Cod.ıo, a.347, i.237, L.VI, 133 , fols. $50 \mathrm{r}-62 \mathrm{v}$.
The text is written in double columns in a clear, easily readable hand with no marginal notations except in a few instances where textual corrections are made, in each case for only one word. The codex contains 72 numbered folios embracing five works plus a fragment. The De proportionibus is the fourth treatise but lacks a title and is anonymous. It has been dated in the fourteenth century by Valentinelli who has described the contents of the codex. ${ }^{9}$ In his description of folios $50 r-72 r$ (it should be 72 v ) Valentinelli has conflated four separate works under the single title Tractatus de velocitate motuum which he believed was divided into two parts. Actually fols. sor-62v are a complete four-chapter version of Oresme's De proportionibus proportionum, which is followed without a break by Oresme's Ad pauca respicientes, extending over fols. $62 \mathrm{v}-65 \mathrm{r}$. The latter treatise is also untitled and anonymous, as is the next work, which is Oresme's Algorismus proportionum covering fols. 6sv-7or. Finally, fols. $70 \mathrm{r}-72 \mathrm{~V}$ contain a fragment of the De proportionibus of Roger Thomas with the incipit "naturalis philosophie completa cognitio absque motus...." ${ }^{\text {io }}$
Since the Ad pauca respicientes follows immediately after the De proportionibus, the same siglum, $V$, has been chosen to designate both treatises. To avoid possible confusion, all references to $V$ will indicate unambiguously the work intended. All four chapters have been collated.
6. $R=$ Vatican Library, Latin MS 4275, fols. 102r-127r.

The De proportionibus proportionum bears no title on fol. 102r and is anonymous. It is written in a single column in a very clear, neat hand. At the conclusion of the treatise on fol. $\mathbf{2} 27 \mathrm{r}$ it is called Tractatus de velocitate motuum. Two other treatises by Oresme precede the De proportionibus. On fols. $90 \mathrm{r}-96 \mathrm{r}$ there is a complete, but anonymous, version of the Algorismus proportionum, which carries the title of Tractatus de additione et

[^16]
## Manuscripts and Editions

subtractione proportionum. ${ }^{\text {II }}$ The third and final part of the De commensurabilitate vel incommensurabilitate motuum celi appears on fols. $96 r-10$ Ir and bears the title Pulchra disputatio: si omnes motus celi sint invicem commensurabiles an non (fol. 96r). ${ }^{12}$

All four chapters of this manuscript of the De proportionibus have been collated for the edition.
7. $S=$ Seville, Biblioteca Colombina, 7-7-13, fols. 114r-122v.

This codex belonged to the very large collection gathered by Fernand Columbus (1489-1 539), second son of Christopher Columbus. The registration or acquisition number of the codex is 10285 , which indicates that it was purchased by Fernand on April io, 1531, in Padua, Italy. ${ }^{13}$ The manuscript is written in double columns in a clear hand. It was probably copied during the second half of the fourteenth century. ${ }^{14}$ Guy Beaujouan reports ${ }^{15}$ that, in addition to the De proportionibus proportionum, Oresme's Questions on the Sphere and Questions on the Geometry of Euclid are in this codex. Also included are Questions on the De generatione by Richard Killington; Questions on the Pbysics by William Collingham; Questions on the Void by Albertinus de Rainaldis de Plaisance; De proportione velocitatum in motibus of Thomas Bradwardine; Pseudo-Oresmian De latitudinibus formarum; and Questions on Logic by Johannes de Wesalia.

The Seville manuscript constitutes a complete four-chapter version of the De proportionibus. No title or author is supplied. Only Chapter I has been fully collated but occasional variant readings have been furnished from the later chapters.
work was identified by means of Thorndike and Kibre, Catalogue of Incipits, c.903.
${ }^{11}$ For a list of eighteen other manuscripts of this popular treatise, see Grant, "Mathematical Theory of Oresme," pp. 309-26.
${ }^{12}$ Anneliese Maier has utilized this manuscript in discussing the third part of Oresme's De commensurabilitate. See Maier, Metaphysische Hintergründe, pp. 28-31.
${ }_{13}$ The connection between Fernand Columbus' itineraries and the acquisition numbers is given by Guy Beaujouan, "Fer-
nand Colomb," Journal des Savants (Oct.Dec., 1960), pp. 145-59. The date and place of purchase for our codex is given on p. 154.
${ }^{14} \mathrm{I}$ am indebted to Guy Beaujouan for this estimate. He stressed, however, that this was only a preliminary evaluation and that further study of the codex was required.
${ }^{15}$ Beaujouan, "Manuscrits scientifiques médiévaux," Actes du dixièm congrès international d'bistoire des sciences, pp. 63 1-34.

## Additional Manuscripts

8. Leipzig, Universitäts-Bibliothek, Latin MS 1480, fols. $135 \mathrm{~V}-153$ r. ${ }^{16}$
9. Dresden, königliche offentliche Bibliothek, MS C 80, fols. 234-44. ${ }^{17}$
10. Erfurt, Wissenschaftliche Bibliothek, Amplonius Q.352, fols. I34v$148 \mathrm{~V} .{ }^{18}$

## Editions

r. Venice edition of 1505 .

This edition is correctly described by Maximilian Curtze who gives the contents of the title page as follows: "Questio de modalibus bassani politi/ Tractatus proportionum introductorius/ ad calculationes suisset/ Tractatus proportionum thoma bradwardini/ Tractatus proportionum nicholai oren/ Tractatus de latitudinibus formarum blasij de parma/ Auctor sex inconvenientum." ${ }^{19}$

The volume is folio size containing seventy-four leaves of which only leaves $2-16$ are numbered. The type is Gothic and the printing is in double columns. On fol. 74 r we find a colophon: "Venetiis mandato et sumptibus heredum quondam No/bilis Viri D. Octaviani Scoti civis

[^17]Moedoetiensis per/ Bonetum Locatellum Bergomensem presbyterum. Kalen/ die Semptembribus. 1505."20
The edition contains two additional works beyond those listed on the the title page. These are given on fol. 74v: "Questio subtilis doctoris Johannis de/Casali de velocitate motus alterationis/ Questio blasij de Parma de tactu cor/ porum duorum/." ${ }^{21}$
Oresme's work on proportions begins on the first unnumbered leaf, 171, and ends on 25r, c.2. It bears the title Proportiones Nicholai horen (17r) and contains the first four chapters. Immediately after, and without any break, the first part of the Ad pauca respicientes commences (25r, c.2) but is called Chapter V of the De proportionibus. The second part of the $A d$ pauca is then labeled Chapter VI (see pp. 72-74). The concluding words of the treatise on 26 v are: "...que in manu dei sunt et ipsa solus novit cuius oculis cuncta sunt nuda et aperta./ Proportionum Nicholai horen. Finis/ Cum dei laude. Amen."
The Venice edition is filled with errors which frequently render it unintelligible. However, it has been fully collated through Chapter I, line 131 and has been designated by the letters $E d$.
2. Paris edition (undated). ${ }^{22}$

This edition of Oresme's De proportionibus is almost identical with the Venice edition of ryos (see p. 130) and it seems reasonable to assume that one was printed from the other since the marginalia are virtually the same in every instance but wholly different from all of the manuscripts listed here. The title page reads as follows: "Tractatus proportionum/ Alberti de Saxonia/ Tractatus proportionum Thome bra-/duardini/ Tractatus proportionum Nicholai horen./ Venales reperiuntur Parisius in vico diui/ Iacobi iuxta templum Sancti yuonis sub/ signo pellicani." ${ }^{23}$ The printer is "de Marnef," whom Curtze calls Godefroy de Marnef.
${ }^{20}$ Ibid.
${ }^{21}$ Ibid.
${ }_{22}$ George Sarton writes that the Tractatus proportionum Nicholai Orem was "first printed in Paris? c. I 500 , then Venice 1505 with the previous item [i.e., the Tractatus de latitudinibus formarum]. Reprinted in Paris c. ifio together with treatises bearing the same title by Albert of Saxony and Thomas Bradwardine. In this edition the author is named Nicholas Horen."-Introduction to the History of Science, Vol. 3, Pt. 2, p. 1496.

The bracketed addition is mine. It is almost certain that the last mentioned edition is the same as that which is under discussion here. Sarton does not explain how he arrived at the approximate date of 1510 . I have no knowledge of a 1500 Paris edition.
${ }^{23}$ Curtze also discusses this edition and quotes the title page but, perhaps through a typographical error, has "Parisiis" for "Parisius."-"Extrait d'une lettre," Bulietin des sciences mathématiques et astronomiques, Vol. 6, 58.

The pages of this edition are unnumbered but Oresme's De propor-tionibus-or Tractatus proportionum as it is called-is the last treatise occupying pages 23-42. However, the De proportionibus proper-i.e., the first four chapters-extends over pages 23-39 c.2. But like the Venice edition, Parts One and Two of the Ad pauca respicientes are incorporated as Chapters V and VI, respectively, of the De proportionibus and these chapters cover pages $39 \mathrm{c} .2-42$. The work terminates with exactly the same concluding words as the Venice edition (see under Venice edition, p. I3I).

## Sigla of Manuscripts and Editions

$C=$ Cambridge, Peterhouse 277, Bibliotheca Pepysiana 2329, fols. $93 \mathrm{~V}-$ ilov.
$E=$ Erfurt, Wissenschaftliche Bibliothek, Amplonius Q.385, fols. 67r82 v .
$E d=$ Questio de modalibus bassani politi, Tractatus proportionum introductorius ad calculationes suisset, Tractatus proportionum thome bradwardini, Tractatus proportionum nicholai oren, Tractatus de latitudinibus formarum blasïy de parma, Auctor sex inconvenientum (Venice, 150s).
$H=$ Paris, Bibliothèque Nationale, fonds latin, 16621, fols. $941 \mathrm{r}-\mathrm{I}$ Ior.
$P=$ Paris, Bibliothèque Nationale, fonds latin, 7371, fols. 269r-278v.
$R=$ Vatican Library, Vat. lat. 4275 , fols. 102r-I 27 r .
$S=$ Seville, Biblioteca Colombina, $7-7-\mathrm{I} 3$, fols. 1 14r-122v.
$V=$ Venice, Bibliotheca Marciana, Cod. 10, a.347, 1.237, L.VI, 133, fols. 50r-62v.

## De proportionibus proportionum

## [Proemium]

. Omnis rationalis opinio de velocitate motuum ponit eam sequi aliquam proportionem: hec quidem proportionem excessus potentie motoris ad resistentiam seu potentiam rei mote, alia vero proportionem resistentiarum manente eadem potentia vel equali vel propor-
5 tionem potentiarum manente eadem resistentia vel equali, tertia proportionem potentie motoris ad resistentiam sive potentiam rei mote quam veram reputo et quam Aristoteles et Averroes tenuerunt. Unde secundum quamlibet istarum opinionum quanto proportio est maior tanto velocitas motus est maior et quanto minor tanto motus est
ıо tardior. Sicut enim velocitas sequitur proportionem sic proportio velocitatum proportionem proportionum consequitur et secundum

Title De proportionibus proportionum om ERSV tractatus magnus et utilis de proportionibus proportionum magistri Nicholai Horesme $C$ tractatus de proportionibus proportionum ab Oresme [in alio manu] $P$ Oresme $H$ Proportiones Nicholai Horen Ed [Proemium] om CEHPRSVEd
I supra Omnis rationalis scr $H$ assit(?) principio virgo(?) maria(?) madonna (?) rationalis: racionabilis $V$ opinio: oppinio $H /$ ante sequi bab $V$ et e | sequi CPRSVEd sequitur $E H$ I-2 aliquam rep $E$
2 hec om $V /$ quidem: quidam(?) $V /$
proportionem ${ }^{2}$ : proportionum $E$
3 motoris: motorum $V /$ seu HPS sive CERVEd
4 resistentiarum CEHPRS resistentie $V E d /$ post potentia add $P$ motiva
$4^{-5}$ vel proportionem...equali $P R S V$ Ed; om CEH
s ante manente add $P$ activarum / eadem om $P$ / post tertia add $R$ vero
5-6 proportionem potentie: potentiam
6 potentie: ponit S / sive CERSVEd seu $H P$
7 Aristoteles: Aristotelem $H$

## On Ratios of Ratios

## [Introduction]

Every reasonable opinion about velocity of motions assumes that it follows some ratio. One opinion is [that velocity of motion follows] a ratio of the excess of the power of the motor to the resistance or power of the thing which is moved;* another opinion supposes that velocity follows the ratio of resistances with the power remaining constant or equal, or [follows] the ratio of the powers with the resistance remaining constant or equal; + a third opinion [assumes that velocity of motions follows] the ratio of the power of the motor to the resistance or power of the thing moved. This last opinion I consider true and the one held by both Aristotle and Averroes. $\ddagger$ Thus, according to any of these opinions, a ratio is greater as the velocity of motion is greater, and as the ratio is less, so the motion is slower. And, indeed, just as velocity depends on a ratio, so also does a

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\(* V_{2} / V_{1}=\left(F_{2}-R_{2}\right) /\left(F_{1}-R_{1}\right)\), where
\(V\) is speed, \(F\) force, and \(R\) resistance.
\(\dagger V_{2} / V_{\mathrm{I}}=R_{\mathrm{I}} / R_{2}\) when \(F_{2}=F_{1}\); or
\(V_{2} / V_{\mathrm{I}}=F_{2} / F_{\mathrm{I}}\) when \(R_{2}=R_{1}\).
\(\ddagger\) This is not yet a full statement of the "true law" (see p. 309) which is \(F_{2} / R_{2}=\) \(\left(F_{\mathrm{I}} / R_{\mathrm{I}}\right)^{n}\), where \(n=V_{2} / V_{\mathrm{I}}\)
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8 ante est add $C$ velocitatum
9 velocitas... maior $E R E d$ motus est
maior $C$ velocitas est maior $H$ motus
est velocior $P$ velocitatis maior motus

9 velocitas...maior $E R E d$ motus est maior $C$ velocitas est maior $H$ motu $V$ velocitas motuum est maior $S$
est ${ }^{2}$ SVEd; om EHPR
9-10 est tardior $\operatorname{tr} C$
II velocitatum: velocitatis $E d /$ conse quitur CSVEd; om $H$ sequitur $E P R$
huiusmodi proportionem proportionum proportio velocitatum in motibus est sumenda.
Ut igitur studiosi in ulteriorem inquisitionem excitentur, utile est 15 de proportione proportionum aliqua dicere quorum notitia non solum ad proportiones motuum sed ad philosophie secreta et ardua negotia prestat inestimabile iuvamentum. Quid sit proportio et qualiter dividitur in proportionem inequalitatis et equalitatis et maioris inequalititatis et minoris et in proportionem rationalem et irrationalem, et
${ }_{20}$ rationalis in quinque genera quorum quodlibet in species dividitur infinitas et cetera, convenientia a pluribus auctoribus iam tradita presuppono. Et ad propositum accedo hunc tractatum per capitalia dividendo.
In quorum primo quedam preambula, velud quedam principia pre-
${ }_{25}$ suppono, premittam sine demonstratione ratione brevis introductionis. Nihil, tamen, in aliis capitulis ex primo supponitur quod non sit per se notum vel sicut ibi tangitur alibi demonstratum. In secundo conclusiones aliquas de proportionibus demonstrabo et subiungam quedam practica documenta. In tertio de proportionibus proportioo num specialius pertractabo. In quarto prius dicta ad proportiones motuum applicabo. In quinto ad velocitates motuum condescendam.

12 huiusmodi CERS huius $H P$ hanc VEd
13 sumenda: attendenda $H$
14 studiosi: studiosum(?) $V /$ in $: \operatorname{ad} H$ inquisitionem excitentur $\operatorname{tr} R /$ inquisitionem: incompositionem $C /$ excitentur CEHS exercitentur PVEd
is proportione: proportionibus $P /$ aliqua: aliqualiter $P /$ dicere $C H P R S E d$ om $E$ dicetur $V /$ quorum $C E H P R S$ quarum $V E d /$ notitia om $V /$ ante non add $V$ sed / post solum add $E$ proportiones
16 ad²: etiam $H$ / philosophie: philosophice $C$ / ante ardua add $C R$ ad
7 inestimabile $C E P R S V$ inextimabile HEd / ante Quid add P igitur / qualiter: equaliter $V$
18 proportionem: partes $E$ / inequalitatis et $S$; om CEHPRVEd / equalitatis et om $R$
19 post minoris addEHP inequalitatis / in om $H$ / proportionem... irrationalem: irrationalem et rationalem $R /$ pro-
portionem CHPSVEd partem E/ post et ${ }^{3}$ add $S$ in / et ${ }^{4}$ om $H$
20 rationalis CHRSVEd; om $E$ rationale P
20-2I dividitur infinitas $\operatorname{tr} C$
21 et CEHPRS; om VEd / cetera convenientia $\operatorname{tr} E d /$ pluribus: peritis $S$ auctoribus om $H /$ tradita: tractata $H$
21-22 presuppono: presuppone $V$
22 accedo $C H P R V E d$ procedo $E$ accedendo $S$
22-24 hunc...quorum om $P$
24 quedam ${ }^{1}$ CEPVEd; om $H$ que $R$ velud $E H R S V$ velut $C E d$ quasi $P$
24-25 presuppono $H$;om $P$ sic $C R S V E d$ sunt sic $E$
25 premittam: premittendo $P$
25-26 sine...introductionis $H$ quodlibet causa brevitatis introductionis aliqua sine demonstratione vera dicam $C$ quod licet causa brevius introductionis sine demonstratione aliqua vera dicam $V$ quodlibet tam brevis introductionis sine demonstratione aliqua vera dicam
ratio of velocities depend on a ratio of ratios; and a ratio of velocities of motions must be taken to vary as such a ratio of ratios.
In order that students may be stimulated to further inquiry, it is useful to say some things about a ratio of ratios. Knowledge of these matters should prove a great help, not only for ratios of motions, but also for the secrets and difficult labors of philosophy. I presuppose some appropriate things already supplied by several authors: [for example,] [I] what a ratio is; and [2] how a ratio of inequality is distinct from one of equality; [3] how one of greater inequality is distinct from one of lesser inequality; and [4] how a rational is distinct from an irrational ratio; and [s] how a rational ratio is classified into five genera, any one of which is subdivided into an infinite number of species, and so forth. I now move on to the matter at hand by dividing this treatise into chapters.
In order to serve as a brief introduction, I shall, in the first chapter, present some preliminary things-for example, I presuppose some prin-ciples-without demonstration. However, nothing from the first chapter is assumed in the other chapters that is not self-evident, or that has not been demonstrated elsewhere. In the second chapter I shall demonstrate some propositions about ratios, adding some practical examples. In the third chapter I shall especially consider "ratios of ratios." In the fourth I shall apply things said previously to ratios of motions. In the fifth I shall

[^18]tionibus demonstrabo CHRSVEd demonstrabo de proportionibus $E$ demonstrabo $P$
28-29 et...documenta om $P$
29 practica $C E H R$ pauca $V$ pulchra $S /$ practica documenta: documenta pauca Ed/In om $P$
29-30 de... pertractabo: quedam practica ponenda subiungam conclusiones aliquas speciales $P$
30 specialius $C E H R S$ specialiter VEd/ pertractabo: pertractando $S$
30-33 In quarto... pertransivi om $P$
31 applicabo $C E H R V E d$ applicando $S$ post quinto scr et del $V$ quinto / ad om $V$ / velocitates $E H V E d$ velocitatem CRS / motuum om Ed

In sexto dicam de incommensurabilitate motuum celestium corrigendo quedam que alias ad pauca aspicientes breviter pertransivi.

## Primum Capitulum

35 Omnes proportiones equalitatis sunt equales nec earum plures species assignantur sed tantum est una. Omnis vero proportio maioris inequalitatis in infinitum excedit proportionem equalitatis et omnis proportio minoris inequalitatis in infinitum exceditur a proportione equalitatis et a qualibet maioris inequalitatis ut postea videbitur. Unde
${ }_{40}$ patet quod nulla est proportio inter proportionem inequalitatis et equalitatis et inter proportionem maioris inequalitatis et proportionem minoris, scilicet unius ad alteram. Quare tantummodo dicendum est de proportionibus maioris inequalitatis inter se et de proportionibus minoris inequalitatis inter se.
45
Proportionem maioris inequalitatis dividere est inter aliquos terminos medium seu media assignare. Verbi gratia, sit $B$ una quantitas maior et $C$ una alia minor et proportio $B$ ad $C$ sit $A$. Dico quod dividere $A$ est invenire seu assignare medium aut media inter $B$ et $C$. Voco, autem, medium generaliter quicquid est maius $C$ et minus

32 ante dicam bab $S$ aliqua / dicam om $H$ / incommensurabilitate $C H S V E d$ commensurabilitate $E R$ / corrigendo $C E$ HRSEd exigendum $V$
33 quedam CHRVEd; om $E$ aliqua $S /$ ante alias bab $E$ ad et $S$ aliquas / aspicientes $E$ aspiciens $C H$ respiciens $S V$ $E d$ inspicientes $R /$ breviter om $S$ / pertransivi HSVEd pertractavi CER | post pertransivi add $V$ et cetera et add Ed Explicit pars prohemialis
34 Primum Capitulum P; om CEH RSVEd
35 equalitatis om $C$ / sunt equales $\operatorname{tr} C$
36 tantum $R$ tamen CEPSVEd/ sed tantum est una om $H$ et rep $V$ sed tamen est una / ante est add $R$ earum / est una $\operatorname{tr} E d /$ maioris om $C$
$36-37$ inequalitatis: equalitatis $V$
37 in om $E$
37-38 omnis proportio: cumque minoris $P$
38 in om $E$ / a proportione om $P$

39 ante equalitatis add $E$ maioris / equalitatis: equaliter $S /$ a om $C$ / qualibet: quolibet $E$ / maioris: maiore $C /$ post inequalitatis add $R$ in infinitum exceditur / postea CEHRSV post PEd
40 patet: pars $C$ / inter proportionem om $E \mid$ ante inequalitatis add $H$ maioris
40-42 inequalitatis...minoris $R$ et proportionem equalitatis nec inter proportionem maioris inequalitatis et minoris $C$ et equalitatis nec inter proportionem maioris inequalitatis et proportio minoris inequalitatis $E$ inequalitatis nec equalitatis nec minoris inequalitatis $H$ equalitatis et proportionem inequalitatis nec inter proportionem maioris inequalitatis et proportionem minoris inequalitatis $P$ equalitatis et proportionem minoris inequalitatis nec inter proportionem equalitatis et proportionem maioris inequalitatis $S$ et proportionem equalitatis nec inter proportionem maioris

## Chapter One

move on to velocities of motions. And in the sixth chapter I shall speak about the incommensurability of celestial motions by correcting certain things which, on another occasion, I had treated briefly in [the course of] reflecting on a few matters.

## Chapter One

All ratios of equality are equal; nor are several species of them assigned, for there is only one. Indeed [when compared exponentially*] every ratio of greater inequality exceeds into infinity a ratio of equality; and every ratio of lesser inequality is exceeded into infinity by a ratio of equality and by any ratio of greater inequality, as will be seen afterward. From this it is obvious that there is no ratio between a ratio of inequality and one of equality, or between a ratio of greater inequality and one of lesser inequalitynamely of one to another. For this reason it is necessary to speak only of the mutual relationships between ratios of greater inequality and, similarly, of the mutual relationships between ratios of lesser inequality.

To divide a ratio is to assign a mean, or means, between some terms. For example, let $B$ be a greater quantity and $C$ another, smaller quantity. Let ratio $B$ to $C$ be $A$. I say that to divide $A$ involves finding or assigning a mean or means between $B$ and $C$. Moreover, I call something a mean that

* See pp. $3^{17}{ }^{17}$ -
nequalitatis et proportionem minoris $V$ equalitatis et inequalitatis nec inter proportionem maioris inequalitatis et minoris inequalitatis $E d$
42 scilicet: et $E /$ scilicet....alteram om $S$ / post alteram hab $E$ excusatio / supra Quare rep $V$ quare / Quare: quia $S$
42-43 tantummodo dicendum est $C H P R$ $V$ dicendum est tantummodo $E S /$ dicendum est tr $E d$
43 proportionibus $C E R$ proportione $H$ PSVEd / ante maioris add HPSVEd proportionum / $\mathrm{de}^{2}$ CPRSVEd; om EH
43-44 proportionibus $C E R$ proportionum $H$ proportionum proportione $P$ proportione proportionum SVEd
44 inequalitatis om $H /$ inter se om $P$ /
post se add $H$ scilicet unius ad alteram 45 ante Proportionem add $P$ et secundo notanda quod ad / aliquos: ambos $E d$ 45-46 ante terminos add CEPRVEd eius / post terminos $m g$ bab $C \mathrm{a} / \mathrm{b} / \mathrm{c}$
46 seu: vel $C /$ media: medias $S / V e r b i$ gratia: ut Ed/sit CEHRVEd; om $S$ si $P$
47 maior om $H / \mathrm{et}^{\mathrm{I}}$ CEHPRV; om SEd | una om $H$ / alia CEHPRV; om $S$ quantitas $E d / \mathrm{B}$ ad $\mathrm{C}: \mathrm{C}$ ad $\mathrm{BC} / \mathrm{ad}$ : et $P$
48 post est hab $P$ inter B et $\mathrm{C} /$ aut $C E H$ $P V E d$ seu $R S$
49 quicquid $C E H P R S$ quod quidem VEd / post maius add $S$ quam / ante minus add $R$ quicquid est
49-50 maius... $\mathrm{B}^{1}$ : minus B et maius $\mathrm{C} P$
so quam $B$. Sit enim $D$ medium inter $B$ et $C$. Cum igitur proportio primi ad tertium componitur ex proportione primi ad secundum et secundi ad tertium ut satis patet ex decima diffinitione et expresse ex commento undecime diffinitionis quinti Euclidis et in principio septimi sequitur quod $A$ proportio, que est inter $B$ et $C$, sit composita ex
55 proportione $B$ ad $D$ et ex proportione $D$ ad $C$. Et quia unumquodque resolvitur et dividitur in ea ex quibus componitur patet quod $A$ proportio dividitur in duas proportiones per $D$ medium assignatum. Quod si inter $B$ et $C$ duo media assignarentur tunc $A$ proportio esset divisa in tres partes vel in tres proportiones et si tria media assigna-
6o rentur tunc esset divisa in quatuor et si quatuor in quinque et sic in infinitum semper in tot partes dividitur quot media assignantur addita unitate quod ex quinto Euclidis posset faciliter ostendi. Et hunc modum dividendi proportiones assignat Jordanus in commento Arismetice sue.
${ }_{6}$
Proportionem maioris inequalitatis augere est ultra terminos eius alium vel alios terminos assignare. Ut si $A$ proportio que est inter $B$ et $C$ debeat augmentari sumendus est unus terminus maior $B$ et comparandus ad $C$, vel terminus minor $C$ ad quem $B$ comparetur, vel unus maior $B$ et alter minor $C$. Verbi gratia, sit $D$ maius $B$ tunc
so enim: igitur $S /$ inter B et $\mathrm{C} C P R S$ $V E d$; om $H$ in B et $\mathrm{C} E$ / proportio: proportionem $P$
sI tertium: ultimum $R \mid$ componitur CHRV compositam $E$ componi $P$ componatur SEd / post componitur bab $P$ oportet / ante primi bab $E \mathrm{~d}(?)$ / secundi: secundo $E$
$5_{2}$ tertium obs $P /$ satis om $P$ / ante et add $H$ quinti/ $\mathrm{ex}^{2}$ : in $S$
\{2- $\{3$ commento om $E$
53 ante undecime scr et del $P$ quinte(?) 1 undecime EPSV quinte $H R$ nec $C$ secundi $E d /$ diffinitionis rep $V$ | Euclidis obs $P /$ in: ex $C /$ septimi CHPRSV; om Ed primi $E /$ post septimi add Ed igitur
$53-54$ sequitur quod om $P$ / post sequitur add $C$ igitur
54 ex om $P$.
ss proportioner : proportionibus $P /$ post et scr et del $E \mathrm{~B}$ ad $\mathrm{D} /$ ex proportione CRSVEd; om EHP / D ad ComE/ $\mathrm{D}^{2}$ : B $S / \mathrm{Et}$ om $E$

56 resolvitur CEHRVEd;obs $P$ dividitur $S$ / et dividitur CEPRVEd; om Het resolvitur $S$ / componitur: composi$\operatorname{tam}($ ? $) E$
57 in duas proportiones obs $P /$ per: patet $P /$ assignatum $C E H P R V$ ad signatum $S$ signatum $E d$
58 Quod: quare(?) C / post duo bab Ed sint I ante assignarentur scr et del $V$ assignantur / assignarentur EHRV; om $E d$ assignentur $C$ signarentur $P S$ / A CPSV; om HREd / esset CHRV $E d$ est $P$ erit $S$
58-59 proportio esset divisa: esset divisa A proportio $E$
59 in $^{1}$ : inter $C /$ in tres partes vel om $S /$ partes: terminos $R$ /partes...proportiones: proportiones vel tres partes $E d$ $\mid$ in $^{2} C E P R V$; om $H /$ tres $^{2}$ om $E$
59-60 assignarentur $E H P R$ assignentur $C$ signarentur $V E d$ assignatur $S$
60 tunc om $S$ /esset: est $P /$ et si quatuor in quinque om $S$ / si CHPREd similiter $E$ sic $V$

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is greater than $C$ and less than $B$. Let $D$ be a mean between $B$ and $C$. Therefore, since a ratio of the first term to the third term is composed of a ratio of the first to the second term, and of the second to the third term -as is sufficiently clear from the tenth definition and explicitly from the comment on the eleventh definition of the fifth book of Euclid and in the beginning of the seventh book of Euclid-it follows that ratio $A$, which equals $B$ to $C$, is composed of ratios $B$ to $D$ and $D$ to $C$.* And since any one thing can be resolved and divided into the things of which it is composed, it is obvious that ratio $A$ can be divided into two ratios by $D$ the assigned mean. But if two means should be assigned between $B$ and $C$, then ratio $A$ would be divided into three parts or three ratios; and if three means were assigned, then it would be divided into four [parts or ratios]; and if four means, then into five [parts or ratios], and so on infinitely. The ratio would always be divided into as many parts as there are means plus one, which can easily be shown by means of the fifth book of Euclid. Jordanus, in a comment in his Arithmetic, designates this method for dividing ratios.

To increase a ratio of greater inequality is to assign beyond its terms another term or terms. In order that ratio $A$, which equals $B$ to $C$, be increased, a term greater than $B$ must be taken and related to $C$; or a term smaller than $C$ must be taken to which $B$ would be related; or one term greater than $B$ and another smaller than $C$. For example, should $D$ be

* $A=B / C=B / D \cdot D / C$.

60-6I in infinitum $H$ ultra in infinitum $S$ ultra CEPRVEd
61 dividitur om $P /$ media CEHPSV in ea $R$ medio $E d /$ assignantur: assignant $C$
62 quod: et $S /$ posset faciliter ostendi EPRSEd faciliter potest ostendi $H$ faciliter ostendi potest $C$ potest faciliter ostendi $V$
63 dividendi proportiones CEHRVEd dividendi $P$ divisionis $S /$ in CHVEd; om EPRS / commento CHPRSV; om $E{ }_{9} E d$
64 post sue scr et del $H$ aliqua verba et mg hab $P$ a proportione(?)
6s ante Proportionem mg hab $H$ tertium notabile et mg hab $P$ notanda / Propor-
tionem: proportiones Ed / post Proportionem add $S$ est / est om $S$ / post ultra $m g h a b E d$ quid sit augere
65-66 terminos...terminos $H R V$ terminum vel terminos vel terminos alium $C$ terminum alium vel alios terminos $E$ terminum vel terminos eius alium vel alios $P$ terminos eius terminalium vel alios $S$ terminos eius alium terminum vel alios terminos $E d$
67 debeat: debet $E d$ /augmentari HPS augeri CERVEd / unus om $R / \mathrm{B}$ om H
68 terminus: unus $P / \mathrm{B}$ comparetur $\operatorname{tr} C$
69 unus: terminus $S / \mathrm{Br}^{\mathrm{r}}$ om $V /$ alter om Ed/Verbi gratia: una $R$ / sit: sicud $E / \mathrm{B}^{2}: \mathrm{D} E$

7o proportio $D$ ad $C$ componitur ex proportione $D$ ad $B$ et $B$ ad $C$, igitur ipsa est maior quam proportio $B$ ad $C$ que est pars eius.

Proportionem, vero, ab alia subtrahere est inter terminos maioris medium assignare quod se habeat ad minorem terminum, vel ad quod maior terminus se habeat secundum proportionem subtrahendam.
75
Proportionem, vero, alteri addere est unam earum in terminis ponere et deinde tertium terminum invenire qui se habeat ad maiorem aliam secundum proportionem quam tu vis addere, vel ad quem minor se habeat in proportione addenda. Ut si proportioni $A$, que est $B$ ad $C$, vis addere proportionem que sit $E$ capias unum terminum $D$ qui
so se habeat ad $B$ in proportione $E$ vel alium terminum $F$ ad quem $C$ sit in proportione $E$ sicut hic si addatur sexquialtera duple.

Inde patet quid sit proportionem duplare, triplare et cetera et hec omnia in quinto Euclidis intelligentibus patefiunt.
Si autem volueris per artem proportionem maioris inequalitatis $8_{5}$ alteri addere tunc oportet denominationem unius per denominationem alterius multiplicare. Et si volueris unam ab altera subtrahere hoc
70 D ad C om $E /$ componitur CERSV composita HPEd | post componitur add $P$ est $/$ ante $\mathrm{D}^{2}$ scr et del $H \mathrm{~B}$ ad
71 igitur om $P /$ estrir $^{1}$ : erit $R$ / ante quam $m g$ hab $P$ notanda
72 vero: notatur C/alia CEHRSEdalio $V$ aliqua $P$ / post alia bab $E$ sit de(?)/ subtrahere: substrahere $C$ / inter om $E$ I post terminos mg hab Ed quid sit subtrahere / ante maioris $m g$ bab $C$ super proportionem(?) / maioris $C E$ HPRV maiores $S E d$
73 ante se bab $S$ sit / habeat: habet $E$ | minorem: minoris $V /$ terminum om $S$ / ante quod mg bab $P$ notanda
74 maior: minor $V /$ terminus: terminis $S /$ habeat: habet $H$
75 vero $E H$ autem CPRSVEd / alteri addere $\operatorname{tr} R$ / post terminis $m g$ hab Ed quid sit addere
76 tertium terminum: secundum $R /$ habeat om H/maiorem CEHRS maiores PVEd | post maiorem add CER maioris
77 aliam om $R$ / aliam...vis: illarum proportionum quavis $S$ / ante secundum add $H$ aliam / secundum proportionem $C E H P R$ et aliorum $V$ in pro-
portione $E d /$ addere: videre $P /$ quem: quam $E d$
77-78 minor se habeat: se habet minor $P$ 78 in proportione addenda: secundum proportionem addendam $H /$ si om $P$ | proportioni: proportio $E /$ ante B add $H$ inter / ad: et $H$
$79 \mathrm{C}: \mathrm{D} S /$ que sit E : $\mathrm{C} S /$ capias: capiat $H$ / terminum: tertium $H / \mathrm{D} C H P$ RVEd; om ES / qui: que $E$
80 habeat: habet $P / \mathrm{E}$ om $R$ / ante alium add $R$ ad
ante C mg hab $\mathrm{H}_{3}{ }^{\text {a e }} \mathrm{e}_{9}$
C: est $V$
80-8I C sit: se habeat C $E / \mathrm{C}$ sit...E: sit proportio C $R$
81 ante E scr et del $H \mathrm{~F}($ ? $) / \mathrm{E}$ om $\mathrm{P} \mid$ addatur: addant $C$ / sexquialtera: sexquialtere $R$ / post duple add $H_{3}$ a 6 e 9 d
et add $C R \mathrm{c} \quad \mathrm{b} \quad \mathrm{d}$
3 a 6 e9
et add $V \quad$ a a d
369
et add $S$ tertie octave(?)
82 ante Inde bab $S$ conclusio / Inde: et

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greater than $B$, then ratio $D$ to $C$, which is composed of ratios $D$ to $B$ and $B$ to $C$, is greater than ratio $B$ to $C$, which is part of it.*

To subtract one ratio from another is to assign a mean between the terms of the greater ratio such that when the mean is related to the smaller term, or the greater term is related to the mean, a ratio is formed that equals the ratio to be subtracted. $\dagger$
To add one ratio to another, [first] express one of themin terms and then find a third term that is related to the greater term as the ratio that you wish to add; or find a third term to which the lesser term is related as the ratio that must be added. That is, if to a ratio $A$, which equals $B$ to $C$, you wish to add the ratio $E$, take a term $D$, which is related to $B$ as ratio $E$; or take another term $F$, to which $C$ is related as ratio $E . \ddagger$ [An example would be this:] if a sesquialterate ratio should be added to a double ratio.

Hence, what is meant by doubling a ratio, or tripling a ratio, etc., is clear. And all these things are revealed to the understanding in the fifth book of Euclid.

If, however, you wish to add a ratio of greater inequality to another by means of algorism, it is necessary to multiply the denomination of one ratio by the denomination of the other. And if you wish to subtract one ratio

* If $A=B / C$ and $D>B$, then $D / C>\quad \ddagger$ To "add" ratio $E$ to $A=B / C$, take (r) $B \mid C$; and if $E<C$, then $B \mid E>B / C$; finally, if $D>B$ and $E<C$, then $D \mid E>$ $B / C$. In each case $B / C$ has been increased. $\dagger$ See p. 313. $D>B$ such that $D / B=E$ which produces $D / B \cdot B / C=D / C$; or (2) assign $F<$ $C$ such that $C / F=E$ yielding $B / C \cdot C / F$ $=B / F$.
tunc Ed | post sit bab P ad / duplare triplare: duplicare triplicare Ed/infra duplare $m g$ hab $C$
proportio proportio

| c | a | b | e |
| :--- | :--- | :--- | :--- |
| 3 |  | 6 |  |

dupla sesquialtera /
et ${ }^{2}$ om $H$
83 omnia: regula $R /$ in $C H P R V$ ex
ERSEd supra Euclidis mg hab C
32
621
intelligentibus $C E H R S V$ in talibus $P$
intelligenti $E d$ / patefiunt $E H P R S V$ patebunt $C$ patent $E d$
84 post Si mg hab $C$ notanda modum addi unum proportionem alteri(?) / Si autem volueris rep $V$
85 alteri om $V$ | addere EHPRS; om CVEd / oportet: oporteret $H /$ denominationem ${ }^{1}$ : denominationes $E d /$ post denominationem ${ }^{1}$ mg bab $C$ notanda modum substrahere proportionem alteri / denominationem ${ }^{2}$ : denominationes $E d$
86 ab: de $P /$ altera $E H P R$ alteram $V$ alia CSEd / hoc: hec C
facies denominationem unius per denominationem alterius dividendo. Denominationum inventio postea docebitur; quarum multiplicatio atque divisio habetur per algorismum.

Divisio, vero, et augmentatio, additio (duplatio, triplatio, et cetera), subtractio in proportionibus minoris inequalitatis fiunt econtrario pro cuius evidentia aliqua sunt notanda.

Primum est quod signatis duobus terminis, sicut verbi gratia $A$ maiore, $B$ vero minore, et sit $C$ differentia unius ad alterum vel ex95 cessus, dico quod proportio maioris inequalitatis que est proportio $A$ ad $B$ augetur per augmentum differentie que est $C$ et per diminutionem diminuitur. Proportio, vero, minoris inequalitatis, scilicet proportio $B$ ad $A$, per augmentum huius differentie minuitur et per diminutionem augetur. Quanto enim differentia est maior tanto proportio maioris inequalitatis est maior et proportio minoris inequalitatis est minor sicut ex octava quinti Euclidis diffinitione postea deducetur. Hoc tamen non est proportionaliter. Si enim $C$ differentia uniformiter augeatur proportio maioris inequalitatis difformiter augebitur et proportio minoris difformiter minueretur, et ita si $C$ mi105 nueretur et cetera.
$C$, vero, differentia potest dupliciter augeri. Uno modo per augmen-
$8_{7}$ facies CPRSV faciet $E H$ facias $E d /$ ante unius $m g$ bab $P$ notanda / post denominationem ${ }^{2} \mathrm{mg}$ bab Ed notabile / alterius om $C$
88 Denominationum CHPR denominationem $E S$ denominationis VEd/ inventio: inventa $S$ / quarum $C E H R$ $E d$ quorum $P$ quare $V /$ multiplicatio CHPRV multiplicanda $E$ multiplicata $S$ multiplex $E d$
89 ante atque bab $V$ ad invicem(?)/atque CEHPSV; om Ed et $R$ / divisio: divisa $S /$ habetur: apparet $E /$ habetur per algorismum: per algebram habetur $E d /$ algorismum: astrologiam(?) $E$
90 vero et: nota $C / \mathrm{et}^{\mathrm{t}}$ om $E /$ augmentatio om $P$ sed bab augmentatio post triplatio / ante additio add $R$ et $/$ additio om $P /$ duplatio triplatio: duplicatio triplicatio $E d /$ et cetera $C E H R$ VEd; om $P$ et etiam $S$
91 subtractio: sub termino $E d /$ inequalitatis: equalitatis $R /$ econtrario $C E H$ RSV econversa $P$ econtra $E d$

92 aliqua sunt $t r R$
93 est om $H$ et rep $S$ / signatis: assignatis $P$ / duobus: et $C$ / infra duobus scr et del $H$ tribus / sicut CEHPVEd; om $S$ aliud $R$ / verbi gratia HPS causa exempli $C$ exempli causa $E R$ exempli gratia VEd
94 post maiore $m g$ bab $C$ 74
similiter et differentiam 621
vero EHPRVEd; om CS / sit CEHR SEd sic $P V /$ alterum $H R V E d$ alium $C S$ alteram $E$
94-95 unius...inequalitatis om $P$
95-96 que...augetur: est augere $E$
96 augetur: augeret $E d /$ est: sit $E$ / et om $R$
96-97 post diminutionem bab $P$ diminionum
97 vero om $E$ et tr $R$ ante diminuitur
97-98 scilicet proportio: que est $S$
98 B ad A om $P /$ per om $E /$ augmentum: augmentationem $P /$ differentie rep $V$
from another, you do this by dividing the denomination of one ratio by the denomination of the other. The [method] of finding denominations will be taught afterward. Multiplication and division of denominations are done by algorism.

In ratios of lesser inequality division, augmentation, addition ([for example] doubling, tripling, etc.), and subtraction are done in an opposite way. As evidence of this some things must be noted.
In the first place, when two terms have been assigned, as, for example, $A$ a greater term and $B$ a lesser term, and $C$ is the difference or excess of one to the other, I say that a ratio of greater inequality-i.e., ratio $A$ to $B$-would be increased by augmenting the difference, i.e., $C$, and decreased by diminishing that difference. However, a ratio of lesser inequality, namely ratio $B$ to $A$, is diminished by increasing the difference and increased by diminishing the difference. In fact, as the difference becomes greater a ratio of greater inequality becomes greater and a ratio of lesser inequality becomes smaller, as will be shown afterwards from the eighth definition of the fifth book of Euclid. However, these [increases and decreases of the ratios] do not occur proportionately. For if $C$, the difference, should be increased uniformly, a ratio of greater inequality would increase non-uniformly and a ratio of lesser inequality would diminish non-uniformly. And the same thing would happen if $C$ were diminished [uniformly], etc.

Now $C$, the difference, can be increased in two ways. One way is to in-
/ minuitur CHPRS diminuitur $E V E d$ / et om $P$
99 diminutionem augetur CEHPREd scilicet(?) augmentum $V$ minutionem augetur $S$ / differentia: differentiam $V$ | post differentia mg bab C432/est om C
99-100 proportio: differentia $R$
100 ante maioris bab $V$ vero / proportio CPRSVEd; om EH
100-101 inequalitatis om $E$
ror est RSV; om CEHPEd/minor obs $P$ | sicut CEHPVEd illud $R$ sic $S$ | quinti CERSEd; om HPV / diffinitione: propositione $E d$
102 deducetur: deducitur $V /$ Hoc: hec $C$ / tamen: enim $E /$ proportionaliter: proponat $E d /$ Si enim obs $P /$ Si: sed $S$ ante C bab Ha

103 uniformiter augeatur $H P R V E d$ sufficienter(?) augetur $E$ proportionaliter augere vel uniformiter $C$ uniformiter augetur $S$ / maioris om $H$
103-5 proportio...et cetera om $E$
104 et proportio obs $P$ / post minoris add $C$ inequalitatis / minueretur $H$ minuetur CPRSVEd/ita om S
104-s et ita...cetera: Si vero C differentia uniformiter minuatur proportio maioris inequalitatis difformiter minuetur et proportio minoris inequalitatis difformiter augebitur $E d$ / minueretur $H R$ minuatur CPSV
ios et cetera CHPRS; om $V$
$106 \mathrm{C} C H R S V E d$; om $E P$ | differentia potest dupliciter obs $P$ / augeri: augmentari $P$
tum $A$ maioris termini. Et tunc ad infinitam augmentationem proportionis maioris inequalitatis requiritur infinitum $C$ augeri et similiter $A$. Alio modo augetur $C$ per diminutionem $B$ minoris termini. Et
nо tunc ad infinitum augeri proportionem non oportet in infinitum $C$ augeri sed oportet in infinitum $B$ minui. Et quia infinita augmentatio non est possibilis naturaliter sicut infinita diminutio patet quod in infinitum possibile est augeri proportionem per diminutionem termini minoris et non est possibile per augmentationem maioris. Cum igitur
115 velocitas motus sequitur proportionem potentie ad resistentiam sequitur quod infinitam velocitatem possibile est esse per diminutionem resistentie et non per augmentationem virtutis. Sed qualiter et in quibus motibus alias declarabo.

Similiter $C$ differentia dupliciter potest diminui. Uno modo per
120 diminutionem $A$ maioris termini, aliomodo per augmentationem $B$ minoris termini. Nec ad diminutionem $C$ in infinitum oportet $A$ minui in infinitum nec in infinitum $B$ augeri. $C$ tamen potest diminui in infinitum ad cuius diminutionem in infinitum sequitur augmentatio proportionis minoris inequalitatis in infinitum et diminutio
${ }_{125}$ proportionis maioris. Et sic quantumcumque $C$ diminueretur dum tamen aliquid remaneret numquam equalitas haberetur. Quantumcumque proportio minoris inequalitatis augeretur numquam ad equa-

107 post maioris mg hab Ed C dupliciter augetur / termini om $P$ / infinitam: infinita $V$
107-8 proportionis: proportiones $C$
108 inequalitatis $S E d$ et cetera $C H R$ et etiam $E P V$ | ante requiritur bab CEd et / requiritur $H P R S V$ reperitur $C$ requiret $E$ sequitur $E d /$ ante infinitum bab RSEd in / infinitum C $\operatorname{tr} P$
108-9 et similiter A om $S$
109 A om $E$ / Alio modo: alteratione $E$ | augetur om $R$ / diminutionem: divisionem $E d$ / termini: A $R$
109-10 Et tunc CEHPREd; om $S$ et $V$ rio in om $E$
rio-1 infinitum ${ }^{2} \ldots$ oportet: infinitam augmentationem C requiritur $E d$
III oportet EHPRSV; om $C$ in infinitum B minui CEPRSV B in infinitum minui $H E d /$ Et quia $C H P R V$ Ed; om $S$ quare quia $E$ / ante infinita bab $R$ in / infinita: in infinitum $H$

112 possibilis: possibile $E$ / possibilis naturaliter $\operatorname{tr} C /$ sicut $C H P R V E d$ sicud $E$ sic $S /$ ante infinita bab $R$ in / diminutio: minutio $S$ / quod: quam $E$
II3 possibile: impossibile $S$ /augeri: augere $C$ / per diminutionem om $P$ / termini om $R$
1I3-14 termini minoris $\operatorname{tr} C$
114 et non est possibile $C H P R$ et non possibile est augeri proportionem $E$ et non sic est possibile $V$ sicut(?) $S$ et non sic possibile est $E d /$ per augmentationem maioris: est proportionem maioris augeri(?) $V /$ ante maioris add $E$ termini
ins sequitur CEPRSV sequatur HEd / ante proportionem $m g$ hab $C$ notabile / post ad add Ed suam
ins-16 sequitur om $S$
II6 post quod add VEdin / infinitam CEH $P R V$ infinita $S$ infinitum $E d /$ velocitatem rep $E$ velocitate $S$ / post velo-
crease $A$, the greater term. Then, in order to increase a ratio of greater inequality to infinity, it is necessary that both $C$ and $A$ be infinitely increased. Another way of increasing $C$ is to diminish $B$, the smaller term. Hence, in order that the ratio be increased to infinity, it might not be necessary that $C$ be increased infinitely; but it would [then] be necessary that $B$ should be diminished infinitely. Now since an infinite increase is not naturally possible as is an infinite diminution, it is clear that an infinite increase of a ratio is possible by decreasing the smaller term and not possible by increasing the greater term. Therefore, since velocity of motion depends on the ratio of power to resistance, it follows that the existence of an infinite velocity is possible by diminishing the resistance and not through an increase of the force. But I shall explain, at another time, how and in which motions [this can occur].

Similarly, $C$, the difference, can be diminished in two ways. In one way by the decrease of $A$, the greater term; in another by increase of $B$, the smaller term. But it is not necessary that $A$ be decreased, or $B$ increased, into infinity in order to decrease $C$ infinitely. Nonetheless, $C$ can be infinitely decreased so that there follows an increase of a ratio of lesser inequality and a decrease of a ratio of greater [inequality] to infinity. And yet, however much $C$ should be diminished, equality would never be reached while something should remain [of $C$ ]. However much a ratio of lesser inequality should be increased, it would never reach equality; and the same
citatem add $V$ sequitur / esse om V | post esse $m g$ hab $P$ primo sequitur
117 augmentationem: augmentum $E d$ ante virtutis add $R$ potentie
118 declarabo: pertractabo $S$
119 differentia: potentia $R /$ dupliciter potest diminui $H V$ dupliciter potest minui $R S$ potest diminui $E$ dupliciter diminui potest $C$ potest dici diminui $P$ diminuitur dupliciter $E d$
120 A CPRSVEd; om EH / maioris: maiori $S /$ augmentationem $E H P R V$ $E d$ augmentum $C S$
121 C om $E$ / infinitum: infinitam $P$
122 ante B add $S$ oportet/B: $\mathrm{A} E /$ diminui: dividi $H$
122-23 nec...infinitum ${ }^{1}$ om $R$
123 in infinitum ${ }^{2}$ EHPRVEd; om CS |
sequitur om $E$
123-24 post augmentatio rep $P$ ad cuius diminutionem infinitum sequitur augmentatio
I24 inequalitatis: equalitatis $S$ / in infinitumCHRS; om EPVEd / ante et hab $E$ in infinitum sequitur augmentatio minoris inequalitatis et hab $V$ in infinitum / et diminutio rep $P$
125 proportionis maioris $H R$; $\operatorname{tr} E P V E d$ minoris proportionis $C$ maioris $S /$ sic CHPS sicut ERVEd/quantumcumque: qualibet $C / \mathrm{C}: \mathrm{D} E d /$ diminueretur CHPRSEd diminuere $E$ diminuetur $V /$ dum: autem $E$
126 remaneret: remanet $C$ / haberetur: habetur $S$ / post haberetur add CEPSV sic et add $R E d$ sicut
litatem attingeret; et ita de diminutione proportionis maioris inequalitatis. Unde patet quod proportio equalitatis excedit in infinitum proportionem minoris inequalitatis et exceditur in infinitum a qualibet proportione inequalitatis maioris quod posset ex dictis faciliter demonstrari.

Si autem $C$ differentia augeatur per augmentationem $A$ et diminutionem $B$ copulate vel etiam si utrumque augeretur $A$, tamen, elocius quam $B$ vel si utrumque minueretur $B$ velocius quam $A$, et ita de diminutione $C$ quod ad propositum non est cura.

Secundo dico quod maiori proportioni maioris inequalitatis correspondet minor proportio minoris inequalitatis et minori maior. Verbi gratia, sicut quadrupla maior est quam dupla ita subquadrupla minor est quam subdupla. Quod probatur ex duobus. Primo quia cuius proportionis denominatio est maior ipsa est maior, cuius vero minor ipsa est minor ut vult Jordanus in secundo Arismetice sue et recitatur in commento sexte decime diffinitionis quinti. Semper loquor de commentis Campani. Denominatio, vero, subquadruple que est
${ }_{145}$ quarti est minor quam denominatio subduple que est secundi. Et quod iste sunt earum denominationes apparebit postea quando docebitur proportionum denominationes invenire. Secundo probatur illud ex secunda parte octave quinti Euclidis per quam habetur quod si

128 attingeret $H P R E d$ attingeretur $C$ perveniet $E V$ attingetur $S /$ de om $V /$ proportionis CEHSVEd; om $P R$
128-29 post inequalitatis bab $P$ proportionum
129 patet: oportet $R /$ in infinitum om $C$
130 minoris inequalitatis $\operatorname{tr} E$ / inequalitatis: equalitatis $R /$ qualibet om $S /$ post qualibet scr et del $E$ parte et supra scr proportione
131 proportione om $C[M S S P$ and $E d$ are not collated beyond this point / inequalitatis maioris HV; $\operatorname{tr} C E R S$ / quod: et $R$
131-32 demonstrari: declarari $V$
133 differentia: dicitur $V$ / augeatur $E H$ $R S$ augere $C$ augetur $V /$ augmentationem: augmentum $S$
133-34 diminutionem: diminutum $S$
134 copulate $H V$; om $C$ copulande(?) $E$ duplando $R$ copulatem $S /$ etiam om $H$ / augeretur CHRV augetur $E$ au-
geatur $S$
135 minueretur CHR diminueretur ESV | ante velocius ${ }^{2}$ add $S$ tamen
I36 de om $R$ / diminutione: minutione $V /$ ante C add CEV de / C om $S$ / quod $E H$; om $C$ quo $R S V$ / propositum... cura: non est cura ad propositum $C 1$ propositum $E H R S$ proportionum(?) $V$ / cura $E H R V$ omnino $S$
137 ante Secundo mg bab $C$ notabile / maiori proportioni $E H R S$; $\operatorname{tr} C V$
139 ante Verbi bab $E$ sicut / Verbi gratia sicut: sicut verbi gratia $H /$ ante ita $m g$ hab $V \mathrm{vb}$ (?)
140 probatur CRSV probo(?) $E$ patet $H /$ Primo: prime $E$ | post Primo hab $E$ ex / quia CHSV; om ER
141 post cuius bab $R$ enim / ipsa est maior EHRS; om CV $/$ ante cuius ${ }^{2}$ add ES et / vero om $S$
142 est om $C$ / vult: innuit $E$ / secundo: secunda $S$ / Arismetice: arismetrice $S$
may be said concerning the diminution of a ratio of greater inequality. From this it is clear that a ratio of equality exceeds a ratio of lesser inequality into infinity, and is exceeded into infinity by any ratio of greater inequality, which could easily be shown from things which have already been stated.
Furthermore, with regard to what has just been proposed, it is not at all relevant whether $C$, the difference, is increased by the augmentation of $A$ and diminution of $B$ simultaneously; or if each were increased, but $A$ more quickly than $B$; or if each should be diminished, $B$, however, more quickly than $A$; and the same applies to the diminution of $C$.
In the second place, I say that to a greater ratio of greater inequality there corresponds a lesser ratio of lesser inequality, and to a lesser ratio of greater inequality a greater ratio of lesser inequality. For example, just as a quadruple ratio is greater than a double ratio, so a subquadruple ratio is smaller than a subdouble ratio. This is shown from two things. First, as the denomination of such a ratio is greater the ratio itself is greater, and the smaller it is the smaller is the ratio, as Jordanus says in the second [book] of his Arithmetic, and which is also stated in the comment on the sixteenth definition of the fifth [book of Euclid]. I refer always to the comments of Campanus. Now the denomination of a subquadruple ratio, which is one-fourth, is less than the denomination of a subdouble ratio, which is one-half. Afterwards, when I shall show [how] to find denominations of ratios, it will be evident that these are their denominations. Second, this very thing is proved from the second part of the eighth [prop-

## / sue CERV; om HS

143 recitatur $C R S V$ recitatione(?) $E$ venitur(?) $H$ | post commento babE igitur / quinti: sexte $E /$ loquor $E H S$ loquitur $V$
143-44 Semper...commentis: per Jordanum de commento $C R$
144 commentis $S V$ commento $H$ quedam $E /$ vero om $S$ / subquadruple $C E H$ subquadrupla $S V$ subduple $R$ | ante que scr et del $E$ que / est om $V$
14) quarti $C E V$ quatuor $H_{4} R$ quadrupli $S /$ minor: maior $E /$ denominatio om $E$ et rep $V$ / subduple $C H R V$ qua-
drupla $E$ subdupli $S$ / secundi $V$ duo $C H R$ dupli $E S$
146 quod iste: hae $E /$ sunt $C E H R$ sint $S V$ / earum: eorum $S$ / earum denominationes $\operatorname{tr} E /$ denominationes: denominationum $H$ /apparebit: patebit C
146-47 docebitur: debitur $V$
147 probatur: secundum $S /$ illud $E R S V$; om $C$ istud $H$
148 post ex hab $E$ octave et ante octave scr et del $E$ A / parte: partis $C$ / octave: sexte $H$ / quinti: quinte $C$ / Euclidis H; om CERSV / si CHRS; om EV
aliqua quantitas ad duas inequales proportionetur ad minorem habebit paretur ad duo et ad quatuor, maior erit proportio unius ad duo que est subdupla quam proportio unius ad quatuor que est subquadrupla. Igitur sicut quadrupla est maior quam dupla ita subquadrupla minor est quam subdupla quod est propositum.

Tertio dico quod sequitur ex dictis quod si fuerint tres termini continue proportionales et maior vocetur primus, medius vero secundus, et minor tertius, tunc proportio primi ad tertium componitur ex proportione primi ad secundum et secundi ad tertium et est proportio primi ad secundum duplicata. Et sic est intelligenda decima diffinitio 6o quinti et dictum Campani in commento undecime dicentis proportionem extremorum componi ex intermediis proportionibus, scilicet quod proportio primi ad ultimum componitur ex proportionibus intermediorum. Et semper per primum debemus intellegere maius et per ultimum minus.
${ }^{165}$
Sed si fuerint tres termini etiam ut prius et primus sit maior etiam proportio secundi ad primum est proportio tertii ad primum duplicata. Unde sicut proportio primi ad secundum est pars et minor proportione primi ad tertium ita proportio secundi ad primum est maior proportione tertii ad primum nec componit proportionem tertii ad o primum nisi diceretur quod minus componitur ex maiori et quod minus est maius duplicatum quod potius verborum abusio videretur.
149 duas: suas $S$ / proportionetur: pro- $156-57$ ante secundus add $H$ vero portionata(?) $V /$ habebit $E H S V$ is7 et CHSV; om $E R /$ minor tertius $\operatorname{tr} C$ habet $C R$
rso et om $E$ / post maiorem ${ }^{2}$ bab $C$ habet / minorem ut si: ut $V$
150-รI ante comparetur bab $V$ comparatum / comparetur: comparatur $V$
Is i ad duor rep $E$ / maior erit proportio: exit proportio maior $E$ / erit: et $V$ / unius: illius $S$
$1 \rho 2$ est $^{1}$ : erit $R /$ est $^{2}$ : erit $R$
1s2- 54 quam...subdupla om $V$
153 Igitur rep $H_{\mid}$sicut: sicud $E$ | ante dupla bab $H$ et
153-54 minor est CV; $\operatorname{tr} S$ minor $E H R$
iss post Tertio mg bab C tertium notabile / dico quod om $H$ | fuerint: sint $H$
156 proportionales: proportiones $C$ / post maior $m g$ hab $C$ numerus / ante medius
add $S$ et add $S$ et
/ ex om $R$
is 8 primi: prima $S / \mathrm{etr}^{\text {om }} \mathrm{V}$
158 primi: prima $S$ eti om $V$ duplicata EHRV dupla CS / est intelligenda $E H R V$; $\operatorname{tr} C S$ / diffinitio om $H$
160 post quinti add C Euclidis / post dictum scr et del V commenti(?) / ante Campani bab $V$ commentum(?) / in commento om $E$ / commento: concepto $C$ / undecime $C E R$ quinte $H S$ secundo $V /$ dicentis: d $H$
161 intermediis proportionibus $\operatorname{tr} S$
162 quod CRSV; om EH / componitur CEHV componatur RS / proportionibus $E H S V$ proportione $C R$
163 intermediorum: intermediatibus(?) $E$ / post Et add V sic / intellegere: intelligere $H$

## Chapter One

191
osition] of the fifth [book] of Euclid, by which we learn that if some quantity were related to two unequal quantities it will form a greater ratio to the lesser quantity and a lesser ratio to the greater quantity. Thus, if i were compared to 2 and 4 , the ratio of $I$ to 2 , which is a subdouble ratio, will be greater than the ratio of $I$ to 4 , which is a subquadruple ratio. Therefore, just as a quadruple is greater than a double ratio, so also is a subquadruple smaller than a subdouble ratio, and this is what has been proposed.*
In the third place, I say it follows from what has already been said that if three terms should be continuously proportional and the greatest term is called the first, the mean term the second, and the smallest term the third, then the ratio of the first to the third term is composed of the ratio of the first to the second and of the second to the third and is, indeed, a ratio of the first to the second squared. $\dagger$ The tenth definition of the fifth [book of Euclid] must be understood this way, as well as the statement of Campanus in his comment on the eleventh [definition of the fifth book] where he says that a ratio of extreme terms is composed of intermediate ratios, namely that a ratio of the first to the last term is composed of ratios of the intermediate terms. By the first term we should always understand the greatest term, and by the last, the smallest term.
But if three terms were arranged as before, and the first is the greatest term, then a ratio of the second to the first term is equal to a ratio of the third to the first squared. $\ddagger$ Hence, just as the ratio of the first to the second is a part of, and smaller than, the ratio of the first to the third term, so is the ratio of the second to the first greater than the ratio of the third to the

* In general, Oresme wishes to show that
if $A>B>C$, then $A / C>A / B$ and $C \mid A$
$<B \mid A$.
$\dagger$ If $A, B$, and $C$ are continuously propor-

165 Sed: similiter $H /$ termini: similiter $S /$ etiam ut $C E V$ etiam $H$ et sicut $R$ sicut $S /$ et: cum $S$
166 ante secundi $b a b E$ primi ad secundum
167 sicut: sicud $E /$ est: et $V /$ post et bab $C$ est
${ }_{168}$ post ita $b a b E$ quod / secundi: tertii $E /$ ante est scr et del $H \mathrm{~cd}(?)$
169 ante nec bab $E$ ut / nec: ut $R$
170 primum: quartum $V /$ ante nisi add $S$
tional terms with $A>B>C$, then $A / C$ $=A|B \cdot B| C$ and $A / C=(A \mid B)^{2}$.
$\ddagger$ Assuming again that $A>B>C$, Oresme insists, incorrectly, that $B / A=(C / A)^{2}$.
ex illa / diceretur $H$ diceremus $E R S V$ dicerem $C /$ minus: minor $E /$ componitur CER componeretur EHS maiori: maiore $E /$ quod $^{2}$ om $R$ / post quod ${ }^{2} m g$ hab $C$ dum tamen
171 duplicatum $C H R V$ dictum quam $E$ duplicato $S$ / post potius bab $E$ quam / verborum abusio $H R V ; \operatorname{tr} S$ verbum abusio $E$ abussit verborum $C$ / videretur $R S$ videbitur $C V$ diceretur $E H$

Exemplum: proportio 4 ad I est dupla proportioni 4 ad 2 , sed proportio 2 ad 4 est proportio I ad 4 duplicata.

Quarto dico ad propositum quod proportionem minoris inequali-
175 tatis augere est medium inter extrema statuere que continue augeretur si medium extremo versus quam erat propinquius signaretur. Ipsam vero diminuere est extremum vel extrema remotius assignare.

Exemplum de diminutione sit proportio subdupla 4 ad 8. Dico quod eam diminuere est extremum vel extrema remotius invenire ut
${ }_{180}$ hic 2, 4, 8. Unde proportio 2 ad 8 est minor quam proportio 4 ad 8 quia est medietas eius. Et si adhuc signes longius ut hic 1, 2, 4, 8, tunc proportio I ad 8 que est suboctupla est tertia pars subduple, scilicet 4 ad 8.
Exemplum de augmentatione vel additione patet per idem. Unde
${ }_{185}$ proportio 2 ad 8 augetur signando 4 in medio et quia est medium proportionale ideo ipsa est duplicata. Et si essent duo media proportionalia signando secundum ipsa est triplicata, si tria signando tertium quadruplicata, et cetera.

Ex istis potest videri quomodo dividitur, quomodo una ab alia 190 subtrahitur, quomodo duplatur, et cetera.

Si autem volueris per artem proportionem minoris inequalitatis alteri addere vel subtrahere oportet econtrario modo agere quo fit in proportionibus maioris inequalitatis. In additione denominatio

172 proportio CHRS primi $E /$ I $C H R_{2}$ $E_{3} S$ / dupla $E H R S$ duplicata $C /$ proportioni $H R S$ proportionis $C$ proportio $E$ / sed CERS; om H
172-73 Exemplum...duplicata om $V$
1732 CHRS duplicata(?) $E$
174 Quarto dico $\operatorname{tr} C$ / dico om $H /$ ad propositum om $S$ / post propositum $m g$ hab $C$ quartum notabile
175 augere: augeri $E /$ statuere $C H R V$ ponere $E$ statuetur $S$ / que: qua $V /$ augeretur $C H R$ augetur $E S V$
176 versus: usque $C /$ quam $H$ quod $C E R(?) V$ | erat $C H$; om $V$ iter $E S$ (?) inter $R$ / propinquius: propinquo $V$ / signaretur: significaretur $S$ / Ipsam: patum $S$
177 vero:idem $E$ /extremum vel extrema : extrema vel extremum $S$ / remotius: remotis(?) $V /$ remotius assignare $t r C$
178 post de scr et del $E$ augmentatione sit
proportio om $E$ / subdupla: subquadrupla $S$
179 remotius $C H R S$ remotus $E$ remotis(?) V
180 hic CES; om $H R V / 8^{1}: 4 \mathrm{~V} /$ minor ...8: proportio minore quam 8 ad $4 E$ 181 quia ...hic om $H$ / signes $E S V$ signares $C$ assignes $R$
182 que om $S$ / suboctupla: subquadrupla $S /$ tertia CHRS tripla $E V$ / pars: penes $S$
184 vel additione om C/additione $S R V$ diminutione $E H /$ per $C E S$; om $R V$ ad(?) $H$
185 augetur om $R$ / est medium $\operatorname{tr} E$
186 essent HRS; om CEV
187 signando secundum $C S$; tr $H$ assignando secundum $E R$ signando tertium $V$ / post secundum bab $E$ tertium / est $E H V$; om $C$ erit $R$ esset $S /$ si: sed $4 S$ / signando tertium om $S$ /post
first term*-and yet it does not compose a ratio of the third to the first term unless it were said that a smaller ratio is composed of a greater ratio and that the smaller ratio is [equal to] the square of the greater ratio. This seems rather an abuse of words. An example: A ratio of 4 to $I$ is the square of a ratio of 4 to 2 , but a ratio of 2 to 4 is [equal to] the square of a ratio of $I$ to 4.

In the fourth place, with reference to what has been proposed, I say that to increase a ratio of lesser inequality is to place a mean between the extremes; and the ratio could be increased continuously if the mean should be assigned closer to an extreme. To diminish [a ratio of lesser inequality] involves assigning an extreme term or more remote extreme terms.

An example of diminution: Let there be a subdouble ratio of 4 to 8 . I say that to diminish it is to find an extreme term, or more remote extremes, as 2, 4, 8. Thus ratio 2 to 8 is smaller than ratio 4 to 8 because it is half of it. And if you should now assign a further extreme $1,2,4,8$, then ratio I to 8 , which is a suboctuple ratio, is the third part of a subdouble, namely 4 to $8 . \dagger$

An example of augmentation or addition is made clear in the same way. Thus a ratio of 2 to 8 could be increased by assigning 4 as a mean, and since it is a mean proportional it [i.e., 2 to 8 ] has been squared. And if there were two mean proportionals assigned in this way, $[2$ to 8$]$ would be cubed; if three mean [proportionals] were assigned, then by assigning the third [ratio, 2 to 8 ] would be raised to its fourth power, etc.

From all this one can see how [a ratio of lesser inequality] is divided, how one [ratio of lesser inequality] is subtracted from another, how it is doubled, etc.

If, however, you wish to add one ratio of lesser inequality to another by algorism or subtract one from another, it is necessary to operate in a way contrary to what was done with ratios of greater inequality. In addi-
*Here, if $A|B<A| C$, then $B|A>C| A$. Oresme holds that $2 / 8=(4 / 8)^{1 / 2}$ and $1 / 8=$ + Consistently repeating the previous error, $\quad(4 / 8)^{1 / 3}$.
tertium add $C$ ipsa esset
187-88 si tria...quadruplicata om $R$
188 quadruplicata $C H S V$ quartum $E /$ et cetera CSV ;om $H$ et $E$ et sic ultra $R$
189 istis: hiis $S$ / potest videri $\operatorname{tr} C$ / quomodo dividitur HRS; om CEV
190 duplatur $E H R S$ duplicatur $C V /$ et cetera CRSV; om EH

191 inequalitatis om $H$
192 alteri addere $\operatorname{tr} R$ / post oportet bab ES etiam bab $H$ vel hab $V$ etiam contrario / modo: magis(?) V | quo CRSV quando $E$ sicut $H$
193 proportionibus: proportione $S /$ maioris CESV; om $H$ minoris $R /$ denominatio: denominante $S$
unius per denominationem alterius dividitur, in subtractione denominatio unius per denominationem alterius multiplicetur. Inventio denominationum postea docebitur quarum multiplicatio et divisio in algerismo docetur.

Quinto dico quod per predicta et per quintum Euclidis satis potest apparere quod quanto proportio maioris inequalitatis est maior tanto proportio minoris inequalitatis sibi correspondens est minor et econverso. Unde proportio proportionum minoris inequalitatis est sicut proportio proportionum maioris inequalitatis sibi correspondens vel quibus opponitur relative mutato tamen nomine relative superpositionis in relativum suppositionis. Verbi gratia si proportio quadrupla sit dupla proportioni duple tunc proportio subquadrupla est subdupla subduple. Si vero tripla est incommensurabilis duple et similiter subtripla erit incommensurabilis subduple. Sufficit igitur tantummodo investigare proportionem proportionum maioris inequalitatis per quam haberi potest proportio proportionum minoris inequalitatis.

Nec de proportionibus minoris inequalitatis quo ad hoc plura dicam.
Medium quantum ad propositum spectat dicitur dupliciter. Primo modo improprie et est medium improportionale. Secundo modo proprie et est medium proportionale quod se habet ad minus extremum in ea proportione in qua maius se habet ad ipsum et loquor semper de proportione geometrica.

Medium improportionale tripliciter dicitur. Quoddam est quod ad utrumque extremorum habet proportionem rationalem sicut 8 est medium inter 9 et 4 . Aliud est quod ad utrumque habet proportionem irrationalem sicut dyameter quadrati inter costam et triplum coste.

195 multiplicetur: multiplicatur $R$
196 denominationum: multiplicationum $R$
198 predicta $H R S V$ dicta $C E / \operatorname{per}^{2} H R S$; om $C E V$ / quintum $H R V$ quintam $C$ per predictum $E ; S /$ satis potest $\operatorname{tr} E$
199 quod CHRV; om ES / proportio: a proportione $V$ / est maior om $E$
200 sibi correspondens om $S$ / correspondens $E H R V$ conveniens $C$
200-201 econverso $C H R V$ converso(?) $E$ econtrario(?) $S$
201 proportio om $H$ / est om $V /$ sicut: sicud $E$
202 proportio rep $H$ / proportionum om $R$ / correspondens $H$ correspondentibus

CERV correspondentis $S /$ vel: ut $S$
203 opponitur: componitur $S$ / mutato: mutatio $C$ / nomine $E H R V$; om $S$ nominetur $C$ / relative CEHV relativo $R S$
203-4 superpositionis $R S V$ superial(?) $H$ suppositionis $C E$
204 relativum: relativis(?) C / suppositionis: superioris(?) $H /$ quadrupla: octupla $V$
$205 \underset{V}{\text { proportioni } C H R S ; ~ o m E \text { proportione }}$ V
206 ante Si mg bab H sufficit / Si: similiter $V /$ vero: vera(?) $E /$ tripla: tertia $S$ / est incommensurabilis $\operatorname{tr} R$ / incommensurabilis: incommensurabile $C$
tion the denomination of one [ratio of lesser inequality] is divided by the denomination of the other; in subtraction the denomination of one [ratio] is multiplied by the denomination of the other. The [method] of finding denominations will be taught afterwards. Multiplication and division [of denominations] are taught in algorism.

In the fifth place, by what has already been said and by the fifth [book] of Euclid, I say that it is clearly apparent that the greater a ratio of greater inequality is, so the ratio of lesser inequality corresponding [or reciprocal] to it is smaller, and conversely. From this it follows that a ratio of ratios of lesser inequality is just like the ratio of ratios of greater inequality that corresponds to it; or it is opposed relatively to [a ratio of ratios of greater inequality] with the name being changed from relative superposition to one of relative subposition. For example, if a quadruple ratio is double to a double ratio, then a subquadruple ratio is half of a subdouble ratio. Furthermore, if a triple ratio is incommensurable to a double ratio, then, similarly, a subtriple ratio will be incommensurable to a subdouble ratio. It is therefore sufficient to examine only a ratio of ratios of greater in equality in order to obtain information about the [corresponding] ratio of ratios of lesser inequality. As far as this is concerned, I shall say no more about ratios of lesser inequality.
[The term] mean may be taken in two ways with reference to what is considered here. In the first way it is taken improperly and is called a mean improportional. In the second way it is taken properly and is a mean proportional because it bears the same relation to the lesser term as the greater term bears to it; and I speak throughout about geometric proportionality.

A mean improportional can be considered in three ways. In one way it forms a rational ratio with each of the extremes, just as 8 is a mean between 9 and 4. In another way it is related to each extreme as an irrational ratio, just as the diagonal of a square between its side and [a length] triple its side.

206-7 et...incommensurabilis: vel $S$
209 haberi potest $\operatorname{tr} R$ / potest $C E H$ poterit $S V /$ proportio: de proportione $S$ / inequalitatis CES; om $H R V$
210 post proportionibus add $S$ proportionum
2 II ante ad add $C$ spectat / propositum: proportionem $S /$ spectat $E R S V$; om $C H$ / dicitur dupliciter $C E R S$; $\operatorname{tr} H$ duplicitur $V /$ Primo $H R S V$ uno $C E$
212 modo $^{1}$ CEV; om HRS / post modo ${ }^{1}$

## add $H$ quidem

213 ante et $m g$ bab $H$ quintum notabile / medium proportionale $\operatorname{tr} R$
214 ante se scr et del $V$ sed
216 tripliciter dicitur $\operatorname{tr} C /$ Quoddam: quod $V$
217 proportionem rationalem $\operatorname{tr} R /$ rationalem $C E H V$ rationaliter $S / 8: 6 \mathrm{~V}$
${ }^{219}$ irrationalem: irrationaliter $S$ / sicut: ut $R$ / quadrati om $S$ cut dyameter quadrati inter costam et duplum coste. Medium multis aliis modis dicitur qui non sunt ad propositum pertinentes.
Pars dicitur uno modo proprie et est pars que vocatur aliquota vel multiplicativa ad quam totum est multiplex et partes sunt plures tales; alio modo improprie et est pars aggregativa vel non aliquota que
${ }_{230}$ multotiens sumpta non precise constituit suum totum sed plus aut minus et hoc habetur in principio quinti Euclidis. Pars si per se sumatur in prima et propria significatione tenetur.
Ex predictis possunt elici quedam diffinitiones. Prima, quid sit proportionem dividere; secunda, quid proportionem augere, quid ${ }_{235}$ addere, duplare, triplare, et cetera; tertia, quid medium; quarta, quid pars.

Possunt etiam poni quedam petitiones. Prima, inter quascumque duas quantitates continuas inequales quotlibet media in infinitum assignare quod sit excessus seu differentiam unius ad alteram divi-

Secunda, inter quoscumque duos numeros inequales solum finitos numeros invenire.
Item ponantur iste suppositiones.
Prima est omnis proportio, tam rationalis quam irrationalis, in
245 quantitatibus continuis reperitur.
Secunda, nulla proportio irrationalis in numeris invenitur.
220 ad: inter C / infra unum scr et del $H$ utrumque / extremum om $S$ / proportionem rationalem: propter rationale $S$
221 ad om $H$ / post aliud add $C E$ proportionem / et sic CHRV sicud ES
224 ante rationalem scr et del $V$ irrationale / I : unitatem $H$
225 sicut $C H V$ sicud $E$ ut $R S /$ costam: coste $V /$ duplum: duplam $E$
226 multis... dicitur EHRS dicitur multis aliis modis $C$ multis aliis dicendum modis $V$ et post aliis scr et del $V$ dicitur
227 vocatur $H S V$ dicitur $C E R /$ vel : et $C$ 228 multiplicativa $C E H R$ multiplicata SV

229 vel $E H V$ et $C S$ seu $R$
230 multotiens sumpta $H$; om $V$ multotiens CERS / non om $H /$ non precise $\operatorname{tr} C /$ precise $H R V$ prescietur(?) $E$ precipue $S$ / constituit: consitueret $C$
231 minus: maius $S /$ hoc $E H R$ hec $C V /$ principio quinti: quinto $S$
232 ante in add $C$ et / prima et propria: propria et prima $C /$ ante propria $b a b$ $H S$ in
233 possunt: ponitur $C$ / ante diffinitiones scr et del $V$ dispositiones / diffinitiones: propositiones $H$
234 post dividere add $R$ vel diminuere / secunda: secundo $S /$ post secunda scr

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[Finally, in] another way it is related to one extreme term in a rational ratio and to the other in an irrational ratio, as [for example] the side is a mean between its half and the diagonal of the square.
A mean proportional is taken in two ways. In one way when it forms rational ratios as 2 between 4 and I ; another way when it forms irrational ratios, as, for example, the diagonal of a square between its side and [a length] double its side. A mean may be considered in many other ways that are not relevant to what has been proposed [for discussion].
A part is taken in one way properly and is called aliquot or multiplicative. [In this sense] the whole is multiple to it; and parts are several such. In another way [part] is taken improperly and is aggregative or non-aliquot. However many times such a part is taken, the result never exactly constitutes its whole, but is always greater or smaller than the whole; and this is found in the beginning of the fifth [book] of Euclid. If part is taken by itself, it must be understood in its first and proper signification.

Some definitions can now be derived from what has already been said. The first, what it is to divide a ratio; the second, what it is to increase, to add, to double, triple, etc.; the third, what a mean is; and the fourth, what a part is.
Some postulates may also be set forth. The first: Between any two continuous unequal quantities any number of means can be assigned into infinity by dividing the excess or difference of one to the other.

The second: Only a finite number of numbers can be found between any two unequal numbers.
These suppositions are also assumed.
The first is: Every ratio, both rational and irrational, can be found in continuous quantities.
The second: No irrational ratio is found in numbers.
et del $V$ quam / quid ${ }^{\mathrm{I}}$ : quam $V /$ post
quid $^{\mathrm{r}}$ bab $R$ sit / quid ${ }^{2}$ : quam $V$
quid ${ }^{1}$ baid $R$ suid $\ldots$ quarta om $C$
235 et cetera om $E$
237 poni quedam: haberi $E /$ quedam: quidam $S$
238 quotlibet $C E H R$ qualibet $S V$
239 seu $E R V$ vel $C$ sequitur $H$ sive $S /$ alteram: alterum $C$
241 Secunda SH; om CERV/quoscumque SHV quoslibet $C E R /$ duos om $C$
/ duos numeros $\operatorname{tr} S$ / ante finitos scr et del $E$ in
242 ante invenire add $R S$ medios / invenire: invenies $C$
243 iste EHRS; om S
243-45 Item...reperitur om $C$
244 est HRSV; om CE / tam om S / quam: vel $S$
246 post Secunda $m g$ bab $C$ petitio / numeris: quantitatibus discretis $E /$ invenitur $C H R V$ reperitur $E S$

Tertia, omnium commensurabilium proportio est rationalis et similiter, econverso, omnis proportio rationalis est commensurabilium. Et omnium incommensurabilium proportio est irrationalis et similiter, ${ }^{50}$ econverso, omnis proportio irrationalis est incommensurabilium.

Totum hoc patet ex quinta et sexta decimi, et ex diffinitionibus commensurabilium et incommensurabilium datis in decima, et ex principiis septimi, et ex commento tertie diffinitionis quinti.
Ex hiis sequuntur alie due.
Una est, et sit quarta, quod quelibet proportio est divisibilis in infinitum quia per primam suppositionem omnis proportio reperitur in quantitatibus continuis et per primam diffinitionem proportionem dividere est media inter extrema assignare et per primam petitionem inter quelibet duo continua inequalia in infinitum possibile est media so assignare.

Alia est, et sit quinta, quod quelibet proportio est sicut quantitas continua in hoc, quod in infinitum est divisibilis sicut quantitas continua et in 2 equalia, et in 3 , et in 4 , et cetera, et per inequalia quomodolibet, et in partes commensurabiles et similiter in partes sibi
${ }_{265}$ invicem incommensurabiles, et cetera et quolibet alio modo quoniam per primam petitionem proportio dividitur secundum divisionem excessus seu differentie maioris termini ad minorem, licet non proportionaliter. Unde non sequitur excessus est divisus per medium, igitur proportio divisa est per medium modo tales excessus et termini pos20 sunt esse quantitas continua per primam suppositionem quia, quidem quantitas continua divisibilis est in infinitum.
247 proportio: proportionum $C$
247-48 similiter om $S$
248 ante rationalis scr et del $V$ ir
249 Et: etiam $V /$ incommensurabilium: commensurabilium $R$ / proportio est $C E R V ; \operatorname{tr} H S$ / irrationalis: irrationabilis $V /$ similiter $C E H R$; om $S V$
250 omnis... incommensurabilium om $S /$ post est bab $V$ proportio
251 quinta: quinto $S /$ decimi: et decima R
252 et incommensurabilium om $C /$ decima: decimi $C$
253 post et $h a b C$ ex diffinitionibus commensurabilium et cetera / tertie corr ex secunde CEHRSV / quinti: quinte $S$
254 post hiis scr et del $V$ consequi con/ sequantur: secutum(?) $V /$ due: re-
gule $S$
255 est $^{1}$ om $H /$ et sit: ut $S /$ quarta $C R S$ tertia $E H$ quinta $V /$ post proportio $s c r$ et del $V$ rationalis / est ${ }^{2}$ : sit $H$ | post est ${ }^{2}$ scr et del $V$ commensurabilis et
257 diffinitionem: suppositionem $E /$ proportionem: proportionale $S$
258 media: medium $V /$ assignare: signare $H$
259 inter om $R$ / quelibet: qualibet $C$ / post duo add $V$ extrema / post continua scr et del $V \mathrm{e}$ / post inequalia scr et del $V$ n / ante in bab $V$ que / possibile est CERV; tr $H$ est $S /$ ante media scr et del $V$ medium
261 et: cum $S$ / quinta CRS quarta et quinta $H$ quarta $E$ sexta $V /$ quod: que $C /$ sicut $C H S V$ sicud $E R$

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The third: A ratio of commensurable [quantities] is rational, and, similarly, the converse, namely every rational ratio is constituted of commensurable quantities. And a ratio of incommensurable quantities is irrational, and, similarly, the converse, namely every irrational ratio is constituted of incommensurable quantities.
All this is clear from the fifth and sixth [propositions] of the tenth [book] of Euclid, from the definition of [the terms] commensurable and incommensurable also given in the tenth book, from the principles of the seventh book, and from the comment on the third definition of the fifth book of Euclid.
Two other [suppositions] follow from these.
One is-and let this be the fourth-that any ratio is divisible into infinity because, by the first supposition, every ratio is found in continuous quantities; and, by the first definition, to divide a ratio is to assign means between the extremes; and, by the first postulate, it is possible to assign means into infinity between any two continuous unequal [quantities].

Another is-and let this be the fifth-that any ratio is as a continuous quantity in the sense that it is divisible into infinity just like a continuous quantity. [It is divisible] into two equal parts, or three, or four, etc.; into unequal parts in any way; into commensurable parts and into parts incommensurable to it, etc.; and, indeed, in any other way because by the first postulate a ratio is divided by division of the excess or difference of the greater to the smaller term-although not proportionately. Thus it does not follow that the excess is divided by the mean [but rather] the ratio has been divided by a mean in a way that such excesses and terms [form] a continuous quantity, by the first supposition, since a continuous quantity is divisible into infinity.

262 post continua bab $V$ et / in hoc om $V$ | quod: et $C /$ est: $\operatorname{sit} C /$ sicut: potentie $S$
262-63 in hoc...continua om $R$ / continua om $E$
$263 \mathrm{et}^{\mathrm{t}}$ om $\mathrm{V} /$ et in $4 \mathrm{om} V / \mathrm{et}^{5}$ om $R$
${ }_{263}-65$ et per...alio modo om $S$
263-64 quomodolibet: quodlibet $V$
$264 \mathrm{in}^{1}$ : per $E /$ similiter: sic $V /$ in partes sibi: si $C$
26s incommensurabiles: commensurabiles $E$ / et cetera $H V$; om CER
266 divisionem EHSV; om $R$ propor-
tionem $C$ / post divisionem bab $H$ excessum vel
267 seu: vel $S /$ seu differentie om $E$
267-68 proportionaliter: proportionabiliter $C$
268 divisus: divisibilis $S$
269 divisa est EHRS; tr CV / modo ymo(?) $H$ | post modo add $E$ omnes / termini: tales $S$
270 quantitas continua: quantitates continue $E$ / quia $H R$ que $C E S V$ / quidem: quacum $V$
271 continua $C H R V$; om $E S$ / est om $H$

Ista suppositio confirmatur per commentum undecime diffinitionis quinti ubi dicitur quod denominatio proportionis duarum quantitatum quibus nullum interponitur medium habet naturam linee; quibus vero
${ }_{275}$ interponitur unum habet naturam superficiei; quibus vero duo naturam corporis, quod non est verisimile quod omnis proportio irrationalis mediate denominatur ab aliquo numero.

Omnis proportio rationalis immediate denominatur ab aliquo numero, aut cum fractione aut fractionibus aut sine fractione. Quarum 880 denominationum inventio docebitur infra.

Proportio, vero, irrationalis dicitur mediate denominari ab aliquo numero quando ipsa est pars aliquota aut partes alicuius proportionis rationalis, aut quando est commensurabilis alicui rationali, quod est idem, sicut proportio dyametri ad costam est medietas duple pro-

Dico, ergo, quod non apparet verum quod omnis proportio irrationalis sit commensurabilis alicui rationali. Et ratio est quia omnis proportio est sicut quantitas continua quo ad divisionem ut patet per ultimam suppositionem. Ergo potest dividi in duo quorum quodlibet
290 est incommensurabile toti per nonam decimi. Igitur erit aliqua proportio que erit pars duple et tamen non erit medietas duple, nec tertia pars, nec quarta, nec due tertie, et cetera, sed erit incommensurabilis duple et per consequens cuicumque commensurabili ipsi duple per commentum octave decimi. Et iterum, pari ratione, aliqua poterit esse

272 suppositio: sic positio $V$ / commentum: conceptum $C /$ undecime: quinte H
273 proportionis CERV proportionum HS
273-74 quantitatum quibus: quantum quando $S$
274 interponitur: interponatur $S /$ naturam linee: numerari secunde ille $S$ / linee: lineam(?) $H /$ vero om $S$
275 interponitur unum: interpositis(?) $C$ habet om $S$ / superficiei $C R V$ superficies $E$ superficiem $H /$ vero: non $S$ duo om $C$
275-76 ante naturam add $C E$ habet
276 ante quod ${ }^{\mathrm{I}} \mathrm{mg}$ hab $S$ alia notanda
277 denominatur $H V$ denominetur $C E R$ denomineretur $S$
278 rationalis: irrationalis $C /$ ab om $V$
278-79 Omnis...numero om $S$

279 aut ${ }^{1}$ : ut $C /$ aut²:om $S /$ aut fractionibus om $H$
280 denominationum: denominationem $V /$ infra $E H S V$ post $C R$
28I irrationalis: rationalis $E$ / mediate denominari: immediate denominatur $E /$ denominari: nominari $S$
283 post rationalis bab $E$ commensurabilis alicui / commensurabilis: commensurabiles $E$ / alicui rationali $\operatorname{tr} S$
283-84 quod est idem CHRV que est irrationalis $E$ quod idem est $S$
284 sicut CRSV sicud $E H$ / medietas: medietate $R$
286 Dico: dicitur $S /$ apparet HRSV oportet esse $C$ est $E$
286-87 omnis proportio irrationalis: non $E$ / omnis...quia om $C$
287 rationali: irrationali $S$ / ratito es $\operatorname{tr} E$ / omnis: dupla $E$

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This supposition is confirmed by the comment on the eleventh definition of the fifth [book of Euclid], where it is stated that the denomination of a ratio of two quantities which have had no mean interposed has the nature of a line; with one mean interposed, [the denomination] has the nature of a surface; and, indeed, with two [such means], it has the nature of a [solid] body because it is not probable that every irrational ratio is mediately denominated by some number.
Every rational ratio is immediately denominated by some number, either with a fraction, or fractions, or without a fraction. The determination of these denominations will be shown below.
An irrational ratio is said to be mediately denominated by some number when it is an aliquot part or parts of some rational ratio, or when it is commensurable to some rational ratio, which is the same thing, as [for example], the ratio of a diagonal to its side is half of a double ratio.*
I say, therefore, that it does not seem true that every irrational ratio is commensurable to some rational ratio. And the reason is that every ratio is just like a continuous quantity with respect to division, which is obvious by the last supposition. Therefore, by the ninth proposition of the tenth [book of Euclid, every ratio] can be divided into two [ratios] any of which is incommensurable to the whole. Thus there will be some ratio which will be part of a double ratio and yet will not be half of a double, nor a third part, or fourth part, or two-thirds part, etc., but it will be incommensurable to a double and, consequently [incommensurable] to any [ratio] commensurable to this double ratio (by the comment on the eighth proposition of the tenth book of Euclid). And further, by the same reasoning there could be some ratio incommensurable to a double and also to a triple ratio and [consequently incommensurable] to any ratios commensurable to these, as

* That is, $(2 /)^{1 / 2}$.

288 sicut: sicud $E$
289 duo: duas $V$
290 est CSV; om $H$ erit $E /$ erit $E H R$; om $S$ et $C$ esset $V /$ aliqua om $R$ 290-91 proportio que erit om V
291 que erit $\operatorname{tr} S /$ eritri$^{1}$ : et $C /$ duple ${ }^{\text {: }}$ duplicabile $C /$ tamen om $E /$ tamen non $\operatorname{tr} V /$ erit ${ }^{2} C H R S$ est $E$ esset $V$

292 et cetera om $C /$ erit: ipsa $H$
293 post consequens add $R$ incommensurabilis / cuicumque $E H S V$ cuilibet $C R$
294 commentum octave EHRV conceptum ergo $C_{4}$ divisionem(?) $S /$ decimi om $H$ / aliqua poterit esse $C H$ poterit esse aliqua $V$ poterit aliqua esse $R /$ poterit: potest(?) $E$

295 incommensurabilis duple et etiam triple et cuilibet commensurabili alicui istarum sicut est medietas sesquitertie, et sic de aliis.

Et sic, forte, poterit esse aliqua irrationalis que est incommensurabilis cuilibet rationali. Nunc videtur ratio si aliqua est incommensurabilis duabus et aliqua tribus et sic ultra quoniam sit aliqua que sit
300 incommensurabilis cuilibet licet non sequatur ex forma argumendi sicut aliqua quantitas continua omnibus quantitatibus unius ordinis est incommensurabilis. Istud, tamen, nescio demonstrare sed si oppositum sit verum est indemonstrabile et ignotum. Hoc etiam patet in commento ultime diffinitionis quinti Euclidis ubi dicitur quod infinite
305 sunt proportiones irrationales quarum denominatio scibilis non est. Quod si locus ab auctoritate valeat sequitur quod non quelibet irrationalis est commensurabilis alicui rationali seu denominabilis ab aliqua rationali et arguitur sic:
Si quelibet est alicui commensurabilis ergo cuiuslibet denominatio
${ }_{310}$ est scibilis. Et arguitur ultra ex opposito consequentis sed probo consequentem.
Quia si non sit ita, sit $B$ una proportio irrationalis cuius denominatio non sit scibilis, et $A$ sit proportio rationalis cui $B$ est commensurabilis, et sit $C$ proportio $B$ ad $A$. Et suppono quod proportio est
${ }_{315}$ scibilis si eius denominatio est scibilis et econtraria. Tunc arguo sic: $A$ est proportio scibilis et $C$ est proportio scibilis ergo $B$ est proportio scibilis. Antecedens patet quia $A$ et $C$ sunt proportiones rationales. Et consequentam probo quia si aliqua quantitas est scibilis seu nota,

295 incommensurabilis: commensurabilis $V /$ etiam om $H$ / cuilibet: cuicumque $S$
296 alicui: cuique $S$ / istarum $H$ istorum $C R V$ costorum(?) $E$ illarum $S$ / est: erit $S$ / sesquitertie om $V$
297 Et om $S$ / sic om $E$ / forte om $C$ | poterit: potest $E$ / esse aliqua $\operatorname{tr} R$ | post aliqua add $E S$ proportio / irrationalis: rationalis $C /$ est $C S$ erit $E V$ sit $H$ esset $R$
298 cuilibet $C E H R$ cuicumlibet $V$ cuiuslibet $S$ / Nunc $E R S V$ nec $C H /$ aliqua: qua $H$ / est scr et del $V$ / supra est bab $H$ sit
299 duabus: duobus $R$ / sic ultra om $E$
300 cuilibet: cuicumlibet $V /$ licet: sed $E /$ sequatur: sequitur $E$ | argumendi

CEH argumenti RSV
301 quantitas continua $\operatorname{tr} E /$ post continua bab C termini(?) / unius: eiusdem $S$ / ordinis om $E$ | post ordinis bab $E$ et
302 incommensurabilis: commensurabilis $E$ / Istud HRSV illud CE / demonstrare CRS; om $H$ demonstrari $E$ dicere $V /$ sed $E H R S$ et $C$ licet $V /$ si sic $V$
303 sit $H R$ est $C E S V$ / verum: unum $R$ Hoc: hec C/etiam: et $C /$ patet $C E R$ apparet $H S V$
304 ultime: tertie $S /$ Euclidis om $\mathrm{H} /$ dicitur: dicit $S$
305 sunt: sint $E /$ non om $R$
306 Quod: quia $S /$ valeat: valet $C /$ quod non rep $V$ / non om $E$ / quelibet: qualibet $S$
306-7 post irrationalis hab $E$ non
[for example], half of a sesquitertian ratio. And the same may be said of other ratios.

And there might be some irrational ratio which is incommensurable to any rational ratio. Now the reason for this seems to be that if some ratio is incommensurable to two [rational ratios], and some ratio is incommensurable to three rational ratios, and so on, then there might be some ratio incommensurable to any rational ratio whatever, though this does not follow from the form of the argument as [it does follow when we say that] some continuous quantity is incommensurable to all quantities of one [geometric] series. However, I do not know how to demonstrate this; but if the opposite should be true, it is indemonstrable and unknown. This is also apparent in the comment on the last definition of the fifth book of Euclid, where it is said that there are infinite irrational ratios whose denominations are not knowable. If the passage from this authority is valid, it follows that not every irrational ratio is commensurable to some rational ratio, or capable of denomination by some rational, and this is argued as follows:

If every [irrational ratio] is commensurable to some [rational], therefore the denomination of any [irrational] is knowable. And I argue further from the opposite of the consequent, but shall prove the consequent.

Now if this should not be so, let $B$ be an irrational ratio whose denomination is not knowable, and $A$ a rational ratio to which $B$ is commensurable; and let $C$ be ratio $B$ to $A$. I assume that a ratio is knowable if its denomination is knowable, and conversely. Then I argue as follows: $A$ is a knowable ratio and $C$ is a knowable ratio; therefore $B$ is a knowable ratio. The antecedent is obvious, because $A$ and $C$ are rational ratios. I prove the consequent because if some quantity is knowable or known, as is $A$,

307 est: sit $E$ / commensurabilis EHSV incommensurabilis $C R$ / seu: sive $C /$ seu denominabilis scr et del $V$
307-8 aliqua: aliquo $R$
309 quelibet: que $R /$ post quelibet add $V$ irrationalis / alicui commensurabilis HR; tr CESV | ante ergo add $V$ rationali
310 arguitur $C E H S$ argumentatur $R$ arguatur $V /$ ultra $C E H R$; om $V$ sic $S$
312 ante Quia hab $S$ quia proportio / sit ${ }^{1}$ est $C$ / ita sit om $E$

313 non sit: est non $R /$ et om $V /$ rationalis: irrationalis $E / \mathrm{B}$ om $H$
314 B ad A: A ad B C/suppono: supponitur $E$ / proportio est: sit $R$
315 scibilis1: scita $H /$ est: sit $S /$ econtraria $E H R V$ econverso CS / arguo om $S$ 316 est proportior $\operatorname{tr} C / \mathrm{B}: \mathrm{C} H$
317 patet: est(?) $E / \mathrm{C}: \mathrm{B} R$
318 si: sit $S /$ post quantitas add $V$ continua / est om $S /$ seu $E H V \operatorname{vel} C$ et $S /$ seu nota om $R$
sicut $A$, et proportio eiusdem ad aliquam aliam sit nota, sicut est $C$, de quantitatibus proponitur de numeris secunda conclusione secundi De numeris datis sicut allegatur et declaratur infra nona suppositione quarti capituli.

Patet, itaque, quod si quelibet proportio irrationalis esset commensurabilis alicui rationali denominatio eius esset scibilis, licet nondum foret scita. Si , autem, dicatur quod auctor intelligit que nondum sint scite, tunc non deberet hoc dicere plus de proportionibus irrationalibus quam de rationalibus quarum similiter alique non sunt scite. Quare, potius, videtur intelligere quod denominationes aliquarum denominatio, vel mediate vel immediate, ab aliquo numero denominetur.

Sufficit, igitur, mihi pro nunc quod ego possum in hoc capitulo ista principaliter facere. Datis quibuscumque proportionibus rationa-
335 libus seu irrationalibus utrum sint commensurabiles, scilicet utrum proportio unius ad alteram sit rationalis vel irrationalis demonstrative ostendere. Item dato quod fuerint commensurabiles et communicantes earum proportionem assignare. Et iste erunt due conclusiones principales huius capituli.

Et posito quod proportiones date sint incommensurabiles non intendo ulterius inquirere utrum proportio unius ad alteram que est irrationalis sit medietas duple aut tertia pars triple vel quadruple et cetera, quia forte esset talis que nullius proportionis rationalis esset

319 et: est $H$ / eiusdem: eius $E$ / aliquam aliam $\operatorname{tr} R$ / sit: est $V /$ est $C E H V$; om RS
320 quantitas om $E /$ est $H S$ erit $E R V$ et $C$ / scibilis seu nota: nota seu scibilis $H$ | scilicet: sicut est $H$ / hoc: hic $V$ / hoc quod dico: sicut $S$
321 proponitur $E S V$ poterit $H$ proponit $R$ / numeris: numero $S$ / post numeris add $H$ ostendi / post secunda bab $R$ scilicet / conclusione $E H R$ conclusio SV
321-22 de ${ }^{1}$...suppositione om $C$ / secundi ...datis om $S$
322 allegatur et om $R /$ nona $H S$; om $V$ secundum $H$ nota $R$ / suppositione HRSV suppositionem $E$

324 itaque: igitur $C$ | post quod scr et del $V$ qul(?) / post proportio bab $S$ sicut allegatur et declaratur / irrationalis: rationalis $C /$ esset: est $R$
326 scita om $E$ / dicatur: dicitur $E /$ quod om $E$ / intelligit: $\mathrm{R}($ ? ) / que CEV quod $H R S / \operatorname{sint} C E H V$ sunt $R$ sit $S$
327 scite: scita $S /$ deberet $E H R V$ deberetur $C$ diceretur $S /$ hoc dicere plus $C V$ dicere $E$ hoc dicere $H R$ hoc $S /$ post hoc scr et del $V \mathrm{dr}$
327-28 irrationalibus: irrationabilibus $H$ 328 quam EV plusquam CHRS / post de add $C$ proportionibus / rationalibus: rationabilibus $H$ / quarum: qua $C$ I similiter alique: alio modo $C$
329 Quare: quia S / intelligere: bab H
and the ratio of $A$ to some other ratio be known, as is $C$, then that other quantity, namely $B$, is knowable or known.* And what I say about quantities is proposed for numbers in the second proposition of the second chapter of On Given Numbers, just as it is mentioned and stated below in the ninth supposition of the fourth chapter.

And so it is clear that if any irrational ratio were commensurable to some rational ratio, the denomination of it would be knowable, although it might not yet be known. If, however, it should be said that the author [i.e., Campanus] understands things which are as yet unknown, then he ought not to say this any more about irrational ratios than of rationals, because, similarly, some [rationals] are not [yet] known. For this reason, he [i.e., Campanus] seems, rather, to understand that the denominations of some [irrationals] are not knowable because they have no denominations at all, since every denomination, whether mediate or immediate, is denominated by some number.

It is, therefore, sufficient for me now to do the following things in this chapter: to show demonstratively whether any given rational or irrational ratios are commensurable, namely whether a ratio of one to the other is rational or irrational; then, having shown that they are commensurable and communicant, to assign their ratio. And these will be the two principle propositions of this chapter.

But having shown that the given ratios are incommensurable, I do not propose to inquire further whether the ratio of one to the other, which is irrational, is half of a double ratio, or a third part of a triple, or quadruple, etc., because it might well be that such would be an aliquot part of no

* See pp. 329-30.
verbum illegibile / quod om $E$ / aliquarum: aliquorum $R$
330 nulle $C H R$ nec $E S V /$ cum: quia $R$
331 denominatio om $V /$ vel $^{1}$ om $C$
333 mihi pro nunc: pro nunc mihi $H$ / possum: possim $R$ / hoc: isto $C$ / post hoc scr et del $V$ a / capitulo: tractatu $H$
334 ista principaliter CHRS; $\operatorname{tr} E V /$ quibuscumque $C H R S$ quibusdam $E$ quibuslibet $V$
335 seu $H$ vel $C R$ sive $E S V$ / ante scilicet bab $R$ vel incommensurabiles et $S$ communicantes earum
$335-36$ scilicet...sit: et pocius alterum
possit esse $E$
336 post vel scr et del $V$ ir et irral / demonstrative: demonstratem $R$
338 ante proportionem $m g$ bab $H$ due conclusiones principales / iste: ille $S$ / due om $V$
339 huius: istius $S$
340 date om $H$
341 alteram: aliam $C$
342 sit $C H S V$ sicud $E$ sicut $R /$ vel: aut $V$ 342-43 et cetera om $H$
343 forte esset $C H R V$; $\operatorname{tr} E S$ / post esset ${ }^{1}$ bab CV aliqua / ante talis bab $S$ aliqua / post nullius $b a b C S$ esset / esset $^{2}$ om $S$
pars aliquota. Et dato quod esset, tamen, foret nimis difficile et forte 345 patet ex auctoritate superius allegata.
De proportione proportionum irrationalium me volo breviter expedire. Sicut iam ex commento quinti Euclidis allegavi infinite sunt proportiones irrationales quarum denominationes sunt ignote et adhuc ${ }^{50}$ cum omnis earum denominatio ex proportione rationali sit assumpta. Si sit aliqua que nulli rationali sit commensurabilis, sicut est verisimile, talis nullam denominationem habebit. Propositis igitur duabus proportionibus irrationalibus per suas denominationes si habeant et sint note statim patet cuiuslibet earum proportio ad proportionem
rationalem a qua denominatur.
Proportio, vero, proportionum rationalium inferius ostendetur et ex istis potest intelligens proportionem earum leviter assignare. Verbi gratia, si queratur de proportione inter medietatem duple proportionis et quartam partem triple dico quod si proportio dupla et tripla o sint incommensurabiles, sicut est rei veritas et infra patebit, similiter et quelibet partes aliquote earum sunt incommensurabiles.

Si , vero, queratur de medietate duple et tertia parte quadruple dico quod quadrupla et dupla sunt commensurabiles, ut post videbitur, ideo quelibet pars aliquota unius est commensurabilis cuilibet parti $6_{5}$ alterius. Proportio vero quadruple ad duplam per docenda patebit et tunc, habita proportione totius ad totum, proportionem partis ad partem faciliter invenies per hunc modum:

Cum, enim, queris de tertia parte quadruple et de medietate duple

344 aliquota: aliquote $V$ | foret: esset $S$ difficile om $C$
345 reperire om $V /$ esset om $E /$ scibilis: subtilis $E$
346 superius $E H V$; om $C R$ prius $S$
347 proportionum om $C$
348 Sicut: sicud $E /$ iam rep $E /$ ex: in $V$
349 denominationes sunt ignote: sunt ignote denominationes $C$
350 earum: eorum $C /$ ex: a $V /$ sit: sicut $E /$ assumpta: sumpta $H$
35 I ante Si scr et del $V$ sit / siti om $E$ sicut: sicud $E$
352 habebit: habuit $C$ / post Propositis hab $V$ de
354 cuiuslibet: cuilibet $S /$ earum om $S$
355 ante rationalem scr et del $V$ ir / a qua

CHSV aliqua $E$ de qua $R$ / post denominatur bab $C$ et cetera
356 proportionum: proportionem $V /$ post proportionum scr et del $E$ ir / rationalium HRSV; om $C$ irrationalium $E /$ post ostendetur $b a b C$ scilicet rationalium /et CER; om $H S V / \mathrm{ex}$ rep $V$
357 istis: isto $R /$ intelligens: intellectus $V$ | earum om $C$ | post leviter bab $C$ terminis(?)
359 post triple add $R$ proportionis / infra si scr et del $H$ est / proportio om $C$ / et tripla om $S$
360 sint $C E R V$ sunt $H$ sit $S /$ incommensurabiles $C H R V$ commensurabiles $E$ incommensurabilis $S /$ et $E H R V$ est
rational ratio. And, finally, if this should be so, it would be very difficult and perhaps impossible to discover because its denomination might not be knowable, as is evident from the authority cited above.
I now wish to speak briefly about a ratio of irrational ratios. From the comment on the fifth book of Euclid, I have already mentioned that there are infinite irrational ratios whose denominations are unknown, but yet when any denomination of them can be taken it must be selected from a rational ratio. If some such ratio should not be commensurable to any rational ratio, as is probable, it will have no denomination. Therefore, if two proposed irrational ratios should have denominations which are known, the ratio of each of them to the rational ratio by which it is denominated is immediately obvious.
Now a ratio of rational ratios will be shown below and with regard to these one can easily understand how to assign their ratio. For example, if one seeks the ratio between half of a double ratio and a fourth part of a triple, I say that if a double and a triple ratio should be incommensurable, as is the truth of the matter and will be shown below, [then] similarly, any aliquot parts of them are incommensurable.

But if one seeks [the ratio] between half of a double ratio and a third part of a quadruple, I say that a quadruple and double ratio are commensurable, as will be seen after; therefore any aliquot part of one ratio is commensurable to any part of the other. A ratio of quadruple to double will be shown in terms of things that are to be taught [later], and then when the ratio of whole to whole is known, you can easily find the ratio of part to part in this manner:

Since, indeed, you seek [the ratio] between a third part of a quadruple
$C$ ut $S /$ patebit: patet $V$
360-61 similiter et: et etiam $C$
${ }_{361}$ et $E H R$; om $S V /$ aliquote earum $\operatorname{tr} E /$ earum: eorum $C$
362 queratur: queritur $S$ / post duple bab $V$ proportionis / tertia: triplo et quadru-
plo $E$ / ante parte scr et del $E$ cua(?)
363 quod: quia $C$ / quadrupla et dupla: dupla et quadrupla $C$ / post $C H R$ prius $E V$ postea $S$ / videbitur CHS videbatur $E V$ videtur $R$
364 pars om $R$ / commensurabilis $S$ commensurabiles $C E H R V /$ cuilibet $C H S$ cuicumlibet $E V$ quelibet $R /$ part
om $S$
36s vero CERS; om $H V$ / quadruple: quadrupla $S$ / duplam: duplum $S$ I docenda $E H R V$ dicta(?) $C$ dicenda $S /$ et: ex $S$
366 proportionem CERS; om HV
366-67 partis ad partem tr $V$ post modum 367 partem: partis $C$ / faciliter $C S$ de facili $E H R V$ / invenies: inveniens $R$
368 Cum: tunc $H$ / enim: ergo $S$ / post enim scr et del $C$ conveneris / queris CSV queritur $E R$ querit $H /$ parte: partis $C$
et proportio quadrupla sit dupla ad proportionem duplam, sicut ad aliquem alium numerum habentem medietatem vel duplam partem. Deinde accipe tertiam partem maioris et medietatem minoris et qualis erit proportio unius istarum partium ad alteram talis erit proportio proportionum predictarum. Et ita poterit in aliis operari.
Verbi gratia, 12 est unus numerus habens tertiam duplus ad 6 qui habet medietatem. Est igitur 12 loco proportionis quadruple, et 6 loco duple. Qualis est itaque proportio 4, que est tertia pars 12, ad 3, que est medietas 6, talis est proportio tertie partis quadruple ad medietatem duple, scilicet proportio sexquitertia. Et eodem modo in so aliis est agendum.

Proportio rationalis potest ymaginari in generali dividi 7 modis, tribus modis in proportiones rationales, et tribus modis in proportiones irrationales, et uno modo in proportionem rationalem et in irrationalem.

Primo modo per rationales equales et sic aliqua possunt dividi, sicut quadrupla in duas duplas. Non omnis, tamen, dividitur hoc modo.

Secundo per rationales inequales quarum quelibet sit pars aut partes. Et non omnis dividitur hoc modo sed aliqua bene sicut sedecupla in octuplam et duplam cuius dupla est una quarta et octupla tres quarte.

Tertio per rationales inequales quarum nulla sit pars aut partes. Et semper capio partem et partes proprie et hoc modo quelibetest
369 et om S/proportio: pars E/ante dupla scr et del $H$ quadruplam / proportionem om $H$ / duplam om $R$ / sicut EHSV ut $C R$
370 patebit post $\operatorname{tr} S /$ numerum CEHS terminum $R V /$ tertiam: tertia $S$
371 aliquem om $S /$ alium om $V /$ numerum om $S$ / partem om $V$
372 tertiam: triplam $V$ / tertiam partem tr $H$ et ante partem scr et del $H$ secundum et ante tertiam scr et del $H$ vel
373 erit ${ }^{1}$ : est $S /$ istarum: illarum $S$ /ante partium scr et del $E$ al / alteram: aliam $C /$ erit $^{2}$ CHR; om $S$ est $E$ esset $V$
374 poterit in aliis: in aliis poterit $C /$ poterit $H$ poteris $E R S V /$ post aliis add $S$ numeris
375 Verbi gratia om $R /$ 12: una 12 $R /$ unus $E H R V$; om $S$ unius $C /$ habens: consequens $S$ / duplus: duplicatas(?)
$S /$ qui: que $E$
376 Est igitur $t r C /$ loco om $E$
377 ante duple add $V$ proportionis / itaque: igitur $S$ / que: qui $C /$ pars om $H / 3$ : tertia $R$
378 que rep $S$ / post medietas add $V$ de / ante talis hab $E$ et $/$ est$^{2}$ : erit $H$
379 Et EHSV; om CR
380 post agendum $m g$ hab $E$ nonum notabile
381 dividi om $E$ / in...modis: dividi 7 modis in generali $C$
382 modis $^{\mathrm{I}}$ EHRS; om CV / modis ${ }^{2}$ CEHV; om $R S$
382-83 proportiones om $S$
383 ante rationalem scr et del $V$ irrationale / in $^{2} C E H V$; om $R S$ | post in ${ }^{2}$ scr et del $V$ irr
385 modo om $R /$ possunt $H$ potest $C E R$ SV
and half of a double, and [since] a quadruple ratio is double to a double ratio, as will be seen after, take a number containing a third [part] which is double to some other number containing a half or double part. Next take that third part of the greater [number] and half of the smaller [number], and as the ratio of one of these parts is to the other, so is the ratio of the aforementioned ratios. And this should be done in other cases.

For example, 12 is a number containing a third double to 6 which contains a half. Therefore, 12 represents the quadruple ratio, and six the double ratio. Now ratio 4 to 3 ( 4 is a third part of $12 ; 3$ is half of 6 ) is as the ratio of a third part of a quadruple ratio to half of a double, namely a sesquitertian ratio.* And the same procedure must be followed in other cases.
A rational ratio can, in general, be imagined to be divisible in seven ways: three ways into rational ratios, three ways into irrational ratios, and one way into one rational and one irrational ratio.

In the first way, [a rational ratio is divisible] into equal rational ratios. Some can be divided this way, as [for example], a quadruple into two double ratios. $\dagger$ However, not every rational is divisible in this manner.

In the second way, it is divisible into unequal rational ratios, any of which would be a part or parts. Not every one is divisible in this way, but some surely are, as [for example] a sedecuple ratio into an octuple and double where the double is one-fourth [part of a sedecuple] and the octuple is three-fourths. $\ddagger$

In the third way, it is divisible into unequal rational ratios, none of which is a part or parts of it. I always take part and parts properly [speaking]

* The ratio $4 / 3$ is the exponent in the ex- $\dagger 4 / \mathrm{y}=4 / 2 \cdot 2 / \mathrm{I}$.
pression $2^{4}=\left(2^{3}\right)^{4 / 3}$; or, in terms of the $\ddagger 16 / \mathrm{I}=16 / 8 \cdot 8 / 4 \cdot 4 / 2 \cdot 2 / \mathrm{I}$ so that $2 / \mathrm{I}=\left({ }^{16} / \mathrm{I}\right)^{1 / 4}$ initial given ratios, $(4)^{1 / 3}=\left[(2 / 1)^{1 / 2}\right]^{4 / 3} . \quad$ and $8 / \mathrm{x}=(16 /)^{3 / 4}$.

386 sicut: sicud $E /$ in: ad $S /$ Non: quadrato(?) $E /$ omnis... modo $E V$ omnes tales dividuntur hoc modo $C$ omnis tamen hoc modo potest dividi $H$ omnes tamen dividuntur hoc modo $R$ tamen omnis dividitur hoc modo $S$
387 post Secundo add $S$ modo / inequales: in rationales(?) $E$
388 dividitur: dividi $H$ / sed aliqua bene om $H$ / sicut: sicud $E /$ post sicut bab $H$ quarta / in CEHR; om $S$
388-89 in octuplam om $V$

389 octuplam et duplam: octupla et dupla $S /$ cuius: eius $V /$ quarta $E R S$ quartum $C$ tertia $H$ quadrupla $V$ /octupla: octuplam $V$ / tres: tria $E$ / quarte: quadruple $C$
390 Tertio HRSV tertie $C$ secundo $E /$ per om $E$
391 semper capio: sic capit $V /$ partem et om $S / \mathrm{et}^{2}$ : vel $C /$ post $\mathrm{et}^{2}$ scr et del $E$ pp / post est scr et del $V$ divid
391-92 est divisibilis $\operatorname{tr} C$
divisibilis sicut dupla in sexquialteram et sexquitertiam. Sexquialtera, vero, in sexquiquintam et sexquiquartam sicut hic $6,5,4$.
Quarto per irrationales equales et quelibet est divisibilis isto modo號 et cetera sed omnia potest dividi in plures irrationales equales quia omnis potest dividi in duas aut in tres aut in quatuor, et cetera, faciendo predicatum disiunctum.
Quinto per irrationales inequales quarum quelibet sit pars aut partes divise proportionis. Et quelibet dividi isto modo intelligendo sicut prius, ut quadrupla dividitur in proportionem dyametri ad costam, que est quarta pars eius, et in proportionem quadruple coste ad dyametrum, que est tres quarte proportionis quadruple.
Sexto per irrationales inequales quarum nulla sit pars aut partes et quelibet potest ita dividi assignando inter eius extrema media improportionalia et incommensurabilia et ea multipliciter variando.
Ista sunt exemplariter et sine demonstratione dicta quia visis sequentibus faciliter apparebunt. Nec ponitur aliquod dubium quod in ro demonstrationibus sequentibus supponatur et quod non clare pateat per dicenda.
De septimo modo statim post dicetur quia facit magis ad propositum quam aliquis aliorum.
Consimiliter dico quod quelibet proportio irrationalis posset yma-

392 ante in $b a b S$ et / et sexquitertiam om $R$
393 sexquiquintam et sexquiquartam $C$ sexquiquartam et sexquiquintam $E R$ sexquiquartam et sexquisertam(?) $H$ sexquioctavam et sexquiquintam $S$ sesquialteram quadruplam et sesquitertiam (?) quintam $V \mid$ ante et scr et del $H$ se / sicut CESV sic $H$ ut $R$ | post hic scr et del $V$ sic
394 quelibet: quilibet $S$ / isto modo CHR illo modo $S$ aliomodo $V$
394-95 irrationales...tamen om $E$
395 duas $H R$ duo CSV
396 duas irrationales: istas in $S /$ quelibet $^{2}$ om S / tres: tertiam(?) $S$
397 omnia CERV omnes RS / equales: inequales $V$
398 aut : vel $S /$ tres: tria $S$ / et cetera EHSV; om CR
399 predicatum: punctum(?) $S /$ predica-
tum disiunctum $\operatorname{tr} R$ / disiunctum: disjuncter $H$
400 irrationales: irrationale $E$
401 Et : in $E /$ dividi $H R$ dividitur $C E V$ dividatur $S /$ isto: illo $V /$ intelligendo: dividendo $R$ / sicut: ut $S$
401-2 sicut prius ut: ut prius sicut $R$
402 dividitur...dyametri: in proportionem dyametri dividitur $V$
403 et $S V$; om $C E H R$ / quadruple $H S$ quadruplici $C$ quadrupli $E R V /$ ad om $H$
403-4 dyametrum ERSV; om $H$ dyameter $C$
404 est om $H /$ tres: tria pars $S$
405 ante Sexto scr et del $H$ sexto(?)
406 ita dividi CERS; tr $H$ dividi $V /$ extrema: extremam $C$
407 et incommensurabilia om C / variando $E H R V$ nominando $C S$
and in this way any rational ratio is divisible, just as a double ratio is divisible into a sesquialterate and a sesquitertian.* Indeed, a sesquialterate ratio is divisible into a sesquiquintan and sesquiquartan thus: $6,5,4 .+$

In the fourth way, the rational ratio is divisible into equal irrational ratios. Any rational is divisible this way, as [for example], a double into two irrationals which are as the diagonal [of a square] to its side. $\ddagger$ However, not every rational can be divided into two equal irrationals, or into three, and so forth; but every rational can be divided into several equal irrationals since each, taken separately, can be divided either into two, or into three, or into four, etc.

In the fifth way, a rational ratio is divisible into unequal irrationals any of which may be a part or parts of the divided ratio. And any rational can be divided in this manner by understanding it as before, just as [for example] a quadruple is divided into a ratio of the diagonal to its side, which is a fourth part of it, and into a ratio of quadruple the side to the diagonal, which is three-fourths of a quadruple ratio. $\S$

In the sixth way, a rational is divisible into unequal irrationals, none of which is a part or parts of it. Any rational can be so divided by assigning mean improportionals and incommensurables between the extremes and by varying them in a multiplicity of ways. II

These things have been stated by way of example and without demonstration because, [once] seen, what follows will be readily apparent. Nor is anything dubious posited which would then be assumed in subsequent demonstrations and which would not be clearly obvious by what will be said.

The seventh way [of dividing a rational ratio] shall be discussed immediately afterward, since it is more pertinent to what is proposed than some of the other ways.

I say, similarly, that any irrational ratio could be imagined as divisible in
$* 2 / \mathrm{I}=4 / 3 \cdot 3 / 2$.
$\S 4 / \mathrm{I}=(4 / \mathrm{I})^{1 / 4} \cdot(4 / \mathrm{I})^{3 / 4}$.
$\dagger 3 / 2=6 / 5 \cdot 5 / 4$.
$\ddagger 2 / 1=2 /(2 / 1)^{1 / 2} \cdot(2 / 1)^{1 / 2 / 1}$.
|| For example, $4 / \mathrm{I}=4 /(3)^{1 / 3} \cdot(3)^{1 / 3} / \mathrm{I}$.

408 sunt: $\sin t R$ / exemplariter: exemplar $V \mid$ ante visis scr et del $E$ sunt(?)
409 ponitur: ponatur $S$ /aliquod dubium: aliquid $R$
410 demonstrationibus sequentibus $E H$ $R S$; $\operatorname{tr} C V$ / supponatur: supponitur

C/et $C E H R$; om SV/ non: tamen $V$ 411 dicenda $H R S V$ dicta $C$ dicende $E$
412 septimo CERV quinto $H$ alio $S /$ modo om $R$ / post: postea $H$ / facit magis $t r S$

415 ginari dividi istis septem modis et de quatuor ultimis dico quod quelibet proportio irrationalis dividitur quolibet illorum quatuor modorum, scilicet tribus modis per proportiones irrationales et uno modo per rationalem et irrationalem seu irrationales. Et de ultimo modo patet quoniam cuiuslibet proportionis irrationalis aliqua proportio 420 rationalis est pars, non tamen aliquota, quia qualibet irrationali aliqua rationalis est minor.
De aliis nec de isto propter brevitatem non plus declaro quia etiam non faciunt ad propositum ut videbitur. De tribus primis modis, scilicet si quelibet vel aliqua proportio irrationalis posset dividi quo-
Istis principiis, preambulis, notabilibus, excusationibus, diffinitionibus, distinctionibus, tanquam quibusdam introductoriis ad intellectum sequentium prelibatis, incipio secundum capitulum conclusiones aliquas demonstrando.

## Secundum Capitulum [Pars prima]

- Prima conclusio. Nulla proportio rationalis est divisibilis septimo modo, scilicet per rationalem et irrationalem, vel per rationales et irrationalem, vel per rationalem vel rationales et irrationales, que non faciant unam rationalem.
Si non est verum sit igitur $A$ una rationalis inter extrema $D$ et $F$ 5 sitque divisa in $B$ rationalem, et $C$ irrationalem per $E$ medium inter extrema data assignatum secundum primam diffinitionem. Tunc pro-

415 post ultimis add $V$ modis
416 proportio om $H /$ quolibet $E H$ qualibet $C$ quodlibet $V$ quorum $S$ / il lorum HSV istorum CE / illorum quatuor: 4 istorum $R /$ quatuor: in s(?) $H$
416-17 quatuor modorum $\operatorname{tr} E$
417 per om S/proportiones irrationales: proportionis irrationalis $S /$ et $C E R$; om HSV
418 ante rationalem add $C$ proportionem post Et add $H$ cetera / modo om $V$
419 quoniam $C E H R$ quia $V$ quando $S$ proportio om $V$
419-22 cuiuslibet...De aliis om $E$
420 non om $V$ / qualibet: quelibet $C$
422 non om $H /$ declaro: declarabo $E /$

## etiam: ibi $V$

423 post ut add $V$ post et $E$ unde / videbitur: patebit $S$ / ante De bab $S$ sed / tribus primis $C H R V ;$ tr $E S$
424 si om $E$ / irrationalis: irrationales $E$ | posset dividi $\operatorname{tr} C$
425 post dicetur hab $S$ ergo 〈s〉equitur secundum capitulum
426 ante Istis bab $H$ a et mg bab $E R$ secundum capitulum / principiis preambulis: premissis principiis $S$ / post principiis add $H$ et / ante -is in preambulis scr et del $V$ am
426-27 diffinitionibus om $H$
427 quibusdam $C H R V$ quibus $E$ quadam $S$ / introductoriis: introductionibus $C$ / intellectum: intellectionem $R$

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these seven ways. But concerning the last four, I say that any irrational ratio is [actually] divisible by any of those four modes, namely three ways into irrational ratios and one way into rational and irrational or irrationals. And the last way is obvious because some rational ratio is part of any irrational ratio-not, however, an aliquot [part]-since some rational ratio is smaller than any irrational.
For the sake of brevity, I shall say no more concerning the other modes since, as will be seen, they are not relevant to what has been proposed. Later, something will be said concerning the first three ways, namely whether any or some irrational ratio could be divided in any or some of those modes.
Having now set forth-by means of principles, preambles, noteworthy things, qualfications, definitions, and distinctions-certain introductory matters for the understanding of what follows, I begin the second chapter by demonstrating some propositions.

## Chapter Two [Part One]

Proposition I. No rational ratio is divisible in the seventh way, namely into a rational and irrational; rationals and an irrational; rational, or rationals, and irrationals which do not constitute one rational ratio.
If this is not true, then let $A$ be a rational ratio formed by terms $D$ and $F$; and let $A$ be divided into $B$, a rational ratio, and $C$, an irrational ratio, by $E$, a mean assigned between the extremes in accordance with the first

428 sequentium: scriptum $S /$ prelibatis: principabatur(?) $E /$ incipio: incipit $H$ 428-29 aliquas om $C$
429 post demonstrando add $C$ ut postea videbitur
[Pars prima] om CEHRV
I Prima conclusio om $C$
2 et irrationalem ${ }^{1}$ om $C$ / irrationalem ${ }^{2}$ : irrationales $E$ | post irrationalem ${ }^{2}$ scr et del $V \mathrm{~s}$
3 vel rationales om C / ante rationales
scr et del $V$ ir / irrationales: irrationalem $E$ | faciant $H R V$ faciunt $C E$
4 est: sit $C$ / una rationalis: proportio $R$ / extrema: extremam $C$ / et om $H$ s ante rationalem scr et del $E$ rationalem / rationalem: rationalis $V / \mathrm{E}: \mathrm{C} C$
6 data $H$ A proportionis date $C$ aut proportionis date $E$ proportionis A date $R$ A proportionis $V$ infra assignatum $m g$ bab $C$ tunc proportio $B /$ primam om $R$
6-7 Tunc...C inter om $R$
portio $B$ attenditur inter $D$ et $E$ et proportio $C$ inter $E$ et $F$ aut econverso, et totalis proportio $A$ inter $D$ et $F$.
Arguatur ergo sic: $E$ est commensurabile ipsi $D$ per tertiam sup${ }_{10}$ positionem quia $B$, eorum proportio, est rationalis; et $F$ est commensurabile ipsi $D$ quia $A$, eorum proportio, est rationalis per tertiam suppositionem. Igitur $F$ est commensurabile ipsi $E$ per octavam decimi (si due quantitates communicant eidem communicabunt inter se). Igitur proportio $F$ ad $E$ est sicut proportio numeri ad numerum
15 per quintam decimi, igitur ipsa est rationalis quia minor terminus est pars aut partes maioris per quartam septimi et patet etiam ex commento secunde diffinitionis quinti. Sed proportio ista que est inter $F$ et $E$ est proportio $C$, igitur $C$ est proportio rationalis quod est contra unum suppositum.
Vel potest sic argui: $E$ est commensurabile $D$ quia $B$ est proportio rationalis, et $F$ est incommensurabile $E$ quia $C$ est proportio irrationalis que est proportio eorum per positum. Ergo $F$ est commensurabile $D$ per commentum octave decimi ubi dicitur quod si alique due quantitates fuerint communicantes, cuicumque una earum communi-
${ }_{25}$ cat et reliqua. Ergo si $F$ esset commensurabile $D$ cum $E$ sit commensurabile $D$ tunc $F$ esset commensurabile ipsi $E$ quod est oppositum minoris. Igitur consequentia fuit bona cuius conclusio est quod $F$ est incommensurabile $D$ ex quo sequitur quod $A$ eorum proportio est irrationalis quod est contra aliud positum. Et sic patet propositum.
${ }^{30}$ Et per idem arguitur quod $A$ proportio non potest dividi in plures rationales et in unam irrationalem, et cetera.

Ex eadem radice potest demonstrari ista conclusio: quod nulla pro-

7 Ei: F V
7-8 et proportio...D et Fom $V$
8 totalis: rationalis $R$
9 Arguatur $H R V$ arguitur $C E /$ est om $R$
ro quia....rationalis om $C$ / eorum: earum $C \mid$ post est ${ }^{1}$ add $E R V$ proportio
ro- II et F...rationalis om $C V$
II D scr et del $H$ / quia $H R$ quod $E$
12 E om E
I3 communicant: communicent $C$ / communicant eidem: eidem communicent $R$
13-14 communicabunt inter se: inter se communicabunt $R$
4 F ad E: E ad F E
is ante decimi scr et del $V$ ergo

16 septimi: decimi $C$
16-17 commento: concepto $C$
${ }_{17}$ quinti $H V$; om $C E R /$ ista : illa $V$ 18 quod: que $E$
19 suppositum $C H R$ positum $E V$
20 Vel $E R V$ ut $C H$ / argui $H R V$ arguere $C E / \mathrm{E}: \mathrm{C} E / \mathrm{E}$ est: est enim E $C$ / proportio: pars $H$
2I et: vel $C$ / incommensurabile: commensurabile $H /$ quia rep $E$
22 ante per scr et del $V \mathrm{pp} /$ commensurabile Hincommensurabile $C E R V$
23 decimi: quinti(?) $E$ / dicitur: dico $V /$ alique om $E$ / due om $R$
24 cuicumque: cui $E$ / earum $C H V$ istarum $R$

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definition. Then ratio $B$ is measured by $D$ and $E$, ratio $C$ by $E$ and $F$, or conversely; and the whole ratio $A$ is measured by $D$ and $F$.
One can then argue as follows: $E$ is commensurable to $D$ by the third supposition, since $B$, their ratio, is rational; and $F$ is commensurable to $D$ since $A$, their ratio, is rational by the third supposition. Therefore, $F$ is commensurable to $E$ by the eighth [proposition] of the tenth [book of Euclid] (if two quantities are commensurable to the same quantity, they will be commensurable to each other). Consequently, ratio $F$ to $E$ is just like a ratio of a number to a number by the fifth proposition of the tenth [book of Euclid], and it is therefore rational since the lesser term is a part or parts of the greater term by the fourth proposition of the seventh [book of Euclid]. This is also evident from the comment on the second definition of the fifth [book of Euclid]. But the ratio relating $F$ and $E$ is $C$, and $C$ therefore is a rational ratio, which is contrary to one of the assumptions.
Or, indeed, it can be argued this way: $E$ is commensurable to $D$ because $B$ is a rational ratio, and $F$ is incommensurable to $E$ since $C$ is an irrational ratio by assumption. Therefore, $F$ is commensurable to $D$, according to the comment on the eighth proposition of the tenth [book of Euclid], where it is stated: If quantities should be commensurable, and one of them is commensurable to any whatever other quantity, then the remaining quantity is also commensurable to it. Thus, if $F$ should be commensurable to $D$ when $E$ is commensurable to $D$, then $F$ would be commensurable to $E$, which is the opposite of the minor [premise]. The consequence therefore holds, the conclusion being that $F$ is incommensurable to $D$ from which it follows that $A$, their ratio, is irrational, which is contrary to another assumption. Hence, what has been proposed is evident. And in the same way one can argue that ratio $A$ cannot be divided into several rational ratios and into one irrational, etc.
On the same basis this proposition can be demonstrated: No irrational

24-25 una...reliqua om $E$ / communicat CHR competit $V$
s esset: sit $E$
26 tunc: et $R /$ esset: sit $E$
27 minoris: maioris $V \mid$ ante cuius scr et del $V$ con / est $H$ erat $C E R$ fuit $V$
28 ante est $a d d C$ sibi / incommensurabile commensurabile $H$ | ante D scr et del $E \mathrm{D}($ ?) / quo $E H R$ qua $C V$ / eorum:
earum $C$
29 est $^{1}$ : sit $E \mid$ post quod mg hab $R$ irrationalis / aliud om $R$
30 idem: illud $E$ / proportio om $E$
31 unam om $E /$ et cetera $E H V$; om $C$ et $R$
32 eadem: e aliquis $E$ / ante demonstrari $m g$ hab $H$ secunda conclusio
32-33 proportio om $H$
portio irrationalis est divisibilis aliquo trium primorum modorum dividendi proportiones in ultimo notabili positorum, scilicet in partes
${ }^{35}$ quarum quelibet sit rationalis, neque per equalia, neque per inequalia, nec aliquo modo dividendi.

Sit enim $A$ irrationalis cuius extrema sint $D$ et $F$ que dividantur per $E$ medium assignatum; et sit $B$ proportio $D$ ad $E$ et sit $C$ proportio $E$ ad $F$.

Tunc arguitur sicut prius: $D$ est commensurabile $E$ quia $B$ est proportio rationalis per tertiam suppositionem, et similiter $F$ est commensurabile $E$ quia $C$ est proportio rationalis. Igitur per octavam decimi $F$ est commensurabile ipsi $D$, igitur $A$ est proportio rationalis cuius oppositum ponebatur. Igitur $A$ non dividitur, et cetera, et ita sive $B$ et $C$ ponantur proportiones equales sive inequales, et cetera, et ita si $A$ ponatur dividi in tres proportiones vel in quatuor, et cetera.
Unde manifestum est quod nulla proportio irrationalis componitur ex rationalibus quamvis sit econverso sicut dupla ex duabus quarum quelibet est sicut dyameter ad costam ut patet in ultimo notabili. Patet
so etiam quod rationalis addita rationali semper facit et reddit rationalem et numquam irrationalem, licet rationalis addita irrationali cumque irrationalem componat.

Secunda conclusio. Si inter duos numeros minores alicuius proportionis rationalis non fuerit numerus medio loco proportionalis sive numeri talis proportio
55 non potest dividi in plures proportiones rationales equales et propter boc nulla proportio rationalis est pars eius aliquota.

Si non sit ita sit igitur $A$ proportio data cuius primi numeri sint $G, H$; signeturque $A$ proportio inter duos quosvis terminos qui sint

33 est om $V /$ primorum: predictorum $H$
34 partes $R$ tales $H$ plures $C E V$
35 sit $E H V$ est $C R /$ neque : nec $E$
36 dividendi $E H V$ dividendo $C R$
37 A om $V / \mathrm{A}$ irrationalis $\operatorname{tr} C /$ sint: sit $R /$ et om $H /$ dividantur $H R$ dividam $C$ dividatur $E V$
38 B proportio $\operatorname{tr} \mathrm{H} / \mathrm{sit} \mathrm{CHR}$; $\operatorname{tr} C V$ sit $E$
39 E: C V
40 arguitur om $H /$ prius: primum(?) $E /$ ante E add $H$ ad / E : C C / quia rep C / B: D $E$
41 et om $E$
42-43 Igitur...rationalis om $R$
43 decimi CHV; om $E /$ ipsi EHV;
om C
45 ponantur proportiones $H$; $\operatorname{tr} C E R V$ | ante inequales scr et del $H$ non / et ${ }_{R}$ cetera $H V$; om $C$ proportiones(?) $E \mathrm{p}$ $R$

46 ante -portiones in proportiones scr et del $V \mathrm{p}(?) /$ vel: sive $R$
47 ante manifestum bab $V$ mai / componitur: composita $C$
48 post ex ${ }^{\mathrm{I}}$ add CERV proportionibus / econverso om $V$
49 est $H R$; om CEV / supra dyameter scr $V \operatorname{dyt}($ ? $) /$ in om $E$
so rationali: irrationali $V \mid$ et om $R \mid$ reddit $C E$; om $H R$
so-s 2 semper...componat: quandoque

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ratio is divisible in any of the first three ways of dividing ratios mentioned in the last of the noteworthy points, namely into parts where any of them is rational-i.e., not into equal rationals, unequal rationals, nor in some other mode of division.

For let $A$ be an irrational whose extreme terms are $D$ and $F$, which are divided by $E$, an assigned mean; and let $B$ be ratio $D$ to $E$ and $C$ ratio $E$ to $F$.

Then one argues as before: $D$ is commensurable to $E$ by the third supposition, since $B$ is a rational ratio; and similarly $F$ is commensurable to $E$ because $C$ is a rational ratio. Therefore, by the eighth [proposition] of the tenth [book of Euclid], $F$ is commensurable to $D$ and, consequently, $A$ is a rational ratio, the opposite of which was assumed. $A$ is not, therefore, divided [into ratios each of which is rational], and this holds whether $B$ and $C$ are assumed equal or unequal ratios, etc.; and it also holds if $A$ should be assumed to have been divided into three ratios, or into four, etc.

From all this it is obvious that no irrational ratio is composed of rational ratios, although, conversely [a rational may be composed of irrationals], just as a double ratio can be composed of two ratios each of which is like a diagonal to its side; and this was made evident in the last of the noteworthy points. It is also clear that a rational ratio added to a rational ratio always produces and yields a rational and never an irrational, although a rational, having had any irrational added to it, would always compose an irrational ratio.

Proposition II. If there should be no mean proportional number or numbers between the prime numbers of some rational ratio, such a ratio cannot be divided into several equal rational ratios and, consequently, no rational ratio is an aliquot part of it.

If this were not so, then let $A$ be the given ratio whose prime numbers are $G$ and $H$, but which is designated by two [other] terms $D$ and $F$. Let
rationalem constituat $V$
si licet $C H R \operatorname{sed} E /$ ante rationalis $b a b$ $H$ alium(?) $/$ rationalis $H$ irrationalis $C E R$ / irrationali $C H R$ rationali $E$
$\varsigma_{1-\xi 2}$ cumque irrationalem $E$ quandoque rationalem $C R$ rationalem cumque $H$
53 Secunda conclusio $V$; om $C$ et Secunda conclusio $m g$ hab E ante irrationalem (linea 52) et tertia conclusio mg hab H ante rationalis (lineae 53-54) et secunda conclusio mg hab $R$ post componat
(linea f2) $^{2} /$ duos om $V /$ minores $H V$; om $E$ umos(?) $C R$
54 post loco hab $E$ positus(?)
ss potest: potuit(?) $E$ / proportiones HRV; om $C E /$ rationales: rationale $V \mid$ rationales equales $\operatorname{tr} R /$ et om $V \mid$ propter: per $C$
57 A proportio data $H R V$ proportio data A $C$ A ut proportio data $E$
58 H om $E$ / signeturque A CHV signetur autem que $E$ signetur A $R$
$D$ et $F$. Dividaturque $A$ proportio in duas proportiones equales per 6o $E$ medium assignatum et sint ille proportiones $B$ et $C$ ita quod proportio $D$ ad $E$ sit $B$ et proportio $E$ ad $F$ sit $C$.
Cum ergo per adversarium utraque sit rationalis, scilicet $A, B, C$, sequitur quod $D, E, F$ sunt sicut tres numeri continue proportionales per quintam decimi. Sunt igitur aliqui tres numeri continue proportionales et proportio extremorum est proportio data, scilicet $A$, inter quos est numerus medio loco proportionalis. Igitur inter aliquos numeros relatos in proportione $A$ est numerus medius proportionalis, ergo et inter quoslibet in eadem proportione relatos. Ista ultima consequentia tenet per octavam octavi. Sed $G$ et $H$ sunt aliqui numeri
70 in proportione $A$ relati quia primi per positum, igitur inter $G$ et $H$ est numerus medius proportionalis cuius oppositum ponebatur, ergo $A$ proportio non potest dividi in duas rationales equales.
Eodem modo arguitur quod non dividatur in tres, nec in quatuor, nec in quinque, et cetera, assignando plura media inter eius extrema 75 quia illa media et illa extrema essent sicut numeri continue proportionales. Et ita inter aliquos numeros relatos in proportione $A$ essent plures numeri medii proportionales, igitur inter primos essent totidem quod est contra positum. Et per octavam octavi patet que est ista: si inter duos numeros numeri quotlibet in continua proportione ce-
so ciderint totidem inter omnes in eadem proportione relatos cadere necesse est. Unde sequitur quod nulla proportio rationalis est pars aliquota alicuius proportionis rationalis inter cuius primos numeros non fuerit numerus medius proportionalis vel numeri. Propter quod dicitur in commento octave septimi et dicit Jordanus in commento
$99 \mathrm{D}: \mathrm{E} R /$ Dividaturque: dividatur igitur $E$ / A proportio $\operatorname{tr} R$ / per: in $R$
60 E om $H$ / assignatum: assignandum $E / \operatorname{sint} C H V$ sit $E R$ / ille $E H V$ iste $C R / \mathrm{et}^{2}$ om $H / \mathrm{Com} C$
61 sit ${ }^{1}$ rep $E /$ sit Com $E / C: E C$
62 post A add CEV et / post B add CERV et
63 post quod bab V de / post D add CERV et / post E add CERV et / sunt: sint H/sicut om $R$
64 tres numeri $\operatorname{tr} C$
66 medio: medius $E$ / loco rep $V$ 66-67 Igitur... proportionalis om $V$ 67 ante A bab H est 68 et $C H R$; om $E V$

69 ante octavam scr et del $H$ octavam 1 post octavi add E Euclidis / post Sed scr et del $E$ verba illegibilia
72 duas: tres $E$ / ante rationales add $R$ proportiones / rationales: proportiones $C$
73 dividatur $H$; om $E$ potest dividi $C$ dividitur $R$ dividi $V$
74 et cetera $E R V$; om $C H$ / plura: talia H
75 illa ${ }^{1} C H V$ nulla $E$ ista $R /$ et illa extrema $H$; om $E$ et extrema $C R$ extrema $V$
77 medii om $H$ / igitur om $V$ | post igitur add EV et
78 quod: que $C /$ per... patet $H$ patet per
ratio $A$ be divided into two equal ratios, $B$ and $C$, by an assigned mean $E$, such that ratio $D$ to $E$ is $B$ and $E$ to $F$ is $C$.
Since, therefore, by the argument each ratio, namely $A, B$, and $C$, is rational, it follows by the fifth of the tenth [book of Euclid] that $D, E$, and $F$ are related as three continuously proportional numbers. Thus some three numbers are continuously proportional where the ratio of extreme terms is the given ratio, namely $A$, between which there is a mean proportional number. Hence between some numbers related as ratio $A$ there is a mean proportional number, and, consequently, between any numbers related in the same ratio there is a mean proportional number. This last consequence is supported by the eighth of the eighth [book of Euclid]. Now $G$ and $H$ are numbers related as ratio $A$ because by hypothesis they are prime numbers of $A$. Consequently, between $G$ and $H$ there is a mean proportional number, the opposite of which was assumed, and thus ratio $A$ cannot be divided into two equal rational ratios.
In the same way it can be argued that by assigning more means between its extremes, ratio $A$ is not divisible into three, four, five, etc. [equal rational ratios], since those means and extremes would be related as continuously proportional numbers. Thus if there were several mean proportional numbers between any numbers related as ratio $A$, there should also be just as many between the prime numbers of ratio $A$, which is, however, contrary to what has been assumed. This is evident by the eighth of the eighth [book of Euclid], which is as follows: If any numbers should fall in continuous proportion between two numbers, then it is necessary that just as many numbers fall between all other numbers related in the same ratio. From this it follows that no rational ratio is an aliquot part of any rational ratio between whose prime numbers there is no mean proportional number or numbers. For this reason it is stated [both] in the comment on the eighth [proposition] of the seventh [book of Euclid] and by Jordanus in a com-
octavam octavi $C R$ per patet octavam
octavi $E$ contra octavam octavi $V /$ ista om $R$
79 quotlibet $C H R$ quodlibet $E V /$ proportione: proportionalitate $R$
80 ante inter add $R$ et
80-82 relatos... rationalis: proportionales E

82 aliquota rep $V$
83 numerus medius $\operatorname{tr} V /$ medius om $E /$ quod $H R$ quid $C E V$
84 dicitur $H R$ ducitur $C$ ducatur $E$ dictum(?) V | post octave scr et del V aut etiam / ante septimi bab $V$ dicit / in ${ }^{2}$. HV; om CER
$8_{5}$ Arismetice sue quod nulla proportio superparticularis potest dividi per medium et intellege in rationales proportiones.

Tertia conclusio. Si aliqua quantitas in due inequalia dividatur quorum quodlibet sit pars eius aut partes illa duo sunt sicut duo numeri minimi. Unde manifestum est quod si minus subtrabatur a maiori et residuum, si fuerit, a
so minori et sic ultra tandem erit devenire ad aliquid quod erit pars utriusque, dividentium et divisi.

Istam conclusionem pono propter sequentem ut aliqua que dicam levius intelligantur. Notandam ergo quod si aliqua quantitas dividatur in duo quorum quodlibet sit pars illa sunt equalia et econverso quia
95 sunt due medietates. Si , vero, in duo inequalia dividatur tunc si illa sint commensurabilia unum est partes totius reliquum, vero, est pars sive partes, et similiter si in tres vel in quatuor, et cetera.

Pro conclusione demonstranda pono nunc tres suppositiones.
Prima est omnis quantitas que est alterius pars vel partes duobus
100 numeris signaretur quorum unus dicitur numerator et alter denominator ut patet ex commento sexte septimi. Et hii numeri quidem sunt contra se primi et in sua proportione minimi sicut dicimus tres quinte. Unde si sunt minimi sunt primi per $22^{\mathrm{am}}$ septimi et econverso per $23^{\mathrm{am}}$ eiusdem.
${ }^{105}$ Secunda est si ab aliqua quantitate dematur aliquid quod sit pars aut partes eius residuum erit similiter pars aut partes eius habens

85 ante Arismetice scr et del $V$ ars / superparticularis: superorum(?) $R$
86 et om $C$ / rationales proportiones HRV; $\operatorname{tr} C E$
87 Tertia conclusio V; om $C$ et ante Si mg hab $E$ conclusio tertia et $R$ tertia conclusio et post illa (linea 88) mg hab $H$ quarta conclusio / due inequalia: duo equalia $R$
88 quodlibet: quelibet $V$ / eius aut partes: aut partes eius $C$ / illa: isti $R$ / sicut CEV; om $H R /$ duo om $R$ / numeri: termini $V$ | post Unde scr et del $V$ ma
89 manifestum: medium $C$ /subtrahatur: abstrahatur $R$ / si fuerit om $H$
90 minori: maiori $H /$ devenire: deveniendum $E$ /aliquid $H$ aliquod $C E R V$ / quod: que $E$ / erit pars $t r R$
91 dividentium: dividendum $V$
92 ante propter scr et del $V \mathrm{pp} /$ propter:
per $R$ / ante ut add $C E V$ et
92-93 que dicam levius $H$ dicenda levius $C V$ dicenda melius $E$ levius dicenda R
93 intelligantur HRV intelligant $C E$ ! Notandum: nota $C /$ dividatur: dividitur $C$
94 quodlibet: quelibet $E$ / ante pars scr et del $V$ pars
95 inequalia: equalia $C$ / dividatur: dividitur $C$ / tunc si $\operatorname{tr} E$
96 ante sint bab $E$ duo / sint: sunt $V$ | partes: pars $R$
97 sive: vel $C$ / et similiter $C H V$ similiter sed $E$ et consimiliter $R /$ si om $R /$ in tres: tria $H$
98 pono: ponam $R /$ nunc $E H V$; om $C R$ | post suppositiones hab $H$ principia(?) 99 post Prima $m g$ bab $H$ tres suppositiones / est ${ }^{1} E R V$; om $C H /$ vel: sive $R$
ioo signaretur: signatur $C$ | et om $H$ |

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ment in his Arithmetic that no superparticular ratio can be divided by a mean-by which you should understand divided into tational ratios.

Proposition III. If any quantity were divided into two unequal parts and one of them is a part or parts of that quantity, those two parts are related as two numbers in their least terms. It is then obvious that if the smaller part were subtracted from the greater part, and the remainder-should there be one-subtracted from the smaller term, and so forth, something will finally be reached which will be part of each of the divisors and the dividend.

I propose this proposition so that some things which I shall discuss later may be more readily understood. It should be observed that if any quantity were divided into two parts any of which is a part, then these are equal parts, and conversely, since they are two halves. If, however, they are divided into two unequal parts which are commensurable, then one is parts of the whole and the other is a part or parts; and this holds similarly for division into three parts, or four, etc.

I now assume three suppositions in order to demonstrate the proposition.

The first is: Every quantity which is a part or parts of another quantity can be assigned two numbers, one of which is called the numerator, and the other the denominator.* This is clear from the comment on the sixth [proposition] of the seventh [book of Euclid]. And, furthermore, these numbers are prime to each other and in their least ratio, as [for example], $3 / 5$. Indeed, if they are in their least ratio they are prime to each other, by the twenty-second proposition of the seventh [book of Euclid]; and conversely, by the twenty-third of the same book.

The second is: If from some quantity something should be taken which is a part or parts of it, the remainder will likewise be a part or parts of it, having the same denominator as the quantity which was initially taken

* For example, $(A)^{1 / s}$ is part of $A$; $(A)^{3 / s}$ is parts of $A$. Numerator and denominator refer to the exponent.

[^19]eandem denominatorem cum eo quod a principio demabatur. Verbi gratia, 6 de to est tres quinte et residuum, scilicet 4 , est due quinte. Et hoc habetur ex octava septimi.

Tertia est omnis quantitas in duo divisa quorum unum sit pars aut partes illa duo partialia sunt sicut numeri numeratores eorum. Unde proportio $2 / 5 \mathrm{ad}^{3} / 5$ est sicut proportio duorum ad tria, et econverso. Hoc etiam satis habetur ex octava septimi et ex quinto Euclidis.
Hiis positis conclusio proposita demonstraretur. Et sit $A$ quoddam totum divisum per inequalia in $B$ maius et $C$ minus quorum unum est pars aut partes. Tunc per secundam suppositionem reliquum est etiam pars aut partes et per eandem idem est numerus denominans $B$ et denominans $C$. Sit itaque ille numerus $D$, et numerus qui numerat $B$ sit $E$ et numerus qui numerat $C$ sit $F$. Tales namque numeros oportet ponere sicut patet ex prima suppositione.

Tunc arguitur sic: $E$ et $D$ sunt contra se primi per primam suppositionem quia $D$ est denominator et $E$ est numerator, similiter $F$ et $D$ sunt contra se primi per eandem. Ergo $E$ et $F$ sunt contra se primi per secundam partem $29^{e}$ septimi que dicit sic: si numerus acervatus 125 ex duobus ad utrumque fuerit primus et illi erunt primi. Modo $D$ est acervatus ex $E$ et $F$ et est primus ad utrumque eorum ut probatum est, igitur $E$ et $F$ sunt primi. Sed $B$ et $C$ sunt sicut $E$ et $F$ per tertiam suppositionem, igitur $B$ et $C$ sunt sicut duo numeri contra se primi et minimi quod principio propositum. Igitur per primam septimi et per primam decimi per subtractionem $C$ minoris a $B$ maiori et iterum

107 denominatorem $E H R$ denominationem $C V /$ demabatur $E R V$ demabitur $C$ denominabatur $H$
108 post 6 bab $E$ et / estri $E H$ sunt $C R$ rep $V \mid$ scilicet $4 C R V 4 E$ illa(?) $H \mid$ est ${ }^{2}$ : sunt $C$
no divisa $H R V$ divise $C E /$ unum om $R$ 110-II aut partes om $E$
111 illa: ista $R$
112 ante $2 / 5$ hab $E$ scilicet $/ 2 / 5$ ad $3 / 5$ CHR $2 / 53 / 5 E$ secunda ad tertiam $V /$ tria HRV tres CE / post econverso add $V$ et
${ }_{11} 3$ Hoc $H R$ hec CEV / habetur: patet $H$ | septimi $C E R$; om $V$ octavi $H$ | et: aut $V$
114 proposita $C H V$ posita $E R / \mathrm{Et}$ : quod $4{ }_{E}$

IIs in: et $R / C$ : in $H$
$116 \mathrm{est}^{\mathrm{t}}$ : erit $C$
116-17 est etiam $E H V ; \operatorname{tr} C R$
117 ante pars add $H$ aut / est numerus $t r R /$ denominans: denominantur $E$
118 C om $V$ / itaque: que C/ille $E H V$ iste $C R$
119 numerat om $H$
I2O patet...suppositione: ex prima sup${ }_{R}^{\text {positione patet } E / \text { patet } C H V \text { apparet }}$ R
121 arguitur sic: maior sit $V / E \ldots$ se $V$ C et D sunt numeri contra se $C$ E et D sunt contra $E$ contra $H \mathrm{E}$ et D sunt numeri contra se $R$ / post sunt scr et del $V \mathrm{~s}($ ? $)$
$122 \mathrm{E} H R V \mathrm{C} C E /$ post $\mathrm{D}^{2}$ hab $C$ et F 123 post sunt ${ }^{\mathrm{I}}$ add $C$ numeri / E: D C / se ${ }^{\mathbf{2}}$

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from the whole. For example, taking six parts from ten is $3 / 5$, and the remainder, namely four parts, is $2 / 5$. ${ }^{*}$ This is shown in the eighth [proposition] of the seventh [book of Euclid].

The third is: The two parts of any quantity that has been divided in two, where one of them is a part or parts of the whole, are related as the numbers representing their numerators. Thus a ratio of $2 / 5$ to $3 / 5$ is as a ratio of 2 to 3 , and conversely. + This is supported by the eighth of the seventh and from the fifth [book] of Euclid.

These suppositions being stated, the proposed proposition must now be demonstrated. Let $A$ be a certain whole which has been divided into unequal parts with $B$ the greater and $C$ the lesser, and one of them is a part or parts of the whole. Then, by the second supposition, the remaining part is also a part or parts; and, by the same supposition, the same number denominates $B$ and $C$. Let this number be $D$, and let $E$ be the number which numbers $B$, and $F$ the number which numbers $C$, for, indeed, one is permitted to assign such numbers by the first supposition.

Then the argument proceeds as follows: $E$ and $D$ are prime to each other by the first supposition, because $D$ is the denominator and $E$ the numerator; likewise, by the same supposition, $F$ and $D$ are prime to each other. Therefore, $E$ and $F$ are prime to each other, by the second part of the twenty-ninth proposition of the seventh [book of Euclid], which says: If a number constituted from two numbers should be prime to each of them, then the two numbers from which it is constituted will be prime to each other. In this way $D$ is constituted from $E$ and $F$ and is prime to each of them, as was shown, and consequently $E$ and $F$ are [mutually] prime. But, by the third supposition, $B$ and $C$ are related as $E$ and $F$; therefore $B$ and $C$ are prime to each other and in their least terms, which was posited at the outset. Therefore, by the first proposition of the seventh [book of Euclid] and by the first of the tenth [book], by the subtraction of $C$, the lesser term, from $B$, the greater, and then the remainder [from the smaller

$$
\begin{aligned}
& * \text { Taking }(A)^{3 / s} \text { from } A \text { leaves }(A)^{2 / s} \text { since } \quad \dagger \text { That is, }(A)^{2 / s}=\left[(A)^{3 / s}\right]^{2 / 3} \text {. } \\
& A=(A)^{1 / s} \cdot(A)^{2 / s} \text {. }
\end{aligned}
$$

CRV; om EH
124 septimi: septime $V /$ que: qui $R$
12s duobus: duabus(?) $E$ / illi: illius(?) $E$ / erunt: essent $C$
126 ante acervatus bab $H$ quo / acervatus EHV coacervatus $C R / \mathrm{E}$ et F : F et E $E$

127 post primi scr et del $V$ et numeri(?) quod est contra
128 B et C : C et $\mathrm{B} E /$ se $C R V$; om $E H$ 129 post quod bab $V$ est contra / propositum $H$ positum $C E V$ positum est $R$ 130 minoris om $E$
residuum, et cetera, tandem erit devenire ad aliquid quod erit sicut unitas respectu utriusque. Et omnis numeri pars est unitas sicut docet una suppositio septimi ex quo patet secundo propositum, scilicet quod per talem detractionem tandem devenietur ad aliquid quod erit pars eorum est sicut unus numerus, et cetera.
Quarta conclusio. Si inter numeros primos alicuius proportionis non fuerit numerus medio loco proportionalis seu numeri nulla proportio rationalis est partes aliquote ipsius.

Si non est ita, sit $A$ proportio rationalis talis cuius $B$ proportio rationalis sit partes aliquote et residuum, quod cum $B$ componit $A$, sit $C$. Tunc per secundam suppositionem precedentem $C$ est pars aut partes ipsius $A$. Et qualitercumque sit necesse est ut $C$ sit proportio rationalis, aliter enim $A$ componeretur ex $B$ rationali et $C$ irrationali quod est impossibile per primam conclusionem.
Si igitur $C$ sit pars $A$, et iam probatum est quod $C$ est proportio rationalis, ergo aliqua proportio rationalis est pars ipsius $A$ inter cuius numeros, scilicet $A$, nullus est numerus medius, et cetera, quod est impossibile per secundam conclusionem.
Si, vero, dicatur quod $C$ est partes ipsius $A$, sicut $B$, tunc $B$ est maius $C$ aut econverso quia si essent equalia iam utrumque esset pars. $A$, scilicet medietas et non partes quod est impossibile per secundam conclusionem.
Sit igitur $B$ maius et $C$ minus. Igitur si minus subtrahatur a maiori ${ }_{153}$ deinde residuum, si fuerit, a minori quotiens potest et cetera, tandem deveniretur ad aliquid quod erit pars ipsius $A$ et pars $B$ et pars ipsius $C$ per precedentem conclusionem. Cumque facta fuerit prima detractio aut remanebit tesiduum aut non. Si nullum est residuum ergo $C$ erat
I3I residuum $C R V$; om $E$ residui $H$ |ante tandem add $R$ et / tandem: eandem $C$ / erit' : et $C /$ devenire: deveniendum $E$
132 numeri $H R$ numerus $C E V$
133 septimi $H R V$ ibi $E$ septimo $C /$ patet $E H V$ apparet $C R /$ secundo: duo $V /$ scilicet: videlicet $R$
134 per: secundum $V /$ erit $C H R$ est $E$ esset $V$
135 ante A scr et del $\mathrm{EAB} / \mathrm{et}^{2}$ om $E$ |post B add $C V$ et / quia $H R V$ et $C$ que $E$ 136 et cetera CHV; om ER
${ }_{137}$ Quarta conclusio mg hab E ante Si et $R$
post numerus (linea 136); om CH quinta conclusio $V /$ post fuerit hab $H$ proportio
I38 seu: sive $C$ / proportio om $V /$ est om H
139 partes aliquote $E H V$ pars aliquota $C R /$ ipsius: ipsorum $R$
140 post A bab $E$ ut
140-41 proportio rationalis: per rationa$\operatorname{lem} E$
141 quod om $H / \mathrm{A} H R$; om CEV
143 est om $E$ | post ut scr et del $V$ s
146 est $^{2} H R V$ sit $C$

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term], etc., something will finally be reached which will function as a unit with respect to each of these. Now every part of a number is a unit, as a supposition of the seventh book [book of Euclid] shows, so that what was proposed in the second part of this proposition is clear, namely that by such a subtraction something will finally be reached that will be part of each of the divisors and the dividend, namely $A, B$, and $C$, since any of them is like a number, etc.

Proposition IV. If there is no mean proportional number or numbers between. the prime numbers of some ratio, no rational ratio is aliquot parts of it.

If this is not so, let $A$ be a rational ratio such that $B$, a rational ratio, is aliquot parts of it; and let the remainder be C , which with $B$ composes $A$. Then, by the preceding second supposition, $C$ is a part or parts of $A$. But be that as it may, it is necessary that $C$ be a rational ratio, for otherwise $A$ would be composed of $B$, a rational, and $C$, an irrational, which is impossible by the first proposition.

Now if $C$ should be a part of $A$-and it has already been proven that $C$ is a rational ratio-then some rational ratio is a part of $A$ between whose numbers, namely $A$ 's, there is no mean number, etc., which is impossible by the second proposition.

If, however, it were held that $C$, like $B$, is parts of $A$, then $B$ is greater than $C$, or the converse; for if they should be equal then each would already be part of $A$, namely half of $A$, and not parts, which is impossible by the second proposition.

Let $B$ be greater than $C$. Then, if the lesser is subtracted from the greater, and the remainder-should there be one-subtracted from the lesser as many times as possible, etc., something should ultimately be reached which will be a part of $A$, part of $B$, and a part of $C$, by the preceding proposition. After the first subtraction has been made, there will either be a remainder or not. If there is no remainder, then $C$ was part of $B$ and, consequently,

146-49 Si...conclusionem om $E$
147 ante aliqua bab $V$ aliquid / est $H$ erit $C R$ esset $V /$ pars om $V$
$148 \mathrm{~A} C H$; om $R V /$ et cetera CHV etiam $R$
1so dicatur: dicitur $V /$ partes: pars $C$ / sicut $H$ sicuti et $C E V$ sicut et $R \mid$ tunc om $E$
is I essent $E H V \operatorname{sint} C R /$ esset $H R V$

## essent $C E$

IS 2 ante scilicet bab $E$ ut / scilicet: si $C$
is 4 maius om $E /$ minus $^{2}$ : verus(?) $E /$ subtrahatur: subtrahitur $C$
iss post si add $R$ necesse / minori: maiori $C$ / potest om $C$ / cetera om $R$
158 est: erit $E$ / est residuum om $H$ | post est scr et del $V$ si
pars ipsius $B$, igitur $C$ est pars ipsius $A$. Patet ultima consequentia :60 quia ex quo: $A$ componitur ex $B$ et $C$ sequitur quod quotiens $C$ reperitur precise in $B$ totiens $C$ reperitur in $A$. Et cum hoc una vice et sic $C$ aliquotiens replicatum precise reddit $A$ et per consequens est pars eius.

Item si nullum sit residuum ergo deveniremus ad aliquid quod est pars utriusque dividentium et divisi iuxta doctrinam precedentem et illud est $C$ quod detrahimus. Igitur $C$ est pars $A$ et iam probatum est quod $C$ est proportio rationalis ergo aliqua proportio rationalis est pars $A$ quod est impossibile per secundam conclusionem.

Si autem residuum fuerit sit illud $D$. Ergo sicut arguebatur de $C$. Oportet quod $D$ sit proportio rationalis quia aliter $B$, proportio rationalis, componeretur ex uno vel pluribus $C$ rationalibus, et $D$ irrationali quod est impossibile per primam conclusionem. Aut igitur $D$ est pars utriusque dividentium et divisi, scilicet $A$ et $B$ et $C$ ita quod non oportet ulterius facere aliquam detractionem quod si con-
175 cedatur iam habetur iterum contra secundam conclusionem quia aliqua proportio rationalis esset pars ipsius $A$.

Si negetur et dicatur quod oportet adhuc detrahere, detrahatur tunc $D$ rationalis a $C$ rationali quotiens potest. Et si non fuerit residuum $D$ erat pars utriusque dividentium et divisi quod prius est improbatum.

Si vero fuerit residuum oportet sicut prius per primam conclusionem quod illud sit proportio rationalis quia aliter $C$ rationalis componeretur ex rationali et irrationali. Et si illud residuum sit pars utriusque et cetera, scilicet $A$ et $B$ et $C$, hoc est sicut prius contra secundam conclusionem. Si sit partes detrahantur ab ipso $D$ et semper
185 oportebit quod residuum sit proportio rationalis sicut poterit semper probari per primam conclusionem. Et quia per precedentem in huius
159 B HRV; omCAE/igitur...ipsius A om $E$
160 ex quo om $E$
160-61 C reperitur $\operatorname{tr} C$
${ }_{161}$ reperitur ${ }^{\text { }} E H R$ reperietur $V /$ reperitur ${ }^{2} H R$ reperiatur $C E V$
162 sic om $V / \mathrm{C}$ : si $C / \mathrm{A}$ om $V$
163 eius: ipsius $C$
164 Item: iterum(?) $V /$ nullum: nullo(?) $H /$ aliquid: aliquod $R /$ est: erit $C$ 165 precedentem $H$ precedentis $C E R V$
166 illud: id $C$ / detrahimus: protrahimus (?) $R \mid$ ante iam add $R$ ita

169 sit: sicut $E$ / ante illud bab $R$ igitur / Ergo: sequitur $E /$ arguebatur: oportebatur $C$
170 ante aliter scr et del $V$ aliter(?) / B: D C
171 componeretur: componitur $V /$ rationalibus $C E$ (?) $H R$ rationali $V$
173 dividentium: dividendium $V$ et ${ }^{1}$ : seu R
174-75 concedatur: conceditur $R$
175 iterum $H R V$; om $C$ utrumque $E$ 176 esset: erit $V$
178 a: et $V /$ potest: fit $R / \operatorname{si}$ om $V$
179 est improbatum $H R V$; $\operatorname{tr} C$ erat pro-

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$C$ is part of $A$. The last consequence is obvious for this reason: [since] $A$ is composed of $B$ and $C$, it follows that $C$ is found a certain number of times exactly in $B$ [and] a certain number of times in $A$. Now when $C$ is multiplied a certain number of times and exactly produces $A$ in the very first try, it must, consequently, be a part of $A$.

Thus, according to what has just been taught, if there were no remainder we would arrive at something that is part of each of the divisors and the dividend, and this would be $C$, which we subtracted. Therefore $C$ is part of $A$-and it has already been proven that $C$ is a rational ratio-so that some rational ratio is a part of $A$, which is impossible by the second proposition.

If, however, there should be a remainder, let it be $D$. Then the argument proceeds just as it did with $C$. It is necessary that $D$ be a rational ratio, for otherwise $B$, a rational ratio, could be composed of one or more rational ratios [equal to] $C$, and of $D$, an irrational, which is impossible by the first proposition. Therefore $D$ is a part of the divisors and the dividend, namely $A, B$, and $C$, so that it is unnecessary to carry out another subtraction; but if this is granted it is again contrary to the second proposition, since some rational ratio would be part of $A$.

If, however, it were denied [that $D$ is a part of $A, B$, and $C$ ], and it is held that another subtraction is necessary, then let $D$, a rational, be subtracted as many times as possible from $C$, a rational. Then if there is no remainder, $D$ was part of the divisors and dividend, which was disproved before.

But if there is a remainder, it is necessary, as before by the first proposition, that it be a rational ratio; for otherwise $C$, a rational, would be composed of a rational and irrational. And if that remainder were part of each, etc., namely $A, B$, and $C$, this, as before, is contrary to the second proposition. If the remainder were parts, they are subtracted from $D$, and it will always be necessary that the remainder be a rational ratio, which can be proved by the first proposition. Now since, by the preceding proposition,

[^20]184 detrahantur $H$ detrahatur $C E R V$
185 ante residuum scr et del $V$ s
189-86 poterit semper probari $E H V$ probari (?) poterit $C$ poterit probari $R$ 186 in: et $V$ / huius: hiis $R / p o s t$ huius add $C E V$ modi
detractionibus non proceditur in infinitum sed erit devenire ad aliquid quod erit pars ipsius $A$ et cetera, et per primam conclusionem probabitur semper quod illud erit proportio rationalis, oportebit tandem 90 concedere quod aliqua proportio rationalis erit pars ipsius $A$ quod est contra secundam conclusionem. Eodem modo arguetur si $C$ ponitur maius et $B$ minus.

Patet itaque qualiter ex tribus primis conclusionibus deducitur quarta quoniam per tertiam habetur quod per continuam detractionem cetera, et in qualibet detractione convincitur per primam conclusio cetera, et in qualibet detractione convincitur per primam conclusionem quod remanens est proportio rationalis, igitur in ultima illud residuum quod erit pars, et cetera, erit proportio rationalis quod est impossibile per secundam conclusionem.
200
Et hoc habetur: posito quod aliqua proportio rationalis sit partes aliquote alicuius rationalis inter cuius primos numeros nullus fuerit numerus medius seu numeri medii igitur impossibile est quod aliqua proportio rationalis sit partes alicuius talis quod est propositum.
Et ut facilius videatur ponatur exemplum in numeris quia si $B$
${ }_{205}$ proportio sit partes $A$ utraque est ut numerus per quintam decimi. Sit igitur $B 3 / 5$ ipsius $A$ et tunc necesse per precedentem quod $C$ sit $2 / 5$. Subtrahendo, igitur, $C$ ab $B$ remanet $1 / 5$ que est pars ipsius quia est $1 / 5$ ipsius $A$. Et de ista parte arguatur sicut prius et sic non sit nisi semel detractio.
Si autem $B$ fuerit $8 /$ II ipsius $A$ tunc $C$ erit $3 /{ }_{\text {II }}$. Subtracto, igitur, $C$ a $B$ quotiens potest remanet $2 / 1$. Et iterum isto residuo subtracto

187 detractionibus $H R V$ detrahentibus $C$ tractionibus $E$ / post sed bab $E$ si(?) / erit devenire: esset denegare $V$
188 ipssius om $R / \mathrm{A}$ om $V / \mathrm{et}^{2}$ om $H$
189 illud: id $C$ / post illud rep $V$ quod erit $C H R$ est $E$ esset $V /$ tandem $C E H$ teandem $V$ tamen $R$
190 ante concedere scr et del $V$ conocedr pars: proportio $R /$ pars ipsius $E H V$ proportio ipsi $C$
191 ante Eodem add $V$ et / arguetur $C H$; om $V$ arguebatur $E$ argueretur $R /$ si Com $V$
191-92 C ponitur $H$ ponatur C $E$ C ponetur $C$ poneretur $R$ ponatur $V$
193 post Patet add $R$ igitur / tribus primis
$\operatorname{tr} V /$ deducitur: distincter(?) $R$
194 ante per ${ }^{1}$ hab $E \mathrm{p}$
195 ad...pars om $C$
196 detractione: detractionibus $E /$ convincitur: convinctum $V$
197 remanens $H R$ remanebit $C E$ remanet $V /$ rationalis om $V /$ igitur: sicut $C /$ illud: id $C$
198 pars om $H$
200 hoc $C H R$ huius $E$ hic $V /$ habetur $H$ sequitur $C E R V /$ posito: ponendo $E$ / quod aliqua: aliter(?) $V$ / ante partes hab $R$ pars sive
201 aliquote: equote(?) $V /$ cuius: quos $R$ 202 seu: sive $C$
203 talis $C H V$; om $R$ tales $E$

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the subtractions do not proceed to infinity but arrive at something which will be a part of $A$, etc., and [since], by the first proposition, it was shown that this remainder will always be a rational ratio, it will be necessary, finally, to concede that some rational ratios will be part of $A$, and this is contrary to the second proposition. One would argue in the same way if $C$ is assumed greater than $B$.
It is clear, therefore, how the fourth proposition is deduced from the first three propositions. By the third proposition one sees that a continuous subtraction of the smaller ratio from the greater results, finally, in something which is a part, etc.; and by the first proposition it is demonstrated that in any subtraction what remains is a rational ratio, from which it follows that in the final subtraction that remainder, which will be a part [of all the previous divisors and dividends], will be a rational ratio, which is impossible by the second proposition.

And [finally] this is what has been shown: it having been assumed that some rational ratio can be aliquot parts of some rational ratio that has no mean number or numbers between its prime numbers, [it was] then [shown that] it is impossible for some rational ratio to be aliquot parts of such a rational ratio, and this is what was proposed.
In order for this to seem easier, I offer an example formulated in numbers, because if ratio $B$ is parts of $A$ then each [i.e., $A$ and $B]$ can be treated as a number, by the fifth [proposition] of the tenth [book of Euclid]. Let $B$ be $3 / 5$ of $A$, then, by the preceding proposition, it is necessary that $C$ be $2 / 5$ of $A$. By subtracting $C$ from $B, 1 / 5$ remains, which is part of $A$ because it is $1 / 5$ of $A$.* With regard to this part, one may argue as before, so that [in this example] there is only one subtraction.

However, if $B$ should be $8 /{ }_{11}$ of $A$, then $C$ will be $3 /{ }_{11}$. After subtracting $C$ from $B$ as many times as possible, ${ }^{2} /$ II $^{\text {II }}$ remains. Subtracting $2 /$ II from $C$

* See p. 340 .

205 numerus: numero $C$
$2063 / 5: 2 / 5 E$ / et om $E /$ per: pre $E$
$207^{2} /{ }_{5}: 3 / 5 \mathrm{~V} /$ Subtrahendo $C H$ subtrahatur $E R V / \mathrm{ab}$ : a $E /$ remanet: remanebit $E / 1 /{ }_{5} C R V^{2} / 5 E H /$ que: qui $H$ / quia $C E V$ que $H$
207-8 ipsius quia est $1 / 5$ om $R$
$2081 / 5 E V H^{2} / 5 C$ / parte: partem(?) C /
arguatur $H R V$ arguebatur $E$ arguitur $C /$ sic om $V /$ non om $E$
210 post fuerit scr et del $V 8 / 8 /{ }_{11} C R 8 / 5$ $H V$ undecimo $E\left|3 /{ }_{11} C E R 3 / 5 H V\right|$ igitur om $R$
211 remanet: remanebit(?) $H / 2 / 1$ II $E 2 / 5$ $H V$ una decima $C$ illa $\mathrm{II}^{\mathrm{a}} R$
a $C$ quantum potest remanet ${ }^{1} /$ II que est pars ipsius $A$ de qua arguitur sicut prius.
Et ita bis sit detractio quandoque vero ter, quandoque quater, et cetera, sed semper devenietur ad aliquid quod erit pars ipsius, sicut dictum est, et deducetur per primam conclusionem quod illud est proportio rationalis quod, tamen, arguetur esse impossibile per secundam ut visum est supra.

Sequitur itaque ex hiis quod si aliqua proportio rationalis sit partes ${ }_{220}$ alterius rationalis ipsa est tales partes quarum quelibet est proportio rationalis. Si enim esset partes quarum quelibet esset proportio irrationalis, sicut oporteret nisi verum esset quod dictum est, tunc per modum detrahendi ante dictum deveniretur ad unam illarum que esset pars totalis proportionis quod arguetur sicut prius et hoc est contra 225 primam conclusionem.

Quinta conclusio. Si fuerit aliqua proportio inter cuius primos numeros nullus fuerit numerus medius proportionalis seu numeri illa erit incommensurabilis cuicumque minori rationali ea et cuilibet maiori que non est multiplex ad ipsam. Utraque pars buius copulative demonstratur deducendo ad impossi-

Sit enim $A$ talis proportio inter cuius primos numeros et cetera, que sit commensurabilis $B$ minori. Igitur utraque earum est sicut unus numerus per quintam decimi, igitur $B$ minor est pars aut partes maioris per quartam septimi. Sed quod $B$ sit pars $A$ est impossibile per se${ }_{235}$ cundam conclusionem. Quod vero sit partes $A$ est impossibile per
$2121 /$ II $C E R$ una quinta $H V$ / arguitur $C H R$ arguebatur $E$ arguatur $V$
214 quandoque ${ }^{\text {: }}$ : quando $H$ / post ter add $H$ et cetera
214-1s quandoque quater et cetera om $R$
215 erit $C H R$ est $E$ esset $V \mid$ ante sicut bab $R$ A
216 illud om $C$
217 arguetur $H R$ arguitur $C E$ argueretur $V$
217-18 per...supra om $R$
218 supra om $V$
219 itaque om $C$ / quod: que $E$ | ante rationalis scr et del $E$ ra
220 alterius: alicuius $H \mid$ ante rationalis add $V$ proportionis / post ipsa mg hab $H$ correlarium / tales: talis $V$
221 esset $^{2}$ : erit $V$

221-22 irrationalis: rationalis $V$
222 sicut $E V$; om $H$ si $C R /$ oporteret nisi $\operatorname{tr} H /$ oporteret $C R$ oportebit(?) $E$ oportum(?) $V /$ ante verum bab $H$ ubi/ esset: est $E$
223-24 illarum...pars: istarum que est $R$
224 totalis: rationalis $E$ / proportionis om $V /$ quod: autem $E /$ arguetur $E H V$ arguitur $C$ argueretur $R /$ hoc $E H R$ hec $C V$ / est contra $E H V$ est etiam per $C$ estiam est $R$
225 primam $C$ secundam $E H V$ per secundam $R$
226 Quinta conclusio mg hab ER post conclusionem (linea 225); om CH sexta conclusio $V$
227 seu om $V$
228 cuicumque $C E H$ cuiuscumque $V$
as many times as possible $1 / 1$ remains, which is part of $A$, and the argument about this part is as before.*

And thus a subtraction might be carried out twice, as often as three times, or four times, etc., but something must always be reached which will be a part [of the whole given ratio], as already stated; and, by the first proposition, it is deduced that that part is a rational ratio, which, nevertheless, must be impossible by the second proposition as seen above.

Futhermore, it follows from all this that if any rational ratio were parts of another rational, it would be such that any one of its parts is a rational ratio. For if any of its parts were an irrational ratio-and this could be granted if the truth of the matter were not as already stated-then by the method of subtracting, stated previously, one of those parts would be reached which would be a part of the whole ratio-and at this point the argument would proceed as before, [showing that an irrational ratio as a part] is contrary to the first proposition.

Proposition V. If there is no mean proportional number or numbers between the prime numbers of some ratio, that ratio will be incommensurable to any smaller rational, and to any greater rational ratio that is not multiple to it. Each part of this proposition is demonstrated by a reductio ad absurdum argument which serves to link the parts.
Let $A$ be a ratio that has [no mean proportional number or numbers] between its prime numbers, and that is commensurable to $B$, a smaller ratio. Then, by the fifth [proposition] of the tenth [book of Euclid], each of these ratios is like a number, so that $B$, the smaller ratio, is a part or parts of the greater, by the fourth [proposition] of the seventh [book of Euclid]. But, by the second proposition, $B$ could not possibly be a part of $A$. Furthermore, it cannot possibly be parts of $A$, by the fourth proposition. Consequently, $A$ is incommensurable to any smaller ratio. Thus the first

* See p. 340.
cuilibet $R /$ minori: minoris $H /$ non om $E$
229 huius om $C$ / demonstratur $E H R$ demonstrari $C$ demonstraretur $V /$ deducendo: ducendo $C$
231 A : aliqua $H$ / talis: totalis $C$ / cuius primos $C H V$; om $E$ quos primos $R /$ post numeros bab $R$ nullus

232 earum: eorum $R$
232-33 unus numerus $\operatorname{tr} R$
233 ante maioris scr et del $V$ maioris
234 septimi: secundi(?) $E /$ ante per $^{2}$ hab $H$ est
234-35 per $^{2}$...impossibile om $H$ | secundam $C E R$ tertiam $V$
quartam. Igitur $A$ est incommensurabilis cuicumque minori. Et sic patet primum et loquor semper de proportionibus rationalibus.
Secundum patet. Et sit $A$ proportio data et sit $C$ una proportio maior quam $A$. Si igitur $C$ sit commensurabile $A$ in alia proportione quam in multiplici tunc $C$ continebit $A$ semel aut plures et cum hoc aliquam eius partem vel aliquas eius partes ut notum est ex diffinitione proportionum in Arismetica Boetii.

Si igitur $C$ contineat $A$ aliquotiens et aliquam eius partem, sit illa pars $D$. Aut igitur $D$ est proportio irrationalis et hoc est impossibile
${ }_{245}$ per primam conclusionem, quia tunc $C$ rationalis componeretur ex $A$ rationali et $D$ irrationali, vel ex pluribus $A$ et $D$ irrationali; vel $D$ est proportio rationalis et hoc iterum est impossibile per secundam conclusionem quia nulla proportio rationalis est pars ipsius $A$.

Et si $C$ contineat aliquotiens $A$ et cum hoc aliquid quod sit partes ${ }^{50}$ ipsius $A$ sit illud $E$. Aut igitur $E$ est proportio irrationalis et hoc est contra primam conclusionem sicut prius, aut $E$ est proportio rationalis et hoc est impossibile per quartam conclusionem quia nulla proportio rationalis est partes ipsius $A$ nec alicuius similis. Et sic patet secundum, scilicet quod nulla proportio rationalis maior quam
a35 $A$ est commensurabilis ipsi $A$ in aliqua proportione que non sit de genere multiplici.

Sexta conclusio. Si proportio maior fuerit multiplex ad minorem ut dupla ad minorem aut tripla et cetera, tot media proportionalia secundum proportionem minorem erunt inter extrema maioris quotiens ipsa maior mi-
${ }_{260}$ norem continet uno dempto et totidem numeros medios secundum proportionem minorem inter primos numeros maioris necesse est inter esse.

Sit $A$ maior, $B$ vero minor. Cum igitur $A$ ponatur multiplex ad $B$, $A$ potest dividi in plura $B$ et hoc divisio fiet per medium seu mediorum assignationem secundum $B$ proportionem ita quod si sit

236 A om $E$ / incommensurabililis: in- $245 \mathrm{Com} H$ commensurabile $H$
237 primum: propositum $V$
238 Et om E / $\mathrm{sit}^{2}$ om E / sit $\mathrm{C} \operatorname{tr} V$
239 maior: data $R /$ alia $H R V$ aliqua $C$ addita(?) $E$
241 aliquam: aliqua $V$ / ante vel hab $H$ et cetera / vel: aut $C$
243 igitur $E H V$ enim $C R$ / contineat: continuat $V /$ aliquam: aliqua $V$
244 Aut om $R$ / igitur om $C$ / irrationalis om $E$

246 vel ex pluribus...irrationali om $R$ pluribus: partibus $C$ / ante A scr et del H A
247 iterum est $\operatorname{tr} R$
248 post proportio scr et del $V$ ra
249 A om $E$ / quod om $E$
$250 E^{2}$ : de $E /$ est $^{2}$ om $V$
251 aut: autem C | post aut scr et del HA est / E om $V$
$251-\varsigma 2$ ante rationalis scr et del $H$ irr(?)
252 post est bab $V$ rationalis

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part of this proposition is evident; and so far I have been speaking about rational ratios.
The second part of the proposition is also evident. Let $A$ be the given ratio and $C$ a greater ratio. If, therefore, $C$ is commensurable to $A$ in some ratio other than a multiple one, $C$, as is known from the definition of ratios in the Arithmetic of Boethius, will contain $A$ one or more times plus some part or parts of $A$.
Now if $C$ should contain $A$ several times plus some part of $A$, then call that part $D$. Then, either $D$ is an irrational ratio, which is impossible by the first proposition because then $C$, a rational, would be composed of $A$, a rational, and $D$, an irrational, or composed of several $A$ 's and $D$, an irrational; or $D$ is a rational ratio, and this again is impossible, by the second proposition, because no rational ratio is part of $A$.
But if $C$ contains $A$ several times plus something which is parts of $A$, then call it $E$. Now $E$ is either an irrational ratio, and this as before is contrary to the first proposition, or $E$ is a rational ratio, and this is impossible by the fourth proposition, since no rational ratio is parts of $A$ nor of any similar ratio. And so the second part of this proposition is clear, namely that no rational ratio greater than $A$ is commensurable to $A$ in any ratio of a non-multiple kind.
Proposition VI. If a greater ratio is multiple to a lesser ratio-i.e., double, or triple, etc.-there will be between the extremes of the greater ratio one less mean proportional than [the number of times] the greater ratio contains the lesser; and it is necessary that there be just as many mean numbers between the prime numbers of the greater [ratio as there are mean proportionals forming the lesser ratio between the extremes of the greater ratio, when expressed in terms that are not mutually prime].
Let $A$ be greater than $B$. Therefore, since $A$ is assumed multiple to $B$, $A$ can be divided into several $B$ 's, and this division could be made by assigning a mean or means which form ratio $B$ so that if there should be

253 alicuius: alicui $V$
254 ante secundum $s c r$ et del $V$ s/proportio om $H$
$259 \mathrm{~A}^{\mathrm{I}}$ : et cetera $V /$ est rep $H$
257 Sexta conclusio mg hab ER ante Si ; om CH septima conclusio $V$
258 dupla ad minorem: tripla $V$ / tot: tunc $R$ / post secundum scr et del $H$ rem(?)

259 erunt: essent $V$ inter om $E$
260 et totidem om $H$
261 numeros om $H /$ inter om $C$
262 post maior scr et del $E$ perbe(?) / B: proportio $E$
263 A om $C$ / hoc: hec $E /$ medium $E H$ medii $C R V$ / seu: vel $C$
264 post si scr et del $V \mathrm{t}$
${ }_{265}$ unum medium $A$ continebit bis ipsam $B$ et erit duplum ad $B$. Sed si sint duo media $A$ componetur ex tribus $B$ et erit triplum et cetera, per primam diffinitionem.

Sed quod totidem numeri et secundum eandem proportionem reperiantur inter primos numeros $A$ proportionis probatur quia inventis talibus mediis continue proportionalibus secundum $B$ proportionem inter aliqua extrema ipsius $A$, illa media et extrema ponantur in numeris postquam, $B$ est proportio rationalis sicut docet secunda octavi que est ista: numeros quotlibet continue proportionales secundum proportionem datam minimos invenire. Igitur proportio primi ad to undecime diffinitionis quinti et in principiis septimi. Sed talis proportio est $A$ que componitur ex pluribus $B$. Igitur inter aliquos numeros relatos secundum $A$ proportionem est numerus medius aut numeri medii secundum $B$ proportionem, igitur inter quoslibet in
${ }_{280}$ eadem proportione relatos per octavam octavi igitur et inter primos et minimos ipsius $A$ et hii sunt totidem quotiens $A$ maior continet $B$ minorem uno dempto ut prius est declaratum.

Septima conclusio. Si proportio maior fuerit commensurabilis minori et non sit multiplex ad eam sed in aliqua proportione quam volueris necesse est ut
${ }_{2} 8_{5}$ inter primos numeros minoris sit numerus aut numeri medii proportionales et quod inter primos numeros maioris sint numeri medii secundum illam proportionem seu proportionalitatem secundum quam inter primos numeros minoris est numerus seu numeri medii.
Propter equivocationem advertendum quod proportio quandoque 290 dicitur multiplex absolute sicut dupla, tripla, et cetera, et tunc hoc
26) medium om $H /$ A: aut $E /$ bis ipsam B: ipsius B bis $C /$ erit: est $R / \operatorname{Sed} H$; om CERV
266 A: aut $E$ / ante erit bab $V$ sic / erit: esset $V$
269 ante quia scr et del $V$ quia
270 talibus om $E$
271 A illa: alia $V$ / illa $C H V$ ista $R$ / ponantur $H R$ ponam $C V(?)$
271-72 in numeris $C R V$; om $H$
271-73 illa ...que om $E$
272 docet CHR; om V
273 proportionales $C H R$ proportionalitas $E$ proportionalis(?) $V$
274 minimos $H V$ numeros $R$ numeros(?) vel minimos(?) $C E$

275 componitur: componetur $V$
276 ante undecime scr et del $V$ lle / undecime: secunde $H$
277 A: autem $E$ / pluribus: proportionibus $R$ / inter: in $C$
278 A: aliquam $V /$ medius: medii $V$
279 ante inter add $V$ et / quoslibet $C E R$ quotlibet $H$ quolibet $V$
280 igitur om $\mathrm{H} /$ et $C H$; om $E R V$
281 et minimos: numeros(?) $H /$ ante maior scr et del $H$ est / continet: continebit $C$
282 ut...declaratum om $H /$ prius est declaratum $C E V$ dictum est $R$
283 Septima conclusio mg hab $E$ ante $\operatorname{Si}$ et $m g$ hab $H$ ante maior et $m g$ hab $R$ post sed in (linea 284); om $C$ octava con-

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one mean, $A$ will contain $B$ twice and will be double to $B$.* But if there are two means, $A$ would be composed of three $B$ 's and will be triple to $B$, etc., by the first definition. $\dagger$
That just as many numbers forming the same ratio $[B]$ are found between the prime numbers of ratio $A$ can be shown, because when such continuously proportional means that form ratio $B$ have been found between the extremes of $A$-afterwards these means and extremes will be considered expressly in numbers- $B$ will be a rational ratio, as shown by the second [proposition] of the eighth [book of Euclid], which says: "To find any continuously proportional numbers in their lowest terms which form a given ratio." Therefore, a ratio of the first to the last term is composed of intermediate ratios, as is clear from the comment on definition eleven of the fifth [book of Euclid] and in the principles of the seventh [book of Euclid]. But $A$, which is composed of several $B$ 's, is such a ratio. Thus between any numbers related as ratio $A$ there is a mean number or numbers forming ratio $B$, and, consequently, by the eighth [proposition] of the eighth [book of Euclid], ratio $B$ can be formed between any numbers related in the same ratio. Hence between the prime and least numbers of $A$ and these [other numbers related as $A$ ], there is the same number of means[namely] as was stated before, one mean less than the number of times $A$, the greater, contains $B$, the lesser.

Proposition VII. If a greater ratio is commensurable to a lesser and is not multiple to it but related in whatever ratio you wish, it is necessary that there be a mean proportional number or numbers between the prime numbers of the lesser; and between the prime numbers of the greater ratio there must be mean numbers which produce the same ratio or proportionality formed by the mean number or numbers found between the prime numbers of the lesser ratio.

Because of equivocation it must be remarked that a ratio is sometimes called absolutely multiple, as [for example], double, triple, etc., and then

* That is, $A=(B)^{2}$. ric means, it follows that $n=m+1$ when $\dagger$ In general, if $m$ is the number of geomet- $\quad A=(B)^{n}$.


## clusio $V$

284 sed...volueris om $V /$ aliqua: alia $R$ quam $C H$ qua $E R$
28s aut: autem $E$ | numeri medii: numeri(?) vel medii (?) H
286 sint: sit $C$ / illam: istam $R$
286-87 illam proportionem $t r H$
287 seu: vel $C /$ numeros om $R$

288 seu: sive $C$
289 ante advertendum bab $R$ est / advertendum: advertendem $V$ / quandoque: aliquam $E$
290 ante tripla scr et del $V$ tertia / ante et ${ }^{1}$ scr et del E et cetera / et cetera om H| tunc om $V$
nomen multiplex est genus et hec nomina dupla, tripla, et cetera, sunt nomina specialia proportionum. Quandoque vero proportio dicitur multiplex comparatione seu relatione ad aliam proportionem. Et ita proportio que non est multiplex absolute est multiplex compa25 rative sicut dupla sexquialtera est dupla sexquialtere vel ad sexquialteram. Et prius accepi primo modo et nunc capio multiplex secundo modo.

Dico, igitur, quod si maior proportio sit commensurabilis minori et non sit multiplex ad ipsam necesse est primos numeros unius cum $\cdots$ primis numeris alterius in mediis convenire eo modo quo predixi. Sit, exempli causa, $A$ proportio maior, $B$ minor que sunt commensurabiles per positum. Igitur $A$ et $B$ sunt ut duo numeri per quintam decimi; igitur $B$ est pars aut partes ipsius $A$ per quartam septimi. Sed non est pars quia tunc $A$ esset multiplex ad $B$ quod est contra positum. Igitur $B$ est partes $A$, igitur $B$ est tales partes ipsius $A$ quarum quelibet est proportio rationalis
Hec consequentia ultima probatur per correlarium quarte et potest sic deduci quia si $B$ est partes $A$ igitur residuum, quod cum $B$ componit $A$, similiter est pars aut partes ipsius $A$ habens eandem denominationem cum $B$ ita quod est talis pars aut partes quales est $B$ per suppositiones factas pro tertia conclusione. Igitur per continuam detractionem minoris a maiori, et cetera, devenietur ad unam illarum partium que probabitur esse proportio rationalis per primam conclusionem que etiam erit pars ipsius $A$ et ipsius $B$ sicut in quarta conclusione deductum est. Igitur $B$ est partes $A$ tales quod quelibet illarum partium est proportio rationalis. Vocetur modo quelibet talis $D$. Igitur $D$ est pars $B$ et similiter est pars $A$ ut probatum est, igitur $B$ est multiplex ad $D$ et similiter $A$ est multiplex ad $D$. Igitur inter

291 post genus hab $E$ ad / hec nomina $H$; om CERV
${ }^{292}$ Quandoque vero $\operatorname{tr} V$
293 multiplex om $E$ / comparatione seu relatione: comparative seu relative $R$ comparatione $E H V$ comparate $C$ seu $H$ vel $C E V$ I relatione $E H V$ relate $C$ / aliam: aliquam $C$
294-95 comparative $H R V$ comparate $C$ comparatione $E$
295 sexquialtera $E$ sexquiquarta CH (?) $R V \mid$ post est add $E$ ex
${ }_{2} 96$ primo modo $R$; om CEHV / nunc:
modo $R$
298 proportio om $R$ / sit CEV est $H R$
299 ante primos bab $V$ inter
300 mediis: medio $R$ / eo modo: econverso $E$ / predixi : dixi $R$ / Sit $E H V$; om $R$ sicut $C$
301 exempli causa: gratia exempli $R /$ causa $E H V$ gratia $C$ | post B add $V$ proportio
302 ut CR; om EHV
303 post quartam bab C quinti / septimi CHR ibi(?) $E$
303-9 per...ipsius A om V

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this name "multiple" is a genus name, and the names "double," "triple," etc., are names of species of ratios. Sometimes, however, a ratio is called multiple with respect to comparison or relation to another ratio. And thus a ratio which is not absolutely multiple is comparatively multiple, as [for example], a double sesquialterate is the double of, or to, a sesquialterate. Previously I accepted multiple in the first way and now I take it in the second way.

I say, therefore, that if a greater ratio is commensurable to a lesser ratio but not multiple to it, it is necessary that the prime numbers of one unite in the means with the prime numbers of the other in the manner stated above. For example, let $A$ be a greater ratio, $B$ a lesser ratio, and assume they are commensurable. Now by the fifth of the tenth [book of Euclid], $A$ and $B$ are related as two numbers so that, by the fourth of the seventh [book of Euclid], $B$ is a part or parts of $A$. But it cannot be a part for then $A$ would be multiple to $B$, which is contrary to the assumption. Therefore $B$ is parts of $A$ such that any whatever of these parts of $A$ is a rational ratio.

This last consequence is proved by a corollary of the fourth proposition and is deduced as follows: If $B$ is parts of $A$, then the remainder which with $B$ composes $A$ is likewise a part or parts of $A$ with the same denomination as $B$, so that by the suppositions made in support of the third proposition it is the same kind of part or parts as is $B$. Therefore, by a continuous subtraction of the lesser from the greater, one of those parts will be reached and will prove to be a rational ratio, by the first proposition, and will also be a part of $A$ and $B$, as was deduced in the fourth proposition. Hence, $B$ is parts of $A$ such that any of the parts is a rational ratio. Let any such part be called $D$. Then $D$ is a part of $B$ and likewise of $A$, as was shown; and consequently $B$ is multiple to $D$ and similarly $A$ is multiple to $D$. Thus, between the prime numbers of $B$ there is a mean number

304 est $^{1} E H R$ erit $C /$ esset $C E H$ est $R$ 305 ipsius $C E H$; om $R$ / ante quarum add $C E$ quo quelibet vel et add $R$ quod quodlibet
307 consequentia ultima $C E H$; $\operatorname{tr} R$
308 post partes add $C$ ipsius / post igitur scr et del $H \mathrm{~b}(?)$ / post cum hab $E$ hoc(?)
$309 \mathrm{~A}^{\mathrm{r}}$ om $C$ / post $\mathrm{A}^{\mathrm{I}}$ bab $E$ ut / est: quarta(?) $H$
310 ante est ${ }^{\mathrm{I}}$ babV A / partes quales $t r V /$ quales: tales qualis $R$ / post quales bab
$C E$ partes / $\mathrm{B}^{2}$ om $E$
311 tertia corr ex quarta CEHRV
312 minoris a maiori: maiores a minori $E$ / et cetera om $R$ / devenietur: devenirem $R /$ illarum $E H V$ istarum $C R$
313 probabitur $E H R$ probatur $C V$ | ante proportio scr et del $E$ continua(?)
314 erit $E H R$ et $C$ esset $V /$ ante A bab $E$ conceptus / quarta: alia $V$
315 deductum: declaratum $R$
316 illarum: istarum $R /$ partium om $V$
primos numeros $B$ est numerus medius aut numeri secundum $D$ ${ }_{320}$ proportionem et consimiliter inter primos numeros $A$ secundum $D$ proportionem per sextam immediate precedentem. Igitur primi numeri $A$ et primi numeri $B$ conveniunt in mediis modo prius dicto quod erat probandum, hoc est quod sunt aliqui numeri medii proportionales inter numeros maioris secundum quamdam proportionem secundum
325 quam proportionem necesse est inter primos numeros minoris numerum vel numeros medios inter esse.

Octava conclusio. Si fuerint due proportiones et inter primos numeros maioris fuerit numerus medius vel numeri medii secundum proportionem minorem aut secundum aliquam proportionem secundum quam inter primos numeros minoris sit numerus aut numeri medii ille due proportiones commensurabiles erunt. Hec est quasi conversa duarum precedentium.

Sit $A$ proportio maior, $B$ minor. Si igitur inter primos numeros $A$ fuerit numerus aut numeri medii secundum $B$ proportionem, cum proportio primi termini ad ultimum componatur ex proportionibus 35 intermediis, sequitur quod $A$ componetur ex pluribus $B$ et per consequens $A$ erit multiplex ad $B$, igitur commensurabilis. Et hec est quasi conversa sexte conclusionis precedentis.
Si , vero, inter primos numeros $A$ fuerit numerus medius seu numeri non tamen secundum $B$ proportionem sed secundum unam aliam
${ }_{340}$ proportionem secundum quam inter primos numeros $B$ est numerus medius aut numeri, et sit illa proportio $C$. Igitur $B$ componitur ex pluribus $C$ quia $C$ est proportio mediorum et $B$ est proportio extremorum modo proportio extremorum componitur ex intermediis, ut sepe dictum est. Igitur $C$ est pars $B$ et per eandem rationem $C$ est

319 primos: positos $C$ / numeros om $V$ / 325 proportionem $C E H$; om $R V$ /priB: quod $E /$ ante D scr et del $E$ verbum $\quad$ mos: positos $C /$ numeros om $E / \mathrm{mi}-$ illegibile
320 consimiliter $E R V$ tamen similiter $C /$ primos: positos $C$ / ante numeros scr et del $E$ numeros(?) / A: aut $E$
320-21 et... proportionem om $H$
321 sextam: quintam(?) $H /$ ante -mediate in immediate scr et del $E \operatorname{mdt}(?)$ l immediate precedentem: immediatis precedentis $C$
$322 \mathrm{~A}: \mathrm{B} E /$ et primi numeri $\mathrm{B} C R V$; om $E$ et numeri $\mathrm{B} H$
323 erat: fuit $V /$ probandum: propositum $C /$ medii $C H R$; om EV
324 maioris CHV maiores $E R$
noris $E H V$ minores $C R$
$325-26$ numerum: numeri(?) $V$
326 ante numeros scr et del $V$ no / medios H; om CERV
327 Octava conclusio mg hab E post esse (linea 326) et mg bab R ante Si ; om CH nona conclusio $V \mid S i \ldots$ inter om $E \mid$ primos om $H$
328 numerus medius vel $C$ numerus seu $E R$ seu fuerit $H$ numerus aut $V$ numeri medii $\operatorname{tr} V /$ aut: autem $E$
329 proportionem om $R /$ primos $H$; om $C E V /$ primos numeros $\operatorname{tr} R$
330 ante ille scr et del E e

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or numbers producing ratio $D$; and similarly between the prime numbers of $A$ [there is a mean number or numbers] producing ratio $D$, according to the immediately preceding sixth proposition. The prime numbers of $A$ and $B$, therefore, unite in means in the manner stated before, and this was to be proved-namely, that there are some mean proportional numbers between the numbers of the greater ratio that form a certain ratio, and that the very same ratio is also formed by the mean number or numbers between the prime numbers of the lesser ratio.

Proposition VIII. Two ratios will be commensurable if between the prime numbers of the greater there is a mean number or numbers forming the lesser ratio; or [if the mean number or numbers between the prime numbers of the greater ratio] form some ratio which is also produced by the mean number or numbers lying between the prime numbers of the lesser ratio. This is the converse of the two preceding propositions.

Let $A$ be the greater ratio, $B$ the lesser. Therefore, if between the prime numbers of $A$ there should be a mean number or numbers that form ratio $B$, then, since a ratio of the first to the last term is composed of the intermediate ratios, it follows that $A$ would be composed of several $B$ 's and, as a consequence, $A$ will be multiple to $B$ and thus commensurable. This is the converse of the preceding sixth proposition.

If, however, between the prime numbers of $A$ there is a mean number or numbers which do not form ratio $B$ but form, rather, another ratio, say $C$, which is also produced by the mean number or numbers lying between the prime numbers of $B$, then $B$ is composed of several $C$ because $C$ is a ratio of means and $B$ is a ratio of extremes and, as has frequently been said, only a ratio of extremes is composed of intermediates. Thus $C$ is part of

[^21]pars ipsius $A$ quia consimiliter $A$ inter primos numeros sunt numeri medii secundum $C$ proportionem, igitur $A$ et $B$ communicant in $C$ et $C$ est mensura communis utrique. Igitur $A$ et $B$ sunt proportiones commensurabiles per diffinitionem commensurabilium in principio decimi datam quod fuit probandum. Et hec est quasi conversa septime
${ }_{350}$ quia in hoc casu ultimo non erunt commensurabiles in proportione multiplici, sed bene in casu primo et hoc est convertere sextam ut dictum est.
Nota quod non sequitur inter istos numeros sunt aliqui numeri medii proportionales igitur inter eosdem est aliquis numerus medius.
355 Ymo sequitur inter istos numeros sunt tantum duo numeri medii ergo nullus numerus medius est inter cosdem, et similiter quatuor igitur nullus, et sic de paribus. Et sic sequitur tantum sunt due linee medio loco proportionales secundum proportionem rationalem inter istas duas igitur nulla linea est et cetera.
Nona conclusio. Datis duabus proportionibus si sint commensurabiles invenire.
Sit ut prius $A$ proportio maior, $B$ minor, tunc utraque earum primitus in primis numeris eius statuere et hoc poteris facere ex practica sequenti. Deinde vide si inter numeros illos iam habitos fuerit $\sigma_{5}$ aliquis numerus medius proportionalis sue numeri et quot fuerint et secundum quam proportionem sicut in sequenti practica saltem pro parte patebit.
Dico, igitur, primo quod si inter primos numeros $A$ maioris nullus

345 ipsius om $R / \mathrm{A}^{1}$ : $\mathrm{B} V /$ consimiliter: similiter $R / \mathrm{A}$ inter primos numeros $H R$ A inter primi numeros $E$ inter primos numeros A $C V$
347 mensura communis $\operatorname{tr} C$
347-48 proportiones commensurabiles tr V
348 per: secundum $R$
349 decimi: et omni(?) $V /$ septime: secunde $E$
350 casu ultimo: capitulo $V /$ erunt: essent V
35 I casu: capitulo $V /$ est convertere $E V$ convertere $E$ convertit $C$
351-52 et hoc...dictum est om $R$
353 Nota: notam C / post numeros $m g$ hab $H$ nota
354 medii $E H V$;om $C R /$ eosdem: $\operatorname{eos} E /$
est aliquis: sunt aliqui numeri vel $C$ 355 numeros $H R$; om $E$ duos numeros $C V /$ tantum rep $E$
356 numerus om $E$
357 post sic ${ }^{1}$ bab $V$ est / paribus: partibus $H / \operatorname{sic}^{2} H$; om $R$ similiter $C E V$
358 ante rationalem scr et del $V$ ir / inter: signate in $C$ / inter istas: signante inter instans $R$
359 duas $C H V$; om $E R /$ nulla linea est $E R$ nulla linea $C$ inter lineam $H$ nulla alia est $V$
360 Nona conclusio mg hab E ante Et (linea 357) et post maior (linea 362) mg bab $R$ principalis(?) conclusio nona; om $C H$ decima conclusio $V /$ Datis: datur(?) in $E$ / post duabus scr et del $H$ quantitatibus / sint: sit $H$

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$B$ and for the same reason $C$ is part of $A$ because similarly $A$ [like $B]$, has between its prime numbers mean numbers that produce ratio $C$ and, therefore, $A$ and $B$ communicate in $C$ and $C$ is a measure common to each.* And so, by the definition of commensurables given in the beginning of the tenth [book of Euclid], $A$ and $B$ are commensurable ratios, which was to be demonstrated. This is like the converse of the seventh proposition because in this last case they will not be commensurables [related] by a multiple ratio as they were in the first case, which, as was said, is the converse of the sixth proposition.
Observe that it does not follow that because there are some mean proportional numbers between these [prime] numbers, there is [at another time] some [one] mean number between the same [prime numbers]. On the contrary, it follows that [if] there are only two mean numbers between those [prime] numbers, then there is no [single] mean number between those same [prime numbers]; similarly, if there are four then there will be no [single] mean; and the same holds true for any even number of mean numbers. And so it follows that if there are only two mean proportional lines that form a rational ratio between two lines, there is no [single mean proportional] line between those two lines [from which rational ratios can be formed].
Proposition IX. [How] to find whether two given ratios are commensurable.
As before, let $A$ be the greater ratio and $B$ the lesser, and express each of them directly in prime numbers, which you can do from instructions given in the next section. Then see if between those [prime] numbers there is some mean proportional number or numbers, and if there are, see how many and what ratio they form. All this will be shown in the instructions reserved for the part following.
I say then, in the first place: If there is no mean number or numbers

* $C$ is their "common measure" because $C=(B)^{1 / m}$ and $C=(A)^{1 / n}$; or $B=(C)^{m}$ and $A=(C)^{n}$.

[^22]
## $C V$ numeros $R$

365 aliquis om $E \mid$ ante proportionalis scr et del $V \mathrm{n}$ / sue numeri om $R$ / quot: quod $V /$ et $^{2} C E V$; om $H R$
366 practica saltem: platica saltim $C$
368 primo om $C$ / primos numeros $\operatorname{tr} E$
fuerit numerus medius seu numeri proportiones date sunt incommen-
Secundo, si inter primos numeros $A$ sit numerus medius seu numeri secundum $B$ proportionem minorem, tunc $A$ erit commensurabilis $B$ et multiplex ad $B$ per octavam.
Tertio, si inter numeros $A$ fuerit numerus, et cetera, non tamen secundum $B$ tunc $A$ non erit multiplex ad $B$ per sextam.
Quarto, si inter numeros $A$ fuerit numerus, et cetera, non tamen secundum $B$ proportionem sed secundum aliquam aliam proportionem secundum quam inter primos numeros $B$ est numerus medius, et cetera, $A$ et $B$ erunt commensurabiles per octavam.
Quinto, si inter numeros $A$ fuerit numerus medius non tamen secundum $B$, nec secundum aliam proportionem secundum quam inter primos numeros $B$ sit numerus, et cetera, ille erunt incommensurabiles per septimam conclusionem.
Sunt igitur quasi quinque conclusiones partiales iuxta quinque ${ }_{3} 8_{5}$ membra divisionis sequentis.

Exemplum de prima sint tripla et dupla. Cum igitur inter numeros maioris, scilicet triple, qui sunt 3 et I , nullus est numerus medius, et cetera, dico quod sunt incommensurabiles.
Exemplum de secundo sint quadrupla et dupla. Quia igitur inter numeros quadruple, qui sunt 4 et I, est medium secundum proportionem duplam, scilicet 2, dico quod sunt commensurabiles et maior est multiplex ad minorem.
Exemplum tertii sint nonacupla et dupla. Cum igitur inter numeros
369 seu: vel $C /$ medius seu numeri: seu numeri medii $R /$ numeri $E H$ medius $C$ medii $V /$ proportiones $C H V$ proportionales $E R$
370 ante per add $E$ et / per: secundum $R$
371 inter om $V /$ post A add $R$ maioris/sit: fuerit $E /$ medius om $R /$ numeri: medius $C$
372 B proportionem $\operatorname{tr} R /$ erit: esset $V /$ commensurabilis: commensurabile $H / B^{2}$ om $V$
373 post octavam add $R$ conclusionem
374 ante numeros $b a b V$ duos / numerus et cetera CH ; om $V$ numerus medius et cetera $E$ terminus et cetera $R$
375 ante tunc add $R$ proportionem
377 aliam om $H$

377-78 proportionem om $E$
378 est: et $V /$ post medius bab $V$ a
378-79 et cetera om $R$
379 post octavam add $H$ et cetera
380 post medius add E et cetera/non CHR nec $E$
380-82 non...numerus om $V$
38 I post B add $R$ proportionem
382 sit CEH est $R$ / post numerus add $R$ medius / erunt: sunt $H$
382-83 incommensurabiles: commensurabiles $R$
383 conclusionem $H$; om CERV
384 quinque ${ }^{2}$ : quintum $C$
385 membra rep $E$
386 sint $H$ sit $C R$ sicut $E V /$ tripla et dupla $E H V$ dupla et tripla $C R$

## Chapter Two, Part One

203
between the prime numbers of the greater ratio $A$, the given ratios are incommensurable, by the fifth proposition.
Second: If between the prime numbers of $A$ there is a mean number or numbers which form the lesser ratio $B$, then $A$ will be commensurable and multiple to $B$, by the eighth proposition.

Third: If between the [prime] numbers of $A$ there is a [mean] number [or numbers] that, however, cannot form ratio $B$, then $A$ will not be multiple to $B$, by the sixth proposition.
Fourth: $A$ and $B$ will be commensurable, by the eighth proposition, if between the [prime] numbers of $A$ there is a [mean] number [or numbers] that, though they cannot form ratio $B$, can, however, form some other ratio which is also formed by a mean number [or numbers] lying between the prime numbers of $B$.
Fifth: By the seventh proposition, $A$ and $B$ will be incommensurable if between the [numbers] of $A$ there is a mean number [or numbers] which can form neither ratio $B$ nor any other ratio that can be formed from the [mean] number [or numbers] lying between the prime numbers of $B$.
These are just like five subsidiary propositions corresponding to the five parts of the following division.
An example of the first [proposition] would be triple and double ratios. I say that they are incommensurable since there is no mean number [or numbers] between the [prime] numbers of the greater ratio, which are 3 and I , namely a triple.*
An example of the second would be quadruple and double ratios. I say that they are commensurable and the greater is multiple to the lesser because between the numbers of the quadruple, which are 4 and I , there is a mean number, namely 2 , that forms a double ratio. +

An example of the third would be nonacuple and double ratios. I say
$* 3 / \mathrm{I}$ and ${ }^{2 / 1}$ are incommensurable because $\quad+4 / \mathrm{I}=4 / 2 \cdot 2 / \mathrm{I}$.
$3 / \mathrm{I} \neq(2 / \mathrm{I})^{n}$, where $n$ is an integer.

387 qui $C H$ que $E R V /$ i: $2 H /$ numerus medius $\operatorname{tr} R$
387-88 et cetera om $C$ tripla dupla / in-
88 post quod add $V$ tripla dupla / in-
commensurabiles: commensurabilis V

389 sint $C H(?) R$ sicut $E V /$ igitur om $E$ 390 qui $C H$ que $E R V$
391-92 et...minorem $H$; om $C E R V$
393 post Exemplum scr et del $E$ de secundo sicut quadrupla / sint $C H R$ sicut $E V$
maioris, qui sunt 9 et I , sit medium secundum proportionem triplam, dilice 3 , et non secundum proportionem minorem propositam, sci licet secundum duplam, dico quod maior non est multiplex ad minorem.

Exemplum quarti sint octupla et quadrupla. Et quia inter utriusque primos numeros est medium, et cetera, secundum eandem proport-
onem, scilicet duplam, dico quod sunt commensurabiles.
Exemplum quinti sint nonacupla et quadrupla. Et inter cuiuslibet numeros est medium sed quia non secundum eandem proportionem sed inter numeros maioris secundum triplam et inter numeros minoris secundum duplam, ideo sunt incommensurabiles.

Et preter istos modos nullus alius modus nec alia dispositio potest ymaginari sicut faciliter potest ostendi per sufficientem divisionem cuius quelibet divisio partialis fiet inter contradictoria.

Divisio sit ista: aut inter numeros $A$ est medium, et cetera, aut non (prima conclusio partialis probata per quintam). Si sit, aut secundum
${ }_{410} B$ (secunda conclusio per octavam), aut non (tertia conclusio per sextam). Si non, aut secundum aliquam proportionem secundum quam inter numeros $B$ est medium (quarta conclusio per octavam) aut non (quinta conclusio per septimam).

394 qui: que $C / 9: 7 V$
395 scilicet 3: qui tertiam $H \mid$ ante se cundum scr et del $V$ ss / ante minorem scr et del V m y la (?) / propositam $H R$ positam CEV
395-96 ante scilicet scr et del $E$ secundum(?)
398 quarti: quadrupli $H$ / sint $C H R$ sicut $E$ sicud $V /$ Et om $R$ / inter om $V /$ utriusque: utrasque $H$
399 primos numeros $\operatorname{tr} R /$ medium om $C$ sed post proportionem (lineae 399-400) bab $C$ medium / et cetera secundum $V$ secundum $C R$ quia secundum $E$ quamdam(?) $H$
399-400 eandem proportionem $\operatorname{tr} R$
400 ante duplam add $R$ secundum / duplam: dupla $C$
401 $\operatorname{sint} C H R$ sicut $E$ sicud $V /$ nonacupla et quadrupla: quadrupla et nonacupla E
402 post non bab $C$ solum
403 maioris: maiores $E /$ et $E H$; om $C R V$ / minoris: minores $E$

404 secundum...incommensurabiles: et duplam proportionem sunt commensurabiles $R /$ post ideo hab $H$ non(?) vel vero(?)
405 istos modos $H R$ istos numeros $C E$ hos modos $V /$ alius $H$ alter $C E V \mid$ alius modus: modus alter $R /$ modus om $H$ / alia : aliqua $C$
406 sicut: sed $E /$ potest: possit $V /$ potest ostendi $\operatorname{tr} C$ / ante divisionem bab $V$ diffinitionem
407 cuius om C / partialis: partibilis $C$ | inter: per $H$
408 sit: si C | post ista bab CRA BethabV A et $\mathrm{B} /$ aut $^{\mathrm{I}}$ : B A $V /$ ante numeros scr et del $E$ numerus(?)
408-9 aut ${ }^{2} \ldots$ quintam $C V$ et sit aut secundum B (secunda conclusio per octavam) $E H$ sed om $E$ et; hab $R$ aut non (ista conclusio probatur per quintam)
409-11 Si....sextam $R$ aut non (quarta conclusio per octavam prima conclusio partialis probata per quintam) $H$ aut
that the greater is not multiple to the lesser since between the numbers of the greater, which are 9 and r , there is a mean, namely 3 , that forms a triple ratio but does not form the proposed lesser ratio, namely a double.*

An example of the fourth would be octuple and quadruple ratios. I say that they are commensurable since there is a mean [number or numbers] between the prime numbers of each that form the same ratio, namely a double. $\dagger$
An example of the fifth would be nonacuple and quadruple ratios. Now between the numbers of each of these there is a mean, but they are incommensurable because the means do not form the same ratio; a triple ratio is formed between the numbers of the greater ratio and a double ratio between the numbers of the lesser. $\ddagger$
Besides these ways no other mode or arrangement can be imagined, as can easily be shown by an adequate division where the subdivisions are made between contradictories.

The division is as follows: Between the [prime] numbers of $A$ there is either a mean [number or numbers] or not (this is the first subproposition proved by the fifth proposition). If there is [a mean or means] it forms ratio $B$ (the second [subproposition proved] by the eighth proposition), or not (the third [subproposition proved] by the sixth). If not, either it forms some ratio which is also formed by a mean [number or numbers] lying between the numbers of $B$ (the fourth [subproposition] proved by the eighth) or not (the fifth [subproposition proved] by the seventh).
$* 9 / 1$ is not multiple to $2 / 1$ since $9 / 1 \neq(2 / 1)^{n}$, where $n$ is an integer.
$\dagger 8 / \mathrm{x}$ and $4 / \mathrm{I}$ are commensurable because $8 /$ $=(2 / \mathrm{I})^{3}$ and $4 / \mathrm{I}=(2 / \mathrm{I})^{2}$-i.e., they have a
non (prima conclusio partialis probata per quintam) aut non si non (tertia conclusio per sextam) $E$
410 tertia $R$ tripla $V$
410-II tertia...sextam om $C$
41 I Si non $C R V$; om $E H$ / post proportionem add $V$ aut
412 inter: aliquando(?) $H$ | inter rep $C$
B est $E R V$ est $C$ sit $H \mid$ post octavam
common base or measure.
$\ddagger 9 / \mathrm{I}$ and $4 / \mathrm{I}$ share no common base and are therefore incommensurable.
$m g$ bab $R$ aut B aut inter numerum(?) ut(?) B(?)
412-13 quarta... septimam $E R V$ (quarta conclusio per octavam) aut non (quinta conclusio per octavam) $C$ aut non si non (tertia conclusio per sextam) aut non (quinta conclusio per septimam) $H$

Decima conclusio. Propositis duabus proportionibus commensurabilibus
Si inter numeros maioris fuerit medium aut media secundum proportionem minorem ipsa continebit minorem totiens quot sunt medii numerii addita unitate ut patet ex prima diffinitione et octava conclusione. Quo scito statim patet proportio.

Si, vero, inter numeros maioris non sit medium in numeris, et cetera, secundum proportionem minorem sed secundum aliquam aliam proportionem secundum quam inter numeros minoris sit medium aut media, tunc illa proportio secundum quam inter numeros utriusque est medium, et cetera, erit pars utriusque et erit sicut unitas 25 que quamlibet earum reddit totiens sumpta, quot inter numeros illius, cuius est pars, sunt media addita unitate sicut ex septima et octava conclusionibus, et prima et secunda diffinitionibus poterit apparere.

Capiatur, igitur, numerus mediorum inter numeros proportionis minoris et addatur unitas, et consimiliter numerus mediorum inter numeros maioris et addatur unitas. Dico quod proportio istarum proportionum erit sicut proportio istorum numerorum. Numerorum vero proportionem per arismeticam investiges. Si , autem, primi numeri dictarum proportionum aliter se habeant, tunc proportiones sunt incommensurabiles per immediate precedentem.

Exemplum primi sint proportio octupla et dupla. Quia igitur inter numeros maioris, qui sunt 8 et I , sunt duo numeri medii, scilicet 4 et 2, secundum proportionem minorem, huic numero mediorum addas unitatem et sunt tres. Dico quod maior continet ter minorem igitur est tripla ad eam quod potest probari sicut prius. Et etiam quoniam

414 Decima conclusio $m g$ hab $E$ post maioris (linea 4I6) et mg bab $R$ ante Propositis; om $\mathrm{CHV} /$ commensurabilibus om $E$
4Is proportionem: proportio $E$
417 ipsa continebit om $C$ / quot: quod $E$ 417-18 medii numerii $E H$; $\operatorname{tr} C R V$ 419 patet: potest patere $H$
421-22 aliquam aliam proportionem: aliam proportionem aliquam $V$
422 secundum om $E$ / ante quam scr et del $V$ maiorem / minoris om $C$
423 illa: ista $R /$ secundum: aic(?) $E /$ numeros om $H$
424 cetera: $\operatorname{ibi}($ ? $) ~ E /$ erit $^{\mathrm{I}}$ : et $C / \mathrm{erit}^{2}:$ et $C$ 425 quamlibet: qualibet $V /$ reddit $E H R$
reddet $C V /$ quot $H$ quotiens $C$ quod $E R V$ / ante illius scr et del $E$ utriusque est medium inter erit pars utriusque / illius: istius $R$
426 sicut: sed $R$
428 Capiatur: capiantur $H$
428-30 proportionis... numeros om $V$ 429 numerus $C H R$ numeros $E$
430 maioris $C E H$ minoris $R$
43 I erit: est $V /$ sicut proportio: maioris (?) $H$ / numerorum: numerus(?) $H$ / Numerorum $E R V$ numeros $C$ numerus $H$
432 proportionem $C H V$; om $E R$ / post primi bab $H$ nullam(?)
433 habeant: haberent $C$

Proposition $X$. [How] to assign a ratio between two proposed commensurable ratios.

If between the numbers of a greater ratio there is a mean or means that form a lesser ratio, the greater will contain the lesser as many times as there are means plus one, as is evident from the first definition and the eighth proposition. Once this is known, the ratio is immediately evident.
If, however, between the numbers of the greater ratio there is no mean number or numbers forming the lesser ratio, but [rather] forming some other ratio that is the same as that produced from the mean or means lying between the [prime] numbers of the lesser ratio, then that ratio formed by the mean or means between the prime numbers of the greater and lesser ratios will be part of each and will be like a unit that produces any of these ratios when taken a certain number of times. [For each of the ratios] between whose numbers it is a part, it can be taken as many times as there are means plus one, as can be shown by the seventh and eighth propositions, and the first and second definitions.

Therefore, take the number of the means between the numbers of the lesser ratio and let a unit be added to it; and, similarly, take the number of the means between the numbers of the greater ratio and add a unit to it. I say that the ratio of these ratios will be as the ratio of those numbers. You can discover the ratio of the numbers by arithmetic. Moreover, if the prime numbers of the said ratios are related otherwise [than by a ratio of numbers], they are incommensurable, by the proposition immediately preceding.

An example of the first [part of this proposition] would be an octuple ratio and a double. Since between the numbers of the greater ratio, which are 8 and 1 , there are two mean numbers, namely 4 and 2 , that form the lesser ratio, add a unit to this number of means and this makes three. I say that the greater contains the lesser three times and is therefore triple to it, which can be proved as before.* Also, when there are four continuously

* Since there are two means in the sequence $8,4,2,1,8 / \mathrm{x}=(2 / \mathrm{I})^{3}$ and is multiple to $2 / \mathrm{I}$.

434 immediate precedentem tr $V$
435 primi: si $\mathrm{H} \mid$ sint CH sicut $E R V$ ante dupla add $E V$ proportio / igitur EHR; om CV
436 qui sunt 8 et 1: que $E$
437 ante addas $b a b V$ addata

437-38 addas unitatem $E R V$; $\operatorname{tr} C$ addat unitatem $H$
438 et $H R$; om CEV | ante ter scr et del $V_{3}$ 439 quod $H$ et $C E V /$ quod potest probari: et probari potest $R /$ etiam om $E$ proportio primi ad ultimum est tripla proportioni primi ad secundam, et cetera, per undecimam diffinitionem quinti Euclidis et per idem patet quod quadrupla est dupla duple, et cetera.

Exemplum secundi sint $32^{\text {la }}$ et 8 la . Cum igitur inter primos numeros maioris, qui sunt 32 et 1 , sint quatuor numeri medii secundum proportionem duplam ut patet disponendo numeros isto modo $32,16,8$, 4, 2, I et inter numeros minoris, scilicet octuple, sunt duo numeri medii secundum eandem proportionem, scilicet duplam, ut prius dicebatur. Capiamus, igitur, numerum mediorum maioris, scilicet quatuor, et addamus unitatem sunt quinque. Et iterum capiamus numerum mediorum minoris cum unitate sunt 3 . Dico, igitur, quod proportio maioris proportionis date ad minorem est sicut 5 ad 3 et est sicut proportio superpartiens duas tertias, $\mathrm{I}^{2} / 3$. Et consimilis proportio est proportionis 243 ad 32 ad proportionem 108 ad 32, scilicet pro-
455 portio septuple superpartientis $19 / 32$ as ad proportionem triplam superpartientam $3 / 8^{\text {as }}$, videlicet $7^{19} / 32$ ad $3^{3 / 8}$ quod patet ex dictis et cetera, et dicendis investigare poteris si tu velis.

Hic finitur prima pars secundi capituli in qua 10 conclusiones continentur de quibus tribus primis septem fit origo supremis.

## [Secunda pars secundi capituli]

In secunda parte huius capituli pono tres practicas regulas utiles ad predictam.

Prima regula seu conclusio est data proportione eius primos numeros invenire.

440 sunt rep $C$ / quatuor $C E R{ }_{3} H$ tres $V$
$44^{1}$ primi ${ }^{\text {I }}$ : prima $E$
442 et cetera om $E /$ cetera om $R /$ undecimam $C E V$ quintam(?) $H R /$ quinti: quinte $V$
443 patet $t r R$ post Euclidis (linea 442)
444 sint $C$; om $H$ sicut $E R V$
445 qui: que $E$
447 minoris: maioris $H /$ numeri: termini C
448 secundum: habet $C$
449 dicebatur: dicebitur $C$
450 et addamus: cum $E$ / ante sunt add $R$ et / capiamus om $E$
451 igitur om $R$

452 sicut ${ }^{2} \mathrm{H}$; om CERV
453 I $2 / 3$ : sicut $2 / 3$ R / ante Et bab H 3 3/8 $454^{-} 5^{6} 243 \ldots 3^{3} / 8 H$ septuple $19 / 32$ ad proportionem $33 / 8 C R$ duple $19 / 32$ ad proportionem $23 / 8 E$ septuple $19 / 32$ ad proportionem $3 / 8 \mathrm{~V}$
456 quod $C H R$ quam $E V /$ patet $H$; om CERV / et cetera $H$; om CERV
457 ante et scr et del $C$ ma / investigare poteris $\operatorname{tr} V /$ poteris $E H R$ poterit $C$ / ante si scr et del $V$ su / tu EHV; om $C R$
458 Hic $C H V$; om $E$ hec $R$ / finitur $E H V$ sit $C R /$ secundi capituli $\operatorname{tr} R /$ qua: quo $R$
proportional terms, as there are in the ratio proposed, the ratio of the first to the last term is triple the ratio of the first to the second term, etc., by the eleventh definition of the fifth [book of Euclid]; and by the same [reasoning] a quadruple ratio is double to a double ratio, etc.

An example of the second [part] would be $32 / \mathrm{I}$ and $8 / \mathrm{I}$. Now between the prime numbers of the greater ratio, namely 32 and I , there are four mean numbers that form a double ratio, as is obvious by arranging the numbers as $32,16,8,4,2,1$. And between the numbers of the lesser ratio, namely the octuple ratio, there are two mean numbers forming the same ratio, namely a double ratio, as was stated before. Therefore, we take the number of means of the greater ratio, namely 4, and add a unit to give 5. And again, we take the number of means of the lesser with a unit added and get 3 . I say, therefore, that the ratio of the greater given ratio to the lesser ratio is as 5 to 3 and is as a superpartient two-thirds ratio, [namely] I $2 / 3 . *$ And the same ratio [of s to 3] is obtained by relating ratio 243 to 32 to ratio 108 to 32 , that is a septuple superpartient ${ }^{19 / 32}$ ratio to a triple superpartient $3 / 8$ ratio, or $7^{19} / 32$ to $3^{3 / 8}$, , which is evident from what has been said, etc., and, if you wish, you can examine the things said.

Here is concluded the first part of the second chapter, which contains ten propositions of which the first three were made the basis of the last seven.

## [Chapter Two, Part Two]

In the second part of this chapter I set forth three practical rules useful in [understanding] what has already been said.

Rule one, or proposition one, is [how] to find the prime numbers of a given ratio.

* $32 / \mathrm{I}$ and $8 / 1$ share a common base, $2 / \mathrm{I}$, so $\dagger$ Since $243 / 32=(3 / 2)^{5}$ and $108 / 32=(3 / 2)^{3}$, it that $(2 / 1)^{5}=\left[(2 / 1)^{3}\right]^{5 / 3}$ and they are related follows that $243 / 32=(108 / 32)^{5 / 3}$, or $(3 / 2)^{5}=$ exponentially as $5 / 3$.

$$
\begin{aligned}
& \text { follows that } 245 / 32=(100 / 32)^{73}, \text { or }(3 / 2)^{5}= \\
& {\left[(3 / 2)^{3}\right]^{3 / 3} \text {. }}
\end{aligned}
$$

459 de... supremis om $E /$ primis $C H R$;om $V /$ fit $C R V$ sit $H$
[Secunda pars secundi capituli] om CEHRV
I ante In $m g$ bab $E$ prima conclusio seu regula et post predictam (linea 2) $m g$ bab $R$ prima conclusio huius partis /
huius capituli om $R$ / post capituli bab $H$ tertii / pono $H$ ponam $C R V$ ponans $E /$ practicas: placitas $C /$ regulas: duas $V /$ utiles om $C$
3 seu: sive $C$ / est om $R /$ primos numeros invenire: invenire numeros eius primos $H$

Primum oportet date proportionis denominationem habere cuius mo3 dum ostendam faciliter per singula genera exemplariter discurrendo.

De genere multiplici dico quod prima species, scilicet dupla, denominatur numero binario. Denominatio vero triple est 3, quadruple 4, quintuple $s$, et cetera.
Superparticularis autem denominatur integro vel unitate et fracquiquinta $I^{1} / 5$, et cetera.
Superpartiens denominatur integro seu unitate et fractionibus ut superpartiens duas tertias $\mathrm{I}^{2} / 3$, superpartiens tres quintas $\mathrm{I}^{3} / 5$, et cetera.
${ }_{55}$ Multiplex superparticularis integris seu numero et fractione isto modo: dupla sexquialtera $2^{1 / 2}$, dupla sexquitertia $2^{1 / 3}$, tripla sexquiquarta $3^{1 / 4}$, et cetera.

Multiplex superpartiens denominatur numero et fractionibus ut dupla superpartiens duas tertias $2 \frac{2}{3}$, tripla superpartiens tertias sep-
${ }_{20}$ timas $3^{3 / 7}$, et sic ultra cuiusvis proportionis denominatione inventa.
Primos eius numeros seu minimos invenies per hunc modum: primo, in multiplicibus non est difficultas cuiuslibet namque proportionis de genere multiplici minor numerus est unitas, maior vero est sua denominatio. Verbi gratia, primi numeri proportionis duple sunt 2 et
25 I , quadruple 4 et I , sextuple 6 et I , et cetera.
In aliis, tamen, generibus taliter est agendum: primo, denominationem proportionis de qua queris scribe per suas figuras. Deinde accipe denominatorem fractionis vel fractionum pro numero minori qui ab aliquibus vocatur comes radicum. Et postea eundem numerum multi-

4 date: dare $C$
4-5 modum: medium $V$
s ostendam: ostendetur $H /$ faciliter om $E$ / exemplariter discurrendo: distinguendo $C$
6 scilicet: seu $H$
7 3: tripla $V$ / quadruple 4: duple quadruple quarta $V$
8 quintuple s om $R / s$ : quinta $V$
9 autem $C E H$; om $R$ vero $V /$ denominatur: denominatio $H /$ vel: et $R /$ et: vel $V$
${ }_{10} 1_{1} /{ }_{2} \mathrm{CEH}_{12} \mathrm{~V}_{1} /{ }_{12} R / \mathrm{I}_{1} /{ }_{3}: 1 /{ }_{13} R$ ${ }^{11} / 4 E V{ }^{12} /{ }_{4} C_{14} H_{1}^{1} /{ }_{14} R$
IO-II sexquiquinta I $1 / 5 \mathrm{om} R$
II I $1 / 5$ : s $H$ / et cetera: tripla sexqui-
quarta $3^{1 / 4} C$
12 Seu: vel $C /$ seu unitate om $E$
13 tres quintas: $s$ tertias $H /$ I $3 / 5 C R$; om VI2/5 $E H$
Is integris: integra $C$ / ante seu bab $C$ denominatur / seu: sive $C$
16 modo om $V / 2^{1 / 2}: 1 / 22 R /$ dupla $^{2}$ om $H_{/ 21 / 3} \mathrm{CEV}_{2}{ }_{3} H^{1} / 23 R$
16-17 tripla sexquiquarta $3^{1 / 4} \mathrm{CEV}$; om $H^{\mathrm{I}} / 34 R$
i8 ut: $\mathrm{d}($ ? $) H$
18-19 denominatur... superpartiens ${ }^{2}$ om V
19 duas CER; om $H$ /ante $2^{2} / 3$ bab E $1 /$ ${ }_{2} 2_{3} C E V{ }_{2}{ }_{3} H^{1} /{ }_{23} R /$ tertias $H R$ tres CEV

It is first necessary to obtain the denomination of the given ratio and I can easily show this by running through the individual genera by way of example.

With respect to a multiple genus [of ratio], I say that the first species, namely a double, is denominated by the number 2. The denomination of a triple is 3 , of a quadruple 4 , quintuple 5 , etc.

The superparticular [genus of ratio] is denominated by the whole [number] or unit plus a fraction, as a sesquialterate by $1 / 2$, a sesquitertian by I $1 / 3$, a sesquiquartan by $\mathrm{I} 1 / 4$, a sesquiquintan by $\mathrm{I} 1 / 5$, etc.

The superpartient genus is denominated by a whole number or unit plus fractions, as a superpartient $2 / 3$ by $\mathrm{I}^{2} / 3$, a superpartient $3 / 5$ by $\mathrm{I} 3 / 5$, etc.

The multiple superparticular genus is denominated by numbers or a [single] number plus a fraction in this way: A double sesquialterate by $21 / 2$, a double sesquitertian by $21 / 3$, a triple sesquiquartan by $31 / 4$, etc.

The multiple superpartient genus is denominated by a number plus fractions, as a double superpartient two-thirds by $2 \frac{2}{3}$, a triple superpartient three-sevenths by $33 / 7$, and so on; and in this way the denomination of any ratio can be found.

You can find the prime or least numbers of a ratio as follows: In the first place, in multiple ratios there is no difficulty, since in any ratio of a multiple genus the smaller number is unity, and the greater is its denomination. For example, the prime numbers of a double ratio are 2 and I , quadruple 4 and I , sextuple 6 and I , etc.

In the other genera, however, one must do the following: First, write down the figures of the ratio whose denomination you seek. Then, take the denominator of the fraction or fractions as the smaller number [of the denomination], which is called by some the "consequent of the roots." Afterward, multiply this same number by the integer or integers in the

[^23]${ }^{30}$ plica per integrum velintegra in denominatione posita et producto adde numeratorem fractionum et tunc habebis numerum maiorem quem aliqui vocant ducem radicum. Verbi gratia, sit proportio sexquialtera que sic scribitur I $1 / 2$. Quia binarius est denominator ideo ipse est numerus minor, ipsum igitur multiplica per unitatem et adde nume-
35 ratorem, scilicet unitatem, et sunt 3, numerus maior. Dico igitur quod primi numeri illius proportionis sunt 3 et 2 .

Aliud exemplum sit data proportio dupla superpartiens quintas septimas que sic scribitur $25 / 7$. Dico quod septem est numerus minor. Multiplica, igitur, 7 per 2 et sunt 14, et adde 5 et sunt 19. Dico, igitur, 40 quod primi numeri date proportionis sunt ig et 7 .

Ex istis, si volueris, poteris accipere denominationes proportionum minoris inequalitatis quarum quelibet denominatur fractione vel fractionibus. Submultiplices fractiones, fractionibus habitis igitur primis numeris alicuius proportionis maioris inequalitatis, illi eidem numeri
45 sunt primi numeri proportionis minoris inequalitatis sibi correlative correspondentes quorum minor est numerator et maior denominator.

Verbi gratia, volo scire denominationem subduple et quia iam habeo quod primi numeri duple sunt 2 et r , ideo denominatio subduple erit 1/2. Item volo scire denominationem subsexquialtere, et iam scio quod
${ }_{50}$ primi numeri sexquialtere, et similiter subsexquialtere, sunt 3 et 2 , dico quod denominatio subsexquialtere est $2 / 3$. Item volo habere denominationem subtriple superpartientis septimas duodecimas. Tunc per predictam inveniam primos numeros triple superpartientis, et cetera, que sibi correspondent in maiori inequalitate et illi numeri sunt

30 vel: ab $C /$ in: pro $R /$ adde $\operatorname{tr} R$ post fractionum (linea 3 I)
3 I habebis: habebit $H$
33 ante $11 / 2$ scr et del $E 11 / 2$ (?) $/$ 1 $1 / 2$ $C E H^{1} / 2 V^{1} /{ }_{12} R /$ est $^{2} E H R$; om $C V$
34 numerus minor $\operatorname{tr} E$ / igitur: enim $R /$ multiplica: multiplicata $E$ / unitatem: unum(?) $E$
353 numerus maior: numerus maior 3 $H$ / quod om $V$
36 illius $E H V$ istius $C R$
37 sit om $H$ | post dupla scr et del $V$ superpartiente(?) / superpartiens quintas: superpartiensquinque(?) $V$
 $V /$ Dico quod septem om $H /$ est om $E$ 39 igitur om $R / \mathrm{et}^{1}$ om $R /$ adde: a $E$ /
et ${ }^{3}$ om $R /$ Dico om $E$
40 date $C H R$ alicuius $E$ data $V /$ 19:9 9
${ }^{41}$ poteris: potest $H /$ accipere $H R$ accipe $C E V$ / denominationes: denominationem $E$ / proportionum $C H$ proportionis $E R$ proportionem $V$
42-44 quarum...inequalitatis om $C$
43 post Sub- in Submultiplices scr et del $E$ mul / fractiones $H$ alie $E V$ fractionem alie $R$ / ante fractionibus bab $H$ aliis(?) 44 eidem $C V$ hiidem $E$
44-45 maioris...proportionis om $R /$ illi... sibi: igitur $H$
4s ante sibi bab $V$ sunt / correlative: correlatum $C$
46 correspondentes $H V$ correspondentis $C$ respondentis(?) $E$ respondentes $R$
given denomination and then add the numerator of the fraction to that product and you will have the greater number, which some call the "antecedent of the roots." For example, let there be a sesquialterate ratio which is written $I / 2$. Since 2 is the denominator, it is the smaller number. Therefore multiply it by the unit and add the numerator, namely a unit, and this makes 3 , the greater number. I say, then, that the prime numbers of this ratio are 3 and 2.
Another example would be where the given ratio is a double superpartient five-sevenths, which is written as $25 / 7$. I say that 7 is the smaller number. Then multiply 7 by 2 which gives 14 , and add $s$ to give 19. Therefore, I say that the prime numbers of the given ratio are 19 and 7 .
From these [examples] you can find, if you wish, the denominations of ratios of lesser inequality, any one of which is denominated by a fraction or fractions. The fractions for submultiple fractions are known from the prime numbers of a ratio of greater inequality, [since] those same numbers are the prime numbers of a ratio of lesser inequality corresponding reciprocally to it; the smaller of the numbers is the numerator and the greater, the denominator.
For example, I wish to find the denomination of a subdouble. Now since I already know that the prime numbers of a double ratio are 2 and I , the denomination of a subdouble will therefore be $\mathrm{I} / 2$. Likewise, I wish to find the denomination of a subsesquialterate. Since I already know that the prime numbers of a "sesquialterate"-and hence of a subsesquialterateare 3 and 2 , I say that the denomination of a subsesquialterate is $2 / 3$. Similarly, I wish to know the denomination of a subtriple superpartient seventwelfths [ratio]. By what has already been said, I should find in a triple superpartient [seven-twelfths ratio]-the reciprocally corresponding ratio of greater inequality-the prime numbers corresponding [to those of a sub-

47 et om $C$ / quia om $R$ / iam om $H$
$4^{8}$ quod om $H$ / duple $C R V$ subduple $E H /$ et om $C / \mathrm{I}$ om $E /$ denominatio $H V$ denominator $C R$ denominans $E$ / erit: sunt $H$
$49^{1 / 2} C_{1}{ }^{1} / 2 E_{2} H^{2} / 2 V^{1} /{ }_{12} R /$ subsexquialtere $C V$ sexquialtere $E H R$ ante et scr et del $V$ est $3^{2}$ item volo habere denominationem subtriple
so et similiter subsexquialtere om $R$
sr subsexquialtere $C E$ sexquialtere $H R$
$V /$ est: et $H / 2 / 3 \mathrm{CV}_{1}{ }^{2} /{ }_{3} E 2$ (?) H 2/ 13 R
$\mathrm{s}_{2}$ subtriple: subduple $R$ / septimas: septe $V \mid$ ante duodecimas scr et del $V$ du
53 per: pre $E$ / inveniam: invenias $H$ | superpartientis $C E R$ superparticularis HV
$s 4$ correspondent $C$ correspondet $E H R$ $V$ / inequalitate $H R V$ inequalitatis $C$ equalitate $E$

5543 et 12 . Dico, itaque, quod denominatio proportionis date est ${ }^{12} / 43$. Et ita de aliis est agendum.
Secunda regula. Propositis duobus numeris si inter eos fuerit unus numerus medius proportionalis invenire. Si ex ductu unius in reliquum fiat numerus quadratus inter eos est unus numerus medius qui est radix illius
6o quadrati ex ductu unius in alteram producti. Hoc potest probari ex $15^{a}$ sexti et $20^{\circ}$ a septimi sicut patet in commento Campani in 20 a septimi. Sed ponatur exemplum sicut 8 et 2 . Et quia ex ductu unius in alterum sit 16 , qui est quadratus, ideo radix eius, scilicet 4 , est medius inter numeros assignatos.
${ }_{6}$
Nota, tamen, quod si inter duos primos numeros illius proportionis reperiatur unus numerus medius necesse est quemlibet eorum esse quadratum. Et similiter inter quoslibet quadratos immediatos reperitur unus numerus medius.

Et iterum per istammet regulam poteris videre si inter numerum 7o medium vel quodlibet aliud extremorum est unum medium. Unde si sint aliquot numeri continue proportionales et inter aliquem eorum et sibi proximum sit aliquis numerus medius inter quemlibet eorum et sibi proximum erit etiam secundum eandem proportionem numerus aliquis medius per octavam octavi. Et sic poteris scire si inter numeros ${ }_{5}$ datos fuerint 3 numeri medii, et similiter si 7 , et si 15 , et sic in infinitum procedendo per tales numeros impares qui oriuntur ex additione numeri immediate sequentis cum numero mediorum numerorum, ut si inveneris is poteris invenire 31 , si sunt, et cetera.

55 ante 43 scr et del $H$ quatuor / ante Dico bab $V$ ita / itaque $C E H$ ita $V$ igitur $R$
s 6 est agendum om $R$
57 Secunda regula $m g$ bab $E$ ante Propositis; om CHV secunda conclusio $m g$ bab $R$ ante Propositis / ante numeris scr et del E verbum illegibile | ante fuerit bab $R$ non
$s 8$ proportionalis om $H /$ in om $V /$ fiat: sit $E /$ numerus: numero $C$
59 unus $t r R$ post medius
60 potest probari $H R V$; tr $C$ potest probare $E$
61 1 $5^{\text {a }}$ : quinta $H$ | ante $20^{0^{2}}$ scret del $V$ Io
62 septimi: primi $E /$ ponatur: ponitur $C /$ quia: patet $E /$ ductu: ductus $V$ 64 medius: medium $C$
6) tamen om $E$ / duos om $E$ / primos numeros EHV; $\operatorname{tr} C R /$ illius $H R V$;
om $E$ iterum $C$
65-66 proportionis om $E$
66 reperiatur: reperitur $V /$ quemlibet: quamlibet $V$
67 eorum: illorum $C$ / Et similiter: aliter $E$ / inter: in $E$
68 reperitur: reperiatur $R$
69 istammet $C E V$ itaque(?) $H$ istam $R /$ regulam: item(?) $E /$ poteris $C R$ potes $E V$ potest $H$ / videre: videri $E$ / numerum om $R$
69-70 numerum medium: numeri medii $E$
70 vel quodlibet aliud $H$ vel quamlibet et aliquod $C$ vel quodlibet aliquod $E$ vel quemlibet quod $R$ et aliquod vel quodlibet $V /$ unum medium $C H V$; $\operatorname{tr} E R$
71 aliquot $C H$ aliqui $E R$ aliquod $V /$ continue proportionales $\operatorname{tr} R$ / post et
triple superpartient seven-twelfths ratio]; the [required] numbers are 43 and 12 . Thus I say that the denomination of the given ratio is ${ }^{12} / 43$. The same must be done for other cases.
Rule Two. [How] to find if there is a mean proportional number between two proposed numbers. If a square number is produced by the multiplication of the one into the other, then there is a mean number between them which is the root of that square produced by multiplying one by the other. This can be proved by the fifteenth of the sixth and the twentieth of the seventh [books of Euclid], which is clear from Campanus' comment on the twentieth [proposition] of the seventh [book]. But let us give an example, say 8 and 2. Since the multiplication of one by the other is 16 , which is a square [number], the root of it, namely 4, is a mean between the numbers assigned.
Observe, however, that if a mean number is found between two prime numbers of this [given] ratio, it is necessary that each of them be a square. And, similarly, a mean number is found between any successive square [numbers].
And again by this very same rule you can see if there is a mean number between a mean number and any one of the extremes. Thus if there should be some continuously proportional numbers and there is a mean number between some one of them and the number proximate to it , then, by the eighth [proposition] of the eighth [book of Euclid], between any one of these numbers and its immediate neighbor there will also be a mean number forming the same ratio. In this way you can discover if between the given numbers there are 3 mean numbers, or 7 , or 15 , and so on ad infinitum, by proceeding through such odd numbers which arise from the addition of the number immediately following [the number of mean numbers] with the number of the mean numbers, so that if you should find is [mean numbers] you can find 3 I-if there are [3I] etc.*

* The general expression for finding the is the initial number of means between two total number of means is $2 n+1$, where $n$ given extremes.


## bab V si

72 post sibi scret del V prox / aliquis $E R V$ aliquod medium vel $C$ / eorum $E R V$; om C
72-73 sit...proximum om $H$
73 erit: esset $V /$ ante etiam scr et del $V$ et
74 poteris: potest(?) $H /$ scire om $C /$ post scire hab $E$ hoc non est demonstratum $757 H R V$ unus $E /$ et $^{2} H R V$ similiter
$E / \mathrm{si}^{2} E H V$ similiter $R$
75-77 et similiter...numeri om $C$
76 tales $H R V$ alios $E /$ qui: que $R$
77 numero: numerorum $R$ / numerorum $E H R$ terminorum $C V /$ ut: et $R$
78 inveneris: invenetur $E /$ poteris $R V$ partes $C$ potes $E$ potest $H \mid$ ante invenire $m g$ hab $E$ conclusio secunda / et cetera $E H V$; om $C R$

Tertia regula. Datis duobus numeris si inter eos fuerint duo numerii medii so investigare. Si fuerint duo numeri minimi vel primi illius proportionis et uterque eorum sit cubicus, tunc inter eos sunt duo numeri medii qui sic inveniuntur: duc radicem maioris in quadratum radicis minoris et habetur minor; deinde ducas radicem minoris in quadratum radicis maioris et habetur maior. Si vero alter non fuerit cubicus non erunt
$8_{5}$ inter eos duo numeri medii.
Idem quoque invenitur si uterque fuerit cubicus quamvis non sunt minimi et ita sive sint proximi sive non. Verbi gratia, in numeris proximis sint 27 et 8 . Capiamus radicem maioris, scilicet 3 , et ducatur in quadratum radicis minoris, qui est 4 , et proveniunt 12 minor nu-
90 merorum mediorum. Deinde ducatur radix minoris, scilicet 2, in quadratum radicis maioris, que est 9 , proveniunt 18 , maior numerorum mediorum. Habebimus, igitur, duos numeros medios inter numeros datos isto modo: $27,18,12,8$, secundum proportionem sexquialteram.

Aliud exemplum in non minimis nec proximis sint 216 et 8 . Ducatur radix maioris, scilicet 6 , in quadratum radicis minoris, scilicet in 4 , proveniunt 24 , minor mediorum; deinde radix minoris, scilicet 2 , in quadratum radicis maioris, scilicet 36 , proveniunt 72 , maior mediorum. Sunt, igitur, quatuor numeri sic dispositi: $216,72,24,8$, et sunt 100 continue proportionales secundum proportionem triplam.

Inventis, itaque, duobus mediis inter duos numeros extremales sunt

79 Tertia regula $m g$ hab $P$ ante medii; om CEHRV
80 numeri minimi : medii numeri(?) $H \mid$ post numeri bab $V$ ure(?) / illius $C H R$ alicuius $E V$
81 corum sit: illorum $R$
82 inveniuntur: in medium $V$ / ante radicem scr et del $V$ radicem
83 ducas: duc $R$ / ante quadratum scr et del $V$ quod
84 ante Si add $C$ et / vero: iterum(?) $H /$ alter $C H R$; om $E V$
84-85 non erunt inter eos: eos non fuerint $E$
85 duo: et $V /$ numeri medii: medii(?) $H$
86 invenitur: invenietur $R /$ uterque: utrique $V$ | post uterque add $C R V$ eorum et $E$ horum / sunt: sint $C$
87 minimi om $H$ / sive... non $H$ si fuerint
proximi sive non sint proximi $E V$ s fuerint proximi sive non proximi $C$ si fuerint proximi sive non $R /$ Verbi sive non $R$ / Verbi grati rep $H /$ numeris: minimis $R$
88 ante proximis add CERV et / sint CHR sicut EV/3 om H
89 in om $H$ / est: sunt $R$ / proveniunt $E H R$ provenit $C V$ 90 ante mediorum bab $H$ figuram

ucatur: ducantur $H /$ minoris $E R V$;
om $H$ maioris $C \mid$ 2: secundum $V$ 91 que $H R V$ qui $C E /$ est: sunt $R$

Rule Three. [How] to investigate if there are two mean numbers between two given numbers. If there should be two least or prime numbers of this [given] ratio and each is a cube [number], then there are two mean numbers between them, which are found in the following way: Multiply the [cube] root of the greater [given number] by the square of the [cube] root of the lesser [given] number and you get the smaller [mean]; then you mutiply the [cube] root of the lesser [given] number by the square of the [cube] root of the greater [given number] and this yields the greater [mean].* If, however, one of them is not a cube [number], there will not be two mean numbers between them.

One can also determine whether each [number] is a cube-even if they are not in their lowest terms-and whether they are proximate or not. For an example in proximate numbers, take 27 and 8 . We take the root of the greater, namely 3 , and it is multiplied by the square of the lesser root, which is 4 , to produce $\mathbf{1 2}$, the lesser of the mean numbers. Then the root of the lesser [number], namely 2 , is multiplied by the square of the root of the greater [number], which is 9 , to produce 18 , the greater of the mean numbers. We will, therefore, have two mean numbers between the given numbers in this arrangement: 27, 18, 12, 8, which form a sesquialterate ratio.

Another example where the [given numbers] are neither in their least terms nor proximate would be 216 and 8 . The [cube] root of the greater, namely 6 , is multiplied by the square of the root of the lesser [number], namely 4 , to produce 24 , the lesser of the means; then the root of the lesser, namely 2 , [is multiplied by] the square of the root of the greater, namely 36 , to produce 72 , the greater of the means. There are, consequently, four numbers arranged as follows: $\mathbf{2 1 6}, \mathbf{7 2}, 24,8$ and they are continuously proportional, forming a triple ratio.

Furthermore, once the two means between the two extreme numbers

* If $A$ and $B$ are cube numbers two means mined by taking $A^{1 / 3} \cdot\left(B^{1 / 3}\right)^{2}$; the greater can be assigned. The lesser mean is deter- by $B^{1 / 3} \cdot\left(A^{1 / 3}\right)^{2}$.

92 Habebimus $C H$ habemus $E R V$ duos om $C$ / post inter add $V$ duos
93 post proportionem scr et del $V$ seq
95 Aliud om $R$ / post non scr et del $E$ numerus(?)/minimis: numeris $C$ /ante nec bab $H \mathrm{f}(?) /$ sint $C R$ sicut $E$ sic $H$ sicud $V$ | ante I in 216 scr et del $H 6$ (?)
96 maioris rep $V$ | ante scilicet scr et del $V$ maioris(?) / quadratum: quadratus $C$ /
radicis om $C /$ minoris: maioris $R$
97 proveniunt: proveniet $V$
98 36:30 C/proveniunt: proveniet $V /$ 72:12 E
99 post 216 scr et del V 72(?) / 72: 12 E roo ante triplam scr et del $V \mathrm{t}$
100-102 secundum... proportionales om $R$ IOI ante extremales scr et del $V$ exe / ante sunt add $C E V$ et
quatuor numeri continue proportionales. Si, igitur, inter primum et secundum sint duo numeri medii secundum aliquam proportionem, sicut potest sciri ex ista regula, sequitur quod similiter inter secundum et tertium. Ymo inter quoslibet in eadem proportione relatos erunt duo numeri medii secundum eandem proportionem secundum quam erant inter primum et secundum per octavam octavi. Igitur sicut per istam regulam scitur si inter numeros datos sint duo alii medii et inveniuntur, ita per eandem regulam inveniuntur 8 , si fuerint, et 26 et 80 et sic in infinitum procedendo per quosdam numeros quorum generatio sic habetur: capta prima radice, scilicet 2, accipiatur numerus qui sequitur uno intermisso, scilicet 4 , et addatur cum duplo illius radicis, qui est 4 , et proveniunt 8 et habetur secundus numerus.

Ad habendum tertium consimiliter est agendum. Capiatur 8, deinde 115 numerus qui sequitur uno intermisso, scilicet 10 , et addatur cum duple ipsius 8, qui est 16 , et proveniunt 26 , qui est tertius numerus mediorum qui possunt per hanc regulam reperiri. Et sic est ulterius operandum.

Iterum, operando semel per istam regulam deinde et semel per regulam precedentem inveniendo unum medium in quolibet intervallo
120 investigatur si fuerint s numeri medii inter numeros assignatos; et agendo semel per istam regulam et bis per secundam si 11 , et semel per istam et ter per secundam si 23 , et bis per istam et semel per secundam si 17 . Et ita multifarie multisque modis istas regulas duas copulando investigatur si inter numeros datos fuerit unus numerus ${ }_{125}$ medius et quis sit ille, et si 2, aut 3, aut 5 , aut 7 , aut 8, aut II, aut 17,

102 continue $C E H$; om $V$ / post primum add $E$ numerum
103 sint $C H R$ sunt $E V$
104 sciri: scire $V /$ quod similiter $C E H$; tr RV
ros Ymo CHV conclusio(?) $E$ et $R /$ in eadem om $V$
107 post primum add $E$ numerum / Igitur: ideo $R$
108 ante datos add $V$ duos / post alii add $E$ numeri / alii medii: numeri medii alii $R /$ medii $C E H$; om $V$
109 inveniuntur: invenitur $V /$ ita CHV itaque $E$ ista $R$
IIo et ${ }^{\text {o }}$ m $V / 80$ corr ex 60 CEHR $26 \mathrm{~V} /$ infinitum: infinito $V /$ quosdam: quosdem $E$
in ante habetur scr et del $V$ heb / accipiatur: accipitur $C$

12 ante uno scr et del $V$ uno / duplo: dupla H
II3 qui: que(?) $H / \mathrm{et}^{\mathrm{t}}$ om $H /$ proveniunt: perveniunt $V$
114 tertium: tertius $E$
i1s 10: $5($ ?) $H /$ post 10 scr et del $V$ decime 116 ipsius: ipsorum $R$ / qui est 16: et sunt $16 R /$ et om $H /$ proveniunt: provenit $C$ lante 26 bab $H s(?) / 26: 16 R / q u i$ est tertius numerus $H V$ numerus tertius $C E R$
117 qui om $H$ / possunt: ponitur $C$ / reperiri $E V$ inveniri $C R$ experiri(?) $H /$ est ulterius $E H V ; \operatorname{tr} C R /$ operandum: agendum et cetera $V$
118 Iterum: item $V$ / ante regulam add $R$ tertiam / et CH; om ERV
i19 regulam: secundam $R$ / inveniendo: inveniendum $E$ / unum: numerum $V$
have been found, there are four continuously proportional numbers. Therefore, if between the first and second [numbers] there should be two mean numbers which form some ratio-and this can be known by this rule-it follows similarly [that the same should obtain] between the second and third [numbers]. Indeed, by the eighth [proposition] of the eighth [book of Euclid], between any numbers related in the same ratio there will be two mean numbers forming the same ratio as was formed between the first and second [numbers]. Hence, just as it can be known by this rule whether there might be two other means between the given numbers and [how] they are found, so also by the same rule one can find 8 means if there are 8 , and 26 , and 80 , and so on ad infinitum, by proceeding through [a sequence of] certain numbers whose generation is determined as follows: The first root, namely 2 , being found, the number following its immediate successor is taken, namely 4 , and is added to twice the root, which is 4 , to produce 8 ; and thus the second number [of total means] is found.

One must operate in a similar way to obtain the third number. Let 8 be taken, then the number which follows its immediate successor, namely io, is added to the double of 8 , which is 16 , to produce 26 , the third number of means that can be found by this rule. And one may proceed further in this manner.*

By operating in turn once by this rule and once by the preceding rule for finding one mean in any interval, one can investigate if there are $s$ mean numbers between the assigned numbers; and operating once by this rule and twice by the second, one can ascertain if there are in [mean numbers]; and once by this [rule] and three times by the second to see if there are 23; and twice by this rule and once by the second to see if there are 17. And thus by combining these two rules, t one can investigate in many different ways if between given numbers there is one mean number and what it is; and if there are $2,3,5,7,8,11,17,23$, or 26 [mean numbers], etc.; and
$\begin{array}{ll}\text { * Assuming, initially, two mean terms be- } & \text { where } n \text { represents the number of means } \\ \text { tween the extremes, the formulation } 3 n+ & \text { and, at the outset, equals } 2 \text {. } \\ 2 \text { will yield the successive totals of means } & + \text { The rules are: } 2 n+1 \text { and } 3 n+2 .\end{array}$

121 semeli om $C$ / regulam $H V$; om $C E R$ /
bis per secundam: per secundam bis
$R / \mathrm{et}^{2}$ om $H$
122 ter: tot $V$
122-23 secundam: tertiam $H$
123 17: $11 E /$ ita $C H R$; om $V$ ibi $E \mid$
istas: illas $V /$ regulas duas $H$; tr

CERV
124 unus om $V$
124-25 numerus medius $C E R$; tr $V$ medius(?) $H$
125 quis: quid $H$ /ille: iste $R / \mathrm{et}^{2}$ : aut $E$ | post si add $C$ sunt / aut 9 : aut 9 (?) $H$ / 17: 15 $E$
aut 23, aut 26, et cetera, et sic per varios numeros multipliciter discurrendo. Si, autem, ad quelibet media proportionalia invenienda habeas regulam generalem placet mihi.

Verumtamen si de aliquo numero mediorum velis temptare, verbi gratia, si vis scire utrum inter $A$ et $B$ sint quatuor numeri medii secundum $C$ proportionem, qui quidem $A$ et $B$ sunt minimi, age sic: invenias sex numeros continue proportionales secundum $C$ proportionem in sua proportionalitate minimos, sicut docet secunda octavi, qui sunt $D E F G H K$. Igitur per tertiam octavi $D$ et $K$ sunt minimi, tur si $D$ est $A$ et $K$ est $B$ inter $A$ et $B$ sunt quatuor numeri medii secundum $C$ proportionem. Sed tamen multa in hiis regulis dicta possent ex arismetica et geometrica demonstrari, sed nolui diutius immorari.

## Tertium capitulum

- In hoc tertio capitulo aliqua magis specialia de proportionibus proportionum adiungam pro quibus quedam suppositiones primitus sunt ponende.

Prima est nulli duo numeri quorum maior est multiplex minoris
5 sunt contra se primi seu in sua proportione minimi. Proba eam ex diffinitione contra se primorum in septimo Euclidis.

Secunda est cuiuslibet proportionis multiplicis alter primorum numerorum est unitas. Hec est consequens ad primam quia eius oppositum infert oppositum prime et patet in prima regula secundi capituli.
${ }_{10}$ Tertia est omnis numerus medio loco proportionalis inter aliquem numerum et unitatem est medius secundum proportionem multipli-

126 23:33 $E$
127 quelibet $C V$ quodlibet $E R$ quotlibet H
128 mihi: michi $V$
129 velis $E R V$ velit $H$ vel $V$
${ }^{1} 30$ vis $E H V$ velis $C R /$ utrum: utrumque (?) $E /$ sint $H R V$ sunt $C E$
13x-33 qui....proportionem om CER
132 invenias $H$ invenies $V /$ continue $H$; om $V$
133 minimos: numeros $R$ / octavi: noni(?) H
133-34 qui sunt $C R V$ qui $\operatorname{sint} E$ que sit $H$
 $C H R$ octavam $E$ triplam $V /$ post
tertiam add $V$ primi / sunt: sint $V /$ minimi: numerum $V$
135 est ${ }^{1}$ : et $E /$ ante $\mathrm{A}^{1}$ scr et del $V \mathrm{~K} /$ ante $\mathrm{B}^{2}$ bab $V$ que / numeri om $C$ / numeri medii $\operatorname{tr} R /$ medii $C E V$ inter(?) medios(?) $H$
136 post proportionem scr et del $V$ in sua proportionalitate minimos / Sed tamen $C$ sui(?) autem cum autem $E V$ seu autem(?) et cum arguitur(?) $H$
136-38 Sed tamen...immorari om $R$
137 ex $E H V$ in $C$ / ante arismetica scr et del $V$ aris / post arismetica bab $E$ probari / et $E H V$ ex $C$ / geometrica $C E$ $V$ geometria $H$ / nolui $C E$ nolluimus
this can be done by proceeding through different numbers in various ways. Moreover, it would please me if you should know a general rule for finding any [number of] mean proportionals whatever.
However, if you wish to try to find any number of means, as, for example, you might desire to know whether there are four mean numbers forming ratio $C$ between $A$ and $B$, where $A$ and $B$ are in their least terms, do as follows: Find six least numbers continuously proportional in that proportionality which forms ratio $C$, as taught in the second [proposition] of the eighth [book of Euclid]; let the terms be $D, E, F, G, H, K$. Therefore, by the third [proposition] of the eighth [book of Euclid], $D$ and $K$ are least numbers so that if $D$ is $A$ and $K$ is $B$ there are four mean numbers which form ratio $C$ between $A$ and $B$. And, finally, many things said in these rules could be demonstrated from arithmetic and geometry, but I wish to delay no longer.

## Chapter Three

In this third chapter I shall add some more special matters concerning ratios of ratios, and for this some suppositions must first be enunciated.

The first is: No two numbers where the greater is multiple to the lesser are mutually prime or in their least terms in that ratio. I prove this by the definition of mutual primes in the seventh [book] of Euclid.
The second is: One of the prime numbers of any multiple ratio is a unit. This is a consequence of the first supposition since the opposite infers the opposite of the first supposition. And this [supposition] is obvious in the first rule of the second chapter.
The third is: Every mean proportional number between some number and the unit is a mean which forms a multiple ratio. The reason is that

[^24]cem. Causa est quia omnis numerus est multiplex unitati quoniam cuiuslibet numeri pars est unitas ut dicitur in principio septimi.

Quarta est nullus est numerus medius proportionaliter seu numeri ${ }_{15}$ inter primos numeros proportionis multiplicis nisi secundum proportionem multiplicem. Ista sequitur ex secunda et tertia.

Quinta est nullius proportionis non multiplicis aliquis numerorum primorum est unitas. Proba sicut tertiam quia aliter numerus esset multiplex unitatis et tunc proportio esset multiplex.
${ }_{20}$ Sexta est nullus est numerus medius seu numeri medii inter primos numeros proportionis non multiplicis secundum proportionem multiplicem, quia se numerus medius esset multiplex minoris et maior multiplex medii tunc maior esset multiplex minoris et sic proportio esset de genere multiplici.
${ }_{25}$ Adverte quod propter brevitatem loquendi voco proportiones in mediis convenire seu participare quando inter primos numeros maioris est numerus medius seu numeri secundum proportionem minorem aut secundum aliquam aliam proportionem secundum quam inter primos numeros minoris sit etiam numerus seu numeri medii eodem
${ }_{30}$ modo quo dicebatur in probatione none secundi capituli. Sequunter conclusiones.
Prima conclusio. Nulla proportio multiplex sive de genere multiplici est commensurabilis proportioni non multiplici vel de alio genere minori ea.

Sit $A$ proportio multiplex, $B$ non multiplex, et sit $B$ minor. Tunc
${ }_{35}$ si $A$ est commensurabilis $B$ igitur $A$ et $B$ sunt duo numeri per quintam decimi. Igitur $B$ minor est pars aut partes maioris, scilicet $A$, per quartam septimi.

I2 ante omnis scr et del $V$ o / numerus om $V /$ unitati: unitatis $E$
13 post dicitur scr et del $E$ in principiis ibi/ principio $H R$ principiis $C E V$ / septimi: primi $E$
14 est $^{1}$ om $V /$ est $^{2}$ om $E \mid$ ante seu scr et del $V \mathrm{~s}$
1s proportionis: per $R$ /ante multiplicis add $E$ non / nisi om $E$
16 Ista $E H V$ ita $C$ illa $R$
${ }_{17}$ est $E H R$; om $C V /$ nullius: nullus $C$ / non multiplicis om $H$ | numerorum: nurnerus $E$
18 primorum: primarum $V /$ Proba: probatur $E$ / ante tertiam scr et del $E$ prius (?)

19 unitatis: unitati $E$
19-22 multiplex ${ }^{1}$...esset om $V$
20 est $^{1} C E H$; om $R /$ est $^{2} C H R$; om $E$ । medius $E H$; om $C R /$ seu $E H R$ vel $C /$ numeri $C H R$; om $E$
2I post proportionis mg hab $H$ sexta suppositio
$22 \mathrm{se} C E H$ si $R$ / ante multiplex scr et del $V \mathrm{mx}$
23 minoris om $C$
25 ante Adverte add RV et / ante loquendi bab $R$ modum
26 convenire: communicare(?) $E /$ seu: vel $C$ / quando: quoniam $C$
27 est om $H$ / seu: vel $C$ / post numeri add $E$ medii / minorem: minoris $C$
every number is multiple to the unit since the unit is part of any number, as stated in the beginning of the seventh [book of Euclid].
The fourth is: There is no mean proportional number or numbers between the prime numbers of a multiple ratio except those that form a multiple ratio. This follows from the second and third suppositions.

The fifth is: None of the prime numbers of a non-multiple ratio is a unit. I show this as in the third [supposition], because otherwise the number [in such a non-multiple ratio] would be multiple to the unit and the ratio would then be multiple.

The sixth is: There is no mean number or numbers that form a multiple ratio between the prime numbers of a non-multiple ratio, because if the mean number were multiple to the lesser and the greater number multiple to the mean, then the greater would be multiple to the lesser and the ratio would be in the multiple genus.*
Note that for the sake of brevity of expression I say that ratios "unite or participate in means" when between the prime numbers of the greater ratio there is a mean number or numbers forming the lesser ratio, $\uparrow$ or forming some other ratio [also] produced by the mean number or numbers lying between the prime numbers of the lesser ratio $\ddagger$ in the same way as described in the proof of the ninth proposition of the second chapter. The propositions follow.

Proposition I. No multiple ratio is commensurable to a smaller non-multiple ratio.
Let $A$ be a multiple ratio, $B$ a non-multiple ratio and $B$ is less than $A$. Then, if $A$ is commensurable to $B, A$ and $B$ are [related as] two numbers, by the fifth [proposition] of the tenth [book of Euclid]. Hence $B$, the lesser, is a part or parts of the greater, namely $A$, by the fourth [proposition] of the seventh [book of Euclid].

* See p. 354.
 then $A=(B)^{n}$, where $n$ is an integer.

28 aliquam om $R$
29 minoris om $H / \operatorname{seu} E H R$ vel $C$ aut $V$ | eodem: eo $R$
30 dicebatur: dicebitur $C$ / probatione proportione $E$ / secundi: quinti $E$
32 ante Prima $m g$ bab $E$ prima conclusio / Prima conclusio $E V$; om $C$ prima $H$ prima est $R$ sed $R m g$ bab prima conclusio / Nulla proportio rep $V$ / post proportio scr et del $V$ in multiplex sive
de genere multiplici est commensurabilis proportioni non multiplici vel de alio genere minori ea / sive: vel $C$ 33 vel om H | ante minori scr et del $E \mathrm{mi}$ 34 post multiplex ${ }^{\mathrm{I}}$ add $C R V$ et / B: B sit $R / \mathrm{B}$ non multiplex om $E$ / et om $H$ | sit om $C$
35 si A est: sicut $H / \mathrm{Br}^{\mathrm{r}}$ om $V /$ duo om $C$ 37 septimi: primi $E$

Si pars igitur inter primos numeros est medium aut media secundum $B$ proportionem per sextam secundi capituli. Sed hoc est impossibile ${ }^{40}$ per quartam suppositionem.

Si, vero, $B$ sit partes ipsius $A$ igitur communicant in mediis per septimam secundi capituli. Aut, ergo, inter utrosque horum numerorum est medium, et cetera, secundum proportionem multiplicem et hoc est impossibile per sextam suppositionem quia $B$ non est de genere
45 multiplici, aut secundum aliam proportionem et hoc est impossibile per quartam suppositionem quia $A$ est de genere multiplici.

Secunda conclusio. Nulla proportio multiplex est commensurabilis alicui non multiplici maiori ea.

Sit $A$ multiplex, $B$ non multiplex maior. Si, igitur, sunt commen-
${ }_{50}$ surabiles, aut $B$ est multiplex ad $A$ igitur inter primos numeros $B$ est medium secundum proportionem $A$, per sextam secundi capituli, et hoc non posset esse per sextam suppositionem, aut $B$ se habet ad $A$ in alia proportione et tunc sequitur quod $A$ sit partes $B$ quod probatur esse impossibile per quartam et sextam suppositiones, sicut prius est argutum.

Tertia conclusio. Nulla proportio de genere multiplici est commensurabilis alicui que non sit de genere multiplici.
Immediate patet quia nulli minori per primam conclusionem, nec alicui maiori per secundam. Unde sequitur quod semper proportio
6o multiplex addita multiplici facit proportionem multiplicem quod etiam patet quia cuiuslibet proportionis multiplicis aliquis numerus est denominatio. Modo si numerus per numerum multiplicetur semper provenit numerus integrorum et omnis numerus alicuius proportionis multiplicis est denominatio. Et additio proportionis ad proportionem sit
$6_{5}$ multiplicando denominationem per denominationem ut in primo capitulo dicebatur.

38 post numeros add $V$ a
39 hoc om $E$
4I B om $H$
42 septimam: secundam $E /$ secundi: quarti(?) $H$ / Aut ergo om $H$ / utrosque $H R$ utrasque $C E V$
44 suppositionem om $H$
45 et HRV; om $E$
45-46 aut... multiplici om $C$
46 post de scr et del $V$ multiplici / post multiplici bab $E$ igitur
47 Secunda conclusio $V$;om $C H$ secunda
conclusio mg bab $E$ ante Nulla et mg bab $R$ post non (linea 49) / proportio multiplex est: est proportio $R$
48 ea om $R$
49 post non add $V$ est
49-50 commensurabiles: commensurabilia $C$
so post aut add $R$ igitur
si proportionem $H V$; om $C E R / \mathrm{A}$ : et $V$ / ante per bab $V$ confirmo
$\varsigma_{2}$ post non bab $E$ per / posset $H R$ potest CEV

Chapter Three
If it is a part of $A$, then between the prime numbers $[$ of $A]$ there is a mean or means forming ratio $B$, by the sixth proposition of the second chapter. But this is impossible by the fourth supposition.

If, however, $B$ is parts of $A$, they "unite in means," by the seventh proposition of the second chapter. Therefore, either there is a mean or means between the numbers [of $A$ and $B$ ] forming a multiple ratio, which is impossible by the sixth supposition since $B$ is not a ratio of a multiple kind, or forming some other [non-multiple] ratio, which is impossible by the fourth supposition since $A$ is a ratio of a multiple kind.

Proposition II. No multiple ratio is commensurable to any greater non-multiple ratio.

Let $A$ be a multiple [ratio] and $B$ a greater non-multiple ratio. If they are commensurable, either $B$ is multiple to $A$, in which event, by the sixth proposition of the second chapter, there is a mean forming ratio $A$ between the prime numbers of $B$, which is impossible by the sixth supposition; or $B$ is related to $A$ in another ratio, from which it follows that $A$ is parts of $B$; and this was shown to be impossible by the fourth and sixth suppositions, as was argued before.

Proposition III. No ratio in a multiple genus is commensurable to any ratio which is not in a multiple genus.

This is immediately obvious because, by the first proposition, it is not commensurable to any lesser ratio, and, by the second proposition, it is not commensurable to any greater ratio. It follows from this that when a multiple ratio is added to a multiple ratio it always produces a multiple ratio. This is also evident because some number is [always] the denomination of any multiple ratio. In this way if a number is multiplied by a number, it produces a number; and every number is the denomination of some multiple ratio. The addition of a ratio to a ratio is achieved by multiplying their denominations, as was said in the first chapter.

53 et om $R$
s3-54 probatur: probabitur $E$
54 suppositiones: suppositionem $V$
55 est: erat $C$
56 Tertia conclusio $V$; om $C H$ tertia conclusio mg bab ER ante Nulla
56-57 est... multiplici om $E$
57 alicui $H V$; om $R$ alicuius $C$
58 post primam bab C s
60 multiplici: unitate $V /$ proportionem multiplicem $\operatorname{tr} C$

6x cuiuslibet $H R V$ cuiuscumque $E$ cuilibet $C$ / proportionis multiplicis $\operatorname{tr} C$ / aliquis: aut $V$
62 si: scilicet $E /$ per numerum om $V$ 63 proportionis om $V$
64 ante est bab $R$ aliquis numerus / sit: facit $V$
65 multiplicando: multitudo $C$ / denominationem ${ }^{2} C H V$ denominatione $E$ denominatorem $R /$ ut om $E$
66 dicebatur: dicebitur $C$

Item sequitur ex dictis quod si aliqua proportio de genere non multiplici duplicetur aut triplicetur aut quomodolibet aliter replicetur numquam provenit proportio de genere multiplici quia aliter sequitur 7o quod talis proportio esset alicuius multiplicis quod est impossibile per primam conclusionem. Unde si denominatio proportionis non multiplicis, que est numerus cum fractione vel cum fractionibus vel numeri cum fractione, et cetera, per se ipsam quotiens multiplicetur numquam haberetur precise numerus integrorum.
Item sequitur quod si aliqua multiplex componitur ex pluribus non multiplicibus, sicut dupla ex sexquialtera et sexquitertia, vel ex multiplici et alia, sicut tripla ex dupla et sexquialtera, quelibet componentium erit incommensurabilis composite et etiam erunt incommensurabiles inter se.
Item patet quod multiplex bene componitur ex non multiplicibus sed numquam aliqua non multiplex ex multiplicibus componetur. Nulla etiam multiplex est multiplex non multiplicis.
Quarta conclusio. Si fuerit aliqua proportio de genere multiplici inter cuius denominationem et unitatem non sit medium seu media ipsa erit incommensura-
$8_{5}$ bilis cuicumque minori et cuilibet maiori que non est multiplex ad eam et de genere multiplici.
Cum enim denominatio eius et unitas sint eius primi numeri ut patet ex prima regula secundi capituli et inter eos nullum sit medium, illa proportio erit incommensurabilis cuicumque minori et cuilibet maiori
90 que non est multiplex ad eam per quintam secundi capituli. Et nulla est multiplex ad eam nec etiam commensurabilis nisi sit de genere multiplici. Per precedentem sequitur itaque propositum.

Ex istis et aliis leviter patet quod proportio proportionum non est sicut proportio suarum denominationum. Iam enim omnes propor-

67 post proportio bab $H$ sit / de genere rep C/ non om $E$
68 duplicetur aut triplicetur HV dupletur aut tripletur $C E R$
69 provenit: proveniet $V$
72 multiplicis: multipliciter $E /$ post fractione scr et del $E$ nu / $\mathrm{cum}^{2} \mathrm{H}$; om $C E R V / \mathrm{vel}^{2}$ om $E$
73 et cetera...ipsam om $R /$ quotiens $C H V$ quotienslibet $E R$
74 haberetur $H R V$ habetur $C E /$ precise numerus $H R V$; tr $C E$
75 componitur $C E H$ componatur $R V /$
post non scr et del $V \mathrm{pl}$
76 et: vel $R$ / ante vel scr et del $V$ ex 77 ex: et $R$
77-78 componentium: componentem $V$
78 incommensurabilis: incommensurabiles $V /$ etiam erunt $\operatorname{tr} E$
80 quod: quia $E /$ multiplex om $R /$ non om H
81 post sed hab $C$ non / componetur: componeretur $V$
82 etiam: enim $C$ / est multiplex om $E /$ post non scr et del $V \mathrm{~m}$
83 Quarta conclusio $V$; om $C$ quarta

It also follows from what has been said that if some ratio of a nonmultiple kind be doubled or tripled, or multiplied in any way whatever, no multiple ratio could ever be produced; for otherwise it follows that such a ratio would be [part] of some multiple ratio, which is impossible by the first proposition. Thus, if the denomination of a non-multiple ratio, which is a number plus a fraction or fractions or numbers plus a fraction, etc., is multiplied by itself a certain number of times, it could never exactly produce an integer.
It also follows that if some multiple is composed of several non-multiples (for example, a double composed of a sesquialterate and sesquitertian),* or composed of a multiple and another kind (a triple composed of a double and a sesquialterate), $\dagger$ any of the component ratios will be incommensurable to the composed ratio and also incommensurable to each other.

It is evident, furthermore, that a multiple can be composed of nonmultiples, but a non-multiple could never be composed of multiple ratios. Also, no multiple ratio is multiple to non-multiple ratios.
Proposition IV. If there were any multiple ratio with no mean or means between its denomination and unity, it will be incommensurable to every lesser ratio and to every greater ratio which is not multiple to it and in the genus of multiple ratios.
Now since its denomination and unity are its prime numbers, as is clear by the first rule of the second chapter, and there is no mean between them, that ratio will be incommensurable to any lesser ratio and to any greater which is not multiple to it, by the fifth proposition of the second chapter. But no ratio is multiple or commensurable to it unless it is in the multiple genus. Hence, from what has just been said, the proposition follows.

From these and other statements it is obvious that a ratio of ratios is not related as the ratio of their denominations. Now all ratios whose de-
$* 2 / \mathrm{I}=4 / 3 \cdot 3 / 2$.
$+3 / \mathrm{I}=3 / 2 \cdot 2 / \mathrm{I}$.
conclusio $m g$ hab $E$ post multiplicis (linea 82) et mg hab H ante est (linea 82) et mg bab $R$ ante Si

84 non: idem $H /$ medium: medius $H /$ seu: vel $C /$ erit $C H V$ est $E$
84-85 erit incommensurabilis: est commensurabilis $R$
85 cuilibet: cuiuslibet $V / \mathrm{et}^{2}$ om $H$
87 Cum: autem $E$

88 prima ... capituli: secundi capituli regula $C /$ eos $E$ ea $H$ eas $C R V /$ illa: ista $R$
89 cuilibet: cuicumque $V$
90 que: qui $E /$ eam $H$ ipsam $C E R V$ 90-91 per...eam om $C$
91 nisi $C H R$ nec $E V /$ de $H R$; om CEV
$9^{2}$ ante Per add $E$ et
93 et om $V$

95 tiones quarum denominationes sunt note erunt commensurabiles. Tripla, quidem, est incommensurabilis duple et, tamen, denominatio eius est sexquialtera ad denominationem duple. Nonacupla vero est dupla triple, et centupla decuple et, tamen, non est talis proportio denominationum, tamen, solum quadrupla et dupla hoc privilegium tenuerunt quod talis est proportio proportionum qualis est proportio denominationum et numquam in aliis reperitur.

Quinta conclusio. Omnis proportio de genere superparticulari est incommensurabilis cuilibet superparticulari et cuilibet alteri que non est multiplex ad ipsam impossibile est esse de genere multiplici.

Cuiuslibet enim proportionis superparticularis primi numeri differunt sola unitate ita quod maior excedit minorem solum per unitatem. Cum igitur inter tales numeros nullus sit numerus medius sequitur quod inter nullius proportionis superparticularis primos numeros est numerus medius. Igitur per quintam secundi capituli ipsa erit incom-
no mensurabilis cuilibet minori et cuilibet maiori que non est multiplex ad eam. Sed nulla superparticularis est multiplex alterius quia iam esset commensurabilis alicui minori cuius oppositum probatum est, igitur nulla superparticularis est commensurabilis alicui superparticulari. Similiter nulla multiplex est multiplex ad eam quia nulla multiplex est
115 multiplex nisi multiplicis, nec commensurabilis nisi multiplici ex tertia huius.

Sexta conclusio. Omnes proportiones de genere multiplici sunt commensurabiles et solum tales quarum denominationes sunt de numero numerorum qui in eadem serie ab unitate continue proportionaliter ordinantur.
Verbi gratia, sit una talis series numerorum secundum duplam pro-

95 erunt $E H V$ in essent $C$ essent $R /$ commensurabiles $C H R$ incommensurabiles $V$ minores(?) $E$
96 quidem: que $E /$ tamen: cum $V$
96-97 eius est: est sicut $C$
97 ad: ac $E /$ vero: non $C$
98 triple: duple $C / \mathrm{et}^{\mathrm{I}}$ om $V$ / centupla: $144^{1 \mathrm{la}} \mathrm{H}$
99 ante solum bab $H$ hoc / quadrupla et dupla $H$ dupla et quadrupla $C E R V /$ hoc om $C$
102 Quinta conclusio $V$; om CH quinta conclusio $m g$ hab $E$ post alteri (linea 103) et mg hab $R$ post denominationum (linea IOI)
102-3 est...superparticulari om $H /$ in-
commensurabilis $C E V$ commensurabilis $R$
103 cuilibet $^{2} C H R$ quilibet $E$ cuiuslibet $V$ / que: qui $E$
104 post ipsam add CRV quam
ros Cuiuslibet: cuilibet $C$ / ante primi bab $V$ denominatores seu / primi rep $V$ ros-6 differunt: differuntur $C$
106 excedit minorem solum: solum excedit minorem $C$
107 numeros om $H$
108 post inter hab H talis / superparticularis om $H$
109 erit $H R$ est $C E V$
ro9-10 post incommensurabilis scr et del $V$ alicui superparticulari
nominations are known will be commensurable. Indeed, a triple ratio is incommensurable to a double and yet the denomination of a triple to a double is a sesquialterate. A nonacuple ratio is double to a triple, and a centuple to a decuple, and yet this not the ratio of their denominations.* Only quadruple and double ratios-and no others-are so privileged that their ratio of ratios is just the same as their ratio of denominations.
Proposition V. Every ratio of a superparticular kind is incommensurable to any other superparticular and to any other ratio which is not multiple to it. [Indeed] it is impossible [for a superparticular ratio] to be in a multiple [genus].

Now the prime numbers of any superparticular differ only by a unit so that the greater [number] exceeds the lesser only by a unit. Therefore, since there is no mean number between such numbers, it follows that there is no mean number between the prime numbers of any superparticular ratio. Consequently, it will be incommensurable to every smaller ratio and to any greater which is not multiple to it. But no superparticular is the multiple of another superparticular because it would then be commensurable to some smaller ratio, the opposite of which has been proved. Therefore, no superparticular is commensurable to any superparticular. Similarly, no multiple [ratio] is multiple to it since no multiple [ratio] is multiple [to any ratio] except to [another] multiple; nor is it [i.e. a multiple ratio] commensurable to any but a multiple ratio, by the third proposition of this chapter.
Proposition VI. In the genus of multiple ratios all ratios are commensurable whose numerical denominations are found in the same series of numbers arranged in continuous proportionality from unity.

For example, let such a series of numbers form a double ratio from unity

* Since $9 / 1=(3 / 1)^{2 / r}$ and their numerical the ratio relating themexponentially, namedenominations are 9 and 3 respectively, it is obvious that $9 / 3=3 / 1$, which differs from
no cuilibet ${ }^{1}$ : cuiuslibet $V /$ cuilibet $^{2}$ : cuiuslibet $V$
in post nulla add $R$ proportio / ante esset scr et del $E$ est
112 ante oppositum scr et del $V$ s / est om $E$ 113 alicui: alteri $E$
114 nulla ${ }^{\mathrm{I}}$ om $R$ / ante multiplex ${ }^{1}$ scr et del $V \mathrm{mx}$ / post eam hab $C$ quia nulla multiplex est multiplex ad eam
IIs nisi ${ }^{1}$ om $E$
the ratio relating themexponentially, name
$\operatorname{ly}^{2} / \mathrm{I}$. The same applies to $100 / \mathrm{t}=(10 / \mathrm{I})^{2 / 1}$.
${ }_{11}$ Sexta conclusio V ; om CH sexta conclusio $m g$ bab $E$ post quarum (linea II8) et mg bab $R$ post commensurabilis (linea 115)
117-18 commensurabiles: incommensurabiles $R$
119 serie: seriem $H$ / continue: continuentur $H$ / proportionaliter ordinantur CHR; $\operatorname{tr} E V$
portionem ab unitate hoc modo: $1,2,4,8,16,32,64$, et sic ultra. Et similiter sit una alia secundum proportionem triplam taliter ordinata: 1, 3, 9, 27, 81, et sic in infinitum et ita de aliis.
Demonstrata una istarum coordinationum, ut puta, prima. Dico 125 quod quelibet proportio que denominatur aliquo istorum numerorum est commensurabilis cuilibet denominate ab aliquo eorundem sicut dupla est commensurabilis quadruple, octuple, et sic de aliis Et quelibet alia proportio que non denominatur aliquo istorum numerorum est incommensurabilis cuilibet istarum sicut sexquialtera, tripla, et cetera.
Dico, primo, quod quelibet istarum sit commensurabilis cuilibet, et cetera, quia cuiuslibet earum primi numeri participant in mediis cum quibuslibet primis numeris cuiuscumque alterius proportionis denominate aliquo numero illius ordinis ut manifeste patet eo quod
${ }_{35}$ cuiuslibet primi numeri sunt denominatio et unitas, ut dictum est. Igitur per nonam secundi capituli ille sunt commensurabiles.
Sed quod nulla alia proportio sit alicui istarum commensurabilis probatur sic quia illa esset multiplex, scilicet de genere multiplici, per tertiam huius. Sed hoc est impossibile quia arguitur sic: nulla alia est - talis quod inter eius primos numeros sit medium aut media secundum aliquam proportionem cuius denominatio sit aliquis numerus istius ordinis; et nulla istarum est talis quod inter numeros eius sit secundum aliam proportionem quam secundum aliquam cuius aliquis istorum sit denominatio. Igitur nulla alia participat in mediis cum aliqua istarum,
${ }_{145}$ igitur nulla alia est commensurabilis alicui istarum per nonam secundi capituli.
Et probatur antecedens quia nullus numerus unius coordinationis est aliquis numerus alterius coordinationis nisi una coordinatio est

121 sic ultra: cetera $R$
122 Et $C E V$; om $R / E t$ similiter om $H$ sit: sicut $E$ / ante triplam add $R$ triplicem vel / taliter: sic $R$
122-23 ordinata: ordinatam $C$
123 81:91 $V / i$ ita: $\operatorname{sic} R$
124 Demonstrata: demonstratur $E /$ prima: primo $V$
125 quelibet: quolibet $E /$ que denominatur: denominata $\mathrm{ab} \mathrm{V} /$ ante aliquo add $R \mathrm{ab}$ / istorum: istarum $H$
126 denominate: denominato $H$ / sicut: si H
127 quadruple octuple $\operatorname{tr} C /$ octuple

EHV; om $R$
129 ante istarum scr et del $V$ et cetera quia cuiuslibet primi numeri participant / tripla CEH triple $R V$
129-30 et cetera om $H$
I 3 I Dico: declaro $R$ / istarum $C H$ talium $E$ talis $R V /$ cuilibet: cuiuslibet $V$
132 et cetera: etiam $E /$ earum om $V$ primi om $H$ / in om $V$
133 quibuslibet $H R V$ quilibet $E$ quibusdam $C$
134 illius ordinis: istius coordinationis $E /$ manifeste patet $\operatorname{tr} R$
135 et unitas rep $V /$ ante dictum add $R$ sepe
in this manner: $1,2,4,8,16,32,64$, and so on. And similarly there could be another forming a triple ratio arranged in this way: $1,3,9,27,81$, and so on into infinity; and the same for other series.
Suppose that the first one of these sequences is represented. I say that any ratio which is denominated by any of these numbers is commensurable to any denominated by others of the same [series of] numbers, as a double is commensurable to a quadruple, octuple; and the same for the others. And any other ratio which is not denominated by any of these numbers is incommensurable to any of them, as a sesquialterate, and a triple, etc.
I say, first, that any of these is commensurable to any other, etc., because the prime numbers of any one of them participate in means with the prime numbers of any other ratio denominated by any number of this series which is manifestly clear because the denomination and unit are the prime numbers, as was said. Therefore, by the ninth proposition of the second chapter, they are commensurable.
And that no other ratio is commensurable to any of these can be proved since it would [then] be multiple-that is, in the multiple genus-by the third propositon of this chapter. But this is impossible because one can argue as follows: no other [ratio] is such that [it would have] between its prime numbers a mean or means forming any ratio whose denomination is some number in this series; and no other of these [ratios] is such that between its [prime] numbers there could be formed another ratio whose denomination is a number in the series. Therefore, no other [ratio] participates in means with any of those [in the series], and hence no other is commensurable to any of them by the ninth proposition of the second chapter.
And the antecedent is proved because no number of one series is a number of another series unless one series is part of another, as [for example],

[^25][^26]pars alterius sicut illa que est secundum proportionem quadruplam est pars eius que est secundum duplam, ut 1,4 , 16 , et cetera. Quilibet istius ordinis est aliquis eius qui est secundum duplam, scilicet 1,2 , $4,8,16$, et cetera, et nulla denominatio recipit medius inter se et unitatem nisi ille numerus medius sit aliquis illius coordinationis. Igitur nulla unius coordinationis participat cum aliqua alterius coordinationis nisi una coordinatio esset pars alterius, ut dictum est.

Ex istis potes videre quod si aliqua proportio multiplex sit dupla ad aliam denominatio maioris est quadrata cuius denominatio minoris est radix, et econverso. Et si una sit tripla ad aliam denominatio maioris est cubica cuius radix est denominatio minoris, et econverso.

Item si aliqua multiplex sit dupla alteri, aut quadrupla, aut sextupla, aut octupla, et sic per denominationes procedendo in paribus intermissis denominatio eius erit quadrata. Si, vero, sit alteri tripla denominatio eius erit cubica et similiter si sextupla et si nonacupla et si duodecupla et sic continue semper duabus intermissis denominationibus. Et sic de
165 utroque ordine ita quod proportio multiplex sit alicui sextupla denominatio eius erit quadrata cubica. Ista possent ex octava noni Euclidis et eius probatione faciliter speculari.

Septima conclusio. Nulla proportio de alio genere quam de multiplici est commensurabilis alteri nisi maior de primis numeris maioris et maior de primis
${ }_{170}$ numeris minoris sunt de numeris aliquorum numerorum qui in eadem ordinatione ab unitate continue proportionaliter ordinantur et similiter cum boc minor maioris et minor minoris sunt de quadam alia serie numerorum qui continue ab

1 50 est ${ }^{\text {I }}$ : et $V /$ post 4 bab $E 8 /$ Quilibet $E H R$ quelibet $C$
150-52 Quilibet...et cetera om $V$
Is I ordinis $E H R$ coordinationis $C /$ scilicet $C E H$ ut $R$
152 et cetera $C E R$; om $H$ / ante recipit $s c r$ et del $E$ rep / medius: medium $R$
153 unitatem $E H R$ unitatis $C$ unitate $V /$ illius $E H$ istius $C R V$
154 Igitur...coordinationis om $C /$ unius $H R$ illius $E V$ / ante participat scr et del $V$ par
154-5s coordinationis om $H$
155 ut dictum est om $R$
is6 potes $V$ potest $C E H R$ / ante videre bab $H$ quis / videre: videri $E$
157 post de- in denominatio ${ }^{\mathbf{I}} \mathrm{mg}$ bab $H$ correlarius / maioris om $E$

158 post econverso add $R$ et cetera / Et CHR; om E
158-59 Et...econverso om $V$
I 99 radix est denominatio $C E H$ denominatio est radix $R$
160 alteri om $E$
162 erit $E H R$ esset $C V /$ quadrata: quadrupla $E$
163 erit: esset $V / \mathrm{et}^{\mathrm{I}}$ om $C / \mathrm{si}^{2}$ om $R /$ duodecupla: I $^{\text {pla }} R$
164 post sic $^{1}$ scr et del $V$ duabus / semper duabus $H$; tr $E V$ duobus semper $C$ semper duobus $R$ / intermissis denominationibus $C R$; tr $V$ intermissis $E H / \operatorname{sic}^{2} H$ si sit $C E R V$
16s ordine: ordinatione $V /$ post quod $s c r$ et del $C$ non / alicui $H R V$ alteri $C E /$ sextupla: octupla $E$
that [series] which forms a quadruple ratio is part of that which forms a double ratio. Thus any [term] of the series $1,4,16$, etc., is part of that [series] which forms a double ratio, namely $1,2,4,8,16$, etc., and no denomination takes a mean between itself and unity unless that mean is a member of the latter series. As was said, then, no members of one series participate with any of another series unless one should be part of the other.
You can see from these statements that if any multiple ratio should be double another, the denomination of the greater is the square of the lesser and the denomination of the lesser is the root, and conversely. And if one should be triple the other, the denomination of the greater is the cube of the lesser and its root is the denomination of the lesser, and conversely.

Also, if any multiple ratio were double another-or quadruple, or sextuple, or octuple, and so on-its denomination will be a square [number] when proceeding through denominations taken in even intervals. If, however [any multiple ratio] were triple another, its denomination will be a cube [number], as it would also be if [any multiple ratio were] six times, nine times, or twelve times [another ratio], and so on, with two intervening denominations omitted. And with reference to each of these series, should a multiple ratio be six times another, its denomination will be a square and cube number.* All this could easily be shown and proved by the eighth [proposition] of the ninth [book] of Euclid.

Proposition VII. No ratio of a kind other than multiple is commensurable to another unless the greater of the prime numbers of the greater ratio and the greater of the prime numbers of the lesser ratio are numbers in the same series arranged in continuous proportionality from unity; and along with these conditions, the lesser [of the prime numbers] of the greater ratio and the lesser [of the prime numbers] of the lesser ratio must belong to another series of numbers arranged in continuous propor-

* See pp. 358-59.

I 66 cubica om $E /$ possent: patent $H$
167 faciliter speculari $E V$ facilius speculari $C R$ facilius speculativiter $H$
168 Septima conclusio $V$; om $C$ septima conclusio $m g$ bab E post maior ${ }^{1}$ (linea 169) et $m g$ hab $H$ post alio et $m g$ hab $R$ ante Nulla
169-70 et... minoris om $C$
170 numeris ${ }^{1} H R$; om $E V /$ sunt $H$ sit $C$
sicut $E R V \mid$ numeris ${ }^{2} H$ numero $C E R V$ I post eadem bab $R$ proportione / ordinatione: coardinatione $C$
171 ordinantur: ordinatur $C /$ minor: minore $H$
172 et minor minoris om $C$ / sunt $H$ sint CRV sicut $E$
${ }^{172-73}$ ab unitate: ad unitatem $R$
unitate proportionaliter ordinantur quod si fuerint continue erunt commensurabiles.
Sit $A$ proportio maior, $B$ minor. Si igitur sunt commensurabiles necesse est eas in mediis convenire per nonam secundi capituli. Aut igitur $A$ est multiplex ad $B$, aut in alia proportione.
Si primo modo igitur inter primos numeros $A$ est numerus medius aut numeri qui conumerati extremis sunt aliquot numeri in sua proportionalitate minimi secundum $B$ proportionem per sextam secundi capituli. Igitur maior de primis $A$ et maior de primis $B$ sunt in eadem serie numerorum ab unitate continue proportionaliter positorum. Et consimiliter minor ipsius $A$ et minor de primis $B$. Hec ultima consequentia probari potest ex probatione secunde octavi.
Et si $A$ fuerit ad $B$ in alia proportione consimiliter arguendum est et iste consequentie convertuntur.
Et ut facilius videatur disposui pro exemplo unam figuram sive quasdam series numerorum quas si diligenter inspexeris cum adiutorio octave et none, decime, duodecime octavi Euclidis poteris propositum prolixius demonstrare et proportiones proportionum facillimum invenire.

$$
\left.\left.\begin{array}{cccccccc}
729 & 486 & 324 & 216 & 144 & 96 & 64 \\
243 & 162 & 108 & 72 & 48 & 32
\end{array}\right] \begin{array}{cccccc} 
\\
81 & 54 & 3 & 6 & 24 & 16
\end{array}\right]
$$

I

Hic enim est una ordinatio lateralis a sinistris secundum proportionem triplam 1, 3, 9, et cetera, et sicut prius dictum est quelibet

173 fuerint $E H$ fuerit $C R V /$ continue om $R$ rep $H$
173-74 commensurabiles: commensurabilis $V$
175 commensurabiles: commensurabilis $C$
176 necesse: minime $V /$ convenire: communicares(?) $E /$ nonam: tertiam $C$
178 inter om $H$ / ante numeros scr et del $V$ nus / est om $H /$ medius: medii $C$

179-80 proportionalitate: proportione $R$ 180 secundi om $C$
181 A om $H /$ maior $^{2}$ : minor $H$
183 consimiliter: similiter $E$ / ipsius: de primis $V /$ et minor $C H$ numero $E$ et $\mathrm{B} R$ minor $V$
184 probatione: parte $R$
185 arguendum est $E H$; $\operatorname{tr} C V$ agendum est $R$
I 86 post iste bab $H$ et
tionality from unity. If these conditions [of continuous proportionality] obtain, the ratios will be commensurable.
Let $A$ be the greater ratio, $B$ the lesser. If they are commensurable, it is necessary that they unite in means by the ninth proposition of the second chapter. Thus, either $A$ is multiple to $B^{*}$ or related in [some] other ratio.
If the former, then between the prime numbers of $A$ there is a mean number or numbers which when taken together with the extremes constitute a series of numbers in their lowest terms which, in that [particular] proportionality, form ratio $B$, by the sixth proposition of the second chapter. Thus the greater of the prime [numbers] of $A$ and $B$ are in the same series of numbers taken in continuous proportionality from unity. And the same can be said of the lesser of the primes of $A$ and $B$. This last consequence can be proved by the second [proposition] of the eighth [book of Euclid].
And if $A$ is related to $B$ in some other ratio, then one must argue in a similar way and these consequents are converted.
So that this might seem easier, I have arranged as an example a figure or certain series of numberst and if you will examine it carefully, with the aid of the eighth, ninth, tenth, and twelfth [propositions] of the eighth [book] of Euclid, you will be able to demonstrate more comprehensively what has been proposed and to find most easily the ratios of ratios.

Here we have, on the left, a lateral series which forms a triple ratio, I, 3, 9 , etc. Now, as stated before, any multiple ratio denominated by any of

* $A=(B)^{n}$, where $n$ is an integer.
$\dagger$ See the figure on facing page, which has not been repeated in the translation.
187 disposui: dispositi $V /$ sive $C H V$ seu $E R$
188 quas si $\operatorname{tr} E /$ diligenter om $E /$ inspexeris: inspexerunt $V$
189 et $H R$; om $C E V /$ none decime $H R V$ et decime et none $C E$ / post decime add $R V$ et / ante poteris add $C$ scilicet
190 prolixius om $E /$ demonstrare: demonstrari $E$ / facillimum $H$ facilius $C R$ faciliter $E$ (?) $V$
190-91 invenire $H$ reperire $C E R V$
192-201 figuram om $C R V$
192 729 Ed; om E $29 \mathrm{H} / 216$ EdEIn $6 \mathrm{H} /$ 144 EdE $244 \mathrm{H} / 96 \mathrm{EH}$; om Ed/64 EdH; om E

$108 \mathrm{EdE}{ }_{168 \mathrm{H} / 72 \mathrm{EH}{ }_{7} \mathrm{Ed} / 48}$ EdH; om E
194 supra 81 scr et del E 34(?) / 24 H 34 EdE / infra 24 scr et del H 36
195 inter 18 et 12 bab $E d($ ? $) E$ quarta
1963 H ; om EdE
$1976 E H$ tres Ed
$1982 H$; om $E d E$
201 I EH; om Ed
202 Hic enim $C H$ hec $E$ hec enim $R /$ Hic enim est bab $V$ in textu sed etiam mg bab $V$ hic enim est infra et proportiones (linea 190)
202-3 proportionem $E R V$ medium $C$ ordinationem $H$
203 9: nono $C /$ dictum est om $R$
proportio multiplex denominata aliquo istorum numerorum est comons cuilibet denominate alquo commensurabilis alicui denominate per aliquem eorundem numerorum. Et idem dico de coordinatione dextra que est secundum proportionem duplam.

Sed deinde ymaginaris ibidem unam coordinationem proportionum primis numeris. Prima est sexquialtera cuius primi numeri sunt primus post unitatem de coordinatione sinistra et primus post unitatem de dextra, scilicet 3 et 2 . Secunda proportio est cuius primi numeri sunt 9 et 4 , tertia cuius primi numeri sunt 27 et 8 , et sic in infinitum quia coordinatio huius posset qualibet ymaginari augeri.

Dico, ergo, quod istarum proportionum secunda est dupla prime, et tertia tripla ad primam, et quarta quadrupla, et cetera. Et quelibet istarum est commensurabilis cuilibet earundem et nulla alia est commensurabilis alicui earum. Omnes enim iste communicant in mediis et nulle alie cum istis nisi fuerit aliqua coordinatio que sit pars istius et inter cuiuslibet harum primos numeros est medium aut media in numeris secundum primam proportionem huius ordinis, scilicet sexquialteram et inter nullius istarum numeros est medium secundum aliquam aliam proportionem nisi ipsa sit multiplex prime, scilicet sexquialtere. Et consimiliter potes unam aliam seriem proportionum
225 ex aliis numeris componere in qua proportio sexquitertia erit inferior sive prima, et ita de qualibet alia proportione non multiplici poteris operari.

Ex ista conclusione, sicut ex precedenti, possunt aliqua elici. Unum est quod si aliqua proportio non multiplex sit dupla alteri maior de

204 proportio: propter $E /$ denominata: denominatio $V$
205 denominate $C R V$ denominato $H$ (?) $E /$ aliquo: aliquid $V$
205-7 et nulla...numerorum om $R$
206 aliquem $E H$ aliquam $C V$
206-7 numerorum CHV; om $E$
207 idem: ita $R$ / que om $R$
207-8 proportionem: coordinationem $H$ 209 Sed om $E$ / ymaginaris: ymaginabis $C$ / ibidem om $E$
209-1 1 proportionum...coordinatione om V
210 primis numeris $E H ; \operatorname{tr} C R$
211 primus': illus(?) $R$ / ante de scr et del
$E$ primi(?) / coordinatione: ordina-
tione $E /$ primus $^{2} H R V$; om $C$ primum E
212 scilicet om $E /$ et om $V /$ est $C V$; om HR
212-13 Secunda...tertia om $E$
213 9...sunt ${ }^{2}$ om $V /$ et $^{1} C R$; om $H$ / sic om $E$
214 huius: cuius $R \mid$ post huius add $C$ modi / posset: potest $R$ / qualibet CH quamcumlibet $E$ quamlibet $V$ quodlibet $R$ / ymaginari augeri $V$ augeri $C$ multiplicari sive augeri $E$ ymaginari $H$ augmentari $R$
2I) istarum proportionum secunda: secunda istarum proportionum $E$
$216 \mathrm{et}^{\mathrm{I}} \mathrm{CHV}$; om E / et quarta quadrupla

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these numbers is commensurable to any [other] ratio denominated by any of the same [numbers], and no other ratio is commensurable to any ratio denominated by any one of these numbers. And I say the same for the series on the right side which forms a double ratio.

Now imagine in the same figure a series of ratios in prime numbers. The first is a sesquialterate whose prime numbers are the first after unity from the left and right series, namely 3 and 2 [respectively]. The second ratio is that whose prime numbers are 9 and 4 ; the third 27 and 8 , and so on ad infinitum, since this series can be imagined to increase as much as you wish.

I say, therefore, that the second of these ratios is double the first; the third is triple the first; the fourth quadruple, and so on.* Any one of these is commensurable to any other [of the same series], and no other ratio is commensurable to any of those in the series. Indeed, all these "communicate in means" but no others with them unless there should be some series which is part of it and between whose prime numbers there is a mean or mean numbers forming the first ratio of this series, namely a sesquialterate, and no other ratio unless it be multiple to a sesquialterate. Similarly you could construct another series of ratios from other numbers in which a sesquitertian ratio will be the lowest or first ratio. You can do this for any other non-multiple ratio.

Some consequences may now be drawn from this and the preceding proposition. One is that if some non-multiple ratio were double another,

* $(3 / 2)^{n}$, where $n=1,2,3,4, \ldots$
om $R$ / quarta $E H V$ quadrupla $C \mid$ quadrupla $C E H$; om $V$
217 istarum: istorum $C$ / est $\ldots$...alia est om $R$ / cuilibet: cuiuslibet $V$
218 earum $H R V$; om $E$ eorum $C$ / iste CHV ille $E$ ista $R$
219 aliqua $H R V$ alia $C E /$ coordinatio $H$ coardinatio $C$ ordinatio $E R V /$ sit: est $R$ / istius: illius $R$
220 harum CEV horum $H$ earum $R$ / post medium sor et del $V$ secundum aliquam aliam proportionem nisi ipse sit multiplex prime scilicet sesquialteram et inter nullius istarum / aut CHV seu $R$
220-21 aut...proportionem: secundum
aliquam proportionem aut media $E$
222 istarum: harum $C$
223 aliquam aliam $H R V ; \operatorname{tr} C$ aliquam $E /$ ipsa: ipse $V$
224 consimiliter: similiter $R /$ potes $C R V$ ponentes $E$ potest $H \mid$ post seriem bab $E$ minorem
225 ex om $E$ / proportio $C R V$ proportione $E H$ / sexquitertia $E H V$ sexquialtera $C R$
226 sive: vel $C$
227 operari: opponi $C$
228 ista: illa $R$
229 est om $E$ / non om $V$ | post non bab $E$
${ }_{230}$ primis numeris maioris erit quadratus cuius maior de primis minoris erit radix, et de minoribus numeris idem dico et econverso, scilicet si primi numeri duarum proportionum ita se habeant maior est dupla minoris. Si , vero, maior proportio sit tripla minoris maior numerus maioris erit cubicus et similiter minor, et eorum primi numeri minoris erunt radices et econverso.
Item, si aliqua proportio non multiplex sit alteri dupla primi numeri eius erunt ambo quadrati et similiter si quadrupla, et si sextupla, et sic ultra per pares denominationes procedendo paribus intermissis. Et econverso, scilicet si primi eius numeri sint quadrati illa proportio ad aliam rationalem erit dupla aut quadrupla, et cetera, et una ad aliam erit dupla. Et si non sint quadrati ad nullam dupla aut quadrupla, et cetera, nec aliqua rationalis erit medietas eius, aut quarta pars, aut sexta, et cetera, sed quelibet talis pars eius denominata numero pari erit proportio irrationaiis.

Item si aliqua proportio non multiplex fuerit ad aliam tripla primi eius numeri sunt cubici et econverso. Et similiter si fuerit ad aliam nonacupla, et si duodecupla, et sic ultra duabus semper denominationibus intermissis, et econverso si primi eius numeri sint cubici ipsa erit ad aliam tripla. Et si non sint cubici ad nullam erit tripla, aut nonacupla, et cetera, nec aliqua rationalis erit tertia pars eius, nec nona, nec duodecima, et cetera, sed quelibet talis pars eius erit proportio irrationalis, et cetera.

Si vero proportio unius proportionis non multiplicis fuerit multi-

230 numeris om $E /$ maioris: maior $E$ erit: et $V /$ maior: maioris $E$ / ante minoris bab $R$ numerus / minoris: numerus $C$
231 dico: dicendum $V /$ et$^{2}$ : quod $E /$ scilicet: et $H$
232 ante proportionum scr et del $V$ proponum / ante maior bab $R$ quod
233 ante maior ${ }^{\mathrm{r}}$ bab $C$ quod / minoris ${ }^{2}$ om $E \mid$ post maior ${ }^{2}$ add $R$ est dupla minoris si vero maior fuerit tripla minoris maior
234 et similiter minor om $V /$ minor $C H R$ quadratus(?) $E$ / et corum om $R$ primi numeri $H ; \operatorname{tr} C R V$ primi $E$
236 Item: et $R /$ alteri: altera $R /$ ante primi bab EH aut
237 eius om $H$ / erunt: essent $C /$ sir $^{1}$ om $V$ /
ante $\mathrm{si}^{2}$ add $E$ similiter / $\mathrm{si}^{2}$ om $H$
238 pares: partes $R$ /ante paribus add $C$ in / paribus $C V$ imparibus $E H R$
239 eius numeri $\operatorname{tr} E /$ illa: ista $E$
240 aliam ${ }^{1} C E H$ aliquam $R V /$ aut om $R /$ una $C V$ vero(?) $E$ prime $H$ similiter $R /$ aliam $^{2} E H V$ aliquam $C R$
241 erit: et per $C /$ si non $\operatorname{tr} V /$ sint: sunt C
242 aliqua: alia $H /$ pars om $R$
243 talis pars eius: eius talis pars $E /$ eius $C H V$; om $R /$ denominata : denominatur $E$ / numero pari tr $R$
244 erit: esset $V$
245 fuerit...tripla: ad aliam fuerit tripla $E$ / aliam $H V$ aliquam $C R$ / tripla HRV duplam $C$
246 aliam $H R$ aliquam $C V$ alteram $E$

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the greater of the prime numbers of the greater ratio will be a square [number] whose root will be the greater of the prime numbers of the lesser [ratio]; and the same can be said of the lesser [prime] numbers.* And the converse also holds, namely if the prime numbers of two ratios are related as just described], the greater is double the lesser. If, however, the greater ratio is triple the lesser, then the greater number of the greater ratio will be a cube [number], and the same holds for the lesser [number of the greater ratio]; and the prime numbers of the lesser ratio will be the roots of these [numbers], and conversely. $\dagger$

Furthermore, if some non-multiple ratio were double another, its prime numbers will both be square; and this applies if some non-multiple is quadruple another, or sextuple, etc., so long as one proceeds through the even [numbered] denominations sequentially. $\ddagger$ And conversely, if the prime numbers [of the non-multiple ratio] were square [numbers], that ratio will be double, or quadruple, etc., another rational; and [in general] one will be double another. But if they were not square [numbers], that non-multiple ratio would not be double, or quadruple, etc., to another ratio, nor will any rational be half of it, or a fourth part, or sixth part of it, etc., but, indeed, any such part of it denominated by an even number will be an irrational ratio.

Again, if some non-multiple ratio were triple another ratio, its prime numbers are cube numbers, and conversely. The same holds if it were nine times another, or twelve times, and so on, with two denominations always omitted.§ And conversely, if its prime numbers are cubes it will be triple another ratio. But if they were not cube numbers, it will not be triple, or nine times, etc., to another [rational ratio]; nor would any rational [ratio] be a third part of it, or a ninth part, or a twelfth part, etc., but any such part will be an irrational ratio. [And this can be extended indefinitely.]

If a ratio of a non-multiple type should be multiple [to some ratio] and

* For example, $9 / 4=(3 / 2)^{2}$.
+ An example would be $27 / 8=(3 / 2)^{3}$.
$\ddagger$ In the sequence $(3 / 2)^{n}$ all numerators and
denominators are square numbers when $n$
247 si om $H$ / duabus $E H R$ duobus $C V$
248 sint $H R V$ sunt $C E /$ ante cubici scr et del $V \mathrm{~s} / \mathrm{ipsa}$ om $R$
249 aliam $E H$ aliquam $C R V$ / si om $E \mid$ sint $E R V$; om $H$ sunt $C /$ erit $^{2}$ : erunt E
$=2,4,6,8,10, \ldots$.
§ In the series $(3 / 2)^{n}$ the numerators and denominators are cube numbers when $n=$ $3,6,9,12,15, \ldots$.

250 et cetera om $V$ / ante tertia scr et del $V$ pars / nec ${ }^{2}$ : aut $C /$ nona: nonupla $E$ 25 I duodecima: duodecupla $E /$ sed: si $R$ / talis om $E$ / eius om $H$ / erit: esset $V$ 252 et cetera om $R$
253 unius om $V /$ non: nec $E$
plex denominata aliquo numero vel denominatione de utroque ordine,
verbi gratia sextupla, tunc primi eius erunt quadrati, cubici.
Harum omnium probationes et exempla poteris habere ex octava noni Euclidis et eius probatione et figura que superius est descripta, et ut brevius hoc transeam nunc dimitto.

Octava conclusio. Si inter primos numeros alicuius proportionis fuerit solum unus numerus medius proportionalis nulla proportio rationalis minor est illi commensurabilis nisi in proportione subdupla nec alia maior nisi in proportione sexquialtera, aut multiplici, aut multiplici sexquialtera.

Cum inter primos numeros eius non est nisi unus numerus medius proportionalis ipsa non dividitur in alias proportiones rationales nisi in duas medietates per sextam secundi capituli. Igitur nulla proportio rationalis erit pars eius nisi sua medietas, nec partes per correlarium quarte secundi capituli, ergo nulla erit sibi commensurabilis nisi aliquo dictorum modorum. Hoc idem potest probari ex nona secundi capituli et ideo cuiuslibet talis illa pars que est tertia eius, vel quarta, vel quinta, ${ }_{270}$ et cetera, est una proportio irrationalis.

Dico etiam quod si inter primos numeros alicuius proportionis fuerint duo numeri medii proportionales et non plures nulla minor est eidem commensurabilis nisi in proportione subtripla aut in proportione subsexquialtera, nec aliqua maior nisi in proportione sex-
275 quitertia, aut superpartiente duas tertias, aut multiplici, aut in multi-

254 utroque: quocumque $R /$ ordine: ordinatione $V$
255 ante verbi scr et del $V$ gratia(?) / tunc om $C$ / primi eius $\operatorname{tr} C$ / eius... quadrati: numeri eius quadrati sunt $R /$ post eius scr et del $V$ erit / ante erunt bab $C E$ numeri / erunt quadrati $E V$; om $C$ quadrati $H$
256 Harum $E H$ horum $C R V$ / probationes: probatione $C$
257 ante Euclidis bab $V$ ecl / et ${ }^{2}$ om $E /$ que om $E$ / est om $E$
258 et ut... dimitto om $R$ / ante ut bab $V$ hic / hoc transeam nunc $H$ nunc transeam hoc $C V$ transeam nec hoc $E$
259 Octava conclusio $V$; om $C H$ octava conclusio $m g$ bab $E$ post dimitto (linea 258) et $m g$ hab $R$ ante $\mathrm{Si} /$ inter: sint $E$ / numeros om $E$
260 proportio rationalis: proportionalis $R /$ est: erit $R$

261 alia $E H$ aliqua $C R V /$ nisi in: nec 6 $R$
262 aut multiplici ${ }^{1} E H V$; om $C R$
263 post Cum add CRV enim / primos numeros eius $H$ eius primos numeros $C R$ primos eius numeros $E V /$ est: sit $E$ / numerus medius $C E H$; $\operatorname{tr} V$ terminus medius $R$
264 alias $H$ aliquas $C V$ aliqua $E$ aliquo $R$ 265 per: secundam $R$
265-67 proportio...ergo om $R$
266 erit $C E H$ esset $V /$ sua medietas $E H$; $\operatorname{tr} V$ suas medietas $C$
267 secundi $C E H$ tertii $V$ / nulla erit $C R$; $\operatorname{tr} E$ nulla esset $V /$ ante sibi add $E$ proportio rationalis pars eius
267-68 ergo... capituli om $H$
268 dictorum $E R V$; om $C /$ potest probari $E R V$; $\operatorname{tr} C$
269 ideo: $\operatorname{ratio}(?) E /$ illa: ista $E /$ vel ${ }^{1}$

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is denominated by some number or denomination from each series-for example [denominated by] a sextuple-then each of its primes will be both a square and cube number.*

In order to move on I shall now leave this, but you can find proofs and examples of all these in the eighth [proposition] of the ninth [book] of Euclid and in the demonstration and figure described above.
Proposition VIII. If there is only one mean proportional number between the prime numbers of any ratio, no smaller rational ratio is commensurable to it except in a subdouble ratio; nor is another greater ratio [commensurable] to it except in a sesquialterate, multiple, or multiple sesquialterate ratio.
Since there is only one mean proportional number between its prime numbers, it is not divided into other rational ratios, except into two halves, by the sixth proposition of the second chapter. Therefore, no rational ratio except its half will be part of it, nor will any be parts of it, by a corrollary of the fourth proposition of the second chapter, and, therefore, no rational ratio will be commensurable to it except in ways just mentioned. $\dagger$ This can be proved by the ninth proposition of the second chapter, and hence that part of any such ratio which is a third, or fourth, or fifth [part], etc., is an irrational ratio.

Also, I say that if there were two-and no more-mean proportional numbers between the prime numbers of any ratio, [then] no smaller ratio is commensurable to it except in a subtriple or subsesquialterate ratio; $\ddagger$ nor is any greater ratio commensurable to it except in a sesquitertian, a superpartient two-thirds, a multiple, multiple sesquitertian, or in a mul-

* When $n=6,12,18,24, \ldots$, the numerators and denominators in $(3 / 2)^{n}$ are both square and cube numbers. Thus $(3 / 2)^{6}=$ $729 / 64$ where $729=(9)^{3}=(27)^{2}$, and $64=$ $(4)^{3}=(8)^{2}$.
$\dagger$ If $A / B$ is a ratio of mutually prime numbers with one mean proportional, then only
$(A \mid B)^{1 / 2}$ can be part of it and commensu rable to it.
$\ddagger$ If $A \mid B$ is a ratio of mutually prime numbers with two mean proportionals, then only $(A \mid B)^{2 / 3}$ and $(A \mid B)^{2 / 3}$ are smaller ratios commensurable to it.

[^27]273-74 post proportione scr et del $V$ seq
plici sexquitertia, aut in multiplici superpartiente duas tertias. Et proportio que est medietas eius, aut quarta, aut quinta, aut sexta, et cetera, est irrationalis. Et si inter primos numeros alicuius fuerint tres numeri medii poterunt etiam de hoc multe poni consimiles propor80 tiones, et ita si quatuor, et cetera, et si quinque, et cetera. Et ita breviter iuxta numerum numerorum mediorum inter primos numeros proportionum possunt de proportionibus multa dici que poterunt per sextam et nonam secundi capituli demonstrari.
Ex dictis etiam patet quod ad quamlibet proportionem rationalem orio rationalis dupla, et similiter tripla, et quadrupla, et sic de aliis secundum genus multiplex. Non tamen ad quamlibet est aliqua rationalis subdupla aut subtripla et cetera. Et ita procedendo similiter non cuilibet est aliqua commensurabilis in proportione sexquialtera, nec cuilibet in sexquitertia, nec in multiplici superparticulari, et ita de aliis generibus et speciebus est dicendum. Sed solum determinate et particulariter ad aliquam est aliqua multiplex et nulla in alia proportione per quintam secundi capituli.

Similiter ad aliquam est aliqua commensurabilis in proportione multiplici, et sexquialtera, et subdupla, et multiplici sexquialtera, et in nulla alia proportione sicut proponit octava huius et sic de aliis.
Item si $A$ linea sit dupla ad $B$ aut in alia quavis proportione inter cuius numeros non sit numerus medius aut numeri nulla linea seu linee medie cum $A$ et $B$ continue proportionaliter ordinate sint com-

276 superpartiente corr ex superparticulari $C E H R V /$ duas tertias $C H R$ tertia $E$ duas duas $V$
277 eius om $C$ /quarta: quatuor $E /$ quinta: quinque $E$ / aut ${ }^{3}$ om $V$ / sexta: sex E
278 est: erit $R$ / primos $H$; om CERV
279 de hoc multe poni $C H$ de hoc poni multe $E$ de poni multe $R$ poni de hoc multe $V$
$280 \mathrm{ita}^{\text {r }}$ : ista $E$ / quatuor: quarta $C$ / et cetera ${ }^{1}$ om $R /$ quinque: quinta $C /$ et cetera ${ }^{2}$ om $H /$ ita $^{2}$ om $H /$ post ita $^{2}$ bab $C$ si
281 ante numerorum scr et del $E$ mo/mediorum om $E$ | ante inter scr et del $E$ i
281-82 proportionum: proportiones $C$
282 poterunt: poterit $R$
284 dictis etiam patet: istis etiam patet scilicet ex dictis $R /$ ad om $V$

28; et ${ }^{1} H R V$; om $C E$ / similiter: aut $E$ / $\mathrm{et}^{2}$ om $V /$ ante quadrupla hab $E$ similiter
286 et sic de aliis: et cetera $R$
287 aliqua: alia $E$ |post aliqua add $E V$ proportio / ante -dupla in subdupla bab $V$ 2 / post cetera bab $C$ nec ad aliquam subtripla aut subquadrupla et cetera $e t$ bab $E$ nec ad aliquam subnonupla aut quadrupla et cetera et hab $V$ nec ad quamlibet subtripla aut subquadrupla et cetera et hab $R$ nec etiam ad quamlibet subtripla aut subquadrupla et cetera / Et om H
288 similiter: et ita $R$ / cuilibet $C H R$ quilibet $E$ cuiuslibet $V /$ est: erit $H$ /aliqua: alia $E$ | post aliqua add $V$ proportio / post in scret del $V$ generibus et speciebus
288-89 proportione sexquialtera $\operatorname{tr} V$
tiple superpartient two-thirds.* But the ratio which is a half, a fourth, a fifth, a sixth, etc., [part] of it is irrational. And if there were [only] three mean numbers between the prime numbers of any ratio, many similar ratios would be [commensurable to it]; and the same applies if there were four means, or five, and so on. And, briefly, depending on how many means there are between the prime numbers, many things can be said about ratios, which are demonstrable by the sixth and ninth propositions of the second chapter.

From what has been said it is also evident that there is some rational ratio double, triple, quadruple, and so forth, to any rational ratio, and this can be said of other ratios forming a multiple type ratio. However, not every rational ratio has a rational subdouble, or subtriple, etc. And by extension it may be said that not every rational ratio has some rational commensurable to it in a sesquialterate ratio; nor [does every rational have another rational commensurable to it] in a sesquitertian ratio and the same may be said for a multiple superparticular; and this can be said of other genera and species [of ratios]. Indeed, [rational] ratios are multiple to other rational ratios only in a prescribed and special way, and are not related in any other ratio, by the fifth proposition of the second chapter.

Similarly, to some rational ratio other [rational] ratios are commensurable in a multiple ratio, in a sesquialterate ratio, in a subdouble, in a multiple sesquialterate, and in no other ratio, as set forth in the eighth proposition of this chapter.

Again, if line $A$ were double $B$, or in any other ratio that had no mean number or numbers between its [prime] numbers, no line or mean lines arranged in continuous proportionality with $A$ and $B$ would be commen-

* All greater ratios of the form $(A / B)^{n / 3}$, where $n=4,5,6,7,8,9, \ldots$, are commensurable to $A \mid B$.

289 cuilibet: quilibet $R /$ post in ${ }^{1}$ add $C$ proportione / sexquitertia $C H R$ sexquialtera $E$ sexquitripla $V$
290-91 determinate et particulariter $C H R$ determinato particulariter $E$ particulariter et determinate $V$
291 aliquam: aliam $E /$ aliqua: alia $E /$ nulla in $\operatorname{tr} E$
294 ante sexquialtera ${ }^{1}$ scr et del $E$ sexquialt / et subdupla...sexquialtera om $E$
295 proponit: ponit $C$ / sic $E H R$ similiter
$C V$
296 A linea: alia $R /$ ad B : aliud $R /$ ante in scr et del $V \mathrm{~b}$
297 sit $C E V$ erit $H$ est $R /$ aut numeri nulla: seu numeri medii seu $R /$ numeri $E H V$ medii $C$ / post linea scr et del $E$ ala et add $C$ media / seu HRV vel $C$ aut $E$
298 post cum bab $V$ linea / sint $E H$ sunt CRV
mensurabiles alicui earundem, scilicet nec $A$ nec $B$. Et si inter numeros proportionis $A$ ad $B$ sit tantum unus numerus medius nulla mediarum proportionaliter est commensurabilis alicui $A$ aut $B$ nisi una, et si duo due, et cetera. Multa poterit intelligens capere per predictam.
Nona conclusio. Nulla proportio de genere multiplici componitur precise ex duabus superparticularibus nisi dupla.

Quod vero dupla fiat precise ex duabus superparticularibus patet quia componitur ex sexquialtera et sexquitertia. Et fit manifestum addendo unam alteri et multiplicando denominationem unius per denominationem alterius sicut in primo capitulo dicebatur. Et patet exemplo: proportio 4 ad 2 , que est dupla, componitur ex proportione o 4 ad 3 , que est sexquitertia, et 3 ad 2, que est sexquialtera. Quod autem nulla alia proportio multiplex sit talis declaratur.

Omnium enim proportionum multiplicium dupla est minor et omnium superparticularium sexquialtera est maior, deinde sexquitertia ut patet per regulam superius allegatam illa proportio est maior cuius 5 denominatio est maior et minor cuius denominatio est minor. Et iam ostensum est quod sexquialtera et sexquitertia componunt precise duplam igitur nulle due minores invicem addite precise redderent duplam, nec maiorem dupla. Sed quelibet alie due superparticulares sunt minores sicut due sexquitertie, aut due sexquiquarte, aut sex-
${ }_{320}$ quitertia et sexquiquarta, exceptis duabus sexquialteris, ex quibus nulla multiplex sit precise. Sed dupla sexquialtera sicut patet ex additione unius ad alteram que tamen dupla sexquialtera licet sit maior dupla est, tamen, minor quam aliqua alia multiplex. Igitur patet quod nulla multiplex, nisi dupla, sit ex duabus superparticularibus quod fuit

299 earundem: eorundem $V$ I scilicet: sicut $E$ / nec A $C H$; om $E$ A $R V$ / si: scilicet $V$
300 proportionis: proportionum $E /$ ad: et $V /$ sit: sicut $V /$ numerus medius $t r$ $R /$ medius $C E H$; om $V$
301 proportionaliter est $H V ; \operatorname{tr} C$ proportio est $E$ proportionum est $R$ । alicui $E H$; om $C R V /$ aut $E H$ ad $C V$ et $R$ / post una bab $E H$ A / et om $C$
302 predictam $H R$ predicta $C E V$
303 Nona conclusio $V$; om CH nona conclusio mg bab E ante Multa (linea 302) et mg hab $R$ post predictam (linea 302)
305 vero: una $C$ / fiat: fuerit $H$ / post pre-
cise bab $E$ fiat / ante duabus scr et del $H$ duabus(?) / ante patet add $V$ sic
306 quia: quod $E /$ componitur: sit $E /$ sexquialtera: sexquialteram $V$ / sexquitertia: hoc $E$ | post Et bab $V$ hoc et $H R$ hec / fit $C R V$ sic $E$ sit(?) $H$ 306-7 manifestum addendo $\operatorname{tr} R$ 308 in: michi $E$ / dicebatur: dicebitur $C$ 309 exemplo: exempla $E /$ que om $E$ 310 ante 2 scr et del $V 3$ / Quod: quia(?) $V$ 3 II alia $C V$; om $E H R$ / multiplex: de genere multiplici $V /$ sit talis om $R$
313 sexquialtera: sexquialteram $V /$ maior: minor $R$
314 superius $E H V$ prius $C R /$ illa: ideo $R$

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surable to $A$ or $B$. And if there were only one mean number between the numbers representing ratio $A$ to $B$, only one mean is proportionately commensurable to $A$ or $B$; and if there were two mean numbers, only two means would be proportionately commensurable to $A$ or $B$, etc. Many things can be understood from what has been said above.

Proposition IX. No ratio of a multiple kind, except a double, is composed of two superparticular ratios.

That a double ratio can be exactly constituted of two superparticulars is evident, since it is composed of a sesquialterate and sesquitertian. But it can be made obvious by adding one to the other and multiplying their denominations, as stated in the first chapter. An example will make this clear: a ratio of 4 to 2 , which is a double, is composed of ratios 4 to 3 , a sesquitertian, and 3 to 2 , a sesquialterate. That no other multiple ratio has this property has already been stated.
Of all multiple ratios a double is the smallest, and of all superparticulars a sesquialterate is greatest, with a sesquitertian next, as is evident by the rule cited above, [namely] that a ratio is greater whose denomination is greater and smaller when its denomination is smaller. Now it has already been shown that a sesquialterate and sesquitertian exactly compose a double ratio, so that no two smaller [superparticulars] when mutually added exactly produce a double or [a ratio] greater than a double. But any other two superparticulars-with the exception of two sesquialterates-are less [than a sesquialterate and sesquitertian], as are two sesquitertians, or two sesquiquartans, or a sesquitertian and a sesquiquartan, from which no multiple [ratio] can be produced exactly. Now a double sesquialterate, although it is greater than a double*-as is evident from the addition of one to the other-is, nonetheless, less than any other multiple. Therefore, it is clear that no multiple except a double is composed of two superparticulars, which was to be proved.

* That is, $3 / 2 \cdot 3 / 2>2 / \mathrm{I}$.

315 est maior: minor est $R /$ est $^{2}$ om $R /$ Et iam: etiam $V$
316 ostensum est $\operatorname{tr} E /$ precise om $E$
317 nulle: nec $C /$ due om $E /$ minores: minorem $C$ / ante invicem bab $E$ in precise redderent $H$ redderent $C E$ componerent $R$ facerent unam $V$
319 due $^{2}$ om $V$
320 et: aut $H /$ ante exceptis add $E$ et $/$
exceptis: ex istis $V /$ ante ex add $E$ et
321 Sed: et $R /$ sexquialtera corr ex sexquiquarta $C E H R V /$ ex: in $R$
322 tamen om $E$ / sexquialtera corr ex sexquitertia $C$ sexquiquarta $E H R V /$ ante dupla ${ }^{2}$ add $E$ quam
323 quam om $C$ / ante Igitur hab $H$ sic(?) / Igitur: ideo $R$
325 post probandum scr et del $H$ si quis

Si quis predicta diligenter consideravit et geometriam et astrologiam sufficienter intellexerit multa de proportionibus consequenter poterit invenire in quibus nolo diutius immorari.
Sed finaliter pono unam aliam conclusionem que videtur sequi ex precedentibus cuius fructus non modicus per dei gratiam in sequentibus apparebit. Et tanto amplius admiraberis quanto circa eam et ea que ex ipsa sequuntur profundius cogitabis. Conclusio est ista:
Decima conclusio. Propositis duabus proportionibus ignotis verisimile est eas incommensurabiles esse; quod si multe proponantur ignote verisimilimum est aliquam alicui incommensurabilem fore.
Sicut inveniebatur in primo capitulo tres sunt modi proportionum. Quedam enim sunt proportiones rationales, alie sunt irrationales habentes denominationes, hoc est rationalibus commensurabiles, et forsitan est tertius modus, scilicet proportiones irrationales que nullam habent denominationem eo quod non sunt commensurabiles aliquibus rationalibus.

Sint, igitur, due proportiones ignote. Aut igitur utraque est rationalis, scilicet de primo modo, et tunc arguitur sic: quibuscumque et quotlibet proportionibus rationalibus secundum unum ordinem denominationum aut secundum plures ordines multo pauciores sunt que sunt invicem commensurabiles et que sunt incommensurabiles multo plures. Igitur duabus earum ignotis propositis verisimile est eas incommensurabiles esse.
Antecedens declaratur. Sumantur enim secundum ordinem suarum denominationum 100 proportiones in genere multiplici sicut dupla, tripla, quadrupla, quintupla, et cetera, usque ad roram et sint sicut roo termini ad invicem comparati. Tunc inter huius terminos seu
326 quis om $C$ / predicta om $E$ / diligenter consideravit $\operatorname{tr} R / \mathrm{et}^{2}$ om $E$
328 in: ex $E /$ nolo: volo $R$
329 unam aliam om $R /$ conclusionem om H
330 modicus: medius $C$
33 I admiraberis $C$ admirabilis $E R$ admiraberit $H$ admirabit $V /$ quanto: quantum $R$ / post quanto add $V$ amplius / post et bab $H$ iam(?) / ante ea add C circa
332 cogitabis CER cogitabit HV/Conclusio est ista $C E H$; om $R V$
333 Decima conclusio $V$; om $C$ prima conclusio $m g$ bab $E$ ante Propositis et de-
cima conclusio $m g$ bab $H$ post multa (linea 327) et mg hab $R$ ante Propositis / ignotis CHV ; om $E$ ignotas $R$
334 incommensurabiles: incommensurabilis $C$
334-35 est aliquam rep $V$
335 incommensurabilem: incommensurabile $C$
337 alie sunt $E H V$ quedam $C R$
338 ante hoc add EV a rationalibus / hoc est rationalibus $C R V$; om $E$ hoc est rationales $H /$ et: nec $H$
339 tertius: tribus(?) $H /$ modus: modos $C$ / scilicet om $H$ / proportiones irrationales $H R V$ proportiones $C$ pro-

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If anyone carefully considered the aforesaid things, and understood geometry and astronomy adequately, he could as a consequence discover many things about ratios, but I wish to delay no longer on these matters.
But, finally, I set forth one other proposition which seems to follow from what has preceded, the fruits of which, by the grace of God, will hardly appear trifling in what follows. Indeed, you will admire it even more as you reflect more deeply upon it and the things which follow from it. The proposition is as follows:
Proposition X. It is probable that two proposed unknown ratios are incommensurable because if many unknown ratios are proposed it is most probable that any [one] would be incommensurable to any [other].
As found in the first chapter, there are three types of ratios. Some, indeed, are tational ratios, others irrational with denominations, i.e., commensurable to rationals, and perhaps there is a third type, namely irrational ratios which have no denomination because they are not commensurable to any rationals.
Let there be two unknown ratios. Now each might be rational, namely of the first type, and the argument would proceed as follows: with any whatever number of rational ratios forming one or more series of denominations, those which are mutually commensurable are much fewer than those which are incommensurable; and therefore it is likely that any two proposed unknown ratios are incommensurable.
The antecedent is [now] demonstrated.* Let roo ratios of the multiple genus be taken according to the sequence of their denominations, as $2 / \mathrm{I}$, ${ }_{3} / \mathrm{I}, 4 / \mathrm{I}, 5 / \mathrm{t}$, etc., up to ${ }^{101} / \mathrm{I}$, and let them be as 100 terms mutually compared. Then by comparing any one of these terms to any other of them

* See p. 41 for full discussion; also pp. 257, 259 .
portionis irrationalis $E$
342 Sint: sunt $E$ / utraque: uterque $H$
343 quibuscumque $C H R$ quibuslibet $E$
343-45 sic...aut om $V$
344 quotlibet $C H R$ quodlibet $E$
345 que: qui $R$
346 sunt invicem om $V /$ et...incommensurabiles: quam se incommensurabiles quia iste sunt $R$
347 est om $C$

348 esse: et cetera(?) $V$
349 suarum om $H$
350 100: tot(?) H / proportiones: proportione $V /$ dupla: quadrupla $E$
351 quintupla $H$; om $E R V$ sextupla $C /$ ad $H R$; om $C E V /$ ioram $C H$ iooam $E R V /$ sint sicut: sicut sunt $R$
352 termini om $C /$ huius $E H V$ huiusmodi $C R /$ seu: vel $C$
proportiones comparando quemlibet cuilibet sunt 4950 proportiones que sunt proportiones proportionum et illarum 25 sunt rationales et plures proportiones rationales tamquam termini sumerentur sicut si vel 300 et acciperentur proportiones eorum adhuc esset proportio irrationalium ad rationales multo maior. Et si capiantur proportiones in alio genere quam multiplici adhuc pauciores erunt invicem comrabiles. Y mo omnes superparticulares sunt invicem incommensurabiles ut patuit supra. Si vero accipiantur quedam de uno genere et quedam de alio adhuc paucissime erunt invicem commensurabiles quia sicut patet ex tertia conclusione huius omnes de genere multiplici sunt incommensurabiles aliis, sicut decupla nulli rationali est
 et cetera. Unde si capiantur omnes multiplices citra centum nulla est commensurabilis alicui rationali minori roo ${ }^{\text {la }}$, exceptis r 6 , ut videbitur postea. Sic igitur patet antecedens declaratum ad cuius declarationem faciunt omnes conclusiones huius preter nonam.
Nunc declaro consequentiam principalem. Videmus enim in numeris quod quibuscumque seu quotlibet per ordinem acceptis numerus perfectorum seu cubicorum multo minor est numero aliorum et quanto plures capiuntur tanto maior est proportio non cubicorum ad cubicos, aut non perfectorum ad perfectos. Ideo si sit aliquis 375 numerus de quo penitus ignoretur, quis est aut quantus, et utrum sit magnus vel parvus, sicut forte numerus horarum omnium que transibunt antequam antichristus, erit verisimile est quod talis numerus

353 quemlibet $H$; om $V$ quamlibet $C E R 1$ cuilibet: cuiuslibet $V /$ sunt $E R V$; om $\mathrm{CH} / 4950: 4550 \mathrm{~V}$
354 illarum $H$ istarum CERV/25:52E
$354-95 \mathrm{et}^{2} \ldots$ irrationales om $E$
355 irrationales $H R V$ irrationalis $C$ / sicut $H R$ ut $C E$ sicud $V$
356 rationales: rationalis $C$ / sumerentur: sumantur $H$
357 vel: et $E /$ acciperentur: acciperent $C /$ eorum $C$ earum $E H R /$ esset $C H R$ conveniunt $E /$ proportio $H R$ proportionum $C$ proportiones $E$
357-58 eorum...proportiones om $V$
358 rationales $E H R$ rationale $C$
359 erunt: essent $C$
359-60 commensurabiles: commensura-
bilis $C$
360 Ymo om $R$ / sunt: erunt $H$
362 adhuc: aut $V /$ paucissime: paucissimes $V$ / commensurabiles: commensurabilis $C$
363 tertia $H R V$ quarta $C$ nona $E$
364 sicut $C$ alicui $E$ similiter etiam $H R V$
$3657^{\text {la }} C H R$ dupla $E$ septima $V / 12^{\text {la }}$ : tripla $E$
366 et cetera om C/ centum: io $C$
367 alicui: alteri $E$ / ante ioola hab $E$ et
367-68 16...postea $H R$ sedecupla ut videbitur postea $C$ i 6 ut videbitur post $V$ ut patuit videtur postea ${ }_{16} E$
369 faciunt: fuerint $C /$ conclusiones: declarationes $H /$ huius om $C /$ nonam: conclusionem(?) $H$

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there would be 4,950 ratios which are ratios of ratios and, of these, 25 -and no more-are rational and all the others are irrational, as I shall show afterwards. Now if more rational ratios were taken as terms, [for example] 200 or 300 , and if ratios of these were taken, the ratio of irrational to rational ratios would be much greater. And if the ratios were taken in a genus other than multiple there will be even fewer mutually commensurable ratios. Indeed, all superparticular ratios are mutually incommensurable, as was seen above. Furthermore, if some ratios are taken from one genus and others from another, this will result in the fewest ratios being mutually commensurable because, as is evident from the third proposition of this [chapter], all ratios in the multiple genus are incommensurable to [ratios of other genera], just as $10 / \mathrm{I}$ is not commensurable to any rational before $100 / \mathrm{I}$, and $7 / \mathrm{I}$ is [commensurable] to [only] one [ratio before ${ }^{100} / \mathrm{I}$ ], and ${ }^{12} / \mathrm{I}, 13 / \mathrm{I}$, etc., are not commensurable to any below ${ }^{100} / \mathrm{r}$. For this reason if all the multiple ratios below $100 / \mathrm{I}$ were taken, only 16 would be commensurable to any rational below ${ }^{100} / \mathrm{I}$, as will be seen afterward. Thus the antecedent has now been stated and all the propositions beyond the ninth of this chapter are designed to make it evident.

I now draw the principle consequent. With regard to numbers, we see that however many numbers are taken in series, the number of perfect or cube numbers is much less than other numbers and as more numbers are taken in the series the greater is the ratio of non-cube to cube numbers or non-perfect to perfect numbers. Thus if there were some number and such information as what it is or how great it is, and whether it is large or small, were wholly unknown-as is the case for the number of all the hours which will pass before anti-Christ-it will be likely that such an unknown number

370 Nunc om $C$ / declaro: declarabo $R$ | consequentiam $C E$ nonam $V$ acutam (?) $H$ nonam conclusionem $R /$ principalem: principale $E$
371 seu quotlibet $H R V$; om $E$ vel quotlibet $C$ / per ordinem acceptis: acceptum per ordinem $V$
372 perfectorum: perfectarum $R /$ seu: vel $C$ / post seu scr et del $E$ cupi / cubicorum: cupicam $E /$ minor: minoris $H$
372-73 est numero...tanto: maiorum et
tanto plures capiuntur quanto $R$
373 non cubicorum om $H$
374 aut: et ad $E /$ si om $C$
375 quis: quo(?) $E /$ est om $E /$ aut $E H V$; om $C R /$ et $H$ est $C$ et est $E$ est et $R$ sit $V /$ sit: si $C$
376 magnus: maius $C /$ vel $H$ aut $C E R V$ horarum omnium $H$; $\operatorname{tr} C R V$ io de horarum $E$
377 antequam om $E /$ talis: tales $H /$ numerus: numera(?) $E$
sit non cubicus; sicut est in ludis si peteretur de numero abscondito utrum sit cubicus vel non tutius est respondere quod non cum hoc 380 probabilius et verisimilius videatur.

Modo sicut est de numeris quantum ad hoc ita est de proportionibus proportionum rationalium, sicut prius est ostensum, quia irrationales sunt aliis multo plures ad sensum prius dictum. Ymo quod plus est si quis diligenter consideraverit inveniret quod inter proportiones pro$3_{5}$ portionum rationalium generaliter rariores sunt ille que sunt rationales quam sint numeri cubici in multitudine numerorum. Igitur si de aliqua proportione proportionum ignota petatur verisimile est illam esse irrationalem et proportiones quarum ipsa est proportio incommensurabiles esse et hoc si proportiones ignote de quibus queritur forent rationales.

Et si forte proportiones proposite essent de secundo modo, scilicet irrationales habentes denominationes et rationalibus commensurabiles, probabitur hoc idem sic. Sint $A$ et $B$, tunc arguitur $A$ est commensurabilis alicui rationali a qua habet denominationem, sit illa $C$. Et $B$ similiter est commensurabilis alicui rationali a qua habet denominationem, sit illa $D$. Et ultra si $C$ et $D$ sunt incommensurabiles $A$ et $B$ sunt incommensurabiles quod arguitur per commentum octave decimi quia sequitur $A$ est commensurabile $C$ et $C$ est incommensurabile $D$, igitur $A$ est incommensurabile $D$. Et ultra, $A$ est incom-
400 mensurabile $D$ et $D$ est commensurabile $B$ igitur $A$ est incommensurabile $B$. Patet, igitur, quod si $C$ et $D$ sunt incommensurabiles $A$ et $B$ sunt incommensurabiles. Sed verisimile est quod $C$ et $D$, que
378 sit non $\operatorname{tr} R$ / cubicus: cupicus $E /$ est H; om C etiam ERV / ante peteretur scr et del $C$ petientur(?) / peteretur $C H R$ petetur $E$ peteret $V /$ abscondito: absconsso $C$
379 utrum $H R V$ sic $C$ si $E /$ ante cubicus scr et del $E$ illa / ante tutius add $R$ cubicus / post non add $R$ cubicus et
38 I supra Modo scr et del $E$ istos(?)/ sicut $C R V$ sicud $E$ sic $H /$ quantum: quam (?) $H$
382 prius est $\operatorname{tr} V$
384 consideraverit: consideravit $V /$ inveniret $H V$ inveniet $C E$ invenient $R$ / inter om $V$
385 rationalium om $E /$ generaliter om $C$ / sunt $^{1}$ om $E /$ ille: iste $E / \operatorname{sunt}^{2} E H R$;

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would not be a cube number. A similar situation is found in games where, if one should inquire whether a hidden number is a cube number, it is safer to reply in the negative since this seems more probable and likely.

Now what has been said here about numbers may be applied to ratios of rational ratios, as was shown before, since there are many more irrationals than others, understood in the previous sense [of ratios of ratios]. What is more, [it is even more applicable to ratios, for] if one reflected carefully he would find that between ratios of rational ratios those which are rational are fewer than cube numbers in an aggregate of numbers. Therefore, if any unknown ratio of ratios were sought, it is probable that it would be irrational and its ratios incommensurable. And all this applies if the unknown ratios which are sought should be rational.

Now if, perchance, the proposed ratios were of the second type, namely irrationals with denominations and commensurable to rationals, the same thing will be proved as follows: Let $A$ and $B$ [be the ratios], and then one can argue that $A$ is commensurable to any rational by which it is denominated, and let us call that $C$. Similarly, $B$ is commensurable to any rational which denominates it, and let us call that $D$. Now if $C$ and $D$ are incommensurable, then $A$ and $B$ are incommensurable-which can be argued by the comment on the eighth [proposition] of the tenth [book of Euclid]since it follows that [if] $A$ is commensurable to $C$ but $C$ is incommensurable to $D$, then $A$ is incommensurable to $D$. And, continuing, [since] $A$ is incommensurable to $D$ and $D$ is commensurable to $B$, then $A$ is incommensurable to $B$. It is clear, therefore, that if $C$ and $D$ are incommensurable, so are $A$ and $B$. But it is probable that $C$ and $D$, which are rational

[^29]mensurabilis $E R$
399 incommensurabile $E H V$ incommensurabilis $C R$
399-400 Et... ${ }^{\text {r }}$ om $V /$ incommensurabile $E H$ incommensurabilis $C R$
$400 \mathrm{D}^{\text {I }} C H R$; om $E /$ commensurabile $E H V$ incommensurabilis $C$ commensurabilis $R$
400-401 igitur...B om $V /$ incommensurabile $E H$ incommensurabilis $C R V$
401 si: se $H$ / incommensurabiles: incommensurabilis $C$
$402 \mathrm{~B}: \mathrm{D} E /$ incommensurabiles: incommensurabilis $C$ / Sed: et $E$
sunt proportiones rationales, sint incommensurabiles ut prius probatum est, igitur verisimile est quod $A$ et $B$ sint incommensurabiles, quod nunc fuit probandum.

Et si forte essent de tertio modo proportionum, si sint alique tales ita quod nullam haberent denominationem ahuc estimandum esset, et verisimile est, quod ita sit de illis sicut de aliis quantum ad hoc, scilicet quod inter proportiones illarum proportionum rationales sunt sitas incommensurabiles esse

Et si forte una esset de uno modo et alia de alio tunc arguitur sic: in quolibet modo per se sumpto proportiones commensurabiles inter se sunt rariores aliis igitur similiter erit in totale multitudine propor-

Ex illis modis aggregata consequentia nota est et antecedens prius probatam est et ex antecedente sequitur propositum igitur et ex consequente.

Et arguitur specialiter. Si $A$ esset de primo modo et $B$ de secundo quia tunc $B$ esset commensurabilis alicui de primo modo, sit illa $C$, tunc ultra $B$ est commensurabile $C$ et $C$ est incommensurabile $A$ igitur $B$ est incommensurabile $A$. Consequentia patet per commentum octave decimi et antecedens est verisimile quia prima pars est vera per positum et secunda est verisimilis, ut prius est probatum, igitur conclusio est verisimilis.
Patet itaque quod duabus proportionibus ignotis propositis quales fuerint sive rationales sive non, verisimile est illas incommensurabiles esse quod fuit primo propositum. Igitur si proponantur multe veri-

403 rationales: irrationales $E /$ sint $H R$ sunt CEV / incommensurabiles: incommensurabile $H$
403-4 probatum est $C E H$; tr $R V$
404 est $^{2}$ om $H /$ sint $H$ sunt $C E R V$ 40 s nunc fuit $H ; \operatorname{tr} C R V$ fuit $E$
406 si$^{1} E H V$; om $C R /$ tertio: secundo $V$ $/$ sint: sunt $R$
407 esset $H$ est $C E R V$
408 et: ut $C$ / ita sit $\operatorname{tr} V /$ illis: istis $C$
409 proportiones: proportione $C$ /illarum $E H V$ istarum $C R /$ proportionum om $V$ | ante rationales add $C V$ proportiones / sunt rep $R$
410 rariores: minores $V$ / ante ideo add $V$ et / ideo om $E$ / esset: est $E$

410-II propositas: proportionalitas $C$
4 II incommensurabiles esse $\operatorname{tr} V$
412 esset $H$ sit $C E R V /$ alia: alio $E$ / post alio add $R$ modo / sic om $H$
413-14 commensurabiles inter se $H R$ inter se commensurabiles $C$ incommensurabiles inter se $E V$
414 sunt om $H$ / post sunt bab $C$ et / rariores $C H R$ maiores $E V /$ similiter erit $H$ ita erit $C$ arguitur est $E$ similiter ita erit $R$ similiter esset $V$ / totale: totali $C /$ multitudine: multiplicatione $E$ 416 Ex illis: istis $R /$ nota: vero $C$
417 probatam: probatum $V / \mathrm{et}^{2}$ om $H$
419 ante arguitur bab $E$ sic / post secundo add $E$ modo
ratios, would be incommensurable, as has already been shown, and consequently it is probable that $A$ and $B$ would be incommensurable, which was to be proved now.

And if it should happen that the proposed ratios belonged to the third type of ratio, if there are any such ratios with no denominations-and it is probable there are, but if not they can be imagined-the same applies to them as to the others with respect to this, namely that among the ratios of these ratios, rationals are fewer than irrationals; and thus it would be probable that the proposed ratios would be incommensurable.

And if, perhaps, one ratio belonged to one kind and the other to another kind, then you argue this way: in any type considered by itself, the ratios commensurable to each other are fewer than the others; and therefore the same will be true for the total number of ratios [resulting from the combination of the two types].

From all these types a collective consequence has been observed and the antecedent was already proved, so that what was proposed follows from the antecedent and, therefore, from the consequent.

Now the above [general assertion] can be argued specifically. If $A$ were a ratio of the first kind and $B$ of the second, then since $B$ could be commensurable to any ratio of the first type, let us call such a ratio $C$. Then, [carrying the argument] further, $B$ is commensurable to $C$ and $C$ is incommensurable to $A$; therefore $B$ is incommensurable to $A$. The consequent is evident, by the comment on the eighth [proposition] of the tenth [book of Euclid]; and the antecedent is probable because the first part [of $\mathrm{it}]$ is true by assumption and, as already shown, the second is probable so that the proposition is probable.

And so it is clear that with two proposed unknown ratios-whether they are rational or not-it is probable that they are incommensurable, which was proposed in the first place. Therefore, if many [unknown ratios] are

420 quia: quod $H /$ tunc: si $R /$ esset $C H$ erit $E R V$ / commensurabilis: incommensurabilis $V$ | ante sit scr et del $V$ sit / illa $E($ ? ) $H R$ ista $C$ ita $V$
421 C est om $E$
422 ante per bab $V$ et
423 vera: una $V$
424 positum: suppositionem $C$ / post secunda add $V$ pars / ut prius est probatum: probatum est prius $C /$ est $^{2} E R$

## fuit $H V$

426 ignotis om $E$ / propositis rep $E$ / quales: qualescumque $E$
427 sive $^{1} H R V$ vel $C$ seu $E /$ sive $^{2} H R V$ vel $C$ seu $E$ / verisimile: verisimiles $E$ / illas $H V$ eas $C$ illa(?) $E$ istas $R$
428 quod fuit primo: proportio fuit primo ad $E$ / post proponantur $m g$ bab $H$ recao(?) conclusionis(?)
428-29 verisimillius: verisimile $V$
simillius est aliquam alicui incommensurabilem fore quod est secundo
${ }_{430}$ propositum. Et quanto plures essent, tanto magis credendum esset quod aliqua sit alicui incommensurabilis, si enim proposita una proportione proportionum verisimile est illam irrationalem esse, propositis pluribus verisimillimum est aliquam irrationalem fore sicut posset in exemplo de numeris cubicis declarari.

In probatione precedentis conclusionis dicebatur quod si capiantur roo proportiones per ordinem in genere multiplici incipiendo a dupla inter eas reperiuntur 4950 proportiones proportionum eas invicem comparando pro cuius declaratione unam conclusionem practicam pono talem.
Undecima conclusio. Quotlibet terminis eiusdem generis propositis quorum quilibet sint inequales quot proportiones inter eos fuerint quemlibet cuilibet comparando invenire.
Numerus propositorum terminorum primitus est sumendus qui multiplicandus est per propinquiorem minorem, scilicet per immediate
445 precedentem. Et numerus productus est numerus proportionum terminorum prius positorum numerando proportiones maioris inequalitatis et minoris. Quod si volueris habere proportiones maioris inequalitatis precise tunc eiusdem numeri producti medietas capiatur et habebis intentum sicut feci in probatione conclusionis precedentis quia ${ }^{50}$ de proportione proportionum maioris inequalitatis erat sermo. Tamen de aliis in primo capitulo fuit expeditum quoniam proportio earum est penitus sicut proportio proportionum maioris inequalitatis sibi correspondentium. Et ideo etiam nunc volo loqui tamen de proportionibus maioris inequalitatis de quibus semper loquuntur auctores
455 quia etiam idem est numerus earum cum numero aliarum.
429 aliquam: aliquem $V /$ alicui: alteri $R /$ incommensurabilem: incommensurabile $C$ / ante quod scr et del $R$ sicut patet in exemplo / est ${ }^{2}$ om $V$
430 esset: est $C$
431 sit alicui $\operatorname{tr} E /$ una om $E$
432 illam: illa $E /$ esse $C R V$; om $E H$
$432-33$ propositis...est $E H R$ propositis pluribus verisimile est $C$ verisimillimum erit propositis pluribus $V$
433 est: erit $R /$ posset $E H V$ possum $C$ patet $R$
434 de: et in $C$
435 probatione: proportione $C$ / capiantur: capiatur $E$

436 roo om $H /$ per ordinem CER; om $H$ / a $C H R$ ad $E /$ dupla $C E H$ duplo $R$ 436-37 per ordinem...proportiones om $V$ 437 reperiuntur $E H R$ inveniuntur $C$
$43^{8}$ practicam $E(?) H R$ placitam $C$ probatam $V$
439 pono om $E$ / talem om $R$
440 Undecima conclusio $m g$ hab $V$ post talem (linea 439) et mg bab R post pono (linea 439); om CEH / Quotlibet: quilibet $E$
440-41 quorum...inequales om $V$
44I $\operatorname{sint} C H R$ sit $E$ / post inequales $m g$ bab $E$ conclusio(?) practica(?) / post eos scr et del $V$ fui / quemlibet $H R$ quam-

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proposed, it is [even] more probable that any one of them would be incommensurable to any other, which was proposed in the second instance. Now the more there are, the more one must believe that any one is incommensurable to any other, for if it is probable that one proposed ratio of ratios is irrational, it is more probable when many are proposed that any one would be irrational, just as could be shown in the example involving cube numbers.

In the proof of the preceding proposition it was said that if roo ratios were taken in a multiple genus commencing with a double ratio and taking them in sequence, 4,950 ratios of ratios could be found between them by a mutual comparison; and to show this I [now] set forth one such practical proposition.

Proposition XI. [How] to find the number of ratios between any proposed number of unequal terms by relating every one of them to every other of them.

The number of proposed terms must first be taken and multiplied by the closest lesser [number], namely by the immediately preceding [number]. The number produced is the [total] number of ratios of greater and lesser inequality [that can be formed] by the proposed terms. But if you desire to have the exact number of ratios of greater inequality, then half the produced number must be taken-and you will understand my purpose as set forth in the proof of the preceding proposition where the discussion concerned a ratio of ratios of greater inequality. However, in the first chapter the connection was made between these and the others [namely ratios of lesser inequality], since every ratio [of ratios of lesser inequality] is exactly like a ratio of ratios of greater inequality corresponding to it. But now I wish to consider only ratios of greater inequality, about which authors always speak, and also because the number of these is identical with the number of the others.
libet $C E V /$ cuilibet $C E H$ cuicumlibet $V$ quilibet $R$
443 est om $E$
444 multiplicandus est: : multiplicando $E \mid$ propinquiorem: propinquationem $C$
445 precedentem $H R V$ precedentis $C$ precedente $E$
446 prius: primo $E$
447 volueris habere $C R V$ volueris $E$ voluerit contrahere $H$ / proportiones: proportionem $R /$ maioris: minoris $H$
449 habebis: habebit $H$ / quia om $H$
4so proportionum CHV;omER/Tamen
$C$ cum $E$ (?) $R V$ et tanto $H$
$45 I$ in om $H$ / fuit: erat $R$ 452 penitus sicut: propositum sicud $E$ $452-54$ sibi...proportionibus om $C$
453 Et ideo $H R V ; \operatorname{tr} E /$ etiam $H R V$; om $E$ / loqui tamen $V ; \operatorname{tr} E$ tamen $H$ loqui $R$ / post tamen add $V$ modo
453-54 post proportionibus add $R$ proportionum
454 ante de hab $R$ immo
45 s etiam om $E$ / numero $E R V$; om $C$ numeris $H$

Sint, igitur, gratia exempli, 4 termini. Multiplicando 4 per 3 proveniunt 12 numerus totalis proportionum in utraque inequalitate cuius medietas vel subduplum est 6 , numerus proportionum maioris inequalitatis inter terminos assignatos. Et totidem linee possunt protrahi
460 de uno ad alterum a 4 punctis dispariter situatis qui sint $A B C D$ et totidem modis possunt quelibet 4 res in 2 combinari. Et ita agendum est si plures res, puncta, seu termini, proponantur.
Ad inveniendum numerum combinationum linearum seu proportionum aliud exemplum sit illud quod ponitur in precedenti conclusio-
465 ne, scilicet in eius declaratione sint itaque 100 proportiones sicut 100 termini. Multiplicando, igitur, istum terminum, scilicet 100, per immediate precedentem, scilicet per 99 , et exibunt 9900 et huius producti capiam medietatem, scilicet 4950 et habebo numerum proportionum maioris inequalitatis 100 proportionum.
470
Si autem iste roo proportiones sint de genere multiplici per ordinem sumpte, sicut prius dicebatur, ut dupla, tripla, quadrupla, quintupla, et cetera, et iste 4950 sunt proportiones earum ostendo quod istarum 4950 proportionum 25 sunt rationales et non plures et omnes alie sunt irrationales. Accipio, primo, $2^{\mathrm{lam}}, 4^{\mathrm{lam}}, 8^{\mathrm{lam}}, 1^{6^{\mathrm{lam}}, 32^{\mathrm{lam}}, 64^{\mathrm{lam}} \text {. Iste }}$
4756 sunt inter se commensurabiles et nulla alia rationalis citra ror $^{\text {lam }}$ est commensurabilis alicui earum per sextam huius capituli. Multiplicamus, ergo, 6 per $s$ et capiamus medietatem producti et habebimus is et iste est numerus proportionum istarum proportionum. Et iste is proportiones proportionum sunt rationales.

456 Sint: sicut $R$ / post igitur $m g$ hab $H$ exemplum / gratia exempli $E H$; om $C$ $\operatorname{tr} R V$ / Multiplicando $E H R$ multiplico $C V /$ per ${ }_{3} C H R$;om $E_{3} \mathrm{~V}$ 457 cuius: illius $V$
458 est om $E /$ proportionum: propor tionis $R$
460 post uno add $V$ termino / alterum:
L. alium $E /$ a: ad $E /$ post 4 mg hab $E$ figuram et post D mg hab H ibid. figuram:

qui $E H$ que $C R V /$ sint: sunt $R$ 461-62 agendum: faciendum $E$
462 est om $R$ / plures: similares $H$ / ante puncta bab $C$ aut / puncta seu $C H$ quanta(?) autem $E$ punctus seu $V /$ puncta...proponantur: termini aut puncta ponantur $R$
463 ante linearum bab $E$ aut / seu: vel $C$ 464 illud $H R V$ id $C E /$ precedenti: precedente $E$
465 ante sint hab $E$ sicud / proportiones: sint $C$ | ante sicut bab $R$ si / post sicut mg hab $H$ ad propositum / ante $100^{2}$ bab $H$ centum et
466 termini om $C$ / Multiplicando: multiplicabo C / igitur: scilicet $R /$ terminum $C H$ numerum $E R V$ / scilicet CHV; om $R$ sicud $E$

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For example, let there be four terms. By multiplying 4 by 3 we get $\mathbf{1 2}$, the total number of ratios in each inequality of which the half, or subdouble, is 6 , the number of ratios of greater inequality between the terms assigned. And just as many lines could be drawn from one to the other of four points, A, B, C, D, situated separately; indeed, any four things can be combined two at a time in just as many ways. If more things, points, or terms are proposed, the same procedure must be followed.
Another example for finding the number of combinations of lines or ratios could be the one set forth in the preceding proposition, namely where there were 100 ratios taken as 100 terms. Hence, by multiplying 100 by the immediately preceding term, namely 99 , we get 9,900 and should I take half this product, namely 4,950 , I shall have the number of ratios of greater inequality of 100 ratios.

Furthermore, if, as was said before, these 100 ratios were taken in sequence from a multiple genus, that is $2 / 1,3 / 1,4 / 1,5 / 1$, etc., and these 4,950 ratios are ratios of these ratios, then I shall show that of these 4,950 ratios, 25 -and no more-are rational and all the others are irrational. First, I take $2 / 1,4 / \mathrm{I}, 8 / \mathrm{I}, 16 / \mathrm{I}, 32 / \mathrm{I}, 64 / \mathrm{I}$. These 6 are mutually commensurable and no other rational below ${ }^{101} / \mathrm{I}$ is commensurable to any of them, by the sixth proposition of this chapter. Therefore, let us multiply 6 by $s$ and takehalf of the product and we will have 15 , which is the number of ratios between these ratios. And these is ratios of ratios are rational.

467 precedentem: precedentis $C /$ per $472-73$ istarum $4950 \operatorname{tr} E$ CHR; om $V$ propositum $E /$ exibunt $47325: 12 E$ $C$ cubicus $E$ exibit $H R V \mid$ ante 9900 bab $V$ scilicet / huius producti rep $H$
468 medietatem $C R V$ medietatus $E$ / post scilicet bab $E$ multe(?) / habebo $E R V$ habeo $C$ / proportionum $C E V$ proportionis $R$
468-69 medietatem... maioris om $H$
469 post inequalitatis scr et del $E$ si autem illa / 100: 200(?) $E$ / proportionum om $E$
470 iste: ille $R / 100 \mathrm{om} H /$ multiplici: multiplicis $E$
471 dicebatur: dicebitur $C$ / quintupla om R
472 et cetera om $E$ / iste CHV; om $E$ ille $R$ / sunt $E V$ sint $C H R$ / post quod bab $E$ iste

474 Accipio: accipiendo $R$ /ante 64 lam scr et del $H_{16} 6$ lam / Iste: ille $R$
$4756 \mathrm{om} \mathrm{H} /$ inter se commensurabiles $C$ inter se $E$ commensurabiles $H$ commensurabiles inter se $R V /$ citra: erit $C /$ 101 $\operatorname{lam} C E$ 100 $H$ 100 lam $R V$
475-76 est...earum: alicui earum est commensutabilis $R$
476 alicui $C H V$ alteri $E$ / earum $E H V$ eorum $C$
476-77 Multiplicamus $H$ multiplicem $C$ multiplicemus $E R V$
477 ergo om $H / \mathrm{et}^{2} C V$; om $E H R \mid$ post habebimus bab $E$ scilicet / Is: $25 R$
478 proportionum istarum proportionum $C R V$ istarum $E$ istarum proportionum $H$ alicui earum per eandem sextam huius. Et sunt 4 et operando sicut prius inveniemus 6 proportiones istarum proportionum que sunt rationales. Item si capiamus $4^{\text {lam }}, 16^{\text {lam }}$, et cetera. De eis dictum est quia ista coordinatio est pars prime. Item $5^{\text {la }}$ et $25^{\text {la }}$ sunt commensurabiles
${ }_{485}$ et nulla eisdem citra $10 I^{\text {lam }}$ quarum est una proportio proportionum rationalis. Item $6^{\text {la }}$ et $3^{6^{\text {la }}}$ dant nobis unam aliam, $7^{\text {la }}, 49^{\text {la }}$ unam aliam, et de $8^{\text {la }}$ et de $9^{\text {la }}$ etiam dictum est. Et item rola et roola unam aliam, summa totalis 25 . Et quia nulla alia sicut $\mathrm{II}^{1 \mathrm{la}}, \mathrm{I} 2^{\text {la }}$, et cetera, est commensurabilis alicui citra ıor ${ }^{\text {lam }}$ nisi sit aliqua predictarum ut patet ex sexta conclusione, manifestum est quod tantum sunt 18 proportiones quarum quelibet est commensurabilis alicui rationali citra $10 I^{\text {lam }}$. Et si quelibet istarum 18 esset commensurabilis cuilibet earundem haberemus 153 proportiones proportionum rationales. Sed quia non ita sed 6 prime sunt commensurabiles inter se, et alie 4 inter se et 495 incommensurabiles primis, et sic de aliis. Ideo tantum sunt 25 proportiones rationales de 4950 proportionibus proportionum, de quibus erat sermo, et relique omnes sunt irrationales. Est igitur proportio irrationalium istarum ad rationales sicut 197 ad I .

Pro conclusione practica prius posita data est una regula sed do unam aliam per quam etiam invenitur numerus proportionum inter terminos quotlibet assignatos.
$48027^{\text {la }}: 7^{\text {la }} E / 8$ I $^{\text {la }} C R V$; om $E 8^{\text {la }} \mathrm{H} \mid$ sunt om $V$ / commensurabiles: commensurabilis $C$ /post commensurabiles hab $E$ et $82^{\text {la }}$
481 earum: earundem $H$ / huius $C R$; om $H$ habemus $V /$ operando CHV comparando $R$
481-82 per...proportionum om $E$
482 inveniemus $C H V$ inveniamus $R$
483 Item: etiam $E /$ si om $H / 4^{\operatorname{lam} C E V}$ ${ }^{2 l a m} H 9^{\text {lam }} R /{ }^{\text {6lam: }}$ : Glam $V /$ eis $E H V$ istis $C$ hiis $R /$ quia: quod $H$
484 ista: illa $R /$ prime: prima $H /$ Item: et $E / 25^{\text {la }}:$ r $^{\text {la }} E /$ commensurabiles om $C$
485 eisdem $C H R$ eis $E$ eiusdem $V /$ rorlam $V$ (?) $C R$ roolam $E$ ioo $H /$ ante quarum hab $E$ prima(?) / post est bab $V$ octava / una proportio $E H R ; \operatorname{tr} C V$
485-86 proportionum rationalis $R V$ proportionum rationalium $C$ rationalium
$E$ rationalis $H$
486 36 la: $30^{\text {la a }} C$ / post aliam ${ }^{\mathrm{r}}$ add $E$ item / $49^{\text {ala }} 4^{4^{\text {la }} E}$
486-87 7 la $\ldots$ aliam om $V$
$487 \mathrm{et}^{\mathrm{t}} \mathrm{E}$; om $\mathrm{CHR} / \mathrm{del}^{\mathrm{I}} \mathrm{CHR}$; om E/et de $9^{\text {la }}$ etiam om $H \mid$ post $\mathrm{et}^{2}$ add $C$ cetera / $\mathrm{de}^{2} C E$; om $R / 9^{\text {la }} C R$ igla $E /$ etiam $E R$; om $C$ | post est bab $C$ et cetera / Et E; om CHR / iola CER 20(?) $H$ / ioola $C E R$ 100 $H$
488 Et quia om $V /$ sicut $C R V$ sicud $E$ sunt $H / \mathrm{I}^{\text {la }}$ : nulla $V /$ post $\mathrm{I}^{\text {la }}$ add $R$ et / $12^{\text {la }}: 2^{\text {la }} E /$ et cetera om $R$ / est om $V$
489 roilam $H V$ rolam $C R$ centuplam $E /$ predictarum: istarum $V$
490 sunt 18: 16 sunt $C / 18$ corr ex 16 CEHRV
491 $10 \mathrm{r}^{\text {lam }}$ corr ex 100 CHRV ioolam $E$
492 si om $C$ /istarum: earum $R / 18$ corr ex 16 CEHRV / cuilibet om $R$

## Chapter Three

In the same way $3 / \mathrm{I}, 9 / \mathrm{I},{ }^{27} / \mathrm{I},{ }^{81} / \mathrm{I}$ are mutually commensurable and no other is commensurable to any of them, by the same sixth proposition of this chapter. Now there are four ratios, and operating as before we find 6 rational ratios of ratios. Likewise, if we should take $4 / \mathrm{I},{ }^{16} / \mathrm{I}$, etc. But these have been covered, since this series is part of the first [series, and so also are the ratios of ratios which they form]. Again, $5 / \mathrm{I}$ and $25 / \mathrm{I}$ are commensurable and no ratio below ${ }^{101} / \mathrm{I}$ is commensurable to them so they form one rational ratio of ratios. In the same manner, $6 / \mathrm{I}$ and $36 / \mathrm{I}$ give us one other [rational ratio of ratios] as do $7 / 1$ and $49 / \mathrm{r}$. [Ratios] $8 / \mathrm{I}$ and $9 / \mathrm{x}$ have been discussed [since they are part of other series]. [Finally], in the same way $10 / \mathbf{r}$ and $100 / \mathrm{I}$ produce one other, for a grand total of 25 . Since other than the aforementioned ratios no other ratio, such as ${ }^{11} / \mathrm{I},{ }^{12} / \mathrm{I}$, etc., could be commensurable to any ratio below ${ }^{101} / \mathrm{I}$, which is clear from the sixth proposition, it is obvious that there are only 18 ratios, each of which is commensurable to some rational ratio below ${ }^{101} / \mathrm{I}$. And if each one of these 18 should be commensurable to any other of the same [18] ratios, we would have 153 rational ratios of ratios. But because this is not so, and the 6 ratios of the first series are only commensurable between themselves, and the 4 others [of the next series commensurable] only between themselves but incommensurable to the first [series], and so on for the remaining series, there are, therefore, only 25 rational ratios [of ratios] out of the 4,950 ratios of ratios under discussion; and the remainder are all irrational. Thus the ratio of irrationals to rationals is as 197 to 1 .

For the practical proposition presented before a rule was given, and [now] I give another one by means of which the number of ratios can be found between any assigned terms.

493 haberemus om $H$ / 153 corr ex 120 CHRV rio(?) E / Sed om(?) H | post non add $R$ est
494 post sed scr et del $V$ quia / sunt commensurabiles $\operatorname{tr} C /$ et alie 4 inter se om $E / \mathrm{et}^{2}$ om $V$
495 Ideo $H V$ non $C$ ergo $E$ igitur $R /$ tantum CHR tamen EV | post sunt scr et del $V$ s
496 rationales: totales $R /$ proportionum om V
497 omnes om(?) $H \mid$ sunt om $E \mid$ post sunt scr et del $H$ incommensurabiles
irrationales: rationales $R /$ proportio $C H R$; om $E$ proportionum $V$
498 irrationalium $C H V$ rationalium $E$ irrationabilium $R /$ ante $\mathrm{ad}^{1}$ scr et del $E$ sicut / 197 correx 198 CHV centum ad octo $E 190 R$
499 practica: placita $C /$ prius om $R /$ ante posita $m g$ hab $E$ secunda regula pro conclusione / posita om $H$ / do: dico V
soo etiam om $R$
sor terminos: numeros $H$

Sit, itaque, numerus terminorum datus $A$. Si , igitur, $A$ fuerit numerus par ab eo deme 2 et per medietatem residui multiplica ipsum $A$ et producto adde medietatem ipsius $A$ et habebis intentum. Si , vero, $A$ est impar ab eo deme I et per medietatem residui multiplica ipsum $A$ et numerus productus erit numerus proportionum, aut combinationum terminorum, aut linearum. Sed si proposita essent puncta dispariter situata.
Exemplum primi: sit 8 numerus terminorum. Deme 2, remanent 6 - cuius medietas est 3 per quam multiplica 8 et sunt 24 cui adde medietatem ipsius 8 et habebis 28 qui est numerus quesitus.
Exemplum secundi: sint termini 7 . Deme 1, remanent 6 cuius medietas est 3 per quam multiplica 7 exibit 21 , numerus qui queritur.
Ex istis duabus regulis, que in uno fine conveniunt quia idem
${ }_{3}{ }^{5}$ haberetur per unam et per aliam, sequitur quod totalis numerus proportionum in una inequalitate, aut combinationum aliquorum terminorum aut linearum inter puncta dispariter situata, non potest esse nisi unus numerorum in hoc ordine positorum 1, 3, 6, 10, 15, 21, 28, 36, et cetera. Que quidem ordinatio componitur ac etiam ulterius quamlibet protenditur in hunc modum: primo et pro primo pones unitatem cui adde 2 et habebis secundum istorum numerorum cui secundo adde 3 et habebis tertium cui adde 4 et habebis quartum cui adde 5 et habebis quintum, et sic ultra. Unde prima differentia est 2, alia 3, alia 4, alia s et sic secundum seriem numerorum. Nullorum itaque terminorum

502 post itaque bab $E$ prima(?)/ numerus terminorum $R V ; \operatorname{tr} C$ numerorum $E$ numerus assignatus terminorum $H^{\prime}$ A $C H R$; om $V$ aut $E /$ A fuerit $\operatorname{tr} E$
s03 et per $H$ et $E$ per $R V /$ multiplica $E H R$ multiplicata $V /$ ante ipsum $b a b$ $E$ per
503-5 $2 \mathrm{et} . . . \mathrm{ab}$ eo om $C$
so4 Ar $E R V$; om $H /$ et $^{1} E H V$; om $R$ adde $E H R$ aliud $V /$ medietatem $E R$ minoris $H$ de medietate $V$ / habebis $E R V$ habebitur $H$
sos est $H R V$ numerus $E$ / medietatem: medietate $E /$ multiplica: multiplicam E
so6 ipsum om $E /$ numerus productus $\operatorname{tr} E$ erit $H R V$ esset $C$ est $E$
so6-7 combinationum: combinationarum $C$
so7 linearum: litterarum $R$ / Sed H; om
$C E R V /$ puncta: producta $V$
sog primi om $C$ / sit 8 numerus terminorum: numerus propositus sit $8 E$ remanent: remanebunt $C / 6 \mathrm{om} C$
f 10 3: tripla $E /$ post per bab $R 8 / 8$ et sunt 24 om $E$ / sunt $C R V$ habebit $H$ / ante 24 scr et del H $28 /$ cui: et $E$
510-II ante medietatem $m g$ bab $H$ exempla
5 II ipsius... $28 C R V$ sunt $28 E$ ipsius 8 et habebit $28 \mathrm{FI} /$ est om $E$ / ante quesitus scr et del $V$ quie
512 sint: si(?) H| ante 7 bab E A et scr et del H A / post 6 bab $E$ est
513 est om $E /$ exibit $C H V$ exibunt $E R$ / 21 numerus $\operatorname{tr} H /$ qui om $E /[M S E$ concludes with word queritur]
514 ante regulis mg bab $H$ sequitur / idem: unum $R$
sis haberetur....aliam $H$ per unam et per aliam habetur $C$ haberetur per unam

Chapter Three
Let the number of given terms be $A$. Then if $A$ is an even number, subtract 2 from it and multiply $A$ by half of the remainder, and then add half of $A$ to the product and you will have the number sought.* However, if $A$ is odd, subtract I from it and multiply $A$ by half of the remainder, and the number produced will be the number of ratios or combination of terms, or lines. $\dagger$ But if points were proposed, they must be separately situated.

An example of the first [case] would be setting the number of terms at 8 . Subtract 2 and 6 remains, half of which is 3 , by which you multiply 8 to get 24 , to which you add half of 8 , which gives 28 , the number that was sought.

An example of the second [case] can be given with 7 terms. Subtract I and 6 remains, half of which is 3 , by which you multiply 7 to get 21 , which is the number sought.

From these two rules, which subserve the same purpose because the same end can be achieved by one and the other, it follows that the total number of ratios in one inequality, or combinations of terms, or lines connecting points separately situated, can only be one of the numbers set forth in the following series: $1,3,6,10,15,21,28,36$, etc. This series can be composed and extended in the following way: To obtain the first term you [simply] take the unit; then, to the unit add 2 and you will have the second number; add 3 to the second term and you have the third term, to which you can add 4 for the fourth term; adding 5 to the fourth term gives the fifth term, and so on. Thus the first difference is 2 , the next 3 , then 4,5 , and so on, through the series of numbers. $\ddagger$ Moreover, of no terms is the

* For an even number of terms $n$, the rule can be formulated as $n[(n-2) / 2]+n / 2$. $\dagger$ For an odd number of terms $n$, the rule is $n[(n-1) / 2]$.
$\ddagger$ The terms in the series of total combina-
quod per aliam $R$ habetur per unam et per aliam $V /$ totalis om $H$
519-16 proportionum: proportio $R$
$s 16$ in una $o m H$ / inequalitate: equalitate $R /$ combinationum: combinatione $R$
$\int_{17}$ ante situata scr et del $V$ siti(?)
${ }_{5} 19$ cetera om $R$ /quidem: quidam $C /$ ordinatio $H$ (?) $C R$ ordinanter $V / \mathrm{ac}$ : aut $H /$ etiam: si in $C /$ ulterius quamlibet $t r V$
s 20 protenditur $C H$ proceditur $R V /$ primo...primo: prius et pro primo nu-
tions can be generated by the arithmetic progression $S_{n}=(n / 2)(a+p)$, where $n$ is the number of terms in the series, $a$ is the first term, and $p$ the $n$th term in the series.
mero $R /$ proprimo $C V$ primo numeros (?) $H$
s21 habebis: habebit $H$ | ante secundo scr et del $V$ secundum / secundo adde $\operatorname{tr} C$ 522 habebis ${ }^{\text { }}$ : habebit $H$ habebis ${ }^{2}$ : habebit $H / s H V$ quintum $C$
522-23 cui...quintum om $R /$ habebis quintum $C$ exibit quintus $H$ habebit quintus $V$
523 ante prima scr et del $V$ sic dicitur / ante differentia bab C dicta(?)

525 totalis numerus proportionum aut combinationum, et cetera, est 2, aut 4 , aut 5 , aut 7 , et sic procedendo per numeros alios a predictis. Et idem dico de lineis factis inter puncta dispariter situata, sed scilicet quod nulla 3 puncta sunt in una linea recta.

## Quartum capitulum

1 Quasdam propositiones de motibus in hoc quarto capitulo demonstrabo pro quibus sunt etiam alique suppositiones premittende.
Prima sit hec: velocitas sequitur proportionem potentie motoris ad mobile seu ad resistentiam eius. Unde proportio unius velocitatis ad
5 alteram est sicut proportio proportionis potentie unius motoris ad suum mobile ad proportionem proportionis alterius motoris ad suum mobile. Ista suppositio patet per Aristotelem secundo celi et per Commentatorem ibidem, et quarto et septimo phisicorum.
Secunda suppositio. Proportio composita ex maiore et minore est ${ }_{10}$ minor quam dupla maioris et est maior quam dupla minoris. Hoc est generaliter verum de qualibet quantitate.
Tertia. Omnes potentie sunt equales que idem mobile vel equalia possunt equali velocitate movere.
Quarta. In quantumcumque potest aliqua potentia, in idem potest 15 quelibet potentia sibi equalis et in quodlibet equale.

Quinta. Omnis pars cui totum est precise duplum est residuo equalis. Et illa cui totum est plusquam duplum residuo est minor; et cui totum est minus quam duplum residuo est maior.
Sexta. Si aliqua pars est commensurabilis suo toti erit etiam com-
s2s et cetera $H V$; om $C$ etiam $R /$ ante es bab $H$ et est
s26 post 7 add $R$ et cetera / alios: aliis $R$
s27 idem: ideo $H$ | sed scilicet $V$; om $H$ scilicet $C R$
$\{283$ : est $V /$ sunt $H V$ sicut $C$ sint $R$
a quarto capitulo $H V ; \operatorname{tr} C$ capitulo $R$
2 sunt etiam $\operatorname{tr} R /$ etiam $H$; om $C$ et sic $V$ / alique: aliqui $H$ | post alique add $C$ et
3-4 potentie...eius: ad mobile vel ad resistentiam eris $C$
4 eius om $R /$ velocitatis: proportionis $R$

6 proportionem: proportionum $V$ / proportionis: proportionum $H$
7 per $^{1}$ CH; om RV
9 Secunda suppositio $H$; om $V$ secunda $C R$ / maiore: minori $H /$ minore: minori $H /$ est: et $C$
10 minoris: minor $V$ / post Hoc add $C$ enim
12 Tertia $C R$; om $V$ tertia suppositio $m g$ bab $H$ ante equales
14 Quarta $C R$; om $V$ suppositio quarta $m g$ bab $H$ ante aliqua / quantumcumque $C H$ quodcumque $R V /$ idem potest $R V$ illud potuit $C$ idem potuit
total number of ratios or combinations 2 , or 4 , or 5 , or 7 , and so on through all the numbers not in the series mentioned above. And I say the same about lines drawn between points separately situated, with the proviso that no 3 points are to be in a straight line.

## Chapter Four

In this fourth chapter I shall demonstrate some propositions concerning motions, for which some suppositions must be set forth.
Supposition I. A velocity varies as the ratio of a motive power to a mobile or the resistance of a mobile. Thus a ratio of one velocity to another varies as a ratio between the ratio of the power of one mover to its mobile and the ratio of another mover to its mobile.* This supposition is evident from Aristotle in On the Heavens, Book II, and from the Commentator's comment on the same section; it is also clear from Aristotle's Pbysics, Books IV and VII.

Supposition II. A ratio composed of one greater and one lesser ratio is less than the greater squared and greater than the square of the lesser. $\dagger$ This is generally true of any kind of quantity.
[Supposition] III. All powers are equal which can move the same mobile, or equal mobiles, with equal velocity.
[Supposition] IV. Whatever some power can do can also be done in the same way and in equal measure by any power equal to it.
[Supposition] V. Every part to which the whole is exactly double is equal to the remainder. And that [part] to which the whole is more than double is less than the remainder; and that part to which the whole is less than double is greater than the remainder. $\ddagger$
[Supposition] VI. If some part is commensurable to its whole, it will also

* This is "Bradwardine's function" which we may represent as $F_{2} / R_{2}=\left(F_{1} / R_{1}\right)^{V_{2} / V_{1}}$. $\dagger$ If $A|C=A| B \cdot B \mid C$ and $A|B>B| C$, then $A / C<(A \mid B)^{2}$ and $A / C>(B / C)^{2}$.
$H$ / potest: potuit $C$
is quelibet potentia $V$ potentia $C$ queli-
bet $H R$ / et om $H$ / post et add RV etiam
${ }_{16}$ Quinta $C R$; om $V$ quinta mg bab $H$ ante duplum
$\ddagger$ If $A / C=A|B \cdot B| C$ and $A \mid C=(B \mid C)^{2}$, then $B|C=A| B$. But if $A \mid C>(B \mid C)^{2}$, then $B / C<A \mid B$; and if $A / C<(B \mid C)^{2}$, then $B / C>A \mid B$.

17 illa om $V /$ cui ${ }^{1}$ : cuius $R /$ minor: maior $C$
7-18 et cui...maior om $C R$
18 est minus $t r V$
19 aliqua om $C$ / erit: esset $V$
${ }_{20}$ mensurabilis residuo quod cum ea componit totum; et si est commensurabilis residuo erit etiam commensurabilis toti. Patet ex eo quod talis portio erit pars aliquota, aut partes, sui totius ex quo est toti commensurabilis. Igitur eadem erit denominatio talis partis et etiam residui, ut patet ex una suppositione facta in probatione tertie con-
25 clusionis secundi capituli. Igitur residuum erit commensurabile tali parti, igitur et toti per octavam decimi.

Septima. Si aliqua pars est incommensurabilis suo toti illa erit incommensurabilis residuo.

Sit $A$ totum, $B$ pars abscissa, $C$ residuum. Tunc arguitur sic: si $B$
${ }_{30}$ et $C$ sint commensurabilia, $A$ erit commensurabile utrique. Igitur, si $A$ non est commensurabile utrique illa non sunt commensurabilia. Consequentia patet a destructione consequentis, et antecedens patet per primam partem none decimi que dicit sic: si sint due quantitates communicantes totum quoque ex hiis confectum utrique earum erit 35 communicans; ergo si $B$ est incommensurabile ipsi $A$ similiter erit incommensurabile ipsi $C$.

Octava. Cognita proportione totius ad aliquam eius partem potest sciri proportio illius partis, seu portionis, ad residuum et etiam proportio totius ad residuum. Si enim proportio totius ad illam portionem ${ }_{40}$ sit rationalis eadem portio et residuum eodem numero denominantur et proportio portionis talis ad residuum est sicut numerator ad numeratorem. Proportio vero totius ad residuum est sicut proportio denominatoris eiusdem residui ad numeratorem eiusdem.

Ista patent ex tribus suppositionibus factis ad tertiam conclusionem

20 ante residuo add $R$ suo / est om $H$
21 erit etiam $C H$ esset $V$ erit $R /$ commensurabilis om $C$ / infra commensurabilis mg hab $H$ patet eo/toti: $H$ (?) / Patet ex om $H$
22 portio $H V$ proportio $C R$ / erit: esset V
23 Igitur: que(?) $V$ / ante eadem add $C$ hec / talis partis $\operatorname{tr} R$
24 tertie: secunde $V$
2s Igitur: ergo $V$ / residuum erit: $H$ (?) | erit commensurabile tali: esset commensurabilis totali $V$
26 igitur: ergo $V$
27 Septima $C R$; om $V$ alia $H$ / incommensurabilis: commensurabilis $V /$ illa CH (?) $V$ ita $R$

27-28 erit incommensurabilis: esset commensurabilis $V$
28 ante residuo add $R$ suo
${ }_{29}$ B pars abscissa C: pars abscissa sit B $C$ / arguitur: dicitur $C$
29-30 B et C: C et B C
$30 \operatorname{sint} H($ ? $) R V$ essent $C /$ erit : esset $V \mid$ commensurabile: commensurabilia $H$ / Igitur: ergo $V$
30-31 A non est: non A $H$
3 I illa om $V$
32 Consequentia: $H($ ? $) /$ patet: apparet $R$
33 que $H V$ qui $C R /$ sint $H$ fuerint $C R V$
34 totum: $H($ ? $) /$ quoque: quam $V$ | confectum: constrictum $C$ / earum: eorum $C$ / erit $C H$ esset $V$
34-35 erit communicans $\operatorname{tr} R$
be commensurable to the remainder that, along with it, composes the whole; and if it is commensurable to the remainder, it will also be commensurable to the whole. From this it is clear that such a part will be an aliquot part or parts of its whole, from which [it follows] that it is commensurable to the whole. Consequently, the denomination of such a part and the remainder will be the same, which is evident from a supposition given in the proof of the third proposition of the second chapter. Thus, the remainder will be commensurable to such a part and to the whole, by the eighth [proposition] of the tenth [book of Euclid].
[Supposition] VII. If some part is incommensurable to its whole, it will be incommensurable to the remainder.

Let $A$ be the whole, $B$ a part which has been separated, and $C$ the remainder. Then it can be argued in the following way: If $B$ and $C$ were commensurable, $A$ will be commensurable to each. Therefore, if $A$ is not commensurable to each, these [two parts] will not be commensurable. The consequent is evident by denial of the consequent, and the antecedent is obvious by the first part of the ninth [proposition] of the tenth [book of Euclid], which says this: If two quantities are commensurable, the whole constituted of these quantities will be commensurable to each of them; therefore, if $B$ is incommensurable to $A$, it will, similarly, be incommensurable to $C$.
[Supposition] VIII. When a ratio of some whole to its part is known, the ratio of this part or portion to the remainder can be known, as can the ratio of whole to remainder. For if the ratio of the whole to that part is rational, the same part and the remainder are denominated by the same number, and the ratio of such a part to the remainder is as a numerator to a numerator. However, the ratio of the whole to the remainder is as a ratio of the denominator of that remainder to the numerator of that same remainder.

These things are clear from the three suppositions made in the third

35 ergo: igitur $R$
36 ipsi: $H(?) /$ ipsi C om $V$
37 Octava om $V /$ eius: sui $H$ / post partem hab $R$ ex hac et bab $H$ ex hoc 37-38 potest sciri: ex qua sciri potest $C$ 38 partis seu portionis $V$ sui proportionis $C$ partis seu proportionis $R /$ seu $H(?) /$ et etiam $H R$ et in $C$ similiter $V$ 38-39 proportio: proportiones $C$
39 illam $C H$ istam $R$ unam $V /$ portionem: proportionem $R$

40 portio $H$ proportio $C R V /$ et: $\operatorname{ad} R /$ numero: modo $H /$ denominantur $H V$ denominatur $C R$
4 I portionis $C H$ proportionis $R V /$ sicut numerator $H V$ numeratoris $C$ sicut proportio numeratorum $R$
42-43 denominatoris eiusdem residui. denominationis eius residuum $C$
43 eiusdem ${ }^{1}$ : eius $R$
44 Ista: illa $R /$ patent $H$ patet $C R V$

45 secundi capituli. Sit $A$ totum, $B$ portio, $C$ residuum, et sit proportio $A$ ad $B$ nota que sit tripla. Igitur $B$ erit $1 / 3$ de $A$, igitur $C$ erit $2 / 3$, igitur proportio $B$ ad $C$ erit sicut 1 ad 2, scilicet subdupla. Et proportio $A$ ad $C$ erit sicut 3 ad 2 , scilicet sexquialtera.
Item ista suppositio habetur ex quarta et quinta conclusionibus
${ }_{50}$ secundi libri De numeris datis. Unde quarta est ista: si totius ad detractum fuerit proportio data et residui ad detractum erit proportio data; quod si residui ad detractum fuerit proportio data, et totius ad detractum simul data erit. Et quinta conclusio est ista: si totius ad detractum fuerit proportio data et totius ad residuum erit proportio
${ }_{55}$ data. Et intelligit per proportionem datam proportionem cuius denominatio nota est ut habetur in principio libri primi. Patet itaque ex prima parte quarte conclusionis et ex ista quinta quod octava suppositio capituli fuerit vera.
Nona suppositio et ultima sit ista: scita proportione duarum quan6o titatum et nota una earum altera poterit esse nota.

Hoc quod dico de quantitatibus, proponitur de numeris secunda conclusio secundi De numeris datis que talis est: si dati numeri ad aliquem fuerit proportio data, et illum datum esse consequitur datum id est notum. Et quia quelibet due quantitates sunt sicut duo numeri, 5 ut patet ex quinta decimi Euclidis, idem erit hoc dicere de numeris aut de quantitabus commensurabilibus quibuscumque.
Item ista suppositio declaratur. Sint enim $A$ et $B$ due quantitates quarum proportio sit nota, que sit sexquialtera. Et $A$ sit quantitas nota, que sit 9 pedum, dico quod $B$ erit 6 , et hoc inveniam isto modo.

45 portio: proportio $R /$ et $C V$; om $H R$
46 A ad B: B ad A $R /$ sit: est $R$
47 erit: esset $V / \mathrm{I}$ : proportio unius $R$
48 erit: esset $V$ | scilicet om $R$ / post scilicet add $C$ proportio
49 Item om $V /$ ista suppositio $\operatorname{tr} C /$ conclusionibus om $C$
so ista om $R$
sı ad: aut $V /$ erit: et $C$
${ }_{51-52}$ et residui...datar ${ }^{\text {o }}$ om $H$
$s_{2}$ quod si...data ${ }^{2}$ om $R$
53 erit: esset $V /$ est: sit $V$ / ista: illa $R$
54 erit: esset $V$
$5 s$ intelligit: intelligitur $C$ / proportion$\mathrm{em}^{2}$ om $H$
56 nota: vero $C$ / habetur...libri: in principio libri habetur $H$ / libri primi
$R V ; \operatorname{tr} C$
57 ista: prima $R$
s7- 98 suppositio...fuerit: prima huius causa fuit $V$
$\varsigma 8$ fuerit: erit $R /$ vera om $C$
59 et ultima om $C$
60 nota ${ }^{\text {I }}$ : vero $C /$ earum: eorum $V /$ post altera add $V$ earum
${ }^{61}$ proponitur $C$ proponit $H($ ? $) R$ componit $V$
62 secundi om $H$
$62-63$ ad aliquem fuerit: fuerint ad aliquem $R$
63 aliquem: aliquam $V$ / illum datum: istum $R$ / esse om $H$
64 est om $R$ / quia om $H$
6s Euclidis H; om CRV | ante idem add

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proposition of the second chapter. Let $A$ be the whole, $B$ a part, $C$ the remainder; and let ratio $A$ to $B$, which is known, be a triple ratio. Therefore, $B$ will be $\frac{1}{3}$ of $A$ so that $C$ will be $\frac{2}{3}$, and, consequently, ratio $B$ to $C$ will be as I to 2, namely a subdouble. Ratio $A$ to $C$ will be as 3 to 2, namely a sesquialterate.*
Moreover, this supposition is based on the fourth and fifth propositions of the second book of On Given Numbers. The fourth proposition says this: If a ratio of a whole to a part were given, then the ratio of the remainder to the initial part will be given; and if the ratio of the remainder to the initial part is given, then the ratio of the whole to the initial part will be given at the same time. Now the fifth proposition is this: If the ratio of the whole to the initial part were given, then the ratio of the whole to the remainder will be given. By a given ratio is meant a ratio whose denomination is known, and this is found in the beginning of the first book [of On Given Numbers]. And so it is evident that the eighth supposition of this chapter is true in virtue of the first part of the fourth proposition and the fifth [of On Given Numbers].
Supposition $I X$, the last one, is this: When the ratio between two quantities is known and one of the quantities is known, the other [quantity] can be known.
I say this about quantities, but in the second proposition of the second [book] of On Given Numbers it is set forth for numbers in this way: If the ratio of a given number to some other number were given, then that [unknown number] would be given, [for] it depends on the given [number] which is known. And since any two quantities are [related] as two numbers, which is obvious from the fifth [proposition] of the tenth [book] of Euclid, the same thing can be said of numbers or any commensurable quantities whatever.

Now this [last] supposition is made clear: Let $A$ and $B$ be two quantities whose ratio, which is a sesquialterate, is known. Should $A$ be a known quantity of 9 feet, I say that $B$ will be 6 feet, and I find it in the following

* Assume that $A=B \cdot C$ and $A=(B)^{3}$. It follows that $B=(A)^{1 / 3}$ and $C=(A)^{2 / 3}$, so that $B=(C)^{1 / 2}$ and $A=(C)^{3 / 2}$.
$H$ hoc / idem...dicere: ideo idem etiam hic dicitur $V$
66 de $C V$; om $H R$ / quibuscumque: quibusdam $C$
67 declaratur: declari $C$ / Sint enim $H$ et
sint $C R V$
68 que sit: scilicet $C / E t$ om $H$
69 pedum om $C$ / post 6 add $R$ pedum / isto: hoc $C$

70 Capiam primos numeros proportionis date, qui sunt 3 et 2 , et dicam sic sicut 3 ad 2 ita $9\langle\mathrm{ad}\rangle 6$, scilicet $A$ ad $B$. Et tunc per communem regulam multiplicabo secundum per tertium, scilicet 2 per 9 , et productum dividam per primum, scilicet per 3, et exibit quartum, scilicet $B$, quod sic. fiet notum et erit 6 . Ista regula est vulgata et alibi de75 monstrata.

Prima conclusio. Quod iste regule sunt false. Si aliqua potentia movet aliquod <mobile> aliqua velocitate dupla potentia movebit idem mobile duplo velocius. Et ista: si aliqua potentia movet aliquod mobile eadem potentiapoterit subduplum movere duplo velocius.
Falsitas prime patet. Et sit $B$ una potentia que moveat $C$ mobile aliqua velocitate, et sit $A$ potentia dupla. Si ergo proportio $B$ ad $C$ sit dupla bene sequitur quod proportio $A$ ad $C$ erit proportio $B$ ad $C$ duplicata per tertium notabile primi capituli. Ergo adhuc valde bene sequitur quod velocitas qua $A$ movet $C$ est dupla ad velocitatem qua
${ }_{85} B$ movet $C$ per primam suppositionem, quia proportio velocitatum est sicut proportio proportionum.
Sed adverte si proportio $B$ ad $C$ sit minor quam dupla, cum proportio $A$ ad $B$ per positum sit dupla, sequitur quod proportio $A \mathrm{ad}$ $C$ erit plusquam dupla ad proportionem $B$ ad $C$ per secundam supposi-
go tionem quia est composita ex proportione $B$ ad $C$, minore, et ex proportione $A$ ad $B$, maiore, scilicet dupla. Ergo $A$ movebit $C$ plusquam in duplo velocius quam $B$ moveat $C$ per primam suppositionem.

Item si proportio $B$ ad $C$ sit maior quam dupla et proportio $A \mathrm{ad}$ $B$ posita est dupla, sequitur quod proportio $A$ ad $C$ erit minor quam 95 dupla ad proportionem $B$ ad $C$ per secundam suppositionem. Igitur

70 qui $R V$ que $C H /$ sunt: sint $C / \mathrm{et}^{2}$ om $C$
$719\langle$ ad $\rangle$ 6:96Cut $9 \mathrm{H}_{9} \mathrm{RV} /$ scilicet om $C / \mathrm{A}$ ad B : A ad aliud puta aliquid B $V$ / tunc om $R$ / per communem: invenit $C$
72 regulam: duplam $V /$ multiplicabo $H V$ multitudo $C R$ / secundum per tertium: 2 per ${ }_{3} H /$ scilicet $H V$ videlicet $C R / 2$ per: pro $H$
73 ante dividam add $C$ tamen / dividam: dividitur $H /$ perr $^{\mathrm{I}}$ om $R / 3$ corr ex 2 $C H R V /$ exibit: exibunt $R /$ quartum 4 H
74 B $H V$ hoc $C 6 R /$ notum: $H($ ? $) /$ alibi $V$ aliunde $C$ alia $H$ aliquando $R$

76 ante Prima conclusio bab $R$ tunc sit Prima conclusio RV ; om $\mathrm{CH} /$ Quod om $R /$ sunt : $\operatorname{sint} C /$ movet $C R$; om $V$ moveat $H$
76-77 aliquod: aliquid $R$
77 〈mobile〉 Ed; om CHRV
78 eadem: eodem modo $C$
78-79poterit subduplum movere: movere poterit subdupla $C$
79 ante movere bab $R$ mobile
80 sit: si $R$ / moveat: movet $C /$ mobile om $C$
${ }^{81}$ potentia: prima $C$
82 bene: unde $R$ / quod: quot $C /$ erit: sit $R$ / B om $V$
83 tertium: secundum $V$

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way: I take the prime numbers of the given ratio, which are 3 and 2 , and I say that as 3 is to 2 so is 9 to 6 , namely as $A$ is to $B$. Then, by a common rule, I multiply the second by the third, namely 2 by 9 , and divide the product by the first number, namely 3 , to get the fourth number, namely $B$, which is now made known and is 6 . This is a common rule and has been demonstrated elsewhere.
Proposition I. That the following rules are false: If a power moves a mobile with a certain velocity, double the power will move the same mobile twice as quickly. And this [rule]: If a power moves a mobile, the same power can move balf the mobile twice as quickly.*
The falsity of the first [rule] is obvious: Now let $B$ be a power that moves mobile $C$ with a certain velocity, and let $A$ be a double power. Then if ratio $B$ to $C$ were a double ratio, it follows, by the third noteworthy point of the first chapter, that ratio $A$ to $C$ will be [equal to] ratio $B$ to $C$ squared. It certainly follows, by the first supposition, that the velocity with which $A$ moves $C$ is double the velocity with which $B$ moves $C$, since a ratio of velocities is just like a ratio of ratios. $\dagger$
But now, on the contrary, if ratio $B$ to $C$ were less than a double when ratio $A$ to $B$ is a double by assumption, it follows, by the second supposition, that ratio $A$ to $C$ would be greater than ratio $B$ to $C$ squared because it is composed of ratio $B$ to $C$, the lesser, and ratio $A$ to $B$, the greater, namely a double. Consequently, by the first supposition, $A$ will move $C$ more than twice as quickly as $B$ will move $C . \ddagger$
Again, if ratio $B$ to $C$ is greater than double and $A$ to $B$ is assumed to be a double ratio, it follows by the second supposition, that ratio $A$ to $C$ will be less than ratio $B$ to $C$ squared. Hence, by the first supposition, $A$
*The first false rule asserts that if $F \mid R \propto V, \quad 4 / 2$, then $F_{2} / R=\left(F_{1} / R\right)^{2 / 1}$, where the then $(2 F \mid R) \propto 2 V$; in the second false rule, if $F / R \propto V$, then $F /(R: 2) \propto 2 V$.
${ }^{+}$Using $F$ 's, $R$ 's, and $V$ 's for $A$ 's, $B$ 's, and $C$ 's, we see that if $F_{1} / R=2 / \mathrm{I}$ and $F_{2} / F_{1}=$ exponent ${ }^{2} /{ }_{1}=V_{2} / V_{\mathrm{I}}$.
$\ddagger$ If $F_{1} / R<2 / \mathrm{I}$ and $F_{2} / F_{1}=4 / 2$, then $F_{2} / R$ $>\left(F_{\mathrm{r}} / R\right)^{2 / 2}\left(\right.$ where $\left.{ }^{2} / /_{\mathrm{I}}=V_{2} / V_{\mathrm{I}}\right)$ since $F_{2} / R=F_{2} / F_{1} \cdot F_{1} / R$.

87 cum: tunc $R$
89 erit: sit $H$ / secundam: duplam $V$
90 minore: minor $H$
91 maiore: minore $R /$ scilicet $C$ quam $H$ quia $R V$

92 moveat: movebit $H$
93 dupla et om $H$
94 B : C $R$ / erit: est $R$
94-96 erit minor... movebit C om $V$
$A$ movebit $C$ velocitate minori quam dupla ad velocitatem qua $B$ movet $C$ per primam suppositionem.
Verbi gratia, sit $A$ 8, $B$ 4. Si igitur proportio $B$ ad $C$ est sicut proportio $A$ ad $B$ ita ut $C$ sit 2, tunc proportio $A$ ad $C$ erit proportio $B$ ad $C$ duplicata, igitur et velocitas duplicata. Si autem proportio $B$ ad $C$ sit minor quam dupla ita ut $C$ sit 3, tunc proportio $A$ ad $C$ erit plusquam proportio $B$ ad $C$ duplicata, ergo velocitas est plusquam duplicata. Si vero proportio $B$ ad $C$ sit maior quam dupla ita ut $C$ sit unitas, tunc proportio $A$ ad $C$ et similiter velocitas erit minor quam 05 duplicata.

Patet igitur quod ex duplicatione potentie non sequitur duplicatio velocitatis nisi in uno casu, scilicet quando potentia prima ponitur ad mobile dupla. Igitur regula est falsa quia ex quo est conditionalis debet esse necessaria. Et antecedens non debet posse esse verum sine conно sequente, et tamen veritas consequentis non stat cum veritate antecedentis nisi in uno casu.

Falsitas secunde regule potest per eadem principia demonstrari quoniam si aliqua potentia movet aliquod mobile aliqua velocitate eadem potentia non movebit subduplum duplo velocius nisi prima 115 velocitas a dupla proportione proveniret. Ymo quandoque illud quod ipsa moveret duplo velocius esset precise subduplum ad primum, quandoque maius quam subduplum, quandoque minus.

Secundo arguo contra secundam regulam sic: quia si sit vera sequitur quod quelibet potentia, quantumcumque debilis, potest movere quodlibet mobile, quantacumque fuerit resistentia.

Et sumatur $A$ potentia que possit movere $C$. Et sit $D$ unum mobile

96 ante quam $m g$ bab $H$ exemplum / ante qua scr et del $V$ quod
96-97 velocitatem...movet $C$ : proportionem B ad C H
97 primam: secundam(?) H / post suppositionem scr et del $H$ igitur a et alium verbum
98 post A scr et del $V 6(?) / 8$ om $H / \mathrm{Br}^{\mathrm{r}}$ : vel $R$
98-100 est...velocitas om $V$
100 et: etiam $R /$ autem: vero $R$
IoI ante 3 add $C$ sicut / $\mathrm{C}^{3}$ : B $V$
102 erit $H$ est $C R V$ | post plusquam ${ }^{\mathrm{r}}$ scret del $H$ dupla / ante proportio add $C$ dupla

103 vero $H R$ nota $C V /$ maior: plus $C /$ ita: vel maior $C$
104 ante unitas add $C$ sicut / unitas: unum $R /$ ante -locitas in velocitas $m g$ bab $H$ conclusio(?) / erit om $V /$ minor corr ex plus CHRV
106 duplicatio: dupla $H$
107 scilicet om $H$ / potentia prima $C V$; tr $H$ potentia primo $R$
Io8 est ${ }^{2}$ om $C / \operatorname{debet} C$ debetur $H$ deberet $R V$
109 necessaria...esse om $R$
in et om $H /$ tamen veritas consequentis: cum velocitas antecedentis $R /$ cum veritate $H V$ converitatis(?) $C$ cum
will move $C$ with a velocity that is less than twice the velocity with which $B$ moves $C$.*
For example, let $A$ be 8 and $B$, 4. Therefore, if ratio $B$ to $C$ is related as ratio $A$ to $B$ so that $C$ would be 2, then ratio $A$ to $C$ will be ratio $B$ to $C$ squared, and thus the velocity is doubled. Furthermore, if ratio $B$ to $C$ were less than double so that $C$ might be 3 , then ratio $A$ to $C$ will be greater than ratio $B$ to $C$ squared, and the velocity is more than doubled. If, however, ratio $B$ to $C$ were greater than a double ratio so that $C$ might be a unit, then ratio $A$ to $C$ [will be less than the square of $B$ to $C$ ], and, similarly, the velocity will be less than doubled.
Thus it is clear that from a doubling of the power, a doubling of the velocity does not follow except in one case, namely when the power is first taken to be double the mobile. Hence the rule is false, because it is conditional and ought to be necessary. And the antecedent ought to be incapable of being true without the consequent [being true], but the truth of the consequent does not agree with the truth of the antecedent except in one case.
The falsity of the second rule can be shown by the same principles, because if a power moves a mobile with a certain velocity, the same power will not move half [the mobile] twice as quickly unless the first velocity should arise from a double ratio. But sometimes what this [power] moves twice as quickly would be exactly half the first [mobile], sometimes greater than half, [and] sometimes less [than half].
In the second place, I argue against the second rule thus: If it were true, it follows that any power, however feeble, can move any mobile, whatever its resistance. $\dagger$
Now let a power $A$ be taken, which can move $C$. And let $D$ be a mobile

* If $F_{1} / R>2 / 1$ and $F_{2} / F_{1}=4 / 2$, then $\dagger$ The entire argument is summarized on $F_{2} / R<\left(F_{1} / R\right)^{2 / \mathrm{I}}$, where ${ }^{2} / \mathrm{I}=V_{2} / V_{1} . \quad$ pp. 44-47.


## velocitate consequentis $R$

II 3 movet: moveat $V$ / aliquod: aliquot $R$
IIs velocitas...proportione:a duplo velo citate $H$ | ante proveniret sor et del $V$ prop / quandoque: quando $V$
116 velocius om $R /$ precise: sicut $H$ subduplum om $R$
117 quam: quandoque $R$ /ante quandoque ${ }^{2}$ add $V$ et

118 sit $H V$ esset $C R$
I 18-19 sequitur: sequeretur $R$
in9 quelibet $H$ quecumque $C R V /$ potest: posset $C$
i20 quodlibet: quodcumque $V$ / quantacumque $H V$ quantocumque $C R /$ resistentia $V$ resistentie $C$ (?) $R$ potentie resistentie $H$
I2I possit: posset $R / \mathrm{Com} C /$ ante D scr et del V de
duplum ad C, et $E$ sit unum aliud duplum ad $D$, et $F$ sit duplum ad $E$, et $G$ ad $F$, et sic ultra. Tunc probatur ista consequentia : $A$ potest movere $C$, igitur potest movere $D$. Et similiter probabitur ista: $A$ vere $E$ igitur et $F$, et sic ultra. Sit itaque $B$ una potentia que possit movere $D$ duplo tardius precise quam $A$ movet $C$, sicut est possibile.
Igitur si regula sit vera $B$ potest movere $C$ duplo velocius quam ipsummet $B$ potest movere $D$ quia $C$ est subduplum ad $D$ et medietas ${ }_{13}$ e eius. Igitur $B$ potest movere $C$ ita velociter precise sicut $A$ potest movere $C$ quia $A$ movet $C$ duplo velocius quam $B$ movet $D$, et $B$ movet $C$ duplo velocius quam $B$ moveat $D$. Igitur $A$ et $B$ eque velociter possunt movere $C$ per nonam quinti-si duo ad tertium habeant eandem proportionem illa sunt equalia. Ergo per tertiam suppositionem $A$ et $B$ sunt equales potentie. Sed $B$ potest movere $D$ per positum, ergo per quartam suppositionem $A$ potest movere $D$, quod fuit probandum.

Et eodem modo probabitur ista consequentia: $A$ potest movere $D$, igitur potest movere $E$ capiendo unam potentiam que possit movere ${ }_{\text {140 }} E$ duplo tardius quam ipsum $A$ possit movere $D$ et probabitur sicut prius quod $A$ et ista potentia data que potest movere $E$ sunt equales ex quo sequitur propositum sicut prius est deductum.
Verumtamen, si $A$ moveat $C$ a proportione quadrupla tunc, gratia materie, bene sequitur $A$ movet $C$. Ergo illa potentia que movet $D$ duplo tardius, hoc est a proportione dupla, scilicet $B$ potentia, potest movere $C$, subduplum de $D$, ita velociter precise sicut $A$ movet $C$. Igitur sunt potentie equales et $A$ potest movere $D$ quia similiter $A$ habet ad $D$ proportionem duplam ita bene sicut $B$. Et hoc est quia sequitur $B$ movet $D$ a proportione dupla, ergo $B$ movet $C$ duplo
$122 \mathrm{E}: \mathrm{C} V /$ unum aliud $C R$; om $H$ unum V
123 E: C $V /$ et Gad Fom $R /$ ista: illa $R /$ A om $C$
124 probabitur ista $H R$ ista probatur $C$ probatur ista $V /$ post movere D scr et del $V$ quia E est subduplum ad D et medietas eius ergo B potest movere
125 EI: C V/ante similiter add $R$ etiam / movere ${ }^{3}$ om $R$
125-26 similiter...et $F$ : et E igitur G $H$ 126 et ${ }^{1} \mathrm{CH}$ potest $R V$ / sic ultra: cetera $C / B$ om $C$ / possit: potest $V$
$127 \mathrm{D} \operatorname{tr} H$ post tardius / precise $R V ; \operatorname{tr} C$ post possit (linea 126); om $H$
128 regula sit: ista est $R$
129 ipsummet $C V$ ipsum $H R$ / subduplum: duplum $R$
$130 \mathrm{~A}: \mathrm{D} H$
$131 \mathrm{C}^{2}$ : in $H /$ movet $^{2}$ : moveat $V /$ et om $H$ 132 ante duplo add $H$ in
132-33 eque velociter: equaliter $R$
133 C : et $C$ / nonam: nonem $R$ / tertium: tertiam $V /$ habeant $H V$ habent $C R$.
135 sunt equales potentie om $R /$ post D scr et del $R$ igitur potest movere A
double to $C$, and $E$ another mobile double to $D$, and $F$ double to $E$, and $G$ to $F$, and so on. Then this consequence is proved: $A$ can move $C$; therefore it can move $D$. And similarly this will be proved: $A$ can move $D$; therefore it can move $E$. And likewise it can move $E$, and therefore $F$, and so on. Moreover, since it is possible, let $B$ be a power which could move $D$ exactly twice as slowly as $A$ can move $C$.

Thus if the rule were true, $B$ can move $C$ twice as quickly as this very same $B$ can move $D$, since $C$ is subdouble to $D$ and half of it. Therefore, $B$ can move $C$ exactly as quickly as $A$ can move $C$, since $A$ moves $C$ twice as quickly as $B$ moves $D$, and $B$ moves $C$ twice as quickly as $B$ can move $D$. Hence $A$ and $B$ can move $C$ equally quickly, by the ninth [proposition] of the fifth [book of Euclid, which says]: If two [quantities] bear the same ratio to a third, they are equal. Consequently, by the third supposition, $A$ and $B$ are equal powers. But by assumption, $B$ can move $D$; therefore, by the fourth supposition, $A$ can move $D$, which was to be proved.

And this consequence will be proved in the same way: $A$ can move $D$; therefore, it can move $E$ by taking a power that can move $E$ twice as slowly as $A$ can move $D$. It can then be proved as before that $A$ and this given power that can move $E$ are equal, from which the proposed consequence follows just as it was deduced before.
However, if $A$ should move $C$ in a quadruple ratio, then in virtue of this it surely follows that $A$ moves $C$. Then that power, namely power $B$, which moves $D$ twice as slowly, that is in a double ratio, can move $C$, the half of $D$, exactly as quickly as $A$ moves $C$. Thus they are equal powers and $A$ can move $D$ because, similarly, $A$, as well as $B$, is related to $D$ in a double ratio. And this is so because it follows that $B$ moves $D$ in a double

136 quartam: octavam $V / D$ om $V$
137 fuit: erat $R$
${ }_{1} 38$ Et $H V$; om $C R$ / ante probabitur add $V$ potest / probabitur: probatur $C$
139 E: C $H$ / capiendo: capido $H$ | post unam add $V$ que / possit: potest $C$ 140 E RVCCH/probabitur: probatur $C$
140-41 duplo...movere E om $R$
141 ista: illa $V / \mathrm{E}$ om $H$

142 ante ex quo mg hab $H$
ef g
48 I6 et cetera
quo: qua $C$
143 Verumtamen: suppositioni(?) $H$
143-44 gratia materie $R V$ gratia mea $C$ igitur materia(?) $H$
144 illa: A $R$
145 dupla om $C$
146 precise om $R$
148 proportionem duplam $\operatorname{tr} R$
velocius sicut prius est concessum quia a proportione quadrupla sicut $A$.
Ideo dicebatur in correlatione quarte tertii capituli quod proportio proportionis quadruple ad duplam est sicut proportio denominationum quod non reperitur in aliis proportionibus. Et propter hoc non sequitur ultra in casu predicto $A$ movet $D$ a proportione dupla ergo illa potentia que potest movere $E$ duplo tardius potest movere $D$ precise ita velociter sicut $A$ movet $D$. Ymo sequitur quod velocius quia sit illa potentia $B$ tunc proportio $B$ ad $E$ erit medietas duple proportionis et cum proportio $E$ ad $D$ sit proportio dupla sequitur
${ }^{160}$ quod proportio $B$ ad $D$ erit composita ex dupla et medietate duple. Igitur erit maior quam dupla, igitur per primam suppositionem $B$ movebit $D$ velocius quam $A$ moveat $D$ quia erat duplumad $D$. Ideo non sequitur amplius $A$ potest movere $D$ igitur movere $E$. Quod, tamen, sequeretur si regula esset vera sicut demonstratum est.
Quid igitur dicemus ad Aristotelem qui in septimo phisicorum videtur ponere huius regulas reprobatas? Dicendum est quod sunt false nisi addatur ad primam: si aliqua potentia movet aliquod mobile a proportione dupla, dupla potentia movebit et cetera. Et similiter ad secundam: si aliqua potentia et cetera, a proportione dupla eadem movebit et cetera. Et ita possimus glosare et dicere quod ita intellegende sunt regule. Et forte quod Aristoteles dixit hoc sed est vitium in translatione. Et si non dixit forte subintellixit.
Secunda conclusio. Qualibet velocitate demonstrata et qua volueris proportione proposita si velocitas demonstrata a proportione proposita, aut a ${ }^{7}$ maiori aut a minori proveniat reperire.

Sit $A$ proportio proposita michi nota, et sit $B$ una proportio a qua proveniat una velocitas demonstrata que quidem $B$ proportio est

I 10 est concessum: ostenssum(?) est $H$
152 ante dicebatur add $C$ sicut / quarte: none $H /$ tertii capituli $\operatorname{tr} V$
154 reperitur: requitur $H$
is6 que om $H$ / duplo: duple $V$
is 8 proportio om $R / \mathrm{EHRCCV} / \mathrm{erit}$ : esset $V$
158-59 duple proportionis $C V$; $\operatorname{tr} R$ dupla proportionis $H$
159 et om C/E: A(?) V
I60 proportio om $R / \mathrm{D}: \mathrm{C} H /$ erit om $H$
161 suppositionem: suppositione $V$
$162 \mathrm{D}^{1}$ om $H \mid$ A om $C /$ moveat: movet $H$ / erat duplum $C R$ erit duplum $H$
erat dupla $V /$ Ideo: igitur $R$
163 ante movere ${ }^{2}$ add $C R$ potest
164 sequeretur $H$ sequitur $C R V /$ esset $C V$ sit $H R /$ sicut: ut $R /$ demonstratum est $C R ; \operatorname{tr} V$ est dicendum $H$
16s Aristotelem: Aristoteles $V /$ qui in $C$ in $H R$ que $V$
166 huius $C H$ huiusmodi $R V /$ reprobatas om $H$ / estom $H$
167 movet: moveat $V$
168 mobile om C / Et om H
169 a om $R$ / post dupla bab $C$ iter
169-70 eadem movebit om $R$ 170 et cetera: iter $C /$ ita $^{1}$ : ista $V /$ possimus

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ratio; therefore, $B$ moves $C$ twice as quickly, as conceded before, because, like $A$, it moves it in a quadruple ratio.

For this reason it was stated in the corollary to the fourth proposition of the third chapter that a ratio of a quadruple to a double ratio is just like a ratio of [their] denominations, but this is not found in other ratios. And because of this it does not follow further in the aforementioned case that [if] $A$ moves $D$ in a double ratio, therefore that power that can move $E$ twice as slowly can move $D$ exactly as quickly as $A$ moves $D$. On the contrary, it follows [that it will move $D$ ] more quickly because that power would be $B$, and then ratio $B$ to $E$ will be half of a double ratio; and since ratio $E$ to $D$ is a double ratio it follows that tatio $B$ to $D$ will be composed of a double and half of a double ratio. It will, consequently, be greater than double, so that by the first supposition $B$ will move $D$ quicker than $A$ moves $D$, since $A$ was double to $D$. Thus it no more follows that [because] $A$ can move $D$, therefore it can move $E$. But, as was shown, this would nevertheless follow if the rule were true.
What, then, should we say to Aristotle who seems to enunciate the repudiated rules in the seventh [book] of the Pbysics? It must be said that they are false unless [the following] is added to the first rule: If a power moves a mobile in a double ratio, a double power will move [the same mobile twice as quickly]. And likewise [this must be added] to the second rule: If a power [moves a mobile] in a double ratio, the same power will move [half of the mobile twice as quickly]. And so we can gloss [Aristotle] and say that these rules ought to be understood [in this way]. Perhaps Aristotle said this but has been poorly translated. But if he did not say it, perhaps he failed to understand [the rules] properly.
Proposition II. When any velocity bas been designated, you wish to find with reference to a proposed ratio whether the designated velocity arises from a ratio that is greater or smaller than the proposed ratio.
Let ratio $A$ be a proposed ratio known to me, and let $B$ be a ratio unknown to me from which there arises a designated velocity. I wish only to

## $C V$ possumus $H R$ | post dicere del $V$ quod in / dicere quod ita $H R$; om $C$ dicere ita quod $V$

${ }_{171}$ quod om $C$ / post quod $m g$ bab $H$ secunda conclusio / dixit $C R$ dicit $H V$ 172 translatione: translatore C / Et om H
173 Secunda conclusio: post ve- in velocitate $m g$ hab $H$ secunda conclusio et idem mg bab $R$ post subintellixit (linea
172) / demonstrata $V$ data CH declarata $R$ / volueris: voluerit $H$
173-74 proportione: proportio $C$
175 maiori: minori $C /$ minori: maiori $C$ 176 ante proposita add $C$ in / michi: et $C$ 176-78 a qua... proportio om $V$
177 proveniat $C R$ proveniret $H /$ una $H$; om $C R /$ B proportio $H R ; \operatorname{tr} C$
michi ignota. Volo modo investigare et scire si $B$ proportio ignota sit equalis $A$ proportioni, michi note sive date, aut si est maior aut minor.

Sit, igitur gratia exempli, $D$ potentia et $E$ resistentia, seu mobile, ita quod $D$ movet $E$ velocitate demonstrata a proportione $B$ ignota. Signetur unum mobile minus $E$ ad quod $E$ se habeat in proportione $A$, nota et data, et sit illud mobile $F$. Tunc habebimus unam propor-
185 tionem $D$ ad $F$ ignotam et sit illa $C$, que erit composita ex intermediis secundum Euclidem in principiis septimi, sicut in primo capitulo sepe est allegatum-composita scilicet ex $A$ proportione notaque attenditur inter $E$ et $F$, et $B$ proportione ignota, que attenditur inter $D$ et $E$.
Applicetur itaque $D$ potentia ad movendum $F$ mobile et moveat
190 ipsum aliqua velocitate. Aut igitur illa velocitas erit prima velocitas duplicata, ita quod $D$ movebit $F$ duplo velocius quam ipsum $D$ movebat $E$-et si sit igitur $C$ proportio, a qua movetur $F$, erit $B$ proportio duplicata a qua $D$ movet $E$ quia velocitas sequitur proportionem et cetera per primam suppositionem, et ultra $C$ proportio
${ }^{195}$ est dupla ad $B$ et est composita ex $B$ et $A$ igitur $B$ est equalis $A$ proportioni per primam partem quinte suppositionis, igitur si velocitas est duplicata ex tali applicatione $D$ ad $F$ iam $B$ proportio scitur esse equalis $A$ proportioni note et prius demonstrata-aut, ex applicatione $D$ ad $F$ velocitas prima erit plusquam duplicata, igitur per
200 primam suppositionem $C$ proportio erit plusquam $B$ proportio duplicata et sic plusquam dupla ad $B$; igitur per secundam partem quinte suppositionis $B$ est minor proportio quam $A$. Aut ex applicatione $D$ ad movendum $F$ velocitas erit minus quam duplicata, igitur per primam suppositionem $C$ proportio erit minor quam dupla ad $B$, suam
${ }_{205}$ partem. Igitur per tertiam partem quinte suppositionis $B$ proportio est maior quam $A$ proportio, que est residuum de $C$ dempto $B$, et hoc est quod volebam scire.

1 78 investigare $H R$ invenire $C$
I 79 sive: aut $C /$ si est $C V$; om $C$ sit $R$
181 igitur om $H /$ et $H R$; om $C$ proposita et $V /$ seu: vel $C$
183 Signetur: signo $H$ / unum mobile $C R ; \operatorname{tr} H V / \mathrm{E}^{2} \operatorname{tr} V$ post habeat
184 A om $R$
18s D om $R / \mathrm{C}: \mathrm{E} H /$ erit: est $R$
I 86 septimi: octava sexti $V$
187 est allegatum: oblegatum est $C /$ notaque $H R$ veroque $C$ dataque $V$

187-88 attenditur: ostenditur $C$
I88 proportione: proportio $C / \mathrm{D} V \mathrm{~A}$ CHR
189 potentia: posita $V /$ movendum $H V$ medium $R$ movens $C$
190 ipsum om $R$ /aliqua: a $C /$ illa: ista $C /$ velocitas ${ }^{\text { }}$ : velocita $V /$ erit: esset $V /$ velocitas ${ }^{2}$ om $C$
191 ante duplo bab $V$ mobile
${ }^{1} 92$ movebat $H R$ movet $C$ movebit $V /$ $\mathrm{C}: \mathrm{D} R /$ erit: esset $V$
investigate and to know if the unknown ratio $B$ is equal to, greater than, or less than ratio $A$, which is known or given to me.
For the sake of an example, let $D$ be a power and $E$ a resistance or mobile so that $D$ moves $E$ with a [certain] velocity [and $D$ to $E$ is] represented by $B$, the unknown ratio. Let there be assigned a mobile $F$, less than $E$, to which $E$ is related in ratio $A$, known and given. We shall then have an unknown ratio $D$ to $F$, which we may designate as $C$, and $C$ will be composed of intermediates, according to Euclid in the principles of the seventh [book] and declared frequently in the first chapter [of this treatise]; that is, $C$ will be composed of the known ratio $A$, which is measured by $E$ to $F$, and the unknown ratio $B$, which is measured by $D$ to $E$.

Then let power $D$ be applied to move mobile $F$, and let it move it with a certain velocity. Therefore, either that velocity will be double the first velocity, or not. [If it is double] then $D$ will move $F$ twice as quickly as $D$ moves $E$, in which event ratio $C$, by which $[D]$ moves $F$, will be the square of ratio $B$, by which $D$ moves $E$, since, by the first supposition, a velocity varies as a ratio, etc. Furthermore, ratio $C$ is the square of $B$ and is composed of $B$ and $A$, so that, by the first part of the fifth supposition, $B$ equals ratio $A$. Consequently, if the velocity is doubled when $D$ is applied to $F$, ratio $B$ is understood to be equal to $A$, the ratio which was known and designated previously. But if on application of $D$ to $F$, the velocity should be more than double the first velocity, then, by the first supposition, ratio $C$ would be greater than the square of ratio $B$ and, hence, more than double to it. Therefore, by the second part of the fifth supposition, $B$ is a smaller ratio than $A$. Or in applying $D$ to move $F$ the velocity will be less than doubled and, therefore, by the first supposition, ratio $C$ will be less than the square of $B$, its part. Then by the third part of the fifth supposition, ratio $B$ is greater than ratio $A$, which is what remains of $C$ when $B$ has been taken away, and this is what I wish to know.

193 D corr ex $\mathrm{B} C H R V /$ movet $C H(?) \quad 200$ ante C bab $V$ et / erit plusquam om $C$ movebat $R V$
194 et cetera om $C$
195 ante $\mathrm{A}^{1}$ bab $V$ sic / post $\mathrm{B}^{3}$ add $V$ proportio / equalis $R V$ equale $C H$
196 primam om $C$
197 B: D $R /$ scitur om $C$
198 post et add $R$ cetera/demonstrata $H$ demonstrate $C R V$
199 erit $C H$ erat $R$ esset $V$

200-204 erit plusquam B...C proportio om CR
202 est $V$ erit $H /$ minor proportio $H$; $\%$ tr V
203 erit $H$ esset $V$
204 erit: esset $V /$ minor: maior $C$
205 partem $^{2}$ om H
206 dempto: $C$ (?)
207 volebam: volebat $R$

Verbi gratia sit $A$ data proportio nota que sit tripla et $B$ sit proportio ignota a qua venit velocitas demonstrata qua $D$ movet $E$. scilictan ad $E$ ad quod $E$ se habet in proportione nota, scilicet tripla. Si igitur $D$ movet $F$ duplo velocius quam $E$ precise, igitur $B$ erat proportio tripla equalis $A$; si plusquam duplo velocius igitur $B$ est minor $A$, scilicet quam tripla; si minus quam in duplo velocius igitur $B$ est maior quam tripla.
Consimiliter, et per idem argumentum, poteris propositum invenire si semper maneat idem mobile et accipias unam potentiam que excedat $D$ secundum proportionem datam. Et sit $F$ et applicetur $E$ mobili primo. Tunc aut movebit $E$ precise duplo velocius quam $D$ movebat $E$, aut minus quam in duplo velocius, aut magis et arguatur sicut in 220 primo casu.

Quando autem velocitas sit duplicata vel plusquam duplicata vel minus patet ex diffinitione velocioris et tardioris posita in sexto phisicorum. Et si sunt circa hoc alique difficultates propter diversa genera motuum nolo modo in eis impediri, sed in proposito volo stare.
Tertia conclusio. Nota proportione duorum mobilium et scito in qua proportione minus movetur velocius ab aliqua potentia quam maius moveatur ab eadem, ad utrumque mobilium proportio potentie fet nota.

Sit $A$ potentia, $B$ maius mobile et $C$ minus, et quia omnis motus provenit a proportione maioris inequalitatis, ut suppono, $A$ erit maius
${ }^{230} \quad B$. Sitque $E$ proportio $A$ ad $B$ et proportio $B$ ad $C$ sit $F$, et proportio $A$ ad $C$ sit $D$.

Igitur $D$ proportio componitur ex proportionibus intermediis que

208 data proportio $C H$; $\operatorname{tr} R V$ / ante nota babV et / sit CR est HV
208-9 proportio om $R$
209 ante ignota bab $R$ sit / ante D bab $R$ scilicet
210 Capiam: capio $R$ / ante F add CRV modo / subtriplam $C H$ subduplam $R$ subduplum $V /$ ad quod $E$ : et quod $R /$ se habet $C H$ se habeat $V$ habeat se $R$
2 II ante duplo bab $H$ in / duplo: triplo $R$
212 ante duplo add $C R V$ in
213 ante A bab $V$ quam
214 B om $H$
215 Consimiliter: similiter $R /$ et $H$; om $C R V /$ poteris propositum $C V ; \operatorname{tr} R$ poterit propositum $H$

216 semper: quando(?) $H$
217 DCH ; om $R \mathrm{~V}$ / secundum: patet $H /$ mobili $C R$ mobile $H V$
218 ante Tunc add $C R V$ et / movebit E: E mobile est $V /$ duplo velocius $\operatorname{tr} R$
219 minus quam in $R V$ in minus $H$ numquam in $C /$ arguatur $H V$ arguitur $C R$
220 casu: capitulo $V$
221 Quando: quod $H$ / duplicata: dupla $C /$ vel: aut $C$
222 velocioris: velociter $H$ / et: aut $C$ / in om $V$
223 sunt circa $R V$ sunt citra $C$ sit circa $H /$ alique: alie $R /$ difficultates $C H$ (?) $V($ ?) diffinitiones $R$
224 nolo: volo $C$ / in proposito volo stare:

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For example, let $A$ be the given and known ratio and let it be a triple ratio; let $B$ be the unknown ratio by which $D$ moves $E$ and from which a designated velocity arises. I shall take $F$ as subtriple to $E$, to which $E$ must be related in a known ratio, namely a triple. Therefore, if D moves $F$ exactly twice as quickly as it moves $E$, then $B$ was a triple ratio [and] equal to $A$; but if $[D$ moves $F$ ] more than twice as quickly [as it moves $E$ ], then $B$ is less than $A$, namely less than a triple ratio; and if [ $D$ moves $F$ ] less than twice as quickly as it moves $E$, then $B$ is greater than a triple ratio.*
In the same manner and by the same argument, you can find what has been proposed if the mobile should remain the same and you take a power which exceeds $D$ in a ratio equal to the given ratio. Let this be $F$ which is applied first to mobile $E$. Then it will either move $E$ exactly twice as quickly as $D$ moves $E$, or less, or more than twice as quickly, just as it was argued in the first case.
Furthermore, when a velocity is doubled, or more or less than doubled, is obvious from the definition of quicker and slower given in the sixth book of the Physics [of Aristotle]. And if, because of diverse kinds of motions, there are some difficulties about this I do not now wish to be detained by them, but wish to stand on what has already been said.
Proposition III. The ratio of a power to each of [two] mobiles can be made known when [botb] the ratio of the two mobiles is known and the ratio by which the same power moves the lesser mobile more quickly than the greater mobile.
Let $A$ be the power, $B$ the greater and $C$ the lesser mobile, and, since I assume that every motion arises from a ratio of greater inequality, $A$ will be greater than $B$. And let $E$ be ratio $A$ to $B, F$ ratio $B$ to $C$, and $D$ ratio $A$ to $C$.
Then ratio $D$ is composed of intermediate ratios, which are $E$ and $F$.

* $D$ is a force and $E, F$ are mobiles or and $B=A=3 / 1$. But if $D / F>(D / E)^{2 / 2}$, resistances; $A=E \mid F=3 / 1$ and $B=D \mid E$. then $D|E<E| F$ and $B<A=3 / 1$; or, if $\begin{array}{ll}\text { resistances, } A=E / 2 / F=(D \mid E) / / \text { then, since } D / F= & D \mid F<(D / E)^{2 / /} \text { then } D|E>E| F \text { and } \\ \text { If } D|F=E| F,\end{array}$ $D / E \cdot E / F$, it follows that $D / E=E / F$ $B>A=3 / \mathrm{I}$.
stare in proposito nolo $C$
225 Tertia conclusio $V$; om $C$ conclusio tertia $m g$ hab $H$ post qua et ante nota $m g$ bab $R$ tertia conclusio / duorum: duarum $H$
226 quam: quamvis $R$
$227 \mathrm{ad}: \mathrm{ab} V /$ proportio: proportione $C$ /
nota: vero $C$
228 et $^{1} H$; om $C R$ /ante minus bab $C$ nota post minus bab $R$ mobile 228-29 mobile...maius om $V$
229 erit $C H$ eritque $R$
230 Sitque: sit quam $H$ | post F bab $H$ ad 232 componitur $H$ componetur $C R V$
sunt $E$ et $F$. Et ultra sit $G$ proportio $D$ ad $E$ ita quod $G$ erit proportio proportionum a quibus velocitates proveniunt. Et quia proportio velocitatum est nota per ypotesim proportio proportionum erit nota, scilicet $G$, per primam suppositionem. Habebimus itaque quattuor proportiones $D, E, F, G$, quarum due sunt note, scilicet $G$ et $F$, et due sunt ignote, scilicet $D$ et $E$, quas volumus esse notas.
Arguatur, igitur, sic: proportio $D$ totius ad $E$, sui partem, est nota, igitur proportio eiusdem partis, scilicet $E$, ad $F$, residuum, erit nota per primam partem octave suppositionis. Et similiter, proportio totius, scilicet $D$, ad $F$, residuum, erit nota per secundam partem eiusdem octave. Et tunc ultra proportio $E$ ad $F$ est nota et $F$ est proportio, sive quantitas, nota, ergo $E$ erit nota per nonam suppositionem. Similiter proportio $D$ ad $F$ est nota et $F$ est proportio vel quantitas nota igitur $D$ erit nota. Igitur due proportiones, scilicet $E$ et $D$, iam sunt note et hoc volebam.
Verbi gratia sit $B$ duplum ad $C$ et velocitas qua $C$ movetur sit tripla ad velocitatem qua $B$ movetur. Igitur per primam suppositionem proportio proportionum, scilicet $G$, erit tripla ita quod $D$ erit triplum ad $E$. Igitur $E$ erit una tertia de $D$, igitur $F$ residuum erit due tertie ipsius $D$. Igitur proportio $D$ ad $F$ erit sicut denominans ad numeratorem scilicet sicut 3 ad 2, videlicet sexquialtera. Et $F$ est proportio dupla per ypotesim igitur $D$ componetur ex dupla et medietate duple et erit tres quarte proportionis quadruple. Et quia $D$ componetur ex $E$ et $F$ et $F$ est dupla $E$ erit medietas duple, vel sit $E$ una tertia et $F$ due tertie. Igitur proportio earum est sicut proportio numeratorum igitur $E$ se habet ad $F$ sicut I ad 2.

Notandum quod licet $B$ sit duplum ad $C$, non oportet propter hoc ${ }_{260}$ quod $A$ moveat $C$ duplo velocius quam moveat $B$ quia hoc est improbatum per primam conclusionem huius.

233 post Et bab $R$ sic / sit om $R$
235 ypotesim: propositum $H$
236 scilicet G; igitur C / Habebimus CV habemus $H R /$ itaque: igitur $R$
$237 \mathrm{D}, \mathrm{E}, \mathrm{F}, \mathrm{G}, H \mathrm{D}$ et E F et G C D et E et F et $\mathrm{G} R V /$ due sunt $V$ due $H R$ duo sunt $C / \mathrm{G}$ et $\mathrm{F}: \mathrm{F}$ et G $C$
239. Arguatur $H V$ arguitur $C R /$ igitur om C/D: B C
240 ad: et $R /$ post F bab $R$ ad / erit: esset V
240-42 erit nota...residuum om $R$

242 scilicet $C H$; om $V$ | residuum $H V$; om C / erit: esset $V$
243 ante Et tunc bab $C$ tocius scilicet D ad F erit nota per secundam partem eiusdem octave
244 sive: vel C/ECH; om VD $R$
244-46 erit nota...nota igitur D om $R V$ 245 est proportio vel $C$ pars sive $H$
$246 \mathrm{D}^{\mathrm{I}} C$; om $H$ / scilicet $C$ videlicet $H R V / \mathrm{E}$ et $\mathrm{D}: \mathrm{D}$ et $\mathrm{E} V$
249 B movetur $\operatorname{tr} C$
$2 \rho 1$ ad E om $R /$ erit ${ }^{\text {tr }} R$ post tertia / F

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Furthermore, let $G$ be ratio $D$ to $E$, so that $G$ will be a ratio of ratios from which the velocities arise. Now since the ratio of velocities is known by hypothesis, the ratio of ratios, namely $G$, will be known, by the first supposition. We will then have four ratios, $D, E, F$, and $G$, of which two are known, namely $G$ and $F$; and two are unknown, namely $D$ and $E$, which we wish to make known.

Let it be argued as follows: The ratio of $D$, the whole, to $E$, its part, is known and therefore the ratio of the same part, namely $E$, to the remainder, $F$, will be known, by the first part of the eighth supposition. And, similarly, the ratio of the whole, namely $D$, to the remainder, $F$, will be known, by the second part of the same eighth supposition. Then, continuing, ratio $E$ to $F$ is known and $F$ is a known ratio or quantity so that $E$ will be known, by the ninth supposition. Likewise, ratio $D$ to $F$ is known and $F$ is a known ratio or quantity and consequently $D$ will be known. Thus the two ratios, namely $E$ and $D$, are now known, and this is what I desired.

For example, let $B$ be double $C$ and let the velocity with which $C$ is moved be triple the velocity with which $B$ is moved. By the first supposition, therefore, the ratio of ratios, namely $G$, will be a triple so that $D$ will be triple to $E$. Hence $E$ will be one-third of $D$, and $F$, the remainder, twothirds of $D$. The ratio of $D$ to $F$ will then be as a denominating number to its numerator, namely 3 to 2, a sesquialterate. Now, by hypothesis, $F$ is a double ratio, and therefore $D$ would be composed of a double and half of a double ratio and will be three-fourths of a quadruple ratio. And since $D$ is composed of $E$ and $F$, and $F$ is a double ratio, $E$ will be half of a double; or $E$ will be one-third [of $D$ ] and $F$ two-thirds [of $D$ ]. Thus their ratio is as a ratio of [their] numerators so that $E$ is related to $F$ as I to 2.*
It must, however, be noted that if $B$ were double $C$, it does not follow for this reason that $A$ could move $C$ twice as quickly as it moves $B$, because this is disproved by the first proposition of this chapter.

* This example is completely summarized on p. 371, under IV.225-70.
om $H$
252 denominans $H$ denominationis $C V$ denominatoris $R$
253 videlicet: scilicet $C$
254 componetur: componitur $R$
259 erit: erunt $R$ / componetur $H V$ componitur $C R$
256 ante una add $R$ est / ante tertia add $V$
proportio / et ${ }^{3}$ : E $V$ 257 ante due add $H$ est / earum: eorum $C$ 259 Notandum: nota $R$ / post oportet bab $R$ quod
260 ante duplo add $H$ in
260-6I hoc est improbatum: illud improbatum est $C$

Sciendum est etiam quod sicut ex proportione duorum mobilium et velocitatum respectu eiusdem potentie ad quodlibet mobilium proportio potentie potest sciri, ita etiam quod econtrario scita pro-
${ }_{26}$ portione duarum potentiarum et velocitatum respectu eiusdem mobilis cuiuslibet potentie ad mobile proportio fiet nota. Et ita proportiones erunt note a quibus velocitates oriuntur ut si ponatur quod $C$ sit maior potentia, $B$ vero minor, et $E$ mobile quod movetur ab utraque successive et sint cetera sicut prius et tunc arguatur ultra penitus sicut supra.
Quarta conclusio. Quavis velocitate demonstrata et qua placet proportione proposita, si proportio ignota a qua venit velocitas sit commensurabilis proportioni proposite investigare, quod si fuerit commensurabilis fiet nota.

Signentur termini sicut in secunda conclusione et sit $A$ proportio
275 nota, $B$ quesita, $D$ potentia et $E$ mobile. Sumatur igitur $F$ mobile minus quam $E$, ad quod $E$ se habeat in proportione nota que est $A$. Sitque $C$ totalis proportio $D$ ad $F$.
Applicetur quoque $D$ potentia $F$ mobili et moveat illud. Aut igitur velocitas qua $D$ movet $F$ est commensurabilis velocitati qua ipsum
${ }_{280} D$ movet $E$ aut non. Si sit, ergo, totalis proportio $C$ erit commensurabilis $B$ sue parti per primam suppositionem, quia ab istis proportionibus veniunt velocitates predicte que sunt commensurabiles. Igitur $B$ proportio erit etiam commensurabilis residuo, scilicet $A$, per sextam suppositionem.
Si vero velocitas maior qua $D$ movet $F$ sit incommensurabilis minori qua $D$ movet $E$, ergo proportio maior a qua venit velocitas maior, scilicet $C$ proportio, est incommensurabilis $B$ proportioni minori a qua venit minor velocitas per primam suppositionem. Et

262 est etiam $\operatorname{tr} H \mid$ ante ex bab $H$ et | duorum: duarum $H$
263 et om $H$ / velocitatum: nolo tamen $C$ / quodlibet: quod $R$
264 potest sciri $\operatorname{tr} C$ / quod: quasi $V$ / post quod bab C illa
265 potentiarum: proportionum $H /$ respectu: $C$ (?)
266 cuiuslibet: cuilibet $C /$ potentie: potentia $H$ / fiet: fuit $V$ / ante ita add $V$ sic
267 ponatur: ponitur $C$
268 vero om C
268-69 ab utraque om $C$

269 sint: sic $R$ / cetera: ceteri $V /$ arguatur $H V$ arguitur $C R /$ ultra $H$; om $R$ ulterius $C V$
271 Quarta conclusio $V$; om CH quarta conclusio mg bab $R$ post proportio (linea 272) / Quavis $C H$ quamvis $R V$ / demonstrata: monstrata $C$ / qua: quam $H$ / proportione rep $R$
272 venit velocitas $\operatorname{tr} R /$ sit: sicut $C$
273 fuerit: fiet $H$ / commensurabilis: incommensurabilis $V$
274 ante termini bab $H$ et
275 nota: vera $C$ / et om $H / F$ om $R$
${ }_{27}$ quam $\mathrm{E} C R($ ? $) \mathrm{E} H$ quam $V / \mathrm{E}^{2}$

It must also be understood that just as the ratio of a power to each of [two] mobiles can be ascertained [if] the ratio of those two mobiles and [the ratio of] the respective velocities produced by that same power are known, so, contrarily, with the ratio of two powers known as well as the [ratio of] velocities they produce with respect to the same mobile, the ratio of each power to that mobile can be made known. And thus the ratios from which velocities arise will be known, so that if it were assumed that $C$ is the greater power, $B$ the lesser power, and $E$ the mobile which is moved by each [power] successively, and if the other things remain as before, one can carry on the argument exactly as above.
Proposition IV. With any velocity designated and any ratio proposed that you please, to investigate if the unknown ratio that gives rise to the velocity is commensurable to the proposed ratio; for if it is, it can be determined.
Let the terms be assigned, as in the second proposition, and let $A$ be the known ratio, $B$ the ratio which is sought, $D$ the power, and $E$ a mobile. Then let $F$, a mobile less than $E$, be taken so that $E$ is related to it in a known ratio, which is $A$. And let the whole ratio $C$ be $D$ to $F$.
Power $D$ is applied to mobile $F$ and moves it. Then, either the velocity with which $D$ moves $F$ is commensurable to the velocity with which $D$ moves $E$, or it is not. If it is, then the whole ratio $C$ will be commensurable to $B$, its part, by the first supposition, since the aforementioned velocities, which are commensurable, come from these ratios. Ratio $B$, therefore, will also be commensurable to the remainder, namely $A$, by the sixth supposition.
If, however, the greater velocity with which $D$ moves $F$ were incommensurable to the smaller velocity with which $D$ moves $E$, then, by the first supposition, the greater ratio from which the greater velocity comes, namely ratio $C$, is incommensurable to the lesser ratio $B$ from which the smaller velocity comes. And, furthermore, by the seventh supposition, part
om C / habeat: habet $H$
${ }_{277} \mathrm{C}$ totalis $R V$; $\operatorname{tr} C$ totalis $H$
278 potentia: potentie $C$ / illud om $V$
279 est: scilicet $H$
280 C om V
28 I B corr ex A CHV om $R$ / sue: sui $R$ /
suppositionem: conclusionem $C$ 283 B corr ex A $C H R V$ / erit $C R$ est $H$
esset $V /$ etiam $C R$; om $H V / \mathrm{A}$ corr ex B CHRV
284 suppositionem: conclusionem $R$
285 qua: quam $H$
287 incommensurabilis: commensurabilis
H|B:CV
288-92 Et... primo propositum om $R$
ultra $B$ pars est incommensurabilis $C$ suo toti, ergo est incommensurabilis $A$ residuo per septimam suppositionem. Patet itaque qualiter invenitur si $B$ proportio est commensurabilis vel incommensurabilis $A$ proportioni et hoc fuit primo propositum.

Si igitur $B$ proportio sit commensurabilis $A$ proportioni, igitur erit commensurabilis $C$ proportioni totali per secundam partem sexte suppositionis. Tunc capiatur proportio velocitatis qua $F$ movetur ab ipso $D$ ad velocitatem qua $E$ movetur ab eodem $D$ et quelibet istarum velocitatum et proportio earum erit nota ex diffinitione velocioris et tardioris posita in sexto phisicorum. Et tunc arguitur sic: proportio velocitatum est nota igitur proportio $C$ proportionis ad $B$ est nota
${ }_{300}$ per primam suppositionem quia ab hiis proportionibus oriuntur velocitates, igitur proportio $B$ ad $A$ erit nota per octavam suppositionem. Sed $A$ est proportio nota, igitur $B$ est nota per ultimam suppositionem.

Aliter arguo sic: proportio velocitatum est nota et proportio mobilium, que est $A$, est nota, ergo ad utrumque mobilium proportio potentie fiet nota per conclusionem immediate precedentem. Igitur $C$ proportio erit nota et similiter $B$ proportio erit nota quod fuit secundo propositum.

Verbi gratia proportio data sit dupla, pono ergo $F$ subduplum ad
${ }_{310} E$. Moveat, itaque, $D E$ mobile in die per unum miliare; et applicetur $D$ ad $F$ et moveat $F$ in equale tempore in unum spacium incommensurabile miliari quod se habet ad miliare sicut dyameter quadrati ad eius costam.

Si igitur ita sit tunc proportio qua movebat $E$ erat incommensurabilis duple et similiter proportio qua movet $F$. Quia tunc $C$ se

289 ante B bab $H$ igitur / pars est $C V$; tr $\mathrm{H} /$ incommensurabilis CH commensurabilis $V /$ est $^{2}$ om $H$
290 A $C V$ sit $H$ / septimam $C H$ aliam $V$
290-91 qualiter invenitur $V$; om $H$ qualibet invenitur $C$
293 igitur ${ }^{\text {r }}$ om $C$ / sit: est $C$ / commensurabilis: incommensurabilis $V /$ erit: esset $V$
294 C: F H/sexte: secunde $R$
295-96 ab ipso: a $R$
297 erit: sit $H /$ velocioris $H V$ velocitatis $C R$
298 in C; om HRV / Et om $H$
299 ad: ab $V /$ est: erit $R$
300-301 velocitates: velocitatem $H$
301 erit $H$ et $C$ est $R$ esset $V /$ post octavam hab $H$ huius
302 Sed: igitur $H$ / igitur...nota om $V$ / est ${ }^{2} H$ erit $C R$
304 arguo: arguitur $C$
305 nota: vero $C$
305-6 proportio...nota: est nota proportio potentie $C$
306 immediate: immedietas $C$
307 erit $^{1} H R$ est $C$ esset $V /$ erit $^{2}$ : est $C /$ secundo $H$; om $V$ secundum $C R$
309 ante proportio add $R \mathrm{~A} /$ pono $R V$

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$B$ is incommensurable to $C$, its whole, and therefore incommensurable to $A$, the remainder. And thus it is clear how one finds whether ratio $B$ is commensurable or incommensurable to ratio $A$, and this was proposed in the first place.

Therefore, if ratio $B$ should be commensurable to ratio $A$, then, by the second part of the sixth supposition, it will be commensurable to the whole ratio $C$. Next, let the ratio be taken of the velocity with which $F$ is moved by $D$ to the velocity with which $E$ is moved by the same $D$, and any of these velocities and their ratios will be known, from the definition of quicker and slower given in the sixth [book] of the Pbysics [of Aristotle]. And then one argues this way: The ratio of velocities is known; therefore, by the first supposition, the ratio of ratio $C$ to $B$ is known, since the velocities arise from these ratios; and consequently, by the eighth supposition, ratio $B$ to $A$ will be known. But $A$ is a known ratio; therefore, by the last supposition, $B$ is known.

I argue in another way in this fashion: The ratio of the velocities is known and the ratio of the mobiles, which is $A$, is known and thus the ratio of the power to each of the mobiles is determined by the immediately preceding proposition. Then ratio $C$ will be known and likewise ratio $B$ will be known, which was proposed in the second place.

For example, should the given ratio be a double, I then assume that $F$ is half of $E$. Moreover, let $D$ move mobile $E$ one mile a day, and let $D$ be applied to $F$, and, in an equal time, move $F$ a distance incommensurable to a mile and related to a mile as the diagonal of a square to its side.*

If this be so, then the ratio with which $E$ moved was incommensurable to a double, and, similarly, the ratio with which $F$ moves [is incommensurable to a double]. Now since $C$ will be related to $B$ as the diagonal to

* Here $A=E / F=2 / \mathrm{and} B$, which is $\quad(D \mid E)^{V} E / V F$, where $V_{F} / V_{E}=(2 / 1)^{1 / 2}$, unknown, equals $D / E$. Mobiles $F$ and $E$ an irrational ratio of velocities. are moved by force $D$ such that $D / F=$

> ponam $C H /$ ergo om $H /$ subduplum: subdupla $C$
> $310 \mathrm{E}^{\mathrm{I}} C R \mathrm{C} H V$
> $310-12$ et applicetur....ad miliare om $C$
> 311 in $V$ per $H R$
> 312 ante quod bab $H$ ita
habebit ad $B$ sicut dyameter ad costam, igitur $B$ erit incommensurabilis residuo, scilicet $A$, per septimam suppositionem.

Si vero $D$ moveat $E$ sicut prius per unum miliare in die et in equali tempore moveat $F$ per tria miliaria, igitur proportio $D$ ad $F$, scilicet
${ }_{320} C$, est tripla ad proportionem $D$ ad $E$, que est $B$. Igitur $B$ est una tertia de $C$ et $A$ est due tertie de ipso $C$, igitur proportio $A$ ad $B$ est sicut 2 ad i. Et $A$ est dupla ut positum est igitur $B$ est medietas duple et hoc volebam scire.
Si vero proportio data, scilicet $A$, fuisset quadrupla et $D$ movet
${ }_{325} E$ per unum miliare, sicut prius, et moveat $F$ eodem tempore per tria miliaria, tunc $C$ esset triplum ad $B$ et per consequens $A$ esset duplum ad $B$. Ergo $B$ esset proportio dupla et $C$ proportio octupla.
Eodem modo habebitur intentum si due potentie uni mobili comparentur, sicut dicebatur de secunda conclusione, ita quod $F$ sit maior
${ }_{330}$ potentia, $E$ minor potentia, et $D$ mobile, et sit proportio $F$ ad $E A$ data. Et postquam $E$ movit $D$ applicetur $F$ ad $D$ et arguatur ut supra. Sciendum quod proportio velocitatum arguitur et scitur ex proportione temporum et spatiorum pertransitorum sive acquisitorum, aut aliquorum talium, ut patet sexto et septimo phisicorum. Ex pro335 portione velocitatum arguitur proportio proportionum, et iste processus est a posteriori. Quando, vero, ex proportione proportionum arguitur proportio velocitatum tunc proceditur a causa et a priori.
Quinta conclusio. Data aliqua velocitate et cognita proportione a qua provenit, de qualibet velocitate sciri poterit a qua proportione oriatur scita tamen proportione velocitatum.

316 habebit: haberet $C$
317 ante septimam bab $V$ scilicet(?)
318 vero: nota de $V / \mathrm{E} H R$ et $C$ C $V /$ et om $R / \mathrm{in}^{2} R V$; om $C H$
319 miliaria om $H$ / scilicet: que $H$
320 C est: erit $H / \mathrm{E}: \mathrm{C} V$
321 ipso: ista $H$ / proportio A $H V$; $\operatorname{tr} R$ proportio $\mathrm{D} C / \operatorname{ad} \mathrm{B} H V \operatorname{ad} \mathrm{C} R \mathrm{~B} C$
322 ante 2 add $C$ proportio / dupla: duplum R
323 hoc: illud $C$
324 proportio: predicta (?) $H /$ data scilicet A $C V$ scilicet A data $R$ scilicet A $H /$ et $\mathrm{D} C R \mathrm{D} H$ et de $V$
325 E R;om $C H V /$ per $^{1}$ om $V /$ et om $V /$ moveat $H$; om $R$ moveret $C V$
325-26 eodem... miliaria $H V$ per tria
miliaria eodem tempore $C$ per tria eodem tempore $R$
326 esset $^{2}$ : est $R$
327 octupla: quadrupla $H$
328 habebitur $V$ habetur $C(?) R$ habebit $H$ / intentum: in tantum $C$ / uni: suo H
$330 \mathrm{et}^{\mathrm{I}}$ om $H / \operatorname{sit}: \operatorname{sic} R / \mathrm{E}^{2}: \mathrm{D} C R$
331 movit $H$ movet $C R$ movebit $V /$ arguatur: arguitur $C$
332 ante Sciendum add $V$ et / scitur: sumatur(?) V
333 aut $C H$ vel $R V /$ ante Ex add $C R V$ et
334 aut $C$ ille $R$
$335-36$ processus: potentia cessus $V$ 337 tunc om $H$
its side, therefore $B$ will be incommensurable to the remainder, namely $A$, by the seventh supposition.*

If, however, $D$ should move $E$ one mile a day, as before, and $F$, in an equal time, moved three miles, then ratio $D$ to $F$, namely $C$, is triple ratio $D$ to $E$, which is $B$. Consequently, $B$ is one-third of $C$, and $A$ is twothirds of $C$, so that ratio $A$ to $B$ is as 2 to 1 . Now $A$, as was assumed, is a double ratio, so that $B$ must be half of a double ratio; and this is what I wished to know.t

Now if the given ratio $A$ were a quadruple and $D$ moves $E$ one mile as before, while in the same time it moves $F$ three miles, then $C$ would be triple $B$, and, as a consequence, $A$ would be double to $B$. Therefore, $B$ should be a double ratio and $C$ an octuple ratio. $\ddagger$

This objective will be achieved in the same way if two powers are related to one mobile, as stated in the second proposition, so that $F$ would be the greater power, $E$ the lesser power, $D$ the mobile, and ratio $F$ to $E$ would be $A$, the given ratio. Then, after $E$ moves $D, F$ would be applied to $D$, and the argument would proceed as [given] above.

It must be understood that from a ratio of times, and from a ratio of distances traversed or acquired, or any such [quantities], one can arrive at and know a ratio of velocities, as is evident from the sixth and seventh [books] of the Pbysics [of Aristotle]. The process of arriving at a ratio of ratios from a ratio of velocities is a posteriori. When, however, a ratio of velocities is derived from a ratio of ratios, the procedure is by way of the cause and is a priori.

Proposition $V$. When any velocity is given and the ratio from which it arises is known, then the ratio that gives rise to any [other] velocity can be known, provided that the ratio of the velocities is known.

* Since $D|F=D| E \cdot E \mid F$ and $D \mid F==(C)^{2 / 3}$, so that $A=(B)^{2}$ or $B=(A)^{1 / 2}$. ( $D / E)^{2} V_{F} / V_{E}$, where $V_{F} / V_{E}$ is irrational, Since by assumption $A=2 / \mathrm{I}$, ratio $B$ can . it follows that $D / E$ and $D / F$ are incommensurable to $E / F$ and, consequently, $B$ ( $=D / E)$ is incommensurable to $A(=$ $E \mid F)$.
$\dagger D / F=(D / E)^{3 / 2}$, where $3 / 1=V_{F} / V_{E}$.
Then, since $D / F=C$ and $D / E=B$, it
follows that $C=(B)^{3}$ and $B=(C)^{1 / 3}, A$
338 Quinta conclusio $V$; om $C$ quinta conclusio mg hab $H$ post provenit (lineae 338-39) et mg hab $R$ post priori (linea
now be determined and equals $(2 / \mathrm{I})^{1 / 2}$.
$\ddagger$ If $A=4 / \mathrm{s}$ and $D / F=(D \mid E)^{1 / 3}$, it is clear that $C=(B)^{3}$ and, as in the preceding note, $B=(A)^{1 / 2}=(4 / \mathrm{I})^{1 / 2}=2 / \mathrm{r}$. Since $C=B \cdot A$, we find that $C=2 / \mathrm{I}$. $4 / 1=8 / 1$.

[^30]Sit $A$ nota proportio a qua venit velocitas data, et sit una alia velocitas isti commensurabilis que oritur ex $B$ proportione. Dico quod $B$ erit nota et arguo sic: proportio proportionum, scilicet $A$ ad $B$, est sicut proportio velocitatum per primam suppositionem; et proportio velocitatum est nota, ut supponitur; igitur proportio $A$ ad $B$ est nota. Sed $A$ est proportio nota per ypotesim ergo $B$ est proportio nota per nonam suppositionem, et hoc est propositum.

Sic igitur sciemus proportionem potentie ad resistentiam, scilicet proportionem a qua venit velocitas, ubi potentia neque a suo mobili
sua resistentia separari nec diversis mobilibus applicari.
Verbi gratia sit $A$ proportio dupla a qua venit velocitas data, et sit aliqua alia velocitas quadrupla ad istam que provenit ex $B$ proportione. Igitur $B$ proportio est quadrupla ad $A$, scilicet ad duplam. Augebo, igitur, proportionem duplam usque ad quadruplam eius, sicut docetur
355 in primo capitulo, et habebo proportionem sedecuplam, igitur $B$ erat proportio sedecupla.

Sexta conclusio. Nota proportione a qua venit velocitas si sit rationalis duos eius primos numeros dare; si, vero, irrationalis duas lineas invenire quarum maior sit sicut potentia motoris, minor vero sicut resistentia rei mote.
Dicam primo de supposito, deinde de proposito. Supponitur quod proportio sit nota a qua venit velocitas et proportio est nota quando denominatio eius scita est. Aliquarum autem proportionum, scilicet omnium rationalium et quarundam irrationalium, denominationes sunt scibiles et aliquarum irrationalium non sunt scibiles, sicut in
${ }_{365}$ primo capitulo et in probatione decime tertii capituli dicebatur. Si igitur fuerit aliqua velocitas que a tali proportione oriatur cuius denominatio scibilis non est impossibile est ut huius proportio fiat nota. Verumtamen de qualibet proportione nobis data, sive danda, poterimus investigare, per secundam conlusionem, utrum ipsa sit maior vel

341 A: autem $C /$ nota proportio $\operatorname{tr} R$ / venit: provenit $R$
342 oritur $C R$ oriuntur $H$ oriatur $V$
346 ypotesim: propositum $V$
$349 \mathrm{a}^{1}$ rep $R$ / velocitas om $C /$ neque: nequit $C$
350 sive $H$ vel $C$ seu $R V$ / sua $H$; om $C R$ a $V /$ mobilibus $C H$ motibus $R V$
352 istam: illam $C$
353 A om $C / A u g e b o C V$ augendo $H$ augmentabo $R$
354 usque om $R /$ quadruplam: quadru-
plum $V$
355 igitur $H$ ergo $C V$ et $R$ / erat: erit $H$
357 Sexta conclusio $V$; om $C$ sexta conclusio $m g$ bab $H$ post proportione et $m g$ bab $R$ post sedecupla (linea 356) / si: sic $C$
358 dare om $H$ / irrationalis: rationalis $C /$ invenire: invenias(?) $H$
359 ante maior bab H scilicet / ante minor bab $R$ et / vero om $R$ / ante sicut bab $V$ sit

Let $A$ be a known ratio from which a given velocity arises, and let there be another velocity, commensurable to this given velocity, which arises from satio $B$. I say that $B$ can be found and argue as follows: A ratio of ratios, namely $A$ to $B$, is like a ratio of velocities, by the first supposition; and the ratio of velocities is known, as is assumed; therefore, ratio $A$ to $B$ is known. But $A$ is known by hypothesis, and consequently $B$ can be made known, by the ninth supposition [of this chapter]; and this is what has been proposed.

In this way we can find the ratio of a power to a resistance, namely the ratio from which a velocity arises, provided the power is not separated from its mobile or resistance, or [simultaneously] applied to diverse mobiles.

For example, let $A$ be a double ratio from which a given velocity arises, and let some other velocity which arises from ratio $B$ be quadruple to it. Therefore, ratio $B$ is quadruple to $A$, namely to a double ratio. Then, as was taught in the first chapter, I shall increase the double ratio to quadruple of itself and shall get a sedecuple ratio. Therefore, $B$ was a sedecuple ratio.*

Proposition VI. If a ratio from which a velocity arises is known and rational, give its prime numbers; but if irrational, find two lines the greater of which is as the power of the mover, the lesser as the resistance of the thing moved.

I shall speak first about something which is taken as an assumption, and then consider what has just been proposed. It is assumed that a ratio which gives rise to a velocity can be known and is known when its denomination is known. The denominations of some ratios are knowable, namely of all rationals and some irrationals, and the denominations of some irrationals are not knowable, as was said in the first chapter and in the proof of the tenth proposition of the third chapter. Thus, if there were some velocity that arises from such a ratio whose denomination is not knowable, it is impossible to make its ratio known. Nevertheless, by the second proposition, we can investigate whether any ratio given, or to be given * If $A=2 / \mathrm{I}$ and $B=(A)^{4 / 1}$, then $B=(2 / \mathrm{I})^{4}={ }^{16} / \mathrm{I}$.

361 est nota $H V ; \operatorname{tr} C$ nota $R$
362 denominatio eius $\operatorname{tr} H$ / autem: aut $R$ 363 et om $C$
364 irrationalium $H R$ aliarum irrationalium $C$ irrationabilium $V$
365 et om $C$ / decime: decimi $H /$ dicebatur: dicebitur $C$

367 ut $R V$ et $C H$ / huius: huiusmodi $R$ / fiat nota: non fiat $C$
368 Verumtamen: veniamus(?) $p$ (?) tamen $V /$ qualibet: quibus $V /$ proportione: proportio $H /$ sive danda $H$; om $C R$ vel danda $V$
369 secundam om $C /$ vel: aut $R$
minor tali proportione irrationali incognoscobili et innominabili. Et sic tandem poterimus invenire duas proportiones satis propinquas ad quas talis proportio ignota se habebit ita quod erit minore maior et maiore minor. Et hoc debet sufficere.
Si , autem, velocitas oriatur a proportione cuius denominatio sit scibilis ad cognoscendum eam ubicumque non invenio regulam generalem. Sed ex quinque conclusionibus precedentibus ad hoc possumus adiuvari multum valde. Per secundam, enim, possumus de qualibet proportione proposita temptare utrum proportio unde venit velocitas sit eidem equalis, sive maior, sive minor. Et ubicumque sciremus pro-
${ }_{380}$ portionem velocitatum atque mobilium respectu eiusdem motoris vel motorum ab eadem potentia, vel velocitatum et potentiarum moventium idem mobile vel equalia, possemus, per tertiam, proportionem invenire.
Ut si $A$, grave, moveatur in aliquo medio et $B$ duplum eius mo-
${ }_{385}$ veatur in eodem medio, ex proportione moventium, scilicet $A$ ad $B$, et velocitatum, de utraque velocitate scietur a qua proportione oriatur nisi fuerit eo quod $B$ plus iuvabitur $A$ medio vel ex difformitate motus vel aliunde, et cetera. Et si aliqua proportio proponitur, scietur si illa de qua queritur, a qua venit velocitas, sit commensurabilis proportioni
${ }_{390}$ proposite; quod si fuerit, fiet nota per quartam conclusionem. Et si per istas tres conclusiones vel per aliquam earum possumus proportionem alicuius velocitatis cognoscere ita quod possumus dicere talis velocitas provenit a tali proportione, verbi gratia velocitas qua aliquod mobile pertransit in hora unum miliare provenit a proportione dupla.
${ }_{395}$ Quod si velis et possis facere placet mihi. Si vero sit defectus ex parte medii non plus possum. Tamen, si hoc possumus sciemus quod omnis
370 proportione: proportioni $C /$ incog- $379-80$ proportionem: proportionum $C$ noscobili: cognoscibili $H$
371 sic $V$; om $C R$ sit $H$ / invenire: investigare $R$
372 talis om $C$ / habebit: habeat $R /$ minore maior: maiore minor $R$
373 maiore minor $C H$ minore maior $R$ maior minor $V$
375 cognoscendum: cognoscendam $R$
375-76 generalem: glacialem(?) $V$
376 hoc: hec $R$ / possumus: possimus $C$
377 adiuvari multum valde $H V$ multum adiuvari valde $C$ adiuvari valde multum $R$
379 eidem om $C /$ sive $^{2}$ : vel $C$

380-8 I atque...velocitatum om $V /$ vel motorum $C$ seu motorum $R$ sive motoris(?) $H$
381 velocitatum et potentiarum: potentiarum velocitatum $R$
382 possemus: possimus $C /$ proportionem $C V$ propositum $H R$
384 ante A bab $H$ quod/duplum eius $H R$ eius $C$ duplum $V$
385 ad: ab $R$
386 velocitate $H$ velocitatum $C R V$ / scietur: scieret $C$
387 eo om $R$ / B om $H$ / plus iuvabitur tr $C /$ motus: motis $C$
to us, is greater or smaller than such an irrational, unknowable, and unnameable ratio. Finally, in this way we can find two ratios sufficiently close so that such an unknown ratio will be greater than the lesser and smaller than the greater. And this ought to suffice.
On the other hand, if the velocity should arise from a ratio whose denomination is knowable, I can discover no general rule for determining it in every case. But we can be greatly aided toward this end by the five preceding propositions. For, by the second proposition, we can test whether a ratio from which a velocity comes is equal to, greater, or less than any proposed ratio. And, by the third proposition, we can find [an unknown] ratio whenever we know the ratio of velocities and the ratio of mobiles in relation to [one and] the same mover, or to [two] movers having the same power; or [whenever we know] the ratio of velocities and the ratio of powers moving the same mobile or equal mobiles.
For example, if $A$, a heavy body, were moved in some medium, and $B$, which is double $A$, is also moved in the same medium, then the ratio that gives rise to each velocity can be determined from the ratio of moving bodies, namely $A$ to $B$, and the ratio of velocities, unless it should happen that $B$ will be propelled more [quickly] than $A$ either by the medium, or because its motion is non-uniform, or for some other reason, etc. And if any ratio is proposed, one can determine if the ratio that is sought and gives rise to a velocity is commensurable to the proposed ratio; and if it is commensurable it can be made known, by the fourth proposition. And so by these three propositions, or by some of them, we can come to know the ratio producing a certain velocity so that we could say, such a velocity arises from such a ratio, as, for example, when we say that the velocity with which a certain mobile traverses one mile in an hour arises from a double ratio. If you wish to do this and are capable of it, it would please me. If, however, the medium were defective, I could do nothing more. But if

[^31]
## $H V$ possimus $C R$

393 a: ex $R$
394 pertransit $H V$ transit $C R$ / ante hora bab $C$ una / a: ex $R$
395 si velis $R V$ velis $C$ si velit $H /$ possis $R V$ possit $C H$
396 medii $V$ medie $H R$ media (?) $C /$ possumus $H V$ possimus $C R /$ sciemus: sciremus $C$
equalis velocitas ab equali proportione procedit et omnes velocitates eiusdem generis ab equalibus proportionibus procedentes sunt equales. Et per quintam conclusionem cuiuslibet velocitatis isti velocitati ${ }^{400}$ commensurabilis proportio fiet nota scita tamen proportione velocitatum. Et ita sciemus proportionem a qua venit velocitas ubi potentia non potest a resistentia separari, nec diversis mobilibus applicari, nec idem mobile pluribus motoribus coaptari.

Sic igitur si alicuius velocitatis circulationis proportio cognoscatur 405 per doctrinam precedentem ita ut possit dici hec velocitas est a proportione dupla vel tripla, et cetera. Et sciatur proportio velocitatis motus alicuius orbis ad istam velocitatem per astrologiam potest sciri ex proportione quantitatum motuum vel circulorum descriptorum, et ex proportione temporum in quo revolvunt. Ex istis duobus, scilicet
$41^{\circ}$ ex notitia proportionis a qua venit velocitas demonstrata et notitia proportionis velocitatis orbis ad velocitatem datam, poterit comprehendi proportio intelligentie moventis ad orbem. Que quidem proportio non debet vocari proportio virtutis ad resistentiam nisi secundum similitudinem sicut puto quia intelligentia movet sola voluntate
415 et nulla alia virtute seu conatu vel difficultate et celum non resistit ei sicut credo fuisse de mente Aristotelis et Averrois.

De hoc alias non plus modo hoc dictum sit de supposito quod fuit in principio positum in his verbis: "nota proportione a qua venit velocitas...."
Nunc restat de proposito disserendum et ibi sunt duo. Primum est si sit rationalis primos eius numeros invenire et hoc iam fuit in primo capitulo expeditum. Secundum est si vero irrationalis duas lineas incommensurabiles dare et cetera, ubi sciendum quod omnis irrationalis

397 equalis velocitas $t r V /$ ab equali proportione: proportione equali $V /$ procedit: provenit $C$ / omnes velocitates: omnis velocitas $R$
398 eiusdem: cuiuscumque $H$ / procedentes $C H$ procedant $R$ proceres $V$
399 cuiuslibet: cuius $V$ / velocitatis CH velocitati $R$ velocitas $V /$ ante isti $m g$ bab $R$ tamen
400 tamen: cum $R$
402 mobilibus $H$ motibus $C R V$
403 motoribus: motibus $C$
404 si om $V /$ circulationis $C H$ circularis RV

40s precedentem $R V$; obs $H$ precedentis C
406 vel: et $R$ / ante tripla add $R V$ a proportione / et cetera om $R$
407 istam: illam $C$ / ante per add $R V$ quod / astrologiam: astronomiam $R /$ potest sciri $R V$; om $H$; $\operatorname{tr} C$
408 motuum corr ex mobilium CHRV | et om V
409 quo: quibus $V /$ revolvunt $H$ revolvuntur CRSV
410 proportionis...notitia om $C$ / venit $H V$; om $R /$ notitia ${ }^{2}$ : nota $R$
4I datam: data $R$ / poterit: possit $V$
we can proceed, we must understand that every equal velocity arises from an equal ratio, and all velocities of the same kind arising from equal ratios are equal. And, by the fifth proposition, [an unknown] ratio giving rise to any velocity which is commensurable to the velocity [arising from a known ratio] can be made known when the ratio of velocities is known. And thus we can know the ratio from which a velocity arises, provided that the power can not be separated from the resistance and is not applied to diverse mobiles, and the same mobile is not assigned to several movers.

Thus if the ratio of any velocity of rotation were known by means of the preceding instruction so that one could say that this velocity is produced by a double or a triple ratio, etc., then the ratio of the velocity of motion of any orb to this velocity can be found in astronomy from the ratio of the quantities of the motions or circles described, and from the ratio of the times in which they revolve. From these two things-namely, knowledge of a ratio from which a designated [or given] velocity arises and knowledge of the ratio of the velocity of an orb to the given velocitythe ratio of a moving intelligence to [its] orb can be expressed. However, I think this ratio ought not to be called a ratio of force to resistance except by analogy, because an intelligence moves by will alone and with no other force, effort, or difficulty, and the heavens do not resist it, as I believe were the opinions of Aristotle and Averroes.

Concerning this more [will be said] at another time, but what has been said [here] must suffice with regard to the assumption made in the enunciation [of this proposition] in these words: "If a ratio from which a velocity arises is known..."

It now remains for us to discuss what has been proposed, and there are two parts to it. First, if the ratio should be rational, find its prime numbers. This was already considered in the first chapter. The second part says that if the ratio is irrational, find two incommensurable lines, etc., where it is

414 puto: puta $V /$ intelligentia: intelligencius $V /$ sola: solam $C /$ voluntate: velocitate $R$
41s nulla of $H /$ seu $H V$ sive $C$ vel $R /$ conatu $H R$ conata $C$ coactu $V /$ vel: et $C$
417 ante De add $R$ et / alias non plus $R V$ non plus sed alias $C$ aut plus non $H /$ modo: media(?) $V /$ ante dictum bab $H$ quibus et bab $R$ igitur et bab $V$ ergo

418 in his $H V$ sub istis $C R /$ nota: data $R$ /a qua: aliqua $R$
420 de proposito: propositum $R /$ disserendum $C S$ disserandum(?) $H$ dicendum $R$ dissendi $V$
42 I primos eius $\operatorname{tr} H$ / iam fuit $\operatorname{tr} R$
422 expeditum $C H$ expositum $R V /$ si om $V \mid$ irrationalis: rationalis $C /$ ante lineas bab $H$ primas
423 et cetera $o m R$ / post sciendum add $R$ est 423-24 irrationalis proportio $\operatorname{tr} V$
proportio cuius denominatio scita est denominatur a proportione tionalis sit, aut a minori.

Si a maiori, tunc illa irrationalis dicitur esse pars illius rationalis sicut una secunda, una tertia, vel una quarta et cetera, aut est partes illius sicut due tertie, vel tres quarte et cetera. Et est unus numerus
${ }_{43}$ numerator, et alter denominator harum partium vel partis. Sumende sunt igitur due linee secundum proportionem rationalem a qua ista denominatur et cuius est pars aut partes. Que quidem rationalis dividenda est in tot partes quantus est numerus denominator illarum partium seu partis dividenda in qua per inventionem linearum medio
435 loco proportionalium et comparanda est una earum ad aliquam totam post eam vel ante in ordine quantus est numerus numerator et maior erit sicut potentia, minor ut resistentia.

Verbi gratia sit proportio data medietas duple, scilicet una secunda, ponam duas lineas unam duplam ad aliam. Et quia duo est denomi-
${ }_{440}$ nator proportionem duplam dividam in duo per inventionem unius linee medie proportionalis inter duas positas. Et quia unitas est numerator, comparabo ad primam post eam, vel ante, et maior erit sicut potentia, et cetera.
Aliud exemplum, sit proportio data due tertie quadruple. Ponam $A$
445 lineam quadruplam ad $B$ et quia 3 est denominator dividam proportionem quadruplam in tres inveniendo duas lineas medias que $\operatorname{sint} C$ et $D$. Eruntque quatuor linee continue proportionales $A C D B$ secundum proportionem que est tertia pars quadruple. Et quia 2 fuit numerator comparabo unam earum ad secundam post eam, vel ante, et maior ${ }_{450}$ erit sicut potentia, et cetera, comparabo enim $A$ ad $D$, vel $C$ ad $B$. Si autem, irrationalis data denominatur ab aliqua rationali minori

424 denominatur: denominata $H$ /a proportione: ab aliqua $H$
425 maiori: maiore $H /$ quam $H R$ quod CV
$425-26$ irrationalis: irrationali $H$
426 minori: minore $H$
427 maiori $C R$ maior $H$ minori $V$ /illa: ista $H$ / dicitur: debet $H /$ esse pars tr $V$
428 una tertia: $2 / 3 \mathrm{C} /$ et cetera om V
429 tres quarte $C S^{2} / 4 R V 4$ quarte $H /$ est om $V$
430 et om $H /$ vel: aut $R /$ partis: partes $C$
43 I sunt igitur: ergo sunt $C$ / ista: illa $R$

432 rationalis: rationales $V$
433 est $^{2} \mathrm{HV}$; om $C R$ /denominator: denominatorum $C$ / illarum $C S$ unum $H$ istarum $R$ illorum $V$
434 seu $H R$ vel $C V$
435 earum: illarum $R$ / aliquam: liquam $C$ 436 ante eam bab $R$ vel
437 erit $H R$ est $C$ et $V /$ ut $C V \operatorname{vel}(?) H$ et $R$
438 post data add C una / secunda: dupla $C$ 439 unam duplam CSV una dupla $H R$
439-40 denominator: numerator $R$
440 duo: duas $R$ / unius om $V$
441 proportionalis: proportionales $V$

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understood that every irrational ratio whose denomination is known is denominated by a rational ratio. Therefore, it is either denominated by a greater or lesser rational ratio.

If it is denominated by a greater rational, then that irrational is said to be a part of that rational ratio, as a second, a third, or fourth part, etc.; or it is parts of it, as two-thirds, or three-fourths, etc. And one number is the numerator, the other the denominator of these parts or this part. There must then be taken two lines which form a rational ratio denominating this irrational and of which it is a part or parts. Next, the rational must be divided into a number of parts equal to the number representing the denominator of these parts or [of this] part, and by finding mean proportional lines and relating one of them to any whole after or before it, the greater line, when taken in order as the number representing the numerator, will be as the power, the lesser line will be as the resistance.

For example, let the given ratio be half of a double, namely a second [part]; I then posit two lines, one double the other. Now since two is the denominator, I divide the double ratio in two by finding a mean proportional line between the two posited lines. And since a unit is numerator, I shall relate [the mean proportional] to the first [line] after or before it, and the greater line will be as the power, etc.*

Another example would be where the given ratio is two-thirds of a quadruple. I assume that line $A$ is quadruple $B$, and since 3 is the denominator [of the exponent], I shall divide the quadruple ratio into three parts by finding two mean lines, say $C$ and $D$. Then $A, C, D$, and $B$ are four continuously proportional lines forming a ratio which is a third part of a quadruple ratio. Since 2 was the numerator [of the exponent], I shall relate one of these lines to the second line after or before it; and the greater line will be as the power, etc., for I shall relate $A$ to $D$, or $C$ to $B .{ }^{\dagger}$

If, however, the given irrational is denominated by some smaller rational,

* See p. 373.
${ }^{+}$See p. 55.

442 maior: minor $V /$ erit: esset $V$
443 et cetera $C V$; om $H R$
444 Aliud exemplum $\operatorname{tr} C$ / data om $H$
446 et om $H$
$447 \mathrm{D}^{\mathrm{I}}: \mathrm{E} R$
$44^{8}$ quadruple om $V$ / post quia $m g$ bab $H$
a ecdb

449 earum: eorum $C$ / eam $H V$; om $C R$ 450 erit: esset $V /$ et cetera om $R / \mathrm{D}: \mathrm{B}$ $C /$ vel: et $R$
45I denominatur: denominetur $C$ /rationali: irrationali $H$
ea, tunc non erit multiplex ad eam ut habetur in probatione prime secundi capituli. Sed indifferenter poterit esse in qualibet alia proportione sicut superparticulari, superpartienti, et cetera. Et ita irrationalis
455 continebit rationalem a qua denominabitur semel, vel pluries, et aliquam vel aliquas eius partes. Et istius partis seu partium unus erit numerus denominator, et alter numerator.
Et ideo sicut prius dicebatur dividenda est rationalis proportio posita in lineis in tot partes quantus est denominator per inventionem
${ }_{460}$ linearum mediarum. Et augenda est ista proportio postea inveniendo ultra istas lineas adhuc alias in continua proportionalitate totidem quantus est numerator. Et omnium istarum maior erit sicut potentia, et cetera.

Verbi gratia sit proportio irrationalis data superpartiens duas tertias ${ }_{465}$ duple. Ponam $A$ lineam duplam ad $B$ et quia 3 est denominator dividam proportionem duplam in 3 assignando duas lineas medias que sunt $C$ et $D$. Erunt itaque quatuor linee continue proportionales $A C$ $D B$. Et quia 2 fuit numerator adhuc inveniam duas alias ultra in continua proportionalitate et non curo utrum sint maiores aut mino-
${ }_{470} \quad$ res. Sint ergo minores et $\operatorname{sint} E$ et $F$ et erunt sex linee continue proportionales $A C D B E F$. Dico, ergo, quod proportio $A$ ad $F$ est proportio irrationalis data et est $A$ sicut potentia, $F$ vero sicut resistentia.

Quod autem proportio $A$ ad $F$ sit irrationalis et quod sit proportio 475 data facillime probabitur ex dictis in primo et secundo capitulis. Primo quia inter primos numeros proportionis duple nullus est numerus seu numeri medii, igitur ipsa est incommensurabilis cuicumque maiori rationali que non est multiplex ad ipsam per quintam secundi capituli. Sed proportio $A$ ad $F$ est maior ea et non est sibi multiplex vel est

452 erit: esset $V$
$45^{2-53}$ ut habetur...esse om $C$
453 post Sed bab $R$ ut / indifferenter $H R$ viderentur(?) $V$
454 post sicut add $R$ in / irrationalis: rationalis $H$
456 eius om $H /$ seu: vel $C /$ unus: unde $C /$ erit: est $H$
457 numerator $H R$ denominator $V$
457-59 et alter... denominator om $C$
458 dividenda $R V$ divida(?) $H /$ est $S V$; om RH
458-s9 proportio posita HV proportio-
nalis proposita $R$
459 post partes $m g$ bab $H$
a 4 cdb 2 ef

460 est om $H$ ista: illa $R$
461 alias: vel(?) $H$ / in continua proportionalitate CH in proportionalitate continua $R$ continua proportione $V$ 462 erit: esset $V /$ potentia obs $H$
463 et cetera $H V$; om $C R$
46s A om V

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then it will not be multiple to the rational, as was seen in the proof of the first proposition of the second chapter. But it can be related to the smaller rational in another ratio, as [for example], a superparticular, a superpartient, and so on indifferently. Thus the irrational will contain the rational by which it is denominated one or more times and some part or parts of it; and of this part or parts one number will be the denominator, and the other the numerator.
And so, as was said before, by finding mean lines the rational ratio represented by lines must be divided into that number of parts equal to the denominator. And this ratio can be increased afterward by finding beyond these lines a total number of lines equal to the numerator and in continuous proportionality. Of all these lines, the greater will be as the power, etc.

For example, let there be a given irrational ratio which is a superpartient two-thirds of a double. I posit that line $A$ is double to line $B$, and, since 3 is the denominator, I shall divide the double ratio into three [parts] by assigning two mean lines, say $C$ and $D$. Now there will be four continuously proportional lines, $A, C, D$, and $B$. And, since 2 is the numerator, I could find two other lines in continuous proportionality beyond these [four lines], and I care not whether they are greater or smaller. Let them be smaller and call them $E$ and $F$, so that there will be six continuously proportional lines, $A, C, D, B, E$, and $F$. Then I say that ratio $A$ to $F$ is the given irrational ratio, and $A$ is as the power, and $F$ as the resistance.*

That ratio $A$ to $F$ is truly irrational and is the given ratio can most easily be shown from statements in the first and second chapters. In the first place, since there is no number or mean numbers between the prime numbers of a double ratio, it follows that it is incommensurable to any greater rational which is not multiple to it, by the fifth [proposition] of the second chapter. Now ratio $A$ to $F$ is greater than it, but is neither

## * See pp. 55-56

466 medias om $V$
467 sunt $H V \operatorname{sint} C$ sit $R$
467-68 A C D B RS C et D et B C A C et D et $\mathrm{B} H$ aut 2 C et D et $\mathrm{B} V$
468 fuit: sint $V$ | post fuit scr et del $R$ denominator/numerator: numeratos $C$ /
duas alias ultra $C R$ ultra duas alias $H V$
469 utrum $C R$ si $H V$

470 Sint: sive $H /$ sex om $R$
472 A om $H$
475 probabitur $R V$ probabiliter $C H$
476 duple om $C /$ seu: vel $C$
477 medii om $R /$ incommensurabilis: commensurabilis $V$ / cuicumque $H S$ cuiuscumque $C R$ cuiuslibet $V$
479 non $H V$ tamen $C R /$ vel $S V$ et $C H R$
${ }_{480}$ sibi commensurabilis, igitur ipsa est irrationalis. Item ut patet proportio $A$ ad $F$ continet duplam et duas tertias duple, igitur ipsa est proportio data.
Si vero proportio data esset multiplex superpartiens vel in alia proportione ad aliquam rationalem, adhuc ex dictis possent due tales
${ }_{4}^{85}$ linee faciliter reperiri. Verbi gratia sit tripla superpartiens tres quartas duple. Ponam $A$ habere ad $B$ proportionem triplam duple, scilicet octuplam. Deinde ponam $C$ subduplum ad $B$ et inveniam tres lineas medias $D E F$, eritque proportio $A$ ad $F$ proportio data. Et idem contingeret si $A$ esset octuplum ad $B$ et $C$ duplum ad $A$, et $D$ et $E$
${ }^{490} \quad$ et $F$ essent medie inter $C$ et $A$. Tunc proportio $D$ ad $B$ erit proportio data composita ex octupla, que est tripla duple, et tribus quartis duple.

Verumtamen supponitur quod tu scias inter duas lineas datas quodlibet medias in continua proportionalitate invenire. Et Euclidis non docuit nisi tantum de una, et habetur ex nona sexti, sed reverendus 495 magister Iohannes de Muris docebat invenire quotlibet sicut credo.

Septima conclusio. Si sit aliqua velocitas que proveniat a proportione rationali inter cuius primos numeros non sit numerus medius proportionalis seu numeri, omnis velocitas minor que est sibi commensurabilis provenit a proportione irrationali; et, similiter, omnis maior que non est multiplex ad ipsam.
Quia proportio velocitatum est sicut proportio proportionum, per primam suppositionem, et quelibet talis proportio inter cuius primos numeros non est, et cetera, est incommensurabilis cuilibet minori et cuilibet maiori que non est multiplex ad ipsam, per quintam secundi capituli, igitur velocitas que ex ea oritur erit incommensurabilis cuili505 bet velocitati minori que sit ex proportione rationali. Igitur si aliqua

480 post ut mg bab $H$ a 16 b ef $g(?)$

48I duplam: duplum $V /$ est om $C$
484 aliquam: aliam $H /$ ex dictis $C H V$; om $R S$
484-85 tales linee $H V ; \operatorname{tr} R$ tales regule $C$
485 faciliter reperiri $\operatorname{tr} C$
486 duple': triple $V /$ ad $B \operatorname{tr} C$ post triplam / proportionem CV proportio $H$ / duple ${ }^{2} S V$; om $C$ quadrupla $H$
486-87 proportionem...et om $R$
487 ponam $C H$ ponatur $V$
488 eritque $H R$ erit itaque $C$ sitque $V /$

A ad $H S \mathrm{D}$ ad $C$ A $V$
490 essent medie: medie erunt $C$ / erit om $V /$ proportio $^{2}$ : proposita $R$ 491 quartis $H V$ quadruplis $C$ quattuor $H$ 492 supponitur: supponatur $R /$ datas om ${ }_{H}$
492-93 quodlibet: quotlibet $R$
493 invenire: reperire $C$
494 ante docuit bab $R$ tamen / tantum om $R /$ et habetur om $V /$ sexti: tertii $H$ 495 docebat: $\operatorname{docet}($ ?) $H$ / quotlibet $C H$ quodlibet $R V$
496 Septima conclusio mg bab H post proveniat et mg hab $R$ ante Si ; om $C V$ proveniat: provenit $R$
497 numeros om $C$ / proportionalis seu
multiple nor commensurable to it and is, therefore, irrational. It is likewise evident that ratio $A$ to $F$ contains a double ratio and two-thirds of a double and, consequently, $A$ to $F$ is the given ratio.

Even if the given ratio were a multiple superpartient or related to some rational in another ratio, two such lines could yet easily be found from what has been said. For example, let the given ratio be a triple superpartient three-fourths of a double ratio. I shall assume that $A$ bears to $B$ a ratio which is triple to a double, namely an octuple. Next, I shall assume that $C$ is half of $B$, and then find three mean lines, $D, E$, and $F$, so that the given ratio $A$ to $F$ will [then] be had. The same thing would occur if $A$ were octuple to $B, C$ double to $A$, and $D, E$, and $F$ were means between $C$ and $A$. Then ratio $D$ to $B$ will be the given ratio composed of an octuple, which is triple to a double ratio, and three-fourths of a double.*

It has indeed been assumed that you would know how to find any number of means in continuous proportionality between two given lines. Now Euclid teaches how to find only one mean, and this is shown in the ninth [proposition] of the sixth [book], but Reverend Master Johannes de Muris has, I believe, shown how to find any number of them.

Proposition VII. If any velocity should arise from a rational ratio that has no mean proportional number or numbers between its prime numbers, then every lesser velocity that is commensurable to it arises from an irrational ratio; and similarly every greater [velocity that is commensurable but] not multiple to it [arises from an irrational ratio].
Since, by the first supposition, a ratio of velocities is like a ratio of ratios, and, by the fifth proposition of the second chapter, any ratio [that has no mean proportional number or numbers] between its prime numbers is incommensurable to any lesser [rational] ratio and to any greater that is not multiple to it, then any velocity which arises from such a ratio will be incommensurable to any lesser velocity which arises from a rational ratio. Therefore, if any smaller velocity is commensurable to such a velocity, it

* See pp. 373-74.
numeri $S$ sive numeri $C$ proportionalis numeri $H$ seu numeri proportionalis vel proportionales $R$ proportionalis seu $V$
498 minor $H S V$ numeri $C$ invicem(?) $H /$ que: qui $V$
499 maior om $V$
soo Quia: quod $C$

SO2 numeros om $H /$ est $^{\mathrm{t}}$ : et $H /$ est $^{2}$ : et $C$ / incommensurabilis: commensurabilis $V$ / cuilibet: cuiuslibet $V$
503 maiori: minori $R /$ ad ipsam $C S$ eius $R V$ cuilibet(?) $H$
504 erit $C H$ est $R$
s04-6 erit...minor om $V$
sos aliqua $H R$ aliquas $C$
velocitas minor est eidem commensurabilis ipsa erit a proportione irrationali et ita est de maiori que non est multiplex ad ipsam.
Verbi gratia sit velocitas a proportione dupla que velocitas sit $B$. Dico quod omnis velocitas sibi commensurabilis provenit a proportisurabilis provenit a proportione rationali et aliqua proportio irrationalis minor dupla est sibi commensurabilis et aliqua incommensurabilis.
Item omnis velocitas maior $B$, si non sit multiplex ad $B$, que provenit a proportione rationali est incommensurabilis ipsi $B$ et omnis commensurabilis ipsi $B$ que non est multiplex eius provenit a proportione irrationali. Sicut si $B$ velocitate pertranseatur leuca in die, et $C$ velocitate pertranseatur leuca cum dimidia, dico quod $C$ provenit a proportione irrationali, sic enim se habent proportiones proportionum ut patet ex quinta secundi capituli et aliis positis in tertio capitulo. Unde ex prima suppositione huius capituli et conclusionibus positis in tertio capitulo poterit intelligens quam plurimas conclusiones de velocitatibus demonstrare.
Verbi gratia omnis velocitas que proveniet ex proportione multiplici est incommensurabilis cuilibet alteri velocitati que non provenit a proportione multiplici quod si utraque fuerit a proportione multiplici non sequitur quod sunt commensurabiles. Hoc patet ex prima suppositione et tertia conclusione tertii capituli. Item omnis velocitas que ortur a proportione superparticulan est incommensurabilis cullibet alteri que proveniet a proportione superparticulari patet per quintam eiusdem tertii capituli.
Et ita iuxta quamlibet conclusionem de tertio capitulo una vel plures conclusiones de velocitatibus poterit demonstrari quas ut brevius 535 transeam pretermitto usque ad decimam eiusdem capituli iuxta quam elicitur talis conclusio.
so6 velocitas $H R$ velocitatas $C /$ erit: esset V
507 irrationali $C R$ rationali $H S V /$ est de: omnis $H$
s08 a proportione: A proportio $R /$ sit $^{2}$ : est $R$
s 10 irrationali: rationali $H$ / proveniens: provenit $H$
sII incommensurabilis $C R$ commensurabilis $H V$
s11-12 Non tamen... provenit om $H$
$\int_{512-13}$ irrationalis minor: rationalis $V$ s15-16 provenit: proveniente $H$ 517 eius: ad B que $V$
518 ante B add $R$ a / leuca: linea(?) $V$ 519 ante $\mathrm{C}^{1}$ add $R$ a / pertranseatur om $H$ s20 habent: habet $V$
521 ante aliis add $V$ in / tertio: secundo $C$ $s_{22}$ post et add $V$ in
${ }^{2} 23$ positis: comparatis $H$
will have been produced by an irrational ratio; and this also applies to any greater velocity which is not multiple to it.

For example, let $B$ be a velocity produced by a double ratio. I say that every velocity commensurable to it arises from an irrational ratio; and every lesser velocity arising from a rational ratio is incommensurable to it. However, not every lesser [velocity] which is incommensurable to it arises from a rational ratio, since some irrational ratio less than a double ratio may be commensurable to it and another incommensurable.

Furthermore, every velocity greater than $B$, but not multiple to it, that arises from a rational ratio is incommensurable to $B$; and every [greater velocity] that is commensurable, but not multiple, to $B$ arises from an irrational ratio. For example, if a league were traversed in one day by velocity $B$, and a league-and-a-half by velocity $C$, I say that $C$ arises from an irrational ratio. For indeed these [velocities] are related as ratios of ratios, which is clear from the fifth proposition of the second chapter and other places in the third chapter. Thus by means of the first supposition of this chapter and the propositions given in the third chapter, one who understands can demonstrate many propositions about velocities.
For example, every velocity that arises from a multiple ratio is incommensurable to any other velocity not arising from a multiple ratio. But even if each velocity were produced by a multiple ratio, it does not follow that they are commensurable. This is obvious from the first supposition and third proposition of the third chapter. Moreover, by the fifth proposition of the same third chapter, it is clear that every velocity that arises from a superparticular ratio is incommensurable to any other arising from a superparticular ratio.

And so, in a like manner, one or more [additional] propositions about velocities can be demonstrated from any proposition of the third chapter, but in order to be briefer so that I may move on, I omit all the propositions up to the tenth of the third chapter, from which latter proposition, however, such a proposition is [now] elicited.

[^32]Propositis duabus velocitatibus quarum proportio sit ignota verisimile est earum proportionem irrationalem esse et illas velocitates incommensurabiles fore. Et maxime propositis pluribus velocitatibus
${ }_{540}$ verisimile est aliquam alicui incommensurabilem esse. Et quanto plures proponuntur tanto verisimillius iudicatur quia sepe dictum est per primam suppositionem quod ita est de proportione velocitatum sicut est de proportione proportionum. Sed proposita una proportione proportionum ignota verisimile est eam incommensurabilem esse et illas
${ }_{545}$ proportiones incommensurabiles fore, quod si plures proportiones proportionum proponantur verisimillimum est aliquam esse irrationalem quia inter proportiones proportionum rariores sunt rationales sicut inter numeros sunt numeri cubici rariores sicut in illa decima conclusione tertii capituli dicebatur. Igitur de proportionibus veloci-
${ }_{550}$ tatum consimiliter est dicendum, scilicet quod propositis duabus velocitatibus et cetera, quod est propositum.

Cumque proportio quantitatum sit sicut proportio velocitatum quibus ille quantitates pertranserentur in eodem tempore vel in equalibus temporibus. Et proportio temporum sicut velocitatum quibus con-
${ }_{535}$ tingeret illis temporibus equalia pertransiri et econverso ut patet ex sexto phisicorum. Sequitur ista conclusio: propositis quibuscumque duobus acquisibilibus per continuum motum quorum proportio sit ignota verisimile est illa esse incommensurabilia. Et si plura proponantur verisimillius est aliquod alicui incommensurabile fore. Et de
${ }_{560}$ duobus temporibus contingit hoc idem affirmare et de quantitatibus continuis quibuscumque.

Verbi gratia sint duo motus inequales quorum proportio sit ignota

538 illas: istas $H$
539 fore: foret(?) $C$ / velocitatibus om $H$
540 aliquam: aliqua(?) $C$ | ante alicui scr et del $V$ ali aliquam / incommensurabilem $H S$ commensurabilem $R V$ incommensutabile $C$ / esse $C H$ fore $R V$
54 1 proponuntur tanto $R V$ proponitur tanto $C$ tanto propositum $H$
542 suppositionem: conclusionem $C$ quod $C R$; om $H V$ / proportione velocitatum: velocitate $H$
542-43 sicut est $H$ sicut $C V$ et $R$
s44 ignota: ignotam $C$ / verisimile: verisimillem $H$ / eam om $H$ / incommensurabilem om $V /$ esse: fore $R /$ illas: istas $H$

545 proportiones ${ }^{1}$ om $H$ / fore: esse $R /$ quod: et $R$
s45-46 proportiones $^{2} \ldots$ proponantur om
s 46 verisimillimum: verisimillius $H$ | est om $H /$ ante esse add $R$ earum
547 rationales: incommensurabiles $H$
548 sicut ${ }^{1}$ : ut $R /$ numeros: numerus $V /$ illa: ista $C$
548-49 decima conclusione HV; tr C decima $R$
s49 proportionibus: proportione $H$
550 est dicendum $\operatorname{tr} C /$ quod $R$; om $H$ quot $C V /$ duabus: et $C$
552 sit: est $R$
533 pertranserentur $C H$ pertransirentur $R$

When two velocities have been proposed whose ratio is unknown, it is probable that their ratio is irrational and that these velocities are incommensurable. And when more velocities are proposed, it is exceedingly probable that any [one of them] would be incommensurable to any [other of them]. And as more are proposed it must be considered even more probable, since it has frequently been said, with reference to the first supposition, that a ratio of velocities is as a ratio of ratios. But when one unknown ratio of ratios has been proposed, it is probable that it is incommensurable and that those ratios would be incommensurable, because if more ratios of ratios are proposed, it is most probable that any one of them would be irrational since there are fewer rationals among ratios of ratios, just as there are fewer cube numbers among numbers, which was stated in that [very] tenth proposition of the third chapter. For this reason, the same thing must be said about ratios of velocities, namely that when two velocities have been proposed [whose ratio is unknown, it is probable that their ratio is irrational and that these velocities are incommensurable], which has been proposed.

Now any ratio of magnitudes [or distances] would be just like the ratio of velocities with which those magnitudes [or distances] were traversed in the same time or in equal times. And a ratio of times is just like a ratio of velocities when it happens that equal distances are traversed in those times, and conversely, which is clear from the sixth [book] of the Physics [of Aristotle]. [From what has been said] this proposition follows: When there have been proposed any two things whatever acquirable [or traversable] by a continuous motion and whose ratio is unknown, it is probable that they are incommensurable. And if more are proposed, it is more probable that any [one of them] is incommensurable to any [other]. The same thing can be said of two times and of any continuous quantities whatever.

For example, let there be two unequal motions which last through an

[^33]qui durent per equale tempus. Dico quod verisimile est quod quantitates pertransite sint incommensurabiles et quelibet alia per motum ${ }_{565}$ huius acquisita vel acquisibilia.

Et si sint duo motus inequales in duratione quorum proportio sit ignota et quibus equalia acquirantur, verisimile est quod huius tempora sint incommensurabilia et de pluribus temporibus ut prius. Est dicendum igitur verisimile est quod dies et annus solaris sint tempora 570 incommensurabilia. Quod si fuerit impossibile est invenire veram anni quantitatem ut si annus duret per aliquos dies et per unam partem diei incommensurabilem diei. Et de aliis consimiliter est dicendum.
Ex predictis etiam sequitur ista conclusio. Propositis duobus motibus corporum celestium verisimile est illos esse incommensurabiles
575 atque verisimillimum est quod aliquis motus celi sit alicui motui alterius orbis incommensurabilis, et oppositum huius si foret verum non posset tamen sciri. Et hoc videtur verum maxime quia ex motibus incommensurabilibus provenit armonia ut postea declarabo.

Quo posito, scilicet quod aliquis motus celi sit alicui motui celesti
580 incommensurabilis, sequuntur conclusiones quamplurime valde pulcre quas alias ordinavi et eas intendo posterius, scilicet in ultimo capitulo, perfectius demonstrare inter quas erunt iste.
Una est: si maxima eclipsis lune semel eveniat, quod potest esse, impossibile est similem alias evenisse et quod amplius futuro eterno
${ }_{585}$ tempore sit ventura. Et semper intelligo naturaliter loquendo et sup-
563 qui: que $V$ / durent: dicuntur quod $V /$ quod $^{2}$ : quam $H$
563-64 quantitates: quantitatum $H$
564 pertransite: pertransire $C$
564-65 motum huius $C R ; \operatorname{tr} V$ motus huius $H$
567 acquirantur $H V$ acquirentur $C R$
568 temporibus: quantitatibus $H$
568-69 Est dicendum $C V$; $\operatorname{tr} H$ dictum est $R$
569 igitur verisimile est om $H$
569-70 tempora incommensurabilia $R S$; $\operatorname{tr} C$ incommensurabilia $H V$
570 veram anni $S V$ unam aliam $C$ anni verum $H$ veri anni $R$
571 ut om $H$
572 incommensurabilem: incommensurabile $C$
573 etiam om $H /$ ista: ita $C$
573-74 motibus: mobilibus $H$
574 verisimile: verisimillem $H$
575 atque om $H$ / verisimillimum: verisimille $C$ / ante sit bab $V$ esse
576 si tr $R$ ante oppositum / foret: fuerit $R$
577 posset tamen sciri $V$ tamen sciri posset $C$ posset sciri $H$ tamen posset sciri $R /$ verum maxime $\operatorname{tr} C$
579 quod...celesti: materiali sit aliter et cetera $S$ / celi: celestis(?) $V$ / celesti $C$; om $V$ celestium(?) $H$ celi $R$
580 sequuntur $R S$ sequitur CH (?) $\mathrm{V} /$ ante conclusiones hab $S$ plures / quamplurime om $S$
581 intendo: intendi $R$ / ante posterius bab $H$ prosequi / posterius scilicet CHR ; om $S$ posterius $V /$ in om $H$
582 perfectius $C R V$; om HS / erunt iste om $S$

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equal time and whose ratio is unknown. I say that it is probable that the magnitudes [or distances] traversed would be incommensurable, as would any other magnitudes [or distances] that are traversed or traversable by these motions.
Now if the ratio of times were unknown between two motions traversing equal magnitudes [or distances] in unequal times, it is probable that the times of this ratio would be incommensurable; and if there were more times [one would argue] as before. Therefore, it must be said that it is probable that a day and the solar year are incommensurable times. And if this be so, it is impossible to discover the true length of the year, for it is just as if the year should last through a certain number of days and one part of a day which is incommensurable to a day. And one can make similar remarks about other things.

From all the things which have been said, this proposition also follows: When two motions of celestial bodies have been proposed, it is probable that they would be incommensurable, and most probable that any celestial motion would be incommensurable to the motion of any other [celestial] sphere; but if the opposite of this were true, it could not be known. And this seems especially true since, as I shall declare afterward, harmony comes from incommensurable motions.
Now that I have declared that any celestial motion might be incommensurable to any other celestial motion, many very beautiful propositions that I arranged at another time follow, and I intend to demonstrate them more perfectly later, in the last chapter, among which will be these.
One [of them] is: If a perfect [or total] eclipse of the moon should occur only once-and this could happen-it is impossible that a like conjunction should have happened at another time and impossible that it happen again during an eternal future time to come. And I always understand this "naturally speaking," and have even assumed an eternity of motion and the

583 maxima: maxime $C /$ lune om $H /$ quod om $R /$ potest esse $H$ potest non esse $C R V$ pono non esse $S$
584 est similem: verisimile est $S /$ alias: vel $H$ / evenisse $H S$ evenire $C$ invenire $R$ invenisse $V$ /futuro $C H V$;om
$R$ futura $S$
s84-8, eterno tempore $\operatorname{tr} C$
585 naturaliter: in $H$
$585-86$ supposita $C V$ supponitur $H$ suppositis $R$
posita adhuc eternitate motus et suppositis principiis Aristoteles que ponit in secundo celi et in aliis locis.

Alia conclusio si duo planete quo ad longitudinem atque latitudinem semel coniungantur ad punctum numquam in perpetuum amplius coniungentur.

Alia conclusio si tres planete semel coniungantur secundum longitudinem ita ut sint simul in eodem meridiano, impossibile est eos, et si in eternum moverentur, iterum coniungi. Et si tantummodo coniungerentur perpetuis temporibus una vice.

Alia erit, nec pro nunc recito plures, in quolibet instanti necesse est corpora celestia taliter se habere quod impossibile est, et fuit, ea aliquando alias taliter se habere ita quod in quolibet instanti talis constellatio erit quod numquam fuit ante nec post erit similis in eternum sicut scriptum est: et veniet tempus quale non fuit ab eo quo gentes esse ceperunt usque ad tempus illud (Danielis duodecimo). Et hoc mediantibus istis corporibus celestibus domino disponente sicut dicit poeta: prima per ipsam quidem regit omnia causa prout vult organa sunt primi, sunt instrumenta supremi.
Multa quidem alia non minus pulcra cum istis ex eodem principio os demonstrabo paucis aliis principiis verisimillimis coassumptis quibus demonstratis. Multi errores in philosophia et in fide ex hiis poterunt impugnari sicut de anno magno quem aliqui posuerunt 36,000 annorum dicentes corpora celestia ad statum pristinum tunc reverti et aspectus preteritos ab antiquo iterum consimiliter ordinari; et cetera talia que alii non demonstrationibus sed iurgiis et garrulationibus sunt assueti reprobare. Bonum enim est ex philosophia philosophos et ex
586 suppositis $C V$; om $H R S /$ principiis: principio et $H$
586-87 Aristoteles...ponit $S$ Aristoteles et sic de aliis que ponit Aristoteles $C$ et regulariter et sic de aliis que ponit Aristoteles $H$ Aristoteles(?) et regulationes(?) sic de aliis que Aristoteles ponit $R$ Aristoteles et regularitate(?) et sic de aliis que ponit Aristoteles $V$
587 in $^{1} S$; om $C V$ multis $H$
${ }_{588}$ planete: plane $V$
589 semel $C R$ solum $H V /$ coniungantur: coniungerentur $H$ / amplius om $C$
590 coniungentur: coniunguntur $R$
591 coniungantur: coniunguntur $V$
s92 eos $H V$ eas $R$ eorum $C$

593 in om $\mathrm{C} /$ coniungi CH reconiungi
s94 coniungerentur $R$ coniungarentur $C$ coniungentur $H V /$ una vice $\operatorname{tr} C$ ante perpetuis
595 nec pro $H R$ ut nunc $C$ nec $V$
596-97 quod...habere ita om $H$ | ea aliquando alias $C R$ alias aliquando $V$
598 quod: quem $H$ / fuit: erat $H /$ ante nec post: aut $H /$ erit $^{2}$ : eris $V$
599 sicut: ut $R /$ et om $H /$ ab eo: ex $H /$ quo: quod $R$
600 ceperunt: inceperunt $C$ | Danielis $H S V$ dari $C$ David $R /$ duodecimo: $22 S /$ Et hoc om $H /$ mediantibus $H$ medietatibus $C R V$

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principles put forth by Aristotle in the second [book] of his On the Heavens and in other places.

Another proposition is: If two planets, with respect to longitude and latitude, should be conjuncted once in a point, they will never again be conjuncted.

Another proposition is: If three planets were conjuncted with respect to longitude so that they were on the same meridian, it is impossible for them to conjunct again even if they were moved eternally. Thus they were in conjunction in only one way through perpetual times.

Another proposition-I shall not relate any more for now-will be this: In any instant it is necessary that celestial bodies be so related that in any moment there will be a configuration such that there never was a similar one before, nor will there be one after in all eternity, just as it has been written in the twelfth [chapter] of Daniel: "And a time shall come such as never was from the time that nations began even until that time." And the Lord disposeth through these celestial bodies, [for] as the poet says: "The First [Cause] rules all things through the heavens as He wishes; [and the heavens] are the organs of the Prime [Being], the instruments of the Supreme [Being]."
Indeed, along with these I shall demonstrate many other no less beautiful propositions based on the same principle, [but] with a few other more probable principles assumed from those which have [already] been demonstrated. Many errors about philosophy and faith could be attacked by the use of these [propositions], as [for example], that [error] about the Great Year which some assert to be 36,000 years, saying that celestial bodies were in an original state and then return [to it in 36,000 years] and that past aspects are arranged again as of old; and other errors of this kind which people have been accustomed to reject not by demonstrations but rather by strife and verbosity. But it is better to attack philosophers with

[^34] poterunt $H$ poterunt ex hiis $R$

612 mathematica mathematicos impugnare ut golias proprio gladio feriatur ${ }_{613}$ manifestetur quoque veritas et falsitas destruatur. Hoc igitur quartum 614 capitulum fineatur.

## CRITICAL NOTES TO TEXT BY LINE NUMBERS

## I.1-13

In this opening passage Oresme proclaims that whatever the substantive disagreements between authors discussing the problem of motion, all are agreed that a velocity must arise from some kind of proportional relationship between force and resistance. Indeed all are agreed that a ratio of velocities must vary as a ratio of ratios, and conversely. In this passage Oresme is not yet using the expression "ratio of ratios" in its restricted exponential sense (after I. 2or it is so used with some frequency throughout the four chapters of the treatise), but only wishes to stress the fact that all who have any opinion agree that the respective ratios of force and resistance are related to one another just as the velocities to which they give rise.

The first of the three opinions seems identical with Bradwardine's erroneous Theory I, which assumes that a ratio of velocities "in motibus sequi excessum potentiae motoris ad potentiam rei motae" (Crosby, Brad., p. 86). This may be symbolized as $V_{2} / V_{I}=F_{2}-R_{2} / F_{I}-R_{1}$, and is, perhaps, traceable to Averroes' commentary on PbysicsIV.8.21 5a.24-21 5b.20, known as Text 7I in the Middle Ages, where Averroes recounts Avempace's in terpretation of the law pertaining to bodies falling freely in a medium and in a void. Ernest A. Moody, in his "Galileo and Avempace: The Dynamics of the Leaning Tower Experiment," Journal of the History of Ideas, Vol. I2 (1951), 186, basing his interpretation on Text 71, says that Avempace's law of falling bodies can be represented by $V=F-R$. Bradwardine, who

## 612 golias: quilibet $H$

6I3 quoque: quocumque $C /$ post destrua tur bab $R$ Amen, Amen. Explicit pul chra tractatus de velocitate motuum
613-14 Hoc...fineatur om $R$
614 fineatur: finatur $V /$ post fineatur bab $C$ in(?) autem domine misserere nostri. Explicit quartum capitulum de
proportionibus huius tractatus editus a reverendo magistro Nicolao Horesme scriptus per me et vocatur trac tatus de proportionibus proportionum et bab $H$ Explicit tractatus de proportionibus datus a magistro Nicolao Oresme
philosophy and mathematicians with mathematics, just as Goliath was struck dead by a suitable weapon, and so also truth is made manifest and falsity destroyed. This, then, ends the fourth chapter.
refutes the theory, mentions that its supporters cited Text 7I in its favor but fails to tell us whether they claimed that Text 7I showed either Avempace or Averroes to be in favor of their opinion. Bradwardine's arguments against the theory are discussed in Crosby, Brad., pp. 32-34 and p. 188, n. 84; the texts and translation appear on $\mathrm{pp} .86-93$.

Oresme offers no refutation of this first opinion and makes no further mention of it. Presumably, he would have agreed with Bradwardine's criticism that it violated Aristotle's dictum (Physics VII.s.250a.4-6) as expressed by Bradwardine: "If a given power moves a given mobile through a given distance in a given time, half that power will move half the mobile through an equal distance in an equal time." Thus if $V_{2}=4-2$ and $V_{I}=$ 2 - 1 , it follows that $V_{2} / V_{\mathrm{I}}=2 / \mathrm{I}$ and they are not equal velocities as required by Aristotle. See Crosby, Brad., p. 87.
The second opinion ( $\mathrm{I} .3-5$ ) is really Bradwardine's erroneous Theory III and is discussed by Oresme at some length in Ch. IV, Prop. I (see above, pp. 43-47, and below, 368-70).
The third view, which Oresme accepts and also attributes to Aristotle and Averroes, is a general statement to the effect that every velocity is produced or determined by some ratio of force and resistance, $(F / R)$. Despite agreement on this point, however, Oresme believes that Aristotle held the second opinion cited in the Proemium (I.3-5), which is actually Bradwardine's erroneous Theory III. In sharp contrast with Oresme, Bradwardine seems to imply that both Aristotle and Averroes accepted his function. In any event, he dissociates Aristotle and Averroes from each of the four erroneous opinions which he refuted and leaves the definite impression that his own true function is a formal, mathematical expression of rules already properly understood by Aristotle (Crosby, Brad., pp. 36-38, III).
I.17-22

The division and subdivision into classes and types of ratios could have been derived from any number of sources, since they were universally accepted in the Middle Ages. Bradwardine, for example, devotes Part I of Chapter I of his Tractatus de proportionibus to enumerating and defining the very types mentioned by Oresme (see Crosby, Brad., pp. 66-71). The five kinds of rational ratios referred to in I. 20 are enumerated by Oresme in II.2.6-20. Indeed, Bradwardine also discussed all of the other specific items which Oresme said he would take from other authors. Thus we find, in Crosby, Brad., p. 67: "what a ratio is" (I.17); "how a ratio of inequality is distinct from one of equality" (I.17-18); "how one of greater inequality is distinct from one of lesser inequality" (I.18-19); "how a rational is distinct from an irrational ratio" (I.19).

## I.22-33

Oresme's description of Chapters V and VI forms part of the overall problem of the connections between these two chapters and the first four. I have discussed this on pp. 72-81, above.
I. 35-44

Although we may write ratios of equality with different terms, such as $A / A$, $B / B, C / C$, and so on, they are all classified in one species by virtue of their equality. The statements in I.36-39 are enlarged upon in I.93-1 32 .

The immediate objective of this paragraph is to justify the subsequent operational procedures of division, augmentation, subtraction, and division of ratios where in any particular case only ratios of greater inequality (i.e., $F / R$ where $F>R$ ) are related, or only ratios of lesser inequality $(F / R$ where $F<R)$, but never ratios of greater inequality with those of lesser inequality, or ratios from either of these categories with those of equality.

The separation of ratios into these three categories and the insistence upon relating ratios only if they belong to the same category was the result of a number of factors.

Since it was axiomatic that motion arises only when the motive force is greater than the resistance (i.e., $F>R$ ), it was necessary that ratios of motion be restricted exclusively to ratios of greater inequality, thereby eliminating from consideration ratios of equality $(F=R)$ and lesser inequal-
ity $(F<R)$. In addition to this physical argument, Bradwardine, in his Tractatus de proportionibus, had furnished a mathematical justification for rigidly separating the three categories of ratios. In Chapter I, Theorem VII, he demonstrates that "no proportion is either greater or less than a proportion of equality" (Crosby, Brad., p. 81), while in Theorem VIII he shows that "no proportion of greater inequality is either greater or less than one of lesser inequality" (Crosby, Brad., p. 85). In effect, Bradwardine shows that since no exponent, $n$, can make a ratio of equality equal to or greater than any ratio of greater inequality, there is, consequently, no mathematical relationship between them that is expressible by any exponent. Thus $(1 / \mathrm{r})^{n}$, where $n$ is any exponent, is unrelatable to $p / q$, where $p>q$ and both are integers, since whatever the value of $n$, it cannot alter the value of $(1 / \mathrm{I})$; and the measure of relationship between ( $1 / \mathrm{I}$ ) and $p / q$ is inexpressible in terms of the exponent. In Bradwardine's words (Crosby, Brad., p. 81): "Nor is any proportion of greater inequality either greater or less than a proportion of equality, for, if it were, then the proportion of equality would be exceeded by a proportion of greater inequality to the extent of some proportion of greater inequality, and, since some proportion of greater inequality would be exceeded by that proportion of greater inequality in that same proportion, it follows (by Axiom 6) that the proportion of equality would be equal to that of inequality. Then (by the same axiom) it follows that a greater and a lesser quantity would be equal to each other."

Pursuing the argument, Bradwardine notes (Crosby, Brad., p. 83): "...a proportion of equality is not less than a proportion of greater inequality for the reason that, if it were, then some multiple of it would either be equal to or greater than that proportion of greater inequality. This consequence is false, for however many times equals are added to equals the proportion of the first pair to the last is no greater than the proportion of the first to the second. Instead, they all remain unchanged as equal proportions of equality." (A similar argument was given by Francischus de Ferraria in his Questio de proportionibus motuum, 1352, edited and translated by M. Clagett in The Science of Mechanics, p. 497 [English] and p. 502 [Latin].) In Theorem VIII the same difficulties are shown to obtain between ratios of greater and lesser inequality. Oresme accepts the same conclusions as Bradwardine, but offers quite different arguments (I.93-132; see also pp. 317-20, below).
Mathematical difficulties and paradoxes also served to preserve the separation between the three types of ratios. This will be borne out by our analysis of I.155-88, where Oresme illustrates the problems medieval mathematicians faced when coping with ratios of lesser inequality. The crux of
the problem lay in the fact that the De proportionibus proportionum was essentially concerned with continuous proportionality and composition of ratios. When applied to ratios of lesser inequality, however, results were obtained which seemed paradoxical and this, no doubt, served to further convince medieval mathematicians that these three categories were basically dissimilar.
In the sixteenth century the rigid separation of the three categories of ratios was maintained by George Lokert in his Tractatus proportionum, where he says:
Ad secundum dubitationem dicitur quod solum proportiones eiusdem rationis sunt proportionabiles vel comparabiles secundum rationem proportionis. Dico omnes proportiones equalitatis esse eiusdem rationis; et ita proportiones maioris inequalitatis adinvicem. Sic ergo nulla proportio equalitatis dicenda est maior minor vel equalis respectu proportionis inequalitatis; nec proportio maioris inequalitatis respectu proportionis minoris inequalitatis.

Lokert's Tractatus proportionum is the last work in a collection of treatises which he himself edited: Questiones et decisiones physicales insignium virorum: Alberti de Saxonia..., Thimonis..., Buridani..., recognitae rursus et emendatae summa accuratione et iudicio Magistri Georgii Lokert Scotia quo sunt tractatus proportionum additi (Paris, IS 18). The folios are unnumbered, but the quotation appears in the first column of the very last page.
I.45-64

Oresme, commencing a section describing operational procedures, explains what he means by "dividing" a ratio of greater inequality $B / C=A$, where $B>C$. Obviously, "mean" is taken here in its broadest signification so that after it has been assigned we have $B / C=B / D \cdot D / C$, where $B / C$, the initial ratio, has been divided into two smaller ratios of greater inequality. If $D$ is a mean proportional, then $B / D=D / C$; if not, $B / D \neq D / C$. Later, Oresme restricts the term "mean" exclusively to geometric mean (I.2I415).

Thus by dividere Oresme means the reduction of a given ratio of greater inequality into two smaller ratios of greater inequality. By composition of the two smaller ratios one can "compose," or reconstitute, the original ratio.
In his references Oresme cites Euclid Bk. V, Def. ıo, which is concerned with geometric proportionality. In his comment on Def. II, Campanus

Critical Notes to Pages $138-142$
distinguishes two senses of "mean," one of which is a geometric or mean proportional, the other a non-geometric or continuous mean improportional. The latter designation is "mean" in the broadest sense, as distinguished in the preceding paragraph (see Euc.-Campanus, p. 109). The reference to in principio septimi (I.s3) is probably to Euclid Bk. VII,Def. I9, of Euc.-Campanus, p. 169, which reads: "Cum continuatae fuerint eaedem vel diversae proportiones, dicetur proportio primi ad ultimum, ex omnibus composita." I have been unable to locate the unspecified reference to Jordanus' Arithmetica concerning the manner of dividing a ratio into $n+$ I parts, or ratios, by assigning $n$ means (I.5 5-64).

## I. $65-71$

Assuming $B>C$, we can increase $B / C$, a ratio of greater inequality, by assigning extreme terms as follows:
I. take a third term $D$, which is greater than $B$, so that $D / C>B / C$;
2. take a thitd term $E$, which is less than $C$, so that $B / E>B / C$;
3. take two terms, $D$ greater than $B$ and $E$ less than $C$, so that $D \mid E>$ $B / C$.

## I. $7^{2-74}$

In order to "subtract" a ratio $E$ from another ratio $B / C$, where $B>C$, one must assign a mean $D$ in the following manner:

1. $D>C$ and $D / C=E<B / C$, in which event $B / C \div D / C=B / D$; or
2. $D<B$ and $B / D=E<B / C$, in which event $B / C \div B / D=D / C$.

Once again Oresme uses the term medium in its widest sense. The term subtrabere signifies the division of one ratio into another. The procedure is related to that of dividing a ratio $B / C$ into two smaller ratios by assigning a mean (I.45-64), such that $B / C=B / D \cdot D / C$. In subtracting we produce first one smaller ratio-either $B / D$ or $D / C$-by assigning mean $D$. Then, by dividing (i.e., subtracting) the lesser ratio into the greater, or part into the whole, the operation is completed.

The method of subtracting ratios by assigning means is needlessly complicated. For example, according to the conditions enunciated by Oresme, to subtract a sesquitertian ratio, namely $4 / 3$, from a sesquialterate ratio, $3 / 2$, one would have to assign a mean $x$ between the terms of the greater ratio $3 / 2$, such that $3 / x=4 / 3$ or $x / 2=4 / 3$. Then one divides $3 / x$, or $x / 2$, into $3 / 2$ and
the quotient is the result of this "subtraction" of ratios. In effect, Oresme is composing ratios since $3 / 2 \div 3 / x=3 / 2 \cdot x / 3=x / 2 ;$ or $3 / 2 \div x / 2=3 / 2 \cdot 2 / x=3 / x$.
A much simpler method is mentioned by Oresme later in this chapter (I.86-89). He says that one ratio can be subtracted from another by simply dividing the denomination of one ratio by the denomination of the other. Such procedures, he says, are taught in treatises on algorism. Indeed, in his Algorismus proportionum, Part I, Rule 2, Oresme shows how "to subtract one rational ratio from another rational ratio" ("proportionem rationalem a proportioni rationali subtrahere"). Here the procedure is no longer a matter of assigning mean terms, but tather of reducing the ratios to their prime numbers and cross-multiplying. Oresme uses the same example which I arbitrarily adapted to illustrate the subtraction of ratios in his De proportionibus. Subtracting $4 / 3$ from $3 / 2$ involves cross-multiplying $3 \cdot 3=9$ and $4 \cdot 2=8$ to obtain $9 / 8$. This operation, which is actually performed by a division of ratios, is called subtraction by Oresme, who offers no clear explanation for the employment of this strange terminology. In Rule 9 of the Algorismus he insists that the actual division of two ratios must be called subtraction, not division. All this may be connected with the fact that in the Algorismus, Oresme "subtracts" (i.e., divides) tatios having exponents to the same base. In these cases there is actually subtraction of exponents. For Part I, Rule 2, of the Algorismus proportionum, see my edition in "The Mathematical Theory of Proportionality of Nicole Oresme (ca. 1320-1382)" (unpublished Ph. D. dissertation, University of Wisconsin, 1957), pp. 33233 ; for Rule 9 , see p. 339. For an earlier and complete edition of all three parts, see Maximilian Curtze, Der Algorismus Proportionum des Nicolaus Oresme. Zum ersten Male nach der Lesart der Handschrift R. $4^{\circ} 2$ der königlichen Gymnasialbibliothek zu Thorn (Berlin, 1868). In Curtze's edition, Rule 2 of Part I appears on p. 14.

## I. $75-86$

The "addition" of a ratio $E$ to another ratio $A$ requires that extreme terms be assigned. If $A=B / C$, then $E$ can be "added" to it in one of two ways:
I. assign $D$, an extreme term greater than $B$, so that $D / B=E$ and $D / B$. $B / C=D / C$; or
2. assign $F$, an extreme term less than $C$, so that $C / F=E$ and $B / C$. $C / F=B / F$.
An example, lacking in detail, is provided by Oresme for the first case,

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where a sesquialterate ratio, $3 / 2$, is added to a double ratio, $2 / 1$. Starting with $2 / 1$, an extreme term, 3 , which is related to 2 in a sesquialterate ratio, is assigned. By composing the ratios we have $3 / 2 \cdot 2 / \mathrm{I}=3 / \mathrm{I}$. This example is poorly chosen, since one can compose these two given ratios directly without assigning any extreme terms. However, we may imagine an example of the following kind. Let us "add" a sesquitertian ratio, $4 / 3$, to a quintuple ratio, $5 / \mathrm{r}$. We must assign an extreme term, $x$, such that $x / 5=4 / 3$, or $1 / x=4 / 3$. Assuming that the first way is chosen, then $\frac{62 / 3}{5}=4 / 3$, and by composition of ratios we get $\frac{62 / 3}{5} \cdot \frac{5}{1}=\frac{6^{2} / 3}{1}$.
In this manner, says Oresme, we can double, triple, quadruple, etc., any ratio. Since he cites Euclid V (I.83), it is cleat that doubling a ratio is to square it; tripling it is to cube it, and so on. According to the rule, to "double" a given ratio, say $3 / 1$, it is necessary to assign an extreme term, 9 , so that $9 / 3 \cdot 3 / \mathrm{I}=9 / \mathrm{I}$. During the Middle Ages terms like duplare, triplare, etc., were used in both an exponential and arithmetic sense.
In Part I, Rule I, of his Algorismus proportionum, Oresme explains how "to add a tational ratio to a rational ratio" ("proportionem rationalem proportioni rationali addere"). The method employed in the Algorismus is much simpler than that used in the De proportionibus. In order to emphasize the difference between the methods, I utilized an example from the Algorismus to illustrate the method of the De proportionibus (see above). In the Algorismus Oresme "adds" a sesquitertian ratio to a quintuple ratio. Having determined the prime numbers of the two ratios- $4 / 3$ and $5 / 1$, respectively-one simply multiplies the numerators and the denominators so that $4 / 3 \cdot 5 / \mathrm{x}=$ $20 / 3=62 / 3$. See Grant, "Mathematical Theory of Oresme," p. 332, or Curtze's edition of Algorismus Proportionum des Oresme, p. 14. This procedure, or method, is alluded to in I.84-86. In the Algorismus the method of assigning extreme terms is ignored in favor of direct multiplication of the numerators and denominators of the prime numbers that represent the respective ratios. By analogy with subtraction of ratios, the addition of ratios in the Algorismus is actually multiplication, but Oresme insists in Rule 9 (p. 339 of my edition) that, properly speaking, it must be called addition. As with subtraction, the rationale underlying this terminology is unclear, but is, perhaps, related to the fact that Oresme does "add" (i.e., multiply) ratios having exponents to the same base, and this does involve the addition of exponents.

We see that Oresme's De proportionibus and Algorismus represent two
different ways of performing mathematical operations on ratios. In the former treatise the primary concern is to treat ratios as continuous magnitudes which are infinitely divisible (I.237-40 and 261-71). The ratios are related almost exclusively by means of geometric proportionality, which accounts for Oresme's concern with assigning mean and extreme terms and ultimately composing the ratios. Only in this way could he develop the concept that one ratio can be an exponential part or parts of another. Representing these ratios numerically was less important, although Oresme does describe how to find the prime numbers of a ratio in II.2.21-56 and utilizes the multiplication of numerical denominations in III.64-66. But in the Algorismus, Oresme's interest was primarily in operations performed on the prime numbers representing the ratios. Hence the arithmetical operations are performed directly on the numbers and there is no need to assign mean or extreme terms.

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\text { I. } 90-136
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In a series of five notanda (I.90-2 10), or noteworthy points, Oresme wishes to show that what has already been related about ratios of greater inequality applies in a contrary way to ratios of lesser inequality.
The first notandum is concerned with the contrary behavior of ratios of greater and lesser inequality. Let $A>B>0$ and $C=A-B$. Then,

1. if $C$ is increased, (a) $A / B$ increases and (b) $B / A$ diminishes;
2. if $C$ is decreased, (a) $A / B$ diminishes and (b) $B / A$ increases.

Since Oresme has already discussed division and augmentation of ratios by assigning mean and extreme terms, it is plausible to assume that $C$ is increased or diminished by assigning means and extremes. He notes, however, that even if $C$ were increased or decreased uniformly the ratios of greater and lesser inequality would increase or diminish non-uniformly.
Having briefly described the consequent variations of ratios attendant upon variations in $C$, Oresme considers next the particular ways in which $C$ itself can be varied. An increase in $A$ or diminution in $B$ will increase $C$. Increasing $A$, and consequently $C$, to infinity will make $A / B$ infinite. Diminishing $B$ infinitely will not increase $C$ infinitely, since $C$ will then approach $A$ as an upper limit. However, it will make $A / B$ infinitely large. Thus $A / B$ can be infinitely increased by infinitely increasing $A$ or infinitely diminishing $B$. But which of these methods of infinitely increasing $A / B$ is naturally possible-that is, in accordance with the laws of nature as distinct
from supernatural action? Oresme, in general agreement with Aristotle and the scholastic tradition, rejects the existence of an infinitely large magnitude and consequently rules out an infinite increase of $A$. Only by an infinite diminution of $B$, the lesser term, is an infinite increase of ratio $A / B$ naturally possible. Therefore, if an infinite velocity were possible, it must be produced by an infinite diminution of the resistance and not by an infinite augmentation of the force. Unfortunately, Oresme breaks off the discussion, promising to take up the matter elsewhere. Whether this promise was fulfilled is unknown to me.
It may seem paradoxical that, after excluding all infinite increase, Oresme does consider how a velocity may be infinitely increased. It must be realized, however, that the increase in velocity does not arise from a direct addition to some magnitude called "velocity," but rather that any infinite velocity is the result of an infinite diminution of a magnitude-namely some resistance or mobile.
Oresme also considers how $C$ can be diminished (I.119-25). This can happen in two ways: (1) by diminishing $A$, the greater term, or ( 2 ) increasing $B$, the smaller term. Although $C$ can be infinitely diminished, quantity $A$, if diminished, only approaches $B$; and $B$, if increased, only approaches $A$. From this it follows that $A \mid B$ and $B \mid A$ both converge to a ratio of equality. But however close they approach, they can never become ratios of equality as long as something remains of $C$, i.e., as long as $C>0$. Therefore, $A / A$ always exceeds $B / A$; and $B \mid B$ is always exceeded by $A / B$ since $C=A-B$, and $C$ will never be diminished to zero.
This discussion is intelligible only if we understand that $C$ is diminished solely by assigning mean terms between $A$ and $B$. This being the case, ratios of greater and lesser inequality can never become ratios of equality ( $\mathrm{I} .36-39$ ), since mean terms can be assigned in infinitum between $A$ and $B$. This appears to be Oresme's justification for his earlier remarks (I.39-44) that ratios of lesser and greater inequality can bear no ratio to ratios of equality; and a fortiori ratios of greater inequality are not relatable to those of lesser inequality. Thus if it were possible to convert any ratio of greater inequality to one of equality or lesser inequality, it would be possible to relate them in some proportional way, since we could then describe the precise operational procedure to achieve the transformation. But there is no way to accomplish this by the method of assigning mean and extreme terms, and Oresme excludes such interrelations between the three categories of ratios.
Indeed this entire procedure is analogous to that expressed in Bk. V,

Def. 4, of Euclid's Elements, which reads, "Magnitudes are said to have a ratio to one another which are capable, when multiplied, of exceeding one another" (Heath's translation of Euclid's Elements, Vol. 2, 114). As Heath observes, this definition not only says that magnitudes forming a ratio must be of the same species, but seems "to exclude the relation of a finite magnitude to a magnitude of the same kind which is either infinitely great or infinitely small..." (Vol.2, 120). Although this particular definition is not included in Campanus' edition of the Elements ${ }^{\text {I }}$ published in 1546 , Oresme's criterion for relating ratios is similar in the sense that mathematical relations between ratios from the three categories are excluded if, after assigning any number of mean or extreme terms, one is unable to alter any ratio belonging to any one of the three categories so that it is made equal to, less than, or greater than some ratio from the other two categories. Convinced that he has demonstrated that no series of finite operations-i.e., no series of mean or extreme terms-can transform ratios from one category to another, Oresme feels justified in asserting that there is no possibility of a mathematical relationship between these types of ratios.
Although Oresme's treatment is more extensive, Bradwardine in a very brief passage, reveals even more clearly how Bk. V, Def. 4, may have served to support the rigid separation between the three categories of ratios. Thus Bradwardine says: "It can likewise be shown that a proportion of equality is not less than a proportion of greater inequality for the reason that, if it were, then some multiple of it would either be equal to or greater than that proportion of greater inequality. This consequence is false, for, however many times equals are added to equals, the proportion of the first pair to the last is no greater than the proportion of the first to the second. Instead,

1 Although omitted from Campanus' edition, Oresme could have read this important definition in at least three places: ( 1 ) Gerard of Cremona's translation of Euclid's Elements; (2) Gerard's translation of An-Nairizi's Commentary on the First Ten Books of Euclid's Elements; (3) the Institutiones of Cassiodorus. I quote the texts of Bk. V, Def. 4, as found in each of these reatises:
(r) "Quantitates quarum quedam ad alias proportionales esse dicuntur sunt quarum quasdam cum multiplicantur super alias addere possibile est."-Gerard of Cremona's translation, MS Paris, BN lat. 7216 , fol. 28 r.
(2) "Quantitates, inter quas dicitur esse proportio, sunt, quarum possibile, cum multiplicantur, alias addere."-Euclidis Opera Omnia, eds. Heiberg and Menge, Vol. 9: Supplementum, Anaritii in Decem Libros Priores Elementorum Euclidis Commentarii, ed. Curtze, p. 16I.
(3) "Proportionem vero ad se invicem magnitudines habere dicuntur, quae possunt sese invicem multiplicatae transcen-dere."-Cassiodori Senatoris Institutiones, ed. Mynors, p. 17 I .
I wish to thank Professor John Murdoch of Harvard University for kindly furnishing these quotations.
they all remain unchanged as equal proportions of equality" (Crosby, Brad., p.83). Thus, if we have $A / A \cdot A / A \cdot A / A \cdots A / A$, and proceed to compose a ratio of the first and last terms, we still get $A / A$. Consequently, however many ratios of equality are taken, i.e., $(A / A)^{n}$ where $n$ is any integer, it is impossible to make $(A / A)^{n}$ equal to or greater than $A / B$, where $A$ $>B$. It seems, then, that Bk. V, Def. 4 , has been taken in an exclusively exponential, rather than arithmetic, sense. For Bradwardine, as for Oresme, two unequal ratios are exponentially relatable only where it is possible to make one of them greater or smaller than the other. This is always done by assigning geometric mean or extreme terms.

An interesting dissent from Oresme's interpretation is presented by Alvarus Thomas in his Liber de triplici motu (Paris, 1509 ; leaves unnumbered). Alvarus notes (sig. c.4r, c.2): "If any ratio of greater inequality were diminished without interruption toward a ratio of equality it is necessary for it to pass continuously and successively through an infinite number of ratios smaller than itself. Thus if a ratio of $8 / 4$ arrived at a ratio of equality by continually diminishing 8 toward 4 , it [i.e., ratio $8 / 4$ ] must pass through all ratios of which it is composed, and these are infinite..." ("...si aliquis proportio maioris inequalitatis diminuatur usque ad proportionem equalitatis necesse est ipsam continuo successive transire per infinitas proportiones minores ea. Ut si proportio 8 ad 4 deveniat ad proportionem equalitatis per diminutionem ipsorum 8 usque ad 4 necesse est eam transire per omnes proportiones ex quibus componitur talis proportio 8 ad 4 , et ille sunt infinite...").

We see that Alvarus implies, in contrast to Oresme, that a ratio of inequality can become a ratio of equality. In the fifth supposition of Chapter V of Part 2 (sig. d.3r, c.1), Alvarus says:

The latitude of a ratio of greater inequality is successively diminishable to zero degree. This is shown, in the first place, because the greater extreme of a ratio of greater inequality is capable of being successively diminished to equality with the lesser extreme. And in such a diminution a ratio of greater inequality is successively diminished to zero degree.... In the second place .... a velocity of motion corresponds to a magnitude of a ratio with respect to equality. But this velocity of motion is diminishable continuously [and] successively to zero degree, and, therefore, the latitude of the ratio corresponding to this [velocity of motion is diminishable] into equality.
(Latitudo proportionis maioris inequalitatis est successive diminuibilis usque ad non gradum. Probatur tum primo quia maius extremum proportionis maioris inequalitatis successive valet diminui usque ad equalitatem minoris extremi, et in
tali diminutione proportio maioris inequalitatis successive diminuitur ad non gradum.... Tum secundo...velocitas motus correspondet magnitudini proportionis quo ad equalitatem. Sed ipsa velocitas motus est diminuibilis continuo successive usque ad non gradum, igitur et latitudo proportionis sibi correspondens in equalitate.)

On the basis of these passages it is likely that Alvarus Thomas would have found no difficulty in relating ratios of greater inequality with ratios of equality. The argument that a velocity moving through successively diminishing degrees will reach zero degree is significant because a zero velocity would occur when $F=R$, and this happens when the ratio of force and resistance has been transformed from a ratio of greater inequality to one of equality.
Thus Alvarus, who accepted Bradwardine's function, seems to hold that one can relate ratios of greater inequality with those of equality. In effect, then, he sides with the opponents of Bradwardine's function who insisted that such ratios were properly relatable if one related their numerical denominations.
Two later writers who took issue with Bradwardine and Oresme were Giovanni Marliani and Alessandro Achillini. Marliani mentions Bradwardine specifically, ${ }^{2}$ and Achillini cites both Bradwardine and Oresme. ${ }^{3}$ After correctly ascribing to Bradwardine the opinion that no proportional relationship could obtain between ratios of greater and lesser inequality, Achillini asserts that Oresme believed that any proportion relating ratios of greater and lesser inequality would be infinitely great ("Thomas Baduardinus [sic!] dixit non esse proportionem inter maioritatem et minoritatem; Nicolaus autem Orem dixit infinite magnam ibi esse proportionem"). Presumably, this is a reference to I.36-39 and I.129-32 of the De proportionibus. Actually, Oresme does not say that the ratio relating a ratio of greater inequality and one of lesser inequality-or the ratio between either of these and a ratio of equality-is infinitely great, but he does insist that a ratio of greater inequality will always remain greater than one of lesser inequality, or equality, however much their difference is diminished in infinitum by assigning means.

## I.137-S4

The second notandum asserts that there exist converse relations between

[^35]Critical Notes to Pages 148 -I ${ }^{2} 2$

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ratios of greater and lesser inequality. Thus if $A>B>C$, then $A / C>$ $A \mid B$ and $C \mid A<B / A$. The reference to Jordanus (I.142) is to the definitions in Bk. II of his Aritbmetica, where the definition of denominatio is as follows: "Denominatio dicitur proportionis minoris quidem ad maiorem pars vel partes minores que in maiore superfluunt. Similes sive una aliquando(?) eadem dicuntur proportiones que eandem recipiunt denominationem maior vero que maiorem et minor que minorem" (Cambridge, Pe terhouse 277, Bibliotheca Pepysiana 2329). My microfilm copy does not reveal any folio numbers, but the definition above appears on $3 \mathrm{v}, \mathrm{c} .2-4 \mathrm{r}$, c.I. All references to Jordanus' Aritbmetica will be to this manuscript, hereafter referred to as Pepys 2329 (see above, p. 127, for a description of this codex). The Aritbmetica covers folios 1-45. The other references may be found in Euc.-Campanus.

## I. $155-73$

If we have three continuously proportional terms, $A, B$, and $C$, and $A>$ $B>C$, then $A / C=A \mid B \cdot B / C$ and $A / C=(A \mid B)^{2}$. Oresme now wishes to show a converse correspondence in ratios of lesser inequality, and in so doing falls into a rather remarkable error. He says, incorrectly, that just as $A / C=(A / B)^{2}$, so also $B / A=(C / A)^{2}($ I.16 $6-66)$; and notes correctly that if $A \mid B<A / C$, then $B|A>C| A$ (I.167-69). But the reciprocal correspondence that Oresme seeks is attained only at the expense of mathematical propriety, for although $A / C=(A / B)^{2}$, where $A / C$, the greater ratio, is composed of $(A \mid B)^{2}$, it cannot be said that $C \mid A=(B \mid A)^{2}$, where $C / A$ is a ratio "composed" of $(B / A)^{2}$. Such an assertion would be an abuse of terms for Oresme, since $B|A>C| A$, and it is senseless to maintain that the lesser ratio is composed of the square of the greater ratio. This seemed absurd and, as a consequence, where the mathematics required that $C / A=(B / A)^{2}$, Oresme insists that $B / A=(C / A)^{2}$. The difficulty is brought out in Oresme's example where the three continuously proportional terms are 4,2 , I. Just as $4 / 1=(4 / 2)^{2}$, so conversely, $2 / 4=(1 / 4)^{2}$. While this kept intact the converse correspondence, which Oresme was anxious to preserve, and also made the greater ratio equal to the square of the lesser ratio, it is of course a false equality, since $2 / 4 \neq(1 / 4)^{2}$. But Oresme would not admit that $1 / 4=(2 / 4)^{2}$ for two reasons. First, it meant that a lesser ratio contained, or was composed of, a greater ratio, and, second, it violated the sort of converse correspondence which Oresme believed to exist between ratios of greater inequality and those of lesser inequality.

It seems that Oresme was guided by physical and geometrical considerations centering on the dictum, "The whole is greater than the part." He appears to have applied such reasoning in the arguments outlined above. As we shall see, this is not the last of Oresme's difficulties with ratios of lesser inequality (I.174-88; see also below, on this page).

Anneliese Maier asserts (Vorläufer Galileis, p. 91, n. 20), without citing any source, that scholastics made a distinction between "composing" and "producing" ratios of lesser inequality. Thus, in our previously cited example, $1 / 4$ cannot be "composed" (componi) of $1 / 2$ and $2 / 4$ because the lesser would be composed of greater ratios. But $1 / 4$ can be "produced" (produci) from greater ratios. Oresme, it should be noted, makes no such distinction.

## I. 174 - 88

In the fourth notandum, Oresme shows that the procedures for increasing and diminishing ratios of lesser inequality are the converse of those which increase and diminish ratios of greater inequality. It is not surprising that Oresme here repeats the same kind of error that we already saw in the third notandum (I.155-73; see also p. 321, above).

Whereas ratios of greater inequality are diminished by assigning mean terms, ratios of lesser inequality are diminished by assigning extreme terms. In an example, Oresme diminishes $4 / 8$ by assigning 2 as an extreme term beyond 4. We now have $2 / 4 \cdot 4 / 8=2 / 8<4 / 8$. Oresme's remark that $2 / 8$ is less than $4 / 8$ and is, indeed, half of it, seems, at first glance, to call for no comment. But after assigning another extreme term, namely 1 , to produce $1 / 8$, which is "the third part of a subdouble, namely 4 to 8 " (I.182), there can be little doubt that for Oresme $1 / 8=(4 / 8)^{1 / 3}$ and in the previous example $2 / 8=(4 / 8)^{1 / 2}$. The correct formulations, namely that $1 / 8=(4 / 8)^{3}$ and $2 / 8=$ $(4 / 8)^{2}$, would have seemed absurd to Oresme since he would have had to admit that a lesser ratio can be composed of a greater ratio that is a part of the lesser ratio.
The same problems arose for Oresme when he increased ratios of lesser inequality by assigning mean terms (I.184-88). Given $2 / 8$ and assigning mean 4, we obtain, in Oresme's scheme, $4 / 8=(2 / 8)^{2}$. Assigning two mean proportionals would cube the given ratio of lesser inequality as, for example, when means 2 and 4 are assigned between I and 8 , and Oresme would say-by analogy with the previous illustration-that $4 / 8=(1 / 8)^{3}$.

## I. 189-97

On the basis of the preceding discussion, Oresme says that we can understand the steps involved in subtracting, dividing, and doubling (i.e., squaring) ratios of lesser inequality. Oresme, however, leaves it to the ingenuity of the reader to puzzle out the specific steps of such operations. This is not surprising in light of his general mishandling of ratios of lesser inequality.
In reconstructing any of these operations, one must proceed in accordance with Oresme's general dictum that operations with ratios of lesser inequality are the converse of those for greater inequality (I.90-92). Let us, briefly, examine how subtracting ratios of lesser inequality might be converse to the "subtraction" of ratios of greater inequality (for the latter, see I.72-74 and above on pp. 313-14). The reader will recall that in my interpretation the essential steps in "subtracting" ratios of greater inequality involved assigning a mean term and then dividing the greater by the lesser ratio. The division was actually effected by composing the ratios. In coping with ratios of lesser inequality, we must, it seems, assign an extreme, rather than a mean, term and directly compose the ratios by multiplying, rather than dividing, them. This would seem to be the extent of the converse operations.
An example may illustrate the basic procedures. First I shall carry out the steps for ratios of greater inequality and then for lesser inequality. Let us subtract $4 / 3$ from $5 / 2$. Assign a mean term, $x$, such that $x / 2=4 / 3$. Therefore, $x=8 / 3$, and $8: 3 / 2=4 / 3$. Dividing the ratios, we get $5 / 2 \div 8: 3 / 2=5 / 8: 3$, which is actually arrived at by composing the ratios wherein $5 / 2 \cdot 2 / 8: 3=$ 5/8:3.

Now let us invert these ratios in order to illustrate the subtraction of ratios of lesser inequality. Subtract $2 / 5$ from $3 / 4$ (we note immediately that by inversion $5 / 2$, previously the greater ratio, becomes $2 / 5$, the smaller of the ratios of lesser inequality). Assign an extreme term, $x$, so that $x / 3=2 / 5$. Therefore, $x=6 / 5$ and $6: 5 / 3=2 / 5$. Multiplying (not dividing as with ratios of greater inequality) the ratios we obtain $6: 5 / 3 \cdot 3 / 4=6: 5 / 4=6 / 20$. Here the composition of ratios is a direct step.

Another operation mentioned by Oresme is that of doubling, tripling, etc.-i.e., squaring, cubing, etc.-ratios of lesser inequality, which must be the converse of squaring, cubing, etc., ratios of greater inequality. Here, the same kind of difficulty would arise, as already mentioned (I. iss-88; see also pp. 321-22, above). It would always be the case that the ratio that is the square of another is itself less than that of which it is the square, just
as, for example, $2 / 8=(4 / 8)^{2}$ where $2 / 8<4 / 8$. Oresme, unable to accept this, insists that $2 / 8=(4 / 8)^{1 / 2}($ I. $178-83)$ and is, consequently, committed to the further absurdity that $4 / 8=(2 / 8)^{2}$.

## I.198-210

Having shown in the second notandum (I.137-54) that if $A>B>C$, then $A / C>A / B$ and, conversely, $C|A<B| A$, Oresme, in the fifth notandum, insists that the same relationships hold between ratios of ratios of greater and lesser inequality (see p. 49, above, for the expression "ratio of ratios"). In his discussion, Oresme commits the same kind of error as in the third and fourth notanda (I.155-88; see also pp. 321-22, above). Thus he says that if $4 / \mathrm{I}=(2 / \mathrm{I})^{2 / 1}$, then $\mathrm{I} / 4=(1 / 2)^{1 / 2}$; it is clear that Oresme again assumes that because $1 / 4<1 / 2$ it must follow that $1 / 4$ is an exponential part of $1 / 2$ in this case the square root of $\mathrm{t} / 2$. This type of converse relationship extends to incommensurables. If a triple ratio, $3 / \mathrm{I}$, is incommensurable to a double ratio, $2 /$, then similarly a subtriple ratio, $1 / 3$, is incommensurable to a subdouble ratio, $1 / 2$. Since Oresme is concerned here with ratios of ratios, the incommensurability of which he speaks is an exponential one where $3 / \mathrm{I} \neq$ $(2 / 5)^{n}$, so that $1 / 3 \neq(1 / 2)^{1 / n}$. In both cases, $n$ is some integer.
With the five notanda completed, and apparently satisfied that he has demonstrated the converse relationships obtaining between reciprocal and corresponding ratios of greater and lesser inequality, Oresme dismisses any further consideration of ratios of lesser inequality. It is hereafter understood that any propositions demonstrated for ratios of greater inequality are true conversely for the reciprocal ratios of lesser inequality.

## I.218-22

The second and third kinds of mean improportionals are respectively $s /\left(2 s^{2}\right)^{1 / 2} \cdot\left(2 s^{2}\right)^{1 / 2} / 3 s$ and $s: 2 / s \cdot s /\left(2 s^{2}\right)^{1 / 2}$, where $s$ is the side of the square.

## I.224-25

In Oresme's example with the irrational mean proportional, we have $25 /\left(2 s^{2}\right)^{1 / 2} \cdot\left(25^{2}\right)^{1 / 2} / s$.

## I. 227-32

Oresme uses the terms pars and partes in an exponential sense in the $D e$ proportionibus. See pp. 25-26, above.

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\text { I. } 233-36
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The four definitions that Oresme says could be derived from what has already been said have, in effect, already been enunciated. For dividing a ratio, see I.45-64; increasing a ratio, I.65-71; adding and doubling (i.e., squaring) ratios, I.75-89; mean, I.211-26; part, I.227-32. Later (I.257s8), Oresme actually refers to the "first definition," for which the reader must turn to I.45-64 since he does not formally give a definition of dividing ratios.

$$
\text { I. } 237-43
$$

What does Oresme mean by the terms petitio (I.237) and suppositio (I.243)? Quite likely, a petitio signifies a postulate and a suppositio, a hypothesis. This usage is found, for example, in Thomas Aquinas' Exposition of the Posterior Analytics, translated by Pierre Conway, O.P. (Quebec, 1956), p. 112, c.I. Campanus of Novara used the term petitiones to designate the Euclidean postulates in the Elements (Euc.-Campanus, p. 3).
But what does Oresme understand by petitio (postulate) and suppositio (hypothesis)? It is probable that Oresme was using the term petitio in the sense of the Euclidean postulates. Just as the latter describe the fundamental character of geometrical space, so Oresme's petitiones tell us something fundamental about the nature of continuous (geometrical) and discontinuous (arithmetical) magnitudes-namely that between any two unequal continuous quantities an infinite number of means can be assigned, and that only a finite number of means are assignable between any two unequal numbers. The suppositiones, on the other hand, are narrower in scope, confined exclusively to ratios as such. Furthermore, the first three suppositiones, at least, are propositions that had been discussed and, in the case of the third supposition (I.247-53), even proved elsewhere (Euclid Bk. X.s and 6). Thus, in part, a suppositio is a proposition that is assumed to be true without proof, but proved elsewhere or at least capable of proof.
In Aristotle's Posterior Analytics I.ıo.76b.26-34, there is a significant passage distinguishing between a postulate and a hypothesis(see Sir Thomas Heath's translation in his Euclid's Elements, Vol. I, 1 18-19). What influence, if any, this passage may have had on Oresme is unknown since he makes no reference to it. However, Oresme's usage seems to conform to Aristotle's most general description of a hypothesis as that "from the truth of which, if assumed, a conclusion can be established" (ibid., p. 119).
I. 25 I-53

Euclid Bk. X.s and 6, along with the first and second definitions of the tenth book, apply to the third supposition. Campanus' comment on Euclid Bk. V, Def. 3 (Euc.-Campanus, pp. 103-4), seems to be relevant to the first and second suppositions. The "principles of the seventh book" (ex principiis septimi) referred to by Oresme are divided by Campanus into definitions, postulates (petitiones), and axioms (communes animi conceptiones) for a grand total of thirty-seven principles (Euc.-Campanus, pp. 168-69). All apply to numbers and ratios of numbers, but which of these Oresme would have the reader single out and to which suppositions they are to be applied is a puzzle.

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\text { I. } 255-60
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The fourth supposition is merely an extension of the first petitio (I.237-40). Since between any two unequal quantities an infinite number of means can be assigned, it follows that by relating any two such quantities in a ratio, we can divide the ratio in infinitum by assigning an infinite number of means.

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\text { I. } 26 \mathrm{I}-7 \mathrm{I}
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See above, pp. 36-37, for a discussion of these lines. In I.268-71, Oresme explains that in any given ratio relating two continuous magnitudes, say $A$ and $B$, where $A>B$ and $A-B=C$, it is not the excess, $C$, that is divided by the successive means-although, of course, $C$ will be dimin-ished-but rather the entire ratio is divided. Thus if some mean, $D$, were assigned, we will have divided $A / B$ into $A / D \cdot D / B$; if two means, $D$ and $E$, were assigned, we divide $A / B$ into three parts $A / D \cdot D / E \cdot E / B$. But however many means are assigned, each one serves to link two successive ratios.

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\text { I. } 272-332
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Oresme offers a lengthy justification for the fifth supposition. At the outset he asserts that the supposition is confirmed by Campanus' comment on Euclid Bk. V, Def. ir. The definition, as given by Campanus, asserts that when four magnitudes, say $A>B>C>D$, are continuously proportional, then $A / D=(A / B)^{3}$. Campanus says that Euclid did not extend
continuous proportionality to more than four terms because natural things do not exceed three dimensions. He then adds this comment:

Denominatio autem proportionis duarum quantitatum quibus nullum interponitur medium habet naturam lineae. Earum vero quibus interponitur unum medium in continua proportionalitate habet naturam superficiei eo quod fit ex multiplicatione denominationis duarum primarum in se. Omne autem quod ex multiplicatione lineae in lineam producitur naturam habet superficiei; si in se quidem quadrati; si vero in alteram parte altera longioris. Sed proportionis earum quantitatum denominatio quibus in continua proportione duo media interponuntur naturam habet solidi quia provenit ex multiplicatione denominationis duarum primarum primo in se ex qua multiplicatione producitur superficies deinde in productum ex qua multiplicatione provenit solidum sive corpus. Omne etenim quod ex multiplicatione lineae in superficiem producitur crescit in solidum. Est ergo ac si diceret proportio duarum quantitatum est simplex intervallum et habens naturam simplicis dimensionis ut lineae; proportionalitas autem triumest duplex intervallum et habens naturam duplicis dimensionis ut superficiei; proportionalitas autem quatuor est triplex intervallum et habens naturam trinae dimensionis ut solidi. Et quia dimensiones ulterius non procedunt ideo non diffinivit proportionem contentam inter extremos proportionalitatis in quinque terminis, aut pluribus, constitutae; vel non diffinivit proportionem in his quia earum proportio habetur ex praedictis diffinitionibus (Euc--Campanus, p. 108).

We have already seen what Oresme means by the mediate numerical denomination of an irrational ratio (I.281-85; see also pp. $3 \mathrm{I}-33$ ). In order that the fifth supposition be valid, it must be possible to divide ratios in such a manner as to produce rational ratios, irrational ratios with mediate numerical denominations, and, finally, irrational ratios lacking mediate numerical denominations-i.e., irrational ratios that have irrational exponents (I.286-332; see also pp. 33-37, above). Ratios of the first two types were easily representable, but those in the third category were mathematically and even verbally inexpressible-except in a negative way (see above, pp. 35-36). But Oresme was convinced of their existence and deemed it necessary to demonstrate the plausibility of this belief for his readers. Indeed it was essential to do this, for otherwise the fifth supposition would not be applicable to all manner of logically conceivable ratios. Oresme undertakes to argue the likelihood of their existence in I.286-308, but prior to this he seeks to show that the existence of the three types of ratios is compatible with the passage quoted above from Campanus. It is not at all obvious how Campanus' remarks are relevant, but the following interpretation is offered to render the connection intelligible.

In linking ratio with dimension, Campanus holds that when we have a ratio of two quantities, say $A / D$ with $A>D$, and no mean is assigned, the denomination of the ratio $A / D$ will have the nature of a line. If one mean is assigned, say $E$, then we have $A / E \cdot E / D$ where the denomination of each ratio has the nature of a line. Multiplying one length by the other produces a surface, from which Campanus concludes that a ratio with one mean assigned has the nature of a surface. Should two means be assigned, say $E$ and $F$, we obtain $A / E \cdot E / F \cdot F / D$. Multiplying $A / E \cdot E / F$ produces a surface and the surface multiplied by $F / D$, representing a line, will produce a third dimension or solid. Thus when two means are assigned, the denomination of $A / D$ has the nature of a solid.
In all this Campanus' purpose is to explain why Euclid did not extend his discussion of proportionality beyond four continually proportional terms. ${ }^{4}$ In the final four lines of the Latin passage quoted above, Campanus says: "And because there are no dimensions beyond [a solid] he [Euclid] did not define a proportion constituted of five or more terms between the extreme terms of a proportionality; or, he did not define a proportion in such terms because their proportion can be had from the definitions stated previously." Oresme's interest in this entire passage is connected with his desire to show that ratio $A / D$ may be any one of the three types of ratio that he will subsequently enumerate (see above, pp. 31-37), namely (i) a rational ratio immediately denominated by a number, (2) an irrational ratio mediately denominated by some number and therefore part or parts of some rational ratio (i.e., $A / D=(G / H) p / q$ where $p / q$ is a rational exponent), and (3) an irrational ratio with no mediate numerical denomination (i.e., $A / D=(G / H)^{p / q}$ where the exponent $p / q$ is irrational). Whatever the type of ratio, the denominations can have the nature of a line, surface, or solid depending on how many means are involved (i.e., none, one, or two). For example, if after assigning one mean the two ratios $A / D \cdot D / E$ are irrational (it is irrelevant whether the exponents are rational or irrational), their multiplication will still produce the nature of a surface. Oresme's motive, then, is to state, at the outset, that operations of the kind described by Campanus can be performed, in principle, with each of the three types of ratios that he will enumerate ( $\operatorname{I} .27^{8-302}$ ) and that in each case the nature
${ }^{4}$ In the modern edition of Euclid (where Campanus' Def. in is Def. io), this definition does go beyond four terms. Heath, in his translation of Euclid's Elements (Vol. 2, 1 14), renders it as follows: "When four
magnitudes are 〈continuously〉 proportional, the first is said to have to the fourth the triplicate ratio of that which it has to the second, and so on continually, whatever be the proportion."
of a line, surface, or solid can be produced. Therefore, ratios in the category of irrational ratios having no mediate numerical denomination can also have natures corresponding to lines, surfaces, and solids, and this, in some sense, is probably intended by Oresme to lend credence to their existence in the mathematical continuum of proportionality.

It is overwhelmingly improbable that Campanus, when he formulated the passage in question, had in mind the distinctions basic to Oresme's approach. We have already seen how Oresme unjustifiably interpreted Campanus' comments on Euclid Bk. V, Def. 16, in support of his own position (see above, pp. 37-38) and the present instance is simply another attempt to make his novel approach compatible with the ideas and concepts of a prominent and widely read mathematician like Campanus.

Oresme now moves to render plausible his contention that there do exist irrational ratios that have no mediate numerical denomination and are incommensurable to every rational ratio (I.286-308; see also pp. 33-37, above). The argument is based on Campanus' comment on Euclid X. 9 (Euc.-Campanus, p. 252), which says that if quantities $A$ and $B$ constitute some whole, $C$, then "if $A$ and $B$ are incommensurable, $C$ will be incommensurable to each of them" ("si autem, $A$ et $B$ sint incommunicantes erit $C$ incommunicans utrique earum"). Now by supposition five a ratio can be divided into two ratios, say $A$ and $B$, that are mutually incommensurable, from which it follows, by Campanus' comment, that the whole ratio $C$ will be incommensurable to each of them. Once again, where Euclid and Campanus take $A, B$, and $C$ as quantities, Oresme reinterprets each quantity as a "ratio of quantities" (see p. 30). Thus ratios $A$ and $B$ will each be incommensurable to ratio $C$, the whole. In symbolizing this, we may say that $A \neq C C^{p l q}$ and $B \neq C^{r l s}$, where $q>p, s>r$, and $p / q, r / s$ are ratios of integers; or, expressed another way, it is possible to represent this as $A=$ $C^{p / q}$ and $B=C^{r / s}$, where $p / q$ and $r / s$ are irrational exponents. If $C$, the whole ratio, were a double ratio, Oresme says (I.289-94) that there is some ratio, say $B$, that is no part of a double ratio but is incommensurable to it and to any ratio commensurable to a double ratio-i.e., to all ratios in the series $(2 / \mathrm{I})^{p}$, where $p$ is any integer or improper fraction. And if it is incommensurable to $(2 / \mathrm{I})^{p}$, it may also be incommensurable to $(3 / \mathrm{I})^{p}$, and so on indefinitely (see above, pp. 36-37, for a further discussion of this argument).

A final argument for the existence of irrational ratios incommensurable to any rational ratio-i.e., irrational ratios with irrational exponents-is found in I.302-23. Here, again, a statement by Campanus is adapted by

Oresme for use in the context of ratios of ratios. Campanus remarks in his comment on Euclid Bk. V, Def. $16{ }^{5}$ (Euc.-Campanus, p. II I), that there are an infinite number of irrational ratios whose denominations are unknowable (see above, pp. 37-38, for a discussion of Oresme's misuse of this remark and p.37, n. $\rho$ o, for the relevant passage from Campanus' comment). Oresme then proves that if Campanus' assertion is true, there must be irrational ratios with irrational exponents since all those with rational exponents are commensurable to some rational ratio, and, consequently, their denominations-i.e., their exponents-are knowable. A summary and explication of the proof now follows:
I. Assume that any irrational ratio is commensurable to some rational ratio. The consequent of this is that the denomination of any irrational ratio is knowable because the exponent of the irrational ratio must itself be rational.
2. In the proof itself, however, Oresme assumes the opposite of the consequent to be true in order to prove the consequent. That is, he assumes that an irrational ratio, $B$, is commensurable to some rational ratio $A$, but that the denomination of $B$ is unknowable.
3. It is also assumed that when a ratio is knowable its denomination is knowable.
4. Oresme assumes that $B=A^{C}$ where ratios $A$ and $C$ are knowable. $A$ is knowable because it is a rational ratio ex bypothesi ( I .313 ) and $C$ because it relates $A$ and $B$ which are commensurable ex bypotbesi. The antecedent, which asserts that $A$ and $C$ are knowable ratios, has now been established. In addition, by (3), their denominations are knowable.
5. But if $A$ and $C$ are knowable ratios, it follows, by the ninth supposition of Ch. IV of the De proportionibus (IV.59-66), that ratio $B$ must be knowable. Hence the opposite of the consequent in ( 2 ) is false and the consequent in ( I ) is true.

We see that all irrational ratios that are commensurable to some rational ratio have denominations that are theoretically knowable. Thus if $B=A^{C}$, the whole irrational ratio is $A^{C}$ and its mediate numerical denomination is the exponent $C$, which is rational. But Campanus states that there are an infinite number of irrational ratios whose denominations are unknowable, which in the context of ratios of ratios means that $C$, the exponent, is irrational.
Oresme notes (I.324-26), however, that even if a rational exponent, $C$,
5 This definition corresponds to Bk. V, Def. 17, in ibid., pp. IIs, 136.

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of an irrational ratio, $A^{C}$, is knowable, it might not as yet be known. He then raises a hypothetical objection to Campanus (I.326-32). Just as there are irrationals with denominations that are as yet unknown, so also are there rational ratios whose denominations are yet unknown (presumably because there are an infinite number of rational ratios). But since rational ratios are all potentially capable of being known, perhaps all irrational ratios are similarly potentially knowable and denominable even though they are at present unknown. But Oresme denies this by interpreting Campanus to mean that it is wholly impossible to know certain irrational ratios because they have no numerical denominations whatever-i.e., they are denominated by irrational exponents and by the very nature of mathematics are unknowable. Thus, as a class, they stand apart from rational ratios and irrational ratios with rational exponents, for these are truly knowable.

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\text { I. } 333-46
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Oresme now sets forth his objective for the remainder of the first chapter. This is twofold:
I. Given any two ratios, $A$ and $B$, related as $B=A p l q$, Oresme says that he will show whether or not $p / q$ is rational. Ratios $A$ and $B$ may be rational or irrational.
2. If ratios $A$ and $B$ are commensurable, i.e., if $p / q$ is a rational ratio or exponent, he will determine $p / q$.
Should $A$ and $B$ be incommensurable, however, Oresme will make no effort to determine $p / q$ since it is possible that $p / q$ is itself "an aliquot part of no rational ratio," in which event it is an irrational exponent and any attempt to determine it would be an exercise in futility.
In 1.337-38, Oresme uses the terms commensurabiles and communicantes as synonymous. He makes this explicit in his Quaestiones super geometriam Euclidis: "Question 7: First, it must be understood that the terms 'commensurable' and 'communicant' are synonymous-that is made clear in the tenth question of this treatise-as are 'incommensurable' and 'incommunicant..." ("Questio 7: Primo sciendum est, quod commensurabile et communicans sunt nomina synonima, ut patet $10^{\circ}$ huius, et eciam incommensurabile et incommunicans..."). ${ }^{6}$ In the Ad paucarespicientes, however, the terms communicans and incommunicans are used to indicate whether points of conjunction or opposition are coincident. In that treatise the terms com-
${ }^{6}$ Oresme: Quaestiones super geometriam Euclidis, ed. Busard, Fasc. I, 16.
municans and commensurabilis are not synonymous (for example, see AP2. 84-87, 166).

## I. 347-80

After reiterating that an irrational ratio might not be denominated by any rational ratio, Oresme remarks that if two irrational ratios have known denominations, then the relationship of each irrational to the rational ratio that serves as its base will be revealed directly by the respective exponents. However, the relationship between two such irrational ratios may be commensurable or incommensurable. In the former case we have a rational ratio of ratios and in the latter, an irrational ratio of ratios (see above, pp. 38-40). Oresme offers an illustration for each of these two types of ratios of ratios.
The first (I.357-6I) involves $(2 / \mathrm{I})^{1 / 2}$ and $(3 / \mathrm{I})^{1 / 4}$, which are incommensurable, and constitute an irrational ratio of ratios since it is impossible to equate these two irrational ratios by means of any rational exponent. Oresme infers this from the fact that $3 / \mathrm{I}$ and $2 / \mathrm{I}$ are incommensurable in an exponential sense, since $3 / 1 \neq(2 / 1)^{p / q}$, where $p>q$ and $p / q$ is a ratio of integers. Therefore, any exponential parts of the base ratios are also incommensurable. However, each of these irrational ratios is commensurable to its respective rational, or base, ratio. That is $(2 / \mathrm{I})^{1 / 2}$, an irrational ratio, is commensurable to, and part of, the rational ratio $2 / 1$. Indeed, as Oresme would express it, it is a "one-half part" of $2 / \mathrm{r}$.
In the second example ( $\mathrm{I} .362-80$ ), the base ratios are commensurable and consequently an aliquot part of one of these ratios is also commensurable to an aliquot part of the other. Therefore, $(4 /)^{1 / 3}$ and $(2 / 5)^{1 / 2}$ are commensurable and constitute a rational ratio of ratios. As in the first example, each of these irrational ratios is commensurable to its rational or base ratio. For example, $(4 / \mathrm{I})^{1 / 3}$ is commensurable to, and part of, the rational ratio $(4 / \mathrm{I})$.
In keeping with his previously stated objective (I.337-38), Oresme seeks the rational exponent that relates the two irrational, but commensurable, ratios $(4 / \mathrm{I})^{1 / 3}$ and $(2 / \mathrm{I})^{1 / 2}$ (see pp. 37, 39, and 346 ). As a first step, he reduces the ratios to the same base, transforming $(4 / 1 /)^{1 / 3}$ to $2^{4}$ and altering $(2 /)^{1 / 2}$ to $2^{3}$. Thus starting with $4^{1 / 3}=2^{2 / 3}$ and $2^{1 / 2}$, Oresme multiplies each of the numerators by the smallest integer that will produce, in each case, a numerator capable of yielding an integral quotient when divided by its denominator. This is obvious by following his steps: $2^{(2 / 3) 6}=2^{12 / 3}=2^{4}$; and $2^{(1 / 2) 6}=2^{6 / 2}=2^{3}$. We now have two ratios $2^{4}$ and $2^{3}$ that are related

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exponentially by a sesquitertian, or $4 / 3$, ratio since $2^{4}=\left(2^{3}\right)^{4 / 3}$; or, in terms of the initial ratios, we get $4^{1 / 3}=\left[\left(2^{1 / 1}\right)^{1 / 2}\right]^{4 / 3}$.

## I. $38 \mathrm{I}-4 \mathrm{II}$

One can imagine that rational ratios are divisible in seven ways. The first six ways can be performed de facto, as the examples will show, but the seventh-which is taken up as the first proposition of Ch. II (II.I.I-3I)is only imaginary and not actually possible. The divisions are made by assigning means:
I. Rational ratios are divisible into equal rational ratios, as for example, $4 / \mathrm{I}$ into $4 / 2 \cdot 2 / \mathrm{I}$, where $4 / \mathrm{I}=(2 / \mathrm{I})^{2}$. Not all rational ratios are divisible in this manner, but only those that can have geometric means assigned. For example, $3 / \mathrm{I}$ is not included in this category.
2. Rational ratios are divisible into unequal rational ratios, any one of which is a part or parts of the initial ratio. As an example, Oresme offers $16 / \mathrm{I}=16 / 8 \cdot 8 / 4 \cdot 4 / 2 \cdot 2 / \mathrm{I}$ where $2 / \mathrm{I}=(16 / \mathrm{I})^{1 / 4}$ and $8 / \mathrm{I}=(16 / \mathrm{I})^{3 / 4}$. Thus $2 / \mathrm{I}$ is a part and $8 / \mathrm{I}$ is parts of $16 / \mathrm{I}$ (see above, pp. 26-27). Not all rationals are divisible in this manner and although Oresme cites no instance, it is clear that $4 / \mathrm{I}$ is excluded from this group since it is only divisible into equal rationals.
3. Rational ratios are divisible into unequal rational ratios, none of which is a part or parts of the given ratio. For example, $2 / 1=4 / 3 \cdot 3 / 2$ but neither $4 / 3$ nor $3 / 2$ is an exponential part or parts (i.e., part taken properly) of $2 / 1$ since they cannot be made equal to $2 / \mathrm{I}$ by any rational exponent. When there is no concern to take parts properly (proprie), any rational ratio is divisible in this third way.
4. Every rational ratio is divisible into equal irrational ratios, as for example, $2 / 1=2 /(2 / \mathrm{I})^{1 / 2} \cdot(2 / 1)^{1 / 2} / \mathrm{I}$. But not every rational ratio is divisible into two equal irrational ratios. Thus $4 / \mathrm{I}$ cannot be so divided because $(4 / \mathrm{I})^{1 / 2}=2 / \mathrm{I}$, a rational ratio. But $4 / \mathrm{I}$ is divisible into three, four, five, etc., equal irrational ratios. Division into three equal irrationals would take the form $4 / I=(4 / I)^{1 / 3} \cdot(4 / I)^{1 / 3} \cdot(4 / I)^{1 / 3}$.
5. Every rational ratio can be divided into unequal irrational ratios each of which is a part or parts of the divided ratio. For example, $4 / \mathrm{I}=(4 / \mathrm{I})^{1 / 4}$. $(4 / \mathrm{I})^{3 / 4}$. Here the means and extremes form a geometric proportionality.
6. In the sixth way, any rational ratio is divisible into unequal irrational ratios by assigning mean improportionals between the extreme terms of the
rational ratio. No example is given, but $4 / \mathrm{I}=4 /(3)^{1 / 3} \cdot(3)^{1 / 3} / \mathrm{I}$ meets the conditions.
7. A final way is imaginable by dividing a rational ratio into rational and irrational ratios. But this is impossible because a rational and irrational ratio taken together cannot compose or reconstitute the initial rational ratio.

## I.414-2I

One can imagine that irrational ratios are also divisible in the same seven ways as rational ratios. But in fact the first three ways (I.385-93) are ruled out since each of these would require that an irrational ratio be divided exclusively into rational ratios, which is impossible since no irrational ratio can be composed solely of rational ratios. This is demonstrated in II.r. 32-52.

Irrational ratios can, however, be divided in the last four ways (I.394413 ), since these involve division into irrational ratios. But their irrelevance to the objectives of the treatise prompt Oresme to dismiss them without further consideration.
II.I.I-3I

This proposition deals with the seventh imaginable way of dividing rational ratios mentioned in the previous chapter ( $\mathrm{I} .38 \mathrm{I}-84,4 \mathrm{I} \mathbf{2 - 1}$ ). Two proofs are given. The first (II.I.4-19) is a reductio ad absurdum, which is now described:
r. Assume that a rational ratio, $A=D / F$, is divisible into rational and irrational ratios. This is accomplished by assigning mean $E$ so that $D / F=$ $D / E \cdot E / F$. Let $D / E=B$ be a rational ratio and $E / F=C$, irrational.
2. $E$ and $D$ are commensurable since $B$ is a rational ratio.
3. $F$ and $D$ are commensurable since $A$ is a rational ratio.
4. Therefore, by Euclid X. $8,{ }^{7} F$ and $E$ are commensurable and $C$ is a rational ratio. But this is contrary to the assumption in (I) that $C$ is irrational. It follows, therefore, that a rational ratio cannot be divided into one rational and one irrational ratio that compose the given rational ratio.
Oresme offers another version of this argument (II.I.20-3 I), which runs as follows:

[^36]
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1. $E$ is commensurable to $D$ since $B$ is a rational ratio.
2. $F$ is incommensurable to $E$ since $C$ is an irrational ratio.
3. But $F$ must be commensurable to $E$ because $A=D / F$ is a rational ratio, so that $D$ and $F$ are commensurable, and $D$ is commensurable to $E$; therefore, $F$ is commensurable to $E$ by Euclid X.8.
4. But $F$ cannot be commensurable to $E$ because this is contrary to the assertion of their incommensurability in (2).
II.1.32-52

Here we have the proof that no irrational ratio is divisible into any combination of rational ratios (I.423-25). The proof is by reductio ad absurdum and is very similar to the preceding proofs of this first proposition (II.r.i-3I).
II.I.53-86

If between the least, or prime, terms of some rational ratio, $A$, no mean proportional numbers can be assigned, then $A$ is not divisible into a series of rational ratios $B, C, D, E, \ldots, \mathrm{~N}$, where $B=C=D=E \ldots=N$. Consequently, none of the rational ratios into which $A$ is divisible is an aliquot-i.e., exponential-part of $A$, namely equal to $A^{1 / n}$ where $n$ is any integer.
Let $G / H$ be the least, or prime, terms of ratio $A$ and assume that $G / H$ $=D / F$. Furthermore, let $A$ be divided by mean $E$ so that $A=B \cdot C$ where $B=D / E, C=E / F$, and $D / E=E / F$. The proof of Prop. II is as follows:
I. Since $E$ is a mean proportional term, we see that $D, E$, and $F$ are three continuously proportional terms.
2. Since $E$ is a mean proportional term between $D$ and $F$, it follows by Euclid VIII. $8^{8}$ that any numbers related as ratio $D / F$ can have a mean proportional number assigned.
3. Now $G / H=D / F$ so that a mean proportional number can be assigned between $G$ and $H$, the prime numbers of $A$.
4. But ex bypothesi no mean proportional number can be assigned between $G$ and $H$ and, consequently, ratio $A$ is not divisible into two equal rational ratios.

8 The enunciation reads: "Si inter duos numeros numeri quotlibet in continua proportionalitate ceciderint, totidem inter omnes duos in eadem proportione relatos
cadere necesse est."-Euc.-Campanus, p. 205. In II.I. 79-8I Oresme quotes it essentially as it appears in Euc.-Campanus.

Nor can $A$ be divided into three, four, five, or more, equal ratios by assigning additional mean terms.

In support of his statement that no superparticular ratio-i.e., $n+1$ where $n$ is any integer equal to or greater than 2-is divisible into equal rational ratios by a mean proportional, Oresme cites Campanus' comment on Euclid VII. 8 (II.r.84) and Jordanus de Nemore's Aritbmetica. The reference to Campanus is simply incorrect because Euclid VII. 8 has no connection with the present problem. I have been unable to locate the unspecified reference in Jordanus' Arithmetica.

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\text { II.土. } 87-\mathrm{I} 36
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In the third proposition, Oresme shows that if a quantity were divided into two unequal parts one of which is a part or parts of the whole, then the two unequal parts would be related as two numbers in their lowest terms. Thus, should $A=B \cdot C$, with $B>C$, then if $B=A^{e l d}, C=A^{f i d}$, and $e+f=d$ (the letters representing the exponents are found in lines II.I. 118-19), Oresme will show that $A^{f i d}=\left(A^{e / d}\right)^{f l e}$. It is clear that tatios $B$ and $C$ are related exponentially as $e / f$, which is to be understood as asserting that $B$ is related to tatio $C$ as $e$ exponential parts of $A$ to $f$ exponential parts of $A$, where each "unit" exponential part is equal to $(A)^{1 / d}$. Oresme's objective is to demonstrate that $e / f$ is in its lowest terms.

Oresme explains (II.I.93-97) that if $A$ is divided into two parts, each of which is a part (i.e., "unit" part), then $A=A^{1 / 2} \cdot A^{1 / 2}$. But if $A$ is divided into two unequal but commensurable parts (see p. 26, above; and I. 387-89, which is discussed on p. 333), we have $A=A^{e / d} . A^{f / d}$; and if $e$ is an integer greater than one, $f$ may be equal to or greater than one (see II.r.ros-9 where this is restated in the second supposition).

Three suppositions are now offered (II.1.98-113) that concern the arithmetical manipulation of exponents. These suppositions and the references to Euclid VII—one of the arithmetical books-have already been discussed (II.r.204-13; see also pp. 26-27 and 340).

By the first supposition the numerator and denominator of each exponent are mutually prime (II.I.121-23), so that $e / d$ and $f / d$ consist of mutually prime terms. Since $e+f=d$ and $d$ is prime to both $e$ and $f$, it follows, by the second part of Euclid VII.29,9 that $e$ and $f$ are mutually prime.
9 "Si fuerint duo numeri contra se primi qui ex ambobus coacervatur ad utrumque eorum erit primus. Si vero ex ambobus coa-
cervatus ad utrumque eorum fuerit primus, duo quoque numeri ad invicem erunt primi."-Euc-Campanus, p. 191. In Heath's

Now, by the third supposition, $B$ and $C$ are related as $e$ and $f$ (II.1.127-28) -i.e., related as two mutually prime numbers. It is clear, then, that $C=$ $B^{f l e}$ or-which is the same thing- $A^{f l d}=\left(A^{e l d}\right)^{f l e}$.
The rest of the proposition (II.r.129-36) is devoted to a discussion of what seems to be a corollary to the proposition as expressed in II.I.88-91. Since it has been established that ratios $C$ and $B$ are related as two prime numbers, namely $f / e$, Oresme refers to Euclid VII. ${ }^{10}$-an arithmetic prop-osition-to show that by continual division (per subtractionem) of $f$ into $e$ (where $f<e$ ), we must arrive at a unit, namely $\mathbf{I}$. Only the unit, or I , can measure the numbers $e, f$, and $d$, each of which is prime to the others. But, thus far we have dealt only with the exponents as numerical relations in abstracto, for which reason Oresme cites Euclid VII.i. However, the number I, as the ultimate numerical unit of the exponents, represents a ratio $A^{1 / d}$, which is the unit ratio with respect to ratios $A^{e l d}, A^{f l d}$, and $A$ itself. That is, just as I is the only numerical unit that can measure numbers $e, f$, and $d$, so is ratio $A^{1 / d}$ the only unit tatio which can measure ratios $A^{e l d}, A^{f / d}$, and $A^{d / d}$ -i.e., ratios $B, C$, and $A$, respectively. Thus having established the unit relationship for the numbers in the exponents, Oresme transfers the same relationships to the ratios of quantities (see pp. 27 and 30 ). This is evident from the concluding statement in the proposition, where Oresme says (II. 1.I3436): "... by such a subtraction something will finally be reached that will be part of each of the divisors and the dividend, namely, $A, B$, and $C$, since any of them is like a number, etc."
We have now seen the role of Euclid VII.r. But in arriving at a unit of measure, Oresme also cited-in the very next line (II.r.i30)-Euclid X.i. Heath translates the enunciation as follows: "Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half; and if this process be repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out." ${ }^{\text {II }}$ What function
translation of Euclid's Elements, Vol. 2, 329, the corresponding proposition is VII. 28. 10 "Si a maiore duorum numerorum minor detrahatur donec minus eo supersit, ac deinde de minore ipsum reliquum donec minus eo relinquatur, itemque a reliquo primo reliquum secundum quousque minus eo supersit, atque in huluscemodi continua detractione nullus fuerit reliquus qui ante relictum numeret usque ad unitatem, eos duos numeros contra se primos esse
necesse est."-Euc.-Campanus, p. 170. The translation by Heath in Euclid's Elements, Vol. 2, 297, is as follows: "Two unequal numbers being set out, and the less being continually subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, the original numbers will be prime to one another.'
${ }^{11}$ Euclid's Elements, Vol. 3, 14. The text as given in Euc.-Campanus, p. 244, is essen-
does this proposition serve? The answer is by no means evident, but it does seem likely that it was intended by Oresme to serve as an analog to VII.i. The question is how close an analog? There are at least two possibilities. The first is that just as VII. i enables us to arrive at the numerical unitnamely I -that measures all the other numbers in the exponents, so Euclid X.i shows with respect to ratios of quantities-rather than numbers-that the greater of two ratios can be diminished until it is less than the smaller of them. Thus if ratio $B$ is greater than $C$, we can reduce $B$ to some ratio less than $C$ by successive "subtractions." Of course the resemblance between Oresme's application of VII.i and X.i ends here, since X.i does not enable us to arrive at a unit ratio-only VII.r permits this. Indeed, in the absence of any elaboration by Oresme, it is wholly unclear as to how one "subtracts" from the greater ratio $B$ to arrive at a ratio less than $C$. It would appear that an exponential, rather than arithmetic, operation is required. One possibility is that more than $B^{1 / 2}$ is "subtracted" from $B$, and then more than $B^{\mathrm{r} / 4}$, and so on, until we arrive at some ratio less than $C$. However, the actual ratio that will be both less than $C$ and also serve as the unit ratio measuring $B, C$, and $A$ must be found by Euclid VII.r.

A second possible interpretation depends on the next-i.e., fourthproposition (II.I.137-225-especially 204-I3). Oresme may have taken X. $I$ in an exponential sense but departed even further from its true intent. He may have understood it as a continuous process of division, beginning with a division of the lesser ratio into the greater and continuing until a unit ratio was obtained. This unit ratio would, of course, be less than the two given ratios and would comply with the general requirement of X.I that a magnitude be reached that is less than the two initial magnitudes. Furthermore, the unit ratio would be a common measure. Thus in II.r. 210-13, Oresme assumes that $B=A^{8 / 11}$ and $C=A^{3 / 11}$ and says that ratio $A^{3 / 11}$ can be "divided" into ratio $A^{8 / 11}$ twice, leaving $A^{2 / 1 r}$ as a remainder (i.e., $\frac{A^{8 / 11}}{A^{3 / 11}}=A^{5 / 11}$ and $\frac{A^{5 / 11}}{A^{3 / 11}}=A^{2 / 11}$;or, if done in one step, $\frac{A^{8 / 11}}{\left(A^{3 / 11}\right)^{2}}=A^{2 / 11}$ ); $A^{2 / 11}$ is then divided into $A^{3 / 1 r}$ leaving $A^{1 / 15}$ as a remainder and ultimate unit ratio that serves as the common measure of ratios $C, B$, and $A$.
If this second interpretation is plausible, Oresme has distorted X . I to force a very close analogy with VII.r, even to the point of arriving at an ultimate unit that is now, however, a unit ratio rather than the numerical

[^37]Critical Notes to Pages 184-190
unit, r. However, if Oresme meant to apply X. 1 to division of ratios in the manner just described, then we must assume that the initial "subtraction" (i.e., division) would have been $\frac{A^{8 / 11}}{\left(A^{3 / 11}\right)^{2}}$, since X.r states that "from the greater there be subtracted a magnitude greater than its half," and so continuously. To have obtained this same result by first dividing $A^{8 / 1 \times}$ by $A^{3 / \mathrm{rr}}$, and then dividing the quotient (Oresme calls it a "remainder"), $A^{5 / \mathrm{rr}}$, by $A^{3 / 11}$ would have violated X.I, for in this instance the initial division would have been by a ratio less than half of the greater ratio.
In interpreting this example from II.r.210-13, I have included an exponent and its base for each ratio. Actually, Oresme may not have conceived of the operation in quite this way. His frequent citation of Euclid VII.I and X.5, and his language in II.I.210-13, indicate that he neglected the base and simply operated arithmetically with the fractional exponents. Thus he would divide $3 /{ }_{11}$ into $8 /{ }_{\text {II }}$ as many times as possible, namely twice, and derive a remainder of $2 / \mathrm{II}$, which must be interpreted as the exponent of base $A$-i.e., $A^{2 / 11}$ (see below, p. 340).
The reader may have observed that Oresme refers to $A, B$, and $C$ as quantities rather than ratios of quantities as I have done. My expression is, however, justified by the fifth supposition of Ch. I (I.26I-7I), where, as we have seen, any ratio is divisible in the same manner as a continuous quantity. Thus if X.I is applicable to quantities, it must, for Oresme, also be applicable to ratios of quantities.

We see that Oresme, in coping with the varied problems arising from his treatment of ratios of ratios, frequently applied to the same case propositions from the arithmetic books of Euclid (especially Bk. VII) and the more general books embracing both number and magnitude ( V and X ). (For other instances, see above on p. 28.) It seems that Oresme, perhaps unknowingly, was ignoring the traditional distinctions between number and magnitude in general and bridging the gap that artificially separated them (see above, pp. 27-28, n. 37). Such moves were essential before mathematics could advance to the development of analytic geometry.

## II.I.137-225

Should a rational ratio be composed of ratios $B$ and $C$, Oresme shows in Prop. IV that ratios $B$ and $C$ cannot be exponential, or aliquot, parts of $A$ if no mean proportional numbers can be assigned between the prime numbers of $A$. He assumes, in a reductio ad absurdum proof, that $B$, a rational
ratio, is parts of $A$ so that $B=A p / q$, where $p / q$ is a ratio of integers in its lowest terms and $q>p>\mathrm{I}$. Then, by the second supposition of Prop. III (II.x.1os-9), it follows that $C=A^{m / q}$ where $m / q$ is a ratio of integers with $q>m \geq \mathrm{I}$. From Ch. II, Prop. I, it is clear that if $A$ and $B$ are rational ratios, $C$ must also be rational.
Oresme now demonstrates that it is impossible for $C$ to be a part( $C=$ $A^{m / q}$ and $m=1$ ) or parts (where $m>1$ ) of $A$ :
I. It cannot be a part of $A$ by Ch. II, Prop. II.
2. If $C$ is parts of $A$, then $C$ must be greater or less than $B$, for if $C=B$ both would be equal to $A^{1 / 2}$ and each would be part of $A$ (II.I.I so-s3). Assuming, then, that $B>C$, Oresme proceeds with a series of divisions (II.I.I $54-92$; he refers to them as "detractions" or "subtractions") and shows that however many times the process is repeated, ultimately one must arrive at some rational ratio that is a part of ratios $A, B$, and $C$. But this ultimate ratio cannot be part of $A$ since this would be contrary to Ch . II, Prop. II.
In furnishing two specific examples (II.r.204-13) in this proposition, Oresme leaves no doubt as to what he means by the "subtractive" process involved in arriving at an ultimate unit ratio. If $B=A^{3 / s}$, then $C=A^{2 / s}$ and $B=C^{3 / 2}$ since $A^{3 / s}=\left(A^{2 / 5}\right)^{3 / 2}$. A basic and initial assumption is that if ratio $B$ is parts of ratio $A$, both $A$ and $B$ can be related as a number to a number. This assumption depends on Euclid X.s (see above, p. 30, for the text and significance of X.s) and enables Oresme to relate $B$ and $A$ as $3 / 5$, which is, of course, the exponent relating $B$ to the base ratio $A$ (see above, pp. 26-27). By dividing (subtrabendo) 2 into 3 (see above, pp. 337-38), representing a division of $2 / 5$ into $3 / 5$, we obtain $1 / 5$, which is understood to represent $A^{1 / s}$. Thus we have arrived at the unit ratio of ratios $B$ and $C$.

The second example involves more than one division before reaching
 as many times as possible we still have a remainder of $2 / \mathrm{II}$. Finally, after dividing $2 /{ }_{\text {II }}$ into $3 /{ }_{\text {II }}$ the remainder is $1 / \mathrm{II}$, which represents the unit ratio $A^{1 / 11}$ for ratios $A, B$, and $C$. It is clear that by Euclid X.s Oresme has related ratios $A, B$, and $C$ numerically as exponential parts. He has then only to perform the divisions with the numbers-i.e., the exponents-to arrive at a unit ratio. That he will arrive at a unit ratio is guaranteed by Euclid VII.r since the exponents, or parts, are in their lowest, or prime, terms. Although Oresme fails to cite VII.I in this proposition, it is clear from the previous proposition that it is required (see above, pp. 337-38).

Prop. V is the first of a sequence of propositions in which the interdependent concepts of "commensurability," "multiple," and "part" are paramount. In each of the two parts of Prop. V , there is a rational ratio $A$ between whose prime numbers there is no mean proportional number. The proofs of both parts of the proposition are derived by reduction to absurdity.

In the first part (II.r.23I-37; see also p. 28, above), ratio $A$ is greater than $B$ and they are assumed commensurable. By Euclid X.s, ratios $A$ and $B$ are related as two numbers and $B$, the lesser, must be a part, or parts of the greater ratio, since Euclid VII. 4 asserts that any smaller number is a part, or parts, of a greater number (II.1.231-34; Euclid X.s and VII. 4 are linked again in II.r.302-3 and III.34-37). Oresme now demonstrates that $B$ is not a part of $A$, i.e., $B \neq A^{1 / n}$, by Ch. I, Prop. II; and $B$ is not parts of $A$, i.e., $B \neq A^{m / n}$, where $m$ and $n$ are integers and $n>m>\mathrm{I}$, by Ch. II, Prop. IV. Ratio $A$ is therefore incommensurable to any smaller rational ratio. But, as will be seen later, $A$ is not incommensurable to every smaller irrational ratio. Indeed any irrational ratio $A^{m / n}$, where $m<n$ and both are integers, is commensurable to $A$.

The second part of the proposition (II.r.238-s6; see also p. 28, above) shows that if ratio $C$ is greater than $A$ and $C \neq A^{n}$, where $n$ is any integer, then $C$ and $A$ must be incommensurable. Oresme assumes that $C \neq A^{n}$ but that $C$ and $A$ are commensurable. Therefore $C$ must contain $A$ one or more times plus some part or parts of $A$. If the former is true, then $C=A$ - $A^{1 / n}$; if the latter, then $C=A \cdot A^{m / n}$, where $n>m>\mathrm{I}$.

Now if $C=A \cdot A^{1 / n}$ and $A^{1 / n}=D$, Oresme shows that $D$ can be neither irrational(Ch. II, Prop. I) nor rational(Ch. II, Prop. II) and yet be a part of $A$. No rational ratio can be part of $A$ because by assumption $A$ has no mean proportional numbers between its prime terms.
If, on the other hand, $C=A \cdot A^{m / n}$ and $A^{m / n}=E$, Oresme demonstrates that whether $E$ is rational or irrational, it cannot be parts of $A$. If $E$ were irrational and parts of $A$, we would have a rational ratio, $A$, composed of rational and irrational ratios, which violates Ch. II, Prop. I; if $E$ were rational it would be contrary to Ch. II, Prop. IV, which asserts that no rational ratio is parts of any ratio that has no mean proportional numbers between its prime terms-and $A$ is such a ratio.

Oresme's reference to the Aritbmetica of Boethius (II.1.24I-42) is of interest because it reveals the same consistent attempt to employ traditional
sources and interpret them as if they had been utilized in a manner identical with, or similar to, his own quite novel usage. Boethius enumerates five types of ratios in his De institutione aritbmetica libri duo (edited by Godofredus Friedlein, Leipzig, 1867). These are multiple (pp. 46-49), superparticular (pp. 49-s 1), superpartient (pp. 57-60), multiple superparticular (pp. 60-64), and multiple superpartient (pp. $65-66$ ). Oresme says that tatio $C$ is greater than $A$, but is related to $A$ in some ratio other than a multiple. For Oresme, however, $C$ could be multiple to $A$ if $C / A=n$ or if $C=A^{n}$. Later, in the very same second chapter, he calls the former "multiple absolutely" and designates the latter exponential form as "multiple comparatively" (II.r.28997; see also p. 29, above). It is clear that in this proposition the term multiple is taken by Oresme in the comparative, or exponential, sense although he does not draw the distinction until Prop. VII (II.r.289-97; see also p. 29, above). For Boethius, on the other hand, a multiple ratio is $C / A=n$ where $C$ and $A$ are simply quantities (not ratios) and $n$ is any integer greater than I .
The same distinction applies also to the last four types of ratios cited above from Boethius. For Boethius they are particular cases of a general form in which $C / A=n \cdot(P / A)$ where $n, P$, and $A$ are integers and $P / A$ is a fraction. It is obvious that Oresme, while using the same designations for such ratios, understands them in an exponential sense, where $C=$ $A^{n(P / A)}$. Thus Oresme extended the designations of the Boethian ratios to cover their usage as exponents. Every ratio distinguished by Boethius may either retain its Boethian arithmetic meaning or be applied as an exponent in Oresme's treatise. The context usually reveals how Oresme is using the Boethian terminology in any particular proposition or example. His reference to the "definition of ratios" in Boethius (II.1.241-42) appears to be a general reference to all five types as they are taken up separately by Boethius (the page numbers have been indicated in this paragraph). Oresme enumerates and defines the five types of ratios in II.2.6-20.

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\text { II.r. } 257-82
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Prop. VI relates two unequal ratios where the greater is multiple to the lesser. Thus if $A>B$ and $A=B^{n}$, where $n$ is any integer, Oresme shows that there will be ( $n-1$ ) mean proportionals assigned between the extreme terms of $A$ (see p. 28). In the context of this proposition, the terms dupla and tripla(III.1.258) mean "square" and "cube" rather than "double" and "triple."

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In what amounts to a second part of Prop. VI, Oresme demonstrates that the same number of mean proportionals will be found between the terms of $A$ even when it is reduced to its prime, or least, terms.
The numerous references to Euclid in this proposition can be located in Euc.-Campanus under the book, proposition, and definition numbers furnished by Oresme. However, the relevant passage in Bk. V, Def. II, is on p. 109. Of the principles in Bk. VII mentioned in II.r.276, Oresme must surely mean Def. 20 (p. 169 of Euc.-Campanus), which considers composition of ratios.

## II.I.283-326

The basic objective and importance of Prop. VII have already been outlined (see p. 29, above). Oresme demonstrates the proposition itself (II.i. 298-304) and a consequence of the proposition (II.r.305-26).
The proposition asserts that if two ratios $A$ and $B$, where $A>B$, are commensurable but not multiple (see above, pp. 29 and 342 , for the two senses of multiple)-i.e., $A \neq B^{n}$-then $B$ must be parts of $A$-i.e., $B=$ $(A)^{m / n}$ where $m$ and $n$ are integers and $n>m>\mathrm{I}$. Although Oresme does not expressly say so, it is obvious that ratios $A$ and $B$ are rational, because they are understood to be in their prime numbers. Since $A$ and $B$ are commensurable they must be related as two numbers by Euclid X.s (see above, pp. 30 and 341), from which it follows that $B$ is an exponential part or parts of $A$. But it cannot be a part of $A$ for then $A=B^{n}$, which is contrary to the assumption that $A$ is not multiple to $B$. Therefore $B$ is parts of $A$, and $B=(A)^{m / n}$.

Oresme now shows (II.I.305-26) that each part of $B$-i.e., $A^{1 / n-m u s t ~}$ be a rational ratio. Since $B=A^{m / n}$ some other ratio, say $C$ (Oresme does not designate it by letter), will also be a part or parts of $A$ (by the second supposition in Prop. III [II.r.ios-9]) so that $C=A p^{\prime n}$, where $p+m=n$, and $n>p \geqq \mathrm{I}$. Assuming implicitly that $B>C$, Oresme makes successive divisions, beginning with the exponent of $C$ into that of $B$ (this is the "subtractive" process mentioned initially in Prop. III [II.r.129-36] and elaborated in Prop. IV) until a single part is obtained, namely $A^{1 / n}$, which Oresme calls $D$. Now $D$ must be a rational ratio (Ch. II, Prop. I) and is a part of $B$ and a part of $A$, so that $B=D^{m}$ and $A=D^{n}$. It follows, therefore, that each part of $B$ is a rational ratio since $D$ is a rational ratio.
The conditions governing the relations between $A, B$, and $D$ are given in the enunciation (II.r.284-88), namely that the mean proportionals as-
signed between the prime numbers of $B$ and $A$ form a series of ratios each equal to the unit ratio $D$. It is in this sense that Oresme says that "the prime numbers of $A$ and $B$, therefore, unite in means" (II.I.321-22; see III. $25-30$ for a definition of the expression "to unite in means"), which proves "that there are some mean proportional numbers between the numbers of the greater ratio that form a certain ratio, and that the very same ratio is also formed by the mean number or numbers between the prime numbers of the lesser ratio" (II.1.323-26).

## II.1.327-59

In Prop. VIII, the criteria for commensurability between any two ratios are presented as converses of Ch. II, Props. VI and VII.
If $A>B$ and between the prime numbers of $A$ one can assign geometric means that form ratios each of which is equal to $B$, then $A=B^{n}$ and $A$ will be multiple and commensurable to $B$ (see Prop. IX [II.I.371-73 and 389-92]). For example (II.1.389-92), if $A=8 / \mathrm{I}$ and $B=2 / \mathrm{I}$, they are commensurable because $8 / \mathrm{I}=8 / 4 \cdot 4 / 2 \cdot 2 / \mathrm{I}$ where each ratio equals $2 / \mathrm{I}$.
Oresme calls this the converse of Ch. II, Prop. VI, even though commensurability is not explicitly discussed in that proposition. Nevertheless, Prop. VI is the converse of this part of Prop. VIII in the sense that Prop. VI assumes $A=B^{n}$-i.e., that $A$ is multiple to $B$-and shows that $A$ can be multiple to $B$ only by assigning geometric means which can form two or more ratios each of which is equal to $B$.
The second part of Prop. VIII asserts that if the means assigned between the prime numbers of $A$ do not form ratio $B$-i.e., $A \neq B^{n}$-but form, rather, a series of ratios each of which is equal to some ratio $C$, where $C$ is the very same ratio formed by the means assigned between the prime numbers of $B$, then it follows that $C=B^{1 / m}$ and $C=A^{1 / n}$; or, $B=C^{m}$ and $A=C^{n}$ so that $A=B^{n / m}$. Thus $A$ and $B$ are commensurable because they "communicate in $C$ and $C$ is a measure common to each" (II.I.346-47). Oresme again cites the Euclidean definition of commensurability (Euclid Bk. X, Def. 1) which, as we know, applies to quantities and not to ratios of quantities related exponentially (see above, pp. 30 and 337). Thus $C$ is explicitly treated as a unit ratio to $B$ and $A$, where $A$ is not multiple to $B$.

This second part of Prop. VIII is the converse of Prop. VII. In the latter proposition, $A$ is assumed to be commensurable but not multiple to $B$. It is then shown that for these conditions to obtain, $A$ and $B$ must "unite in means," i.e., have a unit ratio as a common measure. In Prop. VIII it is

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assumed that the means assigned between the terms of ratios $A$ and $B$ are such that they form a common measure, or unit ratio, and, consequently, that $A$ and $B$ are commensurable, although $A$ is not multiple to $B$.

In II.r.353-59, Oresme remarks that if two geometric means are assignable between the prime numbers of some ratio, then one cannot assign a single mean and yet produce geometric proportionality where equal rational ratios are formed. This can be illustrated by ratio $8 / 1$, where two means must be assigned in order to have a geometric proportionality of the form $8 / 4 \cdot 4 / 2 \cdot 2 / 1$. If only a single mean were assigned, the equal ratios would not be rational.

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\text { II.1.360-4I } 3
$$

Prop. IX is a summary of the conditions for commensurability and incommensurability between two ratios and is based directly upon Props. V-VIII of Ch. II. These conditions are expressed in a series of five subsidiary conclusions or propositions (II.1.368-83) which are then exemplified in proper sequence (II.1.386-404). In summarizing, I shall conflate each subsidiary conclusion with its corresponding example. No line numbers will be cited since the conclusions and examples bear the same ordinal designation.
I. If $A=3 / \mathrm{I}$ and $B=2 / \mathrm{I}$, we have an instance of two ratios being incommensurable, since no mean proportional number can be assigned between the prime numbers of the greater ratio $3 / \mathrm{I}$. Consequently, the lesser ratio $2 / \mathrm{I}$ cannot serve as the unit or base ratio for the greater ratio. In other: words, $3 / \mathrm{I} \neq(2 / \mathrm{I})^{n}$, where $n$ is an integer.
II. If $A=4 / \mathrm{I}$ and $B=2 / \mathrm{I}$, then $A$ and $B$ are commensurable, since we can assign 2, a mean proportional number, between the prime numbers of the greater ratio thereby forming ratios equal to the lesser ratio. Thus the terms 4,2 , and I form ratios $4 / 2$ and $2 / \mathrm{I}$ that are equal to $2 / \mathrm{I}$, the lesser ratio. Ratio $B$ is the base ratio of $A$.
III. If $A=9 / \mathrm{I}$ and $B=2 / 1$, then $A$ is not multiple to $B$. Now 3 is the only assignable mean proportional number between 9 and 1 and permits only triple ratios to be formed. But the lesser ratio is $2 / \mathrm{I}$-not $3 / 1$-for which reason $9 / 1 \neq(2 / \mathrm{I})^{n}$, where $n$ is an integer. Oresme says nothing about commensurability in this third subsidiary conclusion because although a greater ratio is not multiple to a lesser ratio it may yet be commensurable to it. This case is covered in the next subsidiary proposition. It is obvious, however, that $9 / 1$ and $2 / 1$ are also incommensurable since $n$, the exponent relating $9 / 1$ and $2 / 1$ as a ratio of ratios, will be irrational.
IV. If $A=8 / \mathrm{I}$ and $B=4 / \mathrm{I}$, then $A$ and $B$ are commensurable because the mean proportional numbers assignable between the two ratios permit formation of the same base ratio-namely $2 / \mathrm{I}$. That is, $8,4,2$, I and 4,2 , I form only double ratios. It should be noted, however, that ${ }^{8 / I}$ is not multiple to $4 / \mathrm{I}$ since $8 / \mathrm{I} \neq(4 / \mathrm{I})^{n}$, where $n$ is an integer.
V. If $A=9 / \mathrm{I}$ and $B=4 / \mathrm{I}$, we have ratios that are incommensurable. By assigning the appropriate means, we get $9 / 3,3 / 1$ and $4 / 2,2 / 1$, respectively. Ratio $A$ has a base ratio of $3 / \mathrm{I}$ and $B$ a base proportion of $2 / \mathrm{I}$, which makes them incommensurable since $3 / \mathrm{I} \neq(2 / \mathrm{I})^{n}$, where $n$ is rational.

## II.1.414-59

In I. 333-39, Oresme stated that one of the two objectives he wished to achieve in the remaining portions of that chapter was to assign a ratioi.e., an exponent-between two commensurable ratios. This was done for only one particular case ( $\mathrm{I} . \mathbf{3}^{62-80}$ ). But Prop. X of Ch. II is devoted exclusively to this task, drawing upon Props. VII and VIII of Ch. II.

In effect, Oresme shows how to assign the exponent, or ratio of numbers, relating two ratios when the lesser is part of the greater (II.I.416-19 and 435-43) and when it is parts of the greater (II.r.420-27 and 444-57; see also pp. 26-27, above). The examples included in this proposition involve only rational ratios, but earlier Oresme had provided an example for irrational ratios that are commensurable-i.e., related by a rational exponent (I.362-80; see also p. 39, above).

If ratio $A$ is greater than $B$, and between the prime numbers of $A$ there are $(n-1)$ mean proportionals which form ratio $B$, then $A=B^{n / \mathrm{r}}$, where $n$ is an integer. Thus $n / \mathrm{I}$ is the exponent or ratio that must be found in order to relate $A$ and $B$. In this case, $B$ is a part of $A$. A specific example is given (II.1.435-43) where ${ }^{8 / 1}=(2 / \mathrm{I})^{3 / 1}$.

For all cases where $B$ is parts of $A$, Oresme repeats substantially what he said in Prop. VIII (II.r.338-49; see also pp. 344-45, above). Thus if $C$ is the common measure, or unit ratio, of $A$ and $B$, and if $B$ has ( $m-1$ ) geometric means between its prime numbers that can form ratio $C$, then $B=$ $C^{m}$; and if $A$ has ( $n-\mathrm{I}$ ) means between its prime terms, then $A=C^{n}$. From this it follows that $A=B^{n i m}$, where $n$ and $m$ are integers and $n>m$, or, as Oresme puts it, "...the ratio of these ratios will be as the ratio of those numbers" (II.1.430-3I)-i.e., $A$ and $B$ are related as $n / m$, which must be understood as expressing a relationship between unit ratio $C$, taken $n$ times exponentially, to $C$, taken $m$ times exponentially.

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In the first of two examples (II.i.444-53), Oresme shows that $32 / \mathrm{I}$ and $8 / \mathrm{I}$ are related as $5 / 3$, where $2 / \mathrm{I}$ is the unit ratio. That is, $32 / \mathrm{I}=(8 / \mathrm{I})^{5 / 3}$ or $(2 / \mathrm{I})^{5}=\left[(2 / \mathrm{I})^{3}\right]^{5 / 3}$. The exponent $5 / 3$ can be replaced by $\mathrm{I}^{2} / 3$, which is the numerical denomination of $5 / 3$. In the second example (II.I.453-57), Oresme relates $243 / 32$ and $108 / 32$. Oresme does not supply the details in this example but says only that it is like relating $7^{19 / 32}$ to $3^{3 / 8}$ since $243 / 32=7^{19} / 32$ and $108 / 32=33 / 8$. Supplying the steps, we get $243 / 32=(3 / 2)^{5}$ and $108 / 32=$ $3^{3} / 8=27 / 8=(3 / 2)^{3}$ so that $243 / 32=(108 / 32)^{5 / 3}$ or $(3 / 2)^{5}=\left[(3 / 2)^{3}\right]^{5 / 3}$. Thus the ratios of the second example are related exactly as those in the first example.

## II.2.1-2

Part 2 of Ch. II consists of three practical operational rules. In the first, Oresme shows how to find the prime numbers of any given rational ratio. Rules two and three are concerned with determining the number of means between two given extreme terms.

## II. 2.3-40

In order to arrive at the prime numbers of a given ratio, it is necessary to know the numerical denomination of that ratio, and for this reason Oresme lists the traditional fivefold division of rational ratios with their specific denominations. These five species of ratios are found in the Introduction to Aritbmetic by Nicomachus of Gerasa (fl. ca. 100 A.D.) ${ }^{12}$ and in the De institutione arithmetica of Boethius, ${ }^{13}$ which is based almost wholly on Nicomachus. ${ }^{14}$

12 Nicomachus: Introduction to Aritbmetic, trans. into English by D'Ooge, with studies in Greek Arithmetic by Robbins and Karpinski. The five species of ratios of greater inequality are found in Bk. I, Chs. 17-23, pp. 213-29. After examining the evidence, the authors conclude ( $p .72$ ): "The period of Nicomachus' life fell somewhere between the middle of the first century and the middle of the second century after Christ."
${ }_{13}$ Boetii De institutione arithmetica, ed. Friedlein, pp. 46-66.

14 According to Robbins and Karpinski (Nicomachus: Introduction to Arithmetic, p. 132), Boethius' treatise may almost be called a translation of Nicomachus' Introduction;
but "in the composition of his treatise Boethius more often expands than condenses. His method is to intersperse between sections literally translated, or closely paraphrased, others in which the general principles stated by Nicomachus are furnished with exhaustive explanation and copious numerical examples. Nothing is left to the reader to supply. Almost any chapter, compared with the original, will prove to be of this character. Boethius also supplies data in tabular form to a far greater extent than did Nicomachus. The order of the original is preserved for the most part, but occasionally a rearrangement is found" (p. 133).

The five species ${ }^{15}$ of ratios given by Oresme may be presented as follows:

1. Multiple ratios may be represented by $n / \mathbf{I}$, where $n$ is any integer and the actual numerical denomination is $n$. Earlier we saw that Oresme called this type "absolutely multiple" in contrast to "comparatively multiple," where two ratios are related by an exponent that is itself a multiple ratio of the "absolute" type (II.1.289-97; see also pp. 29 and 342, above). The introduction of the term and concept of "comparatively multiple" may be original with Oresme, developing from his overall attempt to utilize and apply customary terminology in his treatment of exponential relations between ratios.
2. Superparticular ratios are of the form $\frac{n+1}{n}$, where $n \geqq 2$ and $n$ is an integer. The form $\frac{n+\mathrm{I}}{n}$ actually constitutes the prime numbers of any superparticular ratio when the value of $n$ is given. Thus if $n=2$ we have a sesquialterate ratio whose prime numbers are $3 / 2$ and whose actual numerical denomination is $\mathrm{I}^{\mathrm{I}} / 2$.
3. Superpartient ratios are of the form $\mathrm{I}+m / n$, where $m$ and $n$ are integers greater than one and $m<n$. Furthermore, $m$ and $n$ must be prime to each other. ${ }^{16}$ Oresme says that a superpartient $2 / 3$ ratio is denominated by $\mathrm{I} 2 / 3$. Its prime numbers are $5 / 3$.
${ }^{15}$ Heath, History of Greek Matbematics, Vol. $I$, $101-4$, presents in tabular form Nicomachus' five types of ratios. In his edition of Bradwardine's Tractatus de proportionibus (pp. 22-24), Crosby summarizes and symbolizes Bradwardine's quite elaborate description of these same five types.
${ }^{16}$ This important qualification is omit ted by Crosby (and Heath, History of Greek Mathematics, Vol. 1, 102), who represents the superpartient ratio as

$$
\frac{" n+m}{n}
$$

[where $n$ and $m$ are integers greater than one, and where $n$ is greater than $m$ ]."Brad., p. 22. The brackets are Crosby's. But should $n=10$ and $m=2$ we have

$$
\frac{10+2}{10}=12 / 10=12 / 10=11 / 5 \text {. }
$$

Now I $2 / 10$ satisfies Crosby's criteria for a superpartient ratio even though it can be reduced to I $1 / 5$ in which event it becomes a superparticular ratio. There is no ques-
tion that both Nicomachus and Boethius insisted upon reducing a ratio to prime terms before classifying it and in the example above would undoubtedly have considered $\mathrm{I} 2 / \mathrm{o}$ as a superparticular ratio, since it can be reduced to I $1 / 5$. Thus Nicomachus, in his Introduction to Aritbmetic, Bk. I, Ch. 20 (D'Ooge, Robbins, and Karpinski version, p .220 ) says in his discussion of superpartient ratios, "It is most necessary to start with two thirds, then two fifths, two sevenths, and after these two ninths, following the advance of the odd numbers; for two quarters, for example, again are a half, two sixths a third, and thus again superparticulars will be produced instead of superpartients, which is not the problem laid before us nor in accord with the systematic construction of our science." Substantially the same judgment is found in Boethius' De institutione aritbmetica, Bk. I, Ch. 28 (see the edition of Friedlein, p. 58 , lines $\mathrm{I}-\mathrm{IO}$ ).
4. Multiple superparticular ratios are of the form $\frac{m n+1}{n}$ (Heath, History of Greek Mathematics, Vol. $r, 103$ ) where $m$ and $n$ are integers greater than one. Thus if $m=n=2$ we obtain $5 / 2$, which are the prime numbers of a double sesquialterate whose numerical denomination is $2^{1} / 2$.
5. Multiple superpartient ratios are of the form $p+m / n$, where $p$ is any integer equal to or greater than one, $m$ and $n$ are mutually prime integers greater than one, and $m<n$. When $p=m=2$, and $n=3$, we have the denomination of a double superpartient two-thirds, or $2^{2} / 3$; if $p=m=3$, and $n=7$, we have a triple superpartient three-sevenths, or $3^{3 / 7}$, and so forth. The prime numbers of these two ratios are $\frac{8 / 3}{}$ and ${ }^{24} / 7$, respectively.

Oresme outlines the steps (II.2.26-40) involved in calculating the prime numbers from the denominations of types $2-5$. The steps are identical to our present mode of altering the form of an improper fraction to that of a ratio. In his description, he calls the numerator, or greater of the prime numbers, the dux radicum (II.2.32) and applies the term comes radicum (II. 2.29) to the denominator, or lesser of the prime numbers. I have translated these terms as "antecedent of the roots" and "consequent of the roots," respectively. Boethius, in discussing superparticular ratios, designates the greater term, or numerator, by $d u x$ and the lesser term, or denominator, as comes. ${ }^{17}$ Nowhere, however, does Boethius use the compound expressions dux radicum and comes radicum, and perhaps this once again reflects the fact that Oresme's De proportionibus is concerned with geometric proportionality and exponents, while Boethius is not truly concerned with this in his De institutione arithmetica. Thus the term dux radicum may mean for Oresme the numerator of an exponential ratio that is itself, qua ratio, in one of the five categories or species of ratios. Thus, for example, a multiple superpartient ratio such as $2 \frac{2 / 3}{}$ is represented in prime numbers by $8 / 3$. Now in $A^{8 / 3}$ the numerator of the exponent, 8 , is the $d u x$ radicum and the denominator, 3 , would be the comes radicum. On this interpretation the use of the term radix in conjunction with $d u x$ and comes signifies that one is considering the traditional five types of ratios as exponents (see p. 342, above).
${ }_{17}$ Boethius says (De institutione arithmetica, ed. Friedlein, pp. 49-so), "Voco autem, maiores numeros duces, minores comites.", The terms "antecedent" and "consequent," for the greater and lesser terms respectively of a ratio of unequal quantities, are used by D'Ooge, Robbins, and Karpinski in their translation of Nicomachus' Introduction to

Aritbmetic. On p. 215 , n.5, they explain: 'The words translated 'antecedent' and
 respectively the larger and smaller terms in a ratio between unequal quantities. Boethius, I.24, adopts the translations duces and comites (Voco autem maiores numeros duces, minores comites)."

Oresme, in effect, tells us that the term was already in use (II.2.29 and 32). In what sort of treatises would such terminology have been appropriate? A likely candidate would be some algebraic work where exponents were indeed used and where such distinctions may have been developed.
II.2.4I-56

The prime numbers of ratios of lesser inequality are the same as those of corresponding ratios of greater inequality (see Oresme's extended discussion on the correspondence between ratios of lesser and greater inequality in I.90-210). If the prime numbers of a ratio of greater inequality are known, it is only required to make the lesser number the numerator and the greater number the denominator in order to arrive at the prime numbers of the corresponding ratio of lesser inequality.
II.2.57-78

In the second rule, Oresme describes how to determine whether there is a mean proportional number between the numbers of a given ratio, say $A / B$. If $A \cdot B$ yields a product that is itself a square number, the root of the product is a mean proportional number. Furthermore, if $A$ and $B$ are prime numbers and contain a mean proportional, then $A$ and $B$ are square numbers; and conversely. In support of these propositions, Oresme cites Euc.Campanus VI.is and VII.2o. These two propositions are concerned with four proportional lines and numbers respectively. Curiously, Oresme does not cite Euc.-Campanus VI.16, which deals with three continuously proportional lines. However, more appropriate in this connection is Campanus' annotatio to his comment on Euclid VII. 20 where he discusses the cases for three lines and three numbers, respectively. He remarks:"Euclid, however, does not propose [the case] for three continuously proportional numbers (as he does in VI.i6 for three lines), [namely] that the product of the first and third [numbers] is equal to the square of the mean; and if that [number] which is produced by multiplying the first and third terms is equal to the square of the mean, those three numbers are continuously proportional just as he proposes for three lines in VI.r6..." ${ }^{18}$ Thus Campanus actually enunciates the very propositions set forth here by Oresme.
${ }^{18}$ "Non proponit autem Euclides de producitur sit aequalis quadrato medii; et tribus numeris continue proportionalibus si ille qui ex primo in tertium producitur quod ille qui ex ductu primi in tertium fuerit aequalis quadrato medii quodillitres

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Having outlined criteria for determining a mean proportional between two numbers, Oresme formulates rules that will yield the total number of means between two numbers. We may represent this as $2 n+1$, where $n$ is the initial number of means. Now if $A / B$ is a given ratio of two numbers between which one mean, say $C$, has been found, we must set $n=1$ so that $2 n+1=3$. Thus if there are other means, they will be found between $A$ and $C, C$ and $B$, and the total between $A$ and $B$ will be three. This computation of mean proportionals is based on Euclid VIII.8, which Oresme summarizes to the effect that in any series of continuously proportional terms if a mean is found between any two successive terms, mean proportionals will also be found between every other pair of successive terms in the series. Carrying the computation a step further, if $n=3$, then $2 n+1=$ 7 , which signifies that if between any two successive numbers in the series we find another mean, a total of seven means can be found by the methods outlined in II.2.57-68.

## II.2.79-100

In the third rule, Oresme explains that at least two mean proportionals can be found between two cube numbers whether they are mutually prime or not. If the given cube numbers are $A$ and $B$, with $A>B$, Oresme's method is as follows: $A^{1 / 3} \cdot\left(B^{1 / 3}\right)^{2}$ produces the lesser mean, and $B^{1 / 3} \cdot\left(A^{1 / 3}\right)^{2}$ yields the greater mean. Oresme gives two examples, the first of which is $27 / 8$ - i.e., $(3 / 2)^{3}$ - which is, for him, a case where the two cube numbers are in their lowest terms and "proximate" numbers. By proximate numbers Oresme seems to mean that 3 and 2 are successive numbers that, when cubed, give $27 / 8$. By the procedure described above, he finds that 18 and 12 are the two mean terms. The second example, $216 / 8$, is one in which the given cube numbers are not mutually prime, nor in their lowest terms, nor proximate. They lack proximity presumably because $216 / 8$ is the cube of $6 / 2,2$ and 6 obviously not being successive numbers.

## II.2.101-17

Having shown how to find two mean proportional numbers between certain given extreme terms $A$ and $B$, Oresme next details a method for deter-
numeri sint continue proportionales sicut proponit in 16 sexti de tribus lineis...." -Euc.-Campanus, p. 184. The modern edition of Euclid does not contain the propo-
sition dealing with three continuously proportional lines, as given by Campanus in VI. 16 of his edition.
mining the possible number of successive means that may be found. If, as before, $n$ is the number of means, then $3 n+2$ will yield the desired results where initially there are at least two means between extreme terms $A$ and $B$. Since at the very least $n=2$, there must be a minimum of four continuously proportional terms. Now should two means be found between any two of the four continuously proportional terms (two such means would be found by the methods outlined in II.2.79-85), it follows from Euclid VIII. 8 that two means will be found between any two successive terms in the series. Our formula tells us that when $n=2,3 n+2=8$, indicating a total of eight means between extremes $A$ and $B$, and now there are ten continuously proportional terms. When $n=8$, and assuming that two mean proportionals have been found between one pair of successive terms in the series of ten continuously proportional numbers, the formula for the next possible total number of means is $3 n+2=26$. The process may be carried on in infinitum.

> II.2.118-23

Up to this point Oresme has been able to determine whether between two terms $A$ and $B$ there can be 1, 3, 7, 15, 31, etc., mean proportional terms, or in general $2 n+1$ means, and he has formulated a method for showing whether there can be $2,8,26,80$, etc., means, or generally $3 n+2$ means (where in both formulations $n$ is the number of means from which one seeks the next total possible number of means). By integrating the two sets of terms we see that Oresme has formulated rules that enable him actually to discover whether there can be $1,2,3,7,8,15,26,31,63,80$, etc., mean proportionals between two given extremes. The two rules are applicable, however, only to those cases where the two given initial, or extreme, terms are square numbers or their product is a square number; and where the terms are cube numbers. Even with these limitations there are possible cases not covered by the separate rules and, indeed, Oresme admits that he has no general rule (II.2.127-28). For this reason he finds it necessary to use various combinations of the two independent rules. Thus to find if there are five mean terms, one must first proceed by rule three (II.2.79-85) and investigate if there are two mean terms between the two given extremes. Should two means be found, we have four continuously proportional terms. Then operating by the procedure outlined in the second rule (II.2. 58-68), we can discover if there is one mean between any two successive terms in the sequence of four continuously proportional terms. If one is

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found, it follows by Euclid VIII. 8 (II.2.70-74) that there will be a mean between every pair of successive terms and three more means will be found, for a total of five. In a similar manner such combinations of the rules can reveal if there are 11 , or 17 , or 23 , etc., means.

Oresme is aware that his rules and combination of rules cannot, for example, yield a case where the total number of means is four, since in that event the extreme terms would not be squares or cubes. For such a contingency, Oresme recommends (II. 2.129-38) generating a series of six continuously proportional terms that form the particular proportion in which one is interested. In this way the six terms include four means and two extremes.
III.1-24

Oresme precedes the propositions of the third chapter with six suppositions. The first (III.4-6) excludes from the category of mutually prime numbers all those pairs of numbers where the greater is the multiple of the lesser. This is proven by an appeal to the definition of mutually prime numbers (Bk. VII, Def. 7, in Euc.-Campanus, p. 168).

Supposition II (III.7-9) is, as Oresme states, a consequence of Supposition I. This is obvious because the latter asserts that when a greater number is multiple to a lesser number, those numbers are not prime to one a nother However, in reducing such ratios to their least or prime terms, the lesser term must, of course, be represented by unity. Such reductions always yield what Oresme calls multiple ratios of the form $n / \mathrm{I}$, where $n$ is an integer. As Oresme indicates, this was already stated in II.2.21-24.
In Supposition III (III.10-13), when Oresme asserts that every mean proportional number between some number and unity forms a multiple ratio, it must be understood that the terms of the successive ratios have been reduced to their least, or prime, terms and that all the ratios are equal, having unity for their respective denominators. This is evident by Supposition II, where he says, "One of the prime numbers of any multiple ratio is a unit" (III. $7-8$ ). Thus, if we have $8 / \mathrm{I}$ and assign means such that $8 / 1=8 / 4$ $\cdot 4 / 2 \cdot 2 / \mathrm{I}$, this reduces to $8 / \mathrm{I}=2 / \mathrm{I} \cdot 2 / \mathrm{I} \cdot 2 / \mathrm{I}$ and in this way each mean has served to form a multiple ratio. It should be noted that Oresme is now using the concept of "absolutely multiple" rather than "comparatively multiple" (II.r.289-97; see also pp. 29 and 342, above). The reference to Euclid(III.13) seems applicable to Bk. VII, Def. 2, which says, "A number is a multitude composed of units." ${ }^{19}$

19 "Numerus est multitudo ex unitatibus composita."-Euc.-Campanus, p. 168.

Suppositions IV and V (III.14-19) require no elucidation. In Supposition VI (III. 20-24), Oresme says that no mean proportional number, say $C$, assigned between the terms of ratios of the form $A / B$, where $A \neq n B$ and $A, B$, and $n$ are integers, can divide $A / B$ into equal multiple ratios. For if a non-multiple ratio such as $A / B$ could be divided into equal multiple ratios by mean $C$, we would have $A / B=A / C \cdot C / B$, in which event $C=k B$ and $A=k C$ (where $k$ is an integer) so that $A=n B$ where $n=k^{2}$ and $n$ is obviously an integer. But this makes $A / B$ a multiple ratio, which is contrary to the assumption that it is non-multiple.
It is important to note that whereas the first part of Ch. II dealt with quantity in general, Ch. III focuses on ratios of numbers. It was in preparation for this that Oresme enumerated and defined the five types of ratios and outlined the procedures for determining their prime numbers in II. 2 . 3- 56 . Furthermore, the basic notion of "part" and "parts" was employed in Ch. II, Part I , whereas in Ch. III the notion of "uniting or participating in means" becomes primary. The latter is a numerical concept (III. $25-30$ and the next paragraph of the Critical Notes), while the notion of part and parts applies to quantity in general and embraces rational and irrational ratios (I.400-404, II.I.92-113, II.I.298-326; see also pp. 26-27, above).

$$
\text { III. } 25-30
$$

Here Oresme furnishes a definition for the expression "to unite or participate in means" (in mediis convenire seu participare; in III. 132 the expression participant in mediis is used; in II.r. 300 , the phrase in mediis convenire; and in II.r.322, conveniunt in mediis; see also pp. 29, 344-45, above). Two cases are distinguished: (r) if $A / B>C / D$ and each ratio is in its prime numbers, these ratios "unite or participate in means" when mean proportionals assigned between $A$ and $B$ generate a sequence of ratios $A / E \cdot E / F \cdot F / G \cdots$. $X \mid B$, where each ratio in the sequence, after reduction to its prime numbers, equals $C / D$; (2) or when $A / B>C / D$ and the means assigned between $A / B$ and $C / D$ form a series of ratios which when reduced to their prime numbers are equal. In both cases the least ratio formed by assigning means is a unit ratio or common exponential base for both of the given ratios.

$$
\text { III. } 32-46
$$

In Prop.I, Oresme demonstrates that no multiple ratio, say $q / \mathrm{I}$, where $q$ is an integer, is related to a smaller non-multiple ratio by any rational ex-
ponent-i.e., they are not commensurable. Thus if $q / \mathrm{I}=A$ and $A>B$, where $A$ is a multiple ratio and $B$, non-multiple, it is shown that $B$ is neither part of $A$-i.e., $B \neq A^{1 / n}$, where $n$ is an integer-nor parts of $A$ i.e., $B \neq A^{m / n}$, where $m$ and $n$ are mutually prime integers and $m<n$.

Using a reductio ad absurdum argument, Oresme draws consequences which contradict earlier conclusions. He shows that if $A$ and $B$ are commensurable, they would be relatable as two numbers, by Euclid X. $\rho$, so that $B=$ $A^{m / n}$ or $A=B^{n / m}$. Now if $A$ and $B$ are related as two numbers $m$ and $n$, then $B$, the lesser ratio, is a part or parts of ratio $A$, by Euclid VII. 4 , where it is shown that "of two unequal numbers, the lesser is part or parts of the greater" (Euc.-Campanus, p. 174). See p. 26, above, for a discussion of the terms pars and partes; for a description of the special interpretation Oresme placed upon Euclid X. 5 and its connections with VII.4, see pp. 28, 341, above, and II.1.231-34, 302-3. Since by "part" or "parts" Oresme is referring to exponents, he shows that an exponent consisting of a ratio of integers cannot relate the types of ratios represented by $A$ and $B$. Any mean proportional numbers assigned between the terms of ratio $A$ can form only multiple ratios (by Supposition IV of Ch. III) and, consequently, cannot produce ratios equal to $B$, which is a non-multiple ratio. Hence $B \neq A^{1 / n}$-i.e., $B$ is not a part of $A$. But neither is $B$ parts of $A$ because this would entail that both $B$ and $A$ are multiple, in an exponential sense, to some ratio that is their common unit measure or base tatio. That is, $B$ and $A$ would "unite in means" (III. $25-30$; see also p. 354, above). This is impossible, however, since $A$ and $B$ cannot both produce multiple ratios nor can non-multiple ratios be derived from both of them. Appeal is made to Supposition VI (III.20-24), where it is stated that means assigned between the terms of $B$ cannot produce a multiple ratio, and to Supposition IV (III.14-16), which asserts that mean proportionals assigned between the terms of a multiple ratio like $A$ cannot produce non-multiple ratios.

## III. $47-55$

In the second proposition, we see that no multiple ratio is commensurable to a greater non-multiple ratio. If $B>A$, where $B$ is a non-multiple and $A$ a multiple ratio, then by a demonstration almost identical with that given in Prop. I of Ch. III, Oresme shows that $B \neq A^{n}$ and consequently $A \neq B^{\prime n}$; similarly $A \neq B^{m / n}$, where $m$ and $n$ are integers and $n>m>\mathrm{I}$.

## III. 56-82

In the third proposition, Oresme draws an obvious general conclusion from the preceding two propositions of Ch. III, namely that multiple ratios are incommensurable to non-multiple ratios. Some corollaries follow.
When multiple ratios are "added" (i.e., multiplied) they produce other multiple ratios. In order to "add" rational ratios, one multiplies their numerical denominations (III.64-66; the reference to Ch. I is to lines $84-89$; see also pp. 315-16, above). For example, if $n / \mathrm{r}$ and $m / \mathrm{r}$ are multiple ratios, their respective numerical denominations are the integers $n$ and $m$. To add $n / \mathrm{I}$ and $m / \mathrm{r}$, we simply multiply $n m$ and the product gives the numerical denomination of the new multiple ratio $\mathrm{nm} / \mathrm{I}$.

Another corollary (III.67-74) shows that non-multiple ratios such as $p / q$, where $p$ and $q$ may be integers in their least terms with $p>q>\mathrm{I}$, can never be transformed into multiple ratios no matter how often they are multiplied by themselves. In short, $(p / q)^{n}$, where $n$ is any integer, can never become a multiple ratio of the form $m / \mathrm{r}$, where $m$ is an integer.
Since all multiple ratios are incommensurable to non-multiples, a third corollary (III.75-79) reveals that non-multiple ratios that are components of multiple ratios are incommensurable to those multiple ratios. For example (III. 76 ), $2 / \mathrm{I}=4 / 3 \cdot 3 / 2$, where $2 / \mathrm{I}$ is a multiple ratio and $4 / 3$ and $3 / 2$ are non-multiples. But $2 / 1$ cannot be commensurable to either $4 / 3$ or $3 / 2$ since $4 / 3 \neq(2 / \mathrm{I})^{m / n}$ and $3 / 2 \neq(2 / \mathrm{I})^{m / n}$, where in each case $m$ and $n$ are mutually prime numbers and $m<n$. The same reasoning applies to an example such as $3 / 1=$ $3 / 2 \cdot 2 / 1$. Indeed $4 / 3$ and $3 / 2$ are mutually incommensurable, as are $3 / 2$ and $2 / 1$. It follows from this (III. 80-82) that multiple ratios can be composed of non-multiple ratios, but that the converse is never true.

$$
\text { III. } 83-\mathrm{IOI}
$$

Oresme, in the fourth proposition, focuses attention on multiple ratios that can have no mean proportionals assigned between their prime terms. Such multiple ratios will be incommensurable to all smaller rational ratios. No illustration is furnished, but $3 / \mathrm{I}$ would be a multiple ratio between whose terms no mean proportional number is assignable, so that it is incommensurable to all smaller rational ratios. That is, $3 / \mathrm{I} \neq(p / q)^{m / n}$, where $3 / \mathrm{I}>p / q$ and $m$ and $n$ are integers with $m>n$.

Such multiple ratios are also incommensurable to any greater rational ratios that are not multiple to them. Using $3 / \mathrm{I}$ again we see that all multiple
ratios of the form $(3 / \mathrm{I})^{n}$, where $n$ is any integer, are multiple and commensurable to $3 / \mathrm{I}$. In general, only multiple ratios of the form $(m / \mathrm{I})^{n}$ are commensurable and multiple to any lesser multiple ratio $m / \mathrm{I}$ that can have no mean proportional number assigned.
Both parts of this proposition had already been demonstrated in Ch. II, Part I, Prop. V (226-56), as Oresme indicates (III.90). However, in Ch. II, Prop. V, Oresme uses the concept of part and parts, while in this proposition he begins with the concept of multiple ratio and assumes that it can have no mean proportional assigned. Now the denomination of such a multiple ratio is some number (numerator) and unity (denominator) which are, therefore, its prime numbers. We now have conditions identical with those in Ch. II, Prop. V, namely a ratio between whose prime numbers there is no mean proportional number. For this reason, Oresme can cite Ch. II, Prop. V, as demonstrating Prop. IV of Ch. III (see also III.ro9, where, in a similar situation, it is used again).

Finally, Oresme draws the crucial distinction (III.93-101) between a "ratio of ratios" and a "ratio of denominations." He notes that all ratios are commensurable when their denominations are known. Thus $3 / 1$ is denominated by 3 and $2 / 1$ by 2 . Now 3 and 2 are commensurable since all numbers are commensurable to one another. If, therefore, we constitute a ratio of $3 / 2$, we have a sesquialterate ratio that is a ratio of denominations, since we have already seen that 3 denominates a triple ratio and 2 a double ratio. However, exponentially, $3 / \mathrm{I}$ and $2 / \mathrm{I}$ are incommensurable since $3 / \mathrm{I} \neq$ $(2 / \mathrm{I})^{m / n}$, where $m$ and $n$ are integers and $m>n$. Thus if two given ratios are related arithmetically by their denominations, we have a ratio of denominations; if related exponentially they constitute a ratio of ratios. With a single exception, for any two given ratios the ratio of denominations will differ from the ratio of ratios. For example (III.97-99), $9 / 1$ and $3 / \mathrm{I}$ form a ratio of denominations of $9 / 3$, or $3 / \mathrm{r}$. But $9 / 1=(3 / \mathrm{I})^{2 / 1}$ when they are related as a ratio of ratios-i.e., related as the exponent $2 / \mathrm{I}$ —and in this case $9 / \mathrm{I}$ is said to be the double (i.e., the square) of $3 / 1$. The exception to this (III.9910I) is $4 / \mathrm{I}$ and $2 / \mathrm{I}$, where $4 / 2=2 / \mathrm{I}$ and $4 / \mathrm{I}=(2 / \mathrm{I})^{2 / \mathrm{I}}$, so that "their ratio of ratios is just the same as their ratio of denominations." See also IV.r y 2-54.

## III.102-16

Every superparticular ratio (II.2.9-11; see also p. 348, above) is incommensurable to every other superparticular ratio and, indeed, to every kind of ratio that is not multiple to it. Thus in the fifth proposition of Ch. III,

Oresme shows that all ratios of the form $\frac{p+1,}{p}$ where $p$ may be any integer, are mutually incommensurable. The proof rests on the fact that no mean proportional numbers can be assigned between the extremes of a superparticular ratio, since the difference between numerator and denominator is always one. For this reason Oresme invokes Ch. II, Prop. V (see also III. 90 and p. 357 , above): "If there is no mean proportional number or numbers between the prime numbers of some ratio, that ratio will be incommensurable to any smaller rational, and to any greater rational ratio that is not multiple to it" (II.r.226-29). Thus if $p$ and $q$ are integers with $p>q$, then $\frac{p+1}{p} \neq\left(\frac{q+1}{q}\right)^{n / m}$, where $m$ and $n$ are integers and $m<n$. Only ratios of the form $\left(\frac{p+1}{p}\right)^{n}$, where $n$ is any integer, are commensurable to $\frac{p+1}{p}$.

## III.117-68

All multiple ratios whose numerical denominations are in the same geometric series are commensurable. For example, since $1,3,9,27,81, \ldots$, or $3^{n}$, where $n=1,2,3,4, \ldots$, are in the same geometric series, it follows that all ratios in the series $(3 / \mathrm{I})^{n}$ are mutually commensurable. Furthermore, any ratio not denominated by a number in the series $3^{n}$ is incommensurable to every ratio in the series $(3 / \mathrm{I})^{n}$. In general, all ratios in a geometric series are mutually commensurable, but are incommensurable to all ratios that are not members of the series. Oresme explains this (III.131-36) by noting that all ratios in a given geometric series "participate in means" (III.132-34); or, in other words, the relationship between any two ratios in such a series is either such that ( 1 ) the lesser is the common measure or unit ratio of the greater, or (2) both ratios can be related to a smaller common unit ratio in the same series that serves as an exponential base (see III. $25-30$ for a definition of the expression in mediis convenire seu participare). Essentially the same idea is expressed by Oresme's concept of pars and partes (see I.400-404, II.r.92-113, and 298-326; see also pp. 26-27 and 343, above).

When we have a series of multiple ratios (see II.2.6-8 and III.7-9) such that $A=B^{n}$, where $n=2,4,6,8, \ldots$, ratio $A$ will be denominated by a square number-i.e., by a number that has a rational square root. Similarly, if $n=3,6,9,12, \ldots$, then $A$ is denominated by a cube number-i.e., by a number that has a rational cube root. And finally, if $n=6,12,18, \ldots$, then $A$ is denominated by both a square and a cube number-i.e., denominated
by a number that has both a square and a cube root, as for example $(2 / \mathrm{I})^{6}=$ 64/1, which has a square root of $8 / 1$ and a cube root of $4 / 1$. As Oresme says (III.166-67), all of this is taken directly from Euclid IX. 8 (Euc.-Campanus, p. 222).

## III.168-258

In the seventh proposition, Oresme gives the conditions of commensurability between any two non-multiple ratios of the form $p / q$, where $p$ and $q$ are mutually prime integers and $p>q>1$. These conditions are (III.16874): (1) the numerators of the two ratios must be in the same geometric series and (2) the denominators must be in the same geometric series. These two series will, of course, differ. With the aid of a figure (III.192-201) and Euclid VIII.8, 9, 10, and 12, Oresme graphically depicts the substance of this proposition. On the left side of the figure we find the series $1,3,9$, $27, \ldots$, and on the tight side the series $\mathrm{I}, 2,4,8,16, \ldots$. All multiple ratios denominated by any terms in the sequence $2,4,8,16, \ldots$, are commensurable and the same holds for multiple ratios denominated by terms in the series 3, 9, 27, 8I, ...

By combining the corresponding terms in each series Oresme forms the sequence of $\operatorname{ratios}(3 / 2)^{n}$, where $n=1,2,3,4,5, \ldots$. All these non-multiple ratios are mutually commensurable and all ratios outside the series are incommensurable to every ratio in the series. This is so, Oresme insists, because all ratios in the series communicate in means (III.218), or unite or participate in means (see III. $25-30$ ). Thus, for example, in the figure, ${ }^{81} / 16$ $={ }^{81} / 54 \cdot 54 / 36 \cdot{ }^{36} / 24 \cdot{ }^{24} / 16$ (see III.194). When these ratios are reduced to their prime numbers, ${ }^{81} / 16=3 / 2 \cdot 3 / 2 \cdot 3 / 2 \cdot 3 / 2=(3 / 2)^{4}$. Similarly, $243 / 32=$ $243 / 162 \cdot 162 / 108 \cdot 108 / 72 \cdot 72 / 48 \cdot 48 / 32$ (III.193) and this series of ratios reduces to ${ }^{243 / 32}=3 / 2 \cdot 3 / 2 \cdot 3 / 2 \cdot 3 / 2 \cdot 3 / 2=(3 / 2)^{5}$. Therefore, $81 / 16$ and $243 / 32$ unite or communicate in means because $3 / 2$ is their common base ratio. All ratios not in the series do not have $3 / 2$ as a base, or unit, ratio and are, consequently, incommensurable to all ratios in the sequence $(3 / 2)^{n}$. One can construct other sequences such as (4/3 $)^{n}$, and so on (III.224-27).

Oresme concludes Prop. VII by considering whether the prime numbers of these non-multiple ratios are square or cube numbers (III.228-55). This section is similar to III. 5 , $6-67$ of the preceding sixth proposition. For example (III. $236-44$ ), in the series $(3 / 2)^{n}$ all numerators and denominators are square numbers when $n=2,4,6,8,10, \ldots$. Thus $(3 / 2)^{4}=81 / 16$, where 9 and 4 are the square roots of square numbers 81 and 16 respectively. But
all ratios such as $(3 / 2)^{3}=27 / 8$, whose prime numbers are not squares, have roots that are irrational-i.e., $(27 / 8)^{1 / 2}$ is an irrational ratio.

But if in the series $(3 / 2)^{n}$ the exponent is $n=3,6,9,12,15, \ldots$ (III.24552 ), the numerators and denominators of each ratio will be cube numbers. Thus $(3 / 2)^{3}=27 / 8$, where $27=(3)^{3}$ and $8=(2)^{3}$; similarly, $(3 / 2)^{6}=729 / 64$, where $729=(9)^{3}$ and $64=(4)^{3}$. Oresme notes that any ratio whose prime numbers are not cube numbers cannot itself be the cube, or sixth power, etc., of any rational ratio; and conversely, no rational ratio can be the cube root, or sixth root, etc., of such a rational ratio. For example, a ratio like $5 / 3$ cannot be the root of any ratio in the series $(3 / 2)^{n}$, where $n=3,6,9$, $12, \ldots$. Indeed, $5 / 3$ can bear only an irrational exponential relationship to any ratio in this series.
Finally, there are ratios whose prime numbers are both square and cube. This situation obtains when we have $(3 / 2)^{n}$, where $n=6, \mathrm{I} 2, \mathrm{I} 8,24, \ldots$ In III. $253-55$, Oresme cites only the case where $n=6$. He says, in effect, that if we have a non-multiple ratio, $3 / 2$, whose terms are drawn from each series in the figure on p. ${ }^{234}$, above, and raise this to the sixth power-Oresme says if we "denominate" it by a sextuple ratio-we have $(3 / 2)^{6}=729 / 64$, where $729=(9)^{3}=(27)^{2}$ and $64=(4)^{3}=(8)^{2}$. Thus 729 and 64 have rational cube and square roots.
III.259-302

In Prop. VIII, Oresme first takes up the case of a ratio of prime numbers that can have only one mean proportional assigned between its terms (III. ${ }^{259-62}$ ). He specifies the kinds of lesser and greater rational ratios to which it can be commensurable. Although no letter designations are employed in this proposition, it will be convenient to let $A$ and $B$ represent the ratio of prime numbers and $C$ the sole mean proportional assignable between its terms. Thus $A / B=A / C \cdot C / B$ and $A / C=(A / B)^{1 / 2}$. Hence $A / C$ is a rational ratio and is commensurable to $A / B$. Indeed, it is the only smaller rational ratio that can be commensurable to $A / B$ since all other ratios of the form $(A / B)^{m / n}$, where $m$ and $n$ are numbers prime to each other and $n>m \geqq 1$, will be irrational (III.263-70).
All those greater rational ratios that are related to $A / B$ in sesquialterate, multiple, and multiple sesquialterate ratios are commensurable to $A / B$ (III. 261-62. The ratios $(A / B)^{3 / 2},(A / B)^{2},(A / B)^{3 / 2},(A / B)^{3},(A / B)^{7 / 2}, \ldots$ are all rational ratios greater than $A / B$ and commensurable to it, because $(A / B)^{1 / 2}$ -a rational ratio-is their common measure. It is clear that the terms "ses-

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quialterate" ( $3 / 2$ ), "multiple" $(2 / 1,3 / 1,4 / 1, \ldots)$, and "multiple sesquialterate" $(5 / 2,7 / 2,9 / 2,11 / 2, \ldots)$ are used in an exponential sense since they represent exponents to the base $A / B$. In general all greater rational ratios commensurable to $A / B$ are representable by $(A \mid B)^{n / 2}$, where $n=3,4,5,6, \ldots$.

If between the prime numbers $A / B$ there should be only two possible mean proportional numbers, then the only smaller rational ratios commensurable to it are $(A / B)^{1 / 3}$ (a "subtriple" ratio) and $(A / B)^{2 / 3}$ (a "subsesquialterate" ratio) (III.271-74). Thus, for example if the means are $C$ and $D$, we have $A / B=A / C \cdot C / D \cdot D / B$, where $A / C=C / D=D / B=(A / B)^{1 / 3}$ and $A / D=C / B=(A / B)^{2 / 3}$. Although Oresme does not repeat his earlier remark concerning the case of one mean (III.265-66), it is clear that all other smaller rational ratios will be exponentially incommensurable to $(A \mid B)$ and, with the exception of $(A / B)^{1 / 3}$ and $(A \mid B)^{2 / 3}$, all smaller ratios of the form $(A \mid B)^{m / n}$, where $m$ and $n$ are numbers prime to each other and $n>m \geqq 1$, are irrational.

All greater rational ratios commensurable to $(A / B)$ are related to it in one of the following ways: a sesquitertian ratio ( $4 / 3$ ); superpartient twothirds $\left(\mathrm{I}^{2} / 3=5 / 3\right)$; multiple ratio; multiple sesquitertian $(7 / 3,10 / 3,13 / 3,16 / 3$, $19 / 3, \ldots)$; and multiple superpartient two-thirds $\left(8 / 3,{ }^{11} / 3,14 / 3,17 / 3,20 / 3, \ldots\right)$. In general all rational ratios $(A / B)^{n / 3}$, where $n=1,2,3,4,5, \ldots$, are commensurable to $A \mid B$ since $(A / B)^{1 / 3}$ is their common measure. All ratios represented by $(A / B)^{1 / n}$, where $n$ is any integer other than 3, are irrational and incommensurable to $A / B$. A similar procedure is applied should there be three, or more, mean proportionals assigned between the prime numbers, $A \mid B$.
Finally, in III.284-90, we are told that for every rational ratio $A / B$, all other ratios $(A / B)^{n}$, where $n$ is any integer, are also rational; but smaller ratios of the form $(A / B)^{1 / n}$ may not be rational. Furthermore, not every rational ratio has a greater rational ratio commensurable to it in the form $(A / B)^{3 / 2}$, or $(A / B)^{4 / 3}$, or for certain values of $m$ and $n$ when our ratio is generally $(A / B)^{\frac{m n+1}{n}}$.
III.303-28

Among multiple ratios only a double ratio, namely $2 / 1$, is composed of two superparticular ratios (III.303-4; for superparticular ratios, see II.2.9-II and above on p. 348). Thus $4 / 2=4 / 3 \cdot 3 / 2$. Now $3 / 2$ (a sesquialterate ratio) and $4 / 3$ (a sesquitertian ratio) are respectively the largest ratios in the superparticular genus of ratios, while $2 / 1$ is the smallest ratio in the multiple class.

Therefore, with one exception, every combination of any other two superparticular ratios will compose, or produce, a ratio less than $2 / \mathrm{I}$ and consequently cannot form any multiple ratio whatever. The exception involves the multiplication of two sesquialterate ratios that produces $9 / 4$ (i.e., $9 / 6$. $6 / 4=9 / 4$ ). Although greater than $2 / 1,9 / 4$ is not a multiple ratio (III. 3 I221).
III.329-498

The substance of this quite significant section has been summarized and discussed in the introduction, pp. 40-42, above.

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\text { III.499-5 } 28
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Oresme gives a second method for determining the possible number of ratios that can be formed from a given number of terms. In an immediately preceding procedural conclusion (Prop. XI, a conclusio practica; see III.43839), Oresme had shown how to find the total number of ratios of greater inequality that could be formed from a given number of terms (III.440-69). This formulation may be expressed as $\frac{n(n-\mathrm{r})}{2}$, where $n$ is the given number of terms. Since each term or number is the denomination of a multiple ratio, Oresme applied the same formulation to a determination of the number of "ratios of ratios" of greater inequality that could be formed from a given number of multiple ratios (III.463-74; see also p. 41, above).
The second method (III.499- 528 ) is subdivided into two procedures, one for an even number of terms and another for an odd number of given terms. The rule for an even number of terms may be formulated as $n[(n-2)$ $/ 2]+n / 2$, where $n$ is the given number of even terms (III. so2-4). For example (III.s09-II), twenty-eight different ratios could be formed from eight terms. For an odd number of terms (III. $504-7$ ), the rule is $n[(n-1) / 2]$. Thus seven terms will form twenty-one ratios or combinations. In both methods (III.440-69 and III.499-5 28), one can find the total number of ratios for both greater inequality and lesser inequality (III.) $14-16$ ) since they will always be equal (III.447-53).
By successively using first one and then the other of these two procedures in the second method, it is possible to enumerate the sequence of total combinations of terms, lines, or points (no more than two points are permitted in a straight line) that can be formed (III.s $14-19$ ). The series of
successive total combinations that can be formed is $1,3,6,10,15,21,28$, $36, \ldots$, etc. Oresme says (III.s 19-23) that the successive terms in this series can be generated by first taking I , then $\mathbf{1}+\mathbf{2}=3$, then $\mathrm{I}+2+3=6$, then $1+2+3+4=10$, and so on. By using the formula for the sum of an arithmetic progression, any one of the successive terms in the series of total combinations can be found. Thus if $S_{n}=\frac{n}{2}(a+p)$-where $n$ denotes the number of terms in the series, $a$ the first term and $p$ the $n$th term in the series-the above sequence can be generated. For example, if $n=8, a=1$, and $p=8$, the number of possible combinations will be $S_{n}=8 / 2(\mathrm{x}+8)=$ 36. Oresme observes that the differences between the successive terms follows the natural number series commencing with 2.

## IV.i-8

The first supposition is actually a statement of Bradwardine's function (see pp. 17-19, above), namely that $F_{2} / R_{2}=\left(F_{1} / R_{1}\right)^{V_{2} / V_{1}}$. This supposition will be cited frequently in Ch . IV to connect a ratio of ratios and a ratio of velocities. Thus if two ratios, $A$ and $B$, are related as $A=(B)^{m / n}$, the ratio $m / n$ is a ratio of ratios (see p. 49). Now if $A=F_{2} / R_{2}, B=F_{1} / R_{1}$, and $m / n=V_{2} / V_{1}$, then $F_{2} / R_{2}=\left(F_{1} / R_{1}\right)^{V_{2} / V_{1}}$ and $V_{2} / V_{1}$ is a ratio of velocities that varies as a ratio of ratios (IV.334-37; see also pp. $\int 1-52$, above). The function of Ch. IV, Supposition I, is to justify the use of ratios of ratios-a concept that emerged from the first three mathematical chapters -in representing ratios of motion.
The fact that Oresme cites Aristotle as the one who provided the basis for this supposition must not be taken as evidence that Oresme believed Aristotle had formulated "Bradwardine's function." On the contrary, in the De proportionibus Oresme is dubious whether Aristotle held the "correct" theory (IV.165-72), while in his Le Livre du ciel et du monde-written in 1377 long after the De proportionibus--he is quite emphatic in accusing the Stagirite of being a proponent of the false theory that was refuted at great length in the De proportionibus (IV.76-164; for the false theory and its refutation by Bradwardine, see pp. 17-19, above; the passage in Le Livre du ciel et du monde attributing the false theory to Aristotle is found below, pp. 368-69).
Why, then, does Oresme cite Aristotle approvingly in connection with a supposition enunciating Bradwardine's function? The answer is simply that it is not with respect to Bradwardine's function that Aristotle is cited with approval, but rather with reference to the opening sentence of Supposition I , which asserts, "A velocity varies as the ratio of a motive power
to a mobile or the resistance of a mobile" (IV.3-4). ${ }^{20}$ It is a general view that all velocities arise from some ratio of force and resistance-i.e., $F / R$. Indeed, Oresme had already expressed agreement with Aristotle on this point (I.s-7; see also p. 309, above). This opinion, though subscribed to by partisans of Bradwardine's function, was not accepted by all scholastics since Bradwardine himself refuted, at considerable length, one opinion that $V=F-R$ (Crosby, Brad., pp. 32, 86-93; Oresme dismisses it in I.2-3; see also pp. 308-9, above).

The extent of familiarity and acceptance of Bradwardine's function is illustrated by the fact that where Bradwardine propounded his function as Theorem I of Ch. III (Crosby, Brad., pp. 112, II3), Oresme expresses it as an assumption (suppositio).
It is difficult to provide precise Bekker number references for the broad citations by Oresme (IV.7-8) to books in the De caelo and Pbysics. I have found no suitable reference in Bk. II of the De caelo, but if Oresme intended Bk. III, the following possibilities are offered: De caelo III.2.3orb.4-s; Pbysics IV.8.215b.1-11; Pbysics VII.5.249b.27-250a.20.

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In Supposition II, Oresme states that if $A / C=A / B \cdot B / C$ and $A / B>B / C$, then $A / C<(A / B)^{2}$ and $A / C>(B / C)^{2}$. In Bradwardine's Tractatus de proportionibus, a particular case of the first part of the second supposition appears as Theorem II and a particular case of the second part as Theorem III of Ch. I, Part 3 (Crosby, Brad., pp. 78-81).

## IV.12-18

Suppositions III and IV require no further exposition. In Supposition V (IV.16-18), if $A / C=A / B \cdot B / C$ and $A / C=(B / C)^{2}$, then $B / C=A / B$.

[^38]But if $A / C>(B / C)^{2}$, then $B / C<A / B$ (Oresme would call $A / B$ the "remainder"); and if $A / C<(B / C)^{2}$, then $B / C>A / B$. If $A / C>(A / B)^{2}$, then $A \mid B<B / C$; and if $A / C<(A \mid B)^{2}$, then $A \mid B>B / C$.

## IV.19-26

Supposition VI is best discussed in terms of the composed ratio $A / C=$ $A / B \cdot B / C$. If, for example, $A / B$ is commensurable to $A / C$-i.e., where $A / B=(A / C)^{m / n}$ with $m$ and $n$ integers in their lowest terms and $m<n$ then $B / C$ must also be commensurable to $A / C$ so that $B / C=(A / C)^{p / n}$, where $p$ and $n$ are mutually prime and $p<n$. Indeed, $m+p=n$. Furthermore, if $A / B=(B / C)^{m / p}$, then $A / B$ is also commensurable to $A / C$, the whole ratio. Oresme's reference to a previous supposition (IV.24-25) is to the second supposition of the first part of Ch. II (II.i.ros-9), which states, in effect, that if $A / B$ is a part, or parts, of $A / C$, then $B / C$ is also a part, or parts, and both ratios are denominated by $n$, the denominator of the exponent.

## IV.27-36

Supposition VII is the negation of the preceding supposition, stating the case for incommensurable relations between part and whole and part to part. In this supposition Oresme actually represents parts and whole by letters. He sets $A=B \cdot C$ and we may interpret these as three ratios. Repeating part of the substance of Supposition VI, he says that if $B$ and $C$, the parts, are commensurable-i.e., $B=C^{m / p}$ (see above), then $A$, the whole, is commensurable to each so that $A=B^{m / n}$ and $A=C^{p / n}$. But if $A \neq B^{m / n}$ and $A \neq C^{p / n}$, then $B$ and $C$ are incommensurable. This follows from denying the consequent (IV.32). That is, by assuming that $A$ is not commensurable to both $B$ and $C$, it follows that $B$ and $C$ are incommensurable. For if $B$ and $C$ were commensurable, it follows from the first part of Euclid X. $9^{2 \mathrm{I}}$ that $A$, the whole, would be commensurable to each. Therefore, from the assertion that $A$ is not commensurable to $B$ and $C$, it is necessary that $B$ and $C$ be incommensurable.
${ }^{21}$ The enunciation of Euclid X. 9 reads "Si fuerint duae quantitates communicantes, totum quoque ex eis confectum utrique earum erit communicans. Si vero fuerit totum utrique commensurabile, erunt am-
bae commensurabiles."-Euc.-Campanus, p. 25 I . It is only the first sentence that Oresme quotes practically verbatim in IV. 33-35.

Having considered the relations between part and whole, and part to part, in terms of commensurability and incommensurability, Oresme, in the eighth supposition, draws upon the three suppositions of II.I.99-113 and states the conditions for actually determining the exponential relations between these entities. If, once again, $A=B \cdot C$ and $A=B^{m / n}$, where $m / n$ is rational and in its lowest terms with $m>n$, then should $m / n$ be known, the exponential relations between $B$ and $C$, as well as those between $A$ and $C$, canalso be known. For example (IV. 45-48), if it is known that $A=B^{3}$, then it follows that $B=A^{1 / 3}$ and $C=A^{2 / 3}$. Consequently, $B=C^{1 / 2}$ and $A=C^{3 / 2}$.

Oresme also cites as fundamental to this supposition Props. IV and V of Bk. 11 of Jordanus de Nemore's De numeris datis. Oresme quotes the enunciations (IV.50-55) substantially as they are given in the text published by Maximilian Curtze. ${ }^{22}$ It will be immediately apparent that Jordanus' arithmetical propositions are related to Oresme's eighth supposition only by analogy and could not possibly represent what Oresme has described.

In Prop. IV Jordanus says, in effect, that if $A=B+C$, then $\frac{B+C}{C}-\mathrm{r}=$ $B / C$. As Jordanus expresses it: "If I is subtracted from the ratio of a whole to a part [of that whole], a ratio of the remaining part to the [initial] part will remain" "Si enim a proportione totius ad detractum tollatur unum, remanebit proportio residui ad detractum"). In an example, where $10=$ $7+3$, we get $10 / 3=3^{1 / 3}$ so that $7 / 3=2^{1 / 3}$.
Although Oresme uses similar language, saying that if we know the relationship between whole to part, we can come to know the ratio between that part and the remaining part, it is perfectly clear that Oresme is relating ratios exponentially, while Jordanus is relating numbers aritbmetically.
The same may be said for Bk. II, Prop. V, of the De numeris datis. There also: $A-B+C$ and Jordanus asserts that if $\frac{B+C}{C}$ is known, one can determine $\frac{B+C}{B}$. By Prop. IV we know that $\frac{B+C}{C}-\mathrm{r}=B / C$ and having found $B / C$ we obviously know $C / B$, so that $\frac{B+C}{B^{-}}=C / B+\mathbf{1}$. As Jordanus states sis.
sit
V/22, Commeritar zu dem Tractatus de Nu- Physik, Vol. 36, 1-23, 41-63, 81-95, 12 Imeris Datis des Jordanus Nemorarius;" ed. 38. Props. IV and V of Bk. II appear on Curtze, in Zeitschrift für Matbematik und pp. 42-43.
it: "If the ratio between a whole and part [of that whole] is given, then the ratio of the remaining part to the [initial] part can be determined, as can the ratio of the [initial] part to the remaining part. Therefore, the ratio of the whole to the remaining part [can be determined]" ("Si enim totius ad detractum proportio fuerit data, et residui ad detractum erit data, quare detracti ad residuum, ergo et totius ad residuum"). As an example we find $10=6+4$, where $10 / 6=1^{2} / 3$ and $4 / 6=2 / 3$ so that $6 / 4=1 \frac{1}{2}$ and, finally, $10 / 4=21 / 2$.

## IV.59-75

The final, and ninth, supposition states that if we know the ratio that obtains between two quantities, $A$ and $B$, and also know the specific value for $A$ or $B$, we can determine the other term in the ratio $A / B$. Thus (IV. 67-74) if $A / B$ is a sesquialterate ratio, then $A / B$, when reduced to its prime terms, is $3 / 2$. Now if $A=9$, then $A / B=9 / 6$ since $3 / 2=9 / 6$. Thus $B$ is 6 and is found by multiplying $9^{\cdot} 2=18$ and dividing 18 by 3 .

In support of the ninth supposition Oresme cites Bk. II, Prop. II, of Jordanus' De numeris datis, which Curtze gives as, "Si dati numeri ad aliquem fuerit proportio data, et illum datum esse consequitur." ${ }^{23}$ (For my translation of this proposition, see p. 267 , above.) Curtze represents the substance of the proposition as follows: (I) if $x / a=b$, then $x=a b$; and if $a / x=b$, then $x=a \mid b$, where $x$ is unknown. Should a whole number and fraction be involved, then $x / a=m+p / q$, where $x=m a+(p / q) a$; and if $a / x=m+p / q$, then $x=a /(m+p / q)$. Oresme's example in Supposition IX (IV.67-74) corresponds to $a / x=b$, where $a=9$, and $b$ is given as a sesquialterate ratio, namely $3 / 2$.

All this seems perfectly proper until we see how Oresme uses the ninth supposition. In Ch. IV, Prop. IlI (IV.243-46), where $E=(F)^{m / p}$ and both $m / p$, the exponent, and $F$, a given ratio, are known, Oresme invokes the ninth supposition in order to determine the unknown ratio, $E$. Similarly, $D=(F)^{n / p}$, where $n \mid p$ and $F$ are known; therefore, by Supposition IX, ratio $D$ can be determined. Oresme tells us (IV.64-66) that Jordanus demonstrated Bk. II, Prop. II, for numbers only, but that by Euclid X.s the proposition applies also to quantities. It is evident, however, that Oresme does not simply mean quantities, but ratios of quantities so that once again Euclid X.s is used in a special exponential sense (see above on p. 30 ).
${ }^{23}$ Ibid., p. 41. Oresme gives an identical (IV.63-64). Oresme's citation is preferable quotation but adds, "datum id est notum"

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\text { IV. } 76-172
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The first proposition has been fully summarized in the introduction (above, pp. 43-47), so it only remains to say something about the last paragraph (IV.165-72) and to offer a brief estimate of the validity of Oresme's criticisms of Aristotle.

Although in IV.165-72 Oresme is not quite correct in saying that Aristotle enunciated the two false rules, ${ }^{24}$ it is plausible to assume that Aristotle would have accepted them. But Oresme was uncertain whether Aristotle meant to add, or had added, certain qualifications that would have saved his rules but which, for some reason, may have been lost in translation. He points out that if the Aristotelian rules had contained something to the effect that the initial ratio $F / R$ must equal $2 / \mathrm{I}$, then his rules would have been converted into particular cases true in both theories (see the tabular figure on P. 21 and Oresme's examples in IV.98-105). If the rules were not qualified in this way, one must conclude that Aristotle was in error.
Some years later, in his Le Livre du ciel et du monde, Oresme abandoned his elaborate argument against the second false rule (IV.in8-64) and his attempt to save Aristotle. No longer concerned whether Aristotle was properly translated, he became convinced that Aristotle held the false rules. In commenting on De caelo I.6.274a.1-3, Oresme says:

He (Aristotle) wishes to say that just as the motive power is greater, so it moves in less time and more quickly, if other things are equal. He says the same thing in the seventh book of the Pbysics (VII.5.249b-250a).
But, saving his reverence, it is not well stated because from this statement it would follow that a power could move a resistance equal to itself and that any power, however small, could move any resistance, however large. I shall demonstrate and prove this, it having been posited that a power moves a resistance with a certain swiftness (isneleté). I make the supposition that it is possible for a power to be just that much less than the original power that it can move this same resistance by a speed which is exactly one-half the posited velocity. And I suppose that there can be another power which can move it with a speed one-fourth of the posited velocity, and another with a speed of one-eighth that velocity and so on. And according to Aristotle here and in the seventh book of the Pbysics, the second power will be one-half of the first, the third will be one-fourth the first, and so on.
${ }^{24}$ The first rule finds no direct counterpart in Physics VII. $\varsigma$, but is clearly implied. The second rule is given in two forms as follows: "If, then, $A$ the movent have moved $B$ a distance $C$ in a time $D$, then in the same time the same force $A$ will move $1 / 2$
$B$ twice the distance $C$, and in $1 / 2 D$ it will move $1 / 2 B$ the whole distance $C$."-Works of Aristotle, ed. Ross, Vol. 2, 249b.30-2 50 oa . 2. It should be noted that Aristotle did not express his rules in terms of velocity, as does Oresme in IV.76-79.

## Critical Notes to Pages 268-274

Thus any power, however small, would move the resistance some degree more slowly than the large power. So, for example, if a power is as 8 and the resistance is as 4 , and the movement takes place in one day, then according to Aristotle the power which would produce such a movement in two days exactly would be 4 , and hence it would be equal to the resistance. Further, that power which would produce this movement in four days would be 2 , and so it would be less than the resistance. A similar situation obtains if we proceed further. This is illogical (inconvenient) and impossible. ${ }^{25}$

Oresme's criticism of Aristotle wholly ignores the latter's qualifications embodied in his shiphauler argument in Physics VII.5.250a.15-20, where he says:

It does not follow that, if a given motive power causes a certain amount of motion, half that power will cause motion either of any particular amount or in any length of time: otherwise one man might move a ship, since both the motive power of the shiphaulers and the distance that they all cause the ship to traverse are divisible into as many parts as there are men.

It is clear that Aristotle placed physical limits to his rules of proportion in the sense that in any case where motion is not produced the rules are suspended. Had Oresme properly taken into account the shiphauler argument, he would not have attributed to Aristotle the absurd consequence that any force, however small, could move any resistance, however large, with some velocity. Oresme's complete silence on the shiphauler passage is puzzling because it is implausible to suppose that he could have been unaware of it. How are we to explain this? Perhaps, as follows. Those who opted for Bradwardine's function would not concede that anyone accepting the Aristotelian law, $V \propto F / R$, might also have been aware of, and accepted, the limitations to the application of this law. Such an admission would have robbed them of their major counterargument. The absurd consequence that any force, however small, could produce a velocity in any resistance, however large, was deducible from an initial condition where $F>R$ and motion occurs-provided that one ignored or reinterpreted the shiphauler illustration. Aristotle's shiphaulers were a potential embarrassment that could not

25 The translation is by Clagett, Science of and Alexander J. Denomy, in Mediaeval Mechanics, pp. 463-64. Clagett's translation is based on the text of Oresme: Le Livre du ciel et du monde, edited by Albert D. Menut

Studies, Vols. 3-5. The particular section quoted here is from Vol. 3, 216.
be taken as intended, namely in physical situations where the rules were evidently inapplicable and contradictory. ${ }^{26}$
IV.I73-224

Although in the enunciation (IV.173-75), Oresme speaks only of determining whether the unknown ratio is greater or less than some proposed given ratio, he does also consider the case where the unknown ratio is equal to the proposed ratio (IV.178-79 and 189-98).
The example illustrating the three cases in IV. 189-207 (see above, p. 48) is as follows (IV.208-14): Assume that $A$ is a known ratio, $B$ unknown, and that $A=E / F=3 / \mathrm{I}$ and $B=D / E$, where $D$ is a force and $E, F$ are mobiles. Now if $D / F=(D \mid E)^{2 / 2}$, then $D / E=E / F$; and consequently $E \mid F=B=3 / \mathrm{I}$ and $B=A$. But if $D / F>(D \mid E)^{2 / \mathrm{r}}$, then $D / E<E \mid F$ and $B<A$ or less than $3 / 1$; and if $D \mid F<(D \mid E)^{2 / 2}$, then $D|E>E| F$ and $B>A$ or $3 / 1$.
In the discussion in IV.181-207, the given ratio, $A$, is equal to $E \mid F$, a ratio of mobiles or resistances. But the same results are obtainable if the mobile is held constant and the given tatio is a ratio of forces (IV.215-20).
In the final paragraph (IV.22I-24), Oresme refers to Aristotle's discussion on the "quicker" and "slower" in Pbysics VI.2, in order to clarify the reader's understanding of what is meant by saying that one body moves with twice the velocity of another; or more or less than twice the velocity of another. Aristotle says (Physics VI.2.232a.25-29): "...it necessarily follows that the quicker of two things traverses a greater magnitude in an equal time, an equal magnitude in less time, and a greater magnitude in less time, in conformity with the definition sometimes given of 'the quicker.' " The reader is expected to apply this statement, as well as the remainder of Aristotle's discussion in Physics VI.2, to Prop. II of Ch. IV. For an excellent treatment of Aristotle's use of the terms "quicker" and "slower," and a translation of the significant passage from a medieval Latin text, see Clagett, The Science of Mechanics in the Middle Ages, pp. 176-78.

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## IV.22 5-70

The specific example illustrating IV.228-47 (see above on p. 49) is given in IV. 248-58. Let $D=A / C, E=A / B$, and $F=B / C$ so that $D$ $=E \cdot F$ ( $A$ is motive force; $B$ and $C$ are resistances). If $B / C=2 / \mathrm{I}$ and $A / C=(A \mid B)^{3 / 1}$, then $D=E^{3 / 1}$, where $3 / \mathrm{I}=G$. Consequently, $E=(D)^{1 / 3}$ and $F=(D)^{2 / 3}$ so that $D=(F)^{3 / 2}$ and $E=(F)^{1 / 2}$. By hypothesis $F=$ $B / C=2 / 1$, which determines that $E=(2 / 1)^{1 / 2}$ and $D=E \cdot F=(2 / \mathrm{I})^{1 / 2}$. $(2 / \mathrm{I})=(4 / \mathrm{I})^{3 / 4}(\mathrm{IV} .253-5 \mathrm{~s})$. We have now found the two ratios, $D$ and $E$, which $\operatorname{are}\left(4 / I^{3}\right)^{3 / 4}$ and $(2 / \mathrm{I})^{1 / 2}$, respectively. When applied to the ratios of force and resistance, we see that $A / C=(4 / \mathrm{I})^{3 / 4}$ and $A / B=(2 / \mathrm{I})^{1 / 2}$ and the ratio of velocities, or "ratio of ratios," to which they give rise is $3 / \mathrm{I}$ since $(4 / \mathrm{I})^{3 / 4}$ $=(2 / 1)^{3 / 2}$ and $(2 / 1)^{3 / 2}=\left[(2 / 1)^{1 / 2}\right]^{3 / 1}$, or $A / C=(A / B)^{3 / 1}$.
Referring to the example above and citing Prop. I of Ch. IV, Oresme warns (IV.259-6I) that simply because the ratio of mobiles is $B / C=2 / \mathrm{I}$ it does not follow that $A$ will move $C$ with twice the velocity with which it moves $B$. Only if $A \mid B=4 / 2$ and $B / C=2 / 1$ would $A$ move $C$ twice as quickly as it moves $B$. In that event, $A / C=4 / \mathrm{I}$ since $A / C=A / B \cdot B / C$ (IV.98-100 and 143-54). But in other cases this does not follow (IV.iriz17, 143-64; see also p. 46, above).

## IV.271-337

The examples in Prop. IV (IV.309-3I) will now be summarized (see also pp. so- 5 I , above). The basic data are the same as given on p . $\varsigma \mathrm{o}$. Let $A$ be the known ratio, where $A=E \mid F=2 / 1$; and $B=D \mid E$ is unknown. Assume that force $D$ moves $E$ one mile in a certain time and, in an equal time, $D$ moves $F$ (where $D / F=C$ ) a distance that is incommensurable to a mile but related to a mile as the diagonal of a square to its side. This may be formulated as follows: $D / F=(D \mid E)^{V_{F / V E}}$, where $V_{F} / V_{E}=(2 /)^{1 / 2}$. Thus the ratio of velocities is actually an irrational exponent (see p. 35). Now because $D / F=C, D \mid E=B$, and $V_{F} / V_{E}=m \mid n$, it is obvious that $C=(B)^{m / n}$, where $m / n=(2 /)^{1 / 2}(m / n$ is, therefore, not a ratio of integers, as it was in the discussion on p. 49) and constitutes an "irrational ratio of ratios" (Oresme does not make this explicit, but see p. 50). Since $C$ and $B$ are incommensurable, it is deducible, by the seventh supposition, that $B$ and $A$ are also incommensurable. That is, since $B \neq(C)^{n m m}$, where $n$ and $m$ are integers and $n<m$, it is also true that $A \neq(C)^{p / m}$ and, consequently, $B \neq$ $(A)^{n p}$ where $n$ and $p$ are integers. It is evident that we cannot know ratio $B$.

In this example (IV.309-17), Oresme speaks of power $D$ moving mobiles $E$ and $F$ over distances rather than with certain velocities. In the formulation, however, we used a ratio of velocities, $V_{F} / V_{E}$, and not a ratio of distances. This is warranted by IV.332-34 (and later by IV.55 2-56), where Oresme says: "It must be understood that from a ratio of times, and from a ratio of distances traversed or acquired, or any such [quantities], one can arrive at and know a ratio of velocities, as is evident from the sixth and seventh [books] of the Physics [of Aristotle]." If, in the present example, we let $S$ represent distance, then $S_{F}\left|S_{E}=V_{F}\right| V_{E}$ when $\mathrm{T}_{F}=T_{E}$, where $T$ is time. Oresme might have couched the example in terms of time, in which event $T_{E} / T_{F}=V_{F} / V_{E}$ when $S_{F}=S_{E}$. In other words if $F$ is force, $R$ resistance, and $V$ velocity, the following expressions, subject to the conditions just specified, are interchangeable: (1) $F_{2} / R_{2}=\left(F_{1} / R_{\mathrm{I}}\right)^{V_{2} / V_{1}}$; (2) $F_{2} / R_{2}=\left(F_{1} / R_{1}\right)^{S_{2} / S_{1}}$; (3) $F_{2} / R_{2}=\left(F_{1} / R_{\mathrm{I}}\right)^{T_{1} / T_{2}}$. The references to Aristotle are probably to Pbysics VI. 2 and VII.5. In the former book, Aristotle discusses the "quicker," while in the latter, he gives his rules of proportion. But in both places he considers motion in terms of distance and time (see above, pp. $368 \mathrm{n} .24,370$ )-not in terms of ratios of velocities. The scholastics, however, usually spoke of tatios of velocities.
In the second example (IV.318-23), Oresme in effect formulates a "rational ratio of ratios." If $S_{F} / S_{E}=3 / \mathrm{I}$, then $V_{F} / V_{E}=3 / \mathrm{I}$ and $D / F=$ $(D \mid E)^{3 / 2}$, from which it follows that $C=(B)^{3 / 4}$. Consequently, $B=(C)^{1 / 3}$ and $A=(C)^{2 / 3}$ so that $A=(B)^{2 / 1}$, or $B=(A)^{1 / 2}$. Now $A=2 / 1$ so that $B=(2 / \mathrm{I})^{1 / 2}$; and ratio $B$ has been made known. If, however (IV.324-27), $A=4 / \mathrm{I}$, then $B=(4 / \mathrm{I})^{1 / 2}=2 / \mathrm{I}$ and $C=(2 / \mathrm{I})^{3 / 1}=8 / \mathrm{I}$.
IV.338-56

In IV. 35 1- 56 Oresme offers an example to illustrate the substance of the proposition that has already been summarized above on p . 52 . It is assumed that ratio $A$ is known and $B$, unknown. If $A=2 / \mathrm{I}$ and the ratio of velocities produced by ratios $B$ and $A$ is $V_{B} / V_{A}=4 / \mathrm{I}$, we have sufficient data to solve for $B$ in the formulation $B=(A)^{V_{B / V A}}$. Thus $B=(2 / \mathrm{I})^{4 / 1}$ so that $B=16 / \mathrm{I}$, or a proportio sedecupla.
The method for expanding $(2 / \mathrm{I})^{4 / 1}$ is given, says Oresme (IV.353-55), in Ch. I. In all probability he is referring to the section on "adding" ratios by assigning extreme terms (I. 75-83; see also pp. 314-15, above).

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\text { IV. } 357-495
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The enumeration of ratios in IV. $362-65$ corresponds to the three categories of ratios specified in III. $336-4 \mathrm{I}$ (seep. 3 II ). Oresme devotes a few lines (IV. 368-73) to the third category, namely to those ratios whose denominations are unknowable, by which he means any ratio $(A \mid B)^{p / q}$ where the exponent $p / q$ is irrational and, presumably, $A / B$ is rational. Such ratios can be dealt with only approximately, in the sense that, by Prop. II of Ch. IV, it is possible to locate any one of them between greater and lesser ratios so that $D \mid E>(A / B)^{p / q}>F / G$, where $D / E$ and $F / G$ are known.
The first example of the first case (IV.438-43; see also above, p. 54) in which Oresme describes how to find two lines denominating an irrational ratio of the form $(A \mid B)^{m / n}$, where $m<n$ and $m|n, A| B$ are rational with $(A \mid B)^{m / n}<A \mid B$, sets $(A / B)^{m / n}=(2 / \mathrm{I})^{1 / 2}$. Two lines are assumed to be related as $A / B$ and a mean proportional line, say $C$, is assigned between them. Since the numerator of the exponent is r , Oresme compares $C$ to either of the lines $A$ or $B$. Thus $A / C=(2) /(2)^{1 / 2}$ and $C / B=(2)^{1 / 2} / \mathbf{1}$. In either event, the longer line represents the force, the shorter line, the resistance. (For the second example of the first case [IV.444- $\rho 0$ ], see above, p . 55.)

In the second example of the second case (IV.485-9I; for the first example in IV. 464-82, see above, pp. $55-56$ ), the irrational ratio is $(A / B)^{m / n}$, where $m>n$ and $A \mid B, m / n$ are rational. Oresme lets $(A \mid B)^{m / n}=(2 / \mathrm{T})^{3^{3 / 4}}$, or $(2 / \mathrm{I})^{15 / 4}$. He first assigns two lines, $A$ and $B$, related as $(2 / \mathrm{I})^{3}=8 / \mathrm{I}$ (an octuple ratio), and then lets $C$, an extreme term, be half of $B$ so that $B / C=$ $2 / \mathrm{I}$. Since $A / B=8 / \mathrm{I}$ we can set it equal to ${ }^{16} / 2$ and we then have $A / B$. $B / C=16 / 2 \cdot 2 / \mathrm{I}$. Now assign three mean proportional lines, $D, E$, and $F$, between $B$ and $C$, producing the series of ratios $A / B \cdot B|D \cdot D| E \cdot E \mid F$. $F / C$, where $A / B=16 / 2$ and $B / D=D|E=E| F=F / C=(2 / \mathrm{I})^{1 / 4}$. That is, we have the series of ratios $16 / 2 \cdot 2 /(2)^{3 / 4} \cdot(2)^{3 / 4} /(2)^{2 / 4} \cdot(2)^{2 / 4} /(2)^{1 / 4} \cdot(2)^{1 / 4} / \mathrm{x}$. Thus $A / F$ is the ratio of lines representing the given ratio $(2 / F)^{3^{3 / 4}}$ since $A \mid B=16 / 2=(2 / 1)^{3}$ and $B / F=(2 / \mathrm{I})^{3 / 4}$. When ratios $A \mid B \cdot B / F$ are composed we obtain $A / F=(2 / \mathrm{r})^{3^{3 / 4}}$, where line $A$ represents motive force and $F$ resistance.
The same result is attainable if $A / B=8 / \mathrm{I}$ and $C=2 A$ or $C / A=16 / 8$. Thus $C / A \cdot A / B=16 / 8 \cdot 8 / \mathrm{r}$. Three mean proportional lines, $D, E$, and $F$, are assigned between $C$ and $A$ to produce the series of ratios $C / D \cdot D / E$. $E|F \cdot F| A \cdot A \mid B$, where $C|D=D| E=E|F=F| A=(2 / \mathrm{I})^{1 / 4}$ and $A \mid B$ $=8 / \mathrm{I}=(2 /)^{3}$. Ratio $D / B$ will be the required ratio since $D / B=D / A$.
$A / B=(2 / \mathrm{I})^{3 / 4} \cdot(2 / \mathrm{I})^{3}$ (note that ratio $C / D$ has been eliminated). Obviously, $D$ will represent force and $B$, resistance.
In this example Oresme has generated a geometric series only between the terms $C$ and $A$ with $A / B$ remaining outside the series, although linked to it by a common term which allows the ratio $D / B$ to be composed.
The sections in Aristotle and Averroes, to which Oresme refers in IV. $415-16$, are uncertain. His statement that the heavens do not resist the motion of the planetary orbs may be based upon Aristotle's description of the unchanging celestial aether (De caelo I.3.269b.14-270b.25) or is, perhaps, a reference to arguments for the regularity of celestial motions ( $D e$ caelo II.6.288a.25-34 and II.6.288b.8-22). In his comments upon these sections in the De caelo, Averroes does not, anymore than Aristotle, state explicitly that the heavens do not resist the motions of the planetary orbs. (See Text 22 of Bk. I and Texts 36 and 37 of Bk. II in Vol. s of Aristotelis opera cum Averrois commentariis, fols. 171-18r, 119v-I20v, and 12If-121v, respectively.)
A popular medieval question was based on Aristotle's discussion of the regularity of celestial motion. It took the form of asking whether the heav-ens-including, it seems, the planets-are weakened in their unceasing motions. Jean Buridan, for example, says:"Aristotle and the Commentator [i.e., Averroes] hold the opposite of this in the beginning of the second book of this treatise [i.e., the De caelo]. For Aristotle says that they [i.e., the planets] are moved without labor, and without any violence or difficulty; and the Commentator says [that they move] without fatigue because there is no contrariety in the heavens." For the Latin text, see E. A. Moody's edition of Iobannis Buridani Quaestiones super libris quattuor de caelo et mundo, Bk. 2, Questio 1, p. 130.

## IV.496-6I4

For a full discussion of Prop. VII, see pp. 56-6r, above.
Oresme's remarks in IV. $552-56$ are essentially the same as those in IV. 332-34 (see p. 372) and are introduced into Prop. VII as a necessary preliminary to a discussion of celestial motion, where distance and time are the prime dimensions involved in the kinematic treatment of heavenly motion (IV.399-416; see also pp. s2-53, above). $^{\text {s }}$.

After enunciating the fundamental proposition that any two celestial motions are probably incommensurable (IV.573-76), Oresme adds: "... this seems especially true since, as I shall declare afterward, harmony comes

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from incommensurable motions" (IV. $577-78$ ). No such discussion appears in either the De proportionibus or Ad pauca respicientes, but this topic is considered in a revised and expanded version of the $A d$ pauca (see above, p. 79), namely in Oresme's De commensurabilitate vel incommensurabilitate motuum celi, Part II, Prop. IV (Vat. lat. 4082, fol. 103v, c. i) and in scattered remarks in Part III (Vat. lat. 4082, fols. ros v-ro8v).
In IV. $579-82$, Oresme says that from the basic assertion "that any celestial motion might be incommensurable to any other celestial motion, many very beautiful propositions that I arranged at another time follow, and I intend to demonstrate them more perfectly later, in the last chapter ...." It seems that prior to, or during the course of, the composition of the De proportionibus, Oresme had already formulated the basic substance of most of the propositions scheduled for inclusion in the final chapter (see p. 76). A sampling of these propositions is given in IV. $83-600$, and they reveal that in Oresme's mind the Ad pauca respicientes, which originally contained these propositions, was to constitute a single "last chapter" (see p. 80). This is clear from the fact that three of the four "sample" propositions find their closest counterparts in the first part of the $A d$ pauca. The first proposition (IV.583-87) appears as a consequence of Part I, Prop. V (API.I53-56); the second (IV.588-90) is similat to Part r, Prop. IV (APr.122-45; see also p. 433, below); the third (IV.591-94) to Part I, Prop. IX (APr.193-211); and the fourth (IV.595-600) is substantially close to a portion of Part 2, Prop. XVIII (AP2.217-19). ${ }^{27}$ These propositions will be discussed in connection with the $A d$ pauca respicientes.
The translation of the passage from Daniel 12:1 (IV.599-600) is that of the Douay version of the Old Testament. ${ }^{28}$ My translation of the somewhat obscure lines in IV. $600-603$ is, I trust, substantially correct. The words organa sunt primi, sunt instrumenta supremi (IV.602-3) are apparently a quotation from a work by Bernard Silvester (fl. II 50 ), since they are repeated by Oresme-with explicit attribution to Bernard-in his Le Livre du ciel et du monde (Bk. II, Ch. 2, 69a), where he says:
Item, en influence et vertu, car par les mouvemens et par les lumieres et par les influences celestielz est gouverné tout cest monde cibas, si comme il appert ou

27 In the De commensurabilitate there For these propositions, see Grant, seems to be no specific proposition cor- ''Oresme: Comm.," pp. 443, 447, and 453, responding to the first one cited above, but there are correspondences with the last three. The second proposition is found in Part II, Prop. I; the third in Part II, Prop. VII; and the fourth in Part II, Prop.XI.
respectively.
${ }_{28}$ The Holy Bible translated from the Latin Vulgate...The Douay Version of the Old Testament; The Confraternity Edition of the New Testament (New York, 1950).
premier de Metheores, et est a entendre excepté ce qui depent de liberté, de volonté et souz Dieu, car les cielz sont instrumens de Dieu par lesquelz Il oeuvre selon ce que il Lui plaist, si comme di(s)t Bernardus Silvester: organa sunt primi, sunt instrumenta suppremi.
Menut and Denomy were unable to identify the work from which the quotation was taken, but cite a similar idea from Bernard's De universitate mundi, II, I3, in their Maistre Nicole Oresme, Le Livre du ciel et du monde, in Mediaeval Studies, Vol. 4, 167. Oresme repeats the same quotation in Bk. II, Ch. 22, 135a, in Mediaeval Studies, Vol. 3, 266. In the De proportionibus it seems that Oresme has quoted more than appears in the Le Livre du ciel et du monde and I have interpreted IV.6oi-3, from "prima" to "supremi," as a direct quotation from "the poet," Bernard Silvester. ${ }^{29}$ For a discussion of Bernard Silvester and his works, see Lynn Thorndike, A History of Magic and Experimental Science, Vol. 2, 99-123.

From his principle that any two celestial motions are probably incommensurable and from the many propositions derivable from it, Oresme says that one can attack "Many errors about philosophy and faith... as [for example], that [error] about the Great Year which some assert to be 36,000 years, saying that celestial bodies were in an original state and then return [to it in 36,000 years] and that past aspects are arranged again as of old;..." (IV.606-9). This appears to be a reference to the following article-one of 219-condemned as contrary to faith in 1277 by Bishop Etienne Tempier of Paris: " 6 . That when all the celestial bodies have returned to the same point-which will happen in 36,000 years-the same effects, now in operation, will be repeated." 30
${ }^{29}$ After reading page proofs for this volume, I discovered the entire quotation from "prima" to "supremi" (IV. 601-3) in the Pseudo-Ovidian De Vetula, probably written by Richard of Fournival (ca. i201-ca. 1260). Barring a possible mistake on Oresme's part, his attribution to Bernard Silvester (see above) of that part of the quotation from "organa" to "supremi" may signify that Richard of Fournival borrowed these words from Bernard and himself furnished the first part of the quotation. On this interpretation, Oresme's source for the entire quotation in the De proportionibus would be the De Vetula, rather than a work of Bernard Silvester. It is, of course, possible that all of this brief passage originated with Bernard and
was subsequently plagiarized in the $D e$ Vetula, so that either Bernard's unidentified work or the De Vetula could have served as Oresme's source. See Brunellus Vigelli et Vetula Ovidii [edited by S. Closius?], Wolfenbüttel, 1662, p. 58 (the $D e$ Vetula is the second work and is separately paginated).
30" 6 . Quod redeuntibus corporibus celestibus omnibus in idem punctum, quod fit in xxx sex milibus annorum, redibunt idem effectus, qui sunt modo."-Denifle and Chatelain, Chartularium, Vol. I, 544 . See also APr.7-10, and below on pp. 42931. Duhem, in Systeme du Monde, Vol. \&, $419-23$, cites a number of articles from the Condemnations of 1277 which were expressly directed against astrology.

## Ad pauca respicientes

# Manuscripts and Editions 

## Manuscripts Used in Establishing Text

The five manuscripts listed below have been collated in their entirety.

1. $A=$ Erfurt, Wissenschaftliche Bibliothek, Amplonius Q.385, fols. 15St-1580.
This codex has already been described, since it also contains Oresme's De proportionibus proportionum (see pp. 126-27). The treatise lacks title and author, but Schum, who was ignorant of Oresme's authorship, referred to it as Collectio conclusionum matbematicarum et naturalium. ${ }^{1}$ The scribe who copied the De proportionibus also copied the $A d$ pauca respicientes. There are no diagrams in this manuscript.
2. $F=$ Paris, Bibliothèque Nationale, fonds latin, 7378 A , fols. $14 \mathrm{~V}-17 \mathrm{v}$.

This appears to be a relatively early manuscript that was probably copied sometime during 1362 in Paris. Duhem has noted that the scribe who copied this treatise also copied the immediately preceding work, the Practica geometriae of Dominicus de Clavasio (or Chivasso). In the colophon of the latter treatise, the scribe has written:
Expliciunt practice geometrie ordinate per magistrum Dominicum de Mastmario de Clavaxio, complete penitus anno ab incarnatione Domini I 346, prima die maij, et scripte Parisius a Jacobo Lectoris Zeelandrino anno Domini 1362 , mense julii. Amen. Amen. ${ }^{2}$
${ }^{1}$ Schum, Amplonianischen Handschriften- $\quad{ }^{2}$ Duhem, Système du monde, Vol. 8, 448. Sammlung, p. 643.

The manuscript is written in a single column in a neat hand. Largeletters are used to set off the enunciations of the propositions from the proofs. A total of twelve diagrams is included. In the colophon the work is ascribed to Oresme and referred to as tractatus brevis et utilis de proportionalitate motuum celestium ( 17 v ; see variant readings for AP2.270). I have refrained from using this title, since it appears in none of the other manuscripts and there is no known reference to it in other works of Oresme to support its authenticity.
3. $H=$ Paris, Bibliothèque Nationale, fonds latin, 16621 , fols. inov-ir 4 r. Since the Ad pauca respicientes follows immediately upon the De proportionibus, the codex has been discussed on pp. 125-26. In total, thirteen figures accompany the text.
4. $B=$ Vatican Library, Palatine Latin MSS, 1354, fols. 233v-237r.

This version of the Ad pauca respicientes is titled De magno anno Platonice (233v) but is not ascribed to Oresme, or for that matter, to any author. It is written in double columns in a fairly readable hand and contains three figures.
5. $V=$ Venice, Bibliotheca Marciana, Cod.ro, a.347, 1.237, L.VI, I33, fols. 62v-65r.

For an earlier description of this codex, see p. 128. No figures appear anywhere in the text or margins. The treatise lacks both title and author.

## Additional Manuscripts

6. London, British Museum, Sloane MSS, 2542, fols. $55 \mathrm{~V}-59 \mathrm{r} .{ }^{3}$

Written in a single column in a somewhat difficult hand, this manuscript contains twelve figures in the margins. The colophon reveals a title of De motibus sperarum and an attribution to Oresme.
${ }^{3}$ Under the title De motibus spherarum,
Menut and Denomy (Oresme. Menut and Denomy (Oresme: Le Livre du ciel, in Mediaeval Studies, Vol. s, 247), in their bibliography of Oresme manuscripts, give the incipit "Ad pauca respicientes de facili..." and proceed to list four manuscripts. However, only MS London, British Museum, Sloane 2542, fols. $5 \varsigma \mathrm{~V}-\varsigma 9 \mathrm{r}$, belongs under this incipit and actually supplied the title from its colophon. The other
three manuscripts (MS Erfurt, Amplonius Q. 299, fols. i13r-126r; MS Bamberg H. J. V. 8, fols. $8 \mathrm{Ir}-98 \mathrm{r}$; and MS Florence, Bibl. Riccardiana 30 , fols. 26r-4or), although indicating by titles or explicits their preoccupation with the sphere, are not versions of the Ad pauca respicientes. No doubt similarity of title explains why these four manuscripts were grouped together.

## Incipit:

Ad pauca respicientes de facili enunciant ut ait Aristoteles. Sic enim aliqui astrologi...

## Explicit and colophon:

... quia ipse solus novit cuius oculis nuda sunt omnia et aperta. Explicit
brevis tractatus et bonus de motibus sperarum et proportio et qualiter possit iudicari eventus rerum(?) aut certum cursus(?) futurorum editus a magistro Nicholo Oresme.

## Editions

The two editions listed have not been collated in establishing the text.
I. Venice edition of 1 505, fols. $25 \mathrm{rc.2-26v}$ (see pp. 130-3I for full description).
2. Paris edition (undated), pages $39 \mathrm{c} .2-42$ (see pp. $13 \mathrm{I}-32$ for description).

## Sigla of Manuscripts

$A=$ Erfurt, Wissenschaftliche Bibliothek, Amplonius Q.385, fols. 1551-158v.
$B=$ Vatican Library, Palatine Latin MSS, 1354, fols. $233 \mathrm{~V}-237 \mathrm{r}$.
$F=$ Paris, Bibliothèque Nationale, fonds latin, 7378 A , fols. $14 \mathrm{v}-17 \mathrm{v}$.
$H=$ Paris, Bibliothèque Nationale, fonds latin, 16621 , fols. irov-II4r.
$V=$ Venice, Bibliotheca Marciana, Cod.io, a.347, i.237, L.VI, i33, fols. 62v-6sr.

## [Ad pauca respicientes]

## [Pars prima]

- Ad pauca respicientes de facili enunciant ut dicit Aristoteles. Sunt enim aliqui astrologi opinantes se ad punctum scire motus, aspectus coniunctiones, oppositiones planetarum et corporum celestium dispositiones credentes se esse sapientes et stulti facti sunt. Posuerunt in
5 celum os suum et lingua eorum transivit in terra de futuris lapsu temerarie iudicando.
Et de istorum numero fuerunt quidam qui propter motum octave spere in 36,000 annis mundum asserebant ad statum pristinum remeare; alii, vero, in 15,000 annis sicut Plato completo peryodo seu
1o anno maiori secundum numerum antedictum.
Ad hanc igitur fatuitatem eradicandam volo modice laborare ad ulteriorem inquisitionem alios exortando ut manifestetur veritas et falsitas destruatur.

Title [Ad pauca respicientes] om $A B F H$ $V$ / [Pars prima] om $A B F H V$
I supra lineam primam hab $H$ verisimillius est maiores partes proportionalium motuum celestium esse irrationales et consimiliter in theorica(?) arguitur verisimilem est quecumque denominata(?)/ respicientes $A B F$ aspicientes $H$ /enunciant $B H F$ enunciantur $A \mid$ Aristoteles: $B(?) /$ Sunt $A B H$ aliud $F$
1-13 Ad pauca...destruatur om $V$
${ }^{2}$ astrologi $B H F$ astrologus $A /$ se $\operatorname{tr}$ $H$ post punctum

3 planetarum $A B F$; om $H$
3-4 dispositiones $B$ dispositionem $F H$ et inter $A$
4 et $A B F$; om $H$ / sunt $A B F$; om $H$ / Posuerunt $B F H$ exuerunt $A$
s eorum $B F H$ earum $A /$ transivit $A B H$ transiunt $F /$ terra $F$ terram $A B H /$ lapsu $F H$ loqui(?) su $B$ sapiente $A$
7 Et de $A B F$; om $H$ / istorum numero $F ; \operatorname{tr} A$ numero illorum $B$ istorum $H$ / quidam qui $B F$ quot(?) qui $A$ aliqui H
8 36,000 BHF 38 A/asserebant $A H$ aspectebant $B F$

## [Concerning some matters] ${ }^{*}$

[Part One]

Concerning some matters, as Aristotle says, there are people who speak out much too readily. For instance, some astrologers who think they know with punctual exactness the motions, aspects, conjunctions, and oppositions of the planets and the dispositions of the celestial bodies believe themselves to be wise men but have been made fools. In judging rashly and erroneously about future events, "they have set their mouth against heaven: and their tongue hath passed through the earth."

And of these there were some who, because of the motion of the eighth sphere in 36,000 years, claimed that the world would return to its original state [in 36,000 years]; but others [held] as did Plato, [that this would occur] in 15,000 years after a complete period or Great Year equal to the aforementioned number.

In order to eliminate this foolishness, I wish to labor modestly by encouraging others to further investigation so that truth is made manifest and falsity destroyed.

* This translates the opening words of the treatise, which, for convenience, have been
adopted as the title (see pp. 77-78 and n. 102).

9 vero in 15,000 annis $B F$; om $A$ vero in $15,000 \mathrm{H} /$ completo $A B F$ completa $H$
Io maiori $B F H$ maximo $A$
I I Ad hanc $B F H$ et iste $A /$ igitur $B F H$;
om $A$ / eradicandam $B F H$ erravit $A$ 12 alios $F$; om $A$ alias $B H /$ exortando $A B F$ motando(?) $H /$ manifestetur $B F H$ manifeste $A$

Hoc termino possibile multipliciter utimur. Uno modo pro con-
15 tingenti aut necessario. Alio modo pro dubio et hoc dupliciter: vel in contingentibus quarum utraque contradictorum est possibilis primo modo; vel in aliis quarum una est necessaria et alia impossibilis et hoc ultimo modo tripliciter. Aut est possibile equaliter, aut est improbabile, aut est probabile.
Exemplum primi: numerus stellarum est par; numerus stellarum est impar. Una est necessaria, alia est impossibilis. Tamen dubium est nobis que sit illa que est necessaria et ideo dicimus de utraque quod est possibilis. Proposito enim quod numerus stellarum est impar, dicet aliquis quod possibile est. Et hoc membrum posset subdividi quia
25 quandoque in talibus non habemus aliquam rationem ad aliquam partem, quandoque autem habemus, et tunc dicitur problema de quo in illo modo opinantur.
Exemplum secundi: numerus stellarum est cubicus. Dicimus enim quod possibile est, non tamen probabile aut opinabile aut verisimile 30 cum tales numeri multo sint aliis pauciores.

Exemplum tertii: numerus stellarum non est cubicus. Dicimus quod possibile est, et probabile et verisimile per oppositum secundi membri. Suppositio prima. Quantitatum quedam sunt invicem commensurabiles, quedam incommensurabiles, et hoc est commune lineis, super${ }_{35}$ ficiebus, corporibus, temporibus, motibus, qualitatibus, et cetera.

Suppositio secunda. Propositis multis quantitatibus quarum proportio

14 termino: circulo $A$
is post aut add $A V$ pro / Alio modo: aut $V$ | ante et hoc $m g$ bab $F$ prima distinctio
16 quarum: quorum $V /$ utraque $A F V$ uterque $B H \mid$ contradictorum $B F$ contradictionum $A$ contradictionarum(?) $H$ contradictoria $V /$ ante primo add $F$ contingenti et $V$ contingens
16-17 primo modo om $A$
17 necessaria obs $H /$ hoc $B F V$; om $A H$
18 ultimo modo $A B F$ ultimo $H$ ultimos $V /$ possibile: probabile $H$
18-19 est improbabile $A F$ est impossibile $B$ improbabile $H$ probabile $V$
19 aut est probabile $F H$ aut probabile $A B$ aut improbabile aut probabile $V$
21 est ${ }^{3} A F$; om $B H /$ ante dubium bab $H$ quia
21-23 Una est...est impar om $V$

22 nobis $A B$; om $F H /$ illa $A F H$ ista $B /$ que est...ideo $A$; om $F H$ necessaria et ideo $B /$ de $B F H$; om $A /$ utraque $A F H$ uterque $B /$ quod $B F H$; om $A$
23 ante est $h a b B$ ipsa / Proposito $F$ proposita $A B$ posite(?) primo(?) modo de posito $H /$ enim $F H$ hac $A$ tamen dicit $B$ / quod FH; om $A B$ / impar $A B H$ par $F$
${ }^{24} \underset{H}{\text { ante }}$ Et hab $V$ hoc / subdividi: dividi H
25 non om $A$ / aliquam ${ }^{1} A B F$ (?); om $H V /$ aliquam $^{2}$ : aliquem $B$
26 quandoque autem $B F H$ quandoque enim $A$ aut $V /$ habemus om $A /$ dicitur problema $B H$ dicitur problema ${ }^{\text {ta(?) }} F$ esset problema $A$ vocamus probabilia $V$
27 in illo $A B$ nullo $F V$ non(?) tertio(?)
$H$

## Part One

We use the term "possible" in many ways. In one way for what is contingent or necessary. In another way to signify doubt, and this use is twofold: either in contingent statements where each of the contradictories is possible in the first way [given above]; or in other statements where one [of the contradictories] is necessary and the other impossible, and this last way is threefold. Either it is equally possible, or it is improbable, or it is probable.

An example of the first way: The number of stars is even; the number of stars is odd. One [of these statements] is necessary, the other impossible. However, we have doubts as to which is necessary, so that we say of each that it is possible. For, once we have proposed, "The number of stars is odd," anyone could say it is possible. And this part could be subdivided, since sometimes in such cases we have no reason for [choosing] one part; and sometimes we do have a reason, and then it is called a "problem" and considered in the way [shown in the next example].
An example of the second way: The number of stars is a cube [number]. Now, indeed, we say that it is possible but not, however, probable or credible or likely, since such numbers are much fewer than others.
An example of the third way: The number of stars is not a cube [number]. We say that it is possible, probable, and likely, by the opposite of the second part.
Supposition I. Some quantities are mutually commensurable, some incommensurable, and this is common to lines, surfaces, bodies, times, motions, qualities, etc.
Supposition II.* If many quantities are proposed and their ratios are un-

* See pp. 85-88.

28-29 enim quod possibile $B F V$ quod possibile enim $A$ quod possibile $H$
29 est non $B F V$; om $A$ non $V /$ opina-
bile: impossibile $V /$ aut $^{2} A B V$ unde
$H$ secundum $F /$ verisimile: verisimilem $F$
30 ante cum add $H$ est et $V$ et / numeri om $B /$ multo sint $A B F ;$ tr $H V /$ ante pauciores add $V$ multo
32 ante $\mathrm{et}^{2}$ add $A$ etiam / ante per add $B$ et / ante membri $m g$ bab $H$ diffinitio
33 Suppositio prima $B$; om $A F H V$ sed ante et verisimile (linea 32) mg bab $F$ alia distinctio / quedam: que $B /$ ante
invicem add $F$ in / invicem $A F H$; om BV
33-34 commensurabiles: incommensurabiles $B$
34 quedam $A F$ que sunt invicem $B$ alie $H$ quedam invicem $V /$ incommensurabiles: commensurabiles $B /$ hocest $B H F$ huius est $A$ hoc esset $V$
35 ante motibus add FV et / et cetera FV; om $A B H$
36 Suppositio secunda $B$; om $A F V$ sed ante proportio $m g$ bab $H$ suppositio secunda / ante Propositis mg bab F prima suppositio
est ignota possibile est, dubium, et verisimile est, aliquam alicui incommensurabilem esse.
Commensurabilia aliquotiens replicata reddunt equalia, et econtraria 40 incommensurabilia numquam. Quia omni proportione data rationali contingit dari numeros primos illius proportionis, ideo si commensurabilia replicentur hiis numeris, maius minori et minus maiori, erunt equalia. Si enim $A$ sit duplum ad $B, A$ semel sumptus et $B$ bis sunt equalia.
Suppositio tertia. Non quelibet proportio omnium quantitatum pertinentium motibus corporum celestium est cognita, sicut proportio circulorum aut magnitudinum pertransitarum non est cognita temporibus equalibus, aut cuiuslibet distantie, et cetera.

Et hoc est satis notum intelligenti hoc enim sciri non potest de
quantitatibus prope nos stantibus propter defectum sensuum
Diffinitio. Tunc est stellarum coniunctio cum semidyameter a centro mundi exiens per eorum corporum centra procedit; aut cum circulus per polos transiens per ipsarum centra progreditur.
55
Prima conclusio. Si duo mobilia moveantur super circulos seu circumferentias commensurabiles et temporibus equalibus pertranseant commensurabilia invicem, et proportio circuli ad circulum non sit ut pertransiti ad pertransitum, aut si circuli sunt equales et inequaliter commensurabiliter moveantur, necesse

37 est $^{2} A F H$ et $B V /$ dubium om $B /$ et $A$; om BFHV / est ${ }^{3}$ om $A$
39 replicata: supra $A$ / econtraria: contraria $A$
40 incommensurabilia: in principalia $A \mid$ numquam $A H V$ vero numquam $F$ numquam et $B$ / ante omni add $V$ in / rationali $A B$; om $F H V$
41 dari $A H$ dare $B F V($ ? $) /$ primos om $F$ / illius $A B H$ istius $F V$ / ideo: igitur $A$
42 replicentur: replicetur $V /$ hiis numeris $A H V$; om $B$ numeris $F /$ maius: maior $A /$ minus maiori: maior minori $\stackrel{c}{\text { maio }}$
43 Si enim: ut si $A$ /ante enim A mg bab $F$ secunda suppositio / sumptus $B V$; om $A$ sumptam $F$ sumptum(?) $H$ l post bis hab A sumptum / sunt BHV erant $A$ sumptam sunt $F$
4) Suppositio tertia $m g$ hab $B$ ante et $B$
bis (linea 43); om AFHV / omnium om $A$
46 corporum celestium $A B F$; $\operatorname{tr} V$ corporum supercelestium $H$ / est: erit $H$
47 aut: vel $A /$ non est cognita $A B$; om $H F$ sed tr $V$ post distantie (linea 48)
48 et cetera FH ; om $B V$ et $A$
$49 \mathrm{Et} B F H$; om $A V /$ hoc est satis $B F V$ huius patet $A$ hoc satis est $H /$ notum om $A$ / intelligenti: consequenti(?) $A$ / post enim add $V$ proprie / sciri non potest: non sciri(?) potest $B$
so quantitatibus: quibus(?) $H$ / stantibus om $H /$ defectum $A F V$ defectus $B H$
${ }_{51}$ quarta $m g$ hab $H$ ante Omnis et hab $F$ tertia suppositio; om $A B V /$ ante unus $m g$ hab $B$ est secundum longitudinem et latitudinem simul/unus motus $\operatorname{tr} H /$ celi $B F H$ circuli $A$ tali celi $V /$ est $A V$; om $F H$ et $B$
52 Diffinitio mg bab $F$ ante nos (linea so);

Part One
known, it is possible, doubtful, and probable that any [one of them] would be incommensurable to any other.
Commensurable quantities when multiplied a certain number of times produce equal quantities and, contrarily, incommensurables never do. Since it happens that two prime numbers can be given for every rational ratio, [it follows that] if the commensurable quantities were multiplied by these [prime] numbers, the greater by the lesser and the lesser by the greater, they would be equal. Thus if $A$ were double $B$, they would be equal when $A$ is taken once and $B$ twice.
Supposition III.* There is no known ratio between any quantities pertaining to the motions of celestial bodies. [For example] we do not know the ratio between circles or magnitudes traversed in equal times; nor [do we know the ratio between the times when] any [given] distance [is traversed], etc.

And it is well understood that we cannot even know about quantities close by because of defective senses.
Supposition IV. Every single celestial motion is uniform.
Definition. There is a conjunction of planets when a semidiameter drawn from the center of the world passes through the centers of [celestial] bodies, or when a circle passing through the poles advances through the centers of the planets.
Proposition I. $\dagger$ If [I] two mobiles are moved on circles or circumferences which are commensurable, [and 2] they traverse mutually commensurable distances in equal times, and [3] the ratio of circle to circle is not as the [ratio] of the distance traversed to the distance traversed, or if the circles are equal and they are moved unequally but commensurably, it is necessary that these two mobiles conjunct in a point in which

* See p. 88.
+ See pp. 88-90.
om $A B H V /$ Tunc: $B(?) /$ ante cum bab $F$ scilicet(?)
53 mundi $A B$; om $F H V /$ exiens $A F H$; om BV | ante eorum scr et del $F$ corpora transiens / eorum $H B V$ earum $A F \mid$ ante procedit $m g$ bab $B$ conclusio secundum longitudinem tantum / procedit: procedunt $F$
54 per polos transiens: transiens per po-
los $A$ ante ipsarum scr et del $V$ ipsam / ipsarum: utraque $A$
5s Prima conclusio mg bab $F$ ante Si et $m g$ bab $H$ ante per $^{2}$ (linea 54); om $A V$. conclusio prima $B$ et ante Si mg hab $B$
prima portis conclusionis / seu $B F H$ vel $A$ vel cum $V$
56 commensurabiles $B F H$ circulorum $A$ commensurabilibus $V /$ et om $V$
$56-57$ invicem FHV ad nos(?) $B$ in invicem $A$
57 ante et add HV et circulus et add B circulus et add $F$ et circulis / ut: sicut $V$ / post ut bab $V$ proportio
58 ante aut add $B$ que et mg bab $B$ secunda pars / sunt: sint $B$ | post inequaliter add $H$ et / commensurabiliter: commensurabilia $B$
est illa coniungi in puncto quo alias coniungentur et quo alias fuerunt coniuncta.
6o In casu primo posito sunt tres conditiones. Due sunt clare et tertia pro tanto subdividitur quia si proportio circuli ad circulum esset ut pertransiti ad pertransitum iam in temporibus equalibus describerent equales angulos supra centrum et tunc semper vel numquam essent coniuncta, sed semper equidistarent.
Hoc exposito ostenditur conclusio. Sint mobilia $A B$ coniuncta in puncto c. Cum igitur circuli et pertransita aliquotiens replicata sint equalia, per secundam suppositionem. Igitur circulus $B$ aliquotiens sumptus et totiens pertransitus est equalis circulo $A$ aliquotiens sumpto et totiens pertransito. Et sic $A$ et $B$ in fine revolutionum erunt in
${ }^{70}$ puncto $c$. Et idem de preterito et idem si circuli ponantur equales et pertransita inequalia commensurabiliter.
Tempus coniunctionis ibidem invenitur arte sic: si circuli sint inequales et moveantur equaliter vide quotam partem vel quotas partes sui circuli pertranseat unumquodque, et si numeri partes denominantes

59 coniungi: coniunctio $V /$ in puncto om $F$ / coniungentur: coniungerentur $F /$ et: aut $B /$ quo $B F$; om $A H$ que $V$ / post coniuncta scr et del $B$ casu primo posito 3 conditiones sunt clare tertia pro tanto subdividitur
60 In $A V$; om $F H /$ casu $F H V$; om $A /$ sunt tres $F H$; $\operatorname{tr} V$ sint tres $A /$ Due sunt clare $A F H$ et sunt clare $3 \mathrm{~V} /$ et $A$; om $F H V /$ tertia $A F H$; om $V$
60-61 In casu...subdividitur om $B$
6r pro tanto subdividitur $F H V$ subdividitur pro tanto $A /$ ante quia $m g$ bab $B$ tertia pars / proportio: proponitur (?) $V /$ esset: fuerit $B /$ ut: sicut $V /$ post ut add $V$ proportio
62 in $V$; om $A B F H /$ temporibus equalibus $\operatorname{tr} V /$ describerent $F H V$ describent $A$ describentur $B$
63 equales angulos $\operatorname{tr} V$ / supra: super $B /$ tunc semper $A B F$ tunc $H$ semper $V$ / vel numquam: numquam vel semper $H$
6s ante Hoc add $H$ et / Hoc: tertio $V$ / exposito $A F H$ facto $B$ proposito $V /$ mobilia A B $A B H$ A B mobilia $F$ duo
mobilia $V$
$6_{5}-66$ coniuncta in puncto $\mathrm{c}:$ in puncto c coniuncta $A$
66 et om $H$ / aliquotiens: aliquota $V /$ sint BHV sunt $A F$
67 secundam: tertiam $V /$ infra aliquotiens $m g$ bab $F$ figuram
68 totiens: totius $F$
69 sic $B H$ sit $A F$ sint $V /$ revolutionum: revolutionis(?) $A /$ erunt $B H V$ sunt $A$ est $F$
70 ponantur: sint $V$
71 inequalia: inequaliter $B$ /post inequalia mg bab H figuram
72 Tempus coniunctionis: tres coniunctiones $V /$ ibidem...arte $A F H$ invenitur arte $B$ arte invenitur ibidem $V$
73 equaliter: inequaliter $V /$ post equaliter bab $B$ vel inequaliter vel si sint equales et moveantur inequaliter / quotas $B F H$; om $A$ quotiens $V$
74 sui circuli $B F H$; om $A$ circuli si $V /$ unumquodque: unus quisque $V /$ ante et $b a b A$ sui circuli / numeri...deno$\underset{B}{\min }$ antes: partium(?) denominatores B

Part One
they will conjunct at other times, and in which they have [aiready] been in conjunction.
In this first case [or proposition], there are three conditions. Two are clear and the third is subdivided, because if the ratio of a circle to a circle were as the distance traversed to the distance traversed, then already in equal times they would describe equal angles around the center and would either always or never be in conjunction but would always be equidistant.
This being expounded, the proposition can now be shown: Let mobiles $A$ and $B$ be in conjunction in point $c$. Then when the circles and the distances traversed have been multiplied [respectively] a certain number of times, they will be equal, by the second supposition. Thus circle $B$ taken


Fig. $2^{*}$
a certain number of times-i.e., as many times as it has been traversed-is equal to circle $A$ taken a certain number of times-i.e., as many times as it has been traversed. And so $A$ and $B$ will be in point $c$ at the end of these revolutions. The same applies to the past and also if the circles are assumed equal and the distances traversed are unequal, but commensurable.
The time of conjunction at point $c$ is found in this way: If the circles are unequal but moved with equal [speeds], see what part or parts of its circle is traversed by any one of the mobiles; and if the numbers denominating the parts are in a multiple ratio, the greater of these numbers num-

* This figure appears in the margin of MS $H$ (see upper left-hand corner of Plate 10) alongside Part I, Prop. IV. Only lower case letters were used in the manuscript figures. But in order to secure agreement between the figures and our edited text, all letters representing mobiles-but not pointshave been capitalized.

Throughout the Adpauca respicientes there is not a single reference by Oresme to any figure or diagram. For this reason no figures have been included within the Latin text, but a few are incorporated in the translation for the reader's convenience. The remainder are omitted.

75 sint in proportione multiplici maior eorum numerat tempus; si non multiplica unum per aliud et productum tempus numerabit.
Exemplum primi: sit circulus $A$ duplus ad circulum $B$. Pertranseatque $B$ una die totum circulum igitur si equaliter movetur $A$ transit sui circuli medietatem; et si $A$ pertransit unum integrum, $B$ duo,
so Et si $A$ unum duplum ad aliud, igitur secunda die coniungentur.
Exemplum secundi: sint circuli ut prius. Moveatur $B$ quolibet die per tertiam, $A$ per quartam. Multiplica 3 per 4 et sunt 12 . Coniungentur in duodecima die in primo puncto.
Si circuli sint equales moveanturque inequaliter fac ut prius vide ${ }_{85}$ quotam, et cetera.

Secunda conclusio. Quecumque mobilia sunt ut prius disposita babent in suis circulis loca seu puncta in quibus coniungentur finito numero numerata per motum eternum infinities replicata, et qui infinities replicabitur in futurum.

Cum enim per precedentem dato puncto coniunctionis ubi alias
go fuerunt vel igitur tempore medio fuerunt alique coniunctiones vel nulle. Si nulle tunc numquam coniungentur nisi in isto puncto sicut in primo exemplo.

Si igitur alique finite et locis finitis fuerunt. Sit ergo una media, igitur per primam conclusionem ibidem fuit alias et erit. Igitur sunt

75 ante maior bab $F$ non semper et hab $V$ semper / si non: sive $V$
76 aliud: illum $F /$ productum $A B F$ per productum $H$ per dictum $V /$ tempus: totum $A$ / numerabit $A B F$; obs $H$ numerabis $V$
77 circulus A FHV; $\operatorname{tr} A B$
77-78 Pertranseatque: $A($ ? $)$
78 totum: unum $V$ / movetur $A B F$ moventur $H$ moveretur(?) $V /$ transit: transibit $B$
79 si $F$; om $B$ sic $A H V / A: B B /$ integrum om $A /$ ante B hab $V$ et $/ \mathrm{B}$ duo om $V$
80 Et si A $F$ et sit $A$ ut sit A $B$ sed $H$ si A $V$ / unum duplum $A$; om $B$ unam duplam $F H V /$ ad aliud $A 1 / 2$ $\mathrm{B} /{ }_{\mathrm{I}}$ et quia et sunt plus iste $B \mathrm{~B}$ duas duplas movetur maior uno $F$ in alio uno $H \mathrm{~B}$ duas in duplo uno maior $V /$ Igitur...coniungentur $A F H$ ideo secunda die completa coniungentur $B$ 80-8I Igitur....sint om $V$
81 secundi $A B H$ si $F$ / ut prius $B F H$;
om VBC $A /$ Moveatur B om V / quolibet die om $A$
82 per tertiam $A B$; om $V$ per tertium $F$ per $3 H /$ post tertiam bab $A$ quales(?) die et bab $F$ sui circuli / A FHVC $A$ A autem $B /$ quartam $A B F_{4} \mathrm{HV} /$ ante Multiplica add $V$ et / 3 per 4: 4 per $3 V \mid$ et $A B F$; om $H V$ / sunt $A H V$ erant $B$ fuerit $F$
${ }^{82-83}$ Coniungentur: coniungetur $V$
83 in $^{1} V$; om $A F H$ igitur $B /$ in primo puncto $F$; om $A B H$ de primo puncto $\stackrel{\text { pun }}{V}$
84 moveanturque $B F H$ motus(?) $A$ moveantur $V / \mathrm{fac} B F V$; om $A$ stat $H /$ vide $H$; om $A$ in $B F V$
85 quotam $H$; om $A$ quodam $F V$ inquirendo(?) $B$ et post inquiren- in inquirendo $m g$ bab $B$ ut in huius duobus capitulis dictum est / et cetera FHV; om $A$ partes(?) $B$
86 Secunda conclusio mg hab $F$ ante Quecumque et $m g$ hab $H$ post ut; om $A V$ conclusio secunda $B /$ sunt ut $B F H$

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bers the time; if they are not in a multiple ratio, multiply one by the other and the product will number the time.

An example of the first [part of this proposition]: Let circle $A$ be double circle $B$. Now should $B$ traverse a whole circle in one day, then, if $A$ is moved with an equal [speed], it traverses half of its circle; and if $A$ traverses one whole circle, $B$ traverses two whole circles. But if $A$ [traverses] a circle double the other, they will be in conjunction on the second day.
An example of the second [part of this proposition]: Let the circles be as before. On any given day $B$ is moved through a third [of its circle] and $A$ through a fourth of its circle. Multiply 3 by 4 and you get 12 . Therefore, they will be in conjunction in the first point on the twelfth day.
If the circles were equal but moved with unequal speeds, do as before: See what [part or parts of its circle is traversed by each of the mobiles], etc.

Proposition II. Any mobiles arranged as before have a finite number of places or points on their circles in which, through an eternal motion, they bave been conjuncted an infinite number of times, and in which they will be conjuncted an infinite number of times in the future.
Since, by the preceding proposition, a point of conjunction is given where the mobiles have been [in conjunction] at other times, then, in the intervening time [between successive conjunctions in that point], either there have been some conjunctions or none at all. If none at all, then the mobiles will never conjunct except in this point as in the first example [of the preceding proposition].
If, however, there were some conjunctions, then there were a finite number in a finite number of places. Let there be one conjunction in the in-
ut B C $A$ ut $V /$ in om $V$
87 loca: loco $V /$ seu: vel $A /$ in om $A$ / quibus coniungentur: coniunctionis A
87-88 numerata per motum om $V$
88 replicata $A$ replicatum $B F$ replicatis $H V /$ qui infinities: quia finities $V$
89 enim: igitur $B /$ ubi alias $V$ quo alias $A$ alias ibi $B$ alias hic(?) $F$ alias hec $H$

90 post medio add $V$ per precedentem alique: alie $V$
9 isto: illo $H$
93 igitur alique $\operatorname{tr} H \mid$ post alique bab $B$ erunt locus finitis fuerunt $F$ locis finitis $A B H$ finitis(?) $V$
94 primam: quartam $A \mid$ post conclusionem $m g$ hab $H$ figuram

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95 tantum duo loca; et si duo fuerint sunt tria loca, et sic ultra. Locorum numerum reperies arte tali quare primo tempus prime coniunctionis quod sic invenies. Divide per differentiam motuum circulum et numerus exiens est tempus prime coniunctionis eorum per quem divide tempus coniunctionis in primo puncto habitum per primam conclu100 sionem et quod exiet coniunctiones differuntes numerabis.

Et loca, vero, poteris invenire ducendo motum unius mobilis in tempus unius coniunctionis et a producto quantum poteris circulum subtrahe et residuum locum ostendit.
Verbi gratia sit circulus 12; $A$ moveatur in die $4, B\langle 9\rangle$. Cum differentia est $s$ per quam divide 12 erunt $22 / 5$, tempus prime coniunctionis, per quem divide tempus coniunctionis in primo puncto revolutione facta quod est 12, ut per primam artem patet. Item habes 5 igitur quinquies coniungentur antequam venient ad primum punctum et tot sunt coniunctionum loca. Si vis habere primum locum duc 1o motum unius sicut 4 per tempus unius coniunctionis, scilicet $2 \frac{2}{5}$, locus prime coniunctionis post primam datam.

Tertia conclusio. In quacumque dispositione fuerint talia mobilia aliquo instanti in eadem fuerunt et erunt infinities ipsis existentibus in eisdem locis.
Quoniam de oppositione, quadraturis, et quocumque aspectum
9) tantum: tantummodo $V /$ et $^{1}$ om $V$ duo ${ }^{2} B F$ due $A H{ }_{2} V /$ fuerint: fuerunt $B /$ tria loca et: et $3 \mathrm{~V} / \mathrm{Lo}$ corum: et loci $A$
96 reperies: invenies $V /$ arte: aut $V /$ quare om $B$
97 invenies: reperies(?) $A /$ circulum: circulus $V /$ et om $A$
98 prime coniunctionis $\operatorname{tr} \mathrm{H} /$ eorum om $B /$ quem $A F V$ quam $B$ quod $H$ / divide: artem $A$
99 per om $H$
99-100 primam conclusionem $A F V$; om $H$ primam coniunctionem $B$
100 ante differuntes add $H$ et / numerabis $H V$ numerabit $A B F$
101 loca rep $B F$ / vero $B F H$; om $A V$ / poteris: poterit $H /$ ducendo motum: duplicando motus $V$
101-2 unius...coniunctionis om $A$
102 a producto $B F H$ per ducto $A$ in ducto $V /$ quantum: quantitas $V$ / poteris: poterit $H$

102-3 circulum subtrahe $A B H$; tr $V$ circulum abtrahe $F$
104 sit: si $A$
ros est om $V$ | ante erunt add $B$ et $/ 22 / 5$ $B F H_{25} A V$
Io6 quem: quam $B$ / post tempus add $A H$ prime / primo om $A$
107 est om $A$ / ante 12 bab $A 1(?) /$ habes $A$ habens $B F$ habet $H V$
108 venient $B F$ veniant $A H V$
ro9 duc: duo $V$
in o motum $A B F$ tempus $H$ motus $V /$ unius...tempus om $V /$ sicut $B F H$ ut A / ante tempus bab $F_{3 / 2 \frac{2}{2} / 5 \text { : duas }}$ ${ }_{2}$ a $^{\text {as }} A$
III ante locus hab $B$ et provenit cum $3 / 5$ qui est
112 Tertia conclusio mg bab $F$ ante In et $m g$ hab $H$ post coniunctionis (linea III); om $A V$ conclusio tertia $B$ / fuerint $A B V$ fuerunt $F H /$ talia mobilia $A B F ; \operatorname{tr} H$ aliqua mobilia $V$ 113 eadem $A F H$ eodem $B V$ / fuerunt:

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tervening time so that by the first proposition there were, and will be, other conjunctions in the same place. Therefore, there are now two places [of conjunction]. And if there were two [conjunctions intervening], there would be [a total of] three places of conjunction, and so on. You find the number of places by means of the time of the first conjunction in the first place, which you find as follows: Divide one circle by the difference of the speeds, and the resulting number is the time of their first conjunction, by which you divide the time of conjunction in the first point, which is found by the first proposition.
Now you will be able to find the places [of conjunction] by multiplying the speed of one mobile by the time of one conjunction, and then subtract the circle as many times as possible from this product, and the remainder will indicate the place [of conjunction].*

For example, let there be a circle of twelve [equal parts], and let $A$ be moved 4 [parts] per day and $B 9$ [parts]. Since the difference is 5 , by which you divide 12 , the time of the first conjunction will be $22 / 5$ [days]; and you divide this into the time it takes to complete a revolution and enter into conjunction in the first point-and this is 12 [days], as is evident by using the first method [in the first proposition]. Thus you have s, and therefore they will be conjuncted five times before they will come to the first point, and there are just as many places of conjunction. If you wish to know the first place [of conjunction], multiply the speed of one [mobile], as [for example], 4 [parts of the circle per day, namely $4 / \mathrm{I}_{2}$ ] by the time of one conjunction, namely $2 \frac{2 / 5}{}$ [days], and you have the place of the first conjunction after the first given conjunction. $\dagger$
Proposition III. $\ddagger$ Whatever the disposition of such mobiles in any instant, they bave been and will be in the same disposition an infinite number of times, with the mobiles being in the same places [an infinite number of times].

Although opposition, quadrature, and any [such] aspect is like conjunc-

[^40] equalia loca, vero, aliarum dispositionum duplo plura.
Ex hiis elicitur quod si sol et luna commensurabiliter moveantur loca ubi coniunguntur et opponuntur sunt finita et loca ubi non coniunguntur infinita, et sic de aliis. Ymo si precise sol cursum suum
120 faceret in uno anno et mars in duobus annis, numquam, nisi in uno loco coniungerentur.
Quarta conclusio. Si duo mobilia moveantur inequaliter incommensurabiliter respectu centri quotienscumque coniungentur in puncto in quo impossibile est ea alias coniungi nec alias fuisse coniuncta.
Et si moverentur in eternum mobilia dicuntur moveri incommensurabiliter quo ad centrum que in temporibus equalibus describunt angulos incommensurabiles. Hoc autem potest contingere quia circumferentie sunt incommensurabiles quibus moventur equaliter aut commensurabiliter; aut quia circumferentie sunt equales seu commen-
${ }_{130}$ surabiles et moventur incommensurabiliter; aut quia circumferentie sunt equales et spatia pertransita inter se incommensurabilia cum circumferentiis; aut quia omnia sunt incommensurabilia. Ex isto patet quod verisimile est ipsa incommensurabiliter moveri cum talis proportio habeat multas causas veritatis.
${ }^{1} 35$
Hoc exposito demonstratur conclusio. Sint $A$ et $B$ coniuncta in puncto $c$ moveanturque equaliter circulis incommensurabilibus. Sint, igitur, post aliquod tempus iterum in $c$. Cum igitur moveantur equa-
ins loca: locis $V /$ coniunctionum: oppositionum $A / \mathrm{et}^{2} A B H$; om $F V \mid$ oppositionum: coniunctionum $A$ sunt: secundum $A$
II6 vero om $A$ / ante duplo bab $B$ in duplo plura: duplariter $H$
117-18 commensurabiliter...ubir om $B$
in 8 coniunguntur: opponunt(?) $A / \mathrm{et}^{1}$ : aut $H /$ opponuntur: coniunguntur $A$
II9 ante infinita add $V$ sunt / Ymo: primo (?) $V$
120 in $^{1} A F V$; om $B H /$ uno anno: duobus annis $A$ /annis $F V$; om $A B H$ / post annis hab $V$ commensurabiliter / in ${ }^{3}$ om $A$
121 coniungerentur: coniungentur $H /$ post coniungerentur bab $B$ et tantum de isto sequitur conclusio quarta et $b a b$ $H$ de hoc in tantum(?) et hab $F$ de hoc in tantum

122 Quarta conclusio mg bab $F$ ante Si et $m g$ bab $H$ post quotienscumque (linea 123); om $A V$ conclusio quarta $B$

123 quotienscumque: quousque $V /$ ante coniungentur $b a b B$ coniungerentur / supra coniungentur mg bab H figuram in quo: aliquo $V$ I post quo bab $H$ alias / ea: centri $H$ / post ea bab $A$ a 124 alias: postea $V$
125 moverentur: moverent(?) $A /$ in eternum mobilia: mobilia mota $A /$ moveri om $A$
I 26 temporibus equalibus $\operatorname{tr} H$
I 27 contingere $B F H$ coniungere $A$ contingetur(?) V/post contingere $m g$ bab $B$ dupliciter / ante quia $m g$ hab $B$ ut
128 incommensurabiles:incommensurabilibus $V$ / post quibus $m g$ bab $B$ mobilia / equaliter: inequaliter $V /$ aut om $A$

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tion, and the [number of] places of conjunction and opposition are equal, the places of other dispositions, however, are twice as many [in number].
From these things one can derive this: If the sun and moon are moved commensurably, the places where they are in conjunction and opposition are finite [in number] and the places where they are not in conjunction are infinite, and this holds for the other [aspects]. Indeed if the sun completed its path in exactly one year and Mars in two years, they would never be conjuncted except in one place.
Proposition IV. If two mobiles are moved unequally and incommensurably with respect to the center, then no matter how many times they will be conjuncted it is impossible that they be conjuncted in a [particular] point at other times, or that they bave been conjuncted in it at other times.

And if they are moved through an eternity, mobiles are said to be moved incommensurably with respect to the center when they describe incommensurable angles in equal times. Now this can occur because the circumferences are incommensurable on which the mobiles are moved with equal speed or commensurable [motions], or because the circumferences are equal or commensurable on which [the mobiles] are moved incommensurably, or because the circumferences are equal but the distances traversed are mutually incommensurable with the circumferences, or because all these are incommensurable.* From all this the probability emerges that these mobiles are moved incommensurably, since there can be many reasons for such a ratio being the way it is.

Now that this has been set forth, the proposition will be demonstrated: Let $A$ and $B$ be in conjunction in point $c$ and be moved with equal speeds on circles which are incommensurable. After some time let them meet once again in $c$. Now since they are moved with uniform speeds, circle $A$ multi-

* See pp. 432-33.

129 commensurabiliter $B F H$ incommensurabiliter $A V /$ aut: et $A$
129-31 seu...equales et spatia $H$ spacia
$A B$ commensurabiles et spacia $F$ omnia loca $V$
13 I inter se $B H$; om $A$ sunt $F$ sunt inter $V$ | post se bab $H$ mobilia(?)/ante cum bab $F$ et inter(?) alia
131-32 cum circumferentiis FHV et(?) $A$ / cum...incommensurabilia om $B$

132 aut $F H$; om $A V \mid$ ante sunt bab $V$ ista 133 est ipsa $B H V$; om $F$ est ista $A$
I35 exposito: expedito $B \mid$ ante A add $H$ mobilia / et $F V$; om $A B H$
136 moveanturque: moveantur est $A$ ! circulis: circulus $V /$ Sint $B F H$ sit AV
137 ante post bab $H$ primo / post BFH; om $A V$ / iterum in om $V$
liter, circulus $A$ sumptus in aliquo numero est equalis circulo $B$ sumpto in aliquo numero. Igitur sunt commensurabilia per secundam supit potest argui de quacum que alia incommensurabilitate quandocumque sint circumferentie dum tamen moveantur incommensurabiliter quo ad centrum.

Sint $A$ et $B$ in puncto $c$. Si igitur post aliquod tempus iterum sint in $c$, tunc quodlibet fecit precise aliquas revolutiones. Igitur circuli sunt commensurabiles quod est contra positum.
Quinta conclusio. Infinita puncta sunt in quibus fuerunt coniuncta, et infinita in quibus erunt coniuncta duo mobilia sic disposita.

Patet quia infinities coniungentur super novo loco, et sic de aliis dispositionibus.
150
Ex isto patet quod si fuerint due circuli intersectantes se sicut in nodis capitis et cauda draconis et mobilia moveantur incommensurabiliter, si semel coniunguntur in nodo numquam hoc alias erit per hoc et alia que sequuntur. Idem posset dici de oppositione. Et ex hoc potest concludi quod est possibile quod eclipsis lune maxima eveniat
is 8 circulo om $A$
${ }^{1} 39$ commensurabilia: incommensurabilia $V /$ secundam: tertiam $V$
140 interimit $B F V$ intendit $A$ intermit $H$ / ita $B F H$ sic $A$ ista $V \mid$ ante potest mg hab Ffiguram
140-41 quacumque: qualibet $V$
141 incommensurabilitate $B F H$ commensurabilitate $A V /$ sint: fuerunt(?) F
141-42 dum tamen $B F V$ non tamen $A$ dummodo $H$
142 moveantur $A$ (?) FHV moventur $B$
143 in...igitur om $F$ / aliquod $A B H$ aliquem $V /$ iterum $A B V$; om $H /$ sint $H V$ fuit $A B$
143-44 post...aliquas om $F$
144 tunc $A B H$; om $V /$ quodlibet $A B$ quelibet $H$ quilibet $V$ / fecit precise $B H$; $\operatorname{tr} A$ reficit(?) precise $V /$ aliquas $A H V$ aliquot $B \mid$ ante Igitur add $V$ et
14s post commensurabiles add $F$ sed pertransierant in eodem tempore igitur motus eorum sunt commensurabiles et add $V$ sed pertransierant in eodem tempore et igitur motus eorum sunt incommensurabilibus

146 Quinta conclusio mg hab $F$ ante Infinita; om $A H V$ conclusio $s B /$ in $A B$; om $F H V$
146-47 et infinita...disposita: talia mobilia sic disposita et infinita quibus erunt coniuncta $V$
147 in $A B$; om $F H$ / erunt $A F H$ exiet(?) B
148 quia: quod $A /$ coniungentur $H V$ coniunguntur $A B F /$ novo: non $V$ 150 isto $B F H$ ista $A$ istis $V$ / fuerint: sint $H /$ intersectantes: intersecantes $A \mid$ ante in mg bab H figuram
Is I nodis $F H$ nodo $A B V /$ capitis: capite $F /$ draconis: traconis $V /$ ante moveantur mg bab $F$ figuram
152 coniunguntur $A F H$ coniungantur $B V \mid$ post nodo add $F$ et / hoci : hic $H$ $\mid$ alias erit $A B ; \operatorname{tr} F$ alias erunt $H$ alias esset $V$
153 Et ex $A F$ ex $B V$ et $H /$ hoc: idem $H$
is 4 potest: possit $F$ / ante concludi add $H$ dici / ante quodr bab $H$ ex hoc / est possibile $F H V$; $\operatorname{tr} A B /$ quod $^{2} B H V$ ut $A F /$ maxima $B F H$ maxime $A V /$ post eveniat bab $V$ propter

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plied by some number is equal to circle $B$ multiplied by some [other] number. They are, therefore, commensurable, by the second supposition, and this destroys the antecedent. One can argue in this manner about any other incommensurability whenever circumferences are involved, provided they are moved incommensurably with respect to the center.

Let $A$ and $B$ be in point $c$. If, after some time, they are again in $c$, then any one of them has made an exact number of revolutions. Therefore, the circles are commensurable, which is contrary to the assumption.

Proposition V.* There are an infinite number of points in which two mobiles disposed [as in the preceding proposition] have coniuncted, and the points in which they will conjunct are [also] infinite [in number].

This is clear because they will be in conjunction an infinite number of


Fig. $3^{\dagger}$
times [but always] at a new place, and this also applies to other dispositions.

It is evident from this that if there were two circles intersecting in the nodes of the head and tail of the dragon and the mobiles are moved incommensurably then, by this proposition and others following, if they are once in conjunction in a node this will never happen at other times. This can also be said about opposition. And from this one can conclude that through eternal times a perfect [or total] lunar eclipse might occur in only

* See p. 94.
figure, oriented horizontally and omitting
$\dagger$ This figure appears in MS $F$. A similar the word nodus, is contained in MS $H$.
perpetuis temporibus una vice precise et similiter de sole et multa inde sequuntur.
Item sint duo quadrata quorum dyameter unius sit costa alterius. Incipiantque $A B$ equaliter moveri ab angulo quodlibet super suum numquam amplius se invenient invicem nec in illo angulo nec in alio.
Sexta conclusio. Si tria mobilia moveantur inequaliter et commensurabiliter quo ad centrum et nunc sint coniuncta alias erunt et fuerunt infinities motu eterno coniuncta et loca coniunctionum trium sunt numerata finite, et sic de aliis dispositionibus.
Hoc potest argui ex principiis ex quibus arguitur ad primam et ${ }^{165}$ secundam conclusiones predictas.

Septima conclusio. Possibile est ut sint tria mobilia quo ad centrum difformiter seu dispariter commensurabiliter mota que numquam coniungentur.

Sint $A B C$. Cum quodlibet cuilibet coniungantur infinities, loca coniunctionis quorumlibet duorum sunt finita per secundam conclu-
i70 sionem. Si igitur loco coniunctionum $A B$ sint alia a locis coniunctionum $B C$ sequitur quod numquam coniungentur.

Verbi gratia sit circulus $A$ ad circulum $C$ in proportione dupla, ad
iss una: nam(?) $V$ / post precise mg hab $B$ tantum
157 ante sint $m g$ bab $H$ duas figuras
is 8 ante B add $V$ et / equaliter moveri $A B F ; \operatorname{tr} H$ moveri $V /$ post angulo add $V$ eodem / super $A B H$ supra $F V$
159 amplius: alias $F /$ illo $A B H$ isto $F V /$ angulo....alio $B F H$ nec alio angulo $A$ nec in alio angulo $V /$ post alio add $B$ et cetera sequitur conclusio sexta
160 Sexta conclusio mg bab $F$ ante Si et $m g$ bab $H$ ante invenient (linea 159); om $A V$ conclusio sexta $B /$ Si: sint $F /$ inequaliter om $V$
161 et ${ }^{1}$ om $B /$ erunt: fuerunt $F / \mathrm{et}^{2}$ om $F$ | fuerunt $A H$ fuerant $B$ erunt $F$ fient V
162 et ${ }^{1}$ om $A$ | post loca bab $A 3$ / coniunctionum: coniunctionis $A \mid$ ante finite $m g$ bab F figuram
162-63 et sic...dispositionibus om $V$
${ }_{1} 63$ post dispositionibus hab $B$ suppositionibus(?) commensurabilibus(?)
164 potest: posset $H /$ argui $A B H$; om $F$ ex premissis $V \mid \mathrm{ex}^{2} B F V$; om $A H \mid$ infra ex quibus mg bab $B$ figuram / et $H V$ et ad $B F$ conclusionem et $A$
r6s conclusiones predictas: conclusionem $H$
166 Septima conclusio $m g$ bab $F$ ante Possible et ante principiis (linea 164) mg bab $H$ septima; om $A V$ conclusio 7 $B /$ ut $F H V$ quod $A B /$ sint om $H /$ quo om $A$ / post centrum add $V$ seu
166-67 difformiter FHV; om $A$ dicuntur (?) $B$
167 seu om $A$ / commensurabiliter mota $A F H ; \operatorname{tr} V$ commensurabiliter tamen nota $B$
168 ante A add $B$ mobilia / Cum om V | cuilibet om $V /$ coniungantur: coniungatur $V /$ ante infinities bab $V$ cum quodlibet
169 coniunctionis: coniunctionum $B /$ quorumlibet om $V /$ sunt: sint $V$
169-70 conclusionem: suppositionem(?) A
170 coniunctionum om $B / \mathrm{AB} \operatorname{sint} A B V$ fuerint $F$ A B sunt $H /$ a locis: locus(?) F
${ }^{171}$ Com $F$
172 A $B F$; om $A H V / \mathrm{C} F H V \mathrm{~B} A B \mid$ dupla: $3 / 2 B /$ ante $\mathrm{ad}^{2}$ add $B$ et

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one place; and something similar could be said about the sun. Many other things follow from this.
Furthermore, let there be two squares where the diagonal of one is the side of the other. Then let $A$ and $B$ be moved with equal speed from any angle on the figure, and never again would they be found in that angle or in another angle.


Fig. $4^{*}$
Proposition VI.t If three mobiles are moved with unequal speeds but commensurably with respect to the center and are now in conjunction, then they will be and bave been in conjunction an infinite number of times through an eternal motion, and the places of conjunction of three mobiles are finite in number; and this applies to the other dispositions.

This can be argued in terms of the principles applied to the first and second propositions stated above.

Proposition VII. $\ddagger$ It is possible there could be three mobiles moving commensurably but non-uniformly, or unequally, with respect to the center and they will never be in conjunction.

Let there be $A, B$, and $C$. Since any one of these would be in conjunction with any other an infinite number of times, the places of conjunction of any two of them are finite in number by the second proposition. If, therefore, the places of conjunction of $A$ and $B$ are other than the places of conjunction of $B$ and $C$, it follows that they will never be in conjunction.

For example, let circle $A$ be related to circle $C$ in a double ratio, and to

* This figure is taken from MS $F$. A variant figure, mobile $A$ moves on the larger, $B$ figure appears in $H$. Note that mobiles $A$ on the smaller square.
and $B$ are shown after departure from con- $\quad+$ See pp. 94-95. junction in some particular angle. In the
$\dagger$ See pp. 94-95.
$\ddagger$ See pp. $95-96$.
circulum $B$ in sexquialtera. Distetque $B$ a linea $d$ per sextam partem sui circuli et incipiant equaliter moveri versus $f$ numquam coniungenin puncto $d$ et $B$ retro per unam sext sui circhi. in puncto $d$ et $B$ retro per unam sextam sui circuli. Incipiantque moveri versus $f$.
Per primam conclusionem $A C$ numquam coniungentur nisi in puncto $d$. Et cum $B A$ coniungantur primo die in puncto opposito $e$, . nisi in puncto $e$, et hoc in sexta die vel sex diebus. Ex hoc sequitur quod tria numquam coniungentur. Totum patet ex prime conclusionis arte et secunde.
Octava conclusio. Possibile est ut sint tria mobilia quo ad centrum incommensurabiliter mota que numquam coniumgentur.
Sint $A B C$, sitque circulus $B$ duplus ad $C$, et $A$ sit sicut dyameter cuius costa esset $B$. Moveanturque equaliter et pertranseat $C$ totum circulum una die. Igitur per primam conclusionem $B C$ semper coniungentur in puncto $d$. Ponatur, igitur, quod $A B$ in eodem puncto aliquando coniungantur quando $B C$ non sunt hic. Igitur per quartam

173 circulum $\mathrm{B} F H V \mathrm{~B} A C B /$ in om $F$ sexquialtera: proportione dupla $B$ Distetque: incipiantque $A / \mathrm{B}^{2}: \mathrm{D} A$ a om $A /$ linea d $B F H ; \operatorname{tr} V$ linee $A /$ sextam $A F H$ quartam $B 6 V$
174 et $B F V$; om $H /$ et incipiant equaliter om $A /$ incipiant $H$ incipiat $F V$ incipient $B /$ ante equaliter $m g$ bab $H$ f guram / equaliter $B F H$ uniformiter $V$
174-77 numquam...versus fom $V$
175 A circulus $A B F ;$ tr $H /$ ante 6 add $F$ ut $/ \mathrm{B}$ circulus $B F$; $\operatorname{tr} H \mathrm{~B} A /$ circulus ${ }^{3} B F$; om $A H /$ et $B F H$; om $A$
if6 et $\mathrm{B} A B F$; om $H /$ unam $B$ unum $A F H$ / sextam corr ex quartam $B$; gradum $A F$ gradus $H$
178 nisi $A B H$; om $F V$
179 B A $\operatorname{tr} B /$ primo $F$ prima $H V$ in opposito primo $A$ in tertio $B /$ in om $A /$ e $F H V$ C $A B$
I80 per eandem...etiam: patet etiam per eandem conclusionem $V /$ eandem $B F H$ primam $A /$ etiam $F H$; om $A B$ / coniungentur $F H V$ coniungantur $A B$

181 e $A H$; om $V$ C $B F /$ in sexta...diebus $A$ sexta die vel de 6 in 6 diebus $F H V$ tertio die vel et deinde(?) de 12 in 12 diebus $B$
182 tria $H_{3} A B$ tres $F /$ tria numquam: numquam $3 \mathrm{~V} /$ Totum patet $A F H$ totaliter(?) hoc patet $B$ hoc sequitur $V /$ prime $H V$ primo $B($ ? $) F$ primam (?) $A$
182-83 conclusionis arte et secunde $B F$ et secunda $A$ secunde conclusionis arte $H$ conclusionis artem et etiam secunda $\stackrel{H}{V}$
184 Octava conclusio $m g$ hab $F$ ante Possibile et $m g$ hab $H$ ante quod tria (linea 182); om $A V$ conclusio octava $B /$ est om $A /$ ut $F H$ quod $A B V$
I86 sitque circulus: circuli sitque $B / \mathrm{B}$ duplus ad C : triplus(?) B ad $\mathrm{C} B / \mathrm{A}^{2}$ om $A$
187 equaliter: taliter(?) $V /$ pertranseat $B F H$ pertranseant $A V$
188 una die: 4 diebus $B$
189 eodem: aliquo $V$

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circle $B$ in a sesquialterate ratio. Now should $B$ be distant from line $d$ by a sixth part of its circle and begin to be moved with uniform speed toward $f$, the three mobiles will never be in conjunction. For let circle $A$ be 6 , circle $B 4$, and circle $C_{3}$; and let $A$ and $C$ be in point $d$ and behind $B$ by one-sixth of its circle. Then let them begin to be moved toward $f$.
By the first proposition, $A$ and $C$ will never be in conjunction except in point $d$. And if $B$ and $A$ should be conjuncted the first day in an opposite point $e$, it is clear, by the same proposition, that they will never be conjuncted except in point $e$, and that will happen on [every] sixth day, or [every]


Fig. s*
six days. It follows from this that the three mobiles will never be in conjunction. All this is evident from the procedures in the first and second propositions.
Proposition VIII.t It is possible there could be three mobiles moved incommensurably with respect to the center wbich will never conjunct.
Let there be $A, B$, and $C$ and let circle $B$ be double to $C$, and $A$ be as the diagonal whose side is $B$. Also let them be moved with equal speeds and let $C$ traverse a whole circle in a day. Then, by the first proposition, $B$ and $C$ will always be conjuncted in $d$. Now let it be assumed that at some time $A$ and $B$ would be conjuncted in that same point $[d]$ when $B$ and $C$ are not there [simultaneously]. By the fourth proposition, therefore, they

* This is a composite figure based on dia- sion of the proposition, see pp. 95-96. grams in MSS $B, F$, and $H$. For a discus- $\dagger$ See p. 96 .
conclusionem numquam alias erunt nec fuerunt coniuncta quare sequitur conclusio.
Nona conclusio. Possibile est ut sint tria mobilia que per totum tempus eternum coniungentur semel et impossibile est ea pluries coniungi, nec alias fuisse coniuncta, nec alias coniungenda.

Sint $A B C$ ut prius disposita, nisi quod sint omnia in puncto $d$. Per primam conclusionem $B$ et $C$ numquam coniungentur nisi in puncto $d$; et cum $A B$ nunc sint in eodem puncto impossibile est ipsa alias fuisse ibi, nec in posterum fore per quartam conclusionem.

Tunc si $B$ et $C$ numquam coniungentur nisi in puncto $d$, et $B$ et $A$ numquam coniungentur in puncto $d$ igitur $A$ et $B$ et $C$ numquam coniungentur de centro; et eodem modo de preterito, et hoc posset multis aliis modis contingere et ista conclusio est valde pulcra.
Si autem plusquam semel in eternitate tota coniungerentur necessario infinities coniungentur quia tunc omnia essent commensurabilia et tunc argueretur per sextam conclusionem.

Patet igitur quod primo casu posito possibile est quod numquam coniungantur; et non est possibile ea coniungi in eternitate nisi semel. Et eodem modo dico quod non est possibile nisi semel quod unum
${ }_{210}$ eorum alteris duobus simul recte opponatur. Et etiam multa alia ex hac conclusione sequuntur.

191 alias om $A$ / nec: vel $V$ / coniuncta V; om AFH ibi $B$
192 post conclusio add $B$ predicta conclusio sequitur
193 Nona conclusio mg hab $F$ ante Possibile et mg hab H post tria; om AV conclusio nona $B /$ est om $F /$ ante sint add $V$ quod / tria: $3 A /$ tempus $F V$; om $A B H$
194 est om $A$ / pluries: plurie $\mathrm{D} V$ / fuisse om $A$
195 nec alias $B F H$ nec $A$ aut $V$
196 nisi: non $A /$ omnia: alia $A$
197 infra nisi $m g$ bab $F$ figuram et $m g$ hab $B$ figuram
198 cum: c $A$ / infra cum mg bab F figuram et mg hab $B$ figuram | post est add $F V$ ea / ipsa om $V$ / post alias add $H$ hoc
199 ibi $A B F$; om $H$ sibi $V /$ quartam: octavam $V$
200 si $F$ sit $A B H$ sicut $V \mid$ et $^{1}$ om $H \mid$ $\mathrm{d} o m F$

201 in puncto d $B F$;om $V$ in d $A H /$ et $^{1}$ $B F$; om $A H V / \mathrm{et}^{2} B F$; om $A H V$
202 coniungentur de centro $F$ contingentur de centra(?) $A$ de centra(?) coniungentur $B$ in centro coniungentur $H$ centro coniungentur $V / \mathrm{et}^{1} A F H$ in $B /$ modo de preterito $A F H$ puncto d $B / \mathrm{et}^{2} B F H$; om $A /$ posset $A B H$ potest $F$
202-5 et eodem...coniungentur om $V$
203 multis $B F$; om $A$ multo $H /$ ista $A B F$ illa $H$ / conclusio est $A B ; \operatorname{tr} F H$
204 tota FH; om $A B$
205 infinities $A B H$ sufficiens(?) $F /$ coniungentur $A$ coniungerentur $B F H \mid$ post quia tunc $m g$ hab $H$ figuram
206 argueretur: arguitur $A /$ sextam $B F H$ $6 A V$
207 quodr ${ }^{\text {: }}$ ex $H$
208 coniungantur $B F H$; om $A$ coniungentur $V$ / post in add $V$ tota
209 Et : in $V /$ possibile nisi semel $B F H$;

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will never be, nor have ever been, conjuncted; and thus the proposition follows.

Proposition IX.* It is possible there could be three mobiles that will be in conjunction only once through all eternity, and it is impossible for them to be conjuncted more than once, or to bave been in conjunction at other times, or to be conjuncted at other times.
Let $A, B$, and $C$ be disposed as before [see Fig. 5, p. 40r], except that [now] they are all in point $d$. By the first proposition, $B$ and $C$ will never be in conjunction except in point $d$; and since $A$ and $B$ are now in the same point, it is impossible for them to have been there at other times or to be there in the future, by the fourth proposition.

Now if $B$ and $C$ will never be in conjunction except in point $d$, and $B$ and $A$ will never be in conjunction in point $d$ [except once], then $A, B$, and $C$ will never be in conjunction with respect to the center; and the same holds for the past, and this could happen in many other ways. Indeed, this is a very wonderful proposition.
Furthermore, should they be conjuncted more than once in all eternity, they will be conjuncted, necessarily, an infinite number of times because then all would be commensurable and one could argue in terms of the sixth proposition.
It is clear, then, that in the first case set forth it is possible that the mobiles might never be in conjunction; and in an eternity it is not possible for them to be in conjunction, except once. And in the same way it is not possible, except once, for one of them to be directly opposed to the other two at the same time. Indeed, many other things also follow from this proposition.

* See p. ${ }^{6}$ 6.


## om $A V$

209-10 quod unum eorum: alterum illorum $V$
210 eorum $A B F$ earum $H /$ alteris duobus $\operatorname{tr} V /$ simul recte $B F H$ directe $A$ recte opponi simul $V$ / opponatur $A$
opponantur BFH / etiam FH; om $A B V$
210-11 ex hac...sequuntur $F H$ sequitur ex hac conclusione $A$ inde sequuntur $B$ sequuntur $V$

- Incipit secunda pars huius operis.

Plura mobilia se habere sicut prius aliquando se habuerunt contingit tribus modis: aut in coniunctione aut in oppositione et tunc non faciunt angulum vel angulos; vel in alia dispositione et tunc circa 5 centrum describunt angulum vel angulos. Coniunctio sit uno solo modo et oppositio similiter. Alia dispositio dupliciter; una vice ante coniunctionem seu oppositionem; alia vice post.

Dispositiones angulares dicuntur proprie similes quando ambe sunt post coniunctiones aut oppositiones; improprie sunt similes quando
${ }_{10}$ una est ante et alia post. Et causa huius est quando una est ante et alia post mobilia prius ibant ad coniunctionem modo recedunt. Item illud quod erat ante est post, sive retro, et econtraria.

Item immediate post primam dispositionem se habent aliter quam immediate post secundam; et immediate ante primam se habuerunt
15 aliter quam se habebunt ante secundam, licet immediate ante secundam se habeant sicut immediate post primam et immediate ante primam sicut immediate post secundam. Quando, autem, ambe sunt post non est ita.

Tria mobilia dicuntur opponi quando non describunt angulum in
20 centro et unum opponitur simul recte duobus. Tria mobilia possunt in centro causare unum angulum aut duos; quatuor autem tres, et sic ultra.
[Pars secunda] om $A B F H V$
I Incipit secunda...operis F; om $A H V$ tractatus secundus $B$; post habuerunt (linea 2) $m g$ bab $A$ secundus pars huius / post operis bab $F$ divisio prima
2 sicut prius aliquando: aliquando sicud prius $A$
2-3 contingit tribus modis: aliquando contingit tripliciter $V$
3 in ${ }^{1}$ om $A /$ aut $^{2}: \operatorname{vel} A /$ in $^{2}$ om $A$
4 vel $^{1} A H V$ nec $B F /$ vel $^{2}$ om $A /$ alia: aliqua $F$ / post alia mg hab $A$ prima dispositio / circa $A B$ in $F H$
$4^{-s}$ circa centrum describunt: faciunt $V$
s centrum $A B$ centro $F H$ /angulum vel angulos $B F H$ angulos $A$ angulos vel angulum $V /$ uno: ergo $H$
6 ante dupliciter bab $V$ media angulorum / ante una add $F$ sit / una vice: uno
modo $A$
7 seu: et ante $H$ | ante alia add $H$ et | alia vice: alio modo $A$ / post post add $A$ oppositionem et coniunctionem
9 post coniunctiones mg bab $H$ diffinitio $/$ aut oppositiones om $V$ / ante improprie add $A$ sed / sunt $B F V$; om $A H$ io et ${ }^{1} A B V$; om $F H /$ ante quando bab $H$ quia / $\mathrm{et}^{3}$ om $H$
${ }^{11}$ prius om $F$ / modo: quando(?) $B \mid$ illud: istud $A$
12 ante est bab $V$ modo / sive retro $B H V$; om $A$ vel retro $F$ / econtraria $F V$ econtrario $A B$ contraria $H$
I3 se habent aliter: aliter se habent $V$
14 ante immediate bab $H$ et similiter primam: secundam $H$ / se habuerunt ABF; om $H$ convertunt(?) $V$
is se habebunt $B$; om $A$ se habuerunt $F$

The second part of this work [now] begins.
It can happen that several mobiles related as before have been related at some time or other in three ways: either in conjunction or opposition, and then they form no angle or angles; or in another disposition, and then they describe an angle or angles. A conjunction can occur in only one way and likewise for an opposition. Any other disposition is twofold: in one way before conjunction or opposition; in the other way after [conjunction or opposition].
Angular dispositions are called "properly similar" when both are after conjunctions or oppositions; they are "improperly similar" when one is before, the other after [conjunctions or oppositions]. And the reason why sometimes one is before and the other after is that the mobiles going to conjunction before are now receding [from conjunction]. Moreover, that which was before is now after, or behind, and conversely.*
Furthermore, immediately after the first [angular] disposition they are related differently than immediately after the second [disposition]; and immediately before the first they are related differently than they will be before the second [disposition], although directly before the second they would be related just as [they were] immediately after the first; and immediately before the first [they would be related] as [they were] immediately after the second [disposition]. But when they are both after, this is not so.

Three mobiles are said to be opposed when they do not describe an angle in the center and one is opposed directly to the [other] two simultaneously. Three mobiles can produce one or two angles in the center; four mobiles, however, can produce [as many as] three, and so on.

* See pp. 97-100
immediate $H$ se habeant $V /$ ante $^{\mathrm{I}}$ : post $H$
15-17 licet...secundam om $A$
I6 habeant $F V$ habebant $B$ habent $H /$ immediate $B F V$; om $H /$ ante $B F V$ post $H$
17 autem: aut $V /$ ambe $A F V$ ambo $B H$

19 post angulum $m g$ bab $H$ secunda diffinitio
21 quatuor autem tres $B$ aut 3 et $4 F V$ aut 3 et cetera(?) $A$ aut 3 et 4 et infra(?) H
21-22 et sic ultra om $A$

Omnia mobilia uno modo coniunguntur. Duo, vero, uno modo opponuntur, tria tribus, quatuor sex, et quinque decem, et cetera. Et 25 hoc scitur arte sic: multiplica numerum mobilium per immediate precedentem et accipe subduplum et tot modis variis opponuntur. Verbi gratia sint 5 . Multiplica per 4 sunt 20 , subduplum est 10 .

Prima conclusio. Inter quascumque dispositiones similes duorum mobilium consequenter se babentes fuit coniunctio aut oppositio in instanti medio.

Quoniam dispositio media quarumcumque dispositionum similium est ut non sint anguli. Si enim describerent angulos essent propinquis uni dispositioni quam alteri, ut patet speculanti. Et ideo inter duas coniunctiones mediat oppositio, et econtraria; et inter alias duas dispositiones oppositio aut coniunctio.
Secunda conclusio. Inter dispositiones angulares improprie similes consequenter se babentes est tantum coniunctio aut oppositio; et inter proprie similes utraque, scilicet coniunctio et oppositio.
Prima pars patet ex prima conclusione. Secunda patet quia dispositiones proprie similes non sunt nisi unum vadat ad unam coniunctio-
${ }^{40}$ nem et aliud ad aliam, aut ambo veniant a duobus. Sed inter duas coniunctiones est oppositio per primam conclusionem. Igitur inter istas dispositiones erit oppositio et coniunctio.

Ex hoc sequitur quod tot sunt loca cuiuscumque dispositionis pro-
23 post mobilia mg bab $H$ suppositio prima
24 et $^{1}$ om $H /$ decem: $4 A /$ et cetera om $A$
25 hoc $A H$; om $B F V$ | ante arte bab $V$ hoc(?)
26 opponuntur $A$ coniungentur $B F H$ opponentur $V$
27 sint $H$; om $A B$ sicut $F V /$ Multiplica: a $6(?) \mathrm{V} / 20: 30 \mathrm{~V} /$ ante subduplum bab $A$ cuius / est to $F H$ sunt $A$ est $B$ sunt io $V$
28 Prima conclusio $m g$ bab $F$ ante Inter et $m g$ bab $H$ post quascumque ; om $A V$ conclusio prima $B /$ similes: consimi les $A$ / duorum mobilium $\operatorname{tr} V$
29 habentes: habentium $F$ / fuit coniunctio $A B F$; tr $V$ est tantum coniunctio $H /$ in $B F V$; om $A /$ in instanti medio: prima pars patet ex prima conclusione secundam patet quia dispositiones proprie similes non sunt nisi unum vadat ad unam $H$

30 quarumcumque $B F V$ quorumcumque $A$ quascumque $H /$ similium: similis $A$
3 I anguli $A B H$ angulum(?) $F$ augeri(?) $V /$ describerent: describunt $V /$ essent: quesirent(?) $B$
32 speculanti: speculationes(?) $V / \mathrm{Et}$ ideo $F H V$ igitur $A$ ideo $B$
33 mediat: erat $V /$ econtraria: contrarium $V /$ et $^{2}$ om $V /$ alias: illas $V /$ duas $A V$; om $B F H$
34 ante oppositio $b a b F$ aut et bab $V$ erat post coniunctio bab $B$ igitur sicut coniunctio et bab $V$ erat oppositio aut erat coniunctio
35 Secunda conclusio mg hab $F$ ante Inter et $m g$ hab $H$ secunda; om $A V$ conclusio secunda $B /$ post Inter add $H$ quascumque
36 et inter: inter se $V /$ ante proprie add $V$ vero / similes: consimiles $B$ | post similes bab $H$ consequenter se

All mobiles are conjuncted in [only] one way. Two mobiles, however, can be opposed in one way; three mobiles in three ways; four mobiles in six ways; five in ten ways, etc.* And this can be known as follows: Multiply the number of mobiles by the immediately preceding [number] and take half [the product]; and they can be opposed in just that many ways. For example, let there be $s$ mobiles. Multiply by 4 which produces 20 , half of which is 10 .
Proposition I.t Between any whatever similar dispositions of two mobiles there is a conjunction or opposition in an intervening instant when the similar dispositions are taken in succession.
Now an intervening disposition between any similar dispositions does not have any angles. For if they should describe angles, the mobiles would be nearer to one disposition than to another, as is clear when one considers the matter. Thus an opposition mediates between two conjunctions, and contrarily. And between two other dispositions there is an opposition or conjunction.
Proposition II.₹ Between "improperly similar" angular dispositions taken in succession there is only a conjunction or an opposition; and between [successive] "properly similar" dispositions both occur, namely a conjunction and an opposition.
The first part is clear from the first proposition. The second is obvious because there are no "properly similar" dispositions unless one [angular disposition] moves to one conjunction and another [angular disposition] to another conjunction, or both come from two [conjunctions]. But between two conjunctions there is an opposition, by the first proposition. Therefore, between these dispositions there will be an opposition and a conjunction.

From this it follows that there are as many places of any properly similar

* By taking the mobiles two at a time, we $\dagger$ See p .100. can represent this as $p(p-1) / 2$, where $p$ is $\ddagger$ See pp. Ioo-ior. the total number of mobiles.
habentes est
37 ante utraque scr et del $V$ tantum coniunctio aut oppositio / scilicet: et $V$ | coniunctio et oppositio $B V$ oppositio et coniunctio $A F H$
38 ante Secunda bab $B$ etiam / ante patet ${ }^{2}$ add $B$ pars

39 proprie: improprie $A /$ vadat om $V$ 40 aut $B F H$ et $A$ ut $V /$ Sed: si $V$
4I coniunctiones: oppositiones $B /$ op positio: coniunctio $B$
2 erit: essent(?) $V /$ coniunctio $A H V$ contraria(?) $B$ econtrario $F$
43 ante Ex add $A B$ et
prie similis quot sunt loca coniunctionis duorum, et totidem sunt loca ${ }_{45}$ dispositionum improprie similium.

Tertia conclusio. Quecumque tria mobilia sive commensurabiliter sive incommensurabiliter mota fuerunt aliquando in aliqua dispositione et nunc communiter sunt in simili improprie, necesse est ut in instanti medio temporis fuerint sine angulo, boc est opposita vel coniuncta.
50 Probatur sicut prima huius secunde partis.
Quarta conclusio. Quecumque tria mobilia sic disposita sunt in dispositione proprie simili bis fuerunt tempore medio sine angulo opposita et coniuncta.

Patet per primam et secundam conclusiones quia sic nunc alia dispositio secunda que est improprie similis non fit donec fuerint sine
55 angulo. Et tertia proprie similis prime et improprie secunde non fit donec fuerit post secundam sine angulo. Igitur inter primam et tertiam, que proprie sunt similes, bis sunt sine angulo.
Quinta conclusio. Tria mobilia commensurabiliter mota in quacumque dispositione sunt nunc, alias erunt et fuerunt in simili proprie et improprie non so tantum semel vel bis sed infinities.

De duobus non est dubium sive commensurabiliter sive incommensurabiliter moveantur. Sed de tribus commensurabiliter motus probatur sic. Sint $A B C, A$ cum $B$ in dispositione in qua est nunc fuit alias ipso existente in hoc loco. Patet per tertiam prime partis igitur $A$ fecit $6_{5}$ aliquot revolutiones. Item per eandem tertiam $A \operatorname{cum} C$ fuit ut nunc

44 quot: quod $V /$ sunt loca $\operatorname{tr} V /$ coniunctionis duorum: dupliciter $V$ post et add $F$ etiam et $V$ inter / totidem: tot $V$
45 similium om $A$
46 Tertia conclusio mg bab F ante Quecumque; om $A H V$ tertia $B$
46-47 commensurabiliter sive incommensurabiliter: commensurabilia sive incommensurabilia $B$
47 post mota bab A que / fuerunt FHV fuerant $A B($ ? ) / aliquando: alii $V$ / et nunc om $V$
$48 \mathrm{in}^{2}$ om $A$
49 fuerint $A H$ fuerit $B V$ fuerunt $F \mid$ sine: sicud $V /$ ante hoc add $A B$ et / est om V / opposita vel coniuncta: oppositio sive coniunctio $B$
so Probatur: probatio $V /$ huius...partis om H
sI Quarta conclusio mg hab Fante Que-
cumque et ante coniuncta (linea 49) mg bab $H$ quarta; om $A V$ quarta $B /$ sic $A F H$ sicut $B V$
52 post fuerunt scr et del $F$ in simili proprie et improprie non tamen semel nec bis sed infinities | ante tempore add $A$ in / tempore om $B$ / ante medio add $B$ in / ante opposita bab $H$ scilicet
53 alia $A F V$ aliqua $B H$
54 est: sit $B$ | ante non bab $B$ prime / post fit hab $A$ hic / fuerint $H V$ fuerunt $A$ fuerit $B F$
is similis: similium $B /$ non om $V$
$\varsigma 6$ fuerit: fuerunt $A$ / secundam: tertiam $V /$ tertiam: secundam $A$
57 proprie sunt $\operatorname{tr} V /$ bis sunt FHV; tr $B$ sunt $A$
58 Quinta conclusio mg bab $F$ ante Tria et ante proprie (linea 57) mg bab $H$ quinta; om $A V$ conclusio quinta $B$ / commensurabiliter: incommensurabi-

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disposition as there are places of conjunction of two [mobiles], and there are just as many places of improperly similar dispositions.

Proposition III.* Any three mobiles-whether moved commensurably or incom-mensurably-tbat have been at some time or other in some disposition and are now together in an improperly similar [disposition], must bave been without an angle in an intervening instant of time-i.e., were in opposition or conjunction.
This is proved just as the first proposition of this second part.
Proposition IV.+ Any three mobiles which bave been in a properly similar disposition twice bave, in the intervening time, been opposed and conjuncted twice without an angle.

This is clear, by the first and second propositions, since now the other second disposition, which is improperly similar, does not occur until the mobiles form no angle. But the third [disposition]-properly similar to the first and improperly to the second-does not occur until after the second [time] that no angle was formed. Therefore, between the first and third, which are properly similar, it happens twice that no angle is formed.

Proposition $V . \ddagger$ Three mobiles that are now moved commensurably in any disposition whatever will be, and bave been, in properly and improperly similar [dispositions] not only once or twice, but an infinite number of times.

Whether two mobiles are moved commensurably or incommensurably there is no doubt [about what happens]. But one may prove in the following way what happens to three mobiles moving commensurably: Let there be $A, B$, and $C$, and $A$ along with $B$ was at other times in the place where it is now. It is obvious, then, by the third proposition of the first part, that $A$ has made a certain number of revolutions. Again, by that same third proposition, $A$ was in this place with $C$ at other times as it is now, and

[^41]alias in isto loco, igitur fecit aliquot revolutiones. Modo revolutiones prime vel sunt equales secundis-et tunc primis factis erunt tres ut prius-vel inequales et tunc cum quilibet numerus aliquotiens replicatus numeret quemlibet aliquotiens replicatum, sequitur quod erunt o aliquotiens sicut nunc.

Si enim $A$ ad habendum dispositionem similem cum $B$ faciat tres revolutiones et ad $C$ quatuor, sequitur quod in duodecima revolutione erunt sicut primus quia quando unus numerus est multiplex ad alium sumendus est maior, quando non unus debet duci in alium et productus 75 erit.

Sexta conclusio. Tria mobilia commensurabiliter mota necesse est ut infinities sint sine angulo.
Quoniam per precedentem dispositiones similes erunt infinite et inter quascumque sunt sine angulo per tertiam conclusionem, igitur
8o oportet ea coniungi et opponi. Aut si non coniungantur sicut ponitur in septima conclusione prime partis, tamen necessario infinities oppo-nentur-non tamen semper eodem modo. Ymo $A$ opponetur $B C$, et $C$ opponetur $B A$.

Ex hoc sequitur quod si loca coniunctionis unius cum alio et cum
$8_{5}$ alio sunt incommunicantia impossibile est quod loca coniunctionis alicuius cum uno et altero sint communicantia. Hoc est consideratio pulcra.

66 isto $F V$; om $H$ illo $A B$ | post loco add $H$ isto / fecit aliquot om $A$ / aliquot $B F H$ aliquas $V$ | post revolutiones ${ }^{1}$ rep $V$ Item per eandem secundam A cum C fecit ut nunc alias in isto loco igitur fecit aliquas revolutiones / ante revolutiones ${ }^{2}$ bab $H$ figuram / revolutiones ${ }^{2}$ : revolutionis $A$
67 prime vel om $V /$ vel $B F H$ non $A$ / factis: finitis(?) $V \mid$ ante erunt bab $A$ aliqui / tres: tertie $V$
68 vel: et $A$ / quilibet om $V$
69 quemlibet: quamlibet $V /$ replicatum: replicatis $V$
${ }^{1}$ dispositionem similem $\operatorname{tr} V$ / faciat: faciet $B$
72 in $H V$; om $A F$ post $B /$ duodecima $F H 12 A B V$ / revolutione $F H$ revolutiones $A B V$
74 sumendus: sumendum $V$ / quando non unus: et $V /$ non $A B F$ uno $H /$
duci: dici $H /$ in: et $B /$ alium: minorem $V$
75 ante erit add $H$ hoc / post erit add $B$ qui queris et add $H$ et hoc est ars huius operis
76 Sexta conclusio mg bab $F$ ante Tria et ante non unus (linea 74) mg hab $H$ sexta; om $A V$ conclusio sexta B/ut FHV; om $A$ quod $B$
76-77 infinities...angulo: sine angulo infinities $V$
77 sint $B F H$ esse $A$
79 per tertiam om $A$ / conclusionem $B F V$;om $A H /$ igitur $A B$ ideo $F H V$
80 ante coniungantur $m g$ bab $F$ figuram $\mid$ sicut $B F V$; om $A$ ut $H /$ ponitur $F H$ ponatur $A B V($ ? $)$
81 tamen $A B H$ cum $F$ igitur $V /$ necessario $F H V$ intentio $A B$
81-82 opponentur $F H$ opponetur(?) $A$ opponuntur $B$ ponentur $V$

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thus it has made a certain number of revolutions. Now the first [series of] revolutions [of $A$ when it arrived at a similar disposition with $B$ ] is either equal to the second [when it arrived at a similar disposition with $C$ ]-and then just as with the completion of the first [series of revolutions] the three [mobiles] will be as before-or unequal; and [in the latter case] since any number multiplied a certain number of times equals any other number multiplied a certain number of times, it follows that they will be just where they are now.

For if $A$ should make three revolutions in order to have a similar disposition with $B$, and four to have a similar disposition with $C$, it follows that in the twelfth revolution they will [all] be as they were at first, because when one number is multiple to another the greater number must be taken, but when [one number is not multiple to another] one must be multiplied by the other to obtain the product.

Proposition VI.* It is necessary that three mobiles moved commensurably should. form no angle an infinite number of [different] times.
[This is evident] because, by the preceding proposition, there will be similar dispositions an infinite number of times and, by the third proposition, the mobiles will form no angle between any [two successive] similar dispositions so that it is necessary for them to be in conjunction or opposition. Or if they did not conjunct, as assumed in the seventh proposition of the first part, they would, nonetheless, be opposed an infinite number of times, [but] not, however, always in the same way. Indeed $A$ would be opposed by $B$ and $C$, and $C$ would be opposed by $B$ and $A$.
From this it follows that if the places of conjunction of one [mobile] with each of the others separately are incommunicant, it is impossible for this mobile to have any common places of conjunction with the other two mobiles. This is a beautiful consideration.

## * See p. 103

82 Ymo: sed $V /$ ante A add $A B$ si opponetur $A F V$; obs $H$ opponeretur $B \mid \mathrm{BC}$ : in $\mathrm{B} B$
83 opponetur $A H$; om $V$ opponeretur $B F / \mathrm{BA} A B$; om $F$ B $H$ A $V$
$84 \operatorname{quod} A B V$; om $F H /$ ante si $m g$ bab $H$ corporum regularum / coniunctionis: oppositionis $A$

8s alio $F H V$ alia $B$ igitur $A /$ sunt $B F H$; om $V$ sint $A /$ coniunctionis. oppositionis $B$
86 alicuius: unius $V /$ et om $V /$ ante Hoc add $A$ et / consideratio om $A$
87 pulcra: pulcram(?) $A$ / post pulcra bab $H$ sed propter veritatem(?) dictum

Septima conclusio. Tria mobilia incommensurabiliter mota, ut in casu none conclusionis prime partis, in quacumque dispositione sunt nunc non fuerunt alias so in biis locis.

Patet quia $B$ cum $C$ fuit alias sicut est in hoc loco; $B$ cum $A$ non fuit alias sicut nunc est in hoc loco, per quintam et tertiam prime partis.

Octava conclusio. Tria mobilia ut prius disposita. Si semel sunt coniuncta impossibile est ipsa opponi.

Quia quando $B C$ coniungantur pertransiverunt aliquotiens suum circulum, sed quanto pertranseunt de suis circulis commensurabiliter $A$ pertransivit incommensurabiliter. Igitur non est in distantia commensurabili puncto coniunctionis trium vel coniunctionis $B C$. Sed punctus oppositus est locus commensurabilis quia dividit circulum roo per medium.

Item arguitur quod $A B$ non coniunguntur quando $C$ eis opponitur. Quia quando $A B$ coniunguntur pertransiverunt de circulis suis incommensurabiliter spatia et quando $B C$ opponuntur pertransiverunt commensurabiliter. Igitur quando $A B$ coniunguntur $B C$ non ros opponuntur.

Nona conclusio. Nulla tria mobilia ut prius disposita aliquando describunt in centro angulos inter se commensurabiles et angulo recto.
Sint $A B C$ in puncto $d$; incipiantque moveri et $B C$ commensurabiliter moveantur, et $A B$ incommensurabiliter quo ad centrum. Tunc

88 Septima conclusio mg bab $F$ ante Tria et $m g$ bab $H$ septima; om $A V$ conclusio septima $B /$ casu: eorundem $B$
89 sunt nunc $A B H ; \operatorname{tr} V$ sint nunc $F \mid$ fuerunt: erunt $H$
90 in: et $V$
9r post sicut mg hab $H$ figuram / in hoc loco: nunc $A$ a ante non add $H$ et
92 nunc est $B F$; $\operatorname{tr} V$ nunc $A H /$ hoc: illo $A$ / ante per add $A$ ut et add $H$ et cetera / per om $B$ / quintam et tertiam : tertiam et quintam $V /$ ante prime add $B$ conclusionis
93 Octava conclusio mg bab $F$ ante Tria; om $A H$ conclusio octava $B /$ coniuncta BFH; om $A$
93-10s Octava conclusio...non opponuntur om $V$
94 est BFH; om $A$ / post opponi bab $A$ est
95 coniungantur $A$ opponuntur $B$ dis-
iunguntur $F$ coniunguntur $H /$ per$\stackrel{\text { transiverunt }}{H} A B(?) F$ pertransierunt H
96 pertranseunt $B$ transiverunt $A F$ pertransierunt $H /$ de $B F H$ in $A /$ circulis $A B F$; om $H$
97 pertransivit $A F H$ pertransit $B /$ Igitur BFH; om $A$ | in $A B F$; om $H$ | ante distantia mg hab $F$ figuram
98 coniunctionis ${ }^{2} A F H$ oppositionis $B$
ror quod $A F H$ quia $B /$ ante B add $H$ et / coniunguntur $A H$ commensuratur(?) $B$ coniungitur $F /$ quando corr ex quia $A B F H$
101-2 ante opponitur bab $B$ non
102 ante B add $H$ et / pertransiverunt $A$ pertransierunt $F H$ pertranseunt $B$ de circulis suis $B F H$ circulos suos $A$
103 spatia $A$; om $B H /$ opponuntur $A B$ opponentur $H$
103-4 spatia...commensurabiliter om $F$ |

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Proposition VII.* Whatever the present disposition of three mobiles moved incommensurably, as in the case of the ninth proposition of the first part, they have not been in those places at other times.
This is clear because $B$ was in this place with $C$ at other times; but $B$ was not in this [same] place at other times with $A$ as it is now. [All this is supported] by the fifth and third propositions of the first part.
Proposition VIII. + Tbree mobiles are arranged as before. If they are once in conjunction, it is impossible for them to be in opposition.
This is so because when $B$ and $C$ are conjuncted they have traversed their [respective] circles a certain number of times. But when they have commensurably traversed their circles, $A$ has traversed [its circle] incommensurably [with respect to $B$ and $C$ ]. Therefore, the distance of $A$ from the point of conjunction of the three mobiles, or from the point of conjunction of $B$ and $C$, is not commensurable [to the distances traversed by $B$ and $C$ ]. But the point opposite [the point of conjunction of $B$ and $C$ ] is a place [whose distance from the point of conjunction] is commensurable [to the distances traversed by $B$ and $C$ ], since it divides the circle in half.

It can be further argued that $A$ and $B$ cannot be in conjunction when $C$ is opposed to them. This is so because when $A$ and $B$ are conjuncted they have traversed, on their [respective] circles, distances which are incommensurable; but when $B$ and $C$ are opposed they have traversed distances which are commensurable. Therefore, when $A$ and $B$ are in conjunction, $B$ and $C$ are not in opposition.

Proposition IX. $\ddagger$ At no time can three mobiles arranged as before describe central angles mutually commensurable and commensurable to a right angle.
Let $A, B$, and $C$ be in point $d$; and let them begin to move with respect to the center, with $B$ and $C$ moving commensurably, $A$ and $B$ incommen-

* See p. 103.
$\ddagger$ See p. ros.
+ See pp. 103-5.
pertransiverunt $A$ pertranserunt $B$ pertransierunt $H$
104 ante $\mathrm{B}^{1}$ add $H$ et
106 Nona conclusio: $m g$ bab $F$ ante Nulla et post in centro (linea 107) mg bab H nona; om $A V$ conclusio nona $B /$

Nulla om $V$ disposita: opposita $A /$ aliquando $A B H$; om $F$ alii $V$
107 in centro om $B$
108 Sint $A B F$ sit $H V$
109 moveantur $A F$; om $H$ moveatur $V$ moventur $B$
no arguitur sic: angulus $C B$ non est recto commensurabilis nisi quando $B d$ est recto commensurabilis. Sed $A B$ numquam est recto commensurabilis quando $B d$ est recto commensurabilis. Igitur $A B$ et $C B$ numquam erunt simul recto commensurabiles.
Maior patet quia angulus $B d$ et $C d$ semper erunt commensurabiles,
115 igitur $C B$ erit eis commensurabilis. Quia maior eorum excedit minorem in $B C$ et quecumque sunt commensurabilia excessus maioris est utrique commensurabilis. Igitur quandocumque $B d$ est recto commensurabilis $C B$ erit recto commensurabilis et non alias.

Quia similiter est econtraria per istam regulam: quecumque uni et
120 eidem sunt commensurabilia inter se sunt commensurabilia. Sed quia $A d$ et $B d$ semper sunt incommensurabilia, igitur $A B$ est eis incommensurabile. Quia maior eorum excedit minorem in $A B$ et quecumque sunt incommensurabilia excessus maioris est utrique incommensurabilis. Igitur quando $B d$ est recto commensurabilis $A B$ non
125 est recto commensurabilis quia tunc essent commensurabilia inter se, quod est negatum.

Decima conclusio. Quandocumque duo talium mobilium coniunguntur, tertium cum eis causat angulum in centro recto incommensurabilem.

Patet faciliter posito quod $A C$ coniungantur, tunc $B C$ et $B A$ sunt
no sic om $B / \mathrm{CB} \operatorname{tr} B /$ est om $V$ | post est mg bab $H$ figuram | recto $B H(?)$; om AFV
in d: C $A /$ est rector ${ }^{\text {: }}$ recte est $V /$ est recto ${ }^{2} A B H ; \operatorname{tr} F V$
112 quando...commensurabilis om $B / \mathrm{d}$ : C $A$
113 erunt simul $A B F$; tr $V$ erunt $H$
114 erunt: erit $A$
is igitur: et ergo $H$ / erit: erunt $A$
$116 \mathrm{BC} \operatorname{tr} F$ / maioris: eorum $H$
117 utrique: uterque $V /$ commensurabilis ...recto om $B /$ quandocumque $A H$ quandoque $F V$
118 B: D $A$ / erit: esset $V$
119 post Quia scret del $B$ similiter est econtraria per primam tantum / similiter... regulam om $B$ / istam $F V$ secundam regulam om $B /$ istam $F V$
(?) $A$ illam $H /$ uni om $H$
119-20 et eidem $F$; om $A B H V$
I20 commensurabilia ${ }^{1}$ : incommensurabilia $A /$ inter se: eidem $H /$ sunt $F H$; om $A$ minor $B$ semper $V /$ commensurabilia ${ }^{2}$ FHV incommensurabilia
$A /$ commensurabilia Sed: patet $B /$ Sed $A F$ inter se minor patet $H$ similiter $V$
121 incommensurabilia $V$ incommensurabiles $B H$ commensurabiles $A($ ? ) $F$
123 maioris est om $H$ / utrique: utticumque $H$
123-24 ante incommensurabilis add $A H$ est
125 ante tunc bab $B$ si sit / essent: esset $V$
126 est om $A$ / post est add $V$ iam / negatum: notis(?) $V \mid$ post negatum add $A$ est et add B prius
127 Decima conclusio mg bab Fante Quandocumque et post Quandocumque mg bab $H$ decima; om $A V$ по $B /$ duo talium $B V ; \operatorname{tr} F$ duo talia $A$ duo trium $H$ / mobilium: mobilia $A$
I28 recto incommensurabilem $A F H$; tr $B$ recto commensurabilem $V \mid$ infra incommensurabilem mg bab $F$ figuram
129 A C $\operatorname{tr} \mathrm{H} /$ coniungantur FHV coniungatur $A$ coniunguntur $B /$ et om $H$

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surably. Then it can be argued as follows: Angle $C B$ is not commensurable to a right angle except when $B d$ is commensurable to a right angle. But [angle] $A B$ is never commensurable to a right angle when $B d$ is commensurable to a right angle. Therefore [angles] $A B$ and $C B$ will never be commensurable to a right angle at the same time.

The major [premise] is clear because angles $B d$ and $C d$ will always be commensurable and therefore $C B$ will be commensurable. Since the greater of these [angles] exceeds the lesser by $B C$, and any of these are commensurable, the excess of the greater angle is commensurable to each [of the other angles]. Thus whenever [angle] $B d$ is commensurable to a right angle, $C B$ will be commensurable to a right angle, but not at other times.


Fig. 6*
Now similarly the contrary is shown by this rule: Any [quantities] whatever commensurable to one and the same [quantity] are commensurable to each other. But since angles $A d$ and $B d$ are always incommensurable, then $A B$ is incommensurable to them. Since the greater of them exceeds the lesser by $A B$, and any of these are incommensurable, the excess of the greater [angle] is incommensurable to each [of the other angles]. Therefore when $B d$ is commensurable to a right angle, $A B$ is not commensurable to a right angle because then they would be commensurable to each other, which has been denied.

Proposition X. + Whenever two such mobiles are in conjunction, the third mobile would produce with them an angle in the center which is incommensurable to a right angle.

This is easily shown, for having assumed that $A$ and $C$ are in conjunction, $\ddagger$ then $B C$ and $B A$ are one angle; but by the preceding proposi-

* This figure appears in MS $F$. An almost identical figure omitting point $d$ is found in MS $H$.
† See p. 106.
$\ddagger$ Fig. $s$ on p. 401 illustrates the relative positions of the mobiles.

130 unus angulus; sed numquam sunt simul uni recto commensurabiles per precedentem. Et sic si $A B$ coniungerentur.
Undecima conclusio. Si duo anguli $A B$ et $C B$ simul sint recto commensurabiles simul talia mobilia numquam coniungentur nec fuerunt coniuncta.

Patet ex conclusione precedentis et sic ulla talia facient angulos tales
${ }_{135}$ commensurabiliter simpliciter. Et sic patet quod si duo coniungantur semel et tertium sit eis in quadratura, vel si faciant duos angulos rectos, vel angulos tres inter se commensurabiles, et sic de aliis, numquam coniungentur.
Duodecima conclusio. Si duo talium mobilium trium aliquando coniungantur
${ }^{140}$ et tertium sit in quadratura, numquam alias coniungentur nec etiam opponentur, et ita de preterito.

Primum probatum est et iterum probo omnia simul. Sint $A B$ coniuncta in $d$ et $C$ sit in quadratura. Tunc arguatur: quandoque $B$ et $C$ facient angulum commensurabilem recto vel coniungentur vel
145 opponentur, pertransitum ab ipso $B$ et ab ipso $C$ erunt commensutabilia atque distantie eorum a puncto $d$. Sed quandocumque $A$ et $C$ facient angulum commensurabilem, et cetera, distantie eorum a

130 uni $A$ et $F H$ in $V$
131 sic: similiter $V /$ post A add $V$ et
132 Undecima conclusio mg hab $F$ ante Si duo et post Si duo mg hab $H$ undecima; om $A V$ conclusio undecima $B /$ post Si duo scr et del $H$ mobilia / $B^{1}$ om $V$ | simul: semel $F$ / sint: sunt $A /$ ante recto add $A$ uni
133 coniungentur: coniungunt(?) $B /$ fuerunt: fuerant $B$
134 ulla corr ex nulla $A B F V$ illam $H \mid$ talia om $H$ | post talia add BFV semel coniuncta et add $A$ simul et semel et add $H$ semel necessaria aliquando $\mid$ facient $F H V$ faciant $A$ faciunt $B$
135 commensurabiliter $A$ converte $B F H$ consequente $V /$ simpliciter om $A / \mathrm{Et}$ sic om $B$ / si om $A$ et rep quod duo coniungantur
136 vel om $H$ / faciant: faciunt $A$
137 vel angulos rep $A$
139 Duodecima conclusio mg bab $F$ ante Siet post qua- in quadratura (linea 140 ) mg bab $H_{12}$; om $A V$ conclusio $12 B /$ coniungantur $A F H$ coniungatur $B V$

140 alias: talia $B /$ ante nec add $H$ nec facient angulos commensurabiles recto / nec etiam opponentur $H$; om $A B F V$
141 et ita $F H V$ et similiter $A$ alicuius et idem intellige $B$
142 Primum: probatum $V /$ post probo mg bab $F$ figuram / omnia: totum $V / \mathrm{A}$ B BFV; $\operatorname{tr} A$ B et $\mathrm{A} H$
143 post coniuncta scr et del $F$ omnia simul / sit: sint $H /$ Tunc $B F V$; om $H$ posset $A \mid$ arguatur $B H V$ arguitur $A F \mid$ quandoque $F H V$ quandocumque $A B$
144 angulum om $A \mid$ ante recto add $B$ angulo / coniungentur: coniunguntur $B$
145 opponentur: opponuntur $B /$ pertransitum: pertransivit(?) $A$
146 post eorum scr et del $V$ ad puncto D erunt commensurabiles /a: ab ipso $V$ | post puncto d bab $H$ duas figuras | Sed BHV; om $F$ / quandocumque $B H$ quandoque $F V /$ ante A bab $V$ est ${ }^{146-48}$ Sed...puncto dom $A$
147 facient $F H V$ faciunt $B /$ et cetera FHV; om B

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tion they are never commensurable at the same time to a right angle. And the same holds if $A$ and $B$ were in conjunction.
Proposition XI.* If the two angles $A B$ and $C B$ were commensurable at the same time to a right angle, such mobiles would never be, nor bave they ever been, in conjunction at the same time.

This is obvious from the preceding proposition, and thus any such mobiles might form angles which are directly commensurable. And this is clear because if two [mobiles] were conjuncted once and the third mobile was in quadrature to them, or if they should make two right angles, or


Fig. $7 \dagger$
three angles commensurable to each other, and so on, [yet] they would never be in conjunction.

Proposition XII. $\ddagger$ If two of three such mobiles were conjuncted at some time while the third was in quadrature, they would never be in conjunction at other times [in the future], nor even in opposition; and all this applies also to the past.

The first [part] has been proved, and once again I prove all these [assertions] at the same time. Let $A$ and $B$ be in conjunction in $d$, and let $C$ be in quadrature. Then one argues as follows: Whenever $B$ and $C$ form an angle commensurable to a right angle, or are in conjunction or opposition, the distance traversed by $B$ and that traversed by $C$ will be commensurable, and so will their distances from point $d$. But whenever $A$ and $C$ form an angle commensurable [to a right angle, or are in conjunction or opposi-

[^42]puncto $d$ erunt incommensurabiles. Igitur numquam simul et semel opponentur, et cetera. Totum potest faciliter speculari precedentibus

Et eodem modo arguitur si $B$ et $C$ coniungantur et $A$ sit eis in quadratura; et ita si $C$ et $A$ sint coniuncta et cetera.
Ex hiis apparet quod est possibile est quod sint tria mobilia que numquam erunt nec fuerunt opposita nec coniuncta, et si motus eorum fuerunt ab eterno.

Tertiadecima conclusio. Si tria mobilia numquam coniungentur nec sunt opposita, ut ponatur in octava conclusione prime partis et in precedenti, in quocumque instanti necesse est illa stare in dispositione tali quod impossibile est ipsa futura esse aut fuisse in simili, nec proprie nec improprie.
Datis enim dispositionibus similibus necesse est illa medio tempore fuisse coniuncta aut opposita, ut patet ex tertia et quarta conclusionibus huius partis. Et tunc summe argumentum a destructione consequentis.

Item loca cuiuscumque dispositionis $B C$ sunt numerata sicut loca 65 coniunctionum; et loca dispositionis $B A$ sunt numerata sicut loca coniunctionum; et hec loca non communicant, ut patet per octavam prime partis.
Quartadecima conclusio. Si dicta mobilia semel solum coniungantur toto eterno, ut ponitur in nona conclusione prime partis, quacumque dispositione

148 puncto d BFH; tr $V$ / incommensurabiles $A B H$ commensurabiles $F V$ / Igitur: media(?) $H$
149 opponentur: opponuntur $B \mid$ ante et bab H facient / et cetera FHV; om $A B /$ potest $A F H$ patet $B V /$ faciliter speculari $A F H$ faciliter speculanti $B$ speculanti faciliter $V$ / precedentibus FHV precedentem $A$ et patet $B$
150 ante intellectis bab $V$ in / intellectis $F H V$ in aliis $A$ intelligenti precedentia $V$
is I Et $A F H$; om $B V$ /arguitur: arguatur $V /$ si: sint $B /$ coniungantur $V$; om $A F$ sint coniuncta $H$ coniuncta $B /$ sit $B H V$ fit(?) $A$ si $F$
192 sint: sunt $A$ cetera om $A$
153 hiis: hoc $V /$ est possibile $\operatorname{tr} V$
${ }^{1} 54$ erunt: fuerunt $V /$ nec $^{1} B F H$ vel $A V$ | fuerunt $B F H$ faciant(?) $A$ erunt $V$ | opposita: coniuncta $V /$ nec $^{2} A F H$
nunc $B$ aut $V /$ coniuncta: opposita $V / \mathrm{et}$ : etiam $H$
155 fuerunt $A F H$ fuisset $B$ sint $V$
is 6 Tertiadecima conclusio mg bab $F$ ante Si et ante mobilia mg hab $H_{13}$;om $A V$ conclusio ${ }_{13} B /$ tria: duo $H /$ coniungentur: coniungantur $H$ / sunt $A H V$ sint $B F$
157 ponatur: ponitur $F$ / octava $B F H$ quarta(?) $A 8 V /$ et: sic $B$
158 quocumque: quacumque $B /$ instanti: distanti $A$ / illa: ista $A$
159 est: in $A /$ futura om $V$ / aut $F H$; om $A$ vel $B V$ / fuisse om $A$
160 enim: in $A$ / illa: talia $A /$ medio tempore $B F V$; tr $A H$
161 coniuncta: opposita $H$ / opposita: coniuncta $\mathrm{H} /$ tertia $\mathrm{BFH}_{3} \mathrm{AV}$ / quarta $B F V 4 A$ quinta $H$
161-62 conclusionibus om $A$
${ }_{162}$ Et om $B$ / summe: summo $B$

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tion], their distances from point $d$ will be incommensurable. Therefore, they could never be opposed at one and the same time, etc. All this can easily be investigated when one understands what has preceded.
And one can argue in the same way if $B$ and $C$ should be conjuncted and $A$ were in quadrature to them; and the same holds if $C$ and $A$ were in conjunction, etc.
It seems from what has been said that it is possible that there could be three mobiles which never will be, nor have been, opposed or conjuncted -even if their motions were eternal.
Proposition XIII.* If three mobiles will never be in conjunction or opposition, as setforth in the eighth proposition of the first part and the preceding proposition, it is necessary in any instant for them to be in such a disposition that it is impossible for them to be, or to have ever been, in a similar [disposition]-whether properly or improperly.

Now, given similar dispositions, it is necessary for the mobiles to have been in conjunction or opposition in the time interval [between two successive similar dispositions], as is clear by the third and fourth propositions of this part. And now you can carry through the argument by denying the consequent.

Furthermore, the places of any disposition of $B$ and $C$ are equal in number to the places of conjunction; but the places of disposition of $B$ and $A$ are not equal in number to the places of conjunction and these places [of conjunction] do not communicate, as is clear by the eighth proposition of the first part.

Proposition XIV. $\dagger$ If the said mobiles were conjuncted only once through all eternity, as set forth in the ninth proposition of the first part, then any disposition

* See pp. 107-8. $\quad \dagger$ See p. 108.
${ }^{1} 64$ Item: ille et $\operatorname{sic} A /$ loca $^{1}$ om $V /$ dispositionis $A H V$ dispositionibus $B F /$ sicut: ut $A$
165 numerata corr ex innumerata $H$
165-66 et loca dispositionis...coniunctionum $H$; om $A B F V$
166 hec: hoc $H$ / communicant: communicat $A /$ ut patet $H$; om $A B F$ sicut
patet $V /$ per octavam: ex 8 V
168 Quartadecima conclusio mg bab $F$ ante Si et mg bab H 14; om $A V$ conclusio $14 B /$ dicta: duo $F /$ semel solum $A F H$; tr $B$ semel $V$
${ }^{1} 69$ ponitur $B F$; om $V$ ponatur $A H /$ in om $A /$ nona conclusione $B F H ; \operatorname{tr} A$ nono conclusione $V$
${ }^{170}$ accepta in qua fuerunt ante coniunctionem erunt in sibi invicem improprie tamdiu post coniunctionem.

Quoniam coniunctio mediat inter dispositiones taliter similes, ut patet per tertiam et quartam conclusiones, unde per unum diem post erunt sicut fuerunt per unum diem ante, et per duos post sicut per ${ }_{75}$ duos ante, et sic sine fine. Et in horum contemplatione locabitur recte planetis.
Quintadecima conclusio. Quacumque dispositione data talia mobilia numquam erunt, nec fuerunt in dispositione proprie simili.
Patet quia numquam opponentur per octavam huius partis, et sic
${ }^{180}$ non sunt sine angulo nisi semel et oporteret quod essent bis sine angulo ut patet ex secunda parte secunde conclusionis huius partis.
Item loca coniunctionum duorum et aliorum duorum sunt incommunicancia nisi semel, igitur et loca aliarum dispositionum sicut prius est argutum.

Sextadecima conclusio. Posita aliqua incommensurabilitate ut prius, impossibile est arte prescire ad punctum locum aut tempus alicuius oppositionis aut coniunctionis aut cuiuscumque alterius aspectus vel cuiusvis dispositionis, preterite vel future.

Hoc patet quia talia non inveniuntur nisi per comparationem aut ${ }^{190}$ commensurationem unius motus ad alterum, ut habetur ex arte prime et secunde conclusionum prime partis. Si igitur non fuerit commensurabilitas totum erit ignotum. Per idem dico quod si tempus quo sol peragrat circulum suum sit incommensurabile diei ita quod annus so-

170 fuerunt: fuerint $H$ / ante coniunctionem om $B /$ in sibi invicem $F H V$ invicem in simili $A$ in simili igitur $B /$ tamdiu $A F H$; om $B$ tam dum $V$
172 similes $B F H$; om $V$
172-73 taliter...patet per om $A$
173 tertiam: $3 \mathrm{~V} /$ et quartam: secundam $A$ / ante unde bab $A$ patet / unde: tantum $B /$ unum: totam $B$
173-74 post erunt sicut: erunt post sicut $B$
174 fuerunt $B F V$ faciant $A$ fuerant $H$ per unum diem ante: ante per unum diem $V /$ unum $A F H$ totum $B /$ post sicut $F H V$ post ut $A$ ante sicut $B$
175 ante: post $B /$ sic om $F /$ sine fine $\operatorname{tr} A$
175-76 recte planetis $A$ annus contemplantis $B F V$ annis sperarum $H$
177 Quintadecima conclusio $m g$ hab $F$ ante

Quacumque et ante talia mg bab $H$ is; om $A V$ is $B /$ talia: tria $V$
178 erunt om $A /$ in dispositione om $B$ proprie $F H V$ propria $A$ improprie $B$
179 per octavam huius partis $F H V$ per huius octavam $A$ sicut patet per octavam $B$
180 non: numquam $B /$ sunt $H$; om $F V$ fuerit $A$ erunt $B /$ semel : scilicet $V /$ et oporteret: ordinares $A /$ quod essent rep $A$ ut esset $V$
181 secunde conclusionis $\operatorname{tr} A /$ ante partis add $V$ secunde
182 Item: sint $A /$ coniunctionum: coniunctionis $H /$ duorum $^{\mathrm{s}} A F H$ duarum $B V /$ aliorum duorum: aliarum duabus $V$
183 igitur et $\operatorname{tr} V /$ post et hab $B$ cetera

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of the mobiles taken [at any point of time] before conjunction will, at the same point of time after conjunction, be in a reverse, or improper, order with respect to that disposition.
Since a conjunction intervenes between such similar dispositions, as evidenced by the third and fourth propositions, it follows that a similar disposition one day after [conjunction] will be exactly as it was one day before conjunction, and two days after will be exactly as two days before, and so on endlessly. And after some reflection, these things can be applied directly to the planets.
Proposition $X V$.* No matter what disposition may be given, such mobiles will never be, and bave never been, in a properly similar disposition.
This is obvious since, by the eighth proposition of this part, they will never be in opposition and thus are without any angle only once and it would be necessary for them to be without an angle twice, which is clear by the second part of the second proposition of this part.
Furthermore, the places of conjunction of the mobiles taken two at a time are not shared except once, and the same applies to the places of the other dispositions, as was argued before.
Proposition XVI.+ Having assumed incommensurability as before, it is impossible to predict scientifically the exact place or time of any past or future opposition, conjunction, aspect, or disposition.
This is evident because [oppositions, conjunctions, aspects, or dispositions] are not found except by comparison or commensuration of one motion to another, as shown in the first and second propositions of the first part. If there should be no commensurability, all [this] will be unknown. By the same argument, I say that if the time in which the sun traverses its circle were incommensurable to a day, so that the solar year would last

* See p. 108.
+ See p. 108.

185 Sextadecima conclusio $m g$ bab $F$ ante Posita et ante nisi (linea I83) mg bab Hi6;om $A V 16 B /$ incommensurabilitate $H$ commensurabilitate $A B F V$
186 arte prescire $A H V$ artem prescire $B$ prescire $F /$ aut: et $H /$ alicuius om $B /$ oppositionis: coniunctionis $V$
187 aut ${ }^{\text {i }} B V$ seu $F H$ vel $A /$ coniunctionis: oppositionis $V /$ aut $^{2}$ : seu $V /$ cuiuscumque: cuiuslibet $V /$ alterius: alicuius $H /$ vel $F H$ et $A B V /$ cuius-
vis: cuius $V /$ dispositionis: disposi tiones $B$

88 nisi $F H V$; om $A$ nec $B /$ aut: vel $A$
91 conclusionum prime om $A /$ fuerit: erit $A$
i92 totum: totus $V /$ erit: esset $V /$ ignotum: ignotis $V /$ Per idem: propter quod $A$ / ante si add $H$ quod
193 circulum suum $A B F$; tr $H V /$ ante sit add $A$ sic / ita : igitur $V$
laris duret per aliquot dies et per partem diei incommensurabilem suo 195 toti, quantitas anni fuit, est, et erit in perpetuum ignota; et eam scire est omnino impossibile atque verum kalendarium invenire. Et eodem modo de anno lunari et de quocumque planeta.
Septimadecima conclusio. Verisimile est corpora celestia in quolibet instanti taliter se babere quod numquam in preterito sic se babuerunt, nec umquam in eternum.

Quia si antecedens est verisimile consequens erit verisimile. Et verisimile est aliquam vel aliquas quantitates, seu circulorum vel distantiarum pertinentium celi motibus, incommensurabilem vel incommensurabiles esse patet ex secunda suppositione prime partis, quoniam multi sunt huius circuli, latitudines, distantie, eccentricitates, et multi motus et multe diversitates.
Et ex hoc sequitur propositum per septimam et tertiamdecimam huius partis, et per alias antedictas.

Duodevicesima conclusio. Possibile est tres planetas, aut quatuor, aut plures coniungi tempore perpetuo solum semel.

Patet satis per conclusionem nonam prime partis.
Et hoc posito si huiusmodi coniunctio sit naturaliter causa generationis alicuius speciei per putrefactionem vel aliquo alio modo, forte 5 aliqua species poterit de novo produci que forte erit durabilis in eternum. Et ita de corruptione.

194 duret: durat $V /$ aliquot $B F H$ aliquos $A$ aliquod $V$ / incommensurabilem: incommensurabile $B$
195 toti: $\operatorname{diei} A /$ erit: esset $V /$ eam: eas(?) A
196 ante est bab $A$ non / est: erit $H$ / omnino om $A$ /impossibile: possibile $A$ kalendarium: talia $B$ / invenire Et om A
${ }^{197} \underset{H}{\text { quocumque }} B F V$ qualibet $A$ quocum H
198 Septimadecima conclusio mg hab $F$ ante Verisimile et mg bab $H_{17}$; om AV conclusio septimadecima $B$ /Verisimile: verisimillem $H$ / infra in quolibet $m g$ hab $F$ figuram
199 umquam $B F$; om $V$ numquam $A H$
200 futuro $B V$ futurum $A F H$ / sic $B H$; om $V$ similiter $F$ / sic se habebunt om $A$ / fuit: esset $V /$ erit: fuit $V$ / seu
dispositio similis $F H$; om $A B V$
202 si om $A$ / antecedens: alias $V$ /ante consequens add $H V$ et $e t$ add $A$ igitur et / erit $B F$ et $A$ est $H$ esset $V$ | verisimile: verisimilem $H$
202-3 Et verisimile est om $A$
203 aliquam vel aliquas: aliquas seu aliquam $A /$ quantitates $F V$; om $A$ ${ }_{B}^{\text {quantitatum } B H / \text { seu om } A / \text { vel: seu }}$
203-4 distantiarum FHV diametrorum $B$ / distantiarum pertinentium $\operatorname{tr} A$
204 pertinentium: pertinentibus $V /$ celi motibus $\operatorname{tr} V /$ incommensurabilem $B H V$ incommensurabiles $A F$
204-5 vel incommensurabiles om $A$
205 ante patet add $B$ et hoc / secunda corr ex prima $A B F H V /$ prime partis: quarte(?) $B$
206 huius: huiusmodi F / post circuli add

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through a certain number of days and a part of a day incommensurable to a whole day, the length of the year was, is, and will be perpetually unknown; and it is wholly impossible to know it and find the true calendar. And the same thing can be said about the lunar year and any planet.
Proposition XVII.* It is probable that in any instant the celestial bodies are related in such a way that they were never so related in the past, nor will be so related at any time in the future; nor was there, nor will there be, a similar configuration or disposition through all eternity.
Now if the antecedent is probable, the consequent will be probable. But that it is probable that any quantity or quantities, whether circles or distances pertaining to celestial motions, would be incommensurable is clear from the second supposition of the first part, since there are many circles, latitudes, distances, eccentricities and many motions and diversities.
And from this what has been proposed follows, by the seventh and thirteenth propositions of this part and other previously mentioned propositions.

Proposition XVIII. $\dagger$ It is possible that three, or four, or more planets could be in conjunction only once through all eternity.

This is quite clear, by the ninth proposition of the first part.
And now if it be assumed that such a [unique] conjunction could be a natural cause of the generation of some species by means of putrefaction or in some other way, perhaps some species can be newly created which will last through eternity. And the same thing may be considered about corruption.

* See pp. 108-9.
$\dagger$ See pp. 109-10.
$V$ et / distantie: instante $A$ / eccentricitates $H$ eccentricitas $B F$ etiam eccentricitas illa $A$ et centricantes $V$ 207 motus: modi $V$
208 Et om $\mathrm{V} /$ septimamettertiamdecimam $F H_{7}$ et $13 B V$ septimam et $13 A$ 208-9 huius partis $F H V$; $\operatorname{tr} A B$
209 et....antedictas: et alias $A$
2 1o Duodevicesima conclusio $m g$ hab $F$ ante Possibile est et ante aut plures $m g$ bab $H$ 18; om $A V$ i8 $B /$ aut $^{1}$ : vel $F$ / aut plures om $A$
2 II solum semel $A B V$; $\operatorname{tr} F H$
212 Patet satis tr et rep $V$ / conclusionem


## nonam $A B F ; \operatorname{tr} H V /$ prime partis $t r$

 H213 hoc posito $F H V$ supposito cum hoc quod $A$ suppono cum hoc quod $B I$ si $F H V$; om $A B /$ huiusmodi $B V$, om $A$ huius $H /$ sit naturaliter: contingit maior $V$
213-14 generationis alicuius $\operatorname{tr} B$
214 vel: aut $B$ /aliquo alio modo $B F$ alio modo aliquo $A$ aliquo $H$ aliquo alio $V /$ forte om $A$
215 ante species add $F$ alia/de novo om $V /$ que: quia $A /$ erit: esset $V$
216 ita: similiter $A$

Et similiter huiusmodi coniunctio forte poterit esse causa alicuius effectus cui non fuit nec erit similis, sicut diluvium vel alicuius huiusmodi. Et forte videbitur alicui mirabile quo modo huiusmodi coniunc-
${ }_{220}$ tio eveniet necessario in instanti quo ipsa fit, ita quod ab eterno fuit verum ipsam futuram esse necessario pro illo mobilibus ad hoc venientibus suis regularibus motibus et ad hoc se disponentibus ab eterno. Nec oportet querere aliam causam quare plus coniunguntur nunc quam tunc aut cur in isto instanti potius quam in alio. Et hoc scito 225 magis credibile videbitur quod agens liberum sicut deus potuit ab eterno disponere et ordinare aliquid fieri aut produci pro aliquo instanti. Nec, sicut prius, oportet querere quare magis in hoc.

Undevicesima conclusio. Supposito quod totus iste mundus inferior virtute celi penitus regeretur, et celum necessario uniformiter moveretur, et omnia eve-
${ }_{230}$ nient de necessitate, et non esset casus nec fortuna nec libertas voluntatis, et mundus esset eternus et motus, adbuc nullus sciret, nec scire posset, recte iudicare de futuris sed esset omnino impossibile nisi per revelationem.

Non enim fit de futuris iudicium nisi per observationes preteritorum et cum sit verisimile ut nulla futura dispositio sit similis alicui prete${ }_{235}$ rite, ut patet ex conclusione decimaseptima et sequitur propositum.

217 similiter $B F H$ principaliter(?) $A$ sic $V /$ huiusmodi $B V$ forte huius $A$ huius $F H$ / coniunctio: corruptionis $V /$ forte poterit $F V$; $t r H$ poterit $A B \mid$ ante poterit scr et del $V$ videbitur alicui numerabile / esse om $A$
2 I8 cui: cuius $A /$ non: nec $V /$ erit: esset $V /$ diluvium $B F H V($ ? $)$ de diluvio $A /$ vel alicuius: et $V$
218-19 huiusmodi $B F V$; on $A$ huius $H$
219 quo modo $A F V$ qualiter $B$ quando $H /$ huiusmodi $V$ huius $A F H$ illa $B$
220 in om $V /$ quo: quod $V /$ ita quod: igitur $V$ | post quod scr et del $A$ ipsa / fuit $F H V$ fuisset $A B$
221 verum: veram(?) $V /$ futuram esse $A F H ; \operatorname{tr} B V /$ necessario pro: utraque probatio(?) $B$ / ante mobilibus bab $A F$ illo et bab Hillo tunc / mobilibus: actibus(?) $A /$ ad hoc: adhuc $B /$ hoc: $\operatorname{hec}(?) B$
222 se: sit $A$
223 oportet $F H V$ erit $A$ oporteret $B \mid$ aliam $B F V$; om $A$ aliquam $H /$ plus:
pocius $B /$ coniunguntur $A F H$ iunguntur $B$ coniungentur $V$
224 quam: quamplus $B$ / isto FHV illo $A B / \mathrm{in}^{2}$ om $A$ / alio $A B$ illo $H V$ isto $F /$ Et om $B /$ scito: scio $V$
225 quod: qua $V$ /agens: angulus $A$ | liberum $B F H$ primum $V /$ liberum sicut: singulum $V \mid$ ante deus add $A$ si
225-26 ab eterno om $V$
226 et $F H$ aut $A B V$ | aliquid: aut $B$ | fieri aut obs $F /$ pro: in $F$
227 Nec $F H V \operatorname{sic}(?) A B /$ sicut $F H V$ quod non $A B$ /ante oportet bab $B$ in alio et non / oportet om $A$ / ante querere bab $A$ in alio nec erit et bab $V$ magis / querere quare magis $H V$ querere quare $A$ querere magis $B$ in hac $F /$ in hoc $B H$; obs $F$ nunc $A$ in isto $V$ / post hoc bab $A$ et non(?) alio et bab $B$ quam in alio et bab $H$ quam in alio ymo minus(?) et bab $V$ instanti quam in alio
228 Undevicesima conclusio mg bab Fante Supposito et post mundus mg hab $H$

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And likewise, such a conjunction might be the cause of some effect which had no similar predecessor nor will have a similar successor, just as [for example] the flood or some other such event. And perhaps it will seem amazing to some how such a conjunction could happen necessarily in the [very] instant in which it does occur, so that it was true through all eternity that this would necessarily occur with mobiles coming to this [configuration] by means of their regular motions and being predisposed for this through all eternity. And one ought not to seek another cause [to explain] why more are conjuncted now than then, or why [it happens] in this instant rather than in another. But knowing this it will seem more credible that a free agent like God could arrange and ordain through eternity that something be done or produced for a certain [particular] instant. Now, just as before, one ought not to seek for an explanation of this.

Proposition XIX.* Assume that this whole lower world is completely ruled by the powver of the beavens, and that the heavens are uniformly moved, and that all things happen from necessity, and there is neither chance nor freedom of the will, and the world and motion are eternal; and yet [conceding all this] no one knows, or could know, how to judge rightly about future events, for it is utterly impossible except by revelation.

Indeed, a judgment about future events could not be made except by observations of past events, but since it would be probable that no future disposition is similar to any past disposition, as made clear in the seventeenth proposition, what has been proposed follows.

* See p. rio.

[^43]232 esset: esse $A /$ revelationem $B F V$ revolutionem $A H$
233 Non obs $F$ / enim fit: igitur $A /$ de futuris iudicium $A F H$ de futuro iudicium $B$ iudicium de futuris $V /$ observationes $A F$ observantias $H V$ observatores $B$ / ante preteritorum $b a b$ $F$ et observantias et bab $B$ vel observationes
234 ante cum bab $V$ si / sit verisimile obs $F /$ ut $B F H$ quod $A V /$ futura dispositio $\operatorname{tr} A$ /similis: commensurabilis $V$ 235 conclusione om $H /$ et $B$; om $A F H V$ 235-36 propositum Item obs $F$

Item posita in motibus aliqua incommensurabilitate, quod est verisimile, omnis futura dispositio, donec de presenti veniat, ignoratur per conclusionem decimamsextam.
Item si omnia essent commensurabilia et in proportione rationali partis neque bene scibilis quia forte in aliqua illarum proportionum oportet venire ad fractiones quasi homini innumerabiles. Et posito quod ad hoc homo veniret adhuc nesciret si haberet intentum nec quando esset inventum propter defectum sensuum.
Item nescitur si motus sunt commensurabiles aut non et qui ignorat antecedens necesse est ipsum ignorare consequens.

Ex hiis sequitur vanari astrologiam; omnia latere preter eum qui numerat multitudinem stellarum, et qui perpetua mundum ratione gubernat. Non igitur presumat ultra aliquis de causa incertis ita faciliter iudicare.

Vicesima conclusio. Nullus propter ista predicta debet scientiam astrologie despicere sive dimittere sive ab ea desperare.

Dicit enim Aristoteles secundo celi quod melius est scire modicum de rebus nobilibus quam multum de re ignobili sive vili. Corpora vero 255 celestia sunt omnium sensibilium corporum nobilissima. Optimum enim est de hoc scire ut homo possit non frivolis garrulationibus sicut ignorantes sed firmis demonstrationibus errores scientifice reprobare;

236 in motibus om $A /$ motibus $H V$ mobili $B$ mobilibus $F$ /aliqua incommensurabilitate quod om $B$ / incommensurabilitate $A H V$ commensurabilitate $F$ / quod $F H V$; om $A$ / est $F H V$ potest $B$
236-37 est...dispositio: omnis dispositio futura erit verisimile $A$
237 dispositio BFH; om V/veniat: eveniat $B$
238 conclusionem decimamsextam: decimasexta conclusionem $A$
239 et om $A$
240 eorum BHV eadem $A$ earum $F$ / est: esset $B /$ tertiam: 3 V
241 neque $B F H$ unde $A$ nec $V /$ bene: sicud(?) $A$ / aliqua: alio $A$ / illarum $B F H$ aliarum $A$ /illarum proportionum $\operatorname{tr} V$
242 oportet: oporteret $H /$ venire: devenire $B$ / quasi: vel $B$

243 quod ad hoc homo $A$ quod adhuc homo $B H$ adhuc quod homo $F$ quod homo ad hoc $V /$ veniret: inveniret $B$ / nesciret: nescium $V$ / si haberet: se habere $B /$ nec: videlicet $A$
245 ante si bab $B$ ut / sunt $F H V \operatorname{sint} B$ est $A /$ aut: vel $A /$ et: sed $B /$ qui $B H V$; om $A$ causa(?) $F$ / ignorat $B H V$ ignoras $A$ ignoratur $F$
246 antecedens...est: necessario antecedens $A$ / ipsum ignorare $B V$ igitur $A$ ignorare et $F /$ ipsum ignorare consequens: consequens ignorare $H$
247 post Ex hiis mg bab $H$ corellarium / vanari: vanam $H /$ omnia $A($ ? $) F$ (?) omnis $B V$ omnes $H$
248 multitudinem: multiplicationem $V /$ qui: in $A$ / perpetua: perpetue $B$
249 gubernat: gubernatum $A /$ Non $B F$ $H$; om $A$ nisi $V /$ igitur: $\operatorname{enim} B$ / presumat $B H$ presumit $A$ presumere

Part Two
Furthermore, assuming any incommensurability in motions, which is probable, every future disposition, as long as it came from the present, would be unknown, by the sixteenth proposition.
And again, even if all [motions] were commensurable and in a rational ratio, a ratio of them might not be known, by the third supposition of the first part, and might not even be knowable, because in some of these ratios one could arrive at fractions which are, as it were, innumerable for men. And assuming that a man could arrive at this [ratio], yet, because of defects of the senses, he might not know if he had gone far enough nor even when it had been found.

Furthermore, one does not know if the motions are commensurable or not; and who is ignorant of the antecedent is necessarily ignorant of the consequent.

It follows from all these things that astrology is vain; that all things lie hidden behind Him who numbers the multitude of stars, and who governs the world by reason everlasting. No one, therefore, should presume to judge so facilely about the cause of uncertain things.

Proposition XX.* Because of what has been said, no one ought to despise the science of astronomy or abandon or despair about it.

For Aristotle, in the second book of On the Heavens, says it is better to know a little about noble things than much about ignoble or vile things. Now celestial bodies are the most noble of all sensible bodies. Indeed, it is best to know about this so that a man might repudiate errors scientifically

* See p. III.
$F$ presumendum $V$ | ante ultra bab $B$ quis / ultra $B V$ quis $A$ velit $F$ aliter $H /$ aliquis $F$; om $A B H V /$ de om $A$
249-50 ita faciliter $B$ partibus(?) $A$ facile $F$ facit $H$ tam facile $V$
250 post iudicare bab $V$ et bene nota hoc
251 Vicesima conclusio mg bab $F$ ante Nullus; om $A H V$ conclusio $20 B /$ ista $A F V$; om $H$ illa $B /$ predicta om $V /$ astrologie: astrologicam(?) $B$
252 despicere sive dimittere om $A /$ sive $^{1}$ $F H$ nec $B$ aut $V /$ sive $^{2} B$ nec $A F H$ seu $V / \mathrm{ab}: \operatorname{de} B$
253 secundo celi om $A /$ quod om $B$
254 de rebus nobilibus $B V$ nobilibus $A$ de nobilibus rebus $F$ duobus nobiibus $H$ / multum: multis $V$ / ante de $^{2}$
bab $A$ secundo celi / re $A H$; om $B F V$ $\mid$ ignobili sive om $B /$ sive $A F H$; om $B$ et $V /$ vili: vilibus $B /$ ante Corpora bab $B$ sed / vero $A F H$; om $B V$
255 sensibilium corporum $A B ; \operatorname{tr} V$ corpora $F$ corpora sensibilium $H$
296 enim: tamen $V /$ est: cum $F /$ ante de bab $H$ tantum / homo: hoc $A /$ possit non FHV; $\operatorname{tr} A$ non solum $B /$ ante frivolis bab $A$ enim(?) / frivolis: sic conceptus $B /$ garrulationibus: garrulationis $A /$ sicut $A B F$ super $V$
256-57 sicut ignorantes om $H$
257 ignorantes $A F$ ignorantis $B$ ignotas $V /$ firmis: firmum $A /$ scientifice $F H$ sciente $A V$ scientur $B$
et reliqua vera seu probabilia ad sobrietatem sapere ut per visibilia dei perfecta opera invisibilem magnificet creatorem. Scriptum est enim opibus operibus tuis et in factis manuum tuarum meditabar; et iterum opera manuum tuarum sunt celi; et alibi celi enarrant gloriam dei, et cetera.
Sufficit etiam bono astologo de motibus et aspectibus prope punctum iudicare et quod sensus non percipiat oppositum iudicare. Et qui ultra vult querere aut opinatur se scire in vanum laborat et affligit spiritum et stulte presumit de prenosticationibus aut effectuum aut eventuum ex constellationibus tantorum. Nisi valde generaliter et dubitative, nullus debet loqui sed potius compescere linguam a talibus que in manu dei sunt. Et ipse solus novit cuius oculis nuda sunt omnia et aperta.

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by solid demonstrations and not by empty babbling as do those who are ignorant; and to understand the remaining truths or probabilities is to acquire prudence so that through visible things the perfect works of God might magnify the invisible creator. For it has been written: "I meditate on all thy works; I meditate upon the works of thy hands"; and again: "The heavens are the works of thy hands"; and elsewhere [it is written]: "The heavens show forth the glory of God," and so on.

Thus it is sufficient for a good astronomer to judge motions and aspects near a point, and that his senses do not observe and judge the opposite. But one who wishes to seek more, or believes he knows, labors in vain and impairs the spirit and is foolishly presumptuous about predictions of effects or events from such configurations. Except in a very general and doubtful way, no one ought to speak but rather to restrain the tongue about things which are in the hands of God, for only He knows to whose eyes "'all things are naked and open."

## CRITICAL NOTES TO TEXT BY LINE NUMBERS

$$
\mathrm{API}_{\mathrm{I} . \mathrm{I}-33}
$$

The concept of a Great Year, mentioned in IV.606-9 as an error in philosophy and faith (see p. 376, above), is referred to again in APr.7-1o in connection with the eighth sphere or sphere of the stars. ${ }^{1}$ It was generally held that the stars made one revolution arround the heavens from west to east in $36,000^{2}$ years, and it was assumed by many that upon completion
${ }^{1}$ There were generally thought to be a total of nine spheres in the heavens. In a brief paragraph, Sacrobosco explains that the heavens are constituted of a "fifth essence" (quinta essentia) that consists of nine spheres, "namely, of the moon, Mercury, Venus, the sun, Mars, Jupiter, Saturn, the fixed stars, and the last heaven. Each of these spheres incloses its inferior spheri-cally."-Spbere of Sacrobosco, ed. and trans. Thorndike, p. 119; the Latin text appears on p. 79. See also p. 118.
${ }_{2}$ The figure of 36,000 years derives from Ptolemy, who reports that Hippar-
chus found a precession of the equinoxes occurred at a rate of at least $\mathrm{I}^{\circ}$ in 100 years. See Heath, Aristarchus of Samos, pp. 17273. In a treatise on the sphere written probably before 1279, John Peckham says," Quod certis experimentis perpendunt celum sydereum varie moveri non tantum super polos mundi motu diurno ab oriente in occidentem sed etiam secundum Ptholomeum moveri alio motu tardo contrario et illo motu secundum ipsum Ptholomeum ut infra dicemus et secundum philosophum libro de proprietatibus elementorum movetur in centum annis gradu uno."-Sphere of
of this motion all the heavenly bodies would be arranged exactly as they were at the beginning of the $36,000-$ year period and that they would then proceed through the same sequence of relationships as in the preceding Great Year. ${ }^{3}$ Oresme notes that some people, agreeing with Plato, held that the Great Year would occur in 15,000 years (API.9-10). In Timaeus ${ }_{39}$ D, Plato does mention a "perfect" year (another name for the Great Year), but nowhere does he give an estimate of its length. ${ }^{4}$
Although Oresme rejects the Great Year as an error in IV. 606-9 and as foolishness in AP ${ }_{\mathrm{I} .1 \mathrm{I}-\mathrm{I} 3}$, he does not formulate specific arguments against it. It seems obvious, however, that had he offered such a refutation, it would have depended on IV. 573 -76 and Suppositions II and III of the Ad pauca respicientes (APr. $36-50$ ), by means of which he could assert the probable incommensurability of celestial motions, deducing therefrom that the celestial bodies would never resume identical positions. In this, Oresme would have agreed with an anonymous fourteenth-century author who repudiated the Great Year (magnus amnus) on grounds that the motions of sun and moon are incommensurable. ${ }^{5}$
A much more elastic conception of the Great Year is found in Oresme's De commensurabilitate. There, in Part I, on the assumption of the commensurability of the celestial motions, he says there would be a Great Year.

[^44]But a Great Year can be applied to one mobile with several simultaneous motions, or to two or more celestial bodies. He reports that some believe that the sun and eighth sphere will enter into a Great Year in 36,000 years, but that a Great Year of all the planets and the eighth sphere will take much longer than 36,000 years ("Unde quodlibet mobile pluribus motibus per se sumptum habet certam periodum que peracta renovatur iterum et sic infinities, et que potest vocari annus magnus istius mobilis. Consimiliter, quelibet duo mobilia celestia simul sumpta complent cursum suum certa periodo temporis, que transacta reincipiunt ut prius, et sic de tribus, sive quotlibet. Et potest dici annus magnus ipsorum, sicut dicunt quidam de sole et octava spera quod annus magnus istorum duorum est 36,000 anni solares. Sed annus magnus omnium planetarum et octave spere esset valde multo maior. Et breviter si omnes motus celi sint commensurabiles invicem, necesse est quod omnium simul una maxima periodus qua, finita, renovatur non eadem sed similis vicibus infinitis, si mundus esset eternus." -Vat. lat. 4082, fol. 102v, c.I). See Grant, "Oresme: Comm.," p. 440, n. 44.

For a discussion of the terms "possible," "doubtful," and "probable," see above, pp. 85-88.

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\text { API. } 33-54
$$

Pierre Duhem, who quotes Supposition II (he calls it Supposition I) in Le Système du monde, Vol. 8, 45 o, uses "Contrapositis" for "Propositis" in API. 36. "Contrapositis" confounds the sense of the supposition. See below, pp. 438-41.

The definition of conjunction (API. $52-59$ ) is twofold. In the first part, a conjunction occurs when a radius extending from the center of the world -i.e., the center of the earth-proceeds through the centers of whatever number of bodies are involved. In this event, all the bodies have the same celestial latitudes and longitudes. In the second part, bodies are said to be in conjunction when a great circle passing through the poles cuts their centers simultaneously. All the bodies then have the same celestial longitude but need not have the same latitude, thereby making the first part of the definition a special case of the second part.

APı. $55-85$
In the first example of Prop. I (APı. 77-80),$\frac{\text { circle } A}{\operatorname{circle} B}=2 /$, the curvilinear
velocities of $A$ and $B$ are equal, and they are assumed to start from conjunction. If $B$ traverses its circle in one day, $A$ will traverse only half of its circle. At the end of two days, $B$ will have moved round its circle twice while $A$ will have completed one circulation, producing a conjunction.
The second example is unclear. Are we to assume that the conditions outlined in APr. $72-73$ still obtain? If so, the circles are unequal but the curvilinear velocities are equal. The discussion on p. 90 is based on this interpretation. When applied to the data of the second example (APı. 8I-83), where $A$ moves $\mathrm{I} / 4$ of its circle daily while $B$ traverses $\mathrm{I} / 3$ of its circle, the relationship of the circles-on the assumption of equal curvilinear veloc-ities-must be circle $A /$ circle $B=4 / 3$. But at the beginning of his second example, Oresme says: "Let the circles be as before" (APr.8i). Is this a reference to the first example where circle $A /$ circle $B=2 / 1$ (API. 77)? If it is, the curvilinear velocities of $A$ and $B$ in the second example cannot be equal, for $B$ will traverse its circle in three days and $A$ in four. This is impossible if the circumferences of the circles are related as 2 to 1 and the curvilinear velocities are equal, since these conditions would merely duplicate the first example (APr.77-80).

$$
\text { API. } 86-\mathrm{I} 49
$$

The four ways in which two mobiles can traverse incommensurable angles in equal times are described in APr.127-32.
I. (API.127-29) Even when the curvilinear velocities (see above, p. 89, n. 8) are equal, or unequal but commensurable, incommensurable angular velocities will result if the circumferences of circles $A$ and $B$ are incommensurable; that is, if circumference $\frac{A}{\text { circumference } B} \neq \frac{p}{q}$, where $p / q$ is rational.
2. (APrif29-30) When the circumferences of the circles are equal or commensurable, incommensurable angular velocities will be produced if mobiles $A$ and $B$ move with incommensurable curvilinear velocities-i.e., in equal times $A$ and $B$ traverse incommensurable distances.
3. (API.130-32) The third way is rather unclear. Does Oresme wish to say that in equal times the angular distance traversed by $A$ is incommensurable to circle $A$ (i.e., to four right angles) and that the angular distance traversed by $B$ is also incommensurable to circle $B$ ? If so, then because the circles are equal, it follows that the angular distances traversed by $A$ and $B$ are incommensurable and, consequently, the velocities are incommensurable.

## Critical Notes to Pages 390-396

4. (API.132) Finally, when the circles are incommensurable and also the curvilinear velocities, it is obvious that the angular velocities will also be incommensurable.
In the De commensurabilitate, written after the $A d$ pauca respicientes, Oresme offers only two conditions, or definitions, for incommensurable velocities. The first conditionarises when mobiles $A$ and $B$ completean equal or unequal number of circulations on their respective circles, butachieve this in incommensurable times. ${ }^{6}$ This criterion, based on traversing distances in incommensurable times, is not even mentioned in the Ad pauca, where in all four cases the times are assumed equal. In the second condition, velocities are incommensurable when in equal times incommensurable central angles ate described. This description, which fails to specify the conditions producing the incommensurable central angles, is so general that it embraces all four conditions distinguished in the $A d$ pauca.
Prop. IV of the Adpauca would certainly represent the proposition enunciated in IV. 8 $^{88-90}$ of the De proportionibus, provided one assumes that in the latter proposition the two planets are moving with incommensurable angular velocities resulting from one of the four conditions outlined above.

## APr.146-92

The "dragon" and the nodes in the "head and tail of the dragon" (in nodis capitis et cauld draconis [APr.1 so-s I]) are described by Sacrobosco in Ch. IV of his Tractatus de spera.
Every planet except the sun has three circles, namely, equant, deferent, and epicycle. The equant of the moon is a circle concentric with the earth and in the plane of the ecliptic. Its deferent is an eccentric circle not in the plane of the eclipticnay, one half of it slants toward the north and the other toward the south-and the deferent intersects the equant in two places, and the figure of that intersection is called the "dragon" because it is wide in the middle and narrow towards the ends. That intersection, then, through which the moon is moved from south to

[^45]north is called the "head of the dragon," while the other intersection through which it is moved from north to south is called the "tail of the dragon."'
The dragon is formed from the intersection of the plane of the moon's orbit (the eccentric deferent circle) and the ecliptic (called the "equant," but surely identical with the ecliptic). The "head of the dragon" is the ascending node, or the point of intersection through which the moon passes as it moves from south to north; the "tail of the dragon" is the descending node, or the point of intersection through which the moon passes as it moves south of the plane of the ecliptic.
Oresme says that if two mobiles-one on each of two intersecting circles -were moving with incommensurable angular velocities and conjuncted once in either of the nodes, they would never again conjunct there. This also holds true for the opposition of two mobiles (APr.IS3). Should it happen that one of the mobiles was in one of the nodal points and the other mobile in opposition to it elsewhere on its circle, they would never again enter into opposition in those same points if their angular velocities are incommensurable.
All this is now applied to perfect, or total, eclipses of the moon and sun (APr.153-5s), since the moon is totally eclipsed (maxima eclipsis) when it is in opposition to the sun and in one of the nodal points. Oresme infers from Prop. V that if the sun and moon were moving with incommensurable angular velocities, a total lunar eclipse would occur only once through all eternity. The same argument is valid for a total eclipse of the sun when it is in conjunction with the moon and the latter is in one of the nodal points.
The example of total lunar eclipse in Prop. V of the Ad pauca respicientes, which is also mentioned in IV. $883-87$ of the De proportionibus, seems based on stationary nodal points. But the nodes have a retrograde motion along the ecliptic, completing one revolution in approximately $181 / 2$ years. Taking into account this regression of the nodes-and Oresme could have known this from Ptolemaic astronomy-it is evident, even on the supposition of incommensurable angular velocities, that a total lunar eclipse could occur repeatedly in an infinite number of different points.
Similar problems involving intersecting circles and a perfect lunar eclipse are taken up in Oresme's Quaestiones super geometriam Euclidis. In Question 7, he discusses "whether the diagonal of a square is commensurable to its

[^46]tially what appears in APr.I $50-56$, which was interpreted above. On the basis of incommensurable angular velocities of sun and moon, he concludes that a total eclipse of the moon could happen only once in all eternity. But his ingenuity squeezes one further consequence from this unique situation. If only one perfect or total eclipse of the moon were possible in an eternally existing world, then prior to that unique total eclipse there must have been parts of the moon's visible surface which were never before eclipsed or darkened. And, of course, after the total eclipse these parts will never again be darkened by eclipse.
The final lines of Prop. V (APr.157-59) are simply a variation on the previous theme. Instead of two intersecting circles related as the diagonal of a square to its side, there are now two squares (see above, p. 399, Fig. 4), where the diagonal of the smaller square is the side of the greater square. Should two mobiles, $A$ and $B$, commence motion from the same angle moving with equal velocities on their respective squares, they will never meet again in either of the two angles in which a meeting would be possible.
APi.I93-2II

Prop. IX, the last in Part I, utilizes the same data as the preceding proposition, i.e., $\frac{\text { circle } B}{\text { circle } C}=\frac{2}{1}$ and $\frac{\text { circle } A}{\text { circle } B}=\frac{\sqrt{2}}{\mathrm{I}}$. Assuming, contrary to the eighth proposition, that mobiles $A, B$, and $C$ are now in conjunction in point $d$, it is obvious that they will never conjunct there again. Mobiles $B$ and $C$ will, of course, conjunct only in $d$, but since $B$ and $A$ have already conjuncted in $d$, it is clear that $A$ will never again be in $d$ when $B$ is there with $C$, because $A$ and $B$ have incommensurable velocities.
Now if it should happen that $A, B$, and $C$ conjuncted more than once in $d$ (APr.204-6), one could infer that their velocities are mutually commensurable, as shown in the sixth proposition (APr.160-65).
Thus Oresme has covered the cases ( $\mathrm{API}_{\mathrm{I} .207-8 \text { ) where three mobiles }}$ moving incommensurably will never conjunct (Prop. VIII), or will conjunct only once (Prop. IX). No more than one conjunction is possible in the same point. ${ }^{12}$
${ }^{12}$ The question as to whether the three mobiles might conjunct in some point other than $d$ is not raised by Oresme. But in the De commensurabilitate, he devotes Part II, Prop. VI, to demonstrate that it is possible for three mobiles with incommensu-
rable velocities to conjunct in other points And in Part II, Prop. IX, he asserts that three such mobiles might conjunct in an infinite number of different points-but only once in each point. See Grant, "Oresme: Comm.," pp. 447, 45 I.

## Critical Notes to Pages 402-410

In the same manner (APr.209-10), if one mobile were in opposition to the other two, it could never again occur in the same points if the velocities were incommensurable.

$$
\mathrm{AP}_{2.1-75}
$$

In AP2.13-18, it must be kept in mind that for Oresme an angular disposition must change after the mobiles pass through conjunction or opposition. In $\mathrm{AP}_{2.13-15}$, we are given a pair of contrasting relationships between a given set of mobiles. Let us assume that immediately after some arbitrarily chosen angular disposition-which we shall call the first angular disposi-tion-the mobiles go into conjunction. Thus-
I. Immediately after the first angular disposition $\longrightarrow$ conjunction.

After conjunction, the mobiles will assume a second angular disposition since a conjunction has intervened. Now before a third angular disposition is produced, an opposition must intervene (by AP2.32-34, 46-49, $51-57$ ). Therefore, immediately after the second angular disposition-and before the third-the mobiles will be in opposition-i.e.,
2. Immediately after the second angular disposition opposition.

We can now understand Oresme's remark that the mobiles are related differently after the first angular disposition than after the second.
Similarly, the mobiles have different relationships before the first and second angular dispositions. In a above, before the mobiles went into the first angular disposition they must have been in opposition. Let us represent this as
3. Immediately before the first angular disposition $\longrightarrow \longrightarrow$ opposition.

By the same reasoning, before entering the second angular disposition, the mobiles will have been in conjunction so that
4. Immediately before the second angular disposition $\longrightarrow$ conjunction.

From these four relationships, we can clarify AP2.15-17. Immediately before the second disposition, the mobiles are related in the same way as immediately after the first angular disposition, since in both cases the mobiles will be in conjunction as represented in I and 4. And when the mobiles are related as in 2 and 3, they will be in opposition. In the case of opposition, however, if more than three mobiles are involved, the position of the mobiles can vary as described in AP2.23-27 and 80-83.

Finally (AP2.17-18), the mobiles will be related differently if the com-
parison is made when they are arranged immediately after the first and second angular dispositions. In this event, the mobiles terminate first in conjunction, and then in opposition, since the comparison involves a and 3. Although Oresme does not mention the final comparison, different results would be obtained if 3 and 4 were compared, since we get first an opposition and then a conjunction.

$$
\mathrm{AP}_{2.7} 6-227
$$

In AP2.84-87, Oresme says that if one mobile, say $A$, conjuncts with $B$ in certain points and with $C$ in other points, then the places of conjunction of $A, B$ and $A, C$ will differ-i.e., "will not communicate"-and the three will never conjunct despite their mutually commensurable motions. They will, however, be in opposition an infinite number of times.

Virtually the same idea is found in AP2.164-67. Here mobiles $A, B$, and $C$ never simultaneously conjunct or oppose, but conjunctions and oppositions do occur when the mobiles are taken two at a time. But the places of conjunction will never be the same for any two of the mobiles, and, consequently, the places of the angular dispositions will also differ. A similar statement is made in AP2.182-84. Thus, where three mobiles are concerned Oresme accounts for their relationships by taking them two at a time.

## AP2.228-70

Parts of the nineteenth ( $\mathrm{AP}_{2.228-35 ; 247-50) ~ a n d ~ a l l ~ o f ~ t h e ~ t w e n t i e t h ~ p r o p-~}^{\text {2 }}$ ositions of Part 2 were translated by Pierre Duhem in Le Système du monde, Vol. 8, 450-5 r. They are the only propositions from the Ad pauca respicientes with which Duhem concerned himself. ${ }^{13}$ In an accompanying interpretation, Duhem not only fails to understand the propositions in question, but also the objectives of the entire treatise, as well as the purpose of Oresme's preoccupation with the problem of the incommensurability of the celestial motions.

In order to argue against the strongest possible position of the astrologers, Oresme, in the final two propositions, asserts that even if the universe were determined, and the celestial motions uniform and eternal, as-
${ }^{13}$ Duhem was unaware of the probable connection between the Ad paucarespicientes, or the Tractatus brevis et utilis de proportionalitate motuum as it is called in the single
manuscript used by Duhem (Paris, BN lat. 7378 A , fols. 14V-17v; see above, pp. 37980), and Chs. I-IV of the De proportionibus proportionum.

Critical Notes to Pages 410-428
trological prediction would be futile because of the improbability that any future celestial disposition would be similar to any past celestial configuration (AP2.228-35).
Duhem insists that any astrologer would have had a ready reply. Oresme, says Duhem, ${ }^{14}$ assumed that the times of any two celestial revolutions are probably incommensurable. But the determination of times, Duhem tells us, depends on observations that Oresme realized could only be approximate (Duhem may have had in mind AP2.263-64). Because of this, Duhem implies that Oresme maintained that any two numerical evaluations of some observable quantity, or quantities, should not be considered as separate and distinct when their difference falls below a certain value. But how was it possible, Duhem queries, for Oresme to determine whether the times of any two celestial revolutions were commensurable or incommensurable? In reply to his own query, Duhem notes first that if we are given an irrational number, it is always possible to find an infinite number of rational numbers that approach and differ from it by as little as you please. Oresme, Duhem seems to argue, would capitalize on this and choose to represent the time of any celestial motion by an irrational, rather than rational, number, on the grounds that the difference between some initially assigned rational value and the successively approaching irrational values could be made sufficiently small as to render indistinct the difference between the rational and irrational values. But the astrologers, according to

14 'Les durées des circulations célestes sont-elles incommensurables entre elles? Oresme déclare que c'est vraisemblable; il eût été bien en peine d'affirmer que c'est vrai. Ces durées ne sont déterminées que par l'observation; or, si précise qu'on la suppose, toute observation n'est qu'approchée; à ses yeux, deux évaluations numériques ne passent plus pour distinctes si leur différence tombe au-dessous d'une certaine grandeur; comment, dès lors, pourrait-1 dire si deux durees de revolution sont, entre elles, commensurables ou incommensurables? Un nombre incommensurable étant donné, ne peut-on toujours trouver une infinité de nombres commensurables qui en diffèrent aussi peu qu'on veut? Toujours, donc, Oresme pourra prétendre que es durées de deux révolutions celestes n'admettent point de commune mesure; mais toujours, aussi, l'astrologue pourra
lui riposter qu'elles en ont une.
"Que les durées des révolutions célestes soient commensurables ou incommensurables, qu'importe, d'ailleurs, à l'astrologue? Si elles sont incommensurables entre elles, jamais, c'est entendu, les astres ne reprendront exactement la configuration qu'ils ont prise une première fois; mais au bout d'un temps suffisant, ils dessineront une constellation qui différera aussi peu qu'on voudra de la constellation autrefois for mée; sans être, dans la seconde circonstance, rigoureusement identiques à ce qu'ils étaient dans la première, les effets que ces astres produisent ici-bas se ressemble ront d'aussi près qu'on le désirera, de si près qu'aucun observateur ne les pourra distinguer; n'est-ce pas, pour l'astrologue, tout comme s'ils se reproduisaient exacte-ment?"-Duhem, Système du monde, Vol. 8, $45^{2}$.

Duhem, could be just as arbitrary, and, by the same reasoning, choose to represent that very same time by a rational number. Thus, Duhem reduces the issue to one of preference and prejudice. The difference between the irrational and rational values could be made so minimal that, with equal reason, one could choose to label the time rational or irrational.
This fanciful argument finds no support in the texts of Oresme-and Duhem supplies none ${ }^{5}$-for the simple reason that Oresme's arguments have nothing whatever to do with observation and approximation. His lack of confidence in evidence of the senses is made explicit in APr.45-50. Indeed, Oresme's lack of concern with observation is apparent from the fact that his propositions depend on punctual exactness-a punctual exactness that he knew was unattainable and that was of no concern to astronomers, as he states in AP2.263-64. This is made perfectly clear in the De commensurabilitate, where he says, "[my intention is] to consider exact and punctual aspects of mobiles that are moved circularly. I do not, however, propose to deal with aspects near a point, which is usually the intention of astronomers who care only that there be no sensible discrepancy-even though a minute, undetectable, error would produce a perceptible discrepancy when multiplied through [a long period of] time." ${ }^{16}$

Thus Oresme assumes exactness in his propositions and then demonstrates, under various conditions and assumptions, the consequences of exact commensurable or incommensurable motions for two or three mobiles. By means of Supposition II (API.36-38), he could convert any proposition where the mobiles move incommensurably into one representing a possible relationship between celestial motions-provided that the celestial bodies are taken to move under the same arbitrary conditions as the mobiles in the abstract propositions.

But why are the celestial motions probably incommensurable? For Du-

[^47]Critical Notes to Pages 424-428
hem, who either did not know, or was unfamiliar with the De proportionibus, this was nothing more than an arbitrary assumption made in order to combat astrology. But the basis for the assumption is a mathematical demonstration in Ch.III, Prop. X, where it is shown that there are more irrational than rational ratios of ratios. Then in Ch. IV, it is shown that ratios of quantities such as time, distance, and velocity vary as ratios of ratios. The ultimate purpose of all this is to show convincingly that even if we cannot determine whether the celestial motions are commensurable or incommensurable, it is mathematically probable that they are incommensurable (see pp. 40-42). It is abundantly clear that whatever else may be said about Oresme's arguments, they are independent of observation and unconnected with approximations. Duhem's interpretation and criticism collapse.
In the second paragraph, Duhem, quite rightly, holds that astrologers would be indifferent to the commensurability or incommensurability of the celestial motions. They could claim that successive configurations resemble each other sufficiently to produce approximately the same effects. But this also is irrelevant to the spirit of Oresme's approach.
Aristotle expresses no sentiment in Book II of De caelo that could be construed as the source of Oresme's reference in AP2.253-54. In his treatise De partibus animalium, Aristotle does say: "The scanty conceptions to which we can attain of celestial things give us, from their excellence, more pleasure than all our knowledge of the world in which we live; just as a half glimpse of persons that we love is more delightful than a leisurely view of other things whatever their number and dimensions" (644b.32-645a.I). But immediately after he praises the study of terrestrial things, emphasizing that in his study of animals he will not omit "any member of the kingdom, however ignoble.... Every realm of nature is marvellous...so we should venture on the study of every kind of animal without distaste; for each and all will reveal to us something natural and something beautiful" ( 645 a. 7-23). ${ }^{17}$

The three quotations from the Vulgate in $\mathrm{AP}_{2.260-61}$ are found in Psalms 142:5, 101:26, and 18:2 respectively. Translations are from the Douay Version of the Old Testament, the edition cited in note 28, p. 375, above. The concluding words of the treatise ( $\mathrm{AP}_{2.269-70 \text { ) are from He- }}$ brews 4:13. Oresme's version differs slightly from the Latin Vulgate, which reads: "Omnia autem nuda et aperta sunt oculis eius...." This variation made it advisable to alter somewhat the translation from the revised Challo-ner-Rheims version.
${ }^{17}$ The translation is that of William Ogle in Works of Aristotle, ed. Ross, Vol. s.

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## A Selective Index of Mathematical Terms and Expressions

In this index, I have sought to include all terms and expressions of genuine interest and significance for the history of medieval mathematics. For obvious reasons, it is unfeasible, and certainly unnecessary, to include every occurrence of every term. But for each term or expression, an effort has been made to incorporate the many different usages and shades of meaning found in the two texts edited in this volume. Terms that occur in the text frequently, or for the most part, in a particular case or verb form are given in the index in that form. But if various forms are used in the text, in the index the nominative singular will represent all forms of a noun and the infinitive all forms of a verb, and the masculine singular will stand for all adjectival forms. Page and line numbers preceded by AP refer to the $A d$ pauca respicientes; otherwise, the reference is to the De proportionibus proportionum. Page numbers are printed in roman type and line numbers in italic.

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[^0]:    ${ }^{28}$ Ibid., pp. 78, 79
    ${ }^{30}$ Ibid., pp. 112, 113.
    ${ }_{29}$ Ibid.

[^1]:    ${ }^{31}$ See below, pp. 72-81, for a full discussion of the connections between Chs. I-IV and V-VI. The dating problem is also discussed.
    ${ }_{32}$ This is Bradwardine's third erroneous

[^2]:    uous humidity, so does a continuous magnitude correspond to a continuous number.' There are, he states, no 'absurd, irrational, irregular, inexplicable or surd numbers.' What he means is that one number, qua number, is not different from any other, 2 is the square root of 4 just as $\sqrt{2}$ is the square root of 2. We can only speak of incommensurability if we consider the ratio of two numbers, $6 \sqrt{2}$ is rational in terms of $\sqrt{2}$, and irrational in terms of 2 ." -Principal Works of Simon Stevin, ed. Crone
    et al., Vol. 2B: Mathematics, ed. Struik, p. 460 . There is an analogy in the way Oresme treats exponential relations between ratios. Any two ratios that could be exponentially related by a rational exponent were commensurable in Oresme's view regardless of whether the two ratios were irrational, rational, or one irrational and the other rational.
    ${ }^{38}$ Euclid X.s and VII. 4 are linked again in II.r.302-3 and III.34-37.

[^3]:    ${ }^{46}$ Cf. Crosby, Brad., pp. 66, 67.

[^4]:    ${ }^{53}$ What we have called Oresme's "principle of mathematical plenitude" seems operative here. See above, p. 36 .

[^5]:    per unam leucam una hora et ita sequitur quod $A$ et $D$ sunt virtutes omnino equales quia, scilicet, eque velociter movent idem mobile, scilicet $B$ [the text has " $D$ "]. Ergo cum $D$ posset movere $C$ sequitur quod etiam poterat ipsum movere" [fol. 107v, c.I].)

    We see, then, that Oresme's argument to this point is essentially the same as Buridan's and it is reasonable to suppose that he is following Buridan. In any event this portion of the argument is not original with Oresme. However, as we shall see,
    the remainder of Oresme's attack agains the false rule is radically different from Buridan's, since the latter is content to rest his case on the fact that if the prima regula is true, then the manifestly absurd rule is also true, with the attendant consequence that motion could arise even when $F<R$, thus violating the self-evident axiom that motion can arise only when $F>R$. Oresme, for the most part, will rely on what he believes to be an internal mathematical inconsistency inherent in the second false rule.

[^6]:    ${ }^{70}$ For the examples given in IV.309-31, see below on pp. 371-72.

[^7]:    97 "Respondeo ponendo quandam propositionem quam ponit doctissimus proportionum indagator magister Nicholaus Horen. Ubicunque occurrit multiplicitas
    proportionum inter quas facile non reperitur proportio consendum est multas earum irrationales esse ad invicem, ..."-libid., sig. q.6v, c.2.

[^8]:    104 " $\ldots$.hunc libellum edidi...in quo premisi quedam ex aliis libris mathematicis supponenda ex quibus conclusiones intuli quarum paucas postquam scripseram alibi reperi."-MS Vat. lat. 4082, fol. 97v, c. 1-2. If the Ad pauca respicientes is the work to which Oresme refers, it is apparent
    from an inspection of the De commensurabilitate that it does not merely contain the older Ad pauca with the newly discovered propositions added here and there, but, indeed, was a completely revised and expanded version of the earlier treatise.

[^9]:    ${ }^{105}$ If Oresme did write such a chapter, author of that treatise. See James F. it cannot be identified with the De propor- McCue, "The Treatise De proportionibus tionibus velocitatum in motibus (MS Paris, velocitatum in motibus Attributed to Nicole Bibliothèque de l'Arsenal 522 , fols. 126r-- Oresme" (unpublished Ph.D. dissertation, 168v), formerly ascribed to him, since Symon de Castello has been revealed as the

[^10]:    I See Clagett, Science of Mechanics, p. 163.
    ${ }_{2}$ See also Prop. XVI of the Ad pauca respicientes (AP2.185-97) and above on p . 65 , where a similar sentiment is expressed in a passage quoted from his Livre de divinacions. In the De commensurabilitate he remarks that astronomers-presumably in contrast to astrologers-are not concerned
    with punctual exactitude, recognizing that this is unattainable (Grant, "Oresme: Comm.," p. 42 I).

    In general, it seems fair to say that Oresme was quite sceptical concerning the human ability to draw precise consequences and inferences from natural phenomena. For this reason, he sometimes offered

[^11]:    25 If the mobiles move commensurably, conjunct in $b$, by Prop. IX of Part I (API. it could be in $b$; but if their motions are incommensurable, they will never again

[^12]:    ${ }^{28}$ This is demonstrated in the next proposition, i.e., XII. See below, pp. 106-7.

[^13]:    Plate 4: Opening page of the De proportionibus. MS Cambridge, Peterhouse 277,

[^14]:    ${ }^{1}$ Review of Der Algorismus Proportio num des Oresme, ed. Curtze, by Charles Thu rot in Revue critique d'bistoire et de littérature, Vol. 3, Pt. 2 (1868), 268.
    ${ }^{2}$ Curtze, Matbematischen Schriften des Oresme, p. 4. Harry Caplan examined the
    immediately following work (fols. 279r290r), also by Oresme, and dated it in the fifteenth century.-Mediaeval Artes Praedicandi(Cornell Studies in Classical Philology, Vol. 24), p. I8, entry ioi.

[^15]:    ${ }^{6}$ Lehmann, Mittelalterliche Bibliothekskataloge, Bayerischen Akademie der Wissenschaften, Vol. 2, 16.
    ${ }_{7}$ Montague James, Catalogue of Manuscripts in the Library of Peterbouse, p. 353.

[^16]:    9 Valentinelli, Bibliotbeca manuscripta, Vol. 4, 222.
    10 Since Valentinelli failed to distinguish this work, it does not appear in his cata-

[^17]:    ${ }^{16}$ Professor Thomas M. Smith of the University of Oklahoma brought this manuscript to my attention and was kind enough to supply a microfilm of it. This manuscript has four chapters but is inferior, containing considerable extraneous interpolated material.
    ${ }_{17}$ This codex is described by Wappler, "Beitrag zur Geschichte der Mathematik," Abbandlungen zur Geschicbte der Mathematik, Vol. 5, 164, n.2. The De proportionibus portion of this codex is wholly unreadable as a result of damage in World War II. Judging from the incipits and explicits given by Wappler, all four chapters were included. For a later description of the entire codex, based largely upon Wappler's account, see Robert of Chester's Latin Translation of the Algebra of Al-Khowarizmi, with an Introduction, Critical Notes, and English Version by Louis C. Karpinski (N.Y., I9rs),
    pp. 53-55.
    ${ }^{18}$ Although this anonymous fourteenthcentury manuscript contains all four chapters of the De proportionibus, it is hopelessly corrupt, for which reason it has not been utilized in establishing the text. For example, the first paragraph ( $\mathrm{I} .1-\mathrm{I} 3$ ) is missing (it has been replaced by something else), and after I.91 it jumps to I. 178 but later reintroduces, in garbled form, some of the omitted lines. The codex is described by Schum, Amplonianischen HandschriftenSammlung, pp. 590-93.
    ${ }_{19}$ Curtze, Mathematischen Scbriften des Oresme, p. 4. In his description of the title page, colophon, and the two additional works not listed on the title page, Curtze gives the abbreviated forms used in the edition. In my quotation they are expanded in full.

[^18]:    $E$ quod licet causa brevis introductionis inde demonstrat aliqua vera dicam $E d$ quodlibet causa brevius introductionis sine demonstratione vera aliqua dicam $R$ quodlibet causa brevitatis sive demonstratione aliqua vera dicam $S$
    $25-27$ sine...In om $P$
    26 primo supponitur $E R S$ primo supposito $C$ prius supponitis $V$ prius presupponitis $E d$
    26-27 in aliis... tangitur: dicam quod non in aliis capitulis vel $H$
    27 ibi CERS; om VEd / tangitur alibi $\operatorname{tr} V \mid$ alibi $C E H S E d$ aliquando $R /$ demonstratum: demonstratur $S /$ secundo: secunda $P$
    28 conclusiones aliquas $\operatorname{tr} P /$ de propor-

[^19]:    alter: alius $R$
    Ior sexte septimi corr ex septime sexti $C E$ $R V$ septimo sexti $H /$ numeri om $H /$ numeri quidem $\operatorname{tr} R /$ quidem $E H$; om $C$ quod(?) $V$
    102 contra se om $V$ | ante minimi scr et del $E$ mil(?) / minimi: numeret(?) V /

[^20]:    batum $E$
    180 post oportet hab E quod
    181 aliter om $V$
    181-82 componeretur: componetur $C$
    182 ante rationali scr et del $V$ ir / sit: fuerit V
    183 et cetera om $H /$ et $^{3}$ om $H /$ est om $E$

[^21]:    331 quasi: prima(?) $H$ / duarum precedentium $\operatorname{tr} R$
    $332 \mathrm{~A}^{2}$ om $E$
    334 primi termini: prima $E /$ componatur $E H R$ componitur $C V$
    335 componetur EHV componitur $C$ componeretur $R /$ pluribus: proportionibus $R$
    336 erit: et $C$ / igitur commensurabilis om $H /$ est om $E$
    337 conversa $C H R$ conversio $E$ conversis
    $V /$ sexte : octave $C /$ precedentis om $R$ 338 primos numeros $\operatorname{tr} V /$ numerus medius seu: medius numerus aut $R$ / seu numeri $E$ seu medii $H$ aut numeri $C V$ 339 unam $E H$; om $C R V /$ aliam: aliquam ${ }^{3} \mathrm{C}$

    340 proportionem om $H$
    342 pluribus: proportionibus $R$
    343 post modo scr et del $E \mathrm{~b}$
    344 eandem: tandem $C$

[^22]:    362 ante Sit add $H$ si / proportio maior $H R ; \operatorname{tr} E$ proportio $C$ maior $V /$ utraque $E H V$ utrumque $C R /$ post utraque bab $E$ quia
    363 numeris eius $H ; \operatorname{tr} C E R V /$ poteris facere $E R V ; \operatorname{tr} C$ poterit facere $H$

[^23]:    19-20 septimas $33 / 7$ EHV $33 / 8 C^{3 / 27} R$
    20 ultra: igitur $R /$ post ultra add $C V$ sit ergo et $E$ sit igitur / cuiusvis: cuius $V /$ proportionis denominatione $\operatorname{tr} R$ 1 denominatione $E H$ denominatio CV
    21 Primos $E V$; om $H$ primum $C$ primum numerum $R$ / eius: seu $R /$ numeros rep $E$ numerum $C$ / seu minimos om $R /$ minimos $E V$ numeros $C$ numeri (?) $H$ / invenies $H$ reperies $C E R V$ per CHR in $E V$
    22 cuiuslibet: cuius $H$

[^24]:    (?) $H$ volui $V /$ diutius $E H V$ diuos $C$ 138 post immorari bab $V$ et cetera

    Tertium capitulum $E V$; om $C H R$
    I tertio capitulo $\operatorname{tr} R \mid$ ante specialia scr et del $V$ p / proportionibus $E H$ proportione $C R$
    $\underset{\text { I-2 }}{\text { proportionibus proportionum } t r} V$
    2 quedam suppositiones $H R V$ quodam
    suppositiones $C$ supponam quedam $E$
    2-3 primitus sunt ponende $E H V$ sunt
    primitus ponendo $C$ sunt ponende primitus $R$
    4 est ${ }^{1}$ om $H$
    5 contra om $C /$ seu: vel $C /$ minimi om
    6 septimo: principio $E$
    8 primam: secundam $C$
    9 oppositum om $R$
    10 est om $R$
    I post unitatem $m g$ bab $H$ tertia suppositio

[^25]:    | post dictum add EV sepe
    136 Igitur: quarti $C$ / nonam: secundam $H$ / ille: iste $R$
    137 alia om C / alicui: alteri C | post istarum add $H$ esset(?)
    138 probatur om $H /$ sic $o m E /$ esset: est $H$
    139 Sed om $H /$ quia $C H$ quod $E R V$ / arguitur: probatur $R$
    140 primos numeros $H R ; \operatorname{tr} C E V$
    141 sit: est $R$
    141-42 istius ordinis $E H V$ istius ordinationis $C$ illius ordinationis $R$

[^26]:    142 istarum: alia $R$
    143 aliam $E R V$ aliquam $C H$ / quam om $H /$ aliquis $C H V$ aliquas $E$ aliqua $R /$ istorum: istarum $V /$ ante sit scr et del $V$ est talis quod inter numeros eius sit secundum aliam proportionem 144 istarum: illarum $V$
    145 alicui $C R V$ alteri $E$ alicuius $H$
    147 probatur antecedens $H$; $\operatorname{tr} C E R V$
    148 aliquis: aliquo(?) $E /$ est $^{2} H$ esset CERV

[^27]:    $C H R$ aut $E V /$ vel quinta $C H V$; om ER
    271 etiam $C R V$; om $E$ igitur $H /$ nume ros: duos $E$
    272 proportionales: proportionalis $V /$ nulla: nisi $H /$ minor: maior $C$
    273 est: erit $C$ / eidem: sibi $E$ / subtripla: sesquitertia $V$

[^28]:    om CV
    386 de om $E$
    387 proportionum om $V$ / ignota om $E$ / petatur $C E R$ petar $H$ petitur $V /$ illam: istam $C$
    388 irrationalem: rationalem $C /$ et: ad $E$ 389 et: ad $E$ / queritur: querit $V$
    390 rationales: irrationales $H$
    391 proposite $E H V$; om $C$ posite $R /$ secundo: primo $V$
    392 et om $V /$ rationalibus: rationales $H$
    393 probabitur: probatur $V /$ sic $C E H$ similiter $V /$ sic Sint: si fuerit $R /$ tunc arguitur om $C$ / post arguitur add $R$ sic 394 alicui: alteri $E /$ qua $C H V$ qui $E$ quo $R$
    395 similiter CHV sibi $E$ / similiter est tr

[^29]:    $R /$ commensurabilis: commensurabile $H$ / alicui: alteri $E$
    395-96 a...denominationem $H$; om $C E R$ V
    396 ante illa scr et del E A / Et: tunc $R /$ si: sic $V /$ incommensurabiles $C V$ commensurabiles $E H R$ / ante A add $R$ et
    397 ante commentum scr et del $V$ commentum
    398 commensurabile CHR commensurabilis $E$ incommensurabile $V /$ est $^{2}$ om $R$
    398-99 incommensurabile CHV incom-

[^30]:    337) / et om $V /$ proportione: proportio $C$
    339 oriatur: oriuntur $C /$ tamen: cum $R$
[^31]:    388 et cetera $C H$; om $R V /$ proponitur $C$ proposita $H$ proponatur $R V /$ ante illa bab $H$ de
    390 fiet om $C$
    391 tres om $V /$ per $^{2} V$; om $C H R /$ aliquam earum $R$ aliquam aliarum $C$ aliam earum $H$ aliquam istarum $V /$ possumus $H V$ possimus $C R$
    $39^{2}$ cognoscere: cognita $H /$ possumus

[^32]:    23-24 conclusiones $R V$; om $C$ alias $H$
    s2s Verbi gratia om $C /$ proveniet: provenit $R$
    26 incommensurabilis: commensurabilis $V /$ provenit: proveniet $H$
    s 27 utraque $C R$ uterque $H$ utriusque $V /$ fuerit om $R$
    s 28 commensurabiles: incommensurabiles $H /$ Hoc $H R$ hec $C V$

[^33]:    pertransirem $V /$ eodem $R V$ eadem CH / tempore om $\mathrm{H} /$ vel om $V$ / in H; om CRV
    5s3-54 equalibus: equalibet $C$
    ss4 temporum sicut velocitatum CH temporum sicut proportio velocitatum $R$ velocitatum sicut temporum $V$
    555 pertransiri $H V$ pertransire $C R /$ ex om $C$
    556 ista conclusio $\operatorname{tr} R$ / propositis: positis

    V
    556-57 quibuscumque duobus $H R$ duabus quibuscumque $C$ quibuscumque duabus $V$
    557 acquisibilibus: acquisitionibus $R /$ continuum motum $\operatorname{tr} C$

    ## 8 illa: illam $C$

    558 illa: illam $C$
    560 duobus: duabus $H$ / idem $R V$ nichil $C$ ubi(?) $H$
    562 inequales: inequalis $C$

[^34]:    601 istis om $H$ / celestibus $C V$ supracelestibus $H R /$ disponente: supponente $R$ / sicut: ut $R$
    602 per ipsam quidem: quidem causa per ipsam $H /$ regit omnia $\operatorname{tr} C$ / causa om H
    Gos ante demonstrabo bab $R$ poterunt demonstrari et alias / aliis: igitur $R$ / post coassumptis $m g$ bab $R$ verisimile(?)
    606 in fide obs $H$ / ex hiis poterunt $C V$

    607 sicut: ut $R$
    608 pristinum: prime scitum $C$ / reverti: reduci $H$
    609 ante aspectus bab $R$ tunc /ante ab bab $R$ iterum / consimiliter: similiter $V$
    610 talia : celestia $R /$ alii $H V$ aliis $C$ aliquando $R /$ post non add $V$ ex
    6II reprobare: reprobari $H$ / Bonum: totum $C$ | est om $H$ | ante ex ${ }^{1}$ hab $R$ quod / et om $H$

[^35]:    ${ }^{2}$ Clagett, Giovanni Marliani, p. 140.
    ${ }^{3}$ In his De proportione motuum, included
    in his collected works.-Achillini...opera omnia in unum collecta, fol. $186 \mathrm{v}, \mathrm{c} .2$.

[^36]:    7 This is Euclid X.12, in Euclid's Elements, trans. Heath, Vol. 3, 34-35.

[^37]:    tially the same. "Si duabus quantitatibus inaequalibus propositis maius dimidio a maiori detrahatur, itemque de reliquo

[^38]:    ${ }^{20}$ Bradwardine offers essentially the same overall theory as the only correct approach. 'Now that these fogs of ignorance, these winds of demonstration, have been put to flight, it remains for the light of knowledge and of truth to shine forth. For true knowledge proposes a fifth theory which states that the proportion of the speeds of motions varies in accordance with the proportion of the power of the mover to the power of the thing moved."
    -Crosby, Brad., p. in I. In support of this theory Bradwardine cites three different passages from the comments of Averroes on Atistotle's De caelo and Pbysics. A1though Aristotle is not explicitly cited as an adherent of this view, it is obvious by implication that Bradwardine links him with Averroes. See above, p. 19, n. 2 I.
    It should be noted that Crosby renders the term proportio as "proportion" where I use "ratio." See above, p. ı6, n. i4.

[^39]:    ${ }^{26}$ Those who accepted the Aristotelian rules-as, for example, Thomas Aquinas, Walter Burley and Domingo Soto-had no difficulty with Aristotle's shiphauler passage. For quotations from these authors

[^40]:    * See pp. 91-92.
    $\ddagger$ See pp. 92-93.
    $\dagger$ See p. $9^{2}$.

[^41]:    * See p. ior.
    $\ddagger$ See pp. 102-3
    + See pp. 101-2.


    ## liter $F$

    59 sunt $B F V \operatorname{sint} A H /$ erunt et fuerunt: fuerunt et erunt $V /$ proprie om $F \mid$ et ${ }^{2}$ om $V$
    60 ante tantum bab $V$ cum / tantum: cum $B /$ vel $B H$ nec $A F V$
    61-62 sive ${ }^{2} \ldots$ moveantur om $V$
    62 Sed $F H V$; om $A B$

    63 sic om $A$
    64 ante Patet add $V$ et
    Gs aliquot $B H$ tantum(?) $A$ aliquotiens $F$ aliquas $V \mid$ ante revolutiones $m g$ bab $F$ figuram / tertiam: secundam $V$ fuit: fecit(?) $V$
    65-66 nunc alias: coniuncta $A$

[^42]:    * See p. ${ }^{106 .}$
    $\ddagger$ See pp. 106-7.
    + This figure appears in MS $F$.

[^43]:    undevicesima; om $A V$ is bab $B$ post mundus / iste mundus $F H V$ ille mundus $B$ mundus ille $A$
    229 regeretur $A H$; om $F$ reguletur $B$ (?) $V / \mathrm{et}^{\mathrm{I}}$ : in $V /$ celum: totum $B /$ necessario: necesse $A$ / ante uniformiter add $B H$ et / moveretur: moventur $V$
    229-30 evenient $B H$ evenirent $F H V$
    230 necessitate obs $F$ / esset: esse $A /$ casus om V
    230-31 et mundus obs $F$
    23I nullus: unus $A /$ sciret om $A /$ nec: non $A /$ posset recte $A B V$; obs $F$ posset recte(?) $H$

[^44]:    Sacrobosco, ed and trans. Thorndike, p. 447 See also p. 24. The Liber de proprietatibus elementorum is a pseudo-Aristotelian treatise. See Martin Grabmann, Forschungen über die lateinischen Aristoteles-Übersetzungen des xiii Jabrbunderts, in Beiträge zur Geschichte der Philosophie des Mittelalters, Vol. 17, Bk. 5/6 (Münster, 1916), pp. 198-204.
    ${ }^{3}$ In his Commentary on the Dream of Scipio, a work influential throughout the Middle Ages, Macrobius describes the concept of a Great Year, or World Year as he called it. "[9] All stars and constellations which seem fixed in the sky and whose peculiar motions no human eye can ever discern nevertheless do move, and in addition to the rotation of the celestial sphere by which they are pulled along they proceed at a pace of their own which is so slow that no mortal's life span is sufficiently long to detect by continuous observation any movement away from the position in which he first saw them. [ro] A world-year will therefore
    be completed when all stars and constellations in the celestial sphere have gone from a definite place and returned to it, so that not a single star is out of the position it previously held at the beginning of the world-year, and when the sun and moon and the five other planets are in the same positions and quarters, that they held at the start of the world-year. [II] This, philosophers tell us, occurs every is,000 years." -Macrobius: Commentary on the Dream of Scipio, trans. Stahl, pp. 220-21.
    ${ }^{4}$ See Plato's Cosmology: The Timaens of Plato, translated by Cornford, p. 117. Oresme's source for a Great Year of 15,000 years duration may have been Macrobius (see concluding sentence of the quotation in preceding note). But on what grounds did he attribute this to Plato?
    ${ }_{5}$ Thorndike, Magic and Experimental Science, Vol. 3, 582. The precise reference is given as MS Paris, BN lat. 6752 , fols. 13 r and I 6 r .

[^45]:    6 The text encompassing both criteria is as follows: "Et circulationes sunt incommensurabiles que in temporibus incommensurabilibus fuerint; et quibus describuntur temporibus equalibus anguli incommensurabiles circa centrum."-MS Vat. lat. 4082, fol. 97v, c.2. In Grant,

[^46]:    7 Sphere of Sacrobosco, ed. and trans. Thorndike, [p. 14I. The Latin text appears
    on p. 114.

[^47]:    ${ }^{15}$ Perhaps Duhem constructed this approximative interpretation from Cusa's statement that one can always get a closer approximation for any given human measure. See above on p. 12I, n.62. However, Duhem's argument may also have been suggested to him by a remark in Oresme's De commensurabilitate, repeated almost verbatim some years later by d'Ailly (both passages are quoted above, p. 120, n.6I), that a part smaller than $1 / 1000$ could transform a rational into an irrational ratio. Actually, Duhem (Système du monde, Vol. 8, 455, 461) conjectured that d'Ailly, not

    Oresme, was the author of the De commensurabilitate. On what grounds, then, might Duhem have thought his elaborate argument was in any way representative of Oresme's attitude? Only a conviction that d'Ailly was echoing Oresme, or that the former was enunciating a view to which Oresme himself would have subscribed. All this, however, is sheer conjecture, since Duhem does not indicate that he was aware of the passages quoted above on p. 120, n.61.
    ${ }^{16}$ The Latin text of this passage is given above on p. 85 , n. 3 .

