FOUNDATIONS OF ISO-DIFFERENTIAL CALCULUS
VOLUME IV
ISO-DYNAMIC EQUATIONS

Foundations of Iso-Differential Calculus Volume IV Iso-Dynamic Equations

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Contents

Preface

This book is intended for readers who have had a course in difference equations, iso-differential calculus and it can be used for a senior undergraduate course.

Chapter 1 deals with the linear first-order iso-difference equations, equilibrium points, eventually equilibrium points, periodic points and cycles.

In Chapter 2 are introduced the iso-difference calculus and the general theory of the linear homogeneous and nonhomogeneous iso-difference equations.

In Chapter 3 are studied the systems of linear iso-difference equations and the linear periodic systems.

Chapter 4 is devoted to the stability theory. They are considered the nonautonomous linear systems, Lyapunov's direct method, stability by linear approximation.

In Chapter 5 is considered the oscillation theory. They are defined the iso-self-adjoint second-order iso-difference equations and they are given some of their properties. They are considered some classes nonlinear iso-difference equations.

In Chapter 6 is studied the asymptotic behavior of some classes iso-difference equations.

Time scales iso-calculus is introduced in Chapter 7. They are given the main properties of the backward and forward jump iso-operators. They are considered the iso-differentiation and iso-integration. They are introduced the iso-Hilger's complex plane and the iso-exponential function.

I will be very grateful to anybody who wants to inform me about errors or just misprints, or wants to express criticism or other comments, to my e-mails svetlingeorgiev1@gmail.com, sgg2000bg@yahoo.com.

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Chapter 1

Introduction

1.1. Linear First-Order Iso-Difference Equations

Throughout of this book we will suppose that $\hat{T} : \mathbb{R} \longrightarrow (0, \infty)$ and n, $n_0 \in \mathbb{N}$.

Definition 1.1.1. *The first-order linear iso-difference equation will be called the equation*

$$\begin{aligned} &\hat{x}^{\wedge}(\widehat{n+1}) = \hat{a}(n)\hat{\times}\hat{x}^{\wedge}(\hat{n}) + \hat{g}(n), \qquad n \geq n_0, \\ &x(n_0) = x_0, \end{aligned} \tag{1}$$

where x is unknown function, $\hat{a}$ and $\hat{g}$ are given iso-functions of first, second, third, fourth or fifth kind on $\mathbb{N}$, x_0 is a given real number.

The equation (1) we can rewrite in the following form.

$$\begin{aligned} &\tfrac{x(n+1)}{\hat{T}(n+1)} = \hat{a}(n)\hat{T}(n)\tfrac{x(n)}{\hat{T}(n)} + \hat{g}(n), \qquad n \geq n_0, \\ &x(n_0) = x_0, \end{aligned}$$

or

$$\begin{aligned} &x(n+1) = \hat{a}(n)\hat{T}(n+1)x(n) + \hat{T}(n+1)\hat{g}(n), \qquad n \geq n_0, \\ &x(n_0) = x_0. \end{aligned}$$

Example 1.1.2. *In the case when $\hat{a}$ and $\hat{g}$ are iso-functions of first kind, the equation (1) we can rewrite in the form.*

$$\begin{aligned} &x(n+1) = \tfrac{a(n)\hat{T}(n+1)}{\hat{T}(n)}x(n) + \hat{T}(n+1)\tfrac{g(n)}{\hat{T}(n)}, \qquad n \geq n_0, \\ &x(n_0) = x_0. \end{aligned}$$

Example 1.1.3. *In the case when $\hat{a}$ is an iso-function of first kind and $\hat{g}$ is an iso-function of fourth kind, the equation (1) we can rewrite in the form.*

$$\begin{aligned} &x(n+1) = \tfrac{a(n)\hat{T}(n+1)}{\hat{T}(n)}x(n) + \hat{T}(n+1)g\left(n\hat{T}(n)\right), \qquad n \geq n_0, \\ &x(n_0) = x_0. \end{aligned}$$

Example 1.1.4. *In the case when $\hat{a}$ is an iso-function of third kind and $\hat{g}$ is an iso-function of fourth kind, the equation* (1) *we can rewrite in the form.*

$$x(n+1) = \frac{a\left(\frac{n}{\hat{T}(n)}\right)\hat{T}(n+1)}{\hat{T}(n)} x(n) + \hat{T}(n+1) g\left(n\hat{T}(n)\right), \qquad n \geq n_0,$$

$$x(n_0) = x_0.$$

One may obtain the solution of (1) by a simple iteration.

$$x(n_0+1) = \hat{a}(n_0)\hat{T}(n_0+1)x(n_0) + \hat{T}(n_0+1)\hat{g}(n_0)$$

$$= \hat{a}(n_0)\hat{T}(n_0+1)x_0 + \hat{T}(n_0+1)\hat{g}(n_0),$$

$$x(n_0+2) = \hat{a}(n_0+1)\hat{T}(n_0+2)x(n_0+1) + \hat{T}(n_0+2)\hat{g}(n_0+1)$$

$$= \hat{a}(n_0+1)\hat{T}(n_0+2)\left(\hat{a}(n_0)\hat{T}(n_0+1)x_0 + \hat{T}(n_0+1)\hat{g}(n_0)\right) + \hat{T}(n_0+2)\hat{g}(n_0+1)$$

$$= \hat{a}(n_0)\hat{a}(n_0+1)\hat{T}(n_0+1)\hat{T}(n_0+2)x_0 + \hat{a}(n_0+1)\hat{T}(n_0+2)\hat{T}(n_0+1)\hat{g}(n_0)$$

$$+\hat{T}(n_0+2)\hat{g}(n_0+1).$$

We suppose that

$$x(n_0+k) = \prod_{i=0}^{k-1} \hat{a}(n_0+i)\hat{T}(n_0+i+1)x_0$$

$$+\hat{T}(n_0+k)\sum_{i=1}^{k-1}\prod_{j=i}^{k-1} \hat{a}(n_0+j)\hat{T}(n_0+j+1)\hat{g}(n_0+i-1) + \hat{T}(n_0+k)\hat{g}(n_0+k-1)$$

for some $k \in \mathbb{N}$, $k \geq 2$.

Hence,

$$x(n_0+k+1) = \hat{a}(n_0+k)\hat{T}(n_0+k+1)x(n_0+k) + \hat{T}(n_0+k+1)\hat{g}(n_0+k)$$

$$= \hat{a}(n_0+k)\hat{T}(n_0+k+1)\Big(\prod_{i=0}^{k-1} \hat{a}(n_0+i)\hat{T}(n_0+i+1)x_0$$

$$+\hat{T}(n_0+k)\sum_{i=1}^{k-1}\prod_{j=i}^{k-1} \hat{a}(n_0+j)\hat{T}(n_0+j+1)\hat{g}(n_0+i-1) + \hat{T}(n_0+k)\hat{g}(n_0+k-1)\Big)$$

$$+\hat{T}(n_0+k+1)\hat{g}(n_0+k)$$

$$= \prod_{i=0}^{k} \hat{a}(n_0+i)\hat{T}(n_0+i+1)x_0$$

$$+\hat{T}(n_0+k+1)\sum_{i=1}^{k}\prod_{j=i}^{k} \hat{a}(n_0+j)\hat{T}(n_0+j+1)\hat{g}(n_0+i-1) + \hat{T}(n_0+k+1)\hat{g}(n_0+k)$$

for some $k \in \mathbb{N}$, $k \geq 2$.

Example 1.1.5. *Let $n_0 = 1$, $x_0 = c$, $c \in \mathbb{R}$, $a(n) = (n+1)\frac{n+2}{n+3}$, $\hat{T}(n) = n+1$, $n \in \mathbb{N}$. We consider the equation* (1) *in the case when $\hat{a}$ is an iso-function of first kind and $\hat{g} \equiv 0$.*

We have

$$\frac{a(n)\hat{T}(n+1)}{\hat{T}(n)} = \frac{(n+1)\frac{n+2}{n+3}(n+3)}{n+2}$$

$$= n+1.$$

Then the equation (1) *takes the form*

$$x(n+1) = (n+1)x(n), \qquad n \in \mathbb{N},$$

$$x(1) = c.$$

We will find a solution of the obtained equation. We have

$$x(2) = 2x(1)$$

$$= 2c,$$

$$x(3) = 3x(2)$$

$$= 2.3c$$

$$= 3!c.$$

We suppose that

$$x(n) = n!c$$

for some $n \in \mathbb{N}$, $n \geq 3$. *Hence,*

$$x(n+1) = (n+1)x(n)$$

$$= (n+1)n!c$$

$$= (n+1)!c, \qquad n \geq 3.$$

Example 1.1.6. *Now we consider the equation* (1) *in the case when* $\hat{a}$ *is an iso-function of first kind,* $\hat{g}$ *is an iso-function of fourth kind. Let* $n_0 = 1$, $x_0 = \frac{1}{2}$, $a(n) = \frac{2n}{n+1}$, $g(n) = \frac{1}{n+1}3^{\sqrt{n}}$, $\hat{T}(n) = n$, $n \in \mathbb{N}$.

Then

$$\frac{a(n)\hat{T}(n+1)}{\hat{T}(n)} = \frac{\frac{2n}{n+1}(n+1)}{n}$$

$$= 2,$$

$$\hat{T}(n+1)g\left(n\hat{T}(n)\right) = \hat{T}(n+1)g(n^2)$$

$$= (n+1)\frac{1}{n+1}3^n$$

$$= 3^n.$$

The equation (1) *takes the form*

$$x(n+1) = 2x(n) + 3^n, \qquad n \geq 1,$$

$$x(1) = \tfrac{1}{2}.$$

We will find a solution of the obtained equation. We have

$$\begin{aligned} x(2) &= 2x(1) + 3^1 \\ &= 2.\tfrac{1}{2} + 3^1 \\ &= 1 + 3^1 \\ &= 3^2 - 5.2^0, \\ x(3) &= 2x(2) + 3^2 \\ &= 2\left(1 + 3^1\right) + 3^2 \\ &= 8 + 3^2 \\ &= 8 + 3^2 + 3^3 - 3^3 \\ &= 3^3 - 10 \\ &= 3^3 - 5.2^1. \end{aligned}$$

We suppose that

$$x(n) = 3^n - 5.2^{n-2}$$

for some $n \geq 3$.

Hence,

$$\begin{aligned} x(n+1) &= 2x(n) + 3^n \\ &= 2\left(3^n - 5.2^{n-2}\right) + 3^n \\ &= 2.3^n + 3^n - 5.2^{n-1} \\ &= 3^{n+1} - 5.2^{n-1} \end{aligned}$$

for $n \in \mathbb{N}$.

Example 1.1.7. *Let* $n_0 = 1$, $x_0 = c$, $c \in \mathbb{R}$, $\hat{T}(n) = n$, $a(n) = \frac{n}{n+1}e^{2n}$, $n \in \mathbb{N}$, $\hat{a}$ *is an iso-function of first kind and* $\hat{g} \equiv 0$. *Then*

$$\begin{aligned} \frac{a(n)\hat{T}(n+1)}{\hat{T}(n)} &= \frac{\frac{n}{n+1}e^{2n}(n+1)}{n} \\ &= e^{2n}. \end{aligned}$$

The equation (1) *takes the form*

$$x(n+1) = e^{2n}x(n), \qquad x(1) = c.$$

We will find a solution of the obtained equation. We have

$$x(2) = e^{2}x(1)$$

$$= ce^{2},$$

$$x(3) = e^{4}x(2)$$

$$= ce^{4}e^{2}$$

$$= ce^{6}$$

$$= ce^{3.2}.$$

We suppose that

$$x(n) = ce^{n(n-1)}$$

for some $n \in \mathbb{N}$. *Then*

$$x(n+1) = e^{2n}x(n)$$

$$= ce^{2n}e^{n(n-1)}$$

$$= ce^{(n+1)n}, \qquad n \in \mathbb{N}.$$

1.2. Equilibrium Points

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$. We define the following functions.

$$f^{\vee}(x(n)) := f\left(\frac{x(n)}{\hat{T}(x(n))}\right), \qquad f^{\wedge}(x(n)) := f\left(\hat{T}(x(n))x(n)\right).$$

Here we will investigate the iso-difference equation

$$x(n+1) = f^{i}(x(n)), \tag{2}$$

which we can rewrite in the form for $i = 1$

$$x(n+1) = f\left(\frac{x(n)}{\hat{T}(x(n))}\right),$$

for $i = 2$

$$x(n+1) = f\left(x(n)\hat{T}(x(n))\right).$$

Definition 1.2.1. *A point x^* in the domain of f^i, $i = 1,2$, and $\hat{T}$ is said to be an equilibrium point of* (2) *if it is a fixed point of f^i, i.e., $f^i(x^*) = x^*$ or*

$$f\left(\frac{x^*}{\hat{T}(x^*)}\right) = x^*, \qquad f(\hat{T}(x^*)x^*) = x^*.$$

Definition 1.2.2. *Let x be a point in the domain of f^i, $i = 1,2$, and $\hat{T}$. If there exists a positive integer r and an equilibrium point x^* of* (2) *such that*

$$f^{ir}(x) = x^*, \qquad f^{ir-1}(x) \neq x^*,$$

then x is an eventually equilibrium point.

Definition 1.2.3. *The equilibrium point x^* of the equation* (2) *is stable if given* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $|x_0 - x^*| < \delta$ *implies*

$$\left|f^{in}(x_0) - x^*\right| < \varepsilon$$

for all $n > 0$. If x^* is not stable, then it is called unstable.

Definition 1.2.4. *The point x^* is said to be attracting if there exists* $\eta > 0$ *such that*

$$|x(0) - x^*| < \eta$$

implies

$$\lim_{n\longrightarrow\infty} x(n) = x^*.$$

If $\eta = \infty$, *x^* is called a global attractor or globally attracting.*

Definition 1.2.5. *The point x^* is an asymptotically stable equilibrium point if it is stable and attracting. If* $\eta = \infty$, *x^* is said to be globally asymptotically stable.*

Theorem 1.2.6. *Let x^* be an equilibrium point of the equation*

$$x(n+1) = f\left(\frac{x(n)}{\hat{T}(x(n))}\right). \tag{3}$$

Let also, f be continuously differentiable at $\frac{x^*}{\hat{T}(x^*)}$, $\hat{T} \in \mathcal{C}^1(\mathbb{R})$, $N \leq \hat{T}(x)$, $|\hat{T}'(x)| \leq N_2$ for all $x \in \mathbb{R}$, $\left|\frac{x^*}{\hat{T}(x^*)}\right| \leq N_3$, $\left|f'\left(\frac{x^*}{\hat{T}(x^*)}\right)\right| < M$, $\frac{M(1+N_2N_3)}{N} < 1$. Then x^* is asymptotically stable.

Proof. We note that there is an interval $J = (x^* - \gamma, x^* + \gamma)$ such that

$$\left|f'\left(\frac{x}{\hat{T}(x)}\right)\right| \leq M$$

for every $x \in J$.

Really, let us suppose that for each open interval $I_n = \left(x^* - \frac{1}{n}, x^* + \frac{1}{n}\right)$, for large n, there is a point $x_n \in I_n$ such that

$$\left|f'\left(\frac{x_n}{\hat{T}(x_n)}\right)\right| > M.$$

Since $\hat{T}$ is a continuous function on $\mathbb{R}$ and

$$\lim_{n\longrightarrow\infty}\frac{x_n}{\hat{T}(x_n)}=\frac{x^*}{\hat{T}(x^*)},$$

and f' is a continuous function at $\frac{x^*}{\hat{T}(x^*)}$, it follows that

$$\lim_{n\longrightarrow\infty}f'\left(\frac{x_n}{\hat{T}(x_n)}\right)=f'\left(\frac{x^*}{\hat{T}(x^*)}\right).$$

Consequently,

$$\begin{aligned} M &\le \lim_{n\longrightarrow\infty}\left|f'\left(\tfrac{x_n}{\hat{T}(x_n)}\right)\right| \\ &= \left|f'\left(\tfrac{x^*}{\hat{T}(x^*)}\right)\right| \\ &< M, \end{aligned}$$

which is a contradiction. This proves the statement.

For $x(0)\in J$ we have

$$\begin{aligned} |x(1)-x^*| &= \left|f\left(\tfrac{x(0)}{\hat{T}(x(0))}\right)-f\left(\tfrac{x^*}{\hat{T}(x^*)}\right)\right| \\ &= |f'(\xi)|\left|\tfrac{x(0)}{\hat{T}(x(0))}-\tfrac{x^*}{\hat{T}(x^*)}\right| \\ &= |f'(\xi)|\left|\tfrac{x(0)}{\hat{T}(x(0))}-\tfrac{x^*}{\hat{T}(x(0))}+\tfrac{x^*}{\hat{T}(x(0))}-\tfrac{x^*}{\hat{T}(x^*)}\right| \\ &\le |f'(\xi)|\tfrac{|x(0)-x^*|}{\hat{T}(x(0))}+|f'(\xi)||x^*|\tfrac{|\hat{T}(x(0))-\hat{T}(x^*)|}{\hat{T}(x(0))\hat{T}(x^*)} \\ &= |f'(\xi)|\tfrac{|x(0)-x^*|}{\hat{T}(x(0))}+\tfrac{|f'(\xi)||x^*||\hat{T}'(\xi_1)||x(0)-x^*|}{\hat{T}(x(0))\hat{T}(x^*)} \\ &= |f'(\xi)|\left(\tfrac{1}{\hat{T}(x(0))}+\tfrac{|x^*||\hat{T}'(\xi_1)|}{\hat{T}(x(0))\hat{T}(x^*)}\right)|x(0)-x^*| \\ &\le M\left(\tfrac{1}{N}+\tfrac{N_2N_3}{N}\right)|x(0)-x^*| \\ &= \tfrac{M(1+N_2N_3)}{N}|x(0)-x^*|, \end{aligned}$$

i.e.,

$$|x(1)-x^*|\le\frac{M(1+N_2N_3)}{N}|x(0)-x^*|, \tag{4}$$

where ξ is between $\frac{x(0)}{\hat{T}(x(0))}$ and $\frac{x^*}{\hat{T}(x^*)}$, ξ_1 is between $x(0)$ and x^*.

Since

$$\frac{M(1+N_2N_3)}{N}<1,$$

then (4) shows that $x(1)$ is closer to x^* than $x(0)$. Consequently, $x(1)\in J$.

Now we suppose that

$$\begin{aligned} |x(n)-x^*| &\leq \tfrac{M^n(1+N_2N_3)^n}{N^n}|x(0)-x^*|, \\ |x(n)-x^*| &\leq \tfrac{M(1+N_2N_3)}{N}|x(n-1)-x^*| \end{aligned} \tag{5}$$

for some $n \in \mathbb{N}$, i.e., we suppose that $x(n)$ is closer to x^* than $x(n-1)$ and $x(0)$.

Now we consider

$$|x(n+1)-x^*|.$$

We have

$$\begin{aligned} |x(n+1)-x^*| &= \left| f\left(\tfrac{x(n)}{\hat{T}(x(n))}\right) - f\left(\tfrac{x^*}{\hat{T}(x^*)}\right)\right| \\ &= |f'(\xi_2)|\left|\tfrac{x(n)}{\hat{T}(x(n))} - \tfrac{x^*}{\hat{T}(x^*)}\right| \\ &= |f'(\xi_2)|\left|\tfrac{x(n)}{\hat{T}(x(n))} - \tfrac{x^*}{\hat{T}(x(n))} + \tfrac{x^*}{\hat{T}(x(n))} - \tfrac{x^*}{\hat{T}(x^*)}\right| \\ &\leq |f'(\xi_2)|\tfrac{|x(n)-x^*|}{\hat{T}(x(n))} + |f'(\xi_2)||x^*|\tfrac{|\hat{T}(x(n))-\hat{T}(x^*)|}{\hat{T}(x(n))\hat{T}(x^*)} \\ &= |f'(\xi_2)|\tfrac{|x(n)-x^*|}{\hat{T}(x(n))} + \tfrac{|f'(\xi)||x^*||\hat{T}'(\xi_3)||x(n)-x^*|}{\hat{T}(x(n))\hat{T}(x^*)} \\ &= |f'(\xi_2)|\left(\tfrac{1}{\hat{T}(x(n))} + \tfrac{|x^*||\hat{T}'(\xi_3)|}{\hat{T}(x(n))\hat{T}(x^*)}\right)|x(n)-x^*| \\ &\leq M\left(\tfrac{1}{N} + \tfrac{N_2N_3}{N}\right)|x(n)-x^*| \\ &= \tfrac{M(1+N_2N_3)}{N}|x(n)-x^*|, \end{aligned}$$

i.e.,

$$|x(n+1)-x^*| \leq \frac{M(1+N_2N_3)}{N}|x(n)-x^*|,$$

and, using (5), we get

$$|x(n+1)-x^*| \leq \frac{M^{n+1}(1+N_2N_3)^{n+1}}{N^{n+1}}|x(0)-x^*|.$$

Here ξ_2 is between $\frac{x(n)}{\hat{T}(x(n))}$ and $\frac{x^*}{\hat{T}(x^*)}$, ξ_3 is between $x(n)$ and x^*.

Consequently, the inequalities (5) are valid for every $n \in \mathbb{N}$.

For $\varepsilon > 0$ we let

$$\delta = \frac{\varepsilon}{\frac{2M(1+N_2N_3)}{N}}.$$

Thus

$$|x(0)-x^*| < \delta$$

implies that

$$|x(n) - x^*| \le \frac{M^n(1+N_2N_3)^n}{N^n} \frac{\varepsilon}{\frac{2M(1+N_2N_3)}{N}}$$

$$\le \frac{M(1+N_2N_3)}{N} \frac{\varepsilon}{\frac{2M(1+N_2N_3)}{N}}$$

$$= \frac{\varepsilon}{2}$$

$$< \varepsilon$$

for all $n \in \mathbb{N}$. This conclusion suggests stability.

Furthermore,

$$\lim_{n \longrightarrow \infty} |x(n) - x^*| = 0,$$

and thus

$$\lim_{n \longrightarrow \infty} x(n) = x^*,$$

we conclude asymptotic stability. □

Theorem 1.2.7. *Let x^* be an equilibrium point of the equation*

$$x(n+1) = f(x(n)\hat{T}(x(n))). \tag{3'}$$

Let also, f is continuously differentiable at $x^*\hat{T}(x^*)$, $|f'(x^*\hat{T}(x^*))| < M$, $\hat{T} \in \mathcal{C}^1(\mathbb{R})$, $\hat{T}(x) \le N_1$, $|\hat{T}'(x)| \le N_2$ for all $x \in \mathbb{R}$, $M(N_1 + |x^*|N_2) < 1$. Then x^* is asymptotically stable.

Proof. We note that there is an interval $J = (x^* - \gamma, x^* + \gamma)$ such that

$$|f'(x\hat{T}(x))| \le M$$

for all $x \in J$.

Really, let us suppose that for each open interval $I_n = \left(x^* - \frac{1}{n}, x^* + \frac{1}{n}\right)$, for large n, there is a point $x_n \in I_n$ such that

$$\left|f'(x_n\hat{T}(x_n))\right| > M.$$

Since f' is a continuous function at $x^*\hat{T}(x^*)$, $\hat{T}$ is continuous on $\mathbb{R}$, we have that

$$\lim_{n \longrightarrow \infty} f'(x_n\hat{T}(x_n)) = f'(x^*\hat{T}(x^*)).$$

Consequently,

$$M \le \lim_{n \longrightarrow \infty} |f'(x_n\hat{T}(x_n))|$$

$$= |f'(x^*\hat{T}(x^*))|$$

$$< M,$$

which is a contradiction.

For $x(0) \in J$ we have

$$\begin{aligned}
|x(1)-x^*| &= |f(x(0)\hat{T}(x(0)) - f(x^*\hat{T}(x^*))| \\
&= |f'(\xi)||x(0)\hat{T}(x(0)) - x^*\hat{T}(x^*)| \\
&= |f'(\xi)||x(0)\hat{T}(x(0)) - x^*\hat{T}(x(0)) + x^*\hat{T}(x(0)) - x^*\hat{T}(x^*)| \\
&\le |f'(\xi)|\left(|x(0)-x^*|\hat{T}(x(0)) + |x^*||\hat{T}(x(0)) - \hat{T}(x^*)|\right) \\
&= |f'(\xi)|\left(|x(0)-x^*|\hat{T}(x(0)) + |x^*||\hat{T}'(\xi_1)||x(0)-x^*|\right) \\
&\le M(N_1 + |x^*|N_2)|x(0)-x^*|,
\end{aligned}$$

i.e.,

$$|x(1)-x^*| \le M(N_1+|x^*|N_2)|x(0)-x^*|. \tag{6}$$

Here ξ is between $x(0)\hat{T}(x(0))$ and $x^*\hat{T}(x^*)$, and ξ_1 is between $x(0)$ and x^*.

Since

$$M(N_1+|x^*|N_2) < 1,$$

then from (6) it follows that $x(1)$ is closer to x^* than $x(0)$.

We suppose

$$\begin{aligned}
|x(n)-x^*| &\le M^n(N_1+|x^*|N_2)^n|x(0)-x^*|, \\
|x(n)-x^*| &\le M(N_1+|x^*|N_2)|x(n-1)-x^*|
\end{aligned} \tag{7}$$

for some $n \in \mathbb{N}$.

Now we consider

$$|x(n+1)-x^*|.$$

We have

$$\begin{aligned}
|x(n+1)-x^*| &= |f(x(n)\hat{T}(x(n))) - x^*| \\
&= |f(x(n)\hat{T}(x(n))) - f(x^*\hat{T}(x^*))| \\
&= |f'(\xi_2)||x(n)\hat{T}(x(n)) - x^*\hat{T}(x^*)| \\
&= |f'(\xi_2)||x(n)\hat{T}(x(n)) - x^*\hat{T}(x(n)) + x^*\hat{T}(x(n)) - x^*\hat{T}(x^*)| \\
&\le |f'(\xi_2)|\left(\hat{T}(x_n)|x(n)-x^*| + |x^*||\hat{T}(x(n)) - \hat{T}(x^*)|\right) \\
&= |f'(\xi_2)|\left(\hat{T}(x(n))|x(n)-x^*| + |x^*||\hat{T}'(\xi_3)||x(n)-x^*|\right) \\
&\le M(N_1+|x^*|N_2)|x(n)-x^*|,
\end{aligned}$$

where ξ_2 is between $x(n)\hat{T}(x(n))$ and $x^*\hat{T}(x^*)$, and ξ_3 is between $x(0)$ and x^*.

Now we apply (7) and we get

$$|x(n+1)-x^*| \leq M^{n+1}(N_1+|x^*|N_2)^{n+1}|x(0)-x^*|.$$

Consequently, (7) are valid for all $n \in \mathbb{N}$.

Let $\varepsilon > 0$ be arbitrarily chosen and let

$$\delta = \frac{\varepsilon}{2M(N_1+|x^*|N_2)}.$$

Thus

$$|x(0)-x^*| < \delta$$

implies that

$$\begin{aligned}
|x(n)-x^*| &\leq M^n(N_1+|x^*|N_2)^n|x(0)-x^*| \\
&< M^n(N_1+|x^*|N_2)^n\delta \\
&= M^n(N_1+|x^*|N_2)^n\tfrac{\varepsilon}{2M(N_1+|x^*|N_2)} \\
&\leq M(N_1+|x^*|N_2)\tfrac{\varepsilon}{2M(N_1+|x^*|N_2)} \\
&= \tfrac{\varepsilon}{2} \\
&< \varepsilon
\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore x^* is stable.

Because

$$\lim_{n\longrightarrow\infty}|x(n)-x^*| = 0,$$

we conclude that

$$\lim_{n\longrightarrow\infty}x(n) = x^*,$$

therefore we have asymptotic stability. □

Theorem 1.2.8. *Let x^* be an equilibrium point of* (3)*. Let also, f be continuously differentiable at $\frac{x^*}{\hat{T}(x^*)}$, $\hat{T} \in \mathcal{C}^1(\mathbb{R})$, $N \leq \hat{T}(x) \leq N_1$, $|\hat{T}'(x)| \leq N_2$ for all $x \in \mathbb{R}$, $\left|\frac{x^*}{\hat{T}(x^*)}\right| \leq N_3$, $M_1 \leq \left|f'\left(\frac{x^*}{\hat{T}(x^*)}\right)\right| \leq M$. If*

$$\frac{M_1}{N_1} - \frac{MN_2N_3}{N} > a$$

for some positive real number a, then x^* is unstable.

Proof. Let us suppose that x^* is stable.

Let $\varepsilon > 0$ be chosen so that $\varepsilon < a$. Then there exists $\delta = \delta(\varepsilon) > 0$ enough small such that

$$M_1 \leq \left|f'\left(\frac{x}{\hat{T}(x)}\right)\right| \leq M$$

for all $x \in (x^* - \delta, x^* + \delta)$ and the inequality

$$|x_0 - x^*| < \delta$$

implies the inequality

$$\left| f\left(\frac{x_0}{\hat{T}(x_0)}\right) - x^* \right| < \varepsilon |x_0 - x^*|.$$

Hence,

$$\begin{aligned}
\varepsilon|x_0 - x^*| &> \left| f\left(\tfrac{x_0}{\hat{T}(x_0)}\right) - x^* \right| \\
&= \left| f\left(\tfrac{x_0}{\hat{T}(x_0)}\right) - f\left(\tfrac{x^*}{\hat{T}(x^*)}\right) \right| \\
&= |f'(\xi)| \left| \tfrac{x_0}{\hat{T}(x_0)} - \tfrac{x^*}{\hat{T}(x^*)} \right| \\
&= |f'(\xi)| \left| \tfrac{x_0 - x^*}{\hat{T}(x_0)} + x^* \tfrac{\hat{T}(x^*) - \hat{T}(x_0)}{\hat{T}(x_0)\hat{T}(x^*)} \right| \\
&= |f'(\xi)| \left| \tfrac{x_0 - x^*}{\hat{T}(x_0)} + x^* \tfrac{\hat{T}'(\xi_1)(x^* - x_0)}{\hat{T}(x_0)\hat{T}(x^*)} \right| \\
&\geq |f'(\xi)| \tfrac{|x_0 - x^*|}{\hat{T}(x_0)} - |f'(\xi)||x^*| \tfrac{|\hat{T}'(\xi_1)||x^* - x_0|}{\hat{T}(x_0)\hat{T}(x^*)} \\
&\geq \tfrac{M_1}{N_1}|x_0 - x^*| - \tfrac{MN_3N_2}{N}|x^* - x_0| \\
&= \left(\tfrac{M_1}{N_1} - \tfrac{MN_2N_3}{N}\right)|x_0 - x^*| \\
&> a|x_0 - x^*|,
\end{aligned}$$

i.e.,

$$\varepsilon > a,$$

which is a contradiction. Here ξ is between $\frac{x_0}{\hat{T}(x_0)}$ and $\frac{x^*}{\hat{T}(x^*)}$, ξ_1 is between x_0 and x^*.

Consequently, the equilibrium point x^* is unstable. □

Theorem 1.2.9. *Let x^* be an equilibrium point of $(3')$. Let also, f be continuously differentiable at $x^*\hat{T}(x^*)$, $\hat{T} \in \mathcal{C}^1(\mathbb{R})$, $N \leq \hat{T}(x) \leq N_1$, $|\hat{T}'(x)| \leq N_2$ for all $x \in \mathbb{R}$, $M_1 \leq \left| f'\left(\frac{x^*}{\hat{T}(x^*)}\right)\right| \leq M$. If*

$$M_1 N - M|x^*|N_2 > a$$

for some positive real number a, then x^* is unstable.

Proof. Let us suppose that x^* is stable.

Let $\varepsilon > 0$ be chosen so that $\varepsilon < a$. Then there exists $\delta > 0$ enough small such that

$$M_1 \leq \left| f'\left(\frac{x}{\hat{T}(x)}\right) \right| \leq M$$

for all $x \in (x^* - \delta, x^* + \delta)$ and the inequality $|x_0 - x^*| < \delta$ implies

$$\left| f(x_0 \hat{T}(x_0)) - x^* \right| < \varepsilon |x_0 - x^*|.$$

Hence,

$$\begin{aligned}
\varepsilon|x_0 - x^*| &> |f(x_0\hat{T}(x_0)) - x^*| \\
&= |f(x_0\hat{T}(x_0)) - f(x^*\hat{T}(x^*))| \\
&= |f'(\xi_2)||x_0\hat{T}(x_0) - x^*\hat{T}(x_0) + x^*\hat{T}(x_0) - x^*\hat{T}(x^*)| \\
&\geq |f'(\xi_2)|\hat{T}(x_0)|x_0 - x^*| - |f'(\xi_2)||x^*||\hat{T}(x_0) - \hat{T}(x^*)| \\
&= |f'(\xi_2)|\hat{T}(x_0)|x_0 - x^*| - |f'(\xi_2)||x^*||\hat{T}'(\xi_3)||x^* - x_0| \\
&\geq (M_1 N - M|x^*|N_2)|x^* - x_0| \\
&> a|x^* - x_0|,
\end{aligned}$$

i.e.,

$$\varepsilon > a,$$

which is a contradiction. Here ξ_2 is between $x_0\hat{T}(x_0)$ and $x^8\hat{T}(x^*)$, ξ_3 is between x_0 and x^*. □

1.3. Periodic Points and Cycles

Definition 1.3.1. *Let b be such that $\frac{b}{\hat{T}(b)}$ is in the domain of the function f. Then*

(i) *b is called a periodic point of $f^\vee$ if for some $k \in \mathbb{N}$ we have*

$$f^k\left(\frac{b}{\hat{T}(b)}\right) = b.$$

The periodic orbit of b

$$O^\vee(b) = \left\{ \frac{b}{\hat{T}(b)}, f\left(\frac{b}{\hat{T}(b)}\right), \ldots, f^{k-1}\left(\frac{b}{\hat{T}(b)}\right) \right\}$$

is often called k-cycle.

(ii) *b is called eventually k-periodic if for some $m \in \mathbb{N}$, $f^m\left(\frac{b}{\hat{T}(b)}\right)$ is a k-periodic point, i.e.,*

$$f^{m+k}\left(\frac{b}{\hat{T}(b)}\right) = f^m\left(\frac{b}{\hat{T}(b)}\right).$$

Definition 1.3.2. *Let b be a k-periodic point of f. Then b is*

(i) *stable if it is stable fixed point of $f^{\vee k}$.*

(ii) *asymptotically stable if it is an asymptotically stable fixed point of* $f^{\vee k}$.

(iii) *unstable if it is an unstable fixed point of* $f^{\vee k}$.

If b possesses a stability property, then we often speak of the stability of a k-cycle or a periodic orbit

$$O^{\vee}(b) = \left\{ x(0) = \frac{b}{\hat{T}(b)}, x(1) = f\left(\frac{b}{\hat{T}(b)}\right), \ldots, x(k-1) = f^{k-1}\left(\frac{b}{\hat{T}(b)}\right) \right\}.$$

Theorem 1.3.3. *Let*

$$O^{\vee}(b) = \{x(0), x(1), \ldots, x(k-1)\}$$

be a k-cycle of a continuously differentiable function $f^{\vee}$. *Let also,* $\hat{T} \in \mathcal{C}^1(\mathbb{R})$, $N \le \hat{T}(x) \le N_1$, $|\hat{T}'(x)| \le N_2$ *for all* $x \in \mathbb{R}$, $\left|\frac{b}{\hat{T}(b)}\right| \le N_3$,

$$M_1 \le |f'(x(0)) f'(x(1)) \ldots f'(x(k-1))| \le M.$$

Then the following statements hold.

(i) *The k-cycle* $O^{\vee}(b)$ *is asymptotically stable if*

$$\frac{M(1+N_2N_3)}{N} < 1.$$

(ii) *The k-cycle* $O^{\vee}(b)$ *is unstable if*

$$\frac{M_1}{N_1} - \frac{MN_2N_3}{N} > a$$

for some positive real number a.

Definition 1.3.4. *Let b be such that* $b\hat{T}(b)$ *is in the domain of the function* f. *Then*

(i) *b is called a periodic point of* $f^{\wedge}$ *if for some* $k \in \mathbb{N}$ *we have*

$$f^k\left(b\hat{T}(b)\right) = b.$$

The periodic orbit of b

$$O^{\wedge}(b) = \left\{ b\hat{T}(b), f\left(b\hat{T}(b)\right), \ldots, f^{k-1}\left(b\hat{T}(b)\right) \right\}$$

is often called k-cycle.

(ii) *b is called eventually k-periodic if for some* $m \in \mathbb{N}$, $f^m\left(b\hat{T}(b)\right)$ *is a k-periodic point, i.e.,*

$$f^{m+k}\left(b\hat{T}(b)\right) = f^m\left(b\hat{T}(b)\right).$$

Definition 1.3.5. *Let b be a k-periodic point of f. Then b is*

(i) *stable if it is stable fixed point of* $f^{\wedge k}$.

(ii) *asymptotically stable if it is an asymptotically stable fixed point of* $f^{\wedge k}$.

(iii) *unstable if it is an unstable fixed point of* $f^{\wedge k}$.

If b possesses a stability property, then we often speak of the stability of a k-cycle or a periodic orbit

$$O^{\wedge}(b) = \left\{x(0) = b\hat{T}(b), x(1) = f\left(b\hat{T}(b)\right), \ldots, x(k-1) = f^{k-1}\left(b\hat{T}(b)\right)\right\}.$$

Theorem 1.3.6. *Let*

$$O^{\wedge}(b) = \{x(0), x(1), \ldots, x(k-1)\}$$

be a k-cycle of a continuously differentiable function $f^{\vee}$. *Let also,* $\hat{T} \in \mathcal{C}^1(\mathbb{R})$, $N \le \hat{T}(x) \le N_1$, $|\hat{T}'(x)| \le N_2$ *for all* $x \in \mathbb{R}$, $\left|b\hat{T}(b)\right| \le N_3$,

$$M_1 \le |f'(x(0))f'(x(1))\ldots f'(x(k-1))| \le M.$$

Then the following statements hold.

(i) *The k-cycle* $O^{\wedge}(b)$ *is asymptotically stable if*

$$M(N_1 + |b|N_2) < 1.$$

(ii) *The k-cycle* $O^{\wedge}(b)$ *is unstable if*

$$M_1N - M|b|N_2 > a$$

for some positive real number a.

Lemma 1.3.7. *Let* $\frac{x}{\hat{T}(x)} \in [a,b]$ *for every* $x \in [a,b]$, $f \in \mathcal{C}([a,b])$. *Let also,* $J = [c,d] \subset [a,b]$ *such that either*

(i) $f\left(\frac{c}{\hat{T}(c)}\right) > c$ *and* $f\left(\frac{d}{\hat{T}(d)}\right) < d$, *or*

(ii) $f\left(\frac{c}{\hat{T}(c)}\right) < c$ *and* $f\left(\frac{d}{\hat{T}(d)}\right) > d$.

Then $f^{\vee}$ *has a fixed point in* (c,d).

Proof. **(i)** Assume that $f\left(\frac{c}{\hat{T}(c)}\right) > c$ and $f\left(\frac{d}{\hat{T}(d)}\right) < d$. We define the map

$$g(x) := f\left(\frac{x}{\hat{T}(x)}\right) - x, \qquad x \in [c,d].$$

Then $g \in \mathcal{C}([c,d])$ and

$$g(c) = f\left(\frac{c}{\hat{T}(c)}\right) - c > 0,$$

$$g(d) = f\left(\frac{d}{\hat{T}(d)}\right) - d < 0.$$

From here and from the intermediate value theorem, it follows that there exists $x^* \in (c,d)$ so that

$$g(x^*) = x^*$$

or

$$f\left(\frac{x^*}{\hat{T}(x^*)}\right) = x^*.$$

(ii) We assume that $f\left(\frac{c}{\hat{T}(c)}\right) < c$ and $f\left(\frac{d}{\hat{T}(d)}\right) > d$. Then

$$g(c) = f\left(\frac{c}{\hat{T}(c)}\right) - c < 0,$$

$$g(d) = f\left(\frac{d}{\hat{T}(d)}\right) - d > 0.$$

Hence, from the intermediate value theorem, we conclude that there exists $x* \in (c,d)$ such that

$$g(x^*) = x^*.$$

Therefore

$$f\left(\frac{x^*}{\hat{T}(x^*)}\right) = x^*.$$

□

Lemma 1.3.8. *Let $x\hat{T}(x) \in [a,b]$ for every $x \in [a,b]$, $f \in \mathcal{C}([a,b])$. Let also, $J = [c,d] \subset [a,b]$ such that either*

(i) $f\left(c\hat{T}(c)\right) > c$ *and* $f\left(d\hat{T}(d)\right) < d$*, or*

(ii) $f\left(c\hat{T}(c)\right) < c$ *and* $f\left(d\hat{T}(d)\right) > d$.

Then $f^\wedge$ has a fixed point in (c,d).

Proof. **(i)** Assume that $f\left(c\hat{T}(c)\right) > c$ and $f\left(d\hat{T}(d)\right) < d$. We define the map

$$g(x) := f\left(x\hat{T}(x)\right) - x, \qquad x \in [c,d].$$

Then $g \in \mathcal{C}([c,d])$ and

$$g(c) = f\left(c\hat{T}(c)\right) - c > 0,$$

$$g(d) = f\left(d\hat{T}(d)\right) - d < 0.$$

From here and from the intermediate value theorem, it follows that there exists $x^* \in (c,d)$ so that

$$g(x^*) = x^*$$

or

$$f\left(x^*\hat{T}(x^*)\right) = x^*.$$

(ii) We assume that $f\left(c\hat{T}(c)\right) < c$ and $f\left(d\hat{T}(d)\right) > d$. Then

$$g(c) = f\left(c\hat{T}(c)\right) - c < 0,$$

$$g(d) = f\left(d\hat{T}(d)\right) - d > 0.$$

Hence, from the intermediate value theorem, we conclude that there exists $x* \in (c,d)$ such that

$$g(x^*) = x^*.$$

Therefore

$$f\left(x^*\hat{T}(x^*)\right) = x^*.$$

□

Lemma 1.3.9. *Let $\frac{x}{\hat{T}(x)} \in [a,b]$ for every $x \in [a,b]$, $f \in \mathcal{C}([a,b])$, $[c,d] \subset [a,b]$. If $f\left(\frac{d}{\hat{T}(d)}\right) > d$ and (c,d) is fixed point-free for $f^\vee$, then $f\left(\frac{x}{\hat{T}(x)}\right) > x$ for all $x \in (c,d)$.*

Proof. Let us suppose that there is $x_1 \in (c,d)$ such that

$$f\left(\frac{x_1}{\hat{T}(x_1)}\right) \leq x_1.$$

If

$$f\left(\frac{x_1}{\hat{T}(x_1)}\right) = x_1,$$

then we get to a contradiction since the interval (c,d) is fixed point-free for the function $f^\vee$. Therefore

$$f\left(\frac{x_1}{\hat{T}(x_1)}\right) < x_1.$$

We define the map

$$g(x) := x - f\left(\frac{x}{\hat{T}(x)}\right), \qquad x \in (x_1, d).$$

Then

$$g(x_1) = x_1 - f\left(\frac{x_1}{\hat{T}(x_1)}\right) > 0,$$

$$g(d) = d - f\left(\frac{d}{\hat{T}(d)}\right) < 0.$$

Hence, using the intermediate value theorem, we conclude that there is a point $x_2 \in (x_1, d)$ such that

$$f\left(\frac{x_2}{\hat{T}(x_2)}\right) = x_2,$$

i.e., x_2 is an equilibrium point for $f^\vee$, which is a contradiction because $x_2 \in (c,d)$ and the interval (c,d) is fixed point-free for $f^\vee$. □

Lemma 1.3.10. *Let $x\hat{T}(x) \in [a,b]$ for every $x \in [a,b]$, $f \in \mathcal{C}([a,b])$, $[c,d] \subset [a,b]$. If $f\left(d\hat{T}(d)\right) > d$ and (c,d) is fixed point-free for $f^\wedge$, then $f\left(x\hat{T}(x)\right) > x$ for all $x \in (c,d)$.*

Proof. We assume that there exists $x_1 \in (c,d)$ so that

$$f\left(x_1\hat{T}(x_1)\right) \leq x_1.$$

If

$$f\left(x_1\hat{T}(x_1)\right) = x_1,$$

then we obtain a contradiction because the interval (c,d) is fixed point-free for the considered function $f^{\wedge}$. Therefore

$$f\left(x_1\hat{T}(x_1)\right) < x_1.$$

We consider the map

$$g(x) := x - f\left(x\hat{T}(x)\right), \qquad x \in (x_1, d).$$

We have

$$g(x_1) = x_1 - f\left(x_1\hat{T}(x_1)\right) > 0,$$

$$g(d) = d - f\left(d\hat{T}(d)\right) < 0.$$

From here, using the intermediate value theorem, we conclude that there exists a point $x_2 \in (x_1, d)$ such that

$$f\left(x_2\hat{T}(x_2)\right) = x_2,$$

i.e., x_2 is an equilibrium point for $f^{\wedge}$, which is a contradiction because $x_2 \in (c,d)$ and the interval (c,d) is fixed point-free for $f^{\wedge}$. $\square$

Definition 1.3.11. *The limit set $\Omega^{\vee}(x_0)$ of the point x_0 is defined by*

$$\Omega^{\vee}(x_0) = \left\{ \frac{y}{\hat{T}(y)} : y \in \mathbb{R}, f^{n_i}\left(\frac{x_0}{\hat{T}(x_0)}\right) \longrightarrow_{n_i \longrightarrow \infty} \frac{y}{\hat{T}(y)} \right\}.$$

Definition 1.3.12. *The limit set $\Omega^{\wedge}(x_0)$ of the point x_0 is defined by*

$$\Omega^{\wedge}(x_0) = \left\{ y\hat{T}(y) : y \in \mathbb{R}, f^{n_i}\left(x_0\hat{T}(x_0)\right) \longrightarrow_{n_i \longrightarrow \infty} y\hat{T}(y) \right\}.$$

Theorem 1.3.13. *Let $x_0 \in \mathbb{R}$, $f \in \mathcal{C}(\mathbb{R})$. Then the following statements hold true.*

(i)

$$\Omega^{\vee}(x_0) = \bigcap_{i=0}^{\infty} \overline{\bigcup_{n=i}^{\infty} \left\{ f^n\left(\frac{x_0}{\hat{T}(x_0)}\right) \right\}}.$$

(ii) *If*

$$f^j\left(\frac{x_0}{\hat{T}(x_0)}\right) = \frac{y_0}{\hat{T}(y_0)}$$

for some $j \in \mathbb{N}$, then

$$\Omega^{\vee}(y_0) = \Omega^{\vee}(x_0).$$

(iii) *$\Omega^{\vee}(x_0)$ is closed and $O^{\vee}(x) \subset \Omega^{\vee}(x_0)$ for every $x \in \Omega^{\vee}(x_0)$.*

(iv) *If $O^{\vee}(x_0)$ is bounded, then $\Omega^{\vee}(x_0)$ is nonempty and bounded.*

Proof. (i) Let $\frac{y}{\hat{T}(y)} \in \Omega^{\vee}(x_0)$ is arbitrarily chosen. Then

$$\frac{f^{n_i}(x_0)}{\hat{T}(x_0)} \longrightarrow_{n_i \longrightarrow \infty} \frac{y}{\hat{T}(y)}.$$

Therefore, for every $i \in \mathbb{N} \cup \{0\}$ there exists enough large $n_i \in \mathbb{N}$ so that

$$\frac{y}{\hat{T}(y)} \in \overline{\left\{ f^{n_i}\left(\frac{x_0}{\hat{T}(x_0)} \right) \right\}},$$

from where

$$\frac{y}{\hat{T}(y)} \in \overline{\bigcup_{n=i}^{\infty} \left\{ f^{n}\left(\frac{x_0}{\hat{T}(x_0)} \right) \right\}},$$

hence,

$$\frac{y}{\hat{T}(y)} \in \bigcap_{i=0}^{\infty} \overline{\bigcup_{n=i}^{\infty} \left\{ f^{n}\left(\frac{x_0}{\hat{T}(x_0)} \right) \right\}}.$$

Because $\frac{y}{\hat{T}(y)} \in \Omega^{\vee}(x_0)$ was arbitrarily chosen, we conclude that

$$\Omega^{\vee}(x_0) \subset \bigcap_{i=0}^{\infty} \overline{\bigcup_{n=i}^{\infty} \left\{ f^{n}\left(\frac{x_0}{\hat{T}(x_0)} \right) \right\}}. \tag{8}$$

Let now

$$\frac{y}{\hat{T}(y)} \in \bigcap_{i=0}^{\infty} \overline{\bigcup_{n=i}^{\infty} \left\{ f^{n}\left(\frac{x_0}{\hat{T}(x_0)} \right) \right\}}.$$

Then for every $i \in \mathbb{N} \cup \{0\}$ we have that

$$\frac{y}{\hat{T}(y)} \in \bigcap_{n=i}^{\infty} \left\{ f^{n}\left(\frac{x_0}{\hat{T}(x_0)} \right) \right\},$$

whereupon

$$f^{n}\left(\frac{x_0}{\hat{T}(x_0)} \right) \longrightarrow_{n \longrightarrow \infty} \frac{y}{\hat{T}(y)},$$

i.e., $\frac{y}{\hat{T}(y)} \in \Omega^{\vee}(x_0)$.

Because

$$\frac{y}{\hat{T}(y)} \in \bigcap_{n=i}^{\infty} \left\{ f^{n}\left(\frac{x_0}{\hat{T}(x_0)} \right) \right\}$$

was arbitrarily chosen, we conclude that

$$\bigcap_{n=i}^{\infty} \left\{ f^{n}\left(\frac{x_0}{\hat{T}(x_0)} \right) \right\} \subset \Omega^{\vee}(x_0). \tag{9}$$

From (8) and (9) we obtain

$$\Omega^{\vee}(x_0) = \bigcap_{n=i}^{\infty} \left\{ f^{n}\left(\frac{x_0}{\hat{T}(x_0)} \right) \right\}.$$

(ii) Let $\frac{y}{\hat{T}(y)} \in \Omega^{\vee}(y_0)$ is arbitrarily chosen. Then

$$f^{n_i}\left(\frac{y_0}{\hat{T}(y_0)}\right) \longrightarrow_{n_i \longrightarrow \infty} \frac{y}{\hat{T}(y)}.$$

Using

$$f^j\left(\frac{x_0}{\hat{T}(x_0)}\right) = \frac{y_0}{\hat{T}(y_0)},$$

we get

$$\begin{aligned} f^{n_i+j}\left(\tfrac{x_0}{\hat{T}(x_0)}\right) &= f^{n_i}\left(f^j\left(\tfrac{x_0}{\hat{T}(x_0)}\right)\right) \\ &= f^{n_i}\left(\tfrac{x_0}{\hat{T}(x_0)}\right) \\ &\longrightarrow_{n_i \longrightarrow \infty} \tfrac{y}{\hat{T}(y)}, \end{aligned}$$

i.e., $\frac{y}{\hat{T}(y)} \in \Omega^{\vee}(x_0)$.

Because $\frac{y}{\hat{T}(y)} \in \Omega^{\vee}(y_0)$ was arbitrarily chosen we get that

$$\Omega^{\vee}(y_0) \subset \Omega^{\vee}(x_0). \tag{10}$$

Let $\frac{x}{\hat{T}(x)} \in \Omega^{\vee}(x_0)$ is arbitrarily chosen. Then

$$f^{n_i}\left(\frac{x_0}{\hat{T}(x_0)}\right) \longrightarrow_{n_i \longrightarrow \infty} \frac{x}{\hat{T}(x)}.$$

Hence,

$$f^{n_i+j}\left(\frac{x_0}{\hat{T}(x_0)}\right) \longrightarrow_{n_i \longrightarrow \infty} \frac{x}{\hat{T}(x)}. \tag{11}$$

Since

$$\begin{aligned} f^{n_i+j}\left(\tfrac{x_0}{\hat{T}(x_0)}\right) &= f^{n_i}\left(f^j\left(\tfrac{x_0}{\hat{T}(x_0)}\right)\right) \\ &= f^{n_i}\left(\tfrac{y_0}{\hat{T}(y_0)}\right), \end{aligned}$$

therefore, using (11), we obtain

$$f^{n_i}\left(\frac{y_0}{\hat{T}(y_0)}\right) \longrightarrow_{n_i \longrightarrow \infty} \frac{x}{\hat{T}(x)}.$$

Because $\frac{x}{\hat{T}(x)} \in \Omega^{\vee}(x_0)$ was arbitrarily chosen, we conclude that

$$\Omega^{\vee}(x_0) \subset \Omega^{\vee}(y_0).$$

From the last relation and from (10) it follows that

$$\Omega^{\vee}(x_0) = \Omega^{\vee}(y_0).$$

(iii) Since

$$\overline{\bigcup_{n=i}^{\infty}\left\{f^n\left(\frac{x_0}{\hat{T}(x_0)}\right)\right\}}$$

is a closed set for every $i \in \mathbb{N}\cup\{0\}$, and the intersection of closed sets is a closed set, we conclude, using (i), that $\Omega^{\vee}(x_0)$ is a closed set.

Let now $\frac{x}{\hat{T}(x)} \in \Omega^{\vee}(x_0)$ is arbitrarily chosen.Then

$$f^{n_i}\left(\frac{x_0}{\hat{T}(x_0)}\right) \longrightarrow_{n_i\longrightarrow\infty} \frac{x}{\hat{T}(x)}. \tag{12}$$

Let $x_1 \in O^{\vee}(x)$ is arbitrarily chosen. Then there exists $k \in \mathbb{N}$ such that

$$x_1 = f^k\left(\frac{x}{\hat{T}(x)}\right).$$

Hence and (12) we find, using that f is continuous,

$$\begin{aligned} f^{n_i+k}\left(\tfrac{x_0}{\hat{T}(x_0)}\right) &= f^k\left(f^{n_i}\left(\tfrac{x_0}{\hat{T}(x_0)}\right)\right) \\ &\longrightarrow_{n_i\longrightarrow\infty} f^k\left(\tfrac{x}{\hat{T}(x)}\right) \\ &= x_1, \end{aligned}$$

i.e., $x_1 \in \Omega^{\vee}(x_0)$.

Because $x_1 \in \Omega^{\vee}(x_0)$ was arbitrarily chosen we conclude that

$$O^{\vee}(x) \subset \Omega^{\vee}(x_0).$$

(iv) We have that $O^{\vee}(x_0)$ is a bounded set. Then there exists a constant $M > 0$ such that

$$\left|f^n\left(\frac{x_0}{\hat{T}(x_0)}\right)\right| < M, \qquad n \in \mathbb{N}. \tag{13}$$

Therefore the sequence $\left\{f^n\left(\frac{x_0}{\hat{T}(x_0)}\right)\right\}$ is a bounded sequence. Hence, there exists a subsequence $\left\{f^{n_k}\left(\frac{x_0}{\hat{T}(x_0)}\right)\right\}$ of the sequence $\left\{f^n\left(\frac{x_0}{\hat{T}(x_0)}\right)\right\}$ which is convergent. Consequently $\Omega^{\vee}(x_0)$ is nonempty.

Let now

$$\frac{y}{\hat{T}(y)} \in \Omega^{\vee}(x_0).$$

Then

$$f^n\left(\frac{x_0}{\hat{T}(x_0)}\right) \longrightarrow_{n\longrightarrow\infty} \frac{y}{\hat{T}(y)}$$

and since (13) holds, we conclude that

$$\left|\frac{y}{\hat{T}(y)}\right| \leq M.$$

Because $\frac{y}{\hat{T}(y)} \in \Omega^{\vee}(x_0)$ was arbitrarily chosen it follows that the set $\Omega^{\vee}(x_0)$ is a bounded set.

$\square$

Similarly, one can prove the following theorem.

Theorem 1.3.14. *Let $x_0 \in \mathbb{R}$, $f \in \mathcal{C}(\mathbb{R})$. Then the following statements hold true.*

(i)

$$\Omega^{\wedge}(x_0) = \bigcap_{i=0}^{\infty} \overline{\bigcup_{n=i}^{\infty} \left\{ f^n\left(x_0 \hat{T}(x_0)\right) \right\}}.$$

(ii) *If*

$$f^j\left(x_0 \hat{T}(x_0)\right) = y_0 \hat{T}(y_0)$$

for some $j \in \mathbb{N}$, then

$$\Omega^{\wedge}(y_0) = \Omega^{\wedge}(x_0).$$

(iii) *$\Omega^{\wedge}(x_0)$ is closed and $O^{\wedge}(x) \subset \Omega^{\wedge}(x_0)$ for every $x \in \Omega^{\wedge}(x_0)$.*

(iv) *If $O^{\wedge}(x_0)$ is bounded, then $\Omega^{\wedge}(x_0)$ is nonempty and bounded.*

MA

Chapter 2

Linear Iso-Difference Equations of Higher Order

2.1. Iso-Difference Calculus

Iso-difference calculus is the iso-discrete analogue of the familiar iso-differential and iso-integral calculus. In this section we introduce some very basic properties of two operators that are essential in the study of iso-difference equations.

Definition 2.1.1. *The iso-difference operator is defined by*

$$\hat{\Delta}x(n) = \frac{x(n+1)}{\hat{T}(n+1)} - \frac{x(n)}{\hat{T}(n)}.$$

Definition 2.1.2. *The iso-shift operator is defined by*

$$\hat{E}x(n) = \frac{x(n+1)}{\hat{T}(n+1)}.$$

We have

$$\hat{E}^2 x(n) = \hat{E}(\hat{E}x(n))$$

$$= \hat{E}\left(\tfrac{x(n+1)}{\hat{T}(n+1)}\right)$$

$$= \tfrac{x(n+2)}{\hat{T}^2(n+2)},$$

$$\hat{E}^3 x(n) = \hat{E}\left(\hat{E}^2 x(n)\right)$$

$$= \hat{E}\left(\tfrac{x(n+2)}{\hat{T}^2(n+2)}\right)$$

$$= \tfrac{x(n+3)}{\hat{T}^3(n+3)}.$$

We suppose that

$$\hat{E}^k x(n) = \frac{x(n+k)}{\hat{T}^k(n+k)} \tag{1}$$

for some $k \in \mathbb{N}$. Then

$$\hat{E}^{k+1}x(n) = \hat{E}\left(\hat{E}^k x(n)\right)$$

$$= \hat{E}\left(\tfrac{x(n+k)}{\hat{T}^k(n+k)}\right)$$

$$= \tfrac{x(n+k+1)}{\hat{T}^{k+1}(n+k+1)}.$$

Consequently, (1) is valid for every $k \in \mathbb{N}$.

One may write

$$\hat{\Delta}x(n) = \hat{E}x(n) - \hat{x}^{\wedge}(\hat{n}).$$

We define the operator

$$\hat{I}^{\wedge\wedge}$$

so that

$$\hat{I}^{\wedge\wedge}x(n) = \hat{x}^{\wedge}(\hat{n})$$

$$= \tfrac{x(n)}{\hat{T}(n)},$$

$$\left(\hat{I}^{\wedge\wedge}\right)^2 x(n) = \hat{I}^{\wedge\wedge}\left(\tfrac{x(n)}{\hat{T}(n)}\right)$$

$$= \tfrac{x(n)}{\hat{T}^2(n)}.$$

Therefore

$$\hat{\Delta}x(n) = \hat{E}x(n) - \hat{I}^{\wedge\wedge}x(n)$$

or

$$\hat{\Delta} = \hat{E} - \hat{I}^{\wedge\wedge}$$

and

$$\hat{E} = \hat{\Delta} + \hat{I}^{\wedge\wedge}.$$

Also,

$$\hat{\Delta}^2 x(n) = \hat{\Delta}\left(\hat{\Delta}x(n)\right)$$

$$= \hat{\Delta}\left(\tfrac{x(n+1)}{\hat{T}(n+1)} - \tfrac{x(n)}{\hat{T}(n)}\right)$$

$$= \hat{\Delta}\left(\tfrac{x(n+1)}{\hat{T}(n+1)}\right) - \hat{\Delta}\tfrac{x(n)}{\hat{T}(n)}$$

$$= \tfrac{1}{\hat{T}(n+2)}\tfrac{x(n+2)}{\hat{T}(n+2)} - \tfrac{x(n+1)}{\hat{T}^2(n+1)} - \tfrac{x(n+1)}{\hat{T}^2(n+1)} + \tfrac{x(n)}{\hat{T}^2(n)}$$

$$= \tfrac{x(n+2)}{\hat{T}^2(n+2)} - 2\tfrac{x(n+1)}{\hat{T}^2(n+1)} + \tfrac{x(n)}{\hat{T}^2(n)}$$

$$= \hat{E}^2 x(n) - 2\hat{E}\hat{I}^{\wedge\wedge}x(n) + \left(\hat{I}^{\wedge\wedge}\right)^2 x(n).$$

We suppose that

$$\hat{\Delta}^k = \left(\hat{E} - \hat{I}^{\wedge\wedge}\right)^k = \sum_{i=0}^{k} \begin{pmatrix} k \\ i \end{pmatrix} (-1)^i \hat{E}^{k-i} \left(\hat{I}^{\wedge\wedge}\right)^i \tag{2}$$

for some $k \in \mathbb{N}$. We consider

$$\hat{\Delta}^{k+1}.$$

We have

$$\hat{\Delta}^{k+1} = \left(\hat{E} - \hat{I}^{\wedge\wedge}\right)^{k+1}$$

$$= \left(\hat{E} - \hat{I}^{\wedge\wedge}\right)\left(\hat{E} - \hat{I}^{\wedge\wedge}\right)^k$$

$$= \left(\hat{E} - \hat{I}^{\wedge\wedge}\right) \sum_{i=0}^{k} \begin{pmatrix} k \\ i \end{pmatrix} (-1)^i \hat{E}^{k=i} \left(\hat{I}^{\wedge\wedge}\right)^i$$

$$= \sum_{i=0}^{k} \begin{pmatrix} k \\ i \end{pmatrix} (-1)^i \hat{E}^{k+1-i} \left(\hat{I}^{\wedge\wedge}\right)^i - \sum_{i=0}^{k} \begin{pmatrix} k \\ i \end{pmatrix} (-1)^i \hat{E}^{k-i} \left(\hat{I}^{\wedge\wedge}\right)^{i+1}$$

$$= \begin{pmatrix} k \\ 0 \end{pmatrix} \hat{E}^{k+1} - \begin{pmatrix} k \\ 1 \end{pmatrix} \hat{E}^k \hat{I}^{\wedge\wedge} + \cdots + (-1)^k \begin{pmatrix} k \\ k \end{pmatrix} \hat{E} \left(\hat{I}^{\wedge\wedge}\right)^k$$

$$- \begin{pmatrix} k \\ 0 \end{pmatrix} \hat{E}^k \hat{I}^{\wedge\wedge} + \begin{pmatrix} k \\ 1 \end{pmatrix} \hat{E}^{k-1} \left(\hat{I}^{\wedge\wedge}\right)^2 - \cdots - (-1)^k \begin{pmatrix} k \\ k \end{pmatrix} \left(\hat{I}^{\wedge\wedge}\right)^{k+1}$$

$$= \begin{pmatrix} k+1 \\ 0 \end{pmatrix} \hat{E}^{k+1} - \begin{pmatrix} k+1 \\ 1 \end{pmatrix} \hat{E}^k \hat{I}^{\wedge\wedge} + \cdots + (-1)^{k+1} \begin{pmatrix} k+1 \\ k+1 \end{pmatrix} \left(\hat{I}^{\wedge\wedge}\right)^{k+1}$$

$$= \sum_{i=0}^{k+1} \begin{pmatrix} k+1 \\ i \end{pmatrix} (-1)^i \hat{E}^{k+1-i} \left(\hat{I}^{\wedge\wedge}\right)^i.$$

Therefore (2) is valid for all $k \in \mathbb{N}$.

Proposition 2.1.3. *We have*

$$\sum_{k=n_0}^{n-1} \hat{\Delta} x(k) = \hat{x}^{\wedge}(\hat{n}) - \hat{x}^{\wedge}(\hat{n}_0).$$

Proof.

$$\sum_{k=n_0}^{n-1} \hat{\Delta} x(k) = \hat{\Delta} x(n_0) + \hat{\Delta} x(n_0+1) + \cdots + \hat{\Delta} x(n-1)$$

$$= \frac{x(n_0+1)}{\hat{T}(n_0+1)} - \frac{x(n_0)}{\hat{T}(n_0)} + \frac{x(n_0+2)}{\hat{T}(n_0+2)} - \frac{x(n_0+1)}{\hat{T}(n_0+1)} + \cdots + \frac{x(n)}{\hat{T}(n)} - \frac{x(n-1)}{\hat{T}(n-1)}$$

$$= \frac{x(n)}{\hat{T}(n)} - \frac{x(n_0)}{\hat{T}(n_0)}$$

$$= \hat{x}^{\wedge}(\hat{n}) - \hat{x}^{\wedge}(\hat{n}_0).$$

□

Let

$$p(\hat{E}) = a_0\hat{E}^k + a_1\hat{E}^{k-1} + \cdots + a_k\hat{I}^{\wedge\wedge}$$

be a polynomial of degree k in $\hat{E}$.

Then

$$p(\hat{E})b^n = \left(a_0\hat{E}^k + a_1\hat{E}^{k-1} + \cdots + a_k\hat{I}^{\wedge\wedge}\right) b^n$$

$$= a_0\hat{E}^k b^n + a_1\hat{E}^{k-1} b^n + \cdots + a_k\hat{I}^{\wedge\wedge} b^n$$

$$= a_0\tfrac{b^{n+k}}{\hat{T}(k)} + a_1\tfrac{b^{n+k-1}}{\hat{T}(k-1)} + \cdots + a_k\tfrac{b^n}{\hat{T}(n)}$$

$$= b^n\hat{I}^{\wedge\wedge}\left(a_0 b^k + a_1 b^{k-1} + \cdots + a_k 1^n\right)$$

$$= b^n\hat{I}^{\wedge\wedge} p(b).$$

Exercise 2.1.4. *Prove that*

$$\hat{\Delta}(x(n)y(n)) = x(n+1)\hat{E}y(n) - x(n)\hat{I}^{\wedge\wedge}y(n)$$

$$= y(n+1)\hat{E}x(n) - y(n)\hat{I}^{\wedge\wedge}x(n)$$

$$= x(n+1)\hat{E}y(n) - y(n)\hat{I}^{\wedge\wedge}x(n)$$

$$= y(n+1)\hat{E}x(n) - x(n)\hat{I}^{\wedge\wedge}y(n).$$

Exercise 2.1.5. *Prove that*

$$\hat{\Delta}\left(\frac{x(n)}{y(n)}\right) = \frac{1}{y(n+1)}\hat{E}x(n) - \frac{1}{y(n)}\hat{I}^{\wedge\wedge}x(n).$$

The iso-discrete analogue of the iso-indefinite integral in calculus is the iso-antidifference operator $\hat{\Delta}^{-1}$, defined as follows.

Definition 2.1.6. *If $\hat{\Delta}F(n) = 0$, then*

$$\hat{\Delta}^{-1}(0) := F(n) = c$$

for some arbitrary constant c.

Moreover, if $\hat{\Delta}F(n) = f(n)$, then

$$\hat{\Delta}^{-1}f(n) := F(n) + c,$$

for some arbitrary constant c. Hence,

$$\hat{\Delta}\hat{\Delta}^{-1}f(n) = \hat{\Delta}F(n) = f(n),$$

$$\hat{\Delta}^{-1}\hat{\Delta}F(n) = \hat{\Delta}^{-1}f(n) = F(n) + c,$$

and

$$\hat{\Delta}\hat{\Delta}^{-1} = I,$$

but

$$\hat{\Delta}^{-1}\hat{\Delta} \neq I.$$

Here I is the identity.

2.2. General Theory of Linear Iso-Difference Equations

Definition 2.2.1. *The normal form of a kth order nonhomogeneous linear iso-difference equation is given by*

$$\hat{y}^{\wedge}(\widehat{n+k}) + p_1(n)\hat{y}^{\wedge}(n+\hat{k}=1) + \cdots + p_k(n)\hat{y}^{\wedge}(\hat{n}) = g(n), \tag{3}$$

where $p_i(n)$, $i=1,2,\ldots,k$, and $g(n)$ are real-valued functions or iso-functions of first, second, third, fourth or fifth kind, defined for $n \geq n_0$, and $p_k(n) \neq 0$ for all $n \geq n_0$. If $g(n)$ is identically zero, then (3) *is said to be a homogeneous equation.*

The equation (3) we can rewrite in the form

$$\frac{y(n+k)}{\hat{T}(n+k)} + p_1(n)\frac{y(n+k-1)}{\hat{T}(n+k-1)} + \cdots + p_k(n)\frac{y(n)}{\hat{T}(n)} = g(n)$$

or

$$y(n+k) + p_1(n)\hat{T}(n+k)\frac{y(n+k-1)}{\hat{T}(n+k-1)} + \cdots + p_k(n)\hat{T}(n+k)\frac{y(n)}{\hat{T}(n)} = \hat{T}(n+k)g(n),$$

or

$$y(n+k) = -p_1(n)\hat{T}(n+k)\frac{y(n+k-1)}{\hat{T}(n+k-1)} - \cdots - p_k(n)\hat{T}(n+k)\frac{y(n)}{\hat{T}(n)} + \hat{T}(n+k)g(n). \tag{4}$$

By letting $n=0$ in (4) we obtain $y(k)$ in terms of $y(k-1)$, $y(k-2)$, ..., $y(0)$. Explicitly, we have

$$y(k) = -p_1(0)\hat{T}(k)\frac{y(k-1)}{\hat{T}(k-1)} - \cdots - p_k(0)\hat{T}(k)\frac{y(0)}{\hat{T}(0)} + \hat{T}(k)g(0).$$

Once $y(k)$ is computed, we can go to the next step and evaluate $y(k+1)$ by letting $n=1$ in (4). This yields

$$y(k+1) = -p_1(1)\hat{T}(k+1)\frac{y(k)}{\hat{T}(k)} - \cdots - p_k(1)\hat{T}(k+1)\frac{y(1)}{\hat{T}(1)} + \hat{T}(k+1)g(1).$$

By repeating the above process, it is possible to evaluate all $y(n)$ for $n \geq k$.

Definition 2.2.2. *A sequence $\{y_n\}_{n=n_0}^{n=\infty}$ or simply $y(n)$ is said to be a solution of* (4) *if it satisfies the equation.*

Observe that if we specify the initial data of the equation, we are led to the corresponding initial value problem.

$$y(n+k)+p_1(n)\hat{T}(n+k)\frac{y(n+k-1)}{\hat{T}(n+k-1)}+\cdots+p_k(n)\hat{T}(n+k)\frac{y(n)}{\hat{T}(n)} = \hat{T}(n+k)g(n), \qquad n \geq n_0, \tag{4}$$

$$y(n_0) = a_0, \qquad y(n_0+1) = a_1, \qquad y(n_0+k-1) = a_{k-1}, \tag{5}$$

where a_i, $i=0,1,2,\ldots,k-1$ are real numbers.

Example 2.2.3. *Let* $k = 3$, $p_1(n) = n+1$, $p_2(n) = n$, $p_3(n) = n-1$, $g(n) = 0$, $n \in \mathbb{N}$, $y(1) = 1$, $y(2) = 1$, $y(3) = 2$. *Then*

$$\begin{aligned}
&\hat{T}(n+3) = n+3+1\\
&= n+4,\\
&\hat{T}(n+3-1) = \hat{T}(n+2)\\
&= n+2+1\\
&= n+3,\\
&\hat{T}(n+3-2) = \hat{T}(n+1)\\
&= n+1+1\\
&= n+2,\\
&\hat{T}(n+3-3) = \hat{T}(n)\\
&= n+1.
\end{aligned}$$

The equation (4) *takes the form*

$$\begin{aligned}
&y(n+3) = -(n+1)(n+4)\tfrac{y(n+2)}{n+3} - n(n+4)\tfrac{y(n+1)}{n+2} - (n-1)(n+4)\tfrac{y(n)}{n+1}\\
&= -\tfrac{(n+1)(n+4)}{n+3}y(n+2) - \tfrac{n(n+4)}{n+2}y(n+1) - \tfrac{(n-1)(n+4)}{(n+1)}y(n).
\end{aligned}$$

We will find $y(4)$ *and* $y(5)$.

We have

$$\begin{aligned}
&y(4) = -\tfrac{2.5}{4}y(3) - \tfrac{1.5}{3}y(2) - \tfrac{0.5}{2}y(1)\\
&= -\tfrac{5}{2}y(3) - \tfrac{5}{3}y(2)\\
&= -\tfrac{5}{2}.2 - \tfrac{5}{3}.1\\
&= -5 - \tfrac{5}{3}\\
&= -\tfrac{20}{3},
\end{aligned}$$

$$y(5) = -\tfrac{3.6}{5}y(4) - \tfrac{2.6}{4}y(3) - \tfrac{1.6}{3}y(2)$$

$$= -\tfrac{18}{5}y(4) - 3y(3) - 2y(2)$$

$$= -\tfrac{18}{5}\left(-\tfrac{20}{3}\right) - 3.2 - 2.1$$

$$= 24 - 6 - 2$$

$$= 24 - 8$$

$$= 16.$$

Example 2.2.4. *Let $k = 2$, $p_1(n) = n+2$, $p_2(n) = n-2$, $g(n) = 0$, $\hat{T}(n) = n^2+1$, $n \in \mathbb{N}$, $y(1) = 1$, $y(2) = 2$. Then*

$$\hat{T}(n+1) = (n+1)^2 + 1$$

$$= n^2 + 2n + 1 + 1$$

$$= n^2 + 2n + 2,$$

$$\hat{T}(n+2) = (n+2)^2 + 1$$

$$= n^2 + 4n + 4 + 1$$

$$= n^2 + 4n + 5.$$

The equation (4) takes the form

$$y(n+2) = -(n+2)(n^2+4n+5)\tfrac{y(n+1)}{n^2+2n+2} - (n-2)(n^2+4n+5)\tfrac{y(n)}{n^2+1}$$

$$= -\tfrac{(n+2)(n^2+4n+5)}{n^2+2n+2}y(n+1) - \tfrac{(n-2)(n^2+4n+5)}{n^2+1}y(n).$$

We will find $y(3)$ and $y(4)$.

We have

$$y(3) = -3.10\tfrac{y(2)}{5} - \tfrac{(-1).10}{2}y(1)$$

$$= -6y(2) + 5y(1)$$

$$= -6.2 + 5.1$$

$$= -12 + 5$$

$$= -7,$$

$$y(4) = -4.\tfrac{29}{10}y(3)$$

$$= -\tfrac{58}{5}(-7)$$

$$= \tfrac{406}{5}.$$

Example 2.2.5. *Let* $k = 3,\ p_1(n) = n+4,\ p_2(n) = n+5,\ p_3(n) = n,\ \hat{T}(n) = n,\ g(n) = 0,$ $n \in \mathbb{N},\ y(1) = 0,\ y(2) = 1,\ y(3) = 3.$ *Then*

$$\hat{T}(n+3) = n+3,$$

$$\hat{T}(n+3-1) = \hat{T}(n+2)$$

$$= n+2,$$

$$\hat{T}(n+3-2) = \hat{T}(n+1)$$

$$= n+1,$$

$$\hat{T}(n+3-3) = \hat{T}(n)$$

$$= n.$$

The equation (4) *takes the form*

$$y(n+3) = -(n+4)(n+3)\tfrac{y(n+2)}{n+2} - (n+5)(n+3)\tfrac{y(n+1)}{n+1} - n(n+3)\tfrac{y(n)}{n}$$

$$= -\tfrac{(n+4)(n+3)}{n+2}y(n+2) - \tfrac{(n+5)(n+3)}{n+1}y(n+1) - (n+3)y(n).$$

We will find $y(4)$*,* $y(5)$ *and* $y(6)$*.*

We have

$$y(4) = -\tfrac{5.4}{3}y(3) - \tfrac{6.4}{2}y(2) - 4y(1)$$

$$= -\tfrac{20}{3}.3 - 12.1 - 4.0$$

$$= -20 - 12$$

$$= -32,$$

$$y(5) = -\tfrac{6.5}{4}y(4) - \tfrac{7.5}{3}y(3) - 5y(2)$$

$$= -\tfrac{15}{2}(-32) - \tfrac{35}{3}.3 - 5.1$$

$$= 240 - 35 - 5$$

$$= 200,$$

$$y(6) = -\tfrac{7.6}{5}y(5) - \tfrac{8.6}{4}y(4) - 6y(3)$$

$$= -\tfrac{42}{5}y(5) - 12y(4) - 6y(3)$$

$$= -\tfrac{42}{5}.200 - 12(-32) - 6.3$$

$$= -1680 + 384 - 18$$

$$= -1314.$$

Exercise 2.2.6. *Prove that the problem* (4), (5) *has unique solution.*

In this section we will develop the general theory of kth-order linear homogeneous iso-difference equations of the form

$$y(n+k) + p_1(n)\hat{T}(n+k)\frac{y(n+k-1)}{\hat{T}(n+k-1)} + \cdots + p_k(n)\hat{T}(n+k)\frac{y(n)}{\hat{T}(n)} = 0, \qquad n \geq n_0. \tag{6}$$

First of all, we will give some important definitions.

Definition 2.2.7. *The functions $f_1(n)$, $f_2(n)$, …, $f_r(n)$ are said to be linearly dependent for $n \geq n_0$ if there are constants a_1, a_2, …, a_r, not all zero, such that*

$$a_1 f_1(n) + a_2 f_2(n) + \cdots + a_r f_r(n) = 0, \qquad n \geq n_0. \tag{7}$$

If $a_j \neq 0$, then we may divide the equality (7) by a_j to obtain

$$f_j(n) = -\tfrac{a_1}{a_j}f_1(n) - \tfrac{a_2}{a_j}f_2(n) - \cdots - \tfrac{a_{j-1}}{a_j}f_{j-1}(n) - \tfrac{a_{j+1}}{a_j}f_{j+1}(n) - \cdots - \tfrac{a_n}{a_j}f_r(n)$$

$$= -\textstyle\sum_{i=1, i\neq j}^{n} \tfrac{a_i}{a_j} f_i(n). \tag{8}$$

The equation (8) says that each f_j with nonzero coefficient is a linear combination of the other f_i's.

Definition 2.2.8. *The negation of linear dependence is linear independence. Explicitly put, the functions $f_1(n)$, $f_2(n)$, …, $f_r(n)$ are said to be linearly independent for $n \geq n_0$ if whenever*

$$a_1 f_1(n) + a_2 f_2(n) + \cdots + a_r f_r(n) = 0$$

for all $n \geq n_0$, then we must have

$$a_1 = a_2 = \cdots = a_r = 0.$$

Example 2.2.9. *Let us consider the functions*

$$2^n, \qquad n2^n, \qquad n^2 2^n, \qquad n \geq 1.$$

We consider the equality

$$a_1 2^n + a_2 n 2^n + a_3 n^2 2^n = 0$$

for real numbers a_1, a_2 and a_3.,

By dividing by 2^n we get

$$a_1 + a_2 n + a_3 n^2 = 0, \qquad n \geq 1.$$

This is impossible unless $a_3 = 0$, since a second-degree equation in n possesses at most two solutions $n \geq 1$. Similarly, $a_2 = 0$, whence $a_1 = 0$, which establishes the linear independence of the considered functions.

Definition 2.2.10. *A set of k-linearly independent solutions of (4) is called a fundamental set of solutions.*

Definition 2.2.11. *The Casoratian $W(n)$ of the solutions $x_1(n)$, $x_2(n)$, ..., $x_r(n)$ is given by*

$$W(n) = \det \begin{pmatrix} x_1(n) & x_2(n) & \dots & x_r(n) \\ x_1(n+1) & x_2(n+1) & \dots & x_r(n+1) \\ \dots & \dots & \dots & \dots \\ x_1(n+r-1) & x_2(n+r-1) & \dots & x_r(n+r-1) \end{pmatrix}.$$

Example 2.2.12. *We will find the Casoratian of the following functions*

$$4^n, \qquad n4^n, \qquad n^2 4^n.$$

We have

$$W(n) = \det \begin{pmatrix} 4^n & n4^n & n^2 4^n \\ 4^{n+1} & (n+1)4^{n+1} & (n+1)^2 4^{n+1} \\ 4^{n+2} & (n+2)4^{n+2} & (n+2)^2 4^{n+2} \end{pmatrix}$$

$$= (n+1)(n+2)^2 4^{3n+3} + n(n+1)^2 4^{3n+3} + n^2(n+2)4^{3n+3}$$

$$-n^2(n+1)4^{3n+3} - (n+2)(n+1)^2 4^{3n+3} - n(n+2)^2 4^{3n+3}$$

$$= 4^{3n+3}\Big((n+1)(n+2)^2 + n(n+1)^2 + n^2(n+2)$$

$$-n^2(n+1) - (n+2)(n+1)^2 - n(n+2)^2\Big)$$

$$= 4^{3n+3}\Big((n+1)(n^2+4n+4) + n(n^2+2n+1) + n^3 + 2n^2$$

$$-n^3 - n^2 - (n+2)(n^2+2n+1) - n(n^2+4n+4)\Big)$$

$$= 4^{3n+3}\Big(n^3 + 4n^2 + 4n + n^2 + 4n + 4 + n^3 + 2n^2 + n + n^2$$

$$-(n^3 + 2n^2 + n + 2n^2 + 4n + 2) - n^3 - 4n^2 - 4n\Big)$$

$$= 4^{3n+3}(n^3 + 4n^2 + 5n - n^3 - 4n^2 - 5n - 2)$$

$$= 4^{3n+3}(-2)$$

$$= -2.2^{6n+6}$$

$$= -2^{6n+7}.$$

Example 2.2.13. *Now we will find the Casoratian of the functions*

$$(-2)^n, \qquad 2^n, \qquad 3.$$

We have

$$W(n) = \det\begin{pmatrix} (-2)^n & 2^n & 3 \\ (-2)^{n+1} & 2^{n+_1} & 3 \\ (-2)^{n+2} & 2^{n+2} & 3 \end{pmatrix}$$

$$= (-2)^n.2^{n+1}.3 + 2^n(-2)^{n+2}.3 + (-2)^{n+1}.2^{n+2}.3$$

$$-(-2)^{n+2}.2^{n+1}.3 - (-2)^n.2^{n+2}.3 - (-2)^{n+1}.2^n.3$$

$$= 3(-1)^n 2^{2n+1} + 3(-1)^n 2^{2n+2} - 3(-1)^n 2^{2n+3}$$

$$-3(-1)^n 2^{2n+3} - 3(-1)^n 2^{2n+2} + 3(-1)^n 2^{2n+1}$$

$$= 3(-1)^n 2^{2n+1}(1 + 2 - 3.2^2 - 3.2^2 - 3.2 + 3)$$

$$= 3(-1)^n 2^{2n+1}(-24)$$

$$= (-1)^{n+1} 3^2 2^{2n+4}.$$

Example 2.2.14. *Now we will find the Casoratian of the functions*

$$0,3^n, \qquad 7^n.$$

We have

$$W(n) = \det\begin{pmatrix} 0,3^n & 7^n \\ 0,3^{n+1} & 7^{n+1} \end{pmatrix}$$

$$= 0,3^n 7^{n+1} - 0,3^{n+1} 7^n$$

$$= 0,3^n 7^n (7 - 0,3)$$

$$= 0,3^n 7^n 6,7.$$

Exercise 2.2.15. *Let* $k = 2$*,*

$$p_1(n) = -\tfrac{(3n-2)(n+2)}{(n+3)(n-1)}, \qquad p_2(n) = \tfrac{2n(n+1)}{(n-1)(n+3)},$$

$$\hat{T}(n) = n + 1, \qquad g(n) = 0, \qquad n \in \mathbb{N}.$$

Verify that $\{n, 2^n\}$ *is a fundamental set of solutions of* (4)*.*

Solution. We have

$$p_1(n)\frac{\hat{T}(n+2)}{\hat{T}(n+1)} = -\frac{(3n-2)(n+2)}{(n+3)(n-1)}\frac{n+3}{n-2}$$

$$= -\frac{3n-2}{n-1},$$

$$p_2(n)\frac{\hat{T}(n+2)}{\hat{T}(n)} = \frac{2n(n+1)}{(n-1)(n+3)}\frac{n+3}{n+1}$$

$$= \frac{2n}{n-1}.$$

Then the equation (4) takes the form.

$$y(n+2) - \frac{3n-2}{n-1}y(n+1) + \frac{2n}{n-1}y(n) = 0. \tag{9}$$

First of all, we will check that n and 2^n satisfy the equation (9).

Really,

$$(n+2) - \frac{3n-2}{n-1}(n+1) + \frac{2n}{n-1}n = \frac{(n+2)(n-1)-(3n-2)(n+1)+2n^2}{n-1}$$

$$= \frac{n^2+n-2-3n^2-n+2+2n^2}{n-1}$$

$$= 0,$$

$$2^{n+2} - \frac{3n-2}{n-1}2^{n+1} + \frac{2n}{n-1}2^n = 2^n\left(4 - 2\frac{3n-2}{n-1} + \frac{2n}{n-1}\right)$$

$$= 2^n\frac{4n-4-6n+4+2n}{n-1}$$

$$= 0.$$

We note that n and 2^n are linearly independent. Therefore $\{n, 2^n\}$ is a fundamental set of solutions of (4).

Exercise 2.2.16. *Let $k = 3$,*

$$p_1(n) = 0, \qquad p_2(n) = -7\frac{n+4}{n+5}, \qquad p_3(n) = 6\frac{n+3}{n+5},$$

$$\hat{T}(n) = n+2, \qquad g(n) = 0, \qquad n \in \mathbb{N}.$$

Prove that $\{1, (-3)^n, 2^n\}$ are solutions of (4) and find their Casoratian.

Solution. We have

$$p_1(n)\frac{\hat{T}(n+3)}{\hat{T}(n+2)} = 0,$$

$$p_2(n)\frac{\hat{T}(n+3)}{\hat{T}(n+1)} = -7\frac{n+4}{n+5}\frac{n+5}{n+4}$$

$$= -7,$$

$$p_3(n)\frac{\hat{T}(n+3)}{\hat{T}(n)} = 6\frac{n+3}{n+5}\frac{n+5}{n+3}$$

$$= 6.$$

Then the equation (4) takes the form

$$y(n+3)-7y(n+1)+6y(n)=0. \tag{10}$$

Now we will check that 1, $(-3)^n$, 2^n are solutions of (10).

We have

$$1-7+6=0,$$

$$(-3)^{n+3}-7(-3)^{n+1}+6(-3)^n=(-3)^n(-27+21+6)$$

$$=0,$$

$$2^{n+3}-7.2^{n+1}+6.2^n=2^n(2^3-7.2+6)$$

$$=0.$$

Now we will compute their Casoratian.

$$W(n)=\det\begin{pmatrix} 1 & (-3)^n & 2^n \\ 1 & (-3)^{n+1} & 2^{n+1} \\ 1 & (-3)^{n+2} & 2^{n+2} \end{pmatrix}$$

$$=(-3)^{n+1}2^{n+2}+(-3)^n2^{n+1}+(-3)^{n+2}2^n$$

$$-(-3)^{n+1}2^n-(-3)^{n+2}2^{n+1}-(-3)^n2^{n+2}$$

$$=(-3)^n.2^n(-12+2+9+3-18-4)$$

$$=(-3)^n2^n(-34+14)$$

$$=(-20)(-3)^n2^n.$$

Exercise 2.2.17. *Let* $k=3$*,*

$$p_1(n)=3\tfrac{n+7}{n+8},\qquad p_2(n)=-4\tfrac{n+6}{n+8},\qquad p_3(n)=-12\tfrac{n+5}{n+8},$$

$$\hat{T}(n)=n+5,\qquad g(n)=0,\qquad n\in\mathbb{N}.$$

Prove that

$$2^n,\qquad (-2)^n,\qquad (-3)^n$$

form a fundamental set of solutions of (4).

Exercise 2.2.18. *Let* $k=2$*,*

$$p_1(n)=0,\qquad p_2(n)=\frac{n^2+1}{n^2+4n+5},\qquad \hat{T}(n)=n^2+1,\qquad g(n)=0,\qquad n\in\mathbb{N}.$$

Prove that the functions $\sin(n)$ *and* $\cos(n)$ *form a fundamental set of solutions of the equation* (4).

Lemma 2.2.19. *(iso-Abel's lemma) Let*

$$y_1(n), \qquad y_2(n), \qquad , y_k(n)$$

be solutions of (6) *and let* $W(n)$ *be their Casoratian. Then for* $n \geq n_0$ *we have the following formula.*

$$W(n_0+l) = (-1)^{kl} \prod_{i=1}^{l} p_k(n_0+l-i) \frac{\hat{T}(n_0+l+k-i)}{\hat{T}(n_0+l-i)} W(n_0) \tag{11}$$

for every $l \in \mathbb{N}$.

Proof. We have

$$W(n_0+1) = \det \begin{pmatrix} y_1(n_0+1) & y_2(n_0+1) & \dots & y_k(n_0+1) \\ y_1(n_0+2) & y_2(n_0+2) & \dots & y_k(n_0+2) \\ \dots & \dots & \dots & \dots \\ y_1(n_0+k) & y_2(n_0+k) & \dots & y_k(n_0+k) \end{pmatrix}.$$

Now we use that

$$y_j(n_0+k) = -\hat{T}(n_0+k) \sum_{i=1}^{k} p_i(n_0) \frac{y_j(n_0+k-1)}{\hat{T}(n_0+k-1)}.$$

Therefore

$$\begin{aligned}
&W(n_0+1) \\
&= \det \begin{pmatrix} y_1(n_0+1) & \dots & y_k(n_0+1) \\ y_1(n_0+2) & \dots & y_k(n_0+2) \\ \dots & \dots & \dots \\ -\hat{T}(n_0+k) \sum_{i=1}^{k} p_i(n_0) \frac{y_1(n_0+k-i)}{\hat{T}(n_0+k-i)} & \dots & -\hat{T}(n_0+k) \sum_{i=1}^{k} p_i(n_0) \frac{y_k(n_0+k-i)}{\hat{T}(n_0+k-i)} \end{pmatrix} \\
&= \det \begin{pmatrix} y_1(n_0+1) & y_2(n_0+1) & \dots & y_k(n_0+1) \\ y_1(n_0+2) & y_2(n_0+2) & \dots & y_k(n_0+2) \\ \dots & \dots & \dots & \dots \\ -\frac{\hat{T}(n_0+k)}{\hat{T}(n_0)} p_k(n_0) y_1(n_0) & -\frac{\hat{T}(n_0+k)}{\hat{T}(n_0)} p_k(n_0) y_2(n_0) & \dots & -\frac{\hat{T}(n_0+k)}{\hat{T}(n_0)} p_k(n_0) y_k(n_0) \end{pmatrix} \\
&= -p_k(n_0) \frac{\hat{T}(n_0+k)}{\hat{T}(n_0)} \det \begin{pmatrix} y_1(n_0+1) & y_2(n_0+1) & \dots & y_k(n_0+1) \\ y_1(n_0+2) & y_2(n_0+2) & \dots & y_k(n_0+2) \\ \dots & \dots & \dots & \dots \\ y_1(n_0) & y_2(n_0) & \dots & y_k(n_0) \end{pmatrix} \\
&= (-1)^k p_k(n_0) \frac{\hat{T}(n_0+k)}{\hat{T}(n_0)} \det \begin{pmatrix} y_1(n_0) & y_2(n_0) & \dots & y_k(n_0) \\ y_1(n_0+1) & y_2(n_0+1) & \dots & y_k(n_0+1) \\ \dots & \dots & \dots & \dots \\ y_1(n_0+k-1) & y_2(n_0+k-1) & \dots & y_k(n_0+k-1) \end{pmatrix} \\
&= (-1)^k p_k(n_0) \frac{\hat{T}(n_0+k)}{\hat{T}(n_0)} W(n_0),
\end{aligned}$$

i.e.,

$$W(n_0+1) = (-1)^k p_k(n_0)\frac{\hat{T}(n_0+k)}{\hat{T}(n_0)}W(n_0).$$

We suppose that

$$W(n_0+l) = (-1)^k p_k(n_0+l-1)\frac{\hat{T}(n_0+l-1+k)}{\hat{T}(n_0+l-1)}W(n_0+l-1)$$

for some $l \in \mathbb{N}$.

We will prove that

$$W(n_0+l+1) = (-1)^k p_k(n_0+l)\frac{\hat{T}(n_0+l+k)}{\hat{T}(n_0+l)}W(n_0+l).$$

Really,

$$W(n_0+l+1) = \det\begin{pmatrix} y_1(n_0+l+1) & y_2(n_0+l+1) & \dots & y_k(n_0+l+1) \\ y_1(n_0+l+2) & y_2(n_0+l+2) & \dots & y_k(n_0+l+2) \\ \dots & \dots & \dots & \dots \\ y_1(n_0+l+k) & y_2(n_0+l+k) & \dots & y_k(n_0+l+k) \end{pmatrix}$$

$$= \det\begin{pmatrix} y_1(n_0+l+1) & \dots & y_k(n_0+l+1) \\ y_1(n_0+l+2) & \dots & y_k(n_0+l+2) \\ \dots & \dots & \dots \\ -\hat{T}(n_0+l+1)\sum_{i=1}^k p_i(n_0+l)\frac{y_1(n_0+l+k-i)}{\hat{T}(n_0+l+k-1)} & \dots & -\hat{T}(n_0+l+1)\sum_{i=1}^k p_i(n_0+l)\frac{y_k(n_0+l+k-i)}{\hat{T}(n_0+l+k-1)} \end{pmatrix}$$

$$= \det\begin{pmatrix} y_1(n_0+l+1) & \dots & y_k(n_0+l+1) \\ y_1(n_0+l+2) & \dots & y_k(n_0+l+2) \\ \dots & \dots & \dots \\ -\hat{T}(n_0+l+k)p_k(n_0+l)\frac{y_1(n_0+l)}{\hat{T}(n_0+l)} & \dots & -\hat{T}(n_0+l+k)p_k(n_0+l)\frac{y_k(n_0+l)}{\hat{T}(n_0+l)} \end{pmatrix}$$

$$= -p_k(n_0+l)\frac{\hat{T}(n_0+l+k)}{\hat{T}(n_0+l)}\det\begin{pmatrix} y_1(n_0+l+1) & y_2(n_0+l+1) & \dots & y_k(n_0+l+1) \\ y_1(n_0+l+2) & y_2(n_0+l+2) & \dots & y_k(n_09+l+2) \\ \dots & \dots & \dots & \dots \\ y_1(n_0+l) & y_2(n_0+l) & \dots & y_k(n_0+l) \end{pmatrix}$$

$$= (-1)^k p_k(n_0+l)\frac{\hat{T}(n_0+l+k)}{\hat{T}(n_0+l)}\det\begin{pmatrix} y_1(n_0+l) & y_2(n_0+l) & \dots & y_k(n_0+l) \\ y_1(n_0+l+1) & y_2(n_0+l+1) & \dots & y_k(n_0+l+1) \\ \dots & \dots & \dots & \dots \\ y_1(n_0+l+k-1) & y_2(n_0+l+k-1) & \dots & y_k(n_0+l+k-1) \end{pmatrix}$$

$$= (-1)^k p_k(n_0+1)\frac{\hat{T}(n_0+l+k)}{\hat{T}(n_0+l)}W(n_0+l),$$

i.e.,

$$W(n_0+l+1) = (-1)^k p_k(n_0+l)\frac{\hat{T}(n_0+l+k)}{\hat{T}(n_0+l)}W(n_0+l).$$

Consequently, for all $l \in \mathbb{N}$

$$W(n_0+l) = (-1)^k p_k(n_0+l-1)\frac{\hat{T}(n_0+l+k-1)}{\hat{T}(n_0+l-1)}W(n_0+l-1).$$

Hence,

$$W(n_0+l) = (-1)^k p_k(n_0+l-1)\tfrac{\hat{T}(n_0+l+k-1)}{\hat{T}(n_0+l-1)}W(n_0+l-1)$$

$$= (-1)^k p_k(n_0+l-1)\tfrac{\hat{T}(n_0+l+k-1)}{\hat{T}(n_0+l-1)}(-1)^k p_k(n_0+l-2)\tfrac{\hat{T}(n_0+l+k-2)}{\hat{T}(n_0+l-2)}W(n_0+l-2)$$

$$= \cdots$$

$$= (-1)^{kl}\textstyle\prod_{i=1}^{l} p_k(n_0+l-1)\tfrac{\hat{T}(n_0+l+k-1)}{\hat{T}(n_0+l-1)}W(n_0).$$

□

Exercise 2.2.20. *Suppose that $p_k(n) \neq 0$ for all $n \geq n_0$. Prove that the Casoratian $W(n) \neq 0$ for all $n \geq n_0$ if and only if $W(n_0) \neq 0$.*

Now we will find the relationship between the linear independence of the solutions and their Casoratian.

Let

$$y_1(n), \qquad y_2(n), \qquad \ldots, y_k(n)$$

be solutions of (6). Suppose that for some constants

$$a_1, \qquad a_2, \qquad \ldots, \qquad a_k \qquad \text{and} \qquad n_0 \in \mathbb{N},$$

we have

$$a_1y_1(n) + a_2y_2(n) + \cdots + a_ky_k(n) = 0$$

for all $n \geq n_0$. Then we generate the following $k-1$ equations.

$$a_1y+_1(n+1_+a_2y_2(n+1_+\cdots+a_ky_k(n+1) = 0$$

$$a_1y_1(n+2) + a_2y_2(n+2) + \cdots + a_ky_k(n+2) = 0$$

$$\ldots$$

$$a_1y_1(n+k-1) + a_2y_2(n+k-1) + \cdots + a_ky_k(n+k-1) = 0.$$

Let

$$Y(n) = \begin{pmatrix} y_1(n+1) & y_2(n+1) & \ldots & y_k(n+1) \\ y_1(n+2) & y_2(n+2) & \ldots & y_k(n+2) \\ \ldots & \ldots & \ldots & \ldots \\ y_1(n+k-1) & y_2(n+k-1) & \ldots & y_k(n+k-1) \end{pmatrix}, \qquad a = \begin{pmatrix} a_1 \\ a_2 \\ \ldots \\ a_k \end{pmatrix}.$$

Thus, we obtain

$$Y(n)a = 0. \tag{12}$$

We observe that

$$W(n) = \det Y(n).,$$

The equation (12) has only the trivial solution, i.e.,

$$a_1 = a_2 = \ldots = a_k = 0,$$

if and only if the matrix $Y(n)$ is nonsingular for $n \geq n_0$, i.e., $W(n) \neq 0$ for all $n \geq n_0$.

In this way we get to the following conclusion.

Theorem 2.2.21. *The set of solutions*

$$y_1(n), \qquad y_2(n), \qquad \ldots, \qquad y_k(n)$$

of the equation (6) *is a fundamental set if and only if for some* $n_0 \in \mathbb{N}$ *the Casoratian* $W(n_0) \neq 0$.

Theorem 2.2.22. *(fundamental theorem) If* $p_k(n) \neq 0$ *for all* $n \geq n_0$. *Then the equation* (6) *has a fundamental set of solutions for* $n \geq n_0$.

Proof. Since the initial problem for the equation (6) has unique solution, then there are solutions

$$y_1(n), \qquad y_2(n), \qquad \ldots, \qquad y_k(n)$$

such that

$$\begin{aligned} & y_i(n_0+i-1) = 1, \\ & y_i(n_0) = y_i(n_0+1) \\ & = y_i(n_0+i-2) = y_i(n_0+i) \\ & = \cdots \\ & = y_i(n_0+k-i) \\ & = 0, \qquad 1 \leq i \leq k. \end{aligned}$$

Hence,

$$\begin{aligned} & y_1(n_0) = 1, \\ & y_2(n_0+1) = 1, \\ & \ldots \\ & y_k(n_0+k-1) = 1. \end{aligned}$$

It follows that

$$W(n_0) = \det I = 1.$$

Consequently, the set

$$\{y_1(n), \qquad y_2(n), \qquad \ldots, \qquad y_k(n)\}$$

is a fundamental set of solutions of the equation (6). $\square$

Exercise 2.2.23. *(superposition principle) If*

$$y_1(n), \qquad y_2(n), \qquad \ldots, \qquad y_r(n)$$

are solutions of the equation (6)*, prove that*

$$y(n) = a_1 y_1(n) + a_2 y_2(n) + \cdots + a_r y_r(n),$$

where a_i*,* $i = 1, 2, \ldots, r$*, are constants, is also solution of the equation* (6)*.*

Now let

$$\{y_1(n), \qquad y_2(n), \qquad \ldots, \qquad y_k(n)\}$$

be a fundamental set of solutions of (6) and let $y(n)$ be any given solution of (6). Then there are constants a_1, a_2, ..., a_k such that

$$y(n) = \sum_{i=1}^{k} a_i y_i(n).$$

To show this, we use the notation (12) to write

$$Y(n)a = \tilde{y}(n),$$

where

$$\tilde{y}(n) = \begin{pmatrix} y(n) \\ y(n+1) \\ \cdots \\ y(n+k-1) \end{pmatrix}.$$

Since $Y(n)$ is nonsingular, it follows that

$$a = Y^{-1}(n)\tilde{y}(n),$$

and, for $n = n_0$,

$$a = Y^{-1}(n_0)\tilde{y}(n_0).$$

Exercise 2.2.24. *Let S be the set of all solutions of the equation* (6) *with the operations* $+$*,* $\cdot$ *defined as follows*

(i) $(x+y)(n) = x(n) + y(n)$*, for* $x, y \in S$*,* $n \in \mathbb{N}$*,*

(ii) $(ax)(n) = ax(n)$*, for* $x \in S$*, a is a constant.*

Prove that

$$(S, +, \cdot)$$

is a linear vector space of dimension k.

2.3. Linear Homogeneous Iso-Difference Equations with Constant Coefficients

Here we will consider the equation (6) in the case when

$$p_i(n)\frac{\hat{T}(n+k)}{\hat{T}(n+k-1)} = a_i = \text{const}, \qquad i = 1, 2, \ldots, k,$$

for some $k \in \mathbb{N}$. In other words, here, in this section, we will consider the equation

$$y(n+k) + a_1 y(n+k-1) + a_2 y(n+k-2) + \cdots + a_k y(n) = 0, \qquad n \geq n_0. \tag{13}$$

We suppose that $a_k \neq 0$.

We assume that the solutions of the equation (13) are in the form λ^n, where λ is a complex number, i.e., we assume that

$$y(n) = \lambda^n.$$

Then

$$y(n+1) = \lambda^{n+1},$$

$$y(n+2) = \lambda^{n+2},$$

$$\ldots$$

$$y(n+k) = \lambda^{n+k}.$$

Hence, using the equation (13), we get

$$\lambda^{n+k} + a_1 \lambda^{n+k-1} + \cdots + a_k \lambda^n = 0$$

or

$$\lambda^k + a_1 \lambda^{k-1} + \cdots + a_k = 0. \tag{14}$$

Definition 2.3.1. *The equation* (14) *will be called the characteristic equation of the equation* (13). *Its roots will be called characteristic roots of the equation* (13).

Because $p_k \neq 0$, none of the characteristic roots is equal to zero.

Example 2.3.2. *Let us consider the equation*

$$y(n+4) + 12y(n+3) + 6y(n+2) + 7y(n+1) + y(n) = 0.$$

We assume that its solutions are in the form $y(n) = \lambda^n$. *We have*

$$y(n+1) = \lambda^{n+1}, \qquad y(n+2) = \lambda^{n+2}, \qquad y(n+3) = \lambda^{n+3}, \qquad y(n+4) = \lambda^{n+4},$$

which we put in the considered equation and we get

$$\lambda^{n+4} + 12\lambda^{n+3} + 6\lambda^{n+2} + 7\lambda^{n+1} + \lambda^n = 0,$$

whereupon the characteristic equation of the considered equation is

$$\lambda^4 + 12\lambda^3 + 6\lambda^2 + 7\lambda + 1 = 0.$$

Example 2.3.3. *Let us consider the equation*

$$y(n+3)-4y(n+2)-5y(n+1)-6y(n)=0.$$

We will suppose that its solutions are in the form $y(n)=\lambda^n$. *We have*

$$y(n+3)=\lambda^{n+3}, \qquad y(n+2)=\lambda^{n+2}, \qquad y(n+1)=\lambda^{n+1}.$$

Hence,

$$\lambda^{n+3}-4\lambda^{n+2}-5\lambda^{n+1}-6\lambda^n=0,$$

from where

$$\lambda^3-4\lambda^2-5\lambda-6=0$$

is the characteristic equation of the considered equation.

Example 2.3.4. *Let us consider the equation*

$$y(n+7)-7y(n+5)-4y(n+4)-3y(n+1)+y(n)=0.$$

We will assume that its solutions are in the form $y(n)=\lambda^n$. *Then*

$$y(n+1)=\lambda^{n+1}, \qquad y(n+2)=\lambda^{n+2}, \qquad y(n+3)=\lambda^{n+3}, \qquad y(n+4)=\lambda^{n+4},$$

$$y(n+5)=\lambda^{n+5}, \qquad y(n+6)=\lambda^{n+6}, \qquad y(n+7)=\lambda^{n+7},$$

which we put in the considered equation and we find

$$\lambda^{n+7}-7\lambda^{n+5}-4\lambda^{n+4}-3\lambda^{n+1}+\lambda^n=0,$$

or

$$\lambda^7-7\lambda^5-4\lambda^4-3\lambda+1=0$$

is the characteristic equation of the considered equation.

Exercise 2.3.5. *Find the characteristic equation of the following equations*

1) $y(n+3)-3y(n+2)-2y(n+1)-10y(n)=0,$

2) $y(n+4)-y(n+3)-y(n+2)-y(n+1)-y(n)=0,$

3) $y(n+2)-10y(n+1)+9y(n)=0,$

4) $y(n+5)-6y(n+3)+y(n+1)=0,$

5) $3y(n+3)-y(n+1)+y(n)=0,$

6) $4y(n+_2)-7y(n)=0.$

Answer.

$$1)\qquad \lambda^3 - 3\lambda^2 - 2\lambda - 10 = 0,$$

$$2)\qquad \lambda^4 - \lambda^3 - \lambda^2 - \lambda - 1 = 0,$$

$$3)\qquad \lambda^2 - 10\lambda + 9 = 0,$$

$$4)\qquad \lambda^5 - 6\lambda^3 + \lambda = 0,$$

$$5)\qquad 3\lambda^3 - \lambda + 1 = 0,$$

$$6)\qquad 4\lambda^2 - 7 = 0.$$

1. Case. Let the characteristic roots

$$\lambda_1, \qquad \lambda_2, \qquad \ldots, \qquad \lambda_k$$

are distinct. We will show that the set

$$\{\lambda_1^n, \qquad \lambda_2^n, \qquad \ldots, \qquad \lambda_k^n\}$$

is a fundamental set of solutions of the equation (13). Since

$$W(0) = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_k \\ \ldots & \ldots & \ldots & \ldots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \ldots & \lambda_k^{k-1} \end{pmatrix},$$

which is the Vandermonde determinant, we have

$$W(0) = \prod_{1 \le i < j \le k} (\lambda_j - \lambda_i).$$

Because all λ_i's are distinct, it follows that

$$W(0) \neq 0.$$

This proves that

$$\{\lambda_1^n, \qquad \lambda_2^n, \qquad \ldots, \qquad \lambda_k^n\}$$

is a fundamental set of solutions of the equation (13). Consequently, the general solution of the equation (13) is

$$y(n) = \sum_{i=1}^{k} a_i \lambda_i^n,$$

where a_i, $i = 1, 2, \ldots, k$, are complex numbers.

Example 2.3.6. *Let us consider the problem*

$$y(n+2)+3y(n+1)+2y(n)=0, \qquad n \geq 2,$$

$$y(0)=1, y(1)=2.$$

The characteristic equation is

$$\lambda^2+3\lambda+2=0,$$

its roots are

$$\lambda_1=-1, \qquad \lambda_2=-2.$$

We have

$$\lambda_1 \neq \lambda_2,$$

from here, the general solution of the considered equation is

$$y(n)=a_1(-1)^n+a_2(-2)^n,$$

a_1 and a_2 are complex numbers. We will find the constants a_1 and a_2 using the initial data. Namely,

$$1=y(0)=a_1+a_2$$

$$2=y(1)=-a_1-2a_2,$$

i.e.,

$$\begin{cases} a_1+a_2=1 \\ -a_1-2a_2=2, \end{cases}$$

therefore

$$a_1=4, \qquad a_2=-3.$$

In this way the general solution is

$$y(n)=4(-1)^n-3(-2)^n, \qquad n \geq 0.$$

Example 2.3.7. *Now we consider the problem*

$$y(n+2)+9y(n)=0, \qquad n \geq 2,$$

$$y(0)=0, \qquad y(1)=3.$$

The characteristic equation is

$$\lambda^2+9=0,$$

its roots are

$$\lambda_{1,2}=\pm 3i.$$

Therefore

$$y(n)=a_1(3i)^n+a_2(-3i)^n.$$

We will find a_1 and a_2 using the initial data. We have

$$y(0) = a_1 + a_2 = 0$$

$$y(1) = 3ia_1 - 3ia_2 = 3,$$

or

$$\begin{cases} a_1 + a_2 = 0 \\ 3ia_1 - 3ia_2 = 3 \end{cases} \implies$$

$$\begin{cases} a_2 = -a_1 \\ 6ia_1 = 3 \end{cases} \implies$$

$$\begin{cases} a_1 = \frac{1}{2i} \\ a_2 = -\frac{1}{2i} \end{cases} \implies$$

$$\begin{cases} a_1 = -\frac{i}{2} \\ a_2 = \frac{i}{2}. \end{cases}$$

From here,

$$y(n) = -\tfrac{i}{2}(3i)^n + \tfrac{i}{2}(-3i)^n$$

$$= \tfrac{3^n}{2}\left(-ii^n + i(-i)^n\right).$$

Now we use that

$$i = \cos\tfrac{\pi}{2} + i\sin\tfrac{\pi}{2},$$

$$-i = \cos\tfrac{\pi}{2} - i\sin\tfrac{\pi}{2},$$

$$i^n = \cos\tfrac{n\pi}{2} + i\sin\tfrac{n\pi}{2},$$

$$(-i)^n \cos\tfrac{n\pi}{2} - i\sin\tfrac{n\pi}{2},$$

$$-ii^n = -i\cos\tfrac{n\pi}{2} + \sin\tfrac{n\pi}{2},$$

$$i(-i)^n i\cos\tfrac{n\pi}{2} + \sin\tfrac{n\pi}{2},$$

$$-ii^n + i(-i)^n = -i\cos\tfrac{n\pi}{2} + \sin\tfrac{n\pi}{2} + i\cos\tfrac{n\pi}{2} + \sin\tfrac{n\pi}{2}$$

$$= 2\sin\tfrac{n\pi}{2}.$$

Consequently,

$$y(n) = 3^n \sin\frac{n\pi}{2}.$$

Example 2.3.8. *Now we consider the problem*

$$y(n+3)-3y(n+2)+4y(n+1)-2y(n)=0, \qquad n\geq 3,$$

$$y(0)=y(1)=0, \qquad y(2)=1.$$

The characteristic equation is

$$\lambda^3-3\lambda^2+4\lambda-2=0 \qquad \Longrightarrow$$

$$\lambda^3-3\lambda^2+3\lambda+\lambda-1-1=0 \qquad \Longrightarrow$$

$$(\lambda^3-1)-3\lambda(\lambda-1)+(\lambda-1)=0 \qquad \Longrightarrow$$

$$(\lambda-1)(\lambda^2+\lambda+1-3\lambda+1)=0 \qquad \Longrightarrow$$

$$(\lambda-1)(\lambda^2-2\lambda+2)=0 \qquad \Longrightarrow$$

$$\lambda_1=1, \qquad \lambda_{2,3}=1\pm i.$$

The general solution is

$$y(n)=a_1(1)^n+a_2(1+i)^n+a_2(1-i)^n$$

$$=a_1+a_2(1+i)^n+a_3(1-i)^n,$$

where a_1, a_2 and a_3 are complex numbers. We will find a_1, a_2 and a_3 using the initial data. We have

$$0=y(0)=a_1+a_2+a_3$$

$$0=y(1)=a_1+a_2(1+i)+a_3(1-i)$$

$$1=y(2)=a_1+a_2(1+i)^2+a_3(1-i)^2$$

or

$$\begin{cases} a_1+a_2+a_3=0 \\ a_1+a_2(1+i)+a_3(1-i)=0 \\ a_1+a_1(1+i)^2+a_3(1-i)^2=1 \end{cases} \qquad \Longrightarrow$$

$$\begin{cases} a_1=-a_2-a_3 \\ -a_2-a_3+a_2(1+i)+a_3(1-i)=0 \\ -a_2-a_3+a_2(1+i)^2+a_3(1-i)^2=1 \end{cases} \qquad \Longrightarrow$$

$$\begin{cases} a_1=-a_2-a_3 \\ ia_2-ia_3=0 \\ -a_2-a_3+a_2(1+i)^2+a_3(1-i)^2=1 \end{cases} \qquad \Longrightarrow$$

$$\begin{cases} a_1 = -a_2 - a_3 \\ a_2 = a_3 \\ -2a_3 + a_3(1+2i-1+1-2i-1) = 1 \end{cases} \implies$$

$$\begin{cases} a_1 = -a_2 - a_3 \\ a_2 = a_3 \\ -2a_3 = 1 \end{cases} \implies$$

$$\begin{cases} a_1 = -a_2 - a_3 \\ a_2 = -\frac{1}{2} \\ a_3 = -\frac{1}{2} \end{cases} \implies$$

$$\begin{cases} a_1 = 1 \\ a_2 = -\frac{1}{2} \\ a_3 = -\frac{1}{2}. \end{cases}$$

Consequently,

$$y(n) = 1 + \left(-\tfrac{1}{2}\right)(1+i)^n - \tfrac{1}{2}(1-i)^n$$

$$= 1 - \tfrac{1}{2}\left((1+i)^n + (1-i)^n\right).$$

Now we use that

$$1+i = \sqrt{2}\left(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4}\right),$$

$$1-i = \sqrt{2}\left(\cos\tfrac{\pi}{4} - i\sin\tfrac{\pi}{4}\right),$$

$$(1+i)^n = \sqrt{2}^n\left(\cos\tfrac{n\pi}{4} + i\sin\tfrac{n\pi}{4}\right),$$

$$(1-i)^n = \sqrt{2}^n\left(\cos\tfrac{n\pi}{4} - i\sin\tfrac{n\pi}{4}\right),$$

$$(1+i)^n + (1-i)^n = 2\sqrt{2}^n\cos\tfrac{n\pi}{4}.$$

Consequently,

$$y(n) = 1 - \sqrt{2}^n\cos\frac{n\pi}{4}.$$

Exercise 2.3.9. *Find the general solution of the following initial problem*

$$\begin{cases} y(n+3) - 3y(n+2) + 2y(n+1) + y(n) = 0 \\ y(0) = 1, \qquad y(1) = 1, \qquad y(2) = 1. \end{cases}$$

2. Case. Suppose that the distinct characteristic roots are λ_1, λ_2, ..., λ_r, with multiplicities m_1, m_2, ..., m_r with

$$\sum_{i=1}^{r} m_i = k,$$

respectively.

Proposition 2.3.10. *The set*

$$G_i = \left\{ \lambda_i^n, \begin{pmatrix} n \\ 1 \end{pmatrix} \lambda_i^{n-1}, \begin{pmatrix} n \\ 2 \end{pmatrix} \lambda_i^{n-2}, \dots, \begin{pmatrix} n \\ m_i - 1 \end{pmatrix} \lambda_i^{n-m_i-1} \right\}$$

is a fundamental set of solutions of the equation (13).

Proof. We have

$$W(0) = \begin{vmatrix} 1 & 0 & \dots & 0 \\ \lambda_i & 1 & \dots & 0 \\ \lambda_i^2 & 2\lambda_i & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \lambda_i^{m_i-1} & \frac{m_i-1}{1!}\lambda_i^{m_i-2} & \dots & \frac{1}{2!3!\dots(m_i-2)!} \end{vmatrix}$$

$$= \frac{1}{2!3!\dots(m_i-2)!} \neq 0.$$

The proof that $\begin{pmatrix} n \\ r \end{pmatrix} \lambda_i^{n-r}$ is a solution of the equation (13) we left to the reader as exercise. $\square$

Theorem 2.3.11. *The set*

$$G = \bigcup_{i=1}^{r} G_i$$

is a fundamental set of solutions of (13).

Proof. We have

$$W(0) = \begin{vmatrix} 1 & 0 & \dots & 1 & 0 & \dots \\ \lambda_1 & 1 & \dots & \lambda_r & 1 & \dots \\ \lambda_1^2 & 2\lambda_1 & \dots & \lambda_r^2 & 2\lambda_r & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \lambda_1^{k-1} & (k-1)\lambda_1^{k-2} & \dots & \lambda_r^{k-1} & (k-1)\lambda_r^{k-2} & \dots \end{vmatrix}$$

$$= \prod_{1 \le i < j \le k} (\lambda_j - \lambda_i)^{m_i m_j}.$$

As $\lambda_i \neq \lambda_j$, we conclude that $W(0) \neq 0$. Hence, the Casoratian $W(n) \neq 0$ for all $n \geq 0$. Thus G is a fundamental set of solutions. $\square$

Corollary 2.3.12. *The general solution of the equation* (13) *is given by*

$$y(n) = \sum_{i=1}^{r} \lambda_i^n \left(a_{i0} + a_{i1}n + \cdots + a_{im_i-1}n^{m_i-1}\right).$$

Example 2.3.13. *Let us consider the initial problem*

$$\begin{cases} y(n+3) - 3y(n+2) + 3y(n+1) - y(n) = 0 \\ y(0) = y(1) = y(2) = 1. \end{cases}$$

The characteristic equation is

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

or

$$(\lambda - 1)^3 = 0,$$

or

$$\lambda_1 = 1, \qquad m_1 = 3.$$

Therefore the general solution is

$$y(n) = (a_1 + a_2 n + a_3 n^2)1^n$$

$$= a_1 + a_2 n + a_3 n^2.$$

We will find a_1, a_2 and a_3 using the initial data.
We have

$$y(0) = a_1 = 1$$

$$y(1) = a_1 + a_2 + a_3 = 1$$

$$y(2) = a_1 + 2a_2 + 4a_3 = 1,$$

from where

$$\begin{cases} a_1 = 1 \\ a_2 + a_3 = 0 \\ 2a_2 + 4a_3 = 0, \end{cases}$$

whereupon

$$\begin{cases} a_1 = 1 \\ a_2 = 0 \\ a_3 = 0. \end{cases}$$

Consequently,

$$y(n) = 1.$$

Example 2.3.14. *Let us consider the initial problem.*

$$\begin{cases} y(n+3)-8y(n+2)+21y(n+1)-18y(n)=0, & n \geq 3, \\ y(0)=0, \quad y(1)=1, \quad y(2)=0. \end{cases}$$

The characteristic equation is

$$\lambda^3-8\lambda^2+21\lambda-18=0 \quad \Longrightarrow$$

$$\lambda^3-8\lambda^2+9\lambda+12\lambda-18=0 \quad \Longrightarrow$$

$$9(\lambda-2)+\lambda(\lambda^2-8\lambda+12)=0 \quad \Longrightarrow$$

$$9(\lambda-2)+\lambda(\lambda-2)(\lambda-6)=0 \quad \Longrightarrow$$

$$(\lambda-2)(\lambda^2-6\lambda+9)=0 \quad \Longrightarrow$$

$$(\lambda-2)(\lambda-3)^2=0.$$

Therefore

$$\lambda_1=2, \quad \lambda_2=3, \quad m_1=1, \quad m_2=2.$$

The general solution is

$$y(n)=a_1 2^n+(a_2+a_3 n)3^n.$$

We will find a_1, a_2 and a_3 using the initial data.

We have

$$\begin{cases} a_1+a_2=0 \\ 2a_1+3(a_2+a_3)=1 \\ 4a_1+9(a_2+2a_3)=0 \end{cases} \quad \Longrightarrow$$

$$\begin{cases} a_2=-a_1 \\ 2a_1-3a_1+3a_3=1 \\ 4a_1-9a_1+18a_3=0 \end{cases} \quad \Longrightarrow$$

$$\begin{cases} a_1=-6 \\ a_2=6 \\ a_3=-\frac{5}{3}. \end{cases}$$

Consequently,

$$y(n)=-6.2^n+\left(6-\frac{5}{3}n\right)3^n.$$

Example 2.3.15. *Let us consider the initial problem*

$$\begin{cases} y(n+3)-5y(n+2)+8y(n+1)-4y(n)=0 \\ y(0)=y(1)=y(2)=1. \end{cases}$$

The characteristic equation is

$$\lambda^3-5\lambda^2+8\lambda-4=0 \qquad \Longrightarrow$$

$$\lambda^3-\lambda^2-4\lambda^2+8\lambda-4=0 \qquad \Longrightarrow$$

$$\lambda^2(\lambda-1)-4(\lambda^2-2\lambda+1)=0 \qquad \Longrightarrow$$

$$\lambda^2(\lambda-1)-4(\lambda-1)^2=0 \qquad \Longrightarrow$$

$$(\lambda-1)(\lambda^2-4\lambda+4)=0 \qquad \Longrightarrow$$

$$(\lambda-1)(\lambda-2)^2=0.$$

Therefore

$$\lambda_1=1, \qquad \lambda_{2,3}=2.$$

The general solution is

$$y(n)=a_1+(a_2+a_3n)2^n.$$

We will find a_1, a_2 and a_3 using the initial data.

We have

$$y(0)=a_1+a_2=1$$

$$y(1)=a_1+2(a_2+a_3)=1$$

$$y(2)=a_1+4(a_2+2a_3)=1,$$

whereupon

$$\begin{cases} a_2=1-a_1 \\ a_1+2-2a_1+2a_3=1 \\ a_1+4-4a_1+8a_3=1 \end{cases} \qquad \Longrightarrow$$

$$\begin{cases} a_2=1-a_1 \\ -a_1+2a_3=-1 \\ -3a_1+8a_3=-3 \end{cases} \qquad \Longrightarrow$$

$$\begin{cases} a_2 = 1 - a_1 \\ a_1 = 2a_3 + 1 \\ -6a_3 - 3 + 8a_3 = -3 \end{cases} \implies$$

$$\begin{cases} a_2 = 1 - a_1 \\ a_1 = 2a_3 + 1 \\ a_3 = 0 \end{cases} \implies$$

$$\begin{cases} a_1 = 1 \\ a_2 = 0 \\ a_3 = 0, \end{cases}$$

i.e.,

$$y(n) = 1.$$

Exercise 2.3.16. *Find the general solution of the initial problem.*

$$\begin{cases} y(n+3) - 9y(n+2) + 27y(n+1) - 27y(n) = 0 \\ y(0) = 1, \qquad y(1) = 2, \qquad y(2) = 3. \end{cases}$$

2.4. Linear Nonhomogeneous Equations

Here we will investigate the equation

$$y(n+k) + p_1(n)\hat{T}(n+k)\frac{y(n+k-1)}{\hat{T}(n+k-1)} + \cdots + p_k(n)\hat{T}(n+k-1)\frac{y(n)}{\hat{T}(n)} = \hat{T}(n+k)g(n), \qquad n \geq n_0. \tag{4}$$

Definition 2.4.1. *$\hat{T}(n+k)g(n)$ is called the iso-forcing term, the iso-external term, the iso-control, or the iso-input of the equation.*

Theorem 2.4.2. *If $y_1(n)$ and $y_2(n)$ are solutions of the equation (4), then*

$$x(n) = y_1(n) - y_2(n)$$

is a solution of the equation

$$y(n+k) + p_1(n)\hat{T}(n+k)\frac{y(n+k-1)}{\hat{T}(n+k-1)} + \cdots + p_k(n)\hat{T}(n+k-1)\frac{y(n)}{\hat{T}(n)} = 0, \qquad n \geq n_0. \tag{6}$$

Proof. We have

$$y_1(n+k)+p_1(n)\hat{T}(n+k)\tfrac{y_1(n+k-1)}{\hat{T}(n+k-1)}+\cdots+p_k(n)\hat{T}(n+k-1)\tfrac{y_1(n)}{\hat{T}(n)}=\hat{T}(n+k)g(n),$$

$$y_2(n+k)+p_1(n)\hat{T}(n+k)\tfrac{y_2(n+k-1)}{\hat{T}(n+k-1)}+\cdots+p_k(n)\hat{T}(n+k-1)\tfrac{y_2(n)}{\hat{T}(n)}=\hat{T}(n+k)g(n), \qquad n\geq n_0.$$

We subtract twice equations and we get

$$(y_1(n+k)-y_2(n+k))+p_1(n)\hat{T}(n+k)\tfrac{y_1(n+k-1)-y_2(n+k-1)}{\hat{T}(n+k-1)}$$

$$+\cdots+p_k(n)\hat{T}(n+k-1)\tfrac{y_1(n)-y_2(n)}{\hat{T}(n)}=\hat{T}(n+k)g(n)-\hat{T}(n+k)g(n),$$

whereupon

$$x(n+k)+p_1(n)\hat{T}(n+k)\frac{x(n+k-1)}{\hat{T}(n+k-1)}+\cdots+p_k(n)\hat{T}(n+k-1)\frac{x(n)}{\hat{T}(n)}=0, \qquad n\geq n_0.$$

□

Theorem 2.4.3. *Any solution $y(n)$ of the equation (4) may be written as*

$$y(n)=y_p(n)+\sum_{i=1}^{k}a_ix_i(n),$$

where $\{x_1(n),x_2(n),\ldots,x_k(n)\}$ is a fundamental set of solutions of the homogeneous equation (6), a_i, $i=1,2,\ldots,k$, are constants, $y_p(n)$ is a particular solution of (4).

Proof. We observe that $y(n)-y_p(n)$ is a solution to the homogeneous equation (6). Thus,

$$y(n)-y_p(n)=\sum_{i=1}^{k}a_ix_i(n),$$

a_i, $i=1,2,\ldots,k$, are constants. □

Example 2.4.4. *Let $k=2$,*

$$p_1(n)=\tfrac{6(n^2+2n+3)}{n^2+4n+6}, \qquad p_2(n)=\tfrac{5(n^2+2)}{n^2+4n+6},$$

$$\hat{T}(n)=n^2+2, \qquad g(n)=32\tfrac{3^n}{n^2+4n+6}, \qquad n\in\mathbb{N}.$$

Then

$$p_1(n)\frac{\hat{T}(n+2)}{\hat{T}(n+1)} = \frac{6(n^2+2n+3)}{n^2+4n+6}\frac{(n+2)^2+2}{(n+1)^2+2}$$

$$= \frac{6(n^2+2n+3)}{n^2+4n+6}\frac{n^2+4n+4+2}{n^2+2n+1+2}$$

$$= \frac{6(n^2+2n+3)}{n^2+4n+6}\frac{n^2+4n+6}{n^2+2n+3}$$

$$= 6,$$

$$p_2(n)\frac{\hat{T}(n+2)}{\hat{T}(n)} = \frac{5(n^2+2)}{n^2+4n+6}\frac{(n+2)^2+2}{n^2+2}$$

$$= \frac{5(n^2+2)}{n^2+4n+6)}\frac{n^2+4n+6}{n^2+2}$$

$$= 5,$$

$$g(n)\hat{T}(n+2) = \frac{32(3^n)}{n^2+4n+6}(n^2+4n+6)$$

$$= 32(3^n).$$

Then the equation (4) *takes the form*

$$yt(n+2)+6y(n+1)+5y(n) = 32(3^n).$$

We will check that

$$y(n) = 3^n$$

is its particular solution.

Indeed,

$$3^{n+2}+6(3^{n+1})+5(3^n) = 3^n(3^2+6.3+5)$$

$$= 32(3^n).$$

Now we will find a fundamental set of solutions of the equation

$$y(n+2)+6y(n+1)+5y(n) = 0. \tag{15}$$

The characteristic equation is

$$\lambda^2+6\lambda+5 = 0.$$

We have

$$\lambda_1 = -5, \qquad \lambda_2 = -1.$$

Then

$$y_1(n) = (-5)^n, \qquad y_2(n) = (-1)^n$$

is a fundamental set of solutions of the equation (15)*. Consequently, the general solution of the considered equation is given by*

$$y(n) = a_1(-5)^n + a_2(-1)^n + 3^n,$$

where a_1 *and* a_2 *are constants.*

Example 2.4.5. *Let* $k = 3$*,*

$$p_1(n) = -\tfrac{3(2n+5)}{2n+7}, \qquad p_2(n) = \tfrac{3(2n+3)}{2n+7}, \qquad p_3(n) = -\tfrac{2n+1}{2n+7},$$

$$\hat{T}(n) = 2n+1, \qquad g(n) = \tfrac{2^n}{2n+7}, \qquad n \in \mathbb{N}.$$

Then

$$p_1(n)\tfrac{\hat{T}(n+3)}{\hat{T}(n+2)} = -\tfrac{3(2n+5)}{2n+7}\tfrac{2(n+3)+1}{2(n+_2)+1}$$

$$= -\tfrac{3(2n+5)}{2n+7}\tfrac{2n+7}{2n+5}$$

$$= -3,$$

$$p_2(n)\tfrac{\hat{T}(n+3)}{\hat{T}(n+1)} = \tfrac{3(2n+3)}{2n+7}\tfrac{2n+7}{2(n+1)+1}$$

$$= \tfrac{3(2n+3)}{2n+7}\tfrac{2n+7}{2n+3}$$

$$= 3,$$

$$p_3(n)\tfrac{\hat{T}(n+3)}{\hat{T}(n)} = -\tfrac{2n+1}{2n+7}\tfrac{2n+7}{2n+1}$$

$$= -1,$$

$$g(n)\hat{T}(n+3) = \tfrac{2^n}{2n+7}(2n+7)$$

$$= 2^n.$$

The equation (4) *takes the form*

$$y(n+3) - 3y(n+2) + 3y(n+1) - y(n) = 2^n.$$

We will check that

$$y(n) = 2^n$$

is its solution.

Indeed,

$$2^{n+3} - 3.2^{n+2} + 3.2^{n+1} - 2^n = 2^n\left(2^3 - 3.2^2 + 3.2 - 1\right)$$

$$= 2^n(8 - 12 + 6 - 1)$$

$$= 2^n.$$

Now we consider the homogeneous equation

$$y(n+3) - 3y(n+2) + 3y(n+1) - y(n) = 0. \tag{16}$$

The characteristic equation is

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0,$$

i.e.,

$$(\lambda-1)^3=0.$$

Therefore

$$\lambda_1=1$$

and a fundamental set of solutions of the equation (16) *is*

$$1, \qquad n, \qquad n^2.$$

Consequently, the general solution of the considered equation is

$$y(n)=a_1+a_2n+a_3n^2+2^n,$$

where a_1, a_2 *and* a_3 *are constants.*

Example 2.4.6. *Let* $k=3$*,*

$$p_1(n)=-9\tfrac{n+9}{n+10}, \qquad p_2(n)=26\tfrac{n+8}{n+10}, \qquad p_3(n)=-24\tfrac{n+7}{n+10},$$

$$g(n)=6\tfrac{5^n}{n+10}, \hat{T}(n)=n+7, \qquad n\in\mathbb{N}.$$

Then

$$\begin{aligned}
p_1(n)\tfrac{\hat{T}(n+3)}{\hat{T}(n+2)} &= -9\tfrac{n+9}{n+10}\tfrac{n+7+3}{n+7+2}\\
&= -9\tfrac{n+9}{n+10}\tfrac{n+10}{n+9}\\
&= -9,\\
p_2(n)\tfrac{\hat{T}(n+3)}{\hat{T}(n+1)} &= 26\tfrac{n+8}{n+10}\tfrac{n+10}{n+7+1}\\
&= 26\tfrac{n+8}{n+10}\tfrac{n+10}{n+8}\\
&= 26,\\
p_3(n)\tfrac{\hat{T}(n+3)}{\hat{T}(n)} &= -24\tfrac{n+7}{n+10}\tfrac{n+10}{n+7}\\
&= -24,\\
g(n)\hat{T}(n+3) &= 6\tfrac{5^n}{n+10}(n+10)\\
&= 6\left(5^n\right).
\end{aligned}$$

The equation (4) *takes the form*

$$y(n+3)-9y(n+2)+26y(n+1)-24y(n)=6\left(5^n\right).$$

We will check that

$$y(n)=5^n$$

is its particular solution.

Indeed,

$$5^{n+3}-9\left(5^{n+2}\right)+26\left(5^{n+1}\right)-24\left(5^{n}\right)=5^{n}\left(125-9.25+26.5-24\right)$$

$$=5^{n}(125-225+130-24)$$

$$=6\left(5^{n}\right).$$

Now we consider the homogeneous equation

$$y(n+3)-9y(n+2)+26y(n+1)-24y(n)=0. \tag{17}$$

The characteristic equation is

$$\lambda^{3}-9\lambda^{2}+26\lambda-24=0 \qquad \Longrightarrow$$

$$\lambda^{3}-9\lambda^{2}+14\lambda+12\lambda-24=0 \qquad \Longrightarrow$$

$$\lambda(\lambda^{2}-9\lambda+14)+12(\lambda-2)=0 \qquad \Longrightarrow$$

$$\lambda(\lambda-2)(\lambda-9)+12(\lambda-2)=0 \qquad \Longrightarrow$$

$$(\lambda-2)(\lambda^{2}-9\lambda+12)=0 \qquad \Longrightarrow$$

$$(\lambda-2)(\lambda-3)(\lambda-4)=0.$$

Hence,

$$\lambda_1=2, \qquad \lambda_2=3, \qquad \lambda_3=4.$$

A fundamental set of solutions of the equation (17) *is*

$$2^{n}, \qquad 3^{n}, \qquad 4^{n}.$$

Consequently, the general solution of the considered equation is

$$y(n)=a_1 2^{n}+a_2 3^{n}+a_3 4^{n}+5^{n},$$

where a_1, a_2 and a_3 are constants.

Exercise 2.4.7. *Let $k=3$,*

$$p_1(n)=-6\tfrac{n+13}{n+14}, \qquad p_2(n)=12\tfrac{n+12}{n+14}, \qquad p_3(n)=-8\tfrac{n+11}{n+14},$$

$$g(n)=-3\tfrac{2^{n+2}}{n+14}, \qquad \hat{T}(n)=n+11.$$

Check if

$$y(n)=2^{n}$$

is a solution to the equation (4). *Find the general solution of the equation* (4).

Definition 2.4.8. *A polynomial operator $N(\hat{E})$, where $\hat{E}$ is the iso-shift operator, is said to be iso-annihilator of $g(n)$ if*

$$N(\hat{E})g(n) = 0.$$

Example 2.4.9. *Let $\hat{T}(n) = n+1$, $g(n) = n^2$,*

$$N(\hat{E}) = \hat{E} - \frac{(n+1)^2}{(n+2)n^2}, \qquad n \in \mathbb{N}.$$

Then

$$N(\hat{E})g(n) = \left(\hat{E} - \tfrac{(n+1)^2}{(n+2)n^2}\right) g(n)$$

$$= \hat{E}g(n) - \tfrac{(n+1)^2}{n^2(n+2)} g(n)$$

$$= \tfrac{g(n+1)}{\hat{T}(n+1)} - \tfrac{(n+1)^2}{(n+2)n^2} g(n)$$

$$= \tfrac{(n+1)^2}{n+1+1} - \tfrac{(n+1)^2}{(n+2)n^2} n^2$$

$$= \tfrac{(n+1)^2}{n+2} - \tfrac{(n+1)^2}{n+2}$$

$$= 0.$$

Example 2.4.10. *Let $\hat{T}(n) = g(n) = e^n$,*

$$N(\hat{E}) = \hat{E} - e^{-n} \qquad n \in \mathbb{N}.$$

Then

$$N(\hat{E})g(n) = \left(\hat{E} - e^{-n}\right) e^n$$

$$= \hat{E}e^n - e^{-n}e^n$$

$$= \tfrac{e^{n+1}}{e^{n+1}} - 1$$

$$= 1 - 1$$

$$= 0.$$

Example 2.4.11. *Let $\hat{T}(n) = n+10$, $a(n) = n+1$,*

$$b(n) = \tfrac{(n+1)(n+10)}{(n+9)(n+11)}, \qquad g(n) = n+9,$$

$$N(\hat{E}) = a(n)\hat{E} - b(n), \qquad n \in \mathbb{N}.$$

Then

$$
\begin{aligned}
N(\hat{E})g(n) &= \left(a(n)\hat{E} - b(n)\right) g(n) \\
&= a(n)\hat{E}g(n) - b(n)g(n) \\
&= a(n)\tfrac{g(n+1)}{\hat{T}(n+1)} - b(n)g(n) \\
&= \tfrac{(n+1)(n+10)}{n+11} - \tfrac{(n+1)(n+10)}{(n+9)(n+11)}(n+9) \\
&= \tfrac{(n+1)(n+10)}{n+11} - \tfrac{(n+1)(n+10)}{n+11} \\
&= 0.
\end{aligned}
$$

Assume now that $N(\hat{E})$ is an annihilator of $g(n)$ in (4). Applying $N(\hat{E})$ on both sides of (4) yields

$$N(\hat{E})p(\hat{E})y(n) = 0.$$

2.5. Method of Variation of Constants

Here we consider the equation

$$y(n+2) + p_1(n)\frac{\hat{T}(n+2)}{\hat{T}(n)}y(n+1) + p_2(n)\frac{\hat{T}(n+2)}{\hat{T}(n)}y(n) = \hat{T}(n+2)g(n), \qquad n \geq n_0, \tag{18}$$

and the corresponding homogeneous equation

$$y(n+2) + p_1(n)\frac{\hat{T}(n+2)}{\hat{T}(n)}y(n+1) + p_2(n)\frac{\hat{T}(n+2)}{\hat{T}(n)}y(n) = 0, \qquad n \geq n_0. \tag{18$'$}$$

We suppose that $y_1(n)$ and $y_2(n)$ are linearly independent solutions of the equation $(18')$.

We will search functions $u_1(n)$ and $u_2(n)$ so that

$$y(n) = u_1(n)y_1(n) + u_2(n)y_2(n), \qquad n \geq n_0,$$

is a solution of the equation (18) and

$$\Delta u_1(n)y_1(n+1) + \Delta u_2(n)y_2(n) = 0, \qquad n \geq n_0,$$

where

$$\Delta u_1(n) = u_1(n+1) - u_1(n), \qquad \Delta u_2(n) = u_2(n+1) - u_2(n).$$

We have

$$y_1(n+2) + p_1(n)\frac{\hat{T}(n+2)}{\hat{T}(n+1)}y_1(n+1) + p_2(n)\frac{\hat{T}(n+2)}{\hat{T}(n)}y_1(n) = 0, \qquad n \geq n_0, \tag{19}$$

$$y_2(n+2) + p_1(n)\frac{\hat{T}(n+2)}{\hat{T}(n+1)}y_2(n+1) + p_2(n)\frac{\hat{T}(n+2)}{\hat{T}(n)}y_2(n) = 0, \qquad n \geq n_0. \tag{20}$$

Also, for $n \geq n_0$,

$$\begin{aligned} y(n+1) &= u_1(n+1)y_1(n+1) + u_2(n+1)y_2(n+1) \\ &= (u_1(n+1) - u_1(n))\, y_1(n+1) + u_1(n)y_1(n+1) \\ &+ (u_2(n+1) - u_2(n))\, y_2(n+1) + u_2(n)y_2(n+1) \\ &= \Delta u_1(n)y_1(n+1) + \Delta u_2(n)y_2(n+1) + u_1(n)y_1(n+1) + u_2(n)y_2(n+1) \\ &= u_1(n)y_1(n+1) + u_2(n)y_2(n+1), \end{aligned}$$

i.e.,

$$y(n+1) = u_1(n)y_1(n+1) + u_2(n)y_2(n+1), \tag{21}$$

$$\begin{aligned} y(n+2) &= u_1(n+2)y_1(n+2) + u_2(n+2)y_2(n+2) \\ &= (u_1(n+2) - u_1(n+1))\, y_1(n+2) + u_1(n+1)y_1(n+2) \\ &+ (u_2(n+2) - u_2(n+1))\, y_2(n+2) + u_2(n+1)y_2(n+2) \\ &= \Delta u_1(n+1)y_1(n+2) + \Delta u_2(n+1)y_2(n+1) + u_1(n+1)y_1(n+2) + u_2(n+1)y_2(n+2) \\ &= u_1(n+1)y_1(n+2) + u_2(n+1)y_2(n+2) \\ &= (u_1(n+1) - u_1(n))\, y_1(n+2) + u_1(n)y_1(n+2) \\ &+ (u_2(n+1) - u_2(n))\, y_2(n+2) + u_2(n)y_2(n+2) \\ &= \Delta u_1(n)y_1(n+2) + \Delta u_2(n)y_2(n+2) + u_1(n)y_1(n+2) + u_2(n)y_2(n+2), \end{aligned}$$

i.e.,

$$y(n+2) = \Delta u_1(n)y_1(n+2) + \Delta u_2(n)y_2(n+2) + u_1(n)y_1(n+2) + u_2(n)y_2(n+2). \tag{22}$$

Now we put (21) and (22) in (18) and we find

$$\begin{aligned} &\Delta u_1(n)y_1(n+2) + \Delta u_2(n)y_2(n+2) \\ &+ u_1(n)y_1(n+2) + u_2(n)y_2(n+2) \\ &+ p_1(n)\tfrac{\hat{T}(n+2)}{\hat{T}(n+1)}\left(u_1(n)y_1(n+1) + u_2(n)y_2(n+1)\right) \\ &+ p_2(n)\tfrac{\hat{T}(n+2)}{\hat{T}(n)}\left(u_1(n)y_1(n) + u_2(n)y_2(n)\right) \\ &= \hat{T}(n+2)g(n), \end{aligned}$$

or

$$\Delta u_1(n)y_1(n+2)+\Delta u_2(n)y_2(n+2)$$

$$+u_1(n)\left(y_1(n+2)+p_1(n)\tfrac{\hat{T}(n+2)}{\hat{T}(n)}y_1(n+1)+p_2(n)\tfrac{\hat{T}(n+2)}{\hat{T}(n)}y_1(n)\right)$$

$$+u_2(n)\left(y_2(n+2)+p_1(n)\tfrac{\hat{T}(n+2)}{\hat{T}(n)}y_2(n+1)+p_2(n)\tfrac{\hat{T}(n+2)}{\hat{T}(n)}y_2(n)\right)$$

$$=\hat{T}(n+2)g(n),$$

now we use (19) and (20), and we get

$$\Delta u_1(n)y_1(n+2)+\Delta u_2(n)y_2(n+2)=\hat{T}(n+2)g(n), \qquad n\geq n_0.$$

In this way for $u_1(n)$ and $u_2(n)$ we get the system

$$\begin{cases} \Delta u_1(n)y_1(n+1)+\Delta u_2(n)y_2(n+1)=0 \\ \Delta u y_1(n)y_1(n+2)+\Delta u_2(n)y_2(n+2)=\hat{T}(n+2)g(n). \end{cases}$$

We have

$$\Delta u_1(n)=\frac{\begin{vmatrix} 0 & y_2(n+1) \\ \hat{T}(n+2)g(n) & y_2(n+2) \end{vmatrix}}{W(n+1)}$$

$$=-\tfrac{\hat{T}(n+1)y_2(n+1)g(n)}{W(n+1)},$$

$$\Delta u_2(n)=\frac{\begin{vmatrix} y_1(n+1) & 0 \\ y_1(n+2) & \hat{T}(n+2)g(n) \end{vmatrix}}{W(n+1)}$$

$$=\tfrac{y_1(n+1)\hat{T}(n+1)g(n)}{W(n+1)},$$

i.e.,

$$\Delta y_1(n)=-\tfrac{\hat{T}(n+1)y_2(n+1)g(n)}{W(n+1)}$$

$$\Delta y_2(n)=\tfrac{y_1(n+1)\hat{T}(n+1)g(n)}{W(n+1)},$$

whereupon

$$u_1(n)=-\textstyle\sum_{r=0}^{n-1}\tfrac{\hat{T}(r+1)y_2(r+1)g(r)}{W(r+1)},$$

$$u_2(n)=\textstyle\sum_{r=0}^{n-1}\tfrac{y_1(r+1)\hat{T}(r+1)g(r)}{W(r+1)}.$$

2.6. Some Nonlinear Iso-Difference Equations

1. Iso-Riccati equations.

Here we consider the equation

$$\hat{y}^{\wedge}(\widehat{n+1})\hat{y}^{\wedge}(\hat{n})+p(n)\hat{y}^{\wedge}(\widehat{n+1})+q(n)\hat{y}^{\wedge}(\hat{n})=0, \tag{23}$$

where p and q are given functions, y is unknown.

The equation (23) we can rewrite in the following form

$$\frac{y(n+1)}{\hat{T}(n+1)}\frac{y(n)}{\hat{T}(n)}+p(n)\frac{y(n+1)}{\hat{T}(n+1)}+q(n)\frac{y(n)}{\hat{T}(n)}=0$$

or

$$y(n+1)y(n)+p(n)\hat{T}(n)y(n+1)+q(n)\hat{T}(n+1)y(n)=0. \tag{24}$$

To solve the equation (24) we let

$$z(n)=\frac{1}{y(n)},$$

whereupon

$$\frac{1}{z(n)z(n+1)}+p(n)\hat{T}(n)\frac{1}{z(n+1)}+q(n)\hat{T}(n+1)\frac{1}{z(n)}=0,$$

or

$$1+p(n)\hat{T}(n)z(n)+q(n)\hat{T}(n+1)z(n+1)=0,$$

i.e., we obtain a linear iso-difference equation.

The corresponding nonhomogeneous iso-Riccati equation is

$$\hat{y}^{\wedge}(\widehat{n+1})\hat{y}^{\wedge}(\hat{n})+p(n)\hat{y}^{\wedge}(\widehat{n+1})+q(n)\hat{y}^{\wedge}(\hat{n})=g(n),$$

where g is a given function. This equation we can rewrite in the form

$$\frac{y(n+1)}{\hat{T}(n+1)}\frac{y(n)}{\hat{T}(n)}+p(n)\frac{y(n+1)}{\hat{T}*(n+1)}+q(n)\frac{y(n)}{\hat{T}(n)}=g(n),$$

or

$$y(n+1)y(n)+p(n)\hat{T}(n)y(n+1)+q(n)\hat{T}(n+1)g(n)=g(n)\hat{T}(n)\hat{T}(n+1). \tag{25}$$

To solve the equation (25) we set

$$y(n)=\frac{z(n+1)}{z(n)}-p(n)\hat{T}(n).$$

Then

$$y(n+1) = \frac{z(n+2)}{z(n+1)} - p(n+1)\hat{T}(n+1),$$

$$y(n)y(n+1) = \left(\frac{z(n+1)}{z(n)} - p(n)\hat{T}(n)\right)\left(\frac{z(n+2)}{z(n+1)} - p(n+1)\hat{T}(n+1)\right)$$

$$= \frac{z(n+2)}{z(n)} - p(n+_1)\hat{T}(n+1)\frac{z(n+1)}{z(n)} - p(n)\hat{T}(n)\frac{z(n+2)}{z(n+1)}$$

$$+p(n)p(n+1)\hat{T}(n)\hat{T}(n+1),$$

whereupon the equation (25) takes the form

$$\frac{z(n+2)}{z(n)} - p(n+_1)\hat{T}(n+1)\frac{z(n+1)}{z(n)} - p(n)\hat{T}(n)\frac{z(n+2)}{z(n+1)}$$

$$+p(n)p(n+1)\hat{T}(n)\hat{T}(n+1)$$

$$= p(n)\hat{T}(n)\left(\frac{z(n+2)}{z(n+1)} - p(n+1)\hat{T}(n+1)\right)$$

$$+q(n)\hat{T}(n+1)\left(\frac{z(n+1)}{z(n)} - p(n)\hat{T}(n)\right)$$

$$= g(n)\hat{T}(n)\hat{T}(n+1),$$

or

$$\frac{z(n+2)}{z(n)} - \hat{T}(n+1)(p(n+1)-q(n))\frac{z(n+1)}{z(n)} = g(n)\hat{T}(n)\hat{T}(n+1),$$

or

$$z(n+2) - \hat{T}(n+1)(p(n+1)-q(n))z(n+1) - g(n)\hat{T}(n)\hat{T}(n+1)z(n) = 0.$$

2. The Iso-Pielou Logistic Equation.

Here we consider the equation

$$\hat{x}^{\wedge}(\widehat{n+1}) = \frac{\alpha\hat{x}^{\wedge}(\hat{n})}{1+_\beta\hat{x}^{\wedge}(\hat{n})}, \tag{26}$$

where α and β are constants.

Definition 2.6.1. *The equation* (26) *is called iso-Pielou logistic equation.*

We can rewrite it in the following form

$$\frac{x(n+1)}{\hat{T}(n+1)} = \frac{\alpha\frac{x(n)}{\hat{T}(n)}}{1+\beta\frac{x(n)}{\hat{T}(n)}},$$

or

$$x(n+1) = \frac{\alpha \frac{\hat{T}(n+1)}{\hat{T}(n)} x(n)}{1 + \frac{\beta}{\hat{T}(n)} x(n)}. \tag{27}$$

Let

$$x(n) = \frac{1}{z(n)}.$$

Then

$$x(n+1) = \frac{1}{z(n+1)},$$

the equation (27) takes the form

$$\frac{1}{z(n+1)} = \frac{\alpha \frac{\hat{T}(n+1)}{\hat{T}(n)} \frac{1}{z(n)}}{1 + \frac{\beta}{\hat{T}(n)} \frac{1}{z(n)}},$$

or

$$\frac{1}{z(n+1)} = \frac{\alpha \frac{\hat{T}(n+1)}{\hat{T}(n)}}{z(n) + \frac{\beta}{\hat{T}(n)}},$$

or

$$z(n) + \frac{\beta}{\hat{T}(n)} = \alpha \frac{\hat{T}(n+1)}{\hat{T}(n)} z(n+1).$$

3. Equations of general Riccati type

Here we consider the equations

$$\hat{x}^{\wedge}(\widehat{n+1}) = \frac{a(n)\hat{x}^{\wedge}(\hat{n}) + b(n)}{c(n)\hat{x}^{\wedge}(\hat{n}) + d(n)}, \tag{28}$$

where $c(n) \neq 0$, $a(n)d(n) - b(n)c(n) \neq 0$ for all $n \geq 0$.

The equation (28) we can rewrite in the form

$$\frac{x(n+1)}{\hat{T}(n+1)} = \frac{a(n)\frac{x(n)}{\hat{T}(n)} + b(n)}{c(n)\frac{x(n)}{\hat{T}(n)} + d(n)},$$

or

$$x(n+1) = \hat{T}(n+1)\frac{a(n)x(n) + b(n)\hat{T}(n)}{c(n)x(n) + d(n)\hat{T}(n)}.$$

To solve this equation we set

$$c(n)x(n) + d(n)\hat{T}(n) = \frac{y(n+1)}{y(n)}.$$

Hence,

$$c(n)x(n) = \frac{y(n+1)}{y(n)} - d(n)\hat{T}(n),$$

or

$$x(n) = \frac{1}{c(n)} \frac{y(n+1)}{y(n)} - \frac{d(n)}{c(n)} \hat{T}(n).$$

Therefore

$$x(n+1) = \frac{1}{c(n+1)} \frac{y(n+2)}{y(n+1)} - \frac{d(n+1)}{c(n+1)} \hat{T}(n+1)$$

and

$$\begin{aligned} &\tfrac{1}{c(n+1)} \tfrac{y(n+2)}{y(n+1)} - \tfrac{d(n+1)}{c(n+1)} \hat{T}(n+1) \\ &= \hat{T}(n+1) \frac{a(n)\left(\frac{1}{c(n)} \frac{y(n+1)}{y(n)} - \frac{d(n)}{c(n)} \hat{T}(n)\right) + b(n)\hat{T}(n)}{\frac{y(n+1)}{y(n)}}, \end{aligned}$$

or

$$\begin{aligned} &y(n+2) - d(n+1)\hat{T}(n+1)y(n+1) \\ &= \hat{T}(n+1)c(n+1)\left(\tfrac{a(n)}{c(n)} y(n+1) - \tfrac{a(n)d(n)}{c(n)} \hat{T}(n)y(n) + b(n)\hat{T}(n)y(n)\right), \end{aligned}$$

or

$$\begin{aligned} &y(n+2) - d(n+1)\hat{T}(n+1)y(n+1) = \hat{T}(n+1) \tfrac{c(n+1)}{c(n)} a(n)y(n+1) \\ &-\hat{T}(n)\hat{T}(n+1) \tfrac{c(n+1)}{c(n)} a(n)d(n)y(n) \\ &+\hat{T}(n)\hat{T}(n+1)b(n)c(n+1)y(n), \end{aligned}$$

or

$$\begin{aligned} &y(n+2) - \hat{T}(n+1)\left(d(n+1) - \tfrac{c(n+1)}{c(n)} a(n)\right) y(n+1) \\ &+\hat{T}(n)\hat{T}(n+1)c(n+1)\left(\tfrac{a(n)d(n)}{c(n)} - b(n)\right) y(n) = 0. \end{aligned}$$

4. Here we consider the iso-difference equation

$$\left(\hat{y}^\wedge(\widehat{n+k})\right)^{r_1} \left(\hat{y}^\wedge(\widehat{n+k-1})\right)^{r_2} \ldots \left(\hat{y}^\wedge(\hat{n})\right)^{r_{k+1}} = g(n),$$

which we can rewrite in the form

$$\left(\frac{y(n+k)}{\hat{T}(n+k)}\right)^{r_1} \left(\frac{y(n+k-1)}{\hat{T}(n+k-1)}\right)^{r_2} \cdots \left(\frac{y(n)}{\hat{T}(n)}\right)^{r_{k+1}} = g(n)$$

or

$$\begin{aligned} &(y(n+k))^{r_1} (y(n+k-1))^{r_2} \ldots (y(n))^{r_{k+1}} \\ &= g(n)\hat{T}(n)\hat{T}(n+1) \ldots \hat{T}(n+k). \end{aligned} \tag{29}$$

Let

$$z(n) = \log(y(n)).,$$

Then

$$z(n+1) = \log(y(n+1)),$$

$$z(n+2)\log(y(n+2)),$$

$$\dots$$

$$z(n+k) = \log(y(n+k)).$$

Hence, using (29),

$$\log\left((y(n+k))^{r_1}(y(n+k-1))^{r_2}\dots(y(n))^{r_{k+1}}\right)$$

$$= \log\left(g(n)\hat{T}(n)\hat{T}(n+1)\dots\hat{T}(n+k)\right),$$

or

$$\log(y(n+k))^{r_1} + \log(y(n+k-1))^{r_2} + \cdots + \log(y(n))^{r_{k+1}}$$

$$= \log\left(g(n)\hat{T}(n)\hat{T}(n+1)\dots\hat{T}(n+k)\right),$$

or

$$r_1\log(y(n+k)) + r_2\log(y(n+k-1)) + \cdots + r_{k+1}\log(y(n))$$

$$= \log\left(g(n)\hat{T}(n)\hat{T}(n+1)\dots\hat{T}(n+k)\right),$$

or

$$r_1 z(n+k) + r_2 z(n+k-1) + \cdots + r_{k+1} z(n)$$

$$= \log\left(g(n)\hat{T}(n)\hat{T}(n+1)\dots\hat{T}(n+k)\right).$$

MA

Chapter 3

Systems of Linear Iso-Difference Equations

3.1. The Basic Theory

Here we consider the system

(1) $\quad \hat{y}^{\wedge}(\hat{n}) = A(n)\hat{y}^{\wedge}(\hat{n}) + g(n),$

where $A(n) = (a_{ij}(n))_{i,j=1}^{k}$ is $k \times k$ matrix function,

$$y(n) = \begin{pmatrix} y_1(n) \\ y_2(n) \\ \dots \\ y_k(n) \end{pmatrix}, \qquad y(n+1) = \begin{pmatrix} y_1(n+1) \\ y_2(n+1) \\ \dots \\ y_k(n+1) \end{pmatrix}, \qquad g(n) = \begin{pmatrix} g_1(n) \\ g_2(n) \\ \dots \\ g_k(n) \end{pmatrix},$$

$g(n)$ is given real-valued function or iso-function of first, second, third, fourth or fifth kind.

The system (1) we can rewrite in the form

$$\frac{y(n+1)}{\hat{T}(n+1)} = A(n)\frac{y(n)}{\hat{T}(n)} + g(n)$$

or

$$y(n+1) = \frac{\hat{T}(n+1)}{\hat{T}(n)}A(n)y(n) + \hat{T}(n+1)g(n). \tag{1}$$

The corresponding homogeneous system is

$$y(n+1) = \frac{\hat{T}(n+1)}{\hat{T}(n)}A(n)y(n). \tag{2}$$

Now we will establish the existence and uniqueness of solutions of (2).

Theorem 3.1.1. *For each $x_0 \in \mathbb{R}^k$ and $n_0 \in \mathbb{N}$ there exists a unique solution $x(n,n_0,x_0)$ of the system (2) such that*

$$x(n_0,n_0,x_0) = x_0.$$

Proof. From (2) we have

$$x(n_0+1,n_0,x_0) = \frac{\hat{T}(n_0+1)}{\hat{T}(n_0)}A(n_0)x(n_0)$$

$$= \frac{\hat{T}(n_0+1)}{\hat{T}(n_0)}A(n_0)x_0,$$

$$x(n_0+2,n_0,x_0) = \frac{\hat{T}(n_0+2)}{\hat{T}(n_0+1)}A(n_0+1)x(n_0+1)$$

$$= \frac{\hat{T}(n_0+2)}{\hat{T}(n_0+1)}\frac{\hat{T}(n_0+1)}{\hat{T}(n_0)}A(n_0+1)A(n_0)x_0$$

$$= \frac{\hat{T}(n_0+2)}{\hat{T}(n_0)}A(n_0+1)A(n_0)x_0.$$

We suppose that

$$x(n_0+k,n_0,x_0) = \frac{\hat{T}(n_0+k)}{\hat{T}(n_0)}\prod_{i=0}^{k-1}A(n_0+i)x_0 \tag{3}$$

for some $k \in \mathbb{N}$.

Now we consider

$$x(n_0+k+1,n_0,x_0).$$

We have, using (3),

$$x(n_0+k+1,n_0,x_0) = \frac{\hat{T}(n_0+k+1)}{\hat{T}(n_0+k)}A(n_0+k)x(n_0+k)$$

$$= \frac{\hat{T}(n_0+k+1)}{\hat{T}(n_0+k)}\frac{\hat{T}(n_0+k)}{\hat{T}(n_0)}A(n_0+k)\prod_{i=0}^{k-1}A(n_0+i)x_0$$

$$= \frac{\hat{T}(n_0+k+1)}{\hat{T}(n_0)}\prod_{i=0}^{k}A(n_0+i)x_0.$$

Consequently, the formula (3) is valid for all $k \in \mathbb{N}$.

The formula (3) gives the unique solution with desired properties. □

Definition 3.1.2. *The solutions*

$$y_1(n), \qquad y_2(n), \qquad \ldots, \qquad y_k(n)$$

of the system (2) *are said to be linearly independent for* $n \geq n_0$ *if whenever*

$$c_1y_1(n)+c_2y_2(n)+\cdots+c_ky_k(n) = 0$$

for all $n \geq n_0$*, then* $c_i = 0$*,* $i = 1,2,\ldots,k$*.*

Let $\Phi(n)$ be a $k \times k$ matrix whose columns are solutions of the system (2). we will write

$$\Phi(n) = [y_1(n),y_2(n),\ldots,y_k(n)].$$

From here,

$$\begin{aligned}
\Phi(n+1) &= [y_1(n+1), y_2(n+1), \ldots, y_k(n+1)] \\
&= \left[\tfrac{\hat{T}(n+1)}{\hat{T}(n)}A(n)y_1(n), \tfrac{\hat{T}(n+1)}{\hat{T}(n)}A(n)y_2(n), \ldots, \tfrac{\hat{T}(n+1)}{\hat{T}(n)}A(n)y_k(n)\right] \\
&= \tfrac{\hat{T}(n+1)}{\hat{T}(n)}A(n)[y_1(n), y_2(n), \ldots, y_k(n)] \\
&= \tfrac{\hat{T}(n+1)}{\hat{T}(n)}A(n)\Phi(n),
\end{aligned}$$

i.e., $\Phi(n)$ satisfies the iso-difference equation

$$\Phi(n+1) = \frac{\hat{T}(n+1)}{\hat{T}(n)}A(n)\Phi(n). \tag{4}$$

The solutions $y_1(n)$, $y_2(n)$, …, $y_k(n)$ are linearly independent for $n \geq n_0$ if and only if the matrix $\Phi(n)$ is nonsingular.

Definition 3.1.3. *If $\Phi(n)$ is a matrix that is nonsingular for all $n \geq n_0$ and satisfies the equation* (4)*, then it is said to be a fundamental matrix for the system* (2)*.*

Theorem 3.1.4. *Let C be a nonsingular matrix. If $\Phi(n)$ is a fundamental matrix of the system* (2)*, then $\Phi(n)C$ is a fundamental matrix of* (2)*.*

Proof. We multiply the equation (4) by the matrix C and we get

$$\Phi(n+1)C = \frac{\hat{T}(n+1)}{\hat{T}(n)}A(n)\Phi(n)C,$$

i.e., $\Phi(n)C$ satisfies the equation (4).

Because $\Phi(n)$ is a fundamental matrix of (2), then it is a nonsingular matrix. Hence, $\Phi(n)C$ is a nonsingular matrix.

Consequently, $\Phi(n)C$ is a fundamental matrix of (2). □

We note that the matrix

$$\Phi(n) = \frac{\hat{T}(n)}{\hat{T}(n_0)} \prod_{i=0}^{n-n_0-1} A(n_0+i)$$

with $\Phi(n_0) = I$, is a fundamental matrix of (2).

Exercise 3.1.5. *Prove that there is a unique solution $\Psi(n)$ of the equation* (4) *with $\Psi(n_0) = I$.*

Let $\Phi(n)$ be a fundamental matrix of the system (2). For m, $n \in \mathbb{N}$ we define the matrix

$$\Phi(n,m) := \Phi(n)\Phi^{-1}(m).$$

Below we will investigate the properties of the matrix $\Phi(n,m)$.

1. $\Phi^{-1}(n,m) = \Phi(m,n)$.

Really,

$$\Phi^{-1}(n,m) = \left(\Phi(n)\Phi^{-1}(m)\right)^{-1}$$

$$= \left(\Phi^{-1}(m)\right)^{-1}(\Phi(n))^{-1}$$

$$= \Phi(m)\Phi^{-1}(n)$$

$$= \Phi(m,n).$$

2. $\Phi(n,m) = \Phi(n,r)\Phi(r,m)$.

Really,

$$\Phi(n,m) = \Phi(n)\Phi^{-1}(m)$$

$$= \Phi(n)\Phi^{-1}(r)\Phi(r)\Phi^{-1}(m)$$

$$= \Phi(n,r)\Phi(r,m).$$

3. For $m < n$ we have

$$\Phi(n,m) = \frac{\hat{T}(n)}{\hat{T}(m)} \prod_{i=0}^{n-m-1} A(m+i).$$

Really,

$$\Phi(n,m) = \Phi(n)\Phi^{-1}(m)$$

$$= \tfrac{\hat{T}(n)}{\hat{T}|(n_0)} \prod_{i=0}^{n-n_0-1} A(n_0+i) \left(\tfrac{\hat{T}(m)}{\hat{T}(n_0)} \prod_{i=0}^{m-n_0-1} A(n_0+i) \right)^{-1}$$

$$= \tfrac{\hat{T}(n)}{\hat{T}(n_0)} A(n_0)A(n_0+1)\dots A(n-1) \tfrac{\hat{T}(n_0)}{\hat{T}(m)} \left(A(n_0)A(n_0+1)\dots A(m-1)\right)^{-1}$$

$$= \tfrac{\hat{T}(n)}{\hat{T}(m)} A(n_0)A^{-1}(n_0)\dots A(m-1)A^{-1}(m-1)A(m)A(m+1)\dots A(n-1)$$

$$= \tfrac{\hat{T}(n)}{\hat{T}(m)} A(m)A(m+1)\dots A(n-1)$$

$$= \tfrac{\hat{T}(n)}{\hat{T}(m)} \prod_{i=0}^{n-m-1} A(m+i).$$

Corollary 3.1.6. *The unique solution $y(n,n_0,y_0)$ of the system* (2) *with $y(n_0,n_0,y_0) = y_0$ is given by*

$$y(n,n_0,y_0) = \Phi(n,n_0)y_0.$$

Corollary 3.1.7. *(iso-Abel's formula) For $k \in \mathbb{N}$ we have*

$$\det\Phi(n+k) = \frac{\hat{T}(n+k)}{\hat{T}(n)} \prod_{i=0}^{k-1} \det(A(n+i))\det\Phi(n). \tag{5}$$

Proof. From the equation (4) we get

$$\det\Phi(n+1)\tfrac{\hat{T}(n+1)}{\hat{T}(n)}\det(A(n)\Phi(n))$$

$$=\tfrac{\hat{T}(n+k)}{\hat{T}(n)}\det(A(n))\det(\Phi(n)).$$

Hence,

$$\det\Phi(n+2)=\tfrac{\hat{T}(n+2)}{\hat{T}(n+1)}\det(A(n+1))\det(\Phi(n+1))$$

$$=\tfrac{\hat{T}(n+2)}{\hat{T}(n+1)}\det(A(n+1))\tfrac{\hat{T}(n+1)}{\hat{T}(n)}\det(A(n))\det(\Phi(n))$$

$$=\tfrac{\hat{T}(n+2)}{\hat{T}(n)}\det(A(n+1))\det(A(n))\det(\Phi(n)).$$

Now we suppose that for some $k\in\mathbb{N}$ we have (5).

We consider

$$\det(\Phi(n+k+1)).$$

We have

$$\det\Phi(n+k+1)=\tfrac{\hat{T}(n+k+1)}{\hat{T}(n+k)}\det(A(n+k))P\det(\Phi(n+k))$$

$$=\tfrac{\hat{T}(n+k+1)}{\hat{T}(n+k)}\tfrac{\hat{T}(n+k)}{\hat{T}(n)}\textstyle\prod_{i=0}^{k-1}\det(A(n+i))\det(A(n+k)\det(\Phi(n))$$

$$=\tfrac{\hat{T}(n+k+1)}{\hat{T}(n)}\textstyle\prod_{i=0}^{k}\det(A(n+i))\det(\Phi(n)).$$

Therefore the iso-Abel's formula (5) is valid for every $k\in\mathbb{N}$. $\square$

Using the iso-Abel's formula one can prove the following corollaries.

Corollary 3.1.8. *The fundamental matrix $\Phi(n)$ is nonsingular for all $n\geq n_0$ if and only if $\Phi(n_0)$ is nonsingular.*

Corollary 3.1.9. *The solutions*

$$y_1(n),\qquad y_2(n),\qquad \ldots,\qquad y_k(n)$$

of the system (2) are linearly independent for $n\geq n_0$ if and only if $\Phi(n_0)$ is nonsingular.

Corollary 3.1.10. *There are k linearly independent solutions of the system (2) for $n\geq n_0$.*

Proposition 3.1.11. *Let $y_1(n)$ and $y_2(n)$ be solutions of the system (2). Then*

$$y_1(n)+y_2(n)$$

is a solution of the system (2).

Proof. Let
$$y(n) = y_1(n) + y_2(n).$$
Then
$$y(n+1) = y_1(n+1) + y_2(n+1)$$
$$= \frac{\hat{T}(n+1)}{\hat{T}(n)} A(n) y_1(n) + \frac{\hat{T}(n+1)}{\hat{T}(n)} A(n) y_2(n)$$
$$= \frac{\hat{T}(n+1)}{\hat{T}(n)} A(n)(y_1(n) + y_2(n))$$
$$= \frac{\hat{T}(n+1)}{\hat{T}(n)} A(n) y(n).$$
□

Proposition 3.1.12. *Let $y_1(n)$ be a solution of the system* (2) *and c is a constant. Then*
$$cy_1(n)$$
is a solution of the system (2).

Proof. Let
$$y(n) = cy_1(n).$$
Then
$$y(n+1) = cy_1(n+1)$$
$$= c\frac{\hat{T}(n+1)}{\hat{T}(n)} A(n) y_1(n)$$
$$= \frac{\hat{T}(n+1)}{\hat{T}(n)} A(n)(cy_1(n))$$
$$= \frac{\hat{T}(n+1)}{\hat{T}(n)} A(n) y(n).$$
□

An immediate sequence of the last two proposition is that if
$$y_1(n), \qquad y_2(n), \qquad \dots, \qquad y_k(n)$$
are also solutions of (2), then so is any linear combination
$$y(n) = c_1 y_1(n) + c_2 y_2(n) + \cdots + c_k y_k(n),$$
where $c_i \in \mathbb{C}$, $i = 1, 2, \dots, k$.

Definition 3.1.13. *Let $\{y_i(n)\}_{i=1}^{k}$ is any linearly independent set of solutions of the system* (2)*, then the general solution of the system* (2) *is defined to be*
$$y(n) = \sum_{i=1}^{n} c_i y_i(n), \tag{6}$$
where $c_i \in \mathbb{C}$, $i = 1, 2, \dots, k$, and at least one $c_i \neq 0$.

The expression (6) may be written

$$y(n) = \Phi(n)c,$$

where

$$\Phi(n) = [y_1(n), y_2(n), \ldots, y_k(n)]$$

is a fundamental matrix,

$$c = \begin{pmatrix} c_1 \\ c_2 \\ \ldots \\ c_k \end{pmatrix} \in \mathbb{C}^k.$$

With $y_p(n)$ we will denote any particular solution of the system (1). The following result gives us an approach to find the general solution of the system (1).

Theorem 3.1.14. *Any solution $y(n)$ of the system (1) can be written as*

$$y(n) = \sum_{i=1}^{k} c_i y_i(n) + y_p(n),$$

where $c_i \in \mathbb{C}$, $i = 1, 2, \ldots, k$.

Proof. Let

$$z(n) = y(n) - y_p(n).$$

Then

$$z(n+1) = y(n+1) - y_p(n+1)$$

$$= \tfrac{\hat{T}(n+1)}{\hat{T}(n)} A(n)y(n) + \hat{T}(n+1)g(n)$$

$$- \tfrac{\hat{T}(n+1)}{\hat{T}(n)} A(n)y_p(n) - \hat{T}(n+1)g(n)$$

$$= \tfrac{\hat{T}(n+1)}{\hat{T}(n)} A(n)(y(n) - y_p(n))$$

$$= \tfrac{\hat{T}(n+1)}{\hat{T}(n)} A(n)z(n),$$

i.e., $z(n)$ is a solution to the system (2). Then

$$z(n) = \sum_{i=1}^{k} c_i y_i(n),$$

where $c_i \in \mathbb{C}$, $i = 1, 2, \ldots, k$.

Consequently,

$$y(n) - y_p(n) = \sum_{i=1}^{n} c_i y_i(n)$$

or

$$y(n) = \sum_{i=1}^{n} c_i y_i(n) + y_p(n).$$

$\square$

Proposition 3.1.15. *A particular solution of the system* (1) *may be written as*

$$y_p(n) = \sum_{r=n_0}^{n} \Phi(n,r+1)\hat{T}(n+1)g(r)$$

with $y_p(n_0) = 0$.

Proof. We have

$$y_p(n+1) = \sum_{r=n_0}^{n} \Phi(n+1,r+1)\hat{T}(n+1)g(r). \tag{7}$$

We note that

$$\begin{aligned}\Phi(n+1,r+1) &= \Phi(n+1)\Phi^{-1}(r+1)\\ &= \tfrac{\hat{T}(n+1)}{\hat{T}(n)}A(n)\Phi(n)\Phi^{-1}(r+1)\\ &= \tfrac{\hat{T}(n+1)}{\hat{T}(n)}A(n)\Phi(n,r+1).\end{aligned}$$

From here and (7), it follows

$$\begin{aligned}y_p(n+1) &= \textstyle\sum_{r=n_0}^{n-1}\Phi(n+1,r+1)\hat{T}(n+1)g(r) + \Phi(n+1,n+1)\hat{T}(n+1)g(n)\\ &= \tfrac{\hat{T}^2(n+1)}{\hat{T}(n)}\textstyle\sum_{r=n_0}^{n-1}A(n)\Phi(n,r+1)g(r) + \hat{T}(n+1)g(n)\\ &= \tfrac{\hat{T}(n+1)}{\hat{T}(n)}A(n)y_p(n) + \hat{T}(n+1)g(n),\end{aligned}$$

i.e., $y_p(n)$ is a solution to the system (1). $\square$

Using the above result we can conclude that the unique solution of the system (1) for which $y(n_0) = y_0$, is given by

$$y(n,n_0,y_0) = \Phi(n,n_0)y_0 + \sum_{r=n_0}^{n-1}\Phi(n,r+1)\hat{T}(n+1)g(r),$$

which we can rewrite in the form

$$\begin{aligned}y(n,n_0,y_0) &= \tfrac{\hat{T}(n)}{\hat{T}(n_0)}\textstyle\prod_{i=0}^{n-n_0-1}A(n_0+i)y_0\\ &+ \textstyle\sum_{r=n_0}^{n-1}\tfrac{\hat{T}(n)}{\hat{T}(r+1)}\prod_{i=0}^{n-r-2}A(r_0+1+i)\hat{T}(n+1)g(r)\end{aligned} \tag{8}$$

Exercise 3.1.16. *Rewrite the last formula in the case when* $A = \text{const}$.

Exercise 3.1.17. *Rewrite the formula* (8) *in the case when* $\hat{T} = \text{const}$, $A \neq \text{const}$.

Exercise 3.1.18. *Rewrite the formula* (8) *in the case when* $\hat{T} = \text{const}$, $A = \text{const}$.

Exercise 3.1.19. *Using the formula* (8) *prove that the initial problem for the system* (1) *has unique solution.*

Exercise 3.1.20. *Rewrite the formula* (8) *in the case when g is an iso-function of third kind.*

Exercise 3.1.21. *Rewrite the formula* (8) *in the case when A is an iso-function of fourth kind.*

Exercise 3.1.22. *Rewrite the formula* (8) *in the case when A and g are iso-functions of third kind.*

Example 3.1.23. *Let* $\hat{T}(n) = n+1$*,*

$$A(n) = \tfrac{n+1}{n+2}\begin{pmatrix} 2 & 2 & -2 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}, \qquad g(n) = \tfrac{1}{n+2}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad n \in \mathbb{N},$$

$$y_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Then

$$\hat{T}(n+1) = n+1+1$$

$$= n+2,$$

$$\tfrac{\hat{T}(n+1)}{\hat{T}(n)}A(n) = \tfrac{n+2}{n+1}\tfrac{n+1}{n+2}\begin{pmatrix} 2 & 2 & -2 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 & -2 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix},$$

$$\hat{T}(n+1)g(n) = (n+2)\tfrac{1}{n+2}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Therefore the system (1) *takes the form*

$$y(n+1) = \begin{pmatrix} 2 & 2 & -2 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix} y(n) + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Let

$$B := \begin{pmatrix} 2 & 2 & -2 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}, \qquad g_1(n) := \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

The general solution of the considered system is given by

$$y(n) = B^n y_0 + \sum_{r=0}^{n-1} B^{n-r-1} g_1(r). \tag{9}$$

We will find B^n using the Putzer algorithm. For more details for the Putzer algorithm we refer the reader to the appendix.

We have

$$B^n = \sum_{j=1}^{3} x_j(n) M(j-1)$$

$$= x_1(n)M(0) + x_2(n)M(1) + x_3(n)M(2)$$

$$= x_1(n)I + x_2(n)M(1) + x_2(n)M(2),$$

i.e.,

$$B^n = x_1(n)I + x_2(n)M(1) + x_3(n)M(2). \tag{10}$$

Now we consider

$$B - \lambda I.$$

We have

$$B - \lambda I = \begin{pmatrix} 2 & 2 & -2 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 & -2 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 2-\lambda & 2 & -2 \\ 0 & 3-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{pmatrix},$$

$$\det(B - \lambda I) = \begin{vmatrix} 2-\lambda & 2 & -2 \\ 0 & 3-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{vmatrix}$$

$$= (2-\lambda)(3-\lambda)^2 - (2-\lambda)$$

$$= (2-\lambda)\left((3-\lambda)^2 - 1\right)$$

$$= (2-\lambda)(3-\lambda-1)(3-\lambda+1)$$

$$= (4-\lambda)(2-\lambda)^2.$$

Therefore

$$\det(B-\lambda I)=0 \qquad \Longleftrightarrow$$

$$(4-\lambda)(2-\lambda)^2=0 \qquad \Longrightarrow$$

$$\lambda_1=4, \qquad \lambda_2=\lambda_3=2.$$

Hence,

$$M(1)=(B-\lambda_1 I)M(0)$$

$$=(B-4I)I$$

$$=B-4I$$

$$=\begin{pmatrix} 2 & 2 & -3 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}-\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$=\begin{pmatrix} -2 & 2 & -2 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix},$$

$$B=M(1)+4I,$$

$$M(2)=(B-\lambda_2 I)M(1)$$

$$=(B-2I)M(1)$$

$$=M(1)M(1)$$

$$=M^2(1)$$

$$=\begin{pmatrix} -2 & 2 & -2 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} -2 & 2 & -2 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$=\begin{pmatrix} 4 & -8 & 8 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix},$$

$$BM(1)=M(2)+2M(1)$$

$$0=M(3)$$

$$=(B-\lambda_3 I)M(2)$$

$$= BM(2) - 2M(2),$$

$$BM(2) = 2M(2).$$

From (10) *we get*

$$\begin{aligned} B^{n+1} &= BB^n \\ &= B(x_1(n)I + x_2(n)M(1) + x_3(n)M(2)) \\ &= x_1(n)B + x_2(n)BM(1) + x_3(n)BM(2) \\ &= x_1(n)(M(1)+4I) + x_2(n)(M(2)+2M(1)) + x_3(n)(2M(2)) \\ &= 4x_1(n)I + x_1(n)M(1) + x_2(n)M(2) + 2x_2(n)M(1) + 2x_3(n)M(2) \\ &= 4x_1(n)I + (x_1(n)+2x_2(n))M(1) + (x_2(n)+2x_3(n))M(2), \end{aligned}$$

i.e.,

$$B^{n+1} = 4x_1(n)I + (x_1(n)+2x_2(n))M(1) + (x_2(n)+2x_3(n))M(2). \tag{11}$$

On the other hand,

$$\begin{aligned} B^{n+1} &= \sum_{j=1}^{3} x_j(n+1)M(j-1) \\ &= x_1(n+1)M(0) + x_2(n+1)M(1) + x_3(n+1)M(2) \\ &= x_1(n+1)I + x_2(n+1)M(1) + x_3(n+1)M(2) \end{aligned}$$

From here and (11) *we get*

$$\begin{cases} x_1(n+1) = 4x_1(n) \\ x_2(n+1) = 2x_2(n) \\ x_3(n+1) = 2x_3(n). \end{cases} \tag{12}$$

Also,

$$\begin{aligned} B^0 &= I \\ &= \sum_{j=0}^{3} x_j(0)M(j-1) \\ &= x_1(0)M(0) + x_2(0)M(1) + x_3(0)M(2) \\ &= x_1(0)I + x_2(0)M(1) + x_3(0)M(2), \end{aligned}$$

whereupon

$$x_1(0) = 1, \qquad x_2(0) = x_3(0) = 0.$$

In this way, using (12)*, we obtain the initial problems*

$$\begin{cases} x_1(n+1) = 4x_1(n) \\ x_2(n+1) = 2x_2(n) \\ x_3(n+1) = 2x_3(n), \\ x_1(0) = 1, x_2(0) = x_3(0) = 0. \end{cases}$$

We consider the initial problem

$$\begin{cases} x_1(n+1) = 4x_1(n) \\ x_1(0) = 1. \end{cases}$$

For its general solution we have

$$x_1(n) = \prod_{i=0}^{n-1} 4$$
$$= 4^n.$$

Now we consider the initial problem

$$\begin{cases} x_2(n+1) = 2x_2(n) + x_1(n) \\ x_2(0) = 0 \end{cases}$$

or

$$\begin{cases} x_2(n+1) = 2x_2(n) + 4^n \\ x_2(0) = 0. \end{cases}$$

For its general solution we have the representation

$$x_2(n) = \sum_{r=0}^{n-1} \left(\prod_{i=r+1}^{n-1} 2\right) 4^r$$
$$= \sum_{r=0}^{n-1} 2^{n-r-1} 2^{2r}$$
$$= 2^{n-1} \sum_{r=0}^{n-1} 2^r$$
$$= 2^{n-1} \frac{1-2^n}{1-2}$$
$$= 2^{n-1}(2^n - 1)$$
$$= 2^{2n-1} - 2^{n-1}.$$

Now we consider the initial problem

$$\begin{cases} x_3(n+1) = 2x_3(n) + x_2(n) \\ x_3(0) = 0 \end{cases}$$

or

$$\begin{cases} x_3(n+1) = 2x_3(n) + 2^{2n-1} - 2^{n-1} \\ \\ x_3(0) = 0. \end{cases}$$

For its general solution we have the following representation

$$\begin{aligned} x_3(n) &= \sum_{r=0}^{n-1} \left(\prod_{i=r+1}^{n-1} 2\right) \left(2^{2r-1} - 2^{r-1}\right) \\ &= \sum_{r=0}^{n-1} 2^{n-r-1} \left(2^{2r-1} - 2^{r-1}\right) \\ &= 2^{n-1} \sum_{r=0}^{n-1} 2^{-r} \left(2^{2r-1} - 2^{r-1}\right) \\ &= 2^{n-1} \sum_{r=0}^{n-1} \left(2^{r-1} - 2^{-1}\right) \\ &= 2^{n-2} \sum_{r=0}^{n-1} 2^r - 2^{n-2} \sum_{r=0}^{n-1} 1 \\ &= 2^{n-2} \frac{2^n-1}{2-1} - n2^{n-2} \\ &= 2^{2n-2} - 2^{n-2} - n2^{n-2} \\ &= 2^{2n-2} - (n+1)2^{n-2}. \end{aligned}$$

Consequently, using (10)*,*

$$\begin{aligned} B^n &= 4^n \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (2^{2n-1} - 2^{n-1}) \begin{pmatrix} -2 & 2 & -2 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \\ &+ \left(2^{2n-2} - (n+1)2^{n-2}\right) \begin{pmatrix} 4 & -8 & 8 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix} \\ &= \begin{pmatrix} \left(2^{2n} - n2^n\right) & \left(-2^{2n} - 2^n(2n+3)\right) & \left(2^{2n} + 2^n(2n+3)\right) \\ 0 & \left(2^{2n} - n2^{n-1}\right) & n2^{n-1} \\ 0 & n2^{n-1} & \left(2^{2n} - n2^{n-1}\right) \end{pmatrix}. \end{aligned}$$

Hence,

$$\begin{aligned} B^n y_0 &= \begin{pmatrix} \left(2^{2n} - n2^n\right) & \left(-2^{2n} - 2^n(2n+3)\right) & \left(2^{2n} + 2^n(2n+3)\right) \\ 0 & \left(2^{2n} - n2^{n-1}\right) & n2^{n-1} \\ 0 & n2^{n-1} & \left(2^{2n} - n2^{n-1}\right) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^{2n} - n2^n \\ 2^{2n} \\ 2^{2n} \end{pmatrix}, \end{aligned} \tag{13}$$

$$B^n g(r) = \begin{pmatrix} \left(2^{2n} - n2^n\right) & \left(-2^{2n} - 2^n(2n+3)\right) & \left(2^{2n} + 2^n(2n+3)\right) \\ 0 & \left(2^{2n} - n2^{n-1}\right) & n2^{n-1} \\ 0 & n2^{n-1} & \left(2^{2n} - n2^{n-1}\right) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 2^{2n} \\ 2^{2n} \end{pmatrix},$$

$$B^{n-r-1} g(r) = \begin{pmatrix} 0 \\ 2^{2(n-r-1)} \\ 2^{2(n-r-1)} \end{pmatrix}$$

$$= 2^{2n-2} \begin{pmatrix} 0 \\ 2^{-2r} \\ 2^{-2r} \end{pmatrix}$$

$$= 2^{2n-2} \begin{pmatrix} 0 \\ 4^{-r} \\ 4^{-r} \end{pmatrix},$$

$$\sum_{r=0}^{n-1} B^{n-r-1} g(r) = 2^{2n-2} \sum_{r=0}^{n-1} \begin{pmatrix} 0 \\ 4^{-r} \\ 4^{-r} \end{pmatrix}$$

$$= 2^{2n-2} \begin{pmatrix} 0 \\ \frac{1-4^{-n}}{1-\frac{1}{4}} \\ \frac{1-4^{-n}}{1-\frac{1}{4}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \frac{4^n-1}{3} \\ \frac{4^n-1}{3} \end{pmatrix}.$$

From here, (9) *and* (13)*, we get*

$$y(n) = \begin{pmatrix} 2^{2n} - n2^n \\ 2^{2n} \\ 2^{2n} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{2^{2n}-1}{3} \\ \frac{2^{2n}-1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} 2^{2n} - n2^n \\ \frac{2^{2n+2}-1}{3} \\ \frac{2^{2n+2}-1}{3} \end{pmatrix}.$$

Exercise 3.1.24. *Let* $\hat{T}(n) = n+1$,

$$A(n) = \tfrac{n+2}{n+3}\begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \qquad g(n) = \tfrac{1}{n+3}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

$$y_0 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \qquad n \in \mathbb{N}.$$

Find the general solution of the system (1).

3.2. Linear Periodic Systems

Here we will investigate the system

$$y(n+1) = \frac{\hat{T}(n+1)}{\hat{T}(n)} A(n) y(n), \qquad n \in \mathbb{Z}, \tag{2}$$

where $\hat{T}(\cdot)$ and $A(\cdot)$ are periodic with a period N, i.e.,

$$\hat{T}(n+N) = \hat{T}(n), \qquad A(n+N) = A(n), \qquad n \in \mathbb{Z}.$$

Proposition 3.2.1. *If* $\Phi(n)$ *is a fundamental matrix of* (2)*, then so is* $\Phi(n+N)$.

Proof. We have, for $n \in \mathbb{Z}$,

$$\Phi(n+N) = \tfrac{\hat{T}(n+N+1)}{\hat{T}(n+N)} A(n+N)\Phi(n+N)$$

$$= \tfrac{\hat{T}(n+1)}{\hat{T}(n)} A(n)\Phi(n+N).$$

□

Proposition 3.2.2. *If* $\Phi(n)$ *is a fundamental matrix of* (2)*, then there exists a nonsingular matrix* C *so that*

$$\Phi(n+N) = \Phi(n)C, \qquad n \in \mathbb{Z}.$$

Proof. Since $\Phi(n+N)$ and $\Phi(n)$ are fundamental matrices of (2), then there exists a nonsingular matrix C such that

$$\Phi(n+N) = \Phi(n)C$$

for $n \in \mathbb{Z}$. □

Proposition 3.2.3. *If* $\Phi(n)$ *is a fundamental matrix of* (2)*, then for every* $n \in \mathbb{Z}$ *we have*

$$\Phi(n+N,N) = \Phi(n,0).$$

Proof. From the last proposition it follows that there exists a nonsingular matrix C such that

$$\Phi(n+N) = \Phi(n)C, \qquad n \in \mathbb{Z}.$$

Hence, for $n = 0$, we get

$$\Phi(N) = \Phi(0)C \qquad \Longrightarrow$$

$$\Phi^{-1}(N) = (\Phi(0)C)^{-1} \qquad \Longrightarrow$$

$$\Phi^{-1}(N) = C^{-1}\Phi^{-1}(0).$$

Therefore

$$\Phi(n+N,N) = \Phi(n+N)\Phi^{-1}(N)$$

$$= \Phi(n)CC^{-1}\Phi^{-1}(0)$$

$$= \Phi(n)\Phi^{-1}(0)$$

$$= \Phi(n,0).$$

□

Theorem 3.2.4. *For every fundamental matrix $\Phi(n)$ of the system (2) there exist a nonsingular periodic matrix $P(n)$ wit period N and a nonsingular matrix B such that*

$$\Phi(n) = P(n)B^n.$$

Proof. Let C be a nonsingular matrix such that

$$\Phi(n+N) = \Phi(n)C.$$

Let now B be a nonsingular matrix such that

$$B^N = C.$$

We define the matrix

$$P(n) = \Phi(n)B^{-n}.$$

Then

$$P(n+N) = \Phi(n+N)B^{-n-N}$$

$$= \Phi(n)CB^{-n-N}$$

$$= \Phi(n)B^N B^{-n-N}$$

$$= \Phi(n)B^{-n}$$

$$= P(n),$$

i.e., $P(n)$ is a nonsingular N periodic matrix. From the definition of P we have

$$\Phi(n) = P(n)B^n.$$

$\square$

Definition 3.2.5. *The matrix $C = B^N$ is called the iso-monodromy matrix.*

Definition 3.2.6. *The eigenvalues λ of the matrix B will be called the iso-Floquet exponents of the system* (2)*. The corresponding eigenvalues λ^N of the matrix B^N will be called the iso-Floquet multipliers of the system* (2)*.*

Proposition 3.2.7. *If $\Phi(n)$ and $\Psi(n)$ are fundamental matrices of the system* (2) *such that*

$$\Phi(n+N) = \Phi(n)C,$$

$$\Psi(n+N) = \Psi(n)E,$$

then C and E are similar.

Proof. We consider the matrices $\Phi(n,n_0)$ and $\Psi(n,n_0)$.

We have

$$\Phi(n,n_0) = \Phi(n)\Phi^{-1}(n_0),$$

$$\Psi(n,n_0) = \Psi(n)\Psi^{-1}(n_0),$$

$$\Phi(n_0,n_0) = \Psi(n_0,n_0) = I.$$

Since $\Phi(n,n_0)$ and $\Psi(n,n_0)$ are fundamental matrices of the system (2) and the problem

$$\kappa(n+1) = \frac{\hat{T}(n+1)}{\hat{T}(n)}A(n)\kappa(n)$$

$$\kappa(n_0) = I$$

has unique solution, we get

$$\Phi(n,n_0) = \Psi(n,n_0).$$

From here,

$$\Phi(n)\Phi^{-1}(n_0) = \Psi(n)\Psi^{-1}(n_0)$$

or

$$\Psi(n) = \Phi(n)\Phi^{-1}(n_0)\Psi(n_0).$$

Hence,

$$\Psi(n+N) = \Phi(n+N)\Phi^{-1}(n_0)\Psi(n_0) \qquad \Longrightarrow$$

$$\Psi(n)C = \Phi(n)E\Phi^{-1}(n_0)\Psi(n_0) \qquad \Longrightarrow$$

$$\Phi(n)\Phi^{-1}(n_0)\Psi(n_0)C = \Phi(n)E\Phi^{-1}(n_0)\Psi(n_0) \qquad \Longrightarrow$$

$$\Phi^{-1}(n_0)\Psi(n_0)C = E\Phi^{-1}(n_0)\Psi(n_0) \qquad \Longrightarrow$$

$$C = \left(\Phi^{-1}(n_0)\Psi(n_0)\right)^{-1}E\Phi^{-1}(n_0)\Psi(n_0).$$

Consequently,

$$C \sim E.$$

□

Theorem 3.2.8. *A complex number λ is a Floquet exponent of the system (2) if and only if there is a nontrivial solution of (2) of the form $\lambda^n q(n)$, $q(n+N) = q(n)$ for all n.*

Proof. Let λ is a Floquet multiplier of the system (2). Then

$$\det(B - \lambda I) = 0$$

Let $x_0 \in \mathbb{R}^k$ be chosen so that

$$(B - \lambda I)x_0 = 0,$$

i.e.,

$$Bx_0 = \lambda x_0.$$

Hence,

$$B^2 x_0 = B(Bx_0)$$

$$= B(\lambda x_0)$$

$$= \lambda B x_0$$

$$= \lambda^2 x_0.$$

We suppose that

$$B^n x_0 = \lambda^n x_0 \tag{14}$$

for some $n \in \mathbb{N}$.

$$B^{n+1} x_0 = B(B^n x_0)$$

$$= B(\lambda^n x_0)$$

$$= \lambda^n B x_0$$

$$= \lambda^{n+1} x_0.$$

Consequently, the equality (14) is valid for all $n \in \mathbb{N}$.

Using (14), we get

$$P(n)B^n x_0 = P(n)\lambda^n x_0$$

$$= \lambda^n P(n) x_0,$$

whereupon

$$x(n, n_0, x_0) = \Phi(n, n_0) x_0$$

$$= P(n) B^n x_0$$

$$= \lambda^n P(n) x_0.$$

Let
$$q(n) = P(n)x_0.$$
Then
$$x(n, n_0, x_0) = \lambda^n q(n).$$
Because $P(n+N) = P(n)$ we have that
$$q(n+N) = q(n),$$
and from here,
$$x(n+N, n_0, x_0) = x(n, n_0, x_0).$$
Let now
$$\lambda^n q(n), \qquad q(n+N) = q(n) \neq 0,$$
is a solution of the system (2).

Then
$$\lambda^n q(n) = \Phi(n, n_0)x_0, \qquad x_0 \neq 0.$$
Hence, using that
$$\Phi(n, n_0) = P(n)B^n,$$
we get
$$\lambda^n q(n) = P(n)B^n x_0,$$
which implies that
$$\lambda^{n+N} q(n) = P(n)B^{n+N} x_0. \tag{15}$$
On the other hand
$$\begin{aligned} \lambda^{n+N} q(n) &= \lambda^N \lambda^n q(n) \\ &= \lambda^N P(n)B^n x_0. \end{aligned}$$
From here and (15) we find
$$P(n)B^{n+N} x_0 = \lambda^N P(n)B^n x_0$$
or
$$P(n)B^n \left(B^N - \lambda^N I\right) x_0 = 0.$$
Since $x_0 \neq 0$, we conclude that
$$\det\left(P(n)B^n \left(B^N - \lambda^N I\right)\right) = 0,$$
hence,
$$\det(B^N - \lambda^N I) = 0,$$
i.e., λ is a Floquet exponent of (2). □

Corollary 3.2.9. *The system* (2) *has a periodic solution of period N if and only if it has a Floquet multiplier equal to* 1.

Proof. Let (2) has a solution $q(n)$ with period N. Then

$$q(n+N) = q(n)$$

and

$$1^n q(n)$$

is a solution of (2). From here and from the previous theorem, it follows that 1 is a Floquet multiplier of (2).

Let now (2) has a Floquet multiplier 1. Then, from the last theorem, it follows that (2) has a solution $1^n q(n)$, $q(n) = q(n+N)$, i.e. (2) has a periodic solution with period N. □

Corollary 3.2.10. *There is a Floquet multiplier of the system* (2) *equal to* -1 *if and only if the system* (2) *has a periodic solution with period* $2N$.

Proof. Let (2) has multiplier equal to -1. Then

$$(-1)^n q(n), \qquad q(n) = q(n+N),$$

is a solution of (2).

Let

$$y(n) = (-1)^n q(n).$$

Then

$$y(n+2N) = (-1)^{n+2N} q(n+2N)$$

$$= (-1)^n (-1)^{2N} q(n+N)$$

$$= (-1)^n q(n)$$

$$= y(n),$$

i.e., y is $2N$ periodic solution of (2).

Let now y is $2N$ periodic solution of (2). Because the system (2) is N periodic, then we can consider it as $2N$ periodic. Since (2) has $2N$ periodic solution, then it has Floquet multiplier equal to 1. Hence, it follows that it has and multiplier equal to -1. □

MA

Chapter 4

Stability Theory

4.1. Basic Notations

Here we consider the system

$$x(n+1) = f(n, x(n)), \qquad x(n_0) = x_0, \tag{1}$$

where $x(n) \in \mathbb{R}^k$, $f : \mathbb{N} \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$. We suppose that $f(n,x)$ is continuous function in x.
Let $\mathbb{R}^k$ is endowed with a norm $||\cdot||$.

Definition 4.1.1. **(i)** *The system* (1) *is said to be autonomous if*

$$f(n, x(n)) = f(x(n)).$$

(ii) *The system* (1) *is said to be periodic if for all* $n \in \mathbb{Z}$

$$f(n+N, x) = f(n, x)$$

for some $N \in \mathbb{N}$.

(iii) *A point* $x^* \in \mathbb{R}^k$ *is called an equilibrium point of the system* (1) *if*

$$f(n, x^*) = x^*$$

for all $n \geq n_0$.

(iv) *The equilibrium point* x^* *of the system* (1) *is said to be stable if given* $\varepsilon > 0$ *and* $n_0 \geq 0$ *there exists* $\delta = \delta(\varepsilon, n_0)$ *such that the inequality*

$$||x_0 - x^*|| < \delta$$

implies

$$||x(n, n_0, x_0) - x^*|| < \varepsilon$$

for all $n \geq n_0$*, uniformly stable if* δ *may be chosen independent of* n_0*, unstable if it is not stable.*

(v) *The equilibrium point x^* of the system* (1) *is said to be attracting if there exists* $\mu = \mu(n_0)$ *so that the inequality*

$$||x_0 - x^*|| < \mu$$

implies

$$\lim_{n \longrightarrow \infty} x(n, n_0, x_0) = x^*,$$

uniformly attracting if the choice of μ is independent of n_0.

(vi) *The equilibrium point x^* of the system* (1) *is said to be asymptotically stable if it is stable and attracting, and uniformly asymptotically stable if it is uniformly stable and uniformly attracting.*

(vii) *The equilibrium point x^* of the system* (1) *is said to be exponentially stable if there exist* $\delta > 0$, $M > 0$, *and* $\eta \in (0,1)$ *such that*

$$||x(n, n_0, x_0) - x^*|| \leq M||x_0 - x^*||\eta^{n-n_0},$$

whenever

$$||x_0 - x^*|| < \delta.$$

(viii) *A solution $x(n, n_0, x_0)$ of the system* (1) *is said to be bounded if for some positive constant M we have*

$$||x(n, n_0, x_0)|| \leq M$$

for all $n \geq n_0$, where the constant M may depend on each solution.

Remark 4.1.2. *If* $\mu = \infty$ *in* (v) *and* (vi) *or* $\delta = \infty$ *in* (vii) *the corresponding stability property is said to be global.*

4.2. Nonautonomous Linear Systems

In this section we will investigate the stability of the system

$$\hat{x}^{\wedge}(\widehat{n+1}) = A(n)\hat{x}^{\wedge}(\hat{n}), \qquad n \geq n_0, \tag{2}$$

or

$$x(n+1) = \frac{\hat{T}(n+1)}{\hat{T}(n)} A(n)x(n), \qquad n \geq n_0, \qquad x(n_0) = x_0. \tag{2}$$

We suppose that $\Phi(n)$ is its fundamental matrix. Without loss of generality we suppose that $\Phi(n_0) = I$. Then the solution $x(n, n_0, x_0)$ of (2) for which $x(n_0, n_0, x_0) = x_0$ is given by

$$x(n, n_0, x_0) = \Phi(n)x_0.$$

Theorem 4.2.1. *Let* $||\Phi(n)|| \leq M$ *for some positive constant M and for every $n \geq n_0$. Then the zero solution of* (2) *is stable.*

Proof. Let $\varepsilon > 0$ be arbitrarily chosen. Let also,

$$\delta = \frac{\varepsilon}{M}.$$

Then the inequality

$$||x_0|| < \delta = \frac{\varepsilon}{M}$$

implies

$$\begin{aligned}
||x(n,n_0,x_0)|| &= ||\Phi(n)x_0|| \\
&\le ||\Phi(n)||||x_0|| \\
&\le M||x_0|| \\
&< M\tfrac{\varepsilon}{M} \\
&= \varepsilon.
\end{aligned}$$

Therefore the zero solution is stable. □

Theorem 4.2.2. *If there exists a constant $M > 0$ such that*

$$||\Phi(n,m)|| \le M \tag{3}$$

for $n_0 \le m \le n < \infty$, then the zero solution of (2) is uniformly stable.

Proof. We will note that for every $n \ge n_0$ we have

$$\Phi(n,n_0) = \Phi(n)\Phi^{-1}(n_0) = \Phi(n).$$

From here and from (3) it follows that for every $n \in \mathbb{N}$, $n \ge n_0$, we have

$$||\Phi(n)|| \le M.$$

Let $\varepsilon > 0$ be arbitrarily chosen. Let also,

$$\delta = \frac{\varepsilon}{M}.$$

Then for every solution $x(n,n_0,x_0)$ the inequality

$$||x_0|| < \frac{\varepsilon}{M}$$

implies

$$\begin{aligned}
||x(n,n_0,x_0)|| &= ||\Phi(n)x_0|| \\
&\le ||\Phi(n)||||x_0|| \\
&< \tfrac{\varepsilon}{M}||\Phi(n)|| \qquad (4)\\
&\le \tfrac{\varepsilon}{M}M \\
&= \varepsilon,
\end{aligned}$$

whereupon, since the choice of δ does not depend on n_0, we conclude that the zero solution is uniformly stable. □

Theorem 4.2.3. *If*

$$\lim_{n\longrightarrow\infty} ||\Phi(n)|| = 0,$$

then the zero solution is asymptotically stable.

Proof. From (4) we get

$$\lim_{n\longrightarrow\infty} ||x(n,n_0,x_0)|| \leq M \lim_{n\longrightarrow} ||\Phi(n)||$$

$$= 0.$$

Therefore the zero solution of (2) is attracting and since it is stable, we conclude that the zero solution is asymptotically stable. □

Theorem 4.2.4. *If there exists positive constant M and $\eta \in (0,1)$ such that*

$$||\Phi(n,m)|| \leq M\eta^{n-m} \qquad \text{for} \qquad n_0 \leq m \leq n < \infty.$$

Then the zero solution is uniformly asymptotically stable.

Proof. From the previous theorems it follows that the zero solution is uniformly stable.
Let now $\varepsilon > 0$ be arbitrarily chosen so that $\varepsilon < M$. We take $\mu = 1$ and N so that

$$\eta^N < \frac{\varepsilon}{M}.$$

Because $\eta \in (0,1)$we can choose N enough large. Consequently, the inequality

$$||x_0|| < 1$$

implies

$$||x(n,n_0,x_0)|| = ||\Phi(n,n_0)x_0||$$

$$\leq M\eta^{n-n_0}$$

$$\leq M\eta^N$$

$$\leq M\frac{\varepsilon}{M}$$

$$= \varepsilon$$

for $n \geq n_0 + N$. Therefore the zero solution is uniformly asymptotically stable solution. □

Theorem 4.2.5. *If the zero solution of the system (2) is stable, then there exists a positive constant $M > 0$ such that*

$$||\Phi(n)|| \leq M$$

for $n \geq n_0 \geq 0$.

Proof. Let $\varepsilon > 0$ be arbitrarily chosen. Because the zero solution is stable, we have that there exists $\delta = \delta(\varepsilon, n_0) > 0$ such that the inequality

$$||x_0|| < \delta$$

implies

$$||x(n, n_0, x_0)|| < \varepsilon$$

or

$$||\Phi(n)x_0|| < \varepsilon$$

for $n \geq n_0 \geq 0$.

We observe that

$$\frac{1}{\delta}||x_0|| < 1.$$

Hence, for $n \geq n_0 \geq 0$,

$$||\Phi(n)|| = \sup_{||\xi|| \leq 1} ||\Phi(n)\xi||$$

$$= \tfrac{1}{\delta} \sup_{||x_0|| \leq \delta} ||\Phi(n)x_0||$$

$$< \tfrac{\varepsilon}{\delta}.$$

Let

$$\varepsilon = M\delta.$$

Then for every $n \geq n_0 \geq 0$ we have that

$$||\Phi(n)|| \leq M.$$

□

Theorem 4.2.6. *If the zero solution of the system* (2) *is uniformly stable, then there exists a positive constant* M *such that*

$$||\Phi(n, m)|| \leq M$$

for every $n \geq m \geq n_0$.

Proof. Since the zero solution is uniformly stable then it is stable. Therefore there exists a positive constant M_1 such that

$$||\Phi(n)|| \leq M_1$$

for every $n \geq n_0$.

Hence, for every $n \geq m \geq n_0$ we have

$$||\Phi(n, m)|| = ||\Phi(n)\Phi^{-1}(m)||$$

$$\leq ||\Phi(n)||||\Phi^{-1}(m)||,$$

from where

$$||\Phi(n, m)||||\Phi(m)|| \leq ||\Phi(n)||$$

and

$$||\Phi(n,m)||||\Phi(m)|| \leq M_1$$

for every $n \geq m \geq n_0$ and because $||\Phi(m)|| \leq M_1$ for every $m \geq n_0$ we conclude that there exists a constant $M > 0$ such that

$$||\Phi(n,m)|| \leq M$$

for every $n \geq m \geq n_0 \geq 0$. □

Theorem 4.2.7. *If the zero solution of the system* (2) *is asymptotically stable, then*

$$\lim_{n \longrightarrow \infty} ||\Phi(n)|| = 0.$$

Proof. Because the zero solution of the system (2) is asymptotically stable, then it is stable and attracting.

Since the zero solution is attracting when there exists $\mu = \mu(n_0) > 0$ so that

$$||x_0|| \leq \mu$$

implies

$$\lim_{n \longrightarrow \infty} x(n, n_0, x_0) = 0$$

or

$$\lim_{n \longrightarrow \infty} \Phi(n)x_0 = 0.$$

We observe that

$$\frac{1}{\mu}||x_0|| \leq 1.$$

Then

$$\begin{aligned} ||\Phi(n)|| &= \sup_{||x_0|| \leq 1} ||\Phi(n)x_0|| \\ &= \tfrac{1}{\mu} \sup_{||x_0|| \leq \mu} ||\Phi(n)x_0|| \\ &= \tfrac{1}{\mu} \sup_{||x_0|| \leq \mu} ||x(n, n_0, x_0)||, \end{aligned}$$

hence,

$$\begin{aligned} &\lim_{n \longrightarrow \infty} ||\Phi(n)|| = \tfrac{1}{\mu} \lim_{n \longrightarrow \infty} \sup_{||x_0|| \leq \mu} ||x(n, n_0, x_0)|| \\ &= 0. \end{aligned}$$

□

Theorem 4.2.8. *If the zero solution of the system* (2) *is uniformly asymptotically stable, then there exist positive constants* M *and* $\eta \in (0,1)$ *such that*

$$||\Phi(n,m)|| \leq M\eta^{n-m}$$

for $n_0 \leq m \leq n < \infty$.

Proof. Because the zero solution of (2) is uniformly asymptotically stable, then it is uniformly stable. Therefore there exists a positive constant M such that

$$||\Phi(n,m)|| \leq M$$

for every $n \geq m \geq n_0$.

From the uniform attractivity, it follows that there exists $\mu > 0$ such that for $\varepsilon \in (0,1)$ there exists N so that

$$||\Phi(n,n_0)|| < \varepsilon$$

for $n \geq n_0 + N$.

Then for $n \in [n_0 + mN, n_0 + (m+1)N]$,

$$\begin{aligned}
&||\Phi(n,n_0)|| = ||\Phi(n,n_0+mN)\Phi(n_0+mN,n_0+Mn-1)\ldots\Phi(n_0+N,n_0)||\\
&\leq ||\Phi(n,n_0+mN)||||\Phi(n_0+mN,n_0+mN-1)||\ldots||\Phi(n_0+N,n_0)||\\
&\leq M\varepsilon^m\\
&\leq \tfrac{M}{\varepsilon}\left(\varepsilon^{\frac{1}{N}}\right)^{(m+1)N}\\
&= \tilde{M}\eta^{(m+1)N}\\
&\leq \tilde{M}\eta^{n-n_0}.
\end{aligned}$$

Here

$$\tilde{M} = \frac{M}{\varepsilon}, \qquad \eta = \varepsilon^{\frac{1}{N}}.$$

□

Corollary 4.2.9. *The zero solution of the system* (2) *is stable if and only if all solutions of* (2) *are bounded.*

Corollary 4.2.10. *The zero solution of the system* (2) *is exponentially stable if and only if it is uniformly asymptotically stable.*

Now we will consider the case when

$$\frac{\hat{T}(n+1)}{\hat{T}(n)}A(n) = A,$$

i.e., we will consider the system

$$y(n+1) = Ay(n). \tag{5}$$

In this case we have

$$\Phi(n) = A^n, \qquad n \in \mathbb{N}.$$

Theorem 4.2.11. *The zero solution of the system* (5) *is stable if and only if* $\rho(A) \leq 1$ *and the eigenvalues of unit modulus are semisimple.*

Proof. Let
$$A = PJP^{-1},$$
where
$$J = \mathrm{diag}(J_1, J_2, \ldots, J_r)$$
is the Jordan form of the matrix A and
$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \ldots & 0 & 0 \\ 0 & \lambda_i & 1 & \ldots & 0 & 0 \\ 0 & 0 & \lambda_i & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \lambda_i & 1 \\ 0 & 0 & 0 & \ldots & 0 & \lambda_i \end{pmatrix}.$$
We have that the zero solution of (5) is stable if and only if there exists a positive constant M_1 such that
$$||\Phi(n)|| \le M_1$$
or
$$||A^n|| = ||(PJP^{-1})^n||$$
$$= ||PJ^nP^{-1}|| \le M_1.$$
Hence, if
$$M = \frac{M_1}{||P||||P^{-1}||},$$
we get
$$||J^n|| \le M.$$
We note that
$$J^n = \mathrm{diag}(J_1^n, J_2^n, \ldots, J_r^n),$$
where
$$J_i^n = \begin{pmatrix} \lambda_i^n & \begin{pmatrix} n \\ 1 \end{pmatrix}\lambda_i^{n-1} & \ldots & \begin{pmatrix} n \\ s_i-1 \end{pmatrix}\lambda_i^{n-s_i+1} \\ 0 & \lambda_i^n & \ldots & \begin{pmatrix} n \\ s_i-2 \end{pmatrix}\lambda_i^{n-s_i+2} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \begin{pmatrix} n \\ 1 \end{pmatrix}\lambda_i^{n-1} \\ 0 & 0 & \ldots & \lambda_i^n \end{pmatrix}.$$
We observe that J_i^n is unbounded if $|\lambda_i| > 1$ or ir $|\lambda_i| = 1$ and J_i is not 1×1 matrix.

Let
$$|\lambda_i| < 1.$$
Also, let $l \in \mathbb{N}$ is arbitrarily chosen. Then
$$|\lambda_i|^n n^l = n^l e^{(\log|\lambda_i|)n}$$
$$\longrightarrow_{n\longrightarrow\infty} 0.$$

Therefore, if $|\lambda_i| < 1$, we get

$$J_i^n \longrightarrow_{n \longrightarrow \infty} 0.$$

$\square$

Using the arguments of the proof of the above theorem, one can prove the following theorem.

Theorem 4.2.12. *The zero solution of* (5) *is asymptotically stable if and only if* $\rho(A) < 1$.

Example 4.2.13. *We consider the case* $k = 2$. *In this case*

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

The characteristic equation of the matrix A is

$$0 = \det(A - \lambda I)$$

$$= \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a - 22 - \lambda \end{vmatrix} \qquad \Longleftrightarrow$$

$$(a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0 \qquad \Longleftrightarrow$$

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0$$

or

$$\lambda^2 - (\mathrm{tr}A)\lambda + \det A = 0. \tag{6}$$

Let

$$p_1 = -\mathrm{tr}A, \qquad p_2 = \det A.$$

Then we get the equation

$$\lambda^2 + p_1\lambda + p_2 = 0.$$

We will search conditions for the parameters p_1 *and* p_2 *so that*

$$|\lambda_1| < 1, \qquad |\lambda_2| < 1,$$

where λ_1 *and* λ_2 *are the roots of the equation* (6).

1. Case. $\lambda_1, \lambda_2 \in \mathbb{R}$, *i.e.,*

$$p_1^2 - 4p_2 \geq 0.$$

We have

$$\lambda_{1,2} = \frac{-p_1 \pm \sqrt{p_2^2 - 4p_2}}{2}.$$

Let

$$\lambda_1 := \frac{-p_1 + \sqrt{p_1^2 - 4p_2}}{2}, \qquad \lambda_2 := \frac{-p_1 - \sqrt{p_1^2 - 4p_2}}{2.}$$

Then

$$|\lambda_1| < 1, \qquad |\lambda_2| < 1 \qquad \Longleftrightarrow$$

$$\begin{cases} p_1^2 - 4p_2 \geq 0 \\ |-p_1 + \sqrt{p_1^2 - 4p_2}| < 2 \\ |-p_1 - \sqrt{p_1^2 - 4p_2}| < 2 \end{cases} \qquad \Longleftrightarrow$$

$$\begin{cases} p_1^2 - 4p_2 \geq 0 \\ -2 + p_1 < \sqrt{p_1^2 - 4p_2} < 2 + p_1 \\ -2 + p_1 < -\sqrt{p_1^2 - 4p_2} < 2 + p_1, \end{cases}$$

whereupon

$$2 + p_1 > 0, \qquad -2 + p_1 < 0$$

or

$$|p_1| < 2.$$

In this way we obtain the system

$$\begin{cases} p_1^2 - 4p_2 \geq 0 \\ |p_1| < 2 \\ \sqrt{p_1^2 - 4p_2} < 2 + p_1 \\ \sqrt{p_1^2 - 4p_2} < 2 - p_1, \end{cases}$$

i.e.,

$$\begin{cases} p_1^2 - 4p_2 \geq 0 \\ |p_1| < 2 \\ p_1^2 - 4p_2 < 4 + 4p_1 + p_1^2 \\ p_1^2 - 4p_2 < 4 - 4p_1 + p_1^2 \end{cases}$$

or

$$\begin{cases} p_1^2 - 4p_2 \geq 0 \\ |p_1| < 2 \\ 0 < p_1 + p_2 + 1 \\ 0 < 1 - p_1 + p_2, \end{cases}$$

or

$$\begin{cases} p_2 \leq \frac{p_1^2}{4} < 1 \\ |p_1| < 2 \\ 1 + p_1 + p_2 > 0 \\ 1 - p_1 + p_2 > 0 \end{cases}$$

or

$$\begin{cases} p_2 < 1 \\ 1 + p_1 + p_2 > 0 \\ 1 - p_1 + p_2 > 0. \end{cases} \tag{7}$$

2. Case. $p_1^2 - 4p_2 < 0$. *Then*

$$\lambda_{1,2} = \frac{-p_1 \pm i\sqrt{4p_2 - p_1^2}}{2},$$

from where

$$\begin{aligned} |\lambda_1| &= |\lambda_2| \\ &= \tfrac{p_1^2}{4} + \tfrac{4p_2 - p_1^2}{4} \\ &= p_2. \end{aligned}$$

Since

$$|\lambda_1| = |\lambda_2| < 1$$

we get

$$\begin{cases} 0 < p_2 < 1 \\ p_1^2 < 4p_2 \end{cases}$$

or

$$\begin{cases} 0 < p_2 < 1 \\ -2\sqrt{p_2} < p_1 < 2\sqrt{p_2}. \end{cases} \tag{8}$$

Now we will see that from the system (8) *we can obtain the system* (7).

Really,

$$
\begin{aligned}
1+p_1+p_2 &> 1-2\sqrt{p_2}+p_2 \\
&= (1-\sqrt{p_2})^2 \\
&> 0, \\
1-p_1+p_2 &> 1-2\sqrt{p_1}+p_2 \\
&= (1-\sqrt{p_2})^2 \\
&> 0.
\end{aligned}
$$

Consequently, the zero solution of the considered system is asymptotically stable if

$$
\begin{cases}
\det A < 1 \\
1+(\mathrm{tr}A)+\det A > 0 \\
1-(\mathrm{tr}A)+\det A > 0
\end{cases}
$$

or

$$
|\mathrm{tr}A| < 1+\det A < 2.
$$

Let λ be an eigenvalue of the matrix A of multiplicity m and $\xi_1, \xi_2, \ldots, \xi_m$ be the generalized eigenvectors which correspond to λ, i.e.,

$$
\begin{cases}
A\xi_1 = \lambda\xi_1 \\
A\xi_2 = \lambda\xi_2+\xi_1 \\
A\xi_3 = \lambda\xi_3+\xi_2 \\
\ldots \\
A\xi_m = \lambda\xi_m+\xi_{m-1}.
\end{cases}
$$

The span of the generalized eigenvectors corresponding to λ will be denoted with E_λ and it is called the generalized eigenspace of the eigenvalue λ of A. If λ_1 and λ_2 are eigenvalues of the matrix A such that $\lambda_1 \neq \lambda_2$, then we have that

$$
E_{\lambda_1} \bigcap E_{\lambda_2} = \emptyset.
$$

We note that each eigenspace E_λ includes the zero vector.

Definition 4.2.14. *The matrix A will be called hyperbolic if it has not any eigenvalues which lie on the unit circle.*

Now we suppose that the matrix A is a hyperbolic matrix.

Let

$$\Delta_s = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$$

such that

$$|\lambda_i| < 1, \qquad 1 \le i \le t,$$

$$\Delta_u = \{\lambda_{r+1}, \lambda_{r+2}, \ldots, \lambda_n\}$$

such that

$$|\lambda_i| > 1, \qquad r+1 \le i \le n.$$

The eigenspace spanned by the eigenvalues in Δ_s is denoted by W^s, where

$$W^s = \bigcup_{i=1}^{r} \lambda_i.$$

The eigenspace spanned by the eigenvalues in Δ_u is denoted by W^u, where

$$W^u = \bigcup_{i=r+1}^{n} \lambda_i.$$

Theorem 4.2.15. *Let A be a hyperbolic matrix. If $x(n)$ is a solution to the system* (5) *with $x(0) \in W^s$, then $x(n) \in W^s$ for every $n \in \mathbb{N}$ and*

$$\lim_{n \longrightarrow \infty} x(n) = 0.$$

Proof. We note that

$$AE_\lambda = E_\lambda.$$

From here we conclude that

$$AW^s = W^s.$$

Therefore

$$x(n) \in W^s$$

for all $n \in \mathbb{N}$.

Because $x(0) \in W^s$, then we have for it the following representation

$$x(0) = \sum_{i=1}^{r} c_i \xi_i,$$

where c_i, $1 \le i \le r$, are constants.

Let

$$J = P^{-1}AP$$

be the Jordan normal form of the matrix A. We can rewrite J in the following form

$$J = \begin{pmatrix} J_s & 0 \\ 0 & J_u \end{pmatrix},$$

where J_s has the eigenvalues in Δ_s and J_u has the eigenvalues in Δ_u.

Also, we have

$$\tilde{\xi}_i = P^{-1}\xi_i = \begin{pmatrix} a_{i1} \\ a_{i2} \\ \dots \\ a_{ir} \\ 0 \\ \dots \\ 0 \end{pmatrix}.$$

Consequently,

$$\begin{aligned} x(n) &= A^n x(0) \\ &= PJP^{-1}\textstyle\sum_{i=1}^r c_i\xi_i \\ &= PJ^n \textstyle\sum_{i=1}^r c_i P^{-1}\xi_i \\ &= PJ^n \textstyle\sum_{i=1}^r c_i\tilde{\xi}_i \\ &= P\textstyle\sum_{i=1}^r \begin{pmatrix} J_s^n & 0 \\ 0 & 0 \end{pmatrix} \\ &\longrightarrow_{n\longrightarrow\infty} 0, \end{aligned}$$

because $J_s^n \longrightarrow_{n\longrightarrow\infty} 0$. $\square$

Theorem 4.2.16. *Let A be a hyperbolic matrix. If $x(n)$ is a solution of the system (5) with $x(0) \in W^u$, then $x(n) \in W^u$ for each n and*

$$\lim_{n\longrightarrow-\infty} x(n) = 0.$$

Proof. We note that

$$AW^u = W^u.$$

Therefore

$$x(n) \in W^u$$

for each n.

Let now

$$J = P^{-1}AP$$

be the Jordan normal form of the matrix A. For it we have the representation

$$J = \begin{pmatrix} J_s & 0 \\ 0 & J_u \end{pmatrix}.$$

Because $x(0) \in W_u$, we have that

$$x(0) = \sum_{i=r+1}^n c_i\xi_i,$$

where ξ_i, $r+1 \leq i \leq n$, are the generalized eigenvectors corresponding to the elements in Δ_u.

Let

$$\tilde{\xi}_i = P^{-1}\xi_i$$

$$= \begin{pmatrix} 0 \\ \dots \\ 0 \\ \xi_{ir+1} \\ \dots \\ \xi_{in} \end{pmatrix}.$$

Then

$$x(n) = A^n x(0)$$

$$= PJ^nP^{-1}x(0)$$

$$= PJ^nP^{-1}\textstyle\sum_{i=r+1}^{n} c_i\xi_i$$

$$= PJ^n \textstyle\sum_{i=r+1}^{n} c_i\tilde{\xi}_i$$

$$= P\textstyle\sum_{i=r+1}^{n} \begin{pmatrix} 0 & 0 \\ 0 & J_u^n \end{pmatrix}$$

$$\longrightarrow_{n\longrightarrow -\infty} 0,$$

because $J_u^n \longrightarrow_{n\longrightarrow -\infty} 0$. □

Now we consider the system (5) in the case when

$$A(n+N) = A(n)$$

for some $N \in \mathbb{N}$.

In this case, if $\Phi(n,n_0)$ is the fundamental matrix for the system (5), then there exist a constant matrix B, whose eigenvalues are called the Floquet exponents, and a nonsingular matrix $P(n,n_0)$ such that

$$P(n+N,n_0) = P(n,n_0).$$

If B^n is bounded, then the fundamental matrix $\Phi(n,n_0)$ is bounded.

If

$$\lim_{n\longrightarrow\infty} B^n = 0,$$

then

$$\lim_{n\longrightarrow\infty} \Phi(n,n_0) = 0.$$

Therefore we have the following result.

Theorem 4.2.17. *The zero solution of the system* (5) *is*

(i) *stable if and only if the Floquet exponents have modulus less than or equal to* 1*, those of modulus* 1 *are semisimple.*

(ii) *asymptotically stable if and only if all the Floquet exponents lie inside the unit disk.*

4.3. Phase Space Analysis

Here we study the stability properties of the second-order linear autonomous systems

$$\begin{cases} x_1(n+1) = a_{11}x_1(n) + a_{12}x_2(n) \\ \\ x_2(n+1) = a_{21}x_1(n) + a_{22}x_2(n) \end{cases} \tag{9}$$

or

$$x(n+1) = Ax(n), \tag{9}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad x(n) = \begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix}, \qquad x(n+1) = \begin{pmatrix} x_1(n+1) \\ x_2(n+1) \end{pmatrix}.$$

Definition 4.3.1. *We will say that the point x^* is an equilibrium point of the system* (9) *if*

$$Ax^* = x^*$$

or

$$(A-I)x^* = 0.$$

If $A-I$ is a nonsingular matrix, then x^* is the unique equilibrium point of the system (9).

If $A-I$ is a singular matrix, then there is a family of equilibrium points for the system (9).

Let

$$J = P^{-1}AP$$

be the Jordan normal form of A.

Let λ_1 and λ_2 be the roots of the equation

$$\det(A-\lambda I) = 0.$$

1. Case. $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq \lambda_2$.

In this case J has the form.

$$J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

2. Case. $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$.

In this case J has the form

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

3. Case. $\lambda_1, \lambda_2 \in \mathbb{C} \backslash \mathbb{R}$,

$$\lambda_{1,2} = \alpha \pm i\beta, \qquad \alpha, \beta \in \mathbb{R}.$$

In this case J has the form.

$$J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

We set

$$y(n) = P^{-1}x(n)$$

or

$$x(n) = Py(n).$$

Then

$$x(n+1) = Py(n+1)$$

and the system (9) takes the form

$$Py(n+1) = APy(n)$$

or

$$y(n+1) = P^{-1}APy(n),$$

or

$$y(n+1) = Jy(n).$$

Let

$$y_0 = \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix}.$$

1. Case. $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq \lambda_2$.

In this case

$$\begin{pmatrix} y_1(n+1) \\ y_2(n+1) \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} y_1(n) \\ y_2(n) \end{pmatrix}$$

or

$$\begin{cases} y_1(n+1) = \lambda_1 y_1(n) \\ \\ y_2(n+1) = \lambda_2 y_2(n). \end{cases}$$

Hence,

$$\begin{pmatrix} y_1(n) \\ y_2(n) \end{pmatrix} = \begin{pmatrix} \lambda_1^n y_{10} \\ \lambda_2^n y_{20} \end{pmatrix}.$$

From here,

$$\frac{y_1(n)}{y_2(n)} = \left(\frac{\lambda_2}{\lambda_1}\right)^n \left(\frac{y_{20}}{y_{10}}\right).$$

If

$$|\lambda_1| > |\lambda_2|,$$

then

$$\lim_{n \longrightarrow \infty} \frac{y_2(n)}{y_1(n)} = 0.$$

If

$$|\lambda_1| < |\lambda_2|,$$

then

$$\lim_{n\longrightarrow\infty} \frac{|y_2(n)|}{|y_1(n)|} = \infty.$$

2. Case. $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$.

In this case

$$\begin{pmatrix} y_1(n) \\ y_2(n) \end{pmatrix} = J^n \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix}$$

$$= \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix} \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix}$$

$$= \begin{pmatrix} \lambda^n y_{10} + n\lambda^{n-1} y_{20} \\ \lambda^n y_{20} \end{pmatrix}$$

or

$$\begin{cases} y_1(n) = \lambda^n y_{10} + n\lambda^{n-1} y_{20} \\ y_2(n) = \lambda^n y_{20}. \end{cases}$$

Thus

$$\lim_{n\longrightarrow\infty} \tfrac{y_2(n)}{y_1(n)} = \lim_{n\longrightarrow\infty} \tfrac{\lambda^n y_{20}}{\lambda^n y_{10} + n\lambda^{n-1} y_{20}}$$

$$= \lim_{n\longrightarrow\infty} \tfrac{\lambda y_{20}}{\lambda y_{10} + n y_{20}}$$

$$= 0.$$

3. Case. $\lambda_1, \lambda_2 \in \mathbb{C}\backslash\mathbb{R}$,

$$\lambda_{1,2} = \alpha \pm i\beta, \qquad \alpha, \beta \in \mathbb{R}.$$

In this case

$$\begin{cases} y_1(n) = |\lambda_1|^n \left(c_1 \cos(nw) + c_2 \sin(nw)\right) \\ y_2(n) = |\lambda_1|^n \left(-c_1 \sin(nw) + c_2 \cos(nw)\right), \end{cases}$$

and using the initial data we find

$$\begin{cases} y_1(n) = |\lambda_1|^n \left(y_{10} \cos(nw) + y_{20} \sin(nw)\right) \\ y_2(n) = |\lambda_1|^n \left(-y_{10} \sin(nw) + y_{20} \cos(nw)\right), \end{cases}$$

where

$$w = \tan^{-1}\left(\frac{\beta}{\alpha}\right).$$

If $|\lambda_1| < 1$, then the equilibrium point is asymptotically stable.

If $|\lambda_1| > 1$, then the equilibrium point is unstable.

4.4. Lyapunov's Direct Method

We begin our investigations with the iso-difference equation

$$x(n+1) = f\left(\frac{x(n)}{\hat{T}(x(n))}\right), \tag{10}$$

where $f : G \longrightarrow \mathbb{R}^k$, $G \subset \mathbb{R}^k$, is continuous.

Example 4.4.1. *Let*

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = \begin{pmatrix} x_1^2 \\ 2x_1x_2 \end{pmatrix},$$

$$\hat{T}(x) = x_1^2 + x_2^2.$$

Then

$$f\left(\tfrac{x}{\hat{T}(x)}\right) = \begin{pmatrix} f_1\left(\frac{x}{\hat{T}(x)}\right) \\ f_2\left(\frac{x}{\hat{T}(x)}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{x_1^2}{\hat{T}^2(x)} \\ 2\frac{x_1x_2}{\hat{T}^2(x)} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{x_1^2}{(x_1^2+x_2^2)^2} \\ 2\frac{x_1x_2}{(x_1^2+x_2^2)^2}. \end{pmatrix}$$

Then the system (10) *takes the form*

$$\begin{cases} x_1(n+1) = \frac{x_1^2(n)}{(x_1^2(n)+x_2^2(n))^2} \\ \\ x_2(n+1) = 2\frac{x_1(n)x_2(n)}{(x_1^2(n)+x_2^2(n))^2}. \end{cases}$$

We assume that x^* is an equilibrium point of the equation (10), i.e.,

$$f\left(\frac{x^*}{\hat{T}(x^*)}\right) = x^*.$$

Example 4.4.2. *Let* $k = 2$ *and*

$$f_1\left(\tfrac{x(n)}{\hat{T}(x(n))}\right) = x_1(n) + 2x_1(n)x_2(n),$$

$$f_2\left(\tfrac{x(n)}{\hat{T}(x(n))}\right) = x_1^2(n) - 2x_1(n)x_2(n).$$

We will find the equilibrium points of the system (10).

Let (x_1^*, x_2^*) *be an equilibrium point of* (10). *Then*

$$\begin{cases} x_1^*(n) + 2x_1^*(n)x_2^*(n) = x_1^*(n) \\ \\ x_1^{*2}(n) - 2x_1^*(n)x_2^*(n) = x_2^*(n), \end{cases}$$

whereupon

$$x_1^*(n) = x_2^*(n) = 0.$$

Example 4.4.3. *Let* $k = 2$ *and*

$$f_1\left(\frac{x(n)}{\hat{T}(x(n))}\right) = \frac{x_2(n)}{1+x_1^2(n)},$$

$$f_2\left(\frac{x(n)}{\hat{T}(x(n))}\right) = \frac{x_1(n)}{1+x_2^2(n)}.$$

We will find the equilibrium points of (10).

Let $(x_1^*(n), x_2^*(n))$ *be an equilibrium point of* (10). *Then*

$$\begin{cases} \frac{x_2^*(n)}{1+x_1^{*2}(n)} = x_1^*(n) \\ \\ \frac{x_1^*(n)}{1+x_2^{*2}(n)} = x_2^*(n) \end{cases} \Longrightarrow$$

$$\begin{cases} x_2^*(n) = x_1^*(n)\left(1+x_1^{*2}(n)\right) \\ \\ \frac{x_1^*(n)}{1+x_1^{*2}(n)\left(1+x_1^{*2}(n)\right)^2} = x_1^*(n)\left(1+x_1^{*2}(n)\right) \end{cases} \Longrightarrow$$

$$\begin{cases} x_1^*(n) = 0 \\ \\ x_2^*(n) = 0 \end{cases} \quad \text{or}$$

$$\begin{cases} x_2^*(n) = x_1^*(n)\left(1+x_1^{*2}(n)\right) \\ \\ 1 = \left(1+x_1^{*2}(n)\right)\left(1+x_1^{*2}(n)\left(1+x_1^{*2}(n)\right)^2\right). \end{cases}$$

We consider

$$1 = \left(1+x_1^{*2}(n)\right)\left(1+x_1^{*2}(n)\left(1+x_1^{*2}(n)\right)^2\right) \Longrightarrow$$

$$1 = 1 + x_1^{*2}(n)\left(1+x_1^{*2}(n)\right)^2 + x_1^{*2}(n) + x_1^{*4}(n)\left(1+x_1^{*2}(n)\right)^2 \Longrightarrow$$

$$0 = x_1^{*2}(n)\left(1+x_1^{*2}(n)\right)^2 + x_1^{*2}(n) + x_1^{*4}(n)\left(1+x_1^{*2}(n)\right)^2 \Longrightarrow$$

$$0 = x_1^{*2}(n)\left(1+\left(1+x_1^{*2}(n)\right)^2 + \left(1+x_1^{*2}(n)\right)^2 x_1^{*2}(n)\right) \Longrightarrow$$

$$x_1^*(n) = 0.$$

Consequently, the unique equilibrium point of (10) *is*

$$x_1^*(n) = x_2^*(n) = 0.$$

Example 4.4.4. *Let* $k = 2$ *and*

$$f_1\left(\frac{x(n)}{\hat{T}(x(n))}\right) = x_1(n)\left(7 + 5x_2(n) + x_2^2(n)\right)$$

$$f_2\left(\frac{x(n)}{\hat{T}(x(n))}\right) = x_2(n)\left(4 + 4x_1(n) + x_1^2(n)\right).$$

We will find the equilibrium points of the system (10).

Let $(x_1^*(n), x_2^*(n))$ *be an equilibrium point of the considered system. Then*

$$\begin{cases} x_1^*(n)\left(7 + 5x_2^*(n) + {x_2^*}^2(n)\right) = x_1^*(n) \\ \\ x_2^*(n)\left(4 + 4x_1^*(n) + {x_1^*}^2(n)\right) = x_2^*(n). \end{cases}$$

From here

$$x_1^*(n) = x_2^*(n) = 0$$

or

$$\begin{cases} 7 + 5x_2^*(n) + {x_2^*}^2(n) = 1 \\ \\ 4 + 4x_1^*(n) + {x_1^*}^2(n) = 1, \end{cases}$$

or

$$x_1^*(n) = x_2^*(n) = 0,$$

or

$$\begin{cases} 6 + 5x_2^*(n) + {x_2^*}^2(n) = 0 \\ \\ 3 + 4x_1^*(n) + {x_1^*}^2(n) = 0, \end{cases}$$

or

$$x_1^*(n) = x_2^*(n) = 0,$$

or

$$\begin{cases} x_1^*(n) = -1, x_1^*(n) = -3 \\ \\ x_2^*(n) = -3, x_2^*(n) = -2. \end{cases}$$

Consequently, the equilibrium points of the considered system are

$$(0,0), \qquad (-3,-1), \qquad (-3,-3), \qquad (-2,-1), \qquad (-2,-3).$$

Exercise 4.4.5. *Let* $k = 2$ *and*

$$f_1\left(\frac{x(n)}{\hat{T}(x(n))}\right) = x_1(n)(2 + x_2(n))$$

$$f_2\left(\frac{x(n)}{\hat{T}(x(n))}\right) = x_2(n)(3 + x_1(n)).$$

Find the equilibrium points of the system (10).

Answer. $(0,0)$, $(-2,-1)$.

Definition 4.4.6. *Let $V:\mathbb{R}^k \longrightarrow \mathbb{R}$ be a real-valued function. The variation of V relative to* (10) *is defined as follows*

$$\Delta V(x) = V\left(f\left(\frac{x}{\hat{T}(x)}\right)\right) - V(x)$$

and

$$\Delta V(x(n)) = V\left(f\left(\tfrac{x(n)}{\hat{T}(x(n))}\right)\right) - V)(x(n))$$

$$= V(x(n+1)) - V(x(n)).$$

We note that if

$$\Delta V(x) \leq 0,$$

then V is nonincreasing along the solutions of (10).

Example 4.4.7. *Let us consider the system*

$$\begin{cases} x_1(n+1) = x_1^2(n) - 2x_1(n)x_2(n) \\ \\ x_2(n+1) = x_1(n)x_2(n). \end{cases}$$

Let also,

$$V(x) = 2x_1^2 + 3x_2^2 - 1.$$

Then

$$\begin{aligned} \Delta V(x(n)) &= V(x(n+1)) - V(x(n) \\ &= 2\left(x_1^2(n) - 2x_1(n)x_2(n)\right)^2 + 3x_1^2(n)x_2^2(n) - 1 \\ &-2x_1^2(n) - 3x_2^2(n) + 1 \\ &= 2x_1^4(n) - 8x_1^3(n)x_2(n) + 8x_1^2(n)x_2^2(n) + 3x_1^2(n)x_2^2(n) \\ &-2x_1^2(n) - 3x_2^2(n) \\ &= 2x_1^4(n) - 8x_1^3(n)x_2(n) + 11x_1^2(n)x_2^2(n) - 2x_1^2(n) - 3x_2^2(n). \end{aligned}$$

Example 4.4.8. *Let us consider the system*

$$\begin{cases} x_1(n+1) = x_1(n) + 3x_2(n) \\ \\ x_2(n+1) = x_2(n). \end{cases}$$

Let also,

$$V(x) = x_1^2 + 2x_1x_2.$$

Then

$$\Delta V(x(n)) = V(x(n+1)) - V(x(n))$$

$$= (x_1(n) + 3x_2(n))^2 + 2(x_1(n) + 3x_2(n))x_2(n)$$

$$-x_1^2(n) - 2x_1(n)x_2(n)$$

$$= x_1^2(n) + 6x_1(n)x_2(n) + 9x_2^2(n) + 2x_1(n)x_2(n)$$

$$+6x_2^2(n) - x_1^2(n) - 2x_1(n)x_2(n)$$

$$= 15x_2^2(n) + 6x_1(n)x_2(n).$$

Definition 4.4.9. *The function V is said to be a Lyapunov function on a subset Y of $\mathbb{R}^k$ if*

(i) *V is continuous on Y,*

(ii) $\Delta V(x) \leq 0$, *whenever $\frac{x}{\hat{T}(x)}$ and $f\left(\frac{x}{\hat{T}(x)}\right)$ belong to Y.*

Example 4.4.10. *Let $k = 2$ and*

$$f_1\left(\frac{x(n)}{\hat{T}(x(n))}\right) = x_2(n),$$

$$f_2\left(\frac{x(n)}{\hat{T}(x(n))}\right) = \frac{\alpha x_1(n)}{1+\beta x_2^2(n)}, \quad \alpha, \beta \in \mathbb{R}, \beta \neq 0.$$

Then the system (10) *takes the form*

$$\begin{cases} x_1(n+1) = x_2(n) \\ x_2(n+1) = \frac{\alpha x_1(n)}{1+\beta x_2^2(n)}. \end{cases}$$

We will find the equilibrium points of the considered system.

Let $x_1^*(n), x_2^*(n)$ is an equilibrium point. Then

$$\begin{cases} x_1^*(n) = x_2^*(n) \\ x_2^*(n) = \frac{\alpha x_1^*(n)}{1+\beta {x_2^*}^2(n)}, \end{cases}$$

whereupon

$$\alpha \frac{x_2^*(n)}{1 + x_2^*(n)^2} = x_2^*(n).$$

Therefore

$$x_2^*(n) = 0$$

or

$$\alpha = 1 + \beta x_2^{*2}(n) \qquad \Longrightarrow$$

$$\alpha - 1 = \beta x_2^{*2}(n) \qquad \Longrightarrow$$

$$x_2^*(n) = \pm\sqrt{\tfrac{\alpha-1}{\beta}}$$

for $(\alpha-1)\beta \geq 0$, $\beta \neq 0$.

Consequently, the equilibrium points of the considered system are

$$\left(0, \pm\sqrt{\frac{\alpha-1}{\beta}}\right)$$

for $(\alpha-1)\beta > 0$, $\beta \neq 0$.

Let

$$V(x) = x_1^2 + x_2^2.$$

Then

$$\Delta V(x(n)) = V(x(n1)) - V(x(n))$$

$$= x_1^2(n+1) + x_2^2(n+1) - x_1^2(n) - x_2^2(n)$$

$$= x_2^2(n) + \alpha^2 \tfrac{x_1^2(n)}{(1+\beta x_2^2(n))^2} - x_1^2(n) - x_2^2(n)$$

$$= x_1^2(n)\left(\tfrac{\alpha^2}{(1+\beta x_2^2(n))^2} - 1\right).$$

Hence,

$$\Delta V(x(n)) \leq 0 \qquad \Longrightarrow$$

$$\tfrac{\alpha^2}{1+\beta x_2^2(n)} - 1 \leq 0.$$

We conclude that for

$$\alpha^2 < 1, \qquad (\alpha-1)\beta \geq 0, \qquad \beta \neq 0,$$

we have $\Delta V(x(n)) \leq 0$*, i.e.,* V *is a Lyapunov function in neighborhoods of the equilibrium points.*

Example 4.4.11. *Let* $k = 2$ *and*

$$f_1\left(\tfrac{x(n)}{\hat{T}(x(n))}\right) = 2x_2(n) - 2x_2(n)x_1^2(n)$$

$$f_2\left(\tfrac{x(n)}{\hat{T}(x(n))}\right) = \tfrac{1}{2}x_1(n) + x_1(n)x_2^2(n).$$

Then the system (10) *takes the form*

$$\begin{cases} x_1(n+1) = 2x_2(n) - 2x_2(n)x_1^2(n) \\ \\ x_2(n+1) = \frac{1}{2}x_1(n) + x_1(n)x_2^2(n). \end{cases}$$

We will find the equilibrium points of the considered system.

Let $(x_1^*(n), x_2^*(n))$ *is an equilibrium point. Then*

$$\begin{cases} 2x_2^*(n) - 2x_2^*(n)x_1^{*2}(n) = x_1^*(n) \\ \frac{1}{2}x_1^*(n) + x_1^*(n)x_2^{*2}(n) = x_2^*(n) \end{cases}$$

or

$$\begin{cases} 2x_2^*(n) - x_1^*(n) = 2x_2^*(n)x_1^{*2}(n) \\ 2x_2^*(n) - x_1^*(n) = 2x_1^*(n)x_2^{*2}(n), \end{cases}$$

whereupon

$$x_1^*(n)x_2^{*2}(n) = x_2^*(n)x_1^{*2}(n).$$

Therefore

$$x_1^*(n) = x_2^*(n) = 0$$

or

$$\begin{cases} x_1^*(n) = x_2^*(n) \\ x_2^{*2}(n) = \frac{1}{2}. \end{cases}$$

The equilibrium points of the considered system are

$$(0,0), \qquad \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \qquad \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

Let

$$V(x) = x_1^2 + 4x_2^2.$$

Then

$$\begin{aligned}
&\Delta V(x(n)) = V(x(n+1)) - V(x(n)) \\
&= x_1^2(n+1) + 4x_2^2(n+1) - x_1^2(n) - 4x_2^2(n) \\
&= (2x_2(n) - 2x_2(n)x_1^2(n))^2 \\
&+4\left(\tfrac{1}{2}x_1(n) + x_1(n)x_2^2(n)\right)^2 \\
&-x_1^2(n) - 4x_2^2(n) \\
&= 4x_2^2(n) - 8x_1^2(n)x_2^2(n) + 4x_2^2(n)x_1^4(n) \\
&+x_1^2(n) + 4x_1^2(n)x_2^2(n) + 4x_1^2(n)x_2^4(n) \\
&-x_1^2(n) - 4x_2^2(n) \\
&= -4x_1^2(n)x_2^2(n) + 4x_1^2(n)x_2^4(n) + 4x_1^4(n)x_2^2(n) \\
&= 4x_1^2(n)x_2^2(n)\left(x_1^2(n) + x_2^2(n) - 1\right).
\end{aligned}$$

Consequently, if

$$x_1^2 + x_2^2 \leq 1,$$

then $\Delta V(x(n)) \leq 0$.

With $B(x,r)$ we will denote the open ball in $\mathbb{R}^k$ of radius r and center x defined by

$$B(x,r) = \left\{ y \in \mathbb{R}^k : ||x-y|| < r \right\}.$$

For convenience, below $B(0,r)$ will be denoted by $B(r)$.

Definition 4.4.12. *We say that the real-valued function V is positive definite at x^* if*

(i) $V(x^*) = 0$,

(ii) $V(x) > 0$ *for all* $x \in B(x^*, r)$, $x \neq x^*$, *for some* $r > 0$.

Example 4.4.13. *Let*

$$V(x) = (x-4)^2.$$

Then

$$V(4) = 0$$

and $V(x) > 0$ *for all* $x \in B(4,2)$, $x \neq 4$. *Therefore V is positive definite at* $x^* = 4$.

Example 4.4.14. *Let* $k = 2$ *and*

$$V(x) = x_1^2 + x_2^2.$$

Then

$$V(0) = 0$$

and $V(x) > 0$ *for every* $x \in B(0,1)$, $x \neq 0$. *Consequently V is positive definite at* 0.

Theorem 4.4.15. *(Iso-Lyapunov stability theorem) Let V is a Lyapunov function for* (10) *in a neighborhood Y of the equilibrium point x^* and V is positive definite at x^*, then x^* is stable. If, in addition,* $\Delta V(x) < 0$ *whenever* $\frac{x}{\hat{T}(x)}$, $f\left(\frac{x}{\hat{T}(x)}\right) \in Y$, $x \neq x^*$, *then x^* is asymptotically stable. Moreover, if* $G = Y = \mathbb{R}^k$ *and*

$$V(x) \longrightarrow \infty \qquad \text{as} \qquad ||x|| \longrightarrow \infty, \tag{11}$$

then x^* is globally asymptotically stable.

Proof. Let $\beta_1 > 0$ be chosen such that

$$B(x^*, \beta_1) \subset G \bigcap Y.$$

We have that

$$f\left(\frac{x^*}{\hat{T}(x^*)}\right) = x^*$$

and since f and $\hat{T}$ are continuous, then there is $\beta_2 > 0$ such that if $x \in B(x^*, \beta_2)$ we have that

$$f\left(\frac{x}{\hat{T}(x)}\right) \in B(x^*, \beta_1).$$

Let now

$$0 < \varepsilon \leq \beta_2.$$

We define the function

$$\psi(\varepsilon) = \min\{V(x) : \varepsilon \leq ||x - x^*|| \leq \beta_1\}.$$

Because V is positive definite at x^* we have $V(x^*) = 0$. Therefore there exists $\delta \in (0, \varepsilon)$ such that

$$V(x) < \psi(\varepsilon)$$

whenever

$$||x - x^*|| < \delta.$$

Now we suppose that there exist $x_0 \in B(x^*, \delta)$ and a natural number m such that $x(r) \in B(x^*, \varepsilon)$ for $1 \leq r \leq m$ and $x(m+1) \notin B(x^*, \varepsilon)$. Since

$$x(m) \in B(x^*, \varepsilon) \subset B(x^*, \beta_2),$$

it follows that $x(m+1) \in B(x^*, \beta_1)$, whereupon, using the definition of the function ψ, we conclude that

$$V(x(m+1)) \geq \psi(\varepsilon).$$

On the other hand, we have

$$V(x(m+1)) \leq V(x(m)) \leq \psi(\varepsilon),$$

which is a contradiction.

Therefore, if $x_0 \in B(x^*, \delta)$, then $x(n) \in B(x^*, \varepsilon)$ for all $n \geq 0$. From here, it follows that x^* is stable.

Now we suppose that $\Delta V(x) < 0$ whenever $\frac{x}{\hat{T}(x)}$, $f\left(\frac{x}{\hat{T}(x)}\right) \in Y$, $x \neq x^*$. We have that if $x_0 \in B(x^*, \delta)$, then $x(n) \in B(x^*, \varepsilon)$ for all $n \in \mathbb{N}$. We assume that the sequence $\{x(n)\}_{n=1}^{\infty}$ does not converge to x^*. Therefore, there exists a subsequence $\{x(n_i)\}_{i=1}^{\infty}$ that converges to $y \in \mathbb{R}^k$. Let $E \subset B(x^*, \beta_1)$ be an open neighborhood of y, $x^* \notin E$. We define on E the function

$$h(x) = \frac{V\left(f\left(\frac{x}{\hat{T}(x)}\right)\right)}{V(x)}$$

and using that $\Delta V(x) < 0$, we have that $h(x) < 1$ for all $x \in E$. Let now $\eta \in (h(y), 1)$. Then there exists $\alpha > 0$ such that $x \in B(y, \alpha)$ implies $h(x) \leq \eta$. In this way

$$\begin{aligned}
V\left(f\left(\frac{x(n_i)}{\hat{T}(x(n_i))}\right)\right) &\leq \eta V(x(n_i)) \\
&\leq \eta^2 V(x(n_i - 1)) \\
&\leq \dots \\
&\leq \eta^{n_i+1} V(x_0).
\end{aligned}$$

Consequently,

$$\lim_{n_i \longrightarrow \infty} V(x(n_i)) = 0.$$

Since V is continuous, we have that

$$\lim_{n_i \longrightarrow \infty} V(x(n_i)) = V(y),$$

therefore

$$V(y) = 0,$$

whereupon

$$y = x^*.$$

Now we suppose that $G = Y = \mathbb{R}^k$.

We assume that there exists an unbounded solution $x(n)$ of (10). Therefore, there exists a subsequence $\{x(n_i)\}_{i=1}^{\infty}$ so that

$$x(n_i) \longrightarrow \infty \qquad \text{as} \qquad n_i \longrightarrow \infty.$$

From here and (11) we conclude that

$$V(x(n_i)) \longrightarrow 0 \qquad \text{as} \qquad n_i \longrightarrow \infty,$$

which is a contradiction because $V(x_0) > V(x(n_i))$ for all $i \in \mathbb{N}$. Therefore x^* is globally asymptotically stable. □

Exercise 4.4.16. *If V is a Lyapunov function on the set*

$$\{x \in \mathbb{R}^k : ||x|| > \alpha\}$$

for some $\alpha > 0$ and (11) holds, prove that all solutions of (10) are bounded.

Hint. Use the last part of the proof of the Lyapunov stability theorem.

Definition 4.4.17. *Let V be a positive Lyapunov function on a subset G of $\mathbb{R}^k$. We define*

$$E = \{x \in \overline{G} : \Delta V(x) = 0\}.$$

Let M be the maximal invariant subset of E, i.e., M is the unit of all invariant subsets of E.

Theorem 4.4.18. *(Iso-LaSalle's invariance principle) We suppose that V is positive definite Lyapunov function for (10) in $G \subset \mathbb{R}^k$. Then for every bounded solution $x(n)$ of (10) which remains in G for all $n \in \mathbb{N}$, there exists a number b such that*

$$\frac{x(n)}{\hat{T}(x(n))} \longrightarrow M \bigcap V^{-1}(b) \qquad \text{as} \qquad n \longrightarrow \infty.$$

Proof. Let $x(n)$ be a bounded solution of (10) in $G \subset \mathbb{R}^k$ with $x(0) = x_0$ and $x(n)$ remains in G for all $n \in \mathbb{N}$. We have that

$$\Omega^{\vee}(x_0) \subset \overline{G}.$$

Also, if $y \in \Omega^{\vee}(x_0)$ then

$$\frac{x(n_i)}{\hat{T}(x(n_i))} \longrightarrow_{n_i \longrightarrow \infty} \frac{y}{\hat{T}(y)}$$

for some subsequence $n_i \in \mathbb{N}$. Since V is nonincreasing and bounded below, we have

$$\lim_{n \longrightarrow \infty} V\left(\frac{x(n)}{\hat{T}(x(n))}\right) = b$$

for some number b. Because V is continuous, we have

$$V\left(\frac{x(n_i)}{\hat{T}(x(n_i))}\right) \longrightarrow_{n_i \longrightarrow \infty} V\left(\frac{y}{\hat{T}(y)}\right).$$

Therefore

$$V\left(\frac{y}{\hat{T}(y)}\right) = b,$$

whereupon, since $\frac{y}{\hat{T}(y)} \in \Omega^{\vee}(x_0)$ was arbitrarily chosen,

$$V\left(\Omega^{\vee}(x_0)\right) = b,$$

i.e.,

$$\Omega^{\vee}(x_0) \subset V^{-1}(b).$$

Also,

$$\Delta V\left(\frac{y}{\hat{T}(y)}\right) = 0$$

for every $\frac{y}{\hat{T}(y)} \in \Omega^{\vee}(x_0)$.

Consequently,

$$\Omega^{\vee}(x_0) \subset E.$$

Since $\Omega^{\vee}(x_0)$ is invariant, we conclude that

$$\Omega^{\vee}(x_0) \subset M.$$

Therefore

$$\frac{x(n)}{\hat{T}(x(n))} \longrightarrow_{n \longrightarrow \infty} \Omega^{\vee}(x_0) \subset M \bigcap V^{-1}(b).$$

□

Theorem 4.4.19. *Let $f(0) = 0$, $V \in \mathcal{C}(Y)$, Y is a neighborhood of the origin, ΔV is positive definite in Y whenever $\frac{x}{\hat{T}(x)}$, $f\left(\frac{x}{\hat{T}(x)}\right) \in Y$, and there exists a sequence $a_i \longrightarrow_{i \longrightarrow \infty} 0$ with $V(a_i) > 0$. Then the zero solution of (10) is unstable.*

Proof. Let $\Delta V(x) > 0$ for $x \in B(r)$, $x \neq 0$, $V(0) = 0$.

We suppose that the zero solution is stable.

Then for $\varepsilon > 0$ there exists $\delta \in (0, \varepsilon)$ such that the inequality

$$||x_0|| < \delta$$

implies

$$||x(n, 0, x_0)|| < \varepsilon, \qquad n \in \mathbb{N}.$$

Let $x_0 = a_j$ for some $j \in \mathbb{N}$ such that $\Delta V(x_0) > 0$ and $||x_0|| < \delta$. Since V is continuous we have that $V(x(n))$ is compact. Therefore $V(x(n))$ is bounded above. Also, from $\Delta V > 0$ in Y we have that $V(x(n))$ is increasing. Therefore there exists

$$\lim_{n \longrightarrow \infty} V(x(n)) = 0,$$

which is impossible because

$$0 < V(x_0) \leq \lim_{n \longrightarrow \infty} V(x(n)).$$

Consequently, the zero solution is unstable. $\square$

Example 4.4.20. *Let*

$$f_1\left(\frac{x(n)}{\hat{T}(x(n))}\right) = 2x_2(n) - x_2(n)x_1^2(n)$$

$$f_2\left(\frac{x(n)}{\hat{T}(x(n))}\right) = x_1(n) + 2x_1(n)x_2^2(n).$$

Then the system (10) *takes the form*

$$\begin{cases} x_1(n+1) = 2x_2(n) - x_2(n)x_1^2(n) \\ \\ x_2(n+1) = x_1(n) + 2x_1(n)x_2^2(n). \end{cases}$$

We will investigate the zero equilibrium point for stability. For this aim we will search a function

$$V(x_1, x_2) = ax_1^2 + bx_2^2,$$

$a, b \in \mathbb{R}$, *so that* $V(x) > 0$ *in an enough small neighborhood of* 0, $x \neq 0$, *and the sign of*

$\Delta V(x(n))$ *to be constant on a neighborhood of the zero. We have*

$$\Delta V(x(n)) = V(x(n+1)) - V(x(n))$$

$$= ax_1^2(n+1) + bx_2^2(n+1) - ax_1^2(n) - bx_2^2(n)$$

$$= a\left(2x_2(n) - x_2(n)x_1^2(n)\right)^2 + b\left(x_1(n) + 2x_1(n)x_2^2(n)\right)^2$$

$$-ax_1^2(n) - bx_2^2(n)$$

$$= 4ax_2^2(n) - 4ax_1^2(n)x_2^2(n) + ax_2^2(n)x_1^4(n)$$

$$+bx_1^2(n) + 4bx_1^2(n)x_2^2(n) + 4bx_1^2(n)x_2^4(n)$$

$$-ax_1^2(n) - bx_2^2(n)$$

$$= (b-a)x_1^2(n) + (4a-b)x_2^2(n) + 4(b-a)x_1^2(n)x_2^2(n)$$

$$+ax_2^2(n)x_1^4(n) + 4bx_1^2(n)x_2^4(n).$$

If $a = b > 0$ *we have that* $V(0) = 0$, $V(x) > 0$ *in a neighborhood of* 0, $x \neq 0$, *and*

$$\Delta V(x(n)) = 3ax_2^2(n) + ax_2^2(n)x_1^4(n) + 4ax_1^2(n)x_2^4(n) > 0$$

for $x \neq 0$.

Consequently, the zero equilibrium point of the considered system is unstable.

If we take $a, b \in \mathbb{R}$ *such that*

$$b \geq a > 0,$$

then

$$4b - a > 0$$

and

$$V(x(n)) > 0, \qquad \Delta V(x(n)) > 0 \qquad \text{for} \qquad x \neq 0.$$

In this case, again the zero equilibrium point is unstable.

Example 4.4.21. *Let* $k = 2$ *and*

$$f_1\left(\frac{x(n)}{\hat{T}(x(n))}\right) = x_1(n) - 2x_1^5(n)x_2^4(n),$$

$$f_2\left(\frac{x(n)}{\hat{T}(x(n))}\right) = x_2(n).$$

The systems (10) *takes the form*

$$\begin{cases} x_1(n+1) = x_1(n) - 2x_1^5(n)x_2^4(n) \\ x_2(n+1) = x_2(n). \end{cases}$$

We will investigate the zero equilibrium point for stability. For this aim we will search a function

$$V(x) = ax_1^2 + bx_2^2, \qquad a, b \in \mathbb{R},$$

so that $V(x) > 0$, $x \neq 0$, *in a neighborhood of the origin, and the sign of* $\Delta V(x(n))$ *to be constant on a neighborhood of the zero.*

We have

$$\Delta V(x(n)) = V(x(n+1)) - V(x(n))$$

$$= ax_1^2(n+1) + bx_2^2(n+1) - ax_1^2(n) - bx_2^2(n)$$

$$= a\left(x_1(n) - 2x_1^5(n)x_2^4(n)\right)^2 + bx_2^2(n)$$

$$-ax_1^2(n) - bx_2^2(n)$$

$$= ax_1^2(n) - 2ax_1^6(n)x_2^4(n) + 4ax_1^{10}(n)x_2^8(n) - ax_1^2(n)$$

$$= 2ax_1^6(n)x_2^4(n)\left(x_1^4(n)x_2^4(n) - 1\right).$$

Consequently, for $a > 0$ *we have that*$V(x) > 0$, $x \neq 0$ *in a neighborhood of the origin, and* $\Delta V(x(n)) \leq 0$ *in an enough small neighborhood of the zero.*

Therefore the zero equilibrium point of the considered system is stable.

Example 4.4.22. *Let* $k = 2$ *and*

$$f_1\left(\frac{x(n)}{\hat{T}(x(n))}\right) = \frac{ax_2^n}{1+x_1^2(n)},$$

$$f_2\left(\frac{x(n)}{\hat{T}(x(n))}\right) = \frac{bx_1^n}{1+x_2^2(n)}, \qquad a, b \in \mathbb{R}.$$

Then the system (10) *takes the form*

$$x_1(n+1) = \frac{ax_2^n}{1+x_1^2(n)},$$

$$x_2(n+1) = \frac{bx_1^n}{1+x_2^2(n)}.$$

We will investigate the zero equilibrium point for stability.

Let

$$V(x) = x_1^2 + x_2^2.$$

Then $V(x) > 0$ *for* $x \neq 0$. *Also,*

$$\Delta V(x(n)) = V(x(n+1)) - V(x(n))$$

$$= x_1^2(n+1) + x_2^2(n+1) - x_1^2(n) - x_2^2(n)$$

$$= a^2\frac{x_2^2(n)}{(1+x_1^2(n))^2} + b^2\frac{x_1^2(n)}{(1+x_2^2(n))^2} - x_1^2(n) - x_2^2(n)$$

$$= x_1^2(n)\left(\frac{b^2}{(1+x_2^2(n))^2} - 1\right) + x_2^2(n)\left(\frac{a^2}{(1+x_1^2(n))^2} - 1\right).$$

Therefore

1. *If* $b^2-1\le 0$, $a^2-1\le 0$, *we have*
$$\Delta V(x)\le 0$$
in a neighborhood of the origin, hence the zero equilibrium point of the considered system is stable.

2. *If* $b^2-1<0$, $a^2-1<0$, *we have that*
$$\Delta V(x)<0, \qquad x\ne 0,$$
in a neighborhood of the origin. In this case the zero equilibrium point is asymptotically stable.

3. *If* U *is an arbitrary chosen neighborhood of the origin with diameter* ε. *then if* $b>1+\frac{2}{\varepsilon}$, $a>1+\varepsilon^2$, *we have that* $\Delta V(x)>0$, $x\ne 0$. *Therefore for* $b>1$ *and* $a>1$ *the zero equilibrium point of the considered system is unstable.*

Exercise 4.4.23. *Let* $k=2$ *and*

$$f_1\left(\frac{x(n)}{\hat{T}(x(n))}\right)=2x_2(n)-2x_2(n)\left(x_1^2(n)+x_2^2(n)\right),$$

$$f_2\left(\frac{x(n)}{\hat{T}(x(n))}\right)=4x_1(n)-4x_1(n)\left(x_1^2(n)+x_2^2(n)\right).$$

Prove that the zero equilibrium point of the system (10) *is asymptotically stable.*

Exercise 4.4.24. *Let* $k=2$ *and*

$$f_1\left(\frac{x(n)}{\hat{T}(x(n))}\right)=x_1(n)-x_1^3(n)x_2^2(n),$$

$$f_2\left(\frac{x(n)}{\hat{T}(x(n))}\right)=x_2(n).$$

Prove that the zero equilibrium point of the system (10) *is stable.*

Now we will investigate the equation

$$x(n+1)=f\left(\hat{T}(x(n))x(n)\right), \tag{12}$$

where $f:G\longrightarrow\mathbb{R}^k$ is continuous function.

Example 4.4.25. *Let* $k=3$ *and*

$$f(x)=\begin{pmatrix} f_1(x)\\ f_2(x)\\ f_3(x)\end{pmatrix}$$

$$=\begin{pmatrix} x_1\\ x_2\\ x_3\end{pmatrix}, \qquad x=(x_1,x_2,x_3),$$

$$\hat{T}(x)=1+x_1^2+x_2^2+x_3^2.$$

Then

$$x_1\hat{T}(x) = x_1(1+x_1^2+x_2^2+x_3^2)$$

$$= x_1 + x_1^3 + x_1x_2^2 + x_1x_3^2,$$

$$x_2\hat{T}(x) = x_2(1+x_1^2+x_2^2+x_3^2)$$

$$= x_2 + x_2x_1^2 + x_2^3 + x_2x_3^2,$$

$$x_3\hat{T}(x) = x_3(1+x_1^2+x_2^2+x_3^2)$$

$$= x_3 + x_1^2x_3 + x_2^2x_3 + x_3^3,$$

$$x\hat{T}(x) = (x_1\hat{T}(x), x_2\hat{T}(x), x_3\hat{T}(x)).$$

Hence,

$$f(x(n)\hat{T}(x(n))) = \begin{pmatrix} f_1(x(n)\hat{T}(x(n))) \\ f_2(x(n)\hat{T}(x(n))) \\ f_3(x(n)\hat{T}(x(n))) \end{pmatrix}$$

$$= \begin{pmatrix} x_1(n) + x_1^3(n) + x_1(n)x_2^2(n) + x_1(n)x_3^2(n) \\ \\ x_2(n) + x_2(n)x_1^2(n) + x_2^3(n) + x_2(n)x_3^2(n) \\ \\ x_3(n) + x_1^2(n)x_3(n) + x_2^2(n)x_3(n) + x_3^3(n) \end{pmatrix}.$$

In this way the equation (12) *takes the form*

$$\begin{cases} x_1(n+1) = x_1(n) + x_1^3(n) + x_1(n)x_2^2(n) + x_1(n)x_3^2(n) \\ \\ x_2(n+1) = x_2(n) + x_2(n)x_1^2(n) + x_2^3(n) + x_2(n)x_3^2(n) \\ \\ x_3(n+1) = x_3(n) + x_1^2(n)x_3(n) + x_2^2(n)x_3(n) + x_3^3(n). \end{cases}$$

Exercise 4.4.26. *Let $k = 2$ and*

$$f(x) = \begin{pmatrix} x_1 - x_2 \\ x_1 + x_2 \end{pmatrix}, \qquad \hat{T}(x) = x_1^2 + 1.$$

Find the equation (12).

Let x^* be an equilibrium point of the equation (12), i.e.,

$$f\left(x^*\hat{T}(x^*)\right) = x^*.$$

Definition 4.4.27. *Let $V : \mathbb{R}^k \longrightarrow \mathbb{R}$ be a real-valued function. The variation of V relative to* (12) *is defined as follows.*

$$\Delta V(x) = V(f(x\hat{T}(x))) - V(x)$$

$$= V(x(n+1)) - V(x(n)).$$

If $\Delta V(x) \leq 0$, then V is nonincreasing along the solutions of the equation (12).

Definition 4.4.28. *The function V is said to be a Lyapunov function on a subset Y of $\mathbb{R}^k$ if*

(i) *V is continuous on Y,*

(ii) *$\Delta V(x) \leq 0$, whenever $x\hat{T}(x)$ and $f(x\hat{T}(x))$ belong to Y.*

Exercise 4.4.29. *Let V is a Lyapunov function for (12) in a neighborhood Y of the equilibrium point x^* and it is positive definite at x^*. Prove that x^* is stable. If, in addition, $\Delta V(x) < 0$ whenever $x\hat{T}(x)$, $f(x\hat{T}(x)) \in Y$, $x \neq x^*$, prove that x^* is asymptotically stable. Moreover, if $G = Y = \mathbb{R}^k$ and*

$$V(x) \longrightarrow \infty \qquad \text{as} \qquad ||x|| \longrightarrow \infty, \tag{13}$$

prove that x^* is globally asymptotically stable.

Exercise 4.4.30. *Let V is a Lyapunov function on the set*

$$\{x \in \mathbb{R}^k : ||x|| > \alpha\}$$

for some $\alpha > 0$ and (13) holds. Prove that all solutions of (12) are bounded.

Definition 4.4.31. *Let V be a positive Lyapunov function for (12) on a subset G of $\mathbb{R}^k$. We define*

$$\tilde{E} = \left\{\overline{G} : \Delta V(x) = 0\right\}.$$

With $\tilde{M}$ we will denote the maximal invariant subset of $\tilde{E}$.

Exercise 4.4.32. *We suppose that V is a positive definite Lyapunov function for (12) in $G \subset \mathbb{R}^k$. Prove that for every bounded solution $x(n)$ of (12) which remains in G for all $n \in \mathbb{N}$, there exists a constant c such that*

$$x(n)\hat{T}(x(n)) \longrightarrow_{n \longrightarrow \infty} \tilde{M} \bigcap V^{-1}(c).$$

Exercise 4.4.33. *Let $f(0) = 0$, $V \in \mathcal{C}(Y)$, Y is a neighborhood of the origin, ΔV is positive definite in Y whenever $x\hat{T}(x)$, $f(x\hat{T}(x)) \in Y$, and there exists a sequence $a_i \longrightarrow_{i \longrightarrow \infty} 0$ with $V(a_i) > 0$. Prove that the zero solution of (12) is unstable.*

4.5. Stability by Linear Approximation

Here we investigate the system

$$\hat{y}^\wedge(\widehat{n+1}) = A(n)\hat{y}^\wedge(\hat{n}) + g\left(n, \hat{y}^\wedge(\hat{n})\right), \tag{14}$$

where $A(n)$ is $k \times k$ matrix for all $n \in \mathbb{N}\bigcup\{0\}$, $g : \mathbb{N}\bigcup\{0\} \times G \longrightarrow \mathbb{R}^k$, $G \subset \mathbb{R}^k$, is a continuous function.

The system (14) we can rewrite in the form

$$\frac{y(n+1)}{\hat{T}(n+1)} = A(n)\frac{y(n)}{\hat{T}(n)} + g\left(n, \frac{n}{\hat{T}(n)}\right)$$

or

$$y(n+1) = \frac{\hat{T}(n+1)}{\hat{T}(n)} A(n)y(n) + \hat{T}(n+1) g\left(n, \frac{y(n)}{\hat{T}(n)}\right).$$

Before our stability analysis of (14) we will consider an important lemma, which is a discrete analogue of the so-called Gronwall inequality.

Lemma 4.5.1. *(discrete analogue of Gronwall inequality) Let $z(n)$ and $h(n)$ be two sequences of real numbers, $n \geq n_0 \geq 0$, and $h(n) \geq 0$. If*

$$z(n) \leq M\left(z(n_0) + \sum_{j=n_0}^{n-1} h(j)z(j)\right)$$

for some $M > 0$, then

$$z(n) \leq z(n_0) \prod_{j=n_0}^{n-1} (1 + Mh(j)),$$

$$z(n) \leq z(n_0) e^{\sum_{j=n_0}^{n-1} Mh(j)}, \qquad n \geq n_0.$$

Proof. Let

$$u(n) = M\left(u(n_0) + \sum_{j=n_0}^{n-1} h(j)u(j)\right), \qquad u(n_0) = z(n_0).$$

We have

$$u(n_0+1) = M(u(n_0) + h(n_0)u(n_0))$$

$$= M(z(n_0) + h(n_0)z(n_0))$$

and since

$$z(n_0+1) \leq M(z(n_0) + h(n_0)z(n_0)),$$

we conclude that

$$z(n_0+1) \leq u(n_0+1).$$

Now we suppose that

$$z(n_0+k) \leq u(n_0+k)$$

for some $k \in \mathbb{N}$.

We will prove that

$$z(n_0+k+1) \leq u(n_0+k+1).$$

Really,

$$z(n_0+k+1) \leq M\left(z(n_0) + \sum_{j=n_0}^{n_0+k} h(j)u(j)\right)$$

$$\leq M\left(u(n_0) + \sum_{j=n_0}^{n_0+k} h(j)u(j)\right)$$

$$= u(n_0+k+1).$$

Therefore

$$z(n) \leq u(n)$$

for all $n \geq n_0$.

Using the definition of $u(n)$ we have

$$u(n+1) - u(n) = M\left(u(n_0) + \sum_{j=n_0}^{n} h(j)u(j)\right) - M\left(u(n_0) + \sum_{j=n_0}^{n-1} h(j)u(j)\right)$$

$$= Mh(n)u(n),$$

or

$$u(n+1) = (1 + Mh(n))\, u(n), \qquad n \geq n_0.$$

Consequently,

$$u(n) = \prod_{j=n_0}^{n-1} (1 + Mh(j))\, u(n_0).$$

Hence, since $z(n) \leq u(n)$ and $z(n_0) = u(n_0)$,

$$z(n) \leq z(n_0) \prod_{j=n_0}^{n-1} (1 + Mh(j))\,.$$

Now, using that

$$1 + Mh(j) \leq e^{Mh(j)},$$

we conclude that

$$z(n) \leq z(n_0) \prod_{j=n_0}^{n-1} e^{Mh(j)}$$

$$= z(n_0) e^{M \sum_{j=n_0}^{n-1} h(j)}.$$

□

Theorem 4.5.2. *We suppose that*

$$g\left(n, \frac{y}{\hat{T}(y)}\right) = \mathrm{o}\left(\left|\left|\frac{y}{\hat{T}(y)}\right|\right|\right)$$

uniformly as $\left|\left|\frac{y}{\hat{T}(y)}\right|\right| \longrightarrow 0$ *and* $\hat{T}(n+1) \leq \hat{T}(n)$, $\hat{T}(n) \geq P$ *for all* $n \geq n_0$. *If the zero solution of the system*

$$z(n+1) = \frac{\hat{T}(n+1)}{\hat{T}(n)} A(n) z(n) \tag{15}$$

is uniformly asymptotically stable, then the zero solution of the system (14) *is exponentially stable.*

Proof. Let $\Phi(n,m)$ is the fundamental matrix of the system (15). Because the zero solution of (15) is uniformly asymptotically stable, we conclude that

$$||\Phi(n,m)|| \leq M\eta^{n-m}, \qquad n \geq m \geq n_0$$

for some $M > 0$ and $\eta \in (0,1)$.

For the solution of the system (14) we have the following representation

$$y(n,n_0,y_0)=\Phi(n,n_0)y_0+\sum_{j=n_0}^{n-1}\Phi(n,j+1)\hat{T}(j+1)g\left(j,\frac{y(j)}{\hat{T}(j)}\right),$$

from where

$$\begin{aligned}
||y(n)|| &= \left|\left|\Phi(n,n_0)y_0+\sum_{j=n_0}^{n-1}\Phi(n,j+1)\hat{T}(j+1)g\left(j,\tfrac{y(j)}{\hat{T}(j)}\right)\right|\right| \\
&\le ||\Phi(n,n_0)y_0||+\left|\left|\sum_{j=n_0}^{n-1}\Phi(n,j+1)\hat{T}(j+1)g\left(j,\tfrac{y(j)}{\hat{T}(j)}\right)\right|\right| \\
&\le ||\Phi(n,n_0)||||y_0||+\sum_{j=n_0}^{n-1}\left|\left|\Phi(n,j+1)\hat{T}(j+1)g\left(j,\tfrac{y(j)}{\hat{T}(j)}\right)\right|\right| \\
&\le ||\Phi(n,n_0)||||y_0||+\sum_{j=n_0}^{n-1}||\Phi(n,j+1)||\left|\left|\hat{T}(j+1)g\left(j,\tfrac{y(j)}{\hat{T}(j)}\right)\right|\right| \\
&\le M\eta^{n-n_0}||y_0||+M\sum_{j=n_0}^{n-1}\eta^{n-j-1}\left|\left|\hat{T}(j+1)g\left(j,\tfrac{y(j)}{\hat{T}(j)}\right)\right|\right| \\
&= M\eta^{n-n_0}||y_0||+\eta^{-1}\sum_{j=n_0}^{n-1}M\eta^{n-j}\left|\left|\hat{T}(j+1)g\left(j,\tfrac{y(j)}{\hat{T}(j)}\right)\right|\right|,
\end{aligned}$$

i.e.,

$$||y(n)||\le M\eta^{n-n_0}||y_0||+\eta^{-1}\sum_{j=n_0}^{n-1}M\eta^{n-j}\left|\left|\hat{T}(j+1)g\left(j,\frac{y(j)}{\hat{T}(j)}\right)\right|\right|. \tag{16}$$

We have that

$$\left|\left|g\left(j,\frac{y(j)}{\hat{T}(j)}\right)\right|\right|\le\varepsilon\left|\left|\frac{y(j)}{\hat{T}(j)}\right|\right|$$

whenever

$$\left|\left|\frac{y(j)}{\hat{T}(j)}\right|\right|<\frac{\delta}{P}\qquad\text{or}\qquad||y(j)||<\delta$$

and since $\hat{T}(j+1)\le\hat{T}(j)$ for all $j\ge n_0$, we conclude that

$$\left|\left|\hat{T}(j+1)g\left(j,\frac{y(j)}{\hat{T}(j)}\right)\right|\right|\le\varepsilon||y(j)||$$

whenever $||y(j)||<\delta$.

From here and (16), we get

$$||y(n)||\le M\eta^{n-n_0}||y_0||+\eta^{-1}\sum_{j=n_0}^{n-1}M\eta^{n-j}\varepsilon||y(j)||$$

whenever $||y(j)||<\delta$, or

$$\eta^{-n}||y(n)||\le M\eta^{-n_0}||y_0||+\eta^{-1}\sum_{j=n_0}^{n-1}M\eta^{-j}\varepsilon||y(j)||$$

whenever $||y(j)|| < \delta$.

We put

$$z(n) := \eta^{-n}||y(n)||, \qquad n \geq n_0.$$

Then

$$z(n) \leq M\left(z(n_0) + \varepsilon\eta^{-1}\sum_{j=n_0}^{n-1} z(j)\right).$$

Now we will use the discrete analogue of the Gronwall inequality, we find

$$z(n) \leq z(n_0)\prod_{j=n_0}^{n-1}\left(1+\varepsilon M\eta^{-1}\right)$$

or

$$\eta^{-n}||y(n)|| \leq \eta^{-n_0}\prod_{j=n_0}^{n-1}\left(1+\varepsilon\eta^{-1}M\right),$$

or

$$||y(n)|| \leq \eta^{n-n_0}||y_0||\tfrac{(\eta+\varepsilon M)^{n-=n_0}}{\eta^{n-n_0}}$$

$$= ||y_0||(\eta+\varepsilon M)^{n-n_0}.$$

Let

$$\varepsilon < \frac{1-\eta}{M}.$$

Then

$$\eta + \varepsilon M < 1.$$

Consequently,

$$||y(n)|| \leq ||y_0||$$

for all $n \geq n_0 \geq 0$. In this way we obtain exponential stability. $\square$

Corollary 4.5.3. *Let*

$$\frac{\hat{T}(n+1)}{\hat{T}(n)}A(n) = B,$$

where B is $k \times k$ constant matrix. If $\rho(B) < 1$, then the zero solution of (14) is exponentially stable.

MA

Chapter 5

Oscillation Theory

5.1. Three-Term Iso-Difference Equations

We consider the three-term iso-difference equation

$$\hat{x}^{\wedge}(\hat{n}) - \hat{x}^{\wedge}(\hat{n}) + p(n)\hat{x}^{\wedge}(\widehat{n-k}) = 0, \qquad n \in \mathbb{N}, \tag{1}$$

$p(n)$ is a sequence defined for $n \in \mathbb{N}$, $k \in \mathbb{N}$.

The equation (1) we can rewrite in the form

$$\frac{x(n+1)}{\hat{T}(n+1)} - \frac{x(n)}{\hat{T}(n)} + p(n)\frac{x(n-k)}{\hat{T}(n-k)} = 0$$

or

$$x(n+1) - \frac{\hat{T}(n+1)}{\hat{T}(n)}x(n) + \frac{\hat{T}(n+1)}{\hat{T}(n-k)}p(n)x(n-k) = 0.$$

Definition 5.1.1. *A nontrivial solution $x(n)$ is said to be oscillatory around zero if for every $N \in \mathbb{N}$ there exists $n \geq N$ such that*

$$x(n)x(n+1) \leq 0.$$

Otherwise, the solution is said to be nonoscillatory.

Definition 5.1.2. *The solution $x(n)$ is said to be oscillatory around an equilibrium point x^* if $x(n) - x^*$ is oscillatory around the zero.*

Firstly we will investigate the solutions of the following associated iso-difference inequalities.

$$x(n+1) - \frac{\hat{T}(n+1)}{\hat{T}(n)}x(n) + \frac{\hat{T}(n+1)}{\hat{T}(n-k)}p(n)x(n-k) \leq 0, \tag{2}$$

$$x(n+1) - \frac{\hat{T}(n+1)}{\hat{T}(n)}x(n) + \frac{\hat{T}(n+1)}{\hat{T}(n-k)}p(n)x(n-k) \geq 0. \tag{3}$$

Theorem 5.1.3. *Let*

$$\limsup_{n\longrightarrow\infty}\hat{T}(n)=q_1, \qquad \liminf_{n\longrightarrow\infty}\hat{T}(n)=q_2,$$
$$\tfrac{q_2}{q_1}\tfrac{k+1}{k}\geq 1. \tag{4'}$$

Let also,

$$\liminf_{n\longrightarrow\infty} p(n)\leq \frac{k^k q_1^{k+1}}{(q_2(k+1))^{k+1}}. \tag{4}$$

Then

(i) *the inequality* (2) *has no eventually positive solution.*

(ii) *the inequality* (3) *has no eventually negative solution.*

Proof. **(i)** We assume the contrary, that is, there exists a solution $x(n)$ of the inequality (2) that is eventually positive. Therefore there exists a natural number N_1 such that $x(n)>0$ for all $n\geq N_1$.

Dividing (2) by $x(n)$, we find, for $n\geq N_1$,

$$\frac{x(n+1)}{x(n)}-\frac{\hat{T}(n+1)}{\hat{T}(n)}+\frac{\hat{T}(n+1)}{\hat{T}(n-k)}p(n)\frac{x(n-k)}{x(n)}\leq 0$$

or

$$\frac{x(n+1)}{x(n)}\leq \hat{T}(n+1)\left(\frac{1}{\hat{T}(n)}-\frac{p(n)}{\hat{T}(n-k)}\frac{x(n-k)}{x(n)}\right). \tag{5}$$

Let

$$z(n):=\frac{x(n)}{x(n+1)}.$$

Then

$$\tfrac{x(n-k)}{x(n)}=\tfrac{x(n-k)}{x(n-k+1)}\tfrac{x(n-k+1)}{x(n-k+2)}\cdots\tfrac{x(n-1)}{x(n)}$$
$$=z(n-k)z(n-k+1)\dots z(n-1).$$

Substituting into the inequality (5) we get

$$\frac{1}{z(n)}\leq \hat{T}(n+1)\left(\frac{1}{\hat{T}(n)}-\frac{p(n)}{\hat{T}(n-k)}z(n-k)z(n-k+1)\dots z(n-1)\right), \qquad n\geq N_1. \tag{6}$$

The condition (4) implies that there exists a positive integer N_2 such that $p(n)>0$ for all $n\geq N_2$.

Let

$$N:=\max\{N_2,N_1+k\}.$$

Then, for $n\geq N$,

$$x(n+1)-\frac{\hat{T}(n+1)}{\hat{T}(n)}x(n)\leq -p(n)\frac{\hat{T}(n+1)}{\hat{T}(n-k)}x(n-k)\leq 0.$$

Consequently, $\frac{x(n)}{\hat{T}(n)}$ is nonincreasing.

Thus

$$z(n) = \frac{\hat{T}(n)}{\hat{T}(n+1)} \frac{\frac{x(n)}{\hat{T}(n)}}{\frac{x(n+1)}{\hat{T}(n+1)}}$$

$$\geq \frac{\hat{T}(n)}{\hat{T}(n+1)}.$$

Let

$$q := \liminf_{n \longrightarrow \infty} z(n).$$

Then, from (6), we get

$$\limsup_{n\longrightarrow\infty} \frac{1}{z(n)} = \frac{1}{\liminf_{n\longrightarrow\infty} z(n)}$$

$$= \frac{1}{q}$$

$$\leq \limsup_{n\longrightarrow\infty} \hat{T}(n+1)\Big(\frac{1}{\liminf_{n\longrightarrow\infty} \hat{T}(n)}$$

$$-\liminf_{n\longrightarrow\infty}\left(\frac{p(n)}{\hat{T}(n-k)} z(n-k-1)\dots z(n-1)\right)\Big)$$

$$= q_1\left(\frac{1}{q_2} - \frac{p}{q_1}q^k\right),$$

i.e.,

$$\frac{1}{q} \leq q_1\left(\frac{1}{q_2} - \frac{p}{q_1}q^k\right)$$

$$= \frac{q_1}{q_2} - pq^k,$$

which yields

$$pq^k \leq \frac{q_1}{q_2} - \frac{1}{q}$$

$$= \frac{q_1 q - q_2}{q_2 q},$$

or

$$p \leq \frac{q_1 q - q_2}{q_2 q^{k+1}}. \tag{7}$$

Let

$$l(q) := \frac{q_1 q - q_2}{q_2 q^{k+1}}.$$

Then

$$l'(q) = \frac{1}{q_2} \frac{q_1 q^{k+1} - (q_1 q - q_2)(k+1)q^k}{q^{2k+2}}$$

$$= \frac{1}{q_2} \frac{q_1 q - (q_1 q - q_2)(k+1)}{q^{k+2}}$$

$$= \frac{1}{q_2} \frac{q_1 q - (k+1)q_1 q + q_2(k+1)}{q^{k+2}}$$

$$= \frac{1}{q_2} \frac{-kq_1 q + q_2(k+1)}{q^{k+2}},$$

$$l'(q) = 0 \qquad \Longleftrightarrow$$

$$\frac{1}{q_2}\frac{-kq_1q+q_2(k+1)}{q^{k+2}} = 0 \qquad \Longrightarrow$$

$$-kq_1q+q_2(k+1) = 0 \qquad \Longrightarrow$$

$$q = \frac{q_2}{q_1}\frac{k+1}{k}.$$

Also,

$$l''(q) = \frac{1}{q_2}\frac{-kq_1q^{k+2}-(kq_1q+q_2(k+1))(k+2)q^{k+1}}{q^{2k+4}}$$

$$= \frac{1}{q_2}\frac{-kq_1q-(kq_1q+q_2(k+1))(k+2)}{q^{k+3}}$$

$$= \frac{1}{q_2}\frac{-kq_1q-k(k+2)q_1q-q_2(k+1)(k+2)}{q^{k+3}}$$

$$= \frac{1}{q_2}\frac{-k(k+3)q_1q-q_2(k+1)(k+2)}{q^{k+3}},$$

$$l''\left(\frac{q_2}{q_1}\frac{k+1}{k}\right) = \frac{1}{q_2}\frac{-(k+1)(k+3)q_2-(k+1)(k+2)q_2}{\left(\frac{q_2}{q_1}\frac{k+1}{k}\right)^{k+3}}$$

$$= -\frac{(k+1)(2k+5)}{\left(\frac{q_2}{q_1}\frac{k+1}{k}\right)^{k+3}}$$

$$< 0.$$

Therefore $l(q)$ attains its maximum at

$$q = \frac{q_2}{q_1}\frac{k+1}{k}.$$

Hence,

$$\max_{q\geq 1} l(q) = l\left(\frac{q_2}{q_1}\frac{k+1}{k}\right)$$

$$= \frac{q_2\frac{k+1}{k}-q_2}{q_2\frac{q_2^{k+1}}{q_1^{k+1}}\left(\frac{k+1}{k}\right)^{k+1}}$$

$$= \frac{\frac{1}{k}}{\frac{q_2^{k+1}}{q_1^{k+1}}\frac{(k+1)^{k+1}}{k^{k+1}}}$$

$$= \frac{k^kq_1^{k+1}}{(q_2(k+1))^{k+1}}.$$

From here and (7) we get

$$p \leq \frac{k^kq_1^{k+1}}{(q_2(k+1))^{k+1}},$$

which is a contradiction.

(ii) The proof of part (ii) is left to the reader as exercise. The proof of this part repeat the main idea of the proof of the part (i).

□

Corollary 5.1.4. *If the conditions* (4) *and* (4′) *hold, then every solution of* (1) *oscillates.*

Proof. We assume the contrary.

Let $x(n)$ be an eventually positive solution of (1). Then the inequality (2) has an eventually positive solution, which is a contradiction with the part (i) of the previous theorem.

Let $x(n)$ be an eventually negative solution of (1). Then the inequality (3) has an eventually negative solution, which is a contradiction with the part (ii) of the previous theorem. □

Theorem 5.1.5. *Let*

$$\limsup_{n\longrightarrow\infty}\hat{T}(n)=q_1,\qquad \liminf_{n\longrightarrow\infty}\hat{T}(n)=q_2,$$

the positive constant a be chosen so that

$$a\frac{q_2}{q_1}-1>0.$$

If

$$\sup_{n\in\mathbb{N}}p(n)\le\frac{aq_2-q_1}{q_2a^{k+1}},$$

then (1) *has a nonoscillatory solution.*

Proof. We divide the equation (1) by $x(n)$ and we get

$$\frac{x(n+1)}{x(n)}-\frac{\hat{T}(n+1)}{\hat{T}(n)}+\frac{\hat{T}(n+1)}{\hat{T}(n-k)}p(n)\frac{x(n-k)}{x(n)}=0.$$

We set

$$z(n):=\frac{x(n)}{x(n+1)}.$$

Then

$$\begin{aligned}\tfrac{x(n-k)}{x(n)}&=\tfrac{x(n-k)}{x(n-k+1)}\tfrac{x(n-k+1)}{x(n)}\\&=\tfrac{x(n-k)}{x(n-k+1)}\tfrac{x(n-k+1)}{x(n-k+2)}\tfrac{x(n-k+2)}{x(n-k+3)}\cdots\tfrac{x(n-1)}{x(n)}\\&=z(n-k)z(n-k+1)\ldots z(n-1).\end{aligned}$$

Thus we get the equation

$$\frac{1}{z(n)}-\frac{\hat{T}(n+1)}{\hat{T}(n)}+\frac{\hat{T}(n+1)}{\hat{T}(n-k)}p(n)z(n-k)z(n-k-1)\ldots z(n-1)=0$$

or

$$\frac{1}{z(n)}=\hat{T}(n+1)\left(\frac{1}{\hat{T}(n)}-\frac{p(n)}{\hat{T}(n-k)}z(n-k)z(n-k+1)\ldots z(n-1)\right).\tag{8}$$

We will prove that the equation (8) has a positive solution. For this aim we will construct such solution.

Let

$$z(1-k) = z(2-k) = \ldots = z(0) = a \geq \frac{q_1}{q_2} \geq \frac{q_2}{q_1}.$$

We have

$$z(n) = \frac{1}{\hat{T}(n+1)\left(\frac{1}{\hat{T}(n)} - \frac{p(n)}{\hat{T}(n-k)} z(n-k)z(n-k+1)\ldots z(n-1)\right)},$$

$$\begin{aligned} z(1) &= \frac{1}{\hat{T}(2)\left(\frac{1}{\hat{T}(1)} - \frac{p(1)}{\hat{T}(1-k)} z(1-k)z(2-k)\ldots z(0)\right)} \\ &= \frac{1}{\hat{T}(2)\left(\frac{1}{\hat{T}(1)} - \frac{p(1)}{\hat{T}(1-k)} a^k\right)} \\ &\geq \frac{1}{q_1\left(\frac{1}{q_2} - \frac{p(1)}{q_2} a^k\right)} \\ &= \frac{1}{\frac{q_1}{q_2}\left(1-p(1)a^k\right)}. \end{aligned} \tag{9}$$

Because

$$\begin{aligned} p(n) &\leq \frac{a\frac{q_2}{q_1} - 1}{\frac{q_2}{q_1} a^{k+1}} \\ &\leq \frac{a\frac{q_2}{q_1}}{\frac{q_2}{q_1} a^{k+1}} \\ &= \frac{1}{a^k} \end{aligned}$$

or

$$p(n)a^k \leq 1 \tag{10}$$

for every $n \in \mathbb{N}$.

From here and (9), we get

$$\begin{aligned} z(1) &\geq \frac{1}{\frac{q_1}{q_2}} \\ &= \frac{q_2}{q_1}. \end{aligned}$$

Also,

$$\begin{aligned} z(1) &= \frac{1}{\hat{T}(2)\left(\frac{1}{\hat{T}(1)} - \frac{p(1)}{\hat{T}(1-k)} z(1-k)z(2-k)\ldots z(0)\right)} \\ &\leq \frac{1}{q_2\left(\frac{1}{q_1} - \frac{p(1)}{q_1} a^k\right)} \\ &= \frac{1}{\frac{q_2}{q_1}\left(1-p(1)a^k\right)}. \end{aligned} \tag{11}$$

Since

$$p(1) \le \frac{a^{\frac{q_2}{q_1}}-1}{\frac{q_2}{q_1}a^{k+1}} \qquad \Longrightarrow$$

$$a^k p(1) \le \frac{a^{\frac{q_2}{q_1}}-1}{a^{\frac{q_2}{q_1}}} \qquad \Longrightarrow$$

$$-a^k p(1) \ge -\frac{a^{\frac{q_2}{q_1}}-1}{a^{\frac{q_2}{q_1}}} \qquad \Longrightarrow$$

$$1-a^k p(1) \ge 1-\frac{a^{\frac{q_2}{q_1}}-1}{a^{\frac{q_2}{q_1}}}$$

$$= \frac{1}{\frac{q_2}{q_1}a} \qquad \Longrightarrow$$

$$\frac{q_2}{q_1}(1-a^k p(1)) \ge \frac{1}{a} \qquad \Longrightarrow$$

$$\frac{1}{\frac{q_2}{q_1}(1-a^k p(1))} \le a.$$

Hence and (11) we get

$$z(1) \le a.$$

Consequently,

$$\frac{q_2}{q_1} \le z(1) \le a.$$

We assume that

$$\frac{q_2}{q_1} \le z(m) \le a$$

for some $m \in \mathbb{N}$.

We will prove that

$$\frac{q_2}{q_1} \le z(m+1) \le a.$$

Really, we have

$$z(m+1) = \frac{1}{\hat{T}(m+2)\Big(\frac{1}{\hat{T}(m+1)}-\frac{p(m+1)}{\hat{T}(m+1+k)}z(m+1-k)\ldots z(m)\Big)}$$

$$\ge \frac{1}{q_1\Big(\frac{1}{q_2}-\frac{p(m+1)}{q_2}a^k\Big)} \tag{12}$$

$$= \frac{1}{\frac{q_1}{q_2}\big(1-p(m+1)a^k\big)}.$$

From (10) we have

$$p(m+1)a^k \le 1.$$

From here and from (12) we obtain

$$z(m+1) \ge \frac{1}{\frac{q_1}{q_2}}$$

$$= \frac{q_2}{q_1}.$$

Also,

$$\begin{aligned} z(m+1) &\leq \frac{1}{q_2\left(\frac{1}{q_1}-\frac{p(m+1)}{q_1}a^k\right)} \\ &= \frac{1}{\frac{q_2}{q_1}(1-p(m+1)a^k)}. \end{aligned} \tag{13}$$

We note that

$$\begin{aligned} 1-p(m+1)a^k &\geq 1-\frac{a\frac{q_2}{q_1}-1}{\frac{q_2}{q_1}a} \\ &= \frac{1}{\frac{q_2}{q_1}a}, \qquad \Longrightarrow \\ \frac{q_2}{q_1}\left(1-p(m+1)a^k\right) &\geq \frac{q_2}{q_1}\frac{1}{\frac{q_2}{q_1}a} \\ &= \frac{1}{a}, \end{aligned}$$

whereupon, using (13),

$$z(m+1) \leq a.$$

Therefore

$$\frac{q_2}{q_1} \leq z(m+1) \leq a.$$

Consequently,

$$\frac{q_2}{q_1} \leq z(n) \leq a$$

for every $n \in \mathbb{N}$.

Now we let

$$\begin{aligned} x(1) &= 1, \\ x(2) &= \frac{x(1)}{z(1)}, \\ x(3) &= \frac{x(2)}{z(2)} \\ &\text{and} \quad \text{so} \quad \text{on}. \end{aligned}$$

In this way we obtain that $x(n)$ is a nonoscillatory solution of (1). □

5.2. Iso-Self-Adjoint Second-Order Equations

In this section we consider the equation

$$\hat{\Delta}\left(p(n-1)\hat{\Delta}x(n-1)\right)+q(n)\hat{x}^{\wedge}(\hat{n})=0, \tag{14}$$

where $p(n)$ and $q(n)$ are given functions for $n \in \mathbb{N}$, $p(n)>0$ for $n \in \mathbb{N}$.

We have

$$\hat{x}^{\wedge}(\hat{n}) = \frac{x(n)}{\hat{T}(n)},$$

$$\hat{\Delta}x(n-1) = \frac{x(n)}{\hat{T}(n)} - \frac{x(n-1)}{\hat{T}(n-1)},$$

$$p(n-1)\hat{\Delta}x(n-1) = p(n-1)\left(\frac{x(n)}{\hat{T}(n)} - \frac{x(n-1)}{\hat{T}(n-1)}\right)$$

$$= \frac{p(n-1)}{\hat{T}(n)}x(n) - \frac{p(n-1)}{\hat{T}(n-1)}x(n-1),$$

$$\hat{\Delta}\left(p(n-1)\hat{\Delta}x(n-1)\right) + q(n)\hat{x}^{\wedge}(\hat{n}) = \hat{\Delta}\left(\frac{p(n-1)}{\hat{T}(n)}x(n) - \frac{p(n-1)}{\hat{T}(n-1)}x(n-1)\right)$$

$$= \frac{p(n)}{\hat{T}^2(n+1)}x(n+1) - \frac{p(n-1)}{\hat{T}^2(n)}x(n) - \frac{p(n)}{\hat{T}^2(n)}x(n) + \frac{p(n-1)}{\hat{T}^2(n-1)}x(n-1).$$

Then the equation (14) we can rewrite in the following form

$$\frac{p(n)}{\hat{T}^2(n+1)}x(n+1) - \frac{p(n-1)}{\hat{T}^2(n)}x(n) - \frac{p(n)}{\hat{T}^2(n)}x(n) + \frac{p(n-1)}{\hat{T}^2(n-1)}x(n-1)$$

$$+\frac{q(n)}{\hat{T}(n)}x(n) = 0$$

or

$$p(n)x(n+1) + \hat{T}^2(n+1)\left(-\frac{p(n-1)+p(n)-q(n)\hat{T}(n)}{\hat{T}^2(n)}\right)x(n)$$

$$+\frac{\hat{T}^2(n+1)p(n-1)}{\hat{T}^2(n-1)}x(n-1) = 0.$$

Let

$$p_1(n) := \hat{T}^2(n+1)\left(-\frac{p(n-1)+p(n)-q(n)\hat{T}(n)}{\hat{T}^2(n)}\right),$$

$$p_2(n) := \frac{\hat{T}^2(n+1)p(n-1)}{\hat{T}^2(n-1)}.$$

Then the equation (14) takes the form

$$p(n)x(n+1) + p_1(n)x(n) + p_2(n)x(n-1) = 0. \tag{14}$$

Theorem 5.2.1. *Let $x_1(n)$ and $x_2(n)$ be two linearly independent solutions of the equation* (14). *Then $x_1(n)$ and $x_2(n)$ cannot have a common zero, that is, if $x_1(r) = 0$ then $x_2(r) \neq 0$.*

Proof. Let us assume that $x_1(n)$ and $x_2(n)$ have a common zero r_1. Then

$$x_1(r_1) = x_2(r_1) = 0$$

and for their Casoratian we have

$$
\begin{aligned}
W(r_1) &= \begin{vmatrix} x_1(r_1) & x_2(r_1) \\ x_1(r_1+1) & x_2(r_1+1) \end{vmatrix} \\
&= \begin{vmatrix} 0 & 0 \\ x_1(r_1+1) & x_2(r_1+1) \end{vmatrix} \\
&= 0,
\end{aligned}
$$

which is a contradiction because $x_1(n)$ and $x_2(n)$ are linearly independent. $\square$

Definition 5.2.2. *A solution $x(n)$, $n \geq n_0 \geq 0$, of the equation* (14) *has a generalized zero at $r > n_0$, if either $x(r) = 0$ or $x(r-1)x(r) < 0$.*

Theorem 5.2.3. *Let $x_1(n)$ and $x_2(n)$ be two linearly independent solutions of the equation* (14). *If $x_1(n)$ has a zero at n_1 and a generalized zero at $n_2 > n_1$, then x_2 must have a generalized zero in $(n_1, n_2]$.*

Proof. We assume that

$$x_1(n_1) = 0$$

and

$$x_1(n_2-1)x_1(n_2) < 0 \qquad \text{or} \qquad x_1(n_2) = 0.$$

We suppose that n_2 is the first generalized zero of $x_1(n)$ greater than n_1. Also, we suppose that

$$
\begin{aligned}
&x_1(n) > 0 \qquad \text{for} \qquad n_1 < n < n_2, \\
&x_1(n_2) \leq 0.
\end{aligned}
$$

Since $x_1(n_1) = 0$ and $x_1(n)$, $x_2(n)$ are linearly independent, then $x_2(n_1) \neq 0$.

We assume that $x_2(n)$ has no generalized zeros in $(n_1, n_2]$. Then $x_2(n)$ is either positive in $[n_1, n_2]$ or negative in $[n_1, n_2]$.

Without loss of generality we suppose that $x_2(n) > 0$ in $[n_1, n_2]$.

Now we choose a positive real number M and $r \in (n_1, n_2)$ such that

$$x_2(r) = Mx_1(r)$$

and

$$x_2(n) \geq Mx_1(n) \qquad \text{in} \qquad [n_1, n_2].$$

We note that

$$x(n) = x_2(n) - Mx_1(n)$$

is also a solution of (14). For it we have

$$
\begin{aligned}
x(r) &= x_2(r) - Mx_1(r) \\
&= Mx_1(r) - Mx_1(r) \\
&= 0,
\end{aligned}
$$

$$x(r-1)x(r+1) = (x_2(r-1) - Mx_1(r-1))\,(x_2(r+1) - Mx_1(r+1))$$

$$\geq 0 \qquad \text{for} \qquad r > n_1.$$

Also,

$$p(r)x(r+1) + \hat{T}^2(r+1)\left(-\tfrac{p(r-1)+p(r)-q(r)\hat{T}(r)}{\hat{T}^2(r)}\right)x(r)$$

$$+\tfrac{\hat{T}^2(r+1)p(r+1)}{\hat{T}^2(r-1)}x(r-1) = 0$$

and since $x(r) = 0$, we get

$$p(r)x(r+1) + \frac{\hat{T}^2(r+1)p(r-1)}{\hat{T}^2(r-1)}x(r-1) = 0,$$

or

$$p(r)x(r+1) = -\frac{\hat{T}^2(r+1)p(r-1)}{\hat{T}^2(r-1)}x(r-1).$$

Because $p(n) > 0$, $\hat{T}^2(n) > 0$, from the last equality we find that

$$x(r+1)x(r-1) < 0,$$

which is a contradiction. □

Exercise 5.2.4. *Let $x_1(n)$ and $x_2(n)$ be two linearly independent solutions of* (14). *If $x_1(n)$ has a generalized zeros at n_1 and $n_2 > n_1$, prove that $x_2(n)$ must have a generalized zero in* $[n_1, n_2]$.

Hint. Use the proof of the previous theorem.

Theorem 5.2.5. *If there exists a subsequence*

$$p(n_k-1) + p(n_k) - q(n_k)\hat{T}(n_k) \leq 0,$$

with $n_k \longrightarrow \infty$ as $k \longrightarrow \infty$, then every solution of the equation (14) *oscillates.*

Proof. We assume the contrary. Then there exists a nonoscillatory solution $x(n)$ of the equation (14). Without loss of generality, we suppose that $x(n) > 0$ for $n \geq N$.

Then, because

$$p(n_k-1) + p(n_k) - q(n_k)\hat{T}(n_k) \leq 0,$$

we get

$$\hat{T}^2(n_k+1)\left(-\frac{p(n_k-1)+p(n_k)-q(n_k)\hat{T}(n_k)}{\hat{T}^2(n_k)}\right)x(n_k) \geq 0,$$

whereupon

$$p(n_k)x(n_k+1) + \hat{T}^2(n_k+1)\left(-\tfrac{p(n_k-1)+p(n_k)-q(n_k)\hat{T}(n_k)}{\hat{T}^2(n_k)}\right)x(n_k)$$

$$+\tfrac{\hat{T}^2(n_k+1)p(n_k-1)}{\hat{T}^2(n_k-1)}x(n_k-1) > 0,$$

which is a contradiction. □

Let
$$p_3(n) := -p_1(n).$$
Then the equation (14) we can rewrite as follows
$$p(n)x(n+1)+p_2(n)x(n-1) = p_3(n)x(n), \tag{14}$$
whereupon
$$\frac{p(n)x(n+1)}{p_3(n)x(n)} + \frac{p_2(n)x(n-1)}{p_3(n)x(n)} = 1. \tag{14}$$

Definition 5.2.6. *The iso-Riccati transformation is defined as follows*
$$z(n) := \frac{p_3(n+1)x(n+1)}{p_2(n+1)x(n)}.$$

Then
$$z(n-1) = \frac{p_3(n)x(n)}{p_2(n)x(n-1)},$$
$$\frac{x(n+1)}{x(n)} = \frac{p_2(n+1)}{p_3(n+1)}z(n).$$
In this way the equation (14) takes the form
$$\frac{p_2(n+1)}{p_3(n+1)}z(n) + \frac{1}{z(n-1)} = 1.$$
Let
$$c(n) := \frac{p_2(n+1)}{p_3(n+1)}.$$
Therefore for the equation (14) we obtain the following representation
$$c(n)z(n) + \frac{1}{z(n-1)} = 1. \tag{15}$$

Theorem 5.2.7. *Let*
$$p(n-1)+p(n)-q(n)\hat{T}(n) > 0$$
for every $n \in \mathbb{N}$. Then every solution of (14) is nonoscillatory if and only if every solution $z(n)$ of (15) is positive for $n \geq N$, for some $N > 0$.

Proof. Since
$$p(n-1)+p(n)-q(n)\hat{T}(n) > 0 \qquad \text{for} \qquad \forall n \in \mathbb{N},$$
then
$$-p_1(n) > 0 \qquad \text{for} \qquad \forall n \in \mathbb{N}$$
and because $p_2(n) > 0$ for every $n \in \mathbb{N}$, we conclude that
$$-\frac{p_1(n+1)}{p_2(n+1)} > 0$$

for every $n \in \mathbb{N}$.

Consequently, every solution of (14) is nonoscillatory if and only if

$$z(n) = -\frac{p_1(n+1)x(n+1)}{p_2(n+1)x(n)}$$

is nonoscillatory for $n \geq N$, for some $N > 0$. $\square$

Theorem 5.2.8. *If*

$$c(n) \geq a(n) > 0$$

for all $n \in \mathbb{N}$, and $z(n) > 0$ is a solution to the equation

$$c(n)z(n) + \frac{1}{z(n-1)} = 1, \tag{15}$$

then the equation

$$a(n)y(n) + \frac{1}{y(n-1)} = 1 \tag{16}$$

has a solution

$$y(n) \geq z(n) > 1$$

for all $n \in \mathbb{N}$.

Proof. Because $c(n) > 0$ and $z(n) > 0$ for every $n \in \mathbb{N}$, we have that, using (15),

$$\frac{1}{z(n-1)} \leq 1$$

for all $n \in \mathbb{N}$, whereupon

$$z(n-1) \geq 1$$

for all $n \in \mathbb{N}$.

We choose $y(0)$ such that

$$y(0) \geq z(0) \geq 1.$$

Then, from (15) and (16), it follows that

$$a(n)y(n) + \frac{1}{y(n-1)} = c(n)z(n) + \frac{1}{z(n-1)} \tag{17}$$

for all $n \in \mathbb{N}$.

We choose $y(1)$ as follows

$$a(1)y(1) + \frac{1}{y(0)} = c(1)z(1) + \frac{1}{z(0)}. \tag{18}$$

Since

$$y(0) \geq z(0),$$

then

$$\frac{1}{y(0)} \leq \frac{1}{z(0)},$$

from here and from (18) we obtain

$$a(1)y(1) \geq c(1)z(1). \tag{19}$$

Because

$$a(1) \leq c(1),$$

from (19) we conclude that

$$y(1) \geq z(1).$$

From (18) we find

$$a(1)y(1) = c(1)z(1) + \frac{1}{z(0)} - \frac{1}{y(0)}$$

or

$$y(1) = \frac{c(1)}{a(1)}z(1) + \frac{1}{a(1)z(0)} - \frac{1}{a(1)y(0)}.$$

Now we put $n = 2$ in (17) and we obtain

$$a(2)y(2) + \frac{1}{y(1)} = c(2)z(2) + \frac{1}{z(1)}. \tag{20}$$

From

$$y(1) \geq z(1)$$

it follows that

$$\frac{1}{y(1)} \leq \frac{1}{z(1)}$$

and from (20) we get

$$a(2)y(2) \geq c(2)z(2),$$

and since

$$a(2) \leq c(2),$$

we conclude that

$$y(2) \geq z(2).$$

From (20) we find

$$a(2)y(2) = c(2)z(2) + \frac{1}{z(1)} - \frac{1}{y(1)}$$

or

$$y(2) = \frac{c(2)}{a(2)}z(2) + \frac{1}{a(2)z(1)} - \frac{1}{a(2)y(1)},$$

and so on, we construct $y(n)$ inductively. □

Theorem 5.2.9. *Let $a > 0$ be arbitrarily chosen. If*

$$\hat{T}^2(n)\left(p(n) + p(n-1) - q(n+1)\hat{T}(n+1)\right) \leq (a-\varepsilon)p(n+2)\hat{T}^2(n+1)$$

for some $\varepsilon > 0$ and for all $n \geq N$, then every solution of (14) is oscillatory.

Proof. If $\varepsilon \geq a$, then the result follows from the previous results.

Let now $\varepsilon \in (0,a)$.

We assume that the equation (14) has a nonoscillatory solution. Then the equation (15) has a positive solution $z(n)$ for $n \geq N$.

From the assumption of the theorem we have

$$\frac{\hat{T}^2(n)\left(p(n)+p(n+1)-q(n+1)\hat{T}(n+1)\right)}{\hat{T}^2(n+1)p(n+2)} \leq a-\varepsilon.$$

Then

$$\begin{aligned} c(n) &= \tfrac{p_2(n+1)}{p_3(n+1)} \\ &= -\tfrac{p_2(n+1)}{p_1(n+1)} \\ &= -\frac{\frac{\hat{T}^2(n+2)p(n+2)}{\hat{T}^2(n)}}{\hat{T}^2(n+2)\left(-\frac{p(n)+p(n+1)-q(n+1)\hat{T}(n+1)}{\hat{T}^2(n+1)}\right)} \\ &= \frac{\hat{T}^2(n+1)p(n+2)}{\hat{T}^2(n)\left(p(n)+p(n+1)-q(n+1)\hat{T}(n+1)\right)} \\ &\geq \tfrac{1}{a-\varepsilon}. \end{aligned}$$

From here and from the previous theorem, it follows that the equation

$$\frac{1}{a-\varepsilon}y(n)+\frac{1}{y(n-1)}=1 \tag{21}$$

has a solution $y(n)$, $n \geq N$, such that

$$y(n) \geq z(n) > 1$$

for all $n \geq N$.

Now we define a positive sequence $x(n)$ inductively, as follows,

$$x(n+1)=\frac{1}{\sqrt{a-\varepsilon}}y(n)x(n), \qquad n \geq N, \qquad x(N)=1.$$

Then

$$y(n)=\sqrt{a-\varepsilon}\frac{x(n+1)}{x(n)},$$

which we substitute in (21) and we get

$$\frac{1}{a-\varepsilon}\sqrt{a-\varepsilon}\frac{x(n+1)}{x(n)}+\frac{x(n-1)}{\sqrt{a-\varepsilon}x(n)}=1$$

or

$$x(n+1)-\sqrt{a-\varepsilon}x(n)+x(n-1)=0, \qquad n \geq N, \tag{22}$$

whose characteristic equation is

$$\lambda^2-\sqrt{a-\varepsilon}\lambda+1=0$$

and its characteristic roots are

$$\lambda_{1,2} = \frac{\sqrt{a-\varepsilon} \pm i\sqrt{\varepsilon}}{2}.$$

Therefore the solutions of the equation (22) are oscillatory.

Consequently, $y(n)$ is oscillatory, which is a contradiction. □

Theorem 5.2.10. *Let $a \geq 4$ be arbitrarily chosen. If*

$$\hat{T}^2(n)\left(p(n)+p(n+1)-q(n+1)\hat{T}(n+1)\right) \geq ap(n+2)\hat{T}^2(n+1)$$

for $n \geq N$, then every solution of (14) is nonoscillatory.

Proof. We have that

$$\begin{aligned} c(n) &= \frac{p_2(n+1)}{p_3(n+1)} \\ &= -\frac{p_2(n+1)}{p_1(n+1)} \\ &= -\frac{\frac{\hat{T}^2(n+2)}{\hat{T}^2(n)}p(n+2)}{\hat{T}^2(n+2)\left(-\frac{p(n)+p(n+1)-q(n+1)\hat{T}(n+1)}{\hat{T}^2(n+1)}\right)} \\ &= \frac{p(n+2)\hat{T}^2(n+1)}{\left(p(n)+p(n+1)-q(n+1)\hat{T}(n+1)\right)\hat{T}^2(n)} \\ &\leq \frac{1}{a}. \end{aligned}$$

Now we will construct inductively a positive solution $z(n)$ of the equation (15) as follows.

Let

$$z(N) = \frac{a+\sqrt{a^2-4a}}{2}$$

and

$$z(n) = \frac{1}{c(n)}\left(1-\frac{1}{z(n-1)}\right), \qquad n \geq N.$$

Then, if we assume that

$$z(N+k) \geq \frac{a+\sqrt{a^2-4a}}{2}$$

for some $k \in \mathbb{N}$, we have

$$z(N+k+1) = \tfrac{1}{c(N+k)}\left(1 - \tfrac{1}{z(N+k)}\right)$$

$$\geq a\left(1 - \tfrac{1}{\frac{a+\sqrt{a^2-4a}}{2}}\right)$$

$$= a\left(1 - \tfrac{2}{a+\sqrt{a^2-4a}}\right)$$

$$= a\left(1 - \tfrac{2\left(a-\sqrt{a^2-4a}\right)}{4a}\right)$$

$$= a\left(1 - \tfrac{a-\sqrt{a^2-4a}}{2a}\right)$$

$$= a\tfrac{2a-a+\sqrt{a^2-4a}}{2a}$$

$$= \tfrac{a+\sqrt{a^2-4a}}{2}.$$

Consequently, for every $n \geq N$, we have

$$z(n) \geq \frac{a+\sqrt{a^2-4a}}{2}.$$

Since we construct for every $N \in \mathbb{N}$ a positive solution of (15) for every $n \geq N$, then every solution of (14) is nonoscillatory. $\square$

5.3. Nonlinear Iso-Difference Equations

In this section we will investigate the oscillatory behavior of the following nonlinear iso-difference equation

$$\frac{x(n+1)}{\hat{T}(n+1)} - \frac{x(n)}{\hat{T}(n)} + p(n) f\left(\frac{x(n-k)}{\hat{T}(x(n-k))}\right) = 0, \tag{23}$$

where $n \geq N$, $k, N \in \mathbb{N}$.

The equation (23) we can rewrite in the following form

$$\frac{\hat{T}(n)x(n+1)}{\hat{T}(n+1)} - x(n) + p(n)\hat{T}(n) f\left(\frac{x(n-k)}{\hat{T}(x(n-k))}\right) = 0. \tag{24}$$

Theorem 5.3.1. *Let $p := \liminf_{n\longrightarrow\infty} p(n) > 0$ and there exists $\lim_{n\longrightarrow\infty} \hat{T}(n) = M < \infty$, $M > 0$. We suppose that f is continuous on $\mathbb{R}$ and satisfies the following conditions*

(i) $x f\left(\frac{x}{\hat{T}(x)}\right) > 0$, $x \neq 0$,

(ii) $\liminf_{x\longrightarrow 0} \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\frac{x}{\hat{T}(x)}} = L$, $0 < L < \infty$,

(iii) *$pL > \frac{k^k}{(k+1)^{k+1}}$ if $k \geq 1$ and $pL > 1$ if $k = 0$.*

Then every solution of (23) oscillates.

Proof. We assume the contrary.

Let $x(n)$ be a nonoscillatory solution of (23).

Without loss of generality we suppose that $x(n) > 0$ for $n \geq N$. Because

$$x(n) f\left(\frac{x(n)}{\hat{T}(x(n))}\right) > 0$$

for $n \geq N$, we conclude that

$$f\left(\frac{x(n)}{\hat{T}(x(n))}\right) > 0 \qquad \text{for} \qquad n \geq N.$$

Therefore

$$\frac{x(n+1)}{\hat{T}(n+1)} - \frac{x(n)}{\hat{T}(n)} = -p(n) f\left(\frac{x(n-k)}{\hat{T}(x(n-k))}\right) < 0,$$

whereupon $\frac{x(n)}{\hat{T}(n)}$ is decreasing.

Hence, there exist

$$\lim_{n \longrightarrow \infty} \frac{x(n)}{\hat{T}(n)} = c_1 \geq 0$$

and

$$\lim_{n \longrightarrow \infty} x(n) = c \geq 0.$$

From here,

$$\lim_{n \longrightarrow \infty} \frac{x(n+1)}{\hat{T}(n+1)} - \lim_{n \longrightarrow \infty} \frac{x(n)}{\hat{T}(n)} = \lim_{n \longrightarrow \infty} p(n) f\left(\frac{x(n-k)}{\hat{T}(x(n-k))}\right)$$

or

$$0 = \lim_{n \longrightarrow \infty} p(n) f\left(\frac{x(n-k)}{\hat{T}(x(n-k))}\right),$$

from where

$$f\left(\frac{c}{\hat{T}(c)}\right) = 0.$$

If $c \neq 0$, then using (i), we conclude that

$$c f\left(\frac{c}{\hat{T}(c)}\right) > 0$$

or

$$f\left(\frac{c}{\hat{T}(c)}\right) > 0.$$

Consequently, $c = 0$ and

$$\lim_{n \longrightarrow \infty} x(n) = 0.$$

We divide (24) by $x(n)$ and we get

$$\frac{\hat{T}(n)x(n+1)}{\hat{T}(n+1)x(n)} = 1 + p(n)\frac{\hat{T}(n)}{x(n)} f\left(\frac{x(n-k)}{\hat{T}(x(n-k))}\right).$$

Let

$$z(n) = \frac{x(n)\hat{T}(n+1)}{\hat{T}(n)x(n+1)}.$$

Then

$$\frac{1}{z(n)} = 1 - p(n)\frac{\hat{T}(n)}{x(n)}\frac{x(n-1)}{\hat{T}(n-1)}\frac{\hat{T}(n-1)}{x(n-1)}\cdots\frac{x(n-k)}{\hat{T}(n-k)}\frac{\hat{T}(n-k)}{x(n-k)} f\left(\frac{x(n-k)}{\hat{T}(x(n-k))}\right)$$

or

$$\frac{1}{z(n)} = 1 - p(n)z(n-1)\ldots z(n-k)\frac{\hat{T}(n-k)}{x(n-k)} f\left(\frac{x(n-k)}{\hat{T}(x(n-k))}\right). \tag{25}$$

Let

$$r := \liminf_{n\longrightarrow\infty} z(n).$$

Then

$$\liminf_{n\longrightarrow\infty}\frac{1}{z(n)} = \liminf_{n\longrightarrow\infty}\left(1 - p(n)z(n-1)\ldots z(n-k)\frac{\hat{T}(n-k)}{x(n-k)} f\left(\frac{x(n-k)}{\hat{T}(x(n-k))}\right)\right)$$

or

$$\frac{1}{r} \le 1 - pr^k L$$

or

$$pr^k L \le \frac{r-1}{r},$$

or

$$pL \le \frac{r-1}{r^{k+1}}. \tag{26}$$

We suppose that $k \ge 1$.

Let

$$l(r) := \frac{r-1}{r^{k+1}}, \qquad k \ge 1.$$

Then

$$l'(r) = \tfrac{r^{k+1}-(k+1)(r-1)r^k}{r^{k+2}}$$

$$= \tfrac{r-(k+1)(r-1)}{r^{k+2}}$$

$$= \tfrac{k+1-kr}{r^{k+2}},$$

$$l'(r) = 0 \qquad \Longrightarrow$$

$$k+1-kr = 0 \qquad \Longrightarrow$$

$$r = \tfrac{k+1}{k}.$$

Also,

$$l''(r) = \left(\frac{k+1-kr}{r^{k+2}}\right)'$$

$$= \frac{-kr^{k+2}-(k+1-kr)(k+2)r^{k+1}}{r^{2k+4}}$$

$$= \frac{-kr-(k+1-kr)(k+2)}{r^{2k+1}}$$

$$= \frac{-(k+1)(k+2)-kr+k(k+2)r}{r^{2k+1}}$$

$$= \frac{-(k+1)(k+2)+k(k+1)r}{r^{2k+1}},$$

$$l''\left(\frac{k+1}{k}\right) = \frac{-(k+1)(k+2)+(k+1)^2}{\left(\frac{k+1}{k}\right)^{2k+1}}$$

$$< 0.$$

Consequently,

$$\max_{r\geq 0} l(r) = l\left(\frac{k+1}{k}\right)$$

$$= \frac{\frac{k+1}{k}-1}{\left(\frac{k+1}{k}\right)^{k+1}}$$

$$= \frac{k^{k+1}}{k(k+1)^{k+1}}$$

$$= \frac{k^k}{(k+1)^{k+1}}.$$

Hence and (26) we conclude that

$$pL \leq \frac{k^k}{(k+1)^{k+1}},$$

which is a contradiction.

Let $k = 0$. Then, using (26),

$$pL \leq \frac{r-1}{r}$$

$$< 1,$$

which is a contradiction. □

Theorem 5.3.2. *Suppose that* $p_i > 0$, $k_i \in \mathbb{N}$, $1 \leq i \leq m$,

$$\sum_{i=1}^{m}(p_i + k_i) \neq m+1,$$

and let

$$\{q_i(n) : 1 \leq i \leq m\}$$

be a set of sequences of real numbers such that

$$\liminf_{n\longrightarrow\infty} q_i(n) \geq p_i, \qquad 1 \leq i \leq m.$$

If the linear difference inequality

$$\frac{x(n+1)}{\hat{T}(n+1)} - \frac{x(n)}{\hat{T}(n)} + \sum_{i=1}^{m} q_i(n)\frac{x(n-k_i)}{\hat{T}(n-k_i)} \leq 0, \qquad n \in \mathbb{N}, \tag{27}$$

has an eventually positive solution $x(n)$, then the corresponding limiting equation

$$y(n+1) - y(n) + \sum_{i=1}^{m} p_i y(n-k_i+1) = 0 \tag{28}$$

also has an eventually positive solution.

Proof. 1. Case. Let $k_i = 1$, $1 \leq i \leq m$. Then (27) and (28) take the forms

$$\tfrac{x(n+1)}{\hat{T}(n+1)} - \tfrac{x(n)}{\hat{T}(n)} + \textstyle\sum_{i=1}^{m} q_i \tfrac{x(n)}{\hat{T}(n)} \leq 0,$$

$$y(n+1) - y(n) + \textstyle\sum_{i=1}^{m} p_i y(n) = 0,$$

or

$$x(n+1) \leq \left(1 - \sum_{i=1}^{m} q_i(n)\right)\frac{\hat{T}(n+1)}{\hat{T}(n)} x(n), \tag{29}$$

$$y(n+1) = \left(1 - \sum_{i=1}^{m} p_i\right) y(n). \tag{30}$$

Let $x(n)$ be an eventually positive solution of the inequality (29). Then, from (29), it follows that for sufficiently large n we have

$$\sum_{i=1}^{m} q_i(n) < 1.$$

Because

$$\liminf_{n\longrightarrow\infty} q_i(n) \geq p_i, \qquad 1 \leq i \leq m,$$

we have that for any $\varepsilon > 0$ there exists $N > 0$ such that

$$0 < p_i \leq q_i(n) + \frac{\varepsilon}{m} \qquad \text{for} \qquad n \geq N.$$

This implies that

$$\begin{aligned} 0 &< \textstyle\sum_{i=1}^{m} p_i \leq \sum_{i=1}^{m} \left(q_i(n) + \frac{\varepsilon}{m}\right) \\ &= \textstyle\sum_{i=1}^{m} q_i(n) + \varepsilon \\ &< 1 + \varepsilon \qquad \text{for} \qquad n \geq N. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrarily chosen, we conclude that

$$0 < \sum_{i=1}^{m} p_i \leq 1.$$

But we have that

$$\sum_{i=1}^{m} p_i \neq 1.$$

Hence,

$$0 < \sum_{i=1}^{m} p_i < 1.$$

Therefore the equation (30) has a positive solution.

2. Case. Let

$$k = \max\{k_1, k_2, \ldots, k_m\} > 1.$$

Let also,

$$u(n) := \frac{\hat{T}(n-1)x(n)}{\hat{T}(n)x(n-1)}.$$

Then

$$\begin{aligned}
\tfrac{x(n-k_i)\hat{T}(n-1)}{x(n-1)\hat{T}(n-k_i)} &= \tfrac{x(n-k_i)\hat{T}(n-k_i+1)}{x(n-k_i+1)\hat{T}(n-k_i)} \tfrac{x(n-k_i+1)\hat{T}(n-k_i+2)}{\hat{T}(n-k_i+1)x(n-k_i+2)} \\
&\cdots \tfrac{x(n-2)\hat{T}(n-1)}{x(n-1)\hat{T}(n-2)} \\
&= \tfrac{1}{u(n-k_i+1)} \tfrac{1}{u(n-k_i+2)} \cdots \tfrac{1}{u(n-1)} \\
&= \prod_{j=1}^{k_i-1} \tfrac{1}{u(n-j)}.
\end{aligned}$$

Then the inequality (27) we can rewrite as follows

$$u(n+1) \leq 1 - \sum_{i=1}^{m} q_i(n) \prod_{j=1}^{k_i-1} \frac{1}{u(n-j)}. \tag{31}$$

Let

$$u := \limsup_{n \longrightarrow \infty} u(n),$$

then from (31) we get

$$\limsup_{n \longrightarrow \infty} u(n+1) \leq \limsup_{n \longrightarrow \infty} \left(1 - \sum_{i=1}^{m} q_i(n) \prod_{j=1}^{k_i-1} \frac{1}{u(n-j)} \right),$$

from where

$$u \leq 1 - \sum_{i=1}^{m} p_i u^{-k_i+1}$$

or

$$u - 1 + \sum_{i=1}^{m} p_i u^{-k_i+1} \leq 0.$$

Let

$$h(\lambda) := \lambda - 1 + \sum_{i=1}^{m} p_i \lambda^{-k_i+1}.$$

Because

$$h(0^+) = \infty \qquad \text{and} \qquad h(\lambda) \le 0,$$

we have that $h(\lambda)$ has a positive root. From here we conclude that (28) has a positive solution

□

Now we consider the equation

$$\frac{x(n+1)}{\hat{T}(n+1)} - \frac{x(n)}{\hat{T}(n)} + \sum_{i=1}^{m} p_i f_i \left(\frac{x(n-k_i)}{\hat{T}(x(n-k_i))} \right) - 0, \tag{32}$$

where $p_i > 0$, $k_i \in \mathbb{N}$, f_i are continuous functions on $\mathbb{R}$, $1 \le i \le m$.

Theorem 5.3.3. *Suppose that there exists*

$$0 < \lim_{n \longrightarrow \infty} \hat{T}(n) < \infty$$

and the following hold

(i) $p_i > 0$, $k_i \in \mathbb{N}$, $1 \le i \le n$,

$$\sum_{i=1}^{m} (p_i + k_i) \ne m + 1,$$

(ii) f_i, $1 \le i \le m$ *are continuous on* $\mathbb{R}$,

$$x f_i \left(\frac{x}{\hat{T}(x)} \right) > 0$$

for $x \ne 0$, $1 \le i \le m$,

(iii)

$$\lim_{x \longrightarrow 0} \frac{f_i\left(\frac{x}{\hat{T}(x)}\right)}{\frac{x}{\hat{T}(x)}} \ge 1, \qquad 1 \le i \le m,$$

(iv)

$$\sum_{i=1}^{m} p_i \frac{(k_i+1)^{k_i+1}}{k_i^{k_i}} > 1.$$

Then every solution of (32) *oscillates.*

Proof. We suppose that the equation (32) has a nonoscillatory solution $x(n)$. Without loss of generality we assume that $x(n)$ is eventually positive solution of the equation (32). Then, from (32), we have

$$\frac{x(n+1)}{\hat{T}(n+1)} - \frac{x(n)}{\hat{T}(n)} = -\sum_{i=1}^{m} p_i f_i \left(\frac{x(n-k_i)}{\hat{T}(x(n-k_i))} \right)$$

$$\le 0,$$

i.e., $\frac{x(n)}{\hat{T}(n)}$ is noinincreasing. Therefore there exists

$$\lim_{n\longrightarrow\infty}\frac{x(n)}{\hat{T}(n)}=c_1,$$

and since there exists

$$0<\lim_{n\longrightarrow\infty}\hat{T}(n)<\infty,$$

we conclude that there exists

$$\lim_{n\longrightarrow\infty}x(n)=c.$$

We assume that $c\neq 0$. Then from (32) we obtain

$$0=\lim_{n\longrightarrow\infty}\tfrac{x(n+1)}{\hat{T}(n+1)}-\lim_{n\longrightarrow\infty}\tfrac{x(n)}{\hat{T}(n)}$$

$$+\textstyle\sum_{i=1}^{m}p_i\lim_{n\longrightarrow\infty}f_i\left(\tfrac{x(n-k_i)}{\hat{T}(x(n-k_i))}\right)$$

$$=\textstyle\sum_{i=1}^{m}p_if_i\left(\tfrac{c}{\hat{T}(c)}\right),$$

therefore

$$f_i\left(\frac{c}{\hat{T}(c)}\right)=0,\qquad 1\leq i\leq m.$$

On the other hand, since $c\neq 0$, we have

$$cf_i\left(\frac{c}{\hat{T}(c)}\right)>0,$$

which is a contradiction.

Consequently,

$$\lim_{n\longrightarrow\infty}x(n)=0.$$

Let now

$$q_i(n)=\frac{p_if_i\left(\frac{x(n-k_i)}{\hat{T}(x(n-k_i))}\right)}{\frac{x(n-k_i)}{\hat{T}(n-k_i)}}.$$

Then the equation (32) takes the following form

$$\frac{x(n+1)}{\hat{T}(n+1)}-\frac{x(n)}{\hat{T}(n)}+\sum_{i=1}^{m}q_i(n)\frac{x(n-k_i)}{\hat{T}(n-k_i)}=0. \tag{33}$$

For $q_i(n)$, $1\leq i\leq m$, we have, using (iii),

$$\liminf_{n\longrightarrow\infty}q_i(n)\geq p_i,\qquad 1\leq i\leq m.$$

From here and from the previous theorem it follows that the corresponding limiting equation has an eventually positive solution.

But from the condition (iv) it follows that every solution of the corresponding limiting equation oscillates, which is a contradiction. $\square$

MA

Chapter 6

Asymptotic Behavior of Iso-Difference Equations

Here we will study the system

$$\hat{y}^{\wedge}(\widehat{n+1}) = (D(n) +_B (n))\,\hat{y}^{\wedge}(\hat{n}) \tag{1}$$

and the unperturbed system

$$\hat{x}^{\wedge}(\widehat{n+1}) = D(n)\hat{x}^{\wedge}(\hat{n}), \tag{2}$$

where

$$D(n) = \mathrm{diag}(\lambda_1(n), \lambda_2(n), \ldots, \lambda_k(n)),$$

$\lambda_i(n) \neq 0$, $i = 1, 2, \ldots, k$, for all $n \geq n_0 \geq 0$, and $B(n)$ is a $k \times k$ matrix defined for $n \geq n_0 \geq 0$.

The systems (1) and (2) we can rewrite in the forms

$$\frac{y(n+1)}{\hat{T}(n+1)} = (D(n) + B(n))\,\frac{y(n)}{\hat{T}(n)}$$

and

$$\frac{x(n+1)}{\hat{T}(n+1)} = D(n)\frac{x(n)}{\hat{T}(n)},$$

or

$$y(n+1) = \frac{\hat{T}(n+1)}{\hat{T}(n)}\,(D(n) + B(n))\,y(n), \tag{1}$$

$$x(n+1) = \frac{\hat{T}(n+1)}{\hat{T}(n)}D(n)x(n). \tag{2}$$

The fundamental matrix of the system (2) is given by

$$\Phi(n) = \mathrm{diag}\left(\prod_{r=n_0}^{n}\frac{\hat{T}(r+1)}{\hat{T}(r)}\lambda_1(r), \prod_{r=n_0}^{n}\frac{\hat{T}(r+1)}{\hat{T}(r)}\lambda_2(r), \ldots, \prod_{r=n_0}^{n}\frac{\hat{T}(r+1)}{\hat{T}(r)}\lambda_n(r)\right).$$

Let S is a subset of the set $\{1, 2, \ldots, k\}$.

We define

$$\Phi_1(n) =:= \operatorname{diag}(\mu_1(n), \mu_2(n), \dots, \mu_k(n)),$$

where

$$\mu_i(n) = \begin{cases} \prod_{r=n_0}^{n-1} \frac{\hat{T}(r+1)}{\hat{T}(r)} \lambda_i(r) & \text{if} \quad i \in S, \\ \\ 0 \quad \text{otherwise.} \end{cases}$$

Also, we define

$$\Phi_2(n) := \Phi(n) - \Phi_1(n).$$

Definition 6.0.4. *The system* (2) *is said to possess an ordinary dichotomy if there exists a constant M such that*

(i) $||\Phi_1(n)\Phi^{-1}(m)|| \leq M$ *for* $n \geq m \geq n_0$,

(ii) $||\Phi_2(n)\Phi^{-1}(m)|| \leq M$ *for* $m \geq n \geq n_0$.

We will note, for $i = 1, 2, \dots, k$, we have

$$\prod_{r=n_0}^{n-1} \frac{\hat{T}(r+1)}{\hat{T}(r)} \lambda_i(r) = \frac{\hat{T}(n_0+1)}{\hat{T}(n_0)} \lambda_1(n_0) \frac{\hat{T}(n_0+2)}{\hat{T}(n_0+1)} \lambda_2(n_0+1) \dots \frac{\hat{T}(n)}{\hat{T}(n-1)} \lambda_i(n-1)$$

$$= \frac{\hat{T}(n)}{\hat{T}(n_0)} \prod_{i=n_0}^{n-1} \lambda_i(r).$$

Therefore we can rewrite $\Phi(n)$ as follows

$$\Phi(n) = \frac{\hat{T}(n)}{\hat{T}(n_0)} \operatorname{diag}\left(\prod_{r=n_0}^{n-1} \lambda_1(r), \prod_{r=n_0}^{n-1} \lambda_2(r), \dots, \prod_{r=n_0}^{n-1} \lambda_k(r) \right).$$

Example 6.0.5. *Let* $k = 3$, $\hat{T}(n) = e^{-n}$, $n_0 = 1$,

$$D(n) = \begin{pmatrix} 1 + \frac{1}{n+2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & n \end{pmatrix}.$$

Here

$$\lambda_1(n) = 1 + \frac{1}{n+2}$$

$$= \frac{n+3}{n+2},$$

$$\lambda_2(n) = 1,$$

$$\lambda_3(n) = n.$$

Then

$$\prod_{r=1}^{n-1}\lambda_1(r)=\prod_{r=1}^{n-1}\tfrac{r+3}{r+2}$$

$$=\tfrac{4}{3}\tfrac{5}{4}\cdots\tfrac{n+2}{n+1}$$

$$=\tfrac{n+2}{3},$$

$$\prod_{r=1}^{n-1}\lambda_2(r)=\prod_{r=1}^{n-1}1$$

$$=1,$$

$$\prod_{r=1}^{n-1}\lambda_3(n)=\prod_{r=1}^{n-1}r$$

$$=1.2.3\ldots(n-1)$$

$$=(n-1)!,$$

$$\tfrac{\hat{T}(n)}{\hat{T}(1)}=\tfrac{e^{-n}}{e^{-1}}$$

$$=e^{-n+1}.$$

From here

$$\Phi(n)=\operatorname{diag}\left(e^{-n+1}\tfrac{n+2}{3},e^{-n+1},e^{-n+1}(n-1)!\right)$$

$$=e^{-n+1}\begin{pmatrix}\frac{n+2}{3} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & (n-1)!\end{pmatrix},$$

$$\det\Phi(n)=e^{-n+1}\tfrac{n+2}{3}(n-1)!$$

Now we will find $\Phi^{-1}(n)$*. For this aim we have a need of the following quantities.*

$$\phi_{11}=e^{-n+1}\begin{vmatrix}1 & 0\\ 0 & (n-1)!\end{vmatrix}$$

$$=(n-1)!e^{-n+1},$$

$$\phi_{12}=-e^{-n+1}\begin{vmatrix}0 & 0\\ 0 & (n-1)!\end{vmatrix}$$

$$=0,$$

$$\phi_{13}=e^{-n+1}\begin{vmatrix}0 & 1\\ 0 & 0\end{vmatrix}$$

$$=0,$$

$$\phi_{21} = -e^{-n+1} \begin{vmatrix} 0 & 0 \\ 0 & (n-1)! \end{vmatrix}$$

$$= 0,$$

$$\phi_{22} = e^{-n+1} \begin{vmatrix} \frac{n+2}{3} & 0 \\ 0 & (n-1)! \end{vmatrix}$$

$$= \frac{n+2}{3}(n-1)!e^{-n+1},$$

$$\phi_{23} = -e^{-n+1} \begin{vmatrix} \frac{n+2}{3} & 0 \\ 0 & 0 \end{vmatrix}$$

$$= 0,$$

$$\phi_{31} = e^{-n+1} \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}$$

$$= 0,$$

$$\phi_{32} = -e^{-n+1} \begin{vmatrix} \frac{n+2}{3} & 0 \\ 0 & 0 \end{vmatrix}$$

$$= 0,$$

$$\phi_{33} = e^{-n+1} \begin{vmatrix} \frac{n+2}{3} & 0 \\ 0 & 1 \end{vmatrix}$$

$$= \frac{n+2}{3}e^{-n+1}.$$

Therefore

$$\Phi^{-1}(n) = \frac{3}{(n+2)(n-1)!e^{-n+1}} \begin{pmatrix} (n-1)!e^{-n+1} & 0 & 0 \\ 0 & \frac{n+2}{3}(n-1)!e^{-n+1} & 0 \\ 0 & 0 & \frac{n+2}{3}e^{-n+1} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{n+2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{(n-1)!} \end{pmatrix}.$$

Let

$$S = \{1\}.$$

Then

$$\Phi_1(n) = \mathrm{diag}(\mu_1(n), \mu_2(n), \mu_3(n)),$$

where

$$\mu_1(n) = \frac{\hat{T}(n)}{\hat{T}(1)} \prod_{r=1}^{n-1} \lambda_1(r)$$

$$= e^{-n+1} \prod_{r=1}^{n-1} \frac{r+3}{r+2}$$

$$= e^{-n+1} \frac{n+2}{3},$$

$$\mu_2(n) = 0,$$

$$\mu_3(n) = 0,$$

because $2,3 \notin S$.

Consequently,

$$\Phi_1(n) = \begin{pmatrix} e^{-n+1}\frac{n+2}{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Phi_2(n) = \Phi(n) - \Phi_1(n)$$

$$= \begin{pmatrix} e^{-n+1}\frac{n+2}{3} & 0 & 0 \\ 0 & e^{-n+1} & 0 \\ 0 & 0 & (n-1)!e^{-n+1} \end{pmatrix} - \begin{pmatrix} e^{-n+1}\frac{n+2}{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & e^{-n+1} & 0 \\ 0 & 0 & (n-1)!e^{-n+1} \end{pmatrix},$$

$$\Phi_1(n)\Phi^{-1}(m) = \begin{pmatrix} e^{-n+1}\frac{n+2}{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{n+2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{(n-1)!} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{n+2}{3(m+2)}e^{-n+1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Phi_2(n)\Phi^{-1}(m) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e^{-n+1} & 0 \\ 0 & 0 & (n-1)!e^{-n+1} \end{pmatrix} \begin{pmatrix} \frac{1}{m+2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{(m-1)!} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & e^{-n+1} & 0 \\ 0 & 0 & \frac{(n-1)!}{(m-1)!}e^{-n+1} \end{pmatrix}.$$

Hence,

$$||\Phi_1(n)\Phi^{-1}(m)|| \le 1 \qquad \text{for} \qquad n \ge m \ge 1,$$

$$||\Phi_2(n)\Phi^{-1}(m)|| \le 1 \qquad \text{for} \qquad m \ge n \ge 1.$$

Exercise 6.0.6. *Let $S = \{2\}$. Find $\Phi_1(n)$ and $\Phi_2(n)$ in the above example.*

Exercise 6.0.7. *Let $S = \{2,3\}$. Find $\Phi_1(n)$ and $\Phi_2(n)$ in the above example.*

Theorem 6.0.8. *We suppose that the system (2) possess an ordinary dichotomy and*

$$\sum_{n=n_0}^{\infty} \left|\left| \frac{\hat{T}(n+1)}{\hat{T}(n)} B(n) \right|\right| < \infty.$$

Then for each bounded solution $x(n)$ of (2) there is a bounded solution of (1) which is given by

$$\begin{aligned} y(n) &= x(n) + \textstyle\sum_{j=n_0}^{n-1} \Phi_1(n)\Phi^{-1}(j+1) \frac{\hat{T}(j+1)}{\hat{T}(j)} B(j)y(j) \\ &- \textstyle\sum_{j=n}^{\infty} \Phi_2(n)\Phi^{-1}(j+1) \frac{\hat{T}(j+1)}{\hat{T}(j)} B(j)y(j). \end{aligned} \tag{3}$$

The converse also holds: for each bounded solution $y(n)$ of (1) there is a bounded solution $x(n)$ of (2).

Proof. Let $x(n)$ be a bounded solution of (2).

Let M be a positive constant such that

$$||\Phi_1(n)\Phi^{-1}(m)|| \leq M \qquad \text{for} \qquad n \geq m \geq n_0,$$

$$||\Phi_2(n)\Phi^{-1}(m)|| \leq M \qquad \text{for} \qquad m \geq n \geq n_0.$$

We construct a sequence $\{y_i(n)\}_{i=1}^{\infty}$ as follows

$$y_1(n) = x(n),$$

$$\begin{aligned} y_{i+1}(n) &= x(n) + \textstyle\sum_{j=n_0}^{n-1} \Phi_1(n)\Phi^{-1}(j+1) \frac{\hat{T}(j+1)}{\hat{T}(j)} B(j)y_i(j) \\ &- \textstyle\sum_{j=n}^{\infty} \Phi_2(n)\Phi^{-1}(j+1) \frac{\hat{T}(j+1)}{\hat{T}(j)} B(j)y_i(j). \end{aligned} \tag{4}$$

Because $x(n)$ is a bounded solution of (2) for $n \in [n_0, \infty)$, we have that there exists a constant c_1 such that

$$|x(n)| \leq c_1 \qquad \text{on} \qquad [n_0, \infty).$$

Hence,

$$|y_1(n)| \leq c_1 \qquad \text{on} \qquad [n_0, \infty).$$

We assume that there exists a positive constant c_i such that

$$|y_i(n)| \leq c_i \qquad \text{on} \qquad [n_0, \infty)$$

for some $i \in \mathbb{N}$.

We will prove that there is a positive constant c_{i+1} such that

$$|y_{i+1}(n)| \leq c_{i+1} \qquad \text{on} \qquad [n_0, \infty).$$

Really,

$$|y_{i+1}(n)| = \left|x(n) + \sum_{j=n_0}^{n-1} \Phi_1(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y_i(j)\right.$$

$$\left.-\sum_{j=n}^{\infty} \Phi_2(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y_i(j)\right|$$

$$\leq |x(n)| + \left|\sum_{j=n_0}^{n-1} \Phi_1(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y_i(j)\right|$$

$$+\left|\sum_{j=n}^{\infty} \Phi_2(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y_i(j)\right|$$

$$\leq |x(n)| + \sum_{j=n_0}^{n-1} \left|\Phi_1(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y_i(j)\right|$$

$$+\sum_{j=n}^{\infty} \left|\Phi_2(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y_i(j)\right|$$

$$\leq |x(n)| + \sum_{j=n_0}^{n-1} ||\Phi_1(n)\Phi^{-1}(j+1)|| \left|\left|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\right|\right| |y_i(j)|$$

$$+\sum_{j=n}^{\infty} ||\Phi_2(n)\Phi^{-1}(j+1)|| \left|\left|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\right|\right| |y_i(j)|$$

$$\leq c_1 + M\sum_{j=n_0}^{n-1} \left|\left|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\right|\right| c_i + M\sum_{j=n}^{\infty} \left|\left|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\right|\right| c_i$$

$$= c_1 + M\sum_{j=n_0}^{\infty} \left|\left|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\right|\right| c_i$$

$$= c_{i+1}.$$

Consequently, $y_i(n)$ is bounded for each i.

Also,

$$y_2(n) = y_1(n) + \sum_{j=n_0}^{n-1} \Phi_1(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y_1(j)$$

$$-\sum_{j=n}^{\infty} \Phi_2(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y_1(j),$$

whereupon

$$y_2(n) - y_1(n) = \sum_{j=n_0}^{n-1} \Phi_1(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y_1(j)$$

$$-\sum_{j=n}^{\infty} \Phi_2(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y_1(j)$$

and

$$\begin{aligned}
|y_2(n)-y_1(n)| &= \Big|\textstyle\sum_{j=n_0}^{n-1}\Phi_1(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y_1(j)\\
&-\textstyle\sum_{j=n}^{\infty}\Phi_2(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y_1(j)\Big|\\
&\leq \Big|\textstyle\sum_{j=n_0}^{n-1}\Phi_1(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y_1(j)\Big|\\
&+\Big|\textstyle\sum_{j=n}^{\infty}\Phi_2(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y_1(j)\Big|\\
&\leq \textstyle\sum_{j=n_0}^{n-1}\Big|\Phi_1(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y_1(j)\Big|\\
&+\textstyle\sum_{j=n}^{\infty}\Big|\Phi_2(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y_1(j)\Big|\\
&\leq \textstyle\sum_{j=n_0}^{n-1}\Big|\Big|\Phi_1(n)\Phi^{-1}(j+1)\Big|\Big|\Big|\Big|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\Big|\Big||y_1(j)|\\
&+\textstyle\sum_{j=n}^{\infty}\Big|\Big|\Phi_2(n)\Phi^{-1}(j+1)\Big|\Big|\Big|\Big|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\Big|\Big||y_1(j)|\\
&\leq c_1M\textstyle\sum_{j=n_0}^{n-1}\Big|\Big|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\Big|\Big|+c_1M\textstyle\sum_{j=n}^{\infty}\Big|\Big|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\Big|\Big|\\
&= c_1M\textstyle\sum_{j=n_0}^{\infty}\Big|\Big|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\Big|\Big|,
\end{aligned}$$

i.e.,

$$|y_2(n)-y_1(n)| \leq c_1M\sum_{j=n_0}^{\infty}\left|\left|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\right|\right|.$$

We assume that

$$|y_{k+1}(n)-y_k(n)| \leq c_1\left(M\sum_{j=n_0}^{\infty}\left|\left|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\right|\right|\right)^k \tag{5}$$

for some $k\in\mathbb{N}$.

We will prove that

$$|y_{k+2}(n)-y_{k+1}(n)| \leq c_1\left(M\sum_{j=n_0}^{\infty}\left|\left|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\right|\right|\right)^{k+1}.$$

Really,

$$y_{k+2}(n)-y_{k+1}(n)=x(n)+\sum_{j=n_0}^{n-1}\Phi_1(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y_{k+1}(j)$$

$$-\sum_{j=n}^{\infty}\Phi_2(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y_{k+1}(j)$$

$$-x(n)-\sum_{j=n_0}^{n-1}\Phi_1(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y_k(j)$$

$$+\sum_{j=n}^{\infty}\Phi_2(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y_k(j)$$

$$=\sum_{j=n_0}^{n-1}\Phi_1(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)(y_{k+1}(j)-y_k(j))$$

$$-\sum_{j=n}^{\infty}\Phi_2(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)(y_{k+1}(j)-y_k(j)),$$

whereupon

$$|y_{k+2}(n)-y_{k+1}(n)|=\Big|\sum_{j=n_0}^{n-1}\Phi_1(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)(y_{k+1}(j)-y_k(j))$$

$$-\sum_{j=n}^{\infty}\Phi_2(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)(y_{k+1}(j)-y_k(j))\Big|$$

$$\leq\Big|\sum_{j=n_0}^{n-1}\Phi_1(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)(y_{k+1}(j)-y_k(j))\Big|$$

$$+\Big|\sum_{j=n}^{\infty}\Phi_2(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)(y_{k+1}(j)-y_k(j))\Big|$$

$$\leq\sum_{j=n_0}^{n-1}\Big|\Phi_1(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)(y_{k+1}(j)-y_k(j))\Big|$$

$$+\sum_{j=n}^{\infty}\Big|\Phi_2(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)(y_{k+1}(j)-y_k(j))\Big|$$

$$\leq\sum_{j=n_0}^{n-1}\Big|\Big|\Phi_1(n)\Phi^{-1}(j+1)\Big|\Big|\Big|\Big|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\Big|\Big||(y_{k+1}(j)-y_k(j))\Big|$$

$$+\sum_{j=n}^{\infty}\Big|\Big|\Phi_2(n)\Phi^{-1}(j+1)\Big|\Big|\Big|\Big|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\Big|\Big|\Big|(y_{k+1}(j)-y_k(j))\Big|$$

$$\leq M\sum_{j=n_0}^{n-1}\Big|\Big|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\Big|\Big||(y_{k+1}(j)-y_k(j))\Big|+M\sum_{j=n}^{\infty}\Big|\Big|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\Big|\Big|\Big|(y_{k+1}(j)-y_k(j))\Big|$$

$$=M\sum_{j=n_0}^{\infty}\Big|\Big|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\Big|\Big|\Big|(y_{k+1}(j)-y_k(j))\Big|$$

$$\leq c_1M\sum_{j=n_0}^{\infty}\Big|\Big|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\Big|\Big|\left(M\sum_{j=n_0}^{\infty}\Big|\Big|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\Big|\Big|\right)^k$$

$$=c_1\left(M\sum_{j=n_0}^{\infty}\Big|\Big|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\Big|\Big|\right)^{k+1}.$$

Consequently, the inequality (5) is valid for every $k\in\mathbb{N}$.

We choose n_0 enough large so that

$$M\sum_{j=n_0}^{\infty}\left|\left|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\right|\right| < \eta < 1.$$

Then, using the inequality (5), we get, for $k\in\mathbb{N}$,

$$|y_{k+1}(n)-y_k(n)| \le c_1\eta^k,$$

from where

$$\sum_{k=1}^{\infty}|y_{k1}(n)-y_k(n)| \le c_1\sum_{k=1}^{\infty}\eta^k$$

$$= c_1\frac{\eta}{\eta-1}$$

$$< \infty.$$

Therefore

$$\sum_{k=1}^{\infty}(y_{k+1}(n)-y_k(n))$$

converges uniformly for $n\ge n_0$.

Let

$$y(n) = y_1(n)+\sum_{i=1}^{\infty}(y_{i+1}(n)-y_i(n))$$

$$= \lim_{i\longrightarrow\infty}y_i(n).$$

Letting $i\longrightarrow\infty$ in (4) we obtain (3).

Let now $y(n)$ is a bounded solution of (1). Then, since

$$\sum_{n=n_0}^{\infty}\left|\left|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\right|\right| < \infty,$$

as in above, it follows that there exists a bounded solution of the system

$$y(n) = \frac{\hat{T}(n+1)}{\hat{T}(n)}(D(n)+B(n))y(n) - \frac{\hat{T}(n+1)}{\hat{T}(n)}B(n)y(n)$$

or

$$y(n) = \frac{\hat{T}(n+1)}{\hat{T}(n)}D(n)y(n).$$

□

Theorem 6.0.9. *Let the system* (2) *possess an ordinary dichotomy and*

$$\lim_{n\longrightarrow\infty}\Phi_1(n) = 0.$$

Let also,

$$\sum_{n=n_0}^{\infty}\left|\left|\frac{\hat{T}(n+1)}{\hat{T}(n)}B(n)\right|\right| < \infty.$$

Then for each bounded solution $x(n)$ of (2) *there corresponds a bounded solution $y(n)$ of* (1) *such that*

$$y(n) = x(n)+o(1).$$

Proof. Let $x(n)$ be bounded solution of the system (2). Then

$$y(n) = x(n) + \sum_{j=n_0}^{n-1} \Phi_1(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y(j)$$

$$-\sum_{j=n}^{\infty} \Phi_2(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y(j)$$

is a bounded solution of the system (1). Then there exists a positive constant L such that

$$||y|| \leq L.$$

Also, because the system (2) possess an ordinary dichotomy, there exists a positive constant M such that

$$||\Phi_1(n)\Phi^{-1}(m)|| \leq M \qquad \text{for} \qquad n \geq m \geq n_0,$$

and

$$||\Phi_2(n)\Phi^{-1}(m)|| \leq M \qquad \text{for} \qquad m \geq n \geq n_0.$$

Let $m \in \mathbb{N}$ which will be determined below. Then

$$y(n) = x(n) + \sum_{j=n_0}^{m-1} \Phi_1(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y(j)$$

$$+\sum_{j=m}^{n-1} \Phi_1(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y(j)$$

$$-\sum_{j=n}^{\infty} \Phi_2(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y(j).$$

Let

$$\Psi(n) = \sum_{j=m}^{n-1} \Phi_1(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y(j)$$

$$-\sum_{j=n}^{\infty} \Phi_2(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y(j).$$

Then

$$y(n) = x(n) + \sum_{j=n_0}^{m-1} \Phi_1(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y(j) + \Psi(n).$$

We will estimate $\Psi(n)$. We have

$$
\begin{aligned}
|\Psi(n)| &= \Big|\textstyle\sum_{j=m}^{n-1}\Phi_1(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y(j)\\
&\quad -\textstyle\sum_{j=n}^{\infty}\Phi_2(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y(j)\Big|\\
&\leq \Big|\textstyle\sum_{j=m}^{n-1}\Phi_1(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y(j)\Big|\\
&\quad +\Big|\textstyle\sum_{j=n}^{\infty}\Phi_2(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y(j)\Big|\\
&\leq \textstyle\sum_{j=m}^{n-1}\Big|\Phi_1(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y(j)\Big|\\
&\quad +\textstyle\sum_{j=n}^{\infty}\Big|\Phi_2(n)\Phi^{-1}(j+1)\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)y(j)\Big|\\
&\leq \textstyle\sum_{j=m}^{n-1}\Big|\Big|\Phi_1(n)\Phi^{-1}(j+1)\Big|\Big|\Big|\Big|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\Big|\Big|\Big|\Big|y(j)\Big|\Big|\\
&\quad +\textstyle\sum_{j=n}^{\infty}\Big|\Big|\Phi_2(n)\Phi^{-1}(j+1)\Big|\Big|\Big|\Big|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\Big|\Big|\Big|\Big|y(j)\Big|\Big|\\
&\leq ML\textstyle\sum_{j=m}^{n-1}\Big|\Big|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\Big|\Big|+ML\sum_{j=n}^{\infty}\Big|\Big|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\Big|\Big|\\
&= ML\textstyle\sum_{j=m}^{\infty}\Big|\Big|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\Big|\Big|,
\end{aligned}
$$

i.e.,

$$|\Psi(n)|\leq ML\sum_{j=m}^{\infty}\Big|\Big|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\Big|\Big|.$$

Let $\varepsilon>0$ be arbitrarily chosen. Because

$$\sum_{j=n_0}^{\infty}\Big|\Big|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\Big|\Big|<\infty,$$

then there exists enough large $m\in\mathbb{N}$ so that

$$ML\sum_{j=m}^{\infty}\Big|\Big|\frac{\hat{T}(j+1)}{\hat{T}(j)}B(j)\Big|\Big|<\frac{\varepsilon}{2}.$$

Consequently, for enough large m we have that

$$|\Psi(n)|<\frac{\varepsilon}{2}.$$

Hence, using that

$$\lim_{n\longrightarrow\infty}\Phi_1(n)=0,$$

we conclude that for enough large n we have

$$y(n) = x(n) + o(1).$$

□

Now we make the following change of the variables.

$$y(n) = \prod_{r=n_0}^{n-1} \lambda_i(r) z(n),$$

for a specific i, $1 \le i \le k$.

Then

$$y(n+1) = \prod_{r=n_0}^{n} \lambda_i(r) z(n+1).$$

In this way the system (1) takes the form.

$$\prod_{r=n_0}^{n} \lambda_i(r) z(n+1) = \frac{\hat{T}(n+1)}{\hat{T}(n)} (D(n) + B(n)) \prod_{r=n_0}^{n-1} \lambda_i(r) z(n)$$

or

$$z(n+1) = \frac{1}{\lambda_i(n)} \frac{\hat{T}(n+1)}{\hat{T}(n)} (D(n) + B(n))\, z(n).$$

Let

$$D_i(n) = \tfrac{1}{\lambda_i(n)} D(n),$$

$$B_i(n) = \tfrac{1}{\lambda_i(n)} B(n).$$

In this way we obtain the system

$$z(n+1) = \frac{\hat{T}(n+1)}{\hat{T}(n)} (D_i(n) + B_i(n))\, z(n)$$

and the associated unperturbed system is

$$z(n+1) = \frac{\hat{T}(n+1)}{\hat{T}(n)} D_i(n) z(n).$$

If there exist constants $\mu > 0$ and $K > 0$ such that for each pair λ_i, λ_j, $i \neq j$, either

$$\prod_{r=0}^{n} \left| \frac{\lambda_i(r)}{\lambda_j(r)} \right| \longrightarrow +\infty \qquad \text{as} \qquad n \longrightarrow +\infty \tag{6}$$

and

$$\prod_{r=n_1}^{n_2} \left| \frac{\lambda_i(r)}{\lambda_j(r)} \right| \ge \mu > 0 \tag{7}$$

for all $0 \le n_1 \le n_2$,

or

$$\prod_{r=n_1}^{n_2} \left| \frac{\lambda_i(r)}{\lambda_j(r)} \right| \leq K \tag{8}$$

for all $0 \leq n_1 \leq n_2$, then the system (2) possess an ordinary dichotomy and

$$\lim_{n \longrightarrow \infty} \Phi_1(n) = 0.$$

Now we suppose (6), (7) and (8). In addition, if

$$\sum_{n=n_0}^{\infty} \frac{1}{|\lambda_i(n)|} \left|\left| \frac{\hat{T}(n+1)}{\hat{T}(n)} B(n) \right|\right| < \infty,$$

then the system (1) has a fundamental set of k solutions such that

$$y_i(n) = (e_i + o(1)) \prod_{r=n_0}^{n-1} \lambda_i(r), \qquad i = 1, 2, \ldots, k,$$

where

$$e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0).$$

Exercise 6.0.10. *Let $k = 2$, $\hat{T}(n) = n + 1$, $n \in \mathbb{N}$,*

$$D(n) = \begin{pmatrix} \frac{3}{n+2} & 0 \\ 0 & n+1 \end{pmatrix}, \qquad B(n) = \begin{pmatrix} \frac{1}{n^4} & \frac{2}{n^3} \\ 0 & \frac{4}{n^7} \end{pmatrix}.$$

Find the asymptotic estimates for a fundamental set of solutions of the system (1).

MA

Chapter 7

Time Scales Iso-Calculus

7.1. Basic Definitions

A time scale is an arbitrary nonempty closed subset of the real numbers.

The sets $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, i.e., the real numbers, the integers, the natural numbers are examples of time scales.

The sets

$$(0,1], \qquad (4,5), \qquad \mathbb{Q}, \qquad \mathbb{C}$$

are not time scales. Here $\mathbb{Q}$ is the set of the rational numbers, $\mathbb{C}$ denotes the set of the complex numbers.

Throughout of this book we will denote a time scale by the symbol $\mathbb{T}$.

Also, we will suppose, throughout of this book, that

$$\hat{T}:\mathbb{T}\longrightarrow(0,\infty), \qquad \frac{l}{\hat{T}(t)}\in\mathbb{T}, \qquad l\hat{T}(t)\in\mathbb{T} \qquad \text{for} \qquad \forall l,t\in\mathbb{T},$$

is a given isotopic element.

Definition 7.1.1. *For $t\in\mathbb{T}$ we define the forward jump iso-operator $\hat{\sigma}:\mathbb{T}\longrightarrow\mathbb{T}$ as follows*

$$\hat{\sigma}(t):=\inf\left\{s\in\mathbb{T}:\hat{s}=\frac{s}{\hat{T}(s)}>\hat{t}=\frac{t}{\hat{T}(t)}\right\}$$

and if $t\neq\pm\infty$ we have to have $\hat{\sigma}(t)\neq\pm\infty$.

Remark 7.1.2. *We will note that for every $t\in\mathbb{T}$ we have that*

$$\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\geq\frac{t}{\hat{T}(t)}.$$

Example 7.1.3. *Let $\mathbb{T}=\mathbb{R}$, $\hat{T}(t)=t^2+1$, $t\in\mathbb{T}$. For $s\in\mathbb{T}$ we consider the inequality*

$$\begin{aligned}
&\tfrac{s}{s^2+1}>\tfrac{t}{t^2+1} \qquad \Longleftrightarrow\\
&s(t^2+1)>t(s^2+1) \qquad \Longleftrightarrow\\
&ts^2-(t^2+1)s+t<0.
\end{aligned}$$

1. Case. $t > 0$.

1.1. Case. $t \in (0,1]$. *In this case*

$$s \in \left(t, \frac{1}{t}\right).$$

Therefore

$$\hat{\sigma}(t) = t.$$

1.2. Case. $t \in [1,\infty)$. *In this case*

$$s \in \left(\frac{1}{t}, t\right).$$

Consequently

$$\hat{\sigma}(t) = \frac{1}{t}.$$

2. Case. $t < 0$.

2.1. Case. $t \in [-1,0)$. *In this case*

$$s \in \left(\frac{1}{t}, t\right)$$

and

$$\hat{\sigma}(t) = \frac{1}{t}.$$

2.2. Case. $t \in (-\infty,-1]$. *In this case*

$$s \in \left(t, \frac{1}{t}\right).$$

Then

$$\hat{\sigma}(t) = t.$$

3. Case. $t = 0$. *In this case*

$$\hat{\sigma}(0) = 0.$$

Example 7.1.4. *Let* $\mathbb{T} = \mathbb{N}\backslash\{1\}$, $\hat{T}(t) = t^2 + 1$. *Then, using the previous example, we have that* $s \in \left(\frac{1}{t}, t\right)$.

Also, we note that $t \geq 2$. *From here* $\frac{1}{t} \notin \mathbb{N}$. *We note that* 1 *is the smallest natural number so that* $\frac{1}{t} < 1$. *Then* $\hat{\sigma}(t) = 1$.

Definition 7.1.5. *For* $t \in \mathbb{T}$ *we define the backward jump iso-operator* $\hat{\rho} : \mathbb{T} \longrightarrow \mathbb{T}$ *as follows*

$$\hat{\rho}(t) := \sup\left\{s \in \mathbb{T} : \hat{s} = \frac{s}{\hat{T}(s)} < \hat{t} = \frac{t}{\hat{T}(t)}\right\}$$

and if $t \neq \pm\infty$ *we have to have* $\hat{\rho}(t) \neq \pm\infty$.

Remark 7.1.6. *We will note that for every $t \in \mathbb{T}$ we have*

$$\frac{\hat{\rho}(t)}{\hat{T}(\hat{\rho}(t))} \leq \frac{t}{\hat{T}(t)}.$$

Example 7.1.7. *Let $\mathbb{T} = [0,\infty)$, $\hat{T}(t) = t^2+1$, $t \in \mathbb{T}$.*

We consider the inequality.

$$\begin{aligned} &\tfrac{s}{s^2+1} < \tfrac{t}{t^2+1} && \iff \\ &s(t^2+1) < (s^2+1)t && \iff \\ &ts^2-(t^2+1)s+t > 0. \end{aligned}$$

We note that

$$ts^2-(t^2+1)s+t = 0$$

if and only if

$$s = t \qquad \text{or} \qquad s = \frac{1}{t}.$$

1. Case. $t \in (0,1]$. *Then $t < \frac{1}{t}$ and*

$$s \in (0,t) \bigcup \left(\frac{1}{t}, +\infty\right).$$

Since $\hat{\rho}(t) \neq \pm\infty$, then

$$\hat{\rho}(t) = t.$$

2. Case. $t \in (1,\infty)$. *Then $\frac{1}{t} < t$. In this case*

$$s \in \left(0, \frac{1}{t}\right) \bigcup \left(\frac{1}{t}, \infty\right).$$

Because $\hat{\rho}(t) \neq \infty$, we conclude that

$$\hat{\rho}(t) = \frac{1}{t}.$$

3. Case. $t = 0$. *Then $\hat{\rho}(0)$ does not exist.*

Example 7.1.8. *Let $\mathbb{T} = \mathbb{N}\backslash\{1\}$, $\hat{T}(t) = t^2+1$, $t \in \mathbb{T}$. Then for every $t \in \mathbb{T}$ we have that $t \geq 2$. Using the previous example, we have that*

$$s \in (0,t) \bigcup \left(\frac{1}{t}, \infty\right).$$

If $t = 2$, then since $1 \notin \mathbb{T}$, we conclude that $\hat{\rho}(t)$ does not exist.

If $t > 2$, then $\hat{\rho}(t) = t-1$.

Definition 7.1.9. *If $\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} > \frac{t}{\hat{T}(t)}$, we will say that t is iso-right-scattered.*

Definition 7.1.10. *If $\frac{\hat{\rho}(t)}{\hat{T}(\hat{\rho}(t))} < \frac{t}{\hat{T}(t)}$, we will say that t is iso-left-scattered.*

Definition 7.1.11. *If $t < \sup \mathbb{T}$ and $\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} = \frac{t}{\hat{T}(t)}$, then t is called iso-right-dense.*

Definition 7.1.12. *If $t > \inf \mathbb{T}$ and $\frac{\hat{\rho}(t)}{\hat{T}(\hat{\rho}(t))} = \frac{t}{\hat{T}(t)}$, then t is called iso-left-dense.*

Definition 7.1.13. *Points that are iso-right-dense and iso-left-dense at the same time are called iso-dense.*

Definition 7.1.14. *If $\mathbb{T}$ has an iso-left-scattered minimum m, then we define*

$$\mathbb{T}^{\kappa} := \mathbb{T} \backslash \{m\}.$$

Otherwise, $\mathbb{T}^{\kappa} := \mathbb{T}$.

Definition 7.1.15. *The graininess iso-function we define as follows*

$$\hat{\mu}(t) = \hat{\sigma}(t) - t.$$

Definition 7.1.16. *If $f : \mathbb{T} \longrightarrow \mathbb{R}$ is a function, then we define the functions*

$$f^{1\hat{\sigma}}(t) := \frac{f(\hat{\sigma}(t))}{\hat{T}(t)},$$

$$f^{2\hat{\sigma}}(t) := f(\hat{\sigma}(t)\hat{T}(t)),$$

$$f^{3\hat{\sigma}}(t) := \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(t)}\right)}{\hat{T}(t)},$$

$$f^{4\hat{\sigma}}(t) := f\left(\frac{\hat{\sigma}(t)}{\hat{T}(t)}\right),$$

$$f^{5\hat{\sigma}}(t) := f(\hat{\sigma}(t)).$$

Example 7.1.17. *Let $\mathbb{T} = [0, \infty)$, $\hat{T}(t) = t^2 + 1$, $f(t) = t^2 + t + 1$, $t \in \mathbb{T}$. We will compute $\hat{\mu}(t)$, $f^{i\hat{\sigma}}(t)$, $i = 1, 2, 3, 4, 5$, $t \in \mathbb{T}$.*

1. Case. $t \in [0, 1]$. *In this case, using the previous examples, we have that*

$$\hat{\sigma}(t) = t.$$

We have

$$\hat{\mu}(t) = \hat{\sigma}(t) - t$$

$$= t - t$$

$$= 0,$$

$$f^{1\hat{\sigma}}(t) = \frac{f(\hat{\sigma}(t))}{\hat{T}(t)}$$

$$= \frac{f(t)}{\hat{T}(t)}$$

$$= \frac{t^2+t+1}{t^2+1},$$

$$f^{2\hat{\sigma}}(t) = f(\hat{T}(t)\hat{\sigma}(t))$$

$$= f(t(t^2+1))$$

$$= f(t^3+t)$$

$$= (t^3+t)^2 + t^3 + t + 1$$

$$= t^6 + 2t^4 + t^2 + t^3 + t + 1$$

$$= t^6 + 2t^4 + t^3 + t^2 + t + 1,$$

$$f^{3\hat{\sigma}}(t) = \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(t)}\right)}{\hat{T}(t)}$$

$$= \frac{f\left(\frac{t}{t^2+1}\right)}{\hat{T}(t)}$$

$$= \frac{\left(\frac{t}{t^2+1}\right)^2 + \frac{t}{t^2+1} + 1}{t^2+1}$$

$$= \frac{t^2+t(t^2+1)+(t^2+1)^2}{(t^2+1)^3}$$

$$= \frac{t^2+t^3+t+t^4+2t^2+1}{(t^2+1)^3}$$

$$= \frac{t^4+t^3+3t^2+t+1}{(t^2+1)^3},$$

$$f^{4\hat{\sigma}}(t) = f\left(\frac{\hat{\sigma}(t)}{\hat{T}(t)}\right)$$

$$= \frac{t^4+t^3+3t^2+t+1}{(t^2+1)^2},$$

$$f^{5\hat{\sigma}}(t) = f(\hat{\sigma}(t))$$

$$= f(t)$$

$$= t^2 + t + 1.$$

2. Case. $t \in [1, \infty)$. *In this case, using the previous examples, we have*

$$\hat{\sigma}(t) = \frac{1}{t}.$$

We have

$$\hat{\mu}(t) = \hat{\sigma}(t) - t$$

$$= \tfrac{1}{t} - t$$

$$= \tfrac{-t^2+1}{t},$$

$$f^{1\hat{\sigma}}(t) = \tfrac{f(\hat{\sigma}(t))}{\hat{T}(t)}$$

$$= \tfrac{f\left(\frac{1}{t}\right)}{\hat{T}(t)}$$

$$= \tfrac{\left(\frac{1}{t}\right)^2 + \frac{1}{t} + 1}{t^2+1}$$

$$= \tfrac{t^2+t+1}{t^2(t^2+1)},$$

$$f^{2\hat{\sigma}}(t) = f(\hat{T}(t)\hat{\sigma}(t))$$

$$= f\left(\tfrac{t^2+1}{t}\right)$$

$$= \left(\tfrac{t^2+1}{t}\right)^2 + \tfrac{t^2+1}{t} + 1$$

$$= \tfrac{(t^2+1)^2 + t(t^2+1) + t^2}{t^2}$$

$$= \tfrac{t^4+2t^2+1+t^3+t+t^2}{t^2}$$

$$= \tfrac{t^4+t^3+3t^2+t+1}{t^2},$$

$$f^{3\hat{\sigma}}(t) = \tfrac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(t)}\right)}{\hat{T}(t)}$$

$$= \tfrac{f\left(\frac{1}{t(t^2+1)}\right)}{\hat{T}(t)}$$

$$= \tfrac{\left(\frac{1}{t(t^2+1)}\right)^2 + \frac{1}{t(t^2+1)} + 1}{t^2+1}$$

$$= \tfrac{t^2(t^2+1)^2 + t(t^2+1) + 1}{t^2(t^2+1)^3}$$

$$= \frac{t^2(t^4+2t^2+1)+t^3+t+1}{t^2(t^2+1)^3}$$

$$= \frac{t^6+2t^4+t^3+t^2+t+1}{t^2(t^2+1)^3},$$

$$f^{4\hat{\sigma}}(t) = f\left(\frac{\hat{\sigma}(t)}{\hat{T}(t)}\right)$$

$$= \frac{t^6+2t^4+t^3+t^2+t+1}{t^2(t^2+1)^2},$$

$$f^{5\hat{\sigma}}(t) = f(\hat{\sigma}(t))$$

$$= f\left(\frac{1}{t}\right)$$

$$= \left(\frac{1}{t}\right)^2 + \frac{1}{t} + 1$$

$$= \frac{t^2+t+1}{t^2}.$$

Example 7.1.18. *Let* $\mathbb{T} = \{2n : n \in \mathbb{N}\}$, $\hat{T}(t) = 7t+5$, $f(t) = t+3$, $t \in \mathbb{T}$. *Then*

$$\hat{\sigma}(t) = \inf\left\{s \in \mathbb{T} : \frac{s}{\hat{T}(s)} > \frac{t}{\hat{T}(t)}\right\} \qquad \Longleftrightarrow$$

$$\hat{\sigma}(t) = \inf\left\{s \in \mathbb{T} : \frac{s}{7s+5} > \frac{t}{7t+5}\right\}.$$

Let us consider the inequality.

$$\frac{s}{7s+5} > \frac{t}{7t+5} \qquad \Longleftrightarrow$$

$$s(7t+5) > t(7s+5) \qquad \Longleftrightarrow$$

$$7st+5s > 7st+5t \qquad \Longleftrightarrow$$

$$s > t.$$

Therefore

$$\hat{\sigma}(t) = t+2.$$

Hence,

$$\frac{\hat{\sigma}(t)}{\hat{T}(t)} = \frac{t+2}{7t+5},$$

$$\hat{\sigma}(t)\hat{T}(t) = (t+2)(7t+5)$$

$$= 7t^2+5t+14t+10$$

$$= 7t^2+19t+10,$$

$$f^{1\hat{\sigma}}(t) = \frac{f(\hat{\sigma}(t))}{\hat{T}(t)}$$

$$= \frac{f(t+2)}{7t+5}$$

$$= \frac{t+2+3}{7t+5}$$

$$= \frac{t+5}{7t+5},$$

$$f^{2\hat{\sigma}}(t) = f(\hat{\sigma}(t)\hat{T}(t))$$

$$= f(7t^2+19t+10)$$

$$= 7t^2+19t+10+3$$

$$= 7t^2+19t+13,$$

$$f^{3\hat{\sigma}}(t) = \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(t)}\right)}{\hat{T}(t)}$$

$$= \frac{f\left(\frac{t+2}{7t+5}\right)}{7t+5}$$

$$= \frac{\frac{t+2}{7t+5}+3}{7t+5}$$

$$= \frac{22t+17}{(7t+5)^2},$$

$$f^{4\hat{\sigma}}(t) = f\left(\frac{\hat{\sigma}(t)}{\hat{T}(t)}\right)$$

$$= f\left(\frac{t+2}{7t+5}\right)$$

$$= \frac{t+2}{7t+5}+3$$

$$= \frac{22t+17}{7t+5},$$

$$f^{5\hat{\sigma}}(t) = f(\hat{\sigma}(t))$$

$$= f(t+2)$$

$$= t+2+3$$

$$= t+5.$$

Example 7.1.19. *Let $\mathbb{T} = \{3n+2 : n \in \mathbb{N}\}$, $\hat{T}(t) = t^2+t$, $f(t) = t+1$. We will find $\hat{\sigma}(t)$ for $n \geq 3$. We have*

$$\hat{\sigma}(t) = \inf\left\{s \in \mathbb{T} : \frac{s}{\hat{T}(s)} > \frac{t}{\hat{T}(t)}\right\} \qquad \Longleftrightarrow$$

$$\hat{\sigma}(t) = \inf\left\{s \in \mathbb{T} : \frac{s}{s^2+s} > \frac{t}{t^2+t}\right\}.$$

We consider the inequality.

$$s(t^2+t) > t(s^2+s) \iff$$

$$st^2+st > ts^2+st \iff$$

$$st(t-s) > 0 \iff$$

$$t > s.$$

Therefore $\hat{\sigma}(t) = 5$. *Now we will find* $f^{1\hat{\sigma}}(t)$ *and* $f^{5\hat{\sigma}}(t)$, $n \geq 3$. *We have*

$$f^{1\hat{\sigma}}(t) = \frac{f(\hat{\sigma}(t))}{\hat{T}(t)}$$

$$= \frac{f(5)}{t^2+t}$$

$$= \frac{6}{t^2+t},$$

$$f^{5\hat{\sigma}}(t) = f(\hat{\sigma}(t))$$

$$= f(5)$$

$$= 6.$$

Example 7.1.20. *Let* $\mathbb{T} = \{7n+2 : n \in \mathbb{N}\}$, $\hat{T}(t) = 2t^2+t$, $f(t) = t+7$, $t \in \mathbb{T}$. *We will find* $\hat{\sigma}(t)$ *for* $n \geq 4$. *We have*

$$\hat{\sigma}(t) = \in \left\{ s \in \mathbb{T} : \frac{s}{\hat{T}(s)} > \frac{t}{\hat{T}(t)} \right\} \iff$$

$$\hat{\sigma}(t) = \inf \left\{ s \in \mathbb{T} : \frac{s}{2s^2+s} > \frac{t}{2t^2+2} \right\}.$$

We consider the inequality.

$$\frac{s}{2s^2+s} > \frac{t}{2t^2+t} \iff$$

$$s(2t^2+t) > t(2s^2+s) \iff$$

$$2st^2+st > 2ts^2+st \iff$$

$$st(t-s) > 0 \iff$$

$$t > s.$$

Consequently $\hat{\sigma}(t) = 9$. *Hence,*

$$\hat{\sigma}(t)\hat{T}(t) = 9(2t^2+t) = 18t^2+9t,$$

$$\frac{\hat{\sigma}(t)}{\hat{T}(t)} = \frac{9}{2t^2+t}.$$

We will find $f^{2\hat{\sigma}}(t)$ and $f^{3\hat{\sigma}}(t)$ for $n \geq 4$. We have

$$
\begin{aligned}
f^{2\hat{\sigma}}(t) &= f(\hat{\sigma}(t)\hat{T}(t)) \\
&= f(18t^2+9t) \\
&= 18t^2+9t+7, \\
f^{3\hat{\sigma}}(t) &= \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(t)}\right)}{\hat{T}(t)} \\
&= \frac{f\left(\frac{9}{2t^2+t}\right)}{2t^2+t} \\
&= \frac{\frac{9}{2t^2+t}+7}{2t^2+t} \\
&= \frac{14t^2+7t+9}{(2t^2+t)^2}.
\end{aligned}
$$

Throughout this book we make the assumption that a, $b \in \mathbb{T}$, often we suppose that $a \leq b$. Then we define the interval $[a,b]$ in $\mathbb{T}$ as follows

$$[a,b] := \{t \in \mathbb{T} : a \leq t \leq b\}.$$

Open interval and half-open intervals, and etc., are defined accordingly.

7.2. Iso-Differentiation

Definition 7.2.1. *We assume that $f : \mathbb{T} \longrightarrow \mathbb{R}$ is a function and let $t \in \mathbb{T}^{\kappa}$. Then we define $f^{\hat{\Delta}}(t)$ to be the number(provided it exists) such that: for every $\varepsilon > 0$, there exists a neighborhood U of t, i.e., $U = (t-\delta, t+\delta)\bigcap\mathbb{T}$ for some $\delta > 0$, so that*

$$\left| \frac{1}{\hat{T}(t)} \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}} - f^{\hat{\Delta}}(t) \right| < \varepsilon \qquad \text{for} \qquad \forall s \in U.$$

We call $f^{\hat{\Delta}}(t)$ the iso-delta(or iso-Hilger) derivative of f at t.

Moreover, we say that f is iso-delta(or iso-Hilger) differentiable(or in short: iso-differentiable) on T^{κ} provided $f^{\hat{\Delta}}$ exists for all $t \in \mathbb{T}^{\kappa}$. The function $f^{\hat{\Delta}} : \mathbb{T}^{\kappa} \longrightarrow \mathbb{R}$ is then called iso-delta derivative of f on $\mathbb{T}^{\kappa}$.

Theorem 7.2.2. *The iso-delta derivative is well defined.*

Proof. Let $t \in \mathbb{T}^{\kappa}$. We suppose that $f : \mathbb{T} \longrightarrow \mathbb{R}$ is a given function and it has two iso-delta derivatives at the point t, $f_1^{\hat{\Delta}}(t)$ and $f_2^{\hat{\Delta}}(t)$.

Let also, $\varepsilon > 0$ be arbitrarily chosen. Then there exist $\delta_1 = \delta_1(\varepsilon) > 0$ and $\delta_2 = \delta_2(\varepsilon) > 0$ such that for every $s_1 \in (t-\delta_1, t+\delta_1)\bigcap\mathbb{T}$ and $s_2 \in (t-\delta_2, t+\delta_2)\bigcap\mathbb{T}$ we have

$$\left| \frac{1}{\hat{T}(t)} \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{s_1}{\hat{T}(s_1)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s_1}{\hat{T}(s_1)}} - f_1^{\hat{\Delta}}(t) \right| < \frac{\varepsilon}{2}, \qquad \left| \frac{1}{\hat{T}(t)} \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{s_2}{\hat{T}(s_2)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s_2}{\hat{T}(s_2)}} - f_2^{\hat{\Delta}}(t) \right| < \frac{\varepsilon}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then for every $s \in (t-\delta, t+\delta)\bigcap \mathbb{T}$ we have

$$\left| \frac{1}{\hat{T}(t)} \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}} - f_1^{\hat{\Delta}}(t) \right| < \frac{\varepsilon}{2}, \qquad \left| \frac{1}{\hat{T}(t)} \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}} - f_2^{\hat{\Delta}}(t) \right| < \frac{\varepsilon}{2}.$$

From here, for every $s \in (t-\delta, t+\delta)\bigcap \mathbb{T}$, we get

$$\begin{aligned}
&\left| f_1^{\hat{\Delta}}(t) - f_2^{\hat{\Delta}}(t) \right| \\
&= \left| f_1^{\hat{\Delta}}(t) - \frac{1}{\hat{T}(t)} \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}} - f_1^{\hat{\Delta}}(t) + \frac{1}{\hat{T}(t)} \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}} - f_1^{\hat{\Delta}}(t) - f_2^{\hat{\Delta}}(t) \right| \\
&\leq \left| \frac{1}{\hat{T}(t)} \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}} - f_1^{\hat{\Delta}}(t) \right| + \left| \frac{1}{\hat{T}(t)} \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}} - f_2^{\hat{\Delta}}(t) \right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Because $\varepsilon > 0$ was arbitrarily chosen, we conclude that

$$f_1^{\hat{\Delta}}(t) = f_2^{\hat{\Delta}}(t).$$

□

Remark 7.2.3. *Sometimes, it is convenient to have $f^{\hat{\Delta}}(t)$ also defined at a point $t \in \mathbb{T}\backslash\mathbb{T}^{\kappa}$. At such a point we use the same definition as in above.*

Example 7.2.4. *Let $f(t) = \alpha = \mathrm{const}$ for every $t \in \mathbb{T}$. We will see that*

$$f^{\hat{\Delta}}(t) = 0 \qquad \text{for} \qquad \forall t \in \mathbb{T}^{\kappa}.$$

Really, for every $\varepsilon > 0$ and for every $\delta > 0$ we have, for $s \in (t-\delta, t+\delta)\bigcap \mathbb{T}$,

$$\begin{aligned}
&\left| \frac{1}{\hat{T}(t)} \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}} - f^{\hat{\Delta}}(t) \right| = \left| \frac{1}{\hat{T}(t)} \frac{\alpha - \alpha}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}} \right| \\
&= |0| < \varepsilon.
\end{aligned}$$

Example 7.2.5. *Let $f(t) = t$, $t \in \mathbb{T}$. Then for every $\varepsilon > 0$ and $\delta > 0$, for $s \in (t-\delta, t+\delta)\bigcap \mathbb{T}$, we have*

$$\begin{aligned}
&\left| \frac{1}{\hat{T}(t)} \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}} - \frac{1}{\hat{T}(t)} \right| = \left| \frac{1}{\hat{T}(t)} \frac{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}} - \frac{1}{\hat{T}(t)} \right| \\
&= \left| \frac{1}{\hat{T}(t)} - \frac{1}{\hat{T}(t)} \right| = 0 < \varepsilon.
\end{aligned}$$

Consequently

$$(t)^{\hat{\Delta}} = \frac{1}{\hat{T}(t)}.$$

Example 7.2.6. *Let $f(t)=t^2$, $t\in\mathbb{T}$, $\hat{T}:\mathbb{T}\longrightarrow(0,\infty)$ be a continuous function. Then*

$$\begin{aligned}
\lim_{s\longrightarrow t}\frac{1}{\hat{T}(t)}\frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)-f\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{s}{\hat{T}(s)}} &= \lim_{s\longrightarrow t}\frac{1}{\hat{T}(t)}\frac{\frac{\hat{\sigma}^2(t)}{\hat{T}^2(\hat{\sigma}(t))}-\frac{s^2}{\hat{T}^2(s)}}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{s}{\hat{T}(s)}}\\
&= \lim_{s\longrightarrow t}\frac{1}{\hat{T}(t)}\frac{\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{s}{\hat{T}(s)}\right)\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}+\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{s}{\hat{T}(s)}}\\
&= \lim_{s\longrightarrow t}\frac{1}{\hat{T}(t)}\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}+\frac{s}{\hat{T}(s)}\right)\\
&= \frac{1}{\hat{T}(t)}\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}+\frac{t}{\hat{T}(t)}\right)
\end{aligned}$$

Consequently

$$\left(t^2\right)^{\hat{\Delta}}=\frac{1}{\hat{T}(t)}\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}+\frac{t}{\hat{T}(t)}\right).$$

Theorem 7.2.7. *Let $f:\mathbb{T}\longrightarrow\mathbb{R}$ is a function, $\hat{T}:\mathbb{T}\longrightarrow(0,\infty)$ is continuous, $t\in\mathbb{T}^{\kappa}$. If f is iso-delta differentiable at t, then f is continuous at $\frac{t}{\hat{T}(t)}$.*

Proof. Let $\varepsilon>0$ be arbitrarily chosen. Then there exists $\delta=\delta(\varepsilon)>0$ such that for every $s\in(t-\delta,t+\delta)\bigcap\mathbb{T}$ we have

$$\begin{aligned}
&\left|f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)-f\left(\frac{s}{\hat{T}(s)}\right)-f^{\hat{\Delta}}(t)\hat{T}(t)\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{s}{\hat{T}(s)}\right)\right|<\varepsilon\left|\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{s}{\hat{T}(s)}\right|,\\
&\left|f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)-f\left(\frac{t}{\hat{T}(t)}\right)-f^{\hat{\Delta}}(t)\hat{T}(t)\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{t}{\hat{T}(t)}\right)\right|<\varepsilon\left|\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{t}{\hat{T}(t)}\right|,\\
&\left|\frac{t}{\hat{T}(t)}-\frac{s}{\hat{T}(s)}\right|<\varepsilon.
\end{aligned}$$

Then for every $s \in (t-\delta, t+\delta)$ we have

$$\left|f\left(\frac{t}{\hat{T}(t)}\right)-f\left(\frac{s}{\hat{T}(s)}\right)\right| = \left|f\left(\frac{t}{\hat{T}(t)}\right)-f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)+f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)-f\left(\frac{s}{\hat{T}(s)}\right)\right|$$

$$= \left|f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)-f\left(\frac{s}{\hat{T}(s)}\right)-f^{\hat{\Delta}}(t)\hat{T}(t)\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{s}{\hat{T}(s)}\right)\right.$$

$$\left.+f^{\hat{\Delta}}(t)\hat{T}(t)\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{s}{\hat{T}(s)}\right)+f\left(\frac{t}{\hat{T}(t)}\right)-f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)\right|$$

$$= \left|f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)-f\left(\frac{s}{\hat{T}(s)}\right)-f^{\hat{\Delta}}(t)\hat{T}(t)\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{s}{\hat{T}(s)}\right)\right.$$

$$-\left(f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)-f\left(\frac{t}{\hat{T}(t)}\right)-f^{\hat{\Delta}}(t)\hat{T}(t)\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{t}{\hat{T}(t)}\right)\right)$$

$$\left.+f^{\hat{\Delta}}(t)\hat{T}(t)\left(\frac{t}{\hat{T}(t)}-\frac{s}{\hat{T}(s)}\right)\right|$$

$$\leq \left|f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)-f\left(\frac{s}{\hat{T}(s)}\right)-f^{\hat{\Delta}}(t)\hat{T}(t)\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{s}{\hat{T}(s)}\right)\right|$$

$$+\left|f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)-f\left(\frac{t}{\hat{T}(t)}\right)-f^{\hat{\Delta}}(t)\hat{T}(t)\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{t}{\hat{T}(t)}\right)\right|$$

$$+\left|f^{\hat{\Delta}}(t)\right|\hat{T}(t)\left|\frac{t}{\hat{\sigma}(t)}-\frac{s}{\hat{\sigma}(s)}\right|$$

$$< \varepsilon\left|\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{(s)}{\hat{T}(s)}\right|+\varepsilon\left|\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{t}{\hat{T}(t)}\right|+\varepsilon|f^{\hat{\Delta}}(t)|\hat{T}(t)$$

$$= \varepsilon\left|\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{t}{\hat{T}(t)}+\frac{t}{\hat{T}(t)}-\frac{s}{\hat{T}(s)}\right|+\varepsilon\left|\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{t}{\hat{T}(t)}\right|+\varepsilon|f^{\hat{\Delta}}(t)|\hat{T}(t)$$

$$\leq \varepsilon\left|\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{t}{\hat{T}(t)}\right|+\varepsilon\left|\frac{t}{\hat{T}(t)}-\frac{s}{\hat{T}(s)}\right|+\varepsilon\left|\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{t}{\hat{T}(t)}\right|+\varepsilon|f^{\hat{\Delta}}(t)|\hat{T}(t)$$

$$< 2\varepsilon\left|\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{t}{\hat{T}(t)}\right|+\varepsilon^2+\varepsilon\left|f^{\hat{\Delta}}(t)\right|\hat{T}(t),$$

i.e.,

$$\left|f\left(\frac{t}{\hat{T}(t)}\right)-f\left(\frac{s}{\hat{T}(s)}\right)\right| < 2\varepsilon\left|\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{t}{\hat{T}(t)}\right|+\varepsilon^2+\varepsilon\left|f^{\hat{\Delta}}(t)\right|\hat{T}(t).$$

Because $\varepsilon > 0$ was arbitrarily chosen, then we conclude that f is continuous at $\frac{t}{\hat{T}(t)}$. □

Theorem 7.2.8. *Let $\hat{T} \in \mathcal{C}(\mathbb{T})$ and $f : \mathbb{T} \longrightarrow \mathbb{R}$ is a function. If f is continuous at $\frac{t}{\hat{T}(t)}$, $t \in \mathbb{T}^{\kappa}$, and t is iso-right-scattered, then f is iso-differentiable at t and*

$$f^{\hat{\Delta}}(t) = \frac{1}{\hat{T}(t)} \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)-f\left(\frac{t}{\hat{T}(t)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{t}{\hat{T}(t)}}.$$

Proof. By continuity

$$\lim_{s\longrightarrow t}\frac{1}{\hat{T}(t)}\frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)-f\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{s}{\hat{T}(s)}}=\frac{1}{\hat{T}(t)}\frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)-f\left(\frac{t}{\hat{T}(t)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{t}{\hat{T}(t)}}.$$

Hence, given $\varepsilon>0$, there is a neighborhood U of t such that

$$\left|\frac{1}{\hat{T}(t)}\frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)-f\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{s}{\hat{T}(s)}}-\frac{1}{\hat{T}(t)}\frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)-f\left(\frac{t}{\hat{T}(t)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{t}{\hat{T}(t)}}\right|<\varepsilon$$

for all $s\in U$. From here, we get the desired result. □

Theorem 7.2.9. *Let $f:\mathbb{T}\longrightarrow\mathbb{R}$ is a function. If $t\in\mathbb{T}^{\kappa}$ is iso-right dense, then f is iso-differentiable at t if and only if the limit*

$$\lim_{s\longrightarrow t}\frac{f\left(\frac{t}{\hat{T}(t)}\right)-f\left(\frac{s}{\hat{T}(s)}\right)}{\frac{t}{\hat{T}(t)}-\frac{s}{\hat{T}(s)}}$$

exists as a finite number. In this case

$$f^{\hat{\Delta}}(t)=\frac{1}{\hat{T}(t)}\lim_{s\longrightarrow t}\frac{f\left(\frac{t}{\hat{T}(t)}\right)-f\left(\frac{s}{\hat{T}(s)}\right)}{\frac{t}{\hat{T}(t)}-\frac{s}{\hat{T}(s)}}.$$

Proof. Assume that f is iso-differentiable at t and t is iso-right-dense. Then

$$\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}=\frac{t}{\hat{T}(t)}.$$

Let $\varepsilon>0$ be arbitrarily chosen. Since f is iso-differentiable at t, there is a neighborhood U of t such that

$$\left|\frac{1}{\hat{T}(t)}\frac{f\left(\frac{t}{\hat{T}(t)}\right)-f\left(\frac{s}{\hat{T}(s)}\right)}{\frac{t}{\hat{T}(t)}-\frac{s}{\hat{T}(s)}}-f^{\hat{\Delta}}(t)\right|$$

for all $s\in U$, $s\neq t$. Hence, we obtain the desired result

$$f^{\hat{\Delta}}(t)=\lim_{s\longrightarrow t}\frac{1}{\hat{T}(t)}\frac{f\left(\frac{t}{\hat{T}(t)}\right)-f\left(\frac{s}{\hat{T}(s)}\right)}{\frac{t}{\hat{T}(t)}-\frac{s}{\hat{T}(s)}}.$$

The remaining part of the proof is left to the reader as exercise. □

Theorem 7.2.10. *Let $f:\mathbb{T}\longrightarrow\mathbb{R}$ is a function and $t\in\mathbb{T}^{\kappa}$. If f is iso-differentiable at t, then*

$$f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)=f\left(\frac{t}{\hat{T}(t)}\right)+\hat{T}(t)\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{t}{\hat{T}(t)}\right)f^{\hat{\Delta}}(t).$$

Proof. We have

$$f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) = f\left(\frac{t}{\hat{T}(t)}\right) + f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{t}{\hat{T}(t)}\right)$$

$$= f\left(\frac{t}{\hat{T}(t)}\right) + \left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{t}{\hat{T}(t)}\right) \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{t}{\hat{T}(t)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{t}{\hat{T}(t)}}$$

$$= f\left(\frac{t}{\hat{T}(t)}\right) + \hat{T}(t)\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{t}{\hat{T}(t)}\right) \frac{1}{\hat{T}(t)} \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{t}{\hat{T}(t)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{t}{\hat{T}(t)}}$$

$$= f\left(\frac{t}{\hat{T}(t)}\right) + \hat{T}(t)\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{t}{\hat{T}(t)}\right) f^{\hat{\Delta}}(t).$$

□

Example 7.2.11. *Let* $\mathbb{T} = [0,1]$, $\hat{T}(t) = t^2+1$, $f(t) = t^2 - 2t + 1$, $t \in \mathbb{T}^\kappa$. *Then*

$$\hat{\sigma}(t) = t$$

and

$$\hat{T}(\hat{\sigma}(t)) = \sigma^2(t) + 1 = t^2 + 1,$$

$$\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} = \frac{t}{t^2+1} = \frac{t}{\hat{T}(t)},$$

i.e., every point $t \in [0,1]$ *is iso-right dense. Therefore*

$$f^{\hat{\Delta}}(t) = \frac{1}{\hat{T}(t)} \lim_{s \longrightarrow t} \frac{f\left(\frac{t}{\hat{T}(t)}\right) - f\left(\frac{s}{\hat{T}(s)}\right)}{\frac{t}{\hat{T}(t)} - \frac{s}{\hat{T}(s)}}$$

$$= \frac{1}{\hat{T}(t)} \lim_{s \longrightarrow t} \frac{\left(\frac{t}{\hat{T}(t)}\right)^2 - 2\frac{t}{\hat{T}(t)} + 1 - \left(\frac{s}{\hat{T}(s)}\right)^2 + 2\frac{s}{\hat{T}(s)} - 1}{\frac{t}{\hat{T}(t)} - \frac{s}{\hat{T}(s)}}$$

$$= \frac{1}{\hat{T}(t)} \lim_{s \longrightarrow t} \frac{\left(\frac{t}{\hat{T}(t)} - \frac{s}{\hat{T}(s)}\right)\left(\frac{t}{\hat{T}(t)} + \frac{s}{\hat{T}(s)}\right) - 2\left(\frac{t}{\hat{T}(t)} - \frac{s}{\hat{T}(s)}\right)}{\frac{t}{\hat{T}(t)} - \frac{s}{\hat{T}(s)}}$$

$$= \frac{1}{\hat{T}(t)} \lim_{s \longrightarrow t} \left(\frac{t}{\hat{T}(t)} + \frac{s}{\hat{T}(s)} - 2\right)$$

$$= \frac{1}{\hat{T}(t)} \left(2\frac{t}{\hat{T}(t)} - 2\right)$$

$$= \frac{1}{t^2+1} \left(2\frac{t}{t^2+1} - 2\right)$$

$$= 2\frac{-t^2+t-1}{(t^2+1)^2}.$$

Example 7.2.12. *Let* $\mathbb{T} = \mathbb{N}$, $\hat{T}(t) = t+1$, $f(t) = t^2 + t$, $t \in \mathbb{T}$, *Firstly, we will find* $\hat{\sigma}(t)$. *We have*

$$\hat{\sigma}(t) = \inf\left\{s \in \mathbb{T} : \frac{s}{\hat{T}(s)} > \frac{t}{\hat{T}(t)}\right\} \qquad \Longleftrightarrow$$

$$\hat{\sigma}(t) = \inf\left\{s \in \mathbb{T} : \frac{s}{s+1} > \frac{t}{t+1}\right\}.$$

Let us consider the inequality.

$$\begin{aligned}
&\tfrac{s}{s+1} > \tfrac{t}{t+1} \qquad \Longleftrightarrow \\
&s(t+1) > t(s+1) \qquad \Longleftrightarrow \\
&st+s > st+t \qquad \Longleftrightarrow \\
&s > t.
\end{aligned}$$

Therefore

$$\hat{\sigma}(t) = t+1.$$

Since

$$\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} = \frac{t+1}{\hat{T}(t+1)} = \frac{t+1}{t+2} \neq \frac{t}{\hat{T}(t)} = \frac{t}{t+1} \qquad \text{for} \qquad \forall t \in \mathbb{T},$$

we conclude that every $t \in \mathbb{T}$ is iso-right scattered.

Now we will find $f^{\hat{\Delta}}(t)$. We have, since all points of $\mathbb{T}$ are iso-right scattered,

$$f^{\hat{\Delta}}(t) = \frac{1}{\hat{T}(t)} \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{t}{\hat{T}(t)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}}. \tag{1}$$

Also,

$$\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} = \frac{t+1}{\hat{T}(t+1)} = \frac{t+1}{t+2},$$

$$\begin{aligned}
f\left(\tfrac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) &= f\left(\tfrac{t+1}{t+2}\right) \\
&= \left(\tfrac{t+1}{t+2}\right)^2 + \tfrac{t+1}{t+2} \\
&= \tfrac{(t+1)^2+(t+1)(t+2)}{(t+2)^2} \\
&= \tfrac{t^2+2t+1+t^2+3t+2}{(t+2)^2} \\
&= \tfrac{2t^2+5t+3}{(t+2)^2},
\end{aligned}$$

$$\begin{aligned}
f\left(\tfrac{t}{\hat{T}(t)}\right) &= f\left(\tfrac{t}{t+1}\right) \\
&= \left(\tfrac{t}{t+1}\right)^2 + \tfrac{t}{t+1} \\
&= \tfrac{t^2+t(t+1)}{(t+1)^2} \\
&= \tfrac{2t^2+t}{(t+1)^2}.
\end{aligned}$$

Consequently, using (1),

$$f^{\hat{\Delta}}(t) = \frac{1}{t+1}\frac{\frac{2t^2+5t+3}{(t+2)^2}-\frac{2t^2+t}{(t+1)^2}}{\frac{t+1}{t+2}-\frac{t}{t+1}}$$

$$= \frac{1}{t+1}\frac{(2t^2+5t+3)(t^2+2t+1)-(2t^2+t)(t^2+4t+4)}{(t+1)^2(t+2)^2\frac{(t+1)^2-t(t+2)}{(t+2)(t+1)}}$$

$$= \frac{2t^4+4t^3+2t^2+5t^3+10t^2+5t+3t^2+6t+3-(2t^4+8t^3+8t^2+t^3+4t^2+4t)}{(t+1)^2(t+2)(t^2+2t+1-t^2-2t)}$$

$$= \frac{2t^4+9t^3+15t^2+11t+3-2t^4-9t^3-12t^2-4t}{(t+1)^2(t+2)}$$

$$= \frac{3t^2+7t+3}{(t+2)(t+1)^2}.$$

Example 7.2.13. *Let* $\mathbb{T} = \mathbb{N}$, $\hat{T}(t) = t+2$, $f(t) = t+1$, $t \in \mathbb{T}$. *Firstly, we will find* $\hat{\sigma}(t)$. *We have*

$$\hat{\sigma}(t) = \left\{s \in \mathbb{T} : \frac{s}{s+2} > \frac{t}{t+2}\right\}.$$

We consider the inequality.

$$\frac{s}{s+2} > \frac{t}{t+2} \iff$$

$$s(t+2) > t(s+2) \iff$$

$$st+2s > st+2t \iff$$

$$s > t.$$

Consequently

$$\hat{\sigma}(t) = t+1.$$

Since

$$\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} = \frac{t+1}{\hat{T}(t+1)} = \frac{t+1}{t+3} \neq \frac{t}{\hat{T}(t)} = \frac{t}{t+2},$$

we conclude that all points of $\mathbb{T}$ *are iso-right scattered. To find* $f^{\hat{\Delta}}(t)$ *we will use* (1). *For this aim we have a need of the following quantities.*

$$f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) = f\left(\frac{t+1}{t+3}\right)$$

$$= \frac{t+1}{t+3}+1$$

$$= \frac{2t+4}{t+3},$$

$$f\left(\frac{t}{\hat{T}(t)}\right) = f\left(\frac{t}{t+2}\right)$$

$$= \frac{t}{t+2}+1$$

$$= \frac{2t+2}{t+2}.$$

Therefore

$$
\begin{aligned}
f^{\hat{\Delta}}(t) &= \frac{1}{t+2}\frac{\frac{2t+4}{t+3}-\frac{2t+2}{t+2}}{\frac{t+1}{t+3}-\frac{t}{t+2}}\\
&= \frac{1}{t+2}\frac{(2t+4)(t+2)-(2t+2)(t+3)}{(t+1)(t+2)-t(t+3)}\\
&= \frac{1}{t+2}\frac{2t^2+4t+4t+8-(2t^2+6t+2t+6)}{t^2+2t+t+2-t^2-3t}\\
&= \frac{2}{2(t+2)}\\
&= \frac{1}{t+2}.
\end{aligned}
$$

Theorem 7.2.14. *Let $f,\ g : \mathbb{T} \longrightarrow \mathbb{R}$ are iso-differentiable at $t \in \mathbb{T}^{\kappa}$. Then the sum $f+g : \mathbb{T} \longrightarrow \mathbb{R}$ is iso-differentiable at t and*

$$
(f+g)^{\hat{\Delta}}(t) = f^{\hat{\Delta}}(t) + g^{\hat{\Delta}}(t).
$$

Proof. Let $\varepsilon > 0$ be arbitrarily chosen. Then there exists a neighborhood of t such that

$$
\left| \frac{1}{\hat{T}(t)} \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}} - f^{\hat{\Delta}}(t) \right| < \frac{\varepsilon}{2},
$$

$$
\left| \frac{1}{\hat{T}(t)} \frac{g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - g\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}} - g^{\hat{\Delta}}(t) \right| < \frac{\varepsilon}{2}
$$

for all $s \in U$.

Therefore for all $s \in \mathbb{T}$ we have

$$
\begin{aligned}
&\left| \frac{1}{\hat{T}(t)} \frac{(f+g)\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - (f+g)\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}} - f^{\hat{\Delta}}(t) - g^{\hat{\Delta}}(t) \right|\\
&= \left| \frac{1}{\hat{T}(t)} \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}} - f^{\hat{\Delta}}(t) + \frac{1}{\hat{T}(t)} \frac{g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - g\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}} - g^{\hat{\Delta}}(t) \right|\\
&\le \left| \frac{1}{\hat{T}(t)} \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}} - f^{\hat{\Delta}}(t) \right| + \left| \frac{1}{\hat{T}(t)} \frac{g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - g\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}} - g^{\hat{\Delta}}(t) \right|\\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\\
&= \varepsilon.
\end{aligned}
$$

□

Theorem 7.2.15. *Let $f : \mathbb{T} \longrightarrow \mathbb{R}$ is iso-differentiable at $t \in \mathbb{T}^{\kappa}$. Then for any constant α, $\alpha \neq 0$, the function $\alpha f : \mathbb{T} \longrightarrow \mathbb{R}$ is iso-differentiable at t and*

$$
(\alpha f)^{\hat{\Delta}}(t) = \alpha f^{\hat{\Delta}}(t).
$$

Proof. Let $\varepsilon > 0$ be arbitrarily chosen. Then there exists a neighborhood U of the point t such that

$$\left| \frac{1}{\hat{T}(t)} \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}} - f^{\hat{\Delta}}(t) \right| < \frac{\varepsilon}{|\alpha|}$$

for every $s \in \mathbb{T}$. Then, for every $s \in U$, we have

$$\begin{aligned}
&\left| \frac{1}{\hat{T}(t)} \frac{(\alpha f)\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - (\alpha f)\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}} - \alpha f^{\hat{\Delta}}(t) \right| \\
&= |\alpha| \left| \frac{1}{\hat{T}(t)} \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}} - f^{\hat{\Delta}}(t) \right| \\
&< |\alpha| \frac{\varepsilon}{|\alpha|} \\
&= \varepsilon.
\end{aligned}$$

□

Exercise 7.2.16. *Let $f, g : \mathbb{T} \longrightarrow \mathbb{R}$ are iso-differentiable at $t \in \mathbb{T}^\kappa$. Prove that $f - g : \mathbb{T} \longrightarrow \mathbb{R}$ is iso-differentiable at t and*

$$(f-g)^{\hat{\Delta}}(t) = f^{\hat{\Delta}}(t) - g^{\hat{\Delta}}(t).$$

Exercise 7.2.17. *Let $f : \mathbb{T} \longrightarrow \mathbb{R}$ is iso-differentiable at $t \in \mathbb{T}^\kappa$. Prove that for every constant α, the functions*

$$\alpha \hat{\times} f, \qquad \hat{\alpha} f, \qquad \hat{\alpha} \hat{\times} f$$

are iso-differentiable at t.

Theorem 7.2.18. *Let $f, g : \mathbb{T} \longrightarrow \mathbb{R}$ be iso-differentiable at t, $t \in \mathbb{T}^\kappa$. Then $fg : \mathbb{T} \longrightarrow \mathbb{R}$ is iso-differentiable at t and*

$$(fg)^{\hat{\Delta}}(t) = f^{\hat{\Delta}}(t) g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) + f\left(\frac{t}{\hat{T}(t)}\right) g^{\hat{\Delta}}(t) = f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) g^{\hat{\Delta}}(t) + f^{\hat{\Delta}}(t) g\left(\frac{t}{\hat{T}(t)}\right).$$

Proof. Since f and g are iso-differentiable at t, then they are continuous at the point $\frac{t}{\hat{T}(t)}$. Therefore

$$\begin{aligned}
&\frac{1}{\hat{T}(t)} \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{s}{\hat{T}(s)}\right) g\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}} \\
&= \frac{1}{\hat{T}(t)} \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}} g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) + f\left(\frac{s}{\hat{T}(s)}\right) \frac{g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - g\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}} \\
&\longrightarrow_{s \longrightarrow t} f^{\hat{\Delta}}(t) g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) + f\left(\frac{t}{\hat{T}(t)}\right) g^{\hat{\Delta}}(t).
\end{aligned}$$

The second part of the proof we left to the reader for exercise. □

Exercise 7.2.19. *Let f, g, $h : \mathbb{T} \longrightarrow \mathbb{R}$ are iso-differentiable at t, $t \in \mathbb{T}^\kappa$. We will find $(fgh)^{\hat{\Delta}}(t)$.*

We have that $fg : \mathbb{T} \longrightarrow \mathbb{R}$ is iso-differentiable at t and from here it follows that $(fg)h : \mathbb{T} \longrightarrow \mathbb{R}$ is iso-differentiable at t. Hence,

$$((fg)h)^{\hat{\Delta}} = (fg)^{\hat{\Delta}} h\left(\tfrac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) + f\left(\tfrac{t}{\hat{T}(t)}\right) g\left(\tfrac{t}{\hat{T}(t)}\right) h^{\hat{\Delta}}(t)$$

$$= f^{\hat{\Delta}}(t) g\left(\tfrac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) h\left(\tfrac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) + f\left(\tfrac{t}{\hat{T}(t)}\right) g^{\hat{\Delta}}(t) h\left(\tfrac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) + f\left(\tfrac{t}{\hat{T}(t)}\right) g\left(\tfrac{t}{\hat{T}(t)}\right) h^{\hat{\Delta}}(t).$$

Exercise 7.2.20. *Let $f\mathbb{T} \longrightarrow \mathbb{R}$ be iso-differentiable at $t \in \mathbb{T}^\kappa$. We will find $(f^2)^{\hat{\Delta}}(t)$.*

We have

$$\begin{aligned}
(f^2)^{\hat{\Delta}}(t) &= (ff)^{\hat{\Delta}}(t) \\
&= f^{\hat{\Delta}}(t) f\left(\tfrac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) + f\left(\tfrac{t}{\hat{T}(t)}\right) f^{\hat{\Delta}}(t) \\
&= f^{\hat{\Delta}}(t)\left(f\left(\tfrac{t}{\hat{T}(t)}\right) + f\left(\tfrac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)\right).
\end{aligned}$$

Exercise 7.2.21. *Let $f : \mathbb{T} \longrightarrow \mathbb{R}$ be iso-differentiable at $t \in \mathbb{T}^\kappa$. We will find $(f^3)^{\hat{\Delta}}(t)$.*

Because $f, f^2 : \mathbb{T} \longrightarrow \mathbb{R}$ are iso-differentiable at t, then $f^3 : \mathbb{T} \longrightarrow \mathbb{R}$ is iso-differentiable at $t \in \mathbb{T}^\kappa$. Therefore

$$\begin{aligned}
(f^3)^{\hat{\Delta}}(t) &= (f^2 f)^{\hat{\Delta}}(t) \\
&= (f^2)^{\hat{\Delta}}(t) f\left(\tfrac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) + f^2\left(\tfrac{t}{\hat{T}(t)}\right) f^{\hat{\Delta}}(t) \\
&= f^{\hat{\Delta}}(t)\left(f\left(\tfrac{t}{\hat{T}(t)}\right) + f\left(\tfrac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)\right) + f^2\left(\tfrac{t}{\hat{T}(t)}\right) f^{\hat{\Delta}}(t) \\
&= f^{\hat{\Delta}}(t)\left(f\left(\tfrac{t}{\hat{T}(t)}\right) + f^2\left(\tfrac{t}{\hat{T}(t)}\right) + f\left(\tfrac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)\right).
\end{aligned}$$

Exercise 7.2.22. *Let $f : \mathbb{T} \longrightarrow \mathbb{R}$ be iso-differentiable at $t \in \mathbb{T}^\kappa$. Prove that*

$$(f^n)^{\hat{\Delta}}(t) = f^{\hat{\Delta}}(t)\left(f\left(\frac{t}{\hat{T}(t)}\right) + \cdots + f^{n-1}\left(\frac{t}{\hat{T}(t)}\right) + f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)\right)$$

for every $n \in \mathbb{N}$, $n \geq 2$.

Theorem 7.2.23. *Let $f : \mathbb{T} \longrightarrow \mathbb{R}$ be iso-differentiable at $t \in \mathbb{T}^\kappa$. Let also,*

$$f\left(\frac{t}{\hat{T}(t)}\right) f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) \neq 0.$$

Then $\frac{1}{f} : \mathbb{T} \longrightarrow \mathbb{R}$ is iso-differentiable at t and

$$\left(\frac{1}{f}\right)^{\hat{\Delta}}(t) = -\frac{1}{f\left(\tfrac{t}{\hat{T}(t)}\right) f\left(\tfrac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)} f^{\hat{\Delta}}(t).$$

Proof. Since f is iso-differentiable at t then f is continuous at $\frac{t}{\hat{T}(t)}$. Then

$$\frac{1}{\hat{T}(t)}\frac{\frac{1}{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)}-\frac{1}{f\left(\frac{s}{\hat{T}(s)}\right)}}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{s}{\hat{T}(s)}} = -\frac{1}{\hat{T}(t)}\frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)-f\left(\frac{s}{\hat{T}(s)}\right)}{f\left(\frac{s}{\hat{T}(s)}\right)f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{s}{\hat{T}(s)}\right)}$$

$$\longrightarrow_{s\longrightarrow t} -\frac{1}{f\left(\frac{t}{\hat{T}(t)}\right)f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)}f^{\hat{\Delta}}(t).$$

□

Theorem 7.2.24. *Let $f,\ g:\mathbb{T}\longrightarrow\mathbb{R}$ are iso-differentiable at $t\in\mathbb{T}^{\kappa}$. Let also,*

$$g\left(\frac{t}{\hat{T}(t)}\right)g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)\neq 0.$$

Then $\frac{f}{g}:\mathbb{T}\longrightarrow\mathbb{R}$ is iso-differentiable at t and

$$\left(\frac{f}{g}\right)^{\hat{\Delta}}(t)=\frac{f^{\hat{\Delta}}(t)g\left(\frac{t}{\hat{T}(t)}\right)-f\left(\frac{t}{\hat{T}(t)}\right)g^{\hat{\Delta}}(t)}{g\left(\frac{t}{\hat{T}(t)}\right)g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)}.$$

Proof. We have that $\frac{1}{g}:\mathbb{T}\longrightarrow\mathbb{R}$ is iso-differentiable at t. From here, $f\frac{1}{g}:\mathbb{T}\longrightarrow\mathbb{R}$ is iso-differentiable at t. Hence,

$$\left(\frac{f}{g}\right)^{\hat{\Delta}}(t)=\left(f\frac{1}{g}\right)^{\hat{\Delta}}(t)$$

$$=f^{\hat{\Delta}}(t)\frac{1}{g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)}+f\left(\frac{t}{\hat{T}(t)}\right)\left(\frac{1}{g}\right)^{\hat{\Delta}}(t)$$

$$=\frac{f^{\hat{\Delta}}(t)}{g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)}-\frac{f\left(\frac{t}{\hat{T}(t)}\right)}{g\left(\frac{t}{\hat{T}(t)}\right)g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)}g^{\hat{\Delta}}(t)$$

$$=\frac{f^{\hat{\Delta}}(t)g\left(\frac{t}{\hat{T}(t)}\right)-f\left(\frac{t}{\hat{T}(t)}\right)g^{\hat{\Delta}}(t)}{g\left(\frac{t}{\hat{T}(t)}\right)g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)}.$$

□

Definition 7.2.25. *For a function $f:\mathbb{T}\longrightarrow\mathbb{R}$ we shall talk about the second iso-derivative $f^{\hat{\Delta}\hat{\Delta}}$ or $f^{\hat{\Delta}^2}$ provided $f^{\hat{\Delta}}$ is iso-differentiable on $\mathbb{T}^{\kappa^2}=(\mathbb{T}^{\kappa})^{\kappa}$ with iso-derivative*

$$f^{\hat{\Delta}^2}=f^{\hat{\Delta}\hat{\Delta}}:=\left(f^{\hat{\Delta}}\right)^{\hat{\Delta}}:\mathbb{T}^{\kappa^2}\longrightarrow\mathbb{R}.$$

Similarly we define higher order iso-derivatives $f^{\hat{\Delta}^n}:\mathbb{T}^{\kappa^n}\longrightarrow\mathbb{R}$, $n\in\mathbb{N}$.

For $t\in\mathbb{T}$ we define

$$\hat{\sigma}^2(t)=\hat{\sigma}(\hat{\sigma}(t)),\qquad \hat{\rho}^2(t)=\hat{\rho}(\hat{\rho}(t)),$$

and $\hat{\sigma}^n(t)$ *and* $\hat{\rho}^n(t)$, $n \in \mathbb{N}$, *are defined accordingly. For convenience, we also put*

$$\hat{\sigma}^0(t) = t, \qquad \hat{\rho}^0(t) = t, \qquad f^{\hat{\Delta}^0} = f, \qquad \mathbb{T}^{\kappa^0} = \mathbb{T}.$$

Example 7.2.26. *Let* $\mathbb{T} = \mathbb{N}$, $\hat{T}(t) = 2t+3$, $t \in \mathbb{T}$. *Firstly we will find* $\hat{\sigma}(t)$ *and* $\hat{\rho}(t)$, $t \geq 2$. *We have*

$$\hat{\sigma}(t) = \inf\left\{s \in \mathbb{T} : \tfrac{s}{\hat{T}(s)} > \tfrac{t}{\hat{T}(t)}\right\} \qquad \Longleftrightarrow$$

$$\hat{\sigma}(t) = \inf\left\{s \in \mathbb{T} : \tfrac{s}{2s+3} > \tfrac{t}{2t+3}\right\}.$$

We consider the inequality.

$$\tfrac{s}{2s+3} > \tfrac{t}{2t+3} \qquad \Longleftrightarrow$$

$$2st + 3s > 2st + 3t \qquad \Longleftrightarrow$$

$$s > t.$$

Therefore $\hat{\sigma}(t) = t+1$.

Also,

$$\hat{\rho}(t) = \sup\left\{s \in \mathbb{T} : \tfrac{s}{\hat{T}(s)} < \tfrac{t}{\hat{T}(t)}\right\} \qquad \Longleftrightarrow$$

$$\hat{\rho}(t) = \sup\left\{s \in \mathbb{T} : \tfrac{s}{2s+3} < \tfrac{t}{2t+3}\right\}.$$

We consider the inequality.

$$\tfrac{s}{2s+3} < \tfrac{t}{2t+3} \qquad \Longleftrightarrow$$

$$2st + 3s < 2st + 3t \qquad \Longleftrightarrow$$

$$s < t.$$

Consequently

$$\hat{\rho}(t) = t-1.$$

Now we will find $\hat{\sigma}^3(t)$ *and* $\hat{\rho}(t)$, $t \geq 3$.

We have

$$\begin{aligned}
\hat{\sigma}^2(t) &= \hat{\sigma}(\hat{\sigma}(t)) \\
&= \hat{\sigma}(t+1) \\
&= t+2, \\
\hat{\sigma}^3(t) &= \hat{\sigma}(\hat{\sigma}^2(t)) \\
&= \hat{\sigma}(t+2) \\
&= t+3,
\end{aligned}$$

and

$$\begin{aligned}\hat{\rho}^2(t) &= \hat{\rho}(\hat{\rho}(t)) \\ &= \hat{\rho}(t-1) \\ &= t-2.\end{aligned}$$

Now we will compute $f^{\hat{\Delta}}(t)$ and $f^{\hat{\Delta}^2}(t)$, $t \geq 2$.

Since

$$\begin{aligned}&\tfrac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} = \tfrac{t+1}{2t+5} > \tfrac{t}{2t+3} \quad \Longleftrightarrow \\ &(t+1)(2t+3) > t(2t+5) \quad \Longleftrightarrow \\ &2t^2+3t+2t+3 > 2t^2+5t \quad \Longleftrightarrow \\ &3 > 0,\end{aligned}$$

we conclude that every point $t \geq 2$, $t \in \mathbb{T}$, is iso-right scattered. From here

$$\begin{aligned}f^{\hat{\Delta}}(t) &= \tfrac{1}{\hat{T}(t)} \tfrac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{t}{\hat{T}(t)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{t}{\hat{T}(t)}} \\ &= \tfrac{1}{2t+3} \tfrac{f\left(\frac{t+1}{2t+5}\right) - f\left(\frac{t}{2t+3}\right)}{\frac{t+1}{2t+5} - \frac{t}{2t+3}} \\ &= \tfrac{1}{2t+3} \tfrac{\frac{t+1}{2t+5} + 2 - \frac{t}{2t+3} - 2}{\frac{t+1}{2t+5} - \frac{t}{2t+3}} \\ &= \tfrac{1}{2t+3},\end{aligned}$$

and

$$\begin{aligned}f^{\hat{\Delta}^2}(t) &= \left(f^{\hat{\Delta}}\right)^{\hat{\Delta}}(t) \\ &= \tfrac{1}{\hat{T}(t)} \tfrac{f^{\hat{\Delta}}\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f^{\hat{\Delta}}\left(\frac{t}{\hat{T}(t)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{t}{\hat{T}(t)}} \qquad (2) \\ &= \tfrac{1}{2t+3} \tfrac{f^{\hat{\Delta}}\left(\frac{t+1}{2t+5}\right) - f^{\hat{\Delta}}\left(\frac{t}{2t+3}\right)}{\frac{t+1}{2t+5} - \frac{t}{2t+3}}.\end{aligned}$$

We note

$$f^{\hat{\Delta}}\left(\frac{t+1}{2t+5}\right) = \frac{1}{2\frac{t+1}{2t+5}+3}$$

$$= \frac{2t+5}{2t+2+6t+15}$$

$$= \frac{2t+5}{8t+17},$$

$$f^{\hat{\Delta}}\left(\frac{t}{2t+3}\right) = \frac{1}{2\frac{t}{2t+3}+3}$$

$$= \frac{2t+3}{2t+6t+9}$$

$$= \frac{2t+3}{8t+9}.$$

From here, using (2), *we get*

$$f^{\hat{\Delta}^2}(t) = \frac{1}{2t+3}\frac{\frac{2t+5}{8t+17}-\frac{2t+3}{8t+9}}{\frac{t+1}{2t+5}-\frac{t}{2t+3}}$$

$$= \frac{2t+5}{(8t+17)(8t+9)}\frac{(2t+5)(8t+9)-(2t+3)(8t+17)}{(t+1)(2t+3)-t(2t+5)}$$

$$= \frac{2t+5}{(8t+17)(8t+9)}\frac{16t^2+18t+40t+45-16t^2-34t-24t-51}{2t^2+3t+2t+3-2t^2-5t}$$

$$= \frac{-4t-10}{(8t+17)(8t+9)}.$$

Example 7.2.27. *Now we will compute* $t^{\hat{\Delta}^2}$ *on arbitrary continuous time-scale* $\mathbb{T}$ *in the case when* t *is iso-right scattered and*

$$\hat{T}(t) - t\hat{T}(\hat{\sigma}(t)) \neq 0.$$

We have

$$t^{\hat{\Delta}^2} = \left(t^{\hat{\Delta}}\right)^{\hat{\Delta}}$$

$$= \left(\frac{1}{\hat{T}(t)}\right)^{\hat{\Delta}}$$

$$= \frac{1}{\hat{T}(t)}\lim_{s\longrightarrow t}\frac{\frac{1}{\hat{T}\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)}-\frac{1}{\hat{T}\left(\frac{s}{\hat{T}(s)}\right)}}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{s}{\hat{T}(s)}}$$

$$= \frac{1}{\hat{T}(t)}\frac{\frac{1}{\hat{T}\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)}-\frac{1}{\hat{T}\left(\frac{t}{\hat{T}(t)}\right)}}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{t}{\hat{T}(t)}}$$

$$= \hat{T}(\hat{\sigma}(t))\frac{\hat{T}\left(\frac{t}{\hat{T}(t)}\right)-\hat{T}\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)}{(\hat{\sigma}(t)\hat{T}(t)-t\hat{T}(\hat{\sigma}(t)))\hat{T}\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)\hat{T}\left(\frac{t}{\hat{T}(t)}\right)}.$$

7.3. Iso-Integration

Definition 7.3.1. *A function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is called iso-regulated provided its right-sided limits exist(finite)at all iso-right-dense points in $\mathbb{T}$ and its left-sided limits exist(finite)at all iso-left-dense points in $\mathbb{T}$.*

Theorem 7.3.2. *Every iso-regulated function on a compact interval is bounded.*

Proof. Let us suppose that $f : [a,b] \longrightarrow \mathbb{R}$ is iso-regulated and unbounded. Then there exists a sequence $\{t_k\}_{k=1}^{\infty}$ such that

$$|f(t_n)| > n. \tag{3}$$

Since $\{t_n\}_{n=1}^{\infty} \subset [a,b]$, then there exists a convergent subsequence $\{t_{n_k}\}_{k=1}^{\infty}$, i.e.,

$$\lim_{k \longrightarrow \infty} t_{n_k} = t_0 \tag{4}$$

for some $t_0 \in [a,b]$. Because $\{t_{n_k}\}_{k=1}^{\infty} \subset \mathbb{T}$ and $\mathbb{T}$ is closed, then we have that $t_0 \in \mathbb{T}$. Since (4), we conclude that t_0 is not isolated point. Therefore there exists either a subsequence that tends to t_0 from above or a subsequence that tends to t_0 from below. Let us assume that there exists a subsequence $\{t_{n_k}\}_{k=1}^{\infty}$ which tends to t_0 from above. Because f is iso-regulated we have that

$$\lim_{k \longrightarrow \infty} f(t_{n_k}) = f(t_0)$$

and $f(t_0)$ is finite. On the other hand, using (3), we get

$$|f(t_{n_k})| > n_k,$$

which is a contradiction. Therefore the function f is bounded. □

Definition 7.3.3. *A function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is called iso-rd-continuous provided it is continuous at iso-right-dense points in $\mathbb{T}$ and its left-sided limits exist(finite) at iso-left-dense points in $\mathbb{T}$. The set of iso-rd-continuous functions $f : \mathbb{T} \longrightarrow \mathbb{R}$ will be denoted with*

$$\hat{C}_{rd} = \hat{C}_{rd}(\mathbb{T}) = \hat{C}_{rd}(\mathbb{T}, \mathbb{R}).$$

The set of functions $f : \mathbb{T} \longrightarrow \mathbb{R}$ that are iso-differentiable and whose iso-derivatives are iso-rd-continuous is denoted by

$$\hat{C}^1_{rd} = \hat{C}^1_{rd}(\mathbb{T}) = \hat{C}^1_{rd}(\mathbb{T}, \mathbb{R}).$$

Exercise 7.3.4. *Let $f : \mathbb{T} \longrightarrow \mathbb{R}$ is continuous. Prove that f is iso-rd-continuous.*

Exercise 7.3.5. *Let $f : \mathbb{T} \longrightarrow \mathbb{R}$ is iso-rd-continuous. Prove that f is iso-regulated.*

Definition 7.3.6. *A continuous function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is called iso-pre-differentiable with region of iso-differentiation D, provided $D \subset \mathbb{T}^{\kappa}$, $\mathbb{T}^{\kappa} \backslash D$ is countable and contains no iso-right-scattered elements of $\mathbb{T}$, and f is iso-differentiable at each $t \in D$.*

Theorem 7.3.7. *(mean value theorem) Let f and g be real-valued functions defined on $\mathbb{T}$, both iso-pre-differentiable with D. Then the inequality*

$$\left|f^{\hat{\Delta}}(t)\right| \leq g^{\hat{\Delta}}(t) \qquad \text{for} \qquad \forall t \in D \tag{5}$$

implies

$$\left|f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{t}{\hat{T}(t)}\right)\right| \leq \left|g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - g\left(\frac{t}{\hat{T}(t)}\right)\right|$$

for every $t \in D$.

Proof. Let $\varepsilon > 0$ and $t \in D$ be arbitrarily chosen. Then there exists a neighborhood U of the point t such that

$$\left|\frac{1}{\hat{T}(t)} \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}} - f^{\hat{\Delta}}(t)\right| < \varepsilon, \tag{6}$$

$$\left|\frac{1}{\hat{T}(t)} \frac{g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - g\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}} - g^{\hat{\Delta}}(t)\right| < \varepsilon \tag{7}$$

for all $s \in U$.

Now we consider the inequality (6). We have

$$\left|\frac{1}{\hat{T}(t)} \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}}\right| - \left|f^{\hat{\Delta}}(t)\right| < \varepsilon$$

for every $s \in U$, from where

$$\left|\frac{1}{\hat{T}(t)} \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}}\right| - \varepsilon < \left|f^{\hat{\Delta}}(t)\right| \tag{8}$$

for any $s \in U$.

From the inequality (7) we get

$$\left|g^{\hat{\Delta}}(t)\right| - \left|\frac{1}{\hat{T}(t)} \frac{g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - g\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}}\right| < \varepsilon$$

for every $s \in U$, whereupon

$$\left|g^{\hat{\Delta}}(t)\right| < \varepsilon + \left|\frac{1}{\hat{T}(t)} \frac{g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - g\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}}\right|$$

for $\forall s \in U$.

From the last inequality, (5) and (8), we find

$$\left| \frac{1}{\hat{T}(t)} \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}} \right| - \varepsilon < \left| f^{\hat{\Delta}}(t) \right| \leq g^{\hat{\Delta}}(t)$$

$$\leq \left| g^{\hat{\Delta}}(t) \right| < \varepsilon + \left| \frac{1}{\hat{T}(t)} \frac{g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - g\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}} \right|$$

for every $s \in U$. Hence, when $s \longrightarrow t$,

$$\left| \frac{1}{\hat{T}(t)} \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{t}{\hat{T}(t)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{t}{\hat{T}(t)}} \right| \leq \left| \frac{1}{\hat{T}(t)} \frac{g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - g\left(\frac{t}{\hat{T}(t)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{t}{\hat{T}(t)}} \right|,$$

from where

$$\left| f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{t}{\hat{T}(t)}\right) \right| \leq \left| g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - g\left(\frac{t}{\hat{T}(t)}\right) \right|.$$

□

Definition 7.3.8. *Assume that f is iso-regulated. Any function F which is iso-pre-differentiable with region of iso-differentiation D such that*

$$F^{\hat{\Delta}}(t) = f(t)$$

holds for all $t \in D$, is called iso-pre-antiderivative of f.

We define indefinite iso-integral of an iso-regulated function f by

$$\hat{\int} f(t)\hat{\Delta}t = F(t) + C,$$

where C is an arbitrary constant and F is an iso-pre-antiderivative of f.

We define the iso-Cauchy integral by

$$\hat{\int}_r^s f(t)\hat{\Delta}t = F(s) - F(r)$$

for all r, $s \in \mathbb{T}$.

Example 7.3.9. *We will find*

$$\hat{\int} \frac{1}{\hat{T}(t)} \hat{\Delta}t.$$

Since

$$(t)^{\hat{\Delta}} = \frac{1}{\hat{T}(t)},$$

we get

$$\hat{\int} \frac{1}{\hat{T}(t)} \hat{\Delta}t = t^2 + C,$$

where C is arbitrary constant.

Example 7.3.10. *We will find*

$$\hat{\int} \frac{1}{\hat{T}(t)} \left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} + \frac{t}{\hat{T}(t)} \right) \hat{\Delta} t.$$

Because

$$\left(t^2\right)^{\hat{\Delta}} = \frac{1}{\hat{T}(t)} \left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} + \frac{t}{\hat{T}(t)} \right),$$

we conclude that

$$\hat{\int} \frac{1}{\hat{T}(t)} \left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} + \frac{t}{\hat{T}(t)} \right) \hat{\Delta} t = t^2 + C,$$

where C is arbitrary constant.

Example 7.3.11. *Let $\mathbb{T} = \{7n+3 : n \in \mathbb{N}\}$, $\hat{T}(t) = t+1$, $f(t) = t+2$, $g(t) = t^2+4$, $t \in \mathbb{T}$.*

Firstly, we will find $\hat{\sigma}(t)$. We have

$$\hat{\sigma}(t) = \inf\left\{ s \in \mathbb{T} : \tfrac{s}{\hat{T}(s)} > \tfrac{t}{\hat{T}(t)} \right\} \qquad \Longleftrightarrow$$

$$\hat{\sigma}(t) = \inf\left\{ s \in \mathbb{T} : \tfrac{s}{s+1} > \tfrac{t}{t+1} \right\}.$$

We consider the inequality.

$$\tfrac{s}{s+1} > \tfrac{t}{t+1} \qquad \Longleftrightarrow$$

$$s(t+1) > t(s+1) \qquad \Longleftrightarrow$$

$$s > t.$$

Consequently

$$\hat{\sigma}(t) = t+7.$$

Therefore

$$\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} = \frac{t+7}{\hat{T}(t+7)} = \frac{t+7}{t+8}.$$

Since

$$\tfrac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} > \tfrac{t}{\hat{T}(t)} \qquad \Longleftrightarrow$$

$$\tfrac{t+7}{t+8} > \tfrac{t}{t+1} \qquad \Longleftrightarrow$$

$$(t+7)(t+1) > t(t+8) \qquad \longrightarrow$$

$$7 > 0,$$

we conclude that all points of the time scale $\mathbb{T}$ are iso-right-scattered.

Now we will find

$$f^{\hat{\Delta}^2}(t), \qquad g^{\hat{\Delta}}(t).$$

Because all points of $\mathbb{T}$ are iso-right-scattered, we have

$$f^{\hat{\Delta}}(t) = \frac{1}{\hat{T}(t)} \frac{f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - f\left(\frac{t}{\hat{T}(t)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{t}{\hat{T}(t)}}$$

$$= \frac{1}{t+1} \frac{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} + 2 - \frac{t}{\hat{T}(t)} - 2}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{t}{\hat{T}(t)}}$$

$$= \frac{1}{t+1} \frac{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{t}{\hat{T}(t)}}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{t}{\hat{T}(t)}}$$

$$= \frac{1}{t+1}.$$

Hence,

$$f^{\hat{\Delta}^2}(t) = \left(f^{\hat{\Delta}}\right)^{\hat{\Delta}}(t)$$

$$= \left(\frac{1}{t+1}\right)^{\hat{\Delta}}$$

$$= \frac{1}{\hat{T}(t)} \frac{\frac{1}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}+1} - \frac{1}{\frac{t}{\hat{T}(t)}+1}}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{t}{\hat{T}(t)}}$$

$$= \frac{1}{t+1} \frac{\frac{1}{\frac{t+7}{t+8}+1} - \frac{1}{\frac{t}{t+1}+1}}{\frac{t+7}{t+8} - \frac{t}{t+1}}$$

$$= \frac{1}{t+1} \frac{\frac{t+8}{2t+15} - \frac{t+1}{2t+1}}{\frac{(t+7)(t+1)-t(t+8)}{(t+8)(t+1)}}$$

$$= (t+8) \frac{(t+8)(2t+1)-(t+1)(2t+15)}{(2t+15)(2t+1)(t^2+8t+7-t^2-8t)}$$

$$= \frac{t+8}{7} \frac{2t^2+t+16t+8-(2t^2+2t+15t+15)}{(2t+1)(2t+15)}$$

$$= \frac{t+8}{7} \frac{-7}{(2t+1)(2t+15)}$$

$$= -\frac{t+8}{(2t+1)(2t+15)}.$$

Also,

$$g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) = g\left(\frac{t+7}{t+8}\right)$$

$$= \left(\frac{t+7}{t+8}\right)^2 + 4$$

$$= \frac{(t+7)^2+4(t+8)^2}{(t+8)^2}$$

$$= \frac{t^2+14t+49+4(t^2+16t+64)}{(t+8)^2}$$

$$= \frac{5t^2+78t+305}{(t+8)^2},$$

$$g\left(\frac{t}{\hat{T}(t)}\right) = g\left(\frac{t}{t+1}\right)$$

$$= \left(\frac{t}{t+1}\right)^2 + 4$$

$$= \frac{t^2+4(t+1)^2}{(t+1)^2}$$

$$= \frac{t^2+4(t^2+2t+1)}{(t+1)^2}$$

$$= \frac{5t^2+8t+4}{(t+1)^2}.$$

Then

$$g^{\hat{\Delta}}(t) = \frac{1}{\hat{T}(t)} \frac{g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - g\left(\frac{t}{\hat{T}(t)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{t}{\hat{T}(t)}}$$

$$= \frac{1}{t+1} \frac{\frac{5t^2+78t+305}{(t+8)^2} - \frac{5t^2+8t+4}{(t+1)^2}}{\frac{t+7}{t+8} - \frac{t}{t+1}}$$

$$= \frac{(5t^2+78t+305)(t+1)^2 - (5t^2+8t+4)(t+8)^2}{7(t+8)(t+1)^2}$$

$$= \frac{(5t^2+78t+305)(t^2+2t+1) - (5t^2+8t+4)(t^2+16t+64)}{7(t+8)(t+1)^2}$$

$$= \frac{5t^4+10t^3+5t^2+78t^3+156t^2+78t+305t^2+610t+305}{7(t+8)(t+1)^2}$$

$$- \frac{5t^4+80t^3+320t^2+8t^3+128t^2+512t+4t^2+64t+256}{7(t+8)(t+1)^2}$$

$$= \frac{14t^2+112t+49}{7(t+8)(t+1)^2}$$

$$= \frac{2t^2+16t+7}{(t+8)(t+1)^2}.$$

From here,

$$\hat{\int} \frac{1}{t+1} \hat{\Delta} t = t + C,$$

$$\hat{\int} \frac{2t^2+16t+7}{(t+8)(t+1)^2} \hat{\Delta} t = t^2 + C.$$

Theorem 7.3.12. *If $f \in \hat{C}_{rd}$ and $t \in \mathbb{T}^{\kappa}$, then*

$$\hat{\int}_{\frac{t}{\hat{T}(t)}}^{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}} f(\tau)\hat{\Delta}\tau = \left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{t}{\hat{T}(t)}\right)\hat{T}(t) f(t).$$

Proof. Let F be an iso-pre-antiderivative of f. Then

$$F^{\hat{\Delta}}(t) = f(t)$$

for all $t \in \mathbb{T}^{\kappa}$. Also,

$$\hat{\int}_{\frac{t}{\hat{T}(t)}}^{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}} f(\tau)\hat{\Delta}\tau = F\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - F\left(\frac{t}{\hat{T}(t)}\right)$$

$$= \left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{t}{\hat{T}(t)}\right)\hat{T}(t)\frac{F\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)-F\left(\frac{t}{\hat{T}(t)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{t}{\hat{T}(t)}}$$

$$= \left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{t}{\hat{T}(t)}\right)\hat{T}(t)F^{\hat{\Delta}}(t)$$

$$= \left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{t}{\hat{T}(t)}\right)\hat{T}(t)f(t).$$

□

Example 7.3.13. *Let* $\mathbb{T} = \{9n+5 : n \in \mathbb{N}\}$, $\hat{T}(t) = 3t+5$, $f(t) = t^2+2t+3$, $t \in \mathbb{T}$.
Firstly, we will find $\hat{\sigma}(t)$. *We have*

$$\hat{\sigma}(t) = \inf\left\{s \in \mathbb{T} : \frac{s}{\hat{T}(s)} > \frac{t}{\hat{T}(t)}\right\} \qquad \Longleftrightarrow$$

$$\hat{\sigma}(t) = \inf\left\{s \in \mathbb{T} : \frac{s}{\hat{T}(s)} > \frac{t}{\hat{T}(t)}\right\}.$$

We consider the inequality.

$$\frac{s}{3s+5} > \frac{t}{3t+5} \qquad \Longleftrightarrow$$

$$s(3t+5) > t(3s+5) \qquad \Longleftrightarrow$$

$$3st+5s > 3st+5t \qquad \Longleftrightarrow$$

$$s > t.$$

Consequently

$$\hat{\sigma}(t) = t+9.$$

Then

$$\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} = \frac{t+9}{\hat{T}(t+9)}$$

$$= \frac{t+9}{3(t+9)+5}$$

$$= \frac{t+9}{3t+32},$$

$$\frac{t}{\hat{T}(t)} = \frac{t}{3t+5}.$$

Hence,

$$\hat{\int}_{\frac{t}{3t+5}}^{\frac{t+9}{3t+32}}(t^2+2t+3)\hat{\Delta}(t) = \left(\frac{t+9}{3t+32}-\frac{t}{3t+5}\right)(t^2+2t+3)(3t+5)$$

$$= \frac{(t+9)(3t+5)-t(3t+32)}{3t+32}(t^2+2t+3)$$

$$= \frac{3t^2+5t+27t+45-3t^2-32t}{3t+32}(t^2+2t+3)$$

$$= \frac{45(t^2+2t+3)}{3t+32}$$

$$= \frac{45t^2+90t+135}{3t+32}.$$

Remark 7.3.14. *We note that if $f \in \hat{C}_{rd}$, then*

$$\hat{\int} f^{\hat{\Delta}}(t)\hat{\Delta}t = f(t),$$

$$\hat{\int}_s^t f^{\hat{\Delta}}(\tau)\hat{\Delta}\tau = f(t)-f(s) \qquad \text{for} \qquad \forall s,t \in \mathbb{T}.$$

Theorem 7.3.15. *If $a,b \in \mathbb{T}$, and $f,\ g \in \hat{C}_{rd}$, then*

$$\hat{\int}_a^b (f(t)+g(t))\hat{\Delta}t = \hat{\int}_a^b f(t)\hat{\Delta}t + \hat{\int}_a^b g(t)\hat{\Delta}t.$$

Proof. Let F and G are iso-pre-antiderivatives of f and g, respectively. Then

$$\hat{\int}_a^b f(t)\hat{\Delta}t = F(b)-F(a),$$

$$\hat{\int}_a^b g(t)\hat{\Delta}t = G(b)-G(a).$$

Also,

$$(F+G)^{\hat{\Delta}}(t) = F^{\hat{\Delta}}(t)+G^{\hat{\Delta}}(t)$$

$$= f(t)+g(t).$$

Consequently $F+G$ is an iso-pre-antiderivative of $f+g$.

Hence,

$$\hat{\int}_a^b (f(t)+g(t))\hat{\Delta}t = (F+g)(t)\Big|_{t=a}^{t=b}$$

$$= (F+G)(b)-(F+G)(a)$$

$$= F(b)+G(b)-F(a)-G(a)$$

$$= F(b)-F(a)+G(b)-G(a)$$

$$= \hat{\int}_a^b f(t)\hat{\Delta}t + \hat{\int}_a^b g(t)\hat{\Delta}t.$$

$\square$

Theorem 7.3.16. *Let a, $b \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and $f \in \hat{C}_{rd}$, then*

$$\hat{\int}_a^b (\alpha f)(t)\hat{\Delta}t = \alpha \hat{\int}_a^b f(t)\hat{\Delta}t.$$

Proof. Let F be an iso-pre-antiderivative of f. Then

$$F^{\hat{\Delta}}(t) = f(t) \tag{9}$$

and

$$\hat{\int}_a^b f(t)\hat{\Delta}t = F(b) - F(a).$$

From the properties of the iso-derivative, we have

$$\alpha F^{\hat{\Delta}}(t) = (\alpha F)^{\hat{\Delta}}(t).$$

From here and (9), we get

$$(\alpha F)^{\hat{\Delta}}(t) = \alpha f(t),$$

i.e., αF is an iso-pre-antiderivative of αf. Consequently

$$\begin{aligned}
&\hat{\int}_a^b (\alpha f)(t)\hat{\Delta}t = (\alpha F)(t)\Big|_{t=a}^{t=b} \\
&= \alpha F(b) - \alpha F(a) \\
&= \alpha(F(b) - F(a)) \\
&= \alpha \hat{\int}_a^b f(t)\hat{\Delta}t.
\end{aligned}$$

□

Theorem 7.3.17. *Let a, $b \in \mathbb{T}$, and $f \in \hat{C}_{rd}$, then*

$$\hat{\int}_a^b f(t)\hat{\Delta}t = -\hat{\int}_b^a f(t)\hat{\Delta}t.$$

Proof. Let F be an iso-pre-antiderivative of f. Then, using the definition for iso-pre-antiderivative, we get

$$\begin{aligned}
&\hat{\int}_a^b f(t)\hat{\Delta}t = F(b) - F(a) \\
&= -(F(a) - F(b)) \\
&= -\hat{\int}_b f(t)\hat{\Delta}t.
\end{aligned}$$

□

Theorem 7.3.18. *Let a, b, $c \in \mathbb{T}$, and $f \in \hat{C}_{rd}$. Then*

$$\hat{\int}_a^b f(t)\hat{\Delta}t = \hat{\int}_a^c f(t)\hat{\Delta}t + \hat{\int}_c^b f(t)\hat{\Delta}t.$$

Proof. Let F be an iso-pre-antiderivative of f. Then

$$\hat{\int}_a^b f(t)\hat{\Delta}t = F(b) - F(a)$$

$$= F(b) - F(c) + F(c) - F(a)$$

$$= \hat{\int}_c^b f(t)\hat{\Delta}t + \hat{\int}_a^c f(t)\hat{\Delta}t.$$

□

Theorem 7.3.19. *If $a \in \mathbb{T}$ and $f \in \hat{\mathcal{C}}_{rd}$, then*

$$\hat{\int}_a f(t)\hat{\Delta}t = 0.$$

Proof. Let F be an iso-pre-antiderivative of f. Then

$$\hat{\int}_a f(t)\hat{\Delta}t = F(a) - F(a)$$

$$= 0.$$

□

Theorem 7.3.20. *Let a, $b \in \mathbb{T}$, and f, $g \in \hat{\mathcal{C}}_{rd}$. Then*

$$\hat{\int}_a^b f^{\hat{\Delta}}(t) g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)\hat{\Delta}t = (fg)(b) - (fg)(a) - \hat{\int}_a^b f\left(\frac{t}{\hat{T}(t)}\right) g^{\hat{\Delta}}(t)\hat{\Delta}t.$$

Proof. Since

$$(fg)^{\hat{\Delta}}(t) = f^{\hat{\Delta}}(t) g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) + f\left(\frac{t}{\hat{T}(t)}\right) g^{\hat{\Delta}}(t),$$

we get

$$\hat{\int}_a^b \left(f^{\hat{\Delta}}(t) g\left(\tfrac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) + f\left(\tfrac{t}{\hat{T}(t)}\right) g^{\hat{\Delta}}(t)\right)\hat{\Delta}t = \hat{\int}_a^b (fg)^{\hat{\Delta}}(t)\hat{\Delta}t$$

$$= (fg)(t)\Big|_{t=a}^{t=b}$$

$$= (fg)(b) - (fg)(a).$$

Hence,

$$\hat{\int}_a^b f^{\hat{\Delta}}(t) g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)\hat{\Delta}t + \hat{\int}_a^b f\left(\frac{t}{\hat{T}(t)}\right) g^{\hat{\Delta}}(t)\hat{\Delta}t = (fg)(b) - (fg)(a),$$

whereupon we get the desired result. □

Theorem 7.3.21. *Let a, $b \in \mathbb{T}$, and f, $g \in \hat{\mathcal{C}}_{rd}$. Then*

$$\hat{\int}_a^b f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) g^{\hat{\Delta}}(t)\hat{\Delta}t = (fg)(b) - (fg)(a) - \hat{\int}_a^b f^{\hat{\Delta}}(t) g\left(\frac{t}{\hat{T}(t)}\right)\hat{\Delta}t.$$

Proof. Since

$$(fg)^{\hat{\Delta}}(t) = f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) g^{\hat{\Delta}}(t) + f^{\hat{\Delta}}(t) g\left(\frac{t}{\hat{T}(t)}\right),$$

we get

$$\begin{aligned} \hat{\int}_a^b \left(f\left(\tfrac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) g^{\hat{\Delta}}(t) + f^{\hat{\Delta}}(t) g\left(\tfrac{t}{\hat{T}(t)}\right)\right)\hat{\Delta}t &= \hat{\int}_a^b (fg)^{\hat{\Delta}}(t)\hat{\Delta}t \\ &= (fg)(t)\Big|_{t=a}^{t=b} \\ &= (fg)(b) - (fg)(a). \end{aligned}$$

Hence,

$$\hat{\int}_a^b f\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) g^{\hat{\Delta}}(t)\hat{\Delta}t + \hat{\int}_a^b f^{\hat{\Delta}}(t) g\left(\frac{t}{\hat{T}(t)}\right)\hat{\Delta}t = (fg)(b) - (fg)(a),$$

from where we get the desired result. $\square$

Theorem 7.3.22. *Let a, $b \in \mathbb{T}$, f, $g \in \hat{C}_{rd}$, and*

$$|f(t)| \le g(t) \qquad \text{on} \qquad [a,b).$$

Then

$$\left|\int_{\frac{t}{\hat{T}(t)}}^{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}} f(\tau)\hat{\Delta}\tau\right| \le \left|\int_{\frac{t}{\hat{T}(t)}}^{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}} g(\tau)\hat{\Delta}(\tau)\right|$$

for every $t \in [a,b)$.

Proof. Let F be an iso-pre-antiderivative of f and G be an iso-pre-antiderivative of g. Then

$$\left|F^{\hat{\Delta}}(t)\right| \le G^{\hat{\Delta}}(t) \qquad \text{on} \qquad t \in [a,b).$$

From the last inequality and from the mean value theorem, it follows

$$\left|F\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - F\left(\frac{t}{\hat{T}(t)}\right)\right| \le \left|G\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - G\left(\frac{t}{\hat{T}(t)}\right)\right|,$$

whereupon

$$\left|\int_{\frac{t}{\hat{T}(t)}}^{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}} f(\tau)\hat{\Delta}\tau\right| \le \left|\int_{\frac{t}{\hat{T}(t)}}^{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}} g(\tau)\hat{\Delta}(\tau)\right|$$

for every $t \in [a,b)$. $\square$

Definition 7.3.23. *If $a \in \mathbb{T}$, $\sup \mathbb{T} = \infty$, and f is iso-rd-continuous on $[a,\infty)$, then we define the improper iso-integral by*

$$\hat{\int}_a^\infty f(t)\hat{\Delta}t := \lim_{b \longrightarrow \infty} \hat{\int}_a^b f(t)\hat{\delta}t$$

provided the limit exists, and we say that the improper iso-integral converges in this case. If this limit does not exist, then we say that the improper iso-integral diverges.

Exercise 7.3.24. *Evaluate the iso-integral*

$$\hat{\int}_a^\infty \frac{1}{t^2}\hat{\Delta}t,$$

when $\mathbb{T} = \mathbb{N}$.

Theorem 7.3.25. *(chain rule) Assume* $g : \mathbb{R} \longrightarrow \mathbb{R}$ *is continuous,* $g : \mathbb{T} \longrightarrow \mathbb{R}$ *is iso-differentiable on* $\mathbb{T}^\kappa$*, and* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *is continuously differentiable. Then there exists c in the real interval* $\left[\frac{t}{\hat{T}(t)}, \frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right]$ *with*

$$(f\circ g)^{\hat{\Delta}}(t) = f'(g(c))g^{\hat{\Delta}}(t). \tag{10}$$

Proof. Let us fix $t \in \mathbb{T}^\kappa$. Firstly, we consider the case when t is iso-right-scattered. In this case

$$(f\circ g)^{\hat{\Delta}}(t) = \frac{1}{\hat{T}(t)} \frac{f\left(g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)\right) - f\left(g\left(\frac{t}{\hat{T}(t)}\right)\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{t}{\hat{T}(t)}}.$$

If

$$g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) = g\left(\frac{t}{\hat{T}(t)}\right),$$

then we get that

$$(f\circ g)^{\hat{\Delta}}(t) = 0 \qquad \text{and} \qquad g^{\hat{\Delta}}(t) = 0,$$

and so (10) holds for any c in the interval $\left[\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}, \frac{t}{\hat{T}(t)}\right]$.

Hence, we can assume that

$$g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) \neq g\left(\frac{t}{\hat{T}(t)}\right).$$

Then

$$\begin{aligned}(f\cdot g)^{\hat{\Delta}}(t) &= \frac{1}{\hat{T}(t)} \frac{f\left(g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)\right) - f\left(g\left(\frac{t}{\hat{T}(t)}\right)\right)}{g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - g\left(\frac{t}{\hat{T}(t)}\right)} \frac{g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - g\left(\frac{t}{\hat{T}(t)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{t}{\hat{T}(t)}} \\ &= f'(\xi)g^{\hat{\Delta}}(t),\end{aligned}$$

where ξ is between $g\left(\frac{t}{\hat{T}(t)}\right)$ and $g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)$. Since $g : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, then there exists $c \in \left[\frac{t}{\hat{T}(t)}, \frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right]$ such that $g(c) = \xi$, which gives the desired result.

Let now t is iso-right-dense. In this case

$$\begin{aligned}(f\cdot g)^{\hat{\Delta}}(t) &= \frac{1}{\hat{T}(t)} \lim_{s\longrightarrow t} \frac{f\left(g\left(\frac{t}{\hat{T}(t)}\right)\right) - f\left(g\left(\frac{s}{\hat{T}(s)}\right)\right)}{\frac{t}{\hat{T}(t)} - \frac{s}{\hat{T}(s)}} \\ &= \frac{1}{\hat{T}(t)} \lim_{s\longrightarrow t} \frac{f\left(g\left(\frac{t}{\hat{T}(t)}\right)\right) - f\left(g\left(\frac{s}{\hat{T}(s)}\right)\right)}{g\left(\frac{t}{\hat{T}(t)}\right) - g\left(\frac{s}{\hat{T}(s)}\right)} \frac{g\left(\frac{t}{\hat{T}(t)}\right) - g\left(\frac{s}{\hat{T}(s)}\right)}{\frac{t}{\hat{T}(t)} - \frac{s}{\hat{T}(s)}} \qquad (11) \\ &= \frac{1}{\hat{T}(t)} \lim_{s\longrightarrow t} f'(\xi_s) \frac{g\left(\frac{t}{\hat{T}(t)}\right) - g\left(\frac{s}{P\hat{T}(s)}\right)}{\frac{t}{\hat{T}(t)} - \frac{s}{\hat{T}(s)}},\end{aligned}$$

where ξ_s is between $g\left(\frac{s}{\hat{T}(s)}\right)$ and $g\left(\frac{t}{\hat{T}(t)}\right)$. By the continuity of g we get that

$$\lim_{s\longrightarrow t}\xi_s = g(t).$$

Hence, using (11), we get

$$(f\circ g)^{\hat{\Delta}}(t) = f'(g(t))g^{\hat{\Delta}}(t).$$

□

Theorem 7.3.26. *(chain rule) Let $\hat{T}\in\mathcal{C}(\mathbb{T})$, $f:\mathbb{R}\longrightarrow\mathbb{R}$ be continuously differentiable and suppose $g:\mathbb{T}\longrightarrow\mathbb{R}$ is iso-differentiable. Then $f\circ g:\mathbb{T}\longrightarrow\mathbb{R}$ is iso-differentiable and the formula*

$$(f\circ g)^{\hat{\Delta}}(t) = g^{\hat{\Delta}}(t)\int_0^1 f'\left(g\left(\frac{t}{\hat{T}(t)}\right)+h\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{t}{\hat{T}(t)}\right)g^{\hat{\Delta}}(t)\right)dh$$

holds.

Proof. First of all we note that

$$f\left(g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)\right)-f\left(g\left(\frac{s}{\hat{T}(s)}\right)\right)=\int_{g\left(\frac{s}{\hat{T}(s)}\right)}^{g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)} f'(\tau)d\tau$$

$$=\left(g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)-g\left(\frac{s}{\hat{T}(s)}\right)\right)\int_0^1 f'\left(hg\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)+(1-h)g\left(\frac{s}{\hat{T}(s)}\right)\right)dh.$$

Let $t\in\mathbb{T}^\kappa$ and $\varepsilon>0$ be given. Since g is iso-differentiable at t, there exists a neighborhood U_1 of t such that

$$\left|\frac{g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)-g\left(\frac{s}{\hat{T}(s)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}-\frac{s}{\hat{T}(s)}}-g^{\hat{\Delta}}(t)\right|<\varepsilon^*$$

for all $s\in U_1$, where

$$\varepsilon^* = \frac{\varepsilon}{2\left(1+\int_0^1|f'(\alpha)|dh+\int_0^1|f'(\beta)|dh\right)},$$

$$\alpha = hg\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)+(1-h)g\left(\frac{s}{\hat{T}(s)}\right),$$

$$\beta = hg\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)+(1-h)g\left(\frac{t}{\hat{T}(t)}\right).$$

Moreover, f' is continuous on $\mathbb{R}$, and therefore it is uniformly continuous on closed subsets of $\mathbb{R}$, and g is continuous as it is iso-differentiable. Hence, there exists a neighborhood U_2 of t such that

$$\left|f'\left(hg\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)+(1-h)g\left(\frac{s}{\hat{T}(s)}\right)\right)-f'\left(hg\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right)+(1-h)g\left(\frac{t}{\hat{T}(t)}\right)\right)\right|$$

$$\leq\frac{\varepsilon}{2(\varepsilon^*+|g^{\hat{\Delta}}(t)|)}$$

or

$$|f'(\alpha)-f'(\beta)|\leq\frac{\varepsilon}{2(\varepsilon^*+|g^{\hat{\Delta}}(t)|)}$$

for all $s \in U_2$.

We then define $U = U_1 \bigcap U_2$ and let $s \in U$. Then we have

$$\left|(f\circ g)\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - (f\circ g)\left(\frac{s}{\hat{T}(s)}\right) - \left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}\right) g^{\hat{\Delta}}(t)\int_0^1 f'(\beta)dh\right|$$

$$= \left|\left(g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - g\left(\frac{s}{\hat{T}(s)}\right)\right)\int_0^1 f'(\alpha)dh - \left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}\right) g^{\hat{\Delta}}(t)\int_0^1 f'(\beta)dh\right|$$

$$= \left|\left(g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - g\left(\frac{s}{\hat{T}(s)}\right) - \left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}\right) g^{\hat{\Delta}}(t)\right)\int_0^1 f'(\alpha)dh\right.$$

$$\left. + \left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}\right) g^{\hat{\Delta}}(t)\int_0^1 \left(f'(\alpha) - f'(\beta)\right)dh\right|$$

$$\leq \left|\left(g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - g\left(\frac{s}{\hat{T}(s)}\right) - \left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}\right) g^{\hat{\Delta}}(t)\right)\right|\int_0^1 |f'(\alpha)|dh$$

$$+ \left|\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}\right|\left|g^{\hat{\Delta}}(t)\right|\int_0^1 |f'(\alpha) - f'(\beta)|\,dh$$

$$\leq \varepsilon^*\left|\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}\right|\int_0^1 |f'(\alpha)|dh + \left|\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}\right|\left|g^{\hat{\Delta}}(t)\right|\int_0^1 |f'(\alpha) - f'(\beta)|\,dh$$

$$\leq \varepsilon^*\left|\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}\right|\int_0^1 |f'(\alpha)|dh + \left|\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}\right|\left(\varepsilon^* + \left|g^{\hat{\Delta}}(t)\right|\right)\int_0^1 |f'(\alpha) - f'(\beta)|\,dh$$

$$\leq \varepsilon^*\left|\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}\right|\int_0^1 |f'(\alpha)|dh + \frac{\varepsilon}{2}\left|\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}\right|$$

$$< \frac{\varepsilon}{2}\left|\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}\right| + \frac{\varepsilon}{2}\left|\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}\right|$$

$$= \varepsilon\left|\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{s}{\hat{T}(s)}\right|.$$

Therefore $f\circ g$ is iso-differentiable at t and its iso-derivative is

$$(f\circ g)^{\hat{\Delta}}(t) = g^{\hat{\Delta}}(t)\int_0^1 f'(\beta)dh$$

$$= g^{\hat{\Delta}}(t)\int_0^1 f'\left(hg\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) + (1-h)g\left(\frac{t}{\hat{T}(t)}\right)\right)dh$$

$$= g^{\hat{\Delta}}(t)\int_0^1 f'\left(g\left(\frac{t}{\hat{T}(t)}\right) + h\frac{g\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))}\right) - g\left(\frac{t}{\hat{T}(t)}\right)}{\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{t}{\hat{T}(t)}}\right)\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{t}{\hat{T}(t)}\right)dh$$

$$= g^{\hat{\Delta}}(t)\int_0^1 f'\left(g\left(\frac{t}{\hat{T}(t)}\right) + hg^{\hat{\Delta}}(t)\left(\frac{\hat{\sigma}(t)}{\hat{T}(\hat{\sigma}(t))} - \frac{t}{\hat{T}(t)}\right)\right)dh.$$

□

Let $\nu : \mathbb{T} \longrightarrow \mathbb{R}$ be a strictly increasing function such that $\tilde{\mathbb{T}} = \nu(\mathbb{T})$ is also time scale. By $\tilde{\hat{\sigma}}$ we denote the iso-jump function on $\tilde{\mathbb{T}}$, and by $\tilde{\hat{\Delta}}$ we denote the iso-derivative on $\tilde{\mathbb{T}}$. Also, $\tilde{\hat{T}} = \nu(\hat{T})$.

Exercise 7.3.27. *Prove that*

$$\nu \circ \hat{\sigma} = \tilde{\hat{\sigma}} \circ \nu$$

under the hypotheses of the above paragraph.

Exercise 7.3.28. *Let* $\mathbb{T} = \{11n+1 : n \in \mathbb{N}\}$*,* $\hat{T}(t) = t+1$*,* $\nu(t) = t+2$*,* $t \in \mathbb{T}$*. Find* $\hat{\sigma}(t)$*,* $\tilde{\hat{\sigma}}(t)$*,* $t \in \mathbb{T}$*. Prove that*

$$\nu \circ \hat{\sigma} = \tilde{\hat{\sigma}} \circ \nu.$$

Exercise 7.3.29. *Let* $\mathbb{T} = \{13n+11 : n \in \mathbb{N}\}$*,* $\hat{T}(t) = t+7$*,* $\nu(t) = t^2+t+2$*,* $t \in \mathbb{T}$*. Find* $\hat{\sigma}(t)$*,* $\tilde{\hat{\sigma}}(t)$*,* $t \in \mathbb{T}$*. Prove that*

$$\nu \circ \hat{\sigma} = \tilde{\hat{\sigma}} \circ \nu.$$

Exercise 7.3.30. *Let* $\mathbb{T} = \{11n+1 : n \in \mathbb{N}\}$*,* $\hat{T}(t) = t^2+1$*,* $\nu(t) = t+2$*,* $t \in \mathbb{T}$*. Find* $\hat{\sigma}(t)$*,* $\tilde{\hat{\sigma}}(t)$*,* $t \in \mathbb{T}$*. Prove that*

$$\nu \circ \hat{\sigma} = \tilde{\hat{\sigma}} \circ \nu.$$

Exercise 7.3.31. *Let* $\mathbb{T} = \{3n+12 : n \in \mathbb{N}\}$*,* $\hat{T}(t) = t+1$*,* $f(t) = t+4$*,* $g(t) = 2t+1$*,* $\nu(t) = 3t+5$*,* $t \in \mathbb{T}$*.*

(i) *Find* $\hat{\sigma}(t)$*,* $\tilde{\hat{\sigma}}(t)$*,* $t \in \mathbb{T}$*.*

(ii) *Find* $f^{\hat{\Delta}}(t)$*,* $g^{\hat{\Delta}^2}(t)$*,* $t \in \mathbb{T}$*.*

(iii) *Find*

$$\nu \circ \hat{\sigma}(t) + 2f^{\hat{\Delta}^2}(t) - 3g^{\hat{\Delta}^3}(t), \qquad t \in \mathbb{T}.$$

(iv) *Prove that*

$$\nu \circ \hat{\sigma} = \tilde{\hat{\sigma}} \circ \nu.$$

Exercise 7.3.32. *Assume that* $\nu : \mathbb{T} \longrightarrow \mathbb{R}$ *is strictly increasing and* $\tilde{\mathbb{T}} = \nu(\mathbb{T})$*.*

(i) *Let* $w : \tilde{\mathbb{T}} \longrightarrow \mathbb{R}$*. If* $w^{\tilde{\hat{\Delta}}}(\nu(t))$*and* $\nu^{\hat{\Delta}}(t)$ *exist for* $t \in \mathbb{T}^{\kappa}$*, then*

$$(w \circ \nu)^{\hat{\Delta}} = \left(w^{\tilde{\hat{\Delta}}} \circ \nu\right) \nu^{\hat{\Delta}}.$$

(ii) *Prove that*

$$\frac{1}{\nu^{\hat{\Delta}}} = \left(\nu^{-1}\right)^{\tilde{\hat{\Delta}}} \circ \nu$$

at points where $\nu^{\hat{\Delta}}$*is different from zero.*

(iii) *If* $f : \mathbb{T} \longrightarrow \mathbb{R}$ *is an iso-rd-continuous function and* ν *is iso-differentiable with iso-rd-continuous iso-derivative, then for* $a, b \in \mathbb{T}$

$$\hat{\int}_a^b f(t)\nu^{\hat{\Delta}}(t)\hat{\Delta}t = \hat{\int}_{\nu(a)}^{\nu(b)} \left(f \circ \nu^{-1}\right)(s)\tilde{\hat{\Delta}}s.$$

7.4. Iso-Hilger's Complex Plane

Let $\hat{T} : \mathbb{C} \longrightarrow (0, \infty)$.

Definition 7.4.1. *For $h > 0$ we define the iso-Hilger complex numbers, the iso-Hilger real axis, the iso-Hilger alternating axis, and the iso-Hilger imaginary circle as*

$$\hat{\mathbb{C}}_h := \left\{ z \in \mathbb{C} : \tfrac{z}{\hat{T}(z)} \neq -\tfrac{1}{h\hat{T}(h)} \right\},$$

$$\hat{\mathbb{R}}_h := \left\{ z \in \hat{\mathbb{C}}_h : \tfrac{z}{\hat{T}(z)} \in \mathbb{R} \quad \text{and} \quad \tfrac{z}{\hat{T}(z)} > -\tfrac{1}{h\hat{T}(h)} \right\},$$

$$\hat{\mathbb{A}}_h := \left\{ z \in \hat{\mathbb{C}}_h : \tfrac{z}{\hat{T}(z)} \in \mathbb{R} \quad \text{and} \quad \tfrac{z}{\hat{T}(z)} < -\tfrac{1}{h\hat{T}(h)} \right\},$$

$$\hat{\mathbb{I}}_h := \left\{ z \in \hat{\mathbb{C}}_h : \left| \tfrac{z}{\hat{T}(z)} + \tfrac{1}{h\hat{T}(h)} \right| = \tfrac{1}{h\hat{T}(h)} \right\},$$

respectively.

Definition 7.4.2. *Let $h > 0$ and $z \in \hat{\mathbb{C}}_h$. We define the iso-Hilger real part of z by*

$$\hat{\mathrm{Re}}_h(z) := \frac{\left| \frac{z}{\hat{T}(z)} h\hat{T}(h) + 1 \right| - 1}{h\hat{T}(h)},$$

and the iso-Hilger imaginary part of z by

$$\hat{\mathrm{Im}}_h(z) := \frac{\mathrm{Arg}\left(\frac{z}{\hat{T}(z)} h\hat{T}(h) + 1 \right)}{h\hat{T}(h)}.$$

We note that $\hat{\mathrm{Re}}_h(z)$ and $\hat{\mathrm{Im}}_h(z)$ satisfy

$$-\frac{1}{h\hat{T}(h)} < \hat{\mathrm{Re}}_h(z) < \infty \qquad \text{and} \qquad -\frac{\pi}{h\hat{T}(h)} < \hat{\mathrm{Im}}_h(z) \leq \frac{\pi}{h\hat{T}(h)}.$$

Definition 7.4.3. *Let $-\frac{\pi}{h\hat{T}(h)} < w \leq \frac{\pi}{h\hat{T}(h)}$. We define the iso-Hilger purely imaginary number $\hat{\mathring{i}}w$ by*

$$\hat{\mathring{i}}w = \hat{T}(h) \frac{e^{iw\frac{h}{\hat{T}(h)}} - 1}{h}.$$

Proposition 7.4.4. *Let $\hat{T} \in \mathcal{C}^1(\mathbb{R})$. Then*

$$\lim_{h \longrightarrow 0} \hat{\mathrm{Re}}_h(z) = \frac{\mathrm{Re}(z)}{\hat{T}(z)}$$

for every $z \in \mathbb{C}$.

Proof. Let $z = a + ib$, $a, b \in \mathbb{R}$. Then

$$\frac{z}{\hat{T}(z)} h\hat{T}(h) = \frac{a+ib}{\hat{T}(z)} h\hat{T}(h)$$

$$= \frac{ah\hat{T}(h)}{\hat{T}(z)} + i\frac{bh\hat{T}(h)}{\hat{T}(z)},$$

$$\frac{z}{\hat{T}(z)} h\hat{T}(h) + 1 = \frac{ah\hat{T}(h)}{\hat{T}(z)} + 1 + i\frac{bh\hat{T}(h)}{\hat{T}(z)},$$

$$\left|\frac{z}{\hat{T}(z)} h\hat{T}(h) + 1\right| = \left|\frac{ah\hat{T}(h)}{\hat{T}(z)} + 1 + i\frac{bh\hat{T}(h)}{\hat{T}(z)}\right|$$

$$= \sqrt{\left(\frac{ah\hat{T}(h)}{\hat{T}(z)} + 1\right)^2 + \frac{b^2h^2\hat{T}^2(h)}{\hat{T}^2(z)}},$$

$$\frac{d}{dh}\left|\frac{z}{\hat{T}(z)} h\hat{T}(h) + 1\right| = \frac{2\left(\frac{a}{\hat{T}(z)}h\hat{T}(h)+1\right)\left(\frac{a}{\hat{T}(z)}\hat{T}(h)+\frac{a}{\hat{T}(z)}h\hat{T}'(h)\right)+2b^2\frac{h\hat{T}^2(h)}{\hat{T}^2(z)}+2b^2\frac{h^2\hat{T}(h)\hat{T}'(h)}{\hat{T}^2(z)}}{2\sqrt{\left(\frac{ah\hat{T}(h)}{\hat{T}(z)}+1\right)^2+\frac{b^2h^2\hat{T}^2(h)}{\hat{T}^2(z)}}}$$

$$\longrightarrow_{h\longrightarrow 0} \frac{2\frac{a}{\hat{T}(z)}\hat{T}(0)}{2}$$

$$= \frac{a}{\hat{T}(z)}\hat{T}(0)$$

$$= \frac{\mathrm{Re}(z)}{\hat{T}(z)}\hat{T}(0).$$

Hence,

$$\lim_{h\longrightarrow 0} \hat{\mathrm{Re}}_h(z) = \lim_{h\longrightarrow 0} \frac{\left|\frac{z}{\hat{T}(z)}h\hat{T}(h)+1\right|-1}{h\hat{T}(h)}$$

$$= \lim_{h\longrightarrow 0} \frac{\frac{d}{dh}\left|\frac{z}{\hat{T}(z)}h\hat{T}(h)+1\right|}{\hat{T}(h)+h\hat{T}'(h)}$$

$$= \frac{\frac{\mathrm{Re}(z)}{\hat{T}(z)}\hat{T}(0)}{\hat{T}(0)}$$

$$= \frac{\mathrm{Re}(z)}{\hat{T}(z)},$$

i.e.,

$$\lim_{h\longrightarrow 0} \hat{\mathrm{Re}}_h(z) = \frac{\mathrm{Re}(z)}{\hat{T}(z)}.$$

$\square$

Proposition 7.4.5. *Let $\hat{T} \in \mathcal{C}^1(\mathbb{R})$. Then*

$$\lim_{h\longrightarrow 0} \hat{\mathring{i}}\hat{\mathrm{Im}}_h(z) = i\frac{\mathrm{Im}(z)}{\hat{T}(z)}$$

for every $z \in \mathbb{C}$.

Proof. Let $z = a + ib$, where $a, b \in \mathbb{C}$. We have

$$\hat{\mathring{i}}\hat{\mathrm{Im}}_h(z) = \hat{T}(h)\frac{e^{i\hat{\mathrm{Im}}_h(z)\frac{h}{\hat{T}(h)}} - 1}{h}$$

$$\begin{aligned}\hat{\mathrm{Im}}_h(z) &= \frac{\mathrm{Arg}\left(\frac{z}{\hat{T}(z)}h\hat{T}(h)+1\right)}{h\hat{T}(h)}\\ &= \frac{\mathrm{Arg}\left(a\frac{h\hat{T}(h)}{\hat{T}(z)}+ib\frac{h\hat{T}(h)}{\hat{T}(z)}\right)}{h\hat{T}(h)}.\end{aligned}$$

Since

$$\mathrm{Re}\left(\frac{z}{\hat{T}(z)}h\hat{T}(h)+1\right) = a\frac{h\hat{T}(h)}{\hat{T}(z)}+1,$$

$$\left|\frac{z}{\hat{T}(z)}h\hat{T}(h)+1\right| = \sqrt{\left(a\frac{h\hat{T}(h)}{\hat{T}(z)}+1\right)^2 + b^2\frac{h^2\hat{T}^2(h)}{\hat{T}^2(z)}},$$

we have

$$\mathrm{Arg}\left(\frac{z}{\hat{T}(z)}h\hat{T}(h)+1\right) = \arccos\frac{a\frac{h\hat{T}(h)}{\hat{T}(z)}+1}{\sqrt{\left(a\frac{h\hat{T}(h)}{\hat{T}(z)}+1\right)^2 + b^2\frac{h^2\hat{T}^2(h)}{\hat{T}^2(z)}}}.$$

From here,

$$\begin{aligned}\frac{d}{dh}\mathrm{Arg}\left(\frac{z}{\hat{T}(z)}h\hat{T}(h)+1\right) &= \frac{d}{dh}\arccos\frac{a\frac{h\hat{T}(h)}{\hat{T}(z)}+1}{\sqrt{\left(a\frac{h\hat{T}(h)}{\hat{T}(z)}+1\right)^2 + b^2\frac{h^2\hat{T}^2(h)}{\hat{T}^2(z)}}}\\
&= -\frac{1}{\sqrt{1-\frac{\left(a\frac{h\hat{T}(h)}{\hat{T}(z)}+1\right)^2}{\left(a\frac{h\hat{T}(h)}{\hat{T}(z)}+1\right)^2 + b^2\frac{h^2\hat{T}^2(h)}{\hat{T}^2(z)}}}}\frac{d}{dh}\frac{a\frac{h\hat{T}(h)}{\hat{T}(z)}+1}{\sqrt{\left(a\frac{h\hat{T}(h)}{\hat{T}(z)}+1\right)^2 + b^2\frac{h^2\hat{T}^2(h)}{\hat{T}^2(z)}}}\\
&= -\frac{1}{\sqrt{1-\frac{\left(a\frac{h\hat{T}(h)}{\hat{T}(z)}+1\right)^2}{\left(a\frac{h\hat{T}(h)}{\hat{T}(z)}+1\right)^2 + b^2\frac{h^2\hat{T}^2(h)}{\hat{T}^2(z)}}}}\\
&\times\frac{\left(a\frac{\hat{T}(h)}{\hat{T}(z)}+a\frac{h\hat{T}'(h)}{\hat{T}(z)}\right)\sqrt{\left(a\frac{h\hat{T}(h)}{\hat{T}(z)}+1\right)^2 + b^2\frac{h^2\hat{T}^2(h)}{\hat{T}^2(z)}} - \frac{\left(a\frac{h\hat{T}(h)}{\hat{T}(z)}+1\right)^2\left(a\frac{\hat{T}(h)}{\hat{T}(z)}+a\frac{h\hat{T}'(h)}{\hat{T}(z)}\right)-\left(a\frac{h\hat{T}(h)}{\hat{T}(z)}+1\right)b^2\left(h\frac{\hat{T}^2(h)}{\hat{T}^2(z)}+h^2\frac{\hat{T}(h)\hat{T}'(h)}{\hat{T}^2(z)}\right)}{\sqrt{\left(a\frac{h\hat{T}(h)}{\hat{T}(z)}+1\right)^2 + b^2\frac{h^2\hat{T}^2(h)}{\hat{T}^2(z)}}}}{\left(a\frac{h\hat{T}(h)}{\hat{T}(z)}+1\right)^2 + b^2\frac{h^2\hat{T}^2(h)}{\hat{T}^2(z)}}\\
&= -\frac{\left(a\frac{\hat{T}(h)}{\hat{T}(z)}+ah\frac{\hat{T}'(h)}{\hat{T}(z)}\right)b^2h^2\frac{\hat{T}^2(h)}{\hat{T}^2(z)} - \left(ah\frac{\hat{T}(h)}{\hat{T}(z)}+1\right)b^2\left(h\frac{\hat{T}^2(h)}{\hat{T}^2(z)}+h^2\frac{\hat{T}(h)\hat{T}'(h)}{\hat{T}^2(z)}\right)}{b\frac{h\hat{T}(h)}{\hat{T}(z)}\left(\left(a\frac{h\hat{T}(h)}{\hat{T}(z)}+1\right)^2 + b^2\frac{h^2\hat{T}^2(h)}{\hat{T}^2(z)}\right)}\\
&\longrightarrow_{h\longrightarrow 0} b\frac{\hat{T}(0)}{\hat{T}(z)},\end{aligned}$$

i.e.,

$$\lim_{h\longrightarrow 0}\frac{d}{dh}\mathrm{Arg}\left(\frac{z}{\hat{T}(z)}h\hat{T}(h)+1\right)=b\frac{\hat{T}(0)}{\hat{T}(z)}.$$

Consequently

$$\lim_{h\longrightarrow 0}\hat{\overset{\circ}{i}}\hat{\mathrm{Im}}_h(z)=\lim_{h\longrightarrow 0}\hat{T}(h)\frac{e^{i\frac{\mathrm{Arg}\left(\frac{z}{\hat{T}(z)}h\hat{T}(h)+1\right)}{\hat{T}^2(h)}}-1}{h}$$

$$=\hat{T}(0)\lim_{h\longrightarrow 0}\frac{e^{i\frac{\mathrm{Arg}\left(\frac{z}{\hat{T}(z)}h\hat{T}(h)+1\right)}{\hat{T}^2(h)}}-1}{h}$$

$$=i\hat{T}(0)\lim_{h\longrightarrow 0}e^{i\frac{\mathrm{Arg}\left(\frac{z}{\hat{T}(z)}h\hat{T}(h)+1\right)}{\hat{T}^2(h)}}\frac{d}{dh}\frac{\mathrm{Arg}\left(\frac{z}{\hat{T}(z)}h\hat{T}(h)+1\right)+1}{\hat{T}^2(h)}$$

$$=i\hat{T}(0)\lim_{h\longrightarrow}\frac{\mathrm{Arg}\left(\frac{z}{\hat{T}(z)}h\hat{T}(h)+1\right)}{\hat{T}^2(h)}$$

$$=i\hat{T}(0)\lim_{h\longrightarrow 0}\frac{\frac{d}{dh}\mathrm{Arg}\left(\frac{z}{\hat{T}(z)}h\hat{T}(h)+1\right)\hat{T}^2(h)-2\hat{T}(h)\hat{T}'(h)\mathrm{Arg}\left(\frac{z}{\hat{T}(z)}h\hat{T}(h)+1\right)}{\hat{T}^4(h)}$$

$$=i\hat{T}(0)\frac{b\frac{\hat{T}(0)}{\hat{T}(z)}}{\hat{T}^2(0)}$$

$$=i\frac{b}{\hat{T}^2(z)}$$

$$=i\frac{\mathrm{Im}(z)}{\hat{T}(z)}.$$

□

Corollary 7.4.6. *Let $\hat{T}\in\mathcal{C}^1(\mathbb{R})$. Then for every $z\in\mathbb{C}$ we have*

$$\lim_{h\longrightarrow 0}\left(\hat{\mathrm{Re}}_h(z)+\hat{\overset{\circ}{i}}\hat{\mathrm{Im}}_h(z)\right)=\frac{z}{\hat{T}(z)}.$$

Example 7.4.7. *Let $h=2$, $\hat{T}(z)=|z|+2$, $z\in\mathbb{C}$.*

Then, if $z=1+i$, we have

$$\hat{T}(z)=|1+i|+2$$

$$=\sqrt{1^2+1^2}+2$$

$$=\sqrt{2}+2,$$

$$\hat{T}(h)=|2|+2$$

$$=4,$$

$$\hat{\mathrm{Re}}_h(z) = \frac{\left|\frac{1+i}{\sqrt{2}+2}8+1\right|-1}{8}$$

$$= \frac{\left|\frac{8(2-\sqrt{2})}{4-2}+1+\frac{8(2-\sqrt{2})}{4-2}i\right|-1}{8}$$

$$= \frac{\left|4(2-\sqrt{2})+1+4(2-\sqrt{2})i\right|-1}{8}$$

$$= \frac{\left|(9-4\sqrt{2})+(8-4\sqrt{2})i\right|-1}{8}$$

$$= \frac{\sqrt{(9-4\sqrt{2})^2+(8-4\sqrt{2})^2}-1}{8}$$

$$= \frac{\sqrt{81-72\sqrt{2}+32+64-64\sqrt{2}+32}-1}{8}$$

$$= \frac{\sqrt{209-136\sqrt{2}}-1}{8},$$

$$\frac{z}{\hat{T}(z)}h\hat{T}(h)+1 = (9-4\sqrt{2})+(8-4\sqrt{2})i,$$

$$\left|\frac{z}{\hat{T}(z)}h\hat{T}(h)+1\right| = \sqrt{209-136\sqrt{2}},$$

$$\operatorname{Arg}\left(\frac{z}{\hat{T}(z)}h\hat{T}(h)+1\right) = \arccos\frac{9-4\sqrt{2}}{\sqrt{209-136\sqrt{2}}},$$

$$\hat{\mathrm{Im}}_h(z) = \frac{\arccos\frac{9-4\sqrt{2}}{\sqrt{209-136\sqrt{2}}}}{8}.$$

If $w = 3$, then

$$\hat{\mathring{i}}3 = 4\frac{e^{i3\frac{2}{4}}-1}{2}$$

$$= 2\left(e^{\frac{3}{2}i}-1\right).$$

Example 7.4.8. *Let $h = 3$, $\hat{T}(z) = |z|^2+1$, $z \in \mathbb{C}$.*

If $z = 2i$, then

$$\hat{T}(z) = |2i|^2+1$$

$$= 4+1$$

$$= 5,$$

$$\hat{T}(h) = |3|^2+1$$

$$= 10,$$

$$h\hat{T}(h) = 3.10$$

$$= 30,$$

$$\frac{z}{\hat{T}(z)}h\hat{T}(h)+1=\frac{2i}{5}3.10+1$$

$$=1+12i,$$

$$\left|\frac{z}{\hat{T}(z)}h\hat{T}(h)+1\right|=|1+12i|$$

$$=\sqrt{1^2+12^2}$$

$$=\sqrt{145},$$

$$\operatorname{Arg}\left(\frac{z}{\hat{T}(z)}h\hat{T}(h)+1\right)=\arccos\frac{1}{\sqrt{145}}.$$

Therefore

$$\hat{\operatorname{Re}}_h(z)=\frac{\sqrt{145}-1}{30},$$

$$\hat{\operatorname{Im}}_h(z)=\frac{\arccos\frac{1}{\sqrt{145}}}{30}.$$

If $w=4$*, then*

$$\hat{\overset{\circ}{i}}=10\frac{e^{i4\frac{3}{10}}-1}{3}$$

$$=\frac{10}{3}\left(e^{\frac{6}{5}i}-1\right).$$

Example 7.4.9. *Let* $h=4$*,* $\hat{T}(z)=|z|^2+2$*,* $z\in\mathbb{C}$*. Let also,* $z=1+2i$*. Then*

$$\hat{T}(z)=|1+2i|^2+2$$

$$=\left|1^2+2^2\right|+2$$

$$=7,$$

$$\hat{T}(h)=\left|4^2\right|+2$$

$$=18,$$

$$h\hat{T}(h)=4.18$$

$$=72,$$

$$\frac{z}{\hat{T}(z)}h\hat{T}(h)+1=\frac{1+2i}{7}72+1$$

$$=\frac{72}{7}+\frac{144}{7}i+1$$

$$=\frac{79}{7}+\frac{144}{7}i,$$

$$\begin{aligned}
\left|\tfrac{z}{\hat{T}(z)}h\hat{T}(h)+1\right| &= \left|\tfrac{79}{7}+\tfrac{144}{7}i\right| \\
&= \sqrt{\left(\tfrac{72}{7}\right)^2+\left(\tfrac{144}{7}\right)^2} \\
&= \tfrac{\sqrt{26977}}{7}, \\
\operatorname{Arg}\left(\tfrac{z}{\hat{T}(z)}h\hat{T}(h)+1\right) &= \arccos\frac{\frac{79}{7}}{\frac{\sqrt{26977}}{7}} \\
&= \arccos\frac{79}{\sqrt{26977}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\hat{\mathrm{Re}}_h(z) &= \frac{\frac{\sqrt{26977}}{7}-1}{72} \\
&= \frac{\sqrt{26977}-7}{504}, \\
\hat{\mathrm{Im}}_h(z) &= \frac{\arccos\frac{79}{\sqrt{26977}}}{72}.
\end{aligned}$$

If $w=5$, then

$$\begin{aligned}
\hat{\mathring{i}}5 &= 18\frac{e^{i5\frac{4}{18}}-1}{4} \\
&= \frac{9}{2}\left(e^{\frac{10}{9}i}-1\right).
\end{aligned}$$

Exercise 7.4.10. *Let $\hat{T}(z)=|z|^4+1$, $z\in\mathbb{C}$, $h=5$. Find*

$$\hat{\mathrm{Re}}_5(3+i), \qquad \hat{\mathrm{Im}}_5(4-i), \qquad \hat{\mathring{i}}10.$$

Theorem 7.4.11. *If $-\frac{\pi}{h\hat{T}(h)}<w\leq\frac{\pi}{h\hat{T}(h)}$, then*

$$\left|\hat{\mathring{i}}w\right|^2 = 4\frac{\hat{T}^2(h)}{h^2}\sin^2\frac{wh}{2\hat{T}(h)}.$$

Proof. We have

$$\begin{aligned}
\left|\hat{\mathring{i}}w\right|^2 &= \left|\hat{T}(h)\frac{e^{iw\frac{h}{\hat{T}(h)}}-1}{h}\right|^2 \\
&= \hat{T}(h)\frac{e^{iw\frac{h}{\hat{T}(h)}}-1}{h}\overline{\left(\frac{e^{iw\frac{h}{\hat{T}(h)}}-1}{h}\right)}
\end{aligned}$$

$$= \hat{T}^2(h)\frac{e^{iw\frac{h}{\hat{T}(h)}}-1}{h}\frac{e^{-iw\frac{h}{\hat{T}(h)}}-1}{h}$$

$$= \hat{T}^2(h)\frac{1-e^{iw\frac{h}{\hat{T}(h)}}-e^{-iw\frac{h}{\hat{T}(h)}}+1}{h^2}$$

$$= \hat{T}^2(h)\frac{2-\cos\frac{wh}{\hat{T}(h)}-i\sin\frac{wh}{\hat{T}(h)}-\cos\frac{wh}{\hat{T}(h)}+i\sin\frac{wh}{\hat{T}(h)}}{h^2}$$

$$= 2\frac{\hat{T}^2(h)}{h^2}\left(1-\cos\frac{wh}{\hat{T}(h)}\right)$$

$$= 4\frac{\hat{T}^2(h)}{h^2}\sin^2\frac{wh}{2\hat{T}(h)}.$$

□

Definition 7.4.12. *For z, $w \in \hat{\mathbb{C}}_h$ we define iso-circle plus as follows*

$$z\hat{\oplus}w = \frac{z}{\hat{T}(z)} + \frac{w}{\hat{T}(w)} + \frac{z}{\hat{T}(z)}\frac{w}{\hat{T}(w)}h\hat{T}(h).$$

Example 7.4.13. *Let $h = 2$, $\hat{T}(z_1) = |z_1|^2 + 1$, $z_1 \in \mathbb{C}$, $z = 1 + i$, $w = 1 - i$. Then*

$$\hat{T}(z) = |1+i|^2 + 1$$

$$= 1^2 + 1^2 + 1$$

$$= 3,$$

$$\frac{z}{\hat{T}(z)} = \frac{1+i}{3}$$

$$= \frac{1}{3} + \frac{1}{3}i,$$

$$\hat{T}(w) = |1-i|^2 + 1$$

$$= 1^2 + 1^2 + 1$$

$$= 3,$$

$$\frac{w}{\hat{T}(w)} = \frac{1-i}{3}$$

$$= \frac{1}{3} - \frac{1}{3}i,$$

$$\hat{T}(h) = |2|^2 + 1$$

$$= 5,$$

$$z\hat{\oplus}w = \frac{1}{3}+\frac{1}{3}i+\frac{1}{3}-\frac{1}{3}i+\left(\frac{1}{3}+\frac{1}{3}i\right)\left(\frac{1}{3}-\frac{1}{3}i\right)2.5$$

$$= \frac{2}{3}+10\left(\frac{1}{9}-\frac{1}{9}i^2\right)$$

$$= \frac{2}{3}+10\left(\frac{1}{9}+\frac{1}{9}\right)$$

$$= \frac{2}{3}+\frac{20}{9}$$

$$= \frac{6}{9}+\frac{20}{9}$$

$$= \frac{26}{9}.$$

Example 7.4.14. *Let* $h=4$, $\hat{T}(z_1)=|z_1|^2+4$, $z_1\in\mathbb{C}$, $z=2-i$, $w=1+i$. *Then*

$$\hat{T}(z) = |2-i|^2+4$$

$$= 2^2+1^2+4$$

$$= 9,$$

$$\frac{z}{\hat{T}(z)} = \frac{2-i}{9}$$

$$= \frac{2}{9}-\frac{1}{9}i,$$

$$\hat{T}(w) = |1+i|^2+4$$

$$= 1^2+1^2+4$$

$$= 6,$$

$$\frac{w}{\hat{T}(w)} = \frac{1+i}{6}$$

$$= \frac{1}{6}+\frac{1}{6}i,$$

$$\hat{T}(h) = |4|^2+4$$

$$= 20,$$

$$z\hat{\oplus}w = \frac{2}{9}-\frac{1}{9}i+\frac{1}{6}+\frac{1}{6}i+\left(\frac{2}{9}-\frac{1}{9}i\right)\left(\frac{1}{6}+\frac{1}{6}i\right)4.20$$

$$= \frac{7}{18}+\frac{1}{18}i+80\left(\frac{1}{27}+\frac{1}{27}i-\frac{1}{54}i-\frac{1}{54}i^2\right)$$

$$= \frac{7}{18}+\frac{1}{18}i+80\left(\frac{1}{18}+\frac{1}{54}i\right)$$

$$= \frac{7}{18}+\frac{1}{18}i+\frac{80}{18}+\frac{40}{27}i$$

$$= \frac{87}{18}+\frac{83}{54}i.$$

Example 7.4.15. *Let $h = 3$, $\hat{T}(z_1) = |z_1| + 1$, $z_1 \in \mathbb{C}$, $z = 3+4i$, $w = 3-4i$. Then*

$$\hat{T}(z) = |3+4i| + 1$$

$$= \sqrt{3^2+4^2} + 1$$

$$= 5+1$$

$$= 6,$$

$$\frac{z}{\hat{T}(z)} = \frac{3+4i}{6}$$

$$= \frac{3}{6} + \frac{4}{6}i$$

$$= \frac{1}{2} + \frac{2}{3}i,$$

$$\hat{T}(w) = |3-4i| + 1 = \sqrt{3^2+(-4)^2} + 1$$

$$= 5+1$$

$$= 6,$$

$$\frac{w}{\hat{T}(w)} = \frac{3-4i}{6}$$

$$= \frac{3}{6} - \frac{4}{6}i$$

$$= \frac{1}{2} - \frac{2}{3}i,$$

$$\hat{T}(h) = |3| + 1$$

$$= 4,$$

$$z \hat{\oplus} w = \frac{1}{2} + \frac{2}{3}i + \frac{1}{2} - \frac{2}{3}i + \left(\frac{1}{2} + \frac{2}{3}i\right)\left(\frac{1}{2} - \frac{2}{3}i\right) 3.4$$

$$= 1 + 12\left(\frac{1}{4} - \frac{4}{9}i^2\right)$$

$$= 1 + 12\left(\frac{1}{4} + \frac{4}{9}\right)$$

$$= 1 + 12.\frac{25}{36}$$

$$= 1 + \frac{25}{3}$$

$$= \frac{28}{3}.$$

Theorem 7.4.16. *We suppose that for every $w \in \mathbb{C}$ the equation $\frac{z}{\hat{T}(z)} = w$ has unique solu-*

tion $z \in \mathbb{C}$. Then $(\hat{\mathbb{C}}_h, \hat{\oplus})$ is an Abelian group.

Proof. Let z, $w \in \hat{\mathbb{C}}_h$ be arbitrarily chosen. Then $z\hat{\oplus}w$ is a complex number and

$$\frac{z}{\hat{T}(z)} \neq -\frac{1}{h\hat{T}(h)}, \qquad \frac{w}{\hat{T}(w)} \neq -\frac{1}{h\hat{T}(h)} \qquad \text{or}$$

$$1 + \frac{z}{\hat{T}(z)} h\hat{T}(h) \neq 0, \qquad 1 + \frac{w}{\hat{T}(w)} h\hat{T}(h) \neq 0.$$

Hence,

$$\begin{aligned} 1 + (z\hat{\oplus}w)\, h\hat{T}(h) &= 1 + \left(\frac{z}{\hat{T}(z)} + \frac{w}{\hat{T}(w)} + \frac{z}{\hat{T}(z)}\frac{w}{\hat{T}(w)} h\hat{T}(h)\right) h\hat{T}(h) \\ &= 1 + h\hat{T}(h)\frac{z}{\hat{T}(z)} + h\hat{T}(h)\frac{w}{\hat{T}(w)}\left(1 + \frac{z}{\hat{T}(z)} h\hat{T}(h)\right) \\ &= \left(1 + h\hat{T}(h)\frac{z}{\hat{T}(z)}\right)\left(1 + \frac{w}{\hat{T}(w)} h\hat{T}(h)\right) \\ &\neq 0, \end{aligned}$$

i.e.,

$$z\hat{\oplus}w \in \hat{\mathbb{C}}_h.$$

Also,

$$\begin{aligned} z\hat{\oplus}w &= \frac{z}{\hat{T}(z)} + \frac{w}{\hat{T}(w)} + \frac{z}{\hat{T}(z)}\frac{w}{\hat{T}(w)} h\hat{T}(h) \\ &= \frac{w}{\hat{T}(w)} + \frac{z}{\hat{T}(z)} + \frac{w}{\hat{T}(w)}\frac{z}{\hat{T}(z)} h\hat{T}(h) \\ &= w\hat{\oplus}z. \end{aligned}$$

Let now $z_1 \in \hat{\mathbb{C}}_h$ be arbitrarily chosen. We will search $w_1 \in \hat{\mathbb{C}}_h$ so that

$$z_1\hat{\oplus}w_1 = 0.$$

We have

$$\begin{aligned} 0 &= \frac{z_1}{\hat{T}(z_1)} + \frac{w_1}{\hat{T}(w_1)} + \frac{z_1}{\hat{T}(z_1)}\frac{w_1}{\hat{T}(w_1)} h\hat{T}(h) \qquad \Longrightarrow \\ -\frac{z_1}{\hat{T}(z_1)} &= \frac{w_1}{\hat{T}(w_1)}\left(1 + \frac{z_1}{\hat{T}(z_1)} h\hat{T}(h)\right) \qquad \Longrightarrow \\ \frac{w_1}{\hat{T}(w_1)} &= -\frac{\frac{z_1}{\hat{T}(z_1)}}{1 + \frac{z_1}{\hat{T}(z_1)} h\hat{T}(h)}. \end{aligned}$$

The last equation has unique solution $w_1 \in \mathbb{C}$. We note that

$$\begin{aligned} 1 + \frac{w_1}{\hat{T}(w_1)} h\hat{T}(h) &= 1 - \frac{\frac{z_1}{\hat{T}(z_1)}}{1 + \frac{z_1}{\hat{T}(z_1)} h\hat{T}(h)} \\ &= \frac{1}{1 + \frac{z_1}{\hat{T}(z_1)} h\hat{T}(h)} \\ &\neq 0. \end{aligned}$$

Consequently $w_1 \in \hat{\mathbb{C}}_h$. □

Exercise 7.4.17. *Show that $\hat{\oplus}$ on $\hat{\mathbb{C}}_h$ satisfies the associative law.*

Throughout of this book we will suppose that $(\hat{C}_h, \hat{\oplus})$ is an Abelian group.

In the last theorem we saw that if $z \in \hat{\mathbb{C}}_h$, then the additive inverse of z under the operation $\hat{\oplus}$ is

$$\hat{\ominus} z := -\frac{\frac{z}{\hat{T}(z)}}{1 + \frac{z}{\hat{T}(z)} h\hat{T}(h)}.$$

Definition 7.4.18. *We define iso-circle minus $\hat{\ominus}$ on $\hat{\mathbb{C}}_h$ as follows*

$$z\hat{\ominus}w := \frac{z}{\hat{T}(z)} + (\hat{\ominus}w) + \frac{z}{\hat{T}(z)} (\hat{\ominus}w)\, h\hat{T}(h).$$

Example 7.4.19. *Let $h = 2$, $\hat{T}(z_1) = |z_1|^2 + 1$, $z_1 \in \mathbb{C}$, $z = 2 + i$, $w = 2 - i$. Then*

$$\hat{T}(z) = |2+i|^2 + 1$$

$$= 2^2 + 1^2 + 1$$

$$= 6,$$

$$\tfrac{z}{\hat{T}(z)} = \tfrac{2+i}{6}$$

$$= \tfrac{1}{3} + \tfrac{1}{6} i,$$

$$\hat{T}(w) = |2-i|^2 + 1$$

$$= 2^2 + 1^2 + 1$$

$$= 6,$$

$$\tfrac{w}{\hat{T}(w)} = \tfrac{2-i}{6}$$

$$= \tfrac{1}{3} - \tfrac{1}{6} i,$$

$$\hat{T}(h) = |2|^2 + 1$$

$$= 5,$$

$$\begin{aligned}
\hat{\ominus} z &= -\frac{\frac{1}{3}+\frac{1}{6}i}{1+\left(\frac{1}{3}+\frac{1}{6}i\right)2.5}\\
&= -\frac{\frac{2+i}{6}}{1+\frac{10}{3}+\frac{5}{3}i}\\
&= -\frac{\frac{2+i}{6}}{\frac{13+5i}{3}}\\
&= -\frac{1}{2}\frac{2+i}{13+5i}\\
&= -\frac{1}{2}\frac{(2+i)(13-5i)}{(13-5i)(13+5i)}\\
&= -\frac{1}{2}\frac{26-10i+13i+5}{169+25}\\
&= -\frac{31+3i}{388},\\
\hat{\ominus} w &= -\frac{\frac{1}{3}-\frac{1}{6}i}{1+\left(\frac{1}{3}-\frac{1}{6}i\right)2.5} = -\frac{\frac{2-i}{6}}{1+\frac{10}{3}-\frac{5}{3}i}\\
&= -\frac{1}{2}\frac{2-i}{13-5i}\\
&= -\frac{1}{2}\frac{(2-i)(13+5i)}{(13-5i)(13+5i)}\\
&= -\frac{1}{2}\frac{26+10i-13i+5}{169+25}\\
&= -\frac{31-3i}{388},\\
z\hat{\ominus} w &= \frac{1}{3}+\frac{1}{6}i+\left(-\frac{31-3i}{388}\right)+\left(\frac{1}{3}+\frac{1}{6}i\right)\left(-\frac{31-3i}{388}\right)2.5\\
&= \frac{1}{3}+\frac{1}{6}i-\frac{31-3i}{388}-\frac{5}{3}\frac{(2+i)(31-3i)}{388}\\
&= \frac{1}{3}+\frac{1}{6}i-\frac{31-3i}{388}-\frac{5}{3}\frac{62-6i+31i+3}{388}\\
&= \frac{1}{3}+\frac{1}{6}i-\frac{31-3i}{388}-\frac{5}{3}\frac{65+25i}{388}\\
&= \frac{1}{3}+\frac{1}{6}i-\frac{93-9i+325+125i}{1164}\\
&= \frac{388+194i-418-116i}{1164}\\
&= \frac{-30+78i}{1164}.
\end{aligned}$$

Example 7.4.20. *Let* $h = 2$, $\hat{T}(z_1) = |z_1|^2 + 2$, $z_1 \in \mathbb{C}$, $z = i$, $w = -i$. *Then*

$$\hat{T}(z) = |i|^2 + 2$$

$$= 1 + 2$$

$$= 3,$$

$$\frac{z}{\hat{T}(z)} = \frac{1}{3}i,$$

$$\hat{T}(w) = |-i|^2 + 2$$

$$= 1 + 2$$

$$= 3,$$

$$\frac{w}{\hat{T}(w)} = -\frac{1}{3}i, \hat{T}(h) = |2|^2 + 2$$

$$= 4 + 2$$

$$= 6,$$

$$\hat{\ominus} w = -\frac{-\frac{i}{3}}{1 - \frac{i}{3} 2.6}$$

$$= \frac{1}{3} \frac{i}{1 - 4i}$$

$$= \frac{1}{3} \frac{i(1+4i)}{(1-4i)(1+4i)}$$

$$= \frac{1}{3} \frac{i + 4i^2}{1 - 16i^2}$$

$$= \frac{1}{3} \frac{-4+i}{17}$$

$$= -\frac{4}{51} + \frac{1}{51}i,$$

$$z \hat{\ominus} w = \frac{1}{3}i + \left(-\frac{4}{51} + \frac{1}{51}i\right) + \frac{1}{3}i\left(-\frac{4}{51} + \frac{1}{51}i\right) 2.6$$

$$= -\frac{4}{51} + \frac{18}{51}i + 4i\left(-\frac{4}{51} + \frac{1}{51}i\right)$$

$$= -\frac{4}{51} + \frac{18}{51}i - \frac{16}{51}i - \frac{4}{51}$$

$$= -\frac{8}{51} + \frac{2}{51}i.$$

Theorem 7.4.21. *Let* $z \in \hat{\mathbb{C}}_h$. *Then*

$$z \hat{\ominus} z = 0.$$

Proof. We have

$$
\begin{aligned}
z\hat{\ominus}z &= \frac{z}{\hat{T}(z)} + \left(\hat{\ominus}z\right) + \frac{z}{\hat{T}(z)}\left(\hat{\ominus}z\right)h\hat{T}(h)\\
&= \frac{z}{\hat{T}(z)} - \frac{\frac{z}{\hat{T}(z)}}{1+\frac{z}{\hat{T}(z)}h\hat{T}(h)} - \frac{z}{\hat{T}(z)}\frac{\frac{z}{\hat{T}(z)}}{1+\frac{z}{\hat{T}(z)}h\hat{T}(h)}h\hat{T}(h)\\
&= \frac{\frac{z}{\hat{T}(z)}+\left(\frac{z}{\hat{T}(z)}\right)^2 h\hat{T}(h)-\frac{z}{\hat{T}(z)}-\left(\frac{z}{\hat{T}(z)}\right)^2 h\hat{T}(h)}{1+\frac{z}{\hat{T}(z)}h\hat{T}(h)}\\
&= 0.
\end{aligned}
$$

□

Theorem 7.4.22. *Let z, $w \in \hat{\mathbb{C}}_h$. Then*

$$
z\hat{\ominus}w = \frac{\frac{z}{\hat{T}(z)} - \frac{w}{\hat{T}(w)}}{1+\frac{w}{\hat{T}(w)}h\hat{T}(h)}.
$$

Proof. We have

$$
\begin{aligned}
z\hat{\ominus}w &= \frac{z}{\hat{T}(z)} + \left(\hat{\ominus}w\right) + \frac{z}{\hat{T}(z)}\left(\hat{\ominus}w\right)h\hat{T}(h)\\
&= \frac{z}{\hat{T}(z)} - \frac{\frac{w}{\hat{T}(w)}}{1+\frac{w}{\hat{T}(w)}h\hat{T}(h)} - \frac{z}{\hat{T}(z)}\frac{\frac{w}{\hat{T}(w)}}{1+\frac{w}{\hat{T}(w)}h\hat{T}(h)}h\hat{T}(h)\\
&= \frac{\frac{z}{\hat{T}(z)}+\frac{z}{\hat{T}(z)}\frac{w}{\hat{T}(w)}-\frac{w}{\hat{T}(w)}-\frac{z}{\hat{T}(z)}\frac{w}{\hat{T}(w)}h\hat{T}(h)}{1+\frac{w}{\hat{T}(w)}h\hat{T}(h)}\\
&= \frac{\frac{z}{\hat{T}(z)}-\frac{w}{\hat{T}(w)}}{1+\frac{w}{\hat{T}(w)}h\hat{T}(h)}.
\end{aligned}
$$

□

Definition 7.4.23. *Let $z \in \hat{\mathbb{C}}_h$. The the iso-generalized square of z is defined by*

$$
z^{\odot} := \left(-\frac{z}{\hat{T}(z)}\right)\left(\hat{\ominus}z\right).
$$

We have the following representation

$$
\begin{aligned}
z^{\odot} &= -\frac{z}{\hat{T}(z)}\frac{-\frac{z}{\hat{T}(z)}}{1+\frac{z}{\hat{T}(z)}h\hat{T}(h)}\\
&= \frac{\left(\frac{z}{\hat{T}(z)}\right)^2}{1+\frac{z}{\hat{T}(z)}h\hat{T}(h)}.
\end{aligned}
$$

Example 7.4.24. *Let $h = 2$, $\hat{T}(z_1) = |z_1|^2 + 10$, $z_1 \in \mathbb{C}$, $z = 1+i$. Then*

$$
\begin{aligned}
\hat{T}(h) &= |2| + 10\\
&= 12,
\end{aligned}
$$

$$\hat{T}(z) = |1+i|^2 + 10$$

$$= 1^2 + 1^2 + 10$$

$$= 12,$$

$$\frac{z}{\hat{T}(z)} = \frac{1+i}{12}.$$

Hence,

$$(1+i)^{\odot} = \frac{\left(\frac{1+i}{12}\right)^2}{1+\frac{1+i}{12}2.12}$$

$$= \frac{1}{144}\frac{(1+i)^2}{1+2+2i}$$

$$= \frac{1}{144}\frac{1+2i+i^2}{3+2i}$$

$$= \frac{1}{144}\frac{2i}{3+2i}$$

$$= \frac{1}{72}\frac{i(3-2i)}{(3+2i)(3-2i)}$$

$$= \frac{1}{72}\frac{3i-2i^2}{9-4i^2}$$

$$= \frac{1}{72}\frac{2+3i}{13}$$

$$= \frac{2+3i}{936}.$$

Example 7.4.25. *Let $z \in \hat{\mathbb{C}}_h$. Then*

$$\frac{z}{\hat{T}(z)} + (\hat{\ominus} z) = \frac{z}{\hat{T}(z)} - \frac{\frac{z}{\hat{T}(z)}}{1+\frac{z}{\hat{T}(z)}h\hat{T}(h)}$$

$$= \frac{\frac{z}{\hat{T}(z)} + \left(\frac{z}{\hat{T}(z)}\right)^2 h\hat{T}(h) - \frac{z}{\hat{T}(z)}}{1+\frac{z}{\hat{T}(z)}h\hat{T}(h)}$$

$$= \frac{\left(\frac{z}{\hat{T}(z)}\right)^2}{1+\frac{z}{\hat{T}(z)}h\hat{T}(h)} h\hat{T}(h)$$

$$= z^{\odot} h\hat{T}(h).$$

Definition 7.4.26. *For $z \in \hat{\mathbb{C}}_h$ we define the iso-cylindrical transformation*

$$\xi_h(z) = \frac{1}{h\hat{T}(h)} \mathrm{Log}\left(1 + \frac{z}{\hat{T}(z)} h\hat{T}(h)\right).$$

For $h = 0$ we define

$$\xi_0(z) = \frac{z}{\hat{T}(z)}.$$

7.5. The Iso Exponential Function

Here we suppose that $h : \mathbb{T} \longrightarrow [0,\infty)$, $\hat{T} : \mathbb{R} \longrightarrow (0,\infty)$.

Definition 7.5.1. *We say that a function $p : \mathbb{T} \longrightarrow \mathbb{R}$ is iso-regressive provided*

$$1+\frac{p(t)}{\hat{T}(p(t))}h(t)\hat{T}(h(t)) \neq 0 \qquad \text{for} \qquad \text{all} \qquad t\in\mathbb{T}^{\kappa}.$$

The set of all iso-regressive and iso-rd-continuous functions $f : \mathbb{T} \longrightarrow \mathbb{R}$ will be denoted in this book with

$$\hat{\mathcal{R}} = \hat{\mathcal{R}}(\mathbb{T}) = \hat{\mathcal{R}}(\mathbb{T},\mathbb{R}).$$

Exercise 7.5.2. *Prove that $\hat{\mathcal{R}}$ is an Abelian group under the iso-circle plus defined by*

$$p\hat{\oplus}q := \frac{p(t)}{\hat{T}(p(t))}+\frac{q(t)}{\hat{T}(q(t))}+\frac{p(t)}{\hat{T}(p(t))}\frac{q(t)}{\hat{T}(q(t))}h(t)\hat{T}(h(t))$$

for all $t\in\mathbb{T}^{\kappa}$, p, $q\in\hat{\mathcal{R}}$. This group will be called the iso-regressive group.

Exercise 7.5.3. *Prove that if p, $q\in\hat{\mathcal{R}}$, then $p\hat{\oplus}q$ and the function $\hat{\ominus}p$ defined by*

$$(\hat{\ominus}p)(t) = -\frac{\frac{p(t)}{\hat{T}(p(t))}}{1+\frac{p(t)}{\hat{T}(p(t))}h(t)\hat{T}(h(t))}$$

for all $t\in\mathbb{T}^{\kappa}$ are also elements of $\hat{\mathcal{R}}$.

Definition 7.5.4. *We define the iso-circle minus $\hat{\ominus}$ on $\hat{\mathcal{R}}$ by*

$$(p\hat{\ominus}q)(t) := \frac{p(t)}{\hat{T}(p(t))}+(\hat{\ominus}q)+\frac{p(t)}{\hat{T}(p(t))}(\hat{\ominus}q)\,h(t)\hat{T}(h(t))$$

for all $t\in\mathbb{T}^{\kappa}$.

Exercise 7.5.5. *Let $p\in\hat{\mathcal{R}}$. Prove that*

$$p\hat{\ominus}p = 0.$$

Definition 7.5.6. *If $p\in\hat{\mathcal{R}}$, then we define the iso-exponential function by*

$$\hat{e}_p(t,s) := \exp\left(\hat{\int_s^t}\xi_{h(\tau)}(p(\tau))\hat{\Delta}\tau\right) \qquad \text{for} \qquad s,t\in\mathbb{T}.$$

We can rewrite the iso-exponential function in the following form

$$\hat{e}_p(t,s) = \exp\left(\hat{\int_s^t}\frac{1}{h(\tau)\hat{T}(h(\tau))}\mathrm{Log}\left(1+\frac{p(\tau)}{\hat{T}(p(\tau))}h(\tau)\hat{T}(h(\tau))\right)\hat{\Delta}\tau\right).$$

Below are some of the properties of the iso-exponential function.

1. If $p \in \hat{\mathcal{R}}$, then the semigroup property

$$\hat{e}_p(t,r)\hat{e}_p(r,s) = \hat{e}_p(t,s) \qquad \text{for} \qquad \text{all} \qquad r,s,t \in \mathbb{T}$$

is satisfied.

Proof. We have

$$\hat{e}_p(t,r)\hat{e}_p(r,s) = \exp\left(\hat{\int}_r^t \xi_{h(\tau)}(p(\tau))\hat{\Delta}\tau\right)\exp\left(\hat{\int}_s^r \xi_{h(\tau)}(p(\tau))\hat{\Delta}\tau\right)$$

$$= \exp\left(\hat{\int}_r^t \xi_{h(\tau)}(p(\tau))\hat{\Delta}\tau + \hat{\int}_s^r \xi_{h(\tau)}(p(\tau))\hat{\Delta}\tau\right)$$

$$= \exp\left(\hat{\int}_s^t \xi_{h(\tau)}(p(\tau))\hat{\Delta}\tau\right)$$

$$= \hat{e}_p(t,s).$$

□

2. $\hat{e}_0(t,s) = 1$ for all t, $s \in \mathbb{T}^\kappa$.

Proof. We have

$$\hat{e}_0(t,s) = \exp\left(\hat{\int}_s^t \xi_{h(\tau)}(0)\hat{\Delta}\tau\right)$$

$$= \exp\left(\hat{\int}_s^t \frac{1}{h(\tau)\hat{T}(h(\tau))}\mathrm{Log}(1)\hat{\Delta}\tau\right)$$

$$= \exp(0)$$

$$= 1 \qquad \text{for} \qquad \text{all} \qquad s,t \in \mathbb{T}^\kappa.$$

□

3. $\hat{e}_p(t,s) = \frac{1}{\hat{e}_p(s,t)}$ for all s, $t \in \mathbb{T}^\kappa$.

Proof. For s, $t \in \mathbb{T}^\kappa$ we have

$$\hat{e}_p(t,s) = \exp\left(\hat{\int}_s^t \xi_{h(\tau)}(p(\tau))\hat{\Delta}\tau\right)$$

$$= \exp\left(-\hat{\int}_t^s \xi_{h(\tau)}(p(\tau))\hat{\Delta}\tau\right)$$

$$= \frac{1}{\exp\left(\hat{\int}_t^s \xi_{h(\tau)}(p(\tau))\hat{\Delta}\tau\right)}$$

$$= \frac{1}{\hat{e}_p(s,t)}.$$

□

Definition 7.5.7. *(iso-trigonometric functions) If $\pm ip \in \hat{\mathcal{C}}_{rd}$, $\pm ip \in \hat{\mathcal{R}}$, then we define the iso-trigonometric functions* $\hat{\cos}_p$ *and* $\hat{\sin}_p$ *by*

$$\hat{\cos}_p := \frac{\hat{e}_{ip} + \hat{e}_{-ip}}{2}, \qquad \hat{\sin}_p := \frac{\hat{e}_{ip} - \hat{e}_{-ip}}{2i}.$$

We have

$$\hat{\cos}_p^2 + \hat{\sin}_p^2 = \left(\frac{\hat{p}e_{ip} + \hat{e}_{-ip}}{2}\right)^2 + \left(\frac{\hat{e}_{ip} - \hat{e}_{-p}}{2i}\right)^2$$

$$= \frac{\hat{e}_{ip}^2 + 2\hat{e}_i\hat{e}_{-ip} + \hat{e}_{-ip}^2}{4} - \frac{\hat{e}_{ip}^2 - 2\hat{e}_{ip}\hat{e}_{-ip} + \hat{e}_{-ip}^2}{4}$$

$$= \hat{e}_{ip}\hat{e}_{-ip}.$$

Exercise 7.5.8. *Show Euler's formula*

$$\hat{e}_{ip}(t, t_0) = \hat{\cos}_p(t, t_0) + i\hat{\sin}_p(t, t_0).$$

Definition 7.5.9. *(iso-hyperbolic functions) If $\pm p \in \hat{\mathcal{C}}_{rd}$, $\pm p \in \hat{\mathcal{R}}$, then we define the iso-hyperbolic functions* $\hat{\cosh}_p$ *and* $\hat{\sinh}_p$ *by*

$$\hat{\cosh}_p = \frac{\hat{e}_p + \hat{e}_{-p}}{2}, \qquad \hat{\sinh}_p = \frac{\hat{e}_p - \hat{e}_{-p}}{2}.$$

We have

$$\hat{\cosh}_p^2 - \hat{\sinh}_p^2 = \left(\frac{\hat{e}_p + \hat{e}_{-p}}{2}\right)^2 - \left(\frac{\hat{e}_p - \hat{e}_{-p}}{2}\right)^2$$

$$= \frac{\hat{e}_p^2 + 2\hat{e}_p\hat{e}_{-p} + \hat{e}_{-p}^2}{4} - \frac{\hat{e}_p^2 - 2\hat{e}_p\hat{e}_{-p} + \hat{e}_{-p}^2}{4}$$

$$= \hat{e}_p\hat{e}_{-p}.$$

Exercise 7.5.10. *Let $\pm p \in \hat{\mathcal{C}}_{rd}$, $\pm p \in \hat{\mathcal{R}}$. Prove that*

$$1) \qquad \hat{\cosh}_p + \hat{\sinh}_p = \hat{e}_p,$$

$$2) \qquad \hat{\cosh}_p - \hat{\sinh}_p = \hat{e}_{-p}.$$

MA

Chapter 8

Appendix

8.1. The Discrete Analogue of the Putzer Algorithm

Let $A = (a_{ij})$ is a real $k \times k$ matrix.

Here we will represent the Putzer algorithm to compute A^n.

An eigenvalue of the matrix A is a real or complex number λ such that

$$A\xi = \lambda\xi$$

for some nonzero $\xi \in \mathbb{C}^k$.

This relation we can rewrite in the form

$$(A - \lambda I)\xi = 0. \tag{1}$$

The equation (1) has a nonzero solution if and only if

$$\det(A - \lambda I) = 0$$

or

$$\lambda^k + a_1\lambda^{k-1} + a_2\lambda^{k-2} + \cdots + a_{k-1}\lambda + a_k = 0. \tag{2}$$

Definition 8.1.1. *The equation (2) is called the characteristic equation of the matrix A, whose roots λ are called the eigenvalues of A.*

If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the eigenvalues of A, then the equation (2) can be written as

$$p(\lambda) = \prod_{j=-1}^{k} (\lambda - \lambda_j).$$

Theorem 8.1.2. *(Cayley-Hamilton theorem) Every matrix A satisfies its characteristic equation, i.e.,*

$$p(A) = \prod_{j=1}^{k} (A - \lambda_j I) = 0$$

or

$$A^k + a_1 A^{k-1} + \cdots + a_{k-1}A + a_k I = 0.$$

We will search a representation of A^n in the following form

$$A^n = \sum_{j=1}^{s} x_j(n) M(j-1),$$

where $x_j(n)$ are scalar functions, which will be determined below, and

$$M(j) = (A - \lambda_j I) M(j-1),$$
$$M(0) = I.$$

Hence,

$$\begin{aligned}
M(n) &= (A - \lambda_n I) M(n-1) \\
&= (A - \lambda_n I)(A - \lambda_{n-1} I) M(n-2) \\
&= \cdots \\
&= (A - \lambda_n I)(A - \lambda_{n-1} I) \ldots (A - \lambda_1 I) M(0) \\
&= (A - \lambda_n I)(A - \lambda_{n-1} I) \ldots (A - \lambda_1 I) \\
&= \textstyle\prod_{i=1}^{n} (A - \lambda_i I).
\end{aligned}$$

From here, using the Cayley-Hamilton theorem, we have

$$M(k) = \prod_{j=1}^{k} (A - \lambda_j I) = 0.$$

Consequently,

$$M(n) = 0 \qquad \text{for} \qquad \text{all} \qquad n \geq k.$$

Therefore

$$A^n = \sum_{j=1}^{k} x_j(n) M(j-1). \tag{3}$$

We set $n = 0$ in (3) and we get

$$\begin{aligned}
A^0 &= I \\
&= \textstyle\sum_{j=1}^{k} x_j(0) M(j-1) \\
&= x_1(0) M(0) + x_2(0) M(1) + \cdots + x_k(0) M(k-1) \\
&= x_1(0) I + x_2(0) M(1) + \cdots + x_k(0) M(k-1),
\end{aligned}$$

whereupon

$$\begin{cases} x_1(0) = 1 \\ x_2(0) = 0 \\ \dots \\ x_k(0) = 0. \end{cases} \tag{4}$$

From (3) we find

$$\begin{aligned} A^{n+1} &= AA^n \\ &= A\textstyle\sum_{j=1}^{k} x_j(n)M(j-1) \\ &= \textstyle\sum_{j=1}^{k} x_j(n)AM(j-1). \end{aligned} \tag{5}$$

From the definition of $M(j)$ we have

$$\begin{aligned} M(j) &= (A - \lambda_j I)M(j-1) \\ &= AM(j-1) - \lambda_j M(j-1) \end{aligned}$$

or

$$AM(j-1) = M(j) + \lambda_j M(j-1).$$

From here and (5) we obtain

$$\begin{aligned} A^{n+1} &= \textstyle\sum_{j=1}^{k} x_j(n)(M(j) + \lambda_j M(j-1)) \\ &= \textstyle\sum_{j=1}^{k} x_j(n)M(j) + \sum_{j=1}^{k} \lambda_j x_j(n)M(j-1) \\ &= x_1(n)M(1) + x_2(n)M(2) + \cdots + x_k(n)M(k) \\ &+ \lambda_1 x_1(n)M(0) + \lambda_2 x_2(n)M(1) + \cdots + \lambda_k x_k(n)M(k-1) \\ &= x_1(n)(M(1) + \lambda_1 M(0)) \\ &+ x_2(n)(M(2) + \lambda_2 M(1)) \\ &+ \cdots \\ &+ x_k(n)(M(k) + \lambda_k M(k-1)). \end{aligned} \tag{6}$$

On the other hand,

$$\begin{aligned} A^{n+1} &= \textstyle\sum_{j=1}^{k} x_j(n+1)M(j-1) \\ &= x_1(n+1)M(0) + x_2(n+1)M(1) + \cdots + x_k(n+1)M(k-1). \end{aligned}$$

From the last equality and (6) we get

$$x_1(n)(M(1)+\lambda_1 M(0))+x_2(n)(M(2)+\lambda_2 M(1))+\cdots+x_k(n)(M(k)+\lambda_k M(k-1))$$

$$=x_1(n+1)M(0)+x_2(n+1)M(1)+\cdots+x_k(n+1)M(k-1),$$

from where

$$\begin{cases} x_1(n+1)=\lambda_1 x_1(n) \\ x_2(n+1)=x_1(n)+\lambda_2 x_2(n) \\ \dots \\ x_k(n+1)=x_{k-1}(n)+\lambda_k x_k(n). \end{cases}$$

Using the last system and (4) we find the functions $x_j(n)$, $j=1,2,\dots,k$.

Example 8.1.3. *Let*

$$A=\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 4 & -4 & 5 \end{pmatrix}.$$

We will find A^n.

We have

$$A-\lambda I=\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 4 & -4 & 5 \end{pmatrix}-\lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 4 & -4 & 5 \end{pmatrix}+\begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix}$$

$$=\begin{pmatrix} 1-\lambda & 2 & -1 \\ 0 & 1-\lambda & 0 \\ 4 & -4 & 5-\lambda \end{pmatrix},$$

$$\det(A-\lambda I)=\begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & 1-\lambda & 0 \\ 4 & -4 & 5-\lambda \end{vmatrix}$$

$$=(1-\lambda)^2(5-\lambda)+4(1-\lambda)$$

$$=(1-\lambda)\left((1-\lambda)(5-\lambda)+4\right)$$

$$=(1-\lambda)\left(5-6\lambda+\lambda^2+4\right)$$

$$=(1-\lambda)(3-\lambda)^2,$$

$\det(A-\lambda I)=0 \qquad \Longrightarrow$

$(1-\lambda)(\lambda-3)^2=0 \qquad \Longrightarrow$

$\lambda_1=1, \qquad \lambda_2=\lambda_3=3,$

$A^n=\sum_{j=1}^{3} x_j(n)M(j-1)$

$=x_1(n)M(0)+x_2(n)M(1)+x_3(n)M(2)$

$=x_1(n)I+x_2(n)M(1)+x_3(n)M(2),$

$M(1)=(A-\lambda_1 I)M(0)$

$=(A-I)I$

$=A-I$

$$=\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 4 & -4 & 5 \end{pmatrix}-\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} 0 & 2 & -1 \\ 0 & 0 & 0 \\ 4 & -4 & 4 \end{pmatrix},$$

$M(2)=(A-\lambda_2 I)M(1)$

$=(A-3I)M(1)$

$$=\begin{pmatrix} -2 & 2 & -1 \\ 0 & -2 & 0 \\ 4 & -4 & 2 \end{pmatrix}\begin{pmatrix} 0 & 2 & -1 \\ 0 & 0 & 0 \\ 4 & -4 & 4 \end{pmatrix}$$

$$=\begin{pmatrix} -4 & 0 & -2 \\ 0 & 0 & 0 \\ 8 & 0 & 4 \end{pmatrix},$$

$A^{n+1}=A.A^n$

$=A\left(x_1(n)I+x_2(n)M(1)+x_3(n)M(2)\right)$

$=x_1(n)A+x_2(n)AM(1)+x_3(n)AM(2),$

$M(1)=(A-I)M(0)$

$=A-I,$

$$A = M(1) + I,$$

$$M(2) = (A - 3I)M(1)$$

$$= AM(1) - 3M(1),$$

$$AM(1) = M(2) + 3M(1).$$

Therefore

$$A^{n+1} = x_1(n)(M(1)+I) + x_2(n)(M(2)+3M(1)) + x_3(n)(M(3)+3M(2))$$

$$= x_1(n)I + x_1(n)M(1) + x_2(n)M(2) + 3x_2(n)M(1) + 3x_3(n)M(2)$$

$$= x_1(n)I + (x_1(n)+3x_2(n))M(1) + (x_2(n)+3x_3(n))M(2),$$

i.e.,

$$A^{n+1} = x_1(n)I + (x_1(n)+3x_2(n))M(1) + (x_2(n)+3x_3(n))M(2). \tag{7}$$

On the other hand,

$$A^{n+1} = \sum_{j=1}^{3} x_j(n+1)M(j-1)$$

$$= x_1(n+1)M(0) + x_2(n+1)M(1) + x_3(n+1)M(2)$$

$$= x_1(n+1)I + x_2(n+1)M(1) + x_3(n+1)M(2).$$

From the last equality and (7) *we get*

$$x_1(n+1)I + x_2(n+1)M(1) + x_3(n+1)M(2)$$

$$= x_1(n)I + (x_1(n)+3x_2(n))M(1) + (x_2(n)+3x_3(n))M(2),$$

whereupon

$$\begin{cases} x_1(n+1) = x_1(n) \\ x_2(n+1) = 3x_2(n) + x_1(n) \\ x_3(n+1) = 3x_3(n) + x_2(n). \end{cases}$$

Also,

$$A^0 = I$$

$$= x_1(0)I + x_2(0)M(1) + x_3(0)M(2).$$

From here

$$x_1(0) = 1, \qquad x_2(0) = 0, \qquad x_3(0) = 0.$$

In this way we obtain the initial problem

$$\begin{cases} x_1(n+1) = x_1(n) \\ x_2(n+1) = 3x_2(n) + x_1(n) \\ x_3(n+1) = 3x_3(n) + x_2(n), \\ x_1(0) = 1, \qquad x_2(0) = 0, \qquad x_3(0) = 0. \end{cases}$$

We consider the initial problem

$$x_1(n+1) = x_1(n)$$

$$x_1(0) = 1.$$

For its general solution we have

$$x_1(n) = \prod_{i=0}^{n-1} 1$$

$$= 1.$$

Now we consider the initial problem

$$\begin{cases} x_2(n+1) = 3x_2(n) + x_1(n) \\ x_2(0) = 0 \end{cases}$$

or

$$\begin{cases} x_2(n+1) = 3x_2(n) + 1 \\ x_2(0) = 0. \end{cases}$$

For its general solution we have

$$x_2(n) = \sum_{i=0}^{n-1} \left(\prod_{i=r+1}^{n-1} 3\right)$$

$$= \sum_{r=0}^{n-1} 3^{n-r-1}$$

$$= 3^{n-1} \sum_{r=0}^{n-1} 3^{-r}$$

$$= 3^{n-1} \frac{1-\frac{1}{3^n}}{1-\frac{1}{3}}$$

$$= 3^{n-1} \frac{3^n-1}{2.3^{n-1}}$$

$$= \frac{3^n-1}{2},$$

i.e.,

$$x_2(n) = \frac{3^n-1}{2}.$$

Now we consider the initial problem

$$\begin{cases} x_3(n+1) = 3x_3(n) + x_2(n) \\ \\ x_3(0) = 0 \end{cases}$$

or

$$\begin{cases} x_3(n+1) = 3x_3(n) + \frac{3^n-1}{2} \\ \\ x_3(0) = 0, \end{cases}$$

from where for its general solution we get

$$\begin{aligned} x_3(n) &= \sum_{r=0}^{n-1} \left(\prod_{i=r+1}^{n-1} 3\right) \tfrac{3^r-1}{2} \\ &= \sum_{r=0}^{n-1} 3^{n-r-1} \tfrac{3^r-1}{2} \\ &= \tfrac{3^{n-1}}{2} \sum_{r=0}^{n-1} (1-3^{-r}) \\ &= \tfrac{3^{n-1}}{2} \left(\sum_{r=0}^{n-1} 1 - \sum_{r=0}^{n-1} 3^{-r}\right) \\ &= \tfrac{3^{n-1}}{2} \left(n - \frac{1-\frac{1}{3^n}}{1-\frac{1}{3}}\right) \\ &= \tfrac{3^{n-1}}{2} \left(n - \tfrac{3^n-1}{2.3^{n-1}}\right) \\ &= \tfrac{n}{2} 3^{n-1} - \tfrac{3^n-1}{4} \\ &= \tfrac{2n-3}{4} 3^{n-1} + \tfrac{1}{4}, \end{aligned}$$

i.e.,

$$x_3(n) = \frac{2n-3}{4} 3^{n-1} + \frac{1}{4}.$$

Consequently,

$$\begin{aligned} A^n &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \tfrac{3^n-1}{2} \begin{pmatrix} 0 & 2 & -1 \\ 0 & 0 & 0 \\ 4 & -4 & 4 \end{pmatrix} \\ &+ \left(\tfrac{2n-3}{4} 3^{n-1} + \tfrac{1}{4}\right) \begin{pmatrix} -4 & 0 & -2 \\ 0 & 0 & 0 \\ 8 & 0 & 4 \end{pmatrix} \\ &= \begin{pmatrix} -(2n-3)3^{n-1} & (3^n-1) & -n3^{n-1} \\ 0 & 1 & 0 \\ 4n3^{n-1} & -2(3^n-1) & \left((2n+3)3^{n-1}-1\right) \end{pmatrix}, \end{aligned}$$

Example 8.1.4. *Let*

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

We will find A^n.

We have

$$A - \lambda I = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{pmatrix},$$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)^3,$$

$$\det(A - \lambda I) = (2-\lambda)^3 = 0 \qquad \Longleftrightarrow$$

$$\lambda_1 = \lambda_2 = \lambda_3 = 2,$$

$$A^n = \textstyle\sum_{j=1}^{3} x_j(n) M(j-1)$$

$$= x_1(n)M(0) + x_2(n)M(1) + x_3(n)M(2)$$

$$= x_1(n)I + x_2(n)M(1) + x_3(n)M(2),$$

$$M(1) = (A - \lambda_1 I)M(0)$$

$$= (A - 2I)I$$

$$= A - 2I$$

$$= \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{aligned}
M(2) &= (A-\lambda_2 I)M(1) \\
&= (A-2I)M(1) \\
&= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
A &= M(1)+2I, \\
AM(1) &= M(2)+2M(1), \\
M(3) &= 0 \\
&= (A-2I)M(2) \\
&= AM(2)-2M(2), \\
AM(2) &= 2M(2), \\
A^{n+1} &= A.A^n \\
&= A\left(x_1(n)I+x_2(n)M(1)+x_3(n)M(2)\right) \\
&= x_1(n)A+x_2(n)AM(1)+x_3(n)AM(2) \\
&= x_1(n)(M(1)+2I)+x_2(n)(M(2)+2M(1))+x_3(n)(2M(2)) \\
&= 2x_1(n)I+x_1(n)M(1)+x_2(n)M(2)+2x_2(n)M(1)+2x_3(n)M(2) \\
&= 2x_1(n)I+(x_1(n)+2x_2(n))M(1)+(x_2(n)+2x_3(n))M(2).
\end{aligned}$$

Therefore

$$\begin{aligned}
& x_1(n+1)I+x_2(n+1)M(1)+x_3(n+1)M(2) \\
&= 2x_1(n)I+(x_1(n)+2x_2(n))M(1)+(x_2(n)+2x_3(n))M(2).,
\end{aligned}$$

whereupon

$$\begin{cases} x_1(n+1) = 2x_1(n) \\ x_2(n+1) = 2x_2(n)+x_1(n) \\ x_3(n+1) = 2x_3(n)+x_2(n). \end{cases}$$

Also,

$$A^0 = I$$

$$= x_1(0)I + x_2(0)M(1) + x_3(0)M(2),$$

from where

$$x_1(0) = 1, \qquad x_2(0) = x_3(0) = 0.$$

In this way we obtain the initial problem

$$\begin{cases} x_1(n+1) = 2x_1(n) \\ x_2(n+1) = 2x_2(n) + x_1(n) \\ x_3(n+1) = 2x_3(n) + x_2(n) \\ x_1(0) = 1, x_2(0) = 0, x_3(0) = 0. \end{cases}$$

We consider the initial problem

$$\begin{cases} x_1(n+1) = 2x_1(n) \\ x_1(0) = 1. \end{cases}$$

For its general solution we have

$$x_1(n) = \prod_{i=0}^{n-1} 2$$

$$= 2^n.$$

Now we consider the problem

$$\begin{cases} x_2(n+1) = 2x_2(n) + x_1(n) \\ x_2(0) = 0 \end{cases}$$

or

$$\begin{cases} x_2(n+1) = 2x_2(n) + 2^n \\ x_2(0) = 0. \end{cases}$$

For its general solution we have

$$x_2(n) = \sum_{r=0}^{n-1} \left(\prod_{i=r+1}^{n-1} 2\right) 2^r$$

$$= \sum_{r=0}^{n-1} 2^{n-r-1} 2^r$$

$$= \sum_{r=0}^{n-1} 2^{n-1}$$

$$= n2^{n-1}.$$

Now we consider the initial problem

$$\begin{cases} x_3(n+1) = 2x_3(n) + x_2(n) \\ \\ x_3(0) = 0 \end{cases}$$

or

$$\begin{cases} x_3(n+1) = 2x_3(n) + n2^{n-1}. \\ \\ x_3(0) = 0 \end{cases}$$

For its general solution we have

$$\begin{aligned} x_3(n) &= \sum_{r=0}^{n-1} \left(\prod_{i=r+1}^{n-1} 2\right) r2^{r-1} \\ &= \sum_{r=0}^{n-1} 2^{n-r-1} r2^{r-1} \\ &= 2^{n-2} \sum_{r=0}^{n-1} r \\ &= 2^{n-2} \frac{n(n-1)}{2} \\ &= n(n-1)2^{n-3}. \end{aligned}$$

Consequently,

$$\begin{aligned} A^n &= 2^n \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + n2^{n-1} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ &+ n(n-1)2^{n-3} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2^n & n2^{n-1} & n(n-1)2^{n-3} \\ 0 & 2^n & n2^{n-1} \\ 0 & 0 & 2^n \end{pmatrix}. \end{aligned}$$

Exercise 8.1.5. *Find A^n, where*

$$1) \qquad A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}, \qquad 2) \qquad A = \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix}.$$

8.2. The Jordan Normal Form

Definition 8.2.1. *We say that the $k \times k$ matrices A and B are similar if there exists a non-singular matrix P such that*

$$P^{-1}AP = B.$$

We will write

$$A \sim B.$$

If $A \sim B$, then they have the same eigenvalues.

Definition 8.2.2. *If a matrix A is similar to the diagonal matrix*

$$D = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_k),$$

then A is said to be diagonalizable. In this case

$$\lambda_1, \lambda_2, \dots, \lambda_k$$

are the eigenvalues of the matrix A.

We suppose that A is a diagonalizable matrix. Then there exists a nonsingular matrix P such that

$$P^{-1}AP = D = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_k).$$

Hence,

$$\begin{aligned} A &= PDP^{-1}, \\ A^2 &= AA \\ &= (PDP^{-1})(PDP^{-1}) \\ &= PD^2P^{-1}. \end{aligned}$$

We suppose that

$$A^n = PD^nP^{-1} \tag{8}$$

for some $n \in \mathbb{N}$.

We consider A^{n+1},

We have

$$\begin{aligned} A^{n+1} &= AA^n \\ &= PDP^{-1}(PD^nP^{-1}) \\ &= PD^{n+1}P^{-1}, \end{aligned}$$

i.e., (8) is valid for all $n \in \mathbb{N}$.

Explicitly,

$$A^n = P \begin{pmatrix} \lambda_1^n & 0 & \dots & 0 \\ 0 & \lambda_2^n & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_k^n \end{pmatrix} P^{-1}.$$

If a $k \times k$ matrix A is not diagonalizable, then it is similar to the Jordan form, i.e.,

$$P^{-1}AP = J,$$

where P is a nonsingular matrix,

$$J = \operatorname{diag}(J_1, J_2, \dots, J_r), \qquad 1 \le r \le k,$$

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ 0 & 0 & \lambda_i & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda_i \end{pmatrix},$$

J_i is $s_i \times s_i$ matrix,

$$\sum_{i=1}^{r} s_i = k.$$

Definition 8.2.3. *J_i is called a Jordan block.*

Definition 8.2.4. *The number r of Jordan blocks corresponding to one eigenvalue λ is called geometric multiplicity of λ, and this number equals the number of linearly independent eigenvectors corresponding to λ.*

Definition 8.2.5. *The algebraic multiplicity of an eigenvalue λ is the number of times it is repeated.*

Definition 8.2.6. *If the algebraic multiplicity is 1, then the eigenvalue is called simple.*

Definition 8.2.7. *If the geometric multiplicity of λ is equal to its algebraic multiplicity, then it is called semisimple.*

Example 8.2.8. *Let us consider the matrix*

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

We will find the eigenvalues of A. We consider

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 0 & 0 & 0 & 0 \\ 0 & 3-\lambda & 0 & 0 & 0 \\ 0 & 0 & 3-\lambda & 0 & 0 \\ 0 & 0 & 0 & 4-\lambda & 1 \\ 0 & 0 & 0 & 0 & 4-\lambda \end{pmatrix},$$

from where

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 & 0 & 0 \\ 0 & 3-\lambda & 0 & 0 & 0 \\ 0 & 0 & 3-\lambda & 0 & 0 \\ 0 & 0 & 0 & 4-\lambda & 1 \\ 0 & 0 & 0 & 0 & 4-\lambda \end{vmatrix}$$

$$= (1-\lambda)(3-\lambda)^2(4-\lambda)^2,$$

$$\det(A - \lambda I) = 0 \qquad \Longleftrightarrow$$

$$(1-\lambda)(3-\lambda)^2(4-\lambda)^2 = 0 \qquad \Longrightarrow$$

$$\lambda_1 = 1, \qquad \lambda_2 = 3, \qquad \lambda_3 = 4.$$

1. $\lambda_1 = 1$. *The algebraic multiplicity of* λ_1 *is* 1. *Consequently,* λ_1 *is a simple eigenvalue. Also,*

$$\operatorname{rang}(A - \lambda_1 I) = \operatorname{rang}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

$$= 4.$$

Therefore the geometric multiplicity of λ_1 *is equal to* $5 - 4 = 1$.

2. $\lambda_2 = 3$. *The algebraic multiplicity of* λ_2 *is equal to* 2. *Also,*

$$\operatorname{rang}(A - \lambda_2 I) = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= 3.$$

Hence, the geometric multiplicity of λ_2 *is equal to* $5 - 3 = 2$. *Consequently,* λ_2 *is a semisimple eigenvalue.*

3. $\lambda_3 = 4$. *The algebraic multiplicity of* λ_3 *is equal to* 2. *Also,*

$$\operatorname{rang}(A - \lambda_3 I) = \operatorname{rang}\begin{pmatrix} -3 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= 4.$$

Therefore the geometric multiplicity of λ_3 *is equal to* $5 - 4 = 1$.

Now, since

$$P^{-1}AP = J,$$

then

$$AP = PJ.$$

Let

$$P = (\xi_1, \xi_2, \ldots, \xi_k).$$

Then

$$A\xi_1 = \lambda_1 \xi_1$$

$$A\xi_2 = \lambda_1 \xi_2 + \xi_1$$

$$\ldots$$

$$A\xi_{s_1} = \lambda_1 \xi_{s_1} + \xi_{s_1 - 1}.$$

Therefore ξ_1 is the only eigenvalue of A in the Jordan chain ξ_1, ξ_2, …, ξ_{s_1}.

Definition 8.2.9. *The vectors ξ_2, ξ_3, …, ξ_{s_1} are called generalized eigenvectors of A, and they may be obtained by using the difference equation*

$$(A - \lambda_1 I)\xi_i = \xi_{i-1}, \qquad i = 2, 3, \ldots, s_1.$$

Repeating this process for the remainder of the Jordan blocks, we can find the generalized eigenvectors to the mth Jordan block using the equation

$$(A - \lambda_m I)\xi_{m_i} = \xi_{m_i - 1}, \qquad i = 2, 3, \ldots, s_m.$$

We note that

$$J^n = \begin{pmatrix} J_1^n & 0 & 0 & \ldots & 0 \\ 0 & J_2^n & 0 & \ldots & 0 \\ 0 & 0 & J_3^n & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & J_r^n \end{pmatrix}.$$

Example 8.2.10. *Let us consider the matrix*

$$A = \begin{pmatrix} 4 & 1 & 2 \\ 0 & 2 & -4 \\ 0 & 1 & 6 \end{pmatrix}.$$

We will find its eigenvalues. We have

$$A - \lambda I = \begin{pmatrix} 4-\lambda & 1 & 2 \\ 0 & 2-\lambda & -4 \\ 0 & 1 & 6-\lambda \end{pmatrix},$$

whereupon

$$\det(A-\lambda I)=\begin{vmatrix} 4-\lambda & 1 & 2\\ 0 & 2-\lambda & -4\\ 0 & 1 & 6-\lambda \end{vmatrix}$$

$$=(4-\lambda)(2-\lambda)(6-\lambda)+4(4-\lambda)$$

$$=(4-\lambda)\left((2-\lambda)(6-\lambda)+4\right)$$

$$=(4-\lambda)(\lambda^2-8\lambda+16)$$

$$=(5-\lambda)^3,$$

$$\det(A-\lambda I)=0 \qquad \Longleftrightarrow$$

$$(4-\lambda)^3=0 \qquad \Longrightarrow$$

$$\lambda_1=\lambda_2=\lambda_3=4.$$

$$\operatorname{rang}(A-\lambda_1 I)=\operatorname{rang}\begin{pmatrix} 0 & 1 & 2\\ 0 & -2 & -4\\ 0 & 1 & 2 \end{pmatrix}$$

$$=1.$$

Consequently, there are two linearly independent eigenvectors which correspond to the eigenvalue $\lambda_1=4$.

Let

$$\xi=\begin{pmatrix} a_1\\ a_2\\ a_3 \end{pmatrix}.$$

Then

$$(A-\lambda_1 I)\xi=(A-4I)\xi$$

$$=\begin{pmatrix} 0 & 1 & 2\\ 0 & -2 & -4\\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} a_1\\ a_2\\ a_3 \end{pmatrix}=0 \qquad \Longrightarrow$$

$$\begin{cases} a_2+2a_3=0\\ -2a_2-4a_3=0\\ a_1+2a_2=0, \end{cases}$$

i.e.,

$$a_2=-2a_3.$$

Therefore

$$\xi = \begin{pmatrix} a_1 \\ -2a_3 \\ a_3 \end{pmatrix}.$$

Let

$$\xi_1 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}, \qquad \xi_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

We will find

$$\xi_3 = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

using the equation

$$(A-4I)\xi_3 = \xi_2 \qquad \Longrightarrow$$

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & -2 & -4 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \qquad \Longrightarrow$$

$$\begin{cases} b_2 + 2b_3 = 1 \\ -2b_2 - 4b_3 = -2 \\ b_2 + 2b_3 = 1. \end{cases}$$

Let

$$b_3 = 1, \qquad b_2 = -1.$$

Then

$$\xi_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

Consequently,

$$P = (\xi_1, \xi_2, \xi_3)$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ -2 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Now we will find P^{-1}. For this aim we have a need of the following quantities.

$$\det(P) == -1+0-0+2-0$$

$$= 1,$$

$$p_{11} = \begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= -2+1$$

$$= -1,$$

$$p_{12} = -\begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= -(-2+1)$$

$$= 1,$$

$$p_{13} = \begin{vmatrix} -2 & -2 \\ 1 & 1 \end{vmatrix}$$

$$= -2+2$$

$$= 0,$$

$$p_{21} = -\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= -1,$$

$$p_{22} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= 0-0$$

$$= 0,$$

$$p_{23} = -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= -(0-1)$$

$$= 1,$$

$$p_{31} = \begin{vmatrix} 1 & 0 \\ -2 & -1 \end{vmatrix}$$

$$= -1 - 0$$

$$= -1,$$

$$p_{32} = -\begin{vmatrix} 0 & 0 \\ -2 & -1 \end{vmatrix}$$

$$= -(0 - 0)$$

$$= 0,$$

$$p_{33} = \begin{vmatrix} 0 & 1 \\ -2 & -2 \end{vmatrix}$$

$$= 0 + 2$$

$$= 2.$$

Consequently,

$$P^{-1} = \tfrac{1}{\det(P)} \begin{pmatrix} p_{11} & p_{21} & p_{31} \\ p_{12} & p_{22} & p_{32} \\ p_{13} & p_{23} & p_{33} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix},$$

$$AP = \begin{pmatrix} 4 & 1 & 2 \\ 0 & 2 & -4 \\ 0 & 1 & 6 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -2 & -2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 4 & 1 \\ -8 & -8 & -6 \\ 4 & 4 & 5 \end{pmatrix},$$

$$J = P^{-1}AP$$

$$= \begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 4 & 1 \\ -8 & -8 & -6 \\ 4 & 4 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

Exercise 8.2.11. *Let*

$$N_i = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Then

$$J_i = \lambda_i I_i + N_i.$$

Prove that

$$N_i^{s_i} = 0.$$

Theorem 8.2.12. *Let A be a $k \times k$ nonsingular matrix and let m be any natural number. Then there exists some $k \times k$ matrix C such that $C^m = A$.*

Proof. Let P a nonsingular $k \times k$ matrix such that

$$P^{-1}AP = J$$

$$= \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & J_r \end{pmatrix}.$$

We have

$$J_i = \lambda_i I_i + N_i$$

$$= \lambda_i \left(I_i + \tfrac{1}{\lambda_i} N_i \right).$$

We observe that

$$L_i = \exp\left(\tfrac{1}{m}\log J_i\right)$$

$$= -\exp\left(\tfrac{1}{m}\log\left(\lambda_i\left(I_i + \tfrac{1}{\lambda_i}N_i\right)\right)\right)$$

$$= \exp\left(\tfrac{1}{m}\left(\log\lambda_i I + \log\left(I_i + \tfrac{1}{\lambda_i}N_i\right)\right)\right)$$

$$= \exp\left(\tfrac{1}{m}\left(\log\lambda_i I + \textstyle\sum_{s=1}^{\infty} \tfrac{(-1)^{s+1}}{s}\left(\tfrac{N_i}{\lambda_i}\right)^s\right)\right)$$

$$N_i^{s_i} = 0$$

$$= \exp\left(\tfrac{1}{m}\left(\log\lambda_i I + \textstyle\sum_{s=1}^{s_i-1} \tfrac{(-1)^{s+1}}{s}\left(\tfrac{N_i}{\lambda_i}\right)^s\right)\right).$$

Consequently, the matrix L_i is well-defined.

Also,

$$L_i^m = J_i.$$

Let now

$$L := \begin{pmatrix} L_1 & 0 & \dots & 0 \\ 0 & L_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & L_r \end{pmatrix}.$$

Then

$$L^m = \begin{pmatrix} L_1^m & 0 & \dots & 0 \\ 0 & L_2^m & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & L_r^m \end{pmatrix}$$

$$= \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & J_r \end{pmatrix}$$

$$= J$$

$$= P^{-1}AP,$$

i.e.,

$$A = PL^m P^{-1}.$$

Let

$$C := PLP^{-1}.$$

Then

$$A = C^m.$$

□

8.3. A Norm of a Matrix

Let X be a vector space.

Definition 8.3.1. *A real-vector function on the vector space X is called a norm, and denoted by $||\cdot||$, if*

(i) $||x|| \geq 0$ *for all* $x \in X$, $||x|| = 0$ *if and only if* $x = 0$,

(ii) $||\lambda x|| = |\lambda| ||x||$ *for all* $x \in X$ *and all scalars* λ,

(iii) $||x+y|| \leq ||x|| + ||y||$ *for all* $x, y \in X$.

Below we suppose that X is a vector space endowed with a norm $||\cdot||$.

Definition 8.3.2. *We will say a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of X is convergent to the element $x \in X$ if*

$$\lim_{n \longrightarrow \infty} ||x_n - x|| = 0.$$

Definition 8.3.3. *Two norms $||\cdot||_1$ and $||\cdot||_2$ in the vector space X are called equivalent if there exist constants α, $\beta > 0$ such that*

$$\alpha||x||_1 \leq ||x||_2 \leq \beta||x||_2$$

for all $x \in X$.

Proposition 8.3.4. *Let $||\cdot||_1$ and $||\cdot||_2$ are two equivalent norms in X. Then, if $\{x_n\}_{n=1}^{\infty}$ is a sequence of elements of X, we have*

$$\lim_{n \longrightarrow \infty} ||x_n||_1 = 0$$

if and only if

$$\lim_{n \longrightarrow \infty} ||x_n||_2 = 0.$$

Let now the vector space $\mathbb{R}^k$ is endowed with the norm $||\cdot||$. Then we can define a norm of a $k \times k$ matrix A as follows

$$||A|| = \max_{||x|| \neq 0} \frac{||Ax||}{||x||}.$$

We have

$$||A|| = \max_{||x|||leq1} ||Ax|| = \max_{||x||=1} ||Ax||.$$

For a $k \times k$ matrix A we define

$$\rho(A) = \max\{|\lambda| : \lambda \quad \text{is} \quad \text{an} \quad \text{eigenvalue} \quad \text{of} \quad A\}.$$

Let λ is an eigenvalue of A. Then

$$Ax = \lambda x,$$

$$||Ax|| = ||\lambda x||$$

$$= |\lambda| ||x||.$$

Hence,

$$\left|\left|A\frac{x}{||x||}\right|\right| \leq ||A|| \qquad \Longrightarrow$$

$$\frac{1}{||x||}||Ax|| \leq ||A|| \qquad \Longrightarrow$$

$$|\lambda| \leq ||A||.$$

Consequently,

$$\rho(A) \leq ||A||.$$

For $x \in \mathbb{R}^k$ we have

1.

$$||x||_1 = \sum_{i=1}^{k} |x_i|, \qquad ||A||_1 = \max_{1\leq j\leq k} \sum_{i=1}^{k} |a_{ij}|.$$

2.

$$||x||_\infty = \max_{1\leq i\leq k} |x_i|, \qquad ||A||_\infty = \max_{1\leq i\leq k} \sum_{j=1}^{k} |a_{ij}|.$$

3.

$$||x||_2 = \left(\sum_{i=1}^{k} x_i^2\right)^{\frac{1}{2}}, \qquad ||A||_2 = \left(\rho(A^T A)\right)^{\frac{1}{2}}.$$

Example 8.3.5. *Let*

$$A = -\begin{pmatrix} 2 & -1 \\ 2 & 3 \end{pmatrix}.$$

Then

$$|a_{11}| + |a_{12}| = 2 + 1 = 3,$$

$$|a_{21}| + |a_{22}| = 2 + 3 = 5.$$

Therefore

$$||A||_1 = 5.$$

Also,

$$|a_{11}| + |a_{21}| = 2 + 2 = 4,$$

$$|a_{12}| + |a_{22}| = 1 + 3 = 4.$$

Consequently,

$$||A||_\infty = 4.$$

$$A^T A = \begin{pmatrix} 2 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 4 \\ 4 & 10 \end{pmatrix},$$

$$\det(A^T A - \lambda I) = \begin{vmatrix} 8-\lambda & 4 \\ 4 & 10-\lambda \end{vmatrix} = (\lambda - 10)(\lambda - 8) - 16$$

$$= \lambda^2 - 18\lambda + 64,$$

$$\det(A^T A - \lambda I) = 0 \qquad \Longleftrightarrow$$

$$\lambda^2 - 18\lambda + 64 = 0 \qquad \Longrightarrow$$

$$\lambda_{1,2} = \frac{9 \pm \sqrt{21}}{2}.$$

Therefore

$$||A||_2 = \left(\rho(A^T A)\right)^{\frac{1}{2}} = \sqrt{\frac{9+\sqrt{21}}{2}}.$$

8.4. Continued Fractions

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of real or complex numbers.

Definition 8.4.1. *A continued fraction is of the form*

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots + \cfrac{a_n}{b_n + \ddots}}}} \tag{9}$$

or in compact form

$$b_0 + \frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\cdots, \tag{9}$$

or

$$b_0 + K\left(\frac{a_n}{b_n}\right), \tag{9}$$

or

$$b_0 + K_{n=1}^{\infty}\left(\frac{a_n}{b_n}\right). \tag{9}$$

Definition 8.4.2. *The nth approximant of a continued fraction is defined as*

$$\begin{aligned} C(n) &= \tfrac{A(n)}{B(n)} \\ &= b_0 + K_{j=1}^{n}\left(\tfrac{a_j}{b_j}\right) \\ &= b_0 + \tfrac{a_1}{b_1+}\tfrac{a_2}{b_2+}\cdots\tfrac{a_n}{b_n} \\ &= b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_n}{b_n}}}. \end{aligned}$$

The sequences $A(n)$ and $B(n)$ are called the nth partial numerator and the nth partial denominator, respectively.

It is always supposed that $\frac{A(n)}{B(n)}$ is in the reduced form, i.e., $A(n)$ and $B(n)$ are coprime, in other words $A(n)$ and $B(n)$ have no common factors.

Definition 8.4.3. *The continued fraction (9) is said to be convergent to a finite limit L if*

$$\lim_{n \longrightarrow \infty} C(n) = L.$$

The continued fraction (9) is said to be divergent otherwise.

Theorem 8.4.4. *For the continued fraction* (9) *with nth approximant* $C(n) = \frac{A(n)}{B(n)}$ *we have that* $A(n)$ *and* $B(n)$ *satisfy the difference equations*

$$A(n) = b_n A(n-1) + a_n A(n-2), \qquad A(-1) = 1, \qquad A(0) = b_0, \tag{10}$$

$$B(n) = b_n B(n-1) + a_n B(n-2), \qquad B(-1) = 0, \qquad B(0) = 1, \tag{11}$$

respectively.

Proof. To prove this result we will use mathematical induction on n.

1. $n = 1$. We have

$$\begin{aligned} C(1) &= \frac{A(1)}{B(1)} \\ &= b_0 + \frac{a_1}{b_1} \\ &= \frac{b_0 b_1 + a_1}{b_1}, \end{aligned}$$

i.e.,

$$\begin{aligned} A(1) &= b_0 b_1 + a_1, \\ B(1) &= b_1. \end{aligned}$$

On the other hand, from (10) we get

$$\begin{aligned} A(1) &= b_1 A(0) + a_1 A(-1) \\ &= b_0 b_1 + a_1. \end{aligned}$$

Also, using (11) we find

$$\begin{aligned} B(1) &= b_1 B(0) + a_1 B(-1) \\ &= b_1. \end{aligned}$$

Therefore the result is valid for $n = 1$.

2. Now we suppose

$$A(m) = b_m A(m-1) + a_m A(m-2), \qquad A(-1) = 1, \qquad A(0) = b_0, \tag{12}$$

$$B(m) = b_m B(m-1) + a_m B(m-2), \qquad B(-1) = 0, \qquad B(0) = 1, \tag{13}$$

for some $m \in \mathbb{N}$.

3. We will prove that

$$A(m+1) = b_{m+1} A(m) + a_{m+1} A(m-1), \qquad A(-1) = 1, \qquad A(0) = b_0, \tag{12$'$}$$

$$B(m+1) = b_{m+1} B(m) + a_{m+1} B(m-1), \qquad B(-1) = 0, \qquad B(0) = 1. \tag{13$'$}$$

Firstly, we will note that $\frac{A(m+1)}{B(m+1)}$ is obtained from $\frac{A(m)}{B(m)}$ by replacing b_m by $b_m + \frac{a_{m+1}}{b_{m+1}}$.

Let
$$\frac{A(m+1)}{B(m+1)} = \frac{A^*(m)}{B^*(m)},$$
where
$$A^*(m) = \left(b_m + \tfrac{a_{m+1}}{b_{m+1}}\right) A(m-1) + a_m A(m-2),$$
$$B^*(m) = \left(b_m + \tfrac{a_{m+1}}{b_{m+1}}\right) B(m-1) + a_m B(m-2).$$
Now we consider $A^*(m)$. For it we have
$$A^*(m) = \tfrac{1}{b_{m+1}} \left(b_m b_{m+1} A(m-1) + a_{m+1} A(m-1) + b_{m+1} a_m A(m-2)\right)$$
$$= \tfrac{1}{b_{m+1}} \left(b_{m+1} \left(b_m A(m-1) + a_m A(m-2)\right) + a_{m+1} A(m+1)\right)$$
$$\text{now} \qquad \text{we} \qquad \text{use} \qquad (12)$$
$$= \tfrac{1}{b_{m+1}} \left(b_{m+1} A(m) + a_{m+1} A(m-1)\right),$$
i.e.,
$$A^*(m) = \frac{1}{b_{m+1}} \left(b_{m+1} A(m) + a_{m+1} A(m-1)\right). \tag{14}$$
For $B^*(m)$ we have
$$B^*(m) = \tfrac{1}{b_{m+1}} \left(b_m b_{m+1} B(m-1) + a_{m+1} B(m-1) + a_m b_{m+1} B(m-2)\right)$$
$$= \tfrac{1}{b_{m+1}} \left(b_{m+1} \left(b_m B(m-1) + a_m B(m-2)\right) + a_{m+1} B(m-1)\right)$$
$$\text{now} \qquad \text{we} \qquad \text{use} \qquad (13)$$
$$= \tfrac{1}{b_{m+1}} \left(b_{m+1} B(m) + a_{m+1} B(m-1)\right),$$
i.e.,
$$B^*(m) = \frac{1}{b_{m+1}} \left(b_{m+1} B(m) + a_{m+1} B(m-1)\right). \tag{15}$$
From (14) and (15) we get
$$C(m+1) = \tfrac{A^*(m)}{B^*(m)}$$
$$= \tfrac{\frac{1}{b_{m+1}} \left(b_{m+1} A(m) + a_{m+1} A(m-1)\right)}{\frac{1}{b_{m+1}} \left(b_{m+1} B(m) + a_{m+1} B(m-1)\right)}$$
$$= \tfrac{b_{m+1} A(m) + a_{m+1} A(m-1)}{b_{m+1} B(m) + a_{m+1} B(m-1)},$$
whereupon we get $(12')$ and $(13')$.

$\square$

Let us consider the difference equation

$$x(n) - b_n x(n-1) - a_n x(n-2) = 0, \qquad a_n \neq 0, \qquad n \in \mathbb{N},$$

whereupon, dividing this difference equation by $x(n-1)$, we get

$$\frac{x(n)}{x(n-1)} - b_n - a_n \frac{x(n-2)}{x(n-1)} = 0.$$

Let

$$y(n) := \frac{x(n)}{x(n-1)}.$$

Then

$$y(n) - b_n - a_n \frac{1}{y(n-1)} = 0$$

or

$$a_n \frac{1}{y(n-1)} = -b_n + y(n),$$

or

$$y(n-1) = \frac{a_n}{-b_n + y(n)}.$$

Applying this formula repeatedly, we obtain

$$y(n-1) = \frac{a_n}{-b_n - 1} \frac{a_{n+1}}{-b_{n+1} +} \frac{a_{n+2}}{-b_{n+2} + \ddots}.$$

In this way we conclude that the converse of the preceding theorem is also true, i.e., every homogeneous second-order difference equation gives rise to an associated fraction.

Now we multiply (10) by $B(n-1)$ and we find

$$A(n)B(n-1) = b_n A(n-1)B(n-1) + a_n A(n-2)B(n-1), \tag{16}$$

we multiply (11) by $A(n-1)$ and we get

$$B(n)A(n-1) = b_n A(n-1)B(n-1) + a_n B(n-2)A(n-1). \tag{17}$$

We subtract (17) from (16) and we obtain

$$A(n)B(n-1) - B(n)A(n-1) = a_n \left(A(n-2)B(n-1) - B(n-2)A(n-1)\right).$$

Let

$$u(n) = A(n)B(n-1) - B(n)A(n-1).$$

Then we get the difference equation

$$u(n) = -a_n u(n-1), \qquad u(0) = -1, \qquad n \in \mathbb{N}.$$

Consequently,

$$\begin{aligned} u(n) &= A(n)B(n-1) - B(n)A(n-1) \\ &= (-1)^{n+1} a_1 a_2 \ldots a_n, \end{aligned}$$

from where dividing both sides by $B(n)B(n-1)$ we find

$$\frac{A(n)}{B(n)} - \frac{A(n-1)}{B(n-1)} = (-1)^{n+1}\frac{a_1a_2\dots a_n}{B(n)B(n-1)}$$

or

$$\Delta\left(\frac{A(n-1)}{B(n-1)}\right) = (-1)^{n+1}\frac{a_1a_2\dots a_n}{B(n)B(n-1)}.$$

Taking the antidifference Δ^{-1} of both sides of the last expression, we get

$$\frac{A(n-1)}{B(n-1)} = \Delta^{-1}\left((-1)^{n+1}\frac{a_1a_2\dots a_n}{B(n)B(n-1)}\right)$$

$$= \frac{A(0)}{B(0)} + \sum_{k=1}^{n-1}(-1)^{k+1}\frac{a_1a_2\dots a_k}{B(k-1)B(k)}.$$

Consequently,

$$C(n) = b_0 + \sum_{k=1}^{n}(-1)^{k+1}\frac{a_1a_2\dots a_k}{B(k-1)B(k)}.$$

Definition 8.4.5. *Two continued fractions $K\left(\frac{a_n}{b_n}\right)$ and $K\left(\frac{a_n^*}{b_n^*}\right)$ are said to be equivalent if they have the same sequence of approximants. We will write*

$$K\left(\frac{a_n}{b_n}\right) \approx K\left(\frac{a_n^*}{b_n^*}\right).$$

Theorem 8.4.6. *Let $b_n > 0$, $n \in \mathbb{N}\bigcup\{0\}$. Then $K\left(\frac{1}{b_n}\right)$ is convergent if and only if the infinite series $\sum_{n=1}^{\infty} b_n$ is divergent.*

Proof. We have that

$$K\left(\frac{1}{b_n}\right) = \sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{B(r-1)B(r)}.$$

Therefore $K\left(\frac{1}{b_n}\right)$ converges if and only if the alternating series

$$\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{B(r-1)B(r)} \tag{17'}$$

converges.

Also, we have that

$$B(n) = B(n-2) + b_nB(n-1), \qquad B(0) = 1, \qquad B(1) = b_1. \tag{18}$$

We observe that

$$B(2) = B(0) + b_2B(1)$$

$$= 1 + b_1b_2$$

$$> 0.$$

We suppose that

$$B(k) > 0$$

for some $k \in \mathbb{N}$.

We will prove that

$$B(k+1) > 0.$$

Really,

$$B(k+1) = B(k-1) + b_{k+1}B(k)$$

$$> 0$$

because $b_{k+1} > 0$, $B(k-1) > 0$, $B(k) > 0$.

Consequently,

$$B(n) > 0$$

for all $n \in \mathbb{N}$.

Hence and (18) we find

$$B(n) - B(n-2) > 0 \qquad \text{for} \qquad \forall n \in \mathbb{N},$$

or

$$B(n+1) > B(n-1) \qquad \text{for} \qquad \forall n \in \mathbb{N},$$

whereupon

$$B(n)B(n+1) > B(n)B(n-1) \qquad \text{for} \qquad \forall n \in \mathbb{N}.$$

Thus

$$\left| \frac{(-1)^{n+1}}{B(n-1)B(n)} \right|$$

is monotonically decreasing.

Therefore the series $(17')$ converges if and only if

$$\lim_{n \longrightarrow \infty} B(n)B(n-1) = \infty.$$

Let

$$\gamma := \min\{1, b_1\}.$$

Then

$$B(0) \geq \gamma, \qquad B(1) \geq \gamma.$$

We suppose that

$$B(k) \geq \gamma$$

for some $k \in \mathbb{N}$. We will prove that

$$B(k+1) \geq \gamma.$$

Really,

$$\begin{aligned}
B(k+1) &= B(k-1) + b_{k+1}B(k)\\
&\geq \gamma + b_{k+1}\gamma\\
&\geq \gamma.
\end{aligned}$$

Therefore

$$B(n) \geq \gamma$$

for every $n \in \mathbb{N}$.

Hence,

$$\begin{aligned}
B(n-1)B(n) &= B(n-1)\left(B(n-2) + b_nB(n-1)\right)\\
&= B(n-1)B(n-2) + b_nB^2(n-1)\\
&\geq B(n-1)B(n-2) + b_n\gamma^2.
\end{aligned}$$

From here,

$$\begin{aligned}
B(1)B(2) &\geq B(1)B(0) + b_2\gamma^2\\
&\geq b_1\gamma + b_2\gamma^2\\
&\geq b_1\gamma^2 + b_2\gamma^2\\
&= (b_1 + b_2)\gamma^2.
\end{aligned}$$

We suppose that

$$B(k-1)B(k) \geq (b_1 + b_2 + \ldots + b_k)\gamma^2$$

for some $k \in \mathbb{N}$.

We will prove that

$$B(k)B(k+1) \geq (b_1 + b_2 + \cdots + b_{k+1})\gamma^2.$$

Really,

$$\begin{aligned}
B(k)B(k+1) &= B(k)\left(B(k-1) + b_{k+1}B(k)\right)\\
&= B(k)B(k-1) + b_{k+1}B(k)\\
&\geq (b_1 + b_2 + \cdots + b_k)\gamma^2 + b_{k+1}\gamma^2\\
&= (b_1 + b_2 + \cdots + b_{k+1})\gamma^2.
\end{aligned}$$

Consequently, for every $n \in \mathbb{N}$ we have

$$B(n-1)B(n) \geq (b_1 + bv_2 + \cdots + b_n)\gamma^2.$$

If

$$\sum_{i=1}^{\infty} b_i$$

is convergent, then

$$\lim_{n\longrightarrow\infty} B(n)B(n-1)=\infty$$

and $K\left(\frac{1}{b_n}\right)$ converges.

On tho other hand, using the difference equation which $B(n)$ satisfies, we have

$$B(n-1)+B(n)=B(n-1)+B(n-2)+b_nB(n-1)$$

$$=B(n-2)+(1+b_n)B(n-1)$$

$$\leq (1+b_n)B(n-2)+(1+b_n)B(n-1)$$

$$=(1+b_n)(B(n-2)+B(n-1)),$$

i.e.,

$$B(n-1)+B(n)\leq (1+b_n)(B(n-2)+B(n-1)).$$

Also,

$$B(n-2)+B(n-1)=B(n-1)+B(n-3)+b_{n-1}B(n-2)$$

$$=B(n-3)+(1+b_{n-1})B(n-2)$$

$$\leq (1+b_{n-1})B(n-3)+(1+b_{n-1})B(n-2)$$

$$=(1+b_{n-1})(B(n-3)+B(n-2)),$$

whereupon

$$B(n-1)+B(n)\leq (1+b_n)(1+b_{n-1})(B(n-3)+B(n-2))$$

and etc.,

$$B(n-1)+B(n)\leq (1+b_n)(1+b_{n-1})\ldots(1+b_2)(B(0)+B(1))$$

$$=(1+b_n)(1+b_{n-1})\ldots(1+b_2)(1+b_1),$$

and using that

$$1+b_i<e^{b_i},\qquad i=1,2,\ldots,n,$$

$$B(n-1)+B(n)<e^{b_1}e^{b_2}\ldots e^{b_n}$$

$$=e^{b_1+b_2+\cdots+b_n}.$$

Thus if

$$\sum_{n=1}^{\infty} b_n$$

converges to L, then

$$B(n-1)+B(n)\leq e^L.$$

Therefore,

$$B(n-1)B(n) \leq \tfrac{1}{4}\left(B(n-1)+B(n)\right)^2$$

$$\leq \tfrac{1}{4}e^{2L}.$$

Consequently, if

$$\lim_{n\longrightarrow\infty} B(n-1)B(n) \neq \infty,$$

then the continued fraction diverges. □

8.5. Tools of Approximation

The main tools of approximating functions are the symbols $\sim$ and O.

We start with the symbol O (big oh)

Definition 8.5.1. *Let $f(t)$ and $g(t)$ be two functions defined on $\mathbb{R}$ or $\mathbb{C}$. We say that*

$$f(t) = O(g(t)), \qquad t \longrightarrow \infty,$$

if there exists a positive constant M such that

$$|f(t)| \leq M|g(t)|$$

for all $t \geq t_0$.

Example 8.5.2. *We will show that*

$$\left(\frac{n^2}{t^4+n^4}\right)^n = O\left(\frac{1}{t^2}\right), \qquad n \longrightarrow \infty,$$

for $t > 1$.

For this aim we have to prove that there exists a positive constant M such that

$$\frac{n^2}{t^4+n^4} \leq M\frac{1}{t^2}$$

for every $n \geq n_0$.

Let $M = 1$. Then

$$\tfrac{n^2}{t^4+n^4} \leq \tfrac{1}{t^2} \qquad \Longleftrightarrow$$

$$n^2t^2 \leq t^4+n^4 \qquad \Longleftrightarrow$$

$$t^4 - n^2t^2 + n^4 \geq 0$$

for every $n \geq n_0$ and for every $t > 1$.

Proposition 8.5.3. *The relation defined by O is not symmetric, i.e., if $f = O(g)$, then it is not necessarily true that $g = O(f)$.*

Proof. We have that

$$x^2 = O(x^4) \qquad \text{when} \qquad x \longrightarrow \infty,$$

from here, it is evident,

$$x^4 \neq O(x^2) \qquad \text{when} \qquad x \longrightarrow \infty.$$

□

Proposition 8.5.4. *Let $f = O(g)$ when $x \longrightarrow \infty$, $g = O(h)$ when $x \longrightarrow \infty$. Then $f = O(h)$ when $x \longrightarrow \infty$.*

Proof. Since $f = O(g)$ when $x \longrightarrow \infty$, then there exists a positive constant M_1 such that

$$|f(x)| \leq M_1 |g(x)| \qquad \text{for} \qquad x \geq x_0.$$

Because $g = O(h)$ when $x \longrightarrow \infty$, then there exists a positive constant M_2 such that

$$|g(x)| \leq M_1 |h(x)| \qquad \text{for} \qquad x \geq x_0.$$

Therefore, for $x \geq x_0$, we have

$$\begin{aligned} & |f(x)| \leq M_1 |g(x)| \\ & \leq M_1 (M_2 |h(x)|) \\ & = M_1 M_2 |h(x)|. \end{aligned}$$

Consequently, $f = O(h)$. □

Definition 8.5.5. *If*

$$\lim_{x \longrightarrow \infty} \frac{f(x)}{g(x)} = 0,$$

then we say that

$$f(x) = o(g(x)), \qquad x \longrightarrow \infty.$$

Example 8.5.6. *We will show that*

$$\sinh\left(\frac{1}{x}\right) = o(1) \qquad \text{as} \qquad x \longrightarrow \infty.$$

Really,

$$\begin{aligned} & \lim_{x \longrightarrow \infty} \frac{\sinh\left(\frac{1}{x}\right)}{1} = \lim_{x \longrightarrow \infty} \left(e^{\frac{1}{x}} - e^{-\frac{1}{x}}\right) \\ & = 1 - 1 \\ & = 0. \end{aligned}$$

Proposition 8.5.7. *We have*

$$o(f(x)) = f(x) o(1) \qquad \text{as} \qquad x \longrightarrow \infty.$$

Proof. Let

$$g(x) := o(f(x)) \qquad \text{as} \qquad x \longrightarrow \infty.$$

Then

$$\lim x \longrightarrow \infty \frac{g(x)}{f(x)} = 0,$$

whereupon

$$\lim_{x\longrightarrow\infty} \frac{\frac{g(x)}{f(x)}}{1} = 0$$

or

$$\frac{g(x)}{f(x)} = o(1) \qquad \text{as} \qquad x \longrightarrow \infty.$$

Therefore

$$g(x) = f(x)o(1) \qquad \text{as} \qquad x \longrightarrow \infty.$$

Using the definition of g we conclude that

$$o(f(x)) = f(x)o(1) \qquad \text{as} \qquad x \longrightarrow \infty.$$

□

Definition 8.5.8. *If*

$$\lim_{x\longrightarrow\infty} \frac{f(x)}{g(x)} = 1,$$

then we say that f is asymptotic to g when $x \longrightarrow \infty$. We will write

$$f \sim g \qquad \text{as} \qquad x \longrightarrow \infty.$$

Proposition 8.5.9. *Let $f \sim g$ as $x \longrightarrow \infty$. Then*

$$\lim_{x\longrightarrow\infty} \frac{f(x)-g(x)}{g(x)} = 0.$$

Proof. Since $f \sim g$ as $x \longrightarrow \infty$, we have that

$$\lim_{x\longrightarrow\infty} \frac{f(x)}{g(x)} = 1,$$

or

$$\lim_{x\longrightarrow\infty} \tfrac{f(x)}{g(x)} - 1 = 0 \qquad \Longrightarrow$$

$$\lim_{x\longrightarrow\infty} \left(\tfrac{f(x)}{g(x)} - 1\right) = 0 \qquad \Longrightarrow$$

$$\lim_{x\longrightarrow\infty} \tfrac{f(x)-g(x)}{g(x)} = 0.$$

□

Proposition 8.5.10. *We have*

$$O(f(x)) = f(x)O(1) \qquad \text{as} \qquad x \longrightarrow \infty.$$

Proof. Let

$$g(x) = O(f(x)) \qquad \text{as} \qquad x \longrightarrow \infty.$$

Then there exists a positive constant M_1 such that

$$|g(x)| \leq M_1 |f(x)| \qquad \text{as} \qquad x \longrightarrow \infty,$$

or

$$\left| \frac{g(x)}{f(x)} \right| \leq M_1 |1| \qquad \text{as} \qquad x \longrightarrow .$$

Consequently,

$$\frac{g(x)}{f(x)} = O(1) \qquad \text{as} \qquad x \longrightarrow \infty,$$

or

$$g(x) = f(x)O(1) \qquad \text{as} \qquad x \longrightarrow \infty,$$

or

$$O(f(x)) = f(x)O(1) \qquad \text{as} \qquad x \longrightarrow \infty,$$

$\square$

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