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Larissa Borissova and Dmitri Rabounski

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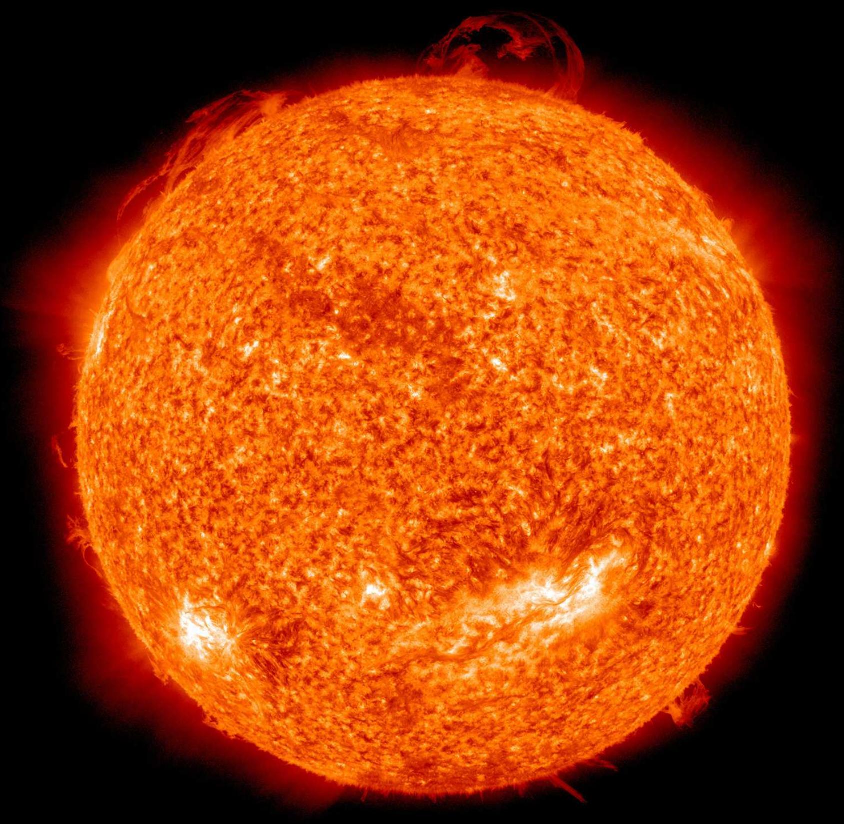
Second edition, expanded with new chapters



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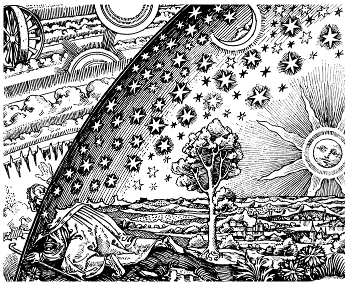
INSIDE STARS

A Theory of the Internal Constitution of Stars,
and the Sources of Stellar Energy
According to General Relativity

Second edition, expanded with new chapters



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Edited by Indranu Suhendro and Suzanne Billharz.
Preface by Pierre Millette.

Titlepage image: The enigmatic woodcut by an unknown artist of the Middle Ages. It is referred to as the *Flammarion Woodcut* because its appearance in page 163 of Camille Flammarion's *L'Atmosphère*: *Météorologie populaire* (Paris, 1888), a work on meteorology for a general audience. The woodcut depicts a man peering through the Earth's atmosphere as if it were a curtain to look at the inner workings of the Universe. The caption "Un missionnaire du moyen âge raconte qu'il avait trouvé le point où le ciel et la Terre se touchent..." translates to "A medieval missionary tells that he has found the point where heaven [the sense here is "sky"] and Earth meet...".

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Preface

A scientist often encounters established ideas that were once the subject of debate, sometimes controversy. Often, we use those ideas with no knowledge of their historical development, nor of the assumptions on which they are based. We rarely stop to ponder the validity of an established idea. This is not surprising as this is how we have been building our edifice of physical theories, by standing on the shoulders of giants, to paraphrase Isaac Newton.

Yet established ideas and theories need to be challenged and revisited when new data or new theories that contradict or shed new light on them, become available. We need not be afraid of new information that risk overturning accepted ideas. After all, this is how new paradigms arise and how progress is achieved.

The question of whether stars are gaseous or liquid is one debate that most scientists are oblivious to. Yet this was a subject of vigorous debate in the late 19th and early 20th centuries, with well-known physicists lining up behind both sides of the question. Larissa Borissova and Dmitri Rabounski provide a summary of the history of this debate and a personal perspective on how they were pulled into it.

Recent evidence for liquid stars, in particular the extensive research performed by Pierre-Marie Robitaille who has proposed the liquid metallic hydrogen model of the Sun*, leads us to revisit this question. Interestingly enough, stellar plasmas are modelled using Magnetohydrodynamics, i.e. magnetic fluid dynamics, a combination of Maxwell's equations of electromagnetism and the Navier-Stokes equations of fluid mechanics†. Magnetohydrodynamics is also used to model liquid metals. This is an indication that the theory of liquid stars is highly plausible as an explanation of solar and stellar astrophysical data.

My interest in this research area stems from the astrophysical research I performed on stellar atmospheres of Wolf-Rayet stars at the University of Ottawa's Department of Physics for my thesis on "Laser Action in C IV, N V and O VI Plasmas Cooled by Adiabatic Expansion". Wolf-Rayet stars exhibit mass loss and an expanding stellar atmosphere. This results in population inversion of certain atomic transitions due to

*Robitaille P.-M. A high temperature liquid plasma model of the Sun. *Progress in Physics*, 2007, vol. 3, issue 1, 70–81.

†Tajima T. and Shibata K. Plasma Astrophysics. Perseus Publishing, Cambridge, 2002; Kulsrud R. M. Plasma Physics for Astrophysics. Princeton University Press, Princeton, 2005.

the rapid cooling of the expanding plasma and the recombination of the free electrons into higher excited ionic states, and laser action in the corresponding emission lines. This physical mechanism has been proposed as the explanation for the prominent spectral lines observed in the spectra of Wolf-Rayet stars.

In this book, Larissa Borissova and Dmitri Rabounski provide a general relativistic theory of the internal constitution of liquid stars, a model that was lacking till now. This they accomplish by using a mathematical formalism first introduced by Abraham Zelmanov for calculating physically observable quantities in a four-dimensional pseudo-Riemannian space, known as the theory of chronometric invariants. This mathematical formalism allows to calculate physically observable (chronometrically invariant) tensors of any rank, based on operators of projection onto the time line and the spatial section of the observer. The basic idea is that physically observable quantities obtained by an observer should be the result of a projection of four-dimensional quantities onto the time line and onto the spatial section of the observer.

This analysis allows them to propose a classification of stars based on three main types: regular stars (covering white dwarfs to super-giants), of which Wolf-Rayet stars are a subtype, neutron stars and pulsars, and collapsars (i.e. black holes). Their theory also provides a model of the internal constitution of our solar system. It provides an explanation for the presence of the asteroid belt, the general structure of the planets inside and outside that orbit, and the net emission of energy by the planet Jupiter.

The ultimate test of any theory of stellar structure is the stellar mass-luminosity relation which is the main empirical relation of observational astrophysics. Using their theory, the authors can calculate the pressure inside stars as a function of radius, including the central pressure. As pointed out by the authors, the temperature of the incompressible liquid star does not depend on the pressure, only on the source of stellar energy (as opposed to a gas, in particular as given by the equation of state of an ideal gas). The authors match the calculated energy production of the suggested mechanism of thermonuclear fusion of the light atomic nuclei in the Hilbert core (the "inner sun") of the stars to the empirical mass-luminosity relation of observational astrophysics, to determine the density of the liquid stellar substance in the Hilbert core.

Pulsars and neutron stars are found to be stars whose physical radius is close to the radius of their Hilbert core. They are modelled by introducing an electromagnetic field in the theory due to their rotation and gravitation. Electromagnetic radiation is found to be emitted only from the poles of those stars, along the axis of rotation of the stars.

This book represents a solid contribution to our understanding of stellar structure from a general relativistic perspective. It provides a general relativistic underpinning to the theory of liquid stars. It raises new ideas on the constitution of stars and planetary systems, and proposes a new approach to stellar structure and evolution which is bound to help us better understand stellar astrophysics.

Ottawa, September 2, 2013

Pierre Millette

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Chapter 1

Problem Statement

§1.1 Introducing a new theory of the internal constitution of stars

In this book, we suggest a new mathematical theory of the internal constitution of stars, and the sources of stellar energy. The theory is based on the common consideration of a star and its field according to the General Theory of Relativity.

This is an alternative to the conventional theory of stars which was introduced in the 1920's by Arthur Eddington [1] and others in the framework of classical mechanics and thermodynamics.

As is known, the conventional theory resulted in the *model of gaseous stars*: stars are considered as gaseous spheres, consisting of mostly hydrogen and a very inhomogeneous interior so that the hydrogen of the extremely hot and dense central region is carried into a process of energy release. It assumes, after Hans Bethe [2], that this exothermic process is thermonuclear fusion producing helium from hydrogen. Two other versions of the gaseous model differ from Eddington's theory in details. Edward Milne [3] conjectured two (or more) different states of matter inside a star. Nikolai Kozyrev [4] arrived at the peculiar picture of low density and temperature inside stars, and a non-nuclear source of stellar energy.

Another theory of the internal constitution of stars was much populated in the 1920–1930's due to James Jeans [5, 6]. This is the *model of liquid stars*. The public discussion between Jeans, who defended the liquid model, and Eddington, the follower of the gaseous model, was fixed in the dozens of short communications they published in the scientific journals. Indeed, Eddington eventually won. Despite the many astrophysical evidences of liquid stars, Jeans' theory did not possess a solid mathematical basis. His theory was based on observational analysis and arguments rather than a mathematical model. In contrast, the theory of gaseous stars was mathematically well-grounded by Eddington. In particular, the mathematical model of gaseous stars gives a theoretical deduction of the mass-luminosity relation, which is one of the main relations of observational astrophysics*. This is a "trump

*The most comprehensive deduction of the mass-luminosity relation in the framework of the model of gaseous stars is given in Part I of Kozyrev's paper [4].

card": once the gaseous model predicts the mass-luminosity relation, the theory is usually claimed to be true in general while all its inconsistencies with observational analysis are merely some "difficulties" to be resolved in the future. Thus the model of gaseous stars became the conventional model for decades to come, until the present time.

We now have to make an important note. As is known, the core of the mathematical theory of the internal constitution of stars consists of two equations: the equation of mechanical equilibrium and the equation of heat equilibrium. The mechanical equilibrium means that the weight of any unit volume of the stellar matter is put into equilibrium with the pressure from within the star. The heat equilibrium (energy balance) means that the energy produced within any unit volume of the stellar matter is put into equilibrium with the energy flow (radiations, convection, or heat conductivity) from within the star onto its surface. These two main equations of the theory come from general physics, and they *do not depend* on whether the stars are out of gas, liquid, or something else. Only then, by introducing the equation of state of ideal gas (and many other particular assumptions) into the main equations, the conventional theory yields gaseous stars and other conclusions including the mass-luminosity relation.

Jeans' theory of liquid stars cannot follow this way. The equation of state of ideal liquid, provided by classical physics, is so simple that it contains not the characteristics of stellar matter which are necessary for further deduction by means of the equations of equilibrium.

Instead of all these considerations of classical mechanics and thermodynamics, we suggest an absolutely different approach to the problem. It is based on the simultaneous consideration of a star and its field according to the General Theory of Relativity. We consider liquid stars: this matches certain new observational evidences for the state of condensed matter inside stars; in particular, that the Sun consists of high-temperature liquid metallic hydrogen (see [7–10]).

In the framework of the General Theory of Relativity, the structure, matter, and field of such a star are characterized by Schwarzschild's metric of a sphere filled with incompressible liquid. The recent theoretical result obtained by one of us [11, 12] showed that, if the Sun is represented as a liquid sphere according to the metric, the Sun's field has a space breaking (discontinuity) in the asteroid strip (this implies that the space breaking impedes the substance to be formed as a planet in this orbit). We are therefore sure, hereby, of following the right path.

We deduce Einstein's field equations in the form that models stars as liquid spheres. This is a particular form of field equations, which may or may not satisfy the particular space metric. Therefore, we then prove that the obtained particular form of the field equations satisfies Schwarzschild's metric of a liquid sphere.

Then, on the basis of the obtained energy-momentum tensor of a perfect liquid (as contained on the right-hand side of the field equations), we deduce the conservation law for the liquid substance of regular stars. Solving the equations of energy-momentum conservation, we obtain the pressure and density of the liquid substance inside the stars. We then obtain the formula for the luminosity of stars in the framework of the liquid model. We study how this theoretical formula can be compatible with the mass-luminosity relation (which is the main empirical relation of observational astrophysics). As a result, we obtain the physical characteristics of the mechanism that produces energy inside the stars.

Concerning the stellar energy mechanism itself, we conclude that it is the conversion of substance into radiation on the surface of the tiny central "core" inside each star. The core can have a different density than that of the other substance of a star (a liquid sphere is homogeneous inside), and is selected by the collapse surface with the radius determined according to the star's mass. Despite almost all the mass of the star is located outside the core (the core is not a black hole), the force of gravity approaches infinity on the surface of the core due to the inner space breaking of the star's field therein. The super-strong force of gravity is sufficient for the transfer of the necessary kinetic energy to the lightweight atomic nuclei of the stellar substance, so that the process of thermonuclear fusion begins. The energy produced by the thermonuclear fusion is that very energy which stars radiate. in other wordss, the tiny core of each star is its luminous "inner sun", while the produced stellar energy is then transferred to the physical surface of the star due to thermal conductivity (which is regular to liquid media).

Neutron stars and pulsars, being rapidly rotating objects, consist a special type of stars. The structure, matter, and field of such stars should be described by another metric, which is that of a rotating liquid sphere under special physical conditions (which are particular to neutron stars and pulsars). We introduce such a metric. According to the metric, the liquid substance of neutron stars and pulsars is in the same state as high-density physical vacuum. We then deduce a particular form of Einstein's field equations which satisfies the metric. We show that the energy-momentum tensor of the obtained field equations satisfies the conservation law only in the case where the energy flow from within the object is very anisotropic, and is directed toward the north and south poles (while the axis of the magnetic field does not coincide with the axis of rotation of the object). This matches the well-known observational data about neutron stars and pulsars.

This is our plan for the upcoming chapters. As a result, we obtain a mathematical theory of liquid stars, and of the sources of stellar energy according to the General Theory of Relativity.

Before proceeding with the steps, in the next §1.2, we survey the space-time metrics we use in our theory. Then we introduce a new classification of stars. This classification is based on the location of the space breaking of a star's field with respect to its surface (the space breaking is calculated according to the metric and proper parameters of the star).

At the end of this chapter, §1.3 gives a survey of the important mathematical apparatus of physically observable quantities in the space-time of the General Theory of Relativity we require for our further calculations.

§1.2 Modelling a star in terms of General Relativity

Stars are spherical bodies filled with substance and light. Their fields are spherically symmetric as well. Therefore, once considering a star in terms the General Theory of Relativity, the structure, matter, and field of such an object should be given by a spherically symmetric space (space-time) metric.

Among the space-time metrics known due to the General Theory of Relativity, three primary metrics describe spherically symmetric fields. These are Schwarzschild's metric of a mass-point, Schwarzschild's metric of a sphere filled with incompressible liquid, and de Sitter's metric which describes a spherical distribution of physical vacuum (λ-field). All these three metrics will be used in our consideration of stars.

1.2.1 The *mass-point metric*

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{1.1}$$

was introduced in 1916 by Karl Schwarzschild [13]. The metric describes the field of a spherically symmetric massive body to so large a distance from it that the physical size of the body is neglected (assuming the body is a mass-point). The metric is written in the spherical coordinates r, ϕ, θ, whose origin meets the mass-point. Also, herein $r_g = \frac{2GM}{c^2}$ is the Hilbert radius of the massive body*, while M is the body's mass (which is the mass of the field source).

*This is not the same as the physical radius of the body. At a distance of the

According to the metric (1.1), such a space does not rotate or deform. The gravitational inertial force (see §1.4 for detail) in the space can be deduced on the basis of the component g_{00} of the fundamental metric tensor. As is seen, g_{00} of the mass-point metric (1.1) has the form

$$g_{00} = 1 - \frac{r_g}{r}\,. \tag{1.2}$$

Differentiating the gravitational potential $\mathrm{w} = c^2(1 - \sqrt{g_{00}})$ with respect to x^i, we obtain

$$\frac{\partial \mathrm{w}}{\partial x^i} = -\frac{c^2}{2\sqrt{g_{00}}}\frac{\partial g_{00}}{\partial x^i}\,. \tag{1.3}$$

We then substitute it into the general formula for gravitational inertial force (1.42) while taking the absence of rotation of the space into account. We obtain the formulae for the covariant and contravariant components of the gravitational inertial force

$$F_1 = -\frac{c^2 r_g}{2r^2}\frac{1}{1 - \frac{r_g}{r}}\,, \qquad F^1 = -\frac{c^2 r_g}{2r^2}\,. \tag{1.4}$$

As is seen from the formulae, the gravitational inertial force in a mass-point space is due Newtonian gravitation, and is reciprocal to the square of the distance r from the gravitating mass.

The curvature of a mass-point space is due to the Newtonian field of gravitation, produced by the massive body in the origin of the coordinates. This is not a constant curvature space; its curvature decreases with distance from the massive body (the field source). At an infinitely large distance from the body the space is flat.

1.2.2 A space filled with a spherically symmetric homogeneous distribution of physical vacuum (the λ-field in Einstein's field equations) without any island of mass presented therein is described by *de Sitter's metric*

$$ds^2 = \left(1 - \frac{\lambda r^2}{3}\right)c^2 dt^2 - \frac{dr^2}{1 - \frac{\lambda r^2}{3}} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{1.5}$$

The metric was introduced in 1918 by Willem de Sitter [16] as a static model of the Universe. It is assumed that $\lambda < 10^{-56}$ in the cosmos, so

Hilbert radius from the center of gravity of the massive body ($r = r_g$), gravitational collapse occurs: in a rotation-free space, this is a state by which the component g_{00} of the fundamental metric tensor $g_{\alpha\beta}$ is zero ($g_{00} = 0$). See §5.1 and §5.2 for details. The *Hilbert radius* was introduced due to David Hilbert (1862–1944) who considered it in 1917 [15] on the basis of Schwarzschild's mass-point metric. It is also known as the *Schwarzschild radius*, despite the fact that Karl Schwarzschild (1873–1916) never considered gravitational collapse in his papers [13, 14].

physical vacuum has a very low density therein. A modern version of the static model of the Universe is presented in [17].

The fundamental metric tensor, via its components according to de Sitter's metric (1.5), manifests that such a space does not deform or rotate. Therefore, the gravitational inertial force (1.42) in such a space is only due to g_{00} which is

$$g_{00} = 1 - \frac{\lambda r^2}{3}\,. \tag{1.6}$$

Accordingly, after the same algebra as previously, we obtain

$$F_1 = \frac{\lambda c^2}{3}\frac{r}{1-\frac{\lambda r^2}{3}}\,, \qquad F^1 = \frac{\lambda c^2}{3}\,r\,. \tag{1.7}$$

This is a non-Newtonian gravitational force, which is proportional to distance r: the force (λ-force) grows along with the distance at which it acts. If $\lambda < 0$ (the observable density of vacuum is positive), this is an attraction force. If $\lambda > 0$ (the observable density of vacuum is negative), this is a repulsion force. See Chapter 5 of our book [18] for details.

The curvature of a de Sitter space is due to the non-Newtonian gravitational field produced by physical vacuum (λ-field), which homogeneously fills the space. The curvature is the same everywhere within the space. This is a constant curvature space, in other words.

1.2.3 The *metric of a sphere filled with incompressible liquid* was originally introduced in 1916 by Karl Schwarzschild [14] in a truncated form containing substantial limitations. He artificially pre-imposed the limitations during the deduction of the metric in order to set the field free of breaking*. The true metric of a sphere filled with incompressible liquid remained unknown until 2009, when one of us deduced it in the most complete form [11, 12], which is free of any limitations and thus takes space breaking into account. The model of stars as liquid spheres plays a key rôle in our theory. We therefore consider the metric of a sphere filled with incompressible liquid in the complete form

$$ds^2 = \frac{1}{4}\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r^2 r_g}{a^3}}\right)^2 c^2 dt^2 - \\ - \frac{dr^2}{1-\frac{r^2 r_g}{a^3}} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{1.8}$$

*Actually, once a limitation is pre-imposed on the metric, the geometry of the metric space is artificially truncated. In this sense, the metric Schwarzschild introduced in 1916 is not the genuine metric of the space of a liquid sphere.

as that deduced in the papers [11, 12]. Herein, $a = const$ is the physical radius of the liquid sphere, while $r_g = \frac{2GM}{c^2}$ is the Hilbert radius calculated according to the mass M of the liquid (which is the field source). The complete deduction of the metric, containing all the necessary details, will be presented in §2.1 of the book wherein we will suggest the metric for regular stars.

The metric (1.8) is written for distances $r < a$: this is the "internal metric" of a sphere filled with incompressible liquid. At the surface of the sphere ($r = a$) the metric coincides with the mass-point metric. Also, as was proven in [12] (this deduction will be repeated in §2.1 of the book), the "outer metric" of the sphere ($r > a$) is as well the same as the mass-point metric: the external field of a liquid sphere is the same as the Newtonian gravitational field of a mass-point.

As is seen from the metric of a liquid sphere (1.8), such a space does not deform or rotate. Therefore, according to the definition of the gravitational inertial force (1.42), the force in such a space is only due to g_{00}. In the metric (1.8), we have

$$g_{00} = \frac{1}{4}\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r^2 r_g}{a^3}}\right)^2. \tag{1.9}$$

After the same algebra as previously, we obtain

$$F_1 = -\frac{c^2 r_g\, r}{a^3}\frac{1}{\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}\right)\sqrt{1-\frac{r_g r^2}{a^3}}}\,, \tag{1.10}$$

$$F^1 = -\frac{c^2 r_g\, r}{a^3}\frac{\sqrt{1-\frac{r_g r^2}{a^3}}}{3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}}\,. \tag{1.11}$$

Since $r < a$ inside the sphere, $F_1 < 0$ therein. Hence, this is a force of attraction. Its numerical value is proportional to distance r. The force is zero at the center of the sphere, and reaches its ultimate-high value on the surface.

It is possible to show that the curvature of such a space, being due to the aforementioned field of attraction, increases with distance from the center of the liquid sphere onto its surface. in other words, the space inside a liquid sphere is not a constant curvature space. (We will provide the proof and discuss both the four-dimensional curvature and the three-dimensional observable curvature of the space in §2.3.)

1.2.4 We now suggest a new modelling of stars in terms of the General Theory of Relativity.

Consider stars as spherical bodies consisting of liquid. In the framework of the liquid model, the internal structure, matter, and field of a star is described by the metric of a sphere filled with incompressible liquid. This is formula (1.8). As was shown above, the force of gravitation increases therein proportionally to the distance from the center of the star. The external field of the star is described by the mass-point metric (1.1). In the external field, the regular Newtonian gravitational force acts. The force is reciprocal to the distance from the star.

The field of a liquid sphere, as such, is not continuous everywhere. According to the external metric (1.1) and internal metric (1.8) of a liquid sphere, its field has *space breaking* which appears at two distances from the center of the liquid sphere. Due to this fact, we now introduce a new classification of stars according to the General Theory of Relativity. We hereby explain how to build it.

The space breaking occurs due to the violation of the *signature prescription conditions* of the space metric. It means that the space has a singularity in that region (surface or volume) wherein at least one of the signature conditions is violated. The signature conditions for a sign-alternating diagonal metric $(+---)$ as that of the four-dimensional pseudo-Riemannian space (which is the basic space-time of the General Theory of Relativity) have the form

$$\left.\begin{aligned} & g_{00} > 0 \\ & g_{00}\, g_{11} < 0 \\ & g_{00}\, g_{11}\, g_{22} > 0 \\ & g = g_{00}\, g_{11}\, g_{22}\, g_{33} < 0 \end{aligned}\right\}. \tag{1.12}$$

The first three are known as the *weak signature conditions.* The fourth is known as the *strong signature condition.* If one or all weak signature conditions are violated, while the strong signature condition is true, this is a *removable singularity.* If the strong signature condition is violated, the space-time has *unremovable singularity*: in this case the solution is regularly failed from consideration, because it "has no physical sense". Actually, one could not see the physical meaning therein. However, it is very meaningful mathematically. We therefore will take any space singularity (space breaking) under consideration.

Consider now the space of a liquid sphere. According to the external metric (1.1) of the sphere, the first signature condition is violated

($g_{00}=0$) at the distance $r=r_g$ from the center:

$$\left.\begin{aligned} g_{00} &= 1-\frac{r_g}{r}=0 \\ g_{00}\,g_{11} &= -1<0 \\ g_{00}\,g_{11}\,g_{22} &= r^2>0 \\ g &= -r^4\sin^2\theta<0 \end{aligned}\right\}. \tag{1.13}$$

The internal metric (1.8) of the sphere manifests that the second, third, and fourth signature conditions are violated at the distance

$$r=r_{br}=\sqrt{\frac{a^3}{r_g}} \tag{1.14}$$

from the center of the sphere, where the aforementioned three functions approach infinity*:

$$\left.\begin{aligned} & g_{00}=\frac{9}{4}\left(1-\frac{r_g}{a}\right)>0 \\ & g_{00}\,g_{11}\to-\infty \\ & g_{00}\,g_{11}\,g_{22}\to\infty \\ & g=g_{00}\,g_{11}\,g_{22}\,g_{33}\to-\infty \end{aligned}\right\}. \tag{1.15}$$

This means that the field of a liquid sphere has space breaking at two distances from the center:

1. The first space breaking occurs on the surface, spherically covering the center of gravity of the liquid sphere at the distance of the Hilbert radius $r=r_g$. This is a surface of gravitational collapse (according to the condition $g_{00}=0$ of this space breaking). In other words, while a liquid sphere itself may not be a collapsar, it always contains a "core" which is selected from the other liquid substance by the surface of gravitational collapse. In the case where the liquid sphere is a star (as in the said model of liquid stars), each star contains such a core. The core is much smaller than the physical radius of regular stars: while the radius of the collapsed core (Hilbert radius) of the Sun is $r_g=2.9\times10^5$ cm (2.9 km), the physical radius of the Sun is 7.0×10^{10} cm (700,000 km). We therefore refer to it as the *inner space breaking*;

*As is known, a function has a breaking when approaching infinity.

2. The second space breaking occurs on the spherical surface covering the liquid body at the distance $r_{br} = \sqrt{a^3/r_g}$. The distance is much larger than the physical radius of regular stars. Thus this is the *outer space breaking* (contrary to the inner space breaking at the Hilbert radius). For example, the second (outer) space breaking of the Sun's field occurs at the distance $r_{br} = 3.4\times10^{13}$ cm $=$ $= 340,000,000$ km $= 2.3$ AU*. This space breaking is located inside the asteroid strip, close to the orbit of the maximal concentration of the asteroids (the asteroid strip is situated from 2.1 AU to 4.3 AU from the Sun). This implies that the space breaking does not permit the substance to be formed into a common physical body (such as a planet) in this orbit.

If the physical radius a of a liquid star is the same as the Hilbert radius $r_g = \frac{2GM}{c^2}$, it is a gravitational collapsar. In this case ($r_g = a$), the internal metric of the liquid sphere (1.8) takes the form

$$ds^2 = \frac{1}{4}\left(1 - \frac{r^2}{a^2}\right)c^2dt^2 - \frac{dr^2}{1 - \frac{r^2}{a^2}} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \qquad (1.16)$$

This metric, under the particular condition $a^2 = \frac{3}{\lambda} > 0$ (thus $\lambda > 0$), has the same form as de Sitter's metric (1.5),

$$ds^2 = \left(1 - \frac{\lambda r^2}{3}\right)c^2dt^2 - \frac{dr^2}{1 - \frac{\lambda r^2}{3}} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \qquad (1.17)$$

which describes a spherical distribution of physical vacuum (the λ-field). This means that such a collapsed object, which is a liquid sphere in the state of gravitational collapse, consists of liquid whose state is close to the state of high-density physical vacuum.

As a result, the new liquid model allows us to introduce a new classification of stars according to the location of the space breaking of a star's field with respect to the physical surface of the star:

TYPE I: REGULAR STARS INCLUDING THE SUN

The collapsed core (Hilbert radius r_g) of a regular star is many orders less than the physical radius a of the star ($r_g \ll a$). The outer space breaking r_{br} of the star's field is located far away from the star, in the cosmos ($r_{br} \gg a$). This, of course, is the list of almost all the visible stars: super-giants, the Sun, brown dwarfs, and even white dwarfs. We will consider regular stars in Chapter 2;

*1 AU $= 1.49\times10^{13}$ cm (Astronomical Unit) is the average distance between the Sun and the Earth.

Object	Mass M, gram	Radius a, cm	Hilbert radius r_g, cm	$\frac{r_g}{a}$	Space breaking r_{br}, cm	$\frac{r_{br}}{a}$	Type
Red super-giant*	4.0×10^{34}	7.0×10^{13}	5.9×10^{6}	8.4×10^{-8}	2.4×10^{17}	3.4×10^{3}	I
White super-giant†	3.4×10^{34}	4.8×10^{12}	5.0×10^{6}	1.0×10^{-6}	4.7×10^{15}	9.8×10^{2}	I
Sun	2.0×10^{33}	7.0×10^{10}	2.9×10^{5}	4.1×10^{-6}	3.4×10^{13}	4.9×10^{2}	I
Jupiter (proto-star)	1.9×10^{30}	7.1×10^{9}	2.8×10^{2}	4.0×10^{-8}	3.4×10^{13}	4.8×10^{3}	I
White dwarf‡	2.0×10^{33}	6.4×10^{8}	3.0×10^{5}	4.7×10^{-4}	2.9×10^{10}	0.45×10^{2}	I
Red dwarfs	6.7×10^{32}	2.3×10^{10}	9.9×10^{4}	4.3×10^{-6}	1.1×10^{13}	4.8×10^{2}	I
Brown dwarfs	1.5×10^{32}	7.0×10^{9}	2.2×10^{4}	3.1×10^{-6}	4.0×10^{14}	5.7×10^{4}	I
Wolf-Rayet stars	1.0×10^{35}	1.4×10^{12}	1.5×10^{7}	1.1×10^{-5}	4.3×10^{14}	3.1×10^{2}	Ia
Neutron stars	2.6×10^{33}	1.0×10^{6}	3.9×10^{5}	0.39	1.6×10^{6}	1.6	II
Pulsar§	3.9×10^{33}	1.6×10^{6}	5.8×10^{5}	0.36	2.7×10^{6}	1.7	II
Black holes	various	various	various	1	1	1	III

*Betelgeuse. †Rigel. ‡Sirius B. §Radio-pulsar J1903+0327.

Table 1.1: Classification of stars according to the General Theory of Relativity. The classification is presented with the numerical values of the parameters we calculated for the typical members of the families of stars.

Type Ia: Wolf-Rayet stars

They are almost the same as regular stars, except that the powerful stellar wind consisting of the particles of the stellar substance, which are permanently erupted from the star, should be taken into account (it is the property characterizing Wolf-Rayet stars). Stellar wind will be considered in Chapter 3;

Type II: Neutron stars and pulsars

For such a star, the radius of the collapsed core is close to the physical radius of the star ($r_g \lesssim a$) but does not reach it (otherwise the star would be invisible for observation). The outer space breaking r_{br} of the star's field is located in the outer cosmos, and is also close to the physical surface of the star but does not reach it ($r_{br} \gtrsim a$). Also, stars of this Type II rotate at high speeds which are close to relativistic velocities. As a result, the metric and energy-momentum tensor of such a star differ from those of regular stars. These are neutron stars and pulsars. We will focus on these stars in Chapter 4;

Type III: Black holes

The Hilbert radius r_g (radius of the inner space breaking) and the radius of the outer space breaking r_{br} of a such an object meet each other on its physical surface ($r_g = r_{br} = a$). These are gravitational collapsars: the condition of gravitational collapse ($g_{00} = 0$) occurs in the physical surface of such an object, so all of its mass is concentrated within the collapsed surface. Black holes will be under focus in Chapter 5 of the book.

This classification is presented in Table 1.1, with the numerical values of the parameters calculated for the typical members of the known families of stars.

The new model of liquid stars according to the General Theory of Relativity, surveyed in the classification of stars, will be a subject to develop in the upcoming chapters.

§1.3 physically observable quantities

Before considering stars in terms of the General Theory of Relativity, we shall outline a theory of physically observable quantities in curved four-dimensional space (space-time). A comprehensive expisition of the said physically observable quantities has already been given in the respective chapters of our books [18, 19]. We now give the necessary theoretical basics of the theory of physically observable quantities according to [19], with some amendments which are required for the current study.

In order to build a descriptive picture of any physical theory, we need to express the results through real physical quantities, which can be measured in experiments (*physically observable quantities*). In the General Theory of Relativity, this problem is not a trivial one at all, because we are looking at objects in a four-dimensional space-time, and so we have to determine which components of the associated four-dimensional tensor quantities are truly physically observable.

Here is the problem in a nutshell. All equations in the General Theory of Relativity are cast in *generally covariant form*, which does not depend on our choice of the frame of reference. The equations, as well as the variables they contain, are four-dimensional. Thus, we ask, which of those four-dimensional variables are truly observable in real physical experiments, i.e. which components are true physically observable quantities? Intuitively we might, at first glance, easily assume that the three-dimensional components of a four-dimensional tensor constitute a physically observable quantity. Yet, at the same time, we cannot be absolutely sure that what we simply observe are truly the three-dimensional components *per se*, if not more complicated variables which depend on other factors, e.g. on the properties of the physical standards of the space of reference.

As is known, a four-dimensional vector (a 1st-rank tensor) has as few as 4 components (1 time component and 3 spatial components). A 2nd-rank tensor, e.g. a rotation or deformation tensor, has 16 components: 1 time component, 9 spatial components, and 6 mixed (time-space) components. Now, are the mixed components truly physically observable quantities? Tensors of higher ranks have even more components; for instance the Riemann-Christoffel curvature tensor has 256 components, so the problem of the heuristic recognition of genuine physically observable components becomes far more complicated. Besides, there is an obstacle related to the recognition of the observable components of covariant tensors (in which indices occupy the lower position) and of mixed-type tensors, which have both lower and upper indices.

We see that the recognition of physically observable quantities in the General Theory of Relativity is not a trivial problem. Ideally we would like to have a mathematical technique to calculate physically observable quantities for tensors of any given ranks *unambiguously*.

Numerous attempts to develop such a mathematical method were made in the 1930's by some of the most outstanding researchers of that time. The goal was nearly attained by Landau and Lifshitz in their famous *The Classical Theory of Fields* [20], first published in Russian in 1939. Aside for discussing the problem of physically observable quantities itself, in §84 of their book, they introduced the interval of physically observable time along with the phsyically observable three-dimensional interval, which depend on the physical properties (physical standards) of the space of reference of an observer. But all such attempts made in the 1930's were very limited to just solving certain particular problems. None of them led to a versatile mathematical apparatus.

A most complete mathematical apparatus for calculating physically observable quantities in a four-dimensional pseudo-Riemannian space was first introduced by Abraham Zelmanov and is known as the *theory of chronometric invariants*, or the *chronometrically invariant formalism.* It was first presented in 1944 in his Ph.D. thesis [21] — then in his condensed papers of 1956–1957 [22, 23].

The essence of Zelmanov's mathematical apparatus of physically observable quantities (chronometric invariants), designed especially for the four-dimensional, curved, non-uniform pseudo-Riemannian space (space-time), is as follows.

At any point of the space-time we can place a three-dimensional *spatial section* $x^0 = ct = const$ (three-dimensional space) orthogonal to a given *time line* $x^i = const$. If a spatial section is everywhere orthogonal to the time lines, which pierce it at each point, such a space is referred to as *holonomic*. Otherwise, if the spatial section is non-orthogonal everywhere to the aforementioned time lines, the space is referred to as *non-holonomic*.

Possible frames of reference of a real observer include a coordinate net spanned over a real physical body (the reference body of the observer, which is located near him) and a real clock located at each point of the coordinate net. Both the coordinate net and clock represent a set of real references to which the observer refers his observations. Therefore, physically observable quantities registered by an observer should be the result of a projection of four-dimensional quantities onto the time line and onto the spatial section of the observer.

The operator of projection onto the time line of an observer is the world-vector of four-dimensional velocity

$$b^{\alpha} = \frac{dx^{\alpha}}{ds} \tag{1.18}$$

of his reference body with respect to him. This world-vector is tangential to the world-line of the observer at each point of his world-trajectory, so this is a unit-length vector

$$b_{\alpha} b^{\alpha} = g_{\alpha\beta} \frac{dx^{\alpha}}{ds} \frac{dx^{\beta}}{ds} = \frac{g_{\alpha\beta}\, dx^{\alpha} dx^{\beta}}{ds^2} = +1\,. \tag{1.19}$$

The operator of projection onto the spatial section of the observer (his local three-dimensional space) is determined as a four-dimensional symmetric tensor $h_{\alpha\beta}$, which is

$$\left.\begin{aligned} h_{\alpha\beta} &= -\, g_{\alpha\beta} + b_\alpha b_\beta \\ h^{\alpha\beta} &= -\, g^{\alpha\beta} + b^\alpha b^\beta \\ h_\alpha^\beta &= -\, g_\alpha^\beta + b_\alpha b^\beta \end{aligned}\right\}. \tag{1.20}$$

The world-vector b^α and the world-tensor $h_{\alpha\beta}$ are orthogonal to each other. Mathematically this means that their common contraction is zero ($h_{\alpha\beta}\, b^\alpha = 0$, $h^{\alpha\beta} b_\alpha = 0$, $h^\alpha_\beta\, b_\alpha = 0$, $h^\beta_\alpha\, b^\alpha = 0$). So, the main properties of the operators of projection onto the time line and the spatial section of the observer are commonly expressed, obviously, as follows:

$$b_\alpha b^\alpha = +1\,, \qquad h^\beta_\alpha\, b^\alpha = 0\,. \tag{1.21}$$

If the observer rests with respect to his reference object (such a case is known as the *accompanying frame of reference*), then $b^i = 0$ in his reference frame. The coordinate nets of the same spatial section are connected to each other through the transformations

$$\left.\begin{aligned} &\tilde{x}^0 = \tilde{x}^0 \left(x^0, x^1, x^2, x^3\right) \\ &\tilde{x}^i = \tilde{x}^i \left(x^1, x^2, x^3\right), \quad \frac{\partial \tilde{x}^i}{\partial x^0} = 0 \end{aligned}\right\}, \tag{1.22}$$

where the third equation displays the fact that the spatial coordinates in the tilde-marked net are independent of the time of the non-tilded net, which is equivalent to a coordinate net where the lines of time are fixed $x^i = const$ at any point. The transformation of the spatial coordinates is nothing but a transition from one coordinate net to another within the same spatial section. The transformation of time means changing the whole set of clocks, so this is a transition to another spatial section (another three-dimensional space of reference). In practice this means replacement of one reference body with all of its physical references with another reference body that has its own physical references. But when using different references, the observer will obtain different results (other observable quantities). Therefore, the physically observable projections in an accompanying frame of reference should be invariant with respect to the transformation of time, which implies invariance with respect to the transformations (1.22). In other words, such quantities should possess the property of *chronometric invariance*.

We therefore refer to the physically observable quantities determined in an accompanying frame of reference as *chronometrically invariant quantities*, or *chronometric invariants* in short.

The tensor $h_{\alpha\beta}$ of projection, being considered in the space of a frame of reference accompanying an observer, possesses all properties attributed to the fundamental metric tensor, namely

$$h_i^{\alpha}\, h_{\alpha}^k = \delta_i^k - b_i b^k = \delta_i^k\,, \qquad \delta_i^k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1.23}$$

where δ_i^k is the unit three-dimensional tensor*. Therefore, in the accompanying frame of reference the three-dimensional tensor h_{ik} can lift or lower indices in chronometrically invariant quantities.

So in the accompanying frame of reference the main properties of the operators of projection are

$$b_{\alpha} b^{\alpha} = +1\,, \qquad h_{\alpha}^i b^{\alpha} = 0\,, \qquad h_i^{\alpha} h_{\alpha}^k = \delta_i^k\,. \tag{1.24}$$

Calculate the components of the operators of projection in the accompanying frame of reference. The component b^0 comes from the obvious condition $b_{\alpha} b^{\alpha} = g_{\alpha\beta}\, b^{\alpha} b^{\beta} = 1$, which in the accompanying frame of reference ($b^i = 0$) is $b_{\alpha} b^{\alpha} = g_{00}\, b^0 b^0 = 1$. This component, in common with the remaining components of b^{α}, is

$$\left.\begin{aligned} b^0 &= \frac{1}{\sqrt{g_{00}}}\,, & b^i &= 0 \\ b_0 &= g_{0\alpha} b^{\alpha} = \sqrt{g_{00}}\,, & b_i &= g_{i\alpha} b^{\alpha} = \frac{g_{i0}}{\sqrt{g_{00}}} \end{aligned}\right\}, \tag{1.25}$$

while the components of $h_{\alpha\beta}$ are

$$\left.\begin{aligned} h_{00} &= 0\,, & h^{00} &= -\,g^{00} + \frac{1}{g_{00}}\,, & h_0^0 &= 0 \\ h_{0i} &= 0\,, & h^{0i} &= -\,g^{0i}, & h_0^i &= \delta_0^i = 0 \\ h_{i0} &= 0\,, & h^{i0} &= -\,g^{i0}, & h_i^0 &= \frac{g_{i0}}{g_{00}} \\ h_{ik} &= -\,g_{ik} + \frac{g_{0i} g_{0k}}{g_{00}}\,, & h^{ik} &= -\,g^{ik}, & h_k^i &= -\,g_k^i = \delta_k^i \end{aligned}\right\}. \tag{1.26}$$

*This tensor δ_i^k is the three-dimensional part of the four-dimensional unit tensor δ_{β}^{α}, which can be used to replace indices in four-dimensional quantities.

Zelmanov created a comprehensive mathematical method for the calculation of the chronometrically invariant (physically observable) projections of any generally covariant (four-dimensional) tensor quantity, and set it forth as a theorem (we refer to it as *Zelmanov's theorem*):

ZELMANOV'S THEOREM

Assume that $Q^{ik\dots p}_{00\dots 0}$ are the components of the four-dimensional tensor $Q^{\mu\nu\dots\rho}_{00\dots 0}$ of the r-th rank, in which all upper indices are not zero, while all m lower indices are zero. Then, the quantities

$$T^{ik\dots p} = (g_{00})^{-\frac{m}{2}}\, Q^{ik\dots p}_{00\dots 0} \tag{1.27}$$

constitute a chronometrically invariant three-dimensional contravariant tensor of $(r-m)$-th rank. Hence the tensor $T^{ik\dots p}$ is a result of m-fold projection onto the time line by the indices $\alpha, \beta \dots \sigma$ and onto the spatial section by $r-m$ the indices $\mu, \nu \dots \rho$ of the initial tensor $Q^{\mu\nu\dots\rho}_{\alpha\beta\dots\sigma}$.

According to the theorem, the chronometrically invariant (physically observable) projections of a four-dimensional vector Q^α are

$$b^\alpha Q_\alpha = \frac{Q_0}{\sqrt{g_{00}}}\,, \qquad h^i_\alpha Q^\alpha = Q^i, \tag{1.28}$$

while the chr.inv.-projections of a symmetric tensor of the 2nd rank $Q^{\alpha\beta}$ are the following quantities:

$$b^\alpha b^\beta Q_{\alpha\beta} = \frac{Q_{00}}{g_{00}}\,, \qquad h^{i\alpha} b^\beta Q_{\alpha\beta} = \frac{Q^i_0}{\sqrt{g_{00}}}\,, \qquad h^i_\alpha h^k_\beta\, Q^{\alpha\beta} = Q^{ik}. \tag{1.29}$$

The chr.inv.-projections of a four-dimensional coordinate interval dx^α are the interval of the physically observable time

$$d\tau = \sqrt{g_{00}}\, dt + \frac{g_{0i}}{c\sqrt{g_{00}}}\, dx^i, \tag{1.30}$$

and the interval of the observable coordinates dx^i which are the same as the spatial coordinates. The physically observable velocity of a particle is the three-dimensional chr.inv.-vector

$$\mathrm{v}^i = \frac{dx^i}{d\tau}\,, \qquad \mathrm{v}_i \mathrm{v}^i = h_{ik}\, \mathrm{v}^i \mathrm{v}^k = \mathrm{v}^2, \tag{1.31}$$

which at isotropic trajectories becomes the three-dimensional chr.inv.-vector of the physically observable velocity of light

$$c^i = \mathrm{v}^i = \frac{dx^i}{d\tau}\,, \quad c_i c^i = h_{ik}\, c^i c^k = c^2. \tag{1.32}$$

Chronometrically projecting the covariant or contravariant fundamental metric tensor onto the spatial section of an accompanying frame of reference ($b^i = 0$)

$$\left.\begin{aligned} h_i^\alpha h_k^\beta \, g_{\alpha\beta} &= g_{ik} - b_i b_k = -\, h_{ik} \\ h_\alpha^i h_\beta^k \, g^{\alpha\beta} &= g^{ik} - b^i b^k = g^{ik} = -\, h^{ik} \end{aligned}\right\}, \tag{1.33}$$

we obtain that the chr.inv.-quantity

$$h_{ik} = -\, g_{ik} + b_i b_k \tag{1.34}$$

is the *chr.inv.-metric tensor* (the *observable metric tensor*), using which we can lift and lower indices of any three-dimensional chr.inv.-tensorial object in the accompanying frame of reference. The contravariant and mixed components of the observable metric tensor are, obviously,

$$h^{ik} = -\, g^{ik}, \qquad h_k^i = -\, g_k^i = \delta_k^i \,. \tag{1.35}$$

Expressing $g_{\alpha\beta}$ through the definition of $h_{\alpha\beta} = - g_{\alpha\beta} + b_\alpha b_\beta$, we obtain the formula for the four-dimensional interval

$$ds^2 = b_\alpha b_\beta \, dx^\alpha dx^\beta - h_{\alpha\beta} \, dx^\alpha dx^\beta, \tag{1.36}$$

expressed through the operators of projection b_α and $h_{\alpha\beta}$. In this formula $b_\alpha dx^\alpha = c\,d\tau$, so the first term is $b_\alpha b_\beta \, dx^\alpha dx^\beta = c^2 d\tau^2$. The second term $h_{\alpha\beta} \, dx^\alpha dx^\beta = d\sigma^2$ in the accompanying frame of reference is the square of the observable three-dimensional interval*

$$d\sigma^2 = h_{ik} \, dx^i dx^k. \tag{1.37}$$

Thus, the four-dimensional interval, represented through the physically observable quantities, is

$$ds^2 = c^2 d\tau^2 - d\sigma^2. \tag{1.38}$$

The main physically observable properties attributed to the accompanying space of reference were deduced by Zelmanov in the framework of the theory, in particular — proceeding from the property of non-commutativity (non-zero difference between the mixed 2nd derivatives with respect to the coordinates)

$$\frac{{}^*\partial^2}{\partial x^i \, \partial t} - \frac{{}^*\partial^2}{\partial t \, \partial x^i} = \frac{1}{c^2} \, F_i \, \frac{{}^*\partial}{\partial t} \,, \tag{1.39}$$

*This is due to the fact that $h_{\alpha\beta}$ in the accompanying frame of reference possesses all properties of the fundamental metric tensor $g_{\alpha\beta}$.

$$\frac{{}^*\partial^2}{\partial x^i\,\partial x^k} - \frac{{}^*\partial^2}{\partial x^k\,\partial x^i} = \frac{2}{c^2}\,A_{ik}\,\frac{{}^*\partial}{\partial t} \tag{1.40}$$

where the chr.inv.-operators of differentiation of Zelmanov are

$$\frac{{}^*\partial}{\partial t} = \frac{1}{\sqrt{g_{00}}}\,\frac{\partial}{\partial t}\,, \qquad \frac{{}^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} - \frac{g_{0i}}{g_{00}}\,\frac{\partial}{\partial x^0}\,. \tag{1.41}$$

The first two physically observable properties are characterized by the following three-dimensional chr.inv.-quantities: the vector of the *gravitational inertial force* F_i and the antisymmetric tensor of the *angular velocities of rotation* of the space of reference A_{ik} which are

$$F_i = \frac{1}{\sqrt{g_{00}}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right), \tag{1.42}$$

$$A_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right) + \frac{1}{2c^2}\left(F_i v_k - F_k v_i\right). \tag{1.43}$$

Here w and v_i characterize the body of reference and the reference space. These are the gravitational potential

$$\mathrm{w} = c^2\,(1 - \sqrt{g_{00}}\,)\,, \qquad 1 - \frac{\mathrm{w}}{c^2} = \sqrt{g_{00}}\,, \tag{1.44}$$

and the linear velocity of rotation of the space

$$\left.\begin{aligned} v_i &= -\,c\,\frac{g_{0i}}{\sqrt{g_{00}}}\,, \quad & v^i &= -\,c\,g^{0i}\sqrt{g_{00}} \\ v_i &= h_{ik}v^k, & v^2 &= v_k v^k = h_{ik}\,v^i v^k \end{aligned}\right\}. \tag{1.45}$$

We note that w and v_i do not possess the property of chronometric invariance, despite $v_i = h_{ik}v^k$ can be obtained as for a chr.inv.-quantity, through lowering the index by the chr.inv.-metric tensor h_{ik}.

Zelmanov also found that the chr.inv.-quantities F_i and A_{ik} are linked to each other by two identities (*Zelmanov's identities*)

$$\frac{{}^*\partial A_{ik}}{\partial t} + \frac{1}{2}\left(\frac{{}^*\partial F_k}{\partial x^i} - \frac{{}^*\partial F_i}{\partial x^k}\right) = 0\,, \tag{1.46}$$

$$\frac{{}^*\partial A_{km}}{\partial x^i} + \frac{{}^*\partial A_{mi}}{\partial x^k} + \frac{{}^*\partial A_{ik}}{\partial x^m} + \frac{1}{2}\left(F_i A_{km} + F_k A_{mi} + F_m A_{ik}\right) = 0\,. \tag{1.47}$$

In the framework of quasi-Newtonian approximation, i.e. in a weak gravitational field at velocities much lower than the velocity of light and in the absence of rotation of space, F_i becomes a regular non-relativistic gravitational force $F_i = \frac{\partial \mathrm{w}}{\partial x^i}$.

Zelmanov also proved the following theorem setting up the condition of holonomity of space:

Zelmanov's theorem on holonomity of space

> Identical equality of the tensor A_{ik} to zero in a four-dimensional region of space (space-time) is the necessary and sufficient condition for the spatial sections to be everywhere orthogonal to the time lines in this region.

in other words, the necessary and sufficient condition of holonomity of a space should be achieved by equating to zero the tensor A_{ik}. Naturally, if the spatial sections are everywhere orthogonal to the time lines (in such a case the space is holonomic), the quantities g_{0i} are zero. Since $g_{0i} = 0$, we have $v_i = 0$ and $A_{ik} = 0$. Therefore, we will also refer to the tensor A_{ik} as the *space non-holonomity tensor*.

If the conditions $F_i = 0$ and $A_{ik} = 0$ are met in common somewhere in the space, the conditions $g_{00} = 1$ and $g_{0i} = 0$ are as well true therein. In such a region, according to (1.30), $d\tau = dt$: the difference between the coordinate time t and the physically observable time τ disappears in the absence of gravitational fields and rotation of the space. In other words, according to the theory of chronometric invariants, the difference between the coordinate time t and the physically observable time τ originates in both gravitation and rotation attributed to the space of reference of the observer (the local space of the Earth in the case of an Earth-bound observer), or in each of these physical factors separately.

On the other hand, it is doubtful to find such a region of the Universe wherein gravitational fields or rotation of the background space would be absent in clear. Therefore, in practice the physically observable time τ and the coordinate time t differ from each other. This means that the real space of our Universe is non-holonomic, and is filled with a gravitational field, while a holonomic space free of gravitation can be only a local approximation to it.

The condition of holonomity of a space (space-time) is linked directly to the problem of integrability of time in it. The formula for the interval of the physically observable time (1.30) has no integrating multiplier. In other words, this formula cannot be reduced to the form

$$d\tau = A\,dt\,, \tag{1.48}$$

where the multiplier A depends on only t and x^i: in a non-holonomic space the formula (1.30) has non-zero second term, depending on the coordinate interval dx^i and g_{0i}. In a holonomic space $A_{ik}=0$, so $g_{0i}=0$. In such a case, the second term of (1.30) is zero, while the first term is the elementary interval of time dt with an integrating multiplier

$$A=\sqrt{g_{00}}=f\left(x^0,x^i\right), \tag{1.49}$$

so we are allowed to write the integral

$$d\tau=\int\sqrt{g_{00}}\,dt\,. \tag{1.50}$$

Hence time is integrable in a holonomic space ($A_{ik}=0$), while it cannot be integrated in the case where the space is non-holonomic ($A_{ik}\neq 0$). In the case where time is integrable (a holonomic space), we can synchronize the clocks in two distantly located points of the space by moving a control clock along the path between these two points. In the case where time cannot be integrated (a non-holonomic space), synchronization of clocks in two distant points is impossible in principle: the larger the distance between these two points is, the more the deviation of time on these clocks is.

The space of our planet, the Earth, is non-holonomic due to the daily rotation of it around the Earth's axis. Hence two clocks located at different points of the surface of the Earth should manifest a deviation between the intervals of time registered on each of them. The larger the distance between these clocks is, the larger the deviation of the physically observable time (expected to be registered on them) is. This effect was surely verified by the well-known Hafele-Keating experiment [24–27] concerned with displacing standard atomic clocks by an airplane around the terrestrial globe, where rotation of the Earth's space sensibly changed the measured time. During a flight along the Earth's rotation, the observer's space on board of the airplane had more rotation than the space of the observer who stayed fixed on the ground. During a flight against the Earth's rotation it was vice versa. An atomic clock on board of the airplane showed a significant deviation of the observed time depending on the velocity of rotation of space.

Because synchronization of clocks at different locations on the surface of the Earth is a highly important problem in marine navigation and also aviation, in an early time de-synchronization corrections were introduced as tables of the empirically obtained corrections which take the Earth's rotation into account. Now, thanks to the theory of chronometric invariants, we know the origin of these corrections, and are able to calculate them on the basis of the General Theory of Relativity.

In addition to gravitation and rotation, the reference body can deform, changing its coordinate nets with time. This fact should also be taken into account in measurements. This can be done by introducing into the equations the three-dimensional symmetric chr.inv.-tensor of the *rate of deformation* of the space of reference

$$\left.\begin{aligned} D_{ik} &= \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t} \\ D^{ik} &= -\frac{1}{2}\frac{{}^*\partial h^{ik}}{\partial t} \\ D = h^{ik}D_{ik} &= \frac{{}^*\partial \ln\sqrt{h}}{\partial t}\,, \qquad h = \det\|\,h_{ik}\| \end{aligned}\right\}. \tag{1.51}$$

The regular Christoffel symbols of the 2nd rank ($\Gamma^{\alpha}_{\mu\nu}$) and the 1st rank ($\Gamma_{\mu\nu,\sigma}$)

$$\Gamma^{\alpha}_{\mu\nu} = g^{\alpha\sigma}\,\Gamma_{\mu\nu,\sigma} = \frac{1}{2}\,g^{\alpha\sigma}\left(\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} + \frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right) \tag{1.52}$$

are linked to the respective *chr.inv.-Christoffel symbols*

$$\Delta^{i}_{jk} = h^{im}\Delta_{jk,m} = \frac{1}{2}\,h^{im}\left(\frac{{}^*\partial h_{jm}}{\partial x^{k}} + \frac{{}^*\partial h_{km}}{\partial x^{j}} - \frac{{}^*\partial h_{jk}}{\partial x^{m}}\right) \tag{1.53}$$

which are determined similarly to $\Gamma^{\alpha}_{\mu\nu}$. The only difference is that here, instead of the fundamental metric tensor $g_{\alpha\beta}$, the chr.inv.-metric tensor h_{ik} is used. The Christoffel symbols characterize the property of *inhomogeneity* of space.

The components of the regular Christoffel symbols are linked to the other chr.inv.-chractersitics of the accompanying space of reference by the following relations:

$$D^{i}_{k} + A^{\cdot i}_{k\cdot} = \frac{c}{\sqrt{g_{00}}}\left(\Gamma^{i}_{0k} - \frac{g_{0k}\,\Gamma^{i}_{00}}{g_{00}}\right), \tag{1.54}$$

$$F^{k} = -\,\frac{c^{2}\,\Gamma^{k}_{00}}{g_{00}}\,, \tag{1.55}$$

$$g^{i\alpha}g^{k\beta}\,\Gamma^{m}_{\alpha\beta} = h^{iq}h^{ks}\Delta^{m}_{qs}\,. \tag{1.56}$$

We now express the chr.inv.-Christoffel symbols through the chr.inv.-properties of the accompanying space of reference. Expressing the components $g^{\alpha\beta}$ and the first derivatives from $g_{\alpha\beta}$ through F_i, A_{ik}, D_{ik}, w, and v_i, after some algebra we obtain

$$\Gamma_{00,0} = -\frac{1}{c^3}\left(1-\frac{\mathrm{w}}{c^2}\right)\frac{\partial \mathrm{w}}{\partial t}\,, \tag{1.57}$$

$$\Gamma_{00,i} = \frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)^2 F_i + \frac{1}{c^4}\, v_i\,\frac{\partial \mathrm{w}}{\partial t}\,, \tag{1.58}$$

$$\Gamma_{0i,0} = -\frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)\frac{\partial \mathrm{w}}{\partial x^i}\,, \tag{1.59}$$

$$\Gamma_{0i,j} = -\frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left(D_{ij} + A_{ij} + \frac{1}{c^2}\,F_j v_i\right) + \frac{1}{c^3}\, v_j\,\frac{\partial \mathrm{w}}{\partial x^i}\,, \tag{1.60}$$

$$\Gamma_{ij,0} = \frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left[D_{ij} - \frac{1}{2}\left(\frac{\partial v_j}{\partial x^i} + \frac{\partial v_i}{\partial x^j}\right) + \frac{1}{2c^2}\left(F_i v_j + F_j v_i\right)\right], \tag{1.61}$$

$$\Gamma_{ij,k} = -\Delta_{ij,k} + \frac{1}{c^2}\left[v_i A_{jk} + v_j A_{ik} + \frac{1}{2}\, v_k\left(\frac{\partial v_j}{\partial x^i} + \frac{\partial v_i}{\partial x^j}\right) - \right.$$
$$\left. - \frac{1}{2c^2}\, v_k\left(F_i v_j + F_j v_i\right)\right] + \frac{1}{c^4}\, F_k v_i v_j\,, \tag{1.62}$$

$$\Gamma_{00}^0 = -\frac{1}{c^3}\left[\frac{1}{1-\frac{\mathrm{w}}{c^2}}\frac{\partial \mathrm{w}}{\partial t} + \left(1-\frac{\mathrm{w}}{c^2}\right) v_k F^k\right], \tag{1.63}$$

$$\Gamma_{00}^k = -\frac{1}{c^2}\left(1-\frac{\mathrm{w}}{c^2}\right)^2 F^k, \tag{1.64}$$

$$\Gamma_{0i}^0 = \frac{1}{c^2}\left[-\frac{1}{1-\frac{\mathrm{w}}{c^2}}\frac{\partial \mathrm{w}}{\partial x^i} + v_k\left(D_i^k + A_{i\cdot}^{\cdot k} + \frac{1}{c^2}\, v_i F^k\right)\right], \tag{1.65}$$

$$\Gamma_{0i}^k = \frac{1}{c}\left(1-\frac{\mathrm{w}}{c^2}\right)\left(D_i^k + A_{i\cdot}^{\cdot k} + \frac{1}{c^2}\, v_i F^k\right), \tag{1.66}$$

$$\Gamma_{ij}^0 = -\frac{1}{c\left(1-\frac{\mathrm{w}}{c^2}\right)}\left\{-D_{ij} + \frac{1}{c^2}\, v_n \times \right.$$
$$\times\left[v_j\left(D_i^n + A_{i\cdot}^{\cdot n}\right) + v_i\left(D_j^n + A_{j\cdot}^{\cdot n}\right) + \frac{1}{c^2}\, v_i v_j F^n\right] +$$
$$\left. + \frac{1}{2}\left(\frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i}\right) - \frac{1}{2c^2}\left(F_i v_j + F_j v_i\right) - \Delta_{ij}^n\, v_n\right\}, \tag{1.67}$$

$$\Gamma_{ij}^k = \Delta_{ij}^k - \frac{1}{c^2}\left[v_i\left(D_j^k + A_{j\cdot}^{\cdot k}\right) + v_j\left(D_i^k + A_{i\cdot}^{\cdot k}\right) + \frac{1}{c^2}\, v_i v_j F^k\right]. \tag{1.68}$$

Zelmanov also deduced formulae for the chr.inv.-projections of the Riemann-Christoffel curvature tensor. He followed the same procedure by which the Riemann-Christoffel tensor was built proceeding from the non-commutativity of the second derivatives of an arbitrary vector Q^α taken in the given space. Taking the non-commutativity of the second chr.inv.-derivatives of an arbitrary vector

$$^*\nabla_i{}^*\nabla_k\,Q_l-{}^*\nabla_k{}^*\nabla_i\,Q_l=\frac{2A_{ik}}{c^2}\frac{^*\partial Q_l}{\partial t}+H_{lki\cdot}^{\cdots j}Q_j\,, \tag{1.69}$$

where the chr.inv.-covariant differential from the vector is

$$^*\nabla_k\,Q^i dx^k=dQ^i+\Delta^i_{kl}\,Q^k dx^l, \tag{1.70}$$

he obtained the chr.inv.-tensor

$$H_{lki\cdot}^{\cdots j}=\frac{^*\partial\Delta^j_{il}}{\partial x^k}-\frac{^*\partial\Delta^j_{kl}}{\partial x^i}+\Delta^m_{il}\,\Delta^j_{km}-\Delta^m_{kl}\Delta^j_{im}\,, \tag{1.71}$$

which is like Schouten's tensor in the theory of non-holonomic manifolds [28]. The tensor $H_{lki\cdot}^{\cdots j}$ differs from the Riemann-Christoffel tensor $R_{\beta\gamma\delta\cdot}^{\cdots\alpha}$ due to the presence of space rotation A_{ik} in the formula (1.69). Nevertheless, its generalization gives the chr.inv.-tensor

$$C_{lkij}=\frac{1}{4}\left(H_{lkij}-H_{jkil}+H_{klji}-H_{iljk}\right), \tag{1.72}$$

which possesses all the algebraic properties of the Riemann-Christoffel tensor in this three-dimensional space. Therefore, Zelmanov called C_{iklj} the *chr.inv.-curvature tensor*, which actually is the tensor of the physically observable curvature of the three-dimensional spatial section of the observer. Its contraction step-by-step

$$C_{kj}=C_{kij\cdot}^{\cdots i}=h^{im}C_{kimj}\,,\qquad C=C^j_j=h^{lj}C_{lj} \tag{1.73}$$

gives the chr.inv.-scalar C which is the *observable three-dimensional curvature* of this space.

The tensor H_{lkij} is connected with the curvature tensor C_{lkij} by

$$H_{lkij}=C_{lkij}+\frac{1}{c^2}\big(2A_{ki}D_{jl}+A_{ij}D_{kl}+A_{jk}D_{il}+ \\ +A_{kl}D_{ij}+A_{li}D_{jk}\big). \tag{1.74}$$

The contracted tensors $H_{lk}=H_{lki\cdot}^{\cdots i}$ and $C_{lk}=C_{lki\cdot}^{\cdots i}$ are connected as

$$H_{lk}=C_{lk}+\frac{1}{c^2}\Big(A_{kj}D^j_l+A_{lj}D^j_k+A_{kl}D\Big)\,. \tag{1.75}$$

In a particular case where the space does not rotate, H_{lkij} and C_{lkij} are the same. This is as well true for H_{lk} and C_{lk}. In this particular case, the tensor $C_{lk} = h^{ij} C_{ilkj}$ has the form

$$C_{lk} = \frac{{}^*\partial}{\partial x^k}\left(\frac{{}^*\partial \ln\sqrt{h}}{\partial x^l}\right) - \frac{{}^*\partial \Delta^i_{kl}}{\partial x^i} + \Delta^m_{il}\Delta^i_{km} - \Delta^m_{kl}\frac{{}^*\partial \ln\sqrt{h}}{\partial x^m}\,. \tag{1.76}$$

The Riemann-Christoffel tensor $R_{\alpha\beta\gamma\delta}$, being a two-pair symmetric tensor (its paired indices are non-symmetric inside each pair, while the pairs are symmetric with respect to each other), has three chr.inv.-projections according to formula (1.29) of the chronometrically invariant formalism. They are as follows:

$$X^{ik} = -\,c^2\,\frac{R_{0\cdot0\cdot}^{\cdot i\cdot k}}{g_{00}}\,, \quad Y^{ijk} = -\,c\,\frac{R_{0\cdots}^{\cdot ijk}}{\sqrt{g_{00}}}\,, \quad Z^{ijkl} = c^2 R^{ijkl}. \tag{1.77}$$

Substituting the necessary components of the Riemann-Christoffel tensor $R_{\alpha\beta\gamma\delta}$ into the formulae for its chr.inv.-projections (1.77), and by lowering indices, Zelmanov obtained the formulae

$$X_{ij} = \frac{{}^*\partial D_{ij}}{\partial t} - \left(D^l_i + A^{\cdot l}_{i\cdot}\right)\left(D_{jl} + A_{jl}\right) + \\ + \left({}^*\nabla_i F_j + {}^*\nabla_j F_i\right) - \frac{1}{c^2}\,F_i F_j\,, \tag{1.78}$$

$$Y_{ijk} = {}^*\nabla_i\left(D_{jk} + A_{jk}\right) - {}^*\nabla_j\left(D_{ik} + A_{ik}\right) + \frac{2}{c^2}\,A_{ij}F_k\,, \tag{1.79}$$

$$Z_{iklj} = D_{ik}D_{lj} - D_{il}D_{kj} + A_{ik}A_{lj} - \\ - A_{il}A_{kj} + 2A_{ij}A_{kl} - c^2 C_{iklj}\,, \tag{1.80}$$

where $Y_{(ijk)} = Y_{ijk} + Y_{jki} + Y_{kij} = 0$ just like in the Riemann-Christoffel tensor. Contraction of the observable spatial projection Z_{iklj} step-by-step as $Z_{il} = h^{kj} Z_{iklj}$ and $Z = h^{il} Z_{il}$ gives

$$Z_{il} = D_{ik}D^k_l - D_{il}D + A_{ik}A^{\cdot k}_{l\cdot} + 2A_{ik}A^{k\cdot}_{\cdot l} - c^2 C_{il}\,, \tag{1.81}$$

$$Z = h^{il}Z_{il} = D_{ik}D^{ik} - D^2 - A_{ik}A^{ik} - c^2 C\,. \tag{1.82}$$

At the end of our survey of the chronometrically invariant formalism, consider Einstein's field equations*

$$R_{\alpha\beta} - \frac{1}{2}\,g_{\alpha\beta}R = -\,\varkappa T_{\alpha\beta} + \lambda g_{\alpha\beta}\,. \tag{1.83}$$

*The left-hand side of the field equations (1.83) is often referred to as the *Einstein tensor* $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}\,g_{\alpha\beta}R$, in notation $G_{\alpha\beta} = -\,\varkappa T_{\alpha\beta} + \lambda g_{\alpha\beta}$.

The field equations, except for the fundamental metric tensor $g_{\alpha\beta}$, include: $R_{\alpha\beta} = R_{\alpha\sigma\beta\cdot}^{\cdots\sigma}$ is Ricci's tensor (the second-rank symmetric tensor coming from contraction of the Riemann-Christoffel curvature tensor), $R = g^{\alpha\beta} R_{\alpha\beta}$ is the curvature scalar, $\varkappa = \frac{8\pi G}{c^2} = 18.6\times10^{-28}$ cm/gram is Einstein's constant of gravitation, $G = 6.672\times10^{-8}$ cm^3/gram$\times$sec^2 is Gauss' constant of gravitation, $T_{\alpha\beta}$ is the energy-momentum tensor of the matter distributed in the space, and λ [cm^{-2}] that describes physical vacuum (see §5.2 of the book [18]).

Landau and Lifshitz [20] use $\varkappa = \frac{8\pi G}{c^4}$ instead of $\varkappa = \frac{8\pi G}{c^2}$ as used by Zelmanov. To understand the reason, set $\varkappa = \frac{8\pi G}{c^4}$ as in our study, and consider the chr.inv.-projections of the energy-momentum tensor

$$\rho = \frac{T_{00}}{g_{00}}\,, \qquad J^i = \frac{c\,T_0^i}{\sqrt{g_{00}}}\,, \qquad U^{ik} = c^2 T^{ik}, \tag{1.84}$$

which come with formula (1.29) as the projections of any second-rank symmetric tensor. They have the following physical meaning: ρ is the *observable density of mass*, J^i is the *observable density of momentum*, and U^{ik} is the *observable stress-tensor*. Ricci's tensor has dimension cm^{-2}. This means that the scalar chr.inv.-projection of the field equations, $\frac{G_{00}}{g_{00}} = -\frac{\varkappa T_{00}}{g_{00}} + \lambda$, and the quantity $\frac{\varkappa T_{00}}{g_{00}} = \frac{8\pi G\rho}{c^2}$ have the same dimension which is cm^{-2}. Hence, the energy-momentum tensor $T_{\alpha\beta}$ has the same dimension as mass density (gram/cm^3). Therefore, once we would use $\varkappa = \frac{8\pi G}{c^4}$ on the right-hand side of the field equations, we would use not the energy-momentum tensor $T_{\alpha\beta}$ but rather $c^2 T_{\alpha\beta}$.

The chr.inv.-projections of Einstein's equations (1.83) are calculated as those of a second-rank tensor (1.29). They have the form (we refer to them as the *chr.inv.-Einstein equations*)

$$\frac{^*\partial D}{\partial t} + D_{jl}D^{lj} + A_{jl}A^{lj} + \left(^*\nabla_j - \frac{1}{c^2}\,F_j\right)F^j = \\ = -\frac{\varkappa}{2}\left(\rho c^2 + U\right) + \lambda c^2, \tag{1.85}$$

$$^*\nabla_j\left(h^{ij}D - D^{ij} - A^{ij}\right) + \frac{2}{c^2}\,F_j A^{ij} = \varkappa J^i, \tag{1.86}$$

$$\frac{^*\partial D_{ik}}{\partial t} - \left(D_{ij} + A_{ij}\right)\left(D_k^j + A_{k\cdot}^{\cdot j}\right) + DD_{ik} - D_{ij}D_k^j + \\ + 3A_{ij}A_{k\cdot}^{\cdot j} + \frac{1}{2}\left(^*\nabla_i F_k + {^*\nabla_k} F_i\right) - \frac{1}{c^2}\,F_i F_k - c^2 C_{ik} = \\ = \frac{\varkappa}{2}\left(\rho c^2 h_{ik} + 2U_{ik} - U h_{ik}\right) + \lambda c^2 h_{ik}\,, \tag{1.87}$$

where $U = h^{ik}U_{ik}$ is the trace of the stress-tensor U_{ik}.

Also, the energy-momentum tensor $T_{\alpha\beta}$ of distributed matter should satisfy the law of conservation which is

$$\nabla_\sigma T^{\alpha\sigma} = 0\,. \tag{1.88}$$

The chr.inv.-projections of the conservation law are calculated as those of a first-rank tensor (1.28). We refer to them as the *conservation law equations*. The equations have the form

$$\frac{{}^*\partial\rho}{\partial t} + D\rho + \frac{1}{c^2}\,D_{ij}U^{ij} + {}^*\widetilde{\nabla}_i\,J^i - \frac{1}{c^2}\,F_i\,J^i = 0\,, \tag{1.89}$$

$$\frac{{}^*\partial J^k}{\partial t} + DJ^k + 2\left(D_i^k + A_{i\cdot}^{\cdot k}\right)J^i + {}^*\widetilde{\nabla}_i\,U^{ik} - \rho F^k = 0\,, \tag{1.90}$$

where the chr.inv.-operator ${}^*\widetilde{\nabla}_i = {}^*\nabla_i - \frac{1}{c^2}F_i$ is created on the basis of the chr.inv.-differential operator ${}^*\nabla_i$ (see *Notations*).

Given these definitions, we can find how any geometric object of a given four-dimensional pseudo-Riemannian space (space-time) is constituted from the viewpoint of any observer whose location is this space. For instance, having any equation obtained in the generally covariant tensor analysis, we can calculate the chr.inv.-projections of it onto the time line and onto the spatial section of any particular body of reference, then formulate the respective chr.inv.-projections in terms of the physically observable properties of the reference space. This way we will arrive at fully qualified equations containing only quantities measurable in practice.

Thus, we now have all the necessary mathematical "equipment" required for our further development of the mathematical theory of the internal constitution of stars, and of the sources of stellar energy, according to the General Theory of Relativity.

Chapter 2

Regular Stars and the Sun

§2.1 Introducing the space metric of a regular star. Einstein's field equations in the form satisfying the metric

In this chapter, we introduce the new mathematical theory of liquid stars being applied to regular stars. This means Type I of stars in terms of the new classification we have just introduced according to the General Theory of Relativity (see §1.2, and Table 1.1 therein). It covers the widest variety of stars, which includes super-giants, sun-like stars (including the Sun), dwarfs, and, white dwarfs*.

The structure, matter, and field of a liquid star are characterized by Schwarzschild's metric of a sphere filled with incompressible liquid. The metric was originally introduced in 1916 by Karl Schwarzschild [14]. He, however, introduced it in a truncated form containing substantial limitations: he artificially pre-imposed these limitations during the deduction in order to set the field free of breaking, thus resulting in the geometry of the metric space artificially truncated. In other words, the metric introduced by Karl Schwarzschild is not quite the genuine metric of the space of a liquid sphere. The true metric of a sphere filled with incompressible liquid, which is free of the said limitations, thus takes space breaking into account, as deduced in 2009 by one of us [11, 12]. We now repeat the deduction here, according to the most detailed explanation [12], along with some recent amendments and comments.

Consider an empty space that houses a spherical island which is a liquid. The structure, matter, and field of such an massive island should be characterized by a space metric which possesses spherical symmetry. As is known, all spherically symmetric metrics have the following general form

$$ds^2 = e^{\nu} c^2 dt^2 - e^{\lambda} dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)\,, \qquad (2.1)$$

where e^{ν} and e^{λ} are functions of r and t.

The matter and field of the spherical island (which is a liquid) should satisfy Einstein's field equations (1.83), which in the case under consid-

*White dwarfs are considered separately in the framework of Eddington's theory of gaseous stars.

eration have the λ-field neglected, i.e.

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta}\, R = -\,\varkappa\, T_{\alpha\beta}\,, \tag{2.2}$$

where $R_{\alpha\beta}$ is Ricci's curvature tensor, R is the curvature scalar, $\varkappa = \frac{8\pi G}{c^2} = 18.6\times10^{-28}$ cm/gram is Einstein's constant of gravitation, and $T_{\alpha\beta}$ is the energy-momentum tensor of the distributed matter (liquid). The energy-momentum tensor (i.e. the distributed matter) should satisfy the conservation law

$$\nabla_\sigma T^{\alpha\sigma} = 0\,, \tag{2.3}$$

where ∇_σ is the four-dimensional symbol of covariant differentiation (see *Notations*).

Einstein's field equations connect the components of the fundamental metric tensor, the space curvature, and distributed matter according to Riemannian geometry. In other words, the invariant square form of Riemannian metric, $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta = inv$, in common with Einstein's field equations characterize Riemannian spaces (the spaces whose geometry is Riemannian). Concerning the General Theory of Relativity, this means as follows. Let us have a Riemannian space having a metric ds^2, and suggest that matter be distributed in it (thus we suggest a particular formula for the energy-momentum tensor $T_{\alpha\beta}$). Then, the components of the fundamental metric tensor $g_{\alpha\beta}$ (known from the formula of the metric ds^2) and the components of the suggested energy-momentum tensor, being commonly substituted into (respectively) the left-hand side and the right-hand side of Einstein's field equations should transform the equations into identities.

This is the way how, on the basis of the general formulae of a spherically symmetric metric (2.1), to deduce the metric of a sphere filled with liquid. We take the energy-momentum tensor of a perfect liquid, then substitute its components into the right-hand side of the field equations. Then we take the components of the fundamental metric tensor from the general spherically symmetric metric (2.1) in their general (non-particular) form containing the coefficients e^ν and e^λ. We substitute the components into the left-hand side of the field equations. Then we look which form of the coefficients e^ν and e^λ makes the left-hand side of the field equations the same as the right-hand side (thus transforming the field equations into identities). Finally, we substitute the obtained particular formulae for the coefficients e^ν and e^λ back into the general formula of spherically symmetric metrics. Voilà! The metric of a sphere filled with perfect liquid has been obtained.

One might as well just ask, why did Schwarzschild himself not do just that? Instead, why did he follow another complicated way, full of assumptions and suppositions? Well... Let us come back to our deduction.

As is known, the energy-momentum tensor of a perfect liquid (which is incompressible and non-viscous) has the form

$$T^{\alpha\beta} = \left(\rho_0 + \frac{p}{c^2}\right) U^{\alpha} U^{\beta} - \frac{p}{c^2}\, g^{\alpha\beta}, \tag{2.4}$$

where $\rho = \rho_0 = const$ is the density of the liquid (which is constant), p is the pressure, while

$$U^{\alpha} = \frac{dx^{\alpha}}{ds}\,, \qquad U_{\alpha} U^{\alpha} = 1 \tag{2.5}$$

is the four-dimensional velocity of the liquid flow with respect to the observer (his reference space coincides with the space of the liquid sphere, with the origin of the coordinates located at the center).

Hence forth we express the field equations in component notations with the physically observable properties of the space selected.

We see that

$$\left.\begin{array}{lll} g_{00} = e^{\nu}, & g_{0i} = 0 & \\ g_{11} = -e^{\lambda}, & g_{22} = -r^2, & g_{33} = -r^2 \sin^2\theta \end{array}\right\} \tag{2.6}$$

in the metric of spherically symmetric spaces (2.1). According to the chronometrically invariant formalism (see §1.3), the gravitational potential in such a space has the following formulation

$$\mathrm{w} = c^2 \left(1 - e^{\frac{\nu}{2}}\right). \tag{2.7}$$

Because $g_{0i} = 0$ in the metric, the space does not rotate. Therefore, the linear velocity of the rotation is $v_i = 0$ as well. Hence the chr.inv.-tensor of the angular velocity of space rotation is zero

$$A_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right) + \frac{1}{2c^2}\left(F_i v_k - F_k v_i\right) = 0\,, \tag{2.8}$$

while the chr.inv.-vector of gravitational inertial force has the form

$$F_i = \frac{c^2}{c^2 - \mathrm{w}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right) = -\frac{c^2}{2}\,\nu'\,, \tag{2.9}$$

where the prime denotes differentiation along the radial coordinate r. With these, the chr.inv.-metric tensor h_{ik} of the space has the non-zero components

$$h_{11} = e^{\lambda}, \quad h_{22} = r^2, \quad h_{33} = r^2 \sin^2\theta\,, \tag{2.10}$$

$$h^{11} = e^{-\lambda}, \quad h^{22} = \frac{1}{r^2}\,, \quad h_{33} = \frac{1}{r^2 \sin^2\theta}\,, \tag{2.11}$$

$$h = \det \|\, h_{ik} \,\| = e^{\lambda} r^4 \sin^2\theta\,. \tag{2.12}$$

Thus, the chr.inv.-tensor of the rate of space deformation, D_{ik}, has only the following non-zero components:

$$D_{11} = \frac{\dot{\lambda}}{2}\, e^{\lambda - \frac{\nu}{2}}, \quad D^{11} = \frac{\dot{\lambda}}{2}\, e^{-\lambda - \frac{\nu}{2}}, \quad D = \frac{\dot{\lambda}}{2}\, e^{-\frac{\nu}{2}}, \tag{2.13}$$

where the upper dot denotes differentiation along the time coordinate t. The chr.inv.-Christoffel symbols, which characterize space inhomogeneity, are calculated according to their definition given in §1.3 with the components of the chr.inv.-metric tensor h_{ik}. After some algebra, we obtain formulae for the non-zero components of $\Delta_{ij,m}$

$$\Delta_{11,1} = \frac{\lambda'}{2}\, e^{\lambda}\,, \quad \Delta_{22,1} = -\,r\,, \quad \Delta_{33,1} = -\,r\, \sin^2\theta\,, \tag{2.14}$$

$$\Delta_{12,2} = r\,, \qquad \Delta_{33,2} = -\,r^2 \sin\theta\cos\theta\,, \tag{2.15}$$

$$\Delta_{13,3} = r\, \sin^2\theta\,, \qquad \Delta_{23,3} = r^2 \sin\theta\cos\theta\,, \tag{2.16}$$

and formulae for the non-zero components of Δ^k_{ij}

$$\Delta^1_{11} = \frac{\lambda'}{2}\,, \quad \Delta^1_{22} = -\,r\, e^{-\lambda}, \quad \Delta^1_{33} = -\,r\, \sin^2\theta\, e^{-\lambda}, \tag{2.17}$$

$$\Delta^2_{12} = \frac{1}{r}\,, \quad \Delta^2_{33} = -\,\sin\theta\cos\theta\,, \tag{2.18}$$

$$\Delta^3_{13} = \frac{1}{r}\,, \qquad \Delta^3_{23} = \cot\theta\,. \tag{2.19}$$

As was shown in §1.3, in a rotation-free space the second-rank chr.inv.-curvature tensor $C_{lk} = h^{ij} C_{ilkj}$, which is the physically observable curvature tensor, has the form (1.76). A rotation-free space is the case under consideration. After some algebra, we obtain the non-zero components of C_{lk} for the spherically symmetric metric (2.1). They are

$$C_{11} = -\,\frac{\lambda'}{r}\,, \qquad C_{22} = \frac{C_{33}}{\sin^2\theta} = e^{-\lambda}\left(1 - \frac{r\lambda'}{2}\right) - 1. \tag{2.20}$$

Let us calculate the chr.inv.-projections of the energy-momentum tensor of a perfect liquid (2.4) according to the associated projections* which are (1.84). With $b^i = 0$ and $b^0 = \frac{1}{\sqrt{g_{00}}}$ (1.25) which characterize an accompanying frame of reference (in the case under consideration, the observer accompanies the liquid sphere), we obtain

$$\rho = \frac{T_{00}}{g_{00}} = \rho_0\,, \quad J^i = \frac{c\,T_0^i}{\sqrt{g_{00}}} = 0\,, \quad U^{ik} = c^2\,T^{ik} = p h^{ik}. \tag{2.21}$$

wherefrom we also have, for $U = h^{ik}U_{ik}$,

$$U = 3p\,. \tag{2.22}$$

The obtained condition $J^i = 0$ means that the liquid is free of flow, while $U^{ik} = ph^{ik}$ means that the observer's reference frame accompanies the liquid medium.

The chr.inv.-Einstein equations (1.85–1.87) in a rotation-free space now take the simplified form

$$\frac{{}^*\partial D}{\partial t} + D_{jl}D^{lj} + \left({}^*\nabla_j - \frac{1}{c^2}\,F_j\right)F^j = -\frac{\varkappa}{2}\left(\rho_0 c^2 + U\right), \tag{2.23}$$

$${}^*\nabla_j\left(h^{ij}D - D^{ij}\right) = 0\,, \tag{2.24}$$

$$\frac{{}^*\partial D_{ik}}{\partial t} - D_{ij}D^j_k + DD_{ik} - D_{ij}D^j_k + \frac{1}{2}\left({}^*\nabla_i F_k + {}^*\nabla_k F_i\right) - \\ - \frac{1}{c^2}\,F_iF_k - c^2 C_{ik} = \frac{\varkappa}{2}\left(\rho_0 c^2 h_{ik} + 2U_{ik} - Uh_{ik}\right), \tag{2.25}$$

where ${}^*\nabla_i$ is the symbol of chr.inv.-differentiation (see *Notations*). The chr.inv.-equations of the conservation law (1.89–1.90) also simplify to

$$D\rho_0 + \frac{1}{c^2}\,D_{ij}U^{ij} = 0\,, \tag{2.26}$$

$${}^*\widetilde{\nabla}_i\,U^{ik} - \rho_0 F^k = 0\,, \tag{2.27}$$

where ${}^*\widetilde{\nabla}_i = {}^*\nabla_i - \frac{1}{c^2}F_i$ (see *Notations*).

Substitute, into the chr.inv.-Einstein equations (2.23–2.25), the chr. inv.-characteristics of the space we have obtained above for the spherically symmetric metric (2.1) and also the obtained chr.inv.-components

*They are the observable density of mass, ρ, the observable density of momentum, J^i, and the observable stress-tensor U^{ik} of the liquid.

of the energy-momentum tensor of a perfect liquid. After some algebra, we obtain the chr.inv.-Einstein equations (2.23–2.25) in component notation (the third tensorial equation splits into three, where the second and third equations remain the same)

$$e^{-\nu}\left(\ddot{\lambda}-\frac{\dot{\lambda}\dot{\nu}}{2}+\frac{\dot{\lambda}^2}{2}\right)-c^2e^{-\lambda}\left[\nu''-\frac{\lambda'\nu'}{2}+\frac{2\nu'}{r}+\frac{(\nu')^2}{2}\right]= \\ =-\varkappa\left(\rho_0c^2+3p\right)e^{\lambda}, \tag{2.28}$$

$$\frac{\dot{\lambda}}{r}\,e^{-\lambda-\frac{\nu}{2}}=0\,, \tag{2.29}$$

$$e^{\lambda-\nu}\left(\ddot{\lambda}-\frac{\dot{\lambda}\dot{\nu}}{2}+\frac{\dot{\lambda}^2}{2}\right)-c^2\left[\nu''-\frac{\lambda'\nu'}{2}+\frac{(\nu')^2}{2}\right]+\frac{2c^2\lambda'}{r}= \\ =\varkappa\left(\rho_0c^2-p\right)e^{\lambda}, \tag{2.30}$$

$$\frac{c^2\left(\lambda'-\nu'\right)}{r}\,e^{-\lambda}+\frac{2c^2}{r^2}\left(1-e^{-\lambda}\right)=\varkappa\left(\rho_0c^2-p\right). \tag{2.31}$$

The second equation manifests that $\dot{\lambda}=0$ in this case. This means that the inner space of the liquid sphere does not deform: with $\dot{\lambda}=0$, we have $D_{11}=0$, $D^{11}=0$, and $D=0$ according to (2.13). Taking this circumstance into account, as well as the stationarity of λ, we reduce the field equations (2.28–2.31) to the final form

$$c^2e^{-\lambda}\left[\nu''-\frac{\lambda'\nu'}{2}+\frac{2\nu'}{r}+\frac{(\nu')^2}{2}\right]=\varkappa\left(\rho_0c^2+3p\right)e^{\lambda}, \tag{2.32}$$

$$\frac{2c^2\lambda'}{r}-c^2\left[\nu''-\frac{\lambda'\nu'}{2}+\frac{(\nu')^2}{2}\right]=\varkappa\left(\rho_0c^2-p\right)e^{\lambda}, \tag{2.33}$$

$$\frac{c^2\left(\lambda'-\nu'\right)}{r}\,e^{-\lambda}+\frac{2c^2}{r^2}\left(1-e^{-\lambda}\right)=\varkappa\left(\rho_0c^2-p\right). \tag{2.34}$$

To solve the field equations (2.32–2.34), we need a formula for the pressure p. To find the formula, we now deal with the conservation equations (2.26–2.27). However, due to the absence of space deformation in the case under consideration ($D_{ik}=0$), the chr.inv.-scalar conservation equation (2.26) vanishes. Only the chr.inv.-vectorial conservation equation (2.27) remains. It takes the form

$${}^*\nabla_i\left(p\,h^{ik}\right)-\left(\rho_0+\frac{p}{c^2}\right)F^k=0\,. \tag{2.35}$$

Since ${}^*\nabla_i\, h^{ik}=0$ is true always (as well as $\nabla_\sigma\, g^{\alpha\sigma}=0$ for the fundamental metric tensor), the remaining conservation equation (2.35) reads

$$h^{ik}\frac{{}^*\partial p}{\partial x^i} - \left(\rho_0 + \frac{p}{c^2}\right) F^k = 0\,. \tag{2.36}$$

Because $\frac{{}^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i}$ in a rotation-free space, this formula reduces to a non-trivial equation which has the form

$$p' e^{-\lambda} + \left(\rho_0 c^2 + p\right) \frac{\nu'}{2}\, e^{-\lambda} = 0\,, \tag{2.37}$$

where $p' = \frac{dp}{dr}$, $\nu' = \frac{d\nu}{dr}$, $e^\lambda \neq 0$. Dividing both parts of (2.37) by $e^{-\lambda}$, we obtain

$$\frac{dp}{\rho_0 c^2 + p} = -\frac{d\nu}{2}\,, \tag{2.38}$$

which is a plain differential equation with separable variables. It easily integrates as

$$\rho_0 c^2 + p = B e^{-\frac{\nu}{2}}\,, \qquad B = const. \tag{2.39}$$

Thus, we obtain the pressure p as a function of ν, which is

$$p = B e^{-\frac{\nu}{2}} - \rho_0 c^2. \tag{2.40}$$

In looking for an r-dependent function $p(r)$, we integrate the field equations (2.32–2.34). Summarizing (2.32) and (2.33), we find

$$\frac{c^2\left(\lambda' + \nu'\right)}{r} = \varkappa B e^{\lambda - \frac{\nu}{2}}. \tag{2.41}$$

Express ν' herefrom, then substitute the result into (2.34). We obtain

$$\frac{2c^2}{r}\,\lambda' + \frac{2c^2}{r^2}\left(e^\lambda - 1\right) - \varkappa B e^{-\lambda - \frac{\nu}{2}} = \varkappa\left(\rho_0 c^2 - p\right) e^\lambda. \tag{2.42}$$

Substituting p from (2.40) into (2.42), we obtain the following differential equation with respect to λ:

$$\lambda' + \frac{e^\lambda - 1}{r} - \varkappa \rho_0\, r e^\lambda = 0\,. \tag{2.43}$$

We introduce a new variable $y = e^\lambda$. Thus $\lambda' = \frac{y'}{y}$. Substituting into this equation y and y', we obtain the Bernoulli equation (see Kamke [29], Part III, Chapter I, §1.34)

$$y' + f(r)\, y^2 + g(r)\, y = 0\,, \tag{2.44}$$

where

$$f(r) = \frac{1}{r} - \varkappa\rho_0 r\,, \qquad g(r) = -\frac{1}{r}\,. \tag{2.45}$$

It has the following solution:

$$\frac{1}{y} = E(r)\int \frac{f(r)\,dr}{E(r)}\,, \tag{2.46}$$

where

$$E(r) = e^{\int g(r)dr}. \tag{2.47}$$

Integrating (2.47), we obtain $E(r)$ which is

$$E(r) = e^{-\int\frac{dr}{r}} = e^{\ln\frac{L}{r}} = \frac{L}{r}\,, \qquad L = const > 0\,, \tag{2.48}$$

thus we obtain $\frac{1}{y} = e^{-\lambda}$ which is

$$e^{-\lambda} = \frac{L}{r}\int \frac{r}{L}\left(\frac{1}{r} - \varkappa\rho_0 r\right) dr = 1 - \frac{\varkappa\rho_0 r^2}{3} + \frac{Q}{r}\,, \quad Q = const. \tag{2.49}$$

To find Q, we re-write equation (2.42) as

$$e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = \varkappa\rho_0\,. \tag{2.50}$$

This equation has a singularity at the point $r = 0$, where the numerical value of the right-hand side term of the equation (the density of the liquid) grows up to infinity by $r \to 0$, i.e. at the center of the sphere. This is a contradiction with the initially assumed condition $\rho_0 = const$, which is specific to incompressible liquids. As a matter of fact, this contradiction should not be in the theory. We remove this contradiction (and the singularity) by re-writting (2.50) in the form

$$e^{-\lambda}(1 - r\lambda') = \frac{d}{dr}\left(re^{-\lambda}\right) = 1 - \varkappa\rho_0 r^2. \tag{2.51}$$

After integration, we obtain

$$re^{-\lambda} = r - \frac{\varkappa\rho_0 r^3}{3} + A\,, \qquad A = const. \tag{2.52}$$

Because $A = 0$ at the central point $r = 0$, it should be zero at any other point as well. Dividing this equation by $r \neq 0$, we obtain

$$e^{-\lambda} = 1 - \frac{\varkappa\rho_0 r^2}{3}\,. \tag{2.53}$$

Comparing this solution with the value $e^{-\lambda}$ obtained earlier (2.49), we see that they meet each other if $Q=0$. Besides, we should suggest that $e^{\lambda_0}=1$ at the central point $r=0$, consequently $\lambda_0=0$.

Thus we have obtained the components $h^{11}=e^{-\lambda}$ and $h_{11}=e^{\lambda}$ of the chr.inv.-metric tensor h_{ik} in the form expressed through the radial coordinate r, i.e.

$$h^{11}=e^{-\lambda}=1-\frac{\varkappa\rho_0 r^2}{3}\,, \qquad h_{11}=e^{\lambda}=\frac{1}{1-\frac{\varkappa\rho_0 r^2}{3}}\,. \tag{2.54}$$

Hence forth, we should introduce a boundary condition on the surface of the sphere. We have $r=a$ on the surface, where a is the radius of the sphere. Thus

$$e^{-\lambda_a}=1-\frac{\varkappa\rho_0 a^2}{3}\,. \tag{2.55}$$

On the other hand, the solution of this function is also the mass-point solution in emptiness. Hence, we have

$$e^{-\lambda_a}=1-\frac{2GM}{c^2 a}\,, \tag{2.56}$$

where M is the mass of the sphere. Comparing both these formulae of $e^{-\lambda_a}$, and taking into account that Einstein's constant of gravitation is $\varkappa=\frac{8\pi G}{c^2}$, we find

$$M=\frac{4\pi a^3\rho_0}{3}=\rho_0 V, \tag{2.57}$$

where $V=\frac{4\pi a^3}{3}$ is the volume of the sphere. Hence, we have obtained the regular relation between the mass and the volume of a homogeneous sphere.

Our next step is the look for the solution $e^{-\lambda}$ outside the sphere, where $r>a$. Since outside the sphere the density of matter (the liquid) is $\rho_0=0$, we obtain, after integration of (2.51),

$$r\,e^{-\lambda}=\int_0^r dr-\int_0^a \varkappa\rho_0\, r^2\, dr=r-\frac{\varkappa\rho_0\, a^3}{3}\,. \tag{2.58}$$

We obtain, from this formula, that

$$e^{-\lambda}=1-\frac{\varkappa\rho_0\, a^3}{3r}\,. \tag{2.59}$$

Taking (2.55) and (2.56) into account, we arrive at the mass-point solution in emptiness

$$e^{-\lambda}=1-\frac{2GM}{c^2 r}\,. \tag{2.60}$$

To obtain ν we use equation (2.41). Substituting

$$\lambda' = \frac{\frac{2\varkappa\rho_0 r}{3}}{1-\frac{\varkappa\rho_0 r^2}{3}} \tag{2.61}$$

and the obtained formula of e^{λ} into (2.41), we obtain, after transformations,

$$\nu' + \frac{\frac{2\varkappa\rho_0 r^2}{3}}{1-\frac{\varkappa\rho_0 r^2}{3}} - \frac{\varkappa B}{c^2}\frac{re^{-\frac{\nu}{2}}}{1-\frac{\varkappa\rho_0 r^2}{3}} = 0\,. \tag{2.62}$$

We introduce a new variable $e^{-\frac{\nu}{2}} = y$. Thus, $\nu' = -\frac{2y'}{y}$. Substituting these into (2.62), we obtain the Bernoulli equation

$$y' + \frac{\varkappa B}{2c^2}\frac{ry^2}{1-\frac{\varkappa\rho_0 r^2}{3}} - \frac{\frac{\varkappa\rho_0 r}{3}\,y}{1-\frac{\varkappa\rho_0 r^2}{3}} = 0\,, \tag{2.63}$$

where

$$f(r) = \frac{\varkappa B}{2c^2}\frac{r}{1-\frac{\varkappa\rho_0 r^2}{3}}\,, \qquad g(r) = -\frac{\frac{\varkappa\rho_0 r}{3}}{1-\frac{\varkappa\rho_0 r^2}{3}}\,. \tag{2.64}$$

Thus, we have the integral

$$\int g(r)\,dr = -\int\frac{\frac{\varkappa\rho_0 r}{3}}{1-\frac{\varkappa\rho_0 r^2}{3}} = \ln N\sqrt{\left|1-\frac{\varkappa\rho_0 r^2}{3}\right|}\,, \quad N = const, \tag{2.65}$$

where

$$E(r) = N\sqrt{\left|1-\frac{\varkappa\rho_0 r^2}{3}\right|}\,. \tag{2.66}$$

In the region where the signature condition $h_{11} = e^{\lambda} > 0$ is satisfied, we have

$$1-\frac{\varkappa\rho_0 r^2}{3} > 0\,, \tag{2.67}$$

therefore we use the modulus of the function here.

Next, we look for $\frac{1}{y} = e^{\frac{\nu}{2}}$, which is

$$e^{\frac{\nu}{2}} = \frac{\varkappa B}{2c^2}\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}\int\frac{r\,dr}{\sqrt{\left(1-\frac{\varkappa\rho_0 r^2}{3}\right)^3}}\,. \tag{2.68}$$

We obtain, after integration,

$$e^{\frac{\nu}{2}} = \frac{\varkappa B}{2c^2}\left(\frac{3}{\varkappa\rho_0} + K\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}\right), \qquad K = const. \tag{2.69}$$

We now find the constants B and K. To find B, we re-write the formula of p by the condition that $p=0$ on the surface of the sphere $(r=a)$. Thus, we obtain

$$B = \rho_0 c^2 e^{\frac{\nu_a}{2}}, \tag{2.70}$$

where $e^{\frac{\nu_a}{2}}$ is the value of the function $e^{\frac{\nu}{2}}$ on the surface. As a result, we have

$$e^{\frac{\nu}{2}} = \frac{\varkappa\rho_0}{2}\, e^{\frac{\nu_a}{2}}\left(\frac{3}{\varkappa\rho_0} + K\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}\right). \tag{2.71}$$

To find K, we take the value of $e^{\frac{\nu}{2}}$ on the surface $(r=a)$

$$e^{\frac{\nu_a}{2}} = \frac{\varkappa\rho_0\, e^{\frac{\nu_a}{2}}}{2}\left(\frac{3}{\varkappa\rho_0} + K\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}}\right). \tag{2.72}$$

We obtain, from this formula, that

$$K = -\frac{1}{\varkappa\rho_0}\frac{1}{\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}}}. \tag{2.73}$$

The quantity $e^{\frac{\nu_a}{2}}$ means the numerical value of $e^{\frac{\nu}{2}}$ by $r=a$ (i.e. on the surface of the sphere). Therefore, we can apply it to the mass-point solution in emptiness at $r=a$, i.e.

$$e^{\frac{\nu_a}{2}} = \sqrt{1-\frac{2GM}{c^2 a}}. \tag{2.74}$$

Taking the formulae of $e^{\frac{\nu_a}{2}}$, (2.55) and (2.56), into account, we obtain

$$e^{\frac{\nu}{2}} = \frac{1}{2}\, e^{\frac{\nu_a}{2}}\left(3 - \sqrt{\frac{1-\frac{\varkappa\rho_0 r^2}{3}}{1-\frac{\varkappa\rho_0 a^2}{3}}}\right) =$$

$$= \frac{1}{2}\left(3\sqrt{1-\frac{2GM}{c^2 a}} - \sqrt{1-\frac{2GMr^2}{c^2 a^3}}\right). \tag{2.75}$$

This formula on the surface $(r = a)$ meets the mass-point solution in emptiness: $e^{\frac{\nu_a}{2}} = \sqrt{1-\frac{2GM}{c^2 a}} = \sqrt{1-\frac{\varkappa\rho_0 a^2}{3}}$.

Thus the metric of the space of a sphere filled with perfect liquid is, since the formulae of ν and λ have already been obtained, as follows:

$$ds^2 = \frac{1}{4}\left(3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}} - \sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}\right)^2 c^2dt^2 - \\ - \frac{dr^2}{1-\frac{\varkappa\rho_0 r^2}{3}} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \quad (2.76)$$

Taking (2.55) and (2.56) into account, we re-write the formula (2.76) as

$$ds^2 = \frac{1}{4}\left(3\sqrt{1-\frac{2GM}{c^2a}} - \sqrt{1-\frac{2GMr^2}{c^2a^3}}\right)^2 c^2dt^2 - \\ - \frac{dr^2}{1-\frac{2GMr^2}{c^2a^3}} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \quad (2.77)$$

Finally, since $\frac{2GM}{c^2} = r_g$ is the Hilbert radius calculated according to the mass M of the liquid sphere, while taking the obtained formula of $e^{\frac{\nu_a}{2}}$ into account, we re-write the metric in the final form

$$ds^2 = \frac{1}{4}\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r^2r_g}{a^3}}\right)^2 c^2dt^2 - \\ - \frac{dr^2}{1-\frac{r^2r_g}{a^3}} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \quad (2.78)$$

This is the final formula for the "inner" metric of the space of a sphere filled with perfect liquid. As is seen, the "inner" metric completely coincides with the mass-point metric in emptiness on the surface of the liquid sphere ($r = a$).

Hence forth, we obtain the space metric outside the liquid sphere ($r > a$). We have already obtained the "external" solution for $e^{-\lambda}$ (2.59), which coincides with the "external" mass-point solution for this function (2.60). Outside the sphere, $B = 0$ (2.39). Hence, (2.41) takes the form

$$\lambda' + \nu' = 0\,, \quad (2.79)$$

where, according to (2.60),

$$\lambda' = \frac{2GM}{c^2r^2}\frac{1}{1-\frac{2GM}{c^2r}}\,. \quad (2.80)$$

Substituting (2.80) into (2.79) then integrating the resulting equation, we obtain

$$\nu = \ln\left(1 - \frac{2GM}{c^2 r}\right) + P, \qquad P = const, \tag{2.81}$$

thus

$$e^{\nu} = P\left(1 - \frac{2GM}{c^2 r}\right). \tag{2.82}$$

Since this function is also

$$e^{\nu} = 1 - \frac{2GM}{c^2 a}\,, \tag{2.83}$$

on the surface $(r = a)$ of the liquid sphere, we obtain $P = 1$. Having the obtained formulae for e^{ν} (2.83) and e^{λ} (2.60) substituted into the spherically symmetric metric (2.1), we obtain that the "outer" space of a sphere filled with perfect liquid is described by the mass-point metric in emptiness (1.1), which is

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{2.84}$$

§2.2 The outer space breaking of the Sun's field matches the asteroid strip

Herein we suggest a new model of the Solar System according to the General Theory of Relativity. Namely — the Sun and the planets will be considered as liquid spheres according to the metric of a liquid sphere (2.78) we have obtained in the foregoing. The metric was also shown in formula (1.8), in §1.2 wherein we surveyed the problem statement of the modelling of a star in terms of the General Theory of Relativity. As was also proven in the previous §2.1, the outer space of a liquid sphere is described by the mass-point metric in emptiness (1.1).

Note that herein we do not discuss whether the internal planets can be represented as liquid spheres or not. Astrophysicists and geologists may simply appeal to the magma, because it is in the state of liquid stone. However, the jovian planets (Jupiter, Saturn, Uranus, and Neptune), according to their density and other parameters, can surely be considered as stars. Herein, we only limit ourselves to the theoretical modelling of the Sun and the planets, without an analysis of their origin or other astrophysical factors. In detail, we focus on the location of the "inner" and "outer" space breaking of their fields: the space breaking of the field within and outside the physical body (liquid sphere). Then we compare the obtained result with the observed distribution of the planets within the Solar System.

Our approach to the Solar System is simple. As is known, given a four-dimensional Riemannian space with a sign-alternating diagonal metric $(+---)$, the breaking occurs in that region (point or surface) of the space wherein at least one of the four signature conditions

$$\left.\begin{aligned} & g_{00} > 0 \\ & g_{00}\, g_{11} < 0 \\ & g_{00}\, g_{11}\, g_{22} > 0 \\ & g = g_{00}\, g_{11}\, g_{22}\, g_{33} < 0 \end{aligned}\right\} \tag{2.85}$$

is violated. The space (space-time) of the General Theory of Relativity is one of this type of space. We therefore consider the signature conditions in the space within and outside the liquid Sun.

2.2.1 In the "inner" space metric of a liquid sphere (2.78), while taking into account that

$$\frac{\varkappa\rho_0\, a^3}{3r} = \frac{2GM}{c^2 r} = \frac{r_g}{r} \tag{2.86}$$

therein*, the fundamental metric tensor has the following non-zero components:

$$\begin{aligned} g_{00} &= \frac{1}{4}\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r^2 r_g}{a^3}}\,\right)^2 = \\ &= \frac{1}{4}\left(3\sqrt{1-\frac{\varkappa\rho_0\, a^2}{3}} - \sqrt{1-\frac{\varkappa\rho_0\, r^2}{3}}\,\right)^2, \end{aligned} \tag{2.87}$$

$$g_{11} = -\frac{1}{1-\frac{r^2 r_g}{a^3}} = -\frac{1}{1-\frac{\varkappa\rho_0 r^2}{3}}\,, \tag{2.88}$$

$$g_{22} = -r^2, \tag{2.89}$$

$$g_{33} = -r^2 \sin^2\theta\,. \tag{2.90}$$

We obtain, from those components, that at a distance from the center of the sphere which is

$$r = r_{br} = \sqrt{\frac{a^3}{r_g}} = \sqrt{\frac{3}{\varkappa\rho_0}} \tag{2.91}$$

*See formulae (2.59) and (2.60) in §2.1.

the second, third, and fourth signature conditions are violated*

$$\left.\begin{aligned} & g_{00} = \frac{9}{4}\left(1 - \frac{r_g}{a}\right) > 0 \\ & g_{00}\,g_{11} \rightarrow -\infty \\ & g_{00}\,g_{11}\,g_{22} \rightarrow \infty \\ & g = g_{00}\,g_{11}\,g_{22}\,g_{33} \rightarrow -\infty \end{aligned}\right\}. \qquad (2.92)$$

This means that the field of the liquid spherical body has space breaking on the spherical surface covering the body at the distance $r_{br} = \sqrt{a^3/r_g}$ from its center.

The Hilbert radius $r_g = \frac{2GM}{c^2}$ (the radius of gravitational collapse) calculated for regular physical bodies is many orders less than their physical sizes. Hence, $a \gg r_g$ for a regular spherical liquid body (thus the body is not a collapsar). Therefore, $r_{br} = \sqrt{a^3/r_g} \gg a$: the spherical surface of the space breaking of the field is located far away from the physical surface of the liquid body (the field source), and hence far away from the inner field. In other words, the inner field and liquid substance of the body produce breaking in the outer space of the body.

What does the outer space breaking of the field mean from the physical viewpoint? Has this space breaking a real action on a physical body appearing in it, or is it only a mathematical fiction? As will be shown in the next §2.3, the space (space-time) of a liquid sphere possesses space breaking in its four-dimensional curvature tensor $R_{\alpha\beta\gamma\delta}$ by the condition $r = r_{br}$. Namely, — the component R_{0101} (2.113), which is the four-dimensional curvature of the space in the (r-t)-direction 0101, possesses breaking at the distance $r = r_{br}$ from the center of the liquid sphere (the curvature function becomes infinite, $R_{0101} \rightarrow \infty$, on the surface of the radius $r = r_{br}$). Because the four-curvature determines the gravitational field which fills the space (and vice versa), the breaking at $r = r_{br}$ implies breaking in the gravitational field of the liquid sphere.

This is the physical sense of the outer space breaking of the field of a liquid sphere.

2.2.2 The outer field of a liquid sphere is due to the same liquid substance, which fills the sphere and produces the field within the sphere itself (its inner field). According to the "outer" space metric (2.84), we see that the fundamental metric tensor of the outer space has the

*Namely — these three functions approach infinity. As is known, a function has such breaking when approaching infinity.

following non-zero components

$$g_{00} = 1 - \frac{r_g}{r}\,, \tag{2.93}$$

$$g_{11} = -\frac{1}{1 - \frac{r_g}{r}}\,, \tag{2.94}$$

$$g_{22} = -r^2, \tag{2.95}$$

$$g_{33} = -r^2 \sin^2\theta\,. \tag{2.96}$$

We see that at a distance of

$$r = r_g = \frac{2GM}{c^2} \tag{2.97}$$

from the center of the body, the first signature condition ($g_{00} > 0$) is violated

$$\left.\begin{aligned} g_{00} &= 1 - \frac{r_g}{r} = 0 \\ g_{00}\,g_{11} &= -1 < 0 \\ g_{00}\,g_{11}\,g_{22} &= r^2 > 0 \\ g &= -r^4 \sin^2\theta < 0 \end{aligned}\right\}. \tag{2.98}$$

In other words, the outer field of a liquid sphere produces space breaking deep within the sphere itself, in its inner space close to the center. For example, the calculated Hilbert radius $r_g = \frac{2GM}{c^2}$ is only 2.9 km for the Sun, while for the Earth it is nothing but only 0.88 cm.

2.2.3 Concerning regular stars, and the Sun in particular, the aforementioned findings imply the following (as per our new model of liquid stars according to the General Theory of Relativity):

1. At the center of each star, a small core exists. The core is separated from the other mass of the star by the said inner space breaking in the star's field, at the distance of the Hilbert radius r_g from the center. The inner space breaking means, physically, that the liquid substance of the star has a singularity on the surface of the Hilbert radius r_g from the center, thus the small core is separated from the major mass (the physical sense of the phenomenon will be more clear from the example of the outer space breaking in the field of the Sun);
2. The field of each star has an outer space breaking surrounding the star by a spherical surface. This "bubble" has a very large radius

of $r_{br}=\sqrt{a^3/r_g}$, which is many orders larger than the physical radius a of the star. Physically, the outer space breaking impedes the near substance such as small stones or dust orbiting the star to be formed as a planet in the orbit of the radius r_{br}.

Let us now calculate the radius of the outer space breaking of the Sun's field by formula (2.91), which is $r_{br}=\sqrt{a^3/r_g}=\sqrt{3/\varkappa\rho_0}$. Substitute the Sun's density $\rho_0=1.41$ gram/cm^3, or the mass $M=2.0\times10^{33}$ gram and the radius $a=6.95\times10^{10}$ cm. We obtain

$$r_{br}=3.4\times10^{13}\ \text{cm}=340,000,000\ \text{km}=2.3\ \text{AU}, \qquad (2.99)$$

where 1 AU $=1.49\times10^{13}$ cm (Astronomical Unit) is the average distance between the Sun and the Earth. We obtain that the spherical surface (bubble) of the outer space breaking of the Sun's field is located within the asteroid strip, very close to the orbit of the maximal concentration of asteroids (as is known, the asteroid strip is located, approximately, 2.1 to 4.3 AU from the Sun).

This truly amazing finding brings us to a conclusion that the internal constitution of the Solar System can be calculated according to the liquid model. Namely, — we consider the Sun and the planets as liquid spheres, then we calculate the outer space breaking r_{br} in the field of each of the cosmic bodies. The results of the calculation are collected altogether in Table 2.1.

These results associated with the planets and the Sun, according to Table 2.1, lead to the next conclusions:

1) The outer space breaking of the Sun's field is located within the asteroid strip, near the maximal concentration of asteroids;
2) The internal planets of the Solar System (Mars, the Earth, Venus, and Mercury) are located within the "bubble" of the outer space breaking of the Sun's field;
3) The "bubbles" of the outer space breaking of the field of each of the internal planets are as well located within the "bubble" of the outer space breaking of the Sun's field;
4) The outer space breaking of the fields of Mars and the Earth reaches the asteroid strip;
5) The outer space breaking of Mars' field is located at 2.9 AU from the Sun. It is within the asteroid strip near the orbit of Phaeton, the hypothetical planet which was once orbiting the Sun according to the Titius–Bode law at $r=2.8$ AU, and whose distraction in the ancient time gave birth to the asteroid strip;

Object	Mass M, gram	Density ρ_0, gram/cm^3	Radius a, cm	Hilbert radius r_g, cm	Orbit, AU	Space breaking r_{br}, AU	Location of r_{br} from the Sun, AU
Sun	1.98×10^{33}	1.41	6.95×10^{10}	2.9×10^{5}	—	2.3	2.3
Internal planets							
Mercury	2.21×10^{26}	4.10	2.36×10^{8}	0.03	0.39	1.3	$-0.9-1.7$
Venus	4.93×10^{27}	5.10	6.19×10^{8}	0.73	0.72	1.2	$-0.5-1.9$
Earth	5.97×10^{27}	5.52	6.38×10^{8}	0.88	1.00	1.1	$-0.1-2.1$
Mars	6.45×10^{26}	3.80	3.44×10^{8}	0.10	1.52	1.4	$0.1-2.9$
Asteroid strip	—	—	—	—	2.5*	—	—
Jovian planets							
Jupiter	1.90×10^{30}	1.38	7.11×10^{9}	280	5.20	2.3	$2.9-7.5$
Saturn	5.68×10^{29}	0.72	6.00×10^{9}	84	9.54	3.2	$6.3-12.7$
Uranus	8.72×10^{28}	1.30	2.55×10^{9}	13	19.2	2.4	$16.8-21.6$
Neptune	1.03×10^{29}	1.20	2.74×10^{9}	15	30.1	2.4	$27.7-32.5$
Pluto	1.31×10^{25}	2.00	1.20×10^{8}	0.002	39.5	1.9	$37.6-41.4$
Kuiper belt	—	—	—	—	30–100	—	—

*The maximal concentration of the asteroids of the asteroid strip is registered at $\sim$2.5 AU from the Sun, while the asteroid strip continues from 2.1 to 4.3 AU (approximately).

Table 2.1: The internal constitution of the Solar System according to the General Theory of Relativity.

6) The "bubble" of the outer space breaking of Jupiter's field meets, from its inner side, that of Mars at $r = 2.9$ AU from the Sun (this is the case of the "parade of the planets"). It is very near 2.8 AU, which is the theoretical orbit of Phaeton according to the Titius-Bode law;
7) The "bubbles" of the outer space breaking of the field of the other jovian planets (Saturn, Uranus, and Neptune) are located within the inner boundary of the Kuiper belt (the strip of the aphelia of the comets orbiting the Sun);
8) The outer space breaking of Neptune's field meets, from the outer side of the "bubble", the inner boundary of the Kuiper belt;
9) The "bubble" of the outer space breaking of the field of Pluto is completely located within the Kuiper belt.

The fact that the outer space breaking of the Sun's field is located within the asteroid strip, near the maximal concentration of asteroids, allows us to say: yes, the space breaking considered in this study has a real physical meaning. Probably, the Sun's space breaking impedes asteroids to be joined into a common physical body (one refers to it as *Phaeton*). Alternatively, if Phaeton was an already existing planet orbiting the Sun near the "space breaking orbit" in the past, the force of gravitation of another massive cosmic body, emerging near the Solar System in the ancient ages (for example, another star passing near it), has displaced Phaeton to the "space breaking orbit" near it, thus leading to the distraction of Phaeton's body.

Thus the internal constitution of the Solar System is formed by the geometric structure of the Sun's field according to Riemannian geometry as manifest within the laws of the General Theory of Relativity.

§2.3 The geometric sense of the outer space breaking

Consider the properties of the curvature of the space of a liquid sphere. First, let us calculate the components of the chr.inv.-curvature tensor C_{lkij}, which is the physically observable curvature tensor of the space.

In a rotation-free space ($A_{ik} = 0$), which is the space of a liquid sphere under consideration, $C_{lkij} = H_{lkij}$ according to the definition of the tensor H_{lkij} (1.74). Therefore, we calculate $C_{lkij} = H_{lkij} = h_{jm} H_{lki\cdot}^{\cdots m}$ by the formula of $H_{lki\cdot}^{\cdots m}$ (1.71), wherein we substitute the respective chr.inv.-Christoffel symbols Δ^{i}_{jk} (2.17–2.19) already obtained for the metric of a liquid sphere (2.78). After some algebra, we obtain that the chr.inv.-curvature tensor C_{lkij} in the space of a liquid sphere has the following non-zero components:

$$C_{1212} = H_{1212} = -\frac{\varkappa\rho_0}{3}\,\frac{r^2}{1-\frac{\varkappa\rho_0 r^2}{3}}\,, \tag{2.100}$$

$$C_{1313} = H_{1313} = -\frac{\varkappa\rho_0}{3}\,\frac{r^2\sin^2\theta}{1-\frac{\varkappa\rho_0 r^2}{3}}\,, \tag{2.101}$$

$$C_{2323} = H_{2323} = -\frac{\varkappa\rho_0}{3}\,r^4\sin^2\theta\,. \tag{2.102}$$

We see that, in the space of a liquid sphere, the non-zero components of the observable space curvature tensor C_{iklj} satisfy the condition

$$C_{iklj} = -\frac{\varkappa\rho_0}{3}\left(h_{kl}h_{ij} - h_{il}h_{kj}\right), \tag{2.103}$$

where the negative constant $-\frac{\varkappa\rho_0}{3}$ is the observable three-dimensional curvature of the space in the respective two-dimensional direction. This means that the *three-dimensional space* of a non-rotating liquid sphere has a *constant negative curvature*. Calculating the observable curvature scalar $C = h^{ik}C_{ik}$, where the non-zero components of C_{ik} are

$$C_{11} = -\frac{2\varkappa\rho_0}{3}\,\frac{1}{1-\frac{\varkappa\rho_0 r^2}{3}}\,, \tag{2.104}$$

$$C_{22} = \frac{C_{33}}{\sin^2\theta} = -\frac{2\varkappa\rho_0 r^2}{3}\,, \tag{2.105}$$

we obtain

$$C = -\,2\varkappa\rho_0 = const < 0\,. \tag{2.106}$$

Hence, according to (2.103), the chr.inv.-curvature tensor C_{iklj} is expressed through the observable curvature scalar C as

$$C_{iklj} = \frac{C}{6}\left(h_{kl}h_{ij} - h_{il}h_{kj}\right). \tag{2.107}$$

Thus, the observable three-dimensional space of a non-rotating liquid sphere is a *constant negative curvature space*. Therefore, the curvature radius $\Re$ of the three-dimensional space is imaginary. It is formulated through the observable curvature scalar C by the relation

$$C = -\,2\varkappa\rho_0 = \frac{1}{\Re^2}\,, \tag{2.108}$$

thus we obtain, finally,

$$\Re = \frac{i}{2\varkappa\rho_0}\,. \tag{2.109}$$

Let us calculate the components of the full Riemann-Christoffel curvature tensor

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\frac{\partial g_{\alpha\delta}}{\partial x^{\beta}x^{\gamma}} + \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}\partial x^{\delta}} - \frac{\partial g_{\beta\delta}}{\partial x^{\alpha}\partial x^{\gamma}} - \frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}\partial x^{\delta}}\right) + \\ + g^{\sigma\tau}\left(\Gamma_{\alpha\delta,\sigma}\Gamma_{\beta\gamma,\tau} - \Gamma_{\beta\delta,\sigma}\Gamma_{\alpha\gamma,\tau}\right). \tag{2.110}$$

According to the metric of a liquid sphere (2.78), we have $g_{ik} = -h_{ik}$ and $\Gamma_{ik,j} = -\Delta_{ik,j}$. Thus, calculating the non-zero components of $\Gamma_{\alpha\beta,\delta}$,

$$\Gamma_{01,0} = -\Gamma_{00,1} = \frac{\varkappa\rho_0 r}{12}\,\frac{3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}} - \sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}}{\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}}\,, \tag{2.111}$$

$$\Gamma_{11,1} = -\frac{\varkappa\rho_0 r}{3}\,\frac{1}{\left(1-\frac{\varkappa\rho_0 r^2}{3}\right)^2}\,, \tag{2.112}$$

and substituting these into (2.110), we obtain

$$R_{0101} = -\frac{\varkappa\rho_0}{12}\,\frac{3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}} - \sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}}{\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}}\,, \tag{2.113}$$

$$R_{1212} = \frac{\varkappa\rho_0}{3}\,\frac{r^2}{1-\frac{\varkappa\rho_0 r^2}{3}} = -C_{1212}\,, \tag{2.114}$$

$$R_{1313} = \frac{\varkappa\rho_0}{3}\,\frac{r^2\sin^2\theta}{1-\frac{\varkappa\rho_0 r^2}{3}} = -C_{1313}\,, \tag{2.115}$$

$$R_{2323} = \frac{\varkappa\rho_0}{3}\,r^4\sin^2\theta = -C_{2323}\,. \tag{2.116}$$

We see that the component R_{0101}, determining the four-dimensional curvature in the (r-t)-direction 0101, does not satisfy the condition of four-dimensional constant curvature spaces which is

$$R_{\alpha\beta\gamma\delta} = Q\left(g_{\beta\gamma}g_{\alpha\delta} - g_{\beta\delta}g_{\alpha\gamma}\right), \qquad Q = const. \tag{2.117}$$

Therefore, we arrive at the following conclusion:

- The *four-dimensional space* of a non-rotating liquid sphere *is not* a constant curvature space. This is in contrast to the *observable three-dimensional space* of the liquid sphere which, as was proven above, is a *constant negative curvature space*.

We see also, from the formulae for C_{1212} (2.100) and C_{1313} (2.101), that the three-dimensional observable curvature C_{iklj} possesses space breaking

$$C_{1212} \rightarrow -\infty\,, \qquad C_{1313} \rightarrow -\infty \tag{2.118}$$

by the condition $r = r_{br} = \sqrt{3/\varkappa\rho_0} = \sqrt{a^3/r_g}$. By the same condition $r = r_{br}$, according to the formula for R_{0101} (2.113), we have

$$R_{0101} \rightarrow -\infty\,. \tag{2.119}$$

In other words, the three-dimensional chr.inv.-curvature C_{iklj} and the four-dimensional Riemannian curvature $R_{\alpha\beta\gamma\delta}$ have space breaking by the condition $r = r_{br}$. Concerning the model of liquid stars, this means:

- In the field of each star, the three-dimensional observable space curvature C_{iklj} and the four-dimensional Riemannian curvature $R_{\alpha\beta\gamma\delta}$ have common space breaking on the spherical surface at the distance $r = r_{br} = \sqrt{3/\varkappa\rho_0} = \sqrt{a^3/r_g}$ from the star.

This is the geometric sense of the outer space breaking of the field of a star (in the framework of the liquid model under consideration).

§2.4 The force of gravity acting inside a liquid star

In a rotation-free space, according to the definition of the gravitational inertial force (1.42), the force is only due to g_{00} (which is determined by the gravitational potential w). Let us calculate that force. Since the gravitational potential is $\mathrm{w} = c^2(1-\sqrt{g_{00}})$, we obtain

$$\frac{\partial \mathrm{w}}{\partial x^i} = -\frac{c^2}{2\sqrt{g_{00}}}\frac{\partial g_{00}}{\partial x^i}\,. \tag{2.120}$$

In the "inner" metric of a non-rotating liquid sphere (2.76),

$$g_{00} = \frac{1}{4}\left(3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}} - \sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}\right)^2, \tag{2.121}$$

or, in the same metric written in the other form (2.78),

$$g_{00} = \frac{1}{4}\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r^2 r_g}{a^3}}\right)^2. \tag{2.122}$$

We therefore obtain that the force acting inside it is

$$F_1 = -\frac{\varkappa\rho_0 c^2 r}{3}\,\frac{1}{\left(3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}}-\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}\right)\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}}\,, \tag{2.123}$$

$$F^1 = -\frac{\varkappa\rho_0 c^2 r}{3}\,\frac{\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}}{3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}}-\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}}\,, \tag{2.124}$$

or, in the other form,

$$F_1 = -\frac{c^2 r_g r}{a^3}\,\frac{1}{\left(3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}\right)\sqrt{1-\frac{r_g r^2}{a^3}}}\,, \tag{2.125}$$

$$F^1 = -\frac{c^2 r_g r}{a^3}\,\frac{\sqrt{1-\frac{r_g r^2}{a^3}}}{3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}}\,. \tag{2.126}$$

This is a force of attraction: since $r < a$ inside the sphere, $F_1 < 0$ therein. The force is proportional to distance r. Its numerical value is zero at the center of the sphere, then it increases with distance upto its ultimate-high value on the surface of the star (where $r = a$)

$$(F_1)_{r=a} = -\frac{\varkappa\rho_0 c^2 a}{6}\,\frac{1}{1-\frac{\varkappa\rho_0 a^2}{3}} = -\frac{c^2 r_g}{2a^2}\,\frac{1}{1-\frac{r_g}{a}}\,, \tag{2.127}$$

$$(F^1)_{r=a} = -\frac{\varkappa\rho_0 c^2 a}{6} = -\frac{c^2 r_g}{2a^2}\,. \tag{2.128}$$

§2.5 Solving the conservation law equations: pressure and density inside the stars

Consider now the pressure p and density ρ_0 inside a regular liquid star. A formula connecting pressure and density inside a medium is the equation of state. It follows as a solution of the conservation law equations.

We have now already obtained almost all that is needed for the formula. In §2.1, we solved the conservation law equations with the energy-momentum tensor of a perfect liquid (2.4), which points to the substance of liquid stars. After substitution of the physically observable components (2.21) of the energy-momentum tensor, the general equations of the conservation law (1.89–1.90) take the particular form (2.26–2.27). In a non-deforming (static) space such as the space of a regular star, only the vectorial conservation equation remains non-vanishing. It has the form (2.36). The equation is solved as the formula (2.40)

$$p = Be^{-\frac{\nu}{2}} - \rho_0 c^2. \tag{2.129}$$

Now, substituting the already found integration constant B (2.70) and function $e^{\frac{\nu}{2}}$ (2.75) into p (2.129), we obtain the final solution connecting pressure p and density ρ_0 inside a regular star

$$p = \rho_0 c^2 \, \frac{\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}} - \sqrt{1-\frac{\varkappa\rho_0 a^2}{3}}}{3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}} - \sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}}\,. \tag{2.130}$$

Find the pressure in the near-surface layer of a star. The constant $\varkappa = 18.6\times10^{-28}$ cm/g is a very small value, while $\rho_0 = 1.4$ gram/cm^3 for the Sun, the yellow dwarf, and is much less than that for larger stars. Therefore, $\varkappa\rho_0 a^2$ is much smaller than 1 for even very large stars. For instance, for Betelgeuse, which is one of the largest red super-giants: $M = 4.0\times10^{34}$ gram, $a = 7.0\times10^{13}$ cm, $\rho_0 = 2.8\times10^{-8}$ gram/cm^3. In this case, we have $\varkappa\rho_0 a^2 = 2.6\times10^{-7}$. As a result, we have, for the values of r and a,

$$\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}} \approx 1 - \frac{\varkappa\rho_0 r^2}{6}\,.$$

Thus, after some algebra, we obtain the approximate formula for the pressure p inside a regular star. It is

$$p \approx \frac{\varkappa\rho_0^2 c^2 \left(a^2 - r^2\right)}{12} = \frac{\rho_0 GM}{2a^2}\left(\frac{a^2 - r^2}{a}\right). \tag{2.131}$$

Let $h = a - r$ be the distance from the surface of the sphere to the point of measurement. Because $h \ll r$ in the near-surface layer, we have

$$a^2 - r^2 = (a - r)(a + r) = h\,(2a + h) \approx 2ah\,. \tag{2.132}$$

Thus, from (2.131), we obtain the regular formula for the pressure in the near-surface layer

$$p = \rho_0 g h\,, \tag{2.133}$$

where $\frac{GM}{a^2} = g$ is the free-fall acceleration in the star's field near its surface.

Object	Mass M, gram	Radius a, cm	Density ρ_0, gram/cm^3	Pressure p_0, dynes/cm^2
Red super-giant*	4.0×10^{34}	7.0×10^{13}	2.8×10^{-8}	5.3×10^{5}
White super-giant†	3.4×10^{34}	4.8×10^{12}	7.3×10^{-5}	1.7×10^{10}
Sun	2.0×10^{33}	7.0×10^{10}	1.4	1.3×10^{15}
Jupiter (proto-star)	1.9×10^{30}	7.1×10^{9}	1.3	1.2×10^{15}
Red dwarfs	6.7×10^{32}	2.3×10^{10}	13	1.3×10^{16}
Brown dwarf‡	4.1×10^{31}	7.0×10^{9}	29	5.7×10^{15}
White dwarf§	2.0×10^{33}	6.4×10^{8}	1.8×10^{6}	1.9×10^{23}

*Betelgeuse. †Rigel. ‡Corot-Exo-3. §Sirius B.

Table 2.2: The main characteristics of the regular stars.

The pressure in the central region of a regular star can easily be found by assuming $r=0$ in the general formula (2.130). Denoting the central pressure as $p_0=p_{r=0}$, we obtain

$$p_0 = \rho_0 c^2\,\frac{1-\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}}}{3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}}-1} \approx \frac{\varkappa\rho_0^2\, a^2 c^2}{12}\,. \tag{2.134}$$

Since $\varkappa=\frac{8\pi G}{c^2}$, we can also re-write this formula in the form

$$p_0 \approx \frac{3GM^2}{8\pi a^4}\,. \tag{2.135}$$

Table 2.2 gives the numerical values of the central pressure p_0 we have calculated according to this formula for the typical members of the known families of regular stars.

We see that, according to our model of liquid stars, the pressure in the central region of Betelgeuse, which is one of largest stars, is only 0.53 atmosphere (1 atm $=10^6$ dynes/cm^2). The smaller the size of a star is, the higher the pressure inside it becomes. The pressure in the central region of Rigel, a white super-giant whose radius is 14.6 times less than that of Betelgeuse, is 1.7×10^4 atm. In dwarfs such as the Sun, the central pressure is $\sim10^9$ atm. However, in white dwarfs, the central pressure reaches 10^{17} atm.

Note that the temperature of condensed matter does not depend on pressure. The incompressible liquid of stars is a sort of condensed matter. Therefore, temperatures inside stars depend solely on the formula of that particular mechanism which produces stellar energy.

This note is important for the understanding of the physical conditions inside stars, and of the sources of stellar energy.

§2.6 The stellar energy mechanism according to the liquid star model and the mass-luminosity relation

First, we make the transition to the dimensionless characteristics of stars, which are expressed in fractions of the respective characteristics of the Sun:

$$\bar{M} = \frac{M}{M_{\odot}}\,, \quad \bar{a} = \frac{a}{a_{\odot}}\,, \quad \bar{\rho} = \frac{\rho}{\rho_{\odot}}\,, \quad \dots \text{ etc.} \tag{2.136}$$

where $\bar{M} = \bar{\rho}_0\,\bar{a}^3$ for a liquid sphere*. For the luminosity L of a star, that is the energy emitted from the entire surface of the star into the outer cosmos per one second, we have

$$\bar{L} = \frac{L}{L_{\odot}}\,. \tag{2.137}$$

With this representation of the characteristics of stars, the analysis becomes much simpler. This is because only the essential factors remain in the formulae while all constant coefficients vanish.

Let us study what mechanism producing stellar energy can now be suggested due to the General Theory of Relativity, so that its productivity satisfies the observed luminosity of stars. In other words, to be the real mechanism that generates energy in stars, the calculated energy production of the suggested mechanism should match the mass-luminosity relation which is the main empirical relation of observational astrophysics.

Consider thus the space metric of a liquid star. As we know already, the space of a liquid star has two primary regions which are described by different metrics:

1) The internal space metric of the star (the metric of a liquid sphere) is valid from the center of the star to its surface. Except on the singular spherical surface of the tiny radius, $r_g = \frac{2GM}{c^2}$ around the center of the star (see below). The internal metric is also valid on the singular spherical surface of the radius $r_{br} = \sqrt{a^3/r_g} = \sqrt{3/\varkappa\rho_0}$, in the far cosmos: the metric produces a breaking of the space curvature at this distance from the star;

*A liquid star has the same density $\rho = \rho_0 = const$ in its entire volume, so its mass is $M = \frac{4}{3}\pi\rho_0 a^3$. In fractions of the Sun's mass, it is $\bar{M} = \bar{\rho}_0\,\bar{a}^3$.

2) The external metric of the star (the mass-point metric) is valid from the surface of the star to infinity. Except on the singular spherical surface of the radius, $r_{br} = \sqrt{a^3/r_g} = \sqrt{3/\varkappa\rho_0}$ which covers the star distantly from its surface in the cosmos (see above). The external metric is also valid deep inside the star, on the singular spherical surface of the tiny radius $r_g = \frac{2GM}{c^2}$ from the center of the star: on this spherical surface, the star's gravitational field possesses space breaking produced due to the external metric.

As was shown in §2.3, the outer space breaking in the far cosmos only implies space curvature breaking. One can show, on the basis of §2.3, that it does not result in an anomaly with respect to the acting force of gravitation.

However, we now show that the force of gravitation has a very strong anomaly on the singular spherical surface of the inner space breaking. Actually, inside the star at the Hilbert radius r_g from its center the external space metric is valid (while the internal metric is valid both inside the Hilbert radius and outside it). Therefore, all calculations for the inner singular surface are processed with the external metric (mass-point metric). This is despite the fact that the singular surface is located deep within the star near its center.

According to the fundamental metric tensor of the external metric of the star (1.1), the chronometrically invariant (physically observable) vector of the force of gravitation F_i has the form (1.4). On the singular spherical surface of the Hilbert radius $r = r_g$, deep inside the star, the observable force of gravity (1.4) reaches an infinitely large magnitude

$$F_1 = -\frac{c^2 r_g}{2r^2}\frac{1}{1-\frac{r_g}{r}} \rightarrow -\infty\,, \tag{2.138}$$

i.e. the gravitational field possesses space breaking on the surface.

Due to its infinitely large magnitude there, the force of gravity, by definition, is sufficient for the transfer of the necessary kinetic energy to the lightweight atomic nuclei of the stellar substance, so that the process of thermonuclear fusion begins. The energy released in the thermonuclear fusion is the energy that the stars radiate.

The singular spherical surface of the Hilbert radius $r_g = \frac{2GM}{c^2}$ surrounds the geometric center of every star. This means that at the center of each star a luminous "inner sun" is located. The "inner sun" is tiny compared to the size of the star. For example, the Hilbert radius of the Sun is only 2.9 km while the physical radius of the Sun is 700,000 km. Therefore, the zone where thermonuclear fusion is processed is not only the surface layer of the radius r_g but all the volume of the "inner sun". In other words, the "inner sun" of the radius r_g is the very place where thermonuclear fusion produces helium from hydrogen, thus providing energy for the luminosity of the star. The energy is then transmitted from the "inner sun" of the star to its surface due to heat conductivity (the conventional transfer of heat in liquids); then it is radiated from the surface into the cosmos.

Since the "inner sun" of a star has a radius equal to the Hilbert radius r_g, we will further refer to it as the *luminous Hilbert core* of a star, or merely — the *Hilbert core.*

The luminosity of a star that shines due to the suggested mechanism of stellar energy depends only on two factors: the volume of the Hilbert core $V = \frac{4}{3}\pi r_g^3$ where energy is released, and the density ρ_g of the stellar substance therein (which can differ from the density ρ_0 of the main mass of the star, see the explanation in the next page). In terms of the dimensionless characteristics of stars, it is

$$\bar{L} = \bar{\rho}_g \bar{r}_g^3 = \bar{\rho}_g \bar{M}^3. \tag{2.139}$$

Recall that the suggested mechanism of stellar energy does not depend on the pressure in the central region of the star: the super-strong force of gravity (2.138) that acts therein provides the conditions necessary for thermonuclear fusion. But the productivity of the mechanism depends on the density of the stellar substance in the Hilbert core.

Calculate the density of the Hilbert core so that the suggested mechanism of stellar energy satisfies the observed mass-luminosity relation.

Proceed from the facts of observational astronomy. It shows the mass-luminosity relation $\bar{L} = \bar{M}^{2.6}$ for the stars whose masses are in the range between $0.2\,M_\odot$ and $0.5\,M_\odot$, $\bar{L} = \bar{M}^{4.5}$ for masses between $0.5\,M_\odot$ and $2\,M_\odot$, $\bar{L} = \bar{M}^{3.6}$ in the range between $2\,M_\odot$ and $10\,M_\odot$, and $\bar{L} = \bar{M}$ for masses much heavier than $10\,M_\odot$. See Table 2.3.

These empirical data of observational astronomy match with our theoretical formula for the luminosity of stars L (2.139), if the stellar substance of the Hilbert core (wherein stellar energy is released) has the density as shown in Table 2.4.

On the basis of the function $\bar{\rho}_g = \bar{M}^y$ according to Table 2.4, we are able to know how dense the Hilbert core of a star is compared to the main mass of the star (known from astronomical observations). We can thus calculate, for some typical stars, the following ratio:

$$\frac{\bar{\rho}_g}{\bar{\rho}_0} = \frac{\bar{M}^y}{\bar{\rho}_0}\,. \tag{2.140}$$

Observed mass-luminosity relation $\bar{L} = \bar{M}^x$	Scale of the stellar masses, in fractions of the Sun's mass $M_\odot$
$\bar{L} = \bar{M}^{2.6}$	$\bar{M} = 0.2 \ldots 0.5$
$\bar{L} = \bar{M}^{4.5}$	$\bar{M} = 0.5 \ldots 2$
$\bar{L} = \bar{M}^{3.6}$	$\bar{M} = 2 \ldots 10$
$\bar{L} = \bar{M}$	$\bar{M} > 10$

Table 2.3: The observed mass-luminosity relation $\bar{L} = \bar{M}^x$.

Density of the Hilbert core $\bar{\rho}_g$	Scale of the stellar masses, in fractions of the Sun's mass $M_\odot$
$\bar{\rho}_g = \bar{M}^{0.4}$	$\bar{M} = 0.2 \ldots 0.5$
$\bar{\rho}_g = \bar{M}^{1.5}$	$\bar{M} = 0.5 \ldots 2$
$\bar{\rho}_g = \bar{M}^{0.6}$	$\bar{M} = 2 \ldots 10$
$\bar{\rho}_g = \bar{M}^{-2}$	$\bar{M} > 10$

Table 2.4: Density of the substance inside the Hilbert core.

Object	Mass $\bar{M}$	Density $\bar{\rho}_0$	Ratio $\bar{\rho}_g/\bar{\rho}_0$
Betelgeuse (red super-giant)	20	2.0×10^{-8}	1.3×10^{9}
Rigel (white super-giant)	17	5.2×10^{-5}	6.7×10^{7}
Jupiter (proto-star)	9.5×10^{-4}	0.9	0.069
Red dwarfs	0.34	9	0.072
Brown dwarf (Corot-Exo-3)	0.021	21	0.010
White dwarf (Sirius B)	1	1.3×10^{6}	7.7×10^{-7}

Table 2.5: Ratio $\bar{\rho}_g/\bar{\rho}_0$ for some typical stars.

The results of the calculations are shown in Table 2.5. On the basis of the calculated ratio $\bar{\rho}_g/\bar{\rho}_0$ as shown in Table 2.5, we arrive at the following conclusion. The luminous Hilbert core of a star — its "inner sun" — can have a density that differs from the density of the main mass of the star. It depends on the particular type of the star. For instance. The stellar substance of the Hilbert core of a giant or super-giant is many orders denser than the main substance of the stars. The Hilbert core of the star that is similar to the Sun has approximately the same density as the star. Concerning the dwarf stars, the Hilbert core of such a star is more rarefied than the main substance of the star. The greater the density of a dwarf star is, the less the density of its core becomes compared to the density of the entire star. In such a star as the white dwarf, the Hilbert core is many orders of magnitude more rarefied than the main substance of the star.

Respectively, the following question arises. All physical bodies have masses, therefore each body should have a Hilbert radius core inside itself. Not only stars, but also planets and even individual elementary particles should have such a core. Yet, why do they not shine like stars?

The answer comes from the state of that substance of which these physical bodies consist. Stars are made up of liquid substance which consists, mostly, of light chemical elements such as hydrogen and helium. Therefore, thermonuclear fusion of light atomic nuclei is possible in the Hilbert core of each star. Due to the fact that the substance is liquid, more and more "nuclear fuel" is delivered from the other regions of the star to its luminous Hilbert core, thus supporting the combustion inside the "nuclear boiler", until the time when all the nuclear fuel of the star ends. Another case — the planets. They consist of mostly heavy elements with only a minor content of hydrogen. Therefore, as soon as the "nuclear boiler" of the Hilbert core has finished all the reserve of the hydrogen fuel in the central region of the planet, it stops producing energy but still remains to exist at the center of the planet, in a latent state.

Astronomers know that the energy emitted by Jupiter exceeds the solar energy absorbed by the entire surface of the planet. The same is as well true for Saturn. This means, according to our theory, that the Hilbert core of each of the planets still processes hydrogen into helium thereby releasing nuclear energy.

Concerning individual elementary particles such as protons, neurons, and electrons: as is known, they are stable and indifferent for a long time as long as they do not interact with other particles. In fact, this means that the Hilbert core of the proton (as well as of the neutron and the electron) does not interact with the main mass of the particle. Why does this happen? We can only guess that either the substance that is inside the particles is in the super-solid state, or there is a layer of the very strong vacuum between the core and the rest mass. On the other hand, the Hilbert core of the proton (and that of the neutron) has a tiny radius of $(r_g)_p = \frac{2Gm_p}{c^2} = 2.48\times10^{-52}$ cm, while the Hilbert core of the electron has even a smaller radius of $(r_g)_e = \frac{2Gm_e}{c^2} = 1.35\times10^{-55}$ cm. As has been stated by Albert Einstein already, the geometric laws (space-time geometry) of the General Theory of Relativity are true, probably, upto the scale of elementary particles. Within the sub-nuclear scale, probably, another geometry works thus stating its own laws which differ from the laws of the General Theory of Relativity. Therefore, we cannot presently state something definite about the physical conditions and processes inside elementary particles.

But as for the regular world of stars and the planets, experimental physics and observational astronomy show that Einstein's theory is correct and works on these scales with high accuracy. Therefore, all our conclusions about the internal constitution of stars, and the mechanism that generates energy in stars should be taken into account.

The particular details of the suggested mechanism of stellar energy are a special theme that is out of the scope of this book (which is mostly on the internal constitution of stars).

§2.7 Conclusion

All the theoretical conclusions about the source of stellar energy, and about the internal constitution of stars that are presented in this Chapter have been obtained in the framework of our model of liquid stars. Our model is based on the presentation about stars as space-time objects, according to the General Theory of Relativity. Below, we list the most important of the conclusions we have thus arrived at:

1. The field of each star possesses space breaking which surrounds the star by a spherical surface. The "bubble" of the outer space breaking of the field has a radius of

$$r_{br} = \sqrt{\frac{3}{\varkappa\rho_0}} = \sqrt{\frac{a^3}{r_g}}\,,$$

which is many orders larger than the physical radius a of the star. The three-dimensional observable space curvature C_{iklj} and the four-dimensional Riemannian curvature $R_{\alpha\beta\gamma\delta}$ have common space breaking on the surface. The outer space breaking impedes the near substance to be formed as a planet in this orbit. The outer space breaking of the Sun's field is located within the asteroid strip, near the maximal concentration of the asteroids;

2. The field of each star possesses inner space breaking, inside the physical body of the star, on the surface of the Hilbert radius
$$r_g = \frac{2GM}{c^2}\,,$$
from the center. This means that there is a small core which is separated, by the singular surface, from the major mass of the star. On the surface of the core, the force of gravity reaches an infinitely large magnitude. The super-strong gravity, by definition, is sufficient for the transfer of the necessary kinetic energy to the lightweight atomic nuclei of the stellar substance, so that thermonuclear fusion begins. Thus, nuclear energy is released. The liquid "nuclear fuel" is delivered from the other regions of the star to the core thus supporting the combustion inside the "nuclear boiler";
3. Every star has a mass M. Therefore, the luminous core of the Hilbert radius $r_g = \frac{2GM}{c^2}$ — the "inner sun" — exists in the center of every star. We refer to it as the *Hilbert core*. This is the place in which thermonuclear fusion produces helium from hydrogen, thus providing energy for the luminosity of the stars. The energy is then transmitted from the "inner sun" of the star to its surface due to heat conductivity (the conventional transfer of heat in liquids) for it then to be radiated into the cosmos;
4. The Hilbert core is tiny compared to the size of stars. For example, for the Sun, $r_g = 2.9$ km;
5. The observed relation "mass-luminosity" of stars is satisfied if the Hilbert core has a density depending on the particular type of the star. The Hilbert core of a giant or super-giant should be many orders denser than the main substance of the stars. The Hilbert core of a star like the Sun should be approximately the same in density as the star. In the dwarf star, the Hilbert core should be more rarefied than the main substance of the star (the core of the white dwarf should be extremely rarefied);
6. Each planet has a mass. Therefore, the Hilbert core exists in the center of every planet. But planets consist of mostly heavy elements with only a minor content of hydrogen. As soon as the "nuclear boiler" of the Hilbert core has finished all the reserve

of the hydrogen fuel in the central region of the planet, it stops producing energy but still remains to exist at the center of the planet, in a latent state.

Chapter 3

Regular Stars. The Description

§3.1 Problem statement. The internal space metric of a regular non-rotating star

To broadly understand the description of a regular star again, recall that in §2.1 we deduced the space metric of a liquid sphere by following the "historical path" as Schwarzschild did it. Namely, — we took the spherically symmetric metric in the general form, then applied the particular conditions of a sphere filled with perfect liquid. The sole difference from Schwarzschild's deduction was that we did not assume any artificial limitations. When following this deduction, we obtained the observable characteristics of the space in the implicit form, as an auxiliary result. It was enough to obtain the space metric of a liquid sphere in the final form. Now, we express the characteristics in the explicit form, through the components of the fundamental metric tensor of the space metric of a liquid sphere which we have obtained in Chapter 2. Then we will study the equations of motion inside the star so as to take the escape velocity into account.

So, the space metric of a liquid sphere has the form (1.8)

$$ds^2 = \frac{1}{4}\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r^2 r_g}{a^3}}\right)^2 c^2 dt^2 - \\ - \frac{dr^2}{1-\frac{r^2 r_g}{a^3}} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \qquad (3.1)$$

We calculate the chr.inv.-characteristics of the space, according to their definitions given in §1.3 and the respective components of the fundamental metric tensor of the metric (3.1). The chr.inv.-metric tensor h_{ik} of the metric (3.1) has the following non-zero components

$$h_{11} = \frac{1}{1-\frac{r^2 r_g}{a^3}}\,, \quad h_{22} = r^2, \quad h_{33} = r^2\sin^2\theta\,, \qquad (3.2)$$

$$h^{11} = 1-\frac{r^2 r_g}{a^3}\,, \quad h^{22} = \frac{1}{r^2}\,, \quad h^{33} = \frac{1}{r^2\sin^2\theta}\,, \qquad (3.3)$$

while its determinant, and the non-zero spatial derivatives of the logarithm of the determinant have the form

$$h = \det \|h_{ik}\| = \frac{r^4 \sin^2\theta}{1 - \frac{r^2 r_g}{a^3}}\,, \tag{3.4}$$

$$\frac{{}^*\partial \ln\sqrt{h}}{\partial r} = \frac{2}{r} + \frac{r_g\, r}{a^3}\frac{1}{1 - \frac{r^2 r_g}{a^3}}\,, \qquad \frac{{}^*\partial \ln\sqrt{h}}{\partial \theta} = \cot\theta\,. \tag{3.5}$$

So forth, after algebra according to the chronometrically invariant formalism (see §1.3 for the definitions of the chr.inv.-quantities), we obtain the following. The chr.inv.-vector of the gravitational inertial force, acting in the space has the form

$$F_1 = -\frac{c^2 r_g}{a^3}\frac{r}{\left(3\sqrt{1 - \frac{r_g}{a}} - \sqrt{1 - \frac{r_g r^2}{a^3}}\right)\sqrt{1 - \frac{r_g r^2}{a^3}}}\,, \tag{3.6}$$

$$F^1 = -\frac{c^2 r_g}{a^3}\frac{r\sqrt{1 - \frac{r_g r^2}{a^3}}}{3\sqrt{1 - \frac{r_g}{a}} - \sqrt{1 - \frac{r_g r^2}{a^3}}}\,, \tag{3.7}$$

where $r < a$ since it is inside the sphere. Therefore, $F_1 < 0$ therein (this means that it is a force of attraction). The non-zero chr.inv.-Christoffel symbols have the form

$$\Delta^1_{11} = \frac{r_g r}{a^3}\frac{1}{1 - \frac{r_g r^2}{a^3}}\,, \quad \Delta^1_{22} = -\frac{\Delta^1_{33}}{\sin^2\theta} = -r\left(1 - \frac{r_g r^2}{a^3}\right), \tag{3.8}$$

$$\Delta^2_{12} = \Delta^3_{13} = \frac{1}{r}\,, \quad \Delta^2_{33} = -\sin\theta\cos\theta\,, \quad \Delta^3_{23} = \cot\theta\,. \tag{3.9}$$

The non-zero components of the chr.inv.-tensor of the three-dimensional observable curvature C_{iklj}, and its contraction C_{ik}, have the form

$$C_{1212} = \frac{C_{1313}}{\sin^2\theta} = -\frac{r_g r^2}{a^3}\frac{1}{1 - \frac{r_g r^2}{a^3}}\,, \quad C_{2323} = -\frac{r_g r^4}{a^3}\sin^2\theta, \tag{3.10}$$

$$C_{11} = -\frac{2r_g}{a^3}\frac{1}{1 - \frac{r_g r^2}{a^3}}\,, \qquad C_{22} = \frac{C_{33}}{\sin^2\theta} = -\frac{2r_g r^2}{a^3}\,. \tag{3.11}$$

So, we now have all that is needed to consider Einstein's equations in the internal field of a regular non-rotating star.

§3.2 Einstein's equations in the internal field of a regular non-rotating star

Consider Einstein's field equations in the space of the metric (3.1). As is known, the energy-momentum tensor of a perfect liquid has the following general form (2.4):

$$T^{\alpha\beta} = \left(\rho_0 + \frac{p}{c^2}\right) U^{\alpha} U^{\beta} - \frac{p}{c^2}\, g^{\alpha\beta}, \tag{3.12}$$

where $\rho_0 = const$ is the density of the liquid, p is the pressure, while U^{α} is the four-dimensional velocity of the flow of the liquid with respect to the observer (the unit four-vector, so $U_{\alpha} U^{\alpha} = 1$). The chr.inv.-projections of the energy-momentum tensor have the form (2.21)

$$\rho = \frac{T_{00}}{g_{00}} = \rho_0\,, \quad J^i = \frac{c\,T_0^i}{\sqrt{g_{00}}} = 0\,, \quad U^{ik} = c^2\,T^{ik} = p h^{ik}, \tag{3.13}$$

where ρ is the observable density of mass, J^i is the observable density of momentum, while U^{ik} is the observable stress tensor. With these formulae, and by taking into account that the space of the particular liquid sphere is free of rotation and deformation ($A_{ik} = 0$, $D_{ik} = 0$), the chr.inv.-Einstein equations (1.85–1.87) take the form

$$^*\nabla_j F^j - \frac{1}{c^2}\, F_j F^j = -\frac{\varkappa}{2}\left(\rho_0 c^2 + U\right), \tag{3.14}$$

$$J^i = 0\,, \tag{3.15}$$

$$\frac{1}{2}\left(^*\nabla_i F_k + {}^*\nabla_k F_i\right) - \frac{1}{c^2}\, F_i F_k - c^2 C_{ik} = = \frac{\varkappa}{2}\left(\rho_0 c^2 h_{ik} + 2U_{ik} - U h_{ik}\right), \tag{3.16}$$

where $^*\nabla_i$ is the symbol of chr.inv.-differentiation (see *Notations*), while $U_{ik} = p h_{ik}$ and $U = 3p$.

Substitute, into the Einstein field equations, the formulae for F_i, C_{ik}, and h_{ik}, which have been calculated for the metric (3.1). We obtain that only two equations remain non-vanishing:

$$\frac{3c^2 r_g}{a^3}\, \frac{\sqrt{1 - \frac{r_g r^2}{a^3}}}{3\sqrt{1 - \frac{r_g}{a}} - \sqrt{1 - \frac{r_g r^2}{a^3}}} = \frac{\varkappa}{2}\left(\rho_0 c^2 + 3p\right), \tag{3.17}$$

$$\frac{3c^2 r_g}{a^3}\, \frac{2\sqrt{1 - \frac{r_g}{a}} - \sqrt{1 - \frac{r_g r^2}{a^3}}}{3\sqrt{1 - \frac{r_g}{a}} - \sqrt{1 - \frac{r_g r^2}{a^3}}} = \frac{\varkappa}{2}\left(\rho_0 c^2 - p\right). \tag{3.18}$$

Multiplying (3.18) by 3 then summarizing the product with (3.17), we obtain

$$\varkappa\rho_0 c^2 = \frac{3c^2 r_g}{a^3}\,. \tag{3.19}$$

Substituting this result back into (3.18), we obtain the *equation of state** for the liquid substance of regular stars

$$p = \rho_0 c^2\, \frac{\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}} - \sqrt{1-\frac{\varkappa\rho_0 a^2}{3}}}{3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}} - \sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}}\,. \tag{3.20}$$

This formula completely coincides with the formula for pressure p (2.130) we have obtained in Chapter 2 as the result of following the path of Schwarzschild.

This formula for pressure p can also be obtained from the conservation equations (2.26–2.27). Because the space of the metric (3.1) does not deform ($h_{ik} \neq f(t)$, hence $D_{ik} = 0$), the chr.inv.-scalar conservation equation (2.26) vanishes. Only the chr.inv.-vectorial conservation equation (2.27) remains. It takes the form

$$^*\nabla_i \left(p\,h^{ik}\right) - \left(\rho_0 + \frac{p}{c^2}\right) F^k = 0\,. \tag{3.21}$$

Herein, $^*\nabla_i\, h^{ik} = 0$ is true always as well as $\nabla_\sigma\, g^{\alpha\sigma} = 0$ for the fundamental metric tensor. Therefore, and because the chr.inv.-derivative with respect to the spatial coordinates coincides with the regular spatial derivative in the case where the space does not rotate, the remaining conservation equation (3.21) has the form

$$h^{ik}\frac{\partial p}{\partial x^i} - \left(\rho_0 + \frac{p}{c^2}\right) F^k = 0\,. \tag{3.22}$$

Substituting the formulae for h^{11} and F^1 we have obtained for the metric (3.1), we transform (3.22) into the differential equation

$$\frac{dp}{\rho_0 c^2 + p} = -\frac{r_g}{a^3}\, \frac{r\,dr}{\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}\right)\sqrt{1-\frac{r_g r^2}{a^3}}}\,. \tag{3.23}$$

This equation can be re-written in the form

$$d\ln\left(\rho_0 c^2 + p\right) = -d\ln\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}\right), \tag{3.24}$$

*The formula connecting pressure and density inside the medium.

which is easy to integrate. After integration, we have

$$p+\rho_0 c^2 = \frac{Q}{3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}}\,, \tag{3.25}$$

where the integration constant Q comes from the obvious condition $p=0$ on the surface of the star (where $r=a$). Then

$$Q = 2\rho_0 c^2\sqrt{1-\frac{r_g}{a}}\,, \tag{3.26}$$

thus we obtain, finally,

$$p+\rho_0 c^2 = 2\rho_0 c^2\,\frac{\sqrt{1-\frac{r_g}{a}}}{3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}}\,. \tag{3.27}$$

It is easy to see that this solution leads to the same formula for p as (3.20), which we have obtained from Einstein's field equations.

§3.3 The internal space metric of a regular rotating star

Consider now the metric of a regular liquid star (3.1) with only the change that the star rotates, at an angular speed ω, along its equatorial axis (the axis ϕ in the spherical coordinates r, θ, ϕ). In this case, the non-rotating metric (3.1) takes the form

$$ds^2 = \frac{1}{4}\left(3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r^2 r_g}{a^3}}\right)^2 c^2dt^2 + \\ + \frac{2\omega r^2\cos\theta}{c}\,cdt\,d\varphi - \frac{dr^2}{1-\frac{r^2 r_g}{a^3}} - r^2\left(d\theta^2+\sin^2\theta\,d\phi^2\right). \tag{3.28}$$

It should be noted that we still consider regular stars, namely those stars whose Hilbert radius is much smaller than their physical radius.

According to the metric (3.28), the star's linear velocity of space rotation is

$$v_1 = v_2 = 0\,, \qquad v_3 = -\frac{2\omega r^2\cos\theta}{3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}}\,. \tag{3.29}$$

As is known from the data of observational astronomy, the majority of stars rotate at linear speeds $v<420$ km/sec. Hence, we have $v^2/c^2<2\times10^{-6}$: most stars rotate slowly compared to the speed of light.

According to the space metric of a regular (slowly rotating) regular star (3.28), represented as a rotating liquid sphere, we have

$$v^2 = h^{ik} v_i v_k = h^{33} v_3 v_3 \,, \qquad h^{33} = -g^{33} = \frac{1}{r^2 \sin^2 \theta} \,. \tag{3.30}$$

So, v^2/c^2 in the space of the metric (3.28) has the form

$$\frac{v^2}{c^2} = \frac{4\omega^2 r^2 \cot^2 \theta}{c^2 \left(3\sqrt{1 - \frac{r_g}{a}} - \sqrt{1 - \frac{r_g r^2}{a^3}}\right)^2} \,. \tag{3.31}$$

Expanding, in this formula, the radicals into series, after elementary transformations we obtain

$$\frac{v^2}{c^2} = \frac{\omega^2 r^2 \cot^2 \theta}{c^2} \left(1 + \frac{3r_g}{2a} - \frac{r_g r^2}{2a^3}\right) . \tag{3.32}$$

Further, we will neglect higher-order terms of the series which are small because $r_g \ll a$ for the regular stars. Therefore, we have

$$v_3 = \omega r^2 \cos \theta \,, \qquad \frac{v^2}{c^2} = \frac{v_3^2}{c^2} = \frac{\omega^2 r^4 \cos^2 \theta}{c^2} \,. \tag{3.33}$$

The non-zero components of the chr.inv.-metric tensor of the metric (3.28) have the form

$$h_{11} = \frac{1}{h^{11}} = \frac{1}{1 - \frac{r^2 r_g}{a^3}} \,, \quad h_{22} = \frac{1}{h^{22}} = r^2, \tag{3.34}$$

$$h_{33} = \frac{1}{h^{33}} = r^2 \sin^2 \theta \left(1 + \frac{\omega^2 r^2 \cot^2 \theta}{c^2}\right) , \tag{3.35}$$

while the determinant of the chr.inv.-metric tensor h_{ik} and the non-zero spatial derivatives of the logarithm of the determinant have the form

$$h = \det \| h_{ik} \| = \frac{r^4 \sin^2 \theta}{1 - \frac{r^2 r_g}{a^3}} \left(1 + \frac{\omega^2 r^2 \cot^2 \theta}{c^2}\right) , \tag{3.36}$$

$$\frac{{}^*\partial \ln \sqrt{h}}{\partial r} = \frac{2}{r} + \frac{r_g r}{a^3} \frac{1}{1 - \frac{r^2 r_g}{a^3}} \,, \tag{3.37}$$

$$\frac{{}^*\partial \ln \sqrt{h}}{\partial \theta} = \cot \theta \left(1 - \frac{\omega^2 r^2}{c^2 \sin^2 \theta} \frac{1}{1 + \frac{\omega^2 r^2 \cot^2 \theta}{c^2}}\right) . \tag{3.38}$$

Respectively, according to the chronometrically invariant formalism (see §1.3 for the definitions of the chr.inv.-quantities), we obtain also the other chr.inv.-characteristics of the space. The chr.inv.-vector of the gravitational inertial force F_i, acting in the space takes the form

$$F_1 = -\frac{c^2 r_g}{a^3}\,\frac{r}{\left(3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}\right)\sqrt{1-\frac{r_g r^2}{a^3}}} < 0\,, \tag{3.39}$$

$$F^1 = -\frac{c^2 r_g}{a^3}\,\frac{r\sqrt{1-\frac{r_g r^2}{a^3}}}{3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}} < 0\,, \tag{3.40}$$

which is a non-Newtonian force of attraction. The approximate formula for the force is

$$F_1 = F^1 \approx -\frac{c^2 r_g r}{2a^3}\,. \tag{3.41}$$

The chr.inv.-tensor of the angular velocity of space, A_{ik}, has the following non-zero components:

$$A_{13} = \frac{2\omega r\cos\theta}{3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}}\,\times \tag{3.42}$$

$$\times\left[\frac{r_g r^2}{a^3}\,\frac{1}{\left(3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}\right)\sqrt{1-\frac{r_g r^2}{a^3}}}-1\right], \tag{3.43}$$

$$A^{13} = \frac{2\omega\left(1-\frac{r_g r^2}{a^3}\right)\cot\theta}{\left(3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}\right) r\sin\theta\left(1+\frac{\omega^2 r^2\cot^2\theta}{c^2}\right)}\,\times \tag{3.44}$$

$$\times\left[\frac{r_g r^2}{a^3}\,\frac{1}{\left(3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}\right)\sqrt{1-\frac{r_g r^2}{a^3}}}-1\right], \tag{3.45}$$

$$A_{23} = \frac{\omega r^2\sin\theta}{3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}}\,, \tag{3.46}$$

$$A^{23} = \frac{\omega}{\left(3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}\right) r^2\sin\theta\left(1+\frac{\omega^2 r^2\cot^2\theta}{c^2}\right)}\,. \tag{3.47}$$

The approximate expressions of these components have the form

$$A_{13} = -\omega r \cos\theta \left(1 + \frac{3r_g}{4a} - \frac{r_g r^2}{a^3}\right), \tag{3.48}$$

$$A^{13} = -\frac{\omega \cot\theta}{r \sin\theta} \left(1 + \frac{3r_g}{4a} - \frac{2r_g r^2}{a^3} - \frac{\omega^2 r^2 \cot^2\theta}{c^2}\right), \tag{3.49}$$

$$A_{23} = \frac{\omega r^2 \sin\theta}{2} \left(1 + \frac{3r_g}{4a} - \frac{r_g r^2}{4a^3}\right), \tag{3.50}$$

$$A^{23} = \frac{\omega}{2r^2 \sin\theta} \left(1 + \frac{3r_g}{4a} - \frac{r_g r^2}{4a^3} - \frac{\omega^2 r^2 \cot^2\theta}{c^2}\right). \tag{3.51}$$

The non-zero chr.inv.-Christoffel symbols for the metric, with higher-order terms $\omega^4 r^4/c^4$ neglected, have the form

$$\Delta^1_{11} = \frac{r_g r}{a^3} \frac{1}{1 - \frac{r_g r^2}{a^3}}, \qquad \Delta^1_{22} = -r\left(1 - \frac{r_g r^2}{a^3}\right), \tag{3.52}$$

$$\Delta^1_{33} = -r \sin^2\theta \left(1 + \frac{2\omega^2 r^2 \cot^2\theta}{c^2}\right)\left(1 - \frac{r_g r^2}{a^3}\right), \tag{3.53}$$

$$\Delta^2_{12} = \frac{1}{r}, \qquad \Delta^3_{13} = \frac{1}{r}\left(1 + \frac{\omega^2 r^2 \cot^2\theta}{c^2}\right), \tag{3.54}$$

$$\Delta^2_{33} = -\sin\theta\cos\theta\left(1 - \frac{\omega^2 r^2}{c^2}\right), \quad \Delta^3_{23} = \cot\theta\left(1 - \frac{\omega^2 r^2}{c^2 \sin^2\theta}\right). \tag{3.55}$$

Neglecting the terms r_g^2/a^2 and the product $\omega^2 r^2/c^2$ to r_g/a, we obtain the approximate formulae for these components

$$\Delta^1_{11} = \frac{r_g r}{a^3}, \qquad \Delta^1_{22} = -r\left(1 - \frac{r_g r^2}{a^3}\right), \tag{3.56}$$

$$\Delta^1_{33} = -r \sin^2\theta \left(1 + \frac{2\omega^2 r^2 \cot^2\theta}{c^2} - \frac{r_g r^2}{a^3}\right), \tag{3.57}$$

$$\Delta^2_{12} = \frac{1}{r}, \qquad \Delta^3_{13} = \frac{1}{r}\left(1 + \frac{\omega^2 r^2 \cot^2\theta}{c^2}\right), \tag{3.58}$$

$$\Delta^2_{33} = -\sin\theta\cos\theta\left(1 - \frac{\omega^2 r^2}{c^2}\right), \quad \Delta^3_{23} = \cot\theta\left(1 - \frac{\omega^2 r^2}{c^2 \sin^2\theta}\right), \tag{3.59}$$

and the non-zero components of the chr.inv.-curvature tensor C_{iklj}, along with the non-zero components of its contraction C_{ik}

$$C_{1212} = -\frac{r_g r^2}{a^3}\,\frac{1}{1-\frac{r_g r^2}{a^3}}\,, \tag{3.60}$$

$$C_{1313} = r^2\sin^2\theta\left(\frac{3\omega^2\cot^2\theta}{c^2} - \frac{r_g}{a^3}\,\frac{1+\frac{\omega^2 r^2\cot^2\theta}{c^2}}{1-\frac{r_g r^2}{a^3}}\right), \tag{3.61}$$

$$C_{2323} = \left[-\frac{r_g r^2}{a^3}\left(1+\frac{\omega^2 r^2\cot^2\theta}{c^2}\right) + \right.$$
$$\left. + \frac{\omega^2 r^2}{c^2}\left(\cot^2\theta + \frac{1}{\sin^4\theta}\right)\right] r^2\sin^2\theta\,, \tag{3.62}$$

$$C_{11} = -\frac{2r_g}{a^3}\,\frac{1}{1-\frac{r_g r^2}{a^3}} + \frac{3\omega^2\cot^2\theta}{c^2}\,, \tag{3.63}$$

$$C_{22} = -\frac{2r_g r^2}{a^3} + \frac{\omega^2 r^2\cot^2\theta}{c^2}\left(\cot^2\theta + \frac{1}{\sin^4\theta}\right), \tag{3.64}$$

$$C_{33} = \left[-\frac{2r_g}{a^3}\left(1+\frac{\omega^2 r^2\cot^2\theta}{c^2}\right) + \left(1-\frac{r_g r^2}{a^3}\right)\times\right.$$
$$\left.\times\,\frac{3\omega^2\cot^2\theta}{c^2} + \frac{\omega^2}{c^2}\left(\cot^2\theta + \frac{1}{\sin^4\theta}\right)\right] r^2\sin^2\theta\,. \tag{3.65}$$

Because $r_g/a \ll 1$ and $\omega^2 r^2/c^2 \ll 1$ for regular stars, we neglect the terms r_g^2/a^2 and the product $\omega^2 r^2/c^2$ to r_g/a. As a result, we obtain the approximate formulae for the chr.inv.-curvature:

$$C_{1212} = -\frac{r_g r^2}{a^3}\,, \quad C_{1313} = r^2\sin^2\theta\left(\frac{3\omega^2\cot^2\theta}{c^2} - \frac{r_g}{a^3}\right), \tag{3.66}$$

$$C_{2323} = \left[-\frac{r_g r^2}{a^3} + \frac{\omega^2 r^2}{c^2}\left(\cot^2\theta + \frac{1}{\sin^4\theta}\right)\right] r^2\sin^2\theta\,, \tag{3.67}$$

$$C_{11} = -\frac{2r_g}{a^3} + \frac{3\omega^2\cot^2\theta}{c^2}\,, \tag{3.68}$$

$$C_{22} = -\frac{2r_g r^2}{a^3} + \frac{\omega^2 r^2\cot^2\theta}{c^2}\left(\cot^2\theta + \frac{1}{\sin^4\theta}\right), \tag{3.69}$$

$$C_{33} = \left(-\frac{2r_g}{a^3} + \frac{4\omega^2\cot^2\theta}{c^2} + \frac{\omega^2}{c^2\sin^4\theta}\right) r^2\sin^2\theta\,. \tag{3.70}$$

§3.4 Einstein's equations in the internal field of a regular rotating star

We now solve Einstein's field equations in the internal space of a rotating regular star, i.e. in the space of the metric (3.28). In the absence of space rotation ($A_{ik}=0$), this problem reduces to the problem that was considered earlier in §5.2 for regular non-rotating star.

So, consider the chr.inv.-Einstein equations (1.85–1.87) in the space of a liquid sphere, which rotates ($A_{ik}\neq 0$) but is free of deformation ($D_{ik}=0$). They take the form

$$A_{jl}A^{lj} + {}^*\nabla_j F^j - \frac{1}{c^2}F_j F^j = -\frac{\varkappa}{2}\left(\rho c^2 + U\right), \tag{3.71}$$

$$-\nabla_j A^{ij} + \frac{2}{c^2}F_j A^{ij} = \varkappa J^i, \tag{3.72}$$

$$2A_{ij}A_{k\cdot}^{\cdot j} + \frac{1}{2}\left({}^*\nabla_i F_k + {}^*\nabla_k F_i\right) - \frac{1}{c^2}F_i F_k - c^2 C_{ik} = \\ = \frac{\varkappa}{2}\left(\rho c^2 h_{ik} + 2U_{ik} - U h_{ik}\right), \tag{3.73}$$

where ${}^*\nabla_i$ is the symbol of chr.inv.-differentiation (see *Notations*). The chr.inv.-quantities ρ, J^i, U^{ik} are the physical observable projections of the energy-momentum tensor $T_{\alpha\beta}$ of the medium that fills the space. (We do not specify the energy-momentum tensor just yet.)

With the obtained components of A_{ik} and F_i (see §3.2 for detail), the chr.inv.-Einstein equations (3.71–3.73) take the form

$$2\omega^2\cot^2\theta\left(1+\frac{3r_g}{2a}-\frac{3r_g r^2}{a^3}-\frac{\omega^2 r^2\cot^2\theta}{c^2}\right)+\frac{\omega^2}{2}\left(1+\frac{3r_g}{2a}-\right. \\ \left. -\frac{r_g r^2}{2a^3}-\frac{\omega^2 r^2\cot^2\theta}{c^2}\right)+\frac{3c^2 r_g}{2a^3}=\frac{\varkappa}{2}\left(\rho c^2+U\right), \tag{3.74}$$

$$\frac{\omega\cot\theta}{r^2\sin\theta}\left(1+\frac{3r_g}{4a}-\frac{4r_g r^2}{a^3}-\frac{3\omega^2 r^2\cot^2\theta}{c^2}\right)=-\varkappa J^3, \tag{3.75}$$

$$2\omega^2\cot^2\theta\left(1+\frac{3r_g}{2a}-\frac{2r_g r^2}{a^3}-\frac{\omega^2 r^2\cot^2\theta}{c^2}\right)+\frac{3c^2 r_g}{2a^3}= \\ =\frac{\varkappa}{2}\left[\left(\rho c^2-U\right)\left(1-\frac{r_g r^2}{a^3}\right)+2U_{11}\right], \tag{3.76}$$

$$\omega^2 r\cot\theta\left(1+\frac{3r_g}{2a}-\frac{5r_g r^2}{4a^3}-\frac{\omega^2 r^2\cot^2\theta}{c^2}\right)=-\varkappa U_{12}, \tag{3.77}$$

$$\frac{\omega^2 r^2}{2}\left(1+\frac{3\,r_g}{2\,a}-\frac{r_g r^2}{2\,a^3}-\frac{\omega^2 r^2\cot^2\theta}{c^2}\right)+\frac{3\,c^2 r_g r^2}{2\,a^3}= \\ =\frac{\varkappa}{2}\Big[\big(\rho c^2-U\big)\,r^2+2U_{22}\Big], \tag{3.78}$$

$$\left[2\omega^2\cot^2\theta\left(1+\frac{3\,r_g}{2\,a}-\frac{3\,r_g r^2}{a^3}\right)+\frac{\omega^2}{2}\left(1+\frac{3\,r_g}{2\,a}-\frac{r_g r^2}{2\,a^3}\right)\right]r^2\sin^2\theta+ \\ +\frac{3\,c^2 r_g r^2\sin^2\theta}{a^3}=\frac{\varkappa}{2}\Big[\big(\rho c^2-U\big)\,r^2\sin^2\theta+2U_{33}\Big]. \tag{3.79}$$

In the framework of our approximation ($r_g/a \ll 1$ and $\omega^2 r^2/c^2 \ll 1$) which holds true for regular stars, we neglect the terms r_g^2/a^2 and the product $\omega^2 r^2/c^2$ to r_g/a. As a result, the obtained chr.inv.-Einstein equations take the much simplified form

$$2\omega^2\cot^2\theta+\frac{\omega^2}{2}=\frac{\varkappa}{2}\big(\rho c^2+U\big), \tag{3.80}$$

$$\frac{\omega\cot\theta}{r^2\sin\theta}=-\varkappa J^3, \tag{3.81}$$

$$2\omega^2\cot^2\theta-\frac{\varkappa}{2}\big(\rho c^2-U\big)=\varkappa U_1^1\,, \tag{3.82}$$

$$\omega^2 r\cot\theta=-\varkappa U_{12}\,, \tag{3.83}$$

$$\frac{\omega^2}{2}-\frac{\varkappa}{2}\big(\rho c^2-U\big)=\varkappa U_2^2\,, \tag{3.84}$$

$$2\omega^2\cot^2\theta+\frac{\omega^2}{2}-\frac{\varkappa}{2}\big(\rho c^2-U\big)=\varkappa U_3^3\,. \tag{3.85}$$

Summarizing (3.82), (3.84), and (3.85), then taking into account that $U_1^1+U_2^2+U_3^3=U$, we obtain

$$4\omega^2\cot^2\theta+\omega^2=\frac{\varkappa}{2}\big(3\,\rho c^2-U\big). \tag{3.86}$$

Summarizing the result (3.86) and (3.80), we obtain

$$3\omega^2\cot^2\theta+\frac{3\omega^2}{4}=\varkappa\rho c^2. \tag{3.87}$$

Multiplying (3.80) by 3 then subtracting the result from (3.86), we obtain

$$\omega^2\cot^2\theta+\frac{\omega^2}{4}=\varkappa U\,. \tag{3.88}$$

As is seen from (3.87) and (3.88),

$$\rho c^2 = 3U\,. \tag{3.89}$$

The energy-momentum tensor $T_{\alpha\beta}$ should also satisfy the conservation equations. The chr.inv.-conservation equation in the rotating space of the star, which is free of deformation ($D_{ik} = 0$), have the form

$$\frac{^*\partial\rho}{\partial t} + {}^*\nabla_i J^i - \frac{2}{c^2}\,F_i J^i = 0\,, \tag{3.90}$$

$$\frac{^*\partial J^k}{\partial t} + 2A_{i\cdot}^{\cdot k}J^i + {}^*\nabla_i U^{ik} - \frac{1}{c^2}\,F_i U^{ik} - \rho F^k = 0\,. \tag{3.91}$$

As follows from the scalar conservation equation (3.90),

$$\frac{^*\partial\rho}{\partial t} = 0\,, \quad \text{i.e.} \quad \rho = const. \tag{3.92}$$

The vectorial conservation equation (3.91) with the index $i = 3$ is satisfied identically. The equations with $i = 1$ and $i = 2$ take the form

$$2A_{3\cdot}^{\cdot 1}J^3 + \frac{\partial U^{11}}{\partial r} + \frac{\partial U^{12}}{\partial\theta} + \frac{\partial\ln\sqrt{h}}{\partial\theta}\,U^{12} + \Delta_{22}^1 U^{22} + \\ + \Delta_{33}^1 U^{33} + \left(\Delta_{11}^1 + \frac{\partial\ln\sqrt{h}}{\partial r} - \frac{1}{c^2}\,F_1\right)U^{11} = \rho F^1, \tag{3.93}$$

$$2A_{3\cdot}^{\cdot 2}J^3 + \frac{\partial U^{12}}{\partial r} + \frac{\partial U^{22}}{\partial\theta} + \frac{\partial\ln\sqrt{h}}{\partial\theta}\,U^{22} + \\ + \Delta_{33}^2 U^{33} + \left(2\Delta_{12}^2 + \frac{\partial\ln\sqrt{h}}{\partial r} - \frac{1}{c^2}\,F_1\right)U^{12} = 0\,. \tag{3.94}$$

In the tensorial chr.inv.-Einstein equations that readily express U_1^1 (3.82), U_{12} (3.83), U_2^2 (3.84), and U_3^3 (3.85), we now take into account that $\rho c^2 = 3U$ (3.89) and the formula for U (3.88). We obtain

$$\varkappa U^{11} = \omega^2\cot^2\theta - \frac{\omega^2}{4}\,, \tag{3.95}$$

$$\varkappa U^{12} = -\frac{\omega^2\cot\theta}{r}\,, \tag{3.96}$$

$$\varkappa U^{22} = \frac{\omega^2}{4r^2} - \frac{\omega^2\cot^2\theta}{r^2}\,, \tag{3.97}$$

$$\varkappa U^{33} = \frac{\omega^2\cot^2\theta}{r^2\sin^2\theta} + \frac{\omega^2}{4r^2\sin^2\theta}\,. \tag{3.98}$$

Substitute these formulae, and also the other necessary quantities, into the remaining conservation equations (3.93) and (3.94). After some algebra, we see that the equations are satisfied identically.

So, Einstein's equations and the conservation equations in the form considered herein satisfy the metric of the internal space of a regular rotating star, that is the space metric (3.28).

§3.5 The stationary vortex-free electromagnetic field of a regular rotating star

A realistic star bears an electromagnetic field. Therefore, we should introduce the electromagnetic field into the theory of liquid stars. Electrodynamics in terms of the chronometrically invariant formalism was introduced in Chapter 3 of our book [18]. We now follow the deduction therefrom, then apply it to the present theory of liquid stars.

So, as is known from the generally covariant formulation of electrodynamics [20], the energy-momentum tensor of an arbitrary electromagnetic field has the form

$$T^{\alpha\beta}_{\text{em}} = \frac{1}{4\pi c^2}\left(-F^{\alpha\cdot}_{\cdot\sigma}F^{\beta\sigma} + \frac{1}{4}\,g^{\alpha\beta}F_{\mu\sigma}F^{\mu\sigma}\right), \tag{3.99}$$

where $F_{\alpha\beta}$ is the electromagnetic field tensor known also as the Maxwell tensor. The field tensor $F_{\alpha\beta}$ is the curl of the four-dimensional electromagnetic potential A^{α}

$$F_{\alpha\beta} = \nabla_{\alpha}A_{\beta} - \nabla_{\beta}A_{\alpha} = \frac{\partial A_{\beta}}{\partial x^{\alpha}} - \frac{\partial A_{\alpha}}{\partial x^{\beta}}\,. \tag{3.100}$$

The physically observable chr.inv.-projections of the four-dimensional electromagnetic potential A^{α} are the scalar electromagnetic potential φ and the vector electromagnetic potential q^i:

$$\varphi = \frac{A_0}{\sqrt{g_{00}}}\,, \qquad q^i = A^i. \tag{3.101}$$

The electromagnetic field tensor $F_{\alpha\beta}$ (5.44) has the following physically observable components

$$\rho_{\text{em}} = \frac{T_{00}}{g_{00}} = \frac{E_i E^i + H_{*i}H^{*i}}{8\pi c^2}\,, \tag{3.102}$$

$$J^i_{\text{em}} = \frac{c\,T^i_0}{\sqrt{g_{00}}} = \frac{1}{4\pi c}\,\varepsilon^{ikm}E_k H_{*m}\,, \tag{3.103}$$

$$U^{ik}_{\text{em}} = c^2 T^{ik} = \rho_{\text{em}}c^2 h^{ik} - \frac{1}{4\pi}\left(E^i E^k + H^{*i}H^{*k}\right), \tag{3.104}$$

where E^i is the three-dimensional chr.inv.-electric strength vector, H^{*i} is the three-dimensional chr.inv.-magnetic strength pseudo-vector of the field, while ε^{imn} is the completely anti-symmetric unitary three-dimensional chr.inv.-pseudo-tensor (see [18], for detail). Namely, —

$$\left.\begin{aligned} E^{*ik} &= -\varepsilon^{ikn}E_n\,, \qquad E_n = \frac{{}^*\partial\varphi}{\partial x^n} + \frac{1}{c}\frac{{}^*\partial q_n}{\partial t} - \frac{\varphi}{c^2}\,F_n \\ H^{*i} &= \frac{1}{2}\,\varepsilon^{imn}H_{mn}\,, \quad H_{mn} = \frac{{}^*\partial q_m}{\partial x^n} - \frac{{}^*\partial q_n}{\partial x^m} - \frac{2\varphi}{c}\,A_{mn} \end{aligned}\right\}. \tag{3.105}$$

As is seen from the definitions (3.105), the chr.inv.-electric field strength and the chr.inv.-magnetic field strength depend on not only the electromagnetic field potentials φ and q^i, but also on the characteristics of the space that is filled with the field. The physical and geometric characteristics of the space that affect the electric and magnetic field strengths are, respectively, the gravitational inertial force F_i acting in the space, and the angular velocity of the space rotation A_{ik}.

Herein, in §5.3, we first consider the electromagnetic field which is stationary and vortex-free. This means that the electromagnetic field potentials are constant ($\varphi = const$, $q_i = const$) while the curl of the vectorial potential is zero ($q_{ik} = 0$). In other words, in the further consideration of the electromagnetic field we will assume that

$$\frac{{}^*\partial\varphi}{\partial x^n} = 0\,, \qquad \frac{{}^*\partial q_n}{\partial t} = 0\,, \qquad q_{ik} = \frac{{}^*\partial q_i}{\partial x^k} - \frac{{}^*\partial q_k}{\partial x^i} = 0 \tag{3.106}$$

in the formulae for the electric and magnetic field strengths (3.105). In this case, we have

$$\left.\begin{aligned} E^i &= -\frac{\varphi}{c^2}\,F^i, \qquad E_i = -\frac{\varphi}{c^2}\,F_i \\ H^{*i} &= -\frac{2\varphi}{c}\,\Omega^{*i}\,, \quad H_{*i} = -\frac{2\varphi}{c}\,\Omega_{*i} \end{aligned}\right\}, \tag{3.107}$$

where Ω^{*i} is the three-dimensional chr.inv.-pseudo-vector of the angular velocity of space rotation

$$\Omega^{*i} = \frac{1}{2}\,\varepsilon^{imn}A_{mn}\,, \qquad \Omega_{*i} = \frac{1}{2}\,\varepsilon_{imn}A^{mn}. \tag{3.108}$$

It is easy to find that, in the internal space of a regular rotating star, that is the space of the metric (3.28), we have

$$\Omega^{*1} = \Omega_{*1} = \frac{\omega}{2}\,, \quad \Omega^{*2} = \frac{\omega\cot\theta}{r}\,, \quad \Omega_{*2} = \omega\,r\cot\theta\,. \tag{3.109}$$

thus

$$\Omega_{*j}\Omega^{*j} = \omega^2\left(\frac{1}{4} + \cot^2\theta\right). \tag{3.110}$$

As is seen from the formulae (3.107), in the stationary vortex-free electromagnetic field the electric field strength E^i is determined by the scalar electromagnetic potential φ and the gravitational inertial force F^i acting in the space. The magnetic field strength H^{*i} is due to the vectorial electromagnetic potential q^i and the angular velocity Ω^{*i} of rotation of space.

Using these formulae for E^i and H^{*i} (3.107), we obtain that the physically observable components (3.102–3.104) of the electromagnetic field tensor $F_{\alpha\beta}$ have the form

$$\rho_{\text{em}} = \frac{\varphi^2}{2\pi c^4}\left(\frac{F_j F^j}{4c^2} + \Omega_{*j}\Omega^{*j}\right), \tag{3.111}$$

$$J^i_{\text{em}} = \frac{\varphi^2}{2\pi c^4}\,\varepsilon^{ikm}F_k\,\Omega_{*m}\,, \tag{3.112}$$

$$U^{ik}_{\text{em}} = \frac{\varphi^2}{2\pi c^2}\left(\frac{F_j F^j}{4c^2} + \Omega_{*j}\Omega^{*j}\right)h^{ik} - \frac{\varphi^2}{\pi c^2}\left(\frac{F^i F^k}{4c^2} + \Omega^{*i}\Omega^{*k}\right). \tag{3.113}$$

It is easy to see that the trace of the electromagnetic field stress tensor, which is expressed as $U_{\text{em}} = h_{ik}U^{ik}_{\text{em}}$, satisfies the condition

$$U_{\text{em}} = \frac{\varphi^2}{2\pi c^2}\left(\frac{F_j F^j}{4c^2} + \Omega_{*j}\Omega^{*j}\right) = \rho_{\text{em}}c^2. \tag{3.114}$$

As follows from the general form of the energy-momentum tensor $T_{\alpha\beta}$ that satisfies the metric (3.28), the tensor should satisfy the condition $\rho c^2 = 3U$ (3.89). This formula differs from $\rho_{\text{em}}c^2 = U_{\text{em}}$ (3.114) we have just obtained for the stationary vortex-free electromagnetic field. We therefore should find such a structure of the electromagnetic field that makes U_{em} satisfying $\rho c^2 = 3U$.

So, according to the condition $\rho c^2 = 3U$ (3.89) and the formula $\omega^2\cot^2\theta + \frac{1}{4}\omega^2 = \varkappa U$ (3.88) obtained from the chr.inv.-Einstein equations, the field density inside a regular rotating star should be

$$\rho = \frac{3\,\Omega_{*j}\Omega^{*j}}{\varkappa c^2}\,, \qquad U = \frac{\Omega_{*j}\Omega^{*j}}{\varkappa}\,. \tag{3.115}$$

Therefore, we substitute the required condition

$$U_{\text{em}} = \frac{\Omega_{*j}\Omega^{*j}}{\varkappa}\,. \tag{3.116}$$

into (3.114) that we obtained for the stationary vortex-free electromagnetic field. We obtain

$$\frac{\varphi^2}{2\pi c^2}\left(\frac{F_j F^j}{4c^2} + \Omega_{*j}\Omega^{*j}\right) = \frac{\Omega_{*j}\Omega^{*j}}{\varkappa}\,, \tag{3.117}$$

or, expanding Einstein's constant of gravitation $\varkappa = \frac{8\pi G}{c^2}$, in the equivalent form

$$c^2\,\Omega_{*j}\Omega^{*j} = \frac{\frac{G\varphi^2}{c^4}}{1-\frac{4G\varphi^2}{c^4}}\,F_j F^j. \tag{3.118}$$

We have considered a stationary electromagnetic field: the scalar and vectorial electromagnetic potentials remain unchanged ($\varphi = const$, $q_i = const$). Therefore, in the formula (3.118), the quantity

$$\frac{G\varphi^2}{c^4} = n\,, \qquad n < \frac{1}{4} \tag{3.119}$$

is a constant dimensionless coefficient depending only on the scalar potential φ of the electromagnetic field of the star.

Using the constant n (3.119), we re-write (3.118) as

$$c^2\,\Omega_{*j}\Omega^{*j} = \frac{n}{1-4n}\,F_i F^i, \quad n < \frac{1}{4}\,. \tag{3.120}$$

Substituting $\Omega_j\Omega^j$ (5.13) and $F_i F^i$ (2.15) into the condition (3.120), we can present it in the alternative (expanded) form

$$\omega^2\left(1+4\cot^2\theta\right) = \frac{4n}{1-4n}\frac{c^2 r_g^2 r^2}{a^6}\,. \tag{3.121}$$

If $n = n_{\max} = \frac{1}{4}$ and thus $\varphi = \varphi_{\max}$, the angular velocity of the star would be $\omega = \infty$ which is non-sense. Therefore, we conclude that $n < \frac{1}{4}$ for all real stars including the Sun. With $n < \frac{1}{4}$, we obtain the upper boundary of the numerical value for the scalar electromagnetic potential possessed by a real star

$$\varphi = \frac{c^2}{2\sqrt{G}} < 1.74\times10^{24}\ \left[\frac{\text{gram}^{1/2}\,\text{cm}^{1/2}}{\text{sec}}\right]. \tag{3.122}$$

With the condition (3.120), the stationary vortex-free electromagnetic field satisfies Einstein's equations and the metric of the internal space of a regular rotating star. In other words, with this condition (3.120), a stationary rotating regular star is a *permanent magnet.*

Because $n = const$ in the stationary electromagnetic field (wherein $\varphi = const$, $q_i = const$), the above condition (3.120) allows us to express the characteristics of the electromagnetic field through the geometric and physical characteristics of the space. In other words, in this particular case, we can geometrize the electromagnetic field.

Substitute the obtained formulae for the gravitational inertial force, and for the angular velocity of space rotation into the physically observable components (3.111–3.113) of the electromagnetic field tensor $F_{\alpha\beta}$. By taking the relations (3.117) and (3.119) into account, we obtain the observable components of $F_{\alpha\beta}$ in the form

$$\rho_{\text{em}} = \frac{n}{2\pi G}\left[\frac{c^2 r_g^2 r^2}{16a^6} + \omega^2\left(\frac{1}{4} + \cot^2\theta\right)\right], \tag{3.123}$$

$$J^3_{\text{em}} = -\frac{nc^2}{4\pi G}\,\frac{\omega r_g}{a^3}\,\frac{\cot\theta}{\sin\theta}\,, \tag{3.124}$$

$$U^{11}_{\text{em}} = -\frac{nc^2}{2\pi G}\left[\frac{c^2 r_g^2 r^2}{16a^6} + \omega^2\left(\frac{1}{4} - \cot^2\theta\right)\right], \tag{3.125}$$

$$U^{12}_{\text{em}} = -\frac{nc^2}{2\pi G}\,\frac{\omega^2\cot\theta}{r}\,, \tag{3.126}$$

$$U^{22}_{\text{em}} = \frac{nc^2}{2\pi G r^2}\left[\frac{c^2 r_g^2 r^2}{16a^6} + \omega^2\left(\frac{1}{4} - \cot^2\theta\right)\right], \tag{3.127}$$

$$U^{33}_{\text{em}} = \frac{nc^2}{2\pi G r^2\sin^2\theta}\left[\frac{c^2 r_g^2 r^2}{16a^6} + \omega^2\left(\frac{1}{4} + \cot^2\theta\right)\right]. \tag{3.128}$$

From these equations we obtain, as previously,

$$U_{\text{em}} = h_{ik}U^{ik}_{\text{em}} = \rho_{\text{em}}c^2. \tag{3.129}$$

The chr.inv.-Einstein equations (3.123–3.128) can further be simplified. In the surface layer of a star ($r \approx a$), the first term in the brackets is

$$\frac{c^2 r_g^2 r^2}{16a^6} \simeq \frac{c^2 r_g^2}{16a^4}\,. \tag{3.130}$$

Consider the Sun as an example. Its surface layer makes one full revolution with a period of 27 days, which is equivalent to the angular velocity of rotation $\omega_\odot \simeq 2.7\times10^{-6}$ sec^{-1}. Therefore, the second term in the brackets is

$$\frac{1}{4}\,\omega_\odot^2 \simeq 1.8\times10^{-12}\ \text{sec}^{-2}. \tag{3.131}$$

The first term in the brackets, while taking the Hilbert radius for the Sun $r_{g\odot} = 2.9\times10^{5}$ cm, and the Sun's physical radius $a_{\odot} = 7.0\times10^{10}$ cm into account, is ten times less:

$$\frac{c^2 r_{g\odot}^2}{16\,a_{\odot}^4} \simeq 2.0\times10^{-13}\ \text{sec}^{-2}. \tag{3.132}$$

Therefore, we can neglect the first term in the brackets for even slowly rotating stars such as the Sun. Thus the chr.inv.-Einstein equations (3.123–3.128) take the simplified form

$$\rho_{\text{em}} = \frac{n\omega^2}{2\pi G}\left(\frac{1}{4} + \cot^2\theta\right), \tag{3.133}$$

$$J^3_{\text{em}} = -\frac{n\,c^2}{4\pi G}\,\frac{\omega\,r_g}{a^3}\,\frac{\cot\theta}{\sin\theta}\,, \tag{3.134}$$

$$U^{11}_{\text{em}} = -\frac{n\,c^2\omega^2}{2\pi G}\left(\frac{1}{4} - \cot^2\theta\right), \tag{3.135}$$

$$U^{12}_{\text{em}} = -\frac{n\,c^2}{2\pi G}\,\frac{\omega^2\cot\theta}{r}\,, \tag{3.136}$$

$$U^{22}_{\text{em}} = \frac{n\,c^2\omega^2}{2\pi G r^2}\left(\frac{1}{4} - \cot^2\theta\right), \tag{3.137}$$

$$U^{33}_{\text{em}} = \frac{n\,c^2\omega^2}{2\pi G r^2\sin^2\theta}\left(\frac{1}{4} + \cot^2\theta\right). \tag{3.138}$$

Nevertheless, the first term in the brackets of the chr.inv.-Einstein equations (3.123–3.128) can be sufficient in the case of small stars such as white dwarfs or brown dwarfs. This is because we have a^6 (the star's radius in the 6th power) in the denominator of the term.

§3.6 Solving Maxwell's equations in the vortex-free electromagnetic field of a regular rotating star

As is known, the electromagnetic field is described by Maxwell's field equations. They consist of two groups of equations. The generally covariant formulation of Maxwell's equations in the four-dimensional pseudo-Riemannian space has the form [20]

$$\nabla_{\sigma} F^{\mu\sigma} = \frac{4\pi}{c}\,j^{\mu}, \qquad \nabla_{\sigma} F^{*\mu\sigma} = 0\,, \tag{3.139}$$

where the first generally covariant equation expresses Group I of Maxwell's equations, while the second generally covariant equation represents Group II. Herein, $F^{*\mu\sigma} = \varepsilon^{\mu\sigma\alpha\beta} F_{\alpha\beta}$ is the pseudo-tensor which is dual to the electromagnetic field tensor $F_{\alpha\beta}$, while j^μ is the four-dimensional current vector.

In terms of the chronometrically invariant formalism of General Relativity, the generally covariant Maxwell equations (3.139) take the following form (see Chapter 3 of the book [18] for details):

$$\left.\begin{aligned} &{}^*\nabla_j E^j - \frac{1}{c} H^{ik} A_{ik} = 4\pi\rho \\ &{}^*\nabla_k H^{ik} - \frac{1}{c^2} F_k H^{ik} - \frac{1}{c}\left(\frac{^*\partial E^i}{\partial t} + DE^i\right) = \frac{4\pi}{c}\, j^i \end{aligned}\right\} \text{I}, \tag{3.140}$$

$$\left.\begin{aligned} &{}^*\nabla_i H^{*i} - \frac{1}{c} E^{*ik} A_{ik} = 0 \\ &{}^*\nabla_k E^{*ik} - \frac{1}{c^2} F_k E^{*ik} - \frac{1}{c}\left(\frac{^*\partial H^{*i}}{\partial t} + DH^{*i}\right) = 0 \end{aligned}\right\} \text{II}. \tag{3.141}$$

Herein, $E^{*ik} = -\varepsilon^{ikn} E_k$ is the pseudo-tensor which is dual to the electric strength tensor E_i, $H^{*i} = \frac{1}{2}\varepsilon^{imn} H_{mn}$ is the pseudo-vector which is dual to the magnetic strength tensor H_{mn}, while $D = h^{ik} D_{ik}$ is the rate of space deformation. (See these definitions in formula (3.105) of §3.5).

The physically observable charge density ρ and the physically observable current vector j^i are the respective chr.inv.-projections of the four-dimensional current vector j^μ

$$\rho = \frac{1}{c}\frac{j_0}{\sqrt{g_{00}}}\,, \qquad j^i = h^i_\mu\, j^\mu. \tag{3.142}$$

Because the space of a liquid sphere under consideration is stationary (the space metric does not depend on time), and also because we assume that the electromagnetic field is stationary, all terms containing the space deformation tensor D_{ik} and the time derivatives of the electromagnetic field strengths vanish. In this particular case, the chr.inv.-Maxwell equations (3.140–3.141) take the simplified form

$$\left.\begin{aligned} &{}^*\nabla_j E^j - \frac{1}{c} H^{ik} A_{ik} = 4\pi\rho \\ &{}^*\nabla_k H^{ik} - \frac{1}{c^2} F_k H^{ik} = \frac{4\pi}{c}\, j^i \end{aligned}\right\} \text{I}, \tag{3.143}$$

$$\left.\begin{aligned} &{}^*\nabla_i H^{*i} - \frac{1}{c} E^{*ik} A_{ik} = 0 \\ &{}^*\nabla_k E^{*ik} - \frac{1}{c^2} F_k E^{*ik} = 0 \end{aligned}\right\} \text{II}. \tag{3.144}$$

Substitute, into the equations (3.143–3.144), the formulae for the gravitational inertial force F_i (3.41) and the angular velocity of space rotation A_{ik} (3.48–3.51) we have obtained in the framework of the space metric of a regular rotating star (3.28). Also, substitute the formulae for the electric strength E^i and the magnetic strength H^{*i} (3.107) of the stationary vortex-free electromagnetic field we have obtained in the space. To simplify the algebra, take into account the following terms for the case if they appear in the brackets commonly with the others. For the Sun ($\omega_\odot \simeq 2.7\times10^{-6}$ sec^{-1}, $r_{g\odot} = 2.9\times10^5$ cm, $a_\odot = 7.0\times10^{10}$ cm), these terms take the following numerical values:

$$\left.\begin{aligned} \frac{r_g}{a} &= 4.1\times10^{-6} \\ \frac{r_g}{a^3} &= 8.5\times10^{-28}\ \text{cm}^{-2} \\ \frac{\omega^2}{c^2} &= 8.1\times10^{-33}\ \text{cm}^{-2} \end{aligned}\right\}. \tag{3.145}$$

For other regular stars, these terms take similar numerical values to within a few orders of magnitude.

As a result, after some algebra we see that the equations of Group II of the chr.inv.-Maxwell-equations (3.143–3.144) vanish, while the equations of Group I take the following form:

$$\frac{3\varphi r_g}{2a^3} = 4\pi\rho, \qquad \frac{\omega\varphi\cot\theta}{r^2\sin\theta} = -2\pi j^3. \tag{3.146}$$

The obtained Maxwell equations characterize the electromagnetic field that originates due to the charges and currents (the field sources). Should the right-hand side of the equations be zero, it would be a source-free electromagnetic field (existing independently of the sources).

The field sources (the charge density ρ and the current vector j^i) are connected to each other through the law of conservation of electric charge. The law has the following general covariant formulation:

$$\nabla_\sigma j^\sigma = 0\,. \tag{3.147}$$

The generally covariant law of conservation of the electric charge (3.147) is also known as the continuity equation. It means that the charge density ρ and the currents j^i, which are two chr.inv.-observable projections of the four-dimensional current vector j^α are conserved within the four-dimensional volume of the field. Maxwell's equations are connected by the generally covariant Lorentz condition

$$\nabla_\sigma A^\sigma = 0\,, \tag{3.148}$$

which means that the scalar electromagnetic potential φ and the vectorial electromagnetic potential q^i (the chr.inv.-observable projections of the four-dimensional potential A^σ of the electromagnetic field) are conserved within the four-dimensional volume of the field.

In the general case of an arbitrary electromagnetic field, the conservation law $\nabla_\sigma j^\sigma = 0$ (3.147) and the Lorentz condition $\nabla_\sigma A^\sigma = 0$ (3.148) have the following chronometrically invariant formulation:

$$\frac{{}^*\partial\rho}{\partial t} + \rho D + {}^*\widetilde{\nabla}_i j^i - \frac{1}{c^2}F_i j^i = 0\,, \tag{3.149}$$

$$\frac{1}{c}\frac{{}^*\partial\varphi}{\partial t} + \frac{\varphi}{c}D + {}^*\widetilde{\nabla}_i q^i - \frac{1}{c^2}F_i q^i = 0\,. \tag{3.150}$$

See Chapter 3 of the book [18] for details. The chr.inv.-differential operators ${}^*\widetilde{\nabla}_i = {}^*\nabla_i - \frac{1}{c^2}F_i$ and ${}^*\nabla_i$ can be found in *Notations*.

Recall that we are considering a stationary vortex-free electromagnetic field. This means that the conditions (3.106) should be true

$$\frac{{}^*\partial\varphi}{\partial x^n} = 0\,, \qquad \frac{{}^*\partial q_n}{\partial t} = 0\,, \qquad q_{ik} = \frac{{}^*\partial q_i}{\partial x^k} - \frac{{}^*\partial q_k}{\partial x^i} = 0\,. \tag{3.151}$$

In this particular case, and in the space which is free of deformation ($D_{ik} = 0$), the conservation equation (3.149) and the Lorentz condition (3.149) are satisfied as identities.

§3.7 Solving Maxwell's equations in the vortical electromagnetic field of a regular rotating star

Consider a regular rotating star whose electromagnetic field is vortical. This means that the curl q_{ik} of the three-dimensional vectorial chr.inv.-potential q_i of the field is non-zero

$$q_{ik} = \frac{{}^*\partial q_i}{\partial x^k} - \frac{{}^*\partial q_k}{\partial x^i} \neq 0\,. \tag{3.152}$$

Assume, according to in Chapter 3 of the book [18] where we considered relativistic electrodynamics, that the four-dimensional electromagnetic field potential A^α has the form

$$A^\alpha = \varphi\frac{dx^\alpha}{ds}\,, \qquad g_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 1. \tag{3.153}$$

In this case, the chr.inv.-projections of A^α have the form

$$\frac{A_0}{\sqrt{g_{00}}} = \widetilde{\varphi}, \quad A^i = q^i = \frac{\widetilde{\varphi}}{c}\,\mathrm{v}^i, \quad \mathrm{v}^i = \frac{dx^i}{d\tau}\,, \tag{3.154}$$

where

$$\widetilde{\varphi} = \frac{\varphi}{\sqrt{1 - \frac{\mathrm{v}^2}{c^2}}}\,, \qquad \mathrm{v}^2 = h_{ik}\,\mathrm{v}^{\mathrm{i}}\mathrm{v}^{\mathrm{k}} \tag{3.155}$$

is the relativistic scalar potential of the electromagnetic field [18]. We assume that the charges move within the star at small velocities ($\mathrm{v}^2 \ll c^2$). Then $\widetilde{\varphi} = \varphi$.

We also assume, according to our consideration of a rotating neutron star, that $\varphi = const$ and $q^1 = q^2 = 0$. Then the chr.inv.-components of A^α take the form

$$\varphi = const\,, \qquad q^3 = \frac{\varphi}{c}\,\mathrm{v}^3, \tag{3.156}$$

where $\mathrm{v}^3 = \frac{d\phi}{d\tau}$, while ϕ is the equatorial coordinate of the spherical polar coordinates r, θ, ϕ. Assuming that the electromagnetic field curl is due to the angular rotation of the star, we have

$$\frac{d\phi}{d\tau} = \omega\,. \tag{3.157}$$

Then the non-zero components of the three-dimensional vector chr.inv.-potential q_i and its curl q_{ik} have the form

$$q^3 = \frac{\varphi\omega}{c}\,, \tag{3.158}$$

$$q_3 = \frac{\varphi\omega}{c}\,r^2\sin^2\theta\,, \tag{3.159}$$

$$q_{31} = \frac{\partial q_3}{\partial r} = \frac{2\varphi\omega}{c}\,r\sin^2\theta\,, \tag{3.160}$$

$$q_{23} = -\frac{\partial q_3}{\partial \theta} = -\frac{2\varphi\omega}{c}\,r^2\sin\theta\cos\theta\,. \tag{3.161}$$

In other words, we consider here only circular motion of the charge along the equatorial coordinate ϕ, which is the geographical longitude of the star (in spherical polar coordinates r, θ, ϕ).

Substitute now the formulae for A_{ik} (3.48–3.51) obtained for the space metric of a regular rotating star (3.28) into the definition of the magnetic strength tensor H_{ik} (3.105). We obtain that the non-zero components of the tensor have the form

$$H_{23} = -\frac{2\varphi\omega r^2\sin\theta}{c}\left[\cos\theta + \frac{1}{2}\left(1 + \frac{3r_g}{4a} - \frac{r_g r^2}{a^3}\right)\right], \tag{3.162}$$

$$H_{31} = \frac{2\varphi\omega r}{c}\left[\sin^2\theta - \cos\theta\left(1 + \frac{3r_g}{4a} - \frac{r_g r^2}{a^3}\right)\right]. \tag{3.163}$$

Using the definition of the magnetic strength preudo-vector (3.105)

$$H^{*i} = \frac{1}{2}\,\varepsilon^{imn}H_{mn} = \frac{1}{2}\,\varepsilon^{imn}q_{mn} - \frac{2\varphi}{c}\,\Omega^{*i}, \tag{3.164}$$

we have, finally,

$$H^{*1} = -\frac{2\varphi\omega}{c}\left[\left(1-\frac{r_g r^2}{2a^3}\right)\cos\theta + \frac{1}{2}\left(1+\frac{3r_g}{4a}-\frac{3r_g r^2}{2a^3}\right)\right], \tag{3.165}$$

$$H^{*2} = \frac{2\varphi\omega}{cr}\left[\left(1-\frac{r_g r^2}{2a^3}\right)\sin\theta - \cot\theta\left(1+\frac{3r_g}{4a}-\frac{3r_g r^2}{2a^3}\right)\right], \tag{3.166}$$

while the covariant (lower-index) versions of the preudo-vector can be calculated as $H_{*1} = h_{11}H^{*1}$ and $H_{*2} = h_{22}H^{*2}$. In the framework of our approximation (with the higher-order terms withheld), we have

$$H^{*1} = -\frac{2\varphi\omega}{c}\left(\cos\theta + \frac{1}{2}\right), \tag{3.167}$$

$$H^{*2} = \frac{2\varphi\omega}{cr}\left(\sin\theta - \cot\theta\right). \tag{3.168}$$

Concerning the chr.inv.-Maxwell equations: in the stationary electromagnetic field, they have the form (3.143–3.144)

$$\left.\begin{aligned} {}^*\nabla_j E^j - \frac{1}{c}\,H^{ik}A_{ik} &= 4\pi\rho \\ {}^*\nabla_k H^{ik} - \frac{1}{c^2}\,F_k H^{ik} &= \frac{4\pi}{c}\,j^i \end{aligned}\right\}\ \text{I}, \tag{3.169}$$

$$\left.\begin{aligned} {}^*\nabla_i H^{*i} - \frac{1}{c}\,E^{*ik}A_{ik} &= 0 \\ {}^*\nabla_k E^{*ik} - \frac{1}{c^2}\,F_k E^{*ik} &= 0 \end{aligned}\right\}\ \text{II}. \tag{3.170}$$

Substitute, into the equations, the obtained formula for the magnetic field strength, and also the formula for the electric field strength (3.105), which holds for the stationary electromagnetic field, has the form

$$E^{*ik} = -\,\varepsilon^{ikn}E_n\,, \qquad E_i = -\frac{\varphi}{c^2}\,F_i = \frac{\varphi r_g r}{2a^3}\,. \tag{3.171}$$

Equations of Group II (3.170) are satisfied identically. Equations of Group I (3.170) take the form

$$\frac{3\varphi r_g}{2a^3} = 4\pi\breve{\rho}, \qquad \frac{\omega\varphi\cot\theta}{r^2\sin\theta} = -2\pi\breve{j}^3. \tag{3.172}$$

where $\breve{\rho}$ and $\breve{j}^3$ are the charge density and the current of the vortical electromagnetic field.

As is easy to see, these solutions are identical to the solutions (3.146) obtained in the vortex-free electromagnetic field. It is easy to obtain that the conservation equation $\nabla_\sigma j^\sigma = 0$ (3.147) and the Lorentz condition $\nabla_\sigma A^\sigma = 0$ (3.148), whose chronometrically invariant formulations are (3.149) and (3.149) respectively, are satisfied as identities.

This means that all the results obtained earlier in the vortex-free electromagnetic field of a regular rotating star are as well true in the present case where the electromagnetic field of the star is vortical.

This happens because all the terms that appear in the equations due to the electromagnetic field curl vanish in the framework of the second-order approximation. In the parlance of physics proper, this means that the presence of a curl in the electromagnetic field does not change the field sources of a regular rotating star. The vortical electromagnetic field can be meaningful only in the case of exotic stars, whose characteristics differ from those of regular stars. We will see the difference in Chapter 5 when considering rapidly rotating neutron stars (pulsars).

§3.8 Geometrization of the electromagnetic field for a regular rotating star

Using the geometric formula for the scalar electromagnetic potential

$$\varphi = c^2 \sqrt{\frac{n}{G}} \tag{3.173}$$

that follows from (3.119), we write down the non-vanishing chr.inv.-Maxwell equations (3.146), or (3.172) which is the same as

$$\breve{\rho} = \rho = \frac{3c^2 r_g}{8\pi a^3}\sqrt{\frac{n}{G}} = \frac{3c^2}{8\pi a^2}\sqrt{\frac{n}{G}}\,\frac{r_g}{a}\,, \tag{3.174}$$

$$\breve{j}^3 = j^3 = -\frac{\omega c^2}{2\pi r^2}\sqrt{\frac{n}{G}}\,\frac{\cot\theta}{\sin\theta}\,, \tag{3.175}$$

$$\breve{j} = j = \sqrt{h_{ik}\, j^i j^k} = \sqrt{h_{33}\, j^3 j^3} = \frac{\omega c^2 \cot\theta}{2\pi r}\sqrt{\frac{n}{G}}\,, \tag{3.176}$$

where $\breve{\rho}$ is the charge density and $\breve{j}^3$ is the current of the vortical electromagnetic field, while ρ and j^3 imply the vortex-free field. The dimensionless numerical coefficient $n = \frac{G\varphi^2}{c^4}$ (3.119) is within the range $0 < n < \frac{1}{4}$. To see why $\frac{1}{4}$, see formula (3.118).

The electromagnetic field sources are expressed herein through only the geometric characteristics of the star's space and the fundamental constants. This means that we have completely geometrized the sources of the stationary electromagnetic field of a regular rotating star.

Now, express the electric and magnetic field strengths through the geometric formula for the scalar electromagnetic potential φ (3.173). Using the formulae for the non-zero components E_1, H^{*1}, and H^{*2} obtained in the stationary electromagnetic field of a regular rotating star, we thus obtain

$$E^1 = E_1 = \sqrt{\frac{n}{G}}\,\frac{c^2 r_g r}{2a^3}\,, \tag{3.177}$$

$$H^{*1} = -\,2\omega\, c\sqrt{\frac{n}{G}}\left(\cos\theta + \frac{1}{2}\right) = -\,2\,c\sqrt{\frac{n}{G}}\left(\omega\cos\theta + \Omega^{*1}\right), \tag{3.178}$$

$$H^{*2} = \frac{2\omega\, c}{r}\sqrt{\frac{n}{G}}\left(\sin\theta - \cot\theta\right) = 2c\sqrt{\frac{n}{G}}\left(\frac{\omega\sin\theta}{r} - \Omega^{*2}\right), \tag{3.179}$$

$$H_{*1} = h_{11}\, H^{*1} = H^{*1}, \tag{3.180}$$

$$H_{*2} = h_{22}\, H^{*2} = r^2\, H^{*2}. \tag{3.181}$$

Here, according to our calculation (3.109–3.110) made in the framework of the space metric of a regular rotating star (3.28),

$$\Omega^{*1} = \Omega_{*1} = \frac{\omega}{2}\,, \qquad \Omega^{*2} = \frac{\omega\cot\theta}{r}\,, \qquad \Omega_{*2} = \omega\, r\cot\theta\,, \tag{3.182}$$

$$\Omega_{*j}\,\Omega^{*j} = \omega^2\left(\frac{1}{4} + \cot^2\theta\right). \tag{3.183}$$

Hence forth, we express the field density $\breve{\rho}_{\,\rm em}$ and the flow of momentum $\breve{J}^3_{\rm em}$ of the vortical electromagnetic field, which are the physically observable projections of the energy-momentum tensor of the field (see Einstein's equations). Using their general formulation made for any electromagnetic field, (3.102) and (3.103), we obtain

$$\breve{\rho}_{\,\rm em} = \frac{n}{2\pi G}\left(\frac{1}{4c^2}\,F_j F^j + \Omega_{*j}\Omega^{*j}\right) + \frac{\omega^2}{2\pi}\,\frac{n}{G} = \\ = \rho_{\rm em} + \frac{\omega^2}{2\pi}\,\frac{n}{G}\,, \tag{3.184}$$

$$\breve{J}^3_{\rm em} = \frac{c^2 r_g r}{4\pi a^3\sin\theta}\,\frac{n}{G}\left(\frac{\omega\sin\theta}{r} - \Omega^{*2}\right) = J^3_{\rm em} + \frac{c^2 r_g\,\omega}{4\pi a^3}\,\frac{n}{G}\,. \tag{3.185}$$

As we can see, the observable characteristics of the electromagnetic field are expressed herein through only the geometric characteristics of the star's space and the fundamental constants. This concerns both the vortical field and the vortex-free field. In the sense of mathematics, this means that the electromagnetic field of a regular rotating star is completely geometrized.

So, in the case of a regular rotating star, both Maxwell's equations and Einstein's field equations are satisfied with the inclusion of the electromagnetic field. This means that they consist a self-consistent system of the Einstein-Maxwell equations that completely describes both gravitational and electromagnetic phenomena inside regular rotating stars.

Finally, we can conclude something pretty interesting for astrophysics by first writing the formula for the charge density ρ (3.174) in the form

$$\rho = \frac{3\,c^2}{8\pi G a^2}\,\sqrt{nG}\,\frac{r_g}{a}\,. \tag{3.186}$$

The first multiplier herein coincides with the formula for the "critical density" of substance in the Universe

$$\rho_{\mathrm{cr}} = \frac{3\,c^2}{8\pi G a^2} = \frac{3H^2}{8\pi G}\,, \tag{3.187}$$

known from observational cosmology. Herein, $H = \frac{c}{a}$ is the Hubble constant, while a is the radius of the observable Universe. In analogy to the Universe, the critical density can be formally introduced for any liquid star. Thus, we can express the charge density ρ of the electromagnetic field of the star as

$$\rho = \rho_{\mathrm{cr}}\sqrt{nG}\,\frac{r_g}{a}\,, \tag{3.188}$$

where $n < \frac{1}{4}$ and, numerically, $\sqrt{G} = 2.6\times 10^{-4}$ cm$^{3/2}$/gram$^{1/2}$ sec.

If the charge density is $\rho = \rho_{\mathrm{cr}}\sqrt{nG}$, the physical radius of the star coincides with the Hilbert radius $a = r_g$. Because $r_g \ll a$ for regular stars, we conclude that the charge density of the electromagnetic field inside any regular rotating star is much less than $\rho_{\mathrm{cr}}\sqrt{nG}$, i.e.

$$\rho \ll \rho_{\mathrm{cr}}\sqrt{nG}\,. \tag{3.189}$$

A few words should be said at the end. As is known, the General Theory of Relativity is the geometric theory of space-time-matter. Its primary task is to express all physical phenomena as the manifestations of space (space-time) geometry. The gravitational field was initially geometrized by Einstein, due to Einstein's field equations. However, the electromagnetic field was not geometrized: as was shown by Einstein, mathematically this problem in a general case is very non-trivial. Nevertheless, it is possible to solve this problem in a particular case where some particular conditions simplify the mathematics. Thus, as was shown above, we have completely geometrized the electromagnetic field in the internal field of a regular rotating star.

§3.9 Conclusion

This Chapter is complementary to the previous Chapter 2, wherein we considered regular stars including the Sun. Three primary tasks were achieved in this Chapter.

First. In Chapter 2, when considering the internal space metric of a liquid star, we followed the historical path as Schwarzschild did when introducing the metric. Namely, — even when we introduced the complete form of the internal space metric of a liquid sphere (which contains singularities), we used the Schwarzschild notation. This notation comes from the general form of a spherically symmetric metric, and thus contains the coefficients e^{ν} and e^{λ} which are the functions of r and t. This is the common method for writing any spherically symmetric metric. But when we calculate the physically observable characteristics directly on the basis of the metric, we obtain them in the firm expressed through e^{ν} and e^{λ} which are unknown. Therefore, we obtain the physically observable characteristics of the space in the incomplete form that needs further calculation of the coefficients e^{ν} and e^{λ}. This makes a huge additional trouble when solving particular problems in the framework of the space metric. Therefore, in this Chapter, we initially introduced the internal space metric of a liquid sphere in the final form, where the coefficients e^{ν} and e^{λ} are already expressed through the main characteristics of the sphere such as its physical radius and the Hilbert radius, and through the radial coordinate r and time t. As a result, we have obtained all the components of the fundamental metric tensor in explicit form, without unknown coefficients. It was the subject of §3.1 and §3.2. Therefore, once we (or someone else) further solve problems in the framework of the internal space metric of a regular liquid star, we initially will have implicit formulae for all the physically observable characteristics of its internal space.

Second. We considered the space metric of a non-rotating liquid sphere. Nevertheless we know that the majority of stars rotate. Most probably, even all stars rotate, but many of them rotate slowly so that the Doppler splitting of the spectral lines that is due to the rotation cannot be registered by modern methods of spectroscopy. In any case, as a matter of fact if we target a liquid star possessing an electromagnetic field, we should consider the internal space metric of a rotating liquid sphere. This metric was introduced in §3.3, then we subsequently introduced Einstein's field equations and Maxwell's equations in the form satisfying the metric. We showed that the electric component of the field primarily originates due to the gravitational field of the star, while the magnetic field component is primarily due to the field rotation. Also, we found that the vortical character of the electromagnetic field does not play a significant rôle in regular rotating stars.

Third. Concerning the most important achievement of this Chapter: in §3.8 we showed that, in the case of the internal space metric of a rotating liquid star, all the physically observable characteristics of the electromagnetic field are expressed through only the geometric characteristics of the star's space and the fundamental constants. This fact means that, in the internal field of a rotating liquid star, the electromagnetic field is completely geometrized.

Chapter 4

Stellar Wind

§4.1 Finding the escape velocity condition for a star

A flow of the particles of the stellar substance is permanently erupted from the surface of any star. A fraction of the flow consists of so rapid particles that they leave the gravitational field of the star forever, for the outer cosmos, thus producing a stellar wind.* In terms of our mathematical theory of liquid stars, this means that the particles of the surface layer of the star are faster than the escape velocity for the star.

Why do the particles of the stellar substance leave the surface of a star? Can this process be likened to the boiling of water in a kettle, or is it entirely something else? Finding the answer to this question constitutes our research task for this Chapter.

To answer this question we should study the motion of the particles of the stellar substance inside a star. To do it, we shall first find the formula for the escape velocity, expressed through the components of the space metric of a liquid star. Then we shall deduce the equations of motion of the particles of the stellar substance inside the star. Thus we shall obtain the physical conditions under which the particles of the surface layer are faster than the escape velocity for the star. Afterwards, we will be able to solve the equations of motion of the particles of the stellar substance.

The said escape velocity, known also as the second cosmic velocity v_{II}, is the velocity at which a test-particle can "leave", forever, the gravitational field of the massive body.†

Let us assume that the particles of the stellar substance travel, radially, from within of the star onto its surface. Let the particles reach the surface then leave the star, forever, for the outer cosmos, thus forming stellar wind. We therefore refer to the formula for the velocity of the particles of the stellar substance, which is expressed through the escape velocity for the star, to as the *escape velocity condition.*

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*Wolf-Rayet stars differ from all the regular stars only by an extremely huge stellar wind: the flow is so powerful that a Wolf-Rayet star loses a substantial part of its mass with the stellar wind.

†This is analogous to the first cosmic velocity, known also as the orbital velocity, which allows the test-particle to be orbiting the massive body without falling down onto its surface.

Thus, for a spherically symmetric body whose mass is M, the escape velocity at the distance r from its center is

$$\mathrm{v}_{\text{II}} = \sqrt{\frac{2GM}{r}}\,. \tag{4.1}$$

This formula comes from the mass-point metric (1.1),

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \tag{4.2}$$

where $r_g = \frac{2GM}{c^2}$, while M is the mass of the body (the field's source).

As was shown in Chapter 2, the field of any liquid star has two primary regions. They are described by two different metrics. The metric of a liquid sphere is valid from the center of the star ($r=0$) to its surface ($r=a$). The mass-point metric is valid from the surface of the star to the outer cosmos. In other words, the particles of the stellar substance travel inside the star along those trajectories which are in accordance with the metric of a liquid sphere. If the particles leave the star (in the case that their velocity exceeds the escape velocity), they travel in the cosmos along those trajectories which are in accordance with the mass-point metric.

Therefore, the velocity of the particles of the stellar substance travelling from the surface of the star for the outer cosmos results as the solution of the equations of motion of a mass-bearing particle according to the mass-point metric. Being expressed in terms of the escape velocity, it is the escape velocity condition for the star.

We derive this formula as a solution of the chr.inv.-equations of non-isotropic geodesics [18, 19]

$$\left.\begin{aligned} &\frac{dm}{d\tau} - \frac{m}{c^2}\,F_i \mathrm{v}^i + \frac{m}{c^2}\,D_{ik}\,\mathrm{v}^i\mathrm{v}^k = 0 \\ &\frac{d\left(m\mathrm{v}^i\right)}{d\tau} + 2m\left(D^i_k + A^{\cdot i}_{k\cdot}\right)\mathrm{v}^k - mF^i + m\Delta^i_{nk}\,\mathrm{v}^n\mathrm{v}^k = 0 \end{aligned}\right\}, \tag{4.3}$$

which are the equations of observable motion of a mass-bearing particle travelling at the observable velocity v^i. The equations result as the observable projections of the well-known generally covariant equations of non-isotropic geodesics (see [18, 19] for detail).

We solve the equations (4.3) for a particle of the stellar substance, which travels only along the radial direction r. Therefore,

$$\mathrm{v}^1 = \frac{dr}{d\tau} \neq 0\,, \qquad \mathrm{v}^2 = \mathrm{v}^3 = 0\,. \tag{4.4}$$

To solve these equations (4.3), we need to formulate the characteristics of the space of the mass-point metric (4.2). As is seen from the metric (4.2), the space is free of rotation and deformation ($A_{ik}=0$, $D_{ik}=0$). Only the gravitational inertial force F_i and the Christoffel symbols Δ^i_{nk} remain non-zero. Calculating these quantities, and also the components of the chr.inv.-metric tensor h_{ik} according to their definitions given in §1.3, we obtain, for the metric (4.2),

$$F_1 = -\frac{c^2 r_g}{2r^2}\frac{1}{1-\frac{r_g}{r}}\,, \qquad F^1 = -\frac{c^2 r_g}{2r^2}\,, \tag{4.5}$$

$$h_{11} = \frac{1}{h^{11}} = 1-\frac{r_g}{r}\,, \quad h_{22} = \frac{1}{h^{22}} = r^2, \quad h_{33} = \frac{1}{h^{33}} = r^2\sin^2\theta\,, \tag{4.6}$$

$$\Delta^1_{11} = -\frac{r_g}{2r^2}\frac{1}{1-\frac{r_g}{r}}\,, \qquad \Delta^1_{22} = \frac{\Delta^1_{33}}{\sin^2\theta} = -r\left(1-\frac{r_g}{r}\right), \tag{4.7}$$

$$\Delta^2_{12} = \Delta^3_{13} = \frac{1}{r}\,, \qquad \Delta^2_{33} = -\sin\theta\cos\theta\,, \qquad \Delta^3_{23} = \cot\theta\,. \tag{4.8}$$

With these, we obtain that the chr.inv.-equations of motion (4.3) in the space of the mass-point metric have the form

$$\left.\begin{aligned} &\frac{1}{m}\frac{dm}{d\tau} = -\frac{r_g}{2r^2}\frac{1}{1-\frac{r_g}{r}}\frac{dr}{d\tau} \\ &\frac{1}{m}\frac{d}{d\tau}\left(m\frac{dr}{d\tau}\right) - \frac{r_g}{2r^2}\frac{1}{1-\frac{r_g}{r}}\left(\frac{dr}{d\tau}\right)^2 + \frac{c^2 r_g}{2r^2} = 0 \end{aligned}\right\}, \tag{4.9}$$

where

$$m = \frac{m_0}{\sqrt{1-\frac{\dot{r}^2}{c^2\left(1-\frac{r_g}{r}\right)}}}\,, \qquad \dot{r} = \frac{dr}{d\tau}\,. \tag{4.10}$$

Here, denoting the relativistic mass of the particle on the surface ($r=a$) of the star as $m_{(0)}$ (this is the "start-mass" of the particle when leaving the star), and denoting the observable velocity of the particle when it leaves the star as $\dot{r}_0$, we have

$$m = m_{(0)}\frac{\sqrt{1-\frac{r_g}{a}}}{\sqrt{1-\frac{r_g}{r}}}\,, \qquad m_{(0)} = \frac{m_0}{\sqrt{1-\frac{\dot{r}_0^2}{c^2\left(1-\frac{r_g^3}{a^3}\right)}}}\,. \tag{4.11}$$

Proceed to solve the chr.inv.-equations of motion (4.9). Substituting the scalar equation into the vectorial equation, we obtain the vectorial equation of motion written with respect to the radial distance r

$$\ddot{r} - \frac{r_g}{r^2}\frac{\dot{r}^2}{1-\frac{r_g}{r}} + \frac{c^2 r_g}{2r^2} = 0\,. \tag{4.12}$$

Denote $\dot{r} = y$, then

$$\ddot{r} = yy'\,, \qquad y' = \frac{dy}{dr}\,,$$

and the equation (4.12) takes the form

$$yy' - \frac{r_g}{r^2}\frac{y^2}{1-\frac{r_g}{r}} + \frac{c^2 r_g}{2r^2} = 0\,. \tag{4.13}$$

Assuming $u(r) = y^2$, we reduce it to the linear differential equation

$$u' - \frac{2r_g}{r^2}\frac{u}{1-\frac{r_g}{r}} + \frac{c^2 r_g}{r^2} = 0\,. \tag{4.14}$$

This equation has the following exact solution:

$$u = \mathrm{e}^{-F}\left(u_0 + \int_r^a g(r)\,\mathrm{e}^F dr\right), \qquad u_0 = y_0^2 = \dot{r}_0^2\,, \tag{4.15}$$

where

$$F(r) = \int_r^a f(r)\,dr\,, \quad f(r) = -\frac{2r_g}{r^2}\frac{1}{1-\frac{r_g}{r}}\,, \quad g(r) = \frac{c^2 r_g}{r^2}\,. \tag{4.16}$$

Integrating the function $f(r)$, we obtain

$$F(r) = \ln\left(\frac{1-\frac{r_g}{a}}{1-\frac{r_g}{r}}\right)^2, \qquad \mathrm{e}^F = \left(\frac{1-\frac{r_g}{a}}{1-\frac{r_g}{r}}\right)^2, \tag{4.17}$$

$$\int_r^a \frac{c^2 r_g\left(1-\frac{r_g}{a}\right)^2 dr}{r^2\left(1-\frac{r_g}{r}\right)^2} = c^2\left(1-\frac{r_g}{a}\right)\left(1-\frac{1-\frac{r_g}{a}}{1-\frac{r_g}{r}}\right). \tag{4.18}$$

Substituting (4.16–4.18) into (4.15), and neglecting the high powers of the term $\frac{r_g}{a}$ (this ratio is tiny for regular stars), we obtain

$$\dot{r}^2 = \dot{r}_0^2\left(1 + \frac{2r_g}{a} - \frac{2r_g}{r}\right) + c^2\left(\frac{r_g}{a} - \frac{r_g}{r}\right). \tag{4.19}$$

From here, we obtain the formula for the radial velocity of the particle of the stellar substance, which leaves the star with stellar wind. Since v_{II} (4.1) on the surface of the star ($r=a$) takes the form

$$\mathrm{v}_{\mathrm{II}} = \sqrt{\frac{2GM}{r}} = c\sqrt{\frac{r_g}{r}} = c\sqrt{\frac{r_g}{a}}\,, \tag{4.20}$$

we obtain

$$\dot{r} = \frac{dr}{d\tau} = c\sqrt{\frac{\dot{r}_0^2 + \mathrm{v}_{\mathrm{II}}^2}{c^2} - \frac{r_g}{r} + \frac{2\dot{r}_0^2}{c^2}\left(\frac{\mathrm{v}_{\mathrm{II}}^2}{c^2} - \frac{c^2 r_g}{r}\right)}\,. \tag{4.21}$$

This is the escape velocity condition we were looking for. If $\dot{r}_0 = 0$, the equation (4.21) manifests the obvious condition

$$\frac{dr}{d\tau} = \sqrt{\mathrm{v}_{\mathrm{II}}^2 - \frac{c^2 r_g}{r}} < \mathrm{v}_{\mathrm{II}}\,. \tag{4.22}$$

According to this condition, the particle of the stellar substance cannot leave the gravitational field of the star, if its start-velocity on the surface of the star is zero. Therefore, when further considering stellar wind, we always assume $\dot{r}_0 \neq 0$ in all the equations of the theory.

Let us obtain the final simplification of the escape velocity condition (4.21). Compare the estimated numerical values of all the terms contained in the radicand. We denote the last term of the radicand as

$$q = \frac{2\dot{r}_0^2}{c^2}\left(\frac{\mathrm{v}_{\mathrm{II}}^2}{c^2} - \frac{c^2 r_g}{r}\right). \tag{4.23}$$

For the Sun, which is a typical regular star, we have: $\mathrm{v}_{\mathrm{II}} = 617$ km/sec, $r_g = 2.9$ km, $\dot{r}_0 = 750$ km/sec*, and $a = 7.0\times10^5$ km. Since $q = 0$ by $r = a$, assume $r > a$ as for stellar wind. We obtain, after some algebra,

$$\frac{\dot{r}_0^2 + \mathrm{v}_{\mathrm{II}}^2}{c^2} \simeq 10^{-5}, \quad \frac{r_g}{r} < 4.1\times10^{-6}, \quad q < 5.3\times10^{-11}. \tag{4.24}$$

For a typical star of the Wolf-Rayet family (see Table 1.1), we have: $\mathrm{v}_{\mathrm{II}} = 982$ km/sec, $r_g = 150$ km, $\dot{r}_0 = 2200$ km/sec, and $a = 1.4\times10^7$ km. Therefore, for a typical Wolf-Rayet star, we obtain

$$\frac{\dot{r}_0^2 + \mathrm{v}_{\mathrm{II}}^2}{c^2} \simeq 6.4\times10^{-5}, \quad \frac{r_g}{r} < 1.1\times10^{-5}, \quad q < 1.2\times10^{-9}. \tag{4.25}$$

*$\dot{r}_0 \simeq 750$ km/sec is typical for the particles of the fast component of the solar wind, whose composition is that of the photosphere. In contrast, the slow component of the solar wind has a composition close to that of the corona. Its particles travel from the Sun at a velocity of about 400 km/sec.

As is seen, the term q has so small a numerical value (four orders less than the other terms in the formula) that can be neglected for stellar wind, which comes from both a regular star and a Wolf-Rayet star. Therefore, the final formula for the escape velocity condition has the form

$$\frac{dr}{d\tau} = c\sqrt{\frac{\dot{r}_0^2}{c^2} + \frac{\mathrm{v}_{\mathrm{II}}^2}{c^2} - \frac{r_g}{r}}\;. \tag{4.26}$$

As follows from the final formula, on the surface of the star $(r = a)$ the velocity of the particle of the stellar substance is $\dot{r}_0$.

§4.2 Light-like (massless) particles inside a regular star

We now consider how particles of the stellar substance and particles of light behave inside the star. (Stars are filled not only with substance but also with light.)

First, consider light-like (massless) particles inside a regular star. Such particles travel along isotropic geodesic lines. The chr.inv.-equations of isotropic geodesics have the form [18, 19]

$$\left.\begin{aligned} &\frac{d\omega}{d\tau} - \frac{\omega}{c^2}\,F_i\,c^i + \frac{\omega}{c^2}\,D_{ik}\,c^i c^k = 0 \\ &\frac{d\left(\omega c^i\right)}{d\tau} + 2\omega\left(D^i_k + A^{\cdot i}_{k\cdot}\right)c^k - \omega F^i + \omega\,\Delta^i_{nk}\,c^n c^k = 0 \end{aligned}\right\}. \tag{4.27}$$

These are the equations of observable motion of a light-like particle — a photon whose frequency is ω, — which travels with the observable velocity of light c^i. These chr.inv.-equations emerge as the observable projections of the well-known generally covariant equations of isotropic geodesics (see [18, 19] for details).

As previously, we assume that regular stars do not rotate or deform ($A_{ik} = 0$, $D_{ik} = 0$). Also, we consider only a photon which travels radially (along the direction $x^1 = r$) from the center of the star to its surface. Therefore, the isotropic geodesic equations (4.27) inside a regular star take the particular form

$$\left.\begin{aligned} &\frac{d\omega}{d\tau} - \frac{\omega}{c^2}\,F_1\,c^1 = 0 \\ &\frac{d\left(\omega c^1\right)}{d\tau} - \omega F^1 + \omega\,\Delta^1_{11}\,c^1 c^1 = 0 \end{aligned}\right\}, \tag{4.28}$$

where the observable (light) velocity of the photon is $c^1 = \frac{dr}{d\tau}$.

Consider the scalar geodesic equation of (4.28). Substituting F_1 (3.6), obtained for the metric of a liquid sphere, we have

$$\frac{1}{\omega}\frac{d\omega}{d\tau} = -\frac{r_g}{a^3}\,\frac{r}{\left(3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}\right)\sqrt{1-\frac{r_g r^2}{a^3}}}\,\frac{dr}{d\tau}\,. \tag{4.29}$$

Re-write this equation in the following form, which can easily be integrated:

$$d\ln\omega = -\frac{d\left|3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}\right|}{\left|3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}\right|} = \\ = d\ln\frac{1}{\left|3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}\right|}\,. \tag{4.30}$$

We study only the photons inside the star. We therefore will look for the solution for the interval $r_g \leqslant r \leqslant a$. We obtain, after integration,

$$\omega = \frac{B}{3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}}\,, \tag{4.31}$$

where B is the integration constant.

Assume that the photons originate from the Hilbert surface ($r_0 = r_g$). Then

$$B = \omega_0\left(3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g^3}{a^3}}\right), \tag{4.32}$$

where ω_0 is the initial value of the photon's frequency (on the Hilbert surface of the star). Since $r_g \ll a$ for regular stars, we neglect the high-power terms of $\frac{r_g}{a}$. Finally, the solution of the scalar geodesic equation, which manifests the photon's frequency (4.31), takes the form

$$\omega = \frac{\omega_0\left(3\sqrt{1-\frac{r_g}{a}}-1\right)}{3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}}\,. \tag{4.33}$$

Now, consider the vectorial geodesic equation of (4.28). With our assumption of the radial motion of the photon, it has the form

$$\frac{d^2 r}{d\tau^2} + \frac{1}{\omega}\frac{d\omega}{d\tau}\frac{dr}{d\tau} + \Delta^1_{11}\left(\frac{dr}{d\tau}\right)^2 - F^1 = 0\,. \tag{4.34}$$

Denote $\ddot{r} = \frac{d^2 r}{d\tau^2}$ and $\dot{r} = \frac{dr}{d\tau}$. Substitute $\frac{1}{\omega}\frac{d\omega}{d\tau}$ (4.29), Δ^1_{11} (3.8), and F^1 (3.7). Thus the vectorial geodesic equation (4.34) transforms into the non-linear differential equation of the second order with respect to r

$$\ddot{r} - \frac{r_g r}{a^3}\frac{\dot{r}^2}{\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}\right)\sqrt{1-\frac{r_g r^2}{a^3}}} + \\ + \frac{r_g r}{a^3}\frac{\dot{r}^2}{1-\frac{r_g r^2}{a^3}} + \frac{c^2 r_g r}{a^3}\frac{\sqrt{1-\frac{r_g r^2}{a^3}}}{3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}} = 0 \,. \qquad (4.35)$$

In this form the equation is simply non-solvable. Therefore, we simplify it by the formula for $\dot{r}^2$ taken from the obvious relation $h_{ik}\, c^i c^k = c^2$, which in the present case has the form

$$\frac{\dot{r}^2}{1-\frac{r_g r^2}{a^3}} = c^2. \qquad (4.36)$$

As a result, the initial equation (4.35) takes the form

$$\ddot{r} + \frac{c^2 r_g r}{a^3} = 0 \,. \qquad (4.37)$$

This is the equation of harmonic oscillation with the frequency

$$\Omega = \frac{c}{a}\sqrt{\frac{r_g}{a}} = \frac{\mathrm{v_{II}}}{a} = \sqrt{\frac{2GM}{a^3}}\,, \qquad (4.38)$$

which is obviously dependent on the escape velocity calculated for the star, $\mathrm{v_{II}}$ (4.20).

In general, the frequency Ω (4.38) only depends on the mass M and radius a, which are the integral characteristics of the star. We therefore refer to it as the *proper frequency of the star*. Table 3.1 gives the numerical values of the proper frequency Ω for the typical members of the known families of stars.

The proper frequency reaches its ultimate-high magnitude $\Omega_{\max} = \frac{c}{a}$ by $r_g = a$. This is the case of gravitational collapsars (black holes), which is also applicable to the Universe as a whole. According to observational estimates, the Universe's radius is $a = 1.3\times10^{28}$ cm, and it is the same as its Hilbert radius r_g: the Universe is a huge gravitational collapsar. Therefore, the proper frequency of the Universe is

$$\Omega_{\max} = \frac{c}{a} = 2.3\times10^{-18}\ \mathrm{sec}^{-1} \qquad (4.39)$$

Object	Mass M, gram	Radius a, cm	Proper frequency Ω, sec^{-1}
Wolf-Rayet stars	1.0×10^{35}	1.4×10^{12}	7.0×10^{-5}
Red super-giant*	4.0×10^{34}	7.0×10^{13}	1.6×10^{-7}
White super-giant†	3.4×10^{34}	4.8×10^{12}	6.4×10^{-6}
Sun	2.0×10^{33}	7.0×10^{10}	8.8×10^{-4}
Jupiter (proto-star)	1.9×10^{30}	7.1×10^{9}	8.4×10^{-4}
Red dwarfs	6.7×10^{32}	2.3×10^{10}	2.7×10^{-3}
Brown dwarf‡	4.1×10^{31}	7.0×10^{9}	7.4×10^{-2}
White dwarf§	2.0×10^{33}	6.4×10^{8}	1.0
Universe	8.8×10^{55}	1.3×10^{28}	2.3×10^{-18}

*Betelgeuse. †Rigel. ‡Corot-Exo-3. §Sirius B.

Table 3.1: The proper frequency Ω for the typical members of the known families of stars, and for the Universe.

which matches the numerical value of the Hubble constant, which is $H=\frac{c}{a}=(2.3\pm0.3)\times10^{-18}$ sec^{-1}. In this case, the integral mass of the Universe should be, according to (4.38),

$$M=\frac{\Omega^2a^3}{2G}=8.8\times10^{55}\ \text{gram} \tag{4.40}$$

which coincides with the observed range of the average density of substance in the Universe, which is from 10^{-28} to 10^{-31} gram/cm^3.

Let us return to the vectorial equation of motion of the light-like particles inside liquid stars.

The vectorial geodesic equation in its final form (4.37) solves as

$$r=B_1\cos\left(\sqrt{\frac{r_g}{a}}\,\frac{c\tau}{a}\right)+B_2\sin\left(\sqrt{\frac{r_g}{a}}\,\frac{c\tau}{a}\right), \tag{4.41}$$

where B_1 and B_2 are the integration constants. Assuming r and $\dot{r}$ at the initial moment of time $\tau_0=0$ to be $r_0=r_g$ and $\dot{r}_0=c$, we obtain

$$B_1=r_g\,,\qquad B_2=a\sqrt{\frac{a}{r_g}}\ . \tag{4.42}$$

As a result, we obtain the final solution for r, which is

$$r=r_g\cos\Omega\tau+a\sqrt{\frac{a}{r_g}}\,\sin\Omega\tau\,,\qquad \Omega=\frac{c}{a}\sqrt{\frac{r_g}{a}}\ , \tag{4.43}$$

i.e. the harmonic oscillation equation $r = A_1 \cos\Omega\tau + A_2 \sin\Omega\tau$. Differentiating (4.43), we obtain the oscillation velocity of the photon

$$\dot{r} = c\cos\Omega\tau - \frac{cr_g}{a}\sin\Omega\tau\,, \qquad \Omega = \frac{c}{a}\sqrt{\frac{r_g}{a}}\,, \tag{4.44}$$

As is seen from the solution (4.43), the light-like matter of each single star oscillates at the frequency Ω (4.38) depending on the mass and radius of the star, and with two primary amplitudes:

a) The oscillation with the amplitude $A_1 = r_g$. In fact this means that the surface of the Hilbert core of the star, wherein the stellar energy is released, oscillates at the proper frequency of the star. Thus the released light-like matter oscillates at the same frequency and amplitude as its source (the Hilbert core);

b) The oscillation with the amplitude $A_2 = \sqrt{a^3/r_g}$, which coincides with the outer space breaking of the star's field (see Chapter 2 for details). The space breaking is located outside the star, in the cosmos. For the Sun ($a = 7.0\times10^{10}$ cm, $r_g = 2.9\times10^5$ cm), we obtain $A_2 = 3.4\times10^{13}$ cm $= 2.3$ AU which is the distance from the Sun to the maximal concentration of asteroids (in the asteroid strip). This means that the spherical surface the outer space breaking of the field of each single star oscillates at the proper frequency of the star, as well as the Hilbert core.

Hence, we arrive at the following fundamental conclusions:

1. The surfaces of both the inner space breaking and the outer space breaking of the star's field oscillate at the same frequency Ω, which is the proper frequency of the star;
2. This frequency and the amplitudes thereof only depend on the mass M and radius a of the star;
3. In fact, this common oscillation of light-like matter comprising the star is due to the gravitational field of the star (originating in the star's mass M).

How does this oscillation affect the photon's frequency? To answer this question, consider the obtained solution for the photon's frequency ω (4.33) in the framework of the two ultimate cases which correspond to two oscillation amplitudes: $r = A_1 = r_g$ and $r = A_2 = \sqrt{a^3/r_g}$. Thus the frequency takes the following numerical values:

$$r = A_1 = r_g: \qquad \omega = \omega_0\,\frac{3\sqrt{1-\frac{r_g}{a}}-1}{3\sqrt{1-\frac{r_g}{a}}-1} = \omega_0\,, \tag{4.45}$$

$$r = A_2 = \frac{a^2}{r_g}: \quad \omega = \omega_0 \, \frac{3\sqrt{1-\frac{r_g}{a}} - 1}{3\sqrt{1-\frac{r_g}{a}}}\,. \tag{4.46}$$

As is seen from these formulae, the primary oscillation of the gravitational field of the star does not affect photons which are close to the Hilbert core (at the center of the star). The oscillation affects only photons at large distances from the Hilbert core.

§4.3 Particles of the stellar substance inside a regular star

Such particles travel along non-isotropic geodesics. The chr.inv.-equations of non-isotropic geodesics (see [18, 19]) have the form (4.3):

$$\left.\begin{aligned} &\frac{dm}{d\tau} - \frac{m}{c^2}\,F_i\,\mathrm{v}^i + \frac{m}{c^2}\,D_{ik}\,\mathrm{v}^i\mathrm{v}^k = 0 \\ &\frac{d\left(m\mathrm{v}^i\right)}{d\tau} + 2m\left(D^i_k + A^{\cdot i}_{k\cdot}\right)\mathrm{v}^k - mF^i + m\Delta^i_{nk}\,\mathrm{v}^n\mathrm{v}^k = 0 \end{aligned}\right\}. \tag{4.47}$$

We assume a regular star to be a liquid sphere, which is free of rotation and deformation ($A_{ik}=0$, $D_{ik}=0$). For a particle of the stellar substance, which travels inside the star radially from the center to the surface, the observable velocity is $\mathrm{v}^1 = \frac{dr}{d\tau}$ while $\mathrm{v}^2 = \mathrm{v}^3 = 0$. In this case, the chr.inv.-equations of non-isotropic geodesics (4.47) take the form

$$\left.\begin{aligned} &\frac{dm}{d\tau} - \frac{m}{c^2}\,F_1\,\mathrm{v}^1 = 0 \\ &\frac{d\left(m\mathrm{v}^1\right)}{d\tau} - mF^1 + m\Delta^1_{11}\,\mathrm{v}^1\mathrm{v}^1 = 0 \end{aligned}\right\}. \tag{4.48}$$

They have the same structure as the chr.inv.-equations of isotropic geodesics (4.28). They solve in the same way. But mass-bearing particles do not possess the light speed condition $h_{ik}\,c^i c^k = c^2$ (4.36) we have used for the isotropic geodesic equations. Therefore, the chr.inv.-equations of non-isotropic geodesics (4.48) will have another solution than that which we have obtained for the chr.inv.-equations of isotropic geodesics (4.28).

Substitute, into the scalar equation of (4.48), F_1 (3.6) which we have obtained for the metric of a liquid sphere. We thus obtain the scalar geodesic equation in the form

$$\frac{1}{m}\frac{dm}{d\tau} = -\frac{r_g}{a^3}\,\frac{r}{\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}\right)\sqrt{1-\frac{r_g r^2}{a^3}}}\,\frac{dr}{d\tau}\,. \tag{4.49}$$

This equation can be re-written in the form

$$d\ln m = -\frac{d\left|3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}\right|}{\left|3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}\right|} = d\ln\frac{1}{\left|3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}\right|}\,, \tag{4.50}$$

which is easy to integrate. With $r_g \leqslant r \leqslant a$ (we study only the particles inside the star) we obtain, after integration,

$$m = \frac{B}{3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}}\,, \tag{4.51}$$

where B is the integration constant.

Let the particles start from the Hilbert surface ($r_0 = r_g$). Then

$$B = m_{(0)}\left(3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g^3}{a^3}}\right), \tag{4.52}$$

where

$$m_{(0)} = \frac{m_0}{\sqrt{1-\frac{\dot{r}_0^2}{c^2\left(1-\frac{r_g^3}{a^3}\right)}}} \tag{4.53}$$

is the initial value of the relativistic mass of the particle on the Hilbert surface of the star. Since $r_g \ll a$ for regular stars, we neglect the high-power terms of $\frac{r_g}{a}$. With all these taken into account, the solution of the scalar geodesic equation, which is (4.51), takes the form

$$m = \frac{m_{(0)}\left(3\sqrt{1-\frac{r_g}{a}}-1\right)}{3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}} = \frac{m_0\left(3\sqrt{1-\frac{r_g}{a}}-1\right)}{\left(3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}\right)\sqrt{1-\frac{\dot{r}_0^2}{c^2}}}\,, \tag{4.54}$$

where

$$m_{(0)} = \frac{m_0}{\sqrt{1-\frac{\dot{r}_0^2}{c^2}}} \tag{4.55}$$

in the framework of our approximation mentioned above.

Now, consider the vectorial geodesic equation of (4.48). With our assumption that the particle of the stellar substance travels radially, from the center of the star to its surface, the equation has the form

$$\frac{d^2r}{d\tau^2} + \frac{1}{m}\frac{dm}{d\tau}\frac{dr}{d\tau} + \Delta^1_{11}\left(\frac{dr}{d\tau}\right)^2 - F^1 = 0\,. \tag{4.56}$$

With $\frac{1}{\omega}\frac{d\omega}{d\tau}$ (4.29), Δ^1_{11} (3.8), and F^1 (3.7) substituted, and with the notations $\ddot{r} = \frac{d^2r}{d\tau^2}$ and $\dot{r} = \frac{dr}{d\tau}$, this equation transforms into the non-linear differential equation of the second order with respect to r

$$\ddot{r} - \frac{r_g r}{a^3}\frac{\dot{r}^2}{\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}\right)\sqrt{1-\frac{r_g r^2}{a^3}}} +$$
$$+ \frac{r_g r}{a^3}\frac{\dot{r}^2}{1-\frac{r_g r^2}{a^3}} + \frac{c^2 r_g r}{a^3}\frac{\sqrt{1-\frac{r_g r^2}{a^3}}}{3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}} = 0\,. \tag{4.57}$$

It is identical to the equation (4.35) we have obtained for the photon, and is non-solvable as well. To simplify the equation, we express $\dot{r}^2$ from the obvious relation $h_{11}\dot{r}\dot{r} = \dot{r}^2$, which takes the form

$$c^2\left(1-\frac{r_g r^2}{a^3}\right)\left(1-\frac{m_0^2}{m^2}\right) = \dot{r}^2, \tag{4.58}$$

where

$$m = \frac{m_0}{\sqrt{1-\frac{\dot{r}^2}{c^2\left(1-\frac{r_g r^2}{a^3}\right)}}}\,. \tag{4.59}$$

It follows, from (4.54), that

$$\frac{m_0}{m} = \frac{\left(3\sqrt{1-\frac{r_g}{a}} - \sqrt{1-\frac{r_g r^2}{a^3}}\right)\sqrt{1-\frac{\dot{r}_0^2}{c^2}}}{3\sqrt{1-\frac{r_g}{a}} - 1}\,. \tag{4.60}$$

Therefore, from (4.58), we obtain

$$\dot{r}^2 = c^2\left(1-\frac{r_g r^2}{a^3}\right)\left[1-\frac{\left(3\sqrt{1-\frac{r_g}{a}}-\sqrt{1-\frac{r_g r^2}{a^3}}\right)^2\left(1-\frac{\dot{r}_0^2}{c^2}\right)}{\left(3\sqrt{1-\frac{r_g}{a}}-1\right)^2}\right]. \quad (4.61)$$

Substituting this formula for $\dot{r}^2$ into the initial differential equation (4.57), and neglecting the high-power terms of $\frac{r_g}{a}$, we obtain the vectorial geodesic equation (4.57) in the solvable form

$$\ddot{r}+\frac{\left(c^2+\dot{r}_0^2\right)r_g r}{2a^3}=0\,. \quad (4.62)$$

This is the equation of harmonic oscillation with the frequency

$$\Omega=\sqrt{\frac{\left(c^2+\dot{r}_0^2\right)r_g}{2a^3}}\,. \quad (4.63)$$

It concerns the particles of the stellar substance. As is easy to see, this formula transforms into the formula for the photon's frequency Ω (4.38) by the ultimate condition $\dot{r}=c$.

The vectorial geodesic equation (4.62) solves as

$$r=Q_1\cos\Omega\tau+Q_2\sin\Omega\tau\,, \quad (4.64)$$

where the integration constant Q_1 and Q_2 results from the conditions $r_0=r_g$ and $\dot{r}_0$ at the initial moment of time $\tau_0=0$. We obtain

$$Q_1=r_g\,,\qquad Q_2=\frac{\dot{r}_0\,a\sqrt{2a}}{\sqrt{(c^2+\dot{r}_0^2)\,r_g}}\,. \quad (4.65)$$

Therefore, the final solution for r has the form

$$r=r_g\cos\Omega\tau+\frac{\dot{r}_0\,a\sqrt{2a}}{\sqrt{(c^2+\dot{r}_0^2)\,r_g}}\sin\Omega\tau\,, \quad (4.66)$$

which is the harmonic oscillation equation $r=A_1\cos\Omega\tau+A_2\sin\Omega\tau$. Differentiating (4.66), we obtain the velocity of the particle

$$\dot{r}=-\sqrt{\frac{\left(c^2+\dot{r}_0^2\right)r_g^3}{2a^3}}\sin\Omega\tau+\dot{r}_0\cos\Omega\tau\,. \quad (4.67)$$

The obtained solution (4.66) manifests that the liquid substance of each single star oscillates at the frequency Ω (4.63) depending on the mass and radius of the star, and with two primary amplitudes:

a) The amplitude $A_1 = r_g$, which is the same as that for photons (see above). The Hilbert core of the star, consisting of liquid substance, oscillates at the proper frequency of the star;

b) The amplitude

$$A_2 = \frac{\dot{r}_0\, a\sqrt{2a}}{\sqrt{(c^2 + \dot{r}_0^2)\, r_g}}\,. \tag{4.68}$$

It depends on the initial velocity of the particles of the stellar substance, $\dot{r}_0$. If $\dot{r}_0 = c$, $A_2 = \sqrt{a^3/r_g}$,which coincides with the amplitude of stellar photons (see above). According to (4.68), the initial velocity for the particles whose oscillation amplitude reaches the surface of the star ($A_2 = a$) is

$$\dot{r}_0 = \frac{c\sqrt{r_g}}{\sqrt{2a - r_g}} = \frac{\mathrm{v_{II}}}{\sqrt{2 - \frac{r_g}{a}}} \simeq \frac{\mathrm{v_{II}}}{\sqrt{2}}\,, \qquad \mathrm{v_{II}} = \sqrt{\frac{2GM}{a}}\,, \tag{4.69}$$

where $\mathrm{v_{II}}$ (4.20) is the escape velocity for the star (with which the particle leaves the gravitational field of the star forever). Thus, by the condition $A_2 \geqslant a$ in (4.68), we obtain the velocity which is necessary for a particle of the stellar substance in order to be erupted from the surface of the star

$$\dot{r}_0 \geqslant \sqrt{\frac{GM}{a}}\,. \tag{4.70}$$

Let us transform the obtained formula for the proper frequency Ω (4.63) in order to express it through the orbital velocity $\mathrm{v_I}$ for the star*

$$\Omega = \frac{c}{a}\sqrt{\frac{r_g}{2a}}\sqrt{1 + \frac{\dot{r}_0^2}{c^2}} = \frac{\mathrm{v_{II}}}{a\sqrt{2}}\sqrt{1 + \frac{\dot{r}_0^2}{c^2}} = \frac{\mathrm{v_I}}{a}\sqrt{1 + \frac{\dot{r}_0^2}{c^2}}\,. \tag{4.71}$$

Using this formula, we express r (4.66) in the form

$$r = r_g \cos\Omega\tau + \frac{\dot{r}_0\, a}{\mathrm{v_I}\sqrt{1 + \frac{\dot{r}_0^2}{c^2}}}\sin\Omega\tau\,, \tag{4.72}$$

which is $r = A_1 \cos\Omega\tau + A_2 \sin\Omega\tau$. So we have

$$A_1 = r_g\,, \qquad A_2 = \frac{\dot{r}_0\, a}{\mathrm{v_I}\sqrt{1 + \frac{\dot{r}_0^2}{c^2}}}\,. \tag{4.73}$$

*The orbital velocity $\mathrm{v_I}$, known also as the first cosmic velocity, allows the test-particle to be orbiting the massive body without falling down onto its surface.

Thus, $\dot{r}$ (4.67) transforms to

$$\dot{r} = -\frac{r_g \mathrm{v_I}}{a}\sqrt{1+\frac{\dot{r}_0^2}{c^2}}\,\sin\Omega\tau + \dot{r}_0\cos\Omega\tau\,. \tag{4.74}$$

Consider now the amplitude A_2 (4.73) for some particular cases, where it has different numerical values:

1) If $\dot{r}_0 = 0$, we have $A_2 = 0$ according to the definition of A_2 (4.73). Hence the particles of the stellar substance oscillate at the amplitude r_g. In other words, if $\dot{r}_0 = 0$, the particles cannot leave the surface of the star;

2) If $\dot{r}_0 = \mathrm{v_I}$, the particles also cannot leave the star. This is because they oscillate at an amplitude less than the physical radius of the star

$$A_2 = \frac{a}{\sqrt{1+\frac{\mathrm{v_I^2}}{c^2}}} < a\,; \tag{4.75}$$

3) If $\dot{r}_0 = \mathrm{v_{II}}$, then the particles leave the star. This is because in the case of $\dot{r}_0 = \mathrm{v_{II}}$ we have

$$A_2 = \frac{a\sqrt{2}}{\sqrt{1+\frac{\mathrm{v_{II}^2}}{c^2}}} \simeq \left(1-\frac{\mathrm{v_{II}^2}}{2c^2}\right) a\sqrt{2} \simeq a\sqrt{2} > a\,; \tag{4.76}$$

4) If $A_2 = a$: the amplitude equals the physical radius of the star. Then we obtain, from the definition of A_2 (4.73), that

$$\dot{r}_0 = \frac{\mathrm{v_I}}{\sqrt{1+\frac{\mathrm{v_I^2}}{c^2}}} \simeq \left(1-\frac{\mathrm{v_I^2}}{2c^2}\right)\mathrm{v_I} < \mathrm{v_I}\,. \tag{4.77}$$

The particles are a little slower than the orbital velocity for the star. This means that, if the amplitude equals the physical radius of the star ($A_2 = a$), the particles may jump up from the surface of the star and yet they do not leave the star for the orbit (they always fall back down on the star).

Thus, the new mathematical theory of liquid stars provides a solid theoretical ground to stellar wind as that consisting of two components. One is a little slower than the orbital velocity for the star, while the other travels faster than the escape velocity. This conclusion matches observational data. For example, the solar wind consists of two components. The slow solar wind travels at a velocity of about 400 km/sec that is slower than the orbital velocity for the Sun, which is $\mathrm{v_I} = 440$ km/sec. The fast solar wind travels with a velocity of about 750 km/sec, which exceeds the escape velocity, $\mathrm{v_{II}} = 617$ km/sec.

§4.4 Conclusion

Let us summarize the main conclusions which we have obtained on the origin of stellar wind. The conclusions are as follows:

1. Given any star, the light-like matter that fills it oscillates at the frequency
$$\Omega = \frac{c}{a}\sqrt{\frac{r_g}{a}} = \frac{\text{v}_{\text{II}}}{a} = \sqrt{\frac{2GM}{a^3}}\,.$$
Each single star has its own frequency Ω according to its mass M and radius a. Therefore, it is the *proper frequency* of the star;
2. The oscillation occurs with two primary amplitudes. The amplitude $A_1 = r_g$ coincides with the surface of the Hilbert core of the star, wherein the stellar energy is released. The amplitude $A_2 = \sqrt{a^3/r_g}$ coincides with the surface of the outer space breaking of the star's field, which is located in the cosmos. For the Sun, $A_2 = 3.4\times10^{13}$ cm $= 2.3$ AU coincides with the maximal concentration of asteroids (in the asteroid strip). This common oscillation of the light-like matter of the star is due to the gravitational field of the star (its source is the star's mass M). In other words, it is the own "breathing" of the star;
3. Particles of the stellar substance oscillate at the frequency
$$\Omega = \sqrt{\frac{\left(c^2+\dot{r}_0^2\right) r_g}{2\,a^3}} = \frac{\text{v}_{\text{II}}}{a\sqrt{2}}\sqrt{1+\frac{\dot{r}_0^2}{c^2}} = \frac{\text{v}_{\text{I}}}{a}\sqrt{1+\frac{\dot{r}_0^2}{c^2}}$$
and with two primary amplitudes. The frequency depends on the initial velocity of the particles, $\dot{r}_0$, and can be expressed through the escape velocity v_{II} and the orbital velocity v_{I} for the star;
4. The amplitude $A_1 = r_g$ is the same as that for photons. This means that the physical surface of the Hilbert core oscillates at the frequency Ω that above. The other amplitude A_2 depends on the initial velocity of the particles
$$A_2 = \frac{\dot{r}_0\, a\sqrt{2a}}{\sqrt{(c^2+\dot{r}_0^2)\, r_g}} = \frac{\dot{r}_0\, a}{\text{v}_{\text{I}}\sqrt{1+\frac{\dot{r}_0^2}{c^2}}}\,;$$
5. Stars emit light (photons) and erupt the particles of the stellar substance (stellar wind) not by the order of special physical conditions, but automatically. The equations of motion of the particles (both the photons and the substance), which travel radially inside

a liquid star, are the harmonic free-oscillation equation

$$\ddot{r} + \Omega^2 r = 0\,, \qquad \Omega^2 = -\frac{2F_1}{r} = \frac{c^2 r_g}{a^3}\,,$$

where $F_1 = -\frac{c^2 r_g r}{2a^3}$ is the linearized form (in the sense of $r_g \ll a$) of the force of gravity acting inside any liquid star. This is a non-Newtonian gravitation (the force is proportional to the distance), which is the cause of the oscillation of both the stellar light-like matter and the stellar substance. Once the oscillation amplitude exceeds the physical radius of the star, the particles come out the star for the cosmos. Therefore, the cause of the emission of stars is the internal structure of these self-gravitating bodies, which are the liquid spheres in the weightless state in the cosmos;

6. According to the theory, stellar wind should consist of two components: slow stellar wind and fast stellar wind. Particles whose oscillation amplitude reaches the star's surface ($A_2 = a$) have the initial velocity

$$\dot{r}_0 = \frac{\mathrm{v_I}}{\sqrt{1 + \frac{\mathrm{v_I^2}}{c^2}}} \simeq \left(1 - \frac{\mathrm{v_I^2}}{2c^2}\right) \mathrm{v_I} < \mathrm{v_I}\,.$$

which does not exceed the orbital velocity $\mathrm{v_I}$ for the star. Particles which are as fast as the escape velocity for the star ($\dot{r}_0 = \mathrm{v_{II}}$) have the oscillation amplitude

$$A_2 = \frac{a\sqrt{2}}{\sqrt{1 + \frac{\mathrm{v_{II}^2}}{c^2}}} \simeq \left(1 - \frac{\mathrm{v_{II}^2}}{2c^2}\right) a\sqrt{2} \simeq a\sqrt{2} > a\,.$$

This means that slow stellar wind is composed of particles whose oscillation amplitude is in the range of $a \leqslant A_2 < a\sqrt{2}$. These particles leave the surface of the star, but not forever. They always fall back down on the star. Fast stellar wind is composed of particles whose oscillation amplitude is $A_2 \geqslant a\sqrt{2}$. Particles of fast stellar wind leave the gravitational field of the star, forever, for the outer cosmos. Naturally, solar wind consists of slow solar wind, which travels at ~ 400 km/sec (slower than $\mathrm{v_{I\odot}} = 440$ km/sec), and of fast solar wind travelling at ~ 750 km/sec (faster than $\mathrm{v_{II\odot}} = 617$ km/sec).

Chapter 5

Neutron Stars and Pulsars

§5.1 Introducing the space metric of a rotating neutron star

This Chapter is most short, and most complicate in the math among the other Chapters of this book. We will apply our model of liquid stars to neutron stars and pulsars. The high level of complexity is due to the fact that, once we introduce, in the space metric, the rotation around even a single coordinate axis, the further calculations become highly problematic. Anyhow, let us begin.

Neutron stars and pulsars are attributed to Type II of our classification of stars according to the General Theory of Relativity (see Table 1.1 in §1.2). This means that the physical radius a of such a star is a little larger than its Hilbert radius r_g: the star is almost a collapsar, but still has a possibility to shine as a regular star. In §1.2 we showed that the space metric of a liquid sphere transforms into de Sitter's metric of a vacuum sphere under the condition of gravitational collapse ($a = r_g$, i.e. the liquid sphere is a collapsar). The metric has the form (1.16)

$$ds^2 = \frac{1}{4}\left(1 - \frac{r^2}{a^2}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{r^2}{a^2}} - r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right). \qquad (5.1)$$

The physical parameters of neutron stars and pulsars are close to the parameters of collapsars, but are not the same (see Table 1.1). Therefore the metric (5.1), which includes the collapse condition, is close to the true metric of a neutron star or a pulsar, but is not.

How to modify the space metric of a collapsar, (5.1), in order to obtain the metric of a neutron star or a pulsar? To leave the collapse condition, but be near it in the same time. Easy.

Remind, the particular condition of gravitational collapse ($g_{00} = 0$) comes from the general condition of gravitational collapse according to which the physical observable time τ (1.30) stops on the surface of the object

$$d\tau = \sqrt{g_{00}}\, dt + \frac{g_{0i}}{c\sqrt{g_{00}}}\, dx^i = 0\,. \qquad (5.2)$$

If the local space of the object does not rotate (all $g_{0i} = 0$), the aforementioned particular condition of collapse ($g_{00} = 0$) occurs.

Once the object rotates (at least one of all three quantities g_{0i} is non-zero), the condition $g_{00}=0$ can remain true on the surface of the object but does not mean gravitational collapse. This is due to the second term of the complete condition of collapse (5.2) which is non-zero in this case. Therefore, once the rotation is introduced into the metric (5.1), the metric describes the sphere which is out the state of gravitational collapse. The faster the sphere rotates, the more its state differs from the state of a collapsed sphere.

If we will find, in addition to the modified metric which contains the rotation, Einstein's field equations in the form containing the strong magnetic field and also, in the same time, satisfying this metric, we will have the complete description of a neutron star or a pulsar. This is our research plan for this Chapter.

First, we add the space rotation to the metric (5.1), according to the theory of chronometric invariant: see formulae (1.45) of §1.3. Assume that the object — the liquid sphere of the radius a — rotates, with an angular speed ω, along its equatorial axis (the axis ϕ in the spherical coordinates r, θ, ϕ). In this case, the initially metric of the collapsed liquid sphere (5.1) takes the following form

$$ds^2 = \frac{1}{4}\left(1-\frac{r^2}{a^2}\right)c^2dt^2 + \frac{2\omega r^2\cos\theta}{c}\,c\,dt\,d\phi - \\ - \frac{dr^2}{1-\frac{r^2}{a^2}} - r^2\left(d\theta^2+\sin^2\theta\,d\phi^2\right), \qquad (5.3)$$

which means that the sphere is not a collapsar (due to the rotation, see the explanation above).

The linear velocity of such a rotation is determined by g_{0i} of the space metric according to the general formula (1.45). In the present case of the metric (5.3), it has the form

$$v_1 = v_2 = 0\,, \qquad v_3 = -\frac{2\omega a r^2\cos\theta}{\sqrt{a^2-r^2}}\,. \qquad (5.4)$$

The ultimate magnitude of the rotation speed of the neutron stars, registered in the astronomical observations, is about 1,000 km/sec. We therefore neglect the terms $\frac{v^2}{c^2}$, where $v^2 = h^{ik}v_i v_k \ll c^2$. The condition $v^2 \ll c^2$ also means that the rotating body remains to be a sphere.

The three-dimensional observable chr.inv.-metric tensor h_{ik} (1.34) of the space, whose metric is (5.3), has the components

$$h_{11} = \frac{1}{h^{11}} = \frac{a^2}{a^2-r^2}\,, \quad h_{22} = \frac{1}{h^{22}} = r^2, \quad h_{33} = \frac{1}{h^{33}} = r^2\sin^2\theta\,, \quad (5.5)$$

while the determinant of the chr.inv.-metric tensor h_{ik}, and its non-zero logarithmic derivatives along the spatial coordinates are

$$h = \det \| h_{ik} \| = \frac{a^2 r^4 \sin^2\theta}{a^2 - r^2}\,, \tag{5.6}$$

$$\frac{{}^*\partial \ln\sqrt{h}}{\partial r} = \frac{2a^2 - r^2}{r\,(a^2 - r^2)}\,, \tag{5.7}$$

$$\frac{{}^*\partial \ln\sqrt{h}}{\partial \theta} = \cot\theta\,. \tag{5.8}$$

Also, due to the assumed condition $v^2 \ll c^2$ (the non-relativistic rotation of the object), the chr.inv.-differential operator along the spatial coordinates (1.41) coincides with the regular differential operator.

With g_{00} and g_{0i} of the metric (5.3), we now deduce the formulae for the chr.inv.-vector of the gravitational inertial force F_i acting in the space, and for the chr.inv.-tensor of the angular rotation of the space, A_{ik}. According to the definitions of these quantities which come from the chronometrically invariant formalism (see §1.3), we obtain

$$F_1 = \frac{c^2 r}{a^2 - r^2}\,, \qquad F^1 = \frac{c^2 r}{a^2}\,, \tag{5.9}$$

$$\left.\begin{aligned} A_{13} &= -\frac{2\omega a^3 r\cos\theta}{(a^2 - r^2)^{3/2}}\,, \quad & A^{13} &= -\frac{2\omega a\cos\theta}{r\sqrt{a^2 - r^2}\,\sin^2\theta} \\ A_{23} &= \frac{\omega a r^2 \sin\theta}{\sqrt{a^2 - r^2}}\,, \quad & A^{23} &= \frac{\omega a}{r^2\sqrt{a^2 - r^2}\,\sin\theta} \end{aligned}\right\}. \tag{5.10}$$

Two other chronometrically invariant (physical observable) quantities will be needed for our further calculations of Einstein's field equations. These are the chr.inv.-Christoffel symbols of the second kind Δ^i_{kn} and the chr.inv.-curvature tensor C_{iklj}. After some algebra according to the general formulae of these quantities (see §1.3 for detail), we obtain that, in the space of the metric (5.3), the chr.inv.-Christoffel symbols Δ^i_{kn} have the following non-zero components

$$\Delta^1_{11} = \frac{r}{a^2 - r^2}\,, \qquad \Delta^1_{22} = -\frac{r\,(a^2 - r^2)}{a^2}\,, \tag{5.11}$$

$$\Delta^1_{33} = -\frac{r\,(a^2 - r^2)}{a^2}\sin^2\theta\,, \quad \Delta^2_{12} = \Delta^3_{13} = \frac{1}{r}\,, \tag{5.12}$$

$$\Delta^2_{33} = -\sin\theta\cos\theta\,, \qquad \Delta^3_{23} = \cot\theta\,, \tag{5.13}$$

while the non-zero components of the chr.inv.-curvature tensor C_{iklj} are

$$C_{1212} = -\frac{r^2}{a^2 - r^2}\,, \tag{5.14}$$

$$C_{1313} = -\frac{r^2}{a^2 - r^2}\,\sin^2\theta\,, \tag{5.15}$$

$$C_{2323} = -\frac{r^4}{a^2}\,\sin^2\theta\,. \tag{5.16}$$

Respectively, the contraction $C_{ik} = h^{mn}C_{imkn}$ (the chr.inv.-analogy of Ricci's tensor) has the following non-zero components

$$C_{11} = -\frac{2}{a^2 - r^2}\,, \quad C_{22} = -\frac{2\,r^2}{a^2}\,, \quad C_{33} = -\frac{2\,r^2}{a^2}\,\sin^2\theta\,. \tag{5.17}$$

With these characteristics of the space of the metric (5.3), we are able to deduce Einstein's field equations in the form satisfying the metric (that is the next step of our research of neutron stars and pulsars).

§5.2 Einstein's field equations and the conservation law equations satisfying the metric

Consider the chr.inv.-Einstein equations in the general form (1.85-1.87). In a stationary space (that means that the space is free of deformations), such as the space of the metric (5.3) that we suggest to neutron stars and pulsars, the chr.inv.-Einstein equations take the simplified form

$$A_{jl}A^{lj} + \left({}^*\nabla_j - \frac{1}{c^2}\,F_j\right)F^j = -\frac{\varkappa}{2}\left(\rho c^2 + U\right) + \lambda c^2, \tag{5.18}$$

$$\frac{2}{c^2}\,F_j A^{ij} - {}^*\nabla_j\,A^{ij} = \varkappa J^i, \tag{5.19}$$

$$2A_{ij}A_{k\cdot}^{\cdot j} + \frac{1}{2}\left({}^*\nabla_i F_k + {}^*\nabla_k F_i\right) - \frac{1}{c^2}\,F_iF_k - c^2 C_{ik} = \\ = \frac{\varkappa}{2}\left(\rho c^2 h_{ik} + 2U_{ik} - U h_{ik}\right) + \lambda c^2 h_{ik}\,. \tag{5.20}$$

Herein, ρ, J^i, and U^{ik} are the chr.inv.-projections (1.84) of the energy-momentum tensor of the continuous matter that fills the space. These are the observable density of mass, the observable density of momentum, and the observable stress-tensor, respectively. While $U = h^{ik}U_{ik}$ is the trace of the observable stress-tensor. Note that the energy-momentum tensor has now an arbitrary form. So, the sort of the distributed matter is not specified for yet.

Substitute, into the equations, the chr.inv.-characteristics of the metric (5.3). While doing so we should take into account the fact that the initially (non-rotating) metric (5.1) was deduced under the conditions $a^2 = \frac{3}{\lambda} > 0$ and $\lambda > 0$ (see §1.2 for detail). As a result, we transform the chr.inv.-Einstein equations (5.18–5.20) to the form

$$\frac{8\omega^2 a^4 \cot^2\theta}{(a^2-r^2)^2} + \frac{2\omega^2 a^2}{a^2-r^2} = \frac{\varkappa}{2}\left(\rho c^2 + U\right), \tag{5.21}$$

$$\frac{2\omega a \cot\theta}{r^2\sqrt{a^2-r^2}\,\sin\theta} = -\varkappa J^3, \tag{5.22}$$

$$\frac{8\omega^2 a^4 \cot^2\theta}{(a^2-r^2)^2} - \frac{\varkappa}{2}\left(\rho c^2 - U\right) = \frac{\varkappa U_{11}\left(a^2-r^2\right)}{a^2}, \tag{5.23}$$

$$\frac{4\omega^2 a^4 r \cot\theta}{(a^2-r^2)^2} = -\varkappa U_{12}, \tag{5.24}$$

$$\frac{2\omega^2 a^2}{a^2-r^2} - \frac{\varkappa}{2}\left(\rho c^2 - U\right) = \frac{\varkappa U_{22}}{r^2}, \tag{5.25}$$

$$\frac{2\omega^2 a^2}{a^2-r^2} + \frac{8\omega^2 a^4 \cot^2\theta}{(a^2-r^2)^2} - \frac{\varkappa}{2}\left(\rho c^2 - U\right) = \frac{\varkappa U_{33}}{r^2\sin^2\theta}. \tag{5.26}$$

By the calculation of $U = h^{ik}U_{ik} = h^{11}U_{11} + h^{22}U_{22} + h^{33}U_{33}$ from the three respective tensorial equations of these, we obtain the relation which connects the quantities ρ and U

$$\frac{16\omega^2 a^4 \cot^2\theta}{(a^2-r^2)^2} + \frac{4\omega^2 a^2}{a^2-r^2} = \frac{\varkappa}{2}\left(3\rho c^2 - U\right). \tag{5.27}$$

Summarizing (5.21) and (5.27), we obtain the formula for the density of the distributed matter that fills the space

$$\frac{12\omega^2 a^4 \cot^2\theta}{(a^2-r^2)^2} + \frac{3\omega^2 a^2}{a^2-r^2} = \varkappa\rho c^2. \tag{5.28}$$

Multiplying (5.21) by 3, then subtracting (5.27) from the obtained product, we obtain the formula for U

$$\frac{4\omega^2 a^4 \cot^2\theta}{(a^2-r^2)^2} + \frac{\omega^2 a^2}{a^2-r^2} = \varkappa U. \tag{5.29}$$

Comparing the obtained formulae (5.28) and (5.29), we see that ρ and U of the distributed matter which fills the space of the metric (5.3) are connected to each other by the relation

$$U = \frac{1}{3}\,\rho c^2. \tag{5.30}$$

Finally, we transform the four tensorial equations of the obtained chr.inv.-Einstein equations (5.21–5.26) so that they express the non-zero contravariant components of the stress-tensor: $U^{11}\!=h^{1m}h^{1n}U_{mn}$, $U^{12}\!=h^{1m}h^{2n}U_{mn}$, $U^{22}\!=h^{2m}h^{2n}U_{mn}$, $U^{33}=h^{3m}h^{3n}U_{mn}$. Taking the obtained relation $U=\frac{1}{3}\rho c^2$ (5.30) and also the obtained formula for ρ (5.28) into account, we obtain

$$\varkappa U^{11} = \frac{8\omega^2 a^2 \cot^2\theta}{a^2-r^2} - \frac{\varkappa\rho c^2 (a^2-r^2)}{3a^2}\,, \tag{5.31}$$

$$\varkappa U^{12} = -\frac{4\omega^2 a^2 \cot\theta}{r\,(a^2-r^2)}\,, \tag{5.32}$$

$$\varkappa U^{22} = \frac{1}{r^2}\left[\frac{2\omega^2 a^2}{a^2-r^2} - \frac{\varkappa\rho c^2}{3}\right], \tag{5.33}$$

$$\varkappa U^{33} = \frac{1}{r^2\sin^2\theta}\left[\frac{2\omega^2 a^2}{a^2-r^2} + \frac{8\omega^2 a^4}{(a^2-r^2)^2} - \frac{\varkappa\rho c^2}{3}\right]. \tag{5.34}$$

Now, we have to check whether the obtained chr.inv.-Einstein equations (i.e. the given particular type of the distributed matter) satisfying the metric (5.3) or not.

How to do it? The terms consisting Einstein's field equations are of two sorts. These are the characteristics of the particular space and the characteristics of the matter which fills the space (the latter are the components of the energy-momentum tensor of the matter). Suppose that we have received, in another way, the formulae for the components of the energy-momentum tensor of the given matter as expressed through the characteristics of the given space. Then, substituting one into the other, we will see: if the equations become identities, they satisfy the particular space; however if not, then not.

To find how ρ, J^i, and U^{ik} of the obtained chr.inv.-Einstein equations are expressed through the geometric characteristics of the space, we consider the conservation law equations (1.89–1.90). The equations are the extended chr.inv.-notation of the conservation law $\nabla_\sigma T^{\alpha\sigma}=0$ for the energy-momentum tensor of the distributed matter.

In the space, which is free of deformations, such as the space of the metric (5.3) we suggested to neutron stars and pulsars, the conservation law equations (1.89–1.90) take the simplified form

$$\frac{{}^*\partial\rho}{\partial t} + {}^*\widetilde{\nabla}_i J^i - \frac{1}{c^2} F_i J^i = 0\,, \tag{5.35}$$

$$\frac{{}^*\partial J^k}{\partial t} + 2A_{i\cdot}^{\cdot k} J^i + {}^*\widetilde{\nabla}_i U^{ik} - \rho F^k = 0\,, \tag{5.36}$$

where ${}^*\widetilde{\nabla}_i = {}^*\nabla_i - \frac{1}{c^2} F_i$ (see *Notations*). According to the obtained chr.inv.-Einstein equations, we have only $J^3 \neq 0$ of all the three components J^i of the observable density of momentum in the rotating liquid sphere. Also, as was shown in §4.2, only $F_1 \neq 0$ therein. Therefore, concerning the scalar conservation law equation (5.35), we have

$$\begin{aligned} {}^*\widetilde{\nabla}_i J^i - \frac{1}{c^2} F_i J^i &= {}^*\widetilde{\nabla}_3 J^3 - \frac{1}{c^2} F_3 J^3 = \\ &= \left(\frac{{}^*\partial J^3}{\partial\phi} + J^3\Delta^j_{j3} - \frac{1}{c^2} F_3 J^3\right) - \frac{1}{c^2} F_3 J^3 = 0\,. \end{aligned} \tag{5.37}$$

As a result, the scalar conservation law equation (5.35) transforms into the condition

$$\frac{{}^*\partial\rho}{\partial t} = 0\,, \tag{5.38}$$

which means that the observable density of the matter (the liquid substance and the fields) that fills the sphere is stationary.

Of the three vectorial conservation law equations (5.36), the equation with the index $k=3$ vanishes. The rest two vectorial equations (with $k=1,2$) take the form, respectively

$$\begin{aligned} &\frac{2A_{31}(a^2-r^2)}{a^2} J^3 + \frac{\partial U^{11}}{\partial r} + \frac{\partial U^{12}}{\partial\theta} + \left(\frac{\partial \ln\sqrt{h}}{\partial\theta}\right) U^{12} + \\ &+ \Delta^1_{22}U^{22} + \Delta^1_{33}U^{33} + \left(\Delta^1_{11} + \frac{\partial \ln\sqrt{h}}{\partial r} - \frac{1}{c^2} F_1\right) U^{11} = \rho F^1, \end{aligned} \tag{5.39}$$

$$\begin{aligned} &\frac{2A_{32}}{r^2} J^3 + \frac{\partial U^{12}}{\partial r} + \frac{\partial U^{22}}{\partial\theta} + \left(\frac{\partial \ln\sqrt{h}}{\partial\theta}\right) U^{22} + \\ &\qquad + \Delta^2_{33}U^{33} + \left(2\Delta^2_{12} + \frac{\partial \ln\sqrt{h}}{\partial r} - \frac{1}{c^2} F_1\right) U^{12} = 0\,. \end{aligned} \tag{5.40}$$

Consider these two conservation law equations (5.39, 5.40) which remain non-vanished for yet. Substitute, into the equations, the characteristics of the space of the rotating liquid sphere and the characteristics of the matter which fills it. The formulae for U^{ik} (5.31–5.34) and the formula for J^3 (5.22) which come from the chr.inv.-Einstein equations. The formulae for the logarithmic derivatives (5.7, 5.8). The obtained formula for ρ (5.28). The formulae for the acting gravitational inertial force F_1 (5.9) and for the non-zero components A_{13}, A_{23} of the tensor of the angular rotation of the space (5.10). When all the formulae have been substituted into the remaining conservation law equations (5.39, 5.40), after some algebra we see that the equations vanish as well.

So, the common solution of Einstein's field equations and the conservation law equations in the space of the rotating liquid sphere showed that the suggested equations are valid in the space. In other words, the space metric (5.3) we have suggested to neutron stars or pulsars satisfies Einstein's field equations (and vice versa).

§5.3 Introducing the electromagnetic field

As is known, every neutron star or pulsar bears the strong magnetic field. Therefore, we go to the next stage in this research. We need to introduce such an energy-momentum tensor that describes the electromagnetic field and satisfies the relation $U = \frac{1}{3}\rho c^2$ (5.30) which follows from the obtained chr.inv.-Einstein equations. The energy-momentum tensor which satisfies the relation $U = \frac{1}{3}\rho c^2$, satisfies the space metric we suggested to neutron stars and pulsars. Once the energy-momentum tensor will be obtained, the equations of the electromagnetic field will be able to be deduced. Then we conclude how the electromagnetic field is distributed inside a neutron star or a pulsar, according to our theory. This is our plan for now.

The energy-momentum tensor of an arbitrary electromagnetic field has the following general form

$$T^{\alpha\beta}_{\text{em}} = \frac{1}{4\pi c^2}\left(-F^{\alpha\cdot}_{\cdot\sigma}F^{\beta\sigma} + \frac{1}{4}\,g^{\alpha\beta}F_{\mu\sigma}F^{\mu\sigma}\right), \tag{5.41}$$

where $F_{\alpha\beta}$ is the electromagnetic field tensor (the Maxwell tensor). It is the curl of the four-dimensional electromagnetic potential A^α

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}\,. \tag{5.42}$$

The physical observable chr.inv.-projections of the vector A^α are the scalar potential φ and the vector potential q^i of the electromagnetic field

$$\varphi = \frac{A_0}{\sqrt{g_{00}}}\,, \qquad q^i = A^i. \tag{5.43}$$

The theory of the electromagnetic field expressed in terms of the chronometrically invariant formalism is well developed in our book [18]. (See Chapter 3 therein.) So forth we follow with the theory, and refer everyone who is curious in the details to the book.

The physical observable components of the electromagnetic field tensor $F_{\alpha\beta}$ (5.42) have the form

$$\rho_{\mathrm{em}} = \frac{T_{00}}{g_{00}} = \frac{E_i E^i + H_{*i} H^{*i}}{8\pi c^2}\,, \tag{5.44}$$

$$J^i_{\mathrm{em}} = \frac{c\,T^i_0}{\sqrt{g_{00}}} = \frac{1}{4\pi c}\,\varepsilon^{ikm} E_k H_{*m}\,, \tag{5.45}$$

$$U^{ik}_{\mathrm{em}} = c^2 T^{ik} = \rho_{\mathrm{em}} c^2 h^{ik} - \frac{1}{4\pi}\left(E^i E^k + H^{*i} H^{*k}\right). \tag{5.46}$$

Herein, E^i and H^{*i} are, respectively, the three-dimensional chr.inv.-vector and chr.inv.-pseudo-vector of the electric and magnetic strengthes of the field, while ε^{imn} is the unit completely antisymmetric three-dimensional chr.inv.-pseudo-tensor. They are expressed as [18]

$$\left.\begin{aligned} E^{*ik} &= -\varepsilon^{ikn} E_n\,, \qquad E_n = \frac{{}^*\partial\varphi}{\partial x^n} + \frac{1}{c}\frac{{}^*\partial q_n}{\partial t} - \frac{\varphi}{c^2}\,F_n \\ H^{*i} &= \frac{1}{2}\,\varepsilon^{imn} H_{mn}\,, \quad H_{mn} = \frac{{}^*\partial q_m}{\partial x^n} - \frac{{}^*\partial q_n}{\partial x^m} - \frac{2\varphi}{c}\,A_{mn} \end{aligned}\right\}. \tag{5.47}$$

We see that the observable electric and magnetic strengthes depend on not only the electromagnetic field itself (the scalar and vectorial potentials, φ and q^i), but also on the acting gravitational inertial force F_i and the angular velocity of the space rotation, A_{ik}.

Pulsars are the massive objects which possess the strong electromagnetic field and the rapid rotation. Therefore, the factors of F_i and A_{ik} are significant in our study. We will, however, neglect the temporal variations and spatial non-uniformities of the electromagnetic potentials by assuming $\varphi = const$ and $q_i = const$. With these assumptions, the observable electric and magnetic strengthes take the form

$$E_i = -\frac{\varphi}{c^2}\,F_i\,, \qquad H^{*i} = -\frac{2\varphi}{c}\,\Omega^{*i}, \tag{5.48}$$

where Ω^{*i} is the three-dimensional chr.inv.-pseudo-vector of the angular velocity that the star rotates

$$\Omega^{*i} = \frac{1}{2}\,\varepsilon^{imn} A_{mn}\,, \qquad \Omega_{*i} = \frac{1}{2}\,\varepsilon_{imn} A^{mn}. \tag{5.49}$$

The components of the chr.inv.-tensor of the angular speed of the space rotation, A_{ik}, are determined by the metric of the space of the rotating star. They, calculated for the metric of a neutron star or a pulsar, are presented in formula (5.10).

Note. The formulae for the electric strength and the magnetic strength (5.48) which we finally suggest to neutron stars and pulsars, manifest that the electromagnetic field of such a star is due to its gravitation and rotation. Namely, — the electric strength E^i of the field is manifested only due to the gravitational field of the star (even if the electromagnetic field potential φ is presented in the star). The magnetic strength H^{*i} is manifested only if the star rotates.

Using these formulae for E^i and H^{*i} (5.48), and all the aforementioned assumptions we suggested to neutron stars and pulsars, we transform the physical observable components of the electromagnetic field tensor $F_{\alpha\beta}$ (5.44–5.45) to the form

$$\rho_{\rm em} = \frac{\varphi^2}{2\pi c^4}\left(\frac{F_j F^j}{4c^2} + \Omega_{*j}\Omega^{*j}\right), \tag{5.50}$$

$$J^i_{\rm em} = \frac{\varphi^2}{2\pi c^4}\,\varepsilon^{ikm}F_k\,\Omega_{*m}\,, \tag{5.51}$$

$$U^{ik}_{\rm em} = \frac{\varphi^2}{2\pi c^2}\left(\frac{F_j F^j}{4c^2} + \Omega_{*j}\Omega^{*j}\right)h^{ik} - \frac{\varphi^2}{\pi c^2}\left(\frac{F^i F^k}{4c^2} + \Omega^{*i}\Omega^{*k}\right), \tag{5.52}$$

which corresponds to the vortex-free electromagnetic field of a neutron star or a pulsar. From here, we obtain the formula for $U_{\rm em} = h_{ik}\,U^{ik}_{\rm em}$

$$U_{\rm em} = \frac{\varphi^2}{2\pi c^2}\left(\frac{F_j F^j}{4c^2} + \Omega_{*j}\Omega^{*j}\right) = \rho_{\rm em}c^2. \tag{5.53}$$

As is seen from this formula, (5.53), in the framework of the assumed conditions of the particular electromagnetic field, we have $U = \rho c^2$. However, as we obtained earlier for the space metric of a liquid neutron star or a liquid pulsar, there should be $U = \frac{1}{3}\rho c^2$ (5.30). In other words, according to the metric, we should have

$$U_{\rm em} = \frac{1}{3}\,\rho_{\rm em}c^2, \tag{5.54}$$

where

$$U_{\rm em} = \frac{\Omega_{*j}\Omega^{*j}}{\varkappa}\,, \qquad \rho_{\rm em} = \frac{3\,\Omega_{*j}\Omega^{*j}}{\varkappa c^2}\,. \tag{5.55}$$

Therefore, our task now is to find such a physical condition under which the electromagnetic field satisfies (5.55), and hence (5.54).

Let us find the condition. With the use of the obtained relationship $U_{\text{em}} = \rho_{\text{em}} c^2$ (5.53), we re-write the (desirable) formula $U_{\text{em}} = \frac{1}{\varkappa}\Omega_{*j}\Omega^{*j}$ (5.55) in the form

$$\frac{\varphi^2}{2\pi c^2}\left(\frac{F_j F^j}{4c^2} + \Omega_{*j}\Omega^{*j}\right) = \frac{\Omega_{*j}\Omega^{*j}}{\varkappa}\,, \tag{5.56}$$

or, because $\varkappa = \frac{8\pi G}{c^2}$, in the equivalent notation

$$c^2\,\Omega_{*j}\Omega^{*j} = \frac{\frac{G\varphi^2}{c^4}}{1 - \frac{4G\varphi^2}{c^4}}\,F_j F^j. \tag{5.57}$$

Note that the quantity $\frac{G\varphi^2}{c^4}$ is dimensionless. The scalar electromagnetic potential is constant, $\varphi = const$, according to our initially assumptions. Therefore, and because the magnetic strength is $H^{*i} = -\frac{2\varphi}{c}\,\Omega^{*i}$ (5.48), the stationary rotating star is a *permanent magnet*.

Denote

$$\frac{G\varphi^2}{c^4} = n\,, \tag{5.58}$$

where $n < \frac{1}{4}$, while c and G are the fundamental constants. Therefore,

$$\varphi = \frac{c^2}{2\sqrt{G}} < 1.74\times10^{24}\ \left[\frac{\text{gram}^{1/2}\,\text{cm}^{1/2}}{\text{sec}}\right]. \tag{5.59}$$

Having the scalar electromagnetic potential φ within this scale of the magnitudes, the electromagnetic field satisfies the space metric of a neutron star or a pulsar.

As a result, we re-write the obtained formula (5.57) in the form

$$c^2\,\Omega_{*j}\Omega^{*j} = \frac{n}{1-4n}\,F_i F^i, \quad n < \frac{1}{4}\,. \tag{5.60}$$

With this particular condition, which connects the acting force of gravitation and the angular velocity of rotation of the space, the electromagnetic field satisfies the space metric and Einstein's field equations which we suggested to neutron stars or pulsars.

§5.4 The distribution of the magnetic strength

To find how the magnetic strength is distributed along the surface of a neutron star or a pulsar, we consider Maxwell's equations. The general formulation of the two groups of Maxwell's equations is

$$\nabla_\sigma F^{\mu\sigma} = \frac{4\pi}{c}\,j^\mu, \qquad \nabla_\sigma F^{*\mu\sigma} = 0\,, \tag{5.61}$$

where $F^{*\mu\sigma} = \varepsilon^{\mu\sigma\alpha\beta} F_{\alpha\beta}$ is the pseudo-tensor which is dual to the electromagnetic field tensor $F_{\alpha\beta}$, while j^μ is the four-dimensional current vector.

This formulation of Maxwell's equations means an arbitrary electromagnetic field. Transform it with taking into account our assumptions which are particular to neutron stars and pulsars. As previously, we neglect the temporal variations and spatial non-uniformities of the electromagnetic potentials by assuming $\varphi = const$ and $q_i = const$. With these assumptions, the current vector is zero ($j^\mu = 0$) thus Maxwell's equations (5.61) take the particular form

$$\nabla_\sigma F^{\mu\sigma} = 0\,, \qquad \nabla_\sigma F^{*\mu\sigma} = 0\,. \tag{5.62}$$

Write down the particular Maxwell equations (5.62) according to the chronometrically invariant formalism. The chr.inv.-Maxwell equations have the form (see Chapter 3 of the book [18] for detail)

$$\left.\begin{aligned} &{}^*\nabla_j E^j - \frac{1}{c}\, H^{ik} A_{ik} = 0 \\ &{}^*\nabla_k H^{ik} - \frac{1}{c^2}\, F_k H^{ik} - \frac{1}{c}\left(\frac{{}^*\partial E^i}{\partial t} + D E^i\right) = 0 \end{aligned}\right\} \text{ I,} \tag{5.63}$$

$$\left.\begin{aligned} &{}^*\nabla_i H^{*i} - \frac{1}{c}\, E^{*ik} A_{ik} = 0 \\ &{}^*\nabla_k E^{*ik} - \frac{1}{c^2}\, F_k E^{*ik} - \frac{1}{c}\left(\frac{{}^*\partial H^{*i}}{\partial t} + D H^{*i}\right) = 0 \end{aligned}\right\} \text{ II,} \tag{5.64}$$

where $E^{*ik} = -\varepsilon^{ikn} E_k$ is the pseudo-tensor which is dual to the electric strength tensor E_i, while $D = h^{ik} D_{ik}$ is the rate of the deformation of the space. Because the space of the rotating liquid sphere which is under our consideration does not deform, and also, according to our initially assumptions, the electric and magnetic strengthes are stationary, the chr.inv.-Maxwell equations take the simplified form

$$\left.\begin{aligned} &{}^*\nabla_j E^j - \frac{1}{c}\, H^{ik} A_{ik} = 0 \\ &{}^*\nabla_k H^{ik} - \frac{1}{c^2}\, F_k H^{ik} = 0 \end{aligned}\right\} \text{ I,} \tag{5.65}$$

$$\left.\begin{aligned} &{}^*\nabla_i H^{*i} - \frac{1}{c}\, E^{*ik} A_{ik} = 0 \\ &{}^*\nabla_k E^{*ik} - \frac{1}{c^2}\, F_k E^{*ik} = 0 \end{aligned}\right\} \text{ II.} \tag{5.66}$$

Substitute, into the chr.inv.-Maxwell equation (5.65, 5.66) which are already adapted to the space metric of a neutron star or a pulsar, the respective formulae for E^i and H^{ik} (5.48) (and those for their dual preudo-tensors), and also the respective characteristics of the space which we have obtained in §4.1.

The first (scalar) equation of Group I (5.65) takes the form

$$\frac{c^2}{a^2}\,\frac{3a^2-2r^2}{a^2-r^2} = 4\,\Omega_{*j}\,\Omega^{*j}. \tag{5.67}$$

Two of the vectorial equations of Group I vanish, while the third vectorial equation takes the form

$$\frac{2\omega a^3}{r^3(a^2-r^2)\sqrt{a^2-r^2}}\,\frac{\cot\theta}{\sin\theta} = 0\,, \tag{5.68}$$

where ω, according to the space metric of the star (5.3), is the angular speed of the rotation of the star along the equatorial axis ϕ. Both the scalar and vectorial equations of Group II (5.66). Therefore, the dry rest which we have from the chr.inv.-Maxwell equations adapted to neutron stars and pulsars, are only the equations (5.67) and (5.68).

Due to the obvious assumption that stars are not point-like objects (so, $a>0$), and that the radial coordinate is positive ($r>0$), we arrive at the solely valid solution of the equation (5.68):

$$\theta = \pm\frac{\pi}{2}\,. \tag{5.69}$$

The solution means that the vectorial equation of Group I of the chr.inv.-Maxwell equations is applicable only to the poles of a neutron star or a pulsar.

The vectorial equation of Group I describes the chr.inv.-function ${}^*\nabla_k H^{ik}$, which means the observable three-dimensional distribution of the magnetic strength H^{ik} of the electromagnetic field of the star. Therefore, the solution (5.69) we have obtained means that the magnetic field of a neutron star or a pulsar manifests itself only at the South Pole and North Pole of the star.

Calculate the magnetic strength $H^{*i} = -\frac{2\varphi}{c}\,\Omega^{*i}$ (5.48) for this case. The components of the unit antisymmentric chr.inv.-pseudo-tensor ε^{ikm} are explained in detail in Chapter 2 of the book [18]. Thus, after algebra we obtain the components of the angular velocity chr.inv.-pseudo-vector Ω^{*i} (5.49) of the star

$$\Omega^{*1} = \frac{A_{23}}{\sqrt{h}} = \omega\,, \qquad \Omega_{*1} = A^{23}\sqrt{h} = \frac{\omega a^2}{a^2-r^2}\,, \tag{5.70}$$

$$\Omega^{*2} = \frac{A_{31}}{\sqrt{h}} = \frac{2\omega a^2 \cot\theta}{r\,(a^2 - r^2)}\,, \quad \Omega_{*2} = A^{31}\sqrt{h} = \frac{2\omega a^2 r \cot\theta}{a^2 - r^2}\,. \quad (5.71)$$

According to the obtained solution of the chr.inv.-Maxwell equations, which is $\theta = \pm\frac{\pi}{2}$ (5.69), we have $\cot\theta = 0$ in the case. Therefore, $\Omega^{*2} = \Omega_{*2} = 0$. This means that the magnetic field of a neutron star or a pulsar has the solely non-zero components $H^{*1} = -\frac{2\varphi}{c}\,\Omega^{*1}$, which thus is the radial r-components directed from the centre of the star to its South Pole and North Pole, then — to the respective polar directions from the star into the outer cosmos.

This result, in common with the solution $\theta = \pm\frac{\pi}{2}$, were obtained on the basis of our mathematical theory of the liquid neutron stars and pulsars. These purely theoretical results completely coincide with the well-known observational data about the pulsars.

§5.5 The frequency and the magnetic strength of a pulsar

The electromagnetic radiation of a pulsar (rapidly rotating neutron star) is the same as the rotational frequency of the star itself. Let us calculate, on the basis of our theory of the liquid neutron stars and pulsars, the frequency of the electromagnetic radiation of a typical pulsar.

Calculate $\Omega_{*j}\Omega^{*j}$ at the South Pole and North Pole of a rotating neutron star (a pulsar), where $\theta = \pm\frac{\pi}{2}$. We obtain

$$\Omega_{*j}\Omega^{*j} = \Omega_{*1}\Omega^{*1} = \frac{\omega^2 a^2}{a^2 - r^2}\,. \quad (5.72)$$

Then the condition (5.60), which connects the angular velocity of rotation of the space and the force of gravity acting in it, takes the form

$$\frac{\omega^2 a^2}{a^2 - r^2} = \frac{n}{1 - 4n}\,\frac{c^2 r^2}{a^2 (a^2 - r^2)}\,. \quad (5.73)$$

The magnetic strength of the electromagnetic field of a neutron star or a pulsar is expressed as $H^{*i} = -\frac{2\varphi}{c}\,\Omega^{*i}$ (5.48). It is due to the rotation of the star. Therefore, studying of the obtained relation (5.73), we can make a conclusion about the electromagnetic radiation of the star.

The relation (5.73) has a breaking at the surface of the star ($r = a$). We therefore assume $r \neq a$. The relation (5.73) thus takes the form

$$r^2 = \frac{1 - 4n}{n}\,\frac{\omega^2 a^4}{c^2}\,, \quad (5.74)$$

where r, with taking the previous solution $\theta = \pm\frac{\pi}{2}$ (5.69) into account, is the radial distance along the polar axis of the rotation of the star.

If the electromagnetic radiation is produced near the surface of the star, inside the surface layer, we have $r \simeq a$. Thus, after some trivial transformations, we obtain the formula for the frequency of the oscillation of the magnetic field of the star

$$\omega = \omega_0 \sqrt{\frac{n}{1-4n}}\,, \qquad \omega_0 = \frac{c}{a}\,, \tag{5.75}$$

where ω_0 is the ultimate high rotational frequency of the star, at which the star rotates with the light speed.

Assume $a = 10^6$ cm which is the typical radius of a neutron star. With this radius, we obtain

$$\omega_0 = 3\times 10^4 \text{ sec}^{-1}. \tag{5.76}$$

It follows, from (5.75), that n is expressed as

$$n = \frac{\omega^2}{\omega_0^2 + 4\omega^2}\,. \tag{5.77}$$

The observed frequencies of the radio-pulsars are in the range between $\omega_{\text{min}} = 0.53$ and $\omega_{\text{max}} = 448.57$ sec^{-1}. This means that $\omega^2 \ll \omega_0^2$. We therefore can neglect ω in the denominator of (5.77). We obtain

$$n = \frac{\omega^2}{\omega_0^2} = \frac{\omega^2 a^2}{c^2}\,. \tag{5.78}$$

Therefore, for the real radio-pulsars, the number n lies in the range

$$3.1\times 10^{-10} < n < 2.2\times 10^{-4}. \tag{5.79}$$

Also, according to formula (5.58) deduced in the framework of our theory, the scalar potential of the electromagnetic field of a pulsar is

$$\varphi = c^2 \sqrt{\frac{n}{G}}\,. \tag{5.80}$$

Consequently, for the real radio-pulsars, we have

$$6.1\times 10^{19} < \varphi < 5.2\times 10^{22}, \tag{5.81}$$

in [gram$^{1/2}$cm$^{1/2}$sec^{-1}]. This interval of the magnitudes of φ satisfies the upper theoretical limitation on the potential, which, according to our theory, is $\varphi < 1.74\times 10^{24}$ gram$^{1/2}$cm$^{1/2}$sec^{-1} (5.59).

Finally, we now will calculate, on the basis of our theory, the expected range of the magnitudes for the magnetic field strength H^{*i} of the pulsars. According to our theory of the liquid neutron stars and pulsars, $H^{*i} = -\frac{2\varphi}{c}\,\Omega^{*i}$ (5.48). With the calculated range of the magnitudes of the scalar electromagnetic potential φ, and with the esteemed range of the rotational frequencies ω of the pulsars, we obtain

$$2.1\times 10^{9} < H^{*1} < 1.5\times 10^{15}, \tag{5.82}$$

in $[\mathrm{gram}^{1/2}\,\mathrm{cm}^{-1/2}\,\mathrm{sec}^{-1}]$. This range of the numerical values very corresponds to the magnitudes of the magnetic field of the radio-pulsars, which are known due to the radio-astronomical observations.

§5.6 Solving Maxwell's equations in the stationary vortex-free magnetic field of a neutron star

Previously, in §5.4–§5.5, we studied Maxwell's equations in the electromagnetic field of a neutron star or a pulsar under the assumption that the four-dimensional current vector j^{α} was zero within the field ($j^{\alpha} = 0$). See (5.62) and so forth. In other words, we assumed that the electromagnetic field was free of currents.

Further, this assumption generates the following problem. Look at the formula for the observable momentum of the electromagnetic field J^{i}_{em} (5.51), which is the Poynting vector of the field. It has the form

$$J^{i}_{\mathrm{em}} = \frac{1}{4\pi c}\,\varepsilon^{ikm} E_k H_{*m} = \frac{\varphi^2}{2\pi c^4}\,\varepsilon^{ikm} F_k\,\Omega_{*m}\,. \tag{5.83}$$

By the current-free assumption $j^{\alpha} = 0$ made in §5.4, we obtained that only the component $H^{*1} = -\frac{2\varphi}{c}\,\Omega^{*1}$ of the magnetic field strength H^{*i} was non-zero at the South Pole and North Pole of the star. In this case the circular momentum of the field, J^{3}_{em}, which should generate the magnetic component H^{*1}, would have been zero: $J^{3}_{\mathrm{em}} = \frac{1}{4\pi c}\,\varepsilon^{312} E_1 H_{*2} = 0$. This causes some trouble, because a model that satisfies astronomical observations of the pulsars should obviously show $H^{*1} = -\frac{2\varphi}{c}\,\Omega^{*1} \neq 0$ and $J^{3}_{\mathrm{em}} = \frac{1}{4\pi c}\,\varepsilon^{312} E_1 H_{*2} \neq 0$.

Recall that we previously arrived at the difficulty that $H^{*1} \neq 0$ but $J^{3}_{\mathrm{em}} = 0$ as a result of our assumption according to which the electromagnetic field was free of currents ($j^{\alpha} = 0$). Therefore, we now will solve Maxwell's equations in common with the condition $j^{\alpha} \neq 0$.

First, we will resolve this problem in the vortex-free electromagnetic field. In §5.7, this problem will be solved in the vortical field.

The space (space-time) metric of the rotating neutron star has the form (5.3). This metric means that the liquid sphere is not a collapsar due to its rotation (see the necessary explanation in the beginning of this Chapter). Physical and geometric characteristics of the space were calculated and presented in §5.1. In addition to these, we only should add that the pseudo-vector of the angular velocity of space rotation, $\Omega^{*i} = \frac{1}{2}\,\varepsilon^{imn} A_{mn}$, has the following components

$$\left.\begin{aligned} \Omega^{*1} &= \omega\,, & \Omega_{*1} &= \frac{\omega a^2}{a^2 - r^2} \\ \Omega^{*2} &= \frac{2\omega a^2 \cot\theta}{r(a^2 - r^2)}\,, & \Omega_{*2} &= \frac{2\omega a^2 r \cot\theta}{a^2 - r^2} \end{aligned}\right\}. \tag{5.84}$$

Respectively, the square of the pseudo-vector of the angular velocity is

$$\Omega_{*j}\Omega^{*j} = \frac{\omega^2 a^2}{a^2 - r^2}\left(1 + \frac{4\,a^2 \cot^2\theta}{a^2 - r^2}\right). \tag{5.85}$$

Assume now that the scalar electromagnetic potential of the field remains unchanged, $\varphi = const$, while the vectorial electromagnetic potential g_i is vortex-free. Then the components of the electric and magnetic field strengths (5.47) take the form

$$\left.\begin{aligned} E^i &= -\frac{\varphi}{c^2}\,F^i, & E_i &= -\frac{\varphi}{c^2}\,F_i \\ H^{*i} &= \frac{1}{2}\,\varepsilon^{imn} H_{mn}\,, & H_{mn} &= -\frac{2\varphi}{c}\,A_{mn} \end{aligned}\right\}. \tag{5.86}$$

Herein, using the definition of Ω^{*i}, which is $\Omega^{*i} = \frac{1}{2}\,\varepsilon^{imn} A_{mn}$, we rewrite the formula for H^{*i} in the form

$$H^{*i} = -\frac{2\varphi}{c}\,\Omega^{*i}, \qquad H_{*i} = -\frac{2\varphi}{c}\,\Omega_{*i}\,. \tag{5.87}$$

Using the formula for F_1 (5.9), then calculating Ω^{*1} and Ω^{*2} from (5.84), we obtain the substantial components of E^i and H^{*i}. They are

$$E_1 = -\frac{\varphi r}{a^2 - r^2}\,, \qquad E^1 = -\frac{\varphi r}{a^2}\,, \tag{5.88}$$

$$H_{*1} = -\frac{2\varphi\omega a^2}{c(a^2 - r^2)}\,, \qquad H^{*1} = -\frac{2\varphi\omega}{c}\,, \tag{5.89}$$

$$H_{*2} = -\frac{4\varphi\omega a^2 r \cot\theta}{c(a^2 - r^2)}\,, \quad H^{*2} = -\frac{4\varphi\omega\, a^2 \cot\theta}{cr\,(a^2 - r^2)}\,. \tag{5.90}$$

Let us find how the magnetic strength is distributed along the surface of the sphere. Consider Maxwell's equations in their full form (5.61)

$$\nabla_\sigma F^{\mu\sigma} = \frac{4\pi}{c}\,j^\mu, \qquad \nabla_\sigma F^{*\mu\sigma} = 0\,, \tag{5.91}$$

which implies the presence of the field current ($j^\alpha \neq 0$). According to the chronometrically invariant formalism, their physically observable (chronometrically invariant) projections — the chr.inv.-Maxwell equations — have the form (see Chapter 3 of the book [18] for detail)

$$\left.\begin{aligned} &{}^*\nabla_j E^j - \frac{1}{c}\,H^{ik}A_{ik} = 4\pi\rho \\ &{}^*\nabla_k H^{ik} - \frac{1}{c^2}\,F_k H^{ik} - \frac{1}{c}\left(\frac{{}^*\partial E^i}{\partial t} + DE^i\right) = \frac{4\pi}{c}\,j^i \end{aligned}\right\} \text{ I,} \tag{5.92}$$

$$\left.\begin{aligned} &{}^*\nabla_i H^{*i} - \frac{1}{c}\,E^{*ik}A_{ik} = 0 \\ &{}^*\nabla_k E^{*ik} - \frac{1}{c^2}\,F_k E^{*ik} - \frac{1}{c}\left(\frac{{}^*\partial H^{*i}}{\partial t} + DH^{*i}\right) = 0 \end{aligned}\right\} \text{ II.} \tag{5.93}$$

Herein, $E^{*ik} = -\varepsilon^{ikn} E_k$ is the pseudo-tensor which is dual to the electric strength tensor E_i, while $D = h^{ik} D_{ik}$ is the rate of space deformation. Because the space of the rotating liquid sphere which we consider does not deform, and also, according to our initial assumptions, the electric and magnetic strengths are stationary, the chr.inv.-Maxwell equations take the simplified form

$$\left.\begin{aligned} &{}^*\nabla_j E^j - \frac{1}{c}\,H^{ik}A_{ik} = 4\pi\rho \\ &{}^*\nabla_k H^{ik} - \frac{1}{c^2}\,F_k H^{ik} = \frac{4\pi}{c}\,j^i \end{aligned}\right\} \text{ I,} \tag{5.94}$$

$$\left.\begin{aligned} &{}^*\nabla_i H^{*i} - \frac{1}{c}\,E^{*ik}A_{ik} = 0 \\ &{}^*\nabla_k E^{*ik} - \frac{1}{c^2}\,F_k E^{*ik} = 0 \end{aligned}\right\} \text{ II.} \tag{5.95}$$

The first equation of Group I (5.94) takes the form

$$\frac{4\varphi\omega^2 a^2}{c^2(a^2 - r^2)}\left(1 + \frac{4a^2\cot^2\theta}{a^2 - r^2}\right) - \frac{\varphi(3a^2 - 2r^2)}{a^2(a^2 - r^2)} = 4\pi\rho. \tag{5.96}$$

It follows from the second equation of Group I that $j^1 = j^2 = 0$ in the framework of our model, while the equation for j^3 takes the form

$$\frac{\varphi\omega a^3}{r^2(a^2 - r^2)\sqrt{a^2 - r^2}}\,\frac{\cot\theta}{\sin\theta} = -\pi j^3 \tag{5.97}$$

and the absolute value of the chr.inv.-current vector j^i is

$$j = \sqrt{j_k\, j^k} = \frac{\varphi\omega a^3 \cot\theta}{\pi r\,(a^2 - r^2)\,\sqrt{a^2 - r^2}}\,. \tag{5.98}$$

Equations of Group II (5.95) satisfy themselves as identities. So, these formulae for ρ, j^3, and j (5.96–5.98) are the exact solutions to the Maxwell equations we have just considered.

So, we have obtained the exact solutions to Maxwell's equations in the internal electromagnetic field of a neutron star or a pulsar, where the field originates due to its sources: the distributed charges ρ and the currents j^i.

The conservation law of the electric charges and the currents sets up a connexion between the sources of the electromagnetic field. The generally covariant form of the conservation law, which is also known as the continuity equation, has the form

$$\nabla_\sigma\, j^\sigma = 0\,. \tag{5.99}$$

It means that the distributed charges ρ and the currents j^i, which are the respective physically observable (chronometrically invariant) projections of the four-dimensional current vector j^α, are conserved within the four-dimensional volume of the field. Also, Maxwell's equations are connected by the Lorenz condition

$$\nabla_\sigma\, A^\sigma = 0\,, \tag{5.100}$$

which is imposed on the four-dimensional electromagnetic potential A^α.

In a general case, the conservation law $\nabla_\sigma\, j^\sigma = 0$ and the Lorentz condition $\nabla_\sigma\, A^\sigma = 0$ written in terms of the chronometrically invariant formalism, have the form (see Chapter 3 of the book [18]), respectively,

$$\frac{^*\partial\rho}{\partial t} + \rho D + {}^*\widetilde{\nabla}_i\, j^i - \frac{1}{c^2}\, F_i\, j^i = 0\,, \tag{5.101}$$

$$\frac{1}{c}\frac{^*\partial\varphi}{\partial t} + \frac{\varphi}{c}\, D + {}^*\widetilde{\nabla}_i\, q^i - \frac{1}{c^2}\, F_i\, q^i = 0\,, \tag{5.102}$$

where ${}^*\widetilde{\nabla}_i = {}^*\nabla_i - \frac{1}{c^2} F_i$ (see *Notations* for definition of ${}^*\nabla_i$).

It is easy to show that, under the particular conditions of the problem we are considering, the chr.inv.-continuity equations (5.101) and the chr.inv.-Lorentz conditions (5.102) are satisfied as identities.

Now, on the basis of the obtained exact solutions (5.96–5.98) of the Maxwell equations, we look for the Poynting vector $J^i_{\rm em}$ that is the observable momentum of the electromagnetic field. We need to know how the Poynting vector is distributed along the surface of the sphere, which is the surface of a neutron star or a pulsar.

The Poynting vector $J^i_{\rm em}$ is the second of the three physically observable projections of the electromagnetic field tensor $F_{\alpha\beta}$ (5.42), which are $\rho_{\rm em}$ (5.44), $J^i_{\rm em}$ (5.45), and $U^{ik}_{\rm em}$ (5.46). We look for the Poynting vector $J^i_{\rm em}$ (5.45) in the framework of the particular conditions according to which the scalar potential φ of the electromagnetic field remains constant, while the vectorial potential q_i of the field is vortex-free, i.e.

$$\varphi = const\,, \qquad \frac{^*\partial q_i}{\partial x^k} - \frac{^*\partial q_k}{\partial x^i} = 0\,.$$

Substituting, into (5.44–5.46), the substantial (non-zero) components of the electric strength E^i and the magnetic strength H^{*i}, which are (5.88–5.90), we obtain

$$\rho_{\rm em} = \frac{\varphi^2}{2\pi c^4}\left(\frac{F_j F^j}{4c^2} + \Omega_{*j}\Omega^{*j}\right) = \frac{\varphi^2}{2\pi c^4}\left[\frac{\omega^2 a^2}{a^2 - r^2} + \frac{4\omega^2 a^4 \cot^2\theta}{(a^2-r^2)^2} + \frac{c^2 r^2}{4a^2(a^2-r^2)}\right], \qquad (5.103)$$

$$J^3_{\rm em} = \frac{\varphi^2}{2\pi c^4}\,\varepsilon^{ikm}F_k\,\Omega_{*m} = \frac{\varphi^2 F_1\,\Omega_{*2}}{2\pi c^4\sqrt{h}} = \frac{\varphi^2\omega\, a}{\pi c^2(a^2-r^2)^{3/2}}\,\frac{\cot\theta}{\sin\theta}\,, \qquad (5.104)$$

$$J_{\rm em} = \left|\sqrt{h_{33}J^3_{\rm em}J^3_{\rm em}}\,\right| = \frac{\varphi}{\pi c^2}\,\frac{\omega\, a\, r\cot\theta}{(a^2-r^2)^{3/2}}\,. \qquad (5.105)$$

Looking at these equations, we can conclude something about the neutron stars and pulsars whose electromagnetic field is vortex-free (the case of the §5.6). So, we see that the electromagnetic field density $\rho_{\rm em}$ is due to the gravitational inertial force, which is the non-Newtonian force of repulsion F_i acting within the sphere, and due to the sphere's rotation. The electromagnetic field density $\rho_{\rm em}$ can be non-zero by the separate condition $F_i \neq 0$ or $A_{ik} \neq 0$, and the common condition $F_i \neq 0$ and $A_{ik} \neq 0$. The density of the field momentum $J^i_{\rm em}$ is non-zero only by the common condition $F_i \neq 0$ and $A_{ik} \neq 0$.

As follows from (5.103), the density $\rho_{\rm em}$ of the vortex-free electromagnetic field of a rotating neutron star (a pulsar) is zero at the equator of the star ($\theta = 0$). Then the field density $\rho_{\rm em}$ increases with the geographic latitude θ to the South Pole and North Pole, where $\theta = \frac{\pi}{2}$ so the density of the vortex-free electromagnetic field takes the ultimately high magnitude $\rho_{\text{em}} = (\rho_{\text{em}})_{\max}$.

Contrarily, the density of the electromagnetic field momentum J^i_{em} (5.105) is ultimately high at the equator, where $\theta = 0$. Then the magnitude of the field momentum J^i_{em} falls down with the geographic latitude θ to the South Pole and North Pole, where it is $J^i_{\text{em}} = 0$.

In addition to these, we can also conclude something about the charge density ρ and the currents j^i within the vortex-free electromagnetic field of a neutron star or a pulsar.

We can initially re-write the respective formulae for the charge density ρ (5.96) and the current j^3 (5.97), obtained from Group I of the chr.inv.-Maxwell equations, as follows:

$$\rho = \frac{\varphi}{\pi c^4}\left(\Omega_{*j}\Omega^{*j} - \frac{1}{4}\,\nabla_j F^j\right), \tag{5.106}$$

$$j^3 = -\frac{\varphi}{\pi}\,\frac{\omega\, a^3}{r^2 (a^2 - r^2)^{3/2}}\,\frac{\cot\theta}{\sin\theta}\,, \tag{5.107}$$

where

$$\nabla_j F^j = \frac{c^2 (3\,a^2 - 2\,r^2)}{a^2 (a^2 - r^2)} > 0\,, \tag{5.108}$$

$$j = \left|\sqrt{h_{33}\, j^3_{\text{em}} j^3_{\text{em}}}\,\right| = \frac{\varphi\omega\, a^3 \cot\theta}{\pi\, r\, (a^2 - r^2)^{3/2}}\,. \tag{5.109}$$

As a result, we see that the electromagnetic charge density within a neutron star or a pulsar is positive $\rho > 0$ (that should be according to the physical sense of a physical field) if

$$\Omega_{*j}\Omega^{*j} > \frac{1}{4}\,\nabla_j F^j. \tag{5.110}$$

Now re-write this inequality with the formula for ρ (5.96). We obtain the following condition which is necessary according to the physical sense:

$$\frac{4\omega^2 a^2}{c^2}\left(1 + \frac{4\cot^2\theta}{a^2 - r^2}\right) > \frac{3\,a^2 - 2\,r^2}{a^2}\,. \tag{5.111}$$

Compare the formulae for the electromagnetic field current j^3 (5.107) and its power j (5.109) to the obtained formulae for the density of the electromagnetic field momentum J^3_{em} (5.104) and the power of the momentum J_{em} (5.105). We have

$$c^2 J^3_{\text{em}} = -\frac{\varphi r^2}{a^2}\, j^3, \qquad c^2 J_{\text{em}} = \frac{\varphi r^2}{a^2}\, j\,. \tag{5.112}$$

Taking (5.57) into account, we express the scalar electromagnetic field potential φ (which is $\varphi = const$, according to our initial assumptions) through the dimensionless constant $n = \frac{G\varphi^2}{c^4}$ (5.58) (the constant is $n < \frac{1}{4}$, see in the end of §5.3, for detail). So, we have

$$\varphi = c^2 \sqrt{\frac{n}{G}}\,, \qquad \varphi^2 = \frac{nc^4}{G}\,, \qquad n < \frac{1}{4}\,. \tag{5.113}$$

With these, we obtain

$$\rho_{\text{em}} = \frac{n}{2\pi G}\left(\Omega_{*j}\Omega^{*j} + \frac{1}{4c^2}\,F_j F^j\right), \tag{5.114}$$

$$J^3_{\text{em}} = \frac{E_1 H_{*2}}{\sqrt{h}} = \frac{4nc^3}{G}\,\frac{\omega\, a}{(a^2 - r^2)^{3/2}}\,\frac{\cot\theta}{\sin\theta}\,, \tag{5.115}$$

$$\rho = \frac{1}{\pi}\sqrt{\frac{n}{G}}\left(\Omega_{*j}\Omega^{*j} - \frac{1}{4}\,\nabla_j F^j\right), \tag{5.116}$$

$$j^3 = -c^2\sqrt{\frac{n}{G}}\,\frac{\omega\, a^3}{r^2(a^2 - r^2)^{3/2}}\,\frac{\cot\theta}{\sin\theta}\,, \tag{5.117}$$

$$J^3_{\text{em}} = -\frac{4\pi r^2 c}{a^2}\sqrt{\frac{n}{G}}\; j^3. \tag{5.118}$$

We see that the greater the scalar electromagnetic potential φ (5.113) of a neutron star or a pulsar, the stronger the three-dimensional circular current j^3 and the three-dimensional circular momentum J^3_{em} of the electromagnetic field. Moreover, the current and momentum flow of the electromagnetic field exist in the star only if the star rotates on the equatorial plane (x^1, x^3), i.e. only if $\Omega_{*2} \neq 0$. If the neutron star does not rotate ($\Omega_{*j}\Omega^{*j} = 0$), the electric charge density of its internal electromagnetic field would be negative ($\rho < 0$).

So, we have arrived at a non-satisfactory result. Both the circular electromagnetic field current j^3 (the current j^i that goes along the longitude coordinate ϕ) and the electromagnetic field momentum J^3_{em} (that is the Poynting vector of the field) are zero at the South Pole and North Pole of the star, where the geographical latitude is $\theta = \frac{\pi}{2}$, and reach the ultimate power at the equator ($\theta = 0$).

Herein, we have assumed that the electromagnetic field of the rotating neutron star is vortex-free. The final feat to match with the observational data will be done with the vortical electromagnetic field of a rotating neutron star (a pulsar). We will do it next, in §5.7.

§5.7 Solving Maxwell's equations in the stationary vortical magnetic field of a neutron star

In analogy to §3.7, consider a rotating neutron star (a pulsar) whose electromagnetic field is vortical. The curl q_{ik} of the three-dimensional vectorial chr.inv.-potential q_i of the field is non-zero

$$q_{ik} = \frac{{}^*\partial q_i}{\partial x^k} - \frac{{}^*\partial q_k}{\partial x^i} \neq 0\,. \tag{5.119}$$

The four-dimensional electromagnetic field potential

$$A^\alpha = \varphi\,\frac{dx^\alpha}{ds}\,, \qquad g_{\alpha\beta}\,\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 1. \tag{5.120}$$

has two chr.inv.-projections

$$\frac{A_0}{\sqrt{g_{00}}} = \widetilde{\varphi}, \quad A^i = q^i = \frac{\widetilde{\varphi}}{c}\,\mathrm{v}^\mathrm{i}, \quad \mathrm{v}^\mathrm{i} = \frac{dx^i}{d\tau}\,, \tag{5.121}$$

where

$$\mathrm{v}^2 = h_{ik}\,\mathrm{v}^\mathrm{i}\mathrm{v}^\mathrm{k}, \quad \mathrm{v}^2 \ll c^2, \quad \widetilde{\varphi} = \frac{\varphi}{\sqrt{1-\frac{\mathrm{v}^2}{c^2}}} = \varphi\,. \tag{5.122}$$

According to our initial assumptions, $\varphi = const$ and $q^1 = q^2 = 0$ in the field. Thus $\mathrm{v}^3 = \frac{d\phi}{d\tau} = \omega$, and the non-zero components of the vectorial electromagnetic potential q^i and the field curl q_{ik} have the form

$$q^3 = \frac{\varphi\omega}{c}\,, \tag{5.123}$$

$$q_3 = \frac{\varphi\omega}{c}\,r^2\sin^2\theta\,, \tag{5.124}$$

$$q_{31} = \frac{\partial q_3}{\partial r} = \frac{2\,\varphi\omega}{c}\,r\sin\theta\,, \tag{5.125}$$

$$q_{23} = -\frac{\partial q_3}{\partial\theta} = -\frac{2\,\varphi\omega}{c}\,r^2\sin\theta\cos\theta\,. \tag{5.126}$$

Using the definition of the field strengths (5.47), we calculate the non-zero components of the magnetic strength of the vortical field

$$H_{23} = -\frac{2\,\varphi\omega\, r^2\sin\theta}{c}\left(\frac{a}{\sqrt{a^2-r^2}} + \cos\theta\right), \tag{5.127}$$

$$H_{31} = \frac{2\,\varphi\omega}{c}\left[\sin^2\theta - \frac{2\,a^3\cos\theta}{(a^2-r^2)\sqrt{a^2-r^2}}\right]. \tag{5.128}$$

Using the relation

$$H^{*i} = \frac{1}{2}\,\varepsilon^{imn}A_{mn} = \frac{1}{2}\,\varepsilon^{imn}q_{mn} - \frac{2\varphi}{c}\,\Omega^{*i}, \tag{5.129}$$

we re-write the components H_{23} (5.127) and H_{31} (5.128) in the form

$$H^{*1} = -\frac{2\varphi\omega}{c}\left(1 + \frac{\sqrt{a^2-r^2}}{a}\cos\theta\right), \tag{5.130}$$

$$H^{*2} = \frac{2\varphi\omega}{cr}\left(\frac{\sqrt{a^2-r^2}\,\sin\theta}{a} - \frac{2\,a^2\cot\theta}{a^2-r^2}\right), \tag{5.131}$$

while their covariant (lower-index) versions can be calculated as follows: $H_{*1} = h_{11}H^{*1}$ and $H_{*2} = h_{22}H^{*2}$.

Let us then find the solution to the chr.inv.-Maxwell equations. In the case where the electromagnetic field is stationary, the chr.inv.-Maxwell equations have the form (5.94–5.95)

$$\left.\begin{aligned} {}^*\nabla_j E^j - \frac{1}{c}\,H^{ik}A_{ik} &= 4\pi\rho \\ {}^*\nabla_k H^{ik} - \frac{1}{c^2}\,F_k H^{ik} &= \frac{4\pi}{c}\,j^i \end{aligned}\right\}\ \text{I}, \tag{5.132}$$

$$\left.\begin{aligned} {}^*\nabla_i H^{*i} - \frac{1}{c}\,E^{*ik}A_{ik} &= 0 \\ {}^*\nabla_k E^{*ik} - \frac{1}{c^2}\,F_k E^{*ik} &= 0 \end{aligned}\right\}\ \text{II}. \tag{5.133}$$

After substituting the electric and magnetic strengths of the vortical electromagnetic field, we see that equations of Group II (5.133) are satisfied as identities. Equations of Group II take the form

$$\frac{4\varphi\omega^2}{c^2}\left[\frac{a^2}{a^2-r^2}\left(1 + \frac{4\,a^2\cot^2\theta}{a^2-r^2}\right) - \frac{a\cos\theta}{\sqrt{a^2-r^2}}\right] - \\ - \frac{\varphi\left(3\,a^2 - 2\,r^2\right)}{a^2\left(a^2-r^2\right)} = 4\pi\breve{\rho}\ , \tag{5.134}$$

$$\frac{3}{2}\frac{\varphi\omega}{a^2} + \frac{\varphi\omega\,a^3}{r^2\left(a^2-r^2\right)\sqrt{a^2-r^2}}\frac{\cot\theta}{\sin\theta} = -\pi\breve{j}^3, \tag{5.135}$$

where $\breve{\rho}$ and $\breve{j}^3$ are the change density and the current of the vortical electromagnetic field. The physical sense of these equations readily looks more understandable when re-written in the form

$$\breve{\rho} = \frac{\varphi}{\pi c^2}\left(\Omega_{*j}\Omega^{*j} - \frac{1}{4}\nabla_j F^j\right) - \frac{\varphi\omega^2}{\pi c^2}\frac{a\cos\theta}{\sqrt{a^2-r^2}}\,, \tag{5.136}$$

$$\breve{j}^3 = -\frac{\varphi\omega a^3}{\pi r^2(a^2-r^2)^{3/2}}\frac{\cot\theta}{\sin\theta} - \frac{3\varphi\omega}{2\pi a^2}\,, \tag{5.137}$$

where $\omega = \Omega^{*1}$. Express now the charge density $\breve{\rho}$ and the current $\breve{j}^3$ of the vortical electromagnetic field through the same characteristics ρ (5.106) and j^3 (5.107) we have calculated in the vortex-free field

$$\breve{\rho} = \rho - \frac{\varphi\omega^2 a\cos\theta}{\pi c^2\sqrt{a^2-r^2}}\,, \tag{5.138}$$

$$\breve{j}^3 = j^3 - \frac{3\varphi\omega}{2\pi a^2}\,. \tag{5.139}$$

As is seen from the equations (5.138) and (5.139), given a rotating neutron star (a pulsar) whose electromagnetic field is vortical, the charge density and currents of the field differ from those of the vortex-free electromagnetic field by those terms depending on the star's rotation.

Respectively, the field density ρ_{em} (5.44) and the circular momentum flow J^3_{em} (5.45) of the vortical electromagnetic field have the form

$$\breve{\rho}_{\text{em}} = \frac{\varphi^2}{2\pi c^4}\left(\frac{1}{4c^2}F_jF^j + \Omega_{*j}\Omega^{*j}\right) + \\ + \frac{\varphi^2}{2\pi c^4}\left[\omega^2\left(1 - \frac{r^2\sin^2\theta}{a^2}\right) - \frac{a\omega^2\cos\theta}{\sqrt{a^2-r^2}}\right], \tag{5.140}$$

$$\breve{J}^3_{\text{em}} = \frac{\varphi^2}{2\pi c^2 a^2}\left[\frac{2\omega a^3}{(a^2-r^2)^{3/2}}\frac{\cot\theta}{\sin\theta} - \omega\right]. \tag{5.141}$$

Or, in the other notation,

$$\breve{\rho}_{\text{em}} = \rho_{\text{em}} + \frac{\varphi^2}{2\pi c^4}\left[\omega^2\left(1 - \frac{r^2\sin^2\theta}{a^2}\right) - \frac{\omega^2 a\cos\theta}{\sqrt{a^2-r^2}}\right], \tag{5.142}$$

$$\breve{J}^3_{\text{em}} = J^3_{\text{em}} - \frac{\varphi^2\omega}{2\pi c^2 a^2}\,. \tag{5.143}$$

To understand the meaning of these resulting formulae, recall that, as follows from the formulae for A_{31} (5.10)

$$A_{31} = \frac{2\omega a^3 r\cos\theta}{(a^2-r^2)^{3/2}}\,, \qquad A^{31} = \frac{2\omega a\cot\theta}{r\sqrt{a^2-r^2}\,\sin\theta}\,, \tag{5.144}$$

this component and hence the pseudo-vector $\Omega^{*2} = \frac{1}{2}\,\varepsilon^{231}A_{31}$ depend on the geographical latitude θ, while $\Omega^{*1} = \frac{1}{2}\,\varepsilon^{123}A_{23} = \omega$ does not.

The obtained formulae for the current vector $\breve{j}^3$ (5.139) and the Poynting vector $\breve{J}^3_{\text{em}}$ (5.143) of the vortical electromagnetic field contain that very term which does not depend on the geographical latitude of the star that possesses the field. This means that, contrary to the vortex-free electromagnetic field, the current vector $\breve{j}^3$ and the flow of momentum $\breve{J}^3_{\text{em}}$ of the vortical electromagnetic field are non-zero at the South Pole and North Pole of the star.

The obtained result, $\breve{J}^3_{\text{em}} \neq 0$ at the South Pole and North Pole, means that a rotating neutron star whose electromagnetic field is vortical can emit electromagnetic radiation along its polar axis, while that possessing a vortex-free electromagnetic field — cannot.

Also, we have to make one more important conclusion in the framework of our mathematical theory of pulsars. Look at the definition of the magnetic field strength $H^{*i} = \frac{1}{2}\,\varepsilon^{imn}H_{mn}$ (5.47), which is

$$H^{*i} = \frac{1}{2}\,\varepsilon^{imn}\left(\frac{{}^*\partial q_m}{\partial x^n} - \frac{{}^*\partial q_n}{\partial x^m} - \frac{2\varphi}{c}\,A_{mn}\right) = \\ = \frac{1}{2}\,\varepsilon^{imn}\left(\frac{{}^*\partial q_m}{\partial x^n} - \frac{{}^*\partial q_n}{\partial x^m}\right) - \frac{2\varphi}{c}\,\Omega^{*i} = \\ = \frac{1}{2}\,\varepsilon^{imn}q_{mn} - \frac{2\varphi}{c}\,\Omega^{*i}. \qquad (5.145)$$

As is seen from this definition, the pseudo-vector of the magnetic field strength H^{*i} is the sum of the pseudo-vector of the electromagnetic field curl $q^{*i} = \frac{1}{2}\,\varepsilon^{imn}q_{mn}$ and the pseudo-vector Ω^{*i} of the angular velocity of the star's rotation. The magnetic field strength H^{*i} coincides with the pseudo-vector of the star's rotation, Ω^{*i}, only if the electromagnetic field curl q_{mn} is zero. If $q_{mn} \neq 0$ which is true in a vortical electromagnetic field, H^{*i} deviates Ω^{*i}. The stronger the curl q_{mn} of the electromagnetic field is, the more the magnetic axis deviates from the axis of the star's rotation.

Astronomers inform us that the electromagnetic field of observed pulsars is very strong. They imply that electromagnetic radiation can leave such a star only at the polar regions, where the latitudinal and longitudinal electromagnetic field components are not so strong as at the equatorial latitudes. Also, according to the oscillating behavior of the signal registered from pulsars, astronomers conclude that the axis of emission and the axis of rotation of the pulsar does not coincide. All these facts from observational astronomy match with our theoretical results about pulsars.

As a result, our mathematical theory of pulsars leads us to the following conclusions that match observational data:

- A rotating neutron star can be a pulsar only if its electromagnetic field is vortical. Moreover, the curl of the electromagnetic field means that the magnetic axis does not coincide with the axis of the star's rotation. Otherwise, in the neutron star whose electromagnetic field is vortex-free, electromagnetic radiation does not come from the South Pole and North Pole of the star.

All these theoretical results have been obtained in the framework of our assumption that the scalar and vectorial electromagnetic field potentials of the star do not depend on time. Of course, some temporal variations of the potentials should pose an effect on the Poynting vector (the flow of momentum) of the field, and thus on the electromagnetic radiation emitted by the pulsar. But now we neglect these effects.

§5.8 Geometrization of the vortical electromagnetic field of a neutron star

Geometrization of the electromagnetic field is one of the primary tasks in the General Theory of Relativity. As was shown already by Einstein, this problem in a general case is very non-trivial from the side of mathematics. So, it is still not resolved in general. Nevertheless, geometrization of the electromagnetic field is possible in particular cases, under some particular conditions that simplify the mathematics.

We now show that in the particular case of a pulsar the electromagnetic field is geometrized. In the language of mathematics this means that once we have Einstein's field equations and Maxwell's equations, the characteristics of the electromagnetic field can be expressed through only the geometric characteristics of the space.

Consider Einstein's field equations (5.18–5.20) and Maxwell's equations (5.132–5.133) we have obtained in the internal field of a rotating neutron star. Note that in the case of the de Sitter-like metric we have derived for neutron stars, the λ-term describes physical vacuum in the state of inflation $\lambda = \kappa\rho$ (see Chapter 1 for detail). Also, as we showed in §5.2, this form of Einstein's equations satisfies the equations of conservation in the space.

We will consider the vortical electromagnetic field. This is because we have shown that only the vortical field gives the result that matches with astronomical observations of pulsars, i.e. the fact that a pulsar emits electromagnetic radiation from only its polar regions.

We have the scalar electromagnetic field potential φ (which remains unchanged for each particular star, according to our initial assumption) expressed through the fundamental constants as $\varphi = c^2\sqrt{\frac{n}{G}}$ (5.58), where $n < \frac{1}{4}$ (see in the end of §5.3). With it, we obtain the electric and magnetic strengths of the vortical field (see §5.7) in the form

$$E^1 = -\sqrt{\frac{n}{G}}\,\frac{c^2 r}{a^2}\,, \tag{5.146}$$

$$E_1 = h_{11}\,E^1 = -\sqrt{\frac{n}{G}}\,\frac{c^2 r}{a^2 - r^2}\,, \tag{5.147}$$

$$H^{*1} = -2\omega c\sqrt{\frac{n}{G}}\left(1 + \frac{\sqrt{a^2 - r^2}}{a}\cos\theta\right) = -2c\sqrt{\frac{n}{G}}\left(\Omega^{*1} + \frac{\omega\sqrt{a^2 - r^2}}{a}\cos\theta\right), \tag{5.148}$$

$$H^{*2} = \frac{2\omega c}{r}\sqrt{\frac{n}{G}}\left(\frac{\sqrt{a^2 - r^2}}{a}\sin\theta - \frac{2a^2\cot\theta}{a^2 - r^2}\right) = -2c\sqrt{\frac{n}{G}}\left(\Omega^{*2} - \frac{\omega\sqrt{a^2 - r^2}}{ar}\sin\theta\right), \tag{5.149}$$

$$H_{*1} = h_{11}\,H^{*1} = \frac{a^2}{a^2 - r^2}\,H^{*1}, \tag{5.150}$$

$$H_{*2} = h_{22}\,H^{*2} = r^2\,H^{*2}. \tag{5.151}$$

Herein, according to the internal space metric of a rotating neutron star or a pulsar, we have Ω^{*1} (5.70), Ω^{*2} (5.71), $\Omega_{*j}\Omega^{*j}$ (5.85):

$$\Omega^{*1} = \frac{A_{23}}{\sqrt{h}} = \omega, \qquad \Omega_{*1} = A^{23}\sqrt{h} = \frac{\omega a^2}{a^2 - r^2}\,, \tag{5.152}$$

$$\Omega^{*2} = \frac{A_{31}}{\sqrt{h}} = \frac{2\omega a^2\cot\theta}{r\,(a^2 - r^2)}\,, \quad \Omega_{*2} = A^{31}\sqrt{h} = \frac{2\omega a^2 r\cot\theta}{a^2 - r^2}\,, \tag{5.153}$$

$$\Omega_{*j}\Omega^{*j} = \frac{\omega^2 a^2}{a^2 - r^2}\left(1 + \frac{4\,a^2\cot^2\theta}{a^2 - r^2}\right). \tag{5.154}$$

As is seen from these formulae, both the electric and magnetic field strengths are expressed here through only the geometric characteristics of the internal space of the pulsar.

Also, in the same way, the charge density $\breve{\rho}$ (5.136) and the current vector $\breve{j}^3$ (5.137) of the vortical electromagnetic field from Maxwell's equations are expressed as:

$$\breve{\rho} = \frac{1}{\pi}\sqrt{\frac{n}{G}}\left(\Omega_{*j}\Omega^{*j} - \frac{1}{4}\nabla_j F^j\right) - \frac{1}{\pi}\sqrt{\frac{n}{G}}\,\frac{\omega^2 a\cos\theta}{\sqrt{a^2-r^2}} = \\ = \rho - \frac{1}{\pi}\sqrt{\frac{n}{G}}\,\frac{\omega^2 a\cos\theta}{\sqrt{a^2-r^2}}\,, \qquad (5.155)$$

$$\breve{j}^3 = -\frac{c^2}{\pi r^2}\sqrt{\frac{n}{G}}\,\frac{\omega a^3}{(a^2-r^2)^{3/2}}\,\frac{\cot\theta}{\sin\theta} + \frac{3c^2\omega}{2a^2}\sqrt{\frac{n}{G}} = \\ = j^3 + \frac{3c^2\omega}{2a^2}\sqrt{\frac{n}{G}}\,. \qquad (5.156)$$

Respectively, the density $\breve{\rho}_{\,\rm em}$ (5.140) and the flow of momentum $\breve{J}^3_{\rm em}$ (5.141) of the vortical electromagnetic field — the characteristics that come from Einstein's equations — are expressed as:

$$\breve{\rho}_{\,\rm em} = \frac{n}{2\pi G}\left(\frac{1}{4c^2}F_jF^j + \Omega_{*j}\Omega^{*j}\right) + \\ + \frac{n}{2\pi G}\left[\omega^2\left(1 - \frac{r^2\sin^2\theta}{a^2}\right) - \frac{a\omega^2\cos\theta}{\sqrt{a^2-r^2}}\right] = \\ = \rho_{\rm em} + \frac{n}{2\pi G}\left[\omega^2\left(1 - \frac{r^2\sin^2\theta}{a^2}\right) - \frac{a\omega^2\cos\theta}{\sqrt{a^2-r^2}}\right], \qquad (5.157)$$

$$\breve{J}^3_{\rm em} = \frac{nc^2}{2\pi Ga^2}\left[\frac{2\omega a^3}{(a^2-r^2)^{3/2}}\,\frac{\cot\theta}{\sin\theta} - \omega\right] = J^3_{\rm em} - \frac{nc^2\omega}{2\pi Ga^2}\,. \qquad (5.158)$$

We see that all the characteristics of the vortical magnetic field are uniquely expressed through only the geometric characteristics of the space inside the pulsar. Therefore, the vortical electromagnetic field of a rotating neutron star (a pulsar) is hereby geometrized.

This fact also means that the system of Einstein's equations and Maxwell's equations in the internal space of a pulsar is a self-consistent system of equations. This self-consistent system of Einstein-Maxwell equations completely describes both gravitational and electromagnetic phenomena inside the pulsar.

However, if the electromagnetic field of a rotating neutron star is vortex-free, Einstein's equations and Maxwell's equations do not comprise a self-consistent system. The electromagnetic field is not geometrized inside the star as such. As was shown in §5.7, such a neutron star cannot emit electromagnetic radiation from its polar regions. Therefore, it cannot be a pulsar.

§5.9 Boundaries of the physical space a pulsar

Consider an observer whose reference frame is connected to the internal space of a star. Where, from his point of view, does the observable physical space of the star end? At which distance from the star?

These questions are answered in the framework of the theory of physically observable (chronometrically invariant) quantities in General Relativity. In terms of physical observables, the real physical space that is allowed to be registered by an observer "ends" at that distance from him where the physical observable time stops: $d\tau = 0$. The physical observable time τ is calculated according to the metric of the observer's space. Therefore, the real physical boundaries of the observer's observable space are determined by the stopped time condition $d\tau = 0$ according to the space metric.

Let us calculate the boundary of the observable space of a pulsar. This is the distance from the center of the pulsar at which, according to the space metric of the pulsar (an observer whose reference frame is connected with the pulsar), the observable time stops ($d\tau = 0$).

As follows from the chronometrically invariant formalism, the observable time interval is formulated as (1.30)

$$d\tau = \sqrt{g_{00}}\, dt + \frac{g_{0i}}{c\sqrt{g_{00}}}\, dx^i = \sqrt{g_{00}}\, dt - \frac{1}{c^2}\, v_i\, dx^i. \tag{5.159}$$

It consists of two terms. The first term is due to the gravitational field potential $\mathrm{w} = c^2\left(1 - \sqrt{g_{00}}\right)$ (1.44). The second term is due to the fact that the space rotates, and is dependent on the linear velocity of the rotation $v_i = -c\,\frac{g_{0i}}{\sqrt{g_{00}}}$ (1.45).

Therefore, the condition $d\tau = 0$ by which the observable time stops in the space of a gravitating and rotating body is expressed as

$$\sqrt{g_{00}}\; dt = \frac{1}{c^2}\, v_i\, dx^i. \tag{5.160}$$

The space (space-time) metric of a rotating neutron star has the form (5.3). See §5.1 for details. In the metric,

$$g_{00} = \frac{1}{4}\left(1 - \frac{r^2}{a^2}\right), \tag{5.161}$$

$$v_1 = v_2 = 0\,, \qquad v_3 = -\frac{2\omega a r^2 \cos\theta}{\sqrt{a^2 - r^2}}\,. \tag{5.162}$$

In this case, the stopped time condition (5.160) takes the form

$$g_{00} = \frac{1}{c^4}\left(v_3\,\frac{dx^3}{dt}\right)^2, \tag{5.163}$$

where $\frac{dx^3}{dt} = \frac{d\phi}{dt} = \omega$. Substituting the formulae for g_{00} (5.161) and v_3 (5.162) into the stopped time condition (5.163), we obtain the formula for the distance r at which the observable time stops

$$r = \frac{a}{\sqrt{1 + \frac{4a^2\omega^2\cos\theta}{c^2}}}\,. \tag{5.164}$$

This formula, (5.164), shows the boundary at which the physical space of the rotating neutron star ends. As is seen from this formula, the boundary r is the same as the neutron star's radius a at the South Pole and North Pole: the geographical latitude is $\theta = \frac{\pi}{2}$ therein, so $\cos\theta = 0$ and thus $r = a$ according to (5.164). Then the boundary r decreases near the equator where it takes the ultimately low numerical value

$$r_{\min} = \frac{a}{\sqrt{1 + \frac{4a^2\omega^2}{c^2}}} \tag{5.165}$$

which depends only on the neutron star's radius a and the angular velocity of its rotation ω.

The more rapid the neutron star rotates, the more oblate the physical space of the star at its equator becomes. According to our formula (5.165), the oblateness manifests itself only at relativistic speeds of rotation, i.e. in pulsars.

Consider PSR J1748-2446ad, that is the fastest-known pulsar discovered in 2004 [30]. It rotates at a period of 0.00139595482(6) sec which means the angular velocity $\omega = \frac{2\pi}{T} = 4,501\ \text{sec}^{-1}$. Its radius a is estimated to be less than 16 km. Proceeding from these observational data, we can calculate the oblateness of the physical space of the pulsar at its equator:

$$\frac{r_{\min}}{a} = \frac{1}{\sqrt{1 + \frac{4a^2\omega^2}{c^2}}} \simeq 0.90. \tag{5.166}$$

§5.10 Conclusion

So, the complete mathematical theory of the liquid neutron stars and pulsars is presented here, in this Chapter. We now repeat the most important conclusions we made on the basis of the theory. They are:

1. As follows from our mathematical theory, the electromagnetic field of a rotating neutron star (a pulsar) is due to its rotation and gravitation. The faster the star rotates, the stronger is the magnetic strength H^{*i} of the field;
2. The magnetic field strength H^{*i} of the pulsar is directed strict along the polar axis of its rotation. Electromagnetic radiation is emitted only from the poles of the star, then comes into the outer cosmos strictly along the axis of the rotation of the star;
3. The electric strength E_i depends on the spatial distribution of the scalar potential and on the temporal variation of the vectorial potential of the electromagnetic field. The magnetic strength H^{*i} depends on the curl of the vector potential the field, and on the angular velocity of the star. Thus the temporal and spatial variations of the electromagnetic field potentials should affect the outcoming electromagnetic impulses emitted by the pulsar;
4. The Poynting vector (the electromagnetic field momentum) is non-zero at the South Pole and North Pole of a rotating neutron star only if its electromagnetic field is vortical. Therefore, a rotating neutron star is a pulsar, thus emitting electromagnetic radiation from the polar regions, only if possesses a vortical electromagnetic field. Also, the presence of the field curls means that the magnetic axis does not coincide with the axis of the star's rotation. A rotating neutron star whose electromagnetic field is vortex-free cannot emit electromagnetic radiation along its polar axis, so it cannot be a pulsar.

All the conclusions are valid only for a rotating star whose physical radius is close to its Hilbert radius. These are rotating neutron stars and pulsars, not the regular stars such as the Sun etc.

Chapter 6

Black Holes

§6.1 Non-rotating liquid collapsars. The main characteristics

Now, we are going to study the collapse condition of a non-rotating sphere of perfect liquid (that is, a collapsed rotation-free star), in terms of our new model of liquid stars. At first sight, this problem statement sounds like non-sense: perfect liquid is incompressible, hence such a liquid body cannot be compressed. Yes, it would be non-sense, if one would consider collapse as the process of compression of a liquid cosmic body. We do not do it that way: in contrast, we do not discuss cosmogony. We merely consider a collapsar as an already existing object. This amounts to the occurrence of the physical conditions, not the evolutionary compression of a liquid cosmic body.

Hence, a cosmic body is a collapsar if the parameters of its field on its physical surface correspond to the condition of gravitational collapse. Namely, — the field of gravity is so strong on the surface of the body that light signals cannot leave the body for the outer cosmos. In terms of the General Theory of Relativity, this means that the physically observable time stops on the surface.

According to the theory of physically observable quantities (chronometrically invariant formalism), the physically observable time interval $d\tau$ (1.30) is formulated in terms of the gravitational potential w and the linear velocity of space rotation v_i as follows:

$$d\tau = \sqrt{g_{00}}\,dt + \frac{g_{0i}}{c\sqrt{g_{00}}}\,dx^i = \left(1 - \frac{\mathrm{w}}{c^2}\right) dt - \frac{1}{c^2}\,v_i\,dx^i. \tag{6.1}$$

Thus the general condition of gravitational collapse has the form

$$d\tau = \sqrt{g_{00}}\,dt + \frac{g_{0i}}{c\sqrt{g_{00}}}\,dx^i = 0\,. \tag{6.2}$$

In a rotation-free space (wherein $v_i = 0$), the general condition of gravitational collapse is as simple as

$$d\tau = \sqrt{g_{00}}\,dt = 0 \tag{6.3}$$

or merely

$$g_{00} = \left(1 - \frac{\mathrm{w}}{c^2}\right)^2 = 0\,. \tag{6.4}$$

Thus, a cosmic object of rotation-free space is a collapsar, if the three-dimensional gravitational potential w on its surface takes the value

$$\mathrm{w} = c^2\,. \tag{6.5}$$

Consider the collapse condition in the case of a non-rotating star consisting of perfect liquid. As is seen from the space metric of such a liquid star, which is (2.76)

$$ds^2 = \frac{1}{4}\left(3\sqrt{1 - \frac{\varkappa\rho_0 a^2}{3}} - \sqrt{1 - \frac{\varkappa\rho_0 r^2}{3}}\right)^2 c^2 dt^2 - \\ - \frac{dr^2}{1 - \frac{\varkappa\rho_0 r^2}{3}} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \tag{6.6}$$

or, in terms of the Hilbert radius r_g (2.78),

$$ds^2 = \frac{1}{4}\left(3\sqrt{1 - \frac{r_g}{a}} - \sqrt{1 - \frac{r^2 r_g}{a^3}}\right)^2 c^2 dt^2 - \\ - \frac{dr^2}{1 - \frac{r^2 r_g}{a^3}} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \tag{6.7}$$

the collapse condition ($g_{00}=0$) in this case has the form

$$3\sqrt{1 - \frac{\varkappa\rho_0 a^2}{3}} - \sqrt{1 - \frac{\varkappa\rho_0 r^2}{3}} = 0\,, \tag{6.8}$$

or, similarly,

$$3\sqrt{1 - \frac{r_g}{a}} - \sqrt{1 - \frac{r^2 r_g}{a^3}} = 0\,. \tag{6.9}$$

Thus, we obtain the radial coordinate r, by which a non-rotating liquid star whose radius is a meets the state of gravitational collapse:

$$r_c = \sqrt{9a^2 - \frac{8a^3}{r_g}}\,. \tag{6.10}$$

Since we keep in mind real cosmic objects, the numerical value of r_c should be real (as well as a and r_g). This requirement is satisfied by the following obvious condition:

$$a \leqslant 1.125\, r_g\,. \tag{6.11}$$

If this condition holds not ($a \geqslant 1.125\, r_g$), the non-rotating liquid body (star) cannot be in the state of gravitational collapse.

The general collapse condition (6.11) includes the particular condition $a = r_g$. Given this particular case of a collapsed non-rotating liquid star, we see that the physical radius a of the star's surface, the Hilbert radius r_g of the star, and the radius of the outer space breaking $r_{br} = \sqrt{a^3/r_g}$ of the star's field coincide:

$$r_c = r_{br} = r_g = a\,. \tag{6.12}$$

The said collapse condition, $a = r_g$, is only a particular case of the general collapse condition (6.11). The general collapse condition (6.11) includes three particular cases, concerning the location of the physical surface of the collapsed liquid star:

1) The collapsed liquid star is larger than the Hilbert radius calculated for the star ($a > r_g$) but less than $1.125\, r_g$;
2) The surface of the collapsed liquid star coincides with its Hilbert radius ($a = r_g$);
3) The collapsed liquid star is completely located within its Hilbert radius ($a < r_g$).

It is obvious that r_c is imaginary for $r_g \ll a$, so the state of gravitational collapse is impossible for such a star. For example, considering the Sun ($a = 7\times10^7$ cm, $M = 2\times10^{33}$ gram, $r_g = 3\times10^5$ cm), we see that r_c (6.10) takes an imaginary numerical value. The same is as well true for other regular stars, ranging from super-giants to white dwarfs. Hence, *regular stars cannot collapse.*

In fact, the particular collapse condition $r_c = r_{br} = r_g = a$ (6.12) formulates the collapse radius r_c as follows*

$$r_c = a = \sqrt{\frac{3}{\varkappa\rho_0}} = \frac{4.0\times10^{13}}{\sqrt{\rho_0}}\ \text{cm}. \tag{6.13}$$

For example, if a collapsed liquid sphere should consist of regular water ($\rho_0 = 1.0$ gram/cm^3), its radius would be $r_c = 4.0\times10^{13}$ cm $= 3.1$ AU, i.e. be located within the asteroid strip (the asteroids are located, approximately, from 2.1 AU to 4.3 AU from the Sun).

*$\varkappa = \frac{8\pi G}{c^2} = 18.6\times10^{-28}$ cm/gram is Einstein's constant of gravitation.

One more example: to be a collapsar whose size is that of neutron stars and pulsars (their radius is $a = (8\text{–}16)\times 10^{5}$ cm $= 8\text{–}16$ km), a liquid body should have $\rho_0 = 2.5\times 10^{15}\text{–}6.3\times 10^{14}$ gram/cm^3, according to the obtained formula for r_c (6.13).

Hence forth, we can calculate the mass of a non-rotating liquid collapsar, proceeding from the formulae $M = \frac{4}{3}\pi a^3 \rho_0$ and $a = r_c = \sqrt{3/\varkappa\rho_0}$ (6.13). We obtain the following dependencies:

$$M = \frac{4\pi a}{\varkappa} = 6.8\times 10^{27}\, a \;\; \text{gram}, \tag{6.14}$$

$$M = \frac{4\sqrt{3}\,\pi}{\varkappa^{3/2}\sqrt{\rho_0}} = \frac{2.7\times 10^{41}}{\sqrt{\rho_0}} \;\; \text{gram}. \tag{6.15}$$

For example, once a collapsed liquid sphere should have the size of neutron stars and pulsars, $a = (8\text{–}16)\times 10^{5}$ cm $= 8\text{–}16$ km, its mass would be $M = (5.4\text{–}11)\times 10^{33}$ gram (i.e. 2.7–5.5 masses of the Sun).

§6.2 The Universe as a huge liquid collapsar

Here is another example: the Universe itself. Astronomers estimate the average density of substance in the Universe to be in the range of 10^{-28} to 10^{-31} gram/cm^3. Also, according to the observational estimates of the Hubble constant $H = \frac{c}{a} = (2.3 \pm 0.3)\times 10^{-18}$ sec^{-1}, the radius of the Universe is $a = 1.3\times 10^{28}$ cm. At the upper boundary of the estimated density $\rho_0 = 10^{-28}$ gram/cm^3, the collapse radius r_c (6.10) meets the field of real numerical values. Respectively, we obtain, according to observational estimates,

$$\left.\begin{aligned} a &= 1.3\times 10^{28}\ \text{cm} \\ \rho_0 &= 10^{-28}\ \text{gram/cm}^3 \\ M &= 9.2\times 10^{56}\ \text{gram} \\ r_g &= 1.4\times 10^{28}\ \text{cm} \\ r_{br} &= 1.3\times 10^{28}\ \text{cm} \\ r_c &= 1.5\times 10^{28}\ \text{cm} \end{aligned}\right\}. \tag{6.16}$$

This is a reason to suggest that the Universe can be considered as a sphere of perfect liquid, which is in the state of gravitational collapse. We will refer to it as the *liquid model of the Universe*. In this case, we should have $r_c = r_{br} = r_g = a$ (6.12). Proceeding from this condition and the numerical value of the radius of the Universe, $a = 1.3\times 10^{28}$ cm,

	M, gram	ρ_0, g/cm^3	a, cm	r_g, cm	r_{br}, cm	r_c, cm
Astron. esteems	9.2×10^{56}	$\sim10^{-28}$	1.3×10^{28}	1.4×10^{28}	1.3×10^{28}	1.5×10^{28}
Liquid model	8.8×10^{55}	9.6×10^{-31}	1.3×10^{28}	1.3×10^{28}	1.3×10^{28}	1.3×10^{28}

Table 5.1: The model of the observable Universe as a non-rotating liquid sphere in the state of gravitational collapse. The calculated parameters of the liquid model are compared to the observational esteems.

obtained from the Hubble constant, we calculate the mass and density which should be attributed to the Universe in the framework of the present liquid model (according to $a = r_g = \frac{2GM}{c^2}$ and $M = \frac{4}{3}\pi a^3 \rho_0$). We obtain

$$\left.\begin{aligned} a &= 1.3\times10^{28}\ \text{cm} \\ \rho_0 &= 9.6\times10^{-31}\ \text{gram/cm}^3 \\ M &= 8.8\times10^{55}\ \text{gram} \\ r_g &= 1.3\times10^{28}\ \text{cm} \\ r_{br} &= 1.3\times10^{28}\ \text{cm} \\ r_c &= 1.3\times10^{28}\ \text{cm} \end{aligned}\right\}. \tag{6.17}$$

The obtained numerical values (6.17) are compared to the estimates of observational astronomy (6.16) in Table 5.1. Since these observational estimates are known very approximately, we can conclude that the observable Universe is a huge collapsar, so all the world we observe, including us, is located within a huge black hole.

In particular, this conclusion meets another made in 1965 by Kyril Stanyukovich [31]. He neither studied the geometric properties of a liquid sphere nor introduced a particular space metric. His analysis was based on the properties of elementary particles. Following this way, Stanyukovich obtained that the Hilbert radius of the Universe is the same as the observed event horizon: the observable Universe is a collapsar. Thus, despite employing another theoretical basis than our own, he arrived at the same conclusion.

§6.3 Pressure and density inside liquid collapsars

Pressure and density inside non-rotating liquid collapsars... The regular formula (2.130) we have obtained for the pressure p inside a sphere of perfect liquid,

$$p = \rho_0 c^2 \frac{\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}} - \sqrt{1-\frac{\varkappa\rho_0 a^2}{3}}}{3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}} - \sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}}\,, \tag{6.18}$$

under the collapse condition $a = \sqrt{3/\varkappa\rho_0}$ takes the simplest form

$$p = -\,\rho_0 c^2 = const, \tag{6.19}$$

where $\rho_0 = const$ by definition inside a sphere filled with perfect liquid. This formula is the *equation of state* of the liquid. This state is known as *inflation*: at positive density of the substance the pressure is negative, so the inner pressure of the substance tries to expand the body from within (despite a liquid body is incompressible).

As is seen, the pressure is constant as well as the density. This means that the liquid substance, which fills a liquid collapsar, is in the state of inflation, and has the same pressure and density throughout the entire volume of the collapsar, from its center to the surface.

§6.4 The inner forces of gravitation. The inner redshift

The formula for the force of gravitation acting inside a non-rotating liquid collapsar can be found from the formula for the force acting inside a non-rotating liquid sphere, once the sphere is in the state of gravitational collapse (in this case, its physical radius is $a = r_g = \sqrt{3/\varkappa\rho_0}$).

Following this way, on the basis of the obtained formulae of the covariant component F_1 (2.123, 2.125) and the contravariant component F^1 (2.124, 2.126) of the gravitational force, we obtain

$$F_1 = \frac{\varkappa\rho_0 c^2 r}{3}\,\frac{1}{1-\frac{\varkappa\rho_0 r^2}{3}} = \frac{c^2 r}{a^2}\,\frac{1}{1-\frac{r^2}{a^2}}\,, \tag{6.20}$$

$$F^1 = \frac{\varkappa\rho_0 c^2 r}{3} = \frac{c^2 r}{a^2}\,. \tag{6.21}$$

Since $r < a$ inside the sphere, $F_1 > 0$. Therefore, this is a force of repulsion. The force increases with distance r, from zero at the center of the liquid collapsar to its ultimate-high value on the surface.

If the observable Universe is really a huge liquid collapsar (at least astronomical data evidence it, as was shown above), the repulsive radial force acting inside the collapsar may cause a frequency shift in travelling photons. To investigate this problem we consider the chr.inv.-equations of isotropic geodesics [18, 19]

$$\left.\begin{aligned} &\frac{d\omega}{d\tau}-\frac{\omega}{c^2}\,F_i\,c^i+\frac{\omega}{c^2}\,D_{ik}\,c^i c^k=0 \\ &\frac{d\left(\omega c^i\right)}{d\tau}+2\omega\left(D^i_k+A^{\cdot i}_{k\cdot}\right)c^k-\omega F^i+\omega\,\Delta^i_{nk}\,c^n c^k=0 \end{aligned}\right\}, \tag{6.22}$$

which are the equations of observable motion of a light-like (massless) particle which travels with the observable velocity of light c^i (a photon whose frequency is ω). The equations of isotropic geodesics result as the observable projections of the well-known generally covariant equations of isotropic geodesics (see [18, 19] for details).

In a rotation-free and non-deforming space ($A_{ik}=0$, $D_{ik}=0$), such as the space of a non-rotating liquid collapsar, the equations (6.22) take the form

$$\left.\begin{aligned} &\frac{d\omega}{d\tau}-\frac{\omega}{c^2}\,F_i\,c^i=0 \\ &\frac{d\left(\omega c^i\right)}{d\tau}-\omega F^i+\omega\,\Delta^i_{nk}\,c^n c^k=0 \end{aligned}\right\}. \tag{6.23}$$

Let a photon travel only along the radial direction $x^1\!=r$. Consider the chr.inv.-scalar geodesic equation (equation of energy) of the photon with the obtained formula for F_1 (6.20) substituted. We also take into account that the photon's observable velocity is the observable velocity of light along the radial direction, $c^1=\frac{dr}{d\tau}$. We obtain

$$\frac{1}{\omega}\,\frac{d\omega}{d\tau}=\frac{r}{a^2-r^2}\,\frac{dr}{d\tau}\,. \tag{6.24}$$

This equation solves as $d\ln\omega=-\frac{1}{2}\,d\ln|a^2-r^2|$, or

$$d\ln\omega=d\ln\frac{1}{\sqrt{a^2-r^2}}\,. \tag{6.25}$$

Herefrom we obtain the function

$$\omega\left(r\right)=\frac{Q}{\sqrt{a^2-r^2}}\,,\qquad Q=const. \tag{6.26}$$

The integration constant Q is found from the obvious boundary condition $\omega\left(r=0\right)=\omega_0$. It is $Q=a^2\omega_0$. Finally, we arrive at the solution

$$\omega=\frac{\omega_0}{\sqrt{1-\frac{r^2}{a^2}}}\,. \tag{6.27}$$

At distances (of the photon's travel) small to the physical radius of the collapsar ($r \ll a$), this formula becomes

$$\omega \simeq \omega_0 \left(1 + \frac{r^2}{2a^2}\right). \tag{6.28}$$

This causes a *square redshift* (we also refer to it as a *parabolic redshift*, due to the parabolic square function)

$$z = \frac{\omega - \omega_0}{\omega_0} = \frac{1}{\sqrt{1 - \frac{r^2}{a^2}}} - 1 > 0 \tag{6.29}$$

in the photon's spectrum: the force of repulsion F_1, acting along the radial coordinate from the observer (in the observer's reference frame), deccelerates photons travelling from a distant object to him. At small distances, of the photon's travel ($r \ll a$), the redshift is

$$z \simeq \frac{r^2}{2a^2}, \tag{6.30}$$

or, formulating this result through the Hubble constant $H = \frac{c}{a}$,

$$z \simeq \frac{H^2 r^2}{2c^2}. \tag{6.31}$$

Thus, the observable parameters of the Universe manifest that it is a huge collapsar. These data match the calculations according to the theory of non-rotating liquid collapsars presented here.

§6.5 The state of the collapsed liquid substance

We now discuss the state of the substance that fills non-rotating liquid collapsars. As easy to see, once a liquid star is in the state of gravitational collapse ($r_g = a$), the space metric of the star (6.7) takes the form

$$ds^2 = \frac{1}{4}\left(1 - \frac{r^2}{a^2}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{r^2}{a^2}} - r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right). \tag{6.32}$$

This metric, under the particular condition $a^2 = \frac{3}{\lambda} > 0$ (thus $\lambda > 0$), has the same form as de Sitter's metric (1.5),

$$ds^2 = \left(1 - \frac{\lambda r^2}{3}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{\lambda r^2}{3}} - r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right), \tag{6.33}$$

which describes a spherical distribution of physical vacuum (the λ-field in Einstein's field equations).

This means that liquid collapsars consist of perfect liquid whose state is similar to the state of physical vacuum. The only difference is that the liquid that fills the collapsars possesses positive density, while the density of physical vacuum is negative with $\lambda > 0$ (see §5.2 and §5.3 of our book [18] for details). Also, regular liquid collapsars have a small size and high density (in contrast to the Universe as a whole). Therefore, the liquid that fills the regular (compact) collapsars is in a state similar to the state of high-density physical vacuum.

What is physical vacuum, known also as the λ-field? It is due to the general formulation of Einstein's field equations

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta} R = -\varkappa\, T_{\alpha\beta} + \lambda g_{\alpha\beta} \tag{6.34}$$

containing the λ-term on the right-hand side. The right-hand side determines distributed matter which fills the space, while the left-hand side determines the space geometry which is Riemannian according to the formulation. Re-write the field equations in the form

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta} R = -\varkappa\, \widetilde{T}_{\alpha\beta}\,, \tag{6.35}$$

where the common energy-momentum tensor $\widetilde{T}_{\alpha\beta} = T_{\alpha\beta} + \breve{T}_{\alpha\beta}$ characterizes both distributed substance and physical vacuum (λ-field).

The energy-momentum tensor of physical vacuum

$$\breve{T}_{\alpha\beta} = -\frac{\lambda}{\varkappa}\, g_{\alpha\beta} \tag{6.36}$$

has the following physically observable projections

$$\breve{\rho} = \frac{\breve{T}_{00}}{g_{00}} = -\frac{\lambda}{\varkappa} = const < 0\,, \tag{6.37}$$

$$\breve{J}^i = \frac{c\,\breve{T}^i_0}{\sqrt{g_{00}}} = 0\,, \tag{6.38}$$

$$\breve{U}^{ik} = c^2 \breve{T}^{ik} = \frac{\lambda}{\varkappa}\, c^2 h^{ik} = -\,\breve{\rho} c^2 h^{ik}. \tag{6.39}$$

which are calculated as well as the observable projections (1.84) of any energy-momentum tensor.

The scalar chr.inv.-projection $\breve{\rho} = -\frac{\lambda}{\varkappa} = const$ implies that physical vacuum is homogeneously distributed in the space, i.e. is a *homogeneous* *medium*. The vectorial chr.inv.-projection $\breve{J}^i = 0$ manifests that the physical vacuum is free of energy flow, i.e. is a *non-emitting medium*.

Let us then find the equation of state of physical vacuum. According to the chronometrically invariant formalism [18, 23], the chr.inv.-stress tensor U^{ik} is expressed through the pressure inside a distributed medium as follows:

$$U_{ik} = p_0 h_{ik} - \alpha_{ik} = p h_{ik} - \beta_{ik}\,, \tag{6.40}$$

where p_0 is the equilibrium pressure known due to the equation of state, p is the true pressure inside the medium, α_{ik} is the chr.inv.-viscous stress tensor, $\beta_{ik} = \alpha_{ik} - \frac{1}{3}\alpha h_{ik}$ is its anisotropic part which reveals itself in anisotropic deformations, while $\alpha = h^{ik}\alpha_{ik}$ is the trace of the viscous stress-tensor α_{ik}. Since a spherically symmetric space is isotropic by definition, we have $\beta_{ik} = 0$ in the present case. Also, by the initial assumption, the medium is non-viscous ($\alpha_{ik} = 0$). Therefore, for physical vacuum, we have

$$\breve{U}_{ik} = \breve{p}\, h_{ik} = -\breve{\rho} c^2 h_{ik}\,. \tag{6.41}$$

Respectively, with the formula of the trace of the observable stress-tensor $U = h^{ik} U_{ik}$, we obtain the equation of state of physical vacuum

$$\breve{p} = -\breve{\rho} c^2, \tag{6.42}$$

which, with negative density $\breve{\rho} = -\frac{\lambda}{\varkappa} < 0$, manifests the state of deflation (the inner pressure of the medium tries to compress the sphere).

Hence forth, we deduce the covariant and contravariant components of the force of gravitation acting inside a vacuum (de Sitter) collapsar. Following the same way of deduction as that for the force acting inside a liquid collapsar (6.20, 6.21), we obtain

$$F_1 = \frac{\lambda c^2 r}{3}\,\frac{1}{1 - \frac{\lambda r^2}{3}}\,, \qquad F^1 = \frac{\lambda c^2 r}{3}\,, \tag{6.43}$$

while for the frequency shift of a photon we obtain

$$\omega = \frac{\omega_0}{\sqrt{1 - \frac{\lambda r^2}{3}}} \simeq \omega_0 \left(1 + \frac{\lambda^2}{6}\right). \tag{6.44}$$

$$z = \frac{\omega - \omega_0}{\omega_0} = \frac{1}{\sqrt{1 - \frac{\lambda r^2}{3}}} - 1 \simeq \frac{\lambda r^2}{6} > 0 \tag{6.45}$$

To understand the results, let us recall that we were able to transform the space metric of a collapsed liquid sphere (6.32) to de Sitter's space metric (6.33) only by the particular condition $a^2 = \frac{3}{\lambda} > 0$. Hence, we have assumed $\lambda > 0$. With $\lambda > 0$ we have obtained a negative density of physical vacuum $\breve{\rho} = -\frac{\lambda}{\varkappa} < 0$ (6.37), the state of inflation $\breve{p} = -\breve{\rho}c^2$ (6.42), the repulsing force $F_1 > 0$ (6.43), and the redshift (6.45).

These are the same results as those obtained for a liquid collapsar, except for the negative density $\breve{\rho} = -\frac{\lambda}{\varkappa} < 0$ (and, hence, the positive pressure $\breve{p} = -\breve{\rho}c^2 > 0$ which gives the state of deflation).

If we should assume a negative value of the λ (i.e. $\lambda < 0$), in order to obtain a positive density of physical vacuum, the collapsar's radius a would be imaginary, which is non-sense for the observed Universe.

However, there is another way to remove this difficulty with respect to the theory. Consider Einstein's field equations (6.34) in a modified form wherein both the energy-momentum tensor of distributed substance and the λ-term are taken with the same sign, i.e.

$$R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta} R = -\varkappa T_{\alpha\beta} - \lambda g_{\alpha\beta}\,. \tag{6.46}$$

In this case, the energy-momentum tensor of physical vacuum is

$$\breve{T}_{\alpha\beta} = \frac{\lambda}{\varkappa}\, g_{\alpha\beta}\,, \tag{6.47}$$

while the physically observable projections of it are

$$\breve{\rho} = \frac{\breve{T}_{00}}{g_{00}} = \frac{\lambda}{\varkappa} = const > 0\,, \tag{6.48}$$

$$\breve{J}^i = \frac{c\,\breve{T}_0^i}{\sqrt{g_{00}}} = 0\,, \tag{6.49}$$

$$\breve{U}^{ik} = c^2 \breve{T}^{ik} = -\frac{\lambda}{\varkappa}\, c^2 h^{ik} = -\breve{\rho} c^2 h^{ik}. \tag{6.50}$$

Given this case, physical vacuum (the λ-field) is as well in the state of inflation ($\breve{p} = -\breve{\rho}c^2$), however its density is positive $\breve{\rho} = \frac{\lambda}{\varkappa} > 0$. Thus, the modified form (6.46) of Einstein's field equations removes the aforementioned contradiction between the theory of liquid collapsars and the observed positive density of substance in the Universe.

Hence, physical vacuum (the λ-field) is a homogeneous, non-viscous, non-emitting medium in the state of inflation.

Concerning the deduced redshift formula (6.45), it depends only on the formulation of the force of repulsion which is deduced from g_{00} of de Sitter's metric (6.33). Since we did not change the space metric, the redshift formula (6.45) remains unchanged as well.

§6.6 Time flows in the opposite direction inside collapsars

In a rotation-free space such as the space of a non-rotating liquid star, the observable time interval $d\tau$ (1.30) has a simplified formulation which is $d\tau = \sqrt{g_{00}}\,dt$. Therefore, $d\tau$ in the field of a non-rotating liquid star, according to g_{00} of the metric (6.6), has the form

$$d\tau = \pm\frac{1}{2}\left(3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}}-\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}\right)dt\,. \qquad (6.51)$$

Under the particular condition $a = r_g = \sqrt{3/\varkappa\rho_0}$ characterizing the star in the state of gravitational collapse, this formula transforms to

$$d\tau = \mp\frac{1}{2}\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}\;dt\,. \qquad (6.52)$$

We see that the sign of the observable time interval $d\tau$ in a liquid star whose state is regular (out of collapse) is opposite to that in the star being a collapsar. In other words, the observable time in the field of regular stars flows in the opposite direction than the observable time inside a collapsar. Just one illustration: we regularly assume that observable time flows from the past to the future. If so, the observable time inside collapsars flows from the future to the past.

§6.7 The boundary conditions of a liquid collapsar

With the condition $a = r_g = \sqrt{3/\varkappa\rho_0}$ characterizing liquid collapsars, the non-zero components of the Riemann-Christoffel curvature tensor $R_{\alpha\beta\gamma\delta}$ (2.113–2.116) obtained in §2.3 take the form

$$R_{0101} = \frac{\varkappa\rho_0}{12} = \frac{1}{4a^2} = const\,, \qquad (6.53)$$

$$R_{1212} = -C_{1212} = \frac{\varkappa\rho_0}{3}\,\frac{r^2}{1-\frac{\varkappa\rho_0 r^2}{3}} = \frac{r^2}{a^2}\,\frac{1}{1-\frac{r^2}{a^2}}\,, \qquad (6.54)$$

$$R_{1313} = -C_{1313} = \frac{\varkappa\rho_0}{3}\,\frac{r^2\sin^2\theta}{1-\frac{\varkappa\rho_0 r^2}{3}} = \frac{r^2}{a^2}\,\frac{\sin^2\theta}{1-\frac{r^2}{a^2}}\,, \qquad (6.55)$$

$$R_{2323} = -C_{2323} = \frac{\varkappa\rho_0}{3}\,r^4\sin^2\theta = \frac{r^4}{a^2}\,\sin^2\theta\,. \qquad (6.56)$$

Since $R_{0101} = \frac{\varkappa\rho_0}{12} = const$ and $R_{0101} > 0$ at the positive density $\rho_0 > 0$ of the liquid, the "inner" space of a liquid collapsar is a four-dimensional *positive constant curvature space.* This is in contrast to our result of §2.3 where we showed that the space of a liquid sphere has a *variable four-curvature* which is *negative.* This means that:

- The state of gravitational collapse is a "bridge" connecting the world of varying four-dimensional negative curvature (world of regular stars) and the world of four-dimensional positive constant curvature (inside those stars in the state of gravitational collapse).

Concerning the three-dimensional observable curvature of the space inside non-rotating liquid collapsars, we calculate C_{11} (2.104), C_{22} (2.105), and the observable curvature scalar $C = h^{ik} C_{ik}$ under the condition $a = r_g = \sqrt{3/\varkappa\rho_0}$ characterizing liquid collapsars. We obtain

$$C_{11} = -\frac{2\varkappa\rho_0}{3}\frac{1}{1-\frac{\varkappa\rho_0 r^2}{3}} = -\frac{2}{a^2}\frac{1}{1-\frac{r^2}{a^2}}\,, \tag{6.57}$$

$$C_{22} = \frac{C_{33}}{\sin^2\theta} = -\frac{2\varkappa\rho_0 r^2}{3} = -\frac{2r^2}{a^2}\,, \tag{6.58}$$

$$C = -\,2\varkappa\rho_0 = -\frac{6}{a^2} = const < 0\,. \tag{6.59}$$

It is a *three-dimensional negative constant curvature space* as well as the space of regular liquid stars (as a matter of fact that regular stars are out of the state of collapse).

Hence forth, we express the force of gravitation acting in the "inner" space of a non-rotating liquid collapsar through the three-dimensional observable curvature of the "inner" space. From the respective formulae for F_1 (6.20) and F^1 (6.21), we obtain

$$F_1 = -\frac{c^2 r}{2}\,C_{11}\,, \qquad F^1 = -\frac{c^2}{2r}\,C_{22}\,. \tag{6.60}$$

we see that both the three-dimensional observable curvature and the force of gravitation possess space breaking:

$$C_{11} \rightarrow -\infty\,, \qquad F_1 = \rightarrow \infty \tag{6.61}$$

by the boundary condition $r = a$ on the surface of the collapsar. This result is, however, trivial.

§6.8 Rotating liquid collapsars

We now reveal rotating liquid collapsars. Let us rotate the space metric of a liquid collapsar (6.32) with an angular velocity ω around the line orthogonal to the equatorial plane. Among the g_{0i}-th components of $g_{\alpha\beta}$, the non-zero component g_{03} characterizes the rotation of space (while $g_{01}=g_{02}=0$). It formulates as

$$g_{03} = -\frac{2\omega r^2 \cos\theta}{c}\,, \tag{6.62}$$

so the linear velocity of space rotation v_i (1.45) is

$$v_3 = \frac{2\omega r^2 \cos\theta}{\sqrt{1-\frac{r^2}{a^2}}}\,, \qquad v_1 = v_2 = 0\,. \tag{6.63}$$

As a result, we obtain the space metric of a rotating liquid collapsar

$$ds^2 = \frac{1}{4}\left(1-\frac{r^2}{a^2}\right)c^2 dt^2 - \frac{dr^2}{1-\frac{r^2}{a^2}} - \\ - \frac{2\omega r^2\cos\theta}{c}\,c\,dt\,d\phi - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{6.64}$$

It is possible to prove that this space metric satisfies Einstein's field equations containing the energy-momentum tensor for perfect liquid (2.4). This means that, once we substitute the particular components of $g_{\alpha\beta}$ taken from the metric (6.64) into the field equations, the left-hand side and right-hand side of the equations are the same: the field equations become identities and are thus satisfied.

The general condition of gravitational collapse means that physical observable time stops ($d\tau=0$). The definition of $d\tau$ (1.30),

$$d\tau = \sqrt{g_{00}}\,dt + \frac{g_{0i}}{c\sqrt{g_{00}}}\,dx^i = \sqrt{g_{00}}\,dt - \frac{1}{c^2}\,v_i dx^i, \tag{6.65}$$

takes both g_{00} and g_{0i} into account. Therefore, with $v_i \neq 0$, the collapse condition is not $d\tau = \sqrt{g_{00}}\,dt = 0$ as that for non-rotating collapsars, but takes, in the present case under consideration, the complete form

$$\sqrt{g_{00}} - \frac{1}{c^2}\,v_3 u^3 = 0\,, \tag{6.66}$$

where $u^3 = \frac{d\phi}{dt} = \omega$. Substituting g_{00} from the metric (6.64), v_3 (6.63), and $u^3=\omega$, we obtain the radius of the collapse surface of a rotating liquid collapsar

$$r_c = \frac{a}{\sqrt{1+\frac{4\,\omega^2 a^2\cos\theta}{c^2}}} \leqslant a\,, \tag{6.67}$$

hence

$$r_c \simeq a\left(1-\frac{2\omega^2 a^2\cos\theta}{c^2}\right) = a - \Delta a\,. \tag{6.68}$$

Assuming $\omega = 10^3$ sec^{-1} and $a = 10^6$ cm for example, we obtain $\Delta a \simeq 22\cos\theta$, i.e. $\Delta a \simeq 22$ meters at the equator of the star, and $\Delta a = 0$ at the South Pole and North Pole.

We see that the collapse surface matches the sphere's radius a only at the poles of the rotation (where the latitude θ is $\pm\frac{\pi}{2}$, so we have $\cos\theta = 0$). In other words, rotating liquid collapsars are not spheres but have an *elliptic form*, which is flattened on the equatorial plane (which is orthogonal to the axis of rotation).

Once the collapsar does not rotate ($\omega = 0$), its form is spherically symmetric ($r_c = a$). Contrarily, at an ultimate relativistic speed of the rotation, the collapsar's elliptic form is highly flattened on the equatorial plane. In the ultimate case, where the collapsar rotates at a speed very close to the speed of light ($\omega^2 a^2 \to c^2$), its form is set up by the equation

$$r_c = \frac{a}{\sqrt{1+4\cos\theta}}\,. \tag{6.69}$$

The other parameters of rotating liquid collapsars we have obtained in the framework of the theory do not change the principal results obtained in §5.1 for non-rotating liquid collapsars. The only difference is that correction for the angular velocity of the collapsar's rotation ω. We therefore omit these results from consideration.

§6.9 Conclusion

Finally, let us recall all the theoretical results of liquid collapsars that have been obtained above:

1. The radial coordinate r_c (6.10) by which a non-rotating liquid sphere of radius a meets gravitational collapse, is

$$r_c = \sqrt{9a^2 - \frac{8a^3}{r_g}}\,.$$

   For regular stars, r_c is in the range of imaginary numerical values. Therefore, regular stars ranging from super-giants to dwarfs and white dwarfs cannot collapse;

2. By the requirement that the collapse radius r_c should be real for real objects, the physical radius a of a collapsar should be

$$a \leqslant 1.125\, r_g\,.$$

If its radius is $a \geqslant 1.125\, r_g$, the non-rotating liquid body (star) cannot be in the state of gravitational collapse.

3. The density of substance is the primary characteristic of non-rotating liquid collapsars. The physical radius a of such a collapsar is reciprocal to the square root of its density ρ_0 (6.13)

$$a = \sqrt{\frac{3}{\varkappa\rho_0}} = \frac{4.0\times10^{13}}{\sqrt{\rho_0}} \text{ cm;}$$

4. The mass M of a non-rotating liquid collapsar is proportional to its physical radius a (6.14)

$$M = \frac{4\pi a}{\varkappa} = 6.8\times10^{27}\, a \text{ gram,}$$

and is reciprocal to the square root of its density ρ_0 (6.15)

$$M = \frac{4\sqrt{3}\,\pi}{\varkappa^{3/2}\sqrt{\rho_0}} = \frac{2.7\times10^{41}}{\sqrt{\rho_0}} \text{ gram;}$$

5. The observable Universe is completely located inside its collapse radius. In other words, it is a gravitational collapsar: all the stars and galaxies, including us, exist within a huge black hole. Its parameters, calculated according to the liquid model, are

$$a = 1.3\times10^{28} \text{ cm,}$$
$$\rho_0 = 9.6\times10^{-31} \text{ gram/cm}^3,$$
$$M = 8.8\times10^{55} \text{ gram;}$$

6. The liquid which fills the collapsars is in the state of inflation

$$p = -\rho_0 c^2 = const,$$

i.e. at the positive density of the substance the pressure is negative, so the inner pressure tries to expand the body from within (while the collapsar does not expand, because a liquid body is incompressible). The pressure and density remain unchanged from the center of the collapsar to its surface;

7. The gravitational inertial force acting inside a non-rotating liquid collapsar is a force of repulsion. It increases with distance, from zero at the center of the collapsar to its ultimate-high value on the surface;

8. The inner force of repulsion produces a square (parabolic) redshift in the photons travelling within the collapsar, to its center;
9. The state of the liquid filling regular (compact) collapsars is similar to the state of high-density physical vacuum (high-density λ-field), which is a homogeneous, non-viscous, non-emitting medium in the state of inflation;
10. The observable time flows in different directions inside and outside collapsars: once we assume that the observable time of our world flows from the past to the future, the observable time inside collapsars flows from the future to the past;
11. The state of gravitational collapse is a "bridge" connecting the world of varying four-dimensional negative curvature (world of regular stars) and the world of four-dimensional positive constant curvature inside gravitational collapsars (black holes);
12. Rotating liquid collapsars are not spheres but possess an *elliptic form* which is flattened on the equatorial plane. The radius r_c of a rotating liquid collapsar is formulated through the sphere's radius a, the latitude θ, and the angular velocity of the respective rotation ω as
$$r_c = \frac{a}{\sqrt{1+\frac{4\,\omega^2 a^2 \cos\theta}{c^2}}}\,.$$

Notations

Ordinary differential of a vector:

$$dA^{\alpha} = \frac{\partial A^{\alpha}}{\partial x^{\sigma}}\, dx^{\sigma}.$$

Absolute differential of a contravariant vector:

$$\mathrm{D}A^{\alpha} = \nabla_{\beta}\, A^{\alpha} dx^{\beta} = dA^{\alpha} + \Gamma^{\alpha}_{\beta\mu}\, A^{\mu} dx^{\beta}.$$

Absolute differential of a covariant vector:

$$\mathrm{D}A_{\alpha} = \nabla_{\beta}\, A_{\alpha} dx^{\beta} = dA_{\alpha} - \Gamma^{\mu}_{\alpha\beta}\, A_{\mu} dx^{\beta}.$$

Absolute derivative of a contravariant vector:

$$\nabla_{\beta}\, A^{\alpha} = \frac{\mathrm{D}A^{\alpha}}{dx^{\beta}} = \frac{\partial A^{\alpha}}{\partial x^{\beta}} + \Gamma^{\alpha}_{\beta\mu}\, A^{\mu}.$$

Absolute derivative of a covariant vector:

$$\nabla_{\beta}\, A_{\alpha} = \frac{\mathrm{D}A_{\alpha}}{dx^{\beta}} = \frac{\partial A_{\alpha}}{\partial x^{\beta}} - \Gamma^{\mu}_{\alpha\beta}\, A_{\mu}\,.$$

Absolute derivative of a 2nd rank contravariant tensor:

$$\nabla_{\beta}\, F^{\sigma\alpha} = \frac{\partial F^{\sigma\alpha}}{\partial x^{\beta}} + \Gamma^{\alpha}_{\beta\mu}\, F^{\sigma\mu} + \Gamma^{\sigma}_{\beta\mu}\, F^{\alpha\mu}.$$

Absolute derivative of a 2nd rank covariant tensor:

$$\nabla_{\beta}\, F_{\sigma\alpha} = \frac{\partial F_{\sigma\alpha}}{\partial x^{\beta}} - \Gamma^{\mu}_{\alpha\beta}\, F_{\sigma\mu} - \Gamma^{\mu}_{\sigma\beta}\, F_{\alpha\mu}\,.$$

Absolute divergence of a vector:

$$\nabla_{\alpha}\, A^{\alpha} = \frac{\partial A^{\alpha}}{\partial x^{\alpha}} + \Gamma^{\alpha}_{\alpha\sigma}\, A^{\sigma}.$$

Chr.inv.-divergence of a chr.inv.-vector:

$$^{*}\nabla_{i}\, q^{i} = \frac{^{*}\partial q^{i}}{\partial x^{i}} + q^{i}\, \frac{^{*}\partial \ln\sqrt{h}}{\partial x^{i}} = \frac{^{*}\partial q^{i}}{\partial x^{i}} + q^{i}\, \Delta^{j}_{ji}\,.$$

Physical chr.inv.-divergence:

$$^{*}\widetilde{\nabla}_{i}\, q^{i} = {}^{*}\nabla_{i}\, q^{i} - \frac{1}{c^{2}}\, F_{i}\, q^{i}.$$

D'Alembert's general covariant operator:

$$\Box = g^{\alpha\beta}\,\nabla_\alpha\nabla_\beta\,.$$

Laplace's ordinary operator:

$$\Delta = -\,g^{ik}\,\nabla_i\nabla_k\,.$$

Chr.inv.-Laplace operator:

$$^*\Delta = h^{ik}\;{}^*\nabla_i\,{}^*\nabla_k\,.$$

Chr.inv.-derivative with respect to the time coordinate and that with respect to the spatial coordinates:

$$\frac{^*\partial}{\partial t} = \frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}\,, \qquad \frac{^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} + \frac{1}{c^2}\,v_i\,\frac{^*\partial}{\partial t}\,.$$

The square of the physically observable velocity:

$$\mathrm{v}^2 = \mathrm{v}_i\,\mathrm{v}^i = h_{ik}\,\mathrm{v}^i\mathrm{v}^k.$$

The linear velocity of the space rotation:

$$v_i = -\,c\,\frac{g_{0i}}{\sqrt{g_{00}}}\,, \quad v^i = -\,c\,g^{0i}\sqrt{g_{00}}\,, \quad v_i = h_{ik}\,v^k.$$

The square of v_i. This is the proof: because of $g_{\alpha\sigma}g^{\sigma\beta} = g_\alpha^\beta$, then under $\alpha = \beta = 0$ we have $g_{0\sigma}g^{\sigma 0} = \delta_0^0 = 1$, hence $v^2 = v_k\,v^k = c^2(1 - g_{00}\,g^{00})$, i.e.:

$$v^2 = h_{ik}\,v^i v^k.$$

The determinants of the metric tensors $g_{\alpha\beta}$ and $h_{\alpha\beta}$ are connected as:

$$\sqrt{-\,g} = \sqrt{h}\,\sqrt{g_{00}}\,.$$

Derivative with respect to the physically observable time:

$$\frac{d}{d\tau} = \frac{^*\partial}{\partial t} + \mathrm{v}^k\frac{^*\partial}{\partial x^k}\,.$$

The 1st derivative with respect to the space-time interval:

$$\frac{d}{ds} = \frac{1}{c\sqrt{1-\frac{\mathrm{v}^2}{c^2}}}\frac{d}{d\tau}\,.$$

The 2nd derivative with respect to the space-time interval:

$$\frac{d^2}{ds^2} = \frac{1}{c^2 - \mathrm{v}^2}\frac{d^2}{d\tau^2} + \frac{1}{\left(c^2 - \mathrm{v}^2\right)^2}\left(D_{ik}\mathrm{v}^i\mathrm{v}^k + \mathrm{v}_i\frac{d\mathrm{v}^i}{d\tau} + \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial x^m}\,\mathrm{v}^i\mathrm{v}^k\mathrm{v}^m\right)\frac{d}{d\tau}\,.$$

The chr.inv.-metric tensor:

$$h_{ik} = -\,g_{ik} + \frac{1}{c^2}\,v_i v_k\,, \quad h^{ik} = -\,g^{ik}, \quad h_i^k = \delta_i^k\,.$$

Zelmanov's relations between the Christoffel regular symbols and the chr.inv.-characteristics of the space of reference:

$$D_k^i + A_{k\cdot}^{\cdot i} = \frac{c}{\sqrt{g_{00}}}\left(\Gamma_{0k}^i - \frac{g_{0k}\Gamma_{00}^i}{g_{00}}\right),$$

$$g^{i\alpha}g^{k\beta}\,\Gamma_{\alpha\beta}^m = h^{iq}h^{ks}\Delta_{qs}^m\,, \quad F^k = -\,\frac{c^2\,\Gamma_{00}^k}{g_{00}}\,.$$

Zelmanov's 1st identity and 2nd identity:

$$\frac{{}^*\partial A_{ik}}{\partial t} + \frac{1}{2}\left(\frac{{}^*\partial F_k}{\partial x^i} - \frac{{}^*\partial F_i}{\partial x^k}\right) = 0\,,$$

$$\frac{{}^*\partial A_{km}}{\partial x^i} + \frac{{}^*\partial A_{mi}}{\partial x^k} + \frac{{}^*\partial A_{ik}}{\partial x^m} + \frac{1}{2}\left(F_i A_{km} + F_k A_{mi} + F_m A_{ik}\right) = 0\,.$$

Derivative from v^2 with respect to the physically observable time:

$$\frac{d}{d\tau}\left(\mathrm{v}^2\right) = \frac{d}{d\tau}\left(h_{ik}\mathrm{v}^i\mathrm{v}^k\right) = 2D_{ik}\mathrm{v}^i\mathrm{v}^k + \frac{{}^*\partial h_{ik}}{\partial x^m}\,\mathrm{v}^i\mathrm{v}^k\mathrm{v}^m + 2\,\mathrm{v}_k\frac{d\mathrm{v}^k}{d\tau}\,.$$

The completely antisymmetric chr.inv.-tensor:

$$\varepsilon^{ikm} = \sqrt{g_{00}}\,E^{0ikm} = \frac{e^{0ikm}}{\sqrt{h}}\,, \quad \varepsilon_{ikm} = \frac{E_{0ikm}}{\sqrt{g_{00}}} = e_{0ikm}\sqrt{h}\,.$$

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Inside Stars

A Theory of the Internal Constitution of Stars, and the Sources of Stellar Energy According to General Relativity

by Larissa Borissova and Dmitri Rabounski

This book announces a mathematical theory of the internal constitution of stars, and the sources of stellar energy according to the General Theory of Relativity. This is an alternative to the conventional theory of gaseous stars which was introduced in the 1920's on the basis of classical mechanics and thermodynamics. In contrast, the common consideration of a star and its field according to the General Theory of Relativity, that presented in this book, comes to the model of liquid stars. Such a star is homogeneous inside, with a tiny core (about a few kilometres in the radius) in the centre. The core is selected from the main mass of the star by the collapse surface with the radius according to the star's mass. Despite almost all mass of the star is located outside the core (the core is not a black hole), the force of gravity approaches to infinity on the surface of the core due to the inner space breaking of the star's field therein. The super-strong force of gravity is sufficient for the transfer of the necessary kinetic energy to the lightweight atomic nuclei of the stellar substance, so that the process of thermonuclear fusion begins. The energy produced by the thermonuclear fusion is that energy which the stars shine: the tiny core of each star is its luminous "inner sun", while the produced stellar energy is then transferred to the physical surface of the star due to the thermal conductivity. A new classification of stars is introduced according to the space breaking of their fields: the regular stars (in the range from the dwarfs to the super-giants), Wolf-Rayet stars, neutron stars (and pulsars), and black holes are considered. The introduced liquid model matches with the new observational evidences for the state of condensed matter inside stars; in particular, that the Sun consists of the high temperature liquid metallic hydrogen.

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