Solid Mechanics and Its Applications

Gérard A. Maugin

Continuum Mechanics Through the Eighteenth and Nineteenth Centuries

Historical Perspectives from John Bernoulli (1727) to Ernst Hellinger (1914)

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Preface

It is not by one way alone that we can arrive at so sublime a mystery
(attributed to Quintus Aurelius Symmachus, 384)

Conceived as a series of more or less autonomous essays, this book exposes the initial developments of continuum thermo-mechanics in a post-Newtonian period extending from the creative works of the Bernoullis to the First World War, i.e., roughly during first the "Age of reason" and next the "Birth of the modern world." The emphasis is rightly placed on contributions from the "Continental" scientists (the Bernoulli family, Euler, d'Alembert, Lagrange, Cauchy, Piola, Duhamel, Neumann, Clebsch, Kirchhoff, Helmholtz, Saint-Venant, Boussinesq, the Cosserat brothers, Caratheodory) in competition with their British peers (Green, Kelvin, Stokes, Maxwell, Rayleigh, Love,..). It underlines the main breakthroughs as well as the secondary ones. It highlights the role of scientists who left essential prints in this history of scientific ideas. It shows how the formidable developments that blossomed in the twentieth century (and perused in a previous book of the author: "Continuum Mechanics Through the Twentieth Century," Springer SMIA series, Dordrecht, 2013) found rich compost in the constructive foundational achievements of the eighteenth and nineteenth centuries. The pre-WWI situation is well summarized by a thorough analysis of treatises (Appell, Hellinger) published at that time. English translations of most critical texts in French or German are given for the benefit of the readers.

Gérard A. Maugin

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Chapter 1
General Introduction: About the Contents and Form of this Book

Abstract This general introduction gives the motivation that led the author to write this book. It provides a general overview of the contents while justifying the peculiar choice of the various essays that reconstruct the history oriented examination of the most fundamental developments from the early eighteenth century to the dawn of the twentieth century. It also gives essential elements of a relevant bibliography concerning the main actors of our play.

In the early 1990s, my wife—who was then a freelance translator for the European Union in Luxemburg and Brussels—and I conceived the following project. Having remarked that many historians of science and epistemologists in the English-speaking world knew less and less foreign languages, we thought of providing an English translation of some landmark scientific texts published before World War One (WWI) in French, German and Italian.[1,2] The subject matter would be the

[1] The reader must know that in the nineteenth century all English scientists knew French and/or German. Also, most American scientists had received a classical education (with Latin and Greek) in good colleges. Even much more recently, when I was a graduate student at Princeton in the late 1960s, I had to pass exams in two foreign languages for my PhD (I took French and German). These obligations have disappeared (or Chinese and Russian have become more popular). Obviously, most papers are now published in English and many books in English are published by non English speakers or are translated from other languages, scientific quality obliged.

[2] In the late seventeenth century and the eighteenth century, the international languages were Latin and French. Swiss scientists such as the Bernoullis could write in French or Latin, Euler could communicate in French, German and Latin. Lagrange could obviously communicate in both Italian and French. The French, as usual, had a difficulty with foreign languages, being convinced of the superiority of their own language. An exception was Voltaire who had spent some (forced) time in England, but his mistress Emilie du Châtelet translated Newton's Principia from the Latin version. English was not practiced outside the UK and the recent American colonies. Newton could only communicate in Latin with foreign scientist visitors such as Abbé (Antonio) Conti during the dispute about the invention of differential calculus by Newton and Leibniz; all this to say that a basic knowledge of Latin is necessary to study the primary sources of the period.

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thermo-mechanics of continua from the eighteenth century to WWI.[3] The original selection of texts comprised texts by d'Alembert, Lagrange, Cauchy, Piola, Kirchhoff, Duhem, the Cosserat brothers, Caratheodory, and Hellinger. We started to work on elements of translation from some of these texts. But our occupations, especially mine with my intense works on configurational forces and heavy administrative duties as Head of a large institute at the University Pierre et Marie Curie, hindered any rapid progress in this project.

In the years 2010–2012 I wrote a kind of short historical perspective of the developments of continuum thermo-mechanics in the second half of the twentieth century (cf. [42]) which I had witnessed and contributed to actively. To introduce the subject I had first to review succinctly what had been achieved in the two foregoing centuries. I started to write short essays on different French scientists (Cauchy, Duhamel, the Cosserat brothers), and this rekindled my interest in the old project of the 1990s. But, in the mean time, the whole landscape had evolved drastically. In particular, English translations had been produced by other people for the relevant works of d'Alembert [16], Lagrange [34], Duhem [19], Caratheodory [11], and the Cosserats [43], and an English translation from the Italian of Piola's collected works is in progress (cf. [46]). Only Cauchy, Duhamel, Kirchhoff, and Hellinger remained non-translated into English. This completely remodelled my strategy which has resulted in the present series of essays that now collectively contribute to a rather complete, albeit condensed, historical view of the development of continuum thermo-mechanics from the early eighteenth century to WWI.

The essays in this volume follow the logic of the flow of time. Most of the essays concern the nineteenth century. But the first essay deals with the development of main ideas in the field between John Bernoulli (1667–1748) [7] and Joseph-Louis Lagrange (1736–1813) [35]. This necessarily involves Daniel Bernoulli (1700–1782) [6], Jean Le Rond d'Alembert (1717–1783), and Leonhard Euler (1707–1783) [21]. With these people most of the solid bases of calculus and the most relevant principles of mechanics are established. The second essay discusses the foundational work [12–14] of Augustin-Louis Cauchy (1789–1857) and its consequences on the future development of continuum mechanics, with an English translation of his seminal paper. The next essay logically concerns further developments based on previous works by d'Alembert, Lagrange and Cauchy with the original viewpoint of Gabrio Piola (1794–1850), George Green (1793–1841) [25], and Gustav Kirchhoff (1824–1887) [33] on the notion of stress, what prepares the way for the theory of nonlinear continuum mechanics to be expanded in the twentieth century. Then a short essay is devoted to the pioneering work considering the coupling between continuum mechanics and heat conduction—basically, thermo-elasticity—by J. M. C. Duhamel (1797–1872) in

[3] It is not by mere chance that the selected period corresponds, in the modern view of global history, to "The Birth of the Modern World 1780–1914" by the Cambridge historian C.A. Bayly (Blackwell History of the World, J. Wiley, UK, 2004; see also J. Osterhammel, Die Verwandlung der Welt: Eine Geschichte des 19. Jahrhunderts, C.H. Beck, München 2009).

France and Franz Neumann (1798–1895) in Germany. The mathematical development of elasticity theory in the expert hands of Gabriel Lamé (1795–1870), Alfred Clebsh (1833–1872), Adhémar Barré de Saint-Venant (1797–1886), and Joseph Boussinesq (1842–1929) is the object of an essay. The vision of von Helmholtz (1821–1894) [28] largely accepted by Pierre Duhem (1861–1916) is discussed in another essay. This was written for the centennial meeting after the death of Helmholtz (1894) held in Berlin in 1994 [40]. The logical bases of thermo-statics and thermo-dynamics as formalized by Constantin Caratheodory (1873–1950) are reported in the next essay. The creative ideas of the Cosserat brothers about generalized continua are then perused on the basis of their book published in 1909. As to the synthetic approach [20, 38] in the framework of energetics proposed by Pierre Duhem following along a line created by W. J. Macquorne Rankine (1820–1872) [47], H. von Helmholtz (1821–1894) [28] and Josiah W. Gibbs (1839–1903) [22, 23], we examine it in a long essay complemented by English translations of essential parts of his texts. This concludes the perusing of the most creative works.

The last two essays have a different nature. They concern syntheses in the form of courses or encyclopaedia articles by Paul Appell (1855–1930) in France [1] and Ernst Hellinger (1883–1950) in Germany [27]. These two essays are in the form of critical and analytical reviews of these two works that are supposed to present globally the achievements of the nineteenth century and therefore allow us to apprehend the contemporary reception of previously discussed works, as of the dawn of the twentieth century for a kind of conclusion. On this occasion we provide an English translation of part of Hellinger's remarkable—but rather neglected—memoir that captures in a few pages most of the landmark advances of the nineteenth century.

The visible recurring themes are finite deformations for deformable solids and variational formulations (principle of virtual work, virtual velocities, and analytic mechanics) [39, 41]. This is not very "Newtonian"—nor very "Truesdellian" in modern terms—, but it reflects the continental line—versus the Newtonian one—that developed with Leibniz and the expansion of the calculus of variations on the continent and survived during an extended period of time. Contributions by British scientists are mentioned and discussed in the course of various essays (but they obviously did not enter our original plans of the 1990s). In all, the present book may be considered a "prequel" to my book of 2013 [42].

In so far as possible we have tried to refer to primary sources, sometimes available in editions of collected works of some of the most famous scientists (e.g., the Bernoullis [6, 7], Euler [21], Lagrange [35], and Cauchy [15]). In gathering useful information as secondary sources, we have benefited from the existing biographies of Cauchy [3], d'Alembert [8, 26], Euler [50], the Bernoullis [4, 5], Lagrange [9, 29], Duhem [30], and Kirchhoff (in [32]), as also from well written historical introductions in some books by scientists such as Lagrange [34], Barré de Saint-Venant [2], and Love [36], and historical discussions on the developments of mechanics by Todhunter [52, 53], Duhem [19], Mach [37], Jouguet [31], Dugas [18], Whittaker [57], Timoshenko [51], Truesdell and Toupin [56], Oravas and

McLean [44, 45], Szabò [49], Truesdell [54, 55], Darrigol [17], Soutas-Little [48] and Capecchi [10], and finally the valuable—but unequal—scientific biographies in Gillispie [24] and the accurately informed MacTutor History of Mathematics Archive at the University of St-Andrews, UK, on the web [58]. For French scientists who were members of the Academy of Sciences in Paris, eulogies have been published by some fellow academicians. This is the case for Barré de Saint-Venant, Duhem, and Boussinesq.

The bibliography that follows lists most primary sources used in this book as well as historical sources of a secondary nature. Short complementary information is given within square brackets. Detailed references are given at the end of each of the other chapters.

References

1. Appell P (1921) Traité de mécanique rationnelle, 3rd edn. Gauthier-Villars, Paris (Facsimile reprint by Gabay, Paris, 1991)
2. Barré de Saint-Venant AJC (1864) Historique abrégé des recherches sur la résistance des matériaux et sur l'élasticité des corps solides. In: Navier's « Résumé des leçons sur l'application de la mécanique ». Dunod, Paris
3. Belhoste B (1991) Augustin-Louis Cauchy: a biography. Springer, New York (English trans: French original "Cauchy, 1789–1857", Editions Belin, Paris)
4. Bernhard H (1938) The Bernoulli family. Double Day, Page & Co, New York
5. Bernhard H (1983) The Bernoulli family. In: Wussing H, Arnold W (eds) Biographien bedeutender Mathematiker, Berlin
6. Bernoulli D (1982) Werke von Daniel Bernoulli. Birkhäuser, Basel
7. Bernoulli J (1742) Johannis Bernoulli opera omnia. Boucquet, Lausannae (reprint Olme, Hildesheim, 1968)
8. Bertrand J (1889) D'Alembert. Paris; available on the web site "Project Guttenberg"
9. Capecchi D (2005) Lagrange e la storia della meccanica. Bari
10. Capecchi D (2012) History of virtual work laws (Science networks historical studies series, vol 42). Birkhäuser-Springer, Milano
11. Caratheodory C (1909) Untersuchungen über die Grundlagen der Thermodynamik. Math Annalen 67, 355–386 (English trans: Kestin J (ed) (1976) The second law of thermodynamics: benchmark papers on energy, vol 5. Dowden, Hutchinson and Ross, Pennsylvania)
12. Cauchy AL (1823) Recherches sur l'équilibre et le mouvement intérieur des corps solides ou fluides, élastiques ou non élastiques. Bull Soc Filomat Paris January issue, 9–13
13. Cauchy AL (1828) Sur les équations qui expriment l'équilibre ou les lois du mouvement intérieur d'un corps solide élastique ou non élastique. Exercices de mathématiques, vol 3, pp 160–187 [This presents in print the ideas originally submitted to the Paris Academy of Sciences on 30th Sept 1822, and abstracted in Cauchy (1823)]
14. Cauchy AL (1828) In: Exercices de Mathématiques, vol 3, pp 188–212 (Sept 1828), and « De la pression ou tension dans un système de points matériels », pp 213–236 (Oct 1828)
15. Cauchy AL (1882–1974) Œuvres complètes, vol 27. Gauthier-Villars, Paris
16. D'Alembert JLR (1743) Traité de dynamique, 1st edn, Paris (reprinted by Gauthier-Villars Publishers, in two volumes, Paris, 1925; also Facsimile reprint by Editions Gabay, Paris, 1998)

17. Darrigol O (2005) Worlds of flows: a history of hydrodynamics from the Bernoullis to Prandtl. Oxford University Press, Oxford
18. Dugas R (1950) History of mechanics. Editions du Griffon, Neuchatel, Switzerland (Dover reprint, New York, 1988)
19. Duhem P (1903) L'évolution de la mécanique (published in seven parts in: Revue générale des sciences, Paris; as a book, A. Joanin, Paris) (English trans: The evolution of mechanics. Sijthoof and Noordhoff, 1980)
20. Duhem P (1911) Traité d'énergétique ou de thermodynamique générale, vol 1 (528 pages), vol 2 (504 pages) Gauthier-Villars, Paris (Reprint by Gabay, Paris, 1997)
21. Euler L (1960) Opera omnia (Complete works). Basel
22. Gibbs JW (1876–1878) On the equilibrium of heterogeneous substances, Parts 1 and 2. Trans Connecticut Acad of Arts and Sciences 3, pp 108–248 (1876); 3, pp 343–520 (1878) (German trans: Ostwald W, French trans: Le Chatelier H, both reputed physico-chemists)
23. Gibbs JW (1928) Collected works of J. Willard Gibbs (Two volumes). In: Bumstead HR, Gibbs-Van Name R (eds) Longmans, Green and Co, New York
24. Gillispie CG (ed) (1970–1990) Dictionary of scientific biography. Scribners, New York
25. Green G (1903) Mathematical papers of the late George Green. Ferrers NM (ed) Mcmillan, London (reprint, Chelsea Publishing Co., New York, 1970)
26. Hankins TL (1970) Jean d'Alembert: science and enlightenment. Clarendon, Oxford, UK (reprint, Gordon and Breach, New York, 1990)
27. Hellinger E (1914) Die allgemein Ansätze der Mechanik der Kontinua. In: Enz. Math.Wiss. Klein F, Wagner K (eds), vol 4, Part 4. Springer, Berlin, pp 602–694
28. von Helmholtz H (1882–1895) Wissenschafliche abhandlungen von Herrmann Helmholtz. JA Barth, Leipzig
29. Hermann D (1963) Joseph Louis Lagrange (1736–1823). Treptow, Berlin
30. Jaki SL (1984) Uneasy genius: the life and works of Pierre Duhem. Martinus Nijhoff, The Hague
31. Jouguet E (1924) Lectures de mécanique (La mécanique enseignée par les auteurs originaux). Gauthier-Villars, Paris (Facsimile reprint, Gabay, Paris, 2007)
32. Jungnickel C, McCormmach R (1986) Intellectual mastery of nature (Two volumes). Chicago University Press, Chicago (especially informative on G. Kirchhoff)
33. Kirchhoff GR (1882) Gesammelte Abhandlungen, Leipzig (Mechanik, 4th edn, Leipzig, 1897)
34. Lagrange JL (1788) Mécanique analytique, 1st edn. Vve Desaint, Paris; 2nd edn. Vve Courcier, Paris, 1811/15 (Facsimile reprint of 4th edn with notes and comments par J. Bertrand and G. Darboux, Gabay, Paris) (English trans: 2nd edn (1811) by Boissonnade A, Vagliente VN. Springer, Dordrecht, 1997)
35. Lagrange JL (1867–1892) Oeuvres complètes (14 volumes). Paris (reprint in 10 volumes, Hildesheim, New York, 1973)
36. Love AEH (1892) A Treatise on the mathematical theory of elasticity, CUP [1944, 4th edn, Dover reprint, New York; originally published in two volumes in 1892–1893]
37. Mach E (1911) The science of mechanics: a critical and historical account of its development. Open Court, Chicago (Paperback 1988) (Original German edition: F.A. Brockhaus, Leipzig, 1883) (analysed by Duhem P (1903) Bulletin des Sciences Mathématiques 27:261–283)
38. Manville O (1927) La physique de Pierre Duhem. Mém Soc Sci Phys Nat Bordeaux, 7ème Série, 171–636 (the most comprehensive and unsurpassed analysis of Duhem's scientific works by one of his direct disciples)
39. Maugin GA (1980) The method of virtual power in continuum mechanics: application to coupled fields. Acta Mechanica 35(1):1–70 (see the historical introduction and references therein)
40. Maugin GA (1994) Helmholtz's influence on Pierre Duhem and our view of mathematical physics. Contribution to: Helmholtz-Feier (Centennial of Helmholtz's Death, Sept 1994, Berlin), Organizers: Ebeling D, Muschik W, p 13, TU Berlin and Humboldt Universität, Berlin (unpublished) (Chap 7 in the present book)

41. Maugin GA (2012) The principle of virtual power: from eliminating metaphysical forces to providing an efficient modelling tool. Cont Mech Thermodynam 25:127–146 (special issue in the honor of Del Piero G)
42. Maugin GA (2013) Continuum mechanics through the twentieth century: a concise historical perspective. Springer SMIA Series, Dordrecht
43. Maugin GA (2013) About the Cosserats' book of 1909 (preprint UPMC, Paris) (Chap 8 in the present book)
44. Oravas GE, McLean L (1966) Historical development of energetical principles in elastomechanics, Part I: From Heraclitos to Maxwell. ASME Appl Mech Rev 19(8):647–658
45. Oravas GE, McLean L (1966) Historical development of energetical principles in elastomechanics, Part II: From Cotterill to Prange. ASME Appl Mech Rev 19(11):919–933
46. Piola G (2014) The complete works of Gabrio Piola, vol I. Springer series Advanced structural materials, vol 18 (commented English translation; to be published)
47. Rankine WJM (1855) On the Science of Energetics. In: Proceedings of the philosophical society of Glasgow, III:381–399 (the original paper on energetics)
48. Soutas-Little RW (2011) History of continuum mechanics. Contribution to the (UNESCO) Encyclopedia of Life Support Systems (EOLSS), Section "Continuum mechanics" Merodio J, Saccomandi G (eds), 14 p, EOLSS Publishers, Oxford (Extremely concise text)
49. Szabò I (1977) Geschichte der mechanischen Prinzipien. Birkhauser, Basel
50. Thiele R (1982) Leonard Euler. Leipzig
51. Timoshenko SP (1953) History of the strength of materials. McGraw Hill, New York (Dover reprint, New York, 1983)
52. Todhunter I (1886) A history of the theory of elasticity and the strength of materials from Galileo to the present time, vol 1. Cambridge University Press, UK (edited and published posthumously in 1886 by Karl Pearson)
53. Todhunter I, Pearson K (1893) A history of the theory of elasticity and the strength of materials from Galilei to the present time, vol 2: Saint-Venant to Lord Kelvin. Cambridge University Press, UK
54. Truesdell CA (1968) Essays in the history of mechanics. Springer, New York
55. Truesdell CA (1984) An idiot's fugitive essays on science. Springer, New York
56. Truesdell CA, Toupin RA (1960) The classical theory of fields. In: Flügge S (ed) Handbuch der Physik, vol III/1. Springer, Berlin, pp 226–858
57. Whittaker ET (1951) A history of the theories of ether and electricity. Nelson, London (originally in two volumes; Dover reprint in one volume, New York, 1989)

Web site

58. MacTutor history of mathematics archive. (The) http://www-history.mcs.st-andrews.ac.uk/index.html

Chapter 2
A Glimpse at the Eighteenth Century: From John Bernoulli to Lagrange

Abstract This essay tries to answer the following question: "What happened to Mechanics between Newton and Lagrange?", hence during what is commonly called the century of the enlightenment or *Age of Reason*. This period where knowledge and learning are the main incentive for intellectuals—of which philosophers and scientists are most representative—, witnessed an evolution that went from the mechanics of point particles (or massive objects seen as such) with Newton to the first elements of continuum mechanics in the hands of Euler and Lagrange. The most famous contributors between these scientists are Jacques (Jacob) Bernoulli, John (Johann) Bernoulli, Daniel Bernoulli, Pierre Varignon, Jean Le Rond d'Alembert, Pierre de Maupertuis, and Leonhard Euler. We peruse the works and the contributions to the formulation and consolidation of various principles by these "mechanicians" on the basis of primary sources—sometimes with original English translations—and accounting for comments by Lagrange and more recent historians–mechanicians such as Jouguet and Truesdell. This period is marked by a great emphasis placed on the notion of living forces, the exploitation of the principle of virtual work, and the expansion of the calculus of variations, which all characterize the "continental" development of the principles of mechanics as opposed to the Newtonian vision. This essay is a prerequisite to the examination in other essays of all what was to be expanded in the thermo-mechanics of continua during the nineteenth century.

2.1 John Bernoulli and the Principle of Virtual Work

In his discussion on the role of mathematics in science, Truesdell [40, p. 99] asks the following question: "Before 1788 (cf. Lagrange [31]) and after 1687 (cf. Newton) something had happened to mechanics. What was it?" This in fact is the subject matter of the present essay. Lagrange [31, 2nd part of his historical introduction] declares that all really started in this period with the work of John Bernoulli (1667–1748) and his apprehending of the principle of virtual work. John (Jean or

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Johann) Bernoulli is the most remarkable member of the Bernoulli family (or dynasty) [4, 22]. He provides a bridge between the Newtonian period of the ending seventeenth century and the new developments that we are going to discuss here. To be fair, however, we must first acknowledge the contribution of his elder brother James (Jacques or Jacob) Bernoulli (1655–1705) who may have been less creative than John, but nonetheless played an essential role in the dissemination of integral calculus (he coined the word "integral"), in the establishment of the theory of probabilities, and in solving critical problems in mechanics (the isochronous curve, the elastica).

John, with all his bad temper and his unhealthy jealousy, was nonetheless an inspiring mentor. He was instrumental in the adoption of Leibniz's successful differential notation for differential calculus on the continent instead of Newton's fluxions. This, unfortunately, created a kind of dichotomy between British and continental developments in mathematics and applications, that was resolved only in the nineteenth century with the adoption of Leibniz's notation by Cambridgians under the influence of French pedagogues. So we can say that John Bernoulli was a "Leibnizian" as opposed to a "Newtonian". In spite of his strabismus toward Newton, Truesdell had to recognize the mathematical genius and creativity of John (cf. [39, 40]). He had an important epistolary exchange with French mathematicians, e.g., Marquis de l'Hôpital (1661–1704)[1] and Pierre Varignon (1654–1722). He defended in 1694 a doctoral thesis in medicine, but this may have been a première in biomechanics since in it John discussed the movement of muscles.

Main discoveries in mathematics by John Bernoulli are: the exponential calculus, trigonometry treated as a branch of analysis, the study of geodesics, the celebrated solution of the brachistochrone (the catenary), an introduction of the treatment of minima and a foundation for the calculus of variations (to inspire Euler), and more important from the viewpoint of this essay, the enunciation of the principle of virtual work (cf. [5, 6]). This matter is thoroughly discussed by Capecchi [11, pp. 199–209] that we do not need to repeat in detail. In this work John was a Leibnizian, reformulating Leibniz's notion of dead force by introducing the elementary increase $f\,dx$, where dx is an infinitesimal virtual displacement of the point of application of the force f in its direction, so that we would write this as the inner product $\mathbf{f} \cdot d\mathbf{x}$ in modern jargon. It is here an

[1] Both de l'Hôpital and Varignon are known French mathematicians, the first one having left his name attached to the famous "rule of l'Hôpital" and the second having created the "law of the parallelogram of forces" in statics, much before the notion of vectors was invented. John Bernoulli taught the first one on calculus (against a good stipend) by visiting him in France and then continuing by correspondence. De l'Hôpital published a book based on the lessons he had received from Bernoulli. This was the natural cause of a dispute for which Bernoulli had a special talent, but here he was probably right. In the case of Varignon, met in Paris in 1692, the mutual friendship seems to have been sincere, in spite of the somewhat touchy character of Bernoulli who did not hesitate to rebuke his friend on occasions. We must also note that Bernoulli had a fruitful correspondence with the great Leibniz. This was the most major correspondence which Leibniz ever carried out. Decidedly, Bernoulli was a great letter-writer like many scientists and intellectuals of the time.

infinitesimal pulse that defines the new quantity which is an infinitesimal energy increase or *mechanical work* in the future work of Coriolis, more than a hundred years later—here we should not oversee that eighteenth century scientists had no notion of what we call mechanical work. Living forces ("vis viva" $= mv^2$) in the sense of Leibniz are the results of the summation of this work over elementary pulses in time, so that, noting that if $f = p = mv$, then $pdx = mv \cdot vdt = mv^2 dt$ we can write an expression of the type

$$\int p\, dx \propto mv^2. \tag{2.1}$$

Depending on the observer this can be viewed as a mathematical theorem or a principle of conservation. According to Capecchi [11, p. 201] it is practically universally acknowledged that what we now call the principle of virtual work finds its origin in John's considerations. This is what Lagrange was referring to as the *principle of virtual velocities* in 1788 [29–31]. As a matter of fact, Bernoulli refers to a law of virtual work in a letter of 1714 to a naval engineer named Bernard Renau d'Elizaray (Capecchi [11], p. 203), and he defines well his notion of virtual velocities in exchanges of letters with Varignon in the period 1714–1715, and in his discourse "on the laws of the communication of motion" of 1728 (Chap. 3, p. 20) reproduced in his collected works [8, vol 3]: "The virtual velocity is the element of velocity that every body gains or loses, of a velocity already acquired during an infinitesimal interval of time, according to its direction" (English translation from the French by Capecchi).

We shall return to John Bernoulli's works in fluid mechanics in the next two sections mostly devoted to his son, Daniel.

2.2 "Bernoulli's Theorem" by Daniel Bernoulli

Daniel Bernoulli (1700–1782) is the second son of John. He is universally known for his "theorem" although he did many other works in hydrodynamics, mathematics, statistics and physics. He studied in Basel, Heidelberg and Strasbourg. Not so strangely for the time, he obtained a doctoral degree in anatomy and botany (1721). He spent most of his professional career in St. Petersburg (for nine years) and Basel (for almost fifty years). He was a close friend of Euler.

Some unavoidable words must be said about this famous theorem because it relies on consequences of the *conservation of energy*. It happens that we know a letter from Daniel Bernoulli to Christian Goldbach (1690–1764) dated from Moscow July 17, 1730 (reproduced in pp. 220–221 in Truedsell [39]), where Daniel develops in French the arguments leading to his "theorem". He appeals there to the conservation of energy to show that there exists a relation (his notation; this looks like a differential equation but it is not)

$$v\,\frac{dv}{dx}=\frac{a-v^2}{2c},\tag{2.2}$$

where $v_0=\sqrt{a}$ is a reference velocity, the acceleration of gravity is equal to 1/2 in the chosen system of units, and the quantity *cvdv*/*dx* is the accelerating force (i.e., an internal pressure per unit density). This is established by relating the speed of a steady flow of an incompressible fluid (water) in a tube to the pressure this fluid exerts on its walls. The increment *dv* is the impulsive increment of speed of the water flowing with speed *v* in traversing the elementary distance *dx* as “if the tube, supposed horizontal, were suddenly to dissolve into thin air at point *x*” (Truesdell’s words). In modern notation this can be transcribed as the well known equation

$$p+\frac{1}{2}\rho\,v^2+\rho\,g\,h=const.,\tag{2.3}$$

thanks to the mentioned identification with internal pressure (a notion that Daniel Bernoulli did not possess). There is no need to emphasize the importance of Eq. (2.3) in hydrodynamics (think of the Venturi effect: when velocity of the flow increases, then the pressure falls) and aerodynamics where the theorem is used in the proof of the existence of lift on an airfoil. This is beautifully exposed by Anderson [2]. This was not the only result of Daniel who produced a fundamental book on hydrodynamics written in 1734 [9]. This scientific achievement caused a burst of envy and jealousy from the shameless John Bernoulli against his own son, publishing a competitor book with the title *Hydraulica* in 1739. He anti-dated the date of writing this opus to 1732 to pretend to (a false) priority! However, it must also be acknowledged that John’s book has many merits. In particular, this book offers the first successful use of the *balance of forces* to determine the motion of a deformable body. This was possible for John because he had recognized that “the fluid on each side of an infinitesimal slice pressed normally upon that slice, with a varying force which was itself a major unknown” [39, p. 121]. With this we are very close to the notion of *internal pressure* and a concrete view of *contiguity* of action in continuum mechanics in a line that both Euler in the period 1749–1752 and Cauchy in the period 1823–1828 will expand. Also, John was the first to practically give the modern form (2.3) to his son’s theorem. Bernoulli [10] also discussed the principle of living forces at the time of the death of his father. To be on the safe side, the paranoid John edited himself his collected works in 1742 [8].

2.3 D’Alembert and the Metaphysical Notion of Force

Daniel Bernoulli’s work also triggered some envy from a young French philosopher-mathematician, Jean Le Rond d’Alembert (1717–1783). Thus in 1744, this gentleman, well educated in the best college in Paris, but mostly self-taught in mathematics, published his own book on the emerging fluid mechanics [16].

According to Truesdell [39, p. 227], this entry of a newcomer in the field "added nothing to the subject".[2]

But several objective facts must be recorded. In contributions that rapidly followed this opus, d'Alembert (e.g., [17]) achieved correct partial differential equations for axially symmetric and plane flows (of the type now called irrotational flows). This is one of the first considerations of a two-dimensional motion of a continuum. He had already introduced for the first time the notion of *partial differential equations* in a previous work of 1743 on the mechanics of a heavy hanging rope. As noticed by Truesdell [39, p. 228], d'Alembert does not speak of "pressure" but of "forces" that are viewed as "reversed accelerations". This fits well in d'Alembert's vision of reducing hydrodynamics to hydrostatics in accord with the general approach to mechanics that he introduced in his celebrated *Treatise on dynamics* [15] written when he was hardly 26. This treatise is a much

[2] Because Euler was his great hero, Truesdell in all his writings (e.g., [39, 40]) has a tendency to belittle the contributions of d'Alembert on whatever subject, sometimes with innuendos verging on defamation. This kind of *idée fixe* is not supported by many other writers, in spite of the admitted unclear, confusing, and somewhat verbose, and probably too hastily written, works of d'Alembert. But he often pioneered new ideas and wrote these works when he was only 26 or 27 years old, and without much formal education in the field. Furthermore, d'Alembert himself had a tendency to grant a higher value to his contributions to the belles-lettres, among the latter, discourses [19] at the *Académie Française* in his mature age when he has become "perpetual" secretary of this academy. His preliminary discourse (i.e., Introduction) of 1751 to the great Encyclopaedia is a remarkable literary piece. Naturally, the French have a tendency to overestimate d'Alembert. I remember receiving as a prize, when in primary school, a book written at the end of the nineteenth century, the purpose of which was to introduce the youth to the oeuvres of the most representative French scientists (they preferably had to be republican and atheist). This included Lagrange, Lazare Carnot, Chevreul, obviously Pasteur, Marcellin Berthelot, and d'Alembert as a first exemplary contributor. My favourites were Pasteur (he saved young kids from the rabies—I did not know he had also improved the production of beer) and the chemist Chevreul (the latter for three reasons: he died when he was 103; he was born in my native city, and he brought light to the house of many people by inventing the artificial stearic candle). I could not understand what was said of d'Alembert. But many years later I discovered that I was scientifically descending in a straight line from d'Alembert according to Mathematics Genealogy. Moreover, in response to the granting of the name of Diderot to our twin University (Paris VII) I decided to call d'Alembert our Institute of Mechanics, Acoustics and Energetics at the University of Paris VI—that had gained the good name of Pierre and Marie Curie in the mean time. Very little is known about d'Alembert's formation in mathematics. From available documents (in particular, one written by d'Alembert late in his life, his personal notes preserved at the Academy of Sciences in Paris, and studies by specialists of Diderot and d'Alembert such as Pfeiffer [38], it is thought that d'Alembert had only a limited formal education—given in Latin by a rather incompetent teacher - in mathematics while an adolescent at Collège Mazarin (also called "Collège des Quatre Nations"). He was caught by an irrepressible taste for mathematics while he was studying law and then started aborted studies in medicine. It is surmised that he studied by himself L'Hôpital, Varignon, John Bernoulli, Maupertuis, and the Principia of Newton (but this is mostly geometrical while d'Alembert will become an analyst). He thus submitted his first (unpublished) memoir on mathematics to the Academy in 1739 (aged 21), and then published in a row famous lengthy works on the bases of dynamics (his famous treatise at age 26), fluid mechanics, the theory of winds, and the vibrations of strings in an interval of some seven years! Not bad for an autodidact, notwithstanding Truesdell's appraisal.

discussed matter, and obviously demolished by Truesdell as incomprehensible by him (probably himself a too much Newtonian adept to start with). It is of interest to note the full (ambitious) title and sort of summary of the treatise (translation from the French): “Treatise on dynamics, in which the laws of equilibrium and motion of bodies are reduced to the smallest number and are proved in a new way, and where a general principle for finding the motion of several bodies which react mutually in any way” is given. Not a bad abstract!

It is true that, following Leibniz, d’Alembert thinks of the notion of “force” (especially in Newton’s gravitation theory) as an obscure, metaphysical and unnecessary primary notion. Thus all is first to be granted to kinematics. With this he is banishing entirely the Newtonian view of mechanics (even the name of Newton is not mentioned). This agrees well with John Bernoulli’s introduction of the principle of virtual work, where forces are nothing but coefficients of virtual variations. But what d’Alembert added was a re-interpretation of inertial forces as the negative of “forces”, thus giving to dynamics the form of statics on the basis of a principle of virtual “powers”. The problem with this author is that, as correctly remarked by Truesdell, he is extremely difficult to read. We thus admire the thorough analysis and deep interpretation that Jouguet [28] could give of the contents of the Treatise on dynamics. To give a taste to the reader we give in Appendix A attached to this essay an English translation of some parts of d’Alembert’s introduction to his treatise. This text shows the central thinking, methodology, and ambition—not always truly satisfied in the treatise—of the project. Anyway, we should remember d’Alembert’s principle according to which: “If we consider a system of material points connected together so that their masses can acquire different respective velocities whenever they move freely or altogether, the quantities of motion gained or lost in the system are equal”. A flavour of d’Alembert’s statement of his principle is given in Appendix B. It is clear that the modern interpretation of such a text is a challenge for most of us. This is nonetheless what is achieved by Jouguet [28, pp. 197–202]. With this d’Alembert provided the bases on which Lagrange was going to build his grand scheme of mechanics. He had also an interest in the theory of music. Perhaps that he was distracted by, interested in, too many fields to compete creatively with his contemporary Euler in mechanics.

Note that the famous *d’Alembert paradox* about the vanishing dragging force on a cylinder placed in a flowing perfect fluid was proved in 1750. This is contrary to common experience. A resolution could be given only with the introduction of discontinuities in the flow field and the notion of wake.

In solid mechanics, d’Alembert also introduced the notion of space-time partial differential equation yielding the wave equation and its paradigmatic solution in 1746. In his later years he published in the form of contributions to some collected works (e.g., [18, 20]). He had much influence on Lagrange whom he mentored in Paris; he obviously discussed with Euler as they often disagreed on many particular points. But, contrary to John Bernoulli, he always remained a gentleman in spite of controversies for which he seems to have developed a special gift for getting involved in. His two main disciples were Pierre-Simon Laplace in analysis and probabilities and Nicolas de Condorcet in economics and statistics. In mathematics he is known for the «theorem of d'Alembert» that says that "any polynomial of degree n with complex coefficients has exactly n (not necessarily distinct) roots in the set of complex numbers" (the theorem was really proved by C.F. Gauss in the nineteenth century) and for his study of the convergence of numerical series. In astronomy, he studied the three-body problem and the precession of equinoxes in 1749. In this he is a precursor of his disciple Laplace.

2.4 The Notion of Internal Pressure and the Fundamental Equations of Hydrodynamics

We certainly agree with Truesdell that Leonhard Euler (1707–1783), the greatest mathematician of the eighteenth century, stands much above d'Alembert in both mathematical creativity and physical intuition. This Euler proved in practice by developing expertly the calculus of variations, solving so many problems, and presenting a theory of fluids that remains intact till our present time, at least for perfect fluids. We cannot peruse the whole work of Euler in the mechanics of continua. This has been achieved by more knowledgeable specialists (among them Truesdell who edited many of the original works in the edition by Orell Füssli Verlag, Basel, and Birkäuser, Basel, in a total of 73 volumes). We are satisfied with a focus on some problems of fluid and solid mechanics.

For us the main two ingredients in fluid mechanics are the notion of *internal pressure*, and the construction of the *field equations* for perfect fluids on the understanding that the notion of *field* has really been introduced. This is indeed the case. For the first ingredient, we may conjecture with Truesdell [39, p. 230] that Euler, while residing at the Berlin Academy since 1741 to become later on its president after the death of Maupertuis, carefully read the prize essays submitted by d'Alembert in 1746 and 1750. This "gave him the final impulse to the creation of the general hydrostatics and hydrodynamics". This was also much influenced by the recent progress in the theory of hydraulics by the Bernoullis, father and son. Thus *pressure* was seen as the action from all sides and from neighbouring elements of fluid on an isolated element of fluid (a "particle"). In modern term, it is *isotropic* and, with Euler, will be viewed as a normal force acting on an element of surface. The notion of *contiguity* is thus definitely reached. Furthermore, it becomes a true *field* that depends on both space and time in the general case of dynamics.

The second argument requires from Euler to think in Newtonian terms to write in 1750 a general *principle of linear momentum* (for whatever body or ensemble of "particles" and not only for a point particle like in Newton), principle that we write here in the condensed form

$$\mathbf{F}(B) = \dot{\mathbf{M}}(B) \tag{2.4}$$

for any portion B of a body, where $\mathbf{F}$ is the resultant force, $\mathbf{M}$ is the total linear momentum, and a superimposed dot denotes the time-rate of change. This was expressed by Euler in differential form, a form that suited the mechanics of continua. Now, we cannot do better than reproduce the original text of Euler (Paragraphs XV and XVI of [25], in the old French orthography, but read just like actual French once the writing conventions are known):

> XV. Maintenant je po*ſ*e pour abréger (there were misprints, here corrected, in two of these component equations. GAM):
>
> $$X = \left(\frac{du}{dt}\right) + u\left(\frac{du}{dx}\right) + v\left(\frac{du}{dy}\right) + w\left(\frac{du}{dz}\right); Y = \ldots; Z = \ldots, \qquad \text{(a)}$$
>
> & l'équation différentielle qui détermine la pre*ſſ*ion p e*ſ*t
>
> $$\frac{dp}{q} = P\,dx + Q\,dy + R\,dz - X\,dx - Y\,dy - Z\,dz\,, \qquad \text{(b)}$$
>
> dans laquelle le tems r est *ſ*uppo*ſ*é con*ſ*tant. Or l'autre équation tirée de la continuité du fluide est:
>
> $$\left(\frac{dq}{dt}\right) + \left(\frac{d.qu}{dx}\right) + \left(\frac{d.qv}{dy}\right) + \left(\frac{d.qw}{dz}\right) = 0, \qquad \text{(c)}$$
>
> & ce *ſ*ont les deux équations qui contiennent toute la Théorie tant de l'équilibre que du mouvement des fluides, dans la plus grande univer*ſ*alité qu'on pui*ſſ*e imaginer.
> XVI. Lorsqu'il e*ſ*t question de l'équilibre, on n'a qu'à faire évanouïr les trois *viteſſes u*, v & w, & puisque alors les quantités X, Y, & Z, évanouï*ſſ*ent aussi, toute la Théorie de l'équilibre des fluides est contenuë dans ces deux équations:
>
> $$\frac{dp}{q} = P\,dx + Q\,dy + Rdz, \qquad \text{(d)}$$
>
> le tems t étant con*ſ*tant, &
>
> $$\left(\frac{dq}{dt}\right) = 0. \qquad \text{(e)}$$

Today's student easily identifies q with the density ρ, (X, Y, Z) as the components of the acceleration and (P, Q, R) as the components of the body force per unit mass, so that Eqs. (a) through (d) are none other than

$$\gamma = \frac{d\mathbf{v}}{dt} := \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}, \qquad \text{(a}'\text{)}$$

$$\rho\,\frac{d\mathbf{v}}{dt} = \rho\mathbf{f} - \nabla p, \qquad \text{(b}'\text{)}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0, \qquad \text{(c}'\text{)}$$

and

$$\nabla p = \rho \mathbf{f}, \tag{d'}$$

in modern intrinsic notation. This says it all. But what about boundary conditions required by this set of equations for complete solution in space? Euler had already pondered this matter before and the only possibility is that pressure corresponds to a *normal* force acting on a surface. We will need the ingenious work of Cauchy in the period 1822–1828 to offer a larger possibility, although the solid case to be discussed also after Euler already hints at a more general situation.

This discussion on fluids may make the reader believe that Euler had no notion of a tangential force. But this is not true because Euler himself dealt with this notion in a problem that had been examined by James (Jacob) Bernoulli a long time before, the elastica (flexible elastic one-dimensional object). On considering the equilibrium of a cut out part of this elastica Euler found that a *shear force* is generally necessary in addition to tension to maintain the balance of this element. He obtained in 1771 the correct governing system of dynamical equations that require considering not only curvature χ (as done by Bernoulli) but simultaneously both normal V and tangential T components of the stress in the form (where s is the arc length)

$$\frac{dT}{ds} - V\chi = -B_t + \rho\ddot{x}_t, \quad \frac{dV}{ds} + T\chi = -B_n + \rho\ddot{x}_n, \quad \frac{dM}{ds} - V = 0. \tag{2.5}$$

Here M is the bending moment and the B's are the components of an applied (body) force. In modern intrinsic notation (2.5) reads (cf. [3])

$$\frac{d\mathbf{S}}{ds} + \mathbf{B} = \rho\,\ddot{\mathbf{x}} \tag{2.5'}$$

with a stress "vector" defined by $\mathbf{S} = T\mathbf{t} + V\mathbf{n}$. More remarkably, Euler in 1774 also proposed for this problem a principle of contiguity (contact action), which we can express in this notation as

$$\mathbf{S}_+ = -\mathbf{S}_-, \tag{2.6}$$

where the plus and minus signs refer to the effect of the material on the opposite sides at the point of junction. This obviously is equivalent to the natural boundary condition (here a junction or continuity condition). Equations (2.5′) and (2.6) anticipate the equations of Cauchy for both the field and (natural) boundary equations. The engineer Charles Augustin de Coulomb (1736–1806) also recognized the notion of shear stress at about the same time as Euler by examining the effect of "both normal and tangential forces acting on the cross section of a beam subject to transverse terminal load" (cf. [39, p. 236]). Anyhow, the reasoning of Euler yielded what is now known as the Euler-Bernoulli theory of beams with a bending moment given by $M = -EId^2w/dx^2$, while plane sections remain plane and there is no shear deformation.

2.5 Linear Momentum and Moment of Momentum: Newtonian Versus Variational Formulations

Equation (2.4) reflects the adoption by Euler of Newton's viewpoint concerning the law of linear momentum. But for a rigid continuous body or a system of rigidly linked point particles (with invariant distances between them), one needs to account for a possible mechanical response in rotation. This is a materialization of Newton's third law ("to each action there is always an equal reaction"). It is Euler [24] who formulated this law of *moment of momentum*, which dynamically involves the notion of *angular momentum* along with the notion of inertia about a certain centre of mass. That is, in addition to a law of the form (2.4) we will have to satisfy a law

$$\mathbf{C} = \dot{\mathbf{J}}, \tag{2.7}$$

where $\mathbf{C}$ is the total torque acting upon the body and $\mathbf{J}$ is the total moment of momentum or angular momentum, both being taken with respect to the same fixed point. Both laws (2.4) and (2.7) are valid for discrete systems or continuous bodies. They constitute the *laws of Mechanics of Euler*; he correctly set forth these as applicable to any part of every body in a memoir published in 1776 [26]. It is the evaluation of $\mathbf{J}$ which requires the introduction of the notion of rotary inertia about the mentioned fixed point. According to Truesdell [39, p. 129], Euler's principle of moment of momentum remains even today a "subtle and often misunderstood" (by physicists, not by Truesdell!) law. However it is of universal and everyday use (think of the orientation of satellites seen as rigid bodies in the first approximation, and the application of this law in the mechanics of robots with the introduction of appropriate kinematic descriptors including, beyond Euler's angles, Cayley-Klein parameters, quaternions, orthogonal matrices, and spinors). This Eulerian mechanics of rigid bodies will be perfected to the utmost by scientists such as Lagrange, Poinsot, Poisson and others. We do not deal further with this matter but note that the symmetry of the Cauchy stress in most of modern continuum mechanics is a consequence of the law (2.7) [41].

We cannot close this perusal of Euler's formidable contributions to mechanics without evoking the fact that, albeit a strict Newtonian from many viewpoints [cf. Eq. (2.4) above, and also (2.7) that complements the original Newtonian view], Euler is also one of the true creators of the calculus of variations, which he never hesitated to use in specific problems (e.g., in the buckling problem). Because of this, he is at the root of the variational approach of Lagrange (see next section).

2.6 Calculus of Variations and Analytical Mechanics: Lagrange

Joseph-Louis Lagrange or Giuseppe Ludovico Lagrangia (1736–1813) is neither Newtonian nor Leibnizian or d'Alembertian; he is above all—if we are allowed the joke—Italian (and perhaps Frenchman by adoption), and also a shy and very quiet man who, contrary to some of his colleagues, succeeded to live through the French revolution without any political involvement. He disliked getting involved in controversies and, according to J.-B. Fourier (of series and heat-conduction fame)—a demanding student in the first year of the *Ecole Normale*—, he always answered by a non-committed "je ne sais pas" (I don't know) to all questions asked by students. But he is only second to Euler in the class of mathematicians of the eighteenth century. His creativity blossomed in all fields of mathematics, mechanics of fluids, solids, and celestial mechanics. Apart from C. F. Gauss (1777–1855), only Cauchy may match his inventiveness in mathematics in the successive Revolution, Empire and Restauration periods (1780–1830). Because of limited space, here we can only focus on some of his contributions to mechanics. His canonical equations of motion in arbitrary systems of coordinates are so beautiful and powerful in all of physics[3] that it is often believed that they were God-given like the Holy Scriptures (but with publication via Lagrange's hands of the celebrated book of 1788 [31]). The real story deviates from this ideal vision in the sense that Lagrange was strongly influenced by, among others, John Bernoulli, Maupertuis, d'Alembert, and Euler. In modern terms, Lagrange's works do not create a new paradigm (in the sense of Thomas Kuhn), neither do they provoke an epistemological rupture (according to the expression of Gaston Bachelard). Furthermore, contrary to the principle of virtual velocities by d'Alembert, his famous equations are restricted to the case of non-dissipative processes (cf. the discussion in [34]) at least until the introduction of a dissipation potential by Lord Rayleigh.

The genesis of Lagrange's equations requires some re-construction, which was more or less told by Lagrange himself in the long historical introduction to the two parts of his book. This greatly simplifies our task. In fact, Lagrange dutifully produces in his introduction a rather extended history of the developments of mechanics through the ages, starting with Archimedes, Stevin, Galilei, Descartes, Huyghens, and Roberval, of course on the basis of what was known in his time (the mechanics of the middle ages will be unearthed and thoroughly examined by Pierre Duhem only at the end of the nineteenth century). Because of the

[3] Some physicists place Lagrange equations simultaneously at the top and the base of all of mechanics; This is the case in the first volume of the celebrated course of theoretical physics by Lev D. Landau and Evgeni F. Lifshitz in the former Soviet Union—with many foreign translations—and the standard, much admired, successful and continuously reprinted book on "Classical Mechanics" by Herbert Goldstein in the USA—first edition 1950—, but unjustly criticized by Truesdell [40, pp. 144–147] as not Newtonian enough. James C. Maxwell was a great admirer of Lagrange and did not hesitate to use Lagrange's formalism to study self and mutual inductance in his treatise on electricity ad magnetism.

composition of his book, Lagrange considers separately the cases of statics and dynamics. Concerning the first half of the eighteenth century, he has to pay his tribute to the works of John Bernoulli, Maupertuis, and d'Alembert. We give in Appendix C the English translation of what we think to be the most important statements of this introduction, which provides a magisterial overview and analysis of the principles of mechanics in the eighteenth century.

In the case of statics, Lagrange emphasizes the role played by the "*principle of the lever*" (in some sense, an ancestor of the principle of virtual displacement) and that of the *composition of forces* ("from which one concludes that any two powers (he means "forces") that act simultaneously on a body (Lagrange means a "point"), are equivalent (equipollent) to one force that is represented, in magnitude (Lagrange uses the word "quantity") and direction, by the diagonal of the parallelogram of which the sides represent the magnitude and direction of the two given powers" [31, p. 10]). Anyway, Lagrange clearly concludes that the principle of virtual work as given by John Bernoulli is the most powerful tool. At the end of his study of statics, he develops the hydrostatics of incompressible fluids.

In the case of dynamics, Lagrange thoroughly scrutinizes the various principles proposed since Newton, avoiding none (see Appendix C). In practice he will combine the principle of virtual work with d'Alembert's astute proposal to view inertial forces as negative applied forces, i.e., a kind of reformulation of Newton's law appropriately multiplied by virtual velocities and summed over all bodies composing the system. The step that will glorify this work among physicists is the consideration of a kinematical description by means of generalized systems of coordinates (see Fourth Section, p. 282 on, in which we witness for the first time the appearance of the functional derivative—in page 285). There follows from this the introduction of the quantity $T - V$, where T is the kinetic energy and V is the potential of interacting forces (this will later be called a Lagrangian L) and, using an argument on the homogeneity of functions (due to Euler), he proves the conservation of the integral $T + V$, which contains the principle of living forces. By invoking the calculus of variations he further shows the validity of Maupertuis' principle of least action. In a sense, with this work Lagrange has unified in a construct typical of an analyst all what concerns the mechanics of systems of points in the absence of dissipative processes. He also provides interesting applications to the oscillations of a linear system of bodies (pp. 320–380).

Lagrange died in 1813, but he had already prepared an extension of his *Analytical mechanics* in a second volume. This was completed (from Lagrange's papers) and edited by J. Binet (1786–1856), G. Prony (1755–1839) and S. F. Lacroix (1765–1843). This is reproduced in Lagrange [33] as reprinted from the third and fourth editions with comments and additions by Joseph Bertrand (1822–1900) and Gaston Darboux (1842–1917)—with more than sixty pages of notes by V. Puiseux (1820–1883), J.-A. Serret (1819–1885), O. Bonnet (1819–1892), J. Bertrand, A. Bravais (1811–1863), and Lagrange himself. What is of highest interest for us here is, apart from many solutions and applications to rotational motions and celestial mechanics, the development of Lagrange's view of the fluid mechanics of incompressible and compressible fluids (pp. 250–312). Readers will not be surprised that

Lagrange adopts here what will become known as the "Lagrangian" kinematical description (p. 253 on). That is, he introduces the initial coordinates (a, b, c) of a fluid "particle" to be later in time at placement (x, y, z). Thus,

$$x = \bar{x}(a, b,\ c;\ t), \text{etc.}$$

He thus write the continuity equation as (in modern notation)

$$dm \ = \ \rho \, dx \, dy \, dz \ = const. \quad \text{or} \quad \rho_0 = \rho J.$$

He introduces a scalar "Lagrange" multiplier to account for incompressibility. He obtains thus the three equations of balance of linear momentum equations in the "Lagrangian" format. But he also shows how to revert to an Eulerian description (Equation F in p. 264). He is quite honest in admitting that it may be easier to deal with Euler's format (p. 265). He does the same for compressible fluids where pressure will now be a constitutive quantity.[4] He concludes this second volume with simple wave problems in one dimension (e.g., in a flute or an organ pipe).

In this book and previous works (e.g., in Mémoires sur le calcul des variations, Torino, 1760), Lagrange greatly contributes to the definite form of what may be called the δ-calculus, that is, the calculus of variations. As already mentioned, this was initiated by Euler in the period 1755–1760 in his study of maxima and minima. This author even introduced what is now called the *Euler-Lagrange equations*, of which the above recalled Lagrange equations are a special case. Dahan-Dalmenico [14] has critically examined this contribution of Lagrange to one of the most useful and efficient tools in theoretical physics.

It is hard not to express admiration in view of Lagrange's book. This is a true monument that is beautifully organized and practically happily concludes the development of the principles of mechanics through the eighteenth century. Lagrange has a style of his own, being fully analytical, somewhat formal (even to the taste of Truedsell [39, p. 132]) and using no argument of geometry. He is even proud of the fact that no figures illustrate his exposition (although a few illustrations may have been welcomed). Lagrange in fact introduced a privileged tool for the "algebraïzation" of Mechanics. The second important point is that Lagrange, after some previous works by Clairaut [13] and Maupertuis [37] but above all with his introduction of generalized coordinates, really inaugurates an era where the recognition of *invariance* in mathematical physics has become fundamental. No wonder that perhaps with some exaggeration W. R. Hamilton called the *Mécanique Analytique* a "kind of scientific poem".

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[4] In the period 1825–1848, Gabrio Piola (1794–1850), an ardent supporter of his "compatriot" Lagrange, will follow the same line of approaching first the equations of motion for finitely deformable bodies (elasticity) in a reference configuration and then transforming them to the actual configuration for the sake of comparison with Cauchy's format of 1828. To do that Piola had to use what is now called an inverse "Piola transformation" of the stress (see [35]). From this we infer that Lagrange had already used a similar transformation (in a rather unidentified manner) for perfect fluids.

2.7 The Age of Reason: Conclusion and Things to Come

The period we spanned in this essay practically corresponds to what is called in history the *Age of Enlightenment* or the *Age of reason* (in French, the "*Siècle des lumières*", in German, the "*Aufklärung*"). What is usually meant by this denomination is a period in which one thinks of reforming society by using reason and to advance knowledge through the scientific method. Scientific thought is promoted together with a challenge of ideas that are too much grounded in tradition and faith. One of our heroes in this essay, d'Alembert, epitomizes the enlightened scientist who simultaneously wants to improve society and teach it through a pharaonic enterprise such as the production of the great "Encyclopédie" (ou *Dictionnaire raisonné des sciences, des arts et des métiers*) directed by him and Denis Diderot (1713–1784), and sold (by subscription) all over Europe in about 20,000 copies in spite of the bulk of thirty-five volumes. About a hundred "philosophers" (today we would say "intellectuals" of all kinds) contributed to this formidable enterprise, including Voltaire (1694–1778), J.-J. Rousseau (1712–1778), and Montesquieu (1689–1755). Many of the contributions on scientific subjects are due to d'Alembert himself.

The enlightenment influenced both American and French revolutions and inspired among others the American Declaration of Independence and the French Declaration of the Rights of Man and of the Citizen. Baruch Spinoza (1632–1677; a philosopher much appreciated by scientists such as Albert Einstein) and John Locke (1632–1704) were inspirations for this movement that spread all over Europe and European colonies in the Americas. In Germany, Immanuel Kant in 1784 tried to answer the question: "Was ist Aufklärung?" A partial answer is "Sapere aude" (dare to know). It has mostly to do with the advance of knowledge in all forms. From the scientific viewpoint, Newton may have sparked the original steps of the movement in the early 1700s. It is symptomatic that we technically concluded the present essay with the publication of Lagrange's "Analytic Mechanics" in 1788, just one year before the French revolution (perhaps not always the best realization of the Enlightenment). We can now summarize the achievements in mechanics in this remarkably active period with the following list:

- the formulation of integral calculus (introduction of the term "integral"; Jacob Bernoulli)
- the principle of virtual work (John Bernoulli, Lagrange [32])
- the parallelogram of forces (Varignon)
- Bernoulli's theorem (Daniel Bernoulli in 1730 [9])
- the general equations of hydraulics [7]
- the concept of shear stress (John Bernoulli, Euler)
- the principle of least action [36, 37]
- two-dimensional motion/partial differential equations [16]
- the wave equation (d'Alembert)
- d'Alembert's principle of virtual velocities [15, 20, 32]
- the notion of internal pressure as a field (D'Alembert; Euler [25])

- the fundamental equations of hydrodynamics [25]
- the principle of linear momentum [24]
- the equations of motion of rigid bodies (Euler)
- the principle of moment of momentum [24]
- the calculus of variations (John Bernoulli [6], Euler [23], Lagrange [29, 30])
- analytic mechanics [31].

In the transition period[5] of the French revolution, Lazare Carnot (1753–1823), both a successful politician (the "organizer" of the victory in 1792 as a kind of Minister of War and scientific adviser to the Convention) and an engineer-scientist by formation at the Military Engineering school of Mézières (the ancestor of the Ecole Polytechnique) pondered the principles of mechanics with a specific interest in their applications to mechanical "machines" (cf. [12]). Carnot was essentially a disciple of d'Alembert since in his book of 1803 (originally published 1783, p. 47) he wrote about "a metaphysical and obscure notion, that of force". He simply states the following alternative [12, Introduction]: "There are two ways to envisage Mechanics, in its principles. The first one is to consider it as the *theory of forces*, i.e., the causes that impress the motions. The second one is to consider it as the *theory of movements* in themselves. Carnot prefers the second avenue. A thorough examination of the principles as enunciated by Carnot is given by Jouguet [28, pp. 72–77, 203–210]. Since Carnot's essay was originally published in 1783, we may consider that its contents somewhat anticipated Lagrange, but certainly not with the same acuity and success. Jouguet [28, pp. 203–210] also discussed the presentation of the principles of mechanics by Fourier [27] in his "Mémoire sur la statique…". Other scientists who pondered the principle of virtual velocities in the early nineteenth century are André-Marie Ampère (1775–1836) and Louis Poinsot (1777–1859) both in 1806 (cf. [1]).

The things to come in the nineteenth century were:

- the general notion of stress (Cauchy in the period 1822–1828) to be perfected by Piola, Kirchhoff and Boussinesq
- nonlinear deformations (Cauchy, Green, Piola, Kirchhoff, Boussinesq)
- the notion of mechanical work (Coriolis)
- the notion of thermo-mechanical couplings (Duhamel, 1837; F. Neumann)
- Heat conduction (Fourier)
- the creation of thermo-statics and thermo-dynamics (Sadi-Carnot, Kelvin, Clausius, Mayer, Helmholtz)
- the mathematics of elasticity (Lamé, Clebsch, Saint-Venant, Boussinesq, Love, the Cosserats)
- the equations for viscous fluids (Navier, Saint-Venant, Stokes)
- criteria of plasticity (Tresca, Lévy, Saint-Venant, 1870s)

[5] This is a transition period because it witnessed the creation of a new way to teach engineering sciences in the class room in combination with solid mathematical bases in the "*Grandes Ecoles*" of which the *Ecole Polytechnique* is the paragon (creation 1794).

- the anisotropy of deformable solids (Duhamel, F. Neumann, Voigt)
- visco-elasticity (Kelvin, Maxwell, Voigt, Boltzmann)
- the science of energetics (Rankine, Duhem, Mach)
- the notion of internal degrees of freedom (Duhem, the Cosserats)

All these are the objects of study of the remaining essays in this book.

Appendix A

Partial English Translation of J. Le Rond d'Alembert, (1743), Traité de dynamique, 1st Edition, David, Paris (1743) [after the reprint of the second augmented edition of 1758 by Gauthier-Villars Publishers, in two volumes, Paris, 1921] by Gérard Maugin. Translator's remarks are placed within square brackets in the main text. This is a verbatim translation without any ambition of literary prowess.

From the preliminary discourse

PP XVIII – XIX [on motion and extension]

....

The motion and its general properties are the first and principal object of mechanics; this science supposes the existence of motion, and we shall assume it as well advocated and recognized by all physicists. Concerning the nature of motion, philosophers, on the contrary, are much more divided on its definition. I admit that nothing seems more natural than to conceive of the motion as the successive application of the mobile to the different parts of infinite space, that we imagine as the locus of bodies [Here d'Alembert thinks essentially of point particles]; but this idea supposes a space of which parts are impenetrable and immobile, but everybody knows that the Cartesians (a sect which, indeed, does not exist any more), do not dissociate space from bodies and that they regard extension and matter as a unique thing. We must admit that starting from such a principle, motion would be the most difficult thing to conceive, and that a Cartesian would soon better come to negate its existence than to try to define its nature. Finally, how much absurd this opinion of philosophers may look, and with so little clarity and precision there are in the metaphysical principle on which they made the effort to lean, we will not try to refute it here; we shall be satisfied with remarking that, to have a clear idea of motion, we cannot avoid distinguishing, at least in thought, two kinds of extension: one, that must be regarded as impenetrable and that properly constitute the bodies; the other which, simply considered as extension, without examining on whether it is penetrable or not, provides the distance from one body to another one, and the parts of which can be considered as fixed and immobile, can serve to judge of the rest or motion of bodies. It is therefore always allowed to us to conceive of an infinite space as the locus of bodies, either real or hypothesized, and to regard motion as the transport of a mobile from one place to another one in space.

…………

PP XXVI – XXVII [on principles and the primary role of motion]

…………

If the principles of the force of inertia, of the compounded [or "composed"] motion and of equilibrium differ essentially from one another, as we cannot avoid to agree, and from another side these three principles suffice to mechanics, then this is to have reduced this science to the smallest possible number of principles that to have built on these principles all the laws of motion of bodies in any circumstances, as I tried to do in this treatise.

Concerning the proofs of these principles in themselves, the scheme I have followed to grant them all the clarity and simplicity that they likely deserve, has always been to deduce them from the consideration of the motion alone [this indicates the marked preference of d'Alembert for kinematics over the notion of forces], viewed in the simplest and clearest manner. All that we perceive distinctly well in the motion of a body is that it traverses a certain space and it takes a certain time to achieve this. This, therefore, is the only idea from which we must extract all principles of mechanics, when we want to prove them in a neat and precise manner; thus we shall not be surprised that, as a consequence of this way of thinking, I have, so to say, distracted the view from the *motive causes* [author's emphasis] to uniquely envisage the motion that produced them; that I have proscribed the forces inherent in the body, obscure and metaphysical beings, that are capable only to dispense darkness on a science so clear by itself.

It is for this reason that I thought appropriate not to enter the examination of the famous question of living forces.

…………….

PP XXIX- XXX [on equilibrium, virtual velocities, and *vis viva*]

…………….

But everybody agrees on the fact there is equilibrium between two bodies when the product of their masses by their virtual velocities, that is, the velocities with which they have a tendency to move, are equal from both sides. Thus, at equilibrium the product of mass by velocity, or what is equivalent, the quantity of motion, can represent the force. All agree also on the fact that in a retarded motion the number of overcome obstacles is like the square of the velocity, so that a body that closed a spring with a certain velocity will, for example, close four springs like the first one with a doubled velocity….From this the partisans of the living forces [*vis viva*] conclude that the force of bodies that actually move is generally the product of mass by the square of the velocity.

....

PP XXXVII – XXXIX [on the contents of the treatise]

....

Having given to the reader a general idea of the object of this work, it remains to say a word about the form that I thought appropriate to give to it. I have tried in the first part, as much as possible, to express the principles of mechanics in a form accessible to the people of the trade; but I could not avoid the use of differential calculus in the theory of varying motions [i.e., non-uniform motions]; I am forced to do that by the nature of the subject. Moreover, I have done in such a way as to encapsulate [obviously not a term in use at the time of d'Alembert] in this first part a sufficiently large number of things in a rather little space, and if I did dot enter any detail in the relevant matter, it is only because I remained focused on the exposition and development of the essential principles of mechanics, and having for purpose to reduce this work to what new ingredients it can contain, I did not thought appropriate to enlarge it with an infinity of particular propositions that can easily be found elsewhere.

The second part, in which I propose to treat of the laws of motion between bodies, is the largest part of the work. That is what led me to give to this book the title of "Traité de dynamique". This name, that properly signifies the science of powers or motive causes, could at first seem inadequate since I envisage Mechanics rather like the science of effects than that of causes [this is the most important point for d'Alembert; exit the notion of force to start with]; nonetheless as the word "*Dynamics*" is very much used among scientists to designate the science of the motion of bodies, which act among themselves in whatever way, I thought to keep the term, in order to announce to geometers by the very title of this treatise that I envisage principally to aim at perfecting and enlarging this part of Mechanics. As it is no less curious than difficult, and relevant problems compose a very large class, the most famous geometers have particularly dealt with it for the last few years; but they succeeded until now to solve only a small number of problems of this class, and only in particular cases. Most of the solutions they provided to us rest on principles that nobody has ever proved in a general manner, such as, for example, that of the living forces. I therefore thought to spend some time on the subject, and show how to solve the questions of dynamics by a unique very simple and direct method which consists only in the already above mentioned combination of the principle of equilibrium and of the compounded motion. I show the exploitation of this in a small number of selected problems, some already known, and others entirely new, and finally others even badly solved by the best mathematicians [notice that d'Alembert is rather avaricious with citations; he does not give names].

The elegance of a solution to a problem consisting above all in exploiting only a few direct principles, one should not be surprised that the uniformity that prevails in all my solutions, and that I have principally in view, renders them a little longer than if I had deduced them from less direct principles. The proof that I would have been obliged to give of these principles would indeed have distracted me from the brevity that I had searched by using them, and the largest part of the present book would have reduced to a shapeless accumulation of problems that did not deserve to see light, in spite of the variety that I tried to expand and the difficulties that accompany each of them.

Also, as this second part is mostly aimed at those who, already learned in differential and integral calculus, have become familiar with the principles established in the first part, or are already familiar with solutions of problems known and ordinary in Mechanics, I must tell that, in order to avoid circumlocutions, I have often used the obscure term of *force* [that the author strongly dislikes], and other terms that are commonly used when treating the motion of bodies; but I never pretended to attach to these terms more ideas than those which result from the principles that I proved, either in the [introductory] Discourse or in the first part of this treatise.

Finally, from the same principle that leads me to the solution of all problems of Dynamics I deduce also several properties of the centre of gravity, some of them entirely new, the others having only been proved in a vague and obscure manner, and I conclude this book with a proof of the principle that is commonly called the *conservation of living forces* [the theorem of the kinetic energy].

Appendix B

English Translation of the "principle of d'Alembert" as given by him in his Traité de dynamique, 1st Edition, David, Paris (1743) [after the reprint of the second augmented edition of 1758 by Gauthier-Villars Publishers, in two volumes, Paris, 1921, pp. 82-83] by Gérard Maugin. Translator's remarks are placed within square brackets in the main text. This is a verbatim translation without any ambition of literary prowess.

Definition. — In what follows I shall call motion of a body [here he means a point] the velocity of this same body considered on account of its direction, and call quantity of motion, in the ordinary sense, the product of mass by velocity.

General problem. — Let a system of bodies be disposed in any manner with respect to one another, and let us suppose that we impose to each of these bodies a particular motion that it cannot follow because of the action exerted by other bodies: find out the motion that each body will take.

Solution. — Let A, B, C, &c. the bodies that compose the system, and suppose that we imposed motions [here d'Alembert means velocities] a, b, c, &c that are forced, because of the mutual action, to change in the motions $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, &c. It is clear that we can regard the motion a impressed on body A as composed of the motion $\mathbf{a}$ that it took and another motion α; that we can equally regard the motions b, c, &c. as composed of motions, $\mathbf{b}$, β, $\mathbf{c}$, γ, &c, from which there follows that the motion of the bodies A, B, C, &c., among them would have been the same if, instead of giving impulses a, b, c, &c., one would have given simultaneously the double impulsions $\mathbf{a}$, α; $\mathbf{b}$, β; $\mathbf{c}$, γ; &c. But, by superimposition, the bodies A, B, C,

&c. have by themselves taken the motions **a**, **b**, **c**, &c. Therefore, the motions α, β, γ, … must be such that they do not disturb the motions **a**, **b**, **c**,…, i.e., if the bodies have received only the motions α, β, γ, &c. these motions should have destroyed themselves mutually and the system would have remained at rest.

From this there results the following principle to find out the motion of several bodies that interact among themselves: *Decompose the motions a*, *b*, *c*, &c *impressed on each body, each in two other motions* **a**, α; **b**, β; **c**, γ; &c *that are such that that if we had imposed to the bodies only motions* **a**, **b**, **c**, &c, *they could have conserved their motion without reciprocal hindrance and that we had impressed only motions* α, β, γ, &c. *the system would have remained at rest; it is clear that* **a**, **b**, **c**, &*c are the motions that the bodies will take by virtue of their action*. Q. E. D.

Appendix C

Partial English translation of J.-L. Lagrange (1788), Méchanique analitique, 1st Edition, Veuve Desaint, Paris (1788) [after the reprint of the second revised edition of 1811/1815 with new title "Mécanique analytique" in two volumes, Veuve Courcier, Paris; reprinted by Gabay, Paris, 1997, from the fourth edition edited by J. Bertand and G. Darboux] by Gérard Maugin. Translator's remarks are placed within square brackets in the main text. This is a verbatim translation without any ambition of literary prowess.

P 10 [on the principle of composition of forces]

………

The second fundamental principle of statics is that of the composition of forces. It is based on the supposition: that if two forces act simultaneously on a body [Lagrange means a "point"], along different directions, then these forces are equivalent to a unique force capable of impressing to the body the same motion as the two forces would give him separately. But a body that we make move uniformly along two different directions simultaneously necessarily takes a path that is along the diagonal of the parallelogram of which it would have followed separately the two sides by virtue of each of the motions. From this one concludes that any two powers [he means "forces"] that act simultaneously on a body [Lagrange means a "point"], are equivalent to one force that is represented, in magnitude [Lagrange uses the word "quantity"] and direction, by the diagonal of the parallelogram of which the sides represent the magnitude and direction of the two given powers". This is in what the principle called the *composition of forces* consists.

This principle suffices by itself to determine the laws of equilibrium in all cases; because, by composing all forces two by two, we must arrive at a unique force that is equivalent to all forces, and this, consequently, must vanish in the case of equilibrium, if there is no fixed point in the system. But if there is such a point, the direction of this unique force must go through this fixed point. This is what we can see in all books on statics, and particularly in the "Nouvelle Mécanique" by Varignon, where the theory of machines [e.g., pulleys] is deduced uniquely from the principle we just spoke about.

…………….

PP17-18 [on the principle of virtual velocities]

…………….

Now I come to the third principle, that of virtual velocities. By "*virtual velocity*" one must understand the velocity that a body in equilibrium is ready to receive, whenever equilibrium is broken, that is, the velocity that the body would really take in the first instant of its motion, and the considered principle consists in that the powers [he means "forces"] are in equilibrium when they are as the inverse ratio of their virtual velocities, estimated along the direction of these powers.

If we examine the conditions of equilibrium in the lever and other machines, it is easy to recognize this law that the weight and the power are always in the inverse ratio of the space that each of these can travel in the same time: however, it seems that the ancients were not aware of this.

……………

PP 20-21 [on the principle of virtual velocities cont'd]

The principle of virtual velocities can be rendered quite general in the following manner:

If any system of bodies or points as we want, is acted upon by any system of powers, is in equilibrium, and we give to this system any small motion, then by virtue of the fact that each point travels an infinitesimally small space that expresses its virtual velocity, the sum over powers each multiplied by the space that the point where it is applied travels along the direction of the same power, will always be equal to zero, regarding as positive the small distances followed in the direction of the powers and as negative those travelled in the opposite direction [Emphasis by Lagrange].

In so far as we know, John Bernoulli was the first to have perceived this great generality of the principle of virtual velocities, and its usefulness to solve problems of statics. This is what we witness in his letters to Varignon, dated 1717, and that the latter placed at the head of New Mechanics, a section entirely devoted to showing the truth and uses of this principle by the different applications.

The same principle then inspired the principle that Maupertuis proposed in the *Mémoires de l'Académie des Sciences de Paris* for the year 1740, under the name of the *law of rest* [Cf. Maupertuis, 1740], and that Euler developed further and rendered more general in the *Mémoires de l'Académie de Berlin* for the year 1751. Finally, it is the same principle that provides a basis for the principle that Courtivron [probably Marquis de Courtivron (1717–1785) specialist of optics] gave in *Mémoires de l'Académie des Sciences de Paris* for the years 1748 and 1749.

And, in general, I believe that I can venture to say that all general principles that could eventually be discovered in the science of equilibrium, will be none other than the same as the principle of virtual velocities, envisaged in a different manner, and of which they will differ only in their expression.

Moreover, this principle is not only very simple and very general by itself, but it also has the precious and unique advantage to be translated in a general formula which comprises all problems that we can propose concerning the equilibrium of bodies. We shall expose this formula in its whole extent: we shall even try to present it in an even more general manner than done until now, and give of it new applications.

As to the nature of the principle of virtual velocities, we must agree with the opinion that it is not sufficiently obvious by itself to be erected as a primary principle, but we can view it as the general expression of equilibrium, deduced from the previously mentioned two principles.

………….

PP 223-224 [on d'Alembert's dynamics and other principles of dynamics]

………….

The *Treatise on dynamics* by d'Alembert, which was published in 1743, put an end to kinds of challenges, by offering a direct and general method to solve, or at least to put in equations, all problems of dynamics that we can imagine. This method reduces all laws of motion of bodies to those of the equilibrium, and thus gives to dynamics the form of statics. We have already remarked that the principle used by Jacques [Jacob] Bernoulli in the search for a centre of oscillations, had the advantage to make this search depend on the condition of equilibrium of a lever; but it was reserved to d'Alembert to envisage this principle in a general manner, and to give to it all the simplicity and fruitfulness that it deserved.

If we impress to several bodies motions that are forced to change as a result of their mutual action, it is clear that we can regard these motions as composed of those they will really take and of motions that are destroyed; from what it follows that the latter must be such that the animated bodies be in equilibrium under their own motions.

This is the principle that d'Alembert gave in *his Treatise on dynamics* and of which he made a fruitful usage in several problems, and above all in that of the precession of equinoxes. This principle does not readily provide the equations needed for the solution of problems of dynamics, but it teaches how to deduce them from conditions of equilibrium. Therefore, by combining this principle with ordinary principles of the equilibrium of the lever, or of the composition of forces, we can always find the equations for each problem; but the difficulty to determine the forces that must be destroyed, as well as the laws of equilibrium among these forces, often renders cumbersome and laborious the application of the principle; and the solutions that follows are most often more complicated than those that would be deduced from less simple and less direct principles, as we can be convinced in the second part of the same *Treatise on Dynamics*.

…………..

In the first part of this work, we have reduced the whole of statics to a unique general formula that provides the laws of equilibrium for any system of bodies acted upon by as many forces as we like. Thus, we shall also reduce to a unique general formula the whole of dynamics; because, to apply to the motion of a system of bodies the formula of its equilibrium, it will suffice to introduce forces that arise from the variations of the motions of each body, and that must be destroyed. The development of this formula, on account of conditions that depend on the nature of the problem, will give all the equations needed for the determination of the motion of each body; and it will remain to integrate these equations, what is a matter of analysis.

One of the advantages of the relevant formula is to offer immediately the general equations which contain the principles or theorems known under the names of conservation of living forces, conservation of the centre of gravity, conservation of the moment of rotation or principle of areas, and the principle of the least quantity of action. These principles must be considered like general results of the laws of dynamics, rather than primary principles in this science; but as they are often employed as such in the solution of problems, we are due to speak about them here, by indicating in what they consist, and to whom scientists, in order to leave nothing untouched in this preliminary exposition of the principles of dynamics.

The first of these four principles, the one concerning the conservation of living forces, was found by Huyghens, but in a form slightly different from the one that it receives now. ……..

Until now this principle was viewed as a simple theorem of mechanics; but when John Bernoulli adopted the distinction made by Leibniz between dead forces or pressures which act without actual motion, and living forces which accompany this motion, as well as the measure of the latter by the product of mass and the square of the velocity, he saw in the principle in question only a consequence of the theory of living forces, and a general law of nature according to which the sum of the living forces of several bodies is conserved while these bodies act mutually on each other by simple pressures, and is constantly equal to the simple living force that results from the action of actual forces that move the bodies. He gave to this principle the name of *conservation of living forces*, and he used it with success to solve some problems which had not been solved before, and of which the solution looked difficult by direct methods.

…………

The great advantage of this principle is to provide immediately a definite equation between the velocities of bodies and the variables that determine their position in space; do that when by the nature of the problem, all these variables reduce to a unique one, this equation suffices to solve it completely, and this is the case of the problem of the centre of oscillations. In general, the conservation of living forces always provides a first integral of the different differential equations for each problem, what is of utmost usefulness on several occasions.

The second principle is due to Newton who, at the beginning of his *Principes mathématiques* [Principia mathematica], proves that the state of rest or motion of the centre of gravity of several bodies, is not altered by the reciprocal action of these bodies, whatever; so that the centre of gravity of the bodies that act on each other in any manner either by means of threads or levers, or by the law of attraction, etc, without any action nor any external obstacles, is always at rest or moves uniformly along a straight line.

Since then, d'Alembert has given to this principle a larger extent by making it known that if each body is acted upon by an accelerating force that acts along parallel lines or are directed toward a fixed point, and acts proportionally to the distance, the centre of gravity must follow the same line as if the bodies were free; to what we can add that the motion of this centre generally is the same as if all forces of bodies, whatever, were applied along its own direction.

It is visible that this principle is used to determine the motion of the centre of gravity, independently of the respective motions of the bodies, and so that it can always provide three definite equations relating the coordinates of bodies and the time, which equations will be the integrals of the differential equations of the problem.

The third principle is much less old than the preceding two ones, and seems to have been discovered simultaneously by Euler, Daniel Bernoulli [son of John] and d'Arcy [Patrice, Comte; 1725–1779], but in different forms. According to the first two of these authors, this principle consists in the fact that in the motion of several bodies about a fixed point, the sum of the product of the mass of each body by it velocity of circulation about this centre and by its distance from this centre, is always independent of the mutual action that the bodies exert on one another, and is conserved in as much there are neither action, nor external obstacles. Daniel Bernoulli gave this principle in the first volume of the *Memoirs of the Berlin Academy* published in 1746, and Euler gave it during the same year in the first volume of his *Opuscules*. ……The principle of d'Arcy [21], as given to the Académie des Sciences in the memoirs of 1647 (published in 1752) ……is a generalization of a beautiful theorem of Newton on the areas described by any centripedal forces. ….. He made of it a kind of metaphysical principle which he called the *conservation of action* to oppose it, or to substitute it to, the *principle of least action*…..

Now I come to the fourth principle that I call the *principle of least action* by analogy with the principle that Maupertuis [37] gave under this denomination….. This principle, only envisaged analytically, consists in the fact that in the motion of bodies interacting with one another, the sum of the masses by the velocities and by the travelled spaces is a *minimum*. This author deduced from it the laws of reflection and refraction of light as well as those of the shock of bodies in two memoirs in 1744 [36] and 1746 [37].

But its applications are much too specific to serve to establish its truth as a general principle. They indeed have something vague and arbitrary that can only render uncertain the consequence that we could deduce from the exactness of the said principle. That is, it seems to me that we should be wrong in granting to this principle the same status as to the other principles that we just exposed….. This principle, combined with that of the living forces and expanded along the rules of the calculus of variations directly provides all equations that are necessary for the solution of each problem; and from this is born an equally simple and general method to treat the questions that concern the motion of bodies; but this method is itself only a corollary of the method that is the object of the second part [on dynamics] of the present book, and has simultaneously the advantage to be extracted from the first principles of Mechanics.

References

1. Ampère AM (1806) Démonstration générale du principe des vitesses virtuelles dégagée de la considération des infiniments petits. J Ecole Polytechnique 13 ème Cahier
2. Anderson JD (1997) A history of aerodynamics and its impact on flying machines. Cambridge University Press, New York
3. Antman SS (1995) Nonlinear problems in elasticity. Springer, New York
4. Bernhard H (1983) The Bernoulli family. In: Wussing H, Arnold W (eds) Biographien bedeutender Mathematiker, Berlin
5. Bernoulli J (John) (1727) Theoremata selecta pro conservatione virium vivarum demonstranda et experimenta confirmanda. Comment Acad Scientarium Imperialis Petropolitanae 2:200–207
6. Bernoulli J (John) (1735) De vera notione virium vivarum earimque usu in dynamicis, ostenso per exemplum, propositumi. Comment Petropolit Acta Eruditorum, Lipsiae (Leipzig), p 21
7. Bernoulli J (John) (1739) Hydraulica, nunc primum detecta ac demonstrata ex fundamentis pure mechanicis. Basel
8. Bernoulli J (John) (ed) (1742) Opera Omnia (Complete works, Four volumes). Bousquet, Lausanne and Geneva
9. Bernoulli D (1738) Hydrodynamica, Basel
10. Bernoulli D (1748) Remarques sur le principe de la conservation des forces vives pris dans un sens général. Histoire Acad Royale des Sci et belles lettres de Berlin 4:356–364
11. Capecchi D (2012) History of virtual work laws. The Science Historical Studies Series, vol 42. Birkhauser-Springer, Milano
12. Carnot LNM (1803) Principes fondamentaux de l'équilibre et du mouvement. Deterville, Paris (Second extended edition of «Essai sur les machines en général, 1783», Deterville, Paris)
13. Clairaut A (1745) Sur quelques principes qui donnent la solution d'un grand nombre de problèmes de dynamique. Mémoires Acad Sci Paris (1742), pp 1–52
14. Dahan-Dalmenico A (1990) Le formalisme variationnel dans les travaux de Lagrange. In: La «mécanique analytique de Lagrange et son héritage», vol I, Acta Acad Scient Taurinensis, Suppl. No. 124:81–108
15. D'Alembert JLR (1743) Traité de dynamique, 1st edn. David, Paris (Reprint of the second augmented edition (1758) by Gauthier-Villars Publishers, in two volumes, Paris, 1925; also Facsimile reprint by Editions Gabay, Paris, 1998)
16. D'Alembert JLR (1744) Traité de l'équilibre et du mouvement des fluides pour servir de suite au traité de dynamique. David, Paris
17. D'Alembert JLR (1752) Essai sur une nouvelle théorie de la résistance des fluides (Competition essay for the 1750 prize of the Academy of Berlin)
18. D'Alembert JLR (1761–1780) Opuscules mathématiques (8 volumes). David, Paris
19. D'Alembert JLR (1779) Eloges lus dans les séances publiques de l'Académie française, Paris
20. D'Alembert JLR (1780) Sur quelques questions de mécanique. Opuscula Matematica, Tomus VIII, Mémoire 56, p 36
21. D'Arcy P (1747) Principe général de dynamique. Mémoires de l'Académie des Sciences, pp 348–356 (published 1752)

22. Dugas R (1950) History of mechanics, Editions du Griffon Neuchatel, Switzerland (Dover reprint, New York, 1988)
23. Euler L (1748) Recherches sur les plus grandes et plus petites qui se trouvent dans les actions des forces. Mémoires Acad Sci Berlin vol 4, p 149
24. Euler L (1752) Découverte d'un nouveau principe de mécanique (1750). Mémoires l'Acad Sci Berlin 6:185–217
25. Euler L (1757) Principes généraux de l'état d'équilibre des fluides. Mémoires de l'Académie des Sciences de Berlin, 1:217–273 (Memoirs presented to this Academy in 1755)
26. Euler L (1776) Nova methodus motum corporum rigidorum. Novi Com Acad Sci Petrop 20(1775):189–207 (also, ibid, 20:208–238)
27. Fourier JB (1798) Mémoire sur la statique contenant la démonstration du principe des vitesses virtuelles et la théorie des moments. J Ecole Polytechnique 5ème cahier:20–65 (Also in Œuvres complètes, Ed. G. Darboux, vol 2, pp 477–521, Gauthier-Villars, Paris)
28. Jouguet E (1924) Lectures de mécanique: La mécanique enseignée par les auteurs originaux. Gauthier-Villars, Paris (Facsimile reprint, Gabay, Paris, 2007)
29. Lagrange JL (1762) Essai d'une nouvelle méthode pour déterminer les maxima et les formules des intégrales minimum indéfinies. In: Serret JA, Darboux G (eds) Œuvres de Lagrange (1867–1892), vol 1, pp 335–362, Gauthier-Villars, Paris
30. Lagrange JL (1762) Application de la méthode exposée dans le mémoire précédent à la solution des problèmes de dynamique différents. In: Serret JA, Darboux G (eds) Œuvres de Lagrange (1867–1892), vol 1, pp 151–316, Gauthier-Villars, Paris
31. Lagrange JL (1788) Mécanique analytique, 1st edn. 1788, Vve Desaint, Paris, 2nd edn. Vve Courcier, Paris, 1811 (Facsimile reprint of 4th edition with notes and comments par J. Bertrand and G. Darboux, Gabay, Paris). English edition: Lagrange JL (1997) (trans: Boissonnade A, Vagliente VN (eds)) 2nd edn. Springer, Dordrecht, 1811
32. Lagrange JL (1798) Sur le principe des vitesses virtuelles. J Ecole Polytechnique 5:115–118
33. Lagrange JL (1965) Reprint of the second volume (3rd and 4th edn.) of Lagrange's Mécanique analytique (exclusively Dynamics) with notes by Joseph Bertrand and Gaston Darboux, Librairie Scientifique Albert Blanchard, Paris
34. Maugin GA (2012) The principle of virtual power: from eliminating metaphysical forces to providing an efficient modelling tool. Cont Mech Thermodynam 25:127–146
35. Maugin GA (2013) Piola and Kirchhoff: On changes of configurations. Preprint, UPMC, Paris (also Chap. 4 in the present book)
36. Maupertuis PLM (de) (1740) Loi de mouvement et de repos de la nature. Mémoires Acad Sciences de Paris. Also: Œuvres de Maupertuis, vol 4, pp 1–28, Bruyset, Lyon (1746)
37. Maupertuis PLM (de) (1744) Des lois du mouvement et du repos. In: Œuvres de Maupertuis, vol 4, pp 29–64. Bruyset, Lyon (1746)
38. Pfeiffer J (2005) Le traité de géométrie de Varignon et l'apprentissage mathématique du jeune d'Alembert. In: Recherches sur Diderot et sur l'Encyclopédie, No.38, Avril 2005, pp 125–150 (edited by Société Diderot, Langres, France)
39. Truesdell CA (1968) Essays in the history of mechanics. Springer, Berlin
40. Truesdell CA (1984) An idiot's fugitive essays on science. Springer, New York
41. Truesdell CA, Toupin RA (1960) The classical theory of fields. In: Flügge S (ed) Handbuch der Physik, Bd III/1, Springer, Berlin

Chapter 3
What Happened on September 30, 1822, and What Were its Implications for the Future of Continuum Mechanics?

Abstract This contribution offers a discussion about the notion of stress in a general continuum as initially proposed in a magisterial paper by Cauchy in 1822 (but published only in 1828) without using arguments involving molecules. This is here presented in its historical context. Cauchy's view is the currently accepted view among mechanicians and engineers although attempts (including by Navier and Cauchy himself) to start from a molecular description in the manner of Newton and Laplace were constantly offered in both nineteenth and twentieth centuries. The discussion introduces other secondary stress definitions such as those by Piola, Kirchhoff, and more recently Eshelby. The question naturally arises of what happens with the possibility to introduce other internal forces such as hyperstresses (in so-called gradient theories) and couple stresses (e.g., in Cosserat continua), and whether some introduced stresses have associated with them a meaningful boundary condition. Also pondered is the question whether one can identify a stress concept in physical approaches still considering interactions between point particles (lattice dynamics, kinetic theory, nonlocal theory, statistical-mechanics approach). The chapter is concluded by a more in depth discussion of the notion of stress-energy-momentum, culminating in that of pseudo-tensor of energy-momentum in gravitation theory.

3.1 Introduction

Augustin Louis Cauchy (1789–1857) was a brilliant French mathematician with an extremely wide scope of interests including mathematical physics and theoretical mechanics as well as pure questions of algebra (theory of permutations) and analysis (complex analysis and theory of residues). Extremely prolific, he had a production that compares well both in quality and quantity with those of Leonard Euler (1707–1783)—a predecessor in many points- and his contemporaries Carl F. Gauss (1777–1855) and A. Cayley (1821–1895). Born in August 1789 practically one month after the fall of the Bastille (July 14, 1789), he certainly did not become

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a revolutionary. On the contrary, politically, he remained all along his life a convinced legitimist—i.e., in favour of a king in the line of the dynasty (the *Bourbons*) of pre-revolution times. The reader may wonder what is the relationship of this political inclination with his scientific works and career. Indeed, although formed as an "ingénieur-savant"[1] in the best schools created by the French Revolution and Republic and strengthened by Napoleon, he benefited from the fall of the latter and the return of the Bourbon kings in 1815 in being given the positions—which he deserved on a purely technical professional basis—of other scientists who lost their positions for political reasons. Among the positions he was granted we single out that of teaching a course on mechanics at the Faculty of Sciences in Paris in 1821. This, according to Belhoste [1, p. 92], may have "provided the inspiration for further research in mathematical physics", in particular the mechanics of continua, although he had already taught some mechanics at the *Ecole Polytechnique* and at the *Collège de France*. The second fact which may a priori seem irrelevant to a scientific discussion, is the marriage of Cauchy with a certain Miss Aloïse de Bure in 1818. This, as we shall see, had a definite consequence on the manner of publishing his works by Cauchy. Now, the title of the present chapter does not question what happened all over the whole world, but more precisely what happened on that precise day of 1822 at the Academy of Sciences in Paris.

3.2 Preliminary Remarks

In 1822, at age thirty three, Cauchy was already an internationally recognized mathematician when, on September 30, he read a memoir on continuum mechanics to the Paris *Académie*. This was to be the foundational paper in that field of mechanics. This may be referred to as his *first* theory (CAUCHY-1) of general continuum mechanics although he had published before papers on fluid mechanics and he was much interested in the possibility of an elastic medium to transmit waves. That now celebrated date of September 30, 1822, was only the presentation of a theoretical framework that was really published in print only six years later in 1828 in an extended and corrected form. As a matter of fact, Cauchy did not even give a true reading or lecture on the contents of his paper in 1822, nor did he leave a copy with the *Académie*.[2] Only a kind of abstract was given in the bulletin of January 1823 of a learned Paris society, the *Société Philomatique* [this

[1] For this notion of ingénieur-savant ("engineer-scientist") see [22, 10].

[2] This has been checked in the files of the session of September 30, 1822 with the kind help of the Librarian (Mrs Florence Greffe) of the Paris Academy of Sciences. This date was mentioned by Truesdell [45], but also much before by Duhem (p. 78, Footnote 2, of Duhem [14]). The original record of this session is reproduced at the end of this chapter. It simply says that Cauchy read about his research (probably just the basic ideas) that was to be printed as a long abstract four months later in the Bulletin of the *Société Philomatique*.

is translated into English in the Appendix]. There might have been quarrels of priority with C.L.M.H. Navier (1785–1836)—another great elastician and fluid dynamicist—that delayed the real publication (see [1, pp. 97–98]). All his life Cauchy, an already mentioned prolific author, flooded the *Académie* with notes and memoirs, so much that the *Académie* at a point decided to fix an upper limit to the number of such contributions that any member could submit! This is where the importance of in-laws should not be overlooked. It happens that Cauchy's wife Aloïse was the daughter and niece of the de Bure brothers, Marie-Jacques and Jean-Jacques, renowned Parisian booksellers and publishers. Cauchy frequently used this possibility an as expedient way to publish in print his own lectures at *Polytechnique* and also many of his memoirs. This was the case of his landmark paper of 1822 on continuum mechanics which was published in 1828 (cf. Cauchy [4]) in the second volume of "Exercices de Mathématiques", a kind of privately produced series published between 1826 and 1830. Cauchy was the only author published in this surprising scientific periodical.

3.3 The Main Contents of Cauchy's 1822/1828 Memoir

What we call the *first* theory (CAUCHY-1) of general continuum mechanics and elasticity of Cauchy is a purely phenomenological continuum theory which does not use the notion of constituent "molecules" and at-a-distance interactions between them (contrary to the *second* theory of Cauchy; see below). Cauchy did not build on an uncultivated ground.[3] Euler had already introduced the (restricted) idea that interactions between parts of a fluid were of the *contact* form and materialized in a single scalar, the hydrostatic pressure p. In modern terms, it is said that the applied traction $\mathbf{t}$ at a point of a regular surface is aligned with the local unit normal $\mathbf{n}$ to that surface, i.e.

$$\mathbf{t} = -p\,\mathbf{n}. \tag{3.1}$$

This applies to so-called Eulerian fluids that present no viscosity. The basic idea propounded by Cauchy in 1822 was to generalize (3.1) to all kinds of continua (see the spot-on general title of the abstract published in Cauchy [3]; full memoir Cauchy [4]. His reasoning is that in this state of generalization the relation (3.1) is replaced by a *linear* relationship (a linear vector relation in the language of Gibbs

[3] The reader will be interested in Truesdell's vision of Cauchy's elaboration of the concept of stress in his Essay "The creation and unfolding of the concept of stress" in pp. 184–238 in Truesdell [45] (this was underlined by J. Casey, private communication). However, in contrast to the present study that emphasizes the story of the concept of stress from and after Cauchy, Truesdell deals with the conceptual stages that led to Cauchy's notion of stress, with works by brilliant predecessors such as Stevins, Galileo Galilei, the Bernoullis, d'Alembert, Euler, Young, and Fresnel. Cauchy himself is parsimonious with citations, and refers to very few scientists with the exception of his contemporary Fresnel.

and Heaviside; cf. Crowe [10]), and not a simple proportionality. That is, in modern intrinsic and Cartesian tensorial notations,

$$\mathbf{t} = \underline{\sigma}.\mathbf{n} \quad \textit{and} \quad t_i = \sigma_{ij} n_j, \quad i,j = 1,2,3. \tag{3.2}$$

Equation (3.1) corresponds to the special case $\underline{\sigma} = -p\,\mathbf{1}\, or\, \sigma_{ij} = -p\,\delta_{ij}$ where δ_{ij} is Kronecker's delta. This can be viewed as a specific constitutive assumption (i.e., the selection of a specific continuum, the Eulerian fluid). The object $\underline{\sigma}$ is the (Cauchy) *stress tensor*. Of course it was identified with the mathematical notion of tensor (which itself smells of its mechanical origin) only much later by Woldemar Voigt. In his generalization Cauchy considers that the applied traction $\mathbf{t}$ can be at any angle to the unit normal to the tangent plane of a surface cut in the material body, thus allowing for a contact action of the *shear* type as well as pressure. The true genial point resides in the absolutely general standpoint and its evident conceptual simplicity. To prove (3.2) Cauchy relied on a reasoning involving a special small volume of matter, in his celebrated *tetrahedron* argument, now reproduced in all introductions to continuum mechanics.[4] It was also proved in most cases (no applied couples) that this Cauchy stress is symmetric, having thus only six independent components at most in standard Euclidean physical space. Relying on an argument already introduced by Euler (by looking for the equilibrium of an elementary parallelepiped) the following local dynamical Cauchy equation of motion could be obtained (here in modern notation) [44, 47, 48]

$$\rho\,\mathbf{a} = \rho\mathbf{f} + div\,\underline{\sigma}, \tag{3.3}$$

where vector $\mathbf{a}$ denotes the acceleration, ρ is the matter density, $\mathbf{f}$ is an external body force per unit mass, and the symbolism *div* denotes the *divergence* operator applied to the tensor $\underline{\sigma}$. Cauchy could not use this vocabulary as the operation of a divergence was essentially introduced by George Green [21] in electromagnetism in the same year in a practically unknown publication. But if we combine Cauchy's lemma (3.2) and Green's divergence theorem we have the following exploitable result:

$$\int_{\partial B} \mathbf{t}\,ds = \int_{\partial B} \underline{\sigma}.\mathbf{n}\,ds = \int_B div\,\underline{\sigma}\,dv, \tag{3.4}$$

for a simply connected volume B bounded by a regular surface ∂B. Applying this to the following global *balance law* of linear momentum,

$$\frac{d}{dt}\int_B \rho\,\mathbf{v}\,dv = \int_B \rho\,\mathbf{f}\,dv + \int_{\partial B} \mathbf{t}\,ds, \tag{3.5}$$

and localizing this on account of an assumed continuity of fields over B, we are led to the local (Cauchy) balance of linear momentum as Eq. (3.3) with $\mathbf{a} = d\mathbf{v}/dt$, what is the modern way of reaching (3.3).

4 See a more technical and rigorous exposition in Noll [39].

Remark 3.1 Very often in the engineering literature the expression *balance laws* and *equations of conservation* are used interchangeably. Like in financial accounting, "balance" carries with it the notion of incoming and outgoing stuff. That is why a quantity like $\underline{\sigma}$ is sometimes referred to as a *flux*.

Remark 3.2 Written in the appropriate coordinate system and in the absence of body source term, an equation such as (3.3) can also be written in the form

$$\frac{\partial}{\partial t}\mathbf{p} - div\,\underline{\sigma} = \mathbf{0}. \tag{3.6}$$

This can be referred to as a *mathematical conservation law*. In particular, with vanishing second term we can say that the quantity $\mathbf{p}$ is strictly conserved in time as

$$\partial \mathbf{p}/\partial t = \mathbf{0}. \tag{3.7}$$

But in statics (no time dependence of fields) or in quasi-statics (possible dependence on time but neglecting acceleration terms), we have the "equilibrium" equation

$$div\,\underline{\sigma} = \mathbf{0}. \tag{3.8}$$

The three possibilities embodied in Eqs. (3.6)–(3.8) can be compared to the Newtonian equation of point-particle motion:

**General Newton equation* (point of constant mass):

$$m\,\mathbf{a} = \sum_{\alpha} \mathbf{F}^{\alpha}, \tag{3.9}$$

where $\mathbf{F}^{\alpha},\ \alpha = 1, 2, \ldots$ is a system of acting forces;

**Statics (Varignon, parallelogram of forces)*:

$$\sum_{\alpha} \mathbf{F}^{\alpha} = \mathbf{0}; \tag{3.10}$$

**Inertial motion (Descartes)*:

$$\frac{d}{dt}\mathbf{p} = \mathbf{0}, \quad \mathbf{p} = m\mathbf{v}, \tag{3.11}$$

**D'Alembert's formulation of* (3.9):

$$\sum_{\alpha} \mathbf{F}^{\alpha} + \mathbf{F}^{a} = \mathbf{0}, \quad \mathbf{F}^{a} = -m\mathbf{a}. \tag{3.12}$$

With these different forms—of which (3.9) and (3.12) are strictly equivalent—we have interpretations at variance depending on the chosen emphasis. With account of the special case (3.10) we have a tendency to consider Newton's equation (3.9) as a definition for the acceleration force. With the special case (3.11) we suffer from another temptation, that of considering (3.9) as a conservation law of linear momentum that is not strictly respected because of the presence of impressed forces. As to (3.12), it is d'Alembert's clever "rewriting" trick to give all "forces" the same status, as understood in his principle of virtual power.

A more or less similar discussion can be envisaged for the continuum Eq. (3.6) through (3.8). What is the primary quantity appearing in these equations? For engineers avoiding a dynamical framework—Eq. (3.8) possibly with an added body force-, the Cauchy stress appears as primary, essentially through the Cauchy Lemma (3.2). But for physicists interested in dynamics, the interpretation of (3.6) as a nonstrict conservation law for linear momentum prevails, the notion of associated flux being only secondary. Finally, with the view of a discriminating physicist and parodying (3.12), we can formally rewrite (3.3) as

$$\mathbf{F}^{ext} + \mathbf{F}^{int} + \mathbf{F}^{a} = \mathbf{0}, \tag{3.13}$$

where

$$\mathbf{F}^{ext} = \rho \mathbf{f},\ \mathbf{F}^{int} = div \underline{\sigma},\ \mathbf{F}^{a} = -\rho \mathbf{a}, \tag{3.14}$$

are volume forces of external, internal, and acceleration origin, respectively. In writing (3.13) we distinguish between *external* forces reserved to *at-a-distance* action (e.g., gravitation, electromagnetism) acting per unit quantity of matter and *internal* forces that account for *contact* action via the second of (3.14) and the notion of Cauchy stress. This "definition" indicates that $\mathbf{F}^{int}$ is defined up to a divergence-free second-order tensor. But the associated natural boundary condition still involves only the initial stress and applied traction. However, this remark makes one ponder the case of an infinite body and the *second* Cauchy's theory (CAUCHY-2) of elasticity proposed by Cauchy in 1828 on the basis of *molecular* considerations [5]. Remember that Cauchy's work of 1822/1828 also proposed a definite theory of linear isotropic elasticity that provides an expression for $\underline{\sigma}$ in terms of infinitesimal strains, with two elasticity coefficients.[5] This was proposed a priori to close the obtained system of differential equations in terms of the elastic displacement gradient. This a priori construct that involves a representation theorem—also attributed to Cauchy—for a second-order tensor is referred to as *Cauchy's elasticity* by Truesdell, Toupin and Noll [47, 48]. Nowadays, this is justified by applying a thermodynamic argument and the consideration of an elasticity potential (in fact following George Green who thereby becomes our second hero).

[5] This was done after correction by Cauchy himself of his initial proposal with only one coefficient; for a general anisotropic body this would yield twenty one independent coefficients at most but its application to specific symmetries requires more group-theoretic reasoning unknown to Cauchy.

3.4 Cauchy's Stress and Hyperstresses

A naturally raised question is what happens to the Cauchy lemma (3.2) when one has to consider a surface that presents irregularities such as an edge where the unit normal may not be uniquely defined. One may also question what happens when in addition to the normal at a regular but curved surface, one tries to account for the geometrical description of the said surface at the second order, thus involving the curvature and the surface variation of the unit normal, hence also the introduction of the tangential derivative. It took some time to ponder these questions and obtain rigorous answers. This was rigorously achieved by Noll and Virga [40] and dell'Isola and Seppecher [11] with an advantage to the latter authors for the brevity of their argument—see also [12] for a more general case. Avoiding the difficult technical points for which we refer the reader to the original authors, we note that these considerations inevitably lead to envisaging the notion of *surface tension* described within the continuum mechanics framework. In addition to the notion of stress à la Cauchy this yields the introduction of the notion of *hyperstress*. This is represented by a third order tensor that is the thermodynamical dual of the second gradient of the displacement vector in elasticity or the second gradient of the velocity in fluids (or the gradient of the density in a "perfect" fluid). This was recognized early in the theory of surface tension by Korteweg [24] and much more recently by Casal [2]. In the case of elasticity, basing on a variational formulation, it was probably Le Roux [28] who first introduced the notion of second gradient theory remarking in the application that the interest for such a formulation appears only for problems with a spatially non-uniform strain. The Cauchy lemma is not used in these energy approaches. As a matter of fact, a formulation such as the principle of virtual power bypasses the Cauchy lemma. Indeed consider that the power of *internal forces* expended in a virtual velocity field $\hat{\mathbf{v}}$ is written as a linear form (for a whole body B)

$$\hat{P}_{int}(B) = \int_B \left(f_i \hat{v}_i + \sigma_{ij}\, \hat{v}_{i,j} + \cdots\right) dB. \tag{3.15}$$

In modern continuum mechanics it is assumed that internal forces for which one needs to construct constitutive equations must be objective (independent of the observer—contrary to externally applied forces and inertial forces that are not subjected to this constraint). So must be the case of their dual partners in (3.15). This rules out the term linear in the velocity itself and the term linear in the skew part of the gradient of the velocity. Being satisfied with a first-order gradient theory, this reduces (3.15) to the expression (here the minus sign is conventional)

$$\hat{P}_{int}(B) = -\int_B \left(\sigma_{ij} \hat{D}_{ji}\right) dB,\ \hat{D}_{ji} := \frac{1}{2}\left(\hat{v}_{j,i} + \hat{v}_{i,j}\right). \tag{3.16}$$

Thus tensor $\underline{\sigma}$ can only be symmetric in this case. In the absence of other internal force (e.g., due to electromagnetic effects), it can therefore be identified to the Cauchy stress when the formulation of the principle of virtual power accounts for the power expended by an applied traction at the regular boundary of B. Pursuing the expansion indicated by the ellipsis in (3.15) will allow one to introduce stresses of higher order, in particular the already mentioned *hyperstress*. We refer to Germain [20] and Maugin [31] for these extensions. This short exercise shows that a weak formulation like the principle of virtual power offers great advantages over the Cauchy type of approach, in particular to obtain a good set of natural boundary conditions at surfaces, corners and apices.

Hyperstresses of another type may be introduced to which a Cauchy type of argument applies. This is the case in media with so-called internal degrees of freedom where each material point, in addition to its translation, is equipped with an internal deformation (called micro-deformation) which in some cases is simply reduced to an internal rigid rotation. This is the case of so-called Cosserat continua. Indeed, the Cosserat brothers were led to consider the possible existence of *internal couples* [6]. They more or less were forced to do that by imposing an invariance (so-called *Euclidean invariance*) on a Lagrangian-Hamiltonian formulation, which invariance treats on an equal footing translations and rotations. This gave rise to the possible existence of a new type of internal force, the *couple stress*, along with that of stress, and as a consequence the possibility to have *non-symmetric* stresses (cf. Le Corre [27]). Such couple stresses also satisfy a lemma of the Cauchy type. But again, the principle of virtual power accounting for the presence of a new velocity field related to the micro-deformation is the most elegant and safe way to deduce all field equations and associated boundary conditions in such a theory.

Gradient theories with hyperstresses and Cosserat media now are part of *generalized continuum mechanics* of which the main characteristic property in effect is to deviate from Cauchy's 1822/1828 pioneering vision (cf. Maugin [33]).

3.5 Stress as a Secondary Notion

In the late 1820s, Cauchy [5], in competition with Navier and Poisson and with a view to envisage anisotropic bodies, decided to construct a linear elasticity theory using arguments involving a molecular picture with kind of interactions *à la* Newton between molecules. This CAUCHY-2 approach bypasses the basic notion of stress tensor. But the general form of the action (repulsion or attraction) of neighbouring molecules on a prototype one must be assumed. The reasoning then consists in making approximations in the infinite series of the involved finite differences to extract no nonsensical continuum equations. This can be achieved only by assuming a specific regular symmetry (a lattice) thus conducing to a generally anisotropic representation of the stress tensor by identification. This recovery of the notion of stress was proposed by Cauchy in his second paper of 1828. However, this does not provide the same number of elasticity coefficients as the first Cauchy's theory because of some constraints brought by the symmetries of molecular interactions (the famous Cauchy-Born relations). Cauchy was to apply his elasticity theory to specific mechanical elements (plates, rods) and to the theory of light propagation in a supposedly elastic medium serving as a support of light vibrations (the ill-fated "ether"). One of his successes was a theory of reflection and refraction at the boundary between two media. This second line fits well in the grand scheme to create a universal molecular physics by Laplace in the manner of Newton in point mechanics and by Ampère in electromagnetism.

Nowadays Cauchy's second theory is considered as obsolete while the first theory is the accepted one. But CAUCHY-2 germinally contains the modern theory of lattice dynamics as developed by Max Born and Theodor von Kármán in the early twentieth century and now a tenet of solid-state physics. This **identification** of a stress in a continuum limit of a particle-like theory is not proper to lattice dynamics. It appears where an internal force can be identified as the divergence of a second order tensor [cf. the second of (3.14)], which will then be called "stress". This is the case in the kinetic theory of fluids where after an appropriate expansion in terms of a small characteristic parameter, a continuum equation of linear momentum can be constructed in the series of moments deduced from the Boltzmann equation (see books on kinetic theory). But we must remember that there is no principle requiring the justification of continuum equations from a molecular description, as a logical continuum theory may be entirely autonomous. Nonetheless, the true physicist will be more than happy to be able to establish such a correspondence. The search for this identification is not vain; it progresses constantly and meets some success in a rigorous mathematical framework.[6]

Still another case where the notion of stress can only be secondary is that of materials of which the response exhibits a strong *nonlocal* nature. That is, in principle, the mechanical (or other) response at a material point depends on the values of independent variables (e.g., strains) at points at a far distance form this point, with a natural decrease of influence with increasing distance. This introduces in continuum mechanics a vision à la Newton-Laplace well illustrated by the book of Eringen [17]. Any cut in the material to apply Cauchy's argument would suppress the prevailing action at-a-distance. No wonder, therefore, that such models are usually first constructed in an infinite body and more than often justified by a lattice-dynamic theory with long-distance (with far-neighbour interaction) forces (See e.g. [25]).

From the above we see that there are cases where a computation from a molecular theory allows one to identify a stress tensor at a bulk point via an equation of the type of the second of (3.14), i.e.,

$$\mathbf{F}^{int} \equiv div\underline{\sigma}. \tag{3.17}$$

[6] This is beautifully demonstrated in the recent book of Murdoch [36] after the statistical-mechanics theory of liquids by John G. Kirkwood (1907–1959) where the liquids' properties are calculated in terms of the interactions between molecules.

This is true only modulo a divergence-free tensor. An example of this is the introduction of *Maxwell stresses* in electromagnetism. This can be first illustrated by a simple field theory in which the basic field equation is none other than the Gauss-Poisson equation for the potential ϕ and electric charge density q:

$$\nabla^2 \phi = -q. \tag{3.18}$$

In multiplying both sides of this equation by the vector $\nabla \phi$ and performing elementary manipulations, we are led to the equation of the electrostatic force acting on q as

$$\mathbf{F}^e = q\mathbf{E} \equiv div\underline{\sigma}^e, \quad \underline{\sigma}^e := \mathbf{E} \otimes \mathbf{E} - \frac{1}{2}\mathbf{E}^2\mathbf{1}, \tag{3.19}$$

where $\mathbf{E} = -\nabla \phi$ is a quasi-static electric field and the symmetric tensor $\underline{\sigma}^e$ may be called the Maxwell stress for such fields [35]. For a vanishing q in vacuum this short proof shows that a divergence-type of *conservation law* with vector components can be associated with the scalar Laplace equation [cf. (3.18)]—a fact more than often ignored, but intimately related to Noether's theorem when (3.18) is deduced from a variational principle (see below). In a general magnetized, electrically polarized and conducting continuum a rather long argument starting with the expression of the elementary force acting on electric charges in—relatively slow—motion (the celebrated Lorentz force) allows one to show that the corresponding "internal force" due to electromagnetic fields in a deformable continuum is formally given by an expression of the type

$$\mathbf{F}^{em} = div\, \underline{\sigma}^{em} - \frac{\partial}{\partial t}\mathbf{p}^{em}, \tag{3.20}$$

where the electromagnetic stress tensor $\underline{\sigma}^{em}$ is generally not symmetric and $\mathbf{p}^{em}$ is a linear electromagnetic momentum for dynamical fields. Expressions of these together with the accompanying energy expression can be found in Maugin [32, Chap. 3] after an evaluation made by Maugin and Collet in 1972 and Maugin and Eringen in 1977. In this approach both quantities $\underline{\sigma}^{em}$ and $\mathbf{p}^{em}$ appear as secondary notions. But the representation (3.20) is a patent mark of the ambiguity in interpretation carried by electromagnetic fields that can alternately be considered as giving rise to at-a-distance (*à la* Newton-Laplace) or contact (*à la* Euler-Cauchy) forces.

3.6 Stress as Part of Stress-Energy-Momentum

It is natural to turn next to a space-time formulation propounded by relativistic studies in the twentieth century. First, a naïve consideration puts us on the right track. For instance, [42, 43] introduced an object, now called the Piola-Kirchhoff stress, from the Cauchy stress by the transport/convection (or “pull back”) definition (so-called Piola transformation)

$$\mathbf{T} = J_F \mathbf{F}^{-1} \underline{\sigma} \quad or \quad T^K_{.i} = J_F X^K_{,j} \sigma_{ji}, \tag{3.21}$$

where $\mathbf{F}^{-1} := \partial \mathbf{X}/\partial \mathbf{x} = \{\partial X^K/\partial x_j\}$ is the inverse deformation gradient, and $J_F := \det \mathbf{F}$, where $\mathbf{F}$ is the direct deformation gradient between a reference configuration K_R (with material coordinates X^K, $K = 1, 2, 3$). Equation (3.3) with $\mathbf{f} = \mathbf{0}$ is then shown to take the following mathematically strict conservation form:

$$\frac{\partial}{\partial t}(\rho_0 v_i) - \frac{\partial}{\partial X^K} T^K_{.i} = 0, \tag{3.22}$$

where $\rho_0 = \rho J_F$ is the matter density at K_R.

The object **T,** not a traditional tensor since having «feet» in two different spaces, stands for a force in the actual configuration K_t computed per unit area of the reference configuration. With (3.22) one is tempted to introduce a space-time parametrization ($X^\alpha = (X^K, X^4 = t)$) such that (3.22) reads equivalently

$$\frac{\partial}{\partial X^\alpha} T^\alpha_{.i} = 0, \quad T^\alpha_{.i} = \left(T^K_{.i}, T^4_{.i} = -\rho_0 v_i\right). \tag{3.23}$$

The first of these has the look of a true (divergence-like) conservation law but it is not really fully space-time in nature since its components still are in three-dimensional physical space. To reach a completely space-time equation one should unite (3.23) with the conservation of energy. As we know now, this was achieved in the first years of the 1900s with Minkoswki’s four-dimensional formulation of special relativity. In modern terms this is introduced by noting $x^\alpha = (x^i, x^4 = ct)$ with a hyperbolic space-time metric $g_{\alpha\beta}$ of signature (+, +, +, −) and noting u^α, $\alpha = 1, 2, 3, 4$, the “world” velocity such that $g_{\alpha\beta} u^\alpha u^\beta + c^2 = 0$. Here c is the velocity of light in vacuum taken as a standard of velocity. A definite step forward was taken by Carl Eckart [15] in a paper that is a real pearl, when he proposed that for general continuous matter energy, momentum and stresses could be accommodated in a single notion, the *stress-energy-momentum space-time tensor* $T^{\alpha\beta}$ within a completely covariant format by using systematically the resolution of any space-time object into “proper” components. That is,

$$T^{\alpha\beta} = c^{-2} \omega u^\alpha u^\beta + c^{-2} u^\alpha q^\beta + p^\alpha u^\beta - t^{\beta\alpha} \tag{3.24}$$

where

$$\omega \equiv c^{-2} u_\alpha T^{\alpha\beta} u_\beta, \; q^\beta \equiv -u_\alpha T^{\alpha\gamma} P^\beta_{.\gamma}, \; p^\alpha \equiv -c^{-2} P^{\,\alpha}_{.\gamma} T^{\gamma\beta} u_\beta, \; t^{\beta\alpha} \equiv -P^\beta_{.\gamma} P^\alpha_{.\delta} T^{\delta\gamma}. \tag{3.25}$$

Here $P_{\alpha\beta}$ is the spatial projector such that

$$P_{\alpha\beta} := g_{\alpha\beta} + c^{-2} u_\alpha u_\beta = P_{\beta\alpha}, \quad u_\alpha P^{\,\alpha}_{.\beta} = 0, \quad P^{\,\alpha}_{.\beta} P^\beta_{.\gamma} = P^\alpha_{..\gamma}. \tag{3.26}$$

These are, respectively, a definition, an orthogonality property,[7] and the condition of idempotence. The four elements present in the canonical decomposition (3.25) are but "spatial" covariant forms of the energy density, energy (heat) flux, momentum density, and (Cauchy) stress. The identification of $t^{\beta\alpha}$ as the relativistic generalization of Cauchy's stress is shown by applying the projector $P^{\gamma}_{.\beta}$ to the general balance law (here a strict conservation law)

$$\frac{\partial}{\partial x^{\alpha}} T^{\alpha\beta} = 0, \tag{3.27}$$

in order to obtain its essentially spatial component, i.e., orthogonal to u^{γ} according to the second of (3.26). Now the identification with Cauchy's stress is not so obvious. The matter was pondered by scientists such as Van Dantzig [49] and Costa de Beauregard [7–9]. We refer the reader to these authors. Expression (3.24) does not generally imply that $t^{\beta\alpha}$ is symmetric. But we note that $T^{\alpha\beta}$ is a powerful generalized notion of rich contents compared to the simple stress notion.

Furthermore, in standard general relativity, *minimal coupling* requires to replace the partial derivative in (3.27) by a covariant derivative. That is,

$$\nabla_{\alpha} T^{\alpha\beta} = 0, \tag{3.28}$$

where ∇_{α} is computed in the space-time varying metric $g_{\alpha\beta}$ which is solution of the celebrated Einstein's gravitation equation

$$A_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \frac{8\pi k}{c^4} T_{\alpha\beta}, \tag{3.29}$$

where $R_{\alpha\beta}$ is the Ricci curvature, R is the scalar curvature of space-time, and k is Newton's gravitation constant. The right-hand side of (3.29) provides the source of energy and momentum (of various origins—including mechanical stresses—in particular from electromagnetism in magnetized and electrically polarized bodies; see [18], Vol. 2, Chap. 15); but note that the unknown $g_{\alpha\beta}$ itself is involved in $T_{\alpha\beta}$ so that only a laborious iteration procedure can help obtain, if ever, a solution of (3.29) for the metric. Equation (3.29) requires that $T_{\alpha\beta}$ be symmetric, since this is the case of the Einstein tensor $A_{\alpha\beta}$.[8]

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[7] As a young researcher I used to call "PU" tensorial objects those that are essentially space-like although written in full covariant form. They satisfy typical orthogonality conditions such as the second of (3.26). The hidden play of words was that PU = "Perpendicular to the world velocity **u**" = "Princeton University" for which the author has a definite affection. It is this property that allows for the identification of the space-time tensor $t_{\alpha\beta}$ with Cauchy's stress of classical continuum mechanics [cf. the last of Eq. (3.25)] [See [30], and papers published between 1971 and 1980 in C.R. Acad. Sci. Paris, Journal of Physics (UK), Ann. Inst. Henri Poincaré (Paris), Journal of Mathematical Physics (USA) and J. General Relativity and Gravitation].

[8] The history of the successive missed and successful steps in the production of Eq. (3.29) in the 1910s is a formidable scientific adventure involving, not only Einstein—as we could believe from modern hagiographic treatments—but also Marcel Grossmann, Max Abraham, Gustav Mie, David Hilbert and Emmy Noether, a story that remains to be fully investigated and understood [In particular,

3.7 The *Nec Plus Ultra*: The Eshelby Stress and the Pseudo Tensor of Energy-Momentum

Both Cauchy stress and the first Piola-Kirchhoff stress present the invaluable feature to have associated with them natural boundary conditions on the stress. This follows directly from Cauchy's fundamental lemma. But there are other stress tensors that are more directly related to the concepts of energy and energy-momentum and with which no direct simple boundary conditions are associated. These tensors are often deduced from the original Cauchy and Piola-Kirchhoff stresses via some manipulation. The first of these is the *second Piola-Kirchhoff stress* deduced from the first through the following definition (complete pull-back of the Cauchy stress to the reference configuration):

$$\mathbf{S} := \mathbf{T}.\mathbf{F}^{-T} i.e., \; S^{KL} = T^{K}_{.i} X^{L}_{,i} = J_F X^{K}_{,i} \sigma^{ij} X^{L}_{,j}. \tag{3.30}$$

The interest for this fully material stress tensor is its "energy" contents as, for Green's elasticity deduced from a potential function W per unit reference volume, it is shown that

$$\mathbf{S} = \frac{\partial \overline{W}}{\partial \mathbf{E}}, \quad W = \overline{W}(\mathbf{E}), \quad \mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{1}_R), \quad \mathbf{C} := \mathbf{F}^T\mathbf{F}. \tag{3.31}$$

But there is no direct meaningful boundary condition involving only $\mathbf{S}$ except in small strains where $\mathbf{S}$ readily reduces to the Cauchy stress.

The second stress in this class is the *material Eshelby stress* $\mathbf{b}$ [19] which can be introduced thus. Apply $F^{i}_{.L}$ to Eq. (3.22) and account for the fact that for an elastic body $W = \tilde{W}(\mathbf{F})$ and $\mathbf{T} = \partial\tilde{W}/\partial\mathbf{F}$. This manipulation provides a mathematically strict conservation law (for homogeneous bodies) in the form

$$\frac{\partial}{\partial t}\mathbf{P} - div_R\,\mathbf{b} = \mathbf{0}, \tag{3.32}$$

wherein we have set

$$\mathbf{P} = -\rho_0 \mathbf{v}.\mathbf{F}, \quad \mathbf{b} := -(L\mathbf{1}_R + \mathbf{T}.\mathbf{F})$$

with

$$L = \frac{1}{2}\rho_0 \mathbf{v}^2 - W. \tag{3.33}$$

(Footnote 8 continued)
were Einstein's equations first written down by Hilbert with the help of Noether since only Eq. (3.29)—with all terms present—could be in agreement with Noether's invariance theorem that associates a conservation laws with a "good" field equation in a variational treatment (see Sect. 3.7 about the Eshelby stress)? Indeed, while the general covariance of the basic Eqs. (3.29) and (3.28) is a tenet (see the discussion in Norton [41]), the Noetherian relationship between these two—field and conservation (in that order)—equations is an acknowledged requirement.

The last quantity is akin to a Lagrangian density. This hints at a possible derivation of $\mathbf{b}$ via a Lagrangian-Hamiltonian variational principle, in which, in effect, application of the Noether's [38] theorem for material space translations provides (3.32) automatically via "Noether's identity". The main interest in the conservation Eq. (3.32) is its role in capturing field singularities—for instance in the theory of fracture—since quantities such as the so-called material momentum (or pseudo-momentum) $\mathbf{P}$ and the Eshelby stress $\mathbf{b}$ are at least second order in the motion and the associated deformation. Tensor $\mathbf{b}$ can be rewritten in two alternate forms as

$$\mathbf{b} = -(L\mathbf{1}_R + \mathbf{C}.\mathbf{S}) = -L\mathbf{1}_R + \mathbf{F}^T.\frac{\partial L}{\partial \mathbf{F}}. \tag{3.34}$$

The first of these shows the relationship of $\mathbf{b}$ with the Mandel stress $\mathbf{M} := \mathbf{F}.\mathbf{T} = \mathbf{C}.\mathbf{S}$ which plays a definite role as driving force in many material structural rearrangements (e.g., in finite-strain plasticity, growth; see [29, 34]). The second of (3.34) exhibits the field-theoretic origin of the notion of Eshelby stress in pure elasticity. This was recognized early by J. D. Eshelby who called $\mathbf{b}$ the *energy-momentum tensor* of elasticity or *Maxwell stress* of elasticity. The last denomination holds good by analogy with a tensor such as in $(3.19)_2$. The first coinage is not entirely correct since $\mathbf{b}$ remains three-dimensional (essentially "spatial"—i.e., 3D—albeit "material") while the notion of energy-momentum (see above) requires a four-dimensional treatment. In spite of its usefulness demonstrated at length in our book [34], the nonsymmetric material tensor $\mathbf{b}$ is not associated with a physically obvious boundary condition.[9]

But the remark concerning the second of (3.34)—a canonical formula in analytical mechanics—naturally takes us back to a relativistic treatment such as in general relativity. It is not difficult to show that $(3.34)_2$ indeed is *minus* the purely space-like part of a four-dimensional energy-momentum tensor as deduced in a four-dimensional formulation.[10] Pondering now the case of Einstein's general relativistic theory of gravitation, we must realize, as emphasized by Landau and Lifshitz in their remarkable "Theory of fields" [26, Section 100] that the covariant form (3.28) does not in general express a conservation law of any truly meaningful physical quantity. The reason for this is that, on account of the expression of the covariant divergence in terms of the usual divergence (3.28) reads [here $g = \det\,(g_{\alpha\beta})$]

[9] Some authors (e.g., [23]) have proposed to consider Eq. (3.32) as autonomous being posited—for any material behaviour—as a general balance law—a "new" equation of physics—by some kind of trick involving a boundary flux of energy together with stresses. The artificiality of this type of reasoning as well as the erroneous concept of the novelty of (3.32) in physics is shown in the Appendix A5.2 of our book [34]. Furthermore, we have also shown that an equation such as (3.32) with a possibly nonvanishing right-hand side could be established without a variational formulation at hand and no application of any Noether theorem (Chap. 5 in Maugin [34])—but with a mimicking of Noether's identity. This fortifies the view of the secondary nature of stresses such as $\mathbf{M}$ or $\mathbf{b}$ compared to the Cauchy stress.

[10] For this see Eq. (4.26) in Maugin [34] and select the ϕ^α there as the three components of the direct motion $\mathbf{x} = \bar{\mathbf{x}}(\mathbf{X}, t)$ between the reference (material) configuration and the actual (physical, i.e., Eulerian) one.

$$\nabla_\alpha T^\alpha_{.\beta} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{-g}T^\alpha_{.\beta}\right) - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\beta}T^{\mu\nu} = 0. \tag{3.35}$$

The quantity which must be conserved is the 4D (four-dimensional) momentum of matter *plus* gravitational field. But the latter is not included in $T^\alpha_{.\beta}$ by its very definition. In other words, this 4D momentum must be the *canonical four-momentum* associated with the *whole* physical system. This means that the corresponding conservation law must present a flux ("stress-energy-momentum tensor") that includes a term that accounts for the gravitational effect. A possible solution is given by Landau and Lifshitz [26] where the additional contribution to the energy-momentum tensor is called the *pseudo-energy momentum tensor of gravitation*. We denote by $G^\alpha_{.\beta}$ this new object so that the looked for local conservation law should read

$$\frac{\partial}{\partial x^\alpha}\left[\sqrt{-g}\left(T^\alpha_{.\beta} + G^\alpha_{.\beta}\right)\right] = 0 \tag{3.36}$$

Accordingly, the four-momentum defined by

$$P^\alpha = \frac{1}{c}\int \sqrt{-g}\left(T^{\alpha\beta}_{.} + G^{\alpha\beta}_{.}\right)dA_\beta \tag{3.37}$$

will be conserved, where dA_β is a space-time surface element and the integration is taken over any infinite space-time hypersurface that contains the whole of three-dimensional space. Here $G^{\alpha\beta}$ is symmetric although not a true tensor (hence the denomination of *pseudo*-tensor). The Landau-Lifshitz definition of $G^{\alpha\beta}$ may be given as

$$G^{\alpha\beta} = \frac{1}{2(-g)}\frac{c^4}{8\pi k}\frac{\partial^2}{\partial x^\mu \partial x^\nu}\left((-g)\left(g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\mu}g^{\beta\nu}\right)\right). \tag{3.38}$$

As usual with his characteristic economy of words and formulas and his dedication to the beauty of the said equations, ([13], Eq. (32.5))[11] Paul Dirac establishes a formula for $G^\alpha_{.\beta}$ by applying Noether's theorem to the Lagrangian density L_g of the gravitational field with

$$G^\alpha_{.\beta}\sqrt{-g} = \frac{\partial L_g}{\partial g_{\gamma\delta,\alpha}}g_{\gamma\delta,\beta} - L_g g^{\alpha}_{.\beta}. \tag{3.39}$$

One obtains thus

$$2\left(8\pi k/c^4\right)G^\alpha_{.\beta}\sqrt{-g} = \left(\Gamma^{\alpha}_{\mu\nu} - g^{\alpha}_{\nu}\Gamma^{\sigma}_{\mu\sigma}\right)\left(g^{\mu\nu}\sqrt{-g}\right)_{,\beta} - L_g g^{\alpha}_{\beta}, \tag{3.40}$$

where the Γ'*s* are Christoffel's symbols. This is Einstein's [16] proposal that may have been inspired by contemporary works by David Hilbert and Emmy Noether.

[11] See also the problem proposed in Landau and Lifshitz [26] at the end of their Section 100.

Contrary to the Landau-Lifshitz definition, the Einstein-Dirac pseudo-tensor $G^{\alpha\beta}$ is not symmetric. Of course (3.39) reminds us of the second of formulas (3.34) since these are canonical definitions in field theory.

In practice, the spatial part of (3.37) is obtained by considering the hypersurface $x^4 = const.$ so that we have the 3D space integral

$$P^i = \frac{1}{c}\int \sqrt{-g}\left(T^{i4} + G^{i4}\right)dV. \tag{3.41}$$

In the absence of gravitational field, this establishes a correspondence with the balance of a canonical momentum obtained in Eq. (3.32). This closes our disquisition about the notion of stress started with Cauchy's 1822/1828 pioneering work.

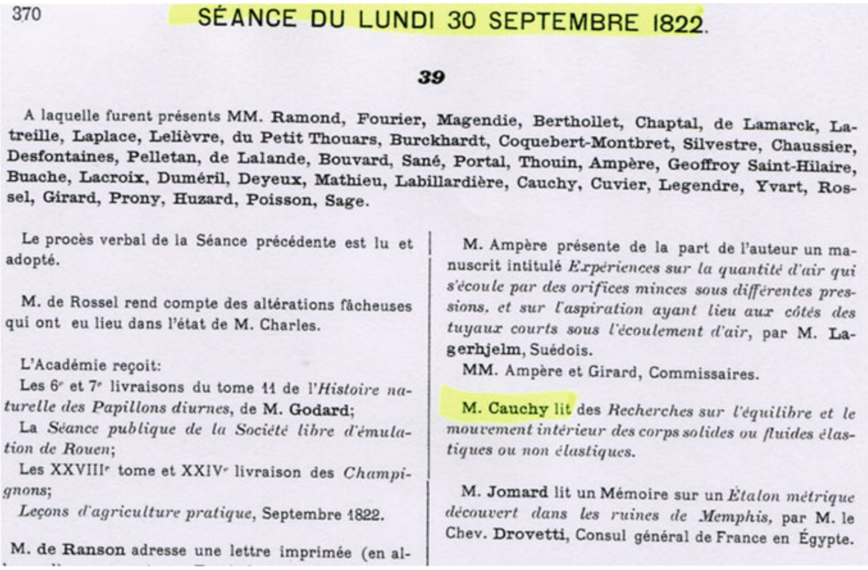
3.8 Conclusion

As witnessed by the above given exposition there is a long way between Cauchy's inception of the stress concept and Einstein-Dirac's pseudo tensor of stress-energy-momentum. We have explored this evolution in a rather pedestrian manner. What fundamentally remains from this is the essential role played by stress tensors or energy-momentum tensors that appear as true fluxes so that a corresponding physically meaningful conserved quantity (momentum) can be constructed. This is satisfied by Cauchy's initial construct of the stress because it provides at once the associated natural mechanical boundary conditions. This also holds for the first Piola-Kirchhoff stress, but not for derived definitions such as those of the second Piola-Kirchhoff stress, Mandel stress and Eshelby's stress in classical continuum mechanics. It is this kind of physical-theoretical argument which materialized in the introduction of the pseudo-tensor of energy-momentum in Einstein's gravitation theory as shown in the foregoing section. Having recurrently emphasized the role of Noether's theorem [37, 38], we also note that Cauchy's original proposal of 1822 and Green's divergence theorem (possibly generalized to space-time) accordingly remain the two basic tenets of continuum theory in spite of all progress achieved since that innocuous—but memorable for our community—day of September 30, 1822. We can say that 1828, with the inception of Green's divergence theorem and Cauchy's detailed presentation of his lemma, was a true *annus mirabilis* for continuum mechanics, providing thus the *fons et origo* of this science.

Acknowledgments Heartful thanks go to Dr Martine Rousseau in Paris and Professor James Casey in Berkeley for their critical careful reading of this contribution that led to much improvement and readability. Mme Florence Greffe ("Conservateur du Patrimoine") from the Archives Library of the Paris Academy of Science is to be thanked for her definite help in providing the "birth certificate" of continuum mechanics.

Appendix A

Augustin L. Cauchy (1789–1857)



370 SÉANCE DU LUNDI 30 SEPTEMBRE 1822.

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A laquelle furent présents MM. **Ramond, Fourier, Magendie, Berthollet, Chaptal, de Lamarck, Latreille, Laplace, Lelièvre, du Petit Thouars, Burckhardt, Coquebert-Montbret, Silvestre, Chaussier, Desfontaines, Pelletan, de Lalande, Bouvard, Sané, Portal, Thouin, Ampère, Geoffroy Saint-Hilaire, Buache, Lacroix, Duméril, Deyeux, Mathieu, Labillardière, Cauchy, Cuvier, Legendre, Yvart, Rossel, Girard, Prony, Huzard, Poisson, Sage.**

Le procès verbal de la Séance précédente est lu et adopté.

M. de Rossel rend compte des altérations fâcheuses qui ont eu lieu dans l'état de M. Charles.

L'Académie reçoit:

Les 6ᵉ et 7ᵉ livraisons du tome 11 de l'*Histoire naturelle des Papillons diurnes*, de M. **Godard**;

La *Séance publique de la Société libre d'émulation de Rouen*;

Les XXVIIIᵉ tome et XXIVᵉ livraison des *Champignons*;

Leçons d'agriculture pratique, Septembre 1822.

M. de **Ranson** adresse une lettre imprimée (en al-

M. Ampère présente de la part de l'auteur un manuscrit intitulé *Expériences sur la quantité d'air qui s'écoule par des orifices minces sous différentes pressions, et sur l'aspiration ayant lieu aux côtés des tuyaux courts sous l'écoulement d'air*, par M. **Lagerhjelm**, Suédois.

MM. Ampère et Girard, Commissaires.

M. **Cauchy** lit des *Recherches sur l'équilibre et le mouvement intérieur des corps solides ou fluides élastiques ou non élastiques*.

M. **Jomard** lit un Mémoire sur un *Étalon métrique découvert dans les ruines de Memphis*, par M. le Chev. **Drovetti**, Consul général de France en Égypte.

The "birth certificate of modern continuum mechanics": Cauchy's reading of his ideas at the September 30, 1822 session of the *Académie des Sciences* in Paris (Procès verbal de l'Académie des Sciences, Tome VII, Décembre 1822, Imprimerie d'Abbadia, Hendaye, 1916; kindly provided by Mrs Florence Greffe, Acad. Sc. Paris; May 2013) The remarkable roster of scientists among the above list of attending academicians is stupendous: e.g., Fourier, Magendie, Berthollet, Chaptal, Lamark, Laplace, Lacroix, Cauchy, Cuvier, Legendre, Prony, Poisson.

Appendix B

A. L. Cauchy—1823: Researches on the equilibrium and internal motion of solid bodies or fluids, whether elastic or non-elastic.

Bulletin of the Société Philomatique, pp. 9–13, 1823, Paris.

(Translation from the French by G.A. Maugin)

The present researches were undertaken on the occasion of the publication of a memoir by M. Navier on August 14, 1820. Its author, with a view to establishing the equilibrium equation of an elastic plane, had considered two kinds of forces, some produced by dilatation or contraction, and the other by the flexion of this plane. Moreover, he had supposed, in his computations, that both these forces are perpendicular to lines or faces on which they are exerted. It seemed to me that these two kinds of forces could be reduced to one kind only, which should be always called tension or pressure, and is of the same nature as the hydrostatic pressure exerted by a fluid at rest on the surface of a solid body. However, the new "pressure" will not always be perpendicular to the faces on which it act, and is not the same in all directions at a given point. Expanding this idea, I arrived soon at the following conclusions.

If in a solid body, whether elastic or not elastic, we succeed to render rigid—in thought. [GAM]—and invariable a small volume element bounded by any surfaces, this small element will be subjected on its different faces and in any point of each of these, to a determined pressure or tension. This pressure or tension will be of the same type as the pressure that a fluid exerts on an element of the boundary of a solid body, save for the difference that the pressure exerted by a fluid at rest on the surface of a solid body is directed normal to this surface, from the outside to the inside, and is independent at each point of the orientation of the surface with respect to the coordinate planes, while—in our case [GAM]—the pressure or tension exerted at a given point can be oriented perpendicularly or obliquely to this surface, sometimes from the outside to the inside if there is condensation [i.e., contraction, GAM] and sometimes from the inside to the outside if there is dilatation, and it can depend on the angle made by the surface with the relevant planes. Furthermore, the pressure or tension exerted on any plane can easily be deduced, in both amplitude and direction, from the pressures or tensions exerted on three given orthogonal planes. I had reached this point when M. Fresnel, who came to me to talk about his works devoted to the study of light and which he had presented only in part to the Institute, told me that, on his own, he had obtained laws in which elasticity varies according to the various directions issued from a unique point, a theorem similar to mine. However, the theorem in question was far from being sufficient for my projected object of study, at that period, that was to formulate the general equations of equilibrium and internal motion of a body; and it is only in recent times that I succeeded to establish the proper new principles that yielded this result, and that, now, I will make known.

From the above mentioned theorem, it follows that the pressure or tension at each point is equivalent to the inverse of the vector radius of an ellipsoid. Three pressures or tensions that we call *principal* correspond to the three axes of this ellipsoid, and we can show [This remark here is in agreement with the last researches of M. Fresnel (See the Bulletin of May 1822)] that each of these is perpendicular to the plane on which its acts. Among these principal pressures or tensions there are a maximum pressure or tension and a minimum one. The other pressures or tensions are distributed symmetrically about these three axes. Moreover, the pressure or tension normal to each plane, i.e., the component, perpendicular to a plane, of the pressure or tension exerted on this plane, is proportional to the inverse of the squared vector radius of a second ellipsoid. Sometimes, this second ellipsoid is replaced by two hyperboloids, one with one sheet, the other with two sheets, which have the same centre, the same axes, and are asymptotic at infinity with a common second- degree surface, of which the edges point in the direction for which pressure or normal tension reduces to zero.

This being said, if we consider a solid body of varying shape and subjected to arbitrary accelerating forces, in order to establish the equilibrium equations of this solid body it will be sufficient to write that there is equilibrium between the motive forces that act on an infinitesimal element along three axes of coordinates, and the orthogonal components of external pressure or tension that act on the faces of this element. We will thus obtain three equations of equilibrium that include, as a particular case, the corresponding equations for the equilibrium of fluids. But, in a general case, these equations contain six unknown functions of the coordinates x, y, z. It remains to determine the value of these six unknown quantities. But the solution of this last problem varies with the nature of the body and its more or less perfect elasticity. Now we shall explain how one can solve this problem for elastic bodies.

When an elastic body is in equilibrium by virtue of arbitrary accelerating forces, one must assume that each molecule has been displaced from the position it occupied when the body was in its natural state. As a consequence of these displacements, there are around each point different condensations or dilatations in different directions. But it is clear that each dilatation produces a tension, and each condensation produces a pressure. Furthermore, I prove that the various condensation or dilatation about this point, decreased by or augmented of the unit, become equal, up to the sign, to the vector radii of an ellipsoid. I call *principal condensations or dilatations* those that occur along the axes of this ellipsoid, about which the others are distributed symmetrically. This being set, it is clear that in an elastic body, tensions or pressures depending only on the condensations or dilatations, are directed in the same directions as the principal condensations or dilatations. In addition, it is natural to assume, at least when the displacements of molecules are small, that the principal tensions or pressures are proportional to the principal condensations and dilatations, respectively. Admitting this principle, we arrive immediately at the equilibrium equations of an elastic body. In the case of very small displacements, the component, perpendicular to a plane, of the pressure or tension exerted on that plane, always is in the same ratio with the condensation or dilatation that occurs in the same direction, and the formulas for equilibrium reduce to four partial differential equations of which each one determine separately the condensation or dilatation in volume, while each of the others serves to fix the displacement parallel to one of the coordinate axes.

The equations of equilibrium of an elastic body being set, it is now easy to deduce by ordinary means the equations of motion. The latter still are four in number, and each of them is a linear partial differential equation with an added variable term. These equations are integrated by use of methods that I exposed in a previous memoir. One of these equations contains only the unknown that represents the condensation or dilatation in volume. In the particular case where the acceleration force becomes constant and keeps everywhere the same direction, this equation reduces to the propagation of sound in air, with the only difference is that the constant it contains, instead of depending on the height of a supposedly homogeneous atmosphere, depends on the linear dilatation or condensation of a body in a given pressure. One must conclude from this that the speed of sound in an elastic body is constant, like in air, but it varies from one body to another one depending on the matter of which it is made. This constancy is all the more remarkable that the displacements of molecules considered successively in fluids and elastic solids obey different laws.

My memoir is concluded by the formation of the equations of the internal motion of solid bodies completely devoid of elasticity. To arrive at this it is sufficient to suppose that in these bodies the pressures or tensions about a point in motion do not depend any more on the total condensations or dilatations that correspond to the absolute displacement measured from the initial positions of the molecules, but only, after any lapse of time, on the very small condensations or dilatations that correspond to the respective displacement of the different points during a short interval of time. One therefore finds that the volume condensation is determined by an equation similar to that governing heat, what establishes a remarkable analogy between the propagation of the caloric [the supposed "fluid" carrying heat. GAM] and the vibrations of a body entirely devoid of elasticity.

In a forthcoming memoir, I shall give the application of the obtained formulas to the theory of elastic plates and strings.

References

1. Belhoste B (1991) Augustin-Louis Cauchy: a biography. Springer, New York (English trans: French original "Cauchy, 1789–1857", Editions Belin, Paris)
2. Casal P (1963) Capillarité interne en mécanique. CR Acad Sci Paris 256:3820–3822

3. Cauchy AL (1823) Recherches sur l'équilibre et le mouvement intérieur des corps solides ou fluides, élastiques ou non élastiques. Bull Soc Filomat Paris 9–13
4. Cauchy AL (1828) Sur les équations qui expriment l'équilibre ou les lois du mouvement intérieur d'un corps solide élastique ou non élastique. Exercices de mathématiques, vol 2, pp 160–187, Sept 1828 (This presents in print the ideas originally submitted to the Paris Academy of Sciences on 30 Sept 1822)
5. Cauchy AL (1828) In: Exercices de Mathématiques, vol 3, Sept 1828, pp 188–212, and «De la pression ou tension dans un système de points matériels» Oct 1828, pp 213–236
6. Cosserat E, Cosserat F (1909) Théorie des corps déformables. Hermann, Paris (reprint, Gabay, Paris, 2008)
7. Costa de Beauregard O (1944) Sur les équations fondamentales, classiques, puis relativistes, de la dynamique des milieux continus. J Math Pures Appl 23:211–217
8. Costa de Beauregard O (1945) Définition covariante de la force. CR Acad Sci Paris 221:743–747
9. Costa de Beauregard O (1946) Sur la théorie des forces élastiques. CR Acad Sci Paris 222:477–479
10. Dahan-Dalmenico A (1984–1985) La mathématisation de la théorie de l'élasticité par A.L. Cauchy et les débats dans la physique mathématique française (1800–1840). Sciences et techniques en perspective 9:1–100
11. Dell'Isola F, Seppecher P (1995) The relationship between edge contact forces, double forces and intersticial working allowed by the principle of virtual power. CR Acad Sci Paris IIb 321:303–308
12. Dell'Isola F, Seppecher P, Madeo A (2012) How contact interactions may depend on the shape of Cauchy cuts in N-th gradient continua: approach *à la* d'Alembert. Zeit angew Math und Physik 63(6):1119–1141
13. Dirac PAM (1975) General theory of relativity. Wiley, New York (reprint: Princeton University Press, Princeton, 1996)
14. Duhem P (1903) L'évolution de la mécanique (published in seven parts in: Revue générale des sciences, Paris; as a book: A. Joanin, Paris, 1906) (English trans: The evolution of mechanics, Sijthoff and Noordhoff, 1980)
15. Eckart CH (1940) The thermodynamics of irreversible processes III: relativistic theory of the simple fluid. Phys Rev 58:919–924
16. Einstein A (1916) Das Hamiltonisches Prinzip und allgemein Relativitätstheorie. Sitzungsber preuss Akad Wiss 2:1111–1115 (also, ibid, 1 (1918): 448–459)
17. Eringen AC (2002) Nonlocal continuum field theories. Springer, New York
18. Eringen AC, Maugin GA (1990) Electrodynamics of continua, (Two volumes), Springer, New York
19. Eshelby JD (1951) Force on an elastic singularity. Phil Trans Roy Soc Lond A244:87–111
20. Germain P (1973) La méthode des puissances virtuelles en mécanique des milieux continus, Première partie: théorie du second gradient. J Mécanique (Paris) 12:235–274
21. Green G (1828) An essay on the mathematical analysis of electricity and magnetism, privately printed, Nottingham (also in: mathematical papers of George Green, Ferrers NM (ed) Macmillan, London, 1871; reprinted by Chelsea, New York, 1970)
22. Grattan-Guinness I (1993) The ingénieur-savant (1800–1830): a neglected figure in the history of French mathematics and science. Sci Context 6(2):405–433
23. Gurtin ME (1999) Configurational forces as basic concepts of continuum physics. Springer, New York
24. Korteweg DJ (1901) Sur la forme que prennent les équations du mouvement des fluides si l'on tient compte des forces capillaires causées par des variations de densité considérables mais continues et sur la théorie de la capillarité dans l'hypothèse d'une variation de la densité. Arch Néer Sci Exactes et Nat Sér. II 6:1–24
25. Kunin IA (1982) Elastic media with microstructure I and II. Springer, Berlin (trans: 1975 Russian edition Kröner E, ed)

26. Landau LD, Lifshitz EM (1971) The classical theory of fields, vol 2 of course on theoretical physics, 3rd edn (trans: Russian). Pergamon Press, Oxford
27. Le Corre Y (1956) La dissymétrie du tenseur des efforts et ses conséquences. J Phys Radium 17:934–939
28. Le Roux J (1911) Etude géométrique de la torsion et de la flexion, dans les déformations infinitésimales d'un milieu continu. Ann Ecole Norm Sup 28:523–579
29. Mandel J (1971) Plasticité classique et visco-plasticité. CISM Lecture Notes, Udine, Italy
30. Maugin GA (1975) On the formulation of constitutive laws in the relativistic mechanics of continua (in French) (Main document of) Thèse de Doctorat ès Sciences Mathématiques, Université de Paris-VI, p 164, Paris
31. Maugin GA (1980) The method of virtual power in continuum mechanics: application to coupled fields. Acta Mech 35:1–70
32. Maugin GA (1988) Continuum mechanics of electromagnetic solids. North-Holland, Amsterdam
33. Maugin GA (2010) Generalized continuum mechanics: what do we understand by that? In: Maugin GA, Metrikine AV (eds) Mechanics of generalized continua: one hundred years after the Cosserats. Springer, New York, pp 3–13
34. Maugin GA (2011) Configurational forces. CRC/ Chapman & Hall/Taylor and Francis, Boca Raton
35. Maxwell JC (1873) Treatise on electricity and magnetism. Clarendon Press, Oxford
36. Murdoch IA (2012) Physical foundations of continuum mechanics. Cambridge University Press, UK
37. Neuenschwander DE (2011) Emmy noether's wonderful theorem. Johns Hopkins University Press, Baltimore
38. Noether EA (1918) Invariante variationsprobleme. Nach Akad Wiss Gïttingen, Math-Phys, Kl. II: 235–257 (English trans: Tavel M, Transp Theory Stat Phys I: 186–207, 1971)
39. Noll W (1974) The foundations of classical mechanics in the light of recent advances in continuum mechanics. Reprinted in W. Noll, The foundations of mechanics and thermodynamics (selected papers of W. Noll). Springer, Berlin, pp 32–47
40. Noll W, Virga E (1990) On edge interactions and surface tension. Arch Rat Mech Anal 111:1–31
41. Norton JD (1993) General covariance and the foundations of general relativity: eight decades of dispute. Rep Prog Phys 56:791–858
42. Piola G (1836) Nuova analisi per tutte le questioni della meccanica moleculare. Mem Mat Fis Soc Ita. Modena 21(1835):155–321
43. Piola G (1848) Intorno alle equazioni fondamentali del movimento di corpi qualsivoglioni considerati secondo la naturale loro forma e costituva. Mem Mat Fiz Soc Ital Modena 24(1):1–186
44. Timoshenko SP (1953) History of the strength of materials. McGraw Hill, New York (Dover reprint, New York, 1983) (review of this book by Truesdell reprinted in Truesdell 1984, pp 251–253)
45. Truesdell CA (1968) Essays in the history of mechanics. Springer, New York
46. Truesdell CA (1984) An idiot's fugitive essays on science. Springer, New York
47. Truesdell CA, Toupin RA (1960) The classical theory of fields. Handbuch der Physik, Ed. S.Flügge, Bd III/1, Springer, Berlin
48. Truesdell CA, Noll W (1965) The Nonlinear field theory of mechanics. Handbuch der Physik, Ed. S. Flügge, Bd III/3, Springer, Berlin
49. Van Dantzig D (1939) Stress tensor and particle density in special relativity. Nature (London) 143:855

Chapter 4
Piola and Kirchhoff: On Changes of Configurations

Abstract The seminal contribution of Gabrio Piola to the foundations of continuum mechanics is critically examined directly on the basis of his publications (1825–1848). This emphasizes the original approach of Piola who favoured a direct projection on the material configuration (where material particles are "labelled"), this yielding the now well known Piola–Kirchhoff stresses in the so-called Piola format of continuum mechanics. Piola is a follower of Lagrange and Poisson, much more than of Cauchy. But he established the connection of his equations with those of the more familiar Euler–Cauchy format (expressed in the actual configuration) of elasticity. Kirchhoff, much more known than Piola because of his renowned works in electricity, spectroscopy and thermo-chemistry, also contributed to the same format as Piola, hence his name attached to that of Piola. The works of Piola acquired a well deserved recognition and an excellent range of applications with the expansion of nonlinear elasticity, the modern theory of material inhomogeneities and the notion of configurational forces.

4.1 Introduction

It is agreed upon [33] that Euler and Lagrange are responsible for the introduction of two kinematical descriptions of the motion of deformable continua, emphasizing the dependence on actual or initial (Lagrangian) coordinates. In their time, this was particularly well exploited in fluid mechanics. However, with the consideration of possible finite deformations, essentially by Cauchy in France [6] and Green in the UK [10], in the framework of elasticity, the relationship between two configurations—the actual one after deformation and perhaps one chosen appropriately to label the "material particles" in a convenient way—became a necessity. It was to be the role of Gabrio Piola in Italy and Gustav Kirchhoff in Germany to clarify this matter, so that the two names are often associated to designate certain entities, e.g., the *Piola–Kirchoff stress tensors*. This possible duality between two kinematic descriptions of course entails the possibility to write the basic equations

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governing the dynamics of continua in two formats, that are now called the Euler–Cauchy and Piola–Kirchhoff formats as they involve the use of different—or "transformed"—tensorial objects (in particular as regards the stress). Of essential importance here is the relationship between tensorial objects expressed in the two different formats. This was practically solved by Piola with the introduction of the (now called) *Piola transformation*, a notion also referred to as pull-back operation (and its inverse the pull-forward [18]). Here we shall critically examine how Piola constructed "his format" of equations by perusing his original works of the period 1825–1848 [24–27] . This format acquires its full importance in the formulation of the theory of material inhomogeneities [20] and the theory of configurational forces [21]. It is in fact our involvement in the expansion of these theories that kindled our interest in the original papers of Piola.

4.2 Piola's Contribution

4.2.1 Some Words of Caution

Gabrio Piola (1794–1850) is an Italian mathematician who was an enthusiastic disciple of Joseph Louis Lagrange (1736–1813), the well known Italian-French mathematician. He is, therefore, an ardent supporter of variational formulations in the Euler–Lagrange tradition. He is the author of generally lengthy memoirs. We must admit that these papers are difficult to read, in reason both of the obsolete mathematical terms and the somewhat antiquated Italian language. We shall focus attention on the memoirs of 1836[1] and 1845 with some comments on that of 1833. In doing so we have decided to translate in modern (intrinsic or indicial) notation many of Piola's mathematical expressions written at a time when neither vector nor tensor notions existed. Thus the motion mapping is given by Piola by the application

$$(a, b, c) \to (x, y, z)$$

at fixed time, and the summation is indicated by a big S (that we replace by a more familiar Σ). However we shall refer to Piola's equations by an indication such as [26, p. 259, Eq. 137]. Note that Piola's notation for motion and deformation is still used by the Cosserat brothers as late as 1909 (who are not much easier to read [22]). Also, it is remarkable that Piola demonstrates an unconscious capture of a hidden algorithm so that he does not always need to write all components of a vector or tensor equation explicitly, but he gives a hint of this matter to the reader. For the sake of simplification, we do not make any distinction between covariant and contravariant tensors, assuming Cartesian systems of coordinates. But this is not altogether correct.

[1] A microfilm copy of the memoirs of 1836 and 1845 was kindly provided to us in 1991 by the Municipal Library of Modena during our stay as a visiting professor of the Italian CNR in Pisa.

4.2.2 The Strategy of Piola

In perusing Piola's works from 1825 to 1848, we can distinguish three main lines that combine together to form a well defined strategy.

The first one is an attempt at avoiding the consideration of infinitesimals—as used by Lagrange—in the conception of the kinematics of moving points, but considering as first principle *the superposition of motions*. This is a line he developed in his competition essay of 1825 [24]. This follows the works of other Italian mathematicians such as Magistrini and Riccardi who criticized Lagrange's use of virtual velocities. This viewpoint studied by Capecchi [3, 4] appears somewhat strange to modern minds, although it does yield the classical form of the equation of motion of a free material point. We shall not dwell further in this matter.

The second line is the a priori consideration of the motion of an ensemble of points in interaction, following Poisson (and also the second theory of continua of Cauchy), and then passing to a limit providing equations for a continuum, with an appropriate definition of what will later on be called the first Piola–Kirchoff stress tensor. This will be examined in greater detail herein below. This is developed at length in Piola [26]. Note that Piola there refers frequently to French mathematicians and mechanicians (Cauchy, Laplace, Poisson, Legendre, Lacroix, and of course Lagrange, hardly a Frenchman to him).

The third line is none other than an application of the Euler–Lagrange variational formulation accounting for possible mathematical constraints such as that due to rigidity. This necessitates the introduction of Lagrange multipliers [26] which are revealed to look like stress tensors. Of course here Piola follows the teaching of his master Lagrange (see the latter's lectures in Lagrange [17]. As shown by Piola [27], in the case of deformable bodies, this formulation in fact leads to the introduction of true stress tensors (Piola stress or Cauchy stress depending on the original definition of rigidity) in a rather formal manner that reminds us of the formulation of the principle of virtual power by Germain [9] or Maugin [19] where linear continuous forms on a set of generalized velocities are introduced a priori with "stresses" as co-factors. Such an approach permits the deduction of the accompanying natural boundary condition involving the stress. As noted by Truesdell and Toupin [33, p. 596, Footnote 3], this was the first deduction of such conditions from a variational principle. Piola's approach will be briefly described in the following paragraphs.

4.2.3 Introduction of the "Piola Format" by Piola

Following Poisson, Piola [26] considers identical point particles of unit mass that we can label (α). Each one is initially at position denoted by (a, b, c) with label (α) and after motion at position (x, y, z) with label (α). In modern notation this would yield the change of position as $x_i^{(\alpha)}$ *or* $\mathbf{x}^{(\alpha)}$ function of $X_K^{(\alpha)}$ *or* $\mathbf{X}^{(\alpha)}$, in Cartesian tensor notation and intrinsic notation, respectively. Thus the kinematic description may be said to be *referential*. With externally applied force $\mathbf{f}^{(\alpha)}$, and a model of interactions between particles (called "molecules") whose exploitation is somewhat obscure, Piola is able to write a variational formulation of the following type (p. 173, Eq. (15))

$$\sum_i \sum_\alpha \left(\frac{d^2 x_i^{(\alpha)}}{dt^2} - f_i^{(\alpha)} \right) \delta x_i^{(\alpha)} + \sum_i \sum_{\alpha,\beta} \phi\left(S_{\alpha,\beta}\right) \delta S_{\alpha,\beta}|_i = 0 \tag{4.1}$$

where the S's—whose details are irrelevant—depend on the relative distances between particles, hence on the x_i. For arbitrary variations of the x_i this formally yields equations of motion of individual particles in the form [26, p. 189, Eq. (41)]

$$f_i^{(\alpha)} - \frac{d^2 x_i^{(\alpha)}}{dt^2} + I_i^{(\alpha)} = 0, \quad \alpha = 1, 2, \ldots \tag{4.2}$$

where $I_i^{(\alpha)}$ is the interaction force with other particles that we do not elaborate further. The "tour de force" of Piola rests in the approximation of these interaction terms (pp. 175–200) and passing to some kind of continuum limit that brings the generic local equation of motion to the vectorial form [26, p. 201, Eq. (56)]

$$\mathbf{f} - \frac{\partial^2 \mathbf{x}}{\partial t^2} + div_X \mathbf{T} = \mathbf{0}, \tag{4.3}$$

where $\mathbf{T}$ is an object with nine independent components (for it has no symmetries) and the modern symbol div_X means the divergence operator with respect to the referential coordinates (a, b, c) i.e., X^K. Obtaining (4.3) involves the neglect of supposedly small terms. Equation (4.3) can also be written as

$$f_i - \frac{\partial^2 x_i}{\partial t^2} - \frac{\partial}{\partial X_K} T_{Ki} = 0, \tag{4.4}$$

in Cartesian tensor analysis.

The change of position and its inverse (assuming invertibility in agreement with Lagrange) can be noted [26, p. 202, Eqs. (58)–(59)]

$$(x, y, z) \text{ functions of } (a, b, c) \text{ and time } t$$

and

$$(a, b, c) \text{ functions of } (x, y, z) \text{ and time } t$$

or in modern notation

$$\mathbf{x} = \bar{\mathbf{x}}(\mathbf{X}, t) \text{ and } \mathbf{X} = \bar{\mathbf{X}}(\mathbf{x}, t). \tag{4.5}$$

Let J denote the Jacobian determinant of the first of these transformations (this is denoted H by Piola, p. 204), i.e.,

$$J = \det \mathbf{F},\ \mathbf{F} = \left\{ F_{iK} = \frac{\partial \bar{x}_i}{\partial X_K} \right\}. \tag{4.6}$$

For a continuum, this is as if (4.3) or (4.4) had been written for a body of referential mass density $\rho_0 = 1$. If this is not the case, ρ_0 has to be introduced and (4.3) has to be rewritten as

$$\rho_0 \left(\mathbf{f} - \frac{\partial^2 \mathbf{x}}{\partial t^2} \right) + div_X \mathbf{T} = \mathbf{0}. \tag{4.7}$$

Then Piola would like to compare his equation of motion with the formulation obtained by Cauchy [6] and Poisson [29] in the actual configuration. To do this he needs some work since he must pass to the spatial parametrization of the Eulerian type in terms of the actual position (x, y, z) or $\mathbf{x} = \{x_i; \quad i = 1, 2, 3\}$. He shows that he can introduce a geometrical object $\boldsymbol{\sigma}$ (noted $\boldsymbol{K}$ by Piola) such that [26, p. 204, Eq. (60)]

$$\sigma_{ij} = J^{-1} \frac{\partial \bar{x}_i}{\partial X_K} T_{Kj} \quad or \quad \underline{\sigma} = J^{-1} \mathbf{F}\ \mathbf{T}. \tag{4.8}$$

Reciprocally (Piola 1936, p. 205, Eq. (63))

$$\mathbf{T} = J\mathbf{F}^{-1} \cdot \underline{\sigma} \quad or \quad T_{Ki} = J \frac{\partial \bar{X}_K}{\partial x_j} \sigma_{ji}. \tag{4.9}$$

He establishes identities like [26, p. 205, Eq. (62)]

$$div_X \mathbf{T} = J\ div_x \underline{\sigma}, \tag{4.10}$$

where div_x means the divergence with respect to the (x, y, z) or $\mathbf{x} = \{x_i;\ i = 1, 2, 3\}$ space parametrization. This involves proving the identities

$$\nabla_x \cdot \left(J^{-1} \mathbf{F} \right) = \mathbf{0} \quad \text{and} \quad \nabla_X \cdot \left(J\mathbf{F}^{-1} \right) = \mathbf{0}. \tag{4.11}$$

Noting that in his format the mass conservation reads (p. 211, Eq. (72) with Γ—which Piola does not yet call density—standing for ρ)

$$\rho_0 = J\rho, \tag{4.12}$$

where ρ is the actual density at (x, y, z), Piola finally shows that Eq. (4.7) above renders the equation of motion (p. 212, Eq. (74))

$$\rho \left(\mathbf{f} - \frac{d^2 \mathbf{x}}{dt^2} \right) + div_x \underline{\sigma} = \mathbf{0}. \tag{4.13}$$

This he identifies with the equation obtained by Cauchy [6, p. 166] or Poisson [29, VIII, p. 387; X, p. 578]. Accordingly, $\boldsymbol{\sigma}$ is none other than the Cauchy stress tensor for any continuum, whether solid or fluid, while $\mathbf{T}$ deserves to be called the *Piola stress* (*first Piola–Kirchhoff stress* in modern jargon). The word density ("densità") here is used for the quantity J^{-1} since $\rho_0 = 1$ for Piola. Equilibrium is obtained by making the acceleration term vanish in Eq. (13)—[26, p. 215, Eq. (79)].

What is original here with Piola is that he has formulated what we call the "Piola format" of the basic equations of continuity—Eq. (4.12)—and of balance of linear momentum. His "format" involves two configurations with a preference for the referential one for the space parametrization. It is sometimes called the *material* formulation [21] since $\mathbf{X}$ refers directly to the material "points" that belong to the "material manifold". The only inconvenience is the appearance of geometrical objects such as $\mathbf{F}$ and $\mathbf{T}$ that have two "feet" in different configurations and will later on be called *two-point tensor fields*—i.e., tensors depending on two "points"—by Einstein or, here precisely *double vectors*. But it must be understood that all computations are effected by Piola with all explicit scalar components of the introduced objects since he has no notion of a tensor (only introduced in the 1880s by Voigt).

The celebrated *Piola transformation* here is represented by Eq. (4.9) that is even made clearer when applied to a vector field. Let $\mathbf{v}$ a vector field with components in the actual framework $\mathbf{x} = \{x_i;\ i = 1, 2, 3\}$. The associated vector field $\mathbf{V}$ in the framework $\mathbf{X} = \{X_K;\ K = 1, 2, 3\}$ is defined by accounting both for the deformation and the volume change; that is:

$$\mathbf{V} = J\mathbf{F}^{-1} \cdot \mathbf{v}. \tag{4.14}$$

This is the Piola transformation—or *pull back* to the reference configuration. The inverse operation is called the *push forward* from reference configuration to actual configuration. Equation (4.9) that defines the first Piola–Kirchhoff stress is thus only a *partial* Piola transformation of the Cauchy stress. This is the rather troubling matter (with many students). But this manipulation allows one to obtain an equation of motion (4.7) with good partial differential derivatives in the space-time parametrization (X_K, t) while this equation still has components in the actual configuration where data in forces are prescribed. For the transformation of boundary conditions on stresses one will have to wait for the formulas obtained by Nanson [23] for the transformation of oriented surface elements. It is not forbidden to construct the full pull-back of the Cauchy stress by completing the transformation (4.9) by defining the fully material stress $\mathbf{S}$ by {T = transposed; [25, Eq. (45)]; [26, Eq. (132)]}

$$\mathbf{S} = \mathbf{T} \cdot \mathbf{F}^{-T} \ or \ S_{KL} = T_{Ki} F_{iL}^{-1} = J \frac{\partial X_K}{\partial x_i} \sigma_{ij} \frac{\partial X_L}{\partial x_j}. \tag{4.15}$$

This is called the *second Piola–Kirchhoff stress* in modern continuum mechanics. It is a true material tensor; it is *symmetric* by construction if the Cauchy stress is symmetric (which is more than often the case). In contrast, it does no make mathematical sense to speak of the symmetry or non-symmetry of $\mathbf{T}$. The thermodynamic importance of $\mathbf{S}$ will be made clear soon in Green's elasticity derived from a potential.

Piola still has to be more precise with the notion of *matter density*. He ponders this notion in his Chap. IV (p. 218 on) where, basically, it is mass divided by volume—valid only for a homogeneous volume—as otherwise the correct definition should involve a limit procedure applied to an infinitesimal element of matter at each point. Note that for Newton it was mass that was defined by density multiplied by volume. Starting from Eq. (4.12) and noticing that in Piola's time ρ_0 may at most be a function of (a, b, c) or $\mathbf{X}$, and not a function of time (this is no longer true in the modern theory of material growth [21, Chap. 10]), a laborious computation leads Piola to the *equation of continuity* in the Eulerian form (Piola, p. 235, Eq. 105):

$$\frac{\partial \rho}{\partial t} + \nabla_x \cdot (\rho \mathbf{v}) = 0, \tag{4.16}$$

where $\mathbf{v} = d\mathbf{x}/dt$ is the velocity. Thus (4.16) and (4.13) correspond to (4.12) and (4.7), respectively.

The rest of the impressive Piola's paper of 1836 concerns the introduction of the displacement for a continuum and the notions of dilatation and condensation (in Cauchy's sense), and many more considerations on the molecular description of the material, whose purpose in principle is to deduce explicit expressions for the interactions introduced in Eq. (4.1) above—in particular with the notion of pressure, and a theory of fluids in concurrence with one expanded by Poisson [30, p. 524]. This goes beyond the present focus.

4.2.4 Stresses as Lagrange Multipliers

As a good disciple of Lagrange, Piola exploits the technique of multipliers to account for constraints. Lagrange had done this for the constraint of incompressibility introducing thus a scalar multiplier that is a mechanical pressure. In Piola [25], the author wants to do it for the constraint of *rigidity* of extended bodies, perhaps as a preparation for the case of deformable bodies [27]. The formulation he offers is quite original albeit a little bit involved. We consider again Piola's original notation (a, b, c) and (x, y, z) for the initial and final positions of any point in the body. The a priori motion is written by Piola as

$$x = f + a_1 a + b_1 b + g_1 c,\ y = g + a_2 a + b_2 b + g_2 c,\ z = h + a_3 a + b_3 b + g_3 c, \tag{4.17}$$

which contains twelve scalar parameters. In modern vector and matrix notation this reads

$$x_i = g_i + \sum_{K=1}^{3} a_{iK} X_K,\ i = 1, 2, 3. \tag{4.18}$$

For a true rigid body motion the number of parameters must be reduced to six (three translations and three rotations). Then by astute manipulations (Piola does not possess the notion of matrix) that look terrible to modern eyes, Piola, by eliminating the parameters, succeeds to express the conditions of orthogonality and normality in differential forms for the function $x_i = \bar{x}_i(X_K)$. Two alternate forms are obtained that we rewrite (here in modern notation) as

$$\bar{e}_{ij} = \sum_{K=1}^{3} F_{iK} F_{jK} - \delta_{ij} = 0 \tag{4.19}$$

and

$$\bar{E}_{KL} = \sum_{i=1}^{3} F_{iK} F_{iL} - \delta_{KL} = 0, \tag{4.20}$$

where the deltas are Kronecker symbols and $F_{iK} := \partial \bar{x}_i / \partial X_K$. In setting

$$e_{ij} = \bar{e}_{ij}/2, \quad E_{KL} = \bar{E}_{KL}/2, \tag{4.21}$$

we recognize with the first part of (4.19) and (4.20) the definition of the *left and right Cauchy–Green strain tensors* of non-linear continuum mechanics up to a factor ½ [33], i.e., in intrinsic notation (with the notation of Maugin [20])

$$\mathbf{e} = \frac{1}{2}\left(\mathbf{c}^{-1} - \mathbf{1}\right), \quad \mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{1}_R) \tag{4.22}$$

with

$$\mathbf{c}^{-1} := \mathbf{F} \cdot \mathbf{F}^T, \quad \mathbf{C} := \mathbf{F}^T \cdot \mathbf{F}. \tag{4.23}$$

The rigid-body conditions (4.19) and (4.20) read

$$\mathbf{c}^{-1} = \mathbf{1}, \quad \mathbf{C} = \mathbf{1}_R, \tag{4.24}$$

where $\mathbf{1}_R$ is the unit tensor in the reference configuration. $\mathbf{C}^{-1}$, the inverse of $\mathbf{C}$, is called the *Piola (material) finite strain*. The tensor $\mathbf{c}$, inverse of $\mathbf{c}^{-1}$, is called the (spatial) *Finger finite strain*. ($\mathbf{c}^{-1}$ is sometimes noted $\mathbf{B}$ so that $\mathbf{c}$ would be $\mathbf{B}^{-1}$).

Conditions (4.24) are integrated forms while it is usual to express the rigidity condition in time differential form (in terms of the rate of strain) equivalent to Killing's theorem, e.g.,

$$\dot{\mathbf{C}} = 0 \tag{4.25}$$

or in variational form $\delta\mathbf{C} = \mathbf{0}$ or $\delta\mathbf{E} = \mathbf{0}$.

The total virtual work of body forces (per unit mass) for a body of referential volume V_0 is given by

$$\delta W^{ext} = \int_{V_0} \rho_0 \mathbf{f} \cdot \delta\mathbf{x} \, dV_0. \tag{4.26}$$

If this body is to be *rigid*, then either one of the mathematical constraints (4.19) and (4.20) must be taken into account. Following Lagrange [17], this is done by introducing *Lagrange multipliers*, here tensors of components λ_{ij} or Λ_{KL} so that the principle of virtual work for equilibrium is written in any of the following two forms (with summation over repeated indices and $dm = \rho_0 dV_0 = \rho dV$):

$$\int_V \rho \mathbf{f} \cdot \delta \mathbf{x}\, dV - \int_V \lambda_{ij} \delta e_{ij} dV = 0 \tag{4.27}$$

or

$$\int_{V_0} \rho_0 \mathbf{f} \cdot \delta \mathbf{x} dV_0 - \int_{V_0} \Lambda_{KL} \delta E_{KL} dV_0 = 0. \tag{4.28}$$

This is the essence of Piola's argument rewritten in modern formalism. Piola prefers the "material" formulation (4.28) over the Eulerian formulation (4.27). Transformation of (4.28) on account of (4.20, 4.21) and localization yield a local *equilibrium* equation in the form

$$\frac{\partial}{\partial X_K} G_{ki} + \rho_0 f_i = 0, \quad G_{Ki} := \Lambda_{KL} F_{iL} \; or \; \mathbf{G} := \Lambda \cdot \mathbf{F}^T. \tag{4.29}$$

Note that λ_{ij} and Λ_{KL} are related by

$$\Lambda_{KL} = J \frac{\partial X_K}{\partial x_i} \lambda_{ij} \frac{\partial X_L}{\partial x_j}, \tag{4.30}$$

a relation similar to (4.15). The introduced tensorial Lagrange multipliers can be interpreted as "reaction internal forces" needed to maintain the rigidity of the body. These internal forces are undetermined for a rigid body.

Of course the natural question is what happens for a deformable body. This was answered by Piola in his long memoir of 1848 (but presented in 1845). This is more formal in the sense that the reaction internal forces become the true *stresses* in action in the body. They are introduced as *coefficients* of the variation of strains in a linear form. That is (4.28) in principle is replaced by

$$\int_{V_0} \rho_0 \mathbf{f} \cdot \delta \mathbf{x}\, dV_0 - \int_{V_0} S_{KL} \delta E_{KL} dV_0 = 0, \tag{4.31}$$

yielding instead of (4.29)

$$\frac{\partial}{\partial X_K} T_{ki} + \rho_0 f_i = 0, \quad T_{Ki} := S_{KL} F_{iL} \; or \; \mathbf{T} := \mathbf{S}\mathbf{F}^T, \tag{4.32}$$

where $\mathbf{T}$ indeed is the first Piola–Kirchhoff stress. But this is not entirely correct because there can be a traction $\mathbf{t}$ acting on the boundary of the body so that, introducing also acceleration forces to obtain the dynamical case (4.31) should be re-written as

$$\int_{V_0} \rho_0 \ddot{\mathbf{x}} \cdot \delta \mathbf{x} dV_0 = \int_{V_0} \rho_0 \mathbf{f} \cdot \delta \mathbf{x}\, dV_0 - \int_{V_0} S_{KL} \delta E_{KL} dV_0 + \int_{\partial V_0} \mathbf{t} \cdot \delta \mathbf{x}\, dS_0. \quad (4.33)$$

This is quite similar to the formulation of the *principle of virtual power* used in modern times by Germain [9] and Maugin [19] —see also Truesdell and Toupin [33, p. 596, Eq. (232.4)]—that considers the right-hand side of Eq. (4.33) as a linear continuous form on virtual velocities including that of the gradient of the motion. Equation (4.33) allows one to obtain the local balance of linear momentum—here in the Piola format (4.7)—as also the accompanying natural boundary condition for stresses. It seems that Piola was really the first to deduce the stress boundary condition from a variational principle [27, Part 2, p. 52]. The original variational principle by Piola goes back to 1833 (Part 3) and 1848 (Part 2, pp. 34–38, 46–50). Hellinger [11, Chap. 4, Paragraph 3d] also dealt with the same variational principle. In addition, Piola formulated analogous variational principles for one-dimensional and two-dimensional systems [27, Chap. 7]. This can be compared with variational formulations by the Cosserat brothers [7]. Pierre Duhem [8] formulated a principle of virtual work that looks very much like the one of Piola for equilibrium [4, p. 390].

To conclude this point, we recall that George Green (1793–1841) introduced in his celebrated memoir of 1839 the same finite-strain tensors as Cauchy, hence the association of the two scientists in the denomination of these tensors. Furthermore, he simultaneously introduced the notion of strain energy W per unit of referential volume, $W = W(\mathbf{E})$, such that

$$\delta \int_{V_0} W(\mathbf{E}) dV_0 = \int_{V_0} \delta W(\mathbf{E}) dV_0 = \int_{V_0} \mathbf{S} : \delta \mathbf{E} dV_0 \quad (4.34)$$

and

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{E}} \; or \; S_{KL} = \frac{\partial W}{\partial E_{KL}}. \quad (4.35)$$

That is why the second Piola–Kirchhoff stress is also referred to as the *energetic stress* while $\mathbf{T}$ may also be called the *nominal stress* because it is evaluated per unit area in the reference configuration although it still behaves as a vector in the actual configuration.

Among the Italian disciples of Piola we must count Eugenio Beltrami (1836–1900) and to a lesser degree Enrico Betti (1823–1892). Like Lamé, Beltrami was a great amateur of curvilinear coordinate systems and his differential methods favoured the early development of tensor calculus in Italy. More to our point, in the application of the principle of virtual work (e.g., in Beltrami [1]), he considered internal forces (stresses) and deformations as dual variables, which is our modern view point with the notion of separating duality between two vector spaces (e.g., in Maugin [19]). As to Betti, although having started as a "Newtonian", he later on based his continuum mechanics on potential energy, strains and the principle of virtual work [2]. This is discussed by Capecchi [4, pp. 392–393].

4.3 The Role of Kirchhoff

Kirchhoff (1824–1887) is one of the German giants in continuum mechanics for the 19th century, although his reputation in electricity, spectroscopy, black-body radiation, and thermo-chemistry is at the same if not higher prestigious level. It is in Königsberg that Kirchhoff took lectures with Neumann, a specialist of the strength of materials. He later became a professor of physics in Breslau, Heidelberg and finally Berlin. Kirchhoff made many important contributions to continuum mechanics and the mechanics of structures [31]. For instance, he proposed a correct model for the bending of plates by means of a variational principle. The two-dimensional equation was deduced from a variational principle (principle of virtual work) in which a reduced potential energy accounts for a set of basic kinematic hypotheses concerning the section of the plate normal to the middle surface and the neglect of any stretching of the elements of the middle plane for small deflections. This much improved the tentative theory proposed earlier by Sophie Germain (1776–1831). This is now referred to as *Kirchhoff–Love theory of plates* after Love (1863–1940) who extended Kirchhoff's approach to the case of thin shells. Kirchhoff also studied theoretically and experimentally the vibrations of plates on the basis of his model. He also subsequently extended his theory of plates to include the case of not too small deflections.

If Kirchhoff is cited here it is because he also considered finite deformations, especially in Kirchhoff [14] —apparently independently of Piola—which is also reported in his lectures on mechanics in Kirchhoff [16]. He was thus led [14, pp. 763–764 and p. 767] to introducing stress tensors similar to those of Piola, hence the two names jointly attached to these geometric objects, even though the role of Kirchhoff in this very subject seems rather minor compared to that of Piola. Both Kirchhoff [14], and later on Poincaré [28; Paragraph 40], explained the "non-symmetry" of $\mathbf{T}$, but the present notation is clear enough. We can also note that if the Cauchy stress is symmetric, then we can also say that $\mathbf{T}$ *is symmetric with respect to* $\mathbf{F}$, because the local equation of moment of momentum (in the absence of internal spin, applied couple, and microstructure [7] $\underline{\sigma} = \underline{\sigma}^T$ also reads

$$\mathbf{F} \cdot \mathbf{T} = (\mathbf{F} \cdot \mathbf{T})^T. \qquad (4.36)$$

It must be emphasized that Kirchhoff's works in elasticity were among his first scientific works in the late 1840s and early 1850s, and then on and off during the rest of his career. Thus in 1850, he wrote down a variational principle which looks somewhat like Piola's one recalled in Eq. (4.31) above. However the second term was expressed in terms of the principal dilatations (in Cauchy's words)—Kirchhoff [13]. This was further expanded in Kirchhoff [15], the most important paper by Kirchhoff in elasticity according to Todhunter [32, p. 63]. Jungnickel and McCormmach [12, p. 295] rightly remark that his work in elasticity provided useful analogies for his works in electricity.

4.4 Conclusion

The interest for the developments recalled in the foregoing two sections mainly rests on two ingredients: one is the importance given to objects defined partially or entirely in the reference configuration such as the Piola–Kirchhoff stress tensors, clearly of utmost interest to Piola. The other is Piola's and Kirchhoff's interest in variational formulations of the type of the principle of virtual work. These formulations remained fashionable and efficient for a large part of the nineteenth century as evidenced by the works and lecture notes of Clebsch [5], Duhem [8], Poincaré [28], the Cosserat brothers [7], and Hellinger [11]. Concerning the first ingredient, it is with the development of *non-linear elasticity* in the 1930s–1950s in Italy, the UK and the USA—as richly illustrated in the encyclopaedia synthesis of Truesdell and Toupin [33]—that the necessity of clearly distinguishing between the actual configuration and a reference one was made clear to all students in continuum mechanics. This has become common practice. More recently, this fruitful approach to the deformation theory in general was enhanced by an inclusive definition of *material inhomogeneity*: this is the possible continuous or discontinuous dependency of material properties (e.g., elasticity) on the material point $\mathbf{X}$ itself, an "element" of the material manifold (this view was forcefully emphasized in Maugin [20]). A corollary of this was the required consideration of the full projection of all field equations on this material manifold leading to a true "material" mechanics of continua in the Piola–Kirchhoff format and the systematic introduction of the (completely material) *Eshelby stress tensor* as the most relevant internal-force concept and its application in the form of *configurational forces* to the theory of defects. This is amply documented in our treatise [21] which owes much in spirit to Eshelby (1916–1981), but also retrospectively much to the original thinking of Piola that we tried to capture in the present contribution.

References

1. Beltrami E (1880–1882) Sulle equazioni generali della elasticità. In: Opere matematiche, vol 3. Hoepli, Milano (1902–1920), pp 383, 407
2. Betti E (1874) Teoria dell'elasticità. Soldaini, Pisa
3. Capecchi D (2003) Gabrio Piola e la meccanica italiana agli inizi dell'ottocento. Atti del XXIII Congresso Nazionale di Storia della Fisica e dell'Astronomia. Bari, 5–7 June 2003, pp 95–108 [available on the web in pdf format]
4. Capecchi D (2012) History of virtual work laws: a history of mechanics prospective. Birkhäuser-Springer, Milan
5. Clebsch RFA (1862) Theorie der Elastizität fester Körper, Teubner, Leipzig [French translation with many additions and comments by A. Barré de Saint-Venant, Paris, 1886]
6. Cauchy AL (1828) Recherches sur les équations qui expriment l'équilibre du mouvement intérieur d'un corps solide, élastique ou non élastique. Exercices de Mathématiques, vol 3, 160–187, Sept 1828 [Also, in Oeuvres complètes, s II, vol 8, pp 195–226, Gauthier-Villars, Paris, 1882–1974]; Sur la pression ou tension dans un corps solide, *ibid*, vol 2, 42–57, March 1827; and in Oeuvres complètes, s II, vol 7, 60–78]

7. Cosserat E, Cosserat F (1909) Théorie des corps déformables. Hermann, Paris [Facsimile reprint by Gabay, Paris, 2009]
8. Duhem P (1891) Hydrodynamique, élasticité, acoustique. Hermann, Paris
9. Germain P (1973) La méthode des puissances virtuelles en mécanique des milieux continus, Première partie: théorie du second gradient. J. Mécanique (Paris) 12:235–274
10. Green G (1839) On the law of the reflection and refraction of light at the common surfaces of two crystallized media. Trans Cambridge Phil Soc 7: 1–24, 121–140 [also in mathematical papers of Green G, In: Ferrers NM (eds) McMillan, London, 1871; reprinted by Chelsea, New York, 1970]
11. Hellinger E (1914) Die allgemein Ansätze der Mechanik der Kontinua. In: Klein F, Wagner K (eds) Enz. Math.Wiss. vol 4, part 4. Springer, Berlin, pp 602–694
12. Jungnickel C, McCormmach E (1986) Intellectual mastery of nature (Theoretical physics from Ohm to Einstein), vol. 1. The torch of mathematics 1800–1870. University of Chicago Press, Chicago
13. Kirchhoff G (1850) Über das Gleichgewicht und die Bewegung einer elastischen Scheibe. Journ f d reine und angewandte Math 40:51–88
14. Kirchhoff G (1852) Über die Gleichungen des Gleichgewichts eines elastischen Körpers bei nicht unendlich kleiner Verschiebungen seiner Teile. Sitzgsber Akad Wiss Wien 9:762–773
15. Kirchhoff G (1859) Über das Gleichgewicht und die Bewegung eines unendlich dünnen elastische Stabes. Journ f d reine und angewandte Math 56:285–313
16. Kirchhoff G (1876) Vorlesungen über mathematische Physik: Mechanik (known as "Lectures on Mechanics"). Teubner, Leipzig [available on the site Gallica of the Bibliothèque Nationale de France (BNF)]
17. Lagrange JL (1806) Leçons sur le calcul des fonctions. Vve Courcier, Paris, p 462
18. Marsden JA, Hughes T (1975) Mathematical theory of elasticity. Academic Press, New York [also as a Dover reprint, New York]
19. Maugin GA (1980) The principle of virtual power in continuum mechanics: application to coupled fields. Acta Mech 35:1–70
20. Maugin GA (1993) Material inhomogeneities in elasticity. Chapman & Hall, London
21. Maugin GA (2011) Configurational forces. Chapman & Hall/Taylor and Francis, New York
22. Maugin GA (2013) About the Cosserats' book of 1909. Preprint UPMC, Paris [see also Chap. 8 in the present book]
23. Nanson EJ (1874) Note on hydrodynamics. Messenger of Mathematics 3: 120–123 [also, ibid, 7: 182–183, 1877–1878]
24. Piola G (1825) Sull'applicazione della meccanica analitica del Lagrange ai principali problemo. Regia Stamperia, Milan [Memoria di Gabrio Piola presentata al concorso del premio e coronata dall'I.R. Istituto di Scienze, ecc. nella solennita del Giorno 4 ottobre 1824]
25. Piola G (1833) La meccanica de'corpi naturalmente estesi tratta col calculo delle variazioni. In: Opusculi matematici e fisici di diversi autori, Giusti, Milan, pp 201–236
26. Piola G (1836) Nueva analisi per tutte le questioni della meccanica moleculare. In: Memorie di matematica e di fisica della Società Italiana delle Scienze, Modena 21: 155–321 (received 1835)
27. Piola G (1848) Intorno alle equazioni fondamentale del movimento dei corpi qualsivogliono, considerati secondo naturale loro forma e costituzione. In: Memorie di matematica e di fisica della Società Italiana delle Scienze, Modena 24(1): 1–186 (received 1845)
28. Poincaré H (1892) Leçons sur la théorie de l'élasticité. Georges Carré, Paris [Facsimile reprint, Gabay, 1990; also University of Michigan, Ann Arbor, 2012]
29. Poisson SD (1829) Mémoire sur l'équilibre et le mouvement de corps élastiques. Mémoires de l'Institut de France, vol VIII, 357–570 (and vol X, p 578)
30. Poisson SD (1833) Traité de mécanique, vol II. Bachelier, Paris
31. Timoshenko SP (1953) History of the strength of materials. McGraw Hill, New York [Dover reprint, New York, 1983]

32. Todhunter I, Pearson K (1893) A history of the theory of elasticity and the strength of materials from Galilei to the present time, vol. 2: Saint-Venant to Lord Kelvin. Cambridge University Press, UK
33. Truesdell C, Toupin RA (1960) The classical field theories. In: Flügge S (ed) Handuch der Physik, Bd III/1. Springer, Berlin

Chapter 5
Duhamel's Pioneering Work in Thermo-elasticity and Its Legacy

Abstract This is a short introduction to the original chapter published in 1837 by the French mathematician J. M. C. Duhamel, in which the first equations of thermo-mechanical couplings or thermo-elasticity are introduced for three-dimensional elastic bodies. This contribution offers a short discussion of the basic ideas behind Duhamel's historical chapter, the strategy applied by Duhamel for combining ideas from Navier's elasticity and Fourier's theory of heat propagation, the illustration by the solution of general equations in some well-chosen problems, and the heritage of Duhamel in thermo-mechanical sciences.

5.1 The Roots of Duhamel's Contribution

Jean-Marie-Constant Duhamel (1797–1872) was, starting in 1830, an assistant and then professor (1834) of analysis and mechanics at the celebrated Ecole Polytechnique in Paris. Contrary to other luminaries such as Cauchy, Navier, Coriolis and others, he was not an "engineer-scientist", having not graduated from this school (he was expelled from it in 1816 with all his class of 1814 for political reasons). That may explain why he became a pure mathematician! His works deal with partial differential equations with special attention to their applications in the theory of heat, rational mechanics, and acoustics. In 1834 he defended a Doctoral thesis in mathematics at the Sorbonne with title "*Théorie mathématique de la chaleur*" (Mathematical theory of heat). This collected the works he had done on this subject matter since the late 1820s. His style was mathematical, in fact ahead of experiments on the subject matter. His main interest was in combining the (then recent) works of Poisson and Navier (on elasticity) and Fourier (on heat) in a

This contribution is a revised and much extended version of a contribution entitled *Duhamel's pioneering work in thermo-elasticity* published in the Springer Encyclopaedia of Thermal stresses (cf. Hetnarski 2014).

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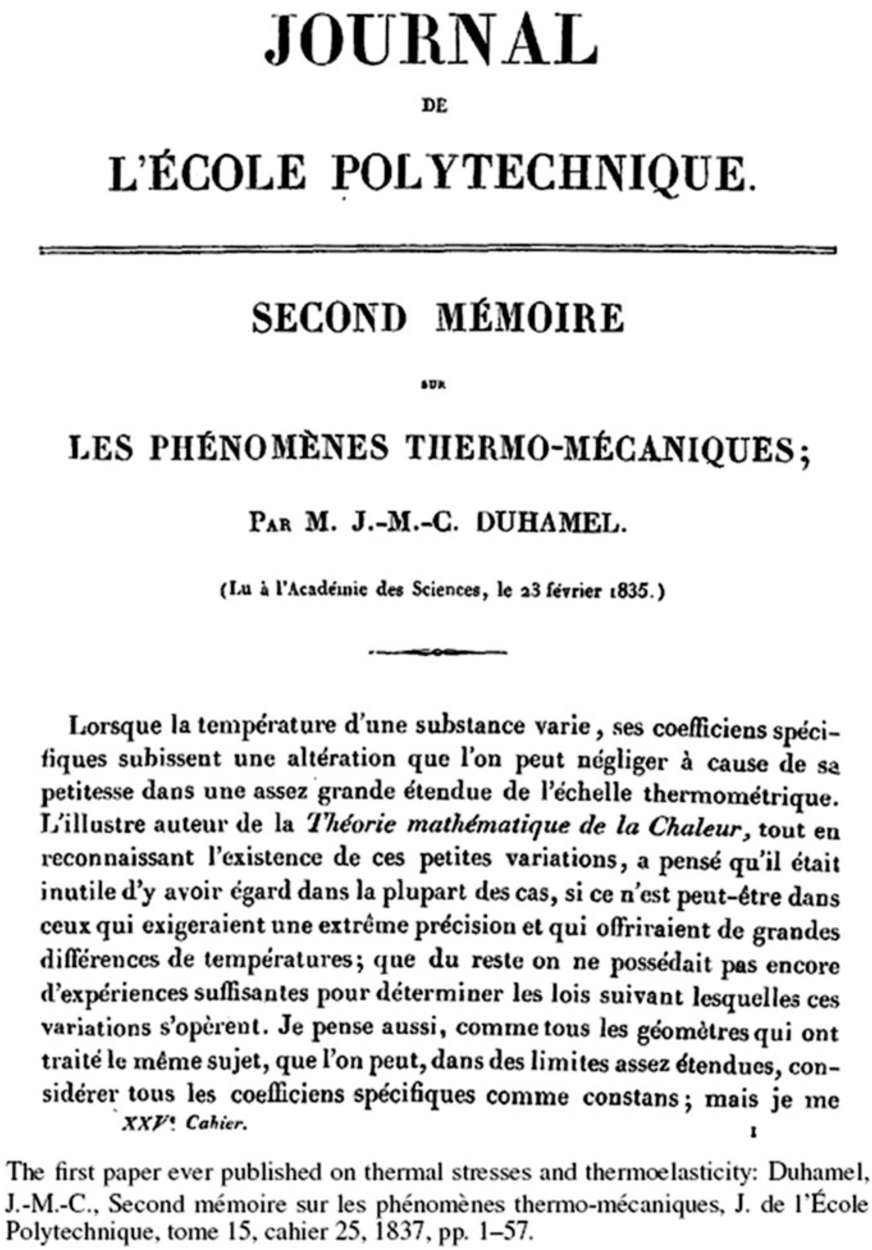
single framework. Thus was born *thermo-elasticity*. He also pioneered in the study of anisotropic continuous media by envisaging anisotropy of heat conduction. In this last case he was influenced by Cauchy and his ellipsoids of strains and stresses by examining what happened along principal axes of the conduction tensor, a property also envisaged by Liouville. His main opus related to the present subject is a celebrated long paper [8]—see Fig. 5.1—in which the full expression of *thermo-elasticity* (of course, not his word; he rightly uses the expression "thermo-mechanical phenomena" or its French equivalent) was given together with the solution of exemplary problems.

5.2 Basic Ideas of Duhamel's 1837 Paper

Nineteenth century scientists explain well in their long introduction what they are going to expose to their readers. This is the case of Duhamel, what greatly simplifies the job of future reviewers. To justify his endeavour Duhamel remarks from the beginning that it is commonly admitted that bodies produce heat when they are compressed and they absorb heat when they are dilated. Accordingly, there should exist a difference between specific heats at fixed volume and at constant pressure. Although Louis Gay-Lussac (1778–1850) and others had done corresponding experiences in air, there did not exist yet at the time such experimental records for solids. Duhamel claims that his theoretical approach suggests such experiments. In harmony with his premises, and contrary to Fourier, he therefore wants to account for the change in distance between "molecules" (now we would say "material points") during heat conduction. He concludes that the evaluation of the observed increase in temperature should be done in two steps, a first one during which all "material points" are kept fixed, and a second step in which the matter density is altered. From this there should follow an alteration in Fourier's heat propagation equation, and a dual coupling between thermal and deformation effects, thus introducing for the first time *thermo-mechanical couplings*, for there are relative changes in position of "molecules" in a non-homogeneously heated solid. The essence of thermo-elasticity resides in these spot-on introductory remarks. He also remarks that in many cases the temperature distribution may be evaluated independently of mechanical quantities, a working hypothesis still used in many modern problems of thermo-elasticity.

5.3 Duhamel's Strategy and Equations of "Thermo-elasticity"

Of course Duhamel never uses terms such as strains and stresses. Neither can he rely on any well-set principles of thermodynamics (that are not yet established, nor even proposed at the time of his writing). What he does is that he carefully applies



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SUR

LES PHÉNOMÈNES THERMO-MÉCANIQUES;

PAR M. J.-M.-C. DUHAMEL.

(Lu à l'Académie des Sciences, le 23 février 1835.)

Lorsque la température d'une substance varie, ses coefficiens spécifiques subissent une altération que l'on peut négliger à cause de sa petitesse dans une assez grande étendue de l'échelle thermométrique. L'illustre auteur de la *Théorie mathématique de la Chaleur*, tout en reconnaissant l'existence de ces petites variations, a pensé qu'il était inutile d'y avoir égard dans la plupart des cas, si ce n'est peut-être dans ceux qui exigeraient une extrême précision et qui offriraient de grandes différences de températures; que du reste on ne possédait pas encore d'expériences suffisantes pour déterminer les lois suivant lesquelles ces variations s'opèrent. Je pense aussi, comme tous les géomètres qui ont traité le même sujet, que l'on peut, dans des limites assez étendues, considérer tous les coefficiens spécifiques comme constans; mais je me

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The first paper ever published on thermal stresses and thermoelasticity: Duhamel, J.-M.-C., Second mémoire sur les phénomènes thermo-mécaniques, J. de l'École Polytechnique, tome 15, cahier 25, 1837, pp. 1–57.

Fig. 5.1 The first paper on thermo-elasticity Duhamel [9] his proposed strategy, attempting with success to find what additional terms should appear in the already known equations of Fourier and Navier. Therefore, he first examines the equilibrium of a body of arbitrary shape, establishing that for a body subjected to the same instantaneous pressure at all its surface points, the value of the observed linear contraction is obtained by dividing the value obtained if there was no produced heat by the ratio of the two specific heats (at fixed pressure and fixed volume). Then he considers the case of a solid thread whose two bases are of arbitrary shape and it is submitted to an almost instantaneous traction without any load on its lateral surface. He determines thus the decrease in temperature that results from this load as well as the change in length and the lateral shrinking of the thread. But this instantaneous equilibrium is immediately altered by the heating of the thread that causes a change in length. The difference between the first and second values of the stretch is in the ratio of unity to the inverse ratio of the two specific heats. This provides a way to determine one of the specific heats knowing the other. The last example in order to fortify his argument concerns a hollow sphere that is subjected to unequal pressures on its inside and outside surfaces. This example offers him an opportunity to exploit his general equations in such a spherical symmetry. He also treats the case of the (slow) cooling of a free sphere, in which one may assume that the whole process occurs through a succession of equilibria. But these equilibria are different from those that would exist if there was no heat expanded during the decrease in distance between the "molecules". The speed of cooling is thus reduced. In solving this problem Duhamel exhibits his dexterity in solving difficult problems in analysis presented by the reciprocal influence of mechanical and thermal effects.

The complete system of equations that Duhamel establishes and treats in the application consists of Navier's equations with additional terms due to dilatation, the corresponding (Cauchy) natural boundary conditions on stresses, and Fourier's equation with coefficient accounting for the ratio of the two specific heats. Although his notation (of course no tensor notation not yet invented at the time) is notably different from the modern one, an experienced "thermo-elastician" would be able to identify his equations.

5.4 Comments and Further Developments

The first remark is that Duhamel's approach, although based on a few acceptable assumptions, is purely mathematical, and thus is in the best tradition of Fourier, Poisson and Cauchy. It is ahead of any clear formulation of the principles of thermodynamics. Duhamel exploits continuity arguments which he justifies by the fact that finite differences between separated "molecules" can be approximated by differentials. The possibility to consider some anisotropy in heat conduction (see another work by Duhamel [7]) pioneers in the theory of anisotropic solids, and thus crystallographic structures.

Duhamel and others' contributions to the theory of thermal propagation in solids have been pondered by Bachelard [2] in a nice critical monograph. As to Truesdell [22, Sect. 44], he misses Duhamel's 1837 paper but he correctly remarks that in the absence of the concept of thermo-elastic energy, we miss the important point whether deformation occurs in isothermal or adiabatic conditions.

It would not be fair to other contributors to attribute to Duhamel the whole merit of the formulation of thermo-elasticity. In particular, it is necessary to acknowledge the excellent work of Franz E. Neumann (1798–1895) in Germany, so that Duhamel's equations are sometimes referred to as the *Duhamel–Neumann equations*, in particular by Timoshenko [21]. Contrary to Duhamel, Neumann was both a theoretician and an experimentalist. Among his many achievements we must emphasize his fruitful work on the double refraction in stressed transparent bodies, which makes him one of the creators of photo-elasticity. He, in fact, used this technique to verify his theoretical results in thermo-elasticity (see Neumann [16]). Furthermore, Neumann mentored remarkable German contributors to mechanics, e.g., Gustav R. Kirchhoff, Alfred Clebsch, and Woldemar Voigt. The latter author, just like his master, contributed to thermo-elasticity by clearly distinguishing between isothermal and isentropic deformations in crystals (see his book: Voigt [24]). It was natural for Pierre Duhem (1861–1916), a passionate supporter of thermodynamics, to clearly specify the various possible conditions whether an elastic solid in finite strains was a heat conductor or not. This he applied in an original study of the propagation of discontinuity waves of any order (in the classification of Hadamard) in thermo-elastic materials [9]. This was revisited by Truesdell [23], and other authors [6]. In the meantime, Léon Brillouin (reported in [4], but works in the 1920s) had also dealt with this matter, while Signorini ([20] 1943 on) produced in the 1940s an extensive definite work on finite-strain thermo-elasticity.

Most of the above works remain at a rather theoretical level. But thermo-mechanical couplings acquired a significant importance with the use of many metallic parts submitted to high temperature in railway technology and aeronautics. This may explain the intense production of solutions of specific problems (concerning structural elements) starting the 1940s. This led to the production of now standard books devoted to *linear* thermo-elasticity, among these, Melan and Parkus [14], Parkus [18], Boley and Weiner [3], Chadwick [5], Nowacki [17], Parkus [19], and more recently Hetnarski [10] and Hetnarski and Eslami [12]. To these should be added the formidable encyclopaedia of thermal stresses edited by Hetnarski [11] that considers all aspects, theoretical and applied, linear and non-linear, as well as generalized laws of heat conduction (allowing for a finite speed of propagation of heat).

To close this brief survey of the evolution of the field over some 170 years, it must be remarked that thermo-elasticity results in the presence of *internal stresses*—or eigenstresses (according to the terminology introduced by E. Kröner in the 1950s)—and is thus a *quasi-plastic* phenomenon (presence of strains and thus elastically associated stresses in the absence of mechanical loading). In the non-linear framework, just like in elasto-plasticity, this led to the consideration of a multiplicative decomposition of the total deformation gradient into elastic and thermal "gradients" (this recurs to an addition for small strains). This is due to Mićunović ([15] 1974 on). Of course, the importance of finite-strain thermo-elasticity for the study of phase transformations cannot be overlooked (cf. Abeyaratne and Knowles [1]. Metallo-thermo-mechanics—according to the term coined by Inoue [13]—is the standard domain of application at the dawn of the twenty-first century.

5.5 Conclusion

Duhamel's paper of 1837 was a breakthrough when it was published, and as one may declare, it was the first paper ever in the field of *coupled fields* in continuum mechanics. Though, it did not contain any significant thermodynamics. But the existence of coupled thermo-mechanical phenomena was an obvious fact. As a faithful follower of Cauchy and other "savants-engineers", Duhamel attacked the problem as a mathematician, although perhaps guided by existing experiments concerning air. Neumann and Voigt were closer to the physical world (and experiments) in their approach to the problem. Yet we must recognize that it took some rather long time for technological applications to require the analytical solution—in the absence of computers—of special situations in a linear theory. It is with Duhem, Brillouin and Signorini, and then the Truesdellian School that a neat non-linear framework was formulated. Modern developments are essential in the treatment of the dynamics of phase transformations, in thermo-plasticity, and now in more complex theories of coupled fields where mechanical, thermal, magnetic, and/or electric fields interact markedly.

References

1. Abeyaratne R, Knowles JK (2001) Evolution of phase transitions. Cambridge University Press, UK
2. Bachelard G (1927) Etude sur l'évolution d'un problème de physique: la propagation thermique dans les solides (Complementary Thesis, Paris, 1927) (reprinted by Librairie philosophique Vrin, Paris, 1973, with a foreword by A. Lichnerowicz)
3. Boley BA, Weiner JH (1960) Theory of thermal stresses. Wiley, New York
4. Brillouin L (1938) Tensors in mechanics and elasticity (original French edn in Paris, Dover reprint 1946; First English translation, Academic Press, New York, 1963)
5. Chadwick P (1960) Thermoelasticity. The dynamical theory. In: Sneddon IN, Hill R (eds) Progress in solid mechanics, vol I. North-Holland, Amsterdam, pp 263–328
6. Coleman BD, Gurtin ME, Herrera I, Truesdell CA (1965) Wave propagation in dissipative materials. Springer, New York (collection of reprints of papers)
7. Duhamel JMC (1832) Sur les équations générales de la propagation de chaleur dans les corps solides dont la conductibilité n'est pas la même dans tous les sens (On the general equations of heat propagation in solid bodies the conductibility of which differs according to the direction). J de l'Ecole Polytechnique, Tome 13, Cahier 21, pp 356–399

8. Duhamel JMC (1837) Second mémoire sur les phénomènes thermo-mécaniques (Second memoir on thermo-mechanical phenomena). J de l'Ecole Polytechnique, Tome 15, Cahier 25, pp 1–57
9. Duhem P (1906) Recherches sur l'élasticité. Gauthier-Villars, Paris (collection of four papers with a total of 218 pages)
10. Hetnarski RB (1986) Thermal stresses, vol 1. Mechanics and mathematical methods. North-Holland, Amsterdam
11. Hetnarski RB (2014) General editor: encyclopaedia of thermal stresses. Springer, Berlin
12. Hetnarski RB, Eslami MR (2009) Thermal stresses—advanced theory and applications. Springer, Dordrecht
13. Inoue T (1997) Metallo-thermo-mechanics. Wiley, New York
14. Melan E, Parkus H (1953) Warmespannungen Infolge stationarer temperaturfelder. Springer, Vienna
15. Mićunović M (1974) A geometrical treatment of thermoelasticity of simple inhomogeneous bodies: I –geometric and kinematic relations. Bull Acad Pol Sci Sér Sci Techn 22:579–588; II (Constitutive equations), & III. (Approximations), 22:633–641, 23:89–97 (1975)
16. Neumann FE (1885) Vorlesung über die Theorie des Elasticität der festen Körper und des Lichtäthers. Teubner, Leipzig
17. Nowacki W (1986) Thermoelasticity. Pergamon Press, Oxford (2nd revised edn; translated from the Polish; original Polish edition, P.W.N., Warsaw, 1962)
18. Parkus H (1959) Instationäre wärmespannungen. Springer, Wien
19. Parkus H (1968) Thermoelasticity. Blaisdell, Waltham
20. Signorini A (1943) Trasfomazioni termoelastiche finite, Memoria 1. Ann di Mat Pura ed Applicata (4) 22:33–143; Memoria 2, ibid, 30:1–72 (1949); Memoria 3, ibid, 39:147–201 (1955)
21. Timoshenko SP (1953) History of the strength of materials. McGraw Hill, New York (Dover reprint, New York, 1983)
22. Truesdell CA (1952) Mechanical foundations of elasticity and fluid mechanics. J Rat Mech Anal 1:125–300 (section 44)
23. Truesdell CA (1961) General and exact theory of waves in finite elastic strain. Arch Rat Mech Anal 8:263–296
24. Voigt W (1910) Lehrbuch der kristallphysik. Teubner, Berlin

Chapter 6
From Cauchy to Boussinesq via Barré de Saint-Venant

Abstract The nineteenth century witnessed a development of continuum mechanics that went from Cauchy's introduction of the notion of stress (tensor) to the introduction of seminal ideas about plasticity and turbulence. Here the emphasis is placed on the works of Barré de Saint-Venant and Boussinesq who contributed mostly to elasticity, but also to the theory and flow of viscous fluids. These works are exemplary of the period, involving little or no thermodynamics, while demonstrating a high dexterity in mathematical manipulations. Saint-Venant is more an engineer-mathematician in the line of engineers-scientists formed at the celebrated Ecole Polytechnique, while Boussinesq appears more as a specialist of astute mathematical modelling, the ancestor of modern phenomenological asymptotic modelling. They received in their time unanimous international recognition from the scientific community. But, in retrospective, these two authors did not receive further the attention they clearly deserve, perhaps due to their style of publication and the irreversible passing of time, though their names remain attached to famous principles and approximations.

6.1 Introduction

If we think of the most important contributors to continuum mechanics in the nineteenth century—being aware of the difficulty and biases of the exercise—, we inevitably list the following scientists: Cauchy, Navier, Lamé, Saint-Venant, and Boussinesq. This comes to mind because they all had a wide expertise in the field (both in fluids and solids) and all with a definite dexterity in mathematics. Remembering our student's time, we note that they all received direct attributions related to some of their achievements, such as: *Cauchy*'s stress tensor, *Navier*-Stokes' equations, *Lamé*'s coefficients of elasticity, *Saint-Venant*'s problem, and *Boussinesq*'s approximation. This is corroborated by the number of citations to elasticians by A. E. H. Love in his celebrated book on the mathematical theory of elasticity (Love [69], 4th edition); in decreasing order: Saint-Venant, Rayleigh,

G. A. Maugin, *Continuum Mechanics Through the Eighteenth and Nineteenth Centuries*, Solid Mechanics and Its Applications 214, DOI: 10.1007/978-3-319-05374-5_6,

Michell, Lamb, Kelvin, Kirchhoff, Boussinesq, Chree, Clebsch, Cauchy, Voigt, Stokes, and Lamé. Here Lord Rayleigh, Lord Kelvin (W. Thomson), Gabriel Stokes and Horace Lamb are well known British mathematical physicists; Kirchhoff, Clebsch and Voigt [89] are interrelated German scientists (they all studied with Franz E. Neumann), and J. H. Michell (1863–1940) is an Australian mathematician—who was first a student of E. J. Nanson in Melbourne (remember Nanson's formulas in non-linear continuum mechanics) and, after studying in Cambridge, published many works on all kinds of elastic structures before 1902 (after which he never published any more papers). Love's citation list may be biased by his marked interest in the elasticity of rods and problems such as torsion and flexion, in the solution of which both Saint-Venant and Michell excelled. This was also the case of Charles Chree (1860–1928) in his young age, but this scientist is better known as a geophysicist and meteorologist.

The fact that all scientists named in our first list are French and very often with the same mixed engineering-science formation is not due to chance. All, save Boussinesq, are "ingénieurs-savants" according to Grattan-Guiness' classification [60] (see Maugin [72], Chap. 1). In addition, Cauchy, undoubtedly the greatest of them, is also a true creative mathematician with an interest in all parts of mathematical sciences. Boussinesq is *à part*, having not benefited from a formation in a "grande école", but having had the chance to be mentored by Saint-Venant. This does not mean that we should disregard other engineers and scientists, especially Green, Kelvin, Stokes and Love in the UK, and Kirchhoff, Clebsch, and von Helmholtz in Germany, some of them in direct competition with their French colleagues, but always with a welcomed inspiration from them and an expressed great admiration. However, there exists an Ariadne's thread that connects the cited French scientists, a thread that we shall unwind or untangle. Since we already discussed the innovative role of Cauchy in continuum mechanics elsewhere [73], the emphasis will be placed on Barré de Saint-Venant and Boussinesq with a brief reminder on Cauchy and Lamé. Other seminal contributors such as Piola and Kirchhoff are examined in a different chapter.

6.2 Cauchy and Lamé

If we consider in succession Cauchy, Lamé, Saint-Venant and Boussinesq, we must recognize the essential differences between these four scientists. Cauchy (1789–1857) is undoubtedly the creator of the notion of stress tensor—the *fons et origo* of continuum mechanics—thus providing a foundation to a general theory of continua, whether fluid or solid, whether elastic or inelastic, as clearly stated in the title of his original contributions [46, 47]. Furthermore, his basic theory of deformations, his introduction of the notions of ellipsoids of strains and stresses and accordingly principal stresses, strains and stretches, and his views on the notions of linearity and isotropy also contribute to our common required background.

The contributions to elasticity problems, both in statics and dynamics, by Cauchy but also by his contemporaries, C. M. L. Navier (1785–1836) and S. D. Poisson (1781–1840), are fundamental. Gabriel Lamé (1795–1870) and Saint-Venant (1797–1886), although born in the same period, had a longer life and thus provide a bridge with the next generation, in particular J. V. Boussinesq (1842–1929). This generation will contribute to a transition to the active period of systematization and consolidation covering the late nineteenth century and the early twentieth century, a period with which A. E. H. Love (1863–1940) in the UK, and Pierre Duhem (1861–1916), Henri Poincaré (1854–1912) and the Cosserat brothers [Francois (1852–1914) and Eugène (1866–1931)]—all contributors to the theory of elasticity in various forms—must be associated.

Another way of qualifying our scientists of interest is to attribute "principles" to Cauchy, "solutions of problems" to Lamé and Saint-Venant, and "modelling" to Boussinesq. But this is an over simplification which nonetheless contains some truth. The rapid development in this overall period of time makes a big difference, if we remember that it is Lamé who published the very first book devoted to elasticity [66], and also introduced systematically the use of curvilinear coordinates [67]. Concerning Lamé's book on elasticity we cannot avoid citing Todhunter from his "History of elasticity" ([85]; as cited by Timoshenko [84], p. 117): "The work of Lamé…affords an example which occurs but rarely of a philosopher of the highest renown condescending to employ his ability in the construction of an elementary treatise on a subject of which he is an eminent cultivator." Thus Lamé became an inescapable reference to all future contributors to elasticity, among them Saint-Venant and Boussinesq, although these last two authors had a broader range of scientific interests than Lamé who nonetheless also published in physics and the theory of light. All authors of course referred to Cauchy with due respect. The reader interested in Cauchy will benefit from reading the beautiful book of Belhoste [44] and the thorough analysis of Cauchy's mathematical style in mechanics by Dahan-Dalmenico [54].

6.3 Barré de Saint-Venant

6.3.1 On his Name

This scientist is always presented as Adhémar Jean Claude Barré de Saint-Venant (1797–1886). But his only correct naming should be (cf. Crowe [53], p. 81): Adhémar Barré, Comte de Saint-Venant, so that the name in a Republican manner simply is Adhémar Barré. Accordingly, calling him simply Saint-Venant is the old aristocratic way of friendly use restricted among (true or false) nobility. Above all, it is *not* St-Venant (as often written by Truesdell). This designates a Catholic saint, whereas Saint-Venant is the name of a small place in France (an ecclesiastic property nearby Niort in west central France). However, we shall follow common (aristocratic) usage damaging thus our democratic reputation. Going even further, for brevity in citation we shall use the abbreviation BSV followed by the year of publication.

6.3.2 Some Biographical Elements

Although educated according to the best standards of the period (*Ecole Polytechnique* and *Ecole Nationale des Ponts et Chausées* (civil engineering)), Saint-Venant had a rather unusual career, apparently initially hindered by some political facts—he was a royalist opposed to the return of Napoleon in 1815. As a matter of fact, he was not granted in time the diploma of Polytechnique in 1816. A direct consequence of this is that he did not join the much wanted *Corps des Ponts et Chaussées* (with an admission to the corresponding school) to which he deserved to belong. In place of this ideal path, he joined the "*Corps des Poudres et Salpêtres*". That sounds a little obsolete and an archaism if we forget the importance of these matters in artillery and the attack of fortresses by "sappers". The Corps of Powders' engineers (the mention of "Saltpeters" had logically disappeared) survived until the twentieth century. As a matter of fact, its engineers were instrumental in devising the French A and H bombs in the 1950–1970s; the *Ecole Nationale Supérieure des Poudres*, a school of specialization for a selected group of some alumni of *Polytechnique,* was incorporated in the 1970s, along with the school of Naval engineers (*Ecole du Génie Maritime*), in a new "grande école", the *Ecole Nationale Supérieure des Techniques Avancées* (for short *ENSTA*)—with dependence on the Ministry of National Defence—, an educational system of which the French have the secret, if not the envied privilege.

This odd formation took Saint-Venant to the world of chemistry for 6 years after what he was granted in 1823 the permission to join the School of Ponts et Chaussées without entrance examination (due to his excellent student's records at Polytechnique) and finally joining the much desired corps. Graduating at the top of his class he then spent some 20 years in various state duties of civil engineering (until 1848). But simultaneously, he succeeded Coriolis as a professor at the School of Ponts et Chaussées and started publishing research papers in mechanics and the strength of materials, an activity that he pursued, never at rest, until his death, aged 89, in 1886. His last publication in the *Comptes Rendus* of the French Academy of Sciences was dated January 02, 1886—probably presented on December 24, 1885—while he died on the 6th of the same month. He had become a member of the Paris Academy of Sciences in 1869, on Poncelet's seat, at a relatively late age of 71 for a scientist of this magnitude.

As already noticed Saint-Venant taught at the *Ponts et Chaussées* school, but for a short time only (1837–1842). More surprising is the fact that he entered a competition to teach at the National Institute of Agronomy where he stayed only for 2 years (1850–1852). From 1852 to his death he devoted his full time to research. From these two aborted experiences we conclude that either he was not successful with students or he finally disliked teaching as too elementary.

6.3.3 *Saint-Venant's Scientific Achievements*

If we remember that Love cited Saint-Venant no less than forty seven times in his book, it is easily realized that reporting Saint-Venant's scientific contributions is not an easy task. Writing a true scientific biography of this engineer-scientist is a required duty that remains to be done. We shall evoke only a few aspects of these contributions and try to delineate some general features that make Saint-Venant an original individual. He published continuously scientific notes and memoirs from 1839 to his death in 1886 (for books, see below). These numerous contributions are a mix of "intuition, pragmatism and mathematical rigour" (as noticed by Maisonneuve [70], p. 187). He worked mainly in mechanics: elasticity, hydrostatics and hydrodynamics. This list provides a template for an organized perusal of his works.

6.3.3.1 Elasticity

This is the domain of continuum mechanics to which his name is attached in priority. First of all is his mathematical formulation of the *compatibility conditions* in small-strain elasticity. Here the basic question is how to determine the three independent components of an elastic displacement from the six components of a symmetric strain (the symmetrized spatial gradient of the displacement): there must exist three conditions satisfied by the strain in order to allow for this spatial integration. Nowadays, these conditions, known as the Navier-Saint-Venant conditions are written in Cartesian tensor notation as (e_{ij} denotes the strain components and a comma followed by a Latin index stands for the partial space derivative):

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0. \tag{6.1}$$

This was later reworked by Michell and Beltrami. But in modern, times, Eq. (6.1) is recognized as the condition that the elastic material manifold be flat, and thus with vanishing Riemannian curvature. It happens that in *three* dimensions this is given by the vanishing of the so-called Einstein tensor A_{ij} defined from the curvature tensor by (ε_{ajk} is Levi-Civita's permutation symbol)

$$A_{ab} = \frac{1}{4}\varepsilon_{ajk}\varepsilon_{bmq}R_{jkmq} = -\varepsilon_{ajk}\varepsilon_{bil}\frac{\partial^2 e_{kl}}{\partial x_j \partial x_i}, \tag{6.2}$$

from which (6.1) follows at once.

Saint-Venant was also involved in the discussion concerning the correct number of elasticity coefficients. Although he considered himself a disciple of Cauchy, Navier and Poisson—not a bad choice—he first followed Navier and Poisson in their choice of one constant only for isotropic elasticity. This error was due to Navier's consideration of a discrete model with too much symmetry in the interactions between constituent "particles" [The great Cauchy faced the same problem with his second theory [48] of elasticity that may be considered the ancestor of "discrete" continuum mechanics]. Similarly, for a general anisotropic (or aelotropic) elastic body, Saint-Venant first favoured the so-called "restricted" theory involving only fifteen constants, while with the energy approach of Green [62]—and also the opinion of Lamé—we know that twenty one is the correct number. Of course Saint-Venant corrected his wrong appraisal but it took him some time to do so.

But the magisterial work of Saint-Venant in elasticity that easily erases this error of appreciation in the number of elasticity coefficients, is provided by his solutions of general problems in torsion and flexure. Here he remains the unique master admired by all mechanicians over Europe. He had a special interest in the mechanical response of structural elements (rods, plates) under prescribed loading on their ends and faces. First, very early, in his course at the Ponts et Chausées in 1842, he introduced the notion of *warping* BSV [6]. That is, normal sections of a loaded rod do not remain plane. This is a much improved sophistication in the description of the potential torsional deformation of elastic structures such as non-circular cylinders and the bending of beams. He approached this problem of combined torsion and flexure of long rods with a very fruitful intuitive idea that we try to describe without the help of formulas: in linear elastostatics it is thought that the difference in effect of two equipollent (i.e., with same resultant) load systems confined to a small part of a boundary becomes negligible away from that boundary. For instance, in the case of the deformation of a prism or a cylinder, we assume that a system of forces is applied only at the two bases. Now, if the transverse dimensions are small compared to the length of the body, the differences in the distribution of forces on the bases produce only local perturbations. As a consequence a practically identical state characterized by longitudinal fibres exerting only tangential actions on their neighbours, or directed along the length, takes place after a short distance from the bases. Exploiting this idea translated into analysis and using the hypothesis of no surface pressure exerted on the lateral faces, Saint-Venant succeeded in justifying from elasticity the accepted laws of flexure and torsion in the strength of materials (see in particular, BSV [9, 11]).

This met an exceptionally good immediate reception, especially outside France. Thus Clebsch in Germany coined the expression "*Saint-Venant problem*" for this approach, and later on, with the enthusiasm of a close disciple, Boussinesq called it "*Saint-Venant's principle*". But with the passing of time we think that the use of "principle" was an exaggeration. We must agree with Maisonneuve [70], himself a mathematician specialist of the problem, that it was more a well thought working hypothesis than anything else, but a very fruitful one indeed. Nowadays, all mathematical considerations of Saint-Venant's problem are necessarily based on reasoning involving asymptotic analysis and good applied functional analysis that provides a rational measure of the considered approximations (e.g., works by R. J. Knops in the UK).

Saint-Venant did not publish a book by himself, but it is noted by Timoshenko [84], p. 108) that the best résumé of Cauchy's theory of elasticity was given by Saint-Venant in the two chapters (pp. 616–762) that he contributed to Moigno's book of (1868).[1] Also, according to the same author ([84], p. 109), Saint-Venant gave the final form of the definition of Cauchy's stress in a short note (BSV [4]). Finally, according to Truesdell ([87], pp. 252–253), the historical perspective on research in the strength of materials and elasticity given by Saint-Venant in his edition of Navier (BSV [13]) remains "the fullest and most accurate history of linear elasticity in the first six decades of the nineteenth century".

6.3.3.2 Anelastic Behaviour

In testing elastic materials to a sufficiently high mechanical load a kind of limit—the elastic limit—appears after which one can hardly control the deformation. In 1D one simply reaches a level, say σ_0, of stress (force per unit area of the section of the sample), at which one loses the control of the elongation. We now say that we observe *plastic flow*, while mathematically in modern terms we formalize this by saying that we lose the uniqueness in the response in deformation (cf. [71]). But real materials are three-dimensional, the stress is a more complex object (tensor) than a scalar, and the datum of one single scalar to characterize the entry into the *plastic regime* is not always sufficient. One must think in terms of a convenient representation of a tensorial state of stress and deformation. Thanks to Cauchy who related this to the representation in terms of ellipsoids, we also know that the length of *principal* axes of the ellipsoids representing stresses and strains are convenient representations of the actual mechanical state. It is at this point that we must introduce the seminal work of Henri E. Tresca[2] who conducted in the early 1870s a series of fine experiments on metals whereby he constructed in an appropriate representation of the principal stresses the elastic limit of the said metals [86].

Three important remarks are in order: first, it is noticed that no change in volume (so called *isochoric* deformation in the modern jargon) is observed during

[1] François Moigno (1804–1884), known as Abbé Moigno, was a catholic priest, who, after being professor of mathematics and leaving the Jesuit order, became Chaplain of the Lycée Louis-le-Grand in Paris (probably the best Lycée in France for the preparation to the entrance examination in "grandes écoles" such as Polytechnique). A prolific and never tired author, he is mostly known as a populariser of science. He was a friend of Cauchy—whom he considered his master—, Ampère and Arago. Among his many books are one on *Lessons on differential and integral calculus*, and one on *Lessons on Analytical Mechanics* (Moigno [75]; the book in which Saint-Venant contributed two additional chapters).

[2] Henri E. Tresca (1814–1885), a French mechanical engineer, was a professor at a Paris institution known as the *Conservatoire National des Arts et Métiers* (for short, *CNAM*) where he conducted experiments on the strength of metals. He is also known as a specialist of metrology.

plastic deformation; second, the directions of the principal stresses coincide with those of the principal strains (this assumes an *isotropic* response); third, the maximum shearing (or tangential) stress at a point is equal to a specific constant. This can be written as $\tau_M = k$. In mathematical terms, we have

$$Sup_{\alpha,\beta}\left|\sigma_\alpha - \sigma_\beta\right| = 2k, \quad \alpha, \beta = 1, 2, 3, \tag{6.3}$$

where the Greek indices label the principal stresses. Introducing the tangential stresses, this can also be expressed by the following set of *three* inequalities:

$$2|\tau_1| \equiv |\sigma_2 - \sigma_3| \leq k, \quad etc, \tag{6.4}$$

by circular permutation. In an astute plane representation this is represented by a hexagon. The interior domain (a convex domain with angular corners) is the domain of elasticity. Although the criterion provided by (6.4) gives good results in the case of metals, this definition of the elastic limit by pieces of intersecting straight lines offers some difficulties in analytic treatment of problems. Nonetheless, Saint-Venant, with his mathematical dexterity, gave a clear mathematical formulation of these results BSV [20, 21, 23]. Moreover, he was able to give the first solution of exemplary problems of elasto-plasticity such as: the torsion of a circular shaft, the plane deformation of a hollow circular cylinder under the action of an internal pressure, etc. These are problems that we still give students to solve without the help of a computer (see, e.g., Maugin [71], Appendix). Saint-Venant also paid special attention to the works and modelling proposed by Maurice Lévy (1838–1910) for ductile materials BSV [19, 24]. Lévy [68] had proposed to discard the elastic behaviour—as negligible for some materials—and to consider only the plastic one, thus in so-called *rigid-plastic* bodies. This is a rather highly singular behaviour since nothing happens to the strain, not even an elastic one, in so far as the plasticity threshold is not reached and then we have an uncontrolled plastic flow occurring along a plateau in stress. Well known mechanicians such as Prandtl and von Kármán were to solve other problems of elasto-plasticity in the early twentieth century.

6.3.3.3 Dynamics and Impact

All along his life Saint-Venant demonstrated an interest for dynamical processes and the theory of impact in particular. We owe to him publications on the lateral or longitudinal impact of bars in the period 1853–1854 [10] and about 1866—see Timoshenko ([84], pp. 179–180) for greater detail. In these he established the resulting vibration modes after impact. He returned to this subject matter in his later life, aided by A. A. Flamant[3] (BSV [29]; also in an Appendix in the French

[3] Alfred-Aimé Flamant (1839–1915) is a French civil engineer (formation: Polytechnique followed by Ponts & Chaussées) who taught in different schools in Lille and Paris. He had an essentially administrative career after supervising large projects. He considered himself a disciple

translation of Clebsch). Flamant published some of the last results in the form of numerical curves after the death of Saint-Venant (cf. [58]). W. Voigt improved the interpretation of the experiments of longitudinal impact of prismatic bars by assuming that the two bars are separated by a layer of transition with ad hoc mechanical properties. Later on it was proved that Saint-Venant's theory is factually pretty good, disagreement between theory and experience being initially due to the difficulty of realization of the experiment ([84], p. 346 and p. 420). The interest of these studies for military problems is obvious.

6.3.3.4 Fluid Mechanics

Nowadays, the name of Saint-Venant is more often associated with the mechanics of deformable solids than with fluid mechanics. But he readily contributed to different aspects of fluid mechanics. Along these we must single out his derivation of the constitutive equations for isotropic viscous fluids. He was a disciple of Navier and Poisson in a primary description by "particles", but he eschewed completely Navier's molecular approach and was able to derive correctly the so-called Navier-Stokes equations in 1843 BSV [3], that is, two years before Stokes whose famous paper was read in [82]. This he humbly pointed to Stokes in a post script to a letter of January 22, 1862 (Cf. [87], pp. 224–228). Accordingly, these celebrated equations should be referred to as the *Navier-Saint-Venant-Stokes* equations. It must however be noted that, contrary to Stokes, Saint-Venant did not give applications of his equations. Still the Saint-Venant paper presents interesting features pointed out by Anderson ([1], p. 91): "That 1843 paper was the first to properly identify the coefficient of viscosity and its role as a multiplying factor for the velocity gradients in the flow. He further identified those products as viscous stresses acting within the fluid because of friction. Saint-Venant got it right and recorded it. Why his name never became associated with these equations is a mystery; certainly it is a miscarriage of technical attribution" (see also [76]).

Other problems of fluid mechanics treated by Saint-Venant belong to a large spectrum of research: the flow of air, the flow from a container through an orifice [26], d'Alembert's paradox, the speed of sound in air, the theory of periodic liquid waves [17], the motion of fluid in shallow water BSV [22], the theory of solitary waves BSV [28], the effects of swell and lapping (in French, "clapotis"; BSV [22]). All these publications are short notes to the *Comptes Rendus*, a common practice of many French scientists at the time. In the case of shallow water and solitary waves, the original equations deduced by Saint-Venant—and therefore presently called "Saint-Venant's equations"—missed a term that Bousisnesq will introduce and

(Footnote 3 continued)
of Saint-Venant—and in fact a co-worker of Saint-Venant during the last years of this scientist; he befriended Boussineq during his stay in Lille in the period 1879–1883. He simplified Boussinesq's theory of granular materials. His name is also attached to the solution of a famous problem (force applied perpendicularly to the edge a semi-infinite elastic plate of unit thickness).

could in this corrected form account for the physical reality of the "long wave" observed in 1834—but reported only in 1844—by John Scott Russell (1808–1882).

6.3.3.5 Vector Calculus

Historian of science Crowe [53] produced an unsurpassed history of the creation and development of vectorial algebra and calculus in the nineteenth century. The great heroes there are Hamilton, Gibbs, Heaviside and Grassmann. The first three authors are related to the creation of quaternions, and vector analysis per se, with the notion of scalar and vector products, dyadics, the gradient, divergence and curl operators, and the nabla symbol. With the German scientist Grassmann[4] and his publication of an original book in [59], we have at hand the masterpiece of a lonely character who creates a whole new field of mathematics, including vector spaces and exterior algebra. True recognition of Grassmann's achievements in this field came only more than 40 years after his original publication with the concept of vector spaces and multi-linear algebra (e.g., by A. N. Whitehead).

Some controversy between Grassmann and Saint-Venant took place when the latter published a kind of vector calculus in BSV [5]. Saint-Venant "had discovered vectorial addition, subtraction, differentiation, and also a multiplication similar to the modern cross product, the major difference being that Saint-Venant's product was, like Grassmann's, not another vector, but a spatially oriented area" (Crowe [53], p. 82). In the discussion of priority—mainly through an exchange of letters, sometimes, via Cauchy—Saint-Venant claimed that his idea came to him as early as 1832, but he did not feel the need to write it down before 1845. However, he seems to have tried the application of his formalism in the lectures he gave at the Agronomical Institute in 1851, obviously not the most appropriate place for the exposition of a new algorithm. This again shows some lack of appreciation and realism from the part of Saint-Venant. He never returned to this matter except in connection with the theory of "algebraic keys" BSV [8], an algebraic method originally developed by Cauchy in 1853, also in competition with Grassmann's idea of "outer multiplication" (Grassmann, perhaps suffering from some paranoia, also made a claim of priority). Remarkably, Saint-Venant made no priority claim for himself or for Grassmann, just granting a geometrical interpretation to the dry manipulations of Cauchy.

[4] Hermann Günther Grassman (1809–1877) is what is commonly called a polymath. That is, he excelled in various branches of knowledge including mathematics, linguistics (with translations from the Sanskrit), physics and general scholarship. Self educated in mathematics and physics, he is now best known for his seminal contribution to the foundation of the theory of vectorial structures with the publication (1844) of his book with self explaining title: "*Die lineale Ausdehnungslehre, ein nuer Zweig der Mathematik*" (that is: The theory of linear extension, a new branch of mathematics"). In spite of his achievements, Grassmann remained a teacher in high school for all his life, but one of his sons (out of eleven children of which seven survived to reach adulthood) became a professor of mathematics at the University of Giessen, thus realizing one of his father's dreams.

Saint-Venant's contribution to this field of "geometric calculus" did not meet great success. Still, Jouguet, writing in 1924 (2nd part, p. 240), cites "Saint-Venant, Grassmann and Hamilton" as the main three contributors to the field to which, personally, we would add Gibbs and Heaviside (see also Coffin [50]).

6.3.3.6 Principles of Mechanics

As mentioned below, Saint-Venant did not write himself a book or complete course of lectures that would have left an organized exposition of his general views on the principles of mechanics. Fortunately, E. Jouguet,[5] in his "Lectures on Mechanics—Mechanics taught from original authors" [64] analysed for us in detail and gave the most important citations from Saint-Venant, from a now inaccessible text published in an eulogy of du Buat (a celebrated eighteen-century (1734–1809) specialist of hydraulics; BSV [15]). The reason of this publication was that at the time Saint-Venant was interested in the flow through an orifice and the flow in canals and rivers, the very area of scientific contributions by du Buat. Contrary to, e.g., Duhem, Saint-Venant was an "atomist". This he clearly demonstrated in his approach to both deformable solids (following Poisson) and fluids (in a manner close to Navier's vision). Accordingly, like G. Kirchhoff [65] and E. Mach, he grants a privileged status to the notions of mass and points: "bodies move like systems of points,… those elementary points of which we suppose bodies are made..", quite a formal viewpoint. As to forces, "they never enter data—that are sensitive things—nor in the looked for solution. We introduce them to solve the problem, and then eliminate them so that in the end we have only times and distances or velocities just as when we started" (BSV cited by Jouguet [64], p. 77), naturally comparing Saint-Venant to d'Alembert and Carnot who, as we know, wanted to evacuate from mechanics this obscure, metaphysical, notion of force). Saint-Venant's first presentation of these ideas was given in a short note BSV [5] and expanded in BSV [7] of which the title emphasizes the basic role of kinematics. We can imagine the surprise of the students of the Agronomical institute where this was delivered; this may explain the short-lived teaching experience of Saint-Venant in this institution, in fact closed in 1852. For more comments we refer the reader to Jouguet for which there exists a recent facsimile reprint.

[5] Emile Jouguet (1871–1943) with a formation acquired at Polytechnique and the Paris School of Mines probably is the most successful direct disciple of Duhem. Complementing the Rankine-Riemann-Hugoniot theory of shock waves, Jouguet has created the theory of *detonation waves* with application to high explosives (Cf. the now well known Chapman-Jouguet condition). He became a professor at both schools where he had studied. The cited book Jouguet [64] of which the relevant part "L'organisation de la mécanique" was first published in 1909 was favourably reviewed by E.B. Wilson in the Bull Amer Math Soc 18: 32–33, 1911. Wilson himself was a former PhD student of Gibbs. He wrote at the early age of 21 the first book on vector analysis (often presented as Gibbs') and became a professor of pure mathematics first at MIT and then at Harvard. He has also reviewed the Cosserats' book of 1909 (see Maugin [72]).

6.3.3.7 Publication of Books

When we peruse the complete roster of publications—with about one hundred notes and memoirs—by Saint-Venant we are struck by two evident facts: like many other French prolific authors of the nineteenth century (including Cauchy, Poincaré and Duhem), he published many short notes in the *Comptes Rendus* of the Paris Academy. Their length and number per year were limited by the rules of the Academy. But unlike other authors who published afterwards much longer memoirs giving details and complements to the short announcement notes, or even lengthy books, Saint-Venant published either in odd places (somewhat like the Cosserat brothers) or incorporating his own results in other authors' book or in the French translation of a foreign book. This demonstrates a kind of rarely met modesty, if not a certain disinterest for an efficient communication. We already mentioned his contribution to Moigno's course in analytical mechanics. The second case is illustrated by Saint-Venant's edition (1864) of the lectures of Navier on the application of mechanics at the Ponts et Chaussées School [13]. Not only did he include there a remarkable historical survey of elasticity, but he also added so many of his own remarks and results that dealt with his work in the elementary theory of the strength of materials (e.g., bending of beams). Comments on the physical theory of elasticity (e.g., about the number of elasticity constants) are given in Appendices III and V to Navier.

A special attention must be paid to Saint-Venant's translation (with the help of A. A. Flamant) from the German of Clebsch's book on elasticity [27], when Saint-Venant was already more than 80 years old. This matter first deserves some comments on the style and interests of Clebsch.[6] Clebsch was destined to become one of the best pure mathematicians in Germany during his short life, mentoring people such as Felix Klein. As a very young scientist he had to work in continuum mechanics (for both fluids and solids). Surprisingly, he wrote (1861–1862) the second book on elasticity ever after Lamé's pioneering book: *Theorie der Elastizität fester Körper*. He did this dutifully including recent results such as those of Kirchhoff on the mechanics of thin bars and Saint-Venant on torsion and bending, the "*Saint-Venant's problem*" (his coining). The contents of the original version of Clebsch's book are carefully discussed by Timoshenko ([84], pp. 255–260) to whom we refer the reader for technical details.

6 Alfred Clebsch (1833–1872) was a student of Franz E. Neumann in Königsberg. He wrote a thesis in fluid mechanics. He became a professor at the Polytechnicum in Karlsruhe when he was only twenty five after spending a short time at the University in Berlin. It is while at Karlsruhe that he wrote his famous book on elasticity (1862)—*Theorie der Elastizität fester Körper*—when there existed only one such book, by Lamé, available. The mathematical inclination of Clebsch and his remarkable gift for it resulted in Clebsch becoming a professor of pure mathematics and ending his brief career as one of the best German mathematicians of his period (works on variational problems, Abelian functions, invariant theory, and algebraic geometry) in Göttingen after teaching in Giessen. His famous students include Max Noether and Felix Klein (1849–1925).

Although Clebsch in principle wrote for engineers, he did it with special emphasis on mathematical methods of solutions, often losing the physical aspect. The book reflects Clebsch's mathematical inclination, very much like Cauchy's, and is much too abstract to promote any interest from engineers. The studied problems often are of little practical interest and, of course, Clebsch shows no interest in numerical applications. In spite of the originality and very existence of a remarkable book—there was none of the type before except Lamé's—it was completely transformed to a true book for the engineering elastician (but still with a good command of mathematics) by Saint-Venant's translation into French. Saint-Venant added so many remarks, justifications, examples (both physical and numerical) and complements that the bulk of the book tripled in translation, becoming more his book than Clebsch's. It must be realized that more than 20 years had passed between the original German publication and the print of the French translation. Many of these new results were indeed due to Saint-Venant as, for instance, the treatment of beams with various shapes of cross section and the study of forced vibrations in bars. Still it must also be acknowledged that Clebsch's original text contained innovative points such as the plane state of stresses in transverse sections of rods and a rather general discussion in the analysis of trusses with the consideration of the displacement of hinges as unknown quantities. In conclusion, it would be fair to attribute co-authorship to Clebsch and Saint-Venant to the French text although A. A. Flamant referred to it casually as the "augmented Clebsch" (other authors call it the "annotated Clebsch").

To the above unusual variety of tackled subjects we should add two that lay a little outside his usual preoccupations: one is an incursion in thermo-mechanical problems (thermo-elasticity as created by J. M. C. Duhamel in France and F. E. Neumann in Germany BSV [25]), and the other where he envisaged the possibility to account for higher-order (than one) gradients of the elastic displacement BSV [16]. This, in some sense, is a *première* in the so-called theory of *generalized* continua to be fully developed only in the 1960–1970s after an attempt by J. Le Roux in the 1910s. This may be justified in a very accurate approach to torsion that necessarily exhibits spatially non-uniform strains.

The never tired Saint-Venant also wrote many reports on a quantity of memoirs submitted by other scientists to the Academy. As noted before, his scientific activity ceased only with his death at the age of 89. Two of his disciples and friends, Boussinesq and Flamant [43], wrote a notice on his life and works (see also [80]). But he receives only a two-column entry in the *Dictionary of Scientific Biography* (General Editor: C. C. Gillispie), again an unmerited lack of attention (as noticed by Truesdell [87], Footnote, p. 519), given the magnitude of Saint-Venant's oeuvre.

6.4 Boussinesq: A Disciple of Saint-Venant

6.4.1 Contributions to Fluid Mechanics

Joseph V. Boussinesq (1842–1929) is nowadays mostly referred to for his work in fluid mechanics, although in his own time, the emphasis was placed on the mechanics of deformable bodies. This shift may be due to the fact that Boussinesq's achievements in solid mechanics have been directly incorporated in the standard engineering curriculum without any further reference to the original works, while his works in fluid mechanics resonate more with some contemporary interests and methods (convection, the "Boussinesq approximation", non-linear waves). A rich and enlightening description of Boussinesq's contribution in fluid mechanics has been given by Bois [30]. It was more than natural for former professors of fluid mechanics in Lille—Bois [30] and Zeytounian [91]—to write such laudatory contributions as Boussinesq had a long and fruitful part of his career there (1873–1886). We in fact refer to them for the case of fluid mechanics in which both are recognized experts. We also refer to Bois for biographical elements, of which some are also available on the web, and to the Dictionary of Scientific Biography of 1970–1990. Nonetheless, we cannot avoid giving a flavour of two of Boussinesq's achievements in this frame.

One is related to the unsatisfactory solution (see above) given by Saint-Venant for the shallow-water and solitary-wave problem. What Boussinesq [32] established by cleverly eliminating the behaviour in depth to the benefit of the longitudinal one, was a wave equation containing both dispersion and non-linearity terms in the form

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \alpha \frac{\partial^4 u}{\partial x^4} - \beta \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)^2 = 0, \tag{6.5}$$

where u is the deviation from horizontal of the fluid surface, x is the longitudinal coordinate along the flow, and α and β are two scalar coefficients. It is the appropriate balance between the effects of dispersion (fourth-order space derivative) and non-linearity which allows for a mathematical reproduction of the long wave, called *solitary wave*, observed by Scott Russell. Other authors (e.g., Lord Rayleigh) worked on the same theme, but it was due to Korteweg and De Vries in the Netherlands to derive in 1895 a non-linear evolution equation—the celebrated *KdV* equation—which is none other than the first-order one-directional equation associated with the Boussinesq equation (in modern terms, by means of the reductive perturbation method). A Boussinesq type of equation was also derived by its author in the dynamics of 1D elastic bodies [41, 90]. This is now most often introduced in the lattice dynamics of crystals. Various generalizations for microstructured crystals and different approximations have led to the introduction of a true "Boussinesq paradigm" (cf. [49]) in non-linear wave propagation.

The second breakthrough of Boussinesq in fluid (thermo-) mechanics is represented by the so-called *Boussinesq approximation*. To explain this in simple words paraphrasing the words of Boussinesq ([41], Foreword), it must be noticed that in the motion of heavy fluids (i.e., fluids under the action of gravity) caused by heat, volumes and densities are approximately conserved although the variation of these quantities is effectively due to these heat phenomena. This results in the—Boussinesq's—observation that it is possible to neglect variations of the density, wherever it is not multiplied by gravity, while keeping their product in the computations. This we may appropriately call the "art of modelling", something hated by Pierre Duhem, but in which Boussinesq revealed himself to be a master. What is really at work in this study of hydrodynamic convection is an excellent appraisal—some would say a flair—of the relative magnitude of different terms in the relevant equations and a special focus on singular cases where relevant parameters enter in competition, something that will later on be incorporated in the consideration of non-dimensional numbers (e.g., Mach's, Froude's and Rayleigh's numbers) and asymptotic analysis. But Boussinesq's purpose is to simplify equations while keeping the essentials of the characteristic and most critical phenomena, and thus the importance of the "art of what to neglect". This was to have a brilliant future in the whole of phenomenological physics in spite of Duhem's unjustified criticism of irrationality.

Other fields of fluid mechanics approached by Boussinesq are: (1) the modelling of viscous flows and (2) the modelling of turbulence.

In the first of these he was led to formulating an expression for the force exerted by a viscous fluid on an immerged moving solid body. Here Boussinesq [36, 37] corrected Stokes' original formula by an additional term. Basset in the UK rediscovered this correction by accounting for a time-varying motion of the solid body, while C. Oseen in Sweden contributed anew to this expression which is nowadays known as the Boussinesq-Basset-Oseen (or *BBO*) equation.

In the second of these adventures in the domain of viscous flows, Boussinesq first tried to express a stress-deformation relationship with a dependence on vortex agitation for relatively rapid flows. With this he had identified the basic problem. However, before Reynolds' pioneering work in the 1880s he formulated a closure hypothesis—involving a turbulent viscosity—for turbulent Navier-Stokes equations. This is known as "*Boussinesq's hypothesis*" [34, 40]. This is documented in Schmitt [81].

More on Boussinesq contribution to fluid mechanics can be found in Darrigol [55].

6.4.2 Contributions to the Mechanics of Deformable Solids

If we are to trust Love and Timoshenko, Boussinesq is primarily a successful contributor to the theory of elasticity and the strength of materials. This is certainly true if we count the twenty citations of works by Boussinesq in Love's classic treatise. Here we must distinguish between works directly related to elasticity and those that tried to model a more involved mechanical response.

In pure linear elasticity, Boussinesq first exhibits an unsurpassed dexterity at bringing the complementary or final mathematical solutions to problems started but left unresolved by former scientists. We cannot list all these problems but examples (after Love) are: solution of the problem of the plane and a theory of local perturbations according to which the effect of a force applied in the neighbourhood of any point of a body falls off very rapidly when the distance from the application point increases; using potential energy to find the response in extension of a longitudinal filament (a problem started by Kirchhoff); the longitudinal impact of a massive body upon one end of a rod after Hugoniot; elastic equivalence of statically equipollent systems of load following Kirchhoff; the problem of isostatic surfaces after Lamé; "simple solutions" (coinage by Boussinesq) with the help of potentials; response of a half elastic space to a pressure load applied over any area; problem of a load applied at a point of a boundary of any shape, in particular straight; analogy with problems of hydrodynamics; distribution of maximum shearing stress on a boundary; uniform stress along a beam; longitudinal impact problem on a rod (after Saint-Venant). Although many of these problems seem at first to be disconnected and are often briefly described in a multitude of short notes to the Academy of Sciences in Paris, their unity resides in the powerful use of potentials and convolution (see the incredible treatise of Boussinesq [37] on the subject—with more than seven hundred pages!). That is, contrary to Saint-Venant, Boussinesq did not hesitate to write bulky volumes offering a synthesis of some works or an exposition of a common mathematical method (such as the exploitation of potentials).

The most popular problem solved in this manner is the "punch" *Boussinesq problem* concerning a concentrated force applied at a point of the otherwise free surface of a semi-infinite elastic space (see [37, 39]). In modern terms this implies a Dirac mass at the application point. Much later on this will be generalized as the "*Mindlin problem*": determine (analytically) the stresses in an elastic half-space subjected to a *sub-surface* point load. It receives applications in geotechnical engineering. This was a kind of "tour de force" by Raymond Mindlin in 1936. Some time still later, Jean Mandel solved the problem of soil deformation under a load accounting for what is called consolidation (expulsion of water from the soil in time), obviously a situation more realistic than the standard Boussinesq problem in many geophysical settings (for these see [74], Chaps. 4 and 7).

It is probably under the influence of Saint-Venant—who acted as some kind of godfather, guide and supporter through the academic system (see the letters exchanged between Saint-Venant and Boussinesq in 1868 in Bois and Verdier [31])—that Boussinesq devoted some of his research to the behaviour and mechanical response of some granular materials involved in soil mechanics [33]. Such media partake of both solid and fluid behaviours. W. Maquorn Rankine had given an unsatisfactory modelling that Boussinesq proposed to improve by considering the Lamé elasticity coefficients to depend on the pressure in the ground. This is but a small correction. But it allows Boussinesq to obtain a definite solution to the problem of the equilibrium of a sand hill that exhibits two cases of collapse of the hill, one (corresponding to the fracture limit with a vanishing external pressure on the external sides of the hill) already obtained by Rankine, and another one by excess of pressure. This remains a historical landmark in soil mechanics and geotechnics.

In addition to the above cited works in the field of continuum mechanics per se, Boussinesq published a lengthy treatise on analysis for applications to mechanics and physics [38]. Like Saint-Venant, but unlike contemporaries like Hertz, Poincaré and Duhem, Boussinesq does not show much interest in the discussion of the principles of mechanics—although he sometimes ponders the right questions (e.g., [42])—, and his main scientific interests are confined to macroscopic physics. Biographic elements are to be found in Picard [77] and Felix [57]; Boussinesq's own partial description of his work is given in Boussinesq [35].

Considered after the passing of time, we see Boussinesq not as a true initiator of theories, but as one of the best analysts of problems of elasticity and fluid mechanics where, with his acute scrutiny of problems, he knows what is most essential and what should be kept in the final analysis to obtain meaningful results. This is mathematical modelling at its utmost level. This vision will be adopted by many mechanicians and applied mathematicians in the second part of the twentieth century with a rigorous application of asymptotic analysis.

6.5 Conclusion

A paper was recently published in the *Notices of the American Mathematical Society* (Vol. 60, no. 7, August 2013, pp. 886–904) with the provocative title: "Is mathematical history written by the victors?" My own answer, whether in mathematics or mechanics—and a fortiori in political history—is an unambiguous YES. The obvious reason for that is that if one tries to publish an original history that falls beside the viewpoint of the "victors", it will never be accepted for publication since the victors usually have at their command all means of publication: specialized journals, review journals, and editorship of series of books.

Thus, if we peruse the lectures, treatises and syntheses published by French scientists at the end of the nineteenth century and the first years of the twentieth century we find many references to Saint-Venant and Bousinesq. This includes authors such as Henri Poincaré (1892), Duhem [56], Brillouin [45], Appell ([2], but first edition in 1900), Jouguet ([64], but first edition in 1909) and the Cosserat brothers [51, 52]. This is also the case of Hellinger [63] and Voigt [89] in Germany and Todhunter [85], Thomson and Tait [83] and Love ([69], but augmented several times) in the UK. Many results of Saint-Venant and Boussinesq were naturally included in the standard expositions of the strength of materials (as evidenced in [84] without the need of precise references. Perhaps that the spirit and works of these two authors became the object of some cult in French *grandes écoles*. Unfortunately, we must also admit that the researches—if any—entertained there in the relevant field were not at the level of this cult. We have commented elsewhere [74, Chap. 7] on this fatal weakness. The flame was taken over by active engineers in the network of Technischen Hochschulen in Germany and the expanding departments of engineering in famous universities and institutes in the USA (Brown, M.I.T., Stanford, Columbia, CALTECH, etc.). Their members had to write the corresponding chapters of this history and they unavoidably took less notice of works produced in French, a language of which the universal character remained only in diplomacy, and this for a limited time only. Cauchy was of course an unavoidable reference, although obviously not read. With the ebb and flow of the ocean on beaches and costal rocks, the statues of Saint-Venant and Boussinesq were cleaned of any unnecessarily attached parasite sea fruits and there finally remained, like in history books for children, almost legendary figures to which expressions like "Saint-Venant principle" and "Boussinesq's approximation" are attached. There was an effort done by Truesdell and Toupin [88] in their monumental encyclopaedia article to correct this tendency.[7] A special case is Boussinesq, whose name was saved from oblivion with the recent expansion of non-linear wave studies.

References

1. Anderson JD (1997) A history of aerodynamics. Cambridge University Press, UK
2. Appell P (1921) Traité de mécanique rationnelle, 3rd edn (1st edn, 1900). Gauthier-Villars, Paris (Facsimilé reprint by Gabay, Paris, 1991)
3. Barré de Saint-Venant AJC (1843) Note à joindre au mémoire sur la dynamique des fluides, présenté le 14 avril 1843. C R Acad Sci Paris 17:1240–1243
4. Barré de Saint-Venant AJC (1845) Note sur la pression dans l'intérieur des corps ou à leur surface de séparation. C R Acad Sci Paris 21:24–26
5. Barré de Saint-Venant AJC (1845) Mémoire sur les sommes et les différences géométriques et leur usage pour simplifier la mécanique. C R Acad Sci Paris 21
6. Barré de Saint-Venant AJC (1847) Mémoire sur la torsion des prismes et sur la forme affectée par leurs sections transversales primitivement planes. C R Acad Sci Paris 24:485–489
7. Barré de Saint-Venant AJC (1851) Principes de mécanique fondés sur la cinématique (Lectures at the institut agronomique). Lithographed (Bachelier, Paris) (available on the site Gallica of the Bibliotèque Nationale de France)
8. Barré de Saint-Venant AJC (1853) De l'interprétation géométrique des clefs algébriques et des déterminants. C R Acad Sci Paris 36:582–585
9. Barré de Saint-Venant AJC (1853) Nouveau mémoire sur la torsion des prismes. C R Acad Sci Paris 36:1028–1030
10. Barré de Saint-Venant AJC (1853–1854) On the impact of prismatical bars (in French). Bull Soc Philomat Paris (Chez Carillon-Goeuvry et Victor Dalmont, Paris)

[7] The works of other famous contributors such as George Green [61], Gabrio Piola [78, 79] and Gustav Kirchhoff [65] is examined in another essay. As to the trio of Cambridgians and immense scientists formed by Kelvin, Stokes and Rayleigh, their work in continuum mechanics and their life are sufficiently well documented in various biographies (see Timoshenko [84], for their contributions to the mechanics of continua).

11. Barré de Saint-Venant AJC (1856) Mémoire sur la flexion des prismes. J Math Pures et Appl 1:89–189
12. Barré de Saint-Venant AJC (1857) Mémoire sur l'impulsion transversale et la résistance vive des barres élastiques appuyées aux deux extrémités. C R Acad Sci Paris 45:204–208
13. Barré de Saint-Venant AJC (1864) Navier's «Résumés des leçons sur l'application de la mécanique à l'établissement des constructions et des machines», 3rd edn Dunod, Paris (available on the site Gallica of the BNF)
14. Barré de Saint-Venant AJC (1864) Historique abrégé des recherches sur la résistance des matériaux et sur l'élasticité des corps solides. In: Navier's «Résumé des leçons sur l'application de la mécanique». Dunod, Paris
15. Barré de Saint-Venant AJC (1865) Notice sur Louis-Joseph comte du Buat. Mémoires de la Société des Sciences, de l'agriculture et des arts de Lille, Vol 1865, 3ème série, Tome II, pp 609–692. Imprimerie L. Danel, Lille (available on the site Gallica of the BNF)
16. Barré de Saint-Venant AJC (1869) Note sur les valeurs que prennent les pressions dans un solide élastique isotrope lorsque l'on tient compte des dérivées d'ordre supérieur des déplacements très petits que leurs points ont prouvés. C R Acad Sci Paris 68:569–571
17. Barré de Saint-Venant AJC (1870) Rapport sur un mémoire de M. Boussinesq, présenté le 19 avril 1969, avec addition le 19 novembre, et relatif à la théorie des ondes liquides périodiques. C R Acad Sci Paris 70:360–367
18. Barré de Saint-Venant AJC (1871) Rapport sur un mémoire de M. Maurice Lévy relatif aux équations générales des mouvements intérieurs des corps solides ductiles, au delà des limites où l'élasticité pourrait les ramener à leur premier état. C R Acad Sci Paris 73:86–91
19. Barré de Saint-Venant AJC (1871) Sur la mécanique des corps ductiles. C R Acad Sci Paris 73:1181–1184
20. Barré de Saint-Venant AJC (1871) Mémoire sur l'établissement des équations différentielles des mouvements intérieurs opérés dans les corps solides ductiles au delà des limites où l'élasticité pourrait les ramener à leur premier état. J Math Pures et Appl 16:308–316
21. Barré de Saint-Venant AJC (1871) Théorie du mouvement non permanent des eaux avec application aux crues des rivières et à l'introduction des marées dans leur lit. C R Acad Sci Paris 73:237–240
22. Barré de Saint-Venant AJC (1871) Sur la houle et le clapotis. C R Acad Sci Paris 73:589–593
23. Barré de Saint-Venant AJC (1872) Sur l'intensité des forces capables de déformer avec continuité des blocs ductiles, cylindriques, pleins ou évidés, et placés dans diverses circonstances. C R Acad Sci Paris 74:1009–1015
24. Barré de Saint-Venant AJC (1872) Sur un complément à donner à une des équations présentées par M. Lévy pour les mouvements plastiques qui sont symétriques autour d'un même axe. C R Acad Sci Paris 74:1083–1087
25. Barré de Saint-Venant AJC (1878) Sur la dilatation des corps échauffés et sur les pressions qu'ils exercent. C R Acad Sci Paris 87:713–718
26. Barré de Saint-Venant AJC (1882) Des mouvements que prennent les diverses parties d'un liquide dans l'intérieur d'un vase ou réservoir d'où il s'écoule par un orifice. C R Acad Sci Paris 94:904–909, 1004–1008, 1139–1144
27. Barré de Saint-Venant AJC (1883) Théorie des corps élastiques (French translation from the German with many comments and additions of Clebsch A (1862) Theorie der Elastizität fester Körper, Leipzig; translation with the help of AA Flamant) (Reprint by Johnson Reprint Corp. New York 1966)
28. Barré de Saint-Venant AJC (1885) Mouvements des molécules de l'onde dite solitaire, propagée à la surface de l'eau d'un canal. C R Acad Sci Paris 101:1101–1105, 1215–1218, 1445–1447
29. Barré de Saint-Venant AJC (1887) Mémoire sur le choc longitudinal de deux barres élastiques. J Math Pures et Appl 12:237–376
30. Bois PA (2007) Joseph Boussinesq (1842–1929) A pioneer of mechanical modelling at the end of the 19th century. C R Mécanique (Acad Sci Paris) 335:479–495

31. Bois PA, Verdier N (2009) Joseph Boussinesq (1842–1929) de Gap à Lille ou de l'élève au savant mécanicien. Contribution au Colloque National de la Recherche CNRIUT 2009, 8 juin 2009, 8 pages
32. Boussinesq JV (1871) Théorie de l'intumescence liquide appelée onde solitaire ou de translation, se propageant dan un canal rectangulaire. C R Acad Sci Paris 72:755–759; 73, 256–260, 1159–1162, 1210–1212
33. Boussinesq JV (1876) Essai théorique sur l'équilibre d'élasticité des massifs pulvérulents et sur la poussée des terres sans cohésion. Acad. Royale de Belgique, (Présenté le 6 juin 1874), Vol 2, p 40 on. Imprimerie F Hayez, Bruxelles
34. Boussinesq JV (1877) Essai sur la théorie des eaux courantes. Mémoires présentés par divers savants à l'Académie des sciences, XXIII(1):1–680 (with an introduction of 22 pages by Saint-Venant)
35. Boussinesq JV (1880) Notice sur les travaux scientifiques de Mr Bousisnesq et suppléments à cette notice. Imprimerie L. Danel, Lille (republished and updated in 1883 and 1885)
36. Boussinesq JV (1885) Sur la résistance qu'oppose un fluide indéfini au repos, sans pesanteur, au mouvement varié d'une sphère solide qu'il mouille sur toute sa surface, quand les vitesses restent bien continues et assez faibles pour que leurs carrés et produits soient négligeables. C R Acad Sci Paris 100:935–937
37. Boussinesq JV (1885) Applications des potentiels à l'étude de l'équilibre et des mouvements des solides élastiques. Gauthier-Villars, Paris
38. Boussinesq JV (1887, 1890) Cours d'analyse infinitésimale en vue des applications mécaniques et physiques, Two volumes. Gauthier-Villars, Paris
39. Boussinesq JV (1888) Equilibre d'élasticité d'un solide sans pesanteur, homogène et isotrope, dont les parties profondes sont maintenues fixes, pendant que la surface éprouve des pressions ou des déplacements connus, s'annulant en dehors d'une région restreinte où ils sont arbitraires. C R Acad Sci Paris 106(1043–1046):1119–1122
40. Boussinesq JV (1897) Théorie de l'écoulement tourbillonnant et tumultueux des liquides. Gauthier-Villars, Paris
41. Boussinesq JV (1903) Théorie analytique de la chaleur mise en harmonie avec la thermodynamique et avec la théorie mécanique de la lumière, Tome II : Refroidissement et échauffement par rayonnement. Conductibilité des tiges, lames et masses cristallines. Courants de convection. Théorie mécanique de la lumière. Gauthier-Villars, Paris (657 pages)
42. Boussinesq JV (1912) Pourquoi les équations différentielles de la mécanique sont du second ordre plutôt que du premier, ou en d'autres termes, déterminent les accélérations des points matériels plutôt que leurs vitesses. C R Acad Sci Paris 155:5–8
43. Boussinesq JV, Flamant AA (1886). «Notice sur la vie et les travaux de Barré de Saint-Venant», Annales des Ponts et Chaussées, 6th series, Vol XII, p 557 on
44. Belhoste B (1991) Augustin-Louis Cauchy: A biography. Springer, New York (English translation from the French original "Cauchy, 1789–1857", Berlin, Paris)
45. Brillouin M (1884–1885) Leçons sur l'élasticité et l'acoustique. Librairie Labouche, ToulouseCapecchi D (2012) History of virtual work laws, Vol 42 in the Science Networks Historical Studies Series. Birkhäuser-Springer, Milano
46. Cauchy AL (1823) Recherches sur l'équilibre et le mouvement intérieur des corps solides ou fluides, élastiques ou non élastiques. Bull Soc Filomat Paris, January issue 9–13 (This is the *fons et origo* of modern continuum mechanics, presenting some long abstract of the ideas originally orally submitted to the Paris Academy of Sciences on 30 Sep 1822)
47. Cauchy AL (1828) Sur les équations qui expriment l'équilibre ou les lois du mouvement intérieur d'un corps solide élastique ou non élastique. Exercices de mathématiques, Vol 3, pp 160–187, Sept 1828 (This presents in print the ideas originally orally presented to the Paris Academy of Sciences on 30 Sep 1822)
48. Cauchy AL (1828) In: Exercices de Mathématiques, Vol 3, Sept 1828, pp 188–212, and «De la pression ou tension dans un système de points matériels». Oct 1828, pp 213–236

49. Christov CI, Maugin GA, Porubov AV (2007) On Boussinesq's paradigm in nonlinear wave propagation. Comptes Rendus Mécanique (Thematic issue devoted to Boussinesq and edited by PA Bois) 335(9–10):521–535
50. Coffin JG (1914) Calcul vectoriel (French translation by M Véronnet of the English original vector calculus: an introduction to vector methods and their various applications to physics and mathematics, 1909). Gauthier-Villars, Paris
51. Cosserat E, Cosserat F (1896) Sur la théorie de l'élasticité. Ann Fac Sci Toulouse 1$^{\text{ère}}$ série 10(3–5) I1–I116
52. Cosserat E, Cosserat F (1909) Théorie des corps déformables, Hermann, Paris
53. Crowe MJ (1967) A history of vector analysis: The evolution of the idea of a vectorial system. University of Notre Dame Press (Dover reprint, New York, 1985)
54. Dahan-Dalmenico A (1984–1985) La mathématisation de la théorie de l'élasticité par A.L. Cauchy et les débats dans la physique mathématique française (1800–1840). Sciences et techniques en perspective 9:1–100
55. Darrigol O (2005) Worlds of flows: a history of hydrodynamics from the Bernoullis to Prandtl. Oxford University Press, Oxford
56. Duhem P (1891) Hydrodynamique, élasticité, acoustique. A. Hermann Editeur, Paris
57. Felix L (1970) Entry: JV. Boussinesq. In: Gillispie CC (ed) Dictionary of scientific biography. General Editor, Scribners & Sons, New York (also: http://www.encyclopedia.com.doc/1G2-2830900573.html)
58. Flamant AA (1889) Courbes représentatives des lois du choc longitudinal et du choc transversal d'un barre prismatique, dressées par feu de Saint-Venant. J Ecole Polytechnique Cahier 59:97–128
59. Grassmann HG (1844) Die lineale Ausdehnungslehre. Wiegand, Leipzig (English translation by L. Kannenberg: a new branch of mathematics. Open Court, Chicago, 1995)
60. Grattan-Guinness I (1993) The ingénieur-savant (1800–1830): a neglected figure in the history of French mathematics and science. Sci Context 6(2):405–433
61. Green G (1828) An essay on the mathematical analysis of electricity and magnetism. Privately printed, Nottingham (Also in: Mathematical papers of George Green, Ed. N.M. Ferrers, Macmillan, London, 1871; reprinted by Chelsea, New York, 1970)
62. Green G (1839) On the laws of reflection and refraction of light at the common surface of two non-crystallized media. Trans Cambridge Phil Soc 7:245–269
63. Hellinger E (1914) Die allgemein Ansätze der Mechanik der Kontinua. In: Klein F, Wagner K (eds) Enz Math Wiss, vol 4, part 4. Springer, Berlin, pp 602–694
64. Jouguet E (1924) Lectures de mécanique (La mécanique enseignée par les auteurs originaux). Gauthier-Villars, Paris (Facsimile reprint, Gabay, Paris, 2007)
65. Kirchhoff G (1876) Vorlesungen über Mechanik, (Lectures on Mechanics), Leipzig
66. Lamé G (1852) Leçons sur la théorie mathématique de l'élasticité des corps solides. Paris, (2nd edn, 1866) (Facsimile reprint, Gabay, Paris, 2006)
67. Lamé G (1859) Leçons sur les coordonnées curvilignes et leurs diverses applications. Paris
68. Lévy M (1871) Extrait du mémoire sur les équations générales des mouvements intérieurs des corps solides ductiles au delà des limites où l'élasticité pourrait les ramener à leur premier état. J Math Pures et Appl 16:369–372
69. Love AEH (1892) A treatise on the mathematical theory of elasticity, (1944, 4th edn, Dover reprint, New York; originally published in two volumes in 1892–1893)
70. Maisonneuve O (2010) Adhémar Barré de Sain-Venant, grand homme de science et de caractère, protecteur d'un saint-andréen célèbre, Joseph Boussinesq. Comm to the Académie des Sciences et Lettres de Montpellier, 31 mai 2010, pp 177–192
71. Maugin GA (1992) The thermomechanics of plasticity and fracture. Cambridge University Press, U.K
72. Maugin GA (2013) On the Cosserat's book of 1909. Preprint UPMC-Paris (herein as Chap. 8)
73. Maugin GA (2013) What happened on September 30, 1822 and what were its implications for the future of continuum mechanics? Preprint UPMC-Paris (herein as Chap. 3)

74. Maugin GA (2013) Continuum mechanics through the twentieth century: a concise historical perspective. Springer, Dordrecht, The Netherlands
75. Moigno F (1868) Leçons de mécanique analytique, Paris (contains two chapters added by Saint-Venant)
76. O'Connor JJ, Robertson EF (2000) Entry: Adhémar Jean Claude Barré de Saint-Venant, MacTutor history of mathematics (http://www-history.mcs.st-andrews.ac.uk/Biogaphies/Saint-Venant.html)
77. Picard E (1933) La vie et l'oeuvre de Joseph Boussinesq. Acad Sci Paris, Discours et notices, Gauthier-Villars, Paris
78. Piola G (1836) Nuova analisi per tutte le questioni della meccanica moleculare. Mem Mat Fis Soc Ital Modena 21(1835):155–321
79. Piola G (1848) Intorno alle equazioni fondamentali del movimento di corpi qualsivoglioni considerati secondo la naturale loro forma e costituva. Mem Mat Fiz Soc Ital Modena 24(1):1–186
80. Sarraut E (1897) Adhémar-Jean-Claude Barré de Saint-Venant (1797–1886). In: Livre du Centenaire de l'Ecole Polytechnique, Paris
81. Schmitt FG (2007) About Boussinesq's turbulent viscosity hypothesis; historical remarks and a direct evaluation of its validity. C R Mécanique 335:617–627
82. Stokes GG (1845) On the theories of internal friction of fluids in motion, and the equilibrium and motion of elastic solids. Trans Cambridge Phil Soc 8:287–319
83. Thomson W (Lord Kelvin), Tait PG (1867). Treatise on natural philosophy (1st Edn) Oxford (2nd edn, Cambridge, 1879–1883]
84. Timoshenko SP (1953) History of the strength of materials, McGraw Hill, New York (Dover reprint, New York, 1983) (Review of this book by Truesdell reprinted in Truesdell, 1984, pp 251–253)
85. Todhunter I (1886) A history of the theory of elasticity and the strength of materials from Galileo to the present time, vol 1 Cambridge University Press, UK (edited and published posthumously in 1886 by Karl Pearson)
86. Tresca HE (1872) Mémoire sur l'écoulement des corps solides. Mémoires présentés par divers savants, Acad Sci Paris 20:75–135
87. Truesdell CA (1984) An idiot's fugitive essays on science. Springer, New York
88. Truesdell CA, Toupin RA (1960) The classical field theories. In: Flügge S (ed) Handbuch der Physik, Bd III/1. Springer, Berlin
89. Voigt W (1887) Teoretische Studien über Elasticitätverhältinsse der Krystalle, I.II. Abh K Ges Wissen Göttingen 34:3–52, 53–100
90. Whittaker ET (1951) A history of the theories of aether and electricity. Nelson, London
91. Zeytounian RK (2003) Il y a cent ans… et aujourd'hui, l'approximation de Boussinesq. Matapli (Paris) 71:71–84

Chapter 7
Helmholtz Interpreted and Applied by Duhem

Abstract Gibbs and Helmholtz provide the strongest scientific influences on Duhem's works in what is now called *mathematical physics.* With the help of examples exhibiting this influence in thermo-mechanics and electrodynamics, it is shown that this conduced Duhem and his followers to a definite style and practice of physical science marked by abstraction and mathematical rigor. This has practically become the rule while helping to classify the numerous, linear or non linear, effects and giving rise to fruitful developments, in continuum physics.

7.1 Hermann von Helmholtz and Pierre Duhem

When Hermann von Helmholtz (1821–1894) dies in Berlin in 1894, the same year as Heinrich Hertz of electrodynamics fame, Pierre Duhem (1861–1916) is only thirty three years old and moving from one teaching position in Rennes to a professorship of theoretical physics at the University of Bordeaux.[1] But he has already published in 1886 one of his most original books [9], a short treatise on thermodynamic potentials and their application to what he calls *"chemical mechanics"* and electric phenomena. This book appears to be some kind of matured rewriting of an aborted thesis project which he wrote and presented in

Unpublished contribution to the Wissenschaftliche Veranstaltungen aus Anlass des 100. Todestages *von* Hermann von Helmholtz. Fachkolloquien zu Themen Helmholtzscher Traditionen, 10 September 1994, Humboldt-Universtät zu Berlin (Thermodynamik: Von den Berliner Anfängen zu Modernen Entwicklungen). Most of the bibliographical material needed in this study was gathered while the author was a member of the Wissenschaftskolleg zu Berlin (1991–92). He received there the help of a formidable library service.

[1] Excellent biographical sketches of P. Duhem are given in Jaki [31], Brouzeng [4], and Miller [42]

G. A. Maugin, *Continuum Mechanics Through the Eighteenth and Nineteenth Centuries*, Solid Mechanics and Its Applications 214, DOI: 10.1007/978-3-319-05374-5_7,

1884 while being a third year student at the celebrated *Ecole Normale Supérieure,* having not yet obtained his formal Master's degree, nor passed the difficult competition exam called the *Agrégation* (formally, Lycée teaching diploma). Unfortunately for Duhem, this project included a somewhat harsh argument against Marcellin Berthelot, then "Pope" of French "republican" physics, but conceptor of the ill-fated principle of *maximal work,* whereby the heat of reaction defines the criterion for the spontaneity of chemical reactions. Rightly, but perhaps without the respect and touch of hypocrisy that would have been more suited in such occasions Duhem, building on of the notion of *free energy* dear to Helmholtz and also known to J. W. Gibbs who called it "*available energy"*, denounced Berthelot's theory as a fraud, and properly defined the required criterion in terms of the free energy. The thesis was rejected. An enmity between Berthelot and Duhem followed from which Duhem never fully recovered, having his academic career impeded for his whole life. Duhem obtained his doctoral degree with another subject (theory of magnetism) and a different jury (involving Henri Poincaré) in 1888.

Pierre Duhem, like many other physicists of the period, was interested in the whole of phenomenological physics, the physics of his time—a pre-quantum time -, and was particularly keen on problems of electromagnetism and electrodynamics. With the discovery of Heinrich Hertz about electromagnetic waves and his will to capture the whole of physics in a somewhat rigorous framework, he turned to the study of the foundations of the theory of electrodynamics. His rapid conclusion was that Maxwell's work was more the work of an inspired artist than that of a logician (which Maxwell had never claimed to be) and if something deserved the respect of a true mathematician or mathematical physicist, it was the electrodynamic theory set forth by Helmholtz, although perhaps there was more in it than necessary (see below). While Poincaré [44] exposed all theories in a detailed, cautious and balanced manner—which announces his *conventionalism*—in his lectures at the Sorbonne, Duhem, true to himself, wrote a pamphlet against Maxwell [14] in which he had to acknowledge Maxwell's obvious genius and ingenuity (in introducing the notion of displacement current) while exposing his sloppiness to the scientific world: Maxwell achieved something great, but not in a form admissible to a professional of mathematical physics. This is the viewpoint that Duhem was to illustrate first on this particular example, and that some irrational nationalistic tendencies were to develop in an open attack of the English way of practicing science, and where German (in particular Helmholtz) and obviously French science got the best share in an analysis which was nonetheless to bring an original and creative viewpoint in epistemology [15].

A few years before his untimely death in 1916, Duhem [16] published a formidable treatise on *energetics* or *general thermodynamics* in which he naturally praised the German school of energetics in which Helmholtz was incorporated as a precursor. Furthermore, basing on the concept of free energy, this beautiful work presented a thorough discussion of stability which would sixty years later be taken over by a whole school of *continuum thermo-mechanics.* That school, under the leadership of Clifford Truesdell, developed, at the image of the best analysts of the nineteenth century and Duhem himself, a *rational approach* which, like all "*rigidifications",* was instantaneously fruitful but may have brought some damage in the long run (think of the Bourbaki style in mathematics). Here also, Duhem had a debt to Helmholtz via Hertz and his enterprise of rationalization of mechanics as shown in his small book on the *principles of mechanics* [27]. From this we gather that Helmholtz inspired or for the least shared, some of the views expressed by Hertz, especially in using Hamilton's dynamic principle. This was to develop in a true *theory of fields* in which the notion of force would ultimately be banished.

In the remainder of this contribution we shall examine in greater detail the four instances and subject matters at which the intellectual trajectories of von Helmholtz and Duhem clearly crossed each other. These views, methods, attitudes toward scientific practice, sometimes only hinted at by Helmholtz and more forcefully expressed by the sanguine Duhem, have necessarily influenced our own practice and our view of *mathematical physics* in general and the way we teach it and write papers in particular; That is, a style has developed which has built on the peculiarities exhibited by our grand predecessors. This is all the more true in a field such as *continuum thermodynamics* which may rightly be considered as a modern expression of *energetism,* but without the limitations and blinkers too often put forward by its contempters.

7.2 Free Energy ("Freie Energie")

Nowadays all our (good) students know the difference between *internal energy and free energy,* the latter being also called *Helmholtz energy,* although no personal name seems to be attached to the former, being more the result of collective thinking and appraisal. They essentially know when each of these prevails in the thermodynamic description of systems, for *isentropic* conditions in the first case, *isothermal* ones in the second case. This obviously has drastic consequences even in the most modern researches and applications of thermo-mechanics. For example, while P. Hugoniot (1851–1887) clearly shows the relationship between his celebrated *jump conditions* across shock waves and the notion of *internal energy* (work generalized by P. Duhem himself in 1901), coherent transition fronts in elasticity, describing transitions of the martensitic-ferroelastic type in conductors of heat, involve the free *energy* of the deformable material, hence a function of strain and temperature, as shown by the author and co-workers.[2] That is, the

[2] In shock waves for one-dimensional motions in fluids the *Hugoniot jump relation* reads

$$H := [e + \langle p \rangle \tau] \tag{a}$$

where $e(\tau, \eta)$ is the *internal energy* per unit mass, a function of specific volume τ and specific entropy η; p is the thermodynamical pressure, $\theta = \partial e / \partial \eta > 0$ is the thermodynamic temperature, <..> is the mean value of a quantity at the shock, and [..] its jump. The best known disciple of Duhem was E. Jouguet, a specialist of shock and detonation waves, and explosives.

driving force acting locally on the transition front is essentially the jump in free energy, or a quantity akin to that. The same holds true for electromechanical or magneto-mechanical transition fronts. In contrast, for *shock waves,* whether in solids or in fluids, it is the internal energy, function of strain or volume and entropy which provides a basis for a discussion of the existence of shocks by means of the so-called *Hugoniot function.* Duhem was instrumental in these developments through his teaching and the introduction of several welcomed concepts. Apparently, he pondered such questions as that of thermodynamic potentials while still in secondary school/gymnasium under the supervision of his physics teacher Jules Moutier. The works that he expanded in the period 1884–1900 in this field clearly are extensions and elaborations of the pioneering work of Gibbs and Helmholtz. He had read both authors while in college, especially the first part of Helmholtz [25]. He was also quite aware of the work of the French geologist François Massieu (1832–1896). It is *him* who gave to Massieu's characteristic functions [34] the name of *thermodynamic potentials,* while he treated systematically all types of thermodynamic systems involving as well thermoelectricity, capillarity, mixtures of perfect gases, those of liquids, solutions in gravitational fields and magnetic fields, freezing points, etc.

Duhem acknowledged his immense debt to Gibbs and Helmholtz when, in Duhem [17], he described in detail his scientific trajectory in the presentation of his works while a candidate for election to the Paris Academy of Sciences. In the 1960s, the Truesdellian-Nollian school of rational thermodynamics took over the *axiomatic* approach of Duhem to introduce a priori in full dynamics notions which are usually defined only in thermostatics, e.g. temperature, entropy, and free energy. Much celebrated papers by Coleman and Noll [8] in fact introduced the statement of the second law of thermodynamics in which the time rate of change of free energy is present as the so-called *Clausius-Duhem* inequality, a mathematical restriction imposed on a large class of material behaviors.[3]

But in his pursuit of a general thermodynamics that started with a full exploitation of Helmholtz' *freie Energie*, Duhem accomplished much more in that he also introduced seminal notions and powerful methods which were to bring efficient results in the second part of the twentieth century. Among the methods we

(Footnote 2 continued)
For one-dimensional *phase-transition fronts* (this dimensionality is chosen for illustrative purpose only) in solids, the *driving force* acting on the front reads:

$$F = -[W(\varepsilon, \theta) - \langle \sigma \rangle : \varepsilon] \tag{b}$$

where W is the free energy per unit volume, a function of strain ε and temperatureθ, the entropy per unit volume is given by $S = -\partial W/\partial \theta$, *and* σ is the stress. $H = 0$ at shocks whereas F is in general not zero at irreversibly progressing phase-transition fronts. Both F and the nonzero propagation velocity V of the front satisfy jointly at the front the second law of thermodynamics in the form $F.V > 0$ or = 0 (for the exact three-dimensional theory in conductors of heat see [39]).

[3] Much more on rational thermodynamics is to be found in Truesdell [46].

should emphasize the systematic use of *Euler's theorem for homogeneous functions.*[4] The celebrated *Gibbs-Duhem equation,* of relatively innocent outlook in our much more mathematically trained society, was the result of this use. It may not be altogether ridiculous to recall that this technique reduces the derivation of relations among partial molar properties of a solution to the repeated application of this theorem. Among the notions unequivocally introduced by Duhem we find those of *normal* variables of state and the embryonic form of what was to become known as *internal variables* of state in recent times. Normal state variables are composed of a set including entropy and a remainder set $\{\chi_\beta; \beta = 1, 2, \ldots, n\}$ according to which the first law of thermodynamics in Gibbs' form can be expressed as

$$dE = \omega + \varphi \;, \tag{7.1}$$

where ω and φ are the elementary work and heat received by the system in such a way that

$$\varphi = \theta\, dS, \quad \omega = \sum_{\beta=1}^{n} \tau_\beta\, d\chi_\beta, \tag{7.2}$$

where S is the entropy and the $\tau'_\beta s$ are thermodynamic forces associated with the $\chi'_\beta s$. That is, there is ***no*** dS in ω and this, together with the positiveness of the dual variable, the temperature θ, makes S singular among the state variables. Indeed, E, the *internal energy,* is a *thermodynamic* potential in the sense of Duhem, in such a way that

$$\theta = \partial E / \partial S, \quad \tau_\beta = \partial E / \partial \chi_\beta \;. \tag{7.3}$$

Manville [32, p.225], in assessing Duhem's contribution to thermodynamics, says that Duhem based on an idea of Helmholtz while introducing the notion of *normal* variables.

Although Duhem was not equipped to solve the problem of what he called the *"nonsensical"* (abérrantes) branches of mechanics—those branches where *dissipation* is the most important mechanism at play, one may find in some of his penetrating writings [12]—cf. Manville [32], p. 303; Truesdell [46], p. 39—the germ of the notion of internal variable of state. Without digressing too much on this we simply note that these are additional variables of state whose introduction reflects our lack of complete control of microscopic mechanisms (e.g. dislocation movement) which are responsible for some macroscopically irreversible manifestations (e.g. plasticity and hardening in metals, magnetic hysteresis in ferromagnets). Although measurable by a "gifted" experimentalist once they have been identified (this is the crux of the matter), these variables are not *controllable* so that they clearly distinguish themselves from the more classical *observable*

[4] This is rightly emphasized by Miller [42], p. 229.

variables of state that are controlled by body or surface actions—cf. Maugin and Muschik [38] for a lengthy analysis.

By gathering the properties of *convexity, Euler's* identity for homogeneous functions of degree *n*, the powerful notion of *Legendre transform* and more generally *Legendre-Fenchel transform, normal* variables and internal *variables* with the local statement of the second law that carries his name, we are now able to formulate in a coherent, mathematically correct, and efficient manner a true thermo-*dynamics* of complex irreversible processes of which Duhem could only dream of. To achieve this, the school of de Donder, Prigogine, Meixner, de Groot and others had, in the mean time, to formulate the second law in an operative form, the celebrated bilinear (in "fluxes" and "velocities") form of the *dissipation inequality* (which is not limited to linear dissipative processes as too often advertised). Also potent in these developments was the axiomatization of *thermostatics* by Caratheodory [6] and Born [2]; here also, with Miller [42, p.229], we must reckon the pioneer role of Duhem [10] who essentially gave the correct definitions relating to the first law [see above Eqs. (7.1)–(7.3)], what marks the true beginnings of the axiomatization of branches of science outside pure mathematics. Duhem paid a special tribute to Helmholtz in this regard (see Duhem [17, p. 75]; also Duhem [16]). Nowadays, most teachers of continuum thermo-mechanics[5] follow in their practice the grand avenue opened by Clausius, Helmholtz, Duhem, Caratheodory and, more recently, among others, P. Bridgman, J. Kestin, C. A. Truesdell, W. Muschik, and I. Müller.

7.3 Helmholtz-Duhem Electrodynamics

One uncompleted project of Pierre Duhem was the incorporation of electricity and magnetism, including nonlinear dissipative effects such as hysteresis, in his broad energetic view. This he never achieved as bears witness his monumental treatise of 1911. But in his search for this development he inevitably faced the various theories that were available in his time.

If we make exception of the early French and Italian works (Coulomb, Poisson, Ampère, Mossotti), we find essentially two avenues along which electromagnetism developed: one led by Faraday in England, which gave rise to W. Thomson's (Kelvin) and Joule's works, and Maxwell's brilliant synthesis and further developments by the "Maxwellians" (Cf. [30]), the other in Germany with, after Gauss, people like Neumann, Kirchhoff, Weber, Riemann, and Helmholtz. In other words,

[5] This is exemplified by the author's course that deals with strongly nonlinear dissipative processes Maugin [36]. This style of thermodynamical exposition is to be found in the *Journal of Non-Equilibrium Thermodynamics*, de Gruyter, Berlin. The book by Bridgman [3] was instrumental in this development, especially in influencing Joseph Kestin from whom we all more or less learned our "thermodynamics". The points of view of Duhem, Bridgman and Kestin are examined in parallel and comparison in the book [37].

in the scrutiny of, at the time, recent works, Duhem was confronted with scientific works expanded in various cultural and educational backgrounds and environments that differed from the French one, different "national styles" following one of his favorite expressions. It is probably during this thorough analysis of available works that he realized how different were the attitude and practice of various scientists in forming ideas, conceiving models in general, and assessing the role of scientific constructs.

Like all potential readers, and the few who indeed passed to the act, Maxwell's [40] treatise on electricity and magnetism seemed to Duhem to be full of contradictions, non rigorous developments, errors in sign, and lacking true experimental foundations (a point Duhem emphasized in spite of his own tenuous contact with experiments). The main question is whether we can reach the mathematical form of physical laws through mere divination, helped in this by a strong inclination towards aesthetics and a love of symmetry, or through a logical unwinding of arguments of which pure mathematics offers a paragon (in its final written form at least) with a view to reflect, but not to explain, physical reality; to *"save the phenomena"* according to Plato's celebrated formula. In Maxwell, his "*bête noire"'*, and perhaps even more in W. Thomson (Lord Kelvin), Duhem detects the prototypically British *"ample and shallow"* mind (Duhem [15, Chap. 4]). According to Duhem, this, obviously, unfavourably compares with the typically French *"narrow and deep"* scientific mind which he naturally considers to be far more superior in so far as scientific development is concerned; this despite all evident successes and creativity of British physics in the nineteen century! The fact that we find both types on both sides of the Channel (to him the French mathematical physicist Boussinesq belongs to the "British" class, op. cit. p.89) did not deter Duhem from his general "theory" which, therefore, must be considered in a true statistical way. The irony of all this is that finding an electromagnetic theory to his own taste in the German background, especially in the form expanded by Helmholtz in the 1870s, Duhem generously classified German science on the French "narrow-deep" minded side: "To a Frenchman or a German, a physical theory is essentially a logical system" [15, p. 78]. This is to be considered a compliment in the mouth of Duhem. With the explosion of World War One, he changed his mind towards most German scientists whom he then relegated in the rigourless ample-shallow type, but for a few exceptions such as Helmholtz. That does not sound very serious at all.

More seriously, it is true that the introduction of the displacement current in electrodynamics always looks like *magic* to the inexperienced observer.[6] Thanks to it, some symmetry is established between electric and magnetic phenomena, from which there results a *wave equation,* with finite speed of propagation for electromagnetic waves. The latter are purely *transverse* as experimentally checked

[6] Duhem [15, English translation, p. 79] claims that Maxwell justifies the introduction of the displacement current by means of two lines:" The variation of the electric displacement should be added to the current in order to obtain the total movement of the electricity".

by Hertz in 1888. By the same token optics was given a full electromagnetic basis. At this point it may be relevant to remind the reader that most researchers in *elasticity theory* in the 1820s–1840s (e.g. G. Green and A. L. Cauchy who were perfect models in Duhem's view of mathematical physics) were motivated by the construction of a model of continuum capable of supporting *transverse vibrations* at it was already known that luminous vibrations in transparent media are essentially transverse (cf. A. Fresnel), while the only continuum whose behavior was well understood before the introduction of the notion of *stress* (tensor) could only support longitudinal vibrations (acoustic waves in the most restricted way).

Now back to Helmholtz and to what Duhem considers a good logico-deductive construct, one that does not disturb the French and German minds. Helmholtz,[7] himself dissatisfied with Maxwell's approach, proposed an electromagnetic theory which precisely allows for the propagation of both transverse *and* longitudinal perturbances. An extra parameter (compared to Maxwell's framework) thus appears in that theory. If it is true (in fact a discussed matter) that, by an appropriate choice of values of parameters, Maxwell's equations appear as a special case of Helmholtz's theory—i.e. transverse fluxes propagate with the velocity of light if one adopts the Faraday-Mossotti hypothesis, Duhem, in the faith of experiments which proved to be wrong, believed that there were experiments showing that longitudinal fluxes can also be propagated, at the velocity of light as well, this fixing the value of Helmholtz's additional parameter. We all know that *"Maxwell's equations"*—as Hertz and Heaviside liked to call Maxwell's theory in a reductive manner—have triumphed. But it is still believed that the logical derivation of Maxwell's equations from a continuum point of view comes best through what Miller [42, p. 231] definitely calls the "*Helmholtz-Duhem"* theory with the proper choice of constants.[8] More than this precise derivation, it is perhaps the general attitude towards theoretical constructs which should bring some lesson

[7] This is mainly exposed in Helmholtz [24]—also (Helmholtz [26], posthumous). In modern times, this theory has been discussed several times, e.g. by Hirosize [29] and Buchwald [5, see Chap. 21]. Strangely enough, none of the modern commentaries cites Duhem's thorough analysis, perhaps because Duhem went through some purgatory period and the original work in French was never reprinted or translated. For the information of the reader, Helmholtz's equations using potentials read (in modern notation).

$$\begin{aligned}
&\nabla^2\mathbf{U} = (1-k)\nabla(\partial\phi/\partial t) - 4\pi\mathbf{J},\\
&\nabla.\mathbf{U} = -k\,\partial\phi/\partial t\,,\\
&\nabla^2\phi = -4\pi\rho_f, \quad \left(\partial\rho_f/\partial t\right) + \nabla.\mathbf{J},
\end{aligned}$$

where $\mathbf{U}$ and ϕ are a vector potential and a scalar potential, and k is a constant to be found by means of experiments conducted on an open circuit. It is to be noted that the time rate of change of ϕ affects $\mathbf{U}$ by virtue of the continuity equation. This, in fact, is a hindrance in the reduction of Helmholtz' to Maxwell's equations. We recommend Buchwald's discussion as very enlightening, especially in so far as the "Maxwell limit" is concerned. Duhem's analysis is also briefly given in his Duhem [17, pp. 147–150].

[8] This is contended by Roy [45], and. O'Rahilly [43, Chap. 5]; See also Buchwald [5, Chap. 21].

especially in teaching practice or in establishing well-framed mathematical-physical theories.

Duhem [17, pp. 147–150] comments that the superiority of Helmholtz's electrodynamic doctrine stems from its application of logical rules of thought, a standpoint that Poincaré [44], Hertz [28], and Boltzmann [1] seem to share to some extent. His attitude towards mathematical rigour and generalization are reflected in many formal approaches to continuum physics, in particular thermodynamics, in the 1960s–1980s, essentially through the works of the Truesdellian-Nollian school and its imitators. A concern to be as general as possible and a fear to miss an effect or coupling, how small it may be, are thus responsible for an inflation in length and breadth which is not commensurate with to the obtained results. The introduction of the so-called "*principle of equipresence*"[9] in the formulation of constitutive equations by C. A. Truesdell, although a useful guideline in several cases, reflects this kind of abuse and often un-necessary generality, the said principle being violated or negated in many cases (hence not a principle at all, at most a precautionary measure). The same holds true of the manifested will to introduce as wide as possible classes of constitutive equations which are *functionals* over elapsed time (hereditary processes) and space (strong nonlocality). In many instances, these can be dispensed with from the beginning with some physical insight which has nothing to do with Maxwell's magic, but is closely related to his ingenuity.

7.4 Stability

Largely under the influence of Gibbs, a life concern and recurrent research theme of Duhem has been the *stability of equilibrium* for a variety of circumstances. He synthesized his results on this and his general theory about it in his treatise on energetics of 1911. In this ambitious endeavour, as we know now in pure mechanics, potential energy plays a fundamental role. Therefore, in a thermodynamic background, it is the *thermodynamic potentials,* Helmholtz's *freie Energie* or the *available energy* of Gibbs in isothermal systems, which capture the essentials of this property of stability. Early in his research Duhem concentrated on isothermal and isentropic stabilities of classical thermodynamics. He was very successful with sufficient conditions but much less with necessary ones. He tried to extend his results to all types of continua including elastic bodies and viscous systems [11]. But the difficulty was beyond the knowledge of the period. Such questions have in fact been rigorously resolved only recently. Coleman [7] and Ericksen [18–20] have been instrumental in dealing with these aspects of stability theory.

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[9] We remind the reader that this "principle" recommends to enter the *whole* set of independent field variables as possible arguments in *all* constitutive equations.

What is perhaps more striking is the visionary insight that Duhem brought to this study by showing familiarity with Lyapunov's works, although with some (forgivable) confusion. In that he pioneered the use of Lyapunov's functions that would become an essential ingredient in the modern studies of Glansdorff and Prigogine [23][10] as the problem of the response of a system to spontaneous fluctuations is related to the Le Châtelier-Braun principle, itself clarified by Duhem's study of the displacement out of equilibrium. Even more striking is the fact that some of his studies on *hysteretic phenomena* clearly anticipate modern formulations of elastoplasticity or magnetic hysteresis by setting forth a local stability criterion of the Drucker type (this gives the sign of the slope at any point of the hysteresis response) and a global one (over a complete cycle—this in fact imposes the sense in which the cycle is described) of the Ilyushin type.[11] These studies, interesting and farsighted as they were, remained practically ignored until scientists interested in the general phenomenon of hysteresis re-discovered them.[12]

7.5 Conclusions

The influence of Gibbs and Helmholtz on Duhem's passionate interest in thermodynamics, on his general view of energetics, and in specific aspects of his research, is obvious.[13] Regarding Helmholtz more particularly, we have already noted that Duhem inherited from him the notion of *free energy,* or more generally,

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[10] See Chap. 4 in Glandsdorff and Prigogine [23] for the stability according to Gibbs and Duhem. The general theory of the stability of thermodynamic equilibrium makes use of the Gibbs-Duhem approach and the balance of entropy. Chapters 6 and 7 deal with systems out of equilibrium. The minimum property of the dissipation function has been established by Helmholtz for a linear viscous fluid. The relationship between the Le Châtelier-Braun principle and Duhem's work on the displacement out of equilibrium is reported in Manville [32, pp. 259–260].

[11] The original works of P. Duhem on hysteretic systems are published in 1901 in the *Zeitschrift für physikalisch Chemie* and in the *Mémoires présentés à la Classe de Sciences de l'Académie de Belgique.* The most relevant equations are best expressed by Manville [32]—apparently the finest and sharpest analyst of Duhem's scientific works—e.g. his Eq. (9) in p. 310, $dA.da > 0$, and his un-numbered equation in p. 313: Integral of $A\ da > 0$ for an isothermal closed cycle, are identical to the expressions of Drucker's and Ilyushin's local and global stability conditions of modern plasticity with hardening—compare to Eqs. (5.75) and (5.88) in Maugin [36], pp. 108 and 111, respectively, where the proof relies on the convexity of the free energy with respect to a, and the convexity of the homogeneous positive dissipation potential in A, the thermodynamical force associated to a. This applies to so-called *generalized standard* (thermodynamic) materials whose two basic potentials (free energy and dissipation) exhibit these properties. Duhem did not possess the last concept but he had a rather clear view of *incremental* laws exhibiting hysteresis as shown by Manville's [32] equations in pp. 307–310.

[12] Thus Duhem's works now belong in all respectable bibliographies on hysteresis, e.g. Visintin [48] and Mayergoz [41].

[13] To Manville [32], p. 197 Gibbs and Helmholtz are not dissociable in Duhem's vision.

thermodynamic potentials, that of normal variables of state, and elements of his general approach to the stability of equilibrium. Duhem saw in Helmholtz not only a forerunner of what he calls "énergétique", but also the prototype of what a good and efficient *mathematical physicist* should be. This is practically manifested in Duhem's reception of competing theories of electromagnetism in which he preferred Helmholtz's ideas, although quite forgotten today (but not in Duhem's time), to the somewhat "amateurish" presentation of Maxwell. This was not so much a matter of contents than one of presentation and interpretation. Duhem's and Helmholtz's views on the need for an abstract and logically ordered theory coincided more or less. Is it not Helmholtz[14] who, writing the foreword to Hertz's *Principles of Mechanics* [27], confesses that "*I remain attached* to *this latter mode of presentation* ("very general representation of facts and their laws by the system of differential equations of physics"), *and I place more confidence in it than in the other* ("mechanical explanations and models à la Maxwell-Thomson").

In [14, 15] Duhem sees Helmholtz's works as the ultimate step in "*the developments which abstract theory has undergone from Scholasticism to Galileo and Descartes; from Huygens, Leibniz and Newton to d'Alembert, Euler, Laplace, and Lagrange; from Sadi Carnot and Clausius to Gibbs and Helmholtz*" (Duhem [15], p. 305). We surmise that this list would have been fully endorsed by C. A. Truesdell with the obvious addition of the marvelous Cauchy. A close examination of this list reveals that all participants concurred, although at different historical times, in creating what may be actually referred to as the *nonlinear theory of fields* in the sense of Truesdell and Noll [47] or *continuum physics* in the sense of Eringen [21]. In this, while adding his personal touch to many arduous points and innovating on specific problems, Duhem was an efficient intermediary between the last great "all-round" scientists of the nineteenth century and the continuum theorists of the late twentieth century, especially in continuum thermomechanics. As a talented and stubborn propagandist in the defense of this cause he was more than influential on our practice and vision of Isaac Newton in his own time. In lower spheres, all our teaching and research practice in continuum thermomechanics is inevitably marked by Gibbs, Helmholtz and Duhem, through the concept of free energy and the constant use of the *Clausius-Duhem inequality* as a constraint imposed on the evolution of all types of phenomena, including severely nonlinear ones. Thermo-elasticity, the theory of defects, the rheology of solutions of macromolecules, and the progress of phase-transition fronts make constant use of thermodynamic restrictions and styles of reasoning that belong to this line of scientists. In a hundred years, a message—that one that Helmholtz expressed vividly in his foreword to Hertz's text—has gone through: we practice mathematical physics in a definite style which is both serene and efficient, in a language that has become accessible to all scientists trained throughout the World. But we would like to add a very personal remark. It is commonly agreed with Duhem that a true "scientific" work in a field of natural science makes use of a concise

[14] Cf. Hertz [27], quoted by Duhem [14, 15], p. 100.

language, with accompanying concentrated objectivity and convincing logic; it must suppress feelings, do not digress into metaphysics, and do not try to find meaning beyond the visible phenomena. But literacy and a good sense of humour are not forbidden, for both author and reader must still share together that exquisite feeling of the true existential pleasure found in the practice of science.

References

1. Boltzmann L (1893) Vorlesungen über Maxwells Theorie der Elektrizität und des Lichtes, 2 volumes. Leipzig
2. Born M (1921) Kritische Betrachtungen zur traditionellen Darstellung der Thermodynamik. Phys Zeit 22:218–224, 249–254, 282–286
3. Bridgman P (1943) The nature of thermodynamics. Harvard University Press, Cambridge
4. Brouzeng P (1987) Duhem: science et providence. Belin, Paris
5. Buchwald JZ (1985) From Maxwell to Microphysics (Aspects of Electromagnetic Theory in the Last Quarter of the Nineteenth Century). The University of Chicago Press, Chicago [see Chap. 21]
6. Caratheorory C (1909) Untersuchungen über die Grundlagen der Thermodynamik. Math Ann 67:355–386
7. Coleman BD (1973) The energy criterion for stability in continuum thermodynamics. Rend Semin Mat Fis Milano 43:85–99
8. Coleman BD, Noll W (1964) Thermodynamics of materials with memory. Arch Rat Mech Anal 17:1–46
9. Duhem P (1886) Le potentiel thermodynamique et ses applications à la mécanique physique et à la théorie des phénomènes électriques. A. Hermann, Paris 248 pp
10. Duhem P (1893) Commentaires aux principes de la thermodynamique, Seconde partie: Le principe de Sadi Carnot et de R. Clausius. Journ Math Pures et Appl 9:293–359
11. Duhem P (1902) Sur les conditions nécessaires pour la stabilité de l'équilibre d un système visqueux. C R Acad Sci Paris 135:1290–1294
12. Duhem P (1903) Considérations sur la stabilité, et plus particulièrement la stabilité des corps élastiques. In: Procès verbaux des séances de la Société des Sciences Physiques et Naturelles de Bordeaux, 25 juin 1903 [also in: Recherches sur l'élasticité, Third part: La stabilité des milieux élastiques, 218 pages. Gauthier-Villars, Paris, 1906]
13. Duhem P (1903) L'évolution de la mécanique, 348 pages. A. Joanin, Paris [Collection of seven papers published in 1903 in the « Revue Générale des sciences Pures et Appliquées »]
14. Duhem P (1906) Les théories électriques de J. Clerk Maxwell - Etude historique et critique, 228 pages. A. Hermann, Paris [This is a collection of papers published in 1900–1901 in the Annales de la Société Scientifique de Bruxelles]
15. Duhem P (1906) La théorie physique, son objet et sa structure, 450 pages. Chevalier et Rivière, Paris (This is a collection of papers published in 1904 and 1905 in the Revue de Philosophie). English translation: The Aim and Structure of Physical Theory, Princeton University Press, 1954; paperback edition 1991; precise references are to this edition; German translation: Autorisierte übersetzung: Ziel und Struktur der Physikalischen Theörien, mit einem Vorwort von Ernst Mach, 367 pages. Johan Ambrosius Barth, Leipzig (1908)]
16. Duhem P (1911) Traité d'énergétique ou de thermodynamique générale, Two volumes, 528 and 504 pages. Gauthier-Villars, Paris
17. Duhem P (1913) Notice sur les Titres et Travaux Scientifiques de Pierre Duhem, rédigée par lui-même lors de sa candidature à l'Académie des Sciences [Reprinted in: Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux, 7ème série, T. 1, pp 40–169 (1917)]

18. Ericksen JL (1966) Thermoelastic stability. In: Proceedings. 5th National Congress Applied Mechanics, pp 187–193
19. Ericksen JL (1977) Special Topics in Elastostatics. In: Advances in Applied Mechanics, Ed. C.-S. Yih, 17:189–244, Academic Press, New York
20. Ericksen JL (1991) Introduction to the thermodynamics of solids. Chapman and Hall, London
21. Eringen AC (Editor 1971–1976) Continuum Physics, Four volumes. Academic Press, New York
22. Eringen AC, Maugin GA (1990) Electrodynamics of Continua, Two volumes. Springer, New York [Soft-cover reprint, Springer, New York, 2012]
23. Glandsdorff P, Prigogine I (1971) Structure, stabilité et fluctuations. Masson, Paris
24. Helmholtz H von (1870) Über die Bewegungsgleichungen der Elektrizität für ruhende leitende Kôrper. (Crelle) J reine angewandte Math 72:57–129
25. Helmholtz H von (1882) Die thermodynamik chemischer Vorgänge. Sitz Deutsch Akad Wiss Berlin 1:22–39
26. Helmholtz H von (1897, posthumous) Vorlesungen über die Theorie des Lichtes. Unter Rucksicht auf. d. elastische und der elektromagnetische Ausschauung, J.A. Barth, Leipzig (Lectures)
27. Hertz H (1894) Die Prinzipien der Mechanik, in neuem Zuzammenhange dargestellt, Leipzig [English translation: The Principles of Mechanics, New York 1899; Dover reprint, New York, 1956]
28. Hertz H (1894–5) Untersuchungen über eine Ausbreitung der elektrischen Kraft. In: Vol. 2 of Gesammelthe Werke, Ed. P. Lenard, Leipzig
29. Hirosize T (1969) Origins of Lorentz Theory of Electrons and the Concept of the Electromagnetic Field. In: Historical Studies in the Physical Sciences, Ed. McCormmach, Vol 1, pp 151–209, Princeton University Press, New Jersey
30. Hunt BR (1991) The maxwellians. Cornell University Press, Ithaca
31. Jaki SL (1984) Uneasy genius: the life and the work of Pierre Duhem. Martinus Nijjhoff, The Hague
32. Manville O (1927) La physique de Pierre Duhem. In: Pierre Duhem, Sa vie - Ses Oeuvres, Mém Soc Sci Phys Nat Bordeaux T.1, 7ème Série, pp 171–636
33. Manville O (1928) L'oeuvre scientifique de Pierre Duhem, Vol l, Bordeaux (see especially pp 71–76 for the scientific heritage of Helmholtz)
34. Massieu F (1869) Sur les fonctions caractéristiques des divers fluides. C R Acad Sci Paris 69(858–864):1057–1061
35. Maugin GA (1988) Continuum mechanics of electromagnetic solids. North-Holland, Amsterdam
36. Maugin GA (1992) Thermomechanics of plasticity and fracture. Cambridge University Press, U.K.
37. Maugin GA (1999) Thermodynamics of nonlinear irreversible behaviors. World Scientific, Singapore
38. Maugin GA, Muschik W (1994) Thermodynamics with internal variables. Part 1: General concepts, Part II. Applications. J Non-Equilibr Thermodynam 19: 217–249, 250–289
39. Maugin GA, Trimarco C (1994) Material Forces and Irreversible Progress of Thermoelastic Phase-transition Fronts. Euromech Colloquium on Phase Transitions in Solids in the Honour of J.L. Ericksen, Udine, May 1994 [Cf. Meccanica (Italy) 30: 605–619, 1995]
40. Maxwell JC (1873) A Treatise on Electricity and Magnetism. Clarendon Press, Oxford (Dover reprint, two volumes, New York, 1954)
41. Mayergoz I (1991) Mathematical models of hysteresis. Springer, New York
42. Miller DG (1970) Entry: "Pierre-Maurice-Marie Duhem". In: Dictionary of Scientific Biographies. Ed. C.C. Gillispie, Vol 3, pp 225–233, Charles Scribner's Sons, New York
43. O'Rahilly A (1931) Electromagnetics. London, (reprinted as Electromagnetic Theory, Two volumes, Dover, New York, 1965)

44. Poincaré H (1890) Electricité et optique, Two volumes. Paris [Lecture notes of the Physics Course at the Sorbonne in 1888, the year electromagnetic waves were experimentally discovered and their speed of propagation shown to be equal to that of light by H. Hertz]
45. Roy L (1923) L'électrodynamique des milieux isotropes en repos d'après Helmholtz et Duhem. Paris
46. Truesdell CA (1986) Rational thermodynamics (revised and enlarged edition). Springer, New York (original edition, McGraw Hill), New York, p 1969
47. Truesdell CA, Noll W (1965) The Nonlinear Field Theories of Mechanics. In: Handbuch der Physik, Bd.III/3, Ed. S. Flügge, Springer, Berlin
48. Visintin A (1991) Bibliography of hysteresis. Preprint, University of Trento, New Jersey

Chapter 8
About the Cosserats' Book of 1909

Abstract The Cosserat brothers published in 1909 an original book where they favour a variational formulation of continuum mechanics together with an invariance which they call "Euclidean invariance" of the Lagrangian-Hamiltonian action. This strategy places on an equal footing translations and possible rotational degrees of freedom, yielding in a natural way what is now commonly called the theory of Cosserat or oriented—or polar—continua with asymmetric stresses and the new notion of couple stresses. Here their landmark work is replaced in its epoch making context underlining the influences they benefited from and the influence they have exerted on their direct contemporaries and much later on (in the second-half of the twentieth century). The sociological scientific environment of the early twentieth century and the typical publication strategy of the time are outlined, explaining thus the Cosserats' own strategy. The further reception of their work from 1909 to the Second World War and the revival of interest in it in the nineteen-fifties are examined critically. Finally, the formalization of their work in a new landscape of continuum thermo-mechanics created essentially by Truesdell is evoked together with other influences and further developments.

8.1 Preliminaries

Year 2009 witnessed the commemoration of the centennial of the publication of their (now) celebrated book on the "Theory of deformable bodies" [13]. Celebrations took place in Paris in the form of a EUROMECH Colloquium with many participants from Germany and Russia although attendants came from the world over (Proceedings edited by Maugin and Metrikine [55]), as also at the National "Ponts et Chaussées" school as one of the authors, François Cosserat (1852–1914), was an alumnus from that school. A fac-simile edition of the original book was published on that occasion (2010) with interesting historical comments by M. Brocato and K. Chatzis. This opus by the Cosserat brothers was their longest common contribution to the science of mechanics. The way this was published as

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also the general approach of the brothers concerning this field and their own professional activities require some comments as it appears that neither François nor his young brother Eugène (1866–1931) were professional mathematicians in the field of mechanics. But they were enlightened amateurs with all technical abilities and background of true professionals. Both became members of the Paris Academy of Sciences (François in 1896, and Eugène in 1919). François was even elected President of the French Society of Mathematics (*Société Mathématique de France*) in 1913 one year before his death. Still the way they published is somewhat unusual and also concentrated in time in the period 1896–1914 with the death of François.

François Cosserat was educated at *the Ecole Polytechnique* in Paris with a further specialization in civil engineering at the *Ecole Nationale des Ponts et Chaussées*. This curriculum in the best mathematical and mechanical tradition was typical of many great French "engineers-scientists" of the nineteenth century (among them, Cauchy, Navier, Lamé, Duhamel, Coriolis, Clapeyron, Poncelet, Liouville, Arago and Barré de Saint-Venant). He had a professional career in the fast growing development of railways with the Nord and then the East companies of Railways in France. Eugène, his younger brother by 14 years, was educated in mathematics at the *Ecole Normale Supérieure* in Paris and became a professional (mathematical) astronomer with a career spent almost entirely in Toulouse in the south–west of France. As such he had to teach courses in analysis, astronomy and celestial mechanics.

From 1896 till the death of François in 1914, the Cosserats published together no less than 21 works in the field of theoretical mechanics. Out of these, 14 were short notes—of three or four pages—to the Paris Academy of Sciences. Apart from their long original memoir of 1896 [12] published in Toulouse in a true serial journal,[1] their other publications in the field are scattered in odd places, often as supplements or comments to books by more acknowledged institutional authors: one is a note in the lecture notes of Gabriel Koenigs (published in 1897)—cf. the review by Lovett [43] and citation below—, one is a note of 37 pages in Vol. 1 of Chwolson's Treatise of Physics in its French translation [10, pp. 236–273], another one is a note of seventy two pages in Appell's Treatise of Rational Mechanics [3, Vol. III, pp. 557–629], still another one is an adaptation in French of an article by Aurel Voss (1845–1931) in German on the principles of rational mechanics in the *Encyclopédie des sciences mathématiques pures et appliquées* (the original is the *Encyklopaedie der mathematischen Wissenschaften*), Vol. IV, pp. 1–187—published in 1915 [14] after Francois's death, and finally their now most celebrated opus is a supplement to Chwolson's Treatise of Physics [11, Vol. II, pp. 953–1173], also published with a new pagination (vi+226 pages) as a separate book by Hermann Editeurs in Paris. As noticed by its American reviewer [77], it is

[1] This original paper considers finite strains following G. Green, Kirchhoff and Boussinesq, and already uses the notion of mobile frame. It is a much cited paper by Appell [3] and Truesdell and Toupin [74].

not clear why the Cosserats included their memoir in the translation of Chwolson's treatise, a treatise that lacks any general theoretical treatment of mechanics while the Cosserats' memoir deals with the foundations of analytical mechanics. They may have used this just as a good opportunity. Accordingly, the book version is preferably considered without the rest of Chwolson's nonetheless highly valuable treatise. NASA had an English translation of it made in 1968 as a result of a revival of interest in generalized continuum mechanics in the 1960s.

> The review of Lovett [43] is particularly enlightening concerning the note added by the Cosserats to the Koenigs' lecture notes of 1897. Citing Lovett: *The introduction of this note is peculiarly fortunate for it is high time that kinematics should comprehend the study of deformation and of deformable spaces. The authors have included in their extract certain generalities on curvilinear coordinates, the deformation of a continuous medium in general, infinitely small deformation, use of the mobile trieder* [sic], *and the case where the non-deformed medium is referred to any curvilinear coordinates.*

It seems that the two brothers, together with the husband, E-V. Davaux, of François' daughter, also an alumnus of *Ecole Polytechnique* with a specialization in naval engineering, were very active in translations from the Russian, German and English (including a translation of J. W. Gibbs's "Elementary principles in statistical mechanics" published only in 1932). For Chwolson's treatise, translation may have been done from the Russian and/or the already existing German translation. Note that it was usual in the nineteenth century and the early twentieth century to include comments and possible personal additions to a translation from an original book. The best example of this usage is provided by Barré de Saint-Venant's [5] French translation of A. Clebsch's *Theorie der Elastizität fester Körper* in such a way that the bulk of the book tripled in translation, resulting in a book that was more his than Clebsch's. But in the case of Chwolson's treatise, the Cosserats' supplements do not shed any light on Chwolson's original contents of Vols. 1 [10] and 2 [11]; they seem out of place, as rightly noted by Wilson [76, 77] who nonetheless emphasized their intrinsic importance.

As to the many Notes to the *Comptes Rendus* of the Paris Academy of Sciences, it was at the time a traditional way to announce a result in brief form so as to provide a priority mark. Cauchy is well known for the flood of such notes that he sent to the Academy. This was also the case of Henri Poincaré and Pierre Duhem among others. Of course, this cannot replace a lengthy well argumented paper with full derivations as many such notes are extremely cryptographic and thus hard to grasp due to their imposed brevity. With all these caveats we can now turn to the real object of this contribution, the "book of 1909".[2]

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[2] Orest Danilovich Chwolson (1852–1924)—also written Khvol'son—was a Professor of physics in St Petersburg. He is the author of a five-volume treatise on physics that was translated into German and French in the early twentieth century. The world renowned theoretical physicist Lev D. Landau had a strongly positive appraisal of this treatise.

8.2 The Main Contents of the Cosserats' Book

According to a recent investigation on *Google Scholar* the Cosserats book is cited about 1,500 times. It is a required citation in the introduction of papers dealing with modern oriented or polar continua. But we can safely assume that very few citers have ever seen the book and, of course, even less have read it, reducing the number of the happy few to small integers. There are good reasons for this. First the language, French, would be a common obstacle. But most of the difficulty comes from the state of mind of the authors and their notation since neither tensor nor direct intrinsic notations are used by these authors. We estimate that the book would be reduced to about 80 pages had a direct notation been used in the modern way. But intrinsic vector notation, not to speak of tensor notation, hardly existed at the time.[3] Also, the bias of the authors to work successively for bodies of one, two and three dimensions, if it may have helped the contemporary readers to grasp the basic ideas of their approach, considerably lengthens the progression. The advantage of working at this rhythm is a possible direct comparison with works by famous engineers of the eighteenth and nineteenth centuries who dealt with elastic rods and surfaces. Indeed, in a very professional manner akin to that entertained by previous authors such as Lagrange or Barré de Saint-Venant, the brothers are very generous in accurately citing previous contributors. As proved in many footnotes, the most cited such authors from a relatively old past are Navier, Poisson, Fresnel, Lamé, Helmholtz, Carnot, F. Reeds and Barré de Saint-Venant. Cauchy, although the universally acknowledged founder of general continuum mechanics, is seldom cited perhaps because he does not use variational principles and therefore is more in the Newtonian tradition of the postulate of balance laws. Gabrio Piola (1794–1850) would have been welcomed in the roster of citations because he uses Lagrangian variational principles and is an aficionado of changes of reference configuration (cf. the Piola transformation). But the Cosserats, like most of the French authors of the period, seem to have ignored him.

The most cited contemporary authors certainly are W. Thomson (alias Lord Kelvin) and P. G. Tait (cf. their "Treatise on Natural Philosophy", [71]), Pierre Duhem (1861–1916; cf. his course on hydrodynamics, elasticity and acoustics, [18]), H. Poincaré (1854–1912), Paul Appell (1855–1930), J. Bertrand (1822–1900), G. Darboux (1842–1917), and sometimes W. Voigt (1850–1919). It is less than anecdotic to note that Darboux, Appell and Koenigs were the three members composing the Jury of the doctoral thesis of Eugène. The deepest influences perceived through the unfolding of the book seem to be those of Lagrange and Hamilton for the variational formulation and the notion of action, Green [30] for the notion of potential energy of deformation, and Darboux [16, 17]

[3] Gibbs' [28] book was the first of its type giving an articulated introduction to vector analysis. This may however be a wrong attribution since the book in fact is E. B. Wilson's redaction with an enriched rendering of Gibbs' lectures in vector analysis at Yale; Wilson was only 22 years old when the book was published (see pp. 228–229 in Crowe [15]).

for the theory of surfaces, curvilinear coordinates and the mobile triad. The employed notion of groups, a *première* in continuum mechanics, is not connected with any obvious citation, although we surmise that the views of S. Lie and H. Poincaré may have been influential concerning this very point. Furthermore, the Cosserats are aware of general discussions on the nature and interpretation of the principles of mechanics (works on this subject by Hertz, Poincaré, Mach and Duhem in the period 1890–1909) as shown in many of their footnotes.

In our opinion the best analysis of the book remains the original review written by Wilson[4] [77] from M.I.T, a luminous text that we shall often paraphrase. Wilson had the right state of mind to capture the essential arguments of the Cosserats. First he considers the book as a contribution to the *analytical* mechanics of continua, and this is spot on. In effect, the very object of the book is the deduction of what we now call "field equations" of continua of one, two or three dimensions, from a Lagrangian-Hamiltonian principle of the general form

$$\delta \int_T \int_V W dV dt = 0, \tag{8.1}$$

where T is a time interval, V is a bounded volume element (a filament, a surface or a volume) in the considered physical space, and W is a known function of well-chosen arguments. In standard variational mechanics W is made explicit in terms of an identified kinetic energy and a potential energy so that W is the Lagrangian "volume" (i.e. lineal, surface or true 3D volume) density where the notion of mass (here density) is a basic one. The Cosserats wanted to remain in a sufficiently general framework that may possibly include various types of dynamics (even the special relativistic one with an appropriate definition of the mass).

The importance of the notion of *action* present in (8.1) was emphasized by William R. Hamilton (1805–1865) and Hermann von Helmholtz (1821–1894). But essential to the Cosserats' presentation is their initial remark that the *action* (energy multiplied by time) as introduced by P. L. Moreau de Maupertuis (1698–1759) is invariant under the group of Euclidean displacements. This requirement systematically applied to (8.1) provides the notion of *Euclidean action* in the Cosserats' formalism. From this should be deduced the basic local balance laws of linear momentum, angular momentum and energy, corresponding to the seven parameters (spatial translation and rotation, time translation) of the Euclidean group in E^3 (completed in a ten-parameter group if we include the definition of the centre of mass). What the Cosserats do is to implement this approach in a well tempered manner with the successive examination of one-dimensional bodies (straight line or curved filament), two-dimensional bodies (deformable surfaces such as plates and thin shells (not their vocabulary)) and three-dimensional bodies, with the possible extension to true dynamics (i.e., accounting for inertial effects). Whether this is a good pedagogical way is a

[4] For this see, e.g., Kelvin, reported in Thomson and Tait, Second edition (1879).

disputed matter in modern continuum mechanics where the equations governing slender bodies are rather deduced from the three-dimensional ones by means of some asymptotic procedure associated with the relative smallness of some dimensions.

As remarked by Wilson [77, p. 242], the Cosserats' book may have proposed "the most general and unifying theory of mechanics" so far (as on 1909). Probably under the influence of Darboux, the Cosserats considered that the "fundamental geometric element in their system is not the point, but the point carrying a system of rectangular axes, that is, the tri-rectangular triedral angle". This is obvious in the case of "an elastic filament that differs from a geometric curve in the way in which a continuous series of rectangular triedral angles differs from the locus of the vertices of the angles". In this case the function W should be "a function of the coordinates of the vertex but also a function of the nine direction cosines of the edges of the angle, and of the first derivatives of these coordinates and direction cosines with respect to time" (in the dynamical case) or the arc length in the case of the elastic filament. All these are Wilson's words.

This function W that is invariant under transformations that belong to the Euclidean group, is said to be an *Euclidean action* density and multiplied by the increment of time dt is the Euclidean action in the time interval dt. For the filament the reasoning can replace dt by an element ds_0 of arc. In this vision the case of straight rods and curves is approached by considering a mobile triad of vectors of which one element is tangent to the line or curve. In the case of two-dimensional bodies the mobile triad has one vector in the plane tangent to the mean surface of the object. In the three-dimensional case the triad has no preferred direction to start with except by convention in a reference configuration. Then the passing from the *motion* of an elastic medium of dimension k to the *equilibrium* of an elastic medium of dimension $k+1$ is, or should be, "well known to all student of mechanics" [77, footnote in p. 243]: "It is this analogy which enables the authors (the Cosserats) to give a uniform treatment to dynamic and static problems of different nature". That is quite remarkable and seldom considered by most of us as Wilson is rather optimistic concerning this point.

We shall not dwell in detail with the Cosserats' treatment which is somewhat repetitive and not very attractive in modern terms.[5] What is also absolutely important is that this enforcement of the Euclidean group structure leads the Cosserats to consider on an equal footing invariance under spatial translations *and* spatial rotations. That is how they are led to considering *nonsymmetric stress tensors* and the presence of body couples and of a new internal force called *coupled stress tensor* in modern jargon. If some notions may have been readily interpreted for the one- and two-dimensional cases in terms of what was known in the strength of structural elements in the nineteenth century, the three-dimensional case comes up as a new notion, although it is remarked that Kelvin and Voigt may

[5] For this unpleasant aspect to modern eyes, see, for instance, the fantastic and frightening aspect of the individual-component equations in pp. 157–172 of the book of 1909.

have hinted at the presence of body couples. For instance, the formidable equilibrium equations printed in page 137 of the Cosserats' book in the 3D case are now written with an inherent economy of symbols as the equations of equilibrium for stresses $\underline{\sigma}$ and couple stresses $\underline{\mu}$ in the form

$$\nabla \cdot \underline{\sigma} + \mathbf{F} = \mathbf{0}, \tag{8.2}$$

and

$$\nabla \cdot \underline{\mu} + \underline{\sigma}_A + \mathbf{C} = \mathbf{0}, \tag{8.3}$$

where $\underline{\sigma}$ is the nonsymmetric stress tensor, $\underline{\sigma}_A$ is its antisymmetric (or skew) part, $\underline{\mu}$ is the third-order couple stress tensor, and $\mathbf{F}$ and $\mathbf{C}$ are volume densities of externally applied force and couple (the latter in tensor skew symmetric form), respectively. The Cosserats' note [13, p. 137] that Eq. (8.3) with $\underline{\mu} = \mathbf{0}$ was evoked by W. Voigt in a work of 1887 that dealt with the elasticity of crystals involving polarized molecules [75]. This may have prompted Ericksen [22] to envisage a modelling of anisotropic fluids and liquid crystals by means of a field of so-called "director" (one unit vector attached to each material point), clearly a special case of Cosserat continuum. The most obvious case of Eq. (8.3) with $\underline{\mu} = \mathbf{0}$ is that obtained in anisotropic electromagnetic continua as amply documented in our book [48]. The Cosserats are aware of Lord Kelvin's former attempts[6] and the contemporary one of Larmor [39] to build a model of elasticity able to transmit transverse (light) waves but they do not seem to know the work of MacCullagh [45] on the same matter. Passing to the dynamic version of Eqs. (8.2) and (8.3), i.e.,

$$\nabla \cdot \underline{\sigma} + \mathbf{F} = \rho \dot{\mathbf{v}} \tag{8.4}$$

and

$$\nabla \cdot \underline{\mu} + \underline{\sigma}_A + \mathbf{C} = \rho \dot{\mathbf{S}}, \tag{8.5}$$

where $\mathbf{v}$ is the matter velocity and $\mathbf{S}$ is an internal spin (angular momentum), and a superimposed dot denotes the time derivative, is not as trivial a matter as thought by Wilson.[7] As a matter of fact, one had to await a work by Eringen [25], to understand that in parallel with the conservation of mass density, the good construction of (8.5) requires the consideration of a law of conservation of rotational inertia (per unit mass).

From the above-given short analysis we can encapsulate the Cosserats' main contribution in their lengthy memoir of 1909 in two main ingredients. One of these is the deduction of field equations such as (8.2) and (8.3)—which were to yield the fruitful notion of *Cosserat continuum* in the 1950s–1970s. The second ingredient,

[6] The reader may consult Maugin [52] for a historical perspective.

[7] To apply his argument we would have needed to know a four-dimensional static case, whatever that may be.

perhaps more important from the general viewpoint of mathematical physics, is the exemplary use of variational principles and a simultaneous application of a *group theoretical argument*, and this before the proof of her famous theorem by Noether [59]. As emphasized by Wilson in his deeply thought review [77, p. 246], an advantage of the Cosserats' approach is the association it provides "with the transfer of any deductive-intuitional physical science to the corresponding formal-deductive mathematical discipline". This is all the best for mathematically inclined mechanicians of the continuum. Correlatively, it yields a loss in the physical intuition while the latter is also a creative asset: mathematical rigorous form and physical innovation may be antagonistic. From the point of view of the kinematics and deformation theory of the continuum, the Cosserats have learnt their lesson in the finite-strain theory from Green, Kirchhoff, Boussinesq and Duhem. Unfortunately, apart from very general formulas for the W function, they have not provided any more information on possible constitutive equations. Apparently they were not so much interested in problem solutions although their original memoir of 1896 and some of their Notes to the *Comptes Rendus* hinted at progress in the solution of elasticity problems, in two-dimensions in particular. In the memoir of 1909 only the one and two-dimensional models are close to engineering concepts as they allow for a representation of the twisting of rods and shells in addition to their bending as noted by Ericksen and Truesdell [24, p. 297]. Bearing in mind these different characteristic properties, it is salient to examine the contemporary reception of their work and what was more useful in it for further developments, much later in the 1950s–1970s.

8.3 Reception and Influence of the Cosserats' Book

Parodying the title of a famous work concerning Leonardo da Vinci by Duhem [20], we could ask "who did the Cosserats read and who read them?" From above made remarks we can safely state that Maupertuis, Lagrange, Hamilton and Kirchhoff must have been primary sources for the bases of the Cosserats' thesis. Much closer to them their contemporaries such as L. Kronecker, G. Koenigs. P. Duhem, H. Poincaré, L. Lecornu (Professor of Mechanics at *Polytechnique*), and G. Darboux have played an essential role in the formation of the authors' background. The same can be said concerning the teachers whom both brothers had in analysis and geometry either at *Polytechnique* or at the *Sorbonne*. Foremost among them is the influence of Darboux with the idea of the mobile triad of vectors. In the case of Duhem, Truesdell had repeatedly pointed out that the idea to attach a triad of rigid vectors (so-called "directors") at each material point in order to describe the orientational changes in some kind of internal rotation goes back to Duhem [19]. But no trace in the Cosserats' opus seems to directly indicate such a borrowing.

The immediate (say in the pre-WWI and early post-WWI period) reception of the Cosserats'works is obvious among mathematically oriented scientists. Of course, Appell who welcomed an addition by the Cosserats in his own treatise of

1909 on rational mechanics easily sided with the Cosserats. Wilson,[8] as a student of Gibbs and a true mathematician, manifests a true enthusiasm for their work as proved by his most favourable review. Cartan [8], the French geometer of Lie-group fame and author of creative developments in modern differential geometry, immediately appreciated the consideration of group arguments in the Cosserats' vision while noting the rich possibility to include the action of distributed couples along with more classical contact forces. This was also true of Ernest Vessiot, another specialist of group theory, who succeeded François Cosserat as president of the French Society of mathematics. On a less "provincial", albeit Parisian level, Heun [34] in his article in the German Encyclopaedia of Mathematics presented a kind of compaction of the Cosserats' arguments for the mechanics of rods. As to Hellinger [33], in a remarkably concise but well informed article to the same encyclopaedia, he correctly captured the new trends in continuum mechanics by accurately citing the most recent works by Boltzmann, Duhem and the Cosserats. But this was published in a tragic period not so favourable to scientific communication. The corresponding volume of this encyclopaedia was never translated into French while all other preceding volumes had been.

One must await a work [68] by Joachim Sudria (1875–1950) to witness an approach truly in the Cosserats' tradition with an unambiguous reference to the notion of Euclidean action. This work was published in Toulouse in a journal in which the Cosserats had published in 1896 and of which Eugène Cosserat was the long time editor (in fact "Secretary") until 1930. It is in this journal that Buhl [7] published an eulogy of Eugène pointing out his role and the influence of Eugene's initial works in geometry in the writing of the papers in common with his older brother François.

Sudria [69] published an up dated version of his memoir as a short monograph. Truesdell told (cf. [4] that it is while perusing works of the 1930s in continuum mechanics that he unburied Sudria's memoir of 1935. Then,—following the (probably unknown to him) advice of Rabbi Rashi of Troyes in Burgundy: "Ask your master his sources" (my citation, GAM)—Truesdell went back in time to uncover the Cosserats' book of 1909. In his usual somewhat grandiloquent style, Truesdell [73] states that "the Cosserats' masterpiece stands as a tower in the field". But he also mentions that "it attracted little attention in its own day and was soon forgotten". This remark may be due to Truesdell's ignorance of citations by French physicists, mathematicians and engineers in the 1920s–1940s [e.g., L.-M.

[8] Edwin Bidwell Wilson (1879–1964) was an American mathematician-physicist who had been a PhD student of J. W. Gibbs at Yale, and became Professor of mathematics first at M.I.T (when he wrote the review of the Cosserats) and then at Harvard. He co-authored a book on vector analysis with Gibbs (first edition, 1901, then several further editions). He was interested in the general principles of physics and mechanics (e.g., relativity), in advanced calculus, and in the differential geometry of surfaces in hyperspaces. Later in his life he contributed much to the developing studies in mathematical economy mentoring Paul A. Samuelson in Keynesian macroeconomy. He was well equipped, both intellectually and technically, to apprehend the quintessence of the Cosserats' works.

Roy, J. Delsarte, J. Pacotte, G. Matisse, P. Sergescu, E. Jouguet, and above all R. L'Hermite who places the Cosserats in the top group together with Lamé, Clebsch, Saint-Venant and Duhem while noting the complexity of the Cosserats' development and the lack of possible direct applications save in the one-dimensional case; cf. [6] (Reprint of the Cosserats' book), p. xxxix]. This takes us directly to the second half of the twentieth century with a frantic rebirth of studies on generalized continuum mechanics.[9]

Following the early considerations by Voigt, French crystallographers showed some interest for the case of nonsymmetric stress tensors in the mid 1950s (cf. [40, 41]). But the first manifestations of the use of "directors", the set of unit vectors attached to each material point in the line of Duhem and the Cosserats, are in works by Ericksen and co-workers [23, 24] dealing with structures of one or two spatial dimensions[10] with explicit reference to the Cosserats' book. This would later on be taken over in works by Green and Naghdi [29]. Then a busy period developed in the 1960s–1970s with the introduction of various models of generalized continua, all more or less first basing on a kind of microscopic description. Among these models, some were identified with the so-called Cosserat continua, as essentially governed by Eqs. (8.4) and (8.5), but also christened with other names such as "oriented media" or "micropolar continua" [50]. It has become traditional to refer to Aero and Kuvshinskii [1], Palmov [62] and German authors such as Günther [32], Neuber [58], and Schaefer [65] as pioneers in the field. It became a moral, more than technical, obligation to refer to the Cosserats' book as demonstrated in practically all contributions to the proceedings [38] of a landmark international symposium held in 1967 in Freudenstadt (Black Forest, Germany). These proceedings were rightly dedicated to the Cosserats and Elie Cartan. Of course this feverish citation business was more paying lip service than anything else since most authors had never read—nor even seen—the Cosserats' book. Grioli [31] appears as an exception in not referring to any Cosserats' work. But we do not know if this was by pure honesty or mere ignorance that this author acted.

Explicit reference to—and exploitation of—Euclidean action is much more rare in continuum mechanics. Here we underline the work of Toupin [72] on oriented (Cosserat) continua and our own work [46] on the more general case of so-called *micromorphic* elastic bodies. This variety of continua was introduced in a landmark paper by Eringen and Suhubi [26]. It is equivalent to a Duhem kind of kinematic description with three deformable "directors" and relative-angle changes between these directors in the course of deformation: the microstructure itself is deformable and is in fact subjected to a homogeneous micro-deformation (represented by six additional internal degrees of freedom). The case of rigid micro-rotation and no micro-deformation then corresponds to the Cosserat continuum. A modelling somewhat equivalent to the Eringen-Suhubi one was proposed by Mindlin [56]. In the case of Cosserat continua there appears the problem of the most convenient

[9] See Maugin [52] for a historical perspective.

[10] Cf. Ericksen and Rivlin [23], Ericksen and Truesdell [24].

mathematical representation of the micro-rotation. This is best solved by considering orthogonal transformations and their own representation by an angle and the unit direction of an axis of rotation in the manner of Gibbs [28]—and "our" Wilson—as shown by Kafadar and Eringen [37]. This was duly exploited by Kafadar [36] in an original approach to the classical problem of the "elastica"—and thus back to the spirit of Cosserats' treatment of one-dimensional elastic curves.

The writer was for the first time exposed to a research course involving "directors" in the lectures delivered by a Serbian scientist, Rastko Stojanovic, in Udine (Italy) in July 1970 (cf. Stojanovic lecture notes at the C.I.S.M. referred to as [66]. This was directed at the continuum representation of defective bodies. This gave him the idea to draw an analogy with deformable continua endowed with a continuous distribution of magnetic spins such as in the micromagnetic theory of ferromagnetism, a fashionable subject matter at the time. Then he applied the Euclidean action method of the Cosserats to deduce all relevant coupled field equations, including an equation formally identical to Eq. (8.5) but with all terms bearing a magnetic interpretation (cf. [47], Chap. III; also [53]). This was recently revisited in C.I.S.M. lecture notes [51]. Cherry on top of the cake, a four-dimensional relativistic theory of oriented media was constructed by complementing the triad of spatial "directors" in the Duhem-Cosserat style by the unit-normalized world velocity into a true four-tuple with a view to incorporate spin effects in relativistic continuum mechanics[11] with local Lorentz invariance replacing the Cosserats' invariance requirement [54].

8.4 Concluding Remarks

In recent times most of the Cosserats' work involving a nonsymmetric stress and couple stresses have been formalized in a modern context often under the title of *asymmetric elasticity* (cf. [60]) or *polar* or *microplar media* of the elastic type (cf. [27], Teodorescu [70]) or of the fluid type (cf. [44, 67]) with mathematical results of the same degree of refinement as those dealing with classical continua.[12] The theory of such polar elastic materials has been fully incorporated in the modern framework of configurational forces [49] with the help of Noether' theorem. The formulation of the deformation nonlinear theory of finite-strain Cosserat elasticity has been much clarified by Pietraszkiewicz and Eremeyev [63]—also Eremeyev and Pietraszkiewicz [21]—, and the most recent lecture notes highlight all fundamental geometrical properties and most interesting applications of Cosserat continua (cf [2]). A rapid search on *Google* provides instantaneously more than

[11] For this 4D generalization see Maugin [47, Chap. VI] and Maugin and Eringen [54].

[12] We use this opportunity to mention the seldom cited book of Jaunzemis [35] where elements of generalized continua are nicely introduced. Jaunzemis' career was interrupted by his untimely death at the age of 48 in 1973.

two hundred thousand entries about the Cosserats, although the most recent references concern a local politician from Amiens, the native city (in Picardie, North West of France) of the Cosserat brothers, thus undoubtedly a family connection. But the Amiens textile company specialized in the production of velvet, created in 1794, and of which the parents of the celebrated brothers were the owners, finally closed down in 2012. As to the heritage of the notion of "Euclidean action", it is more diffuse as the Cosserat notion appears somewhat obsolete in a period of full enforcement of Noether's invariance theorem. Concerning this point, we can cite Levy [42] as a general appraisal of this aspect of the Cosserats' works:

> *Cosserat's theoretical research, designed to include everything in theoretical physics that is directly subject to the laws of mechanics, was founded on the notion of Euclidean action [least action] combined with Lagrange's ideas on the principle of extremality and Lie's ideas on invariance in regard to displacement groups. The bearing of this original and coherent conception was diminished in importance because at the time it was proposed, fundamental ideas were already being called into question by both the theory of relativity and progress in physical theory.*

But, nowadays, "Euclidean action" experiences a flourishing vitality in its acception granted in theoretical physics such as in the functional integrals of quantum physics (cf. [57]).

This concludes the present investigation of the subject. But we note that Pommaret [64], a disciple of Vessiot in group theory, and in a sense a "grand-son" of François Cosserat—with whom he shares the same elite education—has expanded never tired efforts to publicize the works of the Cosserats and their relevance to modern group theory in mathematical physics.

PS. Additional biographic information on the Cosserats and their works can be found in Levy [42] , O'Connor and Robertson [61], and Brocato and Chatzkis ([6]; preceding the reprint of the Cosserats' book). Remarks on the Cosserats' work and Duhem's influence can also be found in Casey and Crochet [9].

Appendix A

Partial English Translation of E. and F. Cosserat, "Théorie des Corps Déformables", Hermann, Paris, 1909, by Gérard A. Maugin (Only the First Chapter on General considerations is translated; original footnotes are reported to the end and numbered consecutively. Translator's remarks are placed within square brackets in the main text. This is a verbatim translation without any ambition of literary prowess (Figs. 8.1, 8.2, 8.3).

THÉORIE

DES

CORPS DÉFORMABLES

par MM. E. et F. COSSERAT

I. — CONSIDÉRATIONS GÉNÉRALES

1. Développement de l'idée de milieu continu. — La notion de corps déformable a joué, au siècle dernier, un rôle important dans le développement de la Physique théorique, et Fresnel (1) doit être regardé, à l'égal de Navier, de Poisson et de Cauchy (2), comme l'un des précurseurs de la théorie actuelle de l'élasticité. Sous l'influence des idées newtoniennes, on ne considérait encore au temps de ces savants que des systèmes discrets de points. Avec les mémorables recherches de G. Green (3), ont apparu les systèmes ponctuels continus. On a essayé depuis d'élargir la conception de Green, qui est insuffisante pour donner à la doctrine des ondes lumineuses toute sa portée. Lord Kelvin (4), en particulier, s'est attaché à définir des milieux continus en chaque point desquels peut s'exercer un moment. La même tendance s'accuse chez Helmholtz (5), dont la controverse avec J. Bertrand (6), à l'égard de la théorie du magnétisme, est très caractéris-

(1) Fresnel. — *Œuvres complètes*, Paris, 1866 ; voir l'introduction de É. Verdet.

(2) Voir Isaac Todhunter et Karl Pearson. — *A History of the Theory of Elasticity and of the Strength of Materials, from* Galilei *to the present time*, Vol. I, Galilei to Saint-Venant, 1886 ; Vol. II, Part I et II, Saint-Venant to Lord Kelvin, 1893. Cet ouvrage remarquable contient une analyse très complète et très précise des travaux des fondateurs de la théorie de l'élasticité.

(3) G. Green. — *Math. Papers*, édited by N. M. Ferrers, fac-similé reprint, Paris, A. Hermann, 1903.

(4) Lord Kelvin. — *Math. and phys. Papers*, volume I, 1882 ; vol. II, 1884 ; vol. III, 1890 ; *Reprint of Papers on Electrostatics and Magnetism*, 2e éd. 1884 ; *Baltimore Lectures on Molecular Dynamics and the Wave Theory of Light*, 1904 ; W. Thomson et P. G. Tait, *Treatise on Natural Philosophy*, 1e éd. Oxford 1867 ; 2e éd. Cambridge 1879-1883.

(5) Helmholtz. — *Vorles. über die Dynamik diskreter Massenpunkte*, Berlin 1897 ; *Vorles. über die elektromagnetische Theorie des Lichtes*, Leipzig 1897 ; *Wiss. Abhandl.*

Fig. 8.1 First page of the Cosserats' book of 1909 (Hermann, Paris, 1909) (Note the reference to Green, Kelvin and the Treatise on Natural Philosophy)

Fig. 8.2 François Cosserat (1852–1914) in his uniform of the *Ecole Polytechnique* around 1871 (*Source* http://www-history.mcs.st-andrews.ac.uk/Cosserat-Francois.html)

Fig. 8.3 Sketchy portrait of Eugène Cosserat (1866–1931) in his fifties (*Source* http://www-history.mcs.st-andrews.ac.uk/PictDisplay.html)

Theory of Deformable Bodies

by MM. E. and F. Cosserat

Preface

This volume contains the development of a note on the Theory of the Euclidean action that Appell has thought appropriate to introduce in the second edition [1909] of his *Treatise on Rational Mechanics.* The reproduction of an appendix to the French edition of the *Treatise of Physics* of Chwolson, explains several peculiarities of the editing and the reference that we make to a previous work on the dynamics of the point and of a rigid body, which is here also combined with the work of the Russian scientist. We took advantage of this new print to correct several mistakes in our text.

Presently, we do not seek to deduce all the consequences of the general results that we will obtain; throughout, we make the effort only to rediscover and clarify the classical theories. In order for this kind of checking of the theory of the Euclidean action to appear more complete, in each part of our exposition we will have to establish the form that the equations of deformable bodies take when one is limited to the consideration of infinitely close states; however, this is a point that we have already addressed, with all necessary details, in our first memoir on the *Theory of elasticity* that we wrote in 1896 (*Annales de la Faculté des Sciences de Toulouse*, Vol. X). Moreover, we suppose that the magisterial lessons of G. Darboux on the *general theory of surfaces* are completely familiar to the reader.

Our researches will make sense only when we have shown how one may envision the theories of heat and electricity by following the already followed path. We devoted two notes to this subject in Volumes III and IV of Chwolson's treatise. The *subdivision*, to use a pragmatic language, appears to be a scientific necessity; nevertheless, one must not loose sight of the fact that it answers deep questions. We have tried to provide an idea of these difficulties in our note on the *Theory of slender bodies* published in 1908 in the *Comptes Rendus* of the *Académie des Sciences* and whose contents were also mentioned by Appell in his treatise.

E. & F. COSSERAT

I.- General considerations

1. **Development of the idea of a continuous medium –** The notion of deformable body has played an important role in the development of theoretical physics during the last century [i.e., 19th century], and Fresnel[1] must be considered as one of the precursors of the present theory of elasticity, on an equal stand with Navier, Poisson and Cauchy[2]. Under the influence of Newtonian ideas, only discrete systems of points were still considered at the time of these scientists. Continuous punctual systems appeared with the memorable researches of G. Green[3]. Since then, one has tried to enlarge the conception of Green, which is not sufficient to provide

its full power to the theory of luminous waves. Lord Kelvin[4], in particular, worked hard to define continuous media at each point of which a moment can be exerted. The same trend is emphasized with Helmholtz[5], of whom the controversy with J. Bertand[6] concerning the theory of magnetism is very characteristic. We can go back to the origin of this evolution, on the one hand, with conceptions introduced in the strength of materials by Bernoulli and Euler[7], and on the other hand, to the theory of "couples" due to Poinsot[8]. Thus we are naturally led to gather, under the same geometrical definition, various concepts of deformable bodies that we meet nowadays in natural philosophy [i.e., physics]. A deformable line is a continuous set equipped with one parameter of trihedrons, a deformable surface with a set of two parameters, and a deformable [3D] medium with three parameters ρ_i. In the presence of motion, one must add the time t to these three geometric parametersρ_i. The mathematical continuity that we assume in such a definition, leaves untouched at each point the trace of an invariable [i.e., rigid] solid; therefore, we can foresee that from a mechanical viewpoint moments will appear that are well known and are studied, since Euler and Bernoulli, along elastic lines and on surfaces, and that Lord Kelvin and Helmholtz have tried to embed in a three-dimensional space.

2. **Difficulties presented by the application of the inductive method in mechanics**.

The primary form of mechanics is inductive; this is what one clearly perceives in the theory of deformable bodies. This theory has first borrowed from the mechanics of invariable [rigid] bodies the propositions relative to the notion of static force, that were applied with the principle of solidification ["rigidification" due to Cauchy]; then the relation between the effort and the deformation was hypothetically first established (generalized Hooke's law), and then only one looked a posteriori under what conditions this was conserved (Green). Carnot[9] already mentioned, one century ago, the defect of this method, where it is constantly called for a priori notions, and where the followed path is not always safe. The static force in fact does not have the effect of a constructive definition, in our classical form of mechanics, and the influence of the reform that Reech[10] proposed regarding that matter in 1852 remained practically unknown until our present time. Perhaps that this is due to the long uncertainty in which elasticians remained concerning the rational foundation that can be attributed to Hooke's law. Analogous hesitations have indeed been manifested, almost in the same form, in other domains of physics[11].

In order to escape from these difficulties, Helmholtz tried to construct what is called an *energetics*, that relies on the principle of least action and on the very idea of energy, the force, whatever its nature, becoming then a secondary notion of deductive origin. But the principle of a minimum in natural phenomena[12] and the concept of energy[13] itself bring us to confront the defects of the inductive method. Why a minimum and what definition to be granted to energy to avoid having simply a physical theory, but a truly mechanical theory? Helmholtz does not seem to have left an answer to these questions. However, he contributed to establish more completely than done before the distinction between the two notions, energy and action, that apparently are identified in classical dynamics. We believe that one must start from the latter [the action] to make perfectly precise the views of Helmholtz and to give to mechanics, or more generally, theoretical physics, a perfectly deductive form.

3. **Theory of Euclidean action**

When we are concerned with the motion of a point, the essential element that enters the definition of action is the Euclidean distance between two infinitesimally close positions of the mobile point. We have shown previously[14] that one can deduce from this single notion all fundamental definitions of classical dynamics, those of quantity of motion [linear momentum], of force and of energy.

Here we propose to establish that we can follow an identical path in the study of the static deformation or the dynamics of discrete systems of points and continuous bodies, and that we arrive thus to the construction of a general theory of action in both extension and motion, that embraces all that, in theoretical physics, is directly governed by the laws of mechanics.

Here also, the action will be the integral of a function of two infinitesimally close elements in time and in the space of the considered medium. Introducing the condition of invariance under the group of Euclidean displacements and defining the medium, as indicated in the first paragraph above, *the density of action at a point will have the same remarkable form that the one already met in the dynamics of the point and of invariable bodies*. Let, with the notations of the *Leçons* of M. Darboux, (ξ_i, η_i, ξ_i), (p_i, q_i, r_i) be the geometric velocities of translation and rotation of the elementary trihedron, and (ξ, η, ξ), $(p, q, r)$the corresponding velocities relative to the motion of this trihedron; The action will be the integral

$$\int_{t_1}^{t_2} \int \ldots \int W(\rho_i, t; \xi_i, \eta_i, \xi_i, p_i, q_i, r_i; \xi, \eta, \xi, p, q, r) d\rho_1, \ldots, d\rho_i \ldots dt.$$

It will suffice to consider the variation of this action to be led to the definition of the quantity of motion, those of efforts and moment of deformation, of the external force and moment, and finally those of the energy of deformation and of motion, via the intermediary of the notion of work.

In this theory, statics will become entirely autonomous, in agreement with the views of Carnot and Reech; we will simply have to take for this purpose a density of action W independent of the velocities (ξ, η, ξ), (p, q, r), that is, to consider a body devoid of inertia, or else a body with inertia on the condition to regard deformation as a *reversible transformation* in the sense of M. Duhem. On the other hand, having recourse to the notion of *hidden arguments* [i.e., arguments that do not appear explicitly in W], we will recover all the concepts of mechanical origin that are employed in physics, for instance, those of flexible and inextensible lines [strings], flexible and inextensible surfaces, invariable [i.e., rigid] bodies, as also less particular definitions as proposed for a deformable line since D. Bernoulli and

Euler till Thomson and Tait, for the deformable surface since Sophie Germain and Lagrange till Lord Rayleigh, and for the deformable media since Navier and Green till Lord Kelvin and W. Voigt.

Envisaging both deformation and motion, we shall arrive in a purely deductive manner at the idea that is contained in the principle of d'Alembert, which relates only to the case where *an action of deformation separates fully from the kinetic action*. Finally, if we suppose that the deformable body is not submitted to any action from the external world and if we introduce, as a consequence, the fundamental notion of *isolated system*, of which M. Duhem[15], and the M. Le Roy[16] have shown the necessity for a rational construction of theoretical physics, we shall naturally be led to the idea of a minimum that Helmholtz had already considered as a starting point, while simultaneously there will appear the principle of conservation of energy, which is at the basis of our present scientific system.

4. **Critique of the principles of mechanics.-** As we just sketched it, the theory of Euclidean action brings a first contribution to the critique of the principles of mechanics.

Its generality allows one to foresee that there are singular phenomena, both in the action on the motion and in the deformation of extension, for example the aspect of solids in a plastic state or near fracture, and that of fluids submitted to large forces. In ordinary circumstances, this generality can be reduced by the consideration of a state that is infinitesimally close to the natural state; this is a point that we already mentioned in our preceding note.

But we can still assume that one or two dimensions of the deformable body become infinitesimally small and then envisage what is called a *slender body*[18]. This notion was developed in 1828 by Poisson, also a short time afterwards, by Cauchy; their aim, like that of all elasticians preoccupied later by this arduous question, was to build a passage between the distinct theories of bodies with one, two and three dimensions. We know that an important part of the works of Barré de Saint-Venant and Kirchhoff is related to a discussion of the researches by Poisson and Cauchy. However, these scientists, and then their followers, did not exhibit the true difficulty of the matter; this difficulty resides, *generally, in the fact that the zero value of the introduced parameter is not an ordinary point, as admitted by Poisson and Cauchy, not even a pole, but an essential singular point*. This important fact justifies the separate studies of lines, surfaces and [3D] media that the reader will find in the present work[19].

[Here the Cosserats touch upon a fundamental problem that concerns *singular perturbations*; this matter will be solved only in the 1960s-1980s for the limit reduction to slender bodies with the correct asymptotic methods (in particular "asymptotic integration" and the "zoom technique") developed by Gol'denveizer in Russia, Ambartsumian in Armenia, Berdichevsky in Moscow, and Ph. Ciarlet and Destuynder in France].

In concluding these preliminary observations, we shall remark that the theory of Euclidean action relies on the notion of *differential invariant* taken in its simplest form. If we enlarge this notion in such a way as to understand the idea of *differential parameter*, modern theoretical physics appears as an immediate extension, from the *Eulerian viewpoint*, of mechanics per se, and we are naturally led to the principles of the theory of heat and to the actual electric doctrines. This new field of research, in which here we start to enter by deducing from the consideration of deformable bodies the idea of radiation of energy, will be explored more completely in a further work [This ambitious programme was never really formulated; F. Cosserat died in 1914]. We shall thus introduce a new precision in the views of H. Lorentz[20] and H. Poincaré[21] in what is called the *principle of reaction* in mechanics.

(Original) Notes

1. Fresnel, *Oeuvres complètes*, Paris, 1866; see the introduction by E. Verdet.
2. See Isaak Todhunter and Karl Pearson, *A history of the theory of elasticity and the strength of materials, from Galilei to the present time*, Vol. I, Galilei to Saint-Venant, 1886; Vol. II, Part I and II, Saint-Venant to Lord Kelvin, 1893. This remarkable work contains a very complete and precise analysis of the works by the founders of the theory of elasticity.
3. G. Green, *Math. Papers*, edited by N.M. Ferrers, facsimile reprint, Paris, A. Hermann, 1903.
4. Lord Kelvin, *Math. and Phys. Papers*, Vol. I, 1882; Vol. II, 1884, Vol. III, 1890; *Reprint of papers in electrostatics and Magnetism*, 2nd edition; *Baltimore Lectures on Molecular Dynamics and the Wave Theory of Light*, 1904; W. Thomson and P.G. Tait, *Treatise on Natural Philosophy*, First edition, Oxford 1867; 2nd edition, Cambridge, 1879-1883.
5. Helmholtz, *Vorles. über die Dynamik diskreter Massenpunkte*, Berlin , 1897; *Vorles. über die elektromagnetische Theorie des Lichtes*, Leipzig, 1897; *Wiss. Abhandl.* Three volumes, Leipzig, 1892-1895.
6. J. Bertrand, *C.R.* 73, p.865; 75, p. 860; 77, p. 1049; See also H. Poincaré, *Electricité et optique, II, Les théories de Helmholtz et les expériences de Hertz*, 1891, p. 51 ; 2nd edition, 1901, p.275.
7. See Todhunter and Pearson – op. cit.
8. Auguste Comte, *Cours de philosophie positive*, 6th edition, Paris, 1907; Vol. I, p. 338: « Whatever, in truth, the fundamental qualities of the conception of M. Poinsot, with respect to statics, one must nonetheless recognize, it seems to me, that it is above all for perfecting dynamics that it is essentially destined; and I can assure you, considering this point, that this conception has not exerted its most important influence so far".
9. Carnot, in his essay of 1783 on "Machines in general" that became in 1793, *Les Principes fondamentaux de l'équilibre et du mouvement* , has searched to reduce mechanics to principles and precise definitions devoid of any metaphysical character and of any vague terms about which philosophers quarrel without reaching any understanding. This reaction led Carnot a little too far, since he went to the point of contesting the legitimacy of the

expression of force, for him an obscure notion, and to which he wanted to substitute exclusively the idea of motion. For the same reason, he could not admit as rigorous any of the known derivations of the rule of the parallelogram of forces, "the very existence of the word force, in its expression, rendering this derivation impossible, by the nature of things itself" (Ch. Combes, Ed. Phillips and Ed. Collignon, *Exposé de la situation de la mécanique appliquée*, Paris, 1867).

10. F. Reech, Cours de mécanique, d'après la nature généralement flexible et élastique des corps, Paris, 1852. This work was written by the illustrious Marine engineer in view of the reform of the teaching of mechanics at Ecole Polytechnique. Since then, his ideas were exposed by J. Andrade, *Leçons de mécanique physique*, Paris, 1898, and by Marbec, Chief engineer in the Navy, in his elementary teaching of mechanics at the school of "Maistrance" [forming non-commissioned officers as mechanical specialists in the French Navy; GAM] in Toulon (1906). See also J. Perrin, *Traité de Chimie physique, les principes*, Paris, 1903.
11. The remark by Lord Kelvin, in his Baltimore Lectures, p. 131, on the work of Blanchet, is particularly of interest in this regard; he mentions that Poisson, Coriolis and Sturm (*C.R.* **7**, p. 1143), as well as Cauchy, Liouville and Duhamel (1841) have accepted without objection the 36 coefficients that Blanchet had admitted in the generalized Hooke law. Lord Kelvin also has opposed from the same viewpoint the law of at-a-distance force of Weber, in the first edition of the *Natural Philosophy*. More recently, the application of the static adiabatic law to the study of waves of finite amplitude has been criticized for the same reason by Lord Rayleigh, and we know that Hugoniot has proposed a dynamic adiabatic law.
12. Maupertuis himself felt the danger of the principle that he introduced when he wrote in 1744: "We do not know enough what is the purpose of nature, and we can misinterpret the quantity of motion that we must regard as its expense, in the production of its effects"; Lagrange first has intended to make of the principle of least action the basis of his analytical mechanics, but later on he recognized the superiority of the method which consists in considering virtual works.
13. Hertz, *Die Prinzipien der Mechanik,* etc., 1894; See especially the introduction.
14. *Note sur la dynamique du point et du corps invariable*, Tome I, page 236.
15. P. Duhem, *Commentaire aux principes de la thermodynamique*, 1892; *La théorie physique, son objet et sa structure*, 1906.
16. E. Le Roy, *La science positive et les philosophies de la liberté*, Congrès int. de philosophie, T.I., 1900.
17. E. and F. Cosserat, *Sur la mécanique générale*, *C.R.*, 145, p. 1139, 1907.
18. E. and F. Cosserat, *Sur la théorie des corps minces*, *C.R.*, 146, p. 169, 1908.
19. It must be that the interest and the importance of the theories of lines and deformable surfaces are today so badly appreciated that the

Encyclopaedia of pure and applied mathematics [*Enz. math. Wiss.*], presently published in Germany, grants them no room. W. Thomson and Tait have avoided to omit them in their *Natural Philosophy*, and they presented them before the theory of three-dimensional elastic bodies. Similarly, P. Duhem, *Hydrodynamique, Elasticité, Acoustique*, Paris, 1891.

20. H. Lorentz – *Versuch einer Theorie der electrischen und optischen Erscheinungen in bewegten Körpern*, Leiden, 1895; reprinted in Leipzig in 1906, *Abhandl. über theoretische Physik*, 1907; *Encycl. der Math. Wissenschaften*, V2, Elektronen Theorie, 1903.
21. H. Poincaré, *Electricité et optique*, 2nd edition, 1901, p. 448.

References

1. Aero EL, Kuvshinskii EV (1961) Fundamental equations of the theory of elastic media with rotationally interacting particles. Soviet Phys Solid State 2:1272–1281 (English translation from the Russian edition, 1960)
2. Altenbach H, Eremeyev VA (eds) (2013) Generalized continua—from the theory to engineering applications (CISM Lecture notes, Udine, 2011). Springer, Wien
3. Appell P (1909) Traité de mécanique rationnelle, T.III, Gauthier-Villars, paris (Note sur la théorie de l'action euclidienne par E. and F. Cosserat, pp 557–629) (reprinted in fac-simile form by Editions Gabay, Paris, 1991)
4. Ball JM, James RD (2002) The scientific life and influence of Clifford Ambrose Truesdell III. Arch Rat Mech Anal 161:1–26
5. Barré de Saint-Venant AJC (1883) Théorie des corps élastiques, French translation from the German with many comments and additions of "Clebsch A (1862) Theorie der Elastizität fester Körper, Leipzig"
6. Brocato M, Chatzis K (2009) Historical essay: In: Reprint of the Cosserats' 1909 book, Hermann Archives, Paris
7. Buhl A (1931) «Eugène Cosserat». Ann Fac Sci Toulouse, 3ème Série, 23, pp v–viii
8. Cartan E (1922) Sur une généralisation de la notion de courbure de Riemann et les espaces à Torsion. C R Acad Sci Paris 174:593–595
9. Casey J, Crochet MJ (1995) Paul M Naghdi (1924–1994). In: Casey J, Crochet MJ (eds) Theoretical, experimental, and numerical contributions to the mechanics of fluids and solids: a collection of papers in honor of Paul M Naghdi. Birkhäuser, Boston, pp S1–S32
10. Chwolson OD (1906) Traité de physique (traduit du Russe), Paris, Hermann (vol I, Note sur la dynamique du point et du corps invariable by E. and F. Cosserat, pp 236–273)
11. Chwolson OD (1909) Traité de physique (traduit du Russe), Paris, Hermann (vol II, Note sur la théorie des corps déformables by E. and F. Cosserat, pp 953–1173)
12. Cosserat E, Cosserat F (1896) Sur la théorie de l'élasticité. Ann Fac Sci Toulouse, 1ère série 10(3–5): I1–I116
13. Cosserat E, Cosserat F (1909) Théorie des corps déformables, Hermann, Paris, 226 pages). Reprint by Editions Gabay, Paris, 2008; Reprint by Hermann Archives, Paris, 2009, with a Preface by G. Capriz and an historical essay by M. Brocato and K. Chatzis. [English Translation: N68-15456: Clearinghouse Federal Scientific and Technical Information, Springfield, Virginia NASA, TT F-11561 (February 1968); another translation by D. Delphenich, 2007]

14. Cosserat E, Cosserat F (1915) Principes de la mécanique rationnelle (after the original article in German by A. Voss). In: Encyclopédie des sciences mathématiques pures et appliquées, vol IV. Gautier-Villars, Paris, pp 1–187
15. Crowe MJ (1967) A history of vector analysis. University of Notre Dame Press (Reprint, Dover, New York, 1985)
16. Darboux G (1887–1896) Leçons sur la théorie générale des surfaces, in 4 volumes. Paris
17. Darboux G (1898) Leçons sur les systèmes orthogonaux et les coordonnées curvilignes, Vol I. Paris
18. Duhem P (1891) Hydrodynamique, élasticité et acoustique. A. Hermann Editeur, Paris
19. Duhem P (1896) Le potentiel thermodynamique et la pression hydrostatique. Ann Ecole Norm Sup 10:187–230
20. Duhem P (1906–1913) Etudes sur Léonard de Vinci—ceux qu'il a lus et ceux qui l'ont lu, 3 volumes. Hermann, Paris
21. Eremeyev AV, Pietraszkiewicz W (2012) Material symmetry group of the non-linear polar-elastic continuum. Int J Solids Struct 49:1993–2005
22. Ericksen JL (1960) Anisotropic fluids. Arch Rat Mech Anal 4:231–237
23. Ericksen JL, Rivlin RS (1954) Large deformations of homogeneous anisotropic materials. Arch Rat Mech Anal 3:281–301
24. Ericksen JL, Truesdell CA (1958) Exact theory of stress and strain in rods and shells. Arch Rat Mech Anal 1:295–323
25. Eringen AC (1966) Theory of micropolar fluids. J Math Mech 16(1):1–18
26. Eringen AC, Suhubi ES (1964) Nonlinear theory of simple microelastic solids I. Int J Eng Sci 2:189–203
27. Eringen AC (1999) Microcontinuum field theories, I—foundations and solids. Springer, New York
28. Gibbs JW (1901) Vector analysis. Yale University Press, New Haven (redacted by Wilson)
29. Green AE, Naghdi PM (1967) Micropolar and director theories of plates. Q J Mech Appl Math 20:183–199
30. Green G (1839) On the laws of reflection and refraction of light at the common surface of two non-crystallized media. Trans Cambridge Phil Soc 7:245–269
31. Grioli G (1960) Elasticità asimmetrica. Ann Mat Pura ed Applicata, Ser. IV, 50:389–417
32. Günther W (1958) Zur Statik und Kinematik des Cosseratschen Kontinuums. Abh Braunschweig Wiss Ges 10:195
33. Hellinger E (1914) Die allgemeinen Ansätze der Mechanik der Kontinua. In: Klein F, Wagner K (eds) Encyclopädia der mathematischen Wissenschaften. vol 4, Part 4. Springer, Berlin, pp 602–694
34. Heun K (1904) Ansätze und allgemeine Methoden der Systemmechanik. In: Klein F, Wagner K (eds) Encyclopädia der mathematischen Wissenschaften, vol 4/2, Article 11. Springer, Berlin, pp 359–504
35. Jaunzemis W (1967) Continuum mechanics. McMilan, New York
36. Kafadar CB (1972) On the nonlinear theory of rods. Int J Eng Sci 10(4):369–391
37. Kafadar CB, Eringen AC (1971) Micropolar media-I—the classical theory. Int J Eng Sci 9:271–308
38. Kröner E (ed) (1968) Generalized continua (Proc IUTAM Symp Freudenstadt, 1967). Springer, Berlin
39. Larmor G (1891) On the propagation of a disturbance in a gyrostatically loaded medium. Proc Roy Soc Lond (Nov 1891)
40. Laval J (1957) L'élasticité du milieu cristallin-I, II, III. J Phys Radium 18(4):247–259; 18(5):289–296; 18(6):369–379
41. Le Corre Y (1956) La dissymétrie du tenseur des efforts et ses conséquences. J Phys Radium 17:934–939
42. Levy JR (1970–1990) Article "Cosserat Eugène-Maurice-Pierre". In: Gillispie CC (ed) Dictionary of scientific biography. Scribner's and Sons, New York. http://www.encyclopedia.com/doc/1G2-2830900999.html

43. Lovett EO (1900) Koenigs' lectures on kinematics. Bull Amer Math Soc 6/7:299–304 (with a reference—emphasizing the role of the mobile frame, curvilinear coordinates and the general introduction of deformations—to the Note added by the Cosserats)
44. Lukaszewicz G (1999) Micropolar fluids—theory and applications. Birkhauser, Boston
45. MacCullagh J (1839) An essay towards a dynamical theory of crystalline reflexion and refraction. Trans Roy Irish Acad Sci 21:17–50
46. Maugin GA (1970) Un principe variationnel pour des milieux micromorphiques non dissipatifs. C R Acad Sci Paris A271:807–810
47. Maugin GA (1971) Micromagnetism and polar media, Ph.D. Thesis, AMS Department, Princeton University, New Jersey, USA (April 1971) (All models are here based on a principle of action, Euclidean action being considered for the nonrelativistic cases)
48. Maugin GA (1988) Continuum mechanics of electromagnetic solids. North-Holland, Amsterdam
49. Maugin GA (1998) On the structure of the theory of polar elasticity. Phil Trans Roy Soc Lond A356:1367–1395
50. Maugin GA (2010) Generalized continuum mechanics: "What do we mean by that?". In: Maugin GA, Metrikine AV (eds) Mechanics of generalized continua: one hundred years after the Cosserats. Springer, New York, pp 3–13
51. Maugin GA (2013) Electromagnetism and generalized continua. In: Altenbach H, Eremeyev VA (eds) Generalized continua—from the theory to engineering applications. Springer, Wien, pp 301–360
52. Maugin GA (2013) Continuum mechanics through the twentieth century: a concise historical perspective. Springer, Dordrecht
53. Maugin GA, Eringen AC (1972) Deformable magnetically saturated media-I–field equations. J Math Phys 13:143–155
54. Maugin GA, Eringen AC (1972) Relativistic continua with directors. J Math Phys 13:1788–1798
55. Maugin GA, Metrikine AV (eds) (2010) Mechanics of generalized continua: one hundred years after the Cosserats. Springer, New York
56. Mindlin RD (1964) Microstructure in linear elasticity. Arch Rat Mech Anal 16:51–78
57. Mukhmadov V, Winitzki S (2007) Introduction to quantum effects in gravity. Cambridge University Press, UK
58. Neuber H (1964) On the general solution of linear elastic problems in isotropic and anisotropic Cosserat Continua. In: Görtler H (ed) Proceedings of 11th international conference of applied mechanics (München, 1964). Springer, Berlin, pp 153–158
59. Noether E (1918) Invariante variationsproblem. Klg-Ges. Wiss Nach Göttingen. Math Phys Kl 2:235–257. (English translation by M. Tavel, Transp. Theory Stat Phys I: 186–207, 1971)
60. Nowacki W (1986) Theory of asymmetric elasticity. Pergamon Press, Oxford, UK (Translation from the Polish)
61. O'Connor JJ, Robertson EF (2012) Article "Eugène Cosserat". In: Mac tutor history of mathematics. Archive, University of St Andrews, Scotland http://www-history.mcs.st-andrews.ac.uk/Biographies/Cosserat.html
62. Palmov A (1964) Fundamental equations of the theory of asymmetric elasticity. Prikl Mat Mekh 28:401–408
63. Pietraszkiewicz W, Eremeyev AV (2009) On natural strain measures of the non-linear micropolar continuum. Int J Solids Struct 46:774–787
64. Pommaret JF (1997) F. Cosserat et le secret de la théorie mathématique de l'élasticité. Ann. Ponts et Chaussées, Nouvelle série, no. 82, 59–66
65. Schaefer H (1967) Das Cosserat-Kontinuum. Z Angew Math Mech 47:34
66. Stojanovic R (1969) Mechanics of polar continua, CISM Lectures, Udine, Italy (Augmented version with a long bibliography: Recent developments in the theory of polar media, Udine, Course no. 27, 1970)
67. Stokes VK (1984) Theories of fluids with microstructure. Springer, Berlin

68. Sudria J (1925) Contribution à la théorie de l'action euclidienne. Ann Fac Sci Toulouse, 3ème série, 17:63–152
69. Sudria J (1935) L'action euclidienne de déformation et de mouvement. Mémorial des Sciences Mathématiques, vol 29. Gauthier-Villars, Paris, 56 p
70. Teodorescu PP (1975) Dynamic of linear elastic bodies. Abacus Press, Tunbridge Wells, Kent, U.K. (Translation from the Romanian)
71. Thomson W (Lord Kelvin), Tait PG (1867) Treatise on natural philosophy, 1st edn, Oxford (2nd edn, Cambridge, 1879–1883)
72. Toupin RA (1964) Theories of elasticity with couple-stress. Arch Rat Mech Anal 17:85–112
73. Truesdell CA (1966) Six lectures on modern natural philosophy. Springer, Berlin
74. Truesdell CA, Toupin RA (1960) The classical theory of fields. In: Flügge S (ed) Handbuch der Physik, Bd. III/1. Springer, Berlin
75. Voigt W (1887) Teoretische Studien über Elasticitätverhältinsse der Krystalle, I.II. Abh K Ges Wissen Göttingen 34:3–52, 53–100
76. Wilson EB (1912) Review of Chwolson's treatise on physics (French edition). Bull Amer Math Soc 18:497–508
77. Wilson EB (1913) An advance in theoretical mechanics: Théorie des corps déformables by E. and F. Cosserat. Bull Amer Math Soc 19(5):242–246

Chapter 9
Caratheodory: Thermodynamics and Topology

Abstract In 1909 there appeared a truly unidentified object in the landscape of thermodynamics. It was an axiomatic formulation of the first and second laws of thermodynamics with geometric and analytic arguments by the mathematician Constantin Caratheodory. This practically ignored the well known experimental factual bases (with heat engines) posed by Carnot and Kelvin, to the benefit of a powerful exploitation of Pfaffian differential forms. This is very convincing to mathematically oriented minds as it introduces the notions of entropy and thermodynamic in a purely mathematical framework, the inverse of temperature playing the role of integrating factor. The present essay analyses the contribution of Caratheodory, its reception by his contemporaries, and the influence it had on some specialists of continuum thermo-mechanics in the second half of the twentieth century.

9.1 Introduction

The reader may be surprised to see here the present essay related to a mathematician who is little known in the world of mechanics, and whose works may seem very abstract and lacking physics as very much axiomatic, while thermodynamics is most often viewed as a physical science based on experience. But to give a personal opinion, I find it much easier to give a course on analytical mechanics (in the rather axiomatic tradition of Lagrange) than to try to present in a very convincing way the principles of thermodynamics to a classroom of students. The precision brought by Caratheodory in his paper of [7] (English translation in Kestin [16])—revisited also in [8]—is of great help. We adopted it in our course [20] as this was also done by Germain [12] following Buchdahl [4, 5], one of the best analysts of Caratheodory's contribution to thermodynamics.[1]

[1] Hans A. Buchdahl (1919–2009) was a theoretical physicist who taught at the University of Tasmania (1942–1962), Australia, and then at the Australian National University (1962–1985). He tried to make Caratheodory's approach to thermodynamics more appetizing to English speaking physicists.

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Caratheodory's presentation did not escape the attention of people like Hellinger who carefully scrutinized the landscape of continuum mechanics in the pre World-War-One period.

For quite a long time thermodynamics dealt only with heat and mechanical work, the latter in the most elementary form, force multiplied by displacement. But the development of continuum mechanics for more complex behaviours, started with thermo-elasticity and continued with the consideration of visibly irreversible processes (very viscous fluids, visco-elasticity, plasticity, creep), that required envisaging constraints brought on these behaviours by the second law. In time, it has become customary to build continuum mechanics on the bases of a general thermodynamic background, hence the coinage of the neologism of continuum *thermo-mechanics* and the general science of *energetics.*

The problem at stake is the following one. Having some notions from the early works [e.g., Sadi Carnot's [10] and Thomson (Kelvin)'s [28] papers], define in a precise mathematical form what are the thermodynamic evolutions that are admitted and what states are accessible or not accessible. In this context, Caratheodory is responsible for the introduction of the notion of *thermodynamic adiabatic accessibility.* His main mathematical instrument is provided by the theory of the differential representation of first order systems as introduced by J. F. Pfaff (1765–1825) in a famous paper published in 1813–1814 [22], and, in modern terms, arguments of topology. We remind the reader that following the rediscovery of Carnot's original (and at first practically ignored) book of 1824 [10] by Thomson (Lord Kelvin 1824–1907) during a stay in Paris, the first law (called *principle* in French or German) of thermodynamics was proposed almost simultaneously by three scientists, Kelvin in the UK, J. R. Mayer (1814–1878) and von Helmholtz (1821–1894) in Germany. This expresses a law of conservation (of *energy*). The second law of thermodynamics was proposed by R. Clausius (1822–1888) and reflects an evolution, said to be irreversible in time, which may have only one direction, toward the future. This is known as the *Arrow of Time*. One special scalar quantity is introduced to represent this effect, the *entropy*.

That is, the *first law* specifies that *energy* can be exchanged between physical systems in the form of heat and thermodynamic work, more precisely: *A change in the internal energy of a closed thermodynamic system is the difference between the heat supplied to the system and the amount of work done by the system on its surroundings.* This is mathematically formulated as the *principle of conservation of energy*. But energy, if indeed conserved, changes its nature in most physical evolutions. This yields the second law according to which energy can only be degraded. Thus: the *second law* that deals with *entropy*, expresses some limitation on the amount of work that can be delivered by a thermodynamic system to an external system.

This represents what is known as irreversibility. It is basically expressed as: *Heat cannot spontaneously flow from a colder region to a hotter location.* This is the expression of a universal principle of decay observable in nature that is mathematically expressed by a forbidden decrease in entropy. The latter is a measure of how much this decay process has progressed. This can also be referred to as the *principle of Carnot* (whose author, nonetheless, had no notion of entropy).

At this point we must recall what is an *axiom* in physics? An axiom—or postulate—here is the statement of self-evident facts and thus, does not need any proof. It is at the starting point of reasoning. This is the definition given by ancient Greeks. In conformity with this definition Caratheodory proposes an axiomatic system for *equilibrium* thermodynamics. In doing so he fulfils his program to provide thermodynamics with "logical order and an intellectual clean up" (his own words), following along the path of the example provided by the axiomatic formulation of geometry by Euclid. As to a *principle*, it is a law or rule that has to be followed such as the law of nature or the way a system is constructed. The principles of such a system are understood by its users as the essential characteristics of the systems. Finally, a *corollary* follows from previously enunciated statements (such as theorems). This reminder of definitions is useful in the following discussion.

9.2 On Caratheodory

Constantin Caratheodory (*Καραθεοδωρή*) (1873–1950) was born in Berlin of Greek parents. He was educated in Belgium (Lycée, Engineering military school). He then worked as a civil engineer in Egypt while educating himself in mathematical analysis. He completed his formal education in mathematics in Berlin and then Göttingen under the supervision of Herrmann Minkowski. He published in 1909 a celebrated axiomatics of thermodynamics that is the object of this essay, introducing the notion of thermodynamic adiabatic accessibility, a work acclaimed by Max Planck and Max Born. He was extremely mobile since during the period 1908–1920 he was a professor successively in Bonn, Hannover, Breslau, Göttingen, and Berlin; then he taught in Smyrna, Athens, Munich, and finally Berlin until 1950. He has published famous mathematical works in analysis with many theorems and conjectures bearing his name. More on his life and the totality of his work is to be found in various biographies published in Greece (cf. [9, 26]).

9.3 The Standard Formulation

This is the formulation of *thermostatics* as accepted in physics and engineering before Caratheodory's axiomatic formulation (cf. [21]). According to Carnot [10], a heat engine is a device that converts heat into mechanical work, and temperature controls the flow of heat between two systems. The notion of efficiency is defined thus. The heat engine works between a high (hot) temperature T_H and a low (cold) temperature T_C. The work per cycle, W_{cycle}, done by the heat engine is equal to the difference between the heat energy q_H put in the system at high temperature and the heat energy q_C ejected at the low temperature in this cycle. The efficiency of the engine is then defined by

$$Eff = \frac{W_{cycle}}{q_H} = \frac{q_H - q_C}{q_H} = 1 - \frac{q_C}{q_H}. \tag{9.1}$$

Carnot's theorem and the fact that any cycle can be composed of any number of smaller cycles result in that the ratio q_C/q_H is a function of the respective temperatures at which they occur. This function is a monotonic function $f(T)$ which by convention can be put equal to T itself. Thus (9.1) yields

$$Eff = 1 - \frac{T_C}{T_H}. \tag{9.2}$$

Then efficiency is 1 or 100 % for $T_C = 0$. Otherwise it is always less than 1. Because of (9.1) and (9.2) we can write

$$\frac{q_H}{T_H} - \frac{q_C}{T_C} = 0. \tag{9.3}$$

The generalization of this result is none-other than the *Carnot-Clausius theorem* which relates entropy, heat and temperature. Indeed, entropy is a well defined quantity at thermodynamic equilibrium (thermo-*statics*) *only*. Inspired by the works of Carnot, Clausius has shown that the line integral $\oint \delta Q/T$—involving a heat increment δQ at temperature T—for *reversible* processes is independent of the path. Accordingly, the differential

$$dS = \delta Q/T = \delta q_{rev}/T$$

defines a state function S which is called the *entropy*. Indeed we have (theorem of Carnot-Clausius)

$$\int_{state\,1}^{state\,2=state\,1} \delta Q/T = \int_{state\,1}^{state\,2} dS = S(state\,2) - S(state\,1) = 0 \tag{9.4}$$

Clausius also found that at each stage of a cycle work and heat are not equal but their difference must be a state function (now called the *internal energy* ε) that vanishes once the cycle is completed. In terms of differentials, if pressure p is the only external parameter associated with volume V, we will have the fundamental relation: $d\varepsilon = TdS - pdV$. This of course shows that entropy and temperature form a conjugate pair of thermodynamic variables. But, in fact, this relation is always true, even if irreversibilities appear in the considered system.

When states at the beginning and end of a cycle differ with a loss of efficiency in the thermal machine, the result (9.4) is invalidated. As a consequence of Carnot's works on the efficiency of thermal machines, Clausius found that for such an evolution

$$S(state\,2) > S(state\,1) \tag{9.5}$$

so that entropy can only grow if $T_2 < T_1$. As a matter of fact, the entropy change between two thermodynamic states, state 1 and state 2, is generally made of two parts: one part $\Delta_{int}S$ relating to a lack of thermal insulation and to mass exchanges (this may also be referred to as entropy change due to interactions with the "surroundings" of the system) and the other part $\Delta_i S$ due to the processes taking place inside the system. That is, instead of (9.5) we should write

$$S(state\,2) - S(state\,1) = \Delta_{\text{int}}S + \Delta_i S, \tag{9.6}$$

with

$$\Delta_{\text{int}}S = 0 \tag{9.7}$$

when the system is thermally insulated (this corresponds to an *adiabatic path* for the state change) and

$$\begin{aligned} &\Delta_i S = 0 \quad \textit{for reversible changes}, \\ &\Delta_i S > 0 \quad \textit{for irreversible changes}, \\ &\Delta_i S < 0 \quad \textit{for impossible changes}. \end{aligned} \tag{9.8}$$

This is the essence of the second law of thermodynamics. For pure adiabatic processes, the $\Delta_i S$ in Eq. (9.8) can be replaced by ΔS by virtue of (9.7). This is the essence of *thermostatics* as at the end of the nineteenth century.

Equivalent to the Carnot-Clausius theorem we can cite the following principle:

Principle of Kelvin [28]: *It is impossible to devise an engine which, working in a cycle, produces no other effect than the extraction of an amount of heat from a reservoir and its complete conversion into mechanical work* [this is equivalent to the impossibility of the *perpetuum mobile* of the second type].

9.4 Caratheodory's Work

The above sketched out arguments have been mathematically formalized by C. Caratheodory in his paper of 1909. Following Caratheodory, we can also say that the reciprocal of temperature plays the role of an *integrating factor* (a mathematical notion) for heat. Accordingly we can write

$$T = \frac{dq_{rev}}{dS}. \tag{9.9}$$

For a system in which entropy S is a function $S(\varepsilon)$ of its energy, then

$$\frac{1}{T} = \frac{dS}{d\varepsilon}. \tag{9.10}$$

Note that Carnot based his reasoning on thermometry and calorimetry. Caratheodory will follow a different path, being foreign to experiments and obviously to imaginary heat engines. Here we scrutinized the main arguments used by

Caratheodory in his original work which relies on the consideration of Pfaffian forms.[2]

We remind the reader that Pfaff's problem consists in the integration of partial differential equations of the form $\omega = 0$ where[3]

$$\omega = \sum_{k=1}^{n} f_k(x_1, \ldots, x_n)\, dx_k. \tag{9.11}$$

Now it must be understood that Caratheodory's approach is *axiomatic*. As such its does not prove the bases of thermodynamics; it postulates them. We briefly analyse this approach constantly keeping in mind this axiomatic nature. In his introduction, he cites a very few predecessors (Mayer, Joule, Clausius, Thomson, Bryan). Because of the axiomatic nature of the work, he has to give precise definitions in his first true section [7, pp. 357–362].[4] He has to rely on Gibbs to introduce the notions of phases and the corresponding "state coordinates" x_i that we now call "state parameters or variables" in an $(n + 1)$-dimensional Cartesian space. These "coordinates" characterize the system S for which we can define equivalent systems. Clearly, x_0 will play a specific role. Then the statements of the first and second laws for systems made of several phases are introduced as AXIOM I and AXIOM II (Sect. 2I).

The first axiom (top of p. 363) tells that in an *adiabatic* change (no heat exchange with the outside), the difference in total internal energy (i.e., summed over phases) between final and initial state is related only to the work A (for "*Arbeit*") received from the outside. The second axiom (bottom of p. 363) concerns the non-accessibility of neighbouring states that can be reached through an *adiabatic* transformation from a known state. We see that in this form both axioms invoke the notion of adiabatic transformations granting to these a paramount

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[2] Johann Friedrich Pfaff (1765–1825) is a German mathematician. He studied mathematics and physics first in Göttingen and then in Berlin. From 1788 to 1810 (closure of the University) he occupied a chair at the University of Helmstedt (in the short-lived Kingdom of Westphalia founded by Napoleon). Then he went to Halle. He was a specialist of analysis with works on partial differential equations, special functions and the theory of series. His most well known work is the one he published on "Pfaffian" forms in 1815 [22]. This is a theory of equations in total differentials. This was later developed by Jacobi, Lie, and others to finally yield the modern Cartan calculus of exterior differential forms. Gauss studied with Pfaff in Helmstedt; he held a high opinion of Pfaff. The term "Pfaffian" was introduced by A. Cayley.

[3] As an example, in 2D let $\omega = Xdx + Ydy$. If the following condition—first established by H. A. Schwarz (1843–1921)—is satisfied: $(\partial X/\partial y) - (\partial Y/\partial x) = 0$, then the Pfaff equation is exact and there exists a function $F(x, y)$ such that $\omega = dF$. This can be generalized to n dimensions. In 3D the condition is that of irrotationality (f_k are the components of a gradient) and F is path independent. For an arbitrary n this line of reasoning leads to exterior differential calculus as expanded by Elie Cartan (1869–1951). When exact integrability is not a priori satisfied, then there still exists the possibility to multiply the ω equation by a so-called integration factor to obtain an integrable expression. This is what is exploited by Caratheodory in his introduction of temperature.

[4] References are to the original German edition.

importance. The notion of neighbourhood obviously connects with *topology* (*analysis situs* as called at the time of Caratheodory). Then we come to the introduction of Pfaffian forms [cf. Eq. (9.11)]. For an adiabatic evolution between time t_0 and time t, Caratheodory writes (p. 386) the first axiom as

$$\varepsilon\{x_0, x_1(t), \ldots, x_n(t)\} - \varepsilon_0 + A(t) = 0. \tag{9.12}$$

The expression for the work A is given by

$$A(t) = \int_{t_0}^{t} DA, \tag{9.13}$$

where DA is the Pfaffian form [cf. Eq. (9.11)]

$$DA = \sum_{i=1}^{n} p_i dx_i. \tag{9.14}$$

Whence (12) takes on the following differential form [Caratheodory's Eq. (14), p. 367]:

$$d\varepsilon + DA = 0. \tag{9.15}$$

Then a discussion is given in terms of parametric curves in the $(n + 1)$-space of "state coordinates" that can represent admissible transformations between two states. We advise the reader to look at technical developments in our book [20, pp. 28–34] that is more readable than Caratheodory's original, and that closely follows Buchdahl. The general adiabatic quasi-static evolution of a system is then written as

$$d\varepsilon + DA = \frac{\partial \varepsilon}{\partial \xi_0} d\xi_0 + \sum_{i=1}^{n} X_i dx_i = 0, \tag{9.16}$$

wherein

$$X_i = \frac{\partial\, \varepsilon}{\partial x_i} + p_i. \tag{9.17}$$

This is equivalent to having introduced the multiplier M, such that

$$M\,\frac{\partial x_0}{\partial \xi_0} = \frac{\partial \varepsilon}{\partial x_0}, \quad x_0 = x_0(\xi_0, x_1, \ldots, x_n). \tag{9.18}$$

The adiabatic quasi-static evolutions correspond to $x_0 = const$. Following a section dealing with the formal introduction of arguments on thermal effects, Sects. 7 and 8 (pp. 375–377) intend to introduce the notions of absolute temperature and entropy (that are dual notions as we know). This is related to the determination of the multiplier M. With proper identification with the absolute temperature T of classical thermodynamics, the final form of the total differential of the energy function is obtained as [13]

$$d\varepsilon = T\,dS - DA, \tag{9.19}$$

where S is identified to the entropy of Clausius. It is noted that ε, S and DA are additive ("extensive" in modern terms) which is note the case of T. The new term is none other than the heat variation dQ so that we have

$$dS = \frac{dQ}{T}, \tag{9.20}$$

if both ε and S are to be state functions in agreement with the classical approach (9.10). Accordingly, the reciprocal of temperature is an integrating factor to make (9.19) a true exact differential form. The crux of this "derivation"—using a *reductio at absurdum* proof—is the application of AXIOM II in the discussion of admissible curves in the state space, followed by an appropriate re-scaling to introduce the thermodynamic temperature. The result can be stated in the form of *Carnot's theorem*:

There exits a universal scaling of temperature T, called thermodynamic temperature or absolute temperature and a function of state $S(x_i)$ *called the entropy of the system, such that*

$$dQ = TdS, \quad S = S(x_i), \quad T > 0, \quad \inf T = 0, \tag{9.21}$$

and the entropy of a combination of thermally simple systems is the sum of the entropy of each of these systems. S is *defined up to an additive constant* (*often considered as equal to zero in the limit as T goes to zero*).

Whenever entropy increases, then we have to face irreversible evolutions. Caratheodory's paper (Sects. 10 through 13) concludes with an attempt at connecting the formally introduced notions of energy, entropy and temperature to experimental facts. This will convince only those permeable to mathematical axiomatic arguments.

9.5 Reception of Caratheodory's Axiomatics

The mathematical dexterity of Caratheodory was recognized at once by his contemporaries. This does not entail a full agreement with his formulation. This was the case of the reception of his works and comments on it by well known physicists such as Max Born [2] and Max Planck [23] who obviously criticized Caratheodory for his lack of physics. In more recent times, apart form Buchdahl who found a real mission in explaining Caratheodory's work to the philistines, we find more serious attacks by those who claim that his axiomatics is de facto not independent of previous principles. Born and Planck view Caratheodory's approach with sympathy; still they express some reserve while comparing it with a more traditional formulation because of the evident lack of contact with experimental facts. Quite typically, the notion of heat is not primary and appears only when adiabatic restriction is removed. Much more important from the point of view of logic is the fact that some authors claim that Caratheodory's approach is *not* independent from previous formulations. This is the case of members of the great Belgian-Dutch school (i.e., of the theory of irreversible thermodynamics) who prove that Caratheodory's axiomatics is *not* independent from the above recalled principle given by Kelvin in his early work [28]. This is the case of Landsberg [18] and Titulaer and van Kampen [27]. What certainly remains is the formal elegance of Caratheodory's formulation (and *not* derivation). As to what led Caratheodory to endeavour this formal treatment—that is now recognized as equivalent to the Carnot-Kelvin-Clausius approach—, this may be due to the friendship with Max Born and the influence of David Hilbert, but we will never know for sure.

9.6 Toward Irreversible Thermodynamics

First as a prerequisite we must acknowledge that the notions of temperature and entropy introduced either in the form of Carnot's theorem or in the axiomatics of Caratheodory are well defined only in thermo-*statics*. The challenging problem in the twentieth century was to go over to true thermo-*dynamics* with the possible occurrence of non-adiabatic transformations and the occurrence of dissipation of energy. Here two main schools are in opposition to solve this challenge in a more or less satisfactory manner. One school may be called that of the *adventurous* scientists (non-pejorative qualification by Paul Germain in oral presentations in the early 1970s), and the other one being that of the *cautious* scientists (but no disdain for this group). This last group builds on the acquisitions of thermo-statics and, therefore, envisages only slight deviations from thermodynamic equilibrium. A basic axiom there is the *axiom of local equilibrium state* according to which "each part of a material system can be approximately considered at each time as being in thermal equilibrium". This will give rise to the theory of (linear) irreversible processes (*T.I.P*) in the 1940s–1960s. The first group feels free from any arguments of thermo-statics: it a priori assumes the existence of entropy and temperature even far outside thermo-dynamic equilibrium. This is represented by the Coleman-Noll-Truesdell approach of the 1960s–1970s—cf. Truesdell [29]. At an intermediate stage, we find those who favour the introduction of internal variables of thermodynamic state to describe dissipative effects. Kestin, the present author, and many other authors who deal with strongly dissipative and singular behaviours such as plasticity, visco-plasticity, damage, creep, and phase transformations, have favoured this approach that is the slightest deviation for the *T.I.P* while bringing to the theory a very high efficiency in practical applications. Of course, the above mentioned axiom of local equilibrium state has to be replaced by one dealing with "local accompanying states" (i.e., for each instantaneous value of the internal variables). For all these we refer the reader to our book [20].

Additional relevant bibliography includes: Antoniou [1], Bridgman [3], Callen [6], Falk and Jung [11], Giles [14], Landé [17], Marshall [19], Pogliani and Berberan-Santos [24], Redlich [25], and Turner [30], some of which fully adhere to Caratheodory's geometric-analytical vision of thermodynamics (as evidenced by the very title of these contributions).

References

1. Antoniou OE (2002) Caratheodory and the foundations of thermodynamics and statistical Physics. Found Phys 32(4):627–641
2. Born M (1921) Kritische Betrachtungen zur traditionellen Dastellung der Thermodynamik. Physik Zeit 22:218–224, 249–254, 282–286
3. Bridgman PW (1943) The nature of thermodynamics. Harvard University Press, Cambridge
4. Buchdahl HA (1960) The concepts of classical thermodynamics. Amer J Physics 28:196–201
5. Buchdahl HA (1966) The concepts of classical thermodynamics. C.U.P, UK
6. Callen HB (1960) Thermodynamics. Wiley, New York
7. Caratheodory C (1909) Untersuchgen über die Grundlagen der Thermodynamik. Math Ann 67:355–386 [There exists an English translation by J. Kestin in: Kestin (ed. 1976) p 206 on]
8. Caratheodory C (1925) Über die Bestimmung der Energie und der absoluten Temperatur mit Hilfe von reversiblen Prozessen. Sitz Preuss Akad Wiss (Berlin) Phys Math Kl, pp 39–47
9. Caratheodory-Rodopoulou S, Vlahostergiou-Vasbateki D (2000) The wise Greek of Munich. Athens
10. Carnot S (1960) Reflections on the motive power of fire and on machines fitted to develop that power (English translation from the French edition of 1824 by R. H. Thurston). Dover, New York
11. Falk G, Jung H (1959) Axiomatik der Thermodynamik. In: Flügge S (ed) Handbuch der Physik, Bd. III/2, Springer, Berlin, pp 119–75
12. Germain P (1973) Mécanique des milieux continus, vol I. Masson, Paris
13. Gibbs JW (1876–1878) On the equilibrium of heterogeneous substances, Parts 1 and 2. Trans Connecticut Acad Arts Sci 3:108–248 (1876); 3:343–520 (1878)
14. Giles R (1964) Mathematical foundations of thermodynamics. Pergamon Press, Oxford
15. Kestin J (1966) A course in thermodynamics. Blaisdell, Waltham, Mass. (Reprint, Hemisphere, Washington, 1979)
16. Kestin J (ed) (1976) The second law of thermodynamics: benchmark papers on energy, vol 5. DH&R, Pennsylvania
17. Landé A (1926) Axiomatische Begründung der Thermodynamik durch Caratheodory. In: Handbuch der Physik, Bd. IX, Chapter 4. Julius Springer, Berlin
18. Landsberg PT (1964) A deduction of Caratheodory's principle from Kelvin's principle. Nature 201:485–486
19. Marshall TW (1978) A simplified version of Caratheodory thermodynamics. Amer J Phys 46:136–237
20. Maugin GA (1999) The thermomechanics of nonlinear irreversible behaviors: an introduction. World Scientific, Singapore
21. Maugin GA (2013–2014) Articles "Classical thermodynamics", "Entropy and thermodynamic temperature", "Continuum thermodynamics". In: Hetnarski RB (General ed) Encyclopedia of thermal stresses. Springer, Berlin-Dordrecht
22. Pfaff JF (1814–1815) Methodus generalis, aequationes differentiarum particularum, necnon aequationes differentiales vulgares, utrasque primi ordinis inter quotcumque variabiles, complete integrandi. Abhandlungen der Königlichen Akademie der Wissenschaften zu Berlin (1814–1815) (there exists a German translation)

23. Planck M (1926) Über die Begrundung des zweiten Hauptsatzes der Thermodynamik. Sitz Berlin Akad Wiss 53:453–463
24. Pogliani L, Berberan-Santos MN (2000) Constantin Caratheodory and the axiomatic thermodynamics. J Math Chem 28(1–3):313–324
25. Redlich O (1968) Fundamental thermodynamics since Caratheodory. Rev Mod Phys 40:556–563
26. Spandagos E (2000) The life and works of Constantine Caratheodory. Aithra, Athens
27. Titulaer UM, Van Kampen NG (1965) On the deduction of Caratheodory's axiom from Kelvin's principle. Physica 31:1029–1032
28. Thomson W (Lord Kelvin) (1851) Trans Roy Soc Edinburgh; Also Lord Kelvin: Mathematical and Physical papers I (1882), p 174
29. Truesdell CA (1984) Rational thermodynamics, 2nd edn. Springer, New York (Enlarged edition)
30. Turner LA (1961) Comments on Buchdahl's treatment of thermodynamics. Amer J Phys 29:40–44

Chapter 10
On Duhem's Energetics or General Thermodynamics

Abstract Pierre Duhem is an unavoidable figure if one wants to scrutinize the progress in the mixed science of mechanics and thermodynamics in the period 1880–1920. He is a prolific writer and a never tired propagandist of the global science of energetics. Here we examine his main contribution, its novelty and its inherent limitations in the light of two remarkable synthetic and/or critical works, his treatise on energetics (Traité d'énergétique ou de thermodynamique générale. Gauthier-Villars, Paris, 528+504 pages, 1911) and his series of papers (Duhem in L'évolution de la mécanique. A. Joanin, Paris, 1903) on the "Evolution of Mechanics" (of which we also provide a partial translation). These works are replaced in their socio-scientific background with its main sources (Gibbs and Helmholtz) and its possible interaction with, and influence on, contemporary scientists. A particular emphasis is put on Duhem's style and interests that are strongly influenced by his combined epistemological, philosophical and historical vision. We concentrate on the specialized fields examined and tentatively improved by Duhem in the "Evolution of mechanics", with a personal interest in those "nonsensical branches"—friction, false equilibria, permanent alterations, hysteresis—that Duhem tries to attach to the former Gibbs' statics and Helmholtz' dynamics by way of subtle generalizations. In this analysis we account for the enlightening comments of contemporaries (E. Picard, J. Hadamard, O. Manville), of his various biographs, and of Duhem's own perusal (Duhem in Notice sur les titres et travaux de Pierre Duhem, 1913) of his oeuvre. We conclude with modern developments which provide answers to queries of Duhem that now appear as too much in advance on their time.

10.1 Introduction

Pierre Duhem (1861–1916), probably one of the most powerful intellects of his period, is a remarkable character. He combines in one person a brilliant and sharp mind, a prolific writer and contributor to phenomenological physics, the champion

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of energetics, a philosopher of science, and the true creator of the history of medieval science. To delineate somewhat the scientific landscape of the relevant period—to fix ideas, 1880–1920—we note that his great contemporaries are, among others, Josiah W. Gibbs (1839–1903), Herrmann von Helmholtz (1821–1894), Henri Poincaré (1854–1912), Ernst Mach (1838–1916) [46], Jacques Hadamard (1865–1963), Henrik A. Lorentz (1853–1928), Albert Einstein (1879–1955), Max Planck (1858–1947), James C. Maxwell (1831–1879) and Lord Kelvin (1824–1907) in the UK, and Joseph V. Boussinesq (1842–1929) in France. Gibbs and Helmholtz are his heroes (cf. [52]). Poincaré and Hadamard are mathematician friends; Mach is a fellow traveller on the road to energetism. Lorentz and Planck are "atomists" and partisans of particles and of molecules (the worst that can be for Duhem). Einstein is beyond the understanding of Duhem (and of many others at the time—but this is not a valid excuse). As to Maxwell (his "bête noire"), Kelvin and Boussinesq, although Duhem had to recognize their creativity, they rely too much on the notion of (mechanical) models to be considered serious and exemplary "theoretical" physicists. In all his works Duhem avoids the use of words such as "particles" and "molecules", hence his a priori rejection of Lorentz and Planck. This agrees with Duhem's profound dislike for hidden motions of hypothetical masses.

Duhem is one of the most prolific authors in his period of activity (roughly 1880–1915); he published twenty-two books in forty-five separate volumes, as well as nearly 400 articles and book reviews in scientific and philosophical journals [56]. He corresponded with about 500 correspondents in all scientifically active countries during his life. His outstanding production can be divided in three large fields: phenomenological physics (elasticity, acoustics, electromagnetism, and physical chemistry—which he usually calls "theoretical physics" and would probably be better designated nowadays as "mathematical physics" because of its definitely marked style), epistemology and philosophy of science, and history of science (with an emphasis on old cosmology—view and explanation of the world—and the sciences of the Middle Ages). Practically, no actual scientist can peruse the whole of this multi-faceted production, because we do not—and cannot—have the vast knowledge and cultural education of these late nineteenth-century scientists.

For the curious reader, we mention that several biographies of Duhem have been written, some hagiographic, some more reasonably scholarly but also biased by some religious and philosophical vision: e.g. Ariew [2], Brouzeng [9], Humbert [39], Jaki [40] and Pierre-Duhem [59]. But to know him better from the scientific viewpoint which is our main concern, we shall mostly refer to the eulogy by Picard [58] and the thorough review of Duhem's work on physics by Manville [47]. Picard was a pure mathematician but with a good understanding of Duhem's works. Manville was a direct disciple of Duhem in Bordeaux; he is sometimes much clearer than Duhem himself. To this we must add Hadamard's [36] kind appraisal of Duhem's contributions to mathematics.

10.2 On Duhem's Style and General View of Science

Here, we can only focus on some aspects of his works on phenomenological physics although one cannot avoid acknowledging the inevitable interrelations between his physics and his epistemological views (his religious opinions and political views, strongly as they may have been expressed and so much influential on his life and vision of the world, are to be left out). Even such a limited perusal is difficult for several reasons: an obvious one is that science and more particularly physics was not written in Duhem's time as they are written now. A second reason is that Duhem, in particular, writes in a very literary (sometimes verbose) style, making extremely long sentences and involving much repetition. He makes an abusive use of semi-colons (as shown in the texts that we translated in the Appendices; Note that this was in the fashion of the time if you consider with sympathy Proust's style). This makes it difficult to follow some of the expanded arguments.

But above all, Duhem, being a pioneer in the field he develops, introduces new words and expressions that have not necessarily passed in our own scientific vocabulary. In what follows, we have to face expressions used in a nonconventional manner such as "mechanics" (when he means a phenomenological theory involving in some sense the notion of "motion"), "doctrines" (where he means types of approach, or simply theories), "motion" (by which he means not only classical local motion but also alterations such as during the fusion of ice, or phase transformations), "velocities" (sometimes meaning reaction rates), and "false equilibria" (that are his own invention). "Gibbs' or Helmholtz' mechanics" does not necessarily refer to mechanics in our accepted sense. Thus some interpretation is necessary. Many of his statements would have been much better expressed with few mathematical symbols. But it happens that Duhem's physics was ahead of the mathematics that would have been best fitting. Duhem was a classical analyst, by what we mean someone trained essentially for dealing with standard ordinary and partial differential equations. He was missing our present knowledge (that is still by force limited) in differential geometry, functional analysis, dynamical systems, and convex analysis. Many of the questions he raised and the horizons he opened would find an answer and a broadening in these fields of mathematics, as amply proved by successful developments in the rational mechanics of continua in the second part of the twentieth century (see [54, 66, 67]).

In the philosophy and methodology of science Duhem wrote two remarkable books, one on the *Aim and Structure of Physical Theory* (original French—Duhem [25, 26]) and the other with a title repeating Plato's moto "*To save the phenomena*" (original French—Duhem [27]—with Greek title). In the first of these he exposes at length the under determination of theory by fact, the rejection of metaphysics and models (as used by, e.g., Kelvin and Maxwell in the UK and Boussinesq in France), and natural classification, rather than explanation, as the very object of physical theory (this can be discussed); according to Duhem, "a physical theory is not an explanation; it is a system of mathematical propositions which has for aim to represent, as simply, completely and exactly as possible an example of experimental laws".

The contents of the second book are clearly explained by its title. Science is elaborated so as to "save the appearance" of actually occurring phenomena; its formulas should not contradict experience. This is the only good criterion of truth. This vision is close to true pragmatism and is not so far from Poincaré's conventionalism. Metaphysics is foreign to pure science. All this had a definite influence on Duhem's own science, the development of which he sees with a high degree of continuity, and as a collective enterprise. This last point justifies his constant use of the plural (collective and not "royal") "we" and not the personal "I", referring [29] appropriately to a citation from Blaise Pascal's Pensées.[1]

About pragmatism, we should ponder the words of Miller [56]: "While for other doctrines a new truth is a discovery, for the pragmatic scientist, it is an invention".[2]

For our main concern in this contribution, the most relevant writing of Duhem is the one on the "evolution of mechanics" [24]. In one chapter of this opus, Duhem examined what, at the time, he called the *"nonsensical" branches of mechanics*. What he means by this somewhat eccentric expression are the fields of physics, mechanics and electromagnetism that do not fit yet in his general framework of thermodynamics. It is interesting to note the list of these fields to which we shall return in Sect. 10.4: so-called false equilibria, friction, hysteresis phenomena, and electro-magnetic theory in materials. These are precisely dissipative phenomena such as thermodynamically irreversible reactions, plasticity, visco-elasticity, memory effects, etc. Now looked upon with our present knowledge, this sounds like a tentative proposal of research programme for the next generation, something quite equivalent in its own field to the Erlangen program (1872) of Felix Klein in geometry and the list (1900–1902) of unsolved—at the time—problems proposed by David Hilbert in pure mathematics—that in fact included the axiomatization of the whole of physics—in particular, Mechanics—as

[1] In the original French: «Certains auteurs, parlant de leur Ouvrages, disent: mon livre, mon commentaire, mon histoire, etc.… Ils sentent leurs bourgeois qui ont pignon sur rue, et toujours un «chez moi» à la bouche. Ils feraient mieux de dire: notre livre, notre commentaire, notre histoire, vu que d'ordinaire il y a plus en cela du bien d'autrui que du leur». In translation: "Certain authors, in speaking of their works, say: my book, my commentary, my history, etc. They smack of these bourgeois homeowners, with "my house" always on their lips. They should rather speak of: our book, our commentary, our history, etc., since, generally speaking, there is far more in them of others than of their own" (cf. [29], in translation).

[2] It is interesting to ponder the confusion between "discovery" and "invention". Usually, "discovery" refers to finding (un-covering) something that pre-existed (e.g., the laws of physics) while "invention" refers to something entirely new (essentially in technology). But in French law the person who "discovers" by chance a treasury (say a box full of old gold coins) is legally called the "inventor" (and not the "discoverer") of this treasure! But the "un-covering" relates to something that was hidden. Think of the first view of the hidden face of the moon, once the human kind could go around the moon. Of course, the so-called hidden face existed before! Was it "discovered" or "invented"? This is a conundrum that we propose to the reader.

Problem no. six. We shall focus on this specific point, providing the reader with partial English translations of the above referred to text and of the introduction to his magisterial treatise on energetics (cf. Appendices A and B).

10.3 Some of Duhem's Creative Works in Thermo-mechanics

Nowadays Duhem is still the subject of many studies in epistemology and philosophy of science in spite of his neglect or belittling by well known historians and philosophers of science such as Clagett [11], Koiré [43], Bachelard [3–5] (and his "epistemological fracture"), and Kuhn [44] (and his "paradigm shift")—all more or less partisans of a "discontinuous" evolution of science. In contrast, his hard science is seldom explicitly discussed, either because its main interest was not captured or because it is implicitly accepted and therefore no longer referred to in detail. Duhem is mostly recognized as one of the creators of *physical chemistry* (with Gibbs) and a forceful contributor to *energetics*, placing the works of William J. Macquorne Rankine (1820–1872) [61], Rudolph Clausius (1822–1888), Herrmann von Helmholtz (1821–1894), Ernst Mach (1838–1916) [46], and Wilhelm Ostwald (1853–1932) at the top of his list of favourite sources and/or competitors. Duhem's view on the unifying role of energetics or general thermodynamics in all of physical sciences (mechanics, electricity and magnetism, heat, etc.) is masterly but quite lengthily expanded in his treatise on "energetics" or "general thermodynamics" [28] which he wrote as some kind of definite treatise. This, as we shall see, has a strong flavour of axiomatic nature that will influence Clifford A. Truesdell and his followers in the 1950s–1960s.

10.3.1 Physical Chemistry

Duhem is undoubtedly one of the creators of physical chemistry and thermochemistry together with Gibbs, van t'Hoff, Ostwald, and Arrhenius. Gibbs and Duhem corresponded; Duhem [16] wrote the first critical analysis of the Gibbsian theory [34] of equilibrium of heterogeneous substances, and one of Duhem's doctoral students came from the USA to pursue the Gibbs-Duhem line, a sufficiently rare fact at the time to deserve being underlined. This culminated in the labelling of a fundamental law for solutions as the *Gibbs-Duhem equation*. Following Helmholtz and F. J. D. Massieu (1832–1896)—an altogether too much neglected scientist (but see Massieu's fundamental work of 1869 [48])—Duhem makes a systematic use of the notion of potential—or characteristic function—where (according to [56, p 229]) "others were still using osmotic pressure as a measure of chemical affinity and using artificial cycle to prove theorems".

One of Duhem's fruitful ideas was to consider reactions rates as (generalized) velocities; this yielded the notion of thermodynamic variables having no inertia, hence governed by first-order differential equations in time, and in the end resulting in forces proportional to the "velocity", a much Aristotelian sounding concept in the view of many. Inspired by the rational mechanics of John Bernoulli, d'Alembert and Lagrange, he could then apply the general notion of virtual variations and virtual work not only in physics but also in chemistry. This also led Duhem to use the expression "motion" for so many evolving situations and that of "mechanics" for many physical alterations. It is in this sense that one must interpret Duhem's general expressions such as "Gibbs' mechanics" [i.e., "chemical mechanics"] and "Helmholtz' mechanics" (see the frequent use of these expressions in Appendices A and B below). But the appropriate introduction of potentials allowed Duhem—by analogy with rational mechanics—to envisage the importance of the problem of maxima and minima in the study of the stability of chemical equilibria. In his typical generalization and application of concepts in various branches of knowledge Duhem did not hesitate to introduce this notion of stability in epistemology while commenting on problems of instability studied by Hadamard.

Duhem was generous in the diffusion of his views in physical chemistry. His first opus on the subject [15] is an epoch making one—although he was still an assistant at the *ENS* in Paris. He also wrote a short introductory course [19], "elementary" lessons for chemists [23], and a so-called "elementary" treatise of chemical mechanics, based on thermodynamics (in four volumes, Duhem [21]) for a total of more that 1,400 pages and dealing with all aspects of the subject (including the continuity of liquid and gaseous states, the dissociation of perfect gases, vaporization, homogeneous solutions, dissolutions, two-component mixtures, and heterogeneous systems).

10.3.2 Fluid Mechanics, Viscous Fluids

Usually Pierre Duhem is not remembered as a contributor to fluid mechanics. But this vision is erroneous. As a matter of fact, he was instrumental in developing some critical points (cf. First volume of Duhem [17]). Among these we note: (1) in the case of nonviscous fluids, problems of hydrodynamic stability with the first application of general theorems going further than Lyapunov and Hadamard, and the problems of buoyancy with much improvement over the former analysis of Bouguer, Euler, and Abbé Bossut; here also Duhem applies the notion of virtual displacement to the problem of the stability of floating bodies going further than the analysis by Lagrange and Lejeune-Dirichlet; (2) in the case of viscous fluids, Duhem paid special attention to the possible propagation of waves. Some time before, Christoffel and Hugoniot had given a precise definition to the notion of waves. J. Hadamard [35] perfected the approach by defining (the order of) a wave front in terms of the quantities that are continuous—so-called invariants across the front—or discontinuous across them. For perfect compressible fluids this yields the two longitudinal and transverse waves. But Duhem, a passionate of thermodynamics, clearly specifies the various possible conditions whether the fluid is heat conductor (velocity according to Newton) or not (in which case it is Laplace's formula that applies). This he extended to discontinuity waves of any order (in the classification of Hadamard).

But the main discovery of Duhem in this field is that no true discontinuity wave can propagate in a viscous fluid; shock waves in the sense of Riemann cannot exist. Accordingly, what we can observe in air (a slightly viscous fluid) are not true discontinuity waves with vanishing thickness across which the derivatives of the velocity would vary abruptly, but are in fact extremely thin layers, that Duhem calls "quasi-waves" (nowadays we say "*structured shock waves*") across which the velocity varies very rapidly although continuously. E. Jouguet (1871–1943), a direct disciple of Duhem, then showed that an almost sudden increase of entropy takes place through this layer. All this input in the theory of fluid mechanics is well abstracted by Duhem [29] himself in his notice written a few years before his untimely death. He also introduced the notion of what we now call contact discontinuities.

Finally, it is generally forgotten that it is Duhem who proved the celebrated inequalities to be satisfied by the viscosity coefficients λ and μ of Navier-Stokes equations ($3\lambda + 2\mu \geq 0$, $\mu \geq 0$) the same year as Stokes, from the non-negativity of the corresponding dissipation.

10.3.3 Deformable Solids

Duhem is probably better known in solid mechanics than in fluid mechanics. A reason for this may be the direct influence he had on some well known contemporaries such as Poincaré and the Cosserat brothers [13] (see also Hellinger [38]), and the fact that he was much interested in large deformations, the study of which had a tremendous development some 50 years later on.

When Duhem reports his investigations in elasticity he is aware of, and praises, the main works by George Green, G. Kirchhoff, W. Thomson (Lords Kelvin), W. Voigt (considering isothermal and adiabatic deformations), and J. V. Boussinesq on elements of nonlinear elasticity; the Cosserat brothers have formulated a rigorous theory (1896), and Hadamard has applied it to obtain a beautiful theorem—Duhem's words [29, p. 100]—concerning wave propagation. Thus Duhem "kept alive a correct finite elasticity inspired by other workers" [56]. In the course of these works, Duhem was the first to study the relationships between waves in isothermal and adiabatic finitely deformed systems without viscosity. Again, the specific interest of Duhem in thermodynamic properties led him to establish conditions of stability and those of the existence of waves (in the sense of Hadamard): no true shock waves can exit in the viscous non-linear thermo-elastic bodies (a result already shown in viscous fluids; see above). In these studies he is also led to introducing the notion of "ondes-cloisons" ["partition (-wall) waves", GAM] through which there is no exchange of matter. These waves are similar to contact discontinuities (no discontinuity in the velocity). They separate the volume of the considered body into cells, so that Duhem notes their resemblance with H. Bénard's cells in a fluid where large differences in temperature generate convection currents and the formation of cells. All this is well documented in notes published in the *Ann. Ecole Normale Supérieure* between 1904 and 1906, but also in the collection of papers gathered in his book on "*Researches in elasticity*" (Duhem [25, 26]; fourth part on waves in viscous and non-viscous media), while his first works in elasticity are presented in the second volume of Duhem [17] which reproduces courses he delivered early in his career in Lille.

Solids in plastic deformation are dissipative with possible permanent alterations and hysteresis. This will be briefly examined in Sect. 10.4 below.

10.3.4 General "Thermo-mechanics" and Thermodynamic Potentials

The general attitude of Pierre Duhem towards what we now call «thermo-mechanics»—emphasizing thus the intimate relationship between thermodynamics and the rational mechanics of continua—is the extensive use of the notion of *potential*.

First among his contribution was his complementary study of Gibbs' relation that is now known as the *Gibbs-Duhem equation*: this thermodynamic relationship expresses changes in the chemical potential of a substance (or mixture of substances in a multi-component system) in terms of changes in the temperature θ and the pressure p of the system. The *chemical potential* μ represents the Gibbs' free energy per molecule of the substance; the change in μ is the amount of energy per molecule available to do work for a process such as in a chemical reaction at constant temperature and pressure. The celebrated Gibbs-Duhem equation reads thus:

$$N\,d\mu = -S\,d\theta + V\,dp,$$

where N is the number of molecules of the substance, S is the entropy of the system, and V is the volume. This equation follows from a combined application of the first and second laws of thermodynamics. In practice, it means that if the chemical potential is known for each substance under one set of conditions, then this equation can be integrated to find the corresponding chemical potential under a different set of conditions. A modern application would be the evaluation of the amount of energy that a car battery can deliver. The relationship is thus of universal application if we account for the rich variety of useful chemical reactions.

The other applications of the notion of potential are more subtle. First, following Helmholtz, and also Massieu [48], Duhem recognized the importance of the notion of *characteristic function*, or potential. Also in the path of Helmholtz, Duhem enforced the notion of *normal variables of state*, according to which entropy is given a special status among the list of independent variables appearing in the relevant potential (nowadays, the internal energy) [this distinction is of importance if we remember that the dual of entropy is none other than the thermodynamic temperature, obviously a very original quantity].

Among the other notions of potential considered by Duhem, we must single out those of "*internal thermodynamic potential*" (equivalent to the potential in rational mechanics), "available energy" (useful energy according to Kelvin and equal to the *free energy* of Helmholtz), "ballistic energy" (cf. [29, p. 87]) (practically equivalent to the modern notion of total potential energy) with the related problem of minimization, and that of "oeuvre" (in fact, total energy). Duhem is one of the first to define heat in terms of energy and work [both Caratheodory, and also Born [6] will follow along this line].

Concerning thermo-mechanical variables, Duhem introduced the notion of *variables exhibiting no inertia*, and thus to be governed by differential equations of the first order in time. In Duhem's own words, chemical "mechanics" is the typical domain of application of this concept. Although this smells of some Aristotelian mechanics ("force" proportional to the "velocity"—i.e., rate of change), this will have a glorious descent with the modern notion of *internal variable of state* (See Sect. 10.5 below). Anyhow, most of these notions, with some unavoidable evolution in the employed vocabulary, permeate the whole of modern continuum thermo-mechanics (cf. [53]).

10.3.5 Contributions to Mathematics

This is just to complement the preceding paragraphs. Hadamard, a friend of Duhem, wrote a very kind and obviously benevolent appraisal of Duhem's contributions to mathematics (cf. [36]). Of course, he does not claim that Duhem is a creative mathematician because Duhem did not create new mathematical concepts, and his mathematics is that of the field of differential and integral calculus (i.e., standard analysis); but Duhem was aware of the recent developments in the field, in particular concerning problems of maxima and minima, e.g., a certain familiarity with Lyapunov's studies.

What is most characteristic of Duhem is his *mathematical style*, in a line that will yield the formulation of mathematical physics in the twentieth century, and provide inspiration to some members of the community of rational continuum physics (including the present writer in his youth). Hadamard, but also Picard [58], emphasizes the rigour given by Duhem to theorems in elasticity, thermo-elasticity, and general theorems for Navier-Stokes fluids and the finite elasticity in (Kelvin-Kirchhoff-Neumann) bodies. As noticed by Miller [56], "No wonder that Duhem's contemporaries often remarked that many of his papers opened with the barest of assumptions followed by a series of theorems, with little motivation for the proposed "axioms" and hardly any appeal to experiment". This style was going to influence a whole school mainly in the USA (cf. [54], Chap. 5, and Sect. 10.5 below). On a more trivial level, it is relevant to note that Duhem was the first good exploiter of Euler's theorem on homogeneous functions in physics. This also will prove useful later on, especially in the thermo-mechanics of elasto-plasticity (cf. [50]).

10.4 Short Analysis of Some Exemplary Writings

For the purpose of illustration and test of our understanding of Duhem's style of writing, we consider in greater detail two of his works for which we provide partial translations into English (Appendices A and B) [Some comments are directly inserted in the text of the English translation, within square brackets].

10.4.1 On the General Treatise on Energetics

First, as an unavoidable obligation, we consider the introduction to Duhem's formidable treatise on energetics of 1911 (Appendix A below). It is no question to envisage a full translation of the whole treatise (a seldom read opus in any case) that would provide a rather dull and boring reading to most modern readers who, anyway, do not have the keys to decipher the code of Duhem's specific vocabulary. As we see it, this introduction defines clearly the aim of the treatise by shedding light on its main purpose. It recalls the objective followed by what Duhem calls "theoretical physics" [this is *not* our current assumption of the term]: *From given physical data, extract new physical laws*. It emphasizes the prevailing role played by *general principles*, for instance, the principle of conservation of energy and the principle of Carnot [a primitive form of the second law for heat machines]. Duhem reminds the reader that *Rational Mechanics* was for a long time the basic "code" for the general principles of physics. But he wants to account, not only for changes in the local motion of objects and in their geometrical form [this was the subject matter of rational mechanics], but also for changes in other "*qualities*" or *states* exhibited by material bodies. This includes thermal, chemical and electromagnetic ones, indeed all what is now the object of phenomenological physics.

The ideal frame for such a global vision is provided by *energetics* (in the sense of Macquorne Rankine who provided an outline of this general framework Rankine [61, 62]) or, in other words, *general thermodynamics* [a term clearly preferred by Duhem] since all physical effects, whatever, must comply with the two basic laws of thermodynamics. Later on, the local form of the second law will be called the *Clausius-Duhem inequality* by Truesdell and his followers, putting thus Duhem on an equal footing with Clausius, the true creator of the second law.

Then Duhem, following his philosophical and epistemological inclination, cannot avoid specifying his views on the relationship between theory and experiments: "theoretical physics" remains free of choosing its own path (see Duhem's [25, 26] "Aim and Structure of Physical Theory"), the relevant principles remaining themselves without logical support, but nonetheless with a historical one.

The reader will find in Capecchi [10, Sect. 18.1]) a brief but deeply thought analysis of the contents of the first part (first "tome", devoted to non-dissipative systems) of Duhem's treatise. This we shall not repeat, noting simply that Capecchi attempts to define what Duhem understands by the rather unclear notion of "oeuvre" (in this author's translation, "activity"): essentially a generalization of mechanical work.

10.4.2 On the Evolution of Mechanics

The second work of Duhem that deserves a closer examination is his series of papers on the *Evolution of Mechanics* because, on the one hand, it is normally written for the layman [we would rather say the enlightened amateur], and on the other hand, it poses queries that will have to be answered in the future. This series of papers was an immediate success with early translations into Polish and German [but the first English translation had to wait for more than seventy years, thus showing the distance between British physics and Duhem's one]. We provide in Appendix B our own English translation of essential parts of Chapter VII of this opus. The whole of this series provides pleasant and informative reading but with the charm of an old-fashioned style.

In Chapter I Duhem presents the various kinds of mechanical explanations with the identification of Aristotelian, Cartesian, Newtonian, and Leibnizian viewpoints. Chapter II is devoted to analytical mechanics with due consideration of virtual velocities, d'Alembert, Lagrange, Poisson, Boscovich, Navier, Cauchy, and Gauss. Chapter III is devoted to the mechanical theories of heat and electricity including the essential contributions of Clausius, Helmholtz, Boltzmann and Gibbs for the theory of heat, and of Faraday, Maxwell and Helmholtz for electromagnetism. We have dealt with Duhem's preference of Helmholtz' approach over that of Maxwell in Maugin [52]—also Chap. 7 in this book. Chapter IV deals with atomism and what Duhem calls the return to Cartesianism. In this line he cites William Thomson (Kelvin), Maxwell, Heinrich Hertz, Boltzmann, and Boussinesq, a line for which he shows no great sympathy (cf. his harsh criticism of the use of mechanical models). In contrast, he sketches the foundations of thermodynamics in Chapter V, rendering full justice to Macquorne Rankine. Here he emphasizes the fact that "*it is possible to speak about physical quantities in the language of algebra*" (p. 301). It is here that he introduces the notion on *virtual variation* ("modification") applied to all kinds of physical quantities, as also the general notion of *alteration* (e.g. change of state) generalizing thus the notions of "motion" and "equilibrium" to non-mechanical concepts. The conservation of energy and the relationship between work and quantity of heat are underlined. "Reversible" modifications are then considered and the limitations brought by the Carnot principle (positive sign of the external work during a cycle) complete this magisterial vision.

Chapter VI on general «statics» and «dynamics» is central in that it defines what Duhem usually calls "Gibbs' mechanics" and "Helmholtz' mechanics". It is badly needed to understand the more thought-provoking last chapter. Advocating the consideration of normal variables of state because the latter provide the simplest formulation of thermodynamics, Duhem rightly emphasizes the role of the characteristic function of Massieu. This is—in varied naming—the "available energy" for Gibbs and Maxwell, the "free energy" for Helmholtz, and the "internal thermodynamic potential" for Duhem. From this quantity one can deduce the necessary and sufficient conditions for a system to remain in equilibrium under the action of external bodies maintained at the *same* temperature as this system. Accordingly, the mathematical form of this assertion is obtained by considering a virtual modification that does not change the temperature. This is in the spirit of Lagrange's variations. With F denoting the free energy and A the external action, this equilibrium condition for a normal variable α typically reads

$$A = \frac{\partial F}{\partial \alpha}. \tag{10.1}$$

Both spirit and methods of Lagrange's statics have been transferred to this Gibbsian formulation, including, if necessary, the introduction of side conditions (mathematical constraints). Its fertility is most obvious in the study of qualitative properties such as in electricity and magnetism. Its greatest success, however, was in the theory of chemical mixtures with the proposal of the *rule of phases* by Gibbs himself. With Eq. (10.1), thermodynamics acquires a large extension, much beyond the mechanical aspects envisaged by Lagrange. In particular, it allows one to determine calorific properties. This follows from the fact that once we know the internal potential of a system, then we can compute its internal energy. Furthermore, an infinitesimally small reversible modification is none other than a virtual modification issued from an equilibrium state. The heat produced in this operation can be determined from the internal potential, and divided by temperature, it (Clausius) provides the associated decrease in entropy (i.e., $\delta S = \delta Q/\theta$).

Equation (10.1) will be completed by an "acceleration term" J_α if the state variable α is endowed with an appreciable inertia and the system is in "motion" [meaning in general by this an "evolution in time"]. This "dynamics" would be obtained by applying *d'Alembert's principle*. But there may also exist viscous phenomena which make that, in the absence of inertia, return to equilibrium is delayed [This is akin to relaxation]. These viscous effects will always result in a non-negative working in any real evolution. This is locally written as $-v_\alpha \delta\alpha \geq 0$ with the sign convention of Duhem for v_α. With both inertia and viscosity—represented by a term v_α—Eq. (10.1) is replaced by

$$A + J_\alpha + v_\alpha = \frac{\partial F}{\partial \alpha}, \tag{10.2}$$

remembering that this is strictly valid only for a constant temperature. This is Duhem's elaboration of "Helmholtz' mechanics" with further works in Krakow in Poland by Władysław Natanson (1864–1937; see [63]) in the period 1896 on. In Eq. (10.2), we must reckon that J_α will involve second-order time derivatives, while the "viscous" contribution v_α will depend, not only on temperature, but also on the generalized velocity associated with variable α. The problem of initial conditions is therefore posed. The latter reduces to knowing only the initial state if the normal variable α has negligible inertia. Such variables are extremely important in many chemical systems or systems exhibiting material changes (e.g., phase transformations).

But the information contained in the above exposed principles and equations is insufficient to solve the whole dynamical problem at hand. The needed additional relations are based on the computation and sign of the produced heat and resulting entropy change. This is nothing but the celebrated *Clausius inequality* that will later be known as the *Clausius-Duhem inequality*. To end with this chapter, it is natural, as done by Duhem, to ask what is the relationship of (10.2) with the equation governing the *kinetic energy* (the "living force" in the old vocabulary)? We can think of multiplying all terms in (10.2) by an infinitesimal change in α, i.e., $\delta\alpha$. This will yield

$$-\delta K_\alpha + A\delta\alpha + v_\alpha \delta\alpha = \frac{\partial F}{\partial \alpha}\delta\alpha, \tag{10.3}$$

remembering that K_α is the kinetic energy associated with α, $-\delta K_\alpha$ is the decrease in kinetic energy, and the partial derivative of F was computed at *constant* temperature so that the right-hand side of (10.3) is not the full variation of F. It is this remark that made Duhem introduce the notion of useful energy. Indeed, we can rewrite (10.3) as

$$\delta K_\alpha - A\delta\alpha = -\delta F - S\delta\theta + v_\alpha \delta\alpha, \tag{10.4a}$$

or

$$\delta K_\alpha - A\delta\alpha = -\delta E + \theta\delta S + v_\alpha \delta\alpha, \tag{10.4b}$$

with

$$S = -\frac{\partial F}{\partial \theta},\ E = F + S\theta. \tag{10.5}$$

The first of these is the definition of *entropy*, while the second introduces the *internal energy*. Duhem calls the left hand side of (10.4a) or (10.4b) the variation of useful (available) energy U. We see that U is none other than F for isothermal evolutions ($\delta\theta = 0$), and none other than E for isentropic evolutions ($\delta S = 0$), both in the absence of viscous effects. If we remember that $-v_\alpha \delta\alpha \geq 0$, then we also see that (10.4a, b) yields[3]

$$-v_\alpha \delta\alpha = A\delta\alpha - \delta(K_\alpha + F) - S\delta\theta \geq 0 \tag{10.6a}$$

or

$$-v_\alpha \delta\alpha = A\delta\alpha - \delta(K_\alpha + E) + \theta\delta S \geq 0. \tag{10.6b}$$

The notion of useful energy receives its whole importance in discussions relating to the stability of an equilibrium state where isothermal and isentropic stabilities are quite different. According to Duhem, it is a postulate due to Helmholtz ("the heat capacity is positive in all systems") that resolves the matter.

10.4.3 On Permanent Alterations and Hysteresis

Now we can have a closer look at Chapter VII of Duhem's "Evolution of Mechanics" for which we give a partial English translation in Appendix B below. This offers a direct continuation and generalization of the contents of Chapter VI. This chapter is quite ahead of its time and introduces queries than could not find an answer in the early twentieth century. But, altogether, we can say that Duhem is on the right track by underlining some branches of mathematical physics which, at the time, appear more or less as nonsensical ("aberrantes"), i.e., not yet fully absorbed in the general science of energetics. We shall concentrate on four kinds of phenomena: friction, false equilibria, permanent alterations, and hysteresis phenomena, leaving aside the case of electromagnetism to which we have contributed many research works and several books certainly in a perspective that Duhem would have appreciated [33, 49, 55] since entirely based on a thermo-mechanical vision.

The phenomena of interest are precisely *dissipative* ones such as thermodynamically irreversible reactions, plasticity, visco-elasticity, and memory effects. The first section of this chapter concerns friction and chemical false equilibria. If

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[3] In modern continuum thermo-mechanics, a transcription of these two equations for a whole body B reads (cf. [50], p. 39, Eqs. 2.54 and 2.55):

$$\Phi_{\text{intr}} = P_{ext} - \frac{d}{dt}\int_B \left[\frac{1}{2}\rho\mathbf{v}^2 + F(.,\theta_0)\right] dv \geq 0 \tag{Na}$$

and

$$\Phi_{\text{intr}} = P_{ext} - \frac{d}{dt}\int_B \left[\frac{1}{2}\rho\mathbf{v}^2 + F(.,S_0)\right] dv \geq 0 \tag{Nb}$$

for isothermal and isentropic transformations, respectively. Here Φ_{intr} is the total intrinsic dissipation and P_{ext} is the power expanded by external forces.

friction is an easily comprehended phenomenon, the phenomenon of *false equilibria*—an expression coined by Duhem—is difficult to grasp; perhaps that they should better be called *quasi-stable* equilibria. In order to introduce this notion, Duhem first clearly defines the notion of reversible changes and then, by way of examples, the theory of mixtures and friction, he develops the notion of *false equilibrium* (see below) for which a new "mechanics" (he means *thermo*-mechanics) is needed. For this one must go farther than Gibbs and Helmholtz who exploited the notion of potential. This will ultimately yield the idea of *irreversible* changes. This irreversibility is defined by negation of reversibility. Miller [56, p. 228] says that Duhem gave "the first precise definition of a reversible process; earlier versions by others (unfortunately often preserved in today's textbooks) are too vague"…"The reversible process between two thermodynamic states A and B of a system is an unrealizable limiting process. The limit of the set of real processes for getting from A to B is obtained by letting the imbalance of forces between the system and the surroundings at each step tend toward zero. Each member of this set of real processes must pass through non-equilibrium states, or else nothing would happen". Duhem also emphasizes the relative appraisal of what is a state of equilibrium. We now return to his original text, p. 418.

To illustrate his discussion, Duhem considers a thermodynamic system with only one *normal* variable of state, say α. We remind the reader that the notion of *normal* variable of state was introduced by Duhem following an idea of Helmholtz: a system of normal variables of state χ_α in the functional dependence of internal energy density E does not include entropy η which, serving to define temperature, is considered as a very special variable, i.e., $E = E(\eta, \chi_\alpha)$; in terms of the free energy, this endows the thermodynamic temperature with a special status and we shall write $F = F(\theta, \chi_\alpha)$. The action of friction—to which we are naturally accustomed—is always positive (with the sign convention of Duhem) and "will depend, just like the action of viscosity, on the absolute temperature, the variable α, and the generalized velocity $\dot{\alpha} = d\alpha/dt$". However, contrary to what happens for viscosity, "it will also depend on the external action A". Also, "it will not vanish with the generalized velocity; the latter going to zero, the action of friction will tend to a positive value g". The additional term to be added to Eq. (10.2) above—written for α—will have a sign that depends on the sign of the generalized velocity. We can summarize Eqs. (10.3) and (10.4b) of Duhem in the single equation:

$$A + J + v - sign(\dot{\alpha})f = \frac{\partial F}{\partial \alpha}. \tag{10.7}$$

With the same working hypotheses, the corresponding equilibrium condition "will no longer be represented by an equality, but by a double inequality that expresses that the absolute value of the difference $A - \partial F/\partial \alpha$ is not larger than g", i.e., [Eq. (5) in Duhem]

$$\left| A - \frac{\partial F}{\partial \alpha} \right| \leq g. \tag{10.8}$$

Here Duhem says only a few words on the equation of living forces [equation of kinetic energy], noting that "it is only necessary to add the work of friction to the work of viscosity. The former, like the latter, is always negative. We also do not deal with the Clausius-inequality which remains exact in the new dynamics. Here also, the work of friction is just being added to the work of viscosity. Other consequences of the laws just formulated, and more particularly the condition of equilibrium, will require a little more attention.

Now, Gibbs' statics, as recalled above, "would require the difference $A - \partial F/\partial \alpha$ to vanish, and therefore having [this is a truism] value between $-g$ and $+g$. The equilibrium states predicted by this Statics, and that are usually called states of *true equilibrium*, are thus among those that are predicted by the new Statics; But the latter announces the existence of an infinity of other equilibrium states, that we designate by the name of *false equilibria*". Citing Duhem in our translation: "If the value of g is large, then the states of false equilibrium spread on both sides of those of true equilibrium, in a large domain. They will shrink close to the states of true equilibrium whenever the value of g is small. If this value becomes sufficiently small, then the states of false equilibrium will be so close to those of true equilibrium that experiments would no longer distinguish them; practically, the Statics of systems with friction would be undistinguishable from Gibbs' statics". We can say with Duhem that "Gibbs' Statics and Helmholtz' Dynamics are limit forms of the Statics and Dynamics of systems with friction; these tend to those when the action of friction becomes infinitesimally small". This remark is not a simple view of the mind; it acquires a particular interest in the study of chemical equilibria. This will be exploited by E. Jouguet in his theory of explosions. But Duhem also tries to illustrate his notion of false equilibrium by some mechanical example, such as the possibility, when considering the problem of the possible rolling of a small ball down a rough hill, that in addition to the true equilibrium at the top of the hill, the small ball may stop for short instants along the rough slope. But in the rest of this section Duhem expands a better example from chemical physics. Miller [56, p. 230] rightly comments that "real false equilibria can also be considered as instances of extremely slow reaction rates". This seems to be the viewpoint adopted nowadays.

Section II of this chapter rings a more familiar bell to our ears, for it deals with *permanent "alterations" and hysteresis effects* that may be closer to our own plasticity, visco-plasticity and creep concepts. Here Duhem first gives a general idea of what permanent alterations are. He emphasizes the role of infinitesimally slow evolutions, adapting in accordance temperature and external actions. Again, one must go beyond Gibbs' statics and Helmholtz' dynamics. That is, one must elaborate on the generalization of Eq. (10.2) or its incremental form that seems to be more appropriate. More precisely, Duhem discusses the possible generalization of the following incremental form that follows from equilibrium [Eq. (10.1)]

$$dA = d\frac{\partial F}{\partial \alpha}. \tag{10.9}$$

Here we remind the reader that $\partial F/\partial \alpha$ is computed at constant temperature while noting that systems exhibiting permanent alterations (e.g., residual strains) are quite different from those exhibiting viscosity [but in modern thermo-mechanics we also envisage a mix of the two effects as visco-plasticity]. The permanent alterations envisaged by Duhem are exhibited when an unloading (decrease in the "cause") following a loading, does not bring the system back to its initial virgin state. We must thus distinguish between the two possibilities of the incremental law that will generalize (10.9) depending on whether we are increasing or decreasing the "cause". For sufficiently slow evolutions, Duhem proposes to generalize (10.9) by [cf. Duhem's Equations (7) and (7b)]

$$dA = d\frac{\partial F}{\partial \alpha} + h\, sign(d\alpha)|d\alpha|, \tag{10.10}$$

where the quantity h may still depend on the state of the system and also on the external action A. Equation (10.10) may seem to be both enigmatic and ad hoc to most readers. It is however in direct line with mathematical works on hysteresis of the 1970s–1990s (see [7, 51]). To find more elaboration by Duhem on (10.10) and applications one should consult the original works of Duhem published in 1901 in a rather odd place (Belgium). Here Duhem is satisfied with the cases of deformations, residual magnetization, magnetic hysteresis, and analogous properties for electric polarisation in dielectrics. Duhem emphasizes the interest of his considerations in metallurgical treatments (tempering, annealing, etc.). But much more is also to be found in the exhaustive and clear analysis of Manville [47]. The latter author remarks that it is difficult to summarize the works of Duhem and his co-workers (doctoral students: Marchis, Saurel, Pélabon, Lenoble) on permanent alterations and the somewhat 400 pages of various memoirs (in particular the seven published in Belgium in 1896, 1897 and 1901) in a short text [20, 22]. But some of the facts and properties recalled by Manville shed light on some aspects that will be of great interest for further comparison with modern developments.

First, for an isothermal infinitesimal transformation we can rewrite (10.10) as

$$dA = \frac{\partial^2 F}{\partial \alpha^2} d\alpha + \bar{h}|d\alpha|. \tag{10.11}$$

For an ever increasing value of α, $d\alpha = |d\alpha|$, and (10.1) yields the differential equation

$$\frac{dA}{d\alpha} = \frac{\partial^2 F}{\partial \alpha^2} + \bar{h}(\alpha, A, \theta), \tag{10.12}$$

which defines a family of ascending curves in the plane (A, α). By the same reasoning, with ever decreasing value of α, we will define a family of descending lines; This vision allows one to build closed cycles and to define a line of "natural states". We note that for an infinitesimal modification that causes the passing of a state (α_0, A_0, θ) to a state (α_1, A_1, θ) (10.11) also yields

$$A_1 - A_0 = \frac{\partial^2 F}{\partial \alpha^2}(\alpha_1 - \alpha_0) + \bar{h} \sum |d\alpha|. \tag{10.13}$$

If the external action recovers its initial value, then $A_1 = A_0$, and (10.13) yields the "permanent" alteration

$$(\alpha_1 - \alpha_0) = -\frac{\bar{h}(\alpha, A, \theta)}{(\partial^2 F/\partial \alpha^2)} \sum |d\alpha|. \tag{10.14}$$

The condition

$$\bar{h}(\alpha, A, \theta) = 0 \tag{10.15}$$

defines the *line of natural states* in the plane (A, α) for a given temperature.

The second important remark is that from the law of displacement of the equilibrium (a *stability* condition about an equilibrium: cause and effect vary in the same sense), the perturbing work in passing from (A, α) to ($A + dA$, $\alpha + d\alpha$) must be non negative (Eq. (9), p. 310, in Manville [47])

$$dA.d\alpha \geq 0. \tag{10.16}$$

This has for immediate consequence that the slopes of the curves in the (A, α) plane must always be in the same sign whether on increasing or decreasing A. Consequently, the end points of a hysteresis cycle in the plane (A, α) are always sharp and cannot be rounded.

A third remark concerns the extension of Clausius' theorem for a non-reversible closed cycle. The quantity

$$\int dS = \int \frac{dQ}{\theta}$$

over a cycle ought to be non-negative. With constant temperature, this reduces to the condition

$$\int dQ \geq 0.$$

Equivalence between heat and work yields that $dQ = Ad\alpha$ for an isothermal cycle, and thus

$$\int_{closed\ cycle} A.d\alpha \geq 0. \tag{10.17}$$

This clearly dictates the sense in which hysteresis cycles are necessarily followed in the plane (A, α).

In practice it was soon discovered that many hysteresis cycles (especially in the mechanics of deformations) are slightly rounded at their ends. The explanation of this phenomenon in contradiction with (10.16) is to be found in the likely presence of small viscous effects. Duhem had several graduate students working out the experimental facets of this research on permanent alterations (Theses of Marchis[4] and Pélabon in 1898, and of Saurel and Lenoble in 1900; see in particular [45]). Results were not very conclusive from the quantitative point of view. Only a general qualitative agreement was found (cf. [47, p. 350]). Sophisticated experimental techniques and accurate measurement possibilities were not yet available at the time, so that these studies were probably too ambitious and in some sense doomed. Here are Duhem's own words: "It seems that no theory of permanent alterations can obtain from experiments more than qualitative and somewhat vague confirmations" (as cited by [47, p. 599]).

In page 423 of his "Evolution of mechanics" Duhem in fact goes more deeply in the notion of natural state and that of residual fields, and then he discusses the possible dynamics—we admit hard to apprehend—in which the standard application of d'Alembert's principle is at fault, referring to works by Henri Bouasse (1866–1958; a French physicist in Toulouse, South–west of France) and the noted German physicist Max Wien (1866–1938).

In the long Section III, pp. 424–427, Duhem considers the general thermodynamics of electromagnetic bodies. He ponders the notions of electromagnetic energy, electric displacement, electromagnetic induction, electrodynamic forces, properties of the system which have no inertia associated with the relevant variables, in spite of the existence of generalized velocities duly associated with such variables, electrodynamic potential, electrokinetic energy, and Ohm's effect considered as a viscosity. He pays an emphatic tribute to Helmholtz. He clearly expresses his irreconcilable appraisal of Maxwell's vision. This, of course, appears to be outdated and certainly not very objective.

The long conclusion (pp. 427–429) of this study is typical of the style of Duhem and of his talent. He provides a rather literary synthesis of the contents of the whole work. He does not hesitate to compare the construction of his successive theories—his *New* "*Mechanics*"—to the "sanguine sketches" of Raphael that are on exhibit at the Louvre museum. There you can examine at leisure the formidable advances of the work of Raphael by successive approximations "starting from a rough sketch and then improving the details at each successive stage and finally producing a masterpiece that finally causes the admiration of all" (p. 428). This may be a bit exaggerated when applied to one's own work. But Duhem is certainly right in defending his progressive complexification of the approach by including more and more "nonsensical" branches that badly require some formalisation and a sound thermodynamic basis. Indeed, the physical world is complex, and its

[4] Lucien Marchis (1863–1941), interested in all means of transformation of energy and transportation, became a professor at the University of Paris in 1910 when a chair financed by the (armament) magnate Basil Zaharoff was created especially for him at the Sorbonne. This was endowed with a substantial amount of money that allowed the collection of a formidable roster of books on aerostation (balloons, Zeppelins) and the beginning of aviation from all over the world. This chair was transformed into a Chair of Aviation and then a Chair of Aerodynamics after the Second World War. Our Institute inherited this formidable collection of which we became the curators.

phenomenological approach cannot be over simplified—in particular mathematically—even though its very nature ignores the notions of particles and molecules. In agreement with Duhem's epistemological vision, the construction of this New Mechanics goes further than Gibbs and Helmholtz but it is still guided by Lagrange and d'Alembert on a rational, mostly mathematical, path. This is the credo of Duhem at a time that is not so ready to accept it. Examining its immediate reception and its delayed legacy is our next endeavour.

10.5 Influence on Contemporaries and Later on

Duhem's original works are seldom directly cited in the beginning of this twenty-first century, i.e. about one century after these works. Indeed, we must realize the extraordinary rapid evolution of his main fields of concern, continuum mechanics in the large and electromagnetism, in this lapse of time. We have surveyed this progress in a recent book [54]. But much of this progress has been made possible thanks to his critical thinking, his embryonic studies, and his deepest works. His own contemporaries followed and applied his works and he did much to advertise and popularise them through his many writings and his huge correspondence with fellow scientists from the world over. The sheer bulk output of his numerous works was of course efficient for this, but the resonance of his works with many interests of the time was also a determining factor. The only obstacle to this swarming of ideas may have been his aerial view, much above the head of many of the less universal thinkers among his contemporaries, and also his personal style, with its inherent complexity and sometimes obscurity already highlighted in Sect. 10.2.

In dealing with this influence we must distinguish between the direct reception by contemporaries, the acceptance and exploitation of seminal ideas, the deepening of the ideas of irreversible thermodynamics, and the renascence of some of Duhem's ideas in the emergence of a true thermo-mechanics with its successful application to the so-called "non-sensical" branches of mechanics.

In the period 1890–1920 the most cited works of Duhem are those on elasticity, waves and physical chemistry. Like for Poincaré, the writing of lecture notes is the occasion for Duhem to revisit the corresponding scientific matter and to introduce new developments. This is particularly the case of his early book (Hydrodynamique, élasticité, acoustique, 1891 [17]). The revisited results and proofs given in this opus are often cited by Poincaré, the Cosserat brothers [13], Appell [1] and Hellinger [38]. The most original part deals with finite strains and, later on, the application of Hadamard's classification of discontinuity waves [35] to elasticity and thermo-elasticity. This will have a glorious descent with works by Truesdell [64] and those gathered in Coleman et al. [12]. Another work of influence was the one devoted to stability and the notions of available and ballistic free energy. This was taken over by Ericksen [31, 32, p. 9].

Physical chemistry, the domain of excellence of Duhem, obviously received most attention. In this line, combining this very field and arguments of wave propagation, E. Jouguet (1871–1943) is probably the most successful direct disciple of Duhem—see in particular Jouguet [41, 42]. Complementing the Rankine-Riemann-Hugoniot theory of shock waves, Jouguet has created the theory of *detonation waves* with application to high explosives. Here the now well known Chapman-Jouguet condition states that the detonation propagates at a velocity at which the reacting gases just reach the sound velocity (in the frame of the leading shock wave) as the reaction ceases. Chapman and Jouguet (circa 1900) in fact stated this condition for an infinitesimally thin detonation front (remember there is no more a true discontinuity front). The physics of this process will be improved by Zel'dovich (USSR), von Neumann (USA) and Döring (Germany) during the Second World War, giving rise to the so-called ZND model.

As to seminal ideas, they are of different magnitude and sometimes relevant more to hearsay than anything else. The idea of *normal* variables of state was readily incorporated in thermo-statics (Caratheodory, Born). The idea of *internal variables of state* to be fully exploited in the 1970–1990s is, according to Truesdell [65, p. 39], to be found in Duhem [28], although we think that Bridgman [8] was more articulate and more successful, in particular with his own influence on the thinking of J. Kestin. It is also said that it is Duhem [18] who may be responsible for the idea to enrich the kinematics of continua by adding to each material point a set of "directors" (unit vectors) in order to provide internal degrees of rotation, an idea that will be fully developed by the Cosserat brothers in their very original book of 1909 [14] (this book introduces the notion of what are now called *Cosserat media*). According to Edelen [30, p. 44], the notion of *non-local continua*—another path to generalized continuum mechanics where the dependence of stress and body force at a point on the state of the *whole* body must be considered—may also be traced back to this decidedly imaginative work of Duhem [18]. Finally, we must record that Duhem was instrumental in fruitfully exploiting the notion of homogeneous functions in thermodynamics (more on this to follow below).

The flame of Duhem's approach to general thermodynamics was successfully carried over by Th. De Donder (1872–1957) and other physicists from the Netherlands and Belgium between 1930 and 1970, resulting in the now commonly admitted *theory of irreversible processes* (S. de Groot, P. Mazur, I. Prigogine)—for short *T.I.P* [60]. However, both Duhem and these scientists did not possess the mathematical tools—such as convex analysis and nonlinear optimization—to deal with some of the properties (plasticity, hysteresis), so that they could deal only with *linear* irreversible processes. The solution would come in the 1970s–1980s for *nonlinear* irreversible processes (see below). We simply note that in *T.I.P* the residual non-negative dissipation is obtained as a bilinear expression in "forces'" and "generalized velocities". The closure hypothesis is a linear expression of one of these in terms of the other with signs respecting the second law, hence the *linear* qualification certainly most often valid for a very small departure from thermodynamic equilibrium (for which the axiom of local thermodynamic state holds good). This makes the dissipation quadratic, i.e. homogeneous of degree *two* in the velocities (think of Rayleigh's dissipation potential). But some of the "nonsensical" effects are of a different nature, often involving the notion of threshold or maximum critical load, and with the strange property of a dissipative effect not depending on the speed at which the cause is applied. Typically, this is what happens in the plastic response of materials not sensitive to the strain rate: "plastic deformation appears to be a process of energy dissipation but at constant state" (Bridgman). As a consequence, the corresponding dissipation is a homogenous function of degree *one* only (in the velocity), obviously a case not manageable by Duhem and standard *T.I.P*.

Now we can examine how Duhem's "nonsensical branches of mechanics" received a satisfactory thermodynamically admissible framework by abandoning some of the working hypotheses of—but deviating the least from—*T.I.P*, but in fact corroborating some of Duhem's mathematical proposals in Chapter VII of his "*Evolution of Mechanics*". Plasticity, visco-plasticity, creep, damage, phase transformations, magnetic and electric hystereses are among these phenomena that attracted most of the attention of mechanicians of materials and applied mathematicians in the last forty years. The thermodynamic answer to the paradoxical situation created by plastic like behaviours was to be found in the consideration of thermodynamics with *internal variables* of state and properties of *convexity* applied to both the internal energy and a pseudo potential of dissipation. Note that the word "convexity" practically never appears in Duhem's writings. But this is an essential notion that appropriately replaces a usual notion such as quadraticity. The *non-linear* thermodynamics of deformable bodies is based on this notion.

As to internal variables of state (Bridgman, Kestin, Gurtin and Coleman), they are identifiable representative parameters of the resulting macroscopically irreversible process. Although being measurable by a gifted experimentalist, they are not directly controllable by external actions, and they obviously are not equipped with an inertia (remember Duhem's variables without inertia). They evolve only under the influence of an evolution of the observable variables, e.g. the local state of stress. Without elaborating further this field of thermo-mechanics (see details in [53]), it suffices to notice that some basic re-interpretation of thermodynamics is necessary, in particular an axiom of *local accompanying state* (Kestin) has to replace the axiom of *local state* of classical thermodynamics. The short but remarkable note by Moreau [57] was essential in introducing the required mathematical notion of convexity, and consequently variational inequalities. Then the French school (Germain, Mandel, and their students; also Ziegler [68] in Switzerland) was instrumental in establishing the corresponding formalism. Of particular efficiency was the model presented by Halphen and Nguyen [37] of so-called "generalized standard materials". Denoting by α the set of relevant internal variables—this denomination cannot be due to pure chance—it was then possible on the basis of convexity arguments and the existence of a pseudo-dissipation function of appropriate degree of homogeneity to prove some of the mathematical results of Duhem, e.g., Eqs. (10.16) and (10.17) above. For example, with the anelastic (plastic) infinitesimal strain ε_{ij}^{p} considered as an internal variable of state (a reasonable hypothesis), equations similar to (10.16) and (10.17) are *proved* in the form

$$\dot{\sigma}_{ij}\dot{\varepsilon}_{ij} \geq 0, \tag{10.18}$$

and

$$\int_{closed\ srain\ cycle} \sigma_{ij} d\varepsilon_{ij} \geq 0. \tag{10.19}$$

The first of these is known as the local stability condition of Drucker and the second as Ilyushin's postulate in the community of plasticity, but now they are proved on a thermodynamic basis. More on this and the cases of magnetic and electric hysteresis is to be found in Maugin [50, 51, 53]. This constitutes a true neo-Duhemian approach.[5]

It appears thus that with appropriate notions of convexity and homogeneity, the proposed programme of Duhem concerning his "nonsensical branches" of mechanics could be fulfilled. The unanswered question here is whether Duhem's writing was very much influential in these developments. We must avoid here the temptation of "precursoritis"; we must admit that we do not know for sure if the thinking of people like J. Kestin—who read French and was the most knowledgeable specialist of all types of thermodynamics—J. Mandel and others was directly influenced by Duhem or by one of his disciples. For instance, Jouguet was a professor of mechanics at both *Polytechnique* and the school mines in Paris when Mandel was a student in these two schools. Jouguet who had a marked interest in the historical development of mechanics and thermodynamics, must have more than simply mentioned the works of his mentor, so that there might have been more continuity than we originally thought.

In retrospective, we must recognize that Duhem's work in physical chemistry "provided a whole generation of French physicists and chemists with their knowledge of chemical thermodynamics" [56, p. 228]. But Miller also wrote [56, p. 232] that "by mid [twentieth] century Duhem's scientific work had been almost completely forgotten". As to the famous "nonsensical branches", he says that "as of this writing [56, p. 229] there is no really adequate thermodynamic theory of such systems, although interest in this subject has recently been revived". We hope to have sufficiently documented the spot on recent developments herein and in various works to disapprove Miller's mild appraisal.

[5] *Personal touch*. In (at the time, 1975) secrete document (report on my French Doctoral Thesis in Mathematics), a well known French mathematician (great geometer also interested in the history of physics), André Lichnerowicz, classified my approach to the thermo-mechanics of relativistic continua as *neo-Duhemian*—but I had not yet read Duhem in those times. When I could read this document after a law was passed giving access to all such personal matters, I felt that this was intended to be derogatory by its author (I may be misinterpreting), who could not figure out how much I later became pleased with such a classification.

10.6 Conclusion

Duhem may have been, with Poincaré, Bouasse and some others, the last scientists to tackle all branches of phenomenological physics simultaneously. This may even be truer of Duhem than of his colleagues because of his all embracing energetic vision. These are bygone days. Furthermore, no doubt that Maxwell's theory of electromagnetism on the one hand and the triumph of atomic theories on the other have put shades on some of his work and diminished the relative importance of his contributions in some fields of science. However, the main purpose of the present contribution was not to exhibit the belated success of his vision—although this is not to be neglected—but to show how his work contributed in a timely and critical manner to the development of phenomenological physics at a turning moment of its history, between the mathematical works of the nineteenth century "giants" (Green, Cauchy, Kelvin, Kirchhoff, Helmholtz, Stokes, Maxwell, Saint-Venant, Boussinesq and Voigt), the consolidation of true engineering sciences (for which Duhem shows no specific interest), the return to experimental bases and molecular considerations, and the emergence of quantum and relativistic physics. The wealth and significance of Duhem's works remain astonishing to our eyes if we accept to have a benevolent look. Surprisingly enough, as shown in the preceding section, some of his most discussed and misunderstood works seem now to be in resonance with fruitful recent developments. This of course teaches a lesson: the full success of a mathematical theory of nature occurs only with the simultaneous availability of ad hoc mathematical tools, the conception of appropriate experimental devices and, now, the exploitation of successful numerical techniques. Unfortunately, the original work on energetics by Duhem could not comply yet with the last three requirements. In contrast, the present theory of the "mechanics of materials" does, although certainly inspired by Duhem's premature achievements.

Appendix A

Partial English Translation of P. Duhem, "Traité d'énergétique ou thermodynamique générale", Gauthier-Villars, Paris, 1911, by Gérard A. Maugin [Only the Introduction and small parts of Chapter I: "Définitions préliminaires", are translated in order to give a flavour of Duhem's style and exposition. Original footnotes, if any, are reported to the end and numbered consecutively. Translator's remarks are placed within square brackets in the main text. This is a verbatim translation without any ambition of literary prowess.

Traité d'énergétique ou thermodynamique générale

Treatise of energetics or general thermodynamics

By Pierre DUHEM

Introduction

1. Of thermodynamics or energetics

Theoretical physics represents by means of quantities (*grandeurs*) the properties of the bodies that it studies. Methods of measurement allow one to place in correspondence, with a more or less broad approximation, each intensity of a property with a particular determination of the quantity that represents this property. Through the methods of measurement, each physical phenomenon corresponds to a set of numbers, each physical law corresponds to one or several algebraic relations between various quantities, each set of concrete bodies to a system of quantities, to an abstract and mathematical scheme.

Theoretical physics has constantly to solve the following problem: *From given physical data, extract new physical laws*; either it proposes to show that the latter, already directly known, are none other than consequences of the former, or it proposes to announce laws that the experimentalist has not yet observed.

To treat this problem theoretical physics combines given laws together, that concern particularly certain physical properties and certain bodies, in agreement with rules issued from *general principles* that are supposed to hold true for all properties and all bodies.

For example, it wants to show that if we know the law of pressure of the saturated vapour of a liquid, and the laws of compressibility and dilatation of a liquid and its vapour, then one can fix the law according to which the heat of vaporisation varies; to this purpose, its combines the first laws along the rules of the *principle of conservation of energy* and the *principle of Carnot*, principles that are supposed to apply to all bodies and all of their properties.

It is the system of these general principles that we propose to expose here.

For a long time, physicists have assumed that all properties of bodies reduced, in the last analysis, to combinations of figures [geometrical forms] and local motions. The general principles to which all physical properties must obey, were none other than the principles that govern the local motions, i.e., the principles that compose *rational mechanics*. Rational mechanics was the code for the general principles of physics.

The reduction of all physical properties to combinations of figures and motion or, following commonly used denomination, the mechanical explanation of the Universe, seems today to be condemned. It is not so for a priori reasons, whether metaphysical or mathematical. It is condemned because it has been so far just a project, a dream, and not a reality. Despite immense efforts, physicists never succeeded to conceive an arrangement of figures and of local motions that, treated following the rules of rational mechanics, give a satisfactory representation of a somewhat extended set of physical laws.

Is the attempt at a reduction of all physics to rational mechanics, an always vain attempt in the past, destined to succeed one day? Only a prophet could answer this question positively or negatively. Without prejudging the meaning of this answer, it appears wiser, provisionally, to renounce their efforts, fruitless until now, towards the mechanical explanation of the Universe.

Therefore, we are going to attempt at a formulation of the corpus of general laws to which all properties must obey, without assuming a priori that these properties are all reducible to a geometrical figure and a local motion. Accordingly, the corpus of these general laws will no longer reduce to rational mechanics.

In truth, the geometrical figure and the local motion remain physical properties; they are in fact those that are the most immediately accessible. Our corpus of general laws must apply to these properties, and, being applied to the latter, it must recover the rules that govern the local motion, the rules of rational mechanics. The latter must, therefore, result from the corpus of general laws that we propose to constitute; it must be what follows when we apply these general laws to particular systems where we account only for the figure of bodies and their local motion.

The code of the general laws of physics is known nowadays under two names: the name of *thermodynamics* and the name of *energetics*.

The name of *thermodynamics* is intimately attached to the history of this science; its two main principles, the principle of Carnot and the principle of conservation of energy, were discovered when studying the motive power in machines exploiting fire. This name is also justified by the fact that the two notions of work and quantity of heat are constantly at play in the reasoning through which this theory develops.

The name of *energetics* is due to Rankine[1]; the idea of *energy* being the first that this theory has to define, the one to which most other used notions are attached; this name seems to us as well chosen as that of *thermodynamics*.

Without deciding which naming is preferable to the other, we shall use both as equivalent to one another.

2. On the logical significance of the principles of energetics

The logical character of the principles that we are going to formulate and group together must be borne in mind[2].

These principles are pure *postulates*; we can state them as we please, on the condition that the statement of each of them is not self contradictory, and that the statements of the various principles are not in reciprocal contradiction.

The character with which we recognize that the *whole set* of the so formulated principles constitute a good theoretical physics is the following one: applying this set of principles to formulas that represent exact experimental laws, we can deduce new formulas which, in turn, represent other exact experimental laws.

The experimental control of the set of principles of energetics is thus the only criterion of truth of this theory.

Indeed, this control can be done only on the whole set of principles of energetics taken in its totality or, for the least, on extended parts of this set. It would be impossible to submit to the control of experiment one isolated among these principles, or even a small number of these principles. Any experiment, simple as it may be, involves in its interpretation very many and diverse principles. We will often have the opportunity to recognize this fact in the course of this exposition.

The experimental control can only concern the whole set of ultimate consequences of the theory; it estimates if this set of consequences provides, or does not provide, a satisfactory representation of these experimental data; but in so far as the theory has not produced the set of these last consequences, we cannot call for this control, as this would be premature; hence the following rule which will be of frequent usage in the sequel of our studies: *In the course of it exposition*, a theoretical physics is free to choose the path that it likes, in so far as it avoids any logical contradiction; in particular, it does not need to account for any experimental fact; it is only when it has reached the end of its development that its ultimate consequences can and must be compared to experimental laws.

To say that the principles of energetics are pure postulates and that no logical constraint limits our right to choose them arbitrarily, is not to say that we will formulate them by chance. On the contrary, we shall be very strictly guided in the choice of these statements, knowing well that it would suffice to alter any thing for the experimental check of the consequences to become at fault at some point.

We are assured by this guideline by our knowledge of the past of science. Principles have been formulated that were proved to be in gross contradiction with experiment; other principles were then substituted to them, which have received a partial confirmation, although an imperfect one; then they were modified, corrected, guaranteeing for each change a more exact agreement of their corollaries with facts. We are assured that the clothing of which we cut the forms will fit exactly the body it must dress because the blueprint has been tried and retouched many times.

Each of the principles that we shall state presents thus no logical proof; but it would carry a historical justification; we could, before stating it, enumerate the principles of differing forms that were tried before it, and which could not fit exactly reality, that we have been forced to reject or to correct until the whole system of energetics adapts in a satisfactory manner to the set of physical laws. The fear of an excessive length will forbid us the exposition of this historical justification.

Notes by Duhem

1. J. Macquorne Rankine, *Outlines of the science of energetics* (*Glagow Philosophical Society Proceedings*, Vol. III, no.6, 2 May 1855). – J. Macquorne Rankine, *Miscellaneous scientific papers*, p. 209.
2. We will limit ourselves to giving here a very concise résumé of what we expanded in the following book: *La théorie physique, son objet et sa structure*, Paris, 1906 [English translation: The aim and structure of physical theory, Princeton University Press, New Jersey, 1954; paperback reprint, 1991]. This work can be viewed as a kind of logical introduction to the present treatise.

Appendix B

Partial English Translation of P. Duhem, "L'évolution de la mécanique – Part VII – Les branches aberrantes de la thermodynamique", Revue genérale des sciences, pp. 416-429, Paris, 1903, by Gérard A. Maugin [Only small parts of this lengthy contribution are translated; Original footnotes, if any, are reported to the end and numbered consecutively. Translator's remarks are placed within square brackets in the main text. This is a verbatim translation without any ambition of literary prowess.

Duhem P. (1903), L'évolution de la mécanique (published in seven parts in: Revue générale des sciences, Paris; as a book, A. Joanin, Paris) [There exists already an English translation: The evolution of mechanics, Sijthoof and Noordhoff, 1980, to which we had no access].

L'évolution de la Mécanique

The Evolution of Mechanics

VII- The nonsensical branches of thermodynamics

I.- Friction and chemical false equilibria

From the original text, p. 418:

That outside systems of which the equilibrium states can always be classified in reversible changes, there exists an infinity of other systems of which the statics is not that of Gibbs, and the dynamics not that of Helmholtz, and that, among such systems, are precisely those exhibiting friction?

Therefore, the laws according to which systems with friction evolve or remain in equilibrium, require a specific formulation. This formulation, we will not ask it to chance. The formulation imposed to statics by Gibbs and to dynamics by Helmholtz was shown to be admirably fruitful; it is natural to conserve its type as much as possible; to deduce the new formulation from the old one by means of additions and modifications as light as possible; this is the idea that guided us when we built the mechanics of systems with friction.

It would not be easy to expose the latter with entering details that the present writing should not involve. However, let us try to draw a summary sketch and, to that purpose, we restrict ourselves to the study of a system such that only one normal variable, apart from temperature, suffices for its definition. Let α represent this unique variable. If F, A, J, v are, respectively, the internal potential [energy], the external action, the inertial force, and the action of viscosity, then according to Helmholtz' dynamics at each instant we can write the equality

$$A + J + v = \frac{\partial F}{\partial \alpha}. \tag{3}$$

This equality, the general law of motion [probably "evolution" would be better] of the system, implies the law of its equilibria, a law in conformity with Gibbs' statics.

The equilibrium of systems with friction does not agree with Gibbs' statics; Equality (3) does not apply to them; but we can try to modify it in such a way that it will be extended to such systems.

To that purpose, we continue to attach to each state of the system a quantity F that is determined without ambiguity through the knowledge of this state. To this quantity that we still call internal potential, we will continue to attach internal energy and entropy by means of previously known relations; the external action, the inertial force and the action of viscosity will remain defined just as before; but these elements will no longer be sufficient to set forth the equation governing the system. It will be necessary to know a new element, the action of friction f.

This action, always positive, will depend, just like the action of viscosity, on the absolute temperature, the variable α and the general velocity $\dot{\alpha} = d\alpha/dt$; but contrary to what happens for the generalized velocity, it will also depend also on the external action A; furthermore, it will not vanish with the generalized velocity; the latter going to zero, the action of friction will tend to a positive value g.

In order to govern the motion of the system, we will no longer have a unique equation, but two distinct equations; the first of these should be used only when the generalized velocity $\dot{\alpha} = d\alpha/dt$ is positive; it will take the following form:

$$A + J + v - f = \frac{\partial F}{\partial \alpha}. \tag{4}$$

The second of these equations will read:

$$A + J + v + f = \frac{\partial F}{\partial \alpha}. \tag{4b}$$

This one will be reserved to the case when the generalized velocity $\dot{\alpha} = d\alpha/dt$ is negative.

As to the equilibrium condition, it will no longer be represented by an equality, but by a double inequality that expresses that the absolute value of the difference $A - \partial F/\partial\alpha$ is not larger than g:

$$-g \leq A - \frac{\partial F}{\partial \alpha} \leq g. \tag{5}$$

We rapidly go over the equation of living forces [equation of kinetic energy]; we can only repeat here practically all what was said when studying the dynamics of Helmholtz; it is only necessary to add the work of friction to the work of viscosity. The former, like the latter, is always negative. We also do not deal with the Clausius inequality which remains exact in the new dynamics. Here also, the work of friction is just being added to the work of viscosity. Other consequences of the laws just formulated, and more particularly the condition of equilibrium, will require a little more attention.

Gibbs' statics would require the difference $A - \partial F/\partial\alpha$ to vanish, and therefore having value between $-g$ and $+g$. The equilibrium states predicted by this Statics, and that are usually called states of *true equilibrium*, are thus among those that are predicted by the new Statics; But the latter announces the existence of an infinity of other equilibrium states, that we designate by the name of *false equilibria*.

If the value of g is large, then the states of false equilibrium spread on both sides of those of true equilibrium, in a large domain. They will shrink close to the states of true equilibrium whenever the value of g is small. If this value becomes sufficiently small, then the states of false equilibrium will be so close to those of true equilibrium that experiments would no longer distinguish them; practically, the Statics of systems with friction would be undistinguishable from Gibbs' statics.

This is only a particular application of the following remark: Gibbs' Statics and Helmholtz' Dynamics are limit forms of the Statics and Dynamics of systems with friction; these tend to those when the action of friction becomes infinitesimally small.

This remark is not a simple view of the mind; it acquires a particular interest in the study of chemical equilibria[1].

Note 1. We have given an exposition of the theory of chemical equilibria accounting for friction and the principal applications of this theory in the following works: *Théorie thermodynamique de la viscosité, du frottement et des faux équilibres chimiques*, Paris, 1896 – *Traité élémentaire de mécanique chimique fondée sur la thermodynamique*, Vol. II, T. I, Paris, 1897; *Thermodynamque et chimie, leçons élémentaires à l'usage des chimistes*; Leçons XVII, XIX and XX, Paris, 1902.

[In the rest of this Chapter Duhem expands an example from chemical physics].

II. Permanent alterations and hysteresis

[Here Duhem first gives a general idea of what permanent alterations are. He emphasizes the role of infinitesimally slow evolutions, adapting accordingly temperature and external actions]

……

[From page 421]

The theory of systems capable of permanent alterations will thus be distinct from general mechanics for which, with Gibbs and Helmholtz, we have sketched the principles; but it will also differ from the mechanics of systems with friction; it will be a new branch of mechanics.

How is this new mechanics to be constituted?

Only the principal thought is of concern to us here; the detail of formulas is not needed; we restrict ourselves to the study of a simple case that will allow for a better transparency of the frame of ideas. As object of our analysis, let us choose a system defined by a single normal variable, apart from temperature; for example a thread [wire] under tension of which the length will be this normal variable, while the pulling weight will be the corresponding external action.

First let us give certain infinitesimal variations to the temperature and the pulling weight; the length of the thread will suffer an infinitesimally small increase. Then let us give to both temperature and pulling weight variations equal in absolute value to the preceding ones, but with opposite sign so that these two quantities recover their original value. The length of the thread is reduced, but this decrease does not have the same absolute value as the preceding lengthening, because the thread suffers a permanent deformation. Thus, in the course of an infinitely slow alteration, a linear algebraic relation determines the infinitely small variation of the length of the thread when we impose infinitely small variations to temperature and pulling weight; but this relation must not have the same form when the thread elongates or when it contracts; a certain equality must be written when the normal variable suffers a positive variation, and another one when this variation is negative.

What guide will help us to discover the form of these two equalities? It is the theory itself, which cannot be sufficient to treat permanent alterations, but which proved to be so fruitful in the study of systems with reversible modifications. We shall look for a construction of this new mechanics in such a way that it is as close as possible to that theory, that it follows from it by a very slight transformation, that it be one of its generalizations, that the Statics and Dynamics of systems admitting no permanent alterations be regarded as limit forms of the Statics and Dynamics of systems with very weak permanent alterations. In a nutshell, we shall follow the same method as that which was given by the theory of systems with friction.

When a system presenting no permanent alteration is subjected to an infinitely slow modification, i.e., a reversible evolution, the equilibrium conditions are satisfied at each instant; if the state of the system depends on a single normal variable α, then at each instant the external action A equals the derivative of the internal potential F with respect to α; this we are taught by equality (1) [i.e., $A = \partial F/\partial \alpha$]

For coordinated infinitely small variations of the temperature, the external action and the normal variable, there therefore exists the relation

$$dA = d\frac{\partial F}{\partial \alpha}, \tag{6}$$

by virtue of the fact that the always equal quantities A and $\partial F/\partial \alpha$ suffer simultaneously equal increases. According to this relation, if we change the sign of these variations without changing their absolute value, then we change the sign of the variation suffered by the normal variable without changing its own absolute value; this way we express the reversibility of the infinitely slow modification.

These peculiarities cannot be met in a system capable of permanent alterations; each of the elements of which the succession composes an infinitely slow modification cannot be governed by equality (6); we must substitute to this equality two distinct relations, one valid when the normal variable increases, and the other valid when this variable decreases.

In the first case, we substitute to equality (6) the relation:

$$dA = d\frac{\partial F}{\partial \alpha} + h\, d\alpha, \tag{7}$$

In the second case, we substitute to (6) the relation

$$dA = d\frac{\partial F}{\partial \alpha} - h\, d\alpha. \tag{7b}$$

The quantity h, of which the introduction in these equations distinguishes systems capable of permanent alterations from those not capable of these, depends on the state of the system, and also on the external action A.

It is obvious that it suffices to give to this quantity h a very small value so that inequalities (7) and (7b) differ very little from the equality (6); the permanent alterations of the system then are very little sensitive, and its infinitely slow modifications are almost reversible; thus, systems without permanent alterations and capable of reversible modifications are precisely limit forms of systems subjected to small permanent alterations.

For systems without permanent alterations, a simple rule allows us to deduce from the internal potential the knowledge of the internal energy, and thus, the quantity of heat involved in an infinitely slow modification. Nothing forbids the extension of this rule to systems with permanent alterations. Joint to what was previously given, it will provide the essential principles on which the Statics of such systems relies[2]. With some accessory hypotheses, all inspired by the desire to make the small branch ("rameau") of Thermodynamics as similar as possible as its master branch, this will complement these principles.

Note 2. We have devoted six memoirs to this Statics under the general title: Les déformations permanentes et l'hystérésis (*Mémoires in-4° de l'Académie de Belgique*, t. LIV, 1895; t. LVL, 1898; t. LXII, 1902) and eight memoirs published under the title: Die dauernden Aenderungen und die Thermodynamik, *Zeitschrift für physikalishe Chemie*, Bd. XXI, XXIII, 1897; XXVIII, 1899; XXXIV, 1900; XXXVII, 1901), etc.

What are the applications of this new Statics?

A first category of permanent alterations is formed by elastic deformations. The traction, torsion and flexion cause deformations that do not disappear with the cause that produced them; these deformations, known and observed at all times since Antiquity, find in the above given principles, their theoretical explanation.

[Here Duhem mentions examples from residual magnetization, magnetic hysteresis, and analogous properties for electric polarisation in dielectrics].

……

[continued from p. 422].

Essential as it is in the study of elasticity and the theory of magnetism, hysteresis is destined to play a very important role in "chemical mechanics" ……

It is without doubt to permanent alterations of this kind that must be attached the effects of tempering, annealing, and hardening that complicate so strangely the study of metals and their industrial combinations. Quite often, these effects result from both elastic hysteresis and chemical hysteresis; only the simultaneous consideration of these two hystereses, can somewhat untangle these phenomena, in appearance inextricable, that present some bodies; such as Nickel based steels of which M. Ch.-E. Guillaume has analyzed the strange properties, or the platinumsilver alloy of which the electric resistance manifests so curious residual variations, as observed by M. H. Chevallier.

This superposition of chemical hysteresis and elastic hysteresis makes the laws of dilatation of glass singularly complex; the observation of the displacement of the zero point of thermometers had not revealed, first to Desprez and then to M. Ch.-Edmond Guillaume much more than this extreme complexity; numerous and patient measurements, guided by the thermodynamics of permanent modifications, have finally allowed M. L. Marchis to put some order in this chaos [Guillaume, Chevallier, Desprez and Marchis were doctoral students of Duhem in Bordeaux].

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[continued from page 423]

……

In a system subjected to permanent alterations, the quantity h, that we name *coefficient of hysteresis*, does not vanish in general. The two equalities (7) and (7b) are thus distinct from one another; if we suppose that the system suffers, with an infinite slowness an infinitely small alteration due to certain variations in the temperature and the external action, we will not be able, by inverting those variations, to reverse the modification and to bring the system back to its initial state.

But what is not generally true may become true in certain particular cases; by associating in an appropriate manner the values of the normal variable, of temperature and of the external action, we can make the coefficient of hysteresis vanish; when these values are thus associated, we say that the system is placed in a *natural state*; in general, if we take the system in any state, defined by a certain value of the normal variable and a certain value of temperature, we can submit it to an external action so that this state becomes a natural one.

[Here Duhem goes more deeply in the notion of natural state and that of residual fields, and then he discusses the possible dynamics in which the standard application of d'Alembert's principle is at fault, referring to works by Henri Bouasse (1866-1958; a French physicist in Toulouse, South-west of France) and the noted German physicist Max Wien (1866-1938)].

III. Electrodynamics and electromagnetism

[In this long section, pp. 424-427, Duhem considers the general thermodynamics of electromagnetic bodies. He ponders the notions of electromagnetic energy, electric displacement, electromagnetic induction, electrodynamic forces, properties of the system which have no inertia associated with the relevant variables, in spite of the existence of generalized velocities duly associated with such variables, electrodynamic potential, electrokinetic energy, Ohm's effect considered as a viscosity. He pays an emphatic tribute to Helmholtz. He clearly expresses his irreconcilable appraisal of Maxwell's vision. This, of course, appears to be outdated and certainly not very objective].

***Conclusion* (pages 427-429).**

[Here Duhem offers a rather literary synthesis of the contents of the whole work. He goes all the way to compare the construction of his successive theories to the "sanguine sketches" of Raphael on exhibit at the Louvre [museum] where you can see the work by successive approximations of this master painter, starting from a rough sketch and then improving the details at each successive picture and finally producing a masterpiece that finally causes the admiration of all. This is the way his new Mechanics, unique though complex, emerges from his own mind].

[continued from page 428, 2nd column]:

The old Mechanics pushed to the extreme the simplification of its fundamental hypotheses. It had condensed these hypotheses in a unique presupposition: All systems are reducible to a set of material points and solid bodies which move in agreement with Lagrange's equations. And even more with Hertz, it went further by erasing real forces from its equations.

The new Mechanics [i.e., Duhem's] is not imbued of such a simplification of its principles; it does not avoid to increase the complication of its fundamental hypotheses; it admits terms of varying nature and form in its equation, terms of viscosity, friction, hysteresis, electro-kinetic energy, while the old Mechanics excluded from its formulas such symbols, as in contradiction with its unique principle.

But reality is more complex, infinitely so; each new improvement in experimental methods, by scrutinizing more thoroughly the facts, discovers in them new complications. Human mind, in its weakness, although trying hard to work toward a simple representation of the external world, suffices to place the image in front of the object and to compare them in good faith in order to realize that this simplicity, so forcefully desired, is an un-captured chimera, an unrealizable utopia.

……

[continued from page 429, first column]:

This capability to mould facts and to capture their finest detail was acquired by the new Physics by giving up some of the requirements that rigidified the old Mechanics. Among these requirements, the first and most essential one was the one that intended to reduce all properties of bodies to quantities, figures and local motions; the new Physics rejects totally this requirement; it admits in its reasoning the consideration of these qualities; it endows the notion of motion with the generality that Aristotle granted to it. This is the secret of its marvellous compliance. Indeed, with this it gives up the consideration of hypothetical mechanisms that the natural philosophy of Newton disliked, the research of the masses and hidden motions of which the only object is to explain the qualities in geometrical terms. Freed from this work that Pascal proclaimed uncertain, painstaking and useless, it can, in all freedom, devotes its efforts to more fruitful endeavours.

……

The creation of this Mechanics, based on thermodynamics, is thus a reaction against atomistic and Cartesian ideas – not foreseen by those who most contributed to it – to the deepest principles of peripatetician doctrines.

……

The expansion of Mechanics is thus a true *evolution*; each stage of this evolution is the natural corollary of previous stages; it is pregnant of future stages. The meditation of this law must me the comfort of the theoretician. It would be presumptuous to imagine that the system toward which he contributes will escape the common fate of systems that were before, and will deserve to last longer than them; But, without any vain verbiage, he is right in thinking that his efforts will not be sterile, that across centuries the ideas that he sowed and made germinate, will continue to grow and bear fruits.

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References

1. Appell P (1921) Traité de mécanique rationnelle, 3rd edn. Gauthier-Villars, Paris (facsimilé reprint by Gabay, Paris, 1991)
2. Ariew R (2007) Entry «Pierre Duhem» in: the open-access Stanford Encyclopedia of Philosophy (revised 2011)
3. Bachelard G (1934) Le nouvel esprit scientifique. Editions Alcan, Paris
4. Bachelard G (1938) La formation de l'esprit scientifique. Librairie Vrin, Paris
5. Bachelard G (1940) La philosophie du non: Essai d'une philosophie du nouvel esprit scientifique. Presses Universitaires de France, Paris
6. Born M (1921) Kritische Betrachtungen zur traditionellen Darstellung der Thermodynamik. Phys Z 22:218–224, 249–254, 282–286
7. Bouc R (1971) Modèle mathématique d'hystérésis. Acustica Acta 24:16–25
8. Bridgman PW (1943) The nature of thermodynamics. Harvard University Press, Cambridge, Mass
9. Brouzeng P (1987) Duhem 1861–1916: science and providence. Editions Belin, Paris
10. Capecchi D (2012) History of virtual work laws. In: The science networks historical studies series, vol 42. Birkhäuser-Springer, Milano
11. Clagett M (1959) Science of mechanics in the Middle Ages. University of Wisconsin Press, Madison, Wis
12. Coleman BD, Gurtin ME, Herrera I, Truesdell C (1965) Wave propagation in dissipative materials. Springer, New York (collection of reprints of papers)
13. Cosserat E, Cosserat F (1896) Sur la théorie de l'élasticité. Ann Fac Sci Toulouse, 1ere série 10(3–5):I1–I116
14. Cosserat E, Cosserat F (1909) Théorie des corps déformables. Hermann, Paris, 226 pages [Reprint by Editions Gabay, Paris, 2008; Reprint by Hermann Archives, Paris, 2009; English translation: N68-15456: Clearinghouse Federal Scientific and Technical Information, Springfield, Virginia NASA, TT F-11561 (February 1968); another translation by D. Delphenich, 2007] (originally published as supplement in pp 953–1173 to: Chwolson O. D., (1909), Traité de physique (traduit du Russe), vol II. Hermann, Paris)
15. Duhem P (1886) Le potentiel thermodynamique et ses applications à la mécanique chimique et à la théorie des phénomènes électriques. Hermann, Paris (xii+248 pages)
16. Duhem P (1887) Etude sur les travaux thermodynamiques de J. Willard Gibbs. Bull Sci Mathématiques, 2nd series 11:122–148, 159–176
17. Duhem P (1891) Hydrodynamique, élasticité, acoustique, vol 2. Hermann, Paris, 688 pages

18. Duhem P (1893) Le potentiel thermodynamique et la pression hydrostatique. Ann Ecole Norm 10:187–230 (Paris)
19. Duhem P (1893) Introduction à la mécanique chimique. Ad. Hoste, Gand, Belgium, 188 pages
20. Duhem P (1896) Théorie thermodynamique de la viscosité, du frottement, et des faux équilibres chimiques. Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux, 5th series 2:1–208
21. Duhem P (1897–1899) Traité élémentaire de mécanique chimique fondée sur la thermodynamique, Four volumes. Hermann, Paris
22. Duhem P (1901) Sur les déformations permanentes et l'hystérésis, Part 6. Mém Acad Roy Belgique 62:1–33; Part 7, ibid, 33–136
23. Duhem P (1902) Thermodynamique et chimie: Leçons élémentaires à l'usage des chimistes. Hermann, Paris, x+496 pages. English edition: Duhem P (1903) Thermodynamics and chemistry: a non-mathematical treatise for chemists and students of chemistry (trans: Burgess GK). Wiley, New York
24. Duhem P (1903) L'évolution de la mécanique (published in seven parts in: revue générale des sciences, Paris; as a book: A. Joanin, Paris). English edition: Duhem P (1980) The evolution of mechanics (trans). Sijthoof and Noordhoff
25. Duhem P (1906) La théorie physique. Editions Chevalier et Rivière, Paris. English edition: Duhem P (1954) The aim and structure of physical theory (trans). Princeton University Press, New Jersey (paperback reprint, 1991)
26. Duhem P (1906) Recherches sur l'élasticité. Gauthier-Villars, Paris (collection of four papers with a total of 218 pages)
27. Duhem P (1908) Sozein ta phainomena, French edition. Hermann, Paris. English edition: Duhem P (1969) To save the phenomena (trans). University Press, Chicago
28. Duhem P (1911) Traité d'énergétique ou de thermodynamique générale, vol 1 (528 pages), vol 2 (504 pages) Gauthier-Villars, Paris (Reprint by Gabay, Paris, 1997)
29. Duhem P (1913) Notice sur les titres et travaux de Pierre Duhem [self-description of his works by P. Duhem when applying for full membership in the Paris Academy of Sciences), Bordeaux (printed posthumously in Mém Soc Sci Phys et Nat de Bordeaux, 7ème série, pp 71–169, 1927). English edition: Duhem P (1987) An account of the scientific titles and works of Pierre Duhem (partial trans: Murciano Y, Schramm L). Sci Context 2:333–348
30. Edelen DGB (1976) A global formulation of continuum physics and the resulting equivalent classes of nonlocal field equations. In: Rogula D (ed) Nonlocal theories of material systems. Ossolineum, Wrocław, Poland, pp 41–79
31. Ericksen JL (1984) Thermodynamics and stability of equilibrium. In: Truesdell CA (ed) Rational thermodynamics, 2nd edn, Appendix G3. Springer, New York
32. Ericksen JL (1991) Introduction to the thermodynamics of solids. Chapman & Hall, London
33. Eringen AC, Maugin GA (1990) Electrodynamics of continua, vol 2. Springer, New York
34. Gibbs JW (1876–1878) On the equilibrium of heterogeneous substances, Parts 1 and 2. Trans Connecticut Acad Arts Sci 3:108–248 (1876); 3:343–520 (1878)
35. Hadamard J (1903) Leçons sur la propagation des ondes et les équations de l'hydrodynamique. Hermann, Paris
36. Hadamard J (1927) L'oeuvre mathématique de Pierre Duhem. In: Special issue of Mém Soc Sci Phys Na. Bordeaux, 7ème série, pp 637–665
37. Halphen B, Nguyen QS (1975) Sur les matériaux standards généralisés. J Mécanique 14:39–63 (Paris)
38. Hellinger E (1914) Die allgemein Ansätze der Mechanik der Kontinua. In: Klein F, Wagner K (eds) Enz Math Wiss, vol 4, Part 4, pp 602–694, Springer, Berlin
39. Humbert P (1932) Pierre Duhem. Paris
40. Jaki SL (1984) Uneasy genius: the life and works of Pierre Duhem. Martinus Nijhoff, The Hague
41. Jouguet E (1905–1906) Sur la propagation des réactions chimiques dans les gaz (On the propagation of chemical reactions in gases). J de Mathématiques Pures et Appliquées, Series 6, 1:347–425 (1905); continued in 2:5–85 (1906)

42. Jouguet E (1917) Mécanique des explosifs, étude de dynamique chimique. Paris
43. Koiré A (1957) From the closed world to the infinite universe (English original). French edition: Koiré A (2003): Du monde clos à l'univers infini (trans: Tarr R). Gallimard, Paris
44. Kuhn TS (1962) The structure of scientific revolutions. University of Chicago Press
45. Lenoble M (1900) Contribution à l'étude des déformations permanentes des fils métalliques; allongement par traction. Doctoral Thesis, Bordeaux
46. Mach E (1911) The science of mechanics; a critical and historical account of its development. Open Court, Chicago (1911; paperback 1988) (Original German edition: F. A. Brockhaus, Leipzig, 1883) [Analysed by P. Duhem in: Bulletin des Sciences Mathématiques, 27:261–283, 1903]
47. Manville O (1927) La physique de Pierre Duhem. Mém Soc Sci Phys Nat Bordeaux, 7ème Série, 171–636 (The most comprehensive and unsurpassed analysis of Duhem's scientific works by one of his direct disciples)
48. Massieu FJD (1869) Sur les fonctions caractéristiques des divers fluides. C R Acad Sci Paris 69(858–864):1057–1061
49. Maugin GA (1988) Continuum mechanics of electromagnetic solids. North-Holland, Amsterdam
50. Maugin GA (1992) The thermomechanics of plasticity and fracture. Cambridge University Press, UK
51. Maugin GA (1992) Thermodynamics of hysteresis. In: Sieniutycz S, Salamon P (eds) Advances in thermodynamics, vol 7, "Extended thermodynamic systems". Taylor and Francis, New York, pp 25–52
52. Maugin GA (1994) Helmholtz's Influence on Pierre Duhem and our view of mathematical physics. In: Ebeling D, Muschik W (eds) Helmholtz-Feier (Centennial of Helmholtz's Death, Sept 1994, Berlin). TU Berlin and Humboldt Universität, Berlin, 13 pages
53. Maugin GA (1999) Thermomechanics of nonlinear irreversible behaviours. World Scientific, Singapore
54. Maugin GA (2013) Continuum mechanics through the twentieth century: a concise historical perspective. Springer, Dordrecht
55. Maugin GA, Pouget J, Drouot R, Collet B (1993) Nonlinear electromechanical couplings. Wiley, New York
56. Miller DG (1970) Duhem, Pierre-Maurice-Marie. In: Gillispie CC (ed) Dictionary of scientific biography, vol 3. Charles Scribner's Sons, New York, pp 225, 233
57. Moreau JJ (1971) Sur les lois de frottement, de viscosité et de plasticité. C R Acad Sci Paris 271:608–611
58. Picard E (1921) La vie et l'oeuvre de Pierre Duhem. Eloge lue dans la séance publique annuelle du, 12 décembre 1921. C R Acad Sci, LVII: C–CXLII
59. Pierre-Duhem H (1936) Un savant français: Pierre Duhem. Paris
60. Prigogine I, Defay R (1954) Chemical thermodynamics. New York
61. Rankine WJM (1853) On the general law of the transformation of energy. Proc Philos Soc Glasgow 3(5):276–280 (Reprinted in: Philosophical magazine, series 4, 5(30):106–117, February 1853)
62. Rankine WJM (1855) On the science of energetics. Proc Philos Soc Glasgow 3:381–399
63. Sredniawa B (2007) Władysław Natanson (1864–1937). Concepts Phys 4:705–723 (Krakow)
64. Truesdell CA (1961) General and exact theory of waves in finite elastic strain. Arch Rat Mech Anal 8:263–296
65. Truesdell CA (1984) Rational thermodynamics, 2nd enlarged edn. Springer, New York
66. Truesdell CA, Noll W (1965) The non-linear field theories of mechanics. In: Flügge S (ed) Handbuch der Physik, Bd.III/3. Springer, Berlin (Referred to for short as *NFTM*)
67. Truesdell CA, Toupin RA (1960) The classical field theories. In: Flügge S (ed) Handbuch der Physik, Bd. III/1. Springer, Berlin (Referred to for short as *CFT*)
68. Ziegler H (1986) An introduction to thermomechanics. North-Holland, Amsterdam

Chapter 11
A Course of Continuum Mechanics at the Dawn of the Twentieth Century (Volume III of Appell's Treatise on Rational Mechanics)

Abstract The treatise on *rational mechanics* published in French by Paul Appell starting in 1900 is a unique monument in the mathematical literature of the Pre-World War One period. Here we critically peruse the volume devoted to continuum mechanics (Volume III). This critical examination is performed in the light of what was known at the time, what were the fashionable themes in continuum mechanics in the early twentieth century, what mathematical techniques were preferred, and what was the naturally influential environment (especially among French mathematicians). All these gave a special tune and contents to a treatise that bears the print of its time, especially with an emphasis on subject matters such as potential theory, the consideration of complex variables, the interest for vortices, barotropic and "barocline" fluids, and new notions such as those put forward by J. Hadamard in wave propagation, by H. Villat and V. Bjerknes in fluid mechanics, and the many references to contemporary works by J.V. Boussinesq, A. Barré de Saint-Venant, H. Poincaré, P. Duhem, and the Cosserat brothers. In contrast, we note the few references to foreign works, the non-exploitation of the then recently proposed vectorial and tensorial concepts, and the lack of interest in dissipative behaviours, whether in fluids or in solids, this in accord with the bannered "rationality" of the treatise.

11.1 Prolegomena: On Paul Appell

The mathematician and mechanician Paul Appell (1855–1930) epitomizes the successful scientist of the French third Republic at the "Belle époque" (i.e. the 20 years before World War One): he presents two parallel careers, one as a true scientist and pedagogue and the other as a man of power in the educational system together with an involvement in some political and socio-cultural matters (see the Appendix). Along the first line, he is not as powerful a creative mathematician as his contemporaries Jacques Hadamard (1865–1963) and Paul Painlevé (1863–1933)—and of course Henri Poincaré (1854–1912), obviously an apart genius. He is more

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traditional in his choice of problems and implementation of solutions than Hadamard who later on became influential in the creation of the Bourbaki School. On the other hand, Painlevé went much farther than Appell in his political involvement going all the way as to become prime minister during World War One and even to be a candidate to the presidency of the Republic. From the point of view of applications Appell was closer to civil engineering while Painlevé, although not himself directly involved in such developments, demonstrated a true interest in the recent progress of aeronautics (he created a Ministry of Aviation). Despite and perhaps because of all these objective facts, Appell wrote the most advanced treatise (the object of this contribution) for the period in what was recognized as the utmost field of application of mathematical analysis, rational mechanics. No other treatise of this magnitude was ever published in France after this opus.

11.2 Setting the Stage

Our object of study here is the celebrated "Treatise on Rational Mechanics" of which the first edition was published in French in 1900 after teaching of this matter by Appell for fifteen years at the Sorbonne. A second edition was published in 1909, a third in 1921, a fourth in 1926, a fifth in 1932, and a sixth published in 1941 and edited by Georges Valiron (1884–1955).[1] A final volume V was published later on but essentially written by René Thiry (1886–1968), himself professor of mechanics at the Sorbonne. The present study refers to the third edition as published by Gauthier-Villars in Paris and reprinted in facsimile form by Editions Gabay, Paris, 1991. This edition, together with the supplement written by the Cosserat brothers for the second edition of 1909 and joined by Gabay to this reprint, is selected as being the most representative one, providing a snapshot of our field as of the relevant period, almost a hundred years ago. We have commented elsewhere [26, Chap. 2] about the state of general mechanics, and more particularly continuum mechanics at the time. This is characterized as a transition period between the contributions of the French, English and German "classics" (e.g., Cauchy, Navier, Duhamel, Lamé, Green, Gauss, Stokes, Maxwell, Kelvin, Kirchhoff, Neumann, Clebsch, Barré de Saint-Venant) and the twentieth century (post WWI). This period is permeated by a feeling of achievement and fulfilment in spite of the emerging quantum physics and relativistic mechanics and the very attractive notion of elementary particle. Furthermore, this is accompanied by a mature reflection on the bases of mechanics by such scientists-philosophers as Hertz, Mach, Poincaré, Duhem and Hamel. Here we focus attention on our specialty, continuum mechanics, as presented in Volume Three entitled "Equilibre et mouvement des milieux continus" ("Equilibrium and motion of continuous

[1] G. Valiron is a scientific "grand father" of the writer via Paul Germain according to Mathematical Genealogy.

media"), an opus of some six hundred seventy pages (without the Cosserats' supplement). These are Chaps. XXVIII–XXXIX of the total treatise. It can only be compared to the long article by C. A. Truesdell and R. A. Toupin published in 1960 in the *Handbuch der Physik* edited in Germany by S. Flügge [34]. These authors in fact often cite Appell's treatise.

Inevitably, Appell's exposition refers to the above cited "classic" scientists of the early and mid nineteenth century as definite contributors. These famous personalities are usually referred to simply by their family name or a name attached to a theorem or a typically solved or simply proposed problem. But Appell must also account for some of his contemporary fellow scientists. Among them we identify Joseph Bertrand (1822–1900; mathematician), Gaston Darboux (1842–1917, specialist of the theory of surfaces), François-Félix Tisserand (1845–1896; specialist of celestial mechanics), Joseph V. Boussinesq (1842–1929; all round mechanician of the continuum), François Cosserat (1852–1914, mathematician and astronomer), Eugène Cosserat (1866–1931; civil engineer), Léon Lecornu (1854–1940; professor of mechanics at *Ecole Polytechnique* and himself author of a treatise on mechanics published in three volumes in the period 1914–1918), Marcel Brillouin (1854–1948; mechanician and physicist), Henri Poincaré (1854–1912; the innovative great mathematician), Emile Picard (1856–1941; famous analyst), Gabriel Koenigs (1858–1931; also the author of a known course on mechanics), Henri Bénard (1874–1933; physicist of "convection" fame), Victor Robin (1855–1897; known for his boundary condition), Pierre Duhem (1861–1916; mathematical physicist, epistemologist and historian of science), Jacques Hadamard (1865–1963; the prototype of absentminded mathematician), and Henri Villat (1879–1972; specialist of fluid mechanics). Poincaré,[2] Duhem, Brillouin, Hadamard, Villat and the Cosserat brothers are the most frequently cited authors. Very few foreign authors of this period are cited, but we note the recurring names of Lord Rayleigh (1842–1919), Elwin B. Christoffel (1829–1900), Eugenio Beltrami (1835–1899), Enrico Betti (1823–1892), Tullio Levi-Civita (1873–1941), Alfred B. Basset (1854–1930), A. E. H. Love (1863–1940; the celebrated elastician), Kazimierz Zorawski (1866–1953), and V. Bjerknes (1862–1951; Norwegian fluid dynamicist, designer of climate models and meteorologist). Appell will welcome in his treatise lengthy contributions by the Cosserats, Villat and Bjerknes. All these male authors are usually referred to in the text by a respectful "M." (for *Monsieur*) and not mentioning their first names. There is no need to introduce a corresponding abbreviation for female authors who are here inexistent (although Sophie Germain and Sofia Kovaleskaya as past authors may have been included at some point).

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[2] The reader may be somewhat surprised to see so many references to works by Poincaré on continuum mechanics. In truth, Poincaré delivered at the Sorbonne many one-semester lecture courses on various aspects of continuum physics, renewing the contents every year and bringing each time a synthetic and critical view illustrated by worked out problems. These lectures were usually put in book form by some of the—obviously very few—auditors (see Poincaré [28–30]) while Poincaré was busy with his own deep mathematical researches.

11.3 The Contents of Appell's Volume on the Mechanics of Continua

11.3.1 Some Words of Introduction

It must be realized that Appell's treatise is a formidable enterprise with very few competitors in the world save for some volumes of the German Encyclopaedia of Mathematics (Enz. Math. Wiss.) edited by Felix Klein and Conrad H. Müller early in the twentieth century—but written by a large group of authors—, and obviously the chapters on mechanics in the *Handbuch der Physik* edited by S. Flügge in the 1950s–1960s, but also written by many contributors (among them: C. A. Truesdell, R. A. Toupin, W. Noll, R. Berker, etc.) [33, 34]. Volume III of Appell's is squeezed between Volume I (devoted to statics and dynamics of the point, about 600 pages) and Volume II (devoted to the dynamics of systems and analytical mechanics; about 570 pages) on the one hand, and Volume IV (concerning equilibrium figures of homogeneous and heterogeneous liquid masses, Figures of the Earth and planets; about 630 pages), on the other hand, all this complemented by a Volume V published later on (1933 and 1955; about 200 pages) and devoted to Elements of Tensor calculus, planned by Appell—for the edition of 1926—but effectively written by René Thiry (1886–1968), a student of H. Villat and a professor of mechanics at the Sorbonne. This last volume of great value includes elements of Riemann, Weyl, Eddington and Cartan geometries, all of interest in general relativity and other geometric generalizations of gravitation theory as also in the theory of structural defects (for Cartan's spaces with torsion). In all, this adds up to some two thousands and five hundred pages! But we analyze here only Volume III and its supplement by the Cosserats.

The overall redaction of this part of the treatise is lengthy, quite detailed, accompanied by exercises with given solutions, and historical comments and up dated references to contemporary works. It is clear that arguments used in derivations were polished in the course of many years of teaching. In the case of the mechanics of deformable solids problems are clearly related to civil engineering. In the case of fluid dynamics there is no allusion to the need of theoretical considerations in the emerging aerodynamics as a mathematical branch of hydrodynamics so that a critical work like that of Prandtl [31] on the boundary layer is completely missed. But there is a sure interest in the theory of barotropic fluids and in the propagation of discontinuity waves of which shock waves are the most well known examples. This is probably due to the influence of Hadamard for whom Appell clearly manifests a strong admiration. Dissipative processes are hardly mentioned (e.g., for viscous fluids or plasticity in the cased of solids). This correlates well with the marked interest of Appell for the theory of potential, the realm of nondissipative mechanical behaviours. All this seems to be the general background and framework in which Appell's treatise must be appreciated and sometimes contrasted with.

11.3.2 Vector Analysis and Potential Theory

The first two chapters of Volume III of the treatise (Chaps. XXVIII and XXIX)—viewed as some kind of prerequisites—bear a strong print of Newtonian and nineteenth-century mathematical physics. They naturally call for the proof of standard theorems of what is now called vector analysis: Green's and Stokes' theorems and Green function, and their application to potential theory, a favourite subject of Appell, with applications to Newtonian attraction, Ampère's theory of magnetism, Faraday's, Maxwell's and Gauss' works. In potential theory proper, emphasis is placed on harmonic functions, Dirichlet's principle, and Dirichlet and Neumann boundary conditions. An example of detailed treatment is that granted to the computation of a potential due to a homogeneous ellipsoid (after Dirichlet). References are often made to the courses on analysis by French contemporary mathematicians such as Poincaré, Boussinesq and Bertrand. Examples of applications are borrowed from Tisserand (in celestial mechanics), Poincaré (theory of the potential) and Kelvin, Maxwell (including from this author's treatise on magnetism and electricity), Lipschitz, Riemann, Sommerfeld, Levi-Civita and Betti (for applications in heat conduction, electromagnetism and mechanics).

These chapters deal with vector fields, but without the relatively new convenient intrinsic vector notation of Gibbs and Heaviside. As a matter of fact, Appell is conscious of this shortcoming as he refers (pp. 26–27) to this "special notation" for which he advises the reader to look at the recent French translation of the book by Coffin [12] (an ardent advocate of this "notation") and also the Italian authors Burali-Forti and Marcolongo.[3] This clearly indicates the little enthusiasm demonstrated by French physicists, engineers and mathematicians—since no true French reference on the subject can be cited by Appell—and the frequent difficulty met by French scientists to readily adopt in their lectures advances and notations proposed by foreigners (this also applies to Maxwell's electrodynamics and Einstein's special relativity, in spite of the works by Poincaré).

11.3.3 Equilibrium and Internal Motion of a Continuous Mass

The very title of this short Chap. XXX (20 pages) of primary importance indicates that the author will closely follow Cauchy's magisterial introduction of the concept of stress (cf. Cauchy [10] with a similar terminology). Only general equations are concerned. In accord with Cauchy—generalizing the case considered by Euler -, the "effort" per unit surface (called "traction" or "tension" by Cauchy, and "stress" in English by Rankine in elasticity) may be obliquely applied. Its linear

[3] For these see the remarkable book on the history of vector analysis by Crowe [15].

relation to the local unit normal yields (in modern terms) the notion of "stress tensor"—a word not yet used by Appell although introduced by W. Voigt some twenty years before. Appell could have used the expression of "linear vector function" in the sense of Gibbs. But he does refer to the expression of the "effort" in terms of the director cosines of the normal—his Eq. (4) in page 134—as a "linear and homogeneous function" of these (p. 133) after the 1828 proof of Cauchy exploiting the epoch-making "tetrahedron argument". Because of this lack of use of a condensed and mathematically justified wording, Appell is bound to repeatedly refer to these famous "six quantities" (in fact the six independent component of a symmetric second-order tensor). Here also we witness the shyness or reluctance with which French scientists adopt terms of foreign origin. The result indeed is the general equilibrium or dynamic equations of continua without further constitutive assumption. Also in the line of Cauchy, Appell considers the "ellipsoid of efforts" and possible changes of coordinates but not alluding to tensor transformations. More recent references are to the Cosserat brothers [13] and Brillouin [8].

11.3.4 Hydrostatics and Stability

The next much longer chapter (Chap. XXXI) with about ninety pages deals with hydrostatics, obviously a very special case already dealt with in the eighteenth century. The main notion here is that of "characteristic equation"—nowadays called "equation of state" or "constitutive relation"—relating pressure, density, and perhaps an additional argument such as temperature or a degree of salinity. Traditional considerations are those relating to level surfaces, barometric formulas, and Archimedes' principle. What are more impressive are the lengthy discussions about figures of equilibrium of masses of fluids and the related question of their stability. Here Appell necessarily refers to the buoyancy problem, the old geometric considerations of Bouguer [7] and Charles Dupin, and the (then) recent studies of Guyou [18], Duhem and Poincaré (in various memoirs for the last two authors).

11.3.5 Deformation of Continua

With this Chap. XXXII Appell returns to general notions based on geometry although the expression "differential geometry" is never used. He naturally bases his presentation on the original work of Cauchy [10, 11], but also frequently refers to recent works by Love, Darboux, Poincaré and the Cosserats [13]. Of course he meets with the "strain tensor" the same wording difficulty as with the stress tensor. Principal dilatation and stretches are casually introduced as well as the notion of homogeneous deformation. The continuity equation is practically proved in the

Lagrange-Piola format ($\rho_0 = \rho J$)—cf. Sect. 664. But in contrast he pays only lip service to the notion of compatibility condition, although dealt with before by Navier and Saint-Venant and more recently by E. Cesaro and V. Volterra—cf. the note in Sect. 673 simply referring to a paper by Riquier [32].

11.3.6 Kinematics of Continua

With the intervention of the time variable combined with the previously introduced sketchy theory of deformation comes the kinematics of continua in Chap. XXXIII. Here again Appell grants (p. 278) to Cauchy the role of founder of this kinematics, although he has to acknowledge the primary role played by Euler and Lagrange. Indeed, in what we think to be a rather modern approach, Appell deals at some length with both sets of Eulerian and Lagrangian variables. Like many Frenchmen, he does not seem to be much aware of the inclusive views expanded by Piola in Italy and Kirchhoff in Germany in the middle of the nineteenth century. In this line he compares the continuity condition in Eulerian and Lagrangian-Piola-an formats. It is also at this point that he introduces for the first time the notion of vorticity as a mean rotation (following Cauchy), but he also cites (Sect. 706) the definition of this vector by Boussinesq (true kinematic definition in terms of the instantaneous rotation of a triad of principal directions) and by Stokes (a mechanical definition related to internal friction in fluids). This manifests a specific interest in a notion that Appell will duly exploit in further chapters, an interest shared by many mechanicians of the period after the pioneering works of Helmholtz, Kelvin and others. Furthermore, irrotational motion brings him back to one of his favourite subjects, potential theory.

What remains most original in this chapter is the introduction of Hadamard's classification of propagating discontinuities in continua. This surprisingly occupies more than twenty pages and follows the remarkable work of Hadamard [19] on general properties of wave propagation. Hadamard cleverly distinguished, in a true three-dimensional framework, between dynamic and kinematic aspects of the phenomenon previously described by Riemann, Christoffel and Hugoniot for essentially one-dimensional motions in a fluid. He classifies moving discontinuities in terms of the order of the space and time derivatives of a field that suffer a true discontinuity. Thus a discontinuity surface across which the medium velocity and the strain are discontinuous is a discontinuity surface of the first order (a shock wave in usual jargon) while one across which the acceleration (second-order time derivative of the motion) is discontinuous is called a discontinuity surface of the second order or "acceleration wave". Jump conditions at the crossing of such surfaces replace the field equations across the surface while usual (continuous) field equations remain valid—but obviously with different field values—on either sides of the surface. As we know now irreversible thermodynamics is a necessary ingredient in a good appraisal of this singular phenomenon of propagation. Truesdell and co-workers will capitalize on this type approach in the 1960s–1970s in order to explain the formation of shock waves from weaker discontinuities such as acceleration waves, especially in deformable solids.[4]

11.3.7 General Theorems of the Dynamics of Perfect Fluids

General theorems that govern the dynamics of perfect fluids are the object of the rather long Chap. XXXIV. Only pressure appears as an internal force in the mass of fluid. Remarkably enough Appell gives the Lagrange format of the equations of motion, together with Lagrange's theorem when a velocity potential exists. He introduces the notion of sound speed by examining the propagation of a second-order discontinuity wave in the classification of Hadamard. As to the Eulerian format of these dynamic equations, it is exposed at length with a beautiful example (spherical layer around a solid spherical nucleus) borrowed from his contemporary Basset [5]. Changes of coordinates are discussed in both Lagrangian and Eulerian formats. So-called permanent motions (with velocity field constant in time) are examined along with fluid filaments and Bernoulli and Torricelli theorems. Irrotational such motions are considered in particular. The chapter concludes with the principle of images (e.g. that of a source with respect to a plane) in the manner of Lord Kelvin.

11.3.8 Theory of Vortices

As already remarked the theory of vortices is a subject of great interest to Appell. No wonder, therefore, that he devotes more than seventy pages to the subject (Chap. XXXV). Of course, Helmholtz is the main contributor with the original theorems that he proved in 1858. But works by Kirchhoff, Kelvin, Rayleigh and Poincaré (see the book on vortices by Poincaré [29]) are also often cited. The considered fluid is viewed as perfect. A primary notion is that of *circulation*. A fundamental result is that the flux of vorticity across a fluid surface is constant in time. The celebrated theorem of Helmholtz states the conservation of vorticity surfaces. Vortex lines play a fundamental role. Problems of connectivity of such lines are much relevant. Geometric proofs are favoured. What is more original (but perhaps in the air at the time of the publication of the treatise) is the strong analogy between vortex loops and electric-field lines (cf. p. 432; and again in p. 459 with the magnetic force). A long discussion about vortex loops follows occupying seventeen pages. The determination of the velocity field from vortices is examined at length. Altogether a rare mention—in the treatise—of experimental works is

[4] The present author will even formulate systematically the relevant kinetic and dynamic compatibility conditions in a covariant relativistic framework (cf. [24]).

given in a short section (Sect. 777) following studies by M. Brillouin. Mathematically oriented considerations on the analytic description of vortex surfaces by the German mathematician Clebsch are paid some attention, while, we admit, we do not know about the geometric problem posed by a certain Transon[5] to which six pages are devoted. In all, this sounds like rather classical material. The accompanying exercises are based on problems by Lagrange, Helmholtz, M. Lévy, H. Poincaré, Beltrami and Clebsch.

11.3.9 Parallel Flows

Fluid motions parallel to a given plane are extensively considered in a Chap. (XXXVI) of almost ninety pages. It must be understood that these are motions that depend only on two planar coordinates, the whole picture being time invariant by translation in the orthogonal direction. The vorticity vector then is orthogonal to the family of parallel planes. The resulting essentially two-dimensional picture lends itself well to the introduction of a velocity potential and the exploitation of complex variables, obviously a technique with intricacies much enjoyed by Appell and his contemporaries (remember that France is the country of Cauchy, the inventor of the theory of residues). But here a short pose shows that Appell—short Sect. 789 in pp. 487–488—is aware of the notion of *solitary wave* as observed in 1834 and reported in 1844 by Scott-Russell, and whose mathematical solution owes much to Boussinesq and Lord Rayleigh.[6]

Of particular interest are the vortex motions of a liquid and the related problem of the evolution of vortex tubes. A remarkable canonical (in the sense of Hamilton's formulation) form is introduced here (pp. 497–503) for a function H that defines the coupling between two vortex tubes. This may look as supererogatory in such a treatise, but it probably reflects a personal interest of the author as also his proximity with Poincaré (cf. [29]). Some simple free-surface wave problems are studied (Gerstner trochoïdal waves for swell). However, in a move typical of Appell, the author then makes a gift of some forty pages to Henri Villat[7] in the third edition of the treatise for an exposition of this author's works on the

5 Probably this is Abel E. Transon (1805–1876) a brilliant student at Ecole Polytechnique and Ecole des Mines in Paris, who became mostly interested in problems of geometry.

6 Boussinesq and Rayleigh are dutifully cited, but Appell does not mention Korteweg and de Vries (1885) who are now given most of the credit for the typical solitary-wave solution, which was to gain an extraordinary renown in the 1960s–1970s with the use of computer simulations and the invention of the inverse-scattering method.

7 H. Villat (1879–1972) became first a professor of rational mechanics in Strasbourg, and then a professor of fluid mechanics at the Faculty of Sciences in Paris, while also teaching at the National School of Aeronautics, nicknamed "Sup' Aero". His course there gave rise to a classic book in fluid mechanics [35]. He was the founder of the Institute of Fluid Mechanics of the University of Paris. This was an ancestor of the Institut Jean Le Rond d'Alembert organized by the writer of this contribution.

irrotational motion of a liquid in contact with a fixed obstacle, and the ensuing problems posed by the notion of *wake*. The importance of such a problem is easily realized for the navigation of ships, and—as we known now—for the forthcoming developments of both theoretical and experimental aerodynamics.[8] Here, due to the essential two-dimensionality of the problem, this is but the realm of the exploitation of complex variables and complex-valued functions for which Villat shows an unmatched dexterity dealing with conformal mappings of various types in agreement with Kirchhoff, Riemann and Picard. The consideration of discontinuous plane motions of a liquid allows one to eliminate d'Alembert's paradox thanks to the introduction of stagnation points, attached streamlines and a rear wake of dead flow. This is magisterially dealt with by Villat who concludes with formulas for the pressure force and moment exerted by the flow on the obstacle. We cannot help but feel a great admiration for the cleverness of these evaluations in hydrodynamics. Most references are of course to Villat's own publications in a remarkably wide range of scientific journals (cf. pp. 559–560)—a rather seldom practice for the period.

11.3.10 Barocline Fluids

Barotropic fluids are those for which the state equation (characteristic equation) reads $f(p, \rho) = 0$. Accordingly level surfaces for pressure and density coincide. Barocline fluids are those for which this characteristic equation contains an additional variable, say, temperature or a degree of salinity. The interest for such a situation is obvious in atmospheric air—the domain of meteorology per se—with the influence of temperature and humidity in altitude and in oceanography with variation in the salinity with depth. This yields the notion of *stratified flows* where circulation and vortex formation are critical parameters. The Norwegian scientist Bjerknes[9] was responsible for a large part (34 pages) of the redaction of this Chap. XXXVII. This author pays a specific attention to analogies with electric and magnetic phenomena (e.g., magnetic induction replacing the fluid velocity).

[8] In this respect see Anderson [1].

[9] Vilhelm Bjerknes (1862–1951) is a Norwegian fluid dynamicist and meteorologist who was first an assistant to Heinrich Hertz in Bonn (1890–1891), and then specialized in hydrodynamics and created a true school of meteorology in Bergen and Oslo. He definitely influenced other fluid dynamicists such as Ekman and Rossby. He is largely responsible for the progress made in meteorology in the first half of the twentieth century. His early contact with Hertz probably kindled his life long side interest in electrodynamics. Together with other Norwegian scientists (cf. [6]), he wrote a splendid lengthy—in all, 864 pages—monograph on physical hydrodynamics and its applications to dynamic meteorology. His father Carl Anton Bjerknes was a precursor in the same field of fluid dynamics, already using hydrodynamic analogies to interpret Maxwell's electromagnetism.

Appell seems to be responsible for the writing of the remaining part of this chapter with extension of the theory of vortices and generalizations of formulas for the fluid circulation, this being complemented by bibliographic recommendations.

11.3.11 Elements of Elasticity Theory

With a chapter of 48 pages, we see that elasticity was not a field of utmost interest for Appell. He in fact first reports the bare essentials often referring the reader to more competent specialists such as Boussinesq, M. Brillouin and the Cosserat brothers for more recent works. The presentation of linear elasticity leans heavily on the original work of Cauchy for the general relationship between infinitesimal strains and stresses—exploiting the relationship between two quadrics. He naturally focuses on the case of homogeneous isotropic linear elastic media with only two elasticity coefficients, the Lamé coefficients. Still he mentions the possibility of anisotropy and shows the reduction to twenty one elasticity coefficients in a general case (cf. pp. 612–615). Duhem [17] are called upon for the proof of the inequalities to be satisfied for the positive definiteness of the quadratic form of the elastic energy. He refers to the Cosserats for the question of mixed boundary conditions and a uniqueness theorem. The only "energy theorem" mentioned is the one on reciprocity due to Betti. Three examples of problems are treated in detail referring to the lessons of Gabriel Lamé on the mathematical theory of elasticity.[10] These are exercises which are still given today to students, e.g., the equilibrium of a cylindrical tube.

The rest of the chapter deals with more advanced matter. This includes in brief form (but over some five pages) some of the considerations by the Cosserats and Poincaré on the "spectrum" of solutions of static equilibrium problems, following the rewriting of these equations for the displacement of Cartesian components u_i by the Cosserats as (in our notation)

$$\Delta u_i + \xi \nabla_i \theta = g_i, \tag{11.1}$$

where θ is the dilatation, $\theta = \nabla_i u_i$, and $\xi = (\lambda + \mu)/\mu$, and with prescribed values at a boundary. By an adequate formal expansion this can be compared to the solutions of Laplace and Helmholtz equations, including with orthogonality properties of two distinct solutions. This type of problems and formulations will wait some sixty-seventy years to be expanded by Russian mathematicians in St Petersburg. Another problem of importance for further research is that of the theory of slender bodies and the basic problem of Saint-Venant (for instance the torsion of a straight cylinder). Here again Appell refers to the works published by

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[10] The book of Lamé [21] was unique in its class before the publication of a book by Clebsch in Germany. The latter was in fact translated into French, commented upon and much enriched (so as to triple in volume) by de Saint-Venant [16].

the Cosserats in various notes at the Comptes Rendus of the Paris Academy in the years 1907–1908. The chapter concludes with the proof of the formulas for the speed of longitudinal and transverse waves in isotropic linear elasticity, the "ether" (no longitudinal waves) and perfect fluids (no transverse waves) being extreme cases. In the listed problems we find again a classic (due to Lamé), that of the equilibrium of a thick spherical layer with different pressures applied on the outer and inner surfaces.

One remark concerns the relationship of elasticity with the more engineering like strength of materials. Appell apologizes (Sect. 834) for not treating this aspect, even superficially, explaining that in this volume he consumed the available space with the contributions of Villat and Bjerknes. What is more surprising is that, even cursorily, he does not mention the possible existence of an elasticity limit, nor does he allude to the existence of plasticity criteria although these were formulated some forty years before (by Tresca in 1872 and Barré de Saint-Venant in 1873), not to speak of the most recent criterion proposed by Huber in Poland in 1904 (but published in French) and von Mises in 1913 in Germany. This is all the more surprising that Maurice Lévy, one of the engineers-scientists who is otherwise often cited for other problems, was also responsible for the introduction of the notion of rigid-plastic behaviour (no elastic response at all) in 1871.[11] Perhaps that Appell considers that this matter does not enter (yet) the framework of rational (continuum) mechanics.

11.3.12 On Viscous Fluids

This Chap. XXXIX on the equations of motion, of a viscous fluid is especially poor. It is extremely brief (less than 5 pages) and the names of Navier, Stokes and Barré de Saint-Venant are hardly cited. In analogy with the introduction of Cauchy's elasticity in the foregoing chapter, the linear (Newtonian) fluid constitutive equations are presented as an expansion limited to the first order in the six independent components of the symmetric part of the velocity gradient. Conditions of symmetry are evoked but not formulated to justify the reduction to two viscosity coefficients for isotropy. Perfect fluids governed by pure pressure (no shear) and Stokesian fluids are noted as a final point. The reader is referred to Boussinesq, Basset, Kirchhoff and Poincaré for further theoretical developments. Poiseuille is mentioned for the flow in capillaries. There is no mention of such notions as the Reynolds number (albeit introduced in 1883) and transition to turbulence. The situation is quite similar to the one of elasticity versus plasticity in the above examined chapter, but here the question resides in the transition between laminar and turbulent flows. In Appell's view, these subject matters did not yet belong to rational mechanics.

[11] See historical remarks in the book of Maugin [25].

11.4 The Cosserats' Theory of Euclidean Action

Although this supplement written by the Cosserat brothers was not reprinted in the considered third edition of the treatise, we appreciate that the publisher of the facsimile reprint thought good to join a copy of this supplement to this reprint. This was a nice move as it offered an opportunity to the reader—who until recently had no access to the 1909 book of the Cosserats—to study in a shortened version the main revolutionary ideas expanded by the brothers in 1909. We do not enter the detail of this contribution since we analysed this famous but unread book in greater detail in a separate work [27]. In a nutshell, however, we remind the reader that the Cosserats propose a kind of analytical mechanics of finitely deformable bodies basing on the notion of Lagrangian-Hamiltonian action and the application to it of a group-theoretical reasoning, the invariance of the action under the group of Euclidean transformations (translations and rotations alike). It is this equal footing between translations and rotations that necessarily led the brothers to envisage the possibility of the existence of so-called couple stresses in parallel with the usual notion of stress, the latter then becoming non-symmetric. This will have a brilliant future and blossom in the second half of the twentieth century. If we mention this supplement here, it is because Appell himself, in the preface—written in 1908—to the second edition of his treatise in 1909, expressed an unbounded enthusiasm for the Cosserats' work on this Euclidean action, an enthusiasm matched by that of the American mathematician who reviewed their work, Wilson [36]. It is therefore strange that Appell, in his preface—written in 1920—to the third edition of his treatise, justifies the suppression of the Cosserat's supplement by claiming that limited available space forced him to this move as he wanted to offer some space to Villat and Bjerknes. We do not think that this is a sustainable argument in a global work of this extensive size. We rather surmise that Appell had realized that in spite of the intrinsic depth of the Cosserats' reasoning, the subject was not so much successful, too hard to be grasped by most mechanicians not trained in mathematical physics at the time, and so probably in advance of its time, while Villat's and Bjerknes' works were more in resonance with the fashionable scientific developments of the period. He may also have thought that there was no longer need for the Cosserats' supplement since practically the same text had appeared in two other media (Chwolson's encyclopaedia and separate book form with a new pagination) and in the same year of 1909 (see [14]), and the book version was still available.

11.5 Concluding Comments

Perusing our foregoing comments we recognize that Appell is very much representative of his own epoch, concerning both the selection of treated subjects and the choice of mathematical techniques. His basic reference in the general notions of continuum mechanics remains the great Cauchy. In comparison, Navier, also a great elastician and fluid dynamicist, receives much less attention. The genius of Cauchy is even more enhanced by the fact that the main contributing mathematical technique is the consideration of complex variables—and the ensuing residues theorem, a technique also due in great part to Cauchy. This is obviously related to the fundamental two-dimensional setting of many problems posed in both fluid dynamics and elasticity. Potential theory also appears as an essential tool in which Appell finds a field at his own level of interest (shared by others scientists of the period, e.g., Poincaré). In comparison, the techniques introduced by J.-B. Fourier, powerful as they were, are hardly mentioned due to the weak interest expressed for dynamics in the wave-frequency domain. Also pregnant in many parts of the volume is the great admiration expressed by Appell for some precursors, e.g., Lamé and Barré de Saint-Venant, but also for some of his fellow mathematicians, in particular, Boussinesq, Duhem, Poincaré and Hadamard. For closer contributors, we must account for the importance granted to the Cosserat brothers (in the foundations of continuum mechanics and many subtle aspects of deformable solids), Bjerknes for hydrodynamics, and Villat for the theory of wakes. He finds in Villat another great specialist of the use of complex-valued functions, while Poincaré, Villat and Bjerknes clearly satisfy his marked interest for the theory of vortices. Dealing with a book on "rational" mechanics we find here very few applications to, and mentions of, experiments. Whatever smells too much of this yet "non-rational" framework is discarded or not even mentioned. This applies to behaviours beyond pure elasticity and perfect fluidity. The contents of the next volume (Volume IV) published in 1921—and not examined in the present contribution—comfort this appraisal with an overemphasis—about 630 pages—on the study of the equilibrium figures of fluid ellipsoids and applications to celestial bodies.

Finally, Appell as an active creative mathematician belongs to a period where vector and tensor analyses are not a subject of teaching in French universities, though there are local researches conducted along these lines following Ricci and Levi-Civita. Probably under the influence of the success met by Einstein's general relativity after the proof of validity of some of its consequences (in particular the deflection of light rays by heavy masses), Appell realized that something on tensor analysis should be added to his treatise if one wants to go further than classical rational mechanics. He was already old and sick when this came up in 1925 so that he commissioned René Thiry to write this additional part as a volume V devoted to "Elements of tensor calculus: geometric and mechanical applications".[12] [13] On perusing the contents of this volume we see the extent of the ambitious framework

[12] See Appell [4].

[13] In France it is only with a book (Brillouin [9] by Léon Brillouin (1889–1969)—a renowned physicist and son of M. Brillouin—that tensors and continuum mechanics were firmly associated. But the book is better known in the USA in its English translation than in France. A definite introduction to tensor analysis was successfully given by Lichnrowicz with a first edition in 1946, and a seventh in [23]. It is in this book that most French students of mathematical physics (including the present writer) were exposed to tensor analysis in the period 1950–1970.

provided by Appell for its redaction. Of course re-writing Volume III in that spirit would have been a formidable enterprise that no immediate successor of Appell would endeavour. One had to await such treatises as those of Truesdell, Toupin and Noll in the new *Handbuch der Physik* edited by S. Flügge after World War II [33, 34] for something more or less equivalent, with a nice complement on tensors by J. L. Ericksen in Truesdell and Toupin [34].

A.1 Appendix: Biography of Paul Appell

The following elements of the biography of Appell may be of enlightening interest. He came from a catholic family from Alsace. He left Alsace for Nancy when Alsace was occupied by the Prussian following the 1870–1871 Franco-Prussian war. He became friend with Henri Poincaré (1854–1912) in Nancy; they remained long life friends. He graduated in mathematics from the famous *Ecole Normale Supérieure* (ENS) in Paris, and obtained his Doctoral degree in this matter in 1876. He followed the teachings of several influential scientists: J.B. Serret, Gaston Darboux, Charles Hermitte, Joseph Bertrand, Maurice Lévy and Urbain Leverrier. After being lecturer at the ENS he obtained the Chair of Rational Mechanics at the Sorbonne in 1885, but also taught (we suppose) more applied matters at the engineering school known as the *Ecole Centrale des Arts et Manufactures* (for short, *Ecole Centrale de Paris* or ECP). His mathematical works (in the order of 200 publications; cf. [3]) were essentially in projective geometry, algebraic functions, differential equations and complex analysis. He occupied at some different or simultaneous times, such powerful positions as: Dean[14] of the Paris Faculty of Sciences (1903–1920), Rector[15] of the Academy of Paris (1920–1925), President of the council of the University of Paris, President of the French Society of Mathematics, and Member of the Paris Academy of Sciences. He was a visiting professor at Harvard and Rome. During WWI he created the "National Research Fund" to be considered the ancestor of the actual CNRS (National Centre for Scientific Research). He founded in 1920 the "Cité Internationale Universitaire de Paris" (CIUP) to accommodate foreign students and visiting scholars.[16] Philosophically he was an atheist while politically he would be

[14] The title of "Dean" practically corresponds to the actual title of President.

[15] The administrative title of "Rector of Academy" corresponds, for education, to that of "Prefect of Department" for other matters, but the "Academy" here refers to a larger region than a single department. He is the local representative in charge of implementing the views of the Minister of Education on the organization of education et al levels in State-own schools.

[16] The "Cité Internationale Universitaire de Paris" (CIUP) is a campus like condominium devised to accommodate foreign students. It comprises a general purpose building equipped with cafeterias, restaurants, concert and theatre halls, swimming pool and accommodations for visiting foreign professors. Most buildings for students were built in the course of time with the help of donations from philanthropists and industrialists or foreign governments. This was a unique

classified as belonging to the left of the political spectrum, having been a "Dreyfusard"[17] and an ardent defender of the separation between the State and the Church (in France this is called "radicalism" not to be mistaken for the meaning granted to this word in the USA). He had strong nationalist feelings (especially concerning his beloved Alsace) but proved also to be a true internationalist (being involved in the *League of Nations* in Geneva, and founding the CIUP). In all, Appell demonstrated an amazing level of activity in teaching, research, editing and public service.

References

1. Anderson JD Jr (1997) A history of aerodynamics. Cambridge University Press, UK
2. Appell P (1921) Traité de mécanique rationnelle, 3rd edn. Gauthier-Villars, Paris (facsimilé reprint by Gabay, Paris, 1991)
3. Appell P (1925) Notice sur les travaux scientifiques. Acta Mathematica 45:161–285
4. Appell P (1955) Traité de mécanique rationnelle, Tome V: «Eléments de calcul tensoriel: applications géométriques et mécaniques» by René Thiry. Gauthier-Villars, Paris; first edition 1926, second edition 1933; new printing 1955. (Editions Gabay, Paris, made the good move to reprint this together with the second edition (1935) of Volume IV in a facsimile reprint in 1991)
5. Basset AB (1890) An elementary treatise on hydrodynamics and sound. Deighton, Bell and Co., Cambridge
6. Bjerknes V, Bjerknes J, Solberg H, Bergeron T (1934) Hydrodynamique physique (avec applications à la météorologie dynamique), 3 volumes. Presses Universitaires de France, Paris
7. Bouguer P (1746) Traité du navire, de sa construction et de ses mouvements. Chez Jombert, Quai des Augustins, Paris
8. Brillouin M (1884–1885) Leçons sur l'élasticité et l'acoustique. Librairie Labouche, Toulouse
9. Brillouin L (1938) Les tenseurs en mécanique et élasticité. Masson, Paris (Dover reprint, New York, 1946; Gabay reprint, Paris, 2012; English Trans. Academic Press, New York, 1963)
10. Cauchy AL (1828) Sur les équations qui expriment l'équilibre ou les lois du mouvement intérieur d'un corps solide élastique ou non élastique. Exercices de mathématiques, vol 3. pp 160–187. (This presents in print the ideas originally submitted to the Paris Academy of Sciences on September 30, 1822)
11. Cauchy AL (1828) In: Exercices de Mathématiques, vol. 3. Pp 188–212. («De la pression ou tension dans un système de points matériels» Oct. 1828, pp 213–236)

(Footnote 16 continued)
structure in France where the notion of campus did not exist. Exceptionally, a few French students were mixed with foreign students to the benefit of both groups. The writer benefited from such a status during three very rich academic years.

[17] The "Dreyfus affair" (1894–1906) was a political scandal involving false accusations of treason (to the benefit of Germany) against a French Jewish officer. This practically split the French population in two parts, "Dreyfusards" (defending Dreyfus) and "anti-Dreyfusards" (who claimed Dreyfus guilty in spite of proved evidence of the contrary).

12. Coffin JG (1914) Calcul vectoriel. Gauthier-Villars, Paris. (French translation by M. Véronnet of the English original “Vector calculus: An introduction to vector methods and their various applications to physics and mathematics”, 1909)
13. Cosserat E, Cosserat F (1896) Sur la théorie de l’élasticité. Ann Fac Sci Toulouse, 1$^{\text{ère}}$ série 10(3–5):I.1–I.116
14. Cosserat E, Cosserat F (1909) Théorie des corps déformables. Hermann, Paris, (226 pages). Reprint by Editions Gabay, Paris, 2008; Reprint by Hermann Archives, Paris, 2009, (English translation: N68-15456: Clearinghouse Federal Scientific and Technical Information, Springfield, Virginia NASA, TT F-11561 (February 1968); another translation by D. Delphenich, 2007) (originally published as supplement in pp 953–1173 to: Chwolson OD (1909), Traité de physique (traduit du Russe), vol II, Paris, Hermann)
15. Crowe MJ (1967) A history of vector analysis: The evolution of the idea of a vectorial system. University of Notre Dame Press (Dover reprint, New York, 1985)
16. de Saint-Venant BAJC (1883) Théorie des corps élastiques. (French translation from the German with many comments and additions of: Clebsch A (1862) Theorie der Elastizität fester Körper, Leipzig)
17. Duhem P (1891) Hydrodynamique, élasticité, acoustique. A. Hermann Editeur, Paris
18. Guyou E (1887). Théorie du navire. Berger-Levrault, Paris (Second edition, 1894)
19. Hadamard J (1903) Leçons sur la propagation des ondes et les équations de l’hydrodynamique. Hermann, Paris
20. Korteweg DJ, de Vries G (1895) On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. Philos Mag 39(Series 3):422–443
21. Lamé G (1852) Leçons sur la théorie mathématique de l’élasticité des corps solides. Paris, (2$^{\text{nd}}$ edition, 1866) (Facsimile reprint, Gabay, Paris, 2006)
22. Lévy M (1890) L’hydrodynamique moderne et l’hypothèse des actions à distance. Revue Générale des Sciences Pures et Appliquées, December 15, 1890
23. Lichnerowicz A (1964) Eléments de calcul tensoriel, 7th edn. Armand Colin, Paris (Reprint by Gabay, Paris, 2011) (English translation by J.W. Leech, “Elements of Tensor Calculus. J. Wiley & Sons, New York, and Methuen & Co Ltd, UK, 1962)
24. Maugin GA (1976) Conditions de compatibilité pour une hypersurface singulière en mécanique relativiste des milieux continus. Ann Inst Henri Poincaré A24:213–241
25. Maugin GA (1992) The thermomechanics of plasticiy and fracture. Cambridge University Press, UK
26. Maugin GA (2013) Continuum mechanics through the twentieth century: a concise historical perspective. Springer, Dordrecht
27. Maugin GA (2013) About the Cosserats’ book of 1909 (preprint UPMC; see also Chapter 8 in the present book)
28. Poincaré H (1892) Leçons sur la théorie de l’élasticité. Georges Carré, Paris (Facsimile reprint, Gabay, 1990; also University of Michigan, Ann Arbor, 2012)
29. Poincaré H (1893) Théorie des tourbillons. Georges Carré, Paris (Facsimile reprint, Gabay, Paris, 1990)
30. Poincaré H (1899) Théorie du potentiel newtonien. Carré et Naud, Paris (Fasimile reprint, Gabay, Paris, 1990)
31. Prandtl L (1904) Ueber Flussigkeitsbewegung bei sehr kleiner Reibung. In: Contribution to the third international mathematical congress, Heidelberg
32. Riquier C (1905) Sur l’intégration d’un système d’équations aux dérivées partielles auquel conduit l’étude des déformations finies d’un milieu continu. Ann Ecole Norm Sup, Third series XXII: 475–538
33. Truesdell CA, Noll W (1965) The non-linear field theories of mechanics. In: Flügge S (ed) Handbuch der Physik, vol III/3. Springer, Berlin, pp 1–602

34. Truesdell CA, Toupin RA (1960) The classical theory of fields. In: Flügge S (ed) Handbuch der Physik, vol III/1. Springer, Berlin, pp 226–858
35. Villat H (1938) Mécanique des fluides, 2nd edn. Gauthier, Paris (First Edition, 1930; Facsimile reprint, Gabay, Paris, 2003)
36. Wilson EB (1913) An advance in theoretical mechanics. Cosserat Bull Am Math Soc 19(5):242–246 (Théorie des corps déformables by E and F)

Chapter 12
A Successful Attempt at a Synthetic View of Continuum Mechanics on the Eve of WWI: Hellinger's Article in the German Encyclopaedia of Mathematics

Abstract This essay analyses the comprehensive nature of a remarkable synthesis published by Hellinger (Die allgemein ansätze der mechanik der kontinua. Springer, Berlin, pp. 602–694, 1914) in a German encyclopaedia. In this contribution Hellinger, a mathematician, succeeds in capturing the progress and subtleties of all what was achieved during the nineteenth century, accounting for most recent works and also pointing at forthcoming developments. On this occasion, the scientific environment of Hellinger is perused and the style of Hellinger and his excellent comprehending of continuum mechanics are evaluated from a document that is a true landmark in the field although often ignored.

12.1 Introduction

In the nineteenth and twentieth centuries German scientists and engineers have developed a special taste for the composition of impressive encyclopaedias and so-called "*Handbucher*" (Handbooks). A typical *Handbuch* (in fact a *Taschenbuch*) in mechanical engineering has been the very popular one by Hütte with many foreign translations, but this was more a catalogue of prescriptions, standards, and elementary formulas of mathematics and strength of materials. Famous collections of the "Handbuch der Physik" have been edited by Geiger and Scheel between 1926 and 1933 [18] and by Flügge between 1955 and 1988 [17]. Mathematicians did not escape this trend. In particular, renowned mathematicians such as Felix Klein (1849–1925) and Conrad H. Mueller (1857–1914) contributed their wide experience and many friendly connections to the creation and edition of a monumental encyclopaedia of mathematics under the German title "*Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen*"—in brief: Enz. Math. Wiss. (EmW)—published by B. G. Teubner (Verlag) in Leipzig

G. A. Maugin, *Continuum Mechanics Through the Eighteenth and Nineteenth Centuries*, Solid Mechanics and Its Applications 214, DOI: 10.1007/978-3-319-05374-5_12,

between 1907 and 1914 [31].[1] Various mathematicians and physicists were called to contribute to this vast enterprise. Part Four of Volume Four was devoted to Mechanics (*Mechanik*).[2] In that Volume the burden of writing Article 30 on the General bases/formulation (principles) of continuum mechanics ("*Die allgemeinen Ansätze der Mechanik der Kontinua*") fell on Ernst Hellinger then in Marburg. The Encyclopaedia is well-documented with scholarly articles. It is aimed at the specialist. Concerning the whole EmW, it is salient to note the following appreciation of I. Grattan–Guiness (2009), the famous historian of sciences: "Many of the articles were the first of their kind on their topic, and several are still the last or the best. Some of them have excellent information on the deeper historical background. This is especially true of articles on applied mathematics, including engineering, which was stressed in its title". This particularly applies to Hellinger's contribution.

Ernst Hellinger (1883–1950) had been educated in Heidelberg, Breslau and Göttingen and was a doctoral student of David Hilbert. This indicates that he was a rather pure mathematician whose most famous mathematical accomplishments were in integral and spectral theories. He became a professor in Frankfurt am Main but he left Germany for the USA in 1939 and then taught at Evanston, Illinois. The writing of this contribution in continuum mechanics in 1913 [26][3] may have been a parenthetical episode in his career. Nonetheless, he was much interested in variational formulations (as shown by the forthcoming perusal of his contribution) and even introduced the notion of two-field variational principle now referred to as the Hellinger-Reissner variational principle in elasticity (cf. [53]). Nonetheless, we surmise that his formation with Hilbert led him to view continuum mechanics as one of the physical sciences to be formalised and given an axiomatic framework, an orientation that will be materialized later on by the Truesdellian school with Noll (cf. [48]). Although not a full time mechanician, Hellinger was able to capture in a rather concise contribution all recent and promising advances by keeping a

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[1] This monumental work was translated into French [43] and edited under the direction of J. Molk—a mathematician specialist of elliptic functions—and P. Appell—a reputed mathematician himself the author of a magisterial treatise on rational mechanics (cf. [1]). A full facsimile reprint of this French translation was produced by Editions Gabay in Paris in the years (1991–1995). But only one volume (exactly IV/4, the presently examined one) was never translated into French, and therefore does not exist in the Gabay reprint. The reason for this phenomenon is not clear. Of course, its date of publication, 1914, was not the most appropriate one given the beginning of World War One. Another possible explanation given by J. Gabay is that P. Langevin, adviser for the translation of the Encyclopaedia after WWI, was not much in favour of phenomenological physics in the sense of Duhem et al. Together with Eleni Maugin, I produced a (non-published) partial translation from the German to English of Hellinger's contribution.

[2] Timoshenko [47, p. v] in his history of the strength of materials refers to this volume for an extended bibliography.

[3] Of course Hellinger's theoretical contribution was complemented by other more specific and applied ones such as those of Heun [29] on the general bases and methods of the mechanics of systems, Voss [52] on the general principles of mechanics, and von Kármán [30] on the physical bases of the mechanics of solids.

sufficiently high standpoint, a balanced neutrality, and an acute insight, and this, in our opinion, much more than some professional mechanicians who kept too much with well established subject matters. In order to help the reader not accustomed with reading in German, a partial translation in English of Hellinger's contribution is provided in an Appendix.

12.2 The Scientific Environment

Although Hellinger was essentially foreign to the engineering spirit, in writing his opus of 1914 he gathered a rich past and contemporary documentation and accounted for most of the recent works in the field of theoretical continuum mechanics. He was not building in a vacuum, but this voluntary embedding in a medium other than his own is altogether remarkable. Of course the influence of his mentor David Hilbert may have played a fundamental role in his clear interest for the general and somewhat axiomatic aspects, so that he must have been familiar with the then recent attempt of Hamel [24] to delineate the structure and principles of mechanics (as of the beginning of the twentieth century), and the recently published treatise on "energetics" by Duhem [12] with its pre-Truesdellian flavour which may have been to his taste. This is corroborated by his frequent citations of these two authors. But he also knows the impressive treatise of Appell [1] on the rational mechanics of deformable bodies and the German synthetic texts of Heun [29, Voss [52], and Voigt [51].

Being basically a mathematician and a great admirer of Lagrange, Hellinger is also very much concerned with variational formulations in the works of W. Thomson (Lord Kelvin), Kirchhoff and, above all, the Cosserat brothers [7, 8, 40]. The last connection may have been through his reading of the Third volume of Appell's treatise [1] in which there is a supplement written by the Cosserats. The appeal to group symmetries in the line of Sophus Lie and Henri Poincaré by the Cosserats may have been very attractive to him. But he also considered the possible occurrence of dissipation with the notion of dissipation potential introduced by Rayleigh, and even time-dependent (memory-like) behaviours in the manner of Boltzmann [2]. The recent work of Hadamard [23] on wave propagation has also left a strong print. Finally, in contrast with many other writers of the period who remain in the classical (Newtonian) framework, Hellinger has already integrated in his views the revolutionary ideas of Einstein in 1905 on relativity and Minkowski [42] on space-time. All these remarks are based on the citations of these authors by Hellinger as checked in the many footnotes to his contribution. This is the general background and favourable scientific environment in which this perspicacious author has framed his article.

12.3 The Contents of Hellinger's Article

12.3.1 Introductory Remark

Hellinger's article is only ninety two pages in print. Nonetheless, it succeeds in providing a rather complete survey of the field both with its established bases, its recent successes and some view of things to come. This friendly neutrality with which the author looks upon his assigned duty—in principle perusing a vast domain of knowledge with about a hundred fifty years of history and a vivid contemporary activity—is conducted with no a priori prejudice as a result—we surmise—of Hellinger being a somewhat outside observer. Hellinger is rather generous but also very accurate with citations. He cites many authors, whatever their nationality, but is clearly most influenced by works published within the thirty years before his synthesis, say in the period 1880–1910. Perhaps because of this "actuality", he does not confine himself to the well established fields (linear elasticity and Eulerian fluids), but he often venture in newly expanded fields of interest such as finite deformations, oriented (Cosserat) bodies, capillarity, formulation of thermo-mechanics, analogy with electrodynamics, and even relativistic continuum mechanics.

From a historical viewpoint, our perusal of this beautiful contribution should not be influenced by our own education in the field (rough period 1960s–1970s) and our knowledge accumulated over an active professional period of some forty five years that witnessed many developments. But it happened that many of these rich developments in a vivid period of research more or less coincide with many of the points touched upon by Hellinger. We do not think that this kind of resonance between Hellinger's approach to our field of interest and our own view is so much due to an influence that this author would have exerted on the generations that followed his own. Indeed, Hellinger's text may have been read by German scientists between the two World Wars. But we must notice that his article was published in an encyclopaedia of *mathematics*, in a style that is permeated by the rigorous thinking of a mathematician—far from engineering interests—and that the text was not translated in any foreign language. It just happened that a spirit close to that of Hellinger re-appeared in our period of activity, and this of course greatly facilitates our apprehending of his exposition.

12.3.2 The Layout and Articulation of the Contribution

Every synthetic work in a field has to respect a definite agenda. This particularly applies to an article in an encyclopaedia of which the readership is not so well delineated. In the present case a tradition has settled that the progression in the presentation of the subject matter follows an almost fixed order (as exemplified in many textbooks on continuum mechanics), geometric background being introduced first, followed by kinematics and the theory of deformations, then kinetics and the general laws of mechanics, general classes of mechanical behaviours and a few more specific examples, and finally (but not always) some more exotic extensions. Hellinger's approach is more difficult to grasp because he is ahead of his time while simultaneously following some masters such as Kirchhoff, Helmholtz, Clebsch and Barré de Saint-Venant, and he has thoroughly gone through the then recent works by W. Voigt, J. V. Boussinesq, E. and F. Cosserat, H. Poincaré, and P. Appell, authors who are very often accurately cited. In reason of the imposed exercise, Hellinger's text is extremely dense. Instead of perusing his contribution just in its order of presentation—the easy way—we have preferred to examine various points, that recur in the whole text and seem to emphasize Hellinger's repeated interest in some specific aspects as an exemplary mathematician (obviously not the point of view of an engineer).

12.4 The Identified Fields of Marked Interest of Hellinger

12.4.1 On General Principles of Mechanics and General Equations

This is not an original point of departure in Sect. 12.2. Hellinger builds on the commonly admitted bases of Newtonian mechanics in the tradition set forth by Euler, Lagrange, Cauchy, but with modern references to Brill [5]; Duhem [12]; Voigt [51], and other contributions to the same encyclopaedia by, e.g., Voss [52] and Heun [29]. He clearly indicates his favoured view of Hilbert and Hamel [24, 25] — later on formalized in Hamel's contribution to the Handbuch der Physik in 1927 — for axiomatization and the consideration of a general thermodynamic framework by Duhem [12]. He also heavily borrows from the treatise of Appell [1] and the recent works by the Cosserat brothers ([7, 8], and their numerous notes in the *Comptes-Rendus* of the Paris Academy of Sciences). But Hellinger does not hesitate to introduce the relativistic Einstein–Minkowski's vision in the last section of his contribution.

Formally, Hellinger is much more attached to the Lagrangian-Hamiltonian variational formulation than to the classical Newtonian type of approach that relies on a statement of laws of equilibrium or dynamics. This he shows even for the bases of statics where he readily implements the principle of virtual work (Sects. 12.3 and 12.4). This may be one of the reasons why this work is not so much cited in the "Anglo-Saxon" literature dominated by Newton's vision and made popular in continuum mechanics by the Truesdellian school in the 1960s. But Hellinger cannot avoid discussing the notion of force as a polar vector (p. 613) and the clever introduction of the concept of stress by Cauchy (*Cauchysche "Drucktheorem"*; p. 615). On this occasion, Hellinger, above all a mathematician, acknowledges the usefulness of the notions of vector analysis and dyadics—linear vector functions—in the line of

J. W. Gibbs (cf. [21]) and the matrix calculus of Cayley (p. 613). He also refers to "tensor components of a dyad" (*Tensorenkomponenten*) after Voigt's lectures on the physics of crystals (p. 624). This is to be contrasted with the rather shy attitude of contemporary authors (e.g., Appell [1]; see my own appraisal in Maugin [41]).

Hellinger's presentation of equilibrium equations in the Eulerian framework with the associated natural boundary conditions (reflecting Cauchy's postulate)—Eqs. (5a) and (5b), p. 617—is rather modern. But he also gives what may be considered the Piola-Kirchhoff format as Eqs. (9a) and (9b) in p. 618, after what looks like a Piola transform for the stress in Eq. (8). He indeed refers to the work of Piola [44] in p. 620. For the symmetry of the Cauchy stress, he refers (p. 619) to Hamel who calls this the "*Boltzmannsches Axiom*" for "*die Symmetrie der Spannungsdyade*". Reductions to the two-dimensional (e.g., plates) and one-dimensional cases (e.g., rods, filaments)—in Eqs. (18a) and (18b) in p. 622 for this last case—are given following the Cosserats.

12.4.2 On Variational Formulations

For a mathematician like Hellinger the attraction to the beauty, economy of thought, and efficacy of variational formulations is inevitable. Hellinger, a follower of Lagrange, Piola, Hamilton, Kirchhoff, Helmholtz and the Cosserats, in fact starts by emphasizing the exploitation of the principle of virtual perturbations ("*virtuellen Verrückungen*"; p. 611 on)—virtual work (a weak formulation in the modern jargon). To the risk of creating an anomalous connection with modern standards, we perceive in these perturbations the notion of test functions (see Maugin [35, 37]). Note that Hellinger gives a mathematically correct definition of what is a material variation by considering an infinitesimal parameter noted σ (and not ε like in modern treatments; cf. pp. 607–608). As a matter of fact Hellinger's statement (7) in p. 612 is, but for different symbols, just the same as in a modern formulation where the principle of virtual powers (for statics) is written for a massive body as

$$P^*_{vol} + P^*_{int} + P^*_{surf} = 0, \tag{12.1}$$

where the three terms refer to volume, internal and surface forces, respectively. The Cauchy stress is introduced in the second term as a co-factor. A power of inertial (acceleration) forces is added in the right-hand side of Eq. (12.1) in the dynamical case. The second term is transformed with the help of Green's divergence theorem [22] to yield a divergence term in the bulk and Cauchy's natural boundary condition at the surface. Hellinger emphasizes the equivalence of the statement (1) with Newton's laws (cf. p. 630).

In dynamics we have D'Alembert's principle per se (*d'Alembertschen Prinzips*, p. 629) and this yields the looked for equations such as Eq. (2) in p. 630. On introducing the kinetic energy, Hellinger is led to the principle of least action (p. 633) of Maupertuis and Hamilton. Gauss' principle of least constraint is also evoked in the same page with the possibility to account for non-holonomic constraints. The general nature of such formulations is clearly acknowledged including with due reference to the Cosserats. With the assumed existence of a strain potential Hellinger touches upon the favourite subject matter of Kirchhoff, Boussinesq, Duhem [10], Poincaré and the Cosserats (pp. 643–651). This led him to examine some questions related to stability in agreement with Dirichet and above all Born [3], as also Italian authors such as Menabrea and Castigliano. He introduces appropriately the notions of canonical transformation (p. 657) and Legendre transformation (function H in p. 654). This leads him to say a few words about minimum principles and stability. Unknown multipliers (interpreted sometimes as stresses or "reaction forces") are introduced wherever a mathematical constraint is imposed (e.g., incompressibility), following ideas of the French mathematician J. Bertrand and also D. Hilbert (see pp. 661–663). Ideal fluids accept a characteristic equation $p = p(\rho)$ when, following Hadamard [23], the potential reduces to a function of the Jacobian of the deformation. In presence of some dissipation Hellinger follows an idea of Rayleigh to consider a potential of dissipation (p. 657). This will later be formalized even for plasticity (dissipation function homogeneous of order one only) in works of the 1970s–1980s (see, e.g., Maugin [36]).

Apart from the extensions to oriented media (his Paragraph 4b, and Paragraph 4.4 below), Hellinger touches two other extensions of the principle of virtual perturbations that were to bear fruits later on. One is the possibility of considering higher-order space derivatives of perturbations. This was envisaged early by Le Roux [33]—apparently unknown to Hellinger—to account for effects of spatially non-uniform strains (such as in torsion) in small-strain elasticity. This would later on be expanded in the so-called gradient theory of continua [Cf. works by R. D. Mindlin in the 1960s, and above all: Germain [19], for the second gradient, and Maugin [35] for a general framework, using the principle of virtual power without knowledge of Hellinger's contribution]. The other is the possibility to account for the existence of unilateral constraints during the variation (Cf. Paragraph 4c). This was to be expanded in the theory of variational inequalities in the mechanics of continua (Cf. e.g., Duvaut and Lions [13, 14]).

Finally, it is often said (cf. Washizu [53]) that Hellinger contributed to the variational formulation of continuum mechanics (elasticity) by introducing before Reissner [45] the notion of *two-field variational principles*. In these both displacement *and* stresses are varied, allowing a relatively easy accounting of boundary conditions of mixed type. Reissner—educated in Germany and himself the son of a reputed mathematical physicist—must have heard of, if not studied, Hellinger's contribution. However, he proudly told the present writer that "he did not see why Hellinger's name was attached to his own name for this notion". It is true that we could not locate where Hellinger introduced this notion. But the association may come from the fact that—as noted above—Hellinger duly considers Legendre transformations of the energy potential, introducing a kind of complementary energy.

12.4.3 On Finite Strains and Elasticity

Hellinger follows the tradition established by Piola, Kirchhoff, Boussinesq and the Cosserats by always considering the case of finite strains, linear elasticity being only an approximation. This is exemplified at many stages in his contribution. First both actual (noted x, y, z) and referential or material (Lagrangian) coordinates (noted a, b, c) are introduced. This later on allows for the introduction of the Piola transformation (8) in p. 618 with a clear algorithm in spite of the absence of tensor notation. The Piola-Kirchhoff form of the equilibrium equations follows at once as Eq. (9a, b). This also applies to Cosserats' media (p. 624–625). In the case of Green elasticity for which there exits a strain potential, the Cauchy-Green's finite strain is duly introduced (cf. Eq. (12.1) in p. 663). An example of higher order (than quadratic) strain energy function is given in p. 665. The exact constitutive equations for Cauchy's stress tensor in finite strains are given as Eq. (5) in p. 645 in Boussinesq's form while Piola's form is given in p. 654 together with Max Born's [4] equations in terms of a potential in stresses—Eqs. (22a, b) in p. 654—after introduction of the complementary energy by means of a Legendre transformation. The resulting compatibility condition for the finite deformation gradient is given in Eq. (24) in p. 655 in a form due to von Kármán. In the case of isotropic materials Hellinger rightfully calls for the invariance under orthogonal transformations (p. 664) and the resulting dependency of the strain energy on the basic invariants that are factors in the Cayley-Hamilton theorem. He evokes on this occasion the possible existence of self-stresses. Citations to Boussinesq, Duhem, Poincaré, the Cosserats, Helmholtz and J. Finger abound. All these now seems quite familiar to students who followed the masters of continuum mechanics in the 1960s–1980s—e.g., in the books of Truesdell and Toupin, Green and Zerna, Leigh, Eringen, etc., in the USA and those of Goldenblatt, Novozhilov, Lurie, Sedov, Ilyushin and others in the Soviet Union—this includes the present writer.

As a true mathematician, Hellinger views small-strain elasticity as a theory of perturbations introducing wherever necessary a small parameter (noted *sigma* and not *epsilon*) that indeed indicates the smallness of strains about an undeformed state [cf. Eq. (3') in p. 608].

[4] The name of Max Born (1882–1970) is most often associated with the matrix formulation of quantum mechanics (with P. Jordan and W. Heisenberg) and his statistical interpretation of the wave function in Schrödinger's equation for which Born received a belated Nobel Prize in 1954. But Born had defended a Ph.D. thesis (1906) on the "stability of the elastica in a plane or space" (to which Hellinger refers). He was also most active in studies related to relativity after 1905 (see here Paragraph 4.8). He was among the initial developers of the lattice dynamics of crystals and contributed much to optics. His friendship with Hellinger dated back to their undergraduate-student years in Breslau ("Wrocław" in Polish) in the early 1900s. He mentored many of the known theoretical physicists of the 1920s and 1930s while in Göttingen. Finally, he was instrumental in the publication by Caratheodory [6] of an axiomatics of thermodynamics (Born suggested a formulation of the second law, the so-called "inaccessibility of states").

12.4.4 On Oriented Media

From the very beginning of his exposition Hellinger envisages the possible existence of internal degrees of freedom of the type proposed by the Cosserats in 1909. For instance, introducing the basic physical parameters of a continuum, together with the notion of density (p. 609), he feels quite natural to consider the possible attachment to each material point (the "Quantum der Materie" with material coordinates in his own language; p. 606) of an oriented trihedron or triad of rigid vectors (*ein "rechtwinkliges Axenkreus"*) likely to represent the varying orientation of "molecules"—as proposed by Voigt [50] and possibly by S.D. Poisson much earlier in 1842 (cf. footnote in p. 609). This yields the notion of "*Medien mit orientierten Teilchen*" (pp. 609–610) in the manner of the Cosserat brothers (and perhaps Duhem [11], p. 206; see Maugin [39]).

Then in considering a variational formulation (principle of virtual work), Hellinger naturally generalizes it to the case including local orientational kinematic properties (pp. 623–627) with specialization to two-dimensional and one-dimensional cases. The concept of couple-stress tensor ["*Drehmoment*" (dyade)] then appears naturally. The author recurs to this framework of "generalized continua" on many occasions, in particular when considering the Green type of elasticity based on the existence of a potential for strains (pp. 646–651) with the application of the Cosserats' concept of "Euclidean action". He returns to the notion of "generalized continuum" while dealing with analogies with the equations of light propagation and electrodynamics (the MacCullagh "ether" of 1839 [34]—an elastic medium able to transmit only transverse waves (light) in agreement with Fresnel's observations—the deduction of Maxwell equations by identifying elastic displacement and electric field on the one hand and vorticity with magnetic induction on the other and as done by authors such as Kelvin or J. Larmor—see pp. 675–681). Of course this is now rather obsolete and was already evaporating in thin air at the time of Hellinger after the works of Lorentz, Poincaré and Einstein. But Hellinger's attitude is above all witness of a marked interest in the rich modelling potentiality offered by continuum mechanics—although sometimes along paths with dead-end—leaving the final choice to true physicists.

We must note that, just like most authors until 1966, Hellinger does not see that, similar to density with its conservation law (cf. Eq. (12.7) in p. 609 in the Lagrange-Piola format), there must exist a conservation law associated with the inertia of the new orientational degrees of freedom. This missed step was resolved much later by Eringen [15].

12.4.5 On One-Dimensional and Two-Dimensional Bodies

Hellinger always considers two-dimensional and one-dimensional material bodies ("Platten und Drähte") as special cases. In this he does not follow the Cosserat brothers who work more with an increase in spatial dimensions than with a successive reduction. Much more than that, in pp. 658–660, he shows his apprehension of the true mathematical problem at the basis of this reduction in dimensions by introducing small parameters (this time noted ε) that are representative of the slenderness in thickness or of two (small) lateral dimensions of the considered material structure. Equation (12.2) in p. 659 is typical of this "asymptotic" approach that will later on be the source of an efficient asymptotic derivation of equations for plates, shells and rods in the expert hands of Gold'denveizer, S.A. Ambartsumian, V.L. Berdichevsky, Ph. Ciarlet and others. Furthermore, Hellinger does not hesitate to introduce the Gaussian parametrization of curved surfaces to treat two-dimensional bodies [cf. pp. 620–621; in particular Eq. (14a, b)]. For one-dimensional elastic bodies, he is naturally led to mentioning the Bernoulli-Euler problem of the *elastica* (pp. 667–668) with the only surviving material coordinate taken as the arc-length along the curve. One had to await the remarkable work by A. E. H. Love (later perfected by R. D. Mindlin) to correctly deduce a quasi-one dimensional dynamical theory of rods with the strange lateral inertia term (the print left by the asymptotic procedure in passing from three dimensions to the rod-like picture).

12.4.6 On Thermodynamics and Dissipative Behaviours

In his introduction (p. 604) Hellinger clearly expresses his opinion that the "mechanics of deformable media, as an autonomous discipline, comprises under formal statements, next to the usual theory of elasticity and hydrodynamics, all the related physical manifestations in the considered continuously extending bodies". The development of these ideas has certainly been influenced by the discipline of *thermodynamics* which, in principle, tries "to embrace the totality of physics" (my translation). Here Hellinger is obviously influenced by his recent reading of "energetists" such as Pierre Duhem with his magisterial treatise of 1911 [12]. The latter may have been read by a handful of happy few.[5] What Hellinger tries in his Sect. 15 (pp. 682–695) is to incorporate the dual notions of entropy and thermodynamic temperature in his fundamental variational formulation. Entropy is considered as an extensive quantity (i.e., proportional to the quantity of matter). Then a term δQ—Equation (12.1) in p. 683—representing the "Wärmezufuhr" with variation of the entropy and co-factor none other than the temperature is to be added to the purely mechanical variation mentioned above at point 4.2. With the introduction of a potential for thermoelastic processes this yields the thermal definition of the temperature (in modern terms: the derivative of internal energy with respect to entropy) and, more surprisingly for the period, Maxwell's

[5] When in 1992, during a one-year stay in Berlin, I borrowed Duhem's [12] opus from the library of the former Kaiser Wilhelm Institute in east Berlin, I discovered that this copy of the books had never been read (pages were not cut out but they were damaged by the water poured by firemen during the fire of the Institute that occurred during the Russian Army take over of Berlin in 1945).

compatibility condition for second-order derivatives of the energy in *thermo-elasticity in finite strains* (Eq. (5) in p. 684; in modern notation this reads

$$\frac{\partial \mathbf{T}}{\partial S} = \frac{\partial \theta}{\partial \mathbf{F}}, \tag{12.2}$$

where **F** is the deformation gradient and **T** is the first Piola-Kirchhoff stress).

In a more general context Hellinger comments on other coupled effects such as temperature and capillarity, pyro-electricity and thermo-chemical processes as considered by J. W. Gibbs in his original works of 1876–1878. He does not mention piezoelectricity although this is already more than thirty years old (experimental discovery by the Curie brothers in 1881) when he writes his contribution.

The above mentioned variational formulation that includes the notion of entropy and temperature is seldom considered. However, Sedov's [46] generalized variational principle—also discussed in Maugin [35]—is along the same line.

For truly dissipative phenomena such as viscosity, in spite of his familiarity with Duhem's treatise which does not propose yet a solution (the future "theory of irreversible processes"), Hellinger is reduced to invoking the notion of dissipation potential à la Rayleigh, as in the case of G.G. Stokes' viscous fluids (cf. p. 671). But he is aware of the existence of more sophisticated models of viscosity. Such a model is the one proposed by Boltzmann [2] in the form—"*elastischen Nachwirkung*"—of hereditary integrals [see p. 641 and Eq. (5) in p. 672] for which Hellinger also cites very recent works, in particular by Vito Volterra up to year 1913 (the year Hellinger completed his manuscript). This shows the concern of this author to be up to date until the last moment. Finally, he also mentions the possible occurrence of a plastic behaviour with a simple plasticity criterion in terms of principal stresses which recalls the Tresca criterion—Inequalities (7) in p. 673—although he refers for these to a work of 1909 by A. Haar and Th. von Kármán. More general or singular behaviours are simply referred to as "halb-platische" oder "vollplastische" Zustände (no need for translation).

What is strange is that Hellinger does not comment on the then recent Caratheodory [6] axiomatization of thermodynamics as suggested by his own friend M. Born, a contribution that is purely in the analytical line and would certainly had been to Hellinger's liking.

12.4.7 On Newly Studied Behaviours

This is just mentioned for the sake of completeness since hereditary materials, half plastic or fully plastic materials, are already evoked in the preceding paragraph. But Hellinger also pays some attention to the phenomenon of *capillarity* in pp. 674–675 for which a rather not commonly referenced work is by the mathematician of "relativity fame", Herrmann Minkowski (see below).

Hellinger, although not pursuing the line further, gives the exact mathematical definition of material inhomogeneity (dependence of material properties on the material coordinates; see top of p. 639). General anisotropic elastic materials (crystals) with at most twenty one independent elasticity coefficients are mentioned for the linear case. In the case of finite strains, like all authors since Cauchy he focuses on the case of isotropy with the resulting introduction of the principal invariants of strains (p. 664) in the strain-energy function. This is purely academic as Hellinger and all other authors of the period could not guess that only rubber-like materials and then finitely-deformable soft biological tissues would provide in time the realm of application of this material description (see Maugin [38]).

12.4.8 On Relativistic Continuum Mechanics

In his attempt at a large conspectus of the State of the Art in 1913, Hellinger included (Sect. 16, pp. 685–694) comments on the most recent developments concerning the relativistic mechanics of continua. This is rather exceptional for the period; in particular if we compare with other well established authors in mechanics (e.g., Appell). This may have aroused his sensibility of mathematician. He seems to be well aware of the original developments by Voigt, Lorentz, and Poincaré on the group structure of special-relativistic transformations. The Lorentz-Poincaré group was a good subject of interest with the works of Minkowski [42], A. Sommerfeld, and F. Klein. His friend Max Born may have had some influence on Hellinger's interest in the field since Born (especially, [4]) and Herglotz [28] seem to be his main sources for the basic definitions and the problem of the possibility of "rigid-body motion" in relativity.

Most of Hellinger's discussion is about the essential differences between the Lorentz-Poincaré group and the Galilean-Newtonian group of space-time transformations. But he is also particularly interested in two points. One is the possible re-formulation of the Cosserats' action principle in space-time in agreement with Minkowski and Herglotz (cf. Eq. (13a, b) in p. 693) with a space-time parametrization that combines material coordinates and a propertime (a parameter along the world line following Minkowski's description) and a total virtual variation for internal forces (components of the energy-momentum tensor). The second point is the possible definition of the notion of *rigid-body motion*, a much discussed matter being given the existing bound on velocities, with the possible local (i.e., differential) solution given by Born and Herglotz in space-time. Allusion to relativistic continuum mechanics will later be given in a bibliographic appendix by Truesdell and Toupin [49, pp. 790–793]. The present writer is one of the very few to have devoted a full albeit brief chapter to relativistic continuum mechanics in a treatise (cf. Eringen and Maugin [16], Vol. 2, Chap. 15; see also the historical perspective in Maugin [38], Chap. 15).

12.5 Conclusion

In his introduction—written in 1913—Hellinger claims that there exists no textbook or monograph in the literature on the specific subject treated in his contribution although there do exist textbooks and treatises of a general nature, but the latter do not emphasize the bases and various possibilities offered by the scheme of continuous matter. He does not intend to treat applications and specific problems. He confines himself to the essentials, “die allgemeinen Ansätze” in his own words. His viewpoint is that analytical mechanics (exploitation of variational formulations) is “the most uniform and efficient manner to approach the general problem of describing a large variety of descriptions of deformable media” in agreement with recent authors like the Cosserats and the initial standpoint of G. Green with an energy potential. This is the type of approach (principle of virtual work, d’Alembert’s principle [9], Lagrange-Hamilton action principle [32], etc.) that suits best his essentially mathematical vision. The pregnant brevity of this approach possesses a “high heuristic value for the exploration of new areas. This is particularly stressed through the intimate relation of such variational principles with thermodynamics”. Furthermore, this allows one to place in evidence the invariant theoretical nature of the considered problems with the notion of transformation groups. This gives a very “modern” print that helps us understand his exposition without too much effort. This “modernity” is striking in spite of the somewhat obsolete notation. Its opens up horizons to many models that will have to wait progress in some branches of pure and applied mathematics for a full blossom (e.g., large deformations, media with internal degrees of freedom, capillarity, hereditary processes, multi-field phenomena).

His mathematical inclination leads him to accept unhesitatingly all new mathematical tools of the period (vector and tensor analysis, matrix calculus, differential geometry, perturbations). The only part that is still missing is convex analysis to be much developed in the 1950s–1970s. But, overall, Hellinger is very successful in his endeavour. This is our appraisal one hundred years later. Unfortunately, we were not able to locate any substantial review or criticism of his contribution in the few years following its publication so that we have no precise idea of the quality and extent of its reception among professional circles, mechanicians and mathematicians. This may exist in some periodical bulletin of a mathematical society. It is therefore with modern eyes, perhaps themselves influenced by Hellinger’s writing—a kind of feedback—that we evaluate it. This is an inevitable bias that we willingly acknowledge.

Following the Cosserat brothers, Hellinger’s view of the domain of interest of continuum mechanics is essentially the mechanics of deformable bodies, by which must be understood the case of deformable solids. This is in contrast with treatises by famous authors such as Appell [1], where most contents rather deal with fluids. At the time fluid mechanics has become a rather autonomous field of study limited to perfect fluids and the Navier-Stokes equations, with specific mathematical techniques of which the use of complex variables has become endemic. But some recent developments of theoretical fluid mechanics such as the asymptotic method involved in the theory of the boundary layer by L. Prandtl could have been to the taste of an analyst like Hellinger. It is only with the birth of the science of *rheology* (concerning whatever can flow to a larger or smaller extent) and the notion of non-Newtonian fluids in the 1920s in the expert hands of E. C. Bingham (1878–1945) and M. Reiner (1888–1976) that fluids will return to the general stage of continuum mechanics. Liquid crystals, with a behaviour clearly classified in 1922 by Georges Friedel (1865–1933) and exhibiting mixed crystal (ordered state) and fluid (flow) characteristics with directional properties, will also enter this general framework with a natural connection with generalized continua of the Duhem-Cosserat type established in the 1960s–1970s. This could not be imagined by Hellinger who remains essentially an analyst, as shown by his other very successful contribution to the same Encyclopaedia of mathematics in co-operation with a friend of student days in Breslau and Göttingen, O. Toeplitz (cf. Hellinger and Toeplitz [27]).

In conclusion, we find in Hellinger's brilliant and very informative contribution all elements and remarks that we would like to deliver—even though superficially—to our mathematically oriented students in an introductory course of high level (e.g., something similar to what Germain tried to do in his course at Ecole Polytechnique, 1986 [20]); many students then thought that this was too much superficial, although all aspects of further developments in specialized short courses were outlined.

Appendix A

Partial translation from the German to English of Hellinger's contribution to the EmW (by Eleni and Gérard A. Maugin, © 2013).

Note: pages of original are indicated at the top left. Modern notations (cf. Maugin [38]) are sometimes given within squared brackets [..] along with Hellinger's notation. Footnotes are not given in full, being just replaced in the main text by a name and a year within brackets for a reference to an author. Some translator's remarks within square brackets are indicated by the initials GAM. Abbreviation EmW means this encyclopaedia.

The general basic laws of continuum mechanics
By E. Hellinger, Marburg A.I.

Contents

1. Introduction
2. The notion of continuum
   a) The continuum and its deformation
   b) Adjunction of physical parameters, density and orientation in particular
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I. **The basic laws of statics**

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Bibliography

For the time being [Hellinger's words, GAM] there are no textbooks or monographs in the literature on the specific subject treated here. In order to avoid repetition we have compiled a list of the most frequently cited works:

A. von Brill (1909).
E. and F. Cosserat (1909).
P. Duhem (1911).
G. Hamel (1912)
J.L. Lagrange (1788) and in Oeuvres complètes, Vol.11 and 12. edited by G. Darboux, Paris 1988/89.
W. Voigt (1895/96).
Cf. also Voss, Stäckel, Heun and Müller-Timpe in the EmW, Vol. IV.

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1. **Introduction**

The purpose of the present work is to give, from a uniform point of view, a comprehensive overview of the various *forms* taken by the different basic laws used in order to determine the evolution in time or even the state of equilibrium in an isolated spatial domain of "continuum mechanics" as a whole, i.e., the mechanics and physics of continuously extending media. Moreover, we shall always keep in mind only those types of continua that do not possess, thanks to restricting conditions, a particularly large number of continuous degrees of freedom. The possibility of expressing in a comparable form the basic equations of various disciplines has already been noticed in the past.

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The "mechanistic" theories of physics which would have reduced the physical existence to the manifestation in the form of motion have considered the quantity of matter from a formal-mathematical point of view, permitting thus to exhibit the equations of physics as special cases of the equations of a general system of varying masses in motion, as also of mass points. They must also make evident these analogies.

Next to the truly mechanical theories, which present more or less detailed pictures of the structure of matter, there has been an attempt, almost from the beginning, but more particularly from the middle of the nineteenth century, to adopt a specific method from analytical mechanics in the manner of J.L. Lagrange; in order to bring under the same general principles all the considered problems, there has been an effort to reduce the fundamental laws of an ever larger number of physical disciplines, to the form of those principles. From a purely phenomenological viewpoint, this could permit the identification of notions - energy, forces, etc. - entering them with certain physical entities. For systems with a finite large number of degrees of freedom, this development is mainly connected with research undertaken on cyclical systems and their applications in the reciprocal laws of mechanics by W. Thomson (Lord Kelvin), J.J. Thomson and H. von Helmholtz.

Eventually, even Lagrange applied his principles to some continuous systems (liquids, flexible strings and plates, etc.). After further elaboration of these approaches, particularly with the development of the theory of elasticity associated with A.L. Cauchy, as well as under the influence exerted by the development of other physical, particularly optical, theories, people became more and more accustomed to considering even continuous systems as autonomous objects of mechanics (with an infinite number of degrees of freedom), since although these systems stand in formal analogy to the old point mechanics, they can perfectly well be treated independently. The "mechanics of deformable continua", as an autonomous discipline, comprises under the formal statements, next to the usual theory of elasticity and hydrodynamics, all the related physical manifestations in the continuously extending media considered here.

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The development of these ideas has certainly been influenced by the discipline of *thermodynamics* which, in principle, tries to embrace the totality of physics and, this way, by putting forward everywhere the general energy function, hence a potential, it naturally yields analogous forms to the fundamental equations of various fields.

All these relations have been treated in the literature of mechanics and physics in many different ways. A lot of what was said in particular in the field of point mechanics, as also of systems with an infinite number of degrees of freedom, can be immediately extended to the continuous systems. Let us mention already the names of only a few authors who have paid special attention to the relations that we will discuss here and that we will often have the occasion to cite in the sequel: W. Voigt (1895–1896), P. Duhem (1911), and E. and F. Cosserat (1909) (For development of a similar kind, what follows has been influenced in many ways by some of the lectures given in Göttingen by D. Hilbert.)

The purpose of this work demands that, in what follows, the *pure* formal-mathematical factor stands in the foreground, by formulating the statements as well as their combinations in a homogeneous and in a, as simple and elegant, way as possible. The research of the mechanical and physical significance of the quantities and equations as well as the proper analytical-mathematical theory are included in various contributions to volumes IV and V—of the present encyclopaedia—where the various disciplines are discussed.

As a uniform mathematical formulation, which is the easiest to apply to the totality of all individual laws, we have used the *variational principle*. However, we find unsatisfactory the form that we observe as a rule in the calculus of variations, and where the unknown functions are determined in such a way that a certain defined integral containing them, acquires an extremal value. Here we find much more preferable the form that yields the variational computation as a necessary condition of the extremal and which has always been expressed by the principle of virtual work: "Let there be an ordered set of quantities $X, \ldots, X_a, \ldots$ dependent on the unknown function x of $a, \ldots, c$ and their derivatives; these functions should satisfy the condition that a determined integral of a linear form represented by these $X, \ldots, X_a, \ldots$ as coefficients, of the arbitrary functions $\delta x, \ldots$ of $a, \ldots, c$ and their derivatives

$$\int \ldots \int \left\{ X\delta x + \ldots + X_a \frac{\partial\, \delta x}{\delta a} + \ldots \right\} \quad da \ldots dc$$

or a sum of such integrals – vanishes identically for all $\delta x, \ldots$ (or else for all those satisfying certain auxiliary conditions)."

The advantage offered by the application of such a variational principle as a basis, as compared to other possible formulations, or even by taking into consideration the fundamental laws, is mainly that the variational principle is able to determine by a single formula the behaviour of the medium under consideration, in all places and at any instant of time, and especially to cover, besides the equations within the enclosed volume, both the boundary conditions and the initial conditions. Moreover, in its pregnant brevity, it is, in a way, much more transparent than the basic laws and, consequently, it possesses a substantial *heuristic* value for the exploration of new areas, for the expression of other generalisations, etc. This is particularly stressed through the intimate relation of the variational principle with thermodynamics. On the other hand, its claim to generalisation is of demonstrative value for the foundations of physical theories. But the variational principle, through the acceptance of coordinate transformations, has also another advantage against the explicit (field) equations; it often permits an easier understanding of the *invariant theoretical nature* of the considered problem, the question about the transformation groups which it leaves unaltered, with no need to introduce any special symbolism.

After an introductory discussion of the notion of continuum and its kinematics we shall present in the first chapter of this work the basic statements of *statics*, and in the second those of the *kinetics*, but regardless of the kind of the force effects that one of them exerts on the continuum. The nature of these force effects, and especially their dependence on the position and the motion of the continuum (*dynamics*), will be discussed in the third chapter, in which we classify the various disciplines; finally, in the same chapter we shall give a short draft of the relation with the laws of thermodynamics on the one hand, and, on the other, we shall stress the behaviour of some statements under transformations of the space and time coordinates and also the interpretation of the *relativistic theory of electrodynamics*.

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2. **The Notion of Continuum**
2a. **The continuum and its deformation**

The general three-dimensional extending continuous medium to which the following considerations apply means - abstraction made of specific properties of matter - a set of material "particles" which (a) are individually identifiable and (b) fill continuously the space within a regular bounded domain. The first property can be expressed by the fact that each particle is identified thanks to three parameters *a, b, c* (in modern terms, a labelling with material coordinates X^K, $K = 1, 2, 3$) so that under any condition that we may consider the medium, they always occupy a different place; the variable volume V_0 of these (particles labelled) *a, b, c* enclosed within the regular surface S_0, characterises the quantity of matter considered here. The second requirement means that the positions of all particles fills (after deformation and motion) a volume V bounded by the regular surface S. If the position of a particle is determined by its Cartesian coordinates $(x, y, z = \{x^i,\ i = 1, 2, 3\})$, then such a condition can be given analytically by the three following functions of *a, b, c*

$$x = x(a, b, c), y = y(a, b, c), z = z(a, b, c) \qquad \left[x^i = x^i\left(X^K\right)\right] \tag{1}$$

which map V_0 into V and whose functional (Jacobian) determinant

$$\Delta = \frac{\partial\,(x, y, z)}{\partial(a, b, c)}\ \left[J = \det\left(\frac{\partial\, x^i}{\partial X^K}\right)\right] \tag{2}$$

inside V_0 does not vanish and is taken positive. We can take a fixed "final" (actual) position for *a, b, c*; then $x - a$, $y - b$, $z - c$, are the components of the translation suffered by each particle in its transition to position (1) and the functions (1) become continuous functions of *a, b, c* as long as we assume that the initially neighbouring particles always remain neighbours. Moreover, we can always suppose that the functions (1) possess enough derivatives with respect to their arguments; disruptions of continuity can be found only at singular points, lines and surfaces (Cf. Voss, Vol 4/1 of EmW, No.9). We shall generally not repeat similar assumptions about further physical occurrences of representative functions.

Each function system (1) fully describes a definite state of deformation of the continuum. Generally speaking, every deformation solution, i.e., every triplet of functions (1) that satisfies the just mentioned continuity conditions, is considered admissible. Restrictions in p. 607 the kind of possible functions will express specific properties of special materials. In any case, the partial derivatives of the functions (1) determine, as we know, the translations, rotations and form changes that suffer every small volume element during deformation (Cf. Abraham, in EmW, IV-14, no. 16).

The basis for the research of the equilibrium solution of any deformation process (1) is obtained by superimposing on it a so-called *infinitesimally small virtual perturbation*, called *virtual* to the extent that it enters arbitrarily in the real existing deformation case [cf. Voss EmW IV-1 No. 30; Voigt (1895–96) and C. Neumann (1879)]. In order to define this notion in a precise mathematical form, without giving up the usual convenient designation and use of the "*infinitesimally small*" quantity, we consider to begin with one of the deformation on which is superimposed another deformation depending upon a parameter σ, with vanishing deformation for $\sigma = 0$, which carries the particle from the original position (x, y, z) to the position

$$\bar{x} = x + \xi(x, y, z; \sigma),$$

[same with (x, y, z) and (ξ, η, ζ)]. This way (ξ, η, ζ) are functions of (x, y, z) and of the parameter σ, which can vary in any small neighbourhood of $\sigma = 0$. Thanks to (1), after elimination of (x, y, z), we can also write the newly introduced deformations in the other form

$$\bar{x} = \bar{x}(a, b, c; \sigma), \; where\, \bar{x}(a, b, c; 0) = x(x, y, z). \tag{3}$$

If f is any of the deformation functions (1) and we consider their derivatives as independent expressions, then we generally note as its "variation" the expression

$$\delta f(x, \ldots, x_a, \ldots) = \left\{ \frac{\partial}{\partial \sigma} f\left(\bar{x}, \ldots, \bar{x}_{a,\ldots}\right) \right\}_{\sigma=0}, \; where\, x_a = \frac{\partial x}{\partial a}, \ldots;$$

yet, during the differentiation *a, b, c* remain constant; the operation δ commutes with the differentiation with respect to *a, b, c*:

$$\delta \frac{\partial f}{\partial a} = \frac{\partial(\delta f)}{\partial a}.$$

If the three functions

$$\left.\frac{\partial \bar{x}}{\partial \sigma}\right|_{\sigma=0} = \left.\frac{\partial \xi}{\partial \sigma}\right|_{\sigma=0} = \delta x(x, y, z), \; same\, for\; (x, y, z)$$

which, thanks to (1), can be considered as function of (x, y, z), do not vanish identically in (x, y, z), then, following the usual stability postulate, we can write

$$\bar{x} = x + \sigma\, \delta x(x, y, z), same\, for\; (x, y, z), \tag{3'}$$

if σ is chosen so small that σ^2 is sufficiently small compared to σ, the so given infinitesimally small virtual perturbation of the continuum is then determined up to the factor σ by the three functions $\delta x, \delta y, \delta z$ of x, y, z. We can immediately classify this perturbation under the notion of "infinitesimally small deformation", as studied in the kinetics of continua (Cf. Abraham, EmW IV-14, No. 18) and we also find that the "*virtual form changes*" ["strains", GAM] of these volume elements derived from it, are determined by the following six quantities

$$\frac{\partial \delta x}{\partial x}, \frac{\partial\, \delta y}{\partial y}, \frac{\partial\, \delta z}{\partial z}, \frac{\partial\, \delta y}{\partial z} + \frac{\partial\, \delta z}{\partial y}, \frac{\partial\, \delta x}{\partial z} + \frac{\partial\, \delta z}{\partial x}\, \frac{\partial\, \delta x}{\partial y} + \frac{\partial\, \delta y}{\partial x} \tag{4}$$

and their "*virtual rotations*" by

$$\frac{1}{2}\left(\frac{\partial \delta z}{\partial y}-\frac{\partial \delta y}{\partial z}\right), \frac{1}{2}\left(\frac{\partial\, \delta x}{\partial z}-\frac{\partial\, \delta z}{\partial x}\right), \frac{1}{2}\left(\frac{\partial \delta y}{\partial x}-\frac{\partial \delta x}{\partial y}\right), \tag{4'}$$

regardless of the σ factor.

A *motion of the continuum* will be interpreted as a consequence of a dependence of the deformation functions upon the time parameter t, and accordingly expressed through the three deformation functions

$$x = x(a, b, c; t),\ y = y(a, b, c; t),\ z = z(a, b, c; t) \quad \left[x^i = x^i\left(X^K, t\right)\right] \tag{5}$$

always depending upon t; these, as functions of all four variables in the necessary neighbourhood, are constant and differentiable. For fixed *a, b, c* (5) represents the trajectory of a certain specific particle.

Just as exposed above, by including in the formulas only the variable t, next to the motion (5) we also introduce the group of motions for $\sigma = 0$, that was omitted in (5),

$$\bar{x} = \ \bar{x}(a, b, c; t; \sigma) = \ x \ + \ \sigma\delta x(x, y, z; t),\ \textit{same for}\ (x, y, z)$$

for small values of the parameter σ and we note δx, δy, δzas the definitions of the *virtual perturbations* superimposed on the motion (5).

2b. **Adjunction of Physical Parameters, Density and Orientation in Particular**

Each physical property of a medium can be described by one or more functions of a, b, c; t which enter in the deformation functions.

In what follows we shall make general use of one such property, the presence of an *invariable mass m* for every volume element V_0 of the medium, which, as an integral over V_0, is expressed as a characteristic density function $\rho_0 = \rho_0(a,\ b,\ c)$ of the medium. By transition to the deformed location (1)

$$\rho = \frac{\rho_0}{\Delta} \qquad \left[\rho = J^{-1}\rho_0\right] \tag{7}$$

results as the true mass density ρ of the distribution of the medium, and the mass in the part V' of V is

$$m = \iiint_{(V')} \rho\, dx\, dy\, dz = \iiint_{\left(V'_0\right)} \rho_0 da\, db\, dc.$$

The variations of the continuum's location in relation to the behaviour of such an adjunction of a physical parameter are not yet firmly laid down. In the meantime, we always leave the mass of such an elementary quantity of matter, i.e., the function $\rho_0(a, b, c)$ unchanged by a virtual perturbation and we replace the density ρ by

$$\bar{\rho} \ = \ \bar{\rho}(x, y, z; \sigma) = \rho + \sigma\delta\rho(x, y, z), \tag{8}$$

so that regarding the continuity condition (cf. EmW IV-15, No.7 p.59 on, A.E.H. Love)

$$\delta\rho_0 = \delta(\rho\Delta)\, or\, \delta\rho + \rho\frac{\partial(\delta x)}{\partial x} + \rho\frac{\partial(\delta y)}{\partial y} + \rho\frac{\partial(\delta z)}{\delta z} = 0.$$

The same thing will be valid in the case of motion, i.e., $\rho_0(a, b, c)$ remains independent of t and ρ will be given as in (7).

There is another basic notion which belongs here and which we will use very often, that is, the idea *that for every particle of the continuum, the various directions attached to it possess different characteristic meanings, and that, for this reason, the specification of its orientation belongs essentially to the description of the situation of the continuum*. This kind of representations was developed in the molecular theory, where the bodies of crystalline structure were viewed as molecules; S.D. Poisson (1842) in particular has applied it in order to establish a better theory of elasticity. Recently, E. and F. Cosserat [1907; Théorie des corps déformables, 1909; Heun in EmW IV-11, Part II]) without any reference to molecular representations have treated extensively such continua equipped with a definite orientation in every particle.

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In a more general way, this notion of oriented particles of the continuum can be formulated analytically [Cf. a remark by P. Duhem 1893 p. 206], since we can think of each particle a, b, c of the continuum as equipped with a *trihedron* (triad; GAM) *of axes at right angles* and these three axes have each director cosines α_i, β_i, γ_i ($i = 1, 2, 3$) in order to describe fully the state of such a medium, next to the functions (1) we must also recognize as functions of a, b, c three independent parameters λ, μ, v (e.g., the Eulerian angles) that define the orientation of such a medium in relation to the coordinate system x, y, z:

$$\lambda = \lambda(a, b, c), \quad \mu = \mu(a, b, c), \quad v = v(a, b, c). \tag{9}$$

Now, every virtual perturbation of the continuum shall be connected with a *virtual rotation* of this trihedron; this way, we get as a basis a group of rotations depending on a parameter σ and with vanishing $\sigma = 0$, starting from the position (9) and replace λ, μ, v, being restricted to sufficiently small values of σ, by

$$\bar{\lambda} = \bar{\lambda}(a, b, c; \sigma) = \lambda + \sigma\delta\lambda(a, b, c) \quad same\, for\, (\lambda, \mu, v). \tag{10}$$

In this manner it is always possible to interpret λ, μ, v as well as $\delta\lambda$, $\delta\mu$, δv either as functions of a, b, c or, with the help of (1), as function of x, y, z. The variations themselves $\delta\alpha_1, \ldots, \delta\gamma_3$ of the director cosines of the three axes are linear homogeneous functions of $\delta\lambda$, $\delta\mu$, δv obtained through the differentiation with respect to σ of the explicit expressions of $\alpha_1, \ldots, \gamma_3$; the components $\delta\pi$, $\delta\kappa$, $\delta\rho$ of the virtual rotation angle velocity in the three axes, are connected with $\delta\alpha_1, \ldots, \delta\gamma_3$ through the formulas

$$\delta\pi = \beta_1\delta\gamma_1 + \beta_2\delta\gamma_2 + \beta_3\delta\gamma_3 = -(\gamma_1\delta\beta_1 + \gamma_2\delta\beta_2 + \gamma_3\delta\beta_3)\, etc \tag{11}$$

$$\delta\alpha_i = \gamma_i\delta\kappa - \beta\delta\rho_i, i = 1, 2, 3, \quad etc; \tag{11'}$$

Incidentally, in contrast with the symbol δ used until now, these are not variations of certain definite functions of a, b, c, but become simultaneously linear homogeneous functions of $\delta\lambda$, $\delta\mu$, $\delta\nu$; we set

$$\delta\lambda = l_1\delta\pi + m_1\delta\kappa + n_1\delta\rho,\ etc. \tag{12}$$

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This way, $\delta\pi$, $\delta\kappa$, $\delta\rho$ (given as functions of a, b, c or x, y, z) define also the virtual rotation of the continuum [These are well known kinematic methods of the theory of surfaces (cf. also EmW, Vol. III D3 No.10; G. Darboux, Leçons sur la théorie générale des surfaces) that E. and F. Cosserat have applied (detailed exposition in their "Théorie des corps déformables", 1909)].

All these formulas can be extended immediately to the case of motion via the inclusion of the time parameter t.

2c. **Two- and one-dimensional continua**

By the suppression of one or two of the three parameters a, b, c, we also obtain immediately the statements for the treatment of two- and one-dimensional continua embedded in three-dimensional space [In a certain sense these problems are simpler than those we meet with in three-dimensional media; in fact some of them belong to the problems of continuum mechanics which have received early a very detailed treatment (cf. P. Stäckel in EmW IV-6, Nos. 22-24, also K. Heun in EmW IV-11, No.19, 20)]. In any case, their position is given by

$$x = x(a,b)\ or\ x = x(a) \quad [same\ for\ (x,y,z)]; \tag{13}$$

The parameters vary in an area S_0 (respectively, along a curve C_0) of the plane $a-b$ (respectively a line of arc length a) which through (13) is based upon a surface S (respectively a curve C). Here also we can assign to each particle a triplet of directions, orthogonal to each other [Cf. E. and F. Cosserat, Chapters II and III, 1909], defined by the functions

$$\lambda = \lambda(a,b),\ respectively\ \lambda = \lambda(a)\ [same\ for\ (\lambda,\mu,\nu)]. \tag{14}$$

12.8 The Basic Laws of Statics

3. **The principle of virtual perturbations**

3a. Forces and stresses

In order to construct the dynamic properties of the continuum upon this kinematic scheme, we shall rely upon the notion of *work*. The totality of the forces and stresses of all kinds which affect the continuum, because of its previous deformation conditions, of its position ["placement", GAM] in space or of some external circumstances - initially considered as a whole without regard to their origin—is in one expression, since they achieve, in every virtual perturbation, a "*virtual work*" δA; this is for us of primary importance and we define it as follows: *let δA be given as a linear homogeneous function of the totality of values of the perturbation components inside the continuum; and let it be a scalar quantity independent from the choice of the coordinate system*. The coefficients, with which each value of δx, δy, δz enters in δA, are the definition parts of the single active force system; the fact that p 612 these are independent from the virtual perturbations (i.e., the linearity of δA) makes us think that, due to their smallness, these perturbations do not modify the usual force effects exerted on each particle. In order to cover the totality of the laws of continuum mechanics, it is necessary to start from the most general expression of the already described types for δA, that consists of the sum of the linear functions of the quantities δx, δy, δz and their derivatives, in any single point of these expressions, on the line, surface and volume integrals which may compose such an expression. We rather consider, at the beginning, an expression - that we shall later elaborate—that consists of a volume integral extending over the whole region V of the continuum, and also an outer-surface integral extending over its surface S; this way, the first one contains a linear form of the nine derivatives of δx, δy, δz with respect to x, y, z [Such statements for the virtual work have been developed earlier, as obvious generalisations of the formulas of point mechanics, for many special problems.....]:

$$\delta A = \delta A_1 + \delta A_2 + \delta A_3, \tag{1}$$

with

$$\delta A_1 = \iiint_{(V)} \rho(X\delta x + Y\delta y + Z\delta z)dV \quad \left[\delta A_1 = \iiint_{(V)} \rho f_i \delta x_i dV\right]$$

$$\delta A_2 = -\iiint_{(V)} \left(X_x \frac{\partial \delta x}{\partial x} + X_y \frac{\partial \delta y}{\partial y} + \ldots + Z_z \frac{\partial \delta z}{\partial z}\right) dV \quad \left[\delta A_2 = -\iiint_{(V)} \sigma_{ij}(\delta x_i)_{,j} dV\right]$$

$$\delta A_3 = \iint_S (\bar{X}\delta x + \bar{Y}\delta y + \bar{Z}\delta z)dS \quad \left[\delta A_3 = \iint_S \bar{t}_i \delta x_i dS\right].$$

The fifteen coefficients present here, - factors of the already discussed perturbation quantities—will be, for every deformation of the considered medium, definite *finite continuous functions of x, y, z or a, b, c, along with their derivatives, everywhere, with the eventual exceptions of certain surfaces*. The obvious meaning of statement (1) then is that, in general, we will only take into consideration the continuously distributed *forces* over space as well over singular surfaces and the continuously distributed *stresses*.

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Initially, the first and last terms in δA are constructed in a very much analogous way with the well known work expressions of point mechanics, except that the factor present now is the mass of the volume element ρdV (respectively the surface element dS; so X, Y, Z are to be thought of as components of the acting forces on the mass unit of the medium, and $\bar{X}, \bar{Y}, \bar{Z}$ as components of the forces acting per unit surface on the outer surface, at the proper point. Since $\delta x, \delta y, \delta z$ are the Cartesian projections of a polar vector and since δA, as a scalar, remains invariant under coordinate transformations, these forces are also polar vectors.

Actually, the integral δA_2 is characteristic of continuum mechanics. The nine coefficients $X_x, \ldots, Z_z$ - in the known designation of Kirchhoff [1855, also works 1882, p.287] that measure the influence of the single determining parts of the virtual deformation by the performed work, will be understood as the *components of the stress state* at the point in question, calculated according to its influence upon the unit volume. Their behaviour, during the coordinate transformations, results from the remark that the nine derivatives $\partial\delta x/\partial x, \ldots, \partial\delta z/\delta z$ of the vector components behave during orthogonal coordinate transformations like the nine products of two vectors (a so-called *dyad*) [Here Hellinger refers to F. Klein, Abraham, Gibbs and Wilson, Heun, and to Cayley's matrix calculus; GAM]

$$X_1.X_2, \ldots, Y_1.Y_2, \ldots, Z_1.Z_2$$

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while the bilinear combination $X_x.\partial\delta x/\partial x + \ldots$ remains invariant. Therefore, if we want to speak of *stress dyads*, the stress components must be transformed again as dyad components. It is possible to decompose any dyad in a (symmetric) component consisting of six elements (a *tensor triple* [Cf. Voigt's terminology; Abraham in EmW IV-14, No.17])

$$X_x, Y_y, Z_z, \frac{1}{2}\left(Y_z + Z_y\right), \frac{1}{2}(Z_x + X_z), \frac{1}{2}\left(X_y + Y_x\right) \quad [\sigma_{(ij)} = \frac{1}{2}\left(\sigma_{ij} + \sigma_{ji}\right)] \qquad (2)$$

and as (skew symmetric) component of three elements

$$Z_y - Y_z, X_z - Z_x, Y_x - X_y \quad [\sigma_{[ij]} = \frac{1}{2}\left(\sigma_{ij} - \sigma_{ji}\right)] \qquad (2')$$

representing an *axial vector*. This splitting corresponds to the emphasis given in Section 2 to the two separate statements (4) and (4') of the virtual deformations of the continuum, and when the integrands of δA_2 are split in the same way

$$\sum_{(xyz,XYZ)} \left\{ X_x \frac{\partial\delta x}{\partial x} + \frac{1}{2}\left(Y_z + Z_y\right)\left(\frac{\partial\delta y}{\partial z} + \frac{\partial\delta z}{\partial y}\right) + \left(Z_y - Y_z\right)\frac{1}{2}\left(\frac{\partial\delta z}{\partial y} - \frac{\partial\delta y}{\partial z}\right)\right\}$$

[where the indication below the summation sign means that the summing expression consists of cyclical exchanges of x, y, z and X, Y, Z].

What follows here in particular is that the six quantities (2) determine that part of the stress that performs work in an infinitesimally small proper form change of the continuum [the strains. GAM] and therefore *the true elastic effects*, while the vector (2') makes possible the determination of the part (that performs work), by the virtual rotation of the volume elements, again without form change, and so the *rotation moment* determined by the stress condition. Moreover, from the negative sign in (1), it results that with positive X_x the performed work is positive even with negative $\partial\delta x/\delta x$, which is then measured as *positive pressure*.

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In order to obtain finally from the statement (1) the meaning of the stress component as surface forces [Cf. C.L. Navier, G. Green], we think of the part of the calculated virtual work reached by the stresses inside a part V_I of the continuum delimited by the closed surface S_I, i.e., the part of the integral δA_2 extended over V_I; if the stress components inside V_I are all, without exception, continuous, then by partial integration and application of the "*Gauss theorem*" (see EmW Chapter IV-14, p.12), this goes over to

$$\iiint_{V_I} \sum_{(xyz,XYZ)} \left(\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}\right) \delta x dV$$
$$+ \iint_{(S_I)} \sum_{(xyz,XYZ)} \big(X_x \cos nx + X_y \cos ny + X_z \cos nz\big) \delta x dS_I,$$

where n means the rotated normal's direction of the surface S_I under V_I at the position of the element dS_I. By comparison with (1), it follows that the stress condition in V_I performs the same virtual work, i.e., it acts exactly as if, next to the volume forces in V_I, upon the surface element dS_I of S_I, computed per unit surface, we had in action the force

$$X_n = X_x \cos nx + X_y \cos ny + X_z \cos nz, (X, Y, Z) \quad [\bar{t}_i = \sigma_{ij} n_j]. \tag{3}$$

This "*pressure theorem*" of Cauchy, by specialisation of the direction of n, yields, as we know, the meaning of the nine components [Cf. Müller-Timpe in EmW IV-23, No.3a; Helmholtz, 1902].

3b. **Survey of the principle of virtual perturbations**

Based on the constructions of the above notions, it is possible to transpose immediately the *Principle of virtual perturbations*, governing the statics of discrete mechanical systems to continuum mechanics: In a determined case of deformation, a continuous medium, in which there are present certain volume forces $X\ldots$ and outer surface forces $\bar{X}\ldots$ and a certain stress condition $X_x\ldots$, is then and only then in equilibrium when the total virtual work of these forces and stresses for each virtual perturbation which is compatible with the conditions somehow imposed on the continuum, vanish:

$$\iiint_{(V)}\left\{\rho\sum_{(xyz,X,Y,Z)}X\delta x-\sum_{(xyz,XYZ)}\left(X_x\frac{\partial\delta x}{\partial x}+X_y\frac{\partial\delta y}{\partial y}+X_z\frac{\partial\delta z}{\partial z}\right)\right\}dV+\iint_{S}\sum_{(xyz,XYZ)}\bar{X}\delta x dS=0. \tag{4}$$

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Actually, J.L. Lagrange had already conducted this transformation, when he established as the basis of his analytical mechanics the [John] *Bernoulli principle of virtual perturbations*; for him, an obvious consequence of the validity of this principle in the point mechanics, is its applicability in his available problems of continuum mechanics, where he always preferred to represent the work expression by a transformation of the limit of the discontinuous system out of or through direct intuition. Ever since, in the further development of the bounding areas of continuum mechanics people have shown a preference for the principle of virtual perturbations; often, they also have, just like Lagrange, relied on the idea that the continuum could be approached through a system of an infinite number of mass points, and that, at the same time, all physical effects in the continuum could be approached through equivalent effects in this approximate system; actually, it seems that the axiomatic specification of this relationship which, for the convertibility of these analogies, needs to postulate, above all, the necessary continuity requirements by strict deduction, does not seem as yet to have been obtained. In the meantime, for continuum mechanics, we prefer and place on top as the *highest axiom* the initially formulated principle itself. And we adopt this standpoint so much more willingly when we consider that the representation of the continuously extending media is much more natural than the abstract "mass points" of the point mechanics [Recently, this view had been particularly supported by G. Hamel, 1908, p.350 - also Hamel's textbook of 1912 where he gives a complete axiomatics of continuum mechanics, that resolves a basic principle like the one used here in a series of independent propositions]. The certainty of the correctness of this axiom is based on one hand on the fact that such a statement corresponds to our general ideas and thinking habits about physics, but mainly on the fact that it is appropriate enough to sufficiently represent the facts of experience.

3c. **Application to continuously deformable continua**

The well known formal operations of the calculus of variation show how easily we can, in many cases, transform the principle of virtual perturbations in a great number of equations between forces and stresses. As a start, if we consider only as typical the sufficiently continuous deformable medium, which is in no way restricted by side conditions, then the condition (4) for every system of continuous functions $\delta x, \delta y, \delta z$ is fulfilled. The transformation of (4) by partial integration, if the forces, stresses and their partial derivatives are always continuous in V, yields then the equations

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1) at every point in the domain V

$$\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}+\rho X=0\,(X,Y,Z)\quad\left[\frac{\partial\sigma_{ij}}{\partial x_j}+\rho f_i=0\right] \tag{5a}$$

2) at every point of the bounding surface S with outer pointing normal directions n

$$X_x\cos\, nx+X_y\cos\, ny+X_z\cos\, nz=\bar{X}\quad\left[\sigma_{ij}n_j=\bar{t}_i\right]. \tag{5b}$$

Therefore, along with the boundary surface condition, we obtain the so-called "stress equations", *that offer necessary and sufficient conditions, so that a determined system of forces and stresses acting at a certain position in a freely deformable continuum be in equilibrium* [These equations are similar to those of A.L. Cauchy, 1828.] Certainly, these conditions are by no means sufficient for us to determine the stress and force components: in order to do this we must introduce the relations that we will treat later, and which emphasize the dependence of the forces and stresses from the actually existing deformation or from other external sources (Cf. Stäkel in EmW IV-6, No.26, and Müller-Timpe in EmW IV-23, No.3b).

In (4) and (5) the independent variable coordinates are in the *deformed* configuration [Hellinger uses "condition". GAM] of the continuum, and the force and stress components find their evident meaning as effects upon mass units and with respect to the surface unit of the medium in a deformed configuration. In contrast to this, following S.D. Poisson's works [Poisson 1829, 1831; This difference has often been overlooked, since at closer examination of infinitesimally small deformations of a stressless quiet state, it actually vanishes so it has only been shown to advantage in the development of the theory of elasticity with finite deformations] people often use a, b, c, interpreted as coordinates at the initial site of the medium, as independent variables; it is true that this leads to components of lesser immediate physical importance, but from the analytical point of view it is more convenient for many purposes. This happens namely when we set [This is Nanson's formula in modern treatments. GAM]

$$kdS_0=dS, \tag{6}$$

and Equation (4) becomes

$$\iiint_{(V_0)}\left\{\rho_0\sum_{(xyz,X,Y,Z)}X\delta x-\sum_{(xyz,XYZ)}\left(X_a\frac{\partial\delta x}{\partial a}+X_b\frac{\partial\delta y}{\partial b}+X_c\frac{\partial\delta z}{\partial cz}\right)\right\}dV_0+\iint_{S_0}\sum_{(xyz,XYZ)}\bar{X}k\delta x dS_0=0 \tag{7}$$

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and therefore

$$\Delta .X = X_a \frac{\partial x}{\partial a} + X_b \frac{\partial y}{\partial b} + X_c \frac{\partial z}{\partial c} \quad (X, Y, Z; x, y, z) \quad \left[\sigma_{ij} = J^{-1} T_i^K \frac{\partial x_j}{\partial X^K}\right]. \qquad (8)$$

Moreover, as it follows by resolution and comparison with (3), X_a, Y_a, Z_a, the components of the surface forces acting upon an element of the surface $a = const.$, thanks to the stress condition in the material lying to the side of increasing a, are calculated upon the unit surface in the actual position in the space $a - b - c$ [CF. Müller-Timpe in EmW IV-23, No.9, and also the elaborate presentation (predicting of course the symmetry of the stress dyad) by E. ad F. Cosserat, 1896]. Just like (5a) and (5b) result from (4), from (7) there results a new form of the equilibrium conditions:

$$\frac{\partial X_a}{\partial a} + \frac{\partial X_b}{\partial b} + \frac{\partial X_c}{\partial c} + \rho_0 X = 0 \, in \, V_0(X, Y, Z) \, [\frac{\partial}{\partial X^K} T^K_{.i} + \rho_0 f_i = 0] \qquad (9a)$$

and

$$X_a \cos n_0 a + X_b \cos n_0 b + X_c \cos n_0 c = k\bar{X} \, on \, S_0, (X, Y, Z) \quad [N_K T_i^K = k\bar{t}_i], \quad (9b)$$

where n_0 means the outer normal direction to the surface element dS_0 in the space $a - b - c$.

[In modern treatments, Equations (9a) and (9b) are referred to as the Piola-Kirchhoff format of the equilibrium equations. GAM].

3d. **Relations with rigid bodies**

It is also possible to derive the equilibrium conditions (5) in a somewhat different manner, from the principle (4). We obtain then the relationship with the "*Rigidification principle*" of A.L. Cauchy [cf. Cauchy, 1822 and 1828; Stäkel in EmW IV-6, No.26, Müller-Timpe in EmW, IV-23, No.3b], often used in the composition of his works. That is, each piece cut off the deformed continuum, under the influence of the intervening volume forces on its inside and of the intervening forces (3) on its outer surface, must be like a rigid body in equilibrium. To this purpose, we only need to consider certain discontinuous perturbations which, of course, will destroy the coherence of the continuously deformable continuum and which initially do not need to make δA vanish; but we can succeed if we approach it through a group of continuous virtual perturbations.

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So we approach a perturbation, which has in a domain V_1 of V constant values $\delta x = \alpha$, $\delta y = \beta$, $\delta z = \gamma$ with the boundary surface S_1, but outside V_1 it vanishes (i.e., a *translation* of the domain V_1) by steady virtual perturbations, while V_1 will be surrounded by any small domain V_2; inside this δx, δy, δz of α, β, γ decrease constantly to zero. For such a virtual perturbation it follows from (4):

$$\iiint_{(V_1)} \rho(X\alpha + Y\beta + Z\gamma)dV_1 + \iint_{(S_1)} (X_n\alpha + Y_n\beta + Z_n\gamma)dS_1 + \iiint_{(V_2)} \sum_{(xyz,XYZ)} \left(\rho X + \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}\right)\delta x\, dV_2 = 0$$

where n denotes a component in dS_1 of V_1. If we let V_2 become smaller and smaller, then the last integral will become sufficiently small as the X, X_x and their derivatives remain finite and since α, β, γ are whichever, there result the three equations

$$\iiint_{(V_1)} \rho X dV_1 + \iint_{(S_1)} X_n dS_1 = 0 \quad (X, Y, Z). \tag{10}$$

These are exactly the equations, in the above mentioned sense - through the application of the so-called *strong-point principle* ("Schwerpunktsatzes") - that govern the piece V_1 seen as rigid and cut out of the continuum. Because of the arbitrariness of the domain V_1, it is easy to obtain from (10) the equations (5a) (Cf. Müller-Timpe in EmW, IV-23m, p.23).

If we proceed in the same manner with a rigid rotation of a part of domain V_1 with the components $qz - ry$, $rx - pz$, $py - qx$, then we have the following equations:

$$\iiint_{(V_1)} \left(\rho(Zy - Yz) + Y_z - Z_y\right) dV_1 + \iint_{(S_1)} (Z_n y - Y_n z)\, dS_1 = 0, \ (X, Y, Z)) \tag{11}$$

This can only fully agree with the equilibrium of a domain V_1 as a rigid body, if we set opposite to the moments of the forces X, Y, Z, distributed in space, and to the surface forces X_n , Y_n, Z_n, another rotation moment affecting directly the volume element, calculated as the vector element (2') of the stress dyad. If then we postulate the surface part in the usual form, so that the sum of moments of the volume and surface forces vanishes, then we obtain immediately the symmetry of the stress dyad [Hamel has included this requirement in his axiomatics of the mechanics of volume elements under the expression "Boltzmann's axiom"].

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In close relationship with this fact, there is another interpretation of the principle of virtual rotations which, from the outset, considers as given only the real *force*, the mass forces X, Y, Z and the surface forces$\bar{X}, \bar{Y}, \bar{Z}$; it is the following easily improved formulation of G. Piola [Modena Mem., 1848]: For the equilibrium it is necessary that the virtual work of the specified forces

$$\iiint_{(V)} (X\delta x + Y\delta y + Z\delta z)dV + \iint_{(S)} (\bar{X}\delta x + \bar{Y}\delta y + \bar{Z}\delta z)dS$$

vanishes for all pure translational virtual perturbations of the entire domain V. These auxiliary conditions for the perturbations are mainly expressed by the nine partial differential equations

$$\frac{\partial \delta x}{\partial x} = 0, \frac{\partial \delta x}{\partial y} = 0, \ldots, \frac{\partial \delta z}{\partial z} = 0.$$

then, according to the well known calculation of variations, we can introduce nine necessary Lagrangian factors [multipliers, GAM] $-X_x, -X_y, \ldots, -Z_z$, and thus we obtain exactly the equations (4) of the former principle, proving this way the components of the stress dyad as Lagrange multipliers of certain rigidity conditions. Of course, they are not determined through this variational principle; they rather play exactly the same role as the internal stresses in the static undetermined problem of the mechanics of rigid bodies [Cf. also Stäckel in EmW IV-6, no.26, p. 550, and Müller-Timpe in EmW IV-23, no.3b, p.24].

If we actually impose the same requirement for all rigid motions of V (instead of for translations only), then we obtain exactly the Piola statement repeated in Vol. IV that according to the six auxiliary conditions it yields only six Lagrangian multipliers and so a symmetric stress dyad.

3e **Two- and one-dimensional continua in three-dimensional space**

All the foregoing statements can be immediately proved for the two- and one-dimensional continua embedded in a three-dimensional space, as it was mentioned at the end of Paragraph 2(32). The only modification is that the dimension of the integration domain changes, and that instead of the derivatives of the virtual perturbations along the three space coordinates, these enter along the two or one coordinates, respectively, inside the deformed medium.

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To begin with, let us consider in detail a two-dimensional continuum that, in the deformed configuration, forms a coherent surface-part S with a border curve C; let there be upon S - for the sake of simplicity – a system of orthogonal parameters u and v that define the length and surface elements given by

$$ds^2 = E\,du^2 + G\,dv^2, \quad dS = h\,du\,dv, \quad h = \sqrt{EG},$$

and ρ denotes the surface density of the mass over S. Then we consider the virtual work

$$\delta A = \iint_{(S)} \sum_{(xyz,XYZ)} \left\{ \rho X \delta x - \left(\frac{X_u}{\sqrt{E}} \frac{\partial \delta x}{\partial u} + \frac{X_v}{\sqrt{G}} \frac{\partial \delta x}{\partial v} \right) \right\} dS + \int_{(C)} \sum_{(xyz,XYZ)} \bar{X}\,\delta x\,ds. \tag{12}$$

Here X, Y, Z and $\bar{X}, \bar{Y}, \bar{Z}$ mean the components of the force attached to the mass unit over S, respectively to the length unit along C; over the surface $X_u, \ldots$ permit the development of expressions very analogous to the $X_x, \ldots$ On the one hand, they produce certain forces attached to the mass distributed over S, and on the other, a stress condition prevailing over S, so that, thanks to the stress condition, a force

$$X_\upsilon = X_u \cos(\upsilon, u) + X_v \cos(\upsilon, v) \tag{13}$$

is exerted on each line element lying along C on one side per unit length; here υ means the normals' orientation of the element.

For media allowing all kinds of continuous perturbations, it is possible to resolve the condition $\delta A = 0$ of the principle of virtual perturbations into six equilibrium conditions; we transform then δA by the well-known methods of partial integration:

$$\frac{1}{h}\left(\frac{\partial\sqrt{G X_u}}{\partial u} + \frac{\partial\sqrt{E X_v}}{\partial v}\right) + \rho X \; = 0 \, on \, S, (X, Y, Z) \tag{14a}$$

$$X_u \cos \upsilon u + X_v \cos \upsilon v = \bar{X} \quad along \, C, \quad (X, Y, Z). \tag{14b}$$

Here υ means the orientation standing normally to C in the surface S, and turned away from the surface-part under consideration. But it is also easy to transform these equations to the initial parameters a, b, when from the transformed equations of the virtual work we obtain

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$$\delta A = \iint_{(S_0)} \sum_{(xyz,XYZ)} \left\{ \rho_0 X - \left(X_a \frac{\partial \delta x}{\partial a} + X_b \frac{\partial \delta x}{\partial b} \right) \right\} da\, db + \int_{(C_0)} \sum_{(xyz,XYZ)} \bar{X} \delta x \frac{ds}{ds_0} ds_0 \tag{15}$$

and so

$$h \, \frac{\partial(u, v)}{\partial(a, b)} X_u = X_a \frac{\partial u}{\partial a} + X_b \, \frac{\partial u}{\partial b}, \quad (X, Y, Z; u, v). \tag{16}$$

By comparing with (13) it follows that $X_a, \ldots$, thanks to the stress condition, means the forces acting on a line element $a = const.$, $b = const$ calculated over the length unit in the $a - b$ domain.

In one-dimensional continua things are presented in much the same way [CF. E. and F. Cosserat, Corps déformables, Chap. II, as well as K. Heun in EmW IV-11, No.19 and P. Stäckel in EmW IV-6, No. 23]. If $s(0 \leq s \leq l)$ is the length of the arc on the curve built in the deformed shape, then we get

$$\delta A = \int_0^l \sum_{(xyz,XYZ)} \left\{ \rho X \delta x - X_s \frac{\partial \delta x}{\partial s} \right\} ds \; + \left[\sum_{(xyz,XYZ)} \bar{X} \delta s \right] \Bigg|_{s=0}^{s=l}, \tag{17}$$

where the meaning of the various quantities is given much as usual, and by arbitrary continuous variations the equilibrium conditions read as

$$\frac{dX_s}{ds} + \rho X = 0 \quad for\, 0 \leq s \leq l,\ (X, Y, Z) \tag{18a}$$

$$X_s = \bar{X}\, at\, s = 0,\ s = l, \quad (X, Y, Z). \tag{18b}$$

Here also, it is sometimes advisable to introduce the initial parameter a as independent, by using the formula

$$\delta A = \int_0^{l_0} \sum_{(xyz,XYZ)} \left\{ \rho_0 X \delta x - X_a \frac{\partial \delta x}{\partial s} \right\} da + \left[\sum_{(xyz,XYZ)} \bar{X} \delta s \right] \Bigg|_{a=0}^{a=l_0}, \quad X_s \frac{ds}{da} = X_a. \tag{19}$$

4. **Extensions of the principle of virtual perturbations**
4a. **Presence of higher perturbation derivatives** (partial translation only)

It is possible to add a whole series of extensions to the statement of the principle of virtual perturbations formulated in Section 3, which allows now, to the greatest extent, to include all the laws concerning continuum mechanics. The first thing consists in admitting in the virtual work the existence of a linear form of the eighteen [spatial] second-order derivatives of the virtual perturbations, e.g., $\partial^2 \delta x / \partial x^2$, per element of volume. In fact, we have introduced here some problems related to these expressions, where it would seem necessary to let the energy functions depend on the second derivatives of the deformation functions. To begin with, this applies to the one- and two-dimensional continua considered (strings and plates [Cf. the discussion of the statement of the potential in Paragraphs 7a, p.645 and also 8a, p.660].

[Here it seems that Hellinger was not aware of such developments by Le Roux in France in 1911–1913; Cf. Maugin [38], Chapter 13. GAM.].

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4b. **Media with oriented particles** (not translated here)

[In this section Hellinger generalizes the presentation of foregoing sections to the case including the Cosserats' trihedron. He essentially relies on the works of W. Voigt (complementing S.D. Poisson's original idea), the Cosserats, J. Larmor, and K. Heun in EmW, IV-11, Nos. 19 and 20. He also considers the special cases of two- and one- dimensional bodies. GAM].

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4c. **Presence of side conditions**

Until now the principle of virtual perturbations has been used mainly in those cases where the continuum was continuously deformable, in every possible way. But in the formulation of the principle there are immediately included such continua *whose mobility is restricted by all kinds of conditions*; actually, some of the first problems treated by Lagrange [Cf. his Mécanique Analytique, 1st part, Section V, Chapter III (non-extensible strings), Section VIII (incompressible fluids).] concern this very case. These conditions are expressed in the first place by equations for the functions (1) and (9) of Section 2, describing the deformations. In these, besides their functions as such, we can also have their derivatives with respect to a, b, c. The equation

$$\omega(a,\, b,\, c;\, x, y,\, z;\, x_a, \ldots, z_c;\, \lambda,\, \mu,\, \nu;\, \lambda_a, \ldots, \nu_c) = 0; \quad x_a = \frac{\partial x}{\partial a} \cdots \tag{13}$$

is then typical for every point in the body V_0. It is then possible to set similar expressions for parts of the body, bounding surfaces, etc. In any case, the possible deformations and the possible rotations (if needed) of the added [Cosserats'] trihedron restricted in this way, or are required to satisfy definite relations between rotations of the trihedron and deformation (for example, a certain orientation of the trihedron relative to space or the medium; see above p. 626). The presence of a, b, c in (13) means that the type of conditions may change from one particle to another. If then we apply to (13) the varied deformation, Section 2, (3) or (10), we obtain through differentiation with respect to σ

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$$\delta\omega = \sum_{(x,y,z)} \left(\frac{\partial \omega}{\partial x} \delta x + \frac{\partial \omega}{\partial x_a} \delta x_a + \ldots \right) + \sum_{(\lambda,\mu,\nu)} \left(\frac{\partial \omega}{\partial \lambda} \delta \lambda + \frac{\partial \omega_a}{\partial \lambda_a}\, \delta \lambda_a + \ldots \right) = 0 \tag{14}$$

and since according to Section 2, p.608, the $\delta x_a \ldots$ agree with the derivatives of $\delta x, \ldots$, thereexists here a *linear homogeneous condition for virtual perturbations*.

So the principle of virtual perturbations requires that δA vanishes for all functions $\delta x, \ldots$ satisfying (14). We can then if, by chance equations (14) do not allow the elimination of one of the perturbation components, replace it by the introduction of a Lagrange multiplier [This treatment was first introduced by Lagrange in his Mécanique Analytique] λ in such a way that

$$\delta A + \iiint_{(V)} \lambda \delta\omega \, dV \; = \; 0 \, for \, all \, \delta x, \ldots, \tag{15}$$

what corresponds exactly to the original principle. Eventually, when (13) applies only at isolated surfaces and curves, or actually the continuum fills only one surface or curve, instead of space integrals in (15) we have then surface or curve integrals. The interpretation of the multiplier λ as a "pressure" will be discussed later on (Paragraph 8b, p. 662).

Finally, we should also consider the possibility, which is also well-known from the mechanics of discrete systems, that there can occur "one-sided ["unilateral" in modern jargon. GAM] accompanying side conditions [constraints. GAM] in the form of inequalities - e.g., let the boundary surfaces of the continuum in their motion be restricted only on one side: let the inside (inner) deformation quantities be subjected to certain inequalities (somehow we think of bodies that allow no compression beyond a certain boundary – or some similar arrangement). Then the equilibrium will be given once more by Fourier's principle of virtual perturbations that, namely, for every system of virtual perturbations satisfying the side conditions, the virtual work is negative or zero:

$$\delta A \leq 0.$$

[CF. Voss in the EmW, IV-1, No. 54; formulation by Gauss in 1830 regarding from the start the extension of continua].

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12.9 The Basic Laws of Kinetics [Dynamics]

5a. **The equations of motion of the continuum**

The task of kinetics is to establish which are the motions of which the continuum is the object, as considered until now, when, somehow, certain force actions are exerted on it in time or, on the opposite, which are the actions necessary for the maintenance of a certain motion. At the same time, the action components are thought of, like in statics, as coefficients of the work expression δA, while the manner in which they depend on the function of motion will remain initially open.

At the beginning we will only be concerned with the ordinary media examined in Section 3. The transition from statics to kinetics can be made exactly as in the mechanics of discrete systems with the help of *d'Alembert's principle* (see Voss in the EmW IV-1, No.36); Passing to continuous systems is almost automatic if, as we did in statics (p. 616), we let ourselves be led by the idea of a limit transition to the continuum, by direct comparison, in the sense of what happens in the analogy between systems of points and continua. Lagrange (cf. Méc. Anal., 2nd part, Section XI, §1) also, when treating the problems of hydrodynamics, considered it form the same point of view. It is possible then to express in terms corresponding to d'Alembert's formulation (Traité de dynamique, Paris, 1743; Voss in the EmW IV-1, p.77) for the general mechanics of continua, the following principle: *If we consider the forces and stresses acting during the motion at a definite instant of time on the volume V_0 of the medium, then they are found to be in static equilibrium, in the earlier sense, in so far as we attach to them, at any time, additional forces whose components, calculated per unit mass of the continuum, are equal, by comparison, to the components of the acceleration*:

$$-\frac{\partial^2 x}{\partial t^2} = -x'', -\frac{\partial^2 y}{\partial t^2} = -y'', -\frac{\partial^2 z}{\partial t^2} = -z''.$$

Even if, in many ways, it proves advisable to place this principle at the summit of kinetics, still the question remains open, in what independent constituents it can be decomposed, and to what extent these are independent from the axioms of statics – a question we faced in exactly the same manner in the mechanics of discrete systems.

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Let us remark briefly that this D'Alembert principle contains essentially, on one hand, a statement equivalent to the second law of Newton, i.e., that the acceleration of a volume element considered as free, is the same as the sum of all the forces acting on it; but, on the other hand – something that Hamel (1908, p. 354; also his *Elementare Mechanik*, p. 306ff) has thoroughly proven – i.e., that one of these first constitutive elements contains, logically, perfectly independent expressions: if the forces acting on a continuum are such the ensuing accelerations on each particle, according to the second *Newtonian law*, are compatible with the kinematic conditions of the system, then these accelerations also really occur. If we cease to introduce the principle of virtual perturbations as an equilibrium condition in the D'Alembert principle, then we obtain the variational principle used by Lagrange (Méc. Anal., 2nd part, Section II) as the basic formulation of dynamics. We imagine the motion for every instant t in Section 3, (6), on which is superimposed an infinitesimally small virtual perturbation compatible with the somehow constituting kinematic conditions at the instant t for the continuum; *then the virtual work performed by the sustaining forces must always vanish*:

$$-\iiint_{(V)} \rho(x''\delta x + y''\delta y + z''\delta z)\, dV + \delta A = 0 \tag{1}$$

and this for every instant of time t in the course of the motion. In the case of a rather continuously deformable body, the equations

$$\rho x'' = \rho X + \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}, \quad (x, y, z; X, Y, Z) \tag{2}$$

follow; in the same way as in Paragraph 3c, as the equations of motion at any point of the continuum and every time, while the boundary conditions (5b) of Sect. 3 persist for every time t. On the other hand, these equations define the motion only when the relationship between the forces and stress components and the motion functions is established [i.e., the constitutive equations, GAM].

Concerning now the kinematic side conditions, we refer exclusively to the case of so-called *holonomic* conditions which contain *no time derivatives* of the motion functions [If we try to handle the problems with non-holonomic conditions by means of d'Alembert's principle, then we must foresee in continuum mechanics, just like in point mechanics, that the varied motion for small σ satisfies the condition – and even more, condition equations for perturbations will clearly be formally written by replacing the time differentiation by an operation; see below p. 633). Cf. Voss in the EmW, IV-1, Nos. 35 and 38, and bibliography there, particularly works by Hölder in 1896 and by Hamel in 1904]. Such a condition differs from the one considered in Paragraph 4c only through the explicit presence of t:

$$\omega(a,\, b,\, c;\, x,\, y, z;\, x_a, \ldots, z_c; t) = 0. \tag{3}$$

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For the virtual perturbations we shall consider no only the form of this condition in time t; the varied position (for any small σ) must satisfy the condition (3) for the considered fixed value of t, so that through differentiation with respect to σ ("variation of motion at fixed t") there follows

$$\sum_{(xyz)} \frac{\partial \omega}{\partial x} \delta x + \sum_{(xyz,abc)} \frac{\partial \omega}{\partial x_a} \delta x_a = 0 \quad \textit{for every}\, t. \tag{3'}$$

From this we obtain the equations of motion in the sense of Paragraph 4c

5b. **Transition to the so-called Hamiltonian principle**

Now we can also convert some very similar well known developments of point mechanics of the d'Alembert principle into variational principles determining the motion. The main object here is to transform the contributions due to the motion (the sustaining forces) in the variation of a unique determined expression for each motion path.

As with Lagrange [Méc. Anal., 2nd part, Section IV, art. 3], the basic identities are

$$x''\delta x = \frac{d}{dt}(x'.\delta x) - \delta\left(\frac{1}{2}x'^2\right), (x, y, z)$$

which follow through repeated differentiation from Section 2, (6), with respect to the independent variables σ and t. If we carry this into (1), and considering that the operation symbols d/dt and δ can be taken out of the integrals, regardless of the factor ρ, since as according to the introduction of $a,\ b,\ c$ as integration variables, the integration domain V_0 as well as the remaining factor ρ_0 are independent from σ and t, we obtain

$$-\frac{d}{dt}\iiint_{(V)} \rho \sum_{(xyz)} x'\delta x.dV + \delta T + \delta A = 0 \tag{4}$$

introducing in this way, by abbreviation, the *total kinetic energy*

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$$T = \frac{1}{2} \iiint_{(V_0)} \rho_0 \sum_{(xyz))} x'^2 dV_0 = \frac{1}{2} \iiint_{(V_0)} \rho \sum_{(xyz)} x'^2 dV. \qquad (5)$$

Equation (4) is the equation used by G. Hamel [Zeit. Math. Phys. 50(1904), p.14] and K. Heun [Lehrbuch der Mechanik, Vol.1, Leipzig, 1906) and in EmW, IV-11, No. 11] under the name of *Lagrangian central equation*, as the basis of the mechanics of systems with a finite number of degrees of freedom, which is then valid in the same sense in continuum mechanics [Cf. Heun in the EmW IV-11, Nos. 19-21)], and is completely equivalent to (1): *The motion takes place so that for every virtual perturbation compatible with the somehow existing conditions at every instant, the time derivative of the virtual work of the quantities of motion ("impulses") x', y', z' per unit mass, is equal to the sum of the variations of the kinetic energy and of the virtual work of the totality of the actions of forces* [if in addition we also vary the time parameter, then it becomes possible to carry over the relation indicated by G. Hamel (Math. Ann., 59 (1904) p. 423, and K, Heun as general central equation to continuum mechanics; cf. Heun, in EmW, IV-11, Nos. 19-21].

If we consider now the motion in the interval of time $t_0 \leq t \leq t_1$, then (4) is valid for every instant, and through integration with respect to t with the assumption that the virtual perturbations vanish at the limits of the interval, it yields the so-called *Hamiltonian principle* [This principle, after it became typical of point mechanics, had been used very early for different specialized fields of continuum mechanics in many different manners (see Voss EmW IV-1, No.42); we can also compare, apart from the bibliography to be mentioned later for each discipline, A. Walter, Diss. Berlin, 1868, as well as the comprehensive presentations in Kirchhoff's Mechanik, p.117ff and W. Voigt's Kompendium, Vol. I, p. 227ff.]: *If over the motion of the continuum we superimpose some virtual perturbations compatible with the existing conditions, which vanish exactly at t_0 and t_1, then the time integral of the sum of virtual work and the variation of the kinetic energy over the interval t_0, t_1, vanishes also*:

$$\int_{t_0}^{t_1} (\delta T + \delta A)\, dt = 0. \qquad (6)$$

Since in (6) the virtual perturbations for every time interval can be chosen arbitrarily, then it is all the more easy to conclude from (6), from (4) or from (1) that *these principles are fully equivalent.*

From this principle it is further possible to derive directly the *principle of least action* in its various forms [As an example, the considerations of O. Hölder in "Die Prinzipien von Hamilton und Maupertuis", Gött. Nach. Math.-Phys. Kl, 1996, p. 122ff, can be immediately extended to continua], but it seems that – regardless of those cases referring to systems with finitely many degrees of freedom – we have not as yet found any substantial application for it.
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5c. **The principle of least constraint**

It is also possible to transfer the inertial contribution of the d'Alembert principle, without integration in time, in the variation, that depends on an expression for each motion condition, determined only from the condition at instant t, where of course the occurrence of second-order time derivatives must be allowed. This way was created the Gauss principle of least constraint [CF. Gauss's Werke V, p. 23. The first analytic formulation of this principle, given only orally by Gauss, was published by R. Lipschitz, J. für Math., 82 (1877), p.321ff; and a little later by J.W. Gibbs in Amer. Journ. 2 (1879) p.49; for further bibliography see Voss in EmW IV-1, No. 39], that recently A. von Brill has chosen as the starting point for a systematic treatment of continuum mechanics (Cf. A. von Brill, 1909).

To reach this principle, we take the virtual perturbation of a group of varied motions Section 2, (6), in the following particular way: Each particle a, b, c will occupy at time t the same position and the same velocity as in the real motion, i.e., the following will be valid for each value of t:

$$\delta x(a, b, c; t) = 0, \delta x'(a, b, c; t) = 0, \quad (x, y, z) \tag{7}$$

while the variations $\delta x''$, $\delta y''$, $\delta z''$ of the accelerations are different from zero. It is now possible to use these three functions in every case as defining parts of the perturbations happening in (1). In the case of a freely deformable continuum, this is evident. But in the conditions of the form (3), this will yield through double differentiation with respect to time,

$$\sum_{(xyz)} \frac{\partial \omega}{\partial x} x'' + \sum_{(xyz,abc)} \frac{\partial \omega}{\partial x_a} x_a'' + \ldots = 0,$$

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where the known functions of $x, \ldots, x_a, \ldots$, and their first time derivatives are indicated by the ellipsis. By variation, i.e., differentiation with respect to σ, thanks to (7) at the chosen time t, there follows

$$\sum_{(xyz)} \frac{\partial \omega}{\partial x} \delta x'' + \sum_{(xyz,abc)} \frac{\partial \omega}{\partial x_a} \delta x_a'' + \ldots = 0,$$

and actually this is exactly the conditions represented above for δx. The introduction of the functions $\delta x'', \ldots$ in (1) is then permitted and it yields, with a light reformulation, the following new principle [Cf. Brill, op. cit.]: *If we alter the real motion of a continuum at a definite instant in such a way that the position and velocity of every one particle remain preserved save that the acceleration of the existing side conditions are modified accordingly, then the following integral sums always vanish*:

$$-\delta \iiint_{(V)} \frac{1}{2}\rho \sum_{(xyz)} x''2 + \iiint_{(V)} \left(\rho \sum_{(XYZ)} X\delta x'' - \sum_{(XYZ)} X_x \frac{\partial \delta x''}{\partial x} \right) dV$$
$$+ \iint_{(S)} \sum_{(XYZ)} \bar{X}\delta x'' dS$$
$$= 0. \tag{8}$$

This can be transformed to a Gaussian form

$$-\delta \iiint_{(V)} \frac{1}{2}\rho \sum_{(xyz,XYZ)} (x'' - X)^2 dV - \iiint_{(V)} \left(\sum_{(XYZ,xyz)} X_x \frac{\partial \delta x''}{\partial x} \right) dV$$
$$+ \iint_{(S)} \sum_{(XYZ)} \bar{X}\delta x'' dS = 0. \tag{8'}$$

The main significance of this principle, just like in point mechanics, consists in the fact that it remains valid and fully unaltered also in systems with *non-holonomic* side conditions. If there exists such a condition, in which next to the motion functions and their spatial derivatives also occur the first time differential quotients:

$$\omega(a, b, c; x, y, z; x_a, \ldots, z_c; x', y'z'; x'_a, \ldots, z'_c; t) = 0$$

Then through single differentiation with respect to t, and by variation (differentiation with respect to σ) we obtain, thanks to (7)

$$\sum_{(xyz)} \frac{\partial \omega}{\partial x'} \delta x'' + \sum_{(xyz,abc)} \frac{\partial \omega}{\partial x'_a} \delta x''_a + \ldots = 0$$

which can be no more added as a side condition.

If ω is especially linear in the velocities $x', \ldots, x'_a, \ldots$, then the result is substantially identical with the form in which, often, one does not consider the d'Alembert principle with non-holonomic conditions and in so doing, instead of the simple virtual perturbations introduced only formally, there also occur the acceleration variations.

A further advantage of this principle as compared to the d'Alembertian one, which however does not seem to have been exploited until now in continuum mechanics, consists in that it offers an appropriate basis even for the treatment of dynamic problems with the inequality type of side conditions: all we need to do is to require that the expression (8) for all admissible variations of the acceleration in agreement with the side condition at instant t, with fixed position and velocity of the individual particles, *be smaller or equal to zero*, exactly like Gauss has already remarked in point mechanics [Cf. Gauss, Werke, Vol. V, p.27].

[The rest of Hellinger's contribution is not translated here].

References

1. Appell P (1909) Traité de mécanique rationnelle, vol 3. Mécanique des milieux Continus, Gauthier-Villars, Paris
2. Boltzmann L (1874) Zur theorie der elastischen Nachwirkung. Sitz K u K Akad Wien 70:275–306
3. Born M (1906) Untersuch, Über die Stabilität der elastischen Linie. Preisschrift, Göttingen
4. Born M (1909) Die theorie des starren elektrons in der kinermatik des relativitätsprinzips. Ann der Physik 30:1–56
5. Brill A (1909) Vorlesungen zur einführung in die mechanik raumerfüllender massen (Lectures on the introduction to mechanics of space-filling—i.e., 3D—masses), Leipzig
6. Caratheodory C (1909) Untersuchungen über die Grundlagen der Thermodynamik. Math Annalen 67:355–386
7. Cosserat E, Cosserat F (1896) Sur la théorie de l'élasticité. Ann Fac Sci Toulouse (1ère série), 10(3-5): I1–I116
8. Cosserat E, Cosserat F (1909). Théorie des corps déformables. Hermann, Paris, 226 pages. (Reprint by Editions Gabay, Paris, 2008; Reprint by Hermann Archives, Paris, 2009 [English translation: N68-15456: Clearinghouse Federal Scientific and Technical Information, Springfield, Virginia NASA, TT F-11561 (February 1968)); another translation by D. Delphenich, 2007] (originally published as a supplement in pp. 953-1173 to: Chwolson OD (1909) Traité de physique (traduit du Russe), Vol. II, Paris, Hermann)
9. D'Alembert J. Le Rond (1743) Traité de dynamique. 1st edn, Paris
10. Duhem P (1891) Hydrodynamique, élasticité, acoustique. A. Hermann Editeur, Paris
11. Duhem P (1896) Le potentiel thermodynamique et la pression hydrostatique. Ann Ecole Norm Sup, 10(3):187–230
12. Duhem P (1911) Traité d'énergétique ou thermodynamique générale. Gauthier-Villars, Paris Two volumes
13. Duvaut G, Lions JL (1972) Inéquations variationnelles en mécanique et en physique. Editons Dunod, Paris
14. Duvaut G, Lions JL (1979) Variational inequations in mechanics and physics. Springer, Berlin
15. Eringen AC (1966) Theory of micropolar fluids. J Math Mech 16(1):1–18
16. Eringen AC, Maugin GA (1990) Electrodynamics of continua. Springer, New York Two volumes
17. Flügge S (ed) (1955–1988) Handbuch der physik. Springer, Berlin
18. Geiger H, Scheel K (eds) (1926–1933) Handbuch der physik. Julius Springer, Berlin
19. Germain P (1973) La méthode des puissances virtuelles en mécanique des milieux continus-I: théorie du second gradient. J de Mécanique 12:235–274 Paris
20. Germain P (1986) Mécanique. Editions de l'ecole polytechnique, Palaiseau, Two volumes
21. Gibbs JW, Wilson EB (1901) Vector analysis. Yale University Press, New Haven [The first book of its kind, this may however be a wrong attribution since the book in fact is E.B. Wilson's redaction with an enriched rendering of Gibbs' lectures in vector analysis at Yale; Wilson was only 22 years old when the book was published (see pp. 228-229 in Crowe MJ (1967) A history of vector analysis, Univ. of Notre Dame Press; reprint Dover, 1985)]
22. Green G (1839) On the laws of reflection and refraction of light at the common surface of two non-crystallized media. Trans Cambridge Phil Soc 7:245–269
23. Hadamard J (1903) Leçons sur la propagation des ondes. Gauthier-Villars, Paris
24. Hamel GWK (1908) Über die grundlagen der mechanik. Math Annalen, 66, 350-397 (This was Hamel's Habilitationsschrift to be further developed in a contribution to the Handbuch der Physik, Die axiome der Mechanik. Geiger and Scheel (eds) Bd. 5. pp.1–42, Springer, Berlin, 1927)
25. Hamel GWK (1912) Elementare Mechanik. Teubner, Berlin

26. Hellinger E (1914) Die allgemein ansätze der mechanik der kontinua. In: Klein F, Müller CH (eds) Enz math wiss, vol 4, part 4, Article 30, Springer, Berlin, pp 602–694
27. Hellinger E, Toeplitz O (1927) Integralgleichungen und gleichungen mit unenlichvielen unbekannten (originally intended for the Enz Math Wiss, reprinted by Chelsea, New York, 1953)
28. Herglotz G (1911) Über die mechanik des deformierbaren körpers vom standpunke der relativitätstheorie. Ann der Physik, 36(4):493–415
29. Heun K (1904) Ansätze und allgemeine methoden der systemmechanik. In: Klein F, Müller CH (eds) Enz math wiss, vol 4/2, Article 11, Springer, Berlin, pp 359–504
30. Kármán Th von (1914) Physikalische grundlagen der festigkeitslehre. In: Klein F, Müller CH (eds) Enz math wiss, vol 4/4, Article 31, Springer, Berlin
31. Klein F, Mueller CH (eds) (1907–1914) Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen (in brief: Enz Math Wiss), B.G. Teubner (Verlag), Leipzig (vol 4, Part 4 on Mechanics is available on the site of the University of Toronto-Gerstein Science Information Centre as archive.org/details/p4encyclopedie04-akaduof)
32. Lagrange JL (1788) Mécanique analytique, First edition. Vve Courcier, Paris
33. Le Roux J (1911) Etude géométrique de la torsion et de la flexion, dans les déformations infinitésimales d'un milieu continu. Ann Ecole Norm Sup 28:523–579
34. MacCullagh J (1839) An essay towards a dynamical theory of crystalline reflection and refraction. Trans Roy Irish Acad Sci 21:17–50
35. Maugin GA (1980) The method of virtual power in continuum mechanics: Application to coupled fields. Acta Mech 35(1):1–70
36. Maugin GA (1992) The thermo-mechanics of plasticity and fracture. Cambridge University Press, UK
37. Maugin GA (2012) The principle of virtual power: from eliminating metaphysical forces to providing an efficient modelling tool. Cont Mech and Thermodynam 25:127–146 (Special issue in the honor of G. Del Piero)
38. Maugin GA (2013) Continuum mechanics through the twentieth century—a concise historical perspective. Springer, Dordrecht
39. Maugin GA (2013) What happened on September 30, 1822, and what were its implications for the future of continuum mechanics? (Preprint UPMC, also Chapter 3 in this book)
40. Maugin GA (2013) About the Cosserats' book of 1909 (Preprint UPMC, also Chapter 8 in this book)
41. Maugin GA (2013) A course of continuum mechanics at the dawn of the twentieth century (Volume III of Appell's Treatise on Rational Mechanics) (Preprint UPMC, also Chapter 11 in the present book)
42. Minkowski H (1908) Die grundgleichungen für die electrodynamischen Vorgänge in bewegten Körpern. Nach Ges d W Gottingen, math-phys KL. p 53
43. Molk J, Appell P (eds) (1904–1916) Encyclopédie des sciences mathématiques pures et appliquées, twenty two volumes, [Facsimile reprint by Gabay, Paris, 1991-1995] (French translation from the German of: F. Klein and C.H. Mueller (eds), Enz Math Wiss, B.G. Teubner (Verlag), Leipzig)
44. Piola G (1848) Intorno alle equazioni fondamentali del movimento di corpi qualsivoglioni considerati secondo la naturale loro forma e costituva. Mem Mat Fiz Soc Ital Modena 24(1):1–186
45. Reissner E (1953) On a variational theorem for finite elastic deformation. J Math Phys 32:129–135 MIT, Cambridge
46. Sedov LI (1968) Variational methods of constructing models of continuous media. In: Parkus H, Sedov LI (eds) Irreversible aspects of continuum mechanics (IUTAM Symposium, Vienna, 1966), Springer, Berlin, pp 346–358
47. Timoshenko SP (1953) History of strength of materials. McGraw Hill, New York (Reprint by Dover, New York, 1985)
48. Truesdell CA, Noll W (1965) The nonlinear field theories of mechanics In: S. Flügge (ed) Handbuch der Physik, vol. III/3. Springer-Verlag, Berlin

49. Truesdell CA, Toupin RA (1960) The classical theories of fields. In: Flügge S (ed) Handbuch der Physik,vol III/1. Springer-Verlag, Berlin
50. Voigt W (1887) Teoretische studien über elasticitätverhältinsse der krystalle, I.II. Abh K Ges Wissen Göttingen 34(3-52):53–100
51. Voigt W (1895/1896) Kompendium der theoretischen physik (Two volumes). Teubner, Leipzig
52. Voss A (1904) Principles of Rational mechanics (in German). In: Klein F, Mueller CH (eds) Enz Math Wiss, vol 4 Article 1, B.G. Teubner (Verlag), Leipzig (translated into French by the Cosserats in Molk and Appell (1904–1916))
53. Washizu K (1968) Variational methods in elasticity and plasticity. Pergamon Press, Oxford

Chapter 13
Epilogue

Abstract This epilogue briefly summarizes the foregoing essays while emphasizing the ways in which this critical perusal was approached, and what appears to be the large themes that received a special magnification and, perhaps, a biased presentation. It underlines the main breakthroughs as well as the secondary ones. It highlights the role of scientists who left essential prints in this history of scientific ideas. It finally outlines the observed timid beginnings of future theories of coupled fields in thermo-mechanics.

13.1 On the Method

Nowadays, two sometimes irreconcilable approaches are considered in political history. One, under the influence of structuralism, favours a global approach epitomizing great movements of ideas and philosophical tendencies (with an emphasis on general themes, sociological and economical background), and the other still basing on chronology, dramatic events, national heroes, great names and even myths for more ancient times, what provides a tempo that is useful to the youth in forming a consolidated view of history.[1] In the present book dealing with

[1] If we compare the examined period with the one considered in our previous book [13] which included the two World Wars of the twentieth century, we find that this was a relatively quiet one. Of course there were wars. An important one, very much similar to World War One in extent and casualties was the Seven-Years war (known as the French and Indian War in the USA) that included England, the Netherlands, and Prussia on one side and France, Austria, Russia, and Sweden on the other with battle fields on three continents (Europe, India, North and Central America) and various seas. It lasted from 1756 to the Treatise of Paris in 1763. This date marks the true birth of a powerful British Empire and the disappearance of French possessions in India and North America (Canada, East of the Mississippi river; the rest of Louisiana, west of the Mississippi river, from New Orleans to the Canadian boarder was sold to the USA by Napoleon in 1803), but with a status quo in Europe. Other conflicts were the Napoleonic wars, the wars of independence in Italy and Greece, the British-Russian war, and the French-Prussian war of 1870, and of course the war of independence in the USA and the unfortunate American Civil War in

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one aspect of the history of scientific ideas—as witnessed by the different essays—, we have preferred a mixed attitude, sometimes emphasizing general themes such as the evolution of the principles of mechanics—or of their wording—, Newtonian versus "continental" viewpoints, combination of continuum mechanics and thermodynamics in a true "thermo-mechanics", and also paying more than justified attention to some remarkable individuals who left their name attached to a theorem or a principle, although these persons are not truly sanctified. Here chronology plays an important role. It is obviously cogent to know that Daniel Bernoulli's celebrated theorem came before Cauchy's postulate on stresses, and the true understanding of the conservation of energy, or that the Navier-Saint-Venant-Stokes equations, although involving dissipation, were written down independently of the second law of thermodynamics that was not yet clearly expressed. In the last case this has left a print since there still are many workers in viscous fluids, and even more in non-Newtonian fluids, who practically do not refer to thermodynamics.[2]

The main result of our mixed attitude has been a series of essays that generally follow the arrow of time while underlining the role played by scientists who brought seminal ideas and contributed the most remarkable breakthroughs. These scientists are not unknown to the majority of students and practicing scientists, because their names are classically attached to familiar theorems or commonly used mathematical objects. Among the names that recur in the above-given essays, we find those of: John and Daniel Bernoulli, Varignon, d'Alembert, Euler, Lagrange, Cauchy, Lamé, Piola, Kirchhoff, Green, Duhamel, Neumann, Carnot, Kelvin, Helmholtz, Clausius, Stokes, Maxwell, Saint-Venant, Boussinesq, the Cosserat brothers, Duhem, Poincaré, and Caratheodory. Parodying what American magazines do for film stars, these individuals form, in some sense, the "hall of fame" of our discipline for the eighteenth and nineteenth centuries. These are all first roles, although some secondary roles may have played a crucial part at some specific time. An unavoidable filtering process took place with our biased

(Footnote 1 continued)
1861–1865. But all these did not alter much the scientific world where international exchanges (e.g., between England and France or France and Germany) continued uninterrupted except in case of physical impossibility (e.g., during a blockade). This is in sharp contrast with what happened in the twentieth century. The great influential event in fact was the French Revolution started in 1789, not because "it did not need savants", [Supposedly, this is what was said by some philistine revolutionaries when the chemist Lavoisier lost his head (but he was not executed because he was a scientist. Remember that Laplace, Lagrange, Monge, Coulomb, Lazare Carnot and others went through this period without physical damage; d'Alembert had died of natural causes in 1783).] but because it instituted a new type of framework for scientific studies with the creation of engineering (polytechnic) schools, a model that was to spread all over the world during the nineteenth century. This was particularly beneficial to the advances in continuum mechanics and its application to mechanical and civil engineering along with implementation of good mathematics (sometimes created for this very purpose) as illustrated by Cauchy, Navier, Fourier, Ampère, Fresnel, Coriolis, Duhamel, Saint-Venant, Poincaré, etc., in France and the disciples of F. Neumann (Kirchhoff, Clebsch, and Voigt) in Germany.

[2] The inequalities to be satisfied by viscosity coefficients in order to guarantee a non-negative dissipation, were in fact proved belatedly by Duhem toward the end of the nineteenth century.

contemporary view since only the fruitful avenues have been retained, those leading to dead ends being altogether ignored, perhaps unjustly because they also play a role in the evolution of ideas. However, this is moderated by the last two essays in which we have examined the appraisal by scientists Appell ([1]—but initially 1909); and Hellinger [6] of the early twentieth century. This does not necessarily coincide with our own appraisal one century later.

13.2 The Main General Themes and Breakthroughs

As shown by the scrutiny of original sources and the offered English translation of some crucial texts, the two most important lines of development exposed in the successive essays are the “continental” vision that emphasizes the consideration of a principle of virtual motion or, in modern terms, a “weak” formulation—and the more “Newtonian” approach based on the postulate of balance laws that follows Euler. The last viewpoint is the approach still favoured by disciples of the late Truesdell, an author who constantly expressed his (justified) admiration for Newton and his successors in the United Kingdom. This is best exemplified by the treatise of Truesdell and Toupin [19]. In contrast, under the influence of Leibniz and John Bernoulli the “variational” approach, in different guises, has been adopted by many scientists in France (d’Alembert, Lagrange, the Cosserats), Italy (Piola), and Germany (Kirchhoff, Helmholtz, Hellinger). This approach had a glorious destiny in mathematical physics, but also in engineering with the creation of numerical methods based on it (finite-element method) and in mathematical proofs based on weak formulations. Among the French exceptions who kept the Newtonian-Eulerian line, we find Cauchy who remained in favour of a postulate of balance laws as proved by his very argument concerning stresses—and also Saint-Venant, Boussinesq and Appell because this has remained the traditional expression of the principles of continuum mechanics in courses to students in engineering. This “postulational” approach of balance laws recently gained some additional favour with the implementation of the numerical method of finite volumes.

In the period extending from John Bernoulli to Hellinger—almost two centuries—breakthroughs have been numerous. They were listed at the end of Chap. 2 either in the form of realizations of the eighteenth century or as things to come in the nineteenth century. But if one has to select among these the few most important ones, then certainly one has to chose the formulation of the principles of linear and angular momenta (for a collection of particles or a global body) by Euler, the introduction of the general concept of stress by Cauchy, and the proposal of the first and second laws of thermodynamics by Sadi Carnot, Kelvin, Mayer, Helmholtz and Clausius. Those are all fundamental principles that still apply today in the fashionable combination known as *thermo-mechanics*. They have found natural extensions within relativistic physics (they even apply to black holes).

According to the already mentioned two possible avenues this led to the following two basic formulations for a deformable body made of a continuous material:

(1) along the "postulational" line: global statement of the two principles of momenta in the Eulerian form, and the two laws of thermodynamics;
(2) along the "variational" line: a global statement of the principle of virtual power, and the two laws of thermodynamics for real evolutions.

For the first line, see more particularly Truesdell and Toupin [19], Eringen [4], and Maugin [11]. For the second line, see the critical essay by the author [12] where it is emphasized that this line has to be preferred for theories of generalized continua where extra balance laws are automatically taken care of by the principle of virtual power. It also does not make use of the Cauchy postulate for stresses and its generalization to the notions of hyperstresses and couple stresses is straightforward. Of course, the reader may find that this line receives an exaggerated magnification in most of the previous chapters. But this emphasis is justified by what permeates from the considered works, mostly in the "Continental" works that were not written in English (see Chap. 1 for this deliberate choice and the initial purpose of this series of essays).

Still, Cauchy certainly is the most remarkable among the cited scientists because not only did he contribute the basic concept of continuum theories (see Chap. 3), but, as a mathematician, he also created some of the most efficient tools in the treatment of problems of continuum mechanics, such as in linear algebra and its geometric representations, elements of group representations, rigorous definitions of integrals and of limits, singular integrals and the notion of principal value, and an invaluable application of complex variables with the theory of residues (see [2]). This was particularly useful in two-dimensional problems of hydrodynamics (see [1 in Chap. 11, 7, 8, 15]) and of linear elasticity (see [9, 14, 16]), and more generally in potential theory.

We note that the most cited authors are, together with Cauchy and in chronological order: John Bernoulli, d'Alembert, Euler and Lagrange for the eighteenth century, and Navier, Lamé, Clebsch, Maxwell, Saint-Venant and Boussinesq for the nineteenth century. This is corroborated by Timoshenko [17] and Todhunter [18] with a bias toward the application of solid mechanics to the strength of materials. Indeed, while many of the perused works bear a strongly mathematical style, applications were not neglected by the same scientists as a result of professional obligations and a new interest in the mechanics of machines, mechanical and civil engineering, and then construction using metallic structural elements. For fluid mechanics which started as a "Swiss" specialty with the Bernoullis and Euler, the nineteenth century witnessed the take over of this field by the British school with stars such as Stokes and the Cambridgians in hydrodynamics. This has remained until now a remarkably fruitful field in the United Kingdom. This is illustrated by the lasting influence exerted by scientists like Lord Kelvin, Lamb [8] and Osborne Reynolds (1842–1912), and the enduring supremacy enjoyed by some journals such as the *Journal of Fluid Mechanics*. On the German side, we cannot overlook the influential works (in particular on vorticity) of Helmholtz—who of course also radiates in other fields of physical and medical science—and of Prandtl with the notion of boundary layer and the revolution it brought in the emerging science of aeronautical flight. The French school is more reduced but we particularly note Navier with his seminal works on viscous flows, Boussinesq with his innovative ideas (modelling and mathematically justified approximations), and Poincaré for his study of the equilibrium shapes of fluid masses. Many of these are described at length and mathematically in the imposing treatise on rational mechanics by Appell [1]. Much more on the history of hydrodynamics for the relevant period can be found in Darrigol [3].

13.3 The Breakthroughs of Second Rank

There is no pejorative or belittling consideration in this classification. It simply is that this is not so much related to principles, except for the laws of virtual work and virtual power and the analytic mechanics of Lagrange (which, probably, would not have existed without the pioneering work of Newton). We rank in this class the formulation of the laws of Eulerian fluids, the laws of linear elasticity by Cauchy and Navier, those of linear viscous fluids by Navier, Saint-Venant and Stokes, the thermo-elasticity of Duhamel and Neumann, and the initial studies on visco-elasticity by Kelvin, Maxwell, Voigt and Boltzmann, and those on plasticity by Tresca, and Saint-Venant. Still in a different class, because of much delayed recognition and applications only in the late twentieth century, we find the proposal of continua with microstructure by Duhem and the Cosserat brothers.

13.4 The Timid Steps in Coupled Fields

We have seen that both Duhamel in France and F. Neumann in Germany pioneered the theory of coupled fields in continuum mechanics by creating practically from nought an embryonic theory of thermo-mechanical interactions. This may have been premature as in fact in advance on the applications of well set laws of thermodynamics. Potential applications were only very few at the time, being limited to some problems posed by the then recent railway technology (over-heating of metallic parts in motion). In so far as electro-magneto-mechanical interactions—a subject matter dear to the writer—are concerned, one must realize that very few such couplings had been identified when electromagnetic effects themselves were not yet fully exposed. Historically, the first coupling is *magnetostriction* discovered by James Prescott Joule in 1847 (the same Joule as the one of the Joule effect in electric conductors). Magnetostriction, an effect quadratic in the magnetic field, in principle exists, but to a rather small extent, in many materials (no specific material symmetry is required for its existence). The second coupling, of electro-mechanical nature, is linear *piezoelectricity* that was discovered in 1881 by the Curie brothers in Paris. This effect, linear in the electric field, requires a material symmetry with no centre of symmetry such as in quartz or Rochelle salt. Technical applications of this effect had to await the First World War with Paul Langevin and the conception of sonars in underwater acoustics for the detection of submarines.

A true nonlinear combination of electrodynamics and continuum mechanics respecting the laws of thermodynamics will be achieved only in the second half of the twentieth century. We have given elsewhere [13, Chap. 12] elements of these developments in a concise historical perspective. Advanced technical treatises dealing at length with this rather complex but extremely rich theory are those of the author [10] and Eringen and Maugin ([5], reprinted in 2012). To be complete we should note that the interaction of light with deformable (transparent) matter was discovered by David Brewster (1781–1868) in 1814–1815 when he found that mechanical stresses induce temporarily in transparent solids directional properties with respect to polarized light. It is the French engineer-scientist Augustin Fresnel (1788–1827) who identified this property with the double refraction of crystals in 1822. This was readily applied by F. Neumann in his experiments on thermo-elasticity. We have here the basis of the technique of photo-elasticity. More along this line had to benefit from laser technology in the twentieth century.

Concerning another flourishing field of application of continuum thermo-mechanics in the last fifty years, biomechanics and mechano-biology, only very few hints at some early development in the perused period are the work of Poiseuille on the flow of blood in 1844 (laminar flow in a cylindrical tube) and the thesis work of John Bernoulli on the movement of muscles in 1694, although Galileo Galilei (in *Two new sciences*) in 1638 had previously pondered the mechanical strength of bones versus the size of animals. Again, one had to await the second half of the twentieth century to see a true blossom of mathematical studies and a realistic mechanical modelling in bio-thermo-mechanics (Cf. some historical remarks in [13]). This concludes our adventures in the realm of continuum thermo-mechanics, between John Bernoulli and Ernst Hellinger, on a more humane tune with the passing from the mechanics of inert matter to that of living matter.

References

1. Appell P (1921) Traité de mécanique rationnelle, 3rd edn. Gauthier-Villars, Paris (Fac simili reprint by Gabay, Paris, 1991)
2. Belhoste B (1991) Augustin-Louis Cauchy: a biography. Springer, New York (English translation from the French original “Cauchy, 1789–1857”, Belin, Paris)
3. Darrigol O (2005) Worlds of flows: a history of hydrodynamics from the Bernoullis to Prandtl. Oxford University Press, Oxford
4. Eringen AC (ed) (1971–1976) Continuum physics, Four volumes. Academic Press, New York

5. Eringen AC, Maugin GA (1990) Electrodynamics of continua, Two volumes. Springer, New York (Soft-cover reprint, Springer, New York, 2012)
6. Hellinger E (1914) Die allgemein Ansätze der Mechanik der Kontinua. In: Klein F, Wagner K (eds) Enz MathWiss, vol 4, Part 4. Springer, Berlin, pp 602–694
7. Jacob C (1959) Introduction mathématique à la mécanique des fluides. Gauthier-Villars, Paris/Ed. Acad. Sci. Romania, Bucarest (original in Romanian, 1952)
8. Lamb H (1879) Hydrodynamics, 1st edn. Cambridge University Press, Cambridge (6th edn, CUP, 1932; Dover reprint, New York, 1945)
9. Love AEH (1892) A Treatise on the mathematical theory of elasticity (1944, 4th edn, Dover reprint, New York; originally published in two volumes in 1892–1893)
10. Maugin GA (1988) Continuum mechanics of electromagnetic solids. North-Holland, Amsterdam
11. Maugin GA (1999) Thermomechanics of nonlinear irreversible behaviours. World Scientific, Singapore
12. Maugin GA (2012) The principle of virtual power: from eliminating metaphysical forces to providing an efficient modelling tool. Cont Mech Thermodynam 25:127–146
13. Maugin GA (2013) Continuum mechanics through the twentieth century: a concise historical perspective. Springer, Dordrecht
14. Mushkelishvili NI (1953) Some basic problems in the mathematical theory of elasticity. Noordhoff, Groningen
15. Sedov LI (1937) Two-dimensional problems in hydrodynamics and aeromechanics (in Russian) Moscow. (English translation: Wiley, New York, 1965)
16. Solomon L (1968) Elasticité linéaire. Masson, Paris
17. Timoshenko SP (1953) History of the strength of materials. McGraw Hill, New York (Dover reprint, New York, 1983)
18. Todhunter I (1886) A history of the theory of elasticity and the strength of materials from Galileo to the present time, vol 1. Cambridge University Press, UK (edited and published posthumously in 1886 by Karl Pearson)
19. Truesdell CA, Toupin RA (1960) The classical theory of fields. In: Flügge S (ed) Handbuch der Physik, vol III/1. Springer, Berlin, pp 226–858

Retrospective
A Gallery of Portraits of the Main Actors

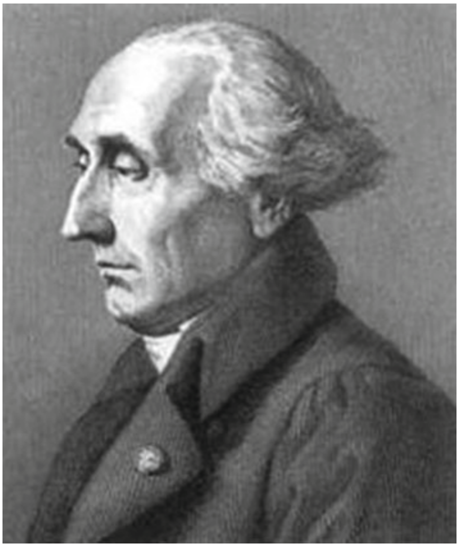
John (Johann) Bernoulli (1667–1748)

G. A. Maugin, *Continuum Mechanics Through the Eighteenth and Nineteenth Centuries*, Solid Mechanics and Its Applications 214, DOI: 10.1007/978-3-319-05374-5,

Daniel Bernoulli (1700–1782)

Leonhard Euler (1707–1783)

Jean Le Rond d'Alembert (1717–1783)



J.L. Lagrange (1736–1813)

Augustin-Louis Cauchy (1789–1857)

J.M.C. Duhamel (1797–1872)

« La statua di Gabrio Piola nel cortile di Brera
a Milano »: G. Piola (1794–1850)

Gustav Kirchhoff (1824–1887)

Franz E. Neumann (1798–1895)

Gabriel Lamé (1795–1870)

Alfred Clebsch (1833–1872)

A.Barré de Saint-Venant (1797–1886)

H. von Helmholtz (1821–1894)

J.V. Boussinesq (1842–1929)

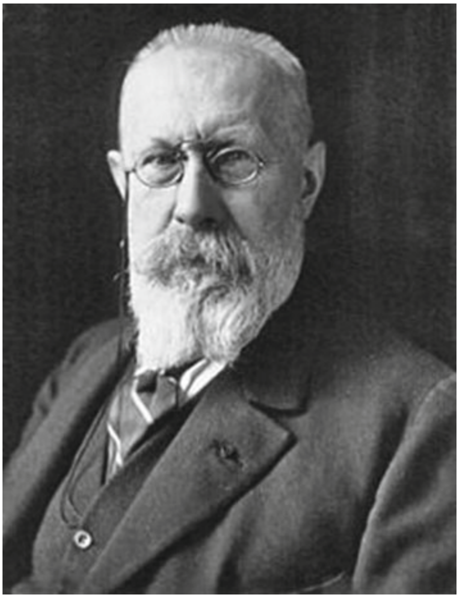
Pierre Duhem (1861–1916)

Emile Jouguet (1871–1943)

Eugène Cosserat (1866–1931)

François Cosserat (1852–1914)

Constantin Caratheodory (1873–1950)



Paul Appell (1855–1930)

Ernst Hellinger (1883–1950)

J.D. Eshelby (1916–1981)