FOUNDATIONS OF ISO-DIFFERENTIAL CALCULUS VOLUME II

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Table of Contents

Preface

This book introduces the main ideas and the fundamental methods of the iso-differential calculus for the iso-functions of several variables.

In Chapter 1 are discussed the structure of the iso-Euclidean spaces, the main conceptions for the iso-functions of the first, the second, the third, the fourth and the fifth kind of n - variables, limits of the iso-real iso-valued iso-functions of several variables, the continuous iso-functions, the main ideas for the iso-partial derivatives of the first, the second, the third, the fourth, the fifth, the sixth and the seventh kind of the iso-functions of several variables, they are introduced the main approaches for the finding of the minima and the maxima of the iso-functions of n variables.

In Chapter 2 are represented some of the most relevant results of the iso-integration theory. The aim is to provide the reader with all that is needed to use the power of the iso-integration.

In Chapter 3 we deal with the line and the surface iso-integrals.

Chapter 4 provides a sufficiently wide introduction to the theory of the iso-Fourier integral.

Chapter 5 is dedicated to some conceptions connected with the iso-Hilbert spaces. They are defined some classes of iso-operators in the iso-Hilbert spaces and given some of their properties.

In Chapter 6 is given a definition for the Santilli-Lie-isotopic power series and they are deducted some of its properties.

I think, in fact, that it is useful for the reader to have a wide spectrum of context in which these ideas play an important role and wherein even the technical and formal aspects play a role. However, I have tried to keep the same spirit, always providing examples and exercises to clarify the main presentation.

I will be very grateful to anybody who wants to inform me about errors or just misprints, or wants to express criticism or other comments, to my e-mails svetlingeorgiev1@gmail.com, sgg2000bg@yahoo.com.

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Chapter 1

Real-Valued Iso-Functions of Several Variables

1.1. Structure of $\hat{F}_{\mathbb{R}^n}$

Let $\hat{T}_1 : \mathbb{R} \longrightarrow \mathbb{R}$, $\hat{T} : \mathbb{R}^n \longrightarrow \mathbb{R}$ be positive functions and $\hat{I}_1 = \frac{1}{\hat{T}_1}$, $\hat{I} = \frac{1}{\hat{T}}$. With $\hat{F}_{\mathbb{R}}$ we will denote the space of the iso-real iso-numbers $\hat{a} = \frac{a}{\hat{T}_1(a)}$, $a \in \mathbb{R}$. Some of the properties of the space $\hat{F}_{\mathbb{R}}$ are studied in "Foundations of Iso-Differential Calculus", Vol. I, [1]. In this chapter we study the iso-functions defined on subsets of the iso-real n - dimensional space $\hat{F}_{\mathbb{R}^n}$, which consists of all ordered n-tuples

$$\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n) = \Big(\frac{x_1}{\hat{T}(x_1, x_2, \ldots, x_n)}, \frac{x_2}{\hat{T}(x_1, x_2, \ldots, x_n)}, \ldots, \frac{x_n}{\hat{T}(x_1, x_2, \ldots, x_n)}\Big),$$

of iso-real iso-numbers, called the iso-coordinates or the iso-components of $\hat{X}$. Here $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. This space sometimes is called iso-Euclidean space.

In this section we introduce an algebraic structure of $\hat{F}_{\mathbb{R}^n}$. We also consider its topological properties, that is, properties that can be described in terms of a special class of subsets, the iso-neighborhood in $\hat{F}_{\mathbb{R}^n}$.

Definition 1.1.1. *The iso-vector sum of*

$$\hat{X} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n) \qquad \text{and} \qquad \hat{Y} = (\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_n)$$

is

$$(A1) \hat{X} + \hat{Y} = (\hat{x}_1 + \hat{y}_1, \hat{x}_2 + \hat{y}_2, \ldots, \hat{x}_n + \hat{y}_n).$$

If $\hat{a}$ is an iso-real iso-number, the iso-scalar multiple of $\hat{X}$ by $\hat{a}$ is

$$\hat{a}\hat{\times}\hat{X} = \tfrac{a}{\hat{T}_1(a)}\hat{T}_1(a)\hat{X} = a\hat{X} = a(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$$

$$= a\Big(\tfrac{x_1}{\hat{T}(x_1,x_2,\ldots,x_n)}, \tfrac{x_2}{\hat{T}(x_1,x_2,\ldots,x_n)}, \ldots, \tfrac{x_n}{\hat{T}(x_1,x_2,\ldots,x_n)}\Big)$$

$$(A2) \quad = \Big(a\tfrac{x_1}{\hat{T}(x_1,x_2,\ldots,x_n)}, a\tfrac{x_2}{\hat{T}(x_1,x_2,\ldots,x_n)}, \ldots, a\tfrac{x_n}{\hat{T}(x_1,x_2,\ldots,x_n)}\Big)$$

$$= \Big(\tfrac{a}{\hat{T}_1(a)}\hat{T}_1(a)\tfrac{x_1}{\hat{T}(x_1,x_2,\ldots,x_n)}, \tfrac{a}{\hat{T}_1(a)}\hat{T}_1(a)\tfrac{x_2}{\hat{T}(x_1,x_2,\ldots,x_n)}, \ldots, \tfrac{a}{\hat{T}_1(a)}\hat{T}_1(a)\tfrac{x_n}{\hat{T}(x_1,x_2,\ldots,x_n)}\Big)$$

$$= (\hat{a}\hat{\times}\hat{x}_1, \hat{a}\hat{\times}\hat{x}_2, \ldots, \hat{a}\hat{\times}\hat{x}_n).$$

Note that "+" stands for the newly defined addition of members of $\hat{F}_{\mathbb{R}^n}$ and, in the right "+", for addition of iso-real iso-numbers. However, this can never lead to confusion, since the meaning of "+" can always be deducted from the symbols on either side of it. A similar comment applies to the use of juxtaposition to indicate iso-scalar multiplication on the left of (A2) and iso-multiplication of iso-real iso-numbers on the right.

Example 1.1.2. *In $\hat{F}_{\mathbb{R}^3}$, let $\hat{T}(x) = x_1^2 + x_2^2 + 3x_3^2$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\hat{T}_1(y) = y^2 + 1$, $y \in \mathbb{R}$. Let also,*

$$X = (-1, 2, 3), \qquad Y = (2, 0, 4).$$

Then

$$\hat{X} = \Big(-\tfrac{1}{32}, \tfrac{2}{32}, \tfrac{3}{32}\Big) = \Big(-\tfrac{1}{32}, \tfrac{1}{16}, \tfrac{3}{32}\Big),$$

$$\hat{Y} = \Big(\tfrac{2}{52}, \tfrac{0}{52}, \tfrac{4}{52}\Big) = \Big(\tfrac{1}{26}, 0, \tfrac{1}{13}\Big),$$

and from here

$$\hat{X} + \hat{Y} = \Big(-\frac{1}{32} + \frac{1}{26}, \frac{1}{16} + 0, \frac{3}{32} + \frac{1}{13}\Big) = \Big(-\frac{7}{416}, \frac{1}{16}, \frac{7}{416}\Big).$$

If $\hat{3} \in \hat{F}_{\mathbb{R}}$ then $\hat{3} = \frac{3}{10}$ and

$$\hat{3}\hat{\times}\hat{X} = \Big(-\frac{21}{416}, \frac{3}{16}, \frac{213}{416}\Big).$$

Exercise 1.1.3. *In $\hat{F}_{\mathbb{R}^2}$, let $\hat{T}(x) = x_1^2 + 1$, $x = (x_1, x_2) \in \mathbb{R}^2$, $\hat{T}_1(y) = y^2 + 1$, $y \in \mathbb{R}$, $X = (1, 2)$, $Y = (-3, 1)$. Find*

$$\hat{X}, \qquad \hat{Y}, \qquad \hat{X} + \hat{Y}, \qquad \hat{2}\hat{\times}\hat{Y}, \qquad \hat{3}\hat{\times}(\hat{X} + \hat{Y}).$$

Answer. $\hat{X} = \Big(\frac{1}{2}, 1\Big)$, $\hat{Y} = \Big(-\frac{3}{10}, \frac{1}{10}\Big)$, $\hat{X} + \hat{Y} = \Big(\frac{1}{5}, \frac{11}{10}\Big)$, $\hat{2}\hat{\times}\hat{Y} = \Big(-\frac{3}{5}, \frac{1}{5}\Big)$, $\hat{2}\hat{\times}(\hat{X} + \hat{Y}) = \Big(\frac{2}{5}, \frac{11}{5}\Big)$.

Definition 1.1.4. *If λ is a real number, the iso-multiplication of $\hat{X}$ by λ is defined as follows*

$$\lambda\hat{\times}\hat{X} = \lambda\hat{T}_1(\lambda)\hat{X} = \lambda\hat{T}_1(\lambda)\Big(\tfrac{x_1}{\hat{T}(x_1,x_2,\dots,x_n)}, \tfrac{x_2}{\hat{T}(x_1,x_2,\dots,x_n)}, \dots, \tfrac{x_n}{\hat{T}(x_1,x_2,\dots,x_n)}\Big)$$

$$= \Big(\lambda\hat{T}_1(\lambda)\tfrac{x_1}{\hat{T}(x_1,x_2,\dots,x_n)}, \lambda\hat{T}_1(\lambda)\tfrac{x_2}{\hat{T}(x_1,x_2,\dots,x_n)}, \dots, \lambda\hat{T}_1(\lambda)\tfrac{x_n}{\hat{T}(x_1,x_2,\dots,x_n)}\Big).$$

Example 1.1.5. *In $\hat{F}_{\mathbb{R}^4}$, let $\hat{T}(x) = x_2^2+3$, $x = (x_1,x_2,x_3,x_4) \in \mathbb{R}^4$, $\hat{T}_1(y) = 1+y^2$, $y \in \mathbb{R}$. Let also, $X = (1,-1,2,3)$. Then*

$$\hat{X} = \Big(\tfrac{1}{4}, -\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}\Big),$$

$$3\hat{\times}\hat{X} = 3\hat{T}_1(3)\Big(\tfrac{1}{4}, -\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}\Big) = 30\Big(\tfrac{1}{4}, -\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}\Big) = \Big(\tfrac{15}{2}, -\tfrac{15}{2}, 15, \tfrac{45}{2}\Big).$$

Exercise 1.1.6. *In $\hat{F}_{\mathbb{R}^2}$, let $\hat{T}(x) = x_1^2+x_2^2+1$, $x = (x_1,x_2) \in \mathbb{R}^2$, $\hat{T}_1(y) = 1+y^4$, $y \in \mathbb{R}$. Let also, $X = (1,0)$, $Y = (-1,-1)$. Find*

$$2\hat{\times}\hat{X} + \hat{3}\hat{\times}\hat{Y}.$$

Answer. $(16,-1)$.

Definition 1.1.7. *If $\hat{\lambda}$ is an iso-real iso-number, the multiplication of $\hat{X}$ by $\hat{\lambda}$ is defined as follows*

$$\hat{\lambda}\hat{X} = \tfrac{\lambda}{\hat{T}_1(x)}\hat{X} = \tfrac{\lambda}{\hat{T}_1(x)}\Big(\tfrac{x_1}{\hat{T}(x_1,x_2,\dots,x_n)}, \tfrac{x_2}{\hat{T}(x_1,x_2,\dots,x_n)}, \dots, \tfrac{x_n}{\hat{T}(x_1,x_2,\dots,x_n)}\Big)$$

$$= \Big(\tfrac{\lambda}{\hat{T}_1(x)}\tfrac{x_1}{\hat{T}(x_1,x_2,\dots,x_n)}, \tfrac{\lambda}{\hat{T}_1(x)}\tfrac{x_2}{\hat{T}(x_1,x_2,\dots,x_n)}, \dots, \tfrac{\lambda}{\hat{T}_1(x)}\tfrac{x_n}{\hat{T}(x_1,x_2,\dots,x_n)}\Big).$$

Example 1.1.8. *In $\hat{F}_{\mathbb{R}^5}$, let $\hat{T}(x) = x_1^2+x_2^4+1$, $x = (x_1,x_2,x_3,x_4,x_5) \in \mathbb{R}^5$, $\hat{T}_1(y) = 1+|y|$, $y \in \mathbb{R}$, $X = (1,-1,-1,0,1)$, $Y = (1,0,1,1,1)$. We will find*

$$4\hat{\times}(\hat{2}\hat{X} + \hat{3}\hat{\times}\hat{Y}).$$

We have

$$\hat{X} = \Big(\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}, 0, \tfrac{1}{3}\Big), \qquad \hat{Y} = \Big(\tfrac{1}{2}, 0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\Big),$$

$$\hat{2}\hat{X} = \tfrac{2}{\hat{T}_1(2)}\Big(\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}, 0, \tfrac{1}{3}\Big) = \tfrac{2}{3}\Big(\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}, 0, \tfrac{1}{3}\Big) = \Big(\tfrac{2}{9}, -\tfrac{2}{9}, -\tfrac{2}{9}, 0, \tfrac{2}{9}\Big),$$

$$\hat{3}\hat{\times}\hat{Y} = \tfrac{3}{\hat{T}_1(3)}T_1\hat{(}3)\Big(\tfrac{1}{2}, 0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\Big) = \Big(\tfrac{3}{2}, 0, \tfrac{3}{2}, \tfrac{3}{2}, \tfrac{3}{2}\Big),$$

$$\hat{4}\hat{\times}(\hat{2}\hat{X} + \hat{3}\hat{\times}\hat{Y}) = 4\hat{T}_1(4)\Big(\Big(\tfrac{2}{9}, -\tfrac{2}{9}, -\tfrac{2}{9}, 0, \tfrac{2}{9}\Big) + \Big(\tfrac{3}{2}, 0, \tfrac{3}{2}, \tfrac{3}{2}, \tfrac{3}{2}\Big)\Big)$$

$$= 20\Big(\tfrac{31}{18}, -\tfrac{2}{9}, \tfrac{23}{18}, \tfrac{3}{2}, \tfrac{31}{18}\Big) = \Big(\tfrac{310}{9}, -\tfrac{40}{9}, \tfrac{230}{9}, 30, \tfrac{310}{9}\Big).$$

Exercise 1.1.9. *In $\hat{F}_{\mathbb{R}^3}$, let $\hat{T}(x) = x_1^2 + |x_2| + 1$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\hat{T}_1(y) = 1 + |y|$, $y \in \mathbb{R}$, $X = (1, -1, -1)$, $Y = (-1, -1, 1)$. Find*

$$\hat{2}\hat{\times}(\hat{3}\hat{X} + 4\hat{\times}\hat{Y}).$$

Answer. $\left(-\frac{77}{6}, -\frac{83}{6}, \frac{77}{6}\right)$.

Definition 1.1.10. *If λ is a real number, the multiplication of $\hat{X}$ by λ is defined as follows*

$$\lambda\hat{X} = \lambda\left(\tfrac{x_1}{\hat{T}(x_1, x_2, \ldots, x_n)}, \tfrac{x_2}{\hat{T}(x_1, x_2, \ldots, x_n)}, \ldots, \tfrac{x_n}{\hat{T}(x_1, x_2, \ldots, x_n)}\right)$$

$$= \left(\lambda\tfrac{x_1}{\hat{T}(x_1, x_2, \ldots, x_n)}, \lambda\tfrac{x_2}{\hat{T}(x_1, x_2, \ldots, x_n)}, \ldots, \lambda\tfrac{x_n}{\hat{T}(x_1, x_2, \ldots, x_n)}\right).$$

Example 1.1.11. *In $\hat{F}_{\mathbb{R}^4}$, let $\hat{T}(x) = |x_1| + |x_2| + |x_3| + |x_4| + 4$, $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, $\hat{T}_1(y) = 3 + |y|$, $y \in \mathbb{R}$, $X = (1, -1, 0, 0)$, $Y = (0, 1, -1, 1)$. We will find*

$$A = 2(4\hat{\times}\hat{Y} + \hat{3}\hat{\times}\hat{X}) + \hat{2}\hat{Y}.$$

We have

$$\hat{X} = \left(\tfrac{1}{6}, -\tfrac{1}{6}, 0, 0\right), \qquad \hat{Y} = \left(0, \tfrac{1}{7}, -\tfrac{1}{7}, \tfrac{1}{7}\right),$$

$$4\hat{\times}\hat{Y} = 4\hat{T}_1(4)\left(0, \tfrac{1}{7}, -\tfrac{1}{7}, \tfrac{1}{7}\right) = (0, 4, -4, 4),$$

$$\hat{3}\hat{\times}\hat{Y} = \tfrac{3}{\hat{T}_1(3)}\hat{T}_1(3)\left(\tfrac{1}{6}, -\tfrac{1}{6}, 0, 0\right) = \left(\tfrac{1}{2}, -\tfrac{1}{2}, 0, 0\right),$$

$$4\hat{\times}\hat{Y} + \hat{3}\hat{\times}\hat{X} = (0, 4, -4, 4) + \left(\tfrac{1}{2}, -\tfrac{1}{2}, 0, 0\right) = \left(\tfrac{1}{2}, \tfrac{7}{2}, -4, 4\right),$$

$$2(4\hat{\times}\hat{Y} + \hat{3}\hat{\times}\hat{X}) = 2\left(\tfrac{1}{2}, \tfrac{7}{2}, -4, 4\right) = (1, 7, -8, 8),$$

$$\hat{2}\hat{Y} = \tfrac{2}{\hat{T}_1(2)}\left(0, \tfrac{1}{7}, -\tfrac{1}{7}, \tfrac{1}{7}\right) = \left(0, \tfrac{2}{35}, -\tfrac{2}{35}, \tfrac{2}{35}\right).$$

Then

$$A = (1, 7, -8, 8) + \left(0, \frac{2}{35}, -\frac{2}{35}, \frac{2}{35}\right) = \left(1, \frac{247}{35}, -\frac{282}{35}, \frac{282}{35}\right).$$

Exercise 1.1.12. *In $\hat{F}_{\mathbb{R}^2}$, let $\hat{T}(x) = |x_1| + |x_2| + 1$, $x = (x_1, x_2) \in \mathbb{R}^2$, $\hat{T}_1(y) = 1 + 2|y|$, $y \in \mathbb{R}$, $X = (1, -1)$, $Y = (1, 1)$. Find*

$$\hat{2}(\hat{3}\hat{\times}\hat{X} + \hat{4}\hat{Y}) - 2\hat{\times}\hat{Y}.$$

Answer. $\left(\frac{512}{135}, \frac{404}{135}\right)$.

If $\hat{a}$ and $\hat{b}$ are elements of $\hat{F}_{\mathbb{R}}$ then

$$\hat{a}\hat{\times}\hat{b} = \frac{a}{\hat{T}_1(a)}\hat{T}_1(a)\frac{b}{\hat{T}_1(b)} = \frac{ab}{\hat{T}_1(b)}$$

and

$$\hat{b}\hat{\times}\hat{a} = \frac{b}{\hat{T}_1(b)}\hat{T}_1(b)\frac{a}{\hat{T}_1(a)} = \frac{ab}{\hat{T}_1(a)}.$$

In other words, when the isotopic element $\hat{T}_1$ does not coincide with some constant, the iso-multiplication of the iso-real iso-numbers is not a commutative operation. Only in the case when $\hat{T}_1 \equiv \text{const}$ we have that the iso-multiplication of the iso-real iso-numbers is commutative.

Below we will suppose that $\hat{T}_1$ is a positive constant.

The defined above operations have the following properties: let $\hat{X}$, $\hat{Y}$, $\hat{Z} \in \hat{F}_{\mathbb{R}^n}$, $\hat{a} \in \hat{F}_{\mathbb{R}}$, $\hat{b} \in \hat{F}_{\mathbb{R}}$, $a \in \mathbb{R}$, $b \in \mathbb{R}$, then

1. $\hat{X} + \hat{Y} = \hat{Y} + \hat{X}$ (the iso-vector addition is commutative),

2. $\hat{X} + (\hat{Y} + \hat{Z}) = (\hat{X} + \hat{Y}) + \hat{Z}$ (the iso-vector addition is associative),

3. $(\hat{X} + \hat{Y}) + \hat{Z} = \hat{X} + (\hat{Y} + \hat{Z})$ (the iso-vector addition is distributive),

4. There is a unique vector $\hat{0} \equiv 0 = (0, 0, \ldots, 0)$, called the zero iso-vector, such that $\hat{X} + 0 = \hat{X}$, $0 = (0, 0, \ldots, 0)$,

5. For each $\hat{X} \in \hat{F}_{\mathbb{R}^n}$ there is a unique iso-vector $-\hat{X}$ such that $\hat{X} + (-\hat{X}) = \hat{0}$,

6. $1\hat{X} = \hat{X}$,

7. $1\hat{\times}\hat{X} = \hat{T}_1(1)\hat{X}$,

8. $\hat{a}\hat{\times}(\hat{b}\hat{\times}\hat{X}) = (\hat{a}\hat{\times}\hat{b})\hat{\times}\hat{X}$,

9. $\hat{a}\hat{\times}(\hat{b}\hat{X}) = (\hat{a}\hat{\times}\hat{b})\hat{X}$,

10. $\hat{a}\hat{\times}(b\hat{\times}\hat{X}) = (\hat{a}\hat{\times}b)\hat{\times}\hat{X}$,

11. $\hat{a}\hat{\times}(b\hat{X}) = (\hat{a}\hat{\times}b)\hat{X}$,

12. $\hat{a}(\hat{b}\hat{\times}\hat{X}) = (\hat{a}\hat{b})\hat{\times}\hat{X}$

13. $\hat{a}(\hat{b}\hat{X}) = (\hat{a}\hat{b})\hat{X}$,

14. $\hat{a}(b\hat{\times}\hat{X}) = (\hat{a}b)\hat{\times}\hat{X}$,

15. $\hat{a}(b\hat{X}) = (\hat{a}b)\hat{X}$,

16. $a\hat{\times}(\hat{b}\hat{\times}\hat{X}) = (a\hat{\times}\hat{b})\hat{\times}\hat{X}$,

17. $a\hat{\times}(\hat{b}\hat{X}) = (a\hat{\times}\hat{b})\hat{X}$,

18. $a\hat{\times}(b\hat{\times}\hat{X}) = (a\hat{\times}b)\hat{\times}\hat{X}$,

19. $a\hat{\times}(b\hat{X}) = (a\hat{\times}b)\hat{X}$,

20. $a(\hat{b}\hat{\times}\hat{X}) = (a\hat{b})\hat{\times}\hat{X}$,

21. $a(\hat{b}\hat{X}) = (a\hat{b})\hat{X}$,

22. $a(b\hat{\times}\hat{X}) = (ab)\hat{\times}\hat{X}$,

23. $a(b\hat{X}) = (ab)\hat{X}$,

24. $(\hat{a}+\hat{b})\hat{\times}\hat{X} = \hat{a}\hat{\times}\hat{X} + \hat{b}\hat{\times}\hat{X}$,

25. $(\hat{a}+\hat{b})\hat{X} = \hat{a}\hat{X} + \hat{b}\hat{X}$,

26. $(a+\hat{b})\hat{\times}\hat{X} = a\hat{\times}\hat{X} + \hat{b}\hat{\times}\hat{X}$,

27. $(a+\hat{b})\hat{X} = a\hat{X} + \hat{b}\hat{X}$,

28. $(\hat{a}+b)\hat{\times}\hat{X} = \hat{a}\hat{\times}\hat{X} + b\hat{\times}\hat{X}$,

29. $(\hat{a}+b)\hat{X} = \hat{a}\hat{X} + b\hat{X}$,

30. $(a+b)\hat{\times}\hat{X} = a\hat{\times}\hat{X} + b\hat{\times}\hat{X}$,

31. $(a+b)\hat{X} = a\hat{X} + b\hat{X}$.

Clearly, $\hat{0} = (0,0,\ldots,0)$ and, if $\hat{X} = (\hat{x}_1,\hat{x}_2,\ldots,\hat{x}_n)$, then

$$-\hat{X} = (-\hat{x}_1,-\hat{x}_2,\ldots,-\hat{x}_n).$$

We write

$$\hat{X} + (-\hat{Y}) = \hat{X} - \hat{Y}.$$

The iso-point $\hat{0}$ is called the iso-origin.

When we wish to emphasize that we are regarding a member of $\hat{F}_{\mathbb{R}^n}$ as part of the algebraic structure, we will speak of it as an iso-vector, otherwise, we will speak of it as an iso-point.

Iso-Length, Iso-Distance and Inner Iso-Product

Definition 1.1.13. *The iso-length of the iso-vector $\hat{X} = (\hat{x}_1,\hat{x}_2,\ldots,\hat{x}_n)$ is*

$$|\hat{X}\hat{|} = \frac{|X|}{\hat{T}(X)}\sqrt{\hat{T}_1}, \qquad X = (x_1,x_2,\ldots,x_n), \qquad |X|\sqrt{x_1^2+x_2^2+\cdots+x_n^2}.$$

The iso-distance between the iso-points $\hat{X}$ and $\hat{Y}$ is

$$|\hat{X}-\hat{Y}\hat{|}.$$

In particular, $|\hat{X}\hat{|}$ is the iso-distance between the iso-point $\hat{X}$ and the iso-origin $\hat{0}$. If $|\hat{X}\hat{|} = \hat{I}_1$, then the iso-vector $\hat{X}$ is an iso-unit iso-vector.

Example 1.1.14. *In* $\hat{F}_{\mathbb{R}^4}$, *let* $\hat{T}(x) = |x_1|^2 + 2$, $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, $\hat{T}_1 = 3$. *Let also* $X = (-1, 2, 1, -3)$, $Y = (1, 1, 0, -1)$. *Then*

$$\hat{X} = \left(-\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{3}, -1\right), \qquad \hat{Y} = \left(\tfrac{1}{3}, \tfrac{1}{3}, 0, -\tfrac{1}{3}\right),$$

$$|\hat{X}\hat{|} = \tfrac{\sqrt{15}}{3}\sqrt{3} = \sqrt{5}, \qquad |\hat{Y}\hat{|} = \tfrac{\sqrt{3}}{3}\sqrt{3} = 1,$$

$$\hat{X} - \hat{Y} = \left(-\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3}, -\tfrac{2}{3}\right), \qquad |\hat{X} - \hat{Y}\hat{|} = \tfrac{\sqrt{10}}{6}\sqrt{3} = \tfrac{\sqrt{30}}{6}.$$

Exercise 1.1.15. *In* $\hat{F}_{\mathbb{R}^2}$, *let* $\hat{T}(x) = |x_1 - x_2|^2 + 2$, $x = (x_1, x_2) \in \mathbb{R}^2$, $\hat{T}_1 = 4$, $X = (-1, 2)$, $Y = (1, -1)$, *Find*

$$|\hat{X}\hat{|}, \qquad |\hat{Y}\hat{|}, \qquad |\hat{X} - \hat{Y}\hat{|}.$$

Answer. $\frac{2\sqrt{5}}{11}$, $\frac{\sqrt{2}}{3}$, $2\frac{\sqrt{13}}{27}$.

Definition 1.1.16. *The inner iso-product of* $\hat{X} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$ *and* $\hat{Y} = (\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_n)$ *is*

$$\hat{X}\hat{\cdot}\hat{Y} = \hat{x}_1 \hat{\times} \hat{y}_1 + \hat{x}_2 \hat{\times} \hat{y}_2 + \cdots + \hat{x}_n \hat{\times} \hat{y}_n$$

$$= \tfrac{x_1}{\hat{T}(X)} \hat{T}_1 \tfrac{y_1}{\hat{T}(Y)} + \tfrac{x_2}{\hat{T}(X)} \hat{T}_1 \tfrac{y_2}{\hat{T}(Y)} + \cdots \tfrac{x_n}{\hat{T}(X)} \hat{T}_1 \tfrac{y_n}{\hat{T}(Y)},$$

$$X = (x_1, x_2, \ldots, x_n), \qquad Y = (y_1, y_2, \ldots, y_n).$$

Example 1.1.17. *In* $\hat{F}_{\mathbb{R}^3}$, *let* $\hat{T}(x) = |x_1 + x_2 + x_3| + 3$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\hat{T}_1 = 2$, $X = (1, -1, 2)$, $Y = (2, -3, 4)$. *We will find* $\hat{X}\hat{\cdot}\hat{Y}$. *We have*

$$\hat{T}(X) = 5, \qquad \hat{T}(Y) = 6,$$

$$\hat{X}\hat{\cdot}\hat{Y} = \tfrac{1}{5}2\tfrac{2}{6} + \tfrac{-1}{5}2\tfrac{-3}{6} + \tfrac{2}{5}2\tfrac{4}{6} = \tfrac{2}{15} + \tfrac{1}{5} + \tfrac{8}{15} = \tfrac{11}{15}.$$

Exercise 1.1.18. *In* $\hat{F}_{\mathbb{R}^2}$, *let* $\hat{T}(x) = |x_1| + |x_2| + 4$, $x = (x_1, x_2) \in \mathbb{R}^2$, $\hat{T}_1 = 3$, $X = (1, -1)$, $Y = (2, 2)$. *Find*

$$\hat{X}\hat{\cdot}\hat{Y}.$$

Answer. 0.

From the definition of the inner iso-product it follows that it can be represented in the form

$$\hat{X}\hat{\cdot}\hat{Y} = \frac{X \cdot Y}{\hat{T}(X)\hat{T}(Y)} \hat{T}_1.$$

Lemma 1.1.19. *(iso- Schwartz's inequality) If* $\hat{X}$ *and* $\hat{Y}$ *are any two iso-vectors in* $\hat{F}_{\mathbb{R}^n}$, *then*

$$|\hat{X}\hat{\cdot}\hat{Y}| \leq |\hat{X}| \hat{\times} |\hat{Y}|,$$

with equality if and only if one of the iso-vectors is an iso-scalar iso-multiple of the other.

Proof. We have

$(A3)\ \ |\hat{X}\hat{\cdot}\hat{Y}| = \left|\frac{X\cdot Y}{\hat{T}(X)\hat{T}(Y)}\hat{T}_1\right| = \frac{|X\cdot Y|}{\hat{T}(X)\hat{T}(Y)}\hat{T}_1,$

after we apply the classical Schwartz's inequality we get

$$|\hat{X}\hat{\cdot}\hat{Y}| \le \frac{|X||Y|}{\hat{T}(X)\hat{T}(Y)}\hat{T}_1 = \left|\frac{X}{\hat{T}(X)}\right|\hat{T}_1\left|\frac{Y}{\hat{T}(Y)}\right| = |\hat{X}|\hat{\times}|\hat{Y}|.$$

If $\hat{X} = \hat{t}\hat{\times}\hat{Y}$ for some $\hat{t} \in \hat{F}_{\mathbb{R}}$, then

$$\hat{X} = t\hat{Y} \qquad \text{or} \qquad \frac{X}{\hat{T}(X)} = t\frac{Y}{\hat{T}(Y)}.$$

From here and (A3) we obtain

$(A4)|\hat{X}\hat{\cdot}\hat{Y}| = \frac{|tY\cdot Y|}{\hat{T}^2(Y)}\hat{T}_1 = |t|\frac{|Y|^2}{\hat{T}^2(Y)}\hat{T}_1.$

On the other hand,

$(A4)|\hat{X}|\hat{\times}|\hat{Y}| = \frac{|X|}{\hat{T}(X)}\hat{T}_1\frac{|Y|}{\hat{T}(Y)} = \left|t\frac{Y}{\hat{T}(Y)}\right|\hat{T}_1\frac{|Y|}{\hat{T}(Y)} = |t|\frac{|Y|^2}{\hat{T}^2(Y)}\hat{T}_1.$

From (A3) and (A4) we conclude that

$(A5)|\hat{X}\hat{\cdot}\hat{Y}| = |\hat{X}|\hat{\times}|\hat{Y}|.$

Now we suppose (A5). Then

$$\frac{|X\cdot Y|}{\hat{T}(X)\hat{T}(Y)}\hat{T}_1 = \frac{|X|}{\hat{T}(X)}\hat{T}_1\frac{|Y|}{\hat{T}(Y)},$$

therefore

$$|X\cdot Y| = |X||Y|,$$

whereupon

$$X = tY$$

for some real number t. Consequently

$$\frac{X}{\hat{T}(X)} = \frac{t}{\hat{T}(X)}\frac{\hat{T}(Y)}{\hat{T}_1}\hat{T}_1\frac{Y}{\hat{T}(Y)} = \widehat{\left(\frac{t\hat{T}(Y)}{\hat{T}(X)}\right)}\hat{\times}\hat{Y}.$$

Let

$$\hat{\lambda} = \widehat{\left(\frac{t\hat{T}(Y)}{\hat{T}(X)}\right)}.$$

Then

$$\hat{X} = \hat{\lambda}\hat{\times}\hat{Y}.$$

$\square$

Lemma 1.1.20. *If $\hat{X}$ and $\hat{Y}$ are any two iso-vectors in $\hat{F}_{\mathbb{R}^n}$, then*

$$|\hat{X}\hat{\cdot}\hat{Y}\hat{|} \leq |\hat{X}|\hat{\times}|\hat{Y}\hat{|} = |\hat{X}\hat{|}\hat{\times}|\hat{Y}|,$$

with equality if and only if one of the iso-vectors is an iso-scalar iso-multiple of the other.

Proof. We have

$$|\hat{X}\hat{\cdot}\hat{Y}\hat{|} = \left|\frac{X\cdot Y}{\hat{T}(X)\hat{T}(Y)}\hat{T}_1\right|\sqrt{\hat{T}_1} = \frac{|X\cdot Y|}{\hat{T}(X)\hat{T}(Y)}\hat{T}_1^{\frac{3}{2}}.$$

Now we apply the Schwartz's inequality and we get

$$|\hat{X}\hat{\cdot}\hat{Y}\hat{|} \leq \frac{|X||Y|}{\hat{T}(X)\hat{T}(Y)}\hat{T}_1^{\frac{3}{2}} = \left|\frac{X}{\hat{T}(X)}\right|\hat{T}_1\left|\frac{Y}{\hat{T}(Y)}\right|\sqrt{\hat{T}_1} = |\hat{X}|\hat{\times}|\hat{Y}\hat{|}$$

$$= \left|\frac{X}{\hat{T}(X)}\right|\sqrt{\hat{T}_1}\hat{T}_1\left|\frac{Y}{\hat{T}(Y)}\right| = |\hat{X}\hat{|}\hat{\times}|\hat{Y}|.$$

Also,

$$|\hat{X}\hat{\cdot}\hat{Y}\hat{|} = |\hat{X}|\hat{\times}|\hat{Y}\hat{|} \qquad \Longleftrightarrow$$

$$|\hat{X}\hat{\cdot}\hat{Y}|\sqrt{\hat{T}_1} = |\hat{X}|\hat{\times}|\hat{Y}|\sqrt{\hat{T}_1} \qquad \Longleftrightarrow$$

$$|\hat{X}\hat{\cdot}\hat{Y}| = |\hat{X}|\hat{\times}|\hat{Y}|$$

if and only if one of the iso-vectors is an iso-scalar iso-multiple of the other. $\square$

Lemma 1.1.21. *If $\hat{X}$ and $\hat{Y}$ are any two iso-vectors in $\hat{F}_{\mathbb{R}^n}$, then*

$$|\hat{X}\cdot\hat{Y}\hat{|} \leq |\hat{X}||\hat{Y}\hat{|} = |\hat{X}\hat{|}|\hat{Y}|,$$

with equality if and only if one of the iso-vectors is an iso-scalar iso-multiple of the other.

Proof. We have

$$|\hat{X}\cdot\hat{Y}\hat{|} = \left|\frac{X}{\hat{T}(X)}\cdot\frac{Y}{\hat{T}(Y)}\right|\sqrt{\hat{T}_1} = \frac{|X\cdot Y|}{\hat{T}(X)\hat{T}(Y)}\sqrt{\hat{T}_1} \leq \frac{|X||Y|}{\hat{T}(X)\hat{T}(Y)}\sqrt{\hat{T}_1}$$

$$= \left|\frac{X}{\hat{T}(X)}\right|\left|\frac{Y}{\hat{T}(Y)}\right|\sqrt{\hat{T}_1} = |\hat{X}||\hat{Y}\hat{|} = \left|\frac{X}{\hat{T}(X)}\right|\sqrt{\hat{T}_1}\left|\frac{Y}{\hat{T}(Y)}\right| = |\hat{X}\hat{|}|\hat{Y}|.$$

Also,

$$|\hat{X}\cdot\hat{Y}\hat{|} = |\hat{X}||\hat{Y}\hat{|} \qquad \Longleftrightarrow$$

$$\left|\frac{X}{\hat{T}(X)}\cdot\frac{Y}{\hat{T}(Y)}\right|\sqrt{\hat{T}_1} = \left|\frac{X}{\hat{T}(X)}\right|\left|\frac{Y}{\hat{T}(Y)}\right|\sqrt{\hat{T}_1} \qquad \Longleftrightarrow$$

$$|X\cdot Y| = |X||Y|$$

if and only if there exists $t\in\mathbb{R}$ such that

$$X = tY \qquad \Longleftrightarrow$$

$$\frac{X}{\hat{T}(X)} = \frac{t\hat{T}(y)}{\hat{T}(X)}\frac{1}{\hat{T}_1}\hat{T}_1\frac{Y}{\hat{T}(Y)} \qquad \Longleftrightarrow$$

$$\hat{X} = \widehat{\left(\frac{t\hat{T}(y)}{\hat{T}(X)}\right)}\hat{\times}\hat{Y}.$$

We note that $\widehat{\left(\frac{t\hat{T}(y)}{\hat{T}(X)}\right)} \in \hat{F}_{\mathbb{R}}$. $\square$

Exercise 1.1.22. *If $\hat{X}$ and $\hat{Y}$ are any two iso-vectors in $\hat{F}_{\mathbb{R}^n}$, then*

$$|\hat{X} \cdot \hat{Y}| \leq |\hat{X}||\hat{Y}|,$$

with equality if and only if one of the iso-vectors is an iso-scalar iso-multiple of the other.

Theorem 1.1.23. *(Iso-Triangle Inequality) If $\hat{X}$ and $\hat{Y}$ are in $\hat{F}_{\mathbb{R}^n}$, then*

$$|\hat{X}+\hat{Y}\hat{|} \leq |\hat{X}\hat{|}+|\hat{Y}\hat{|},$$

with equality if and only if one of the iso-vectors is a nonegative iso-scalar iso-multiple of the other.

Proof. We have

$$|\hat{X}+\hat{Y}\hat{|} = \left|\frac{X}{\hat{T}(X)}+\frac{Y}{\hat{T}(Y)}\right|\sqrt{\hat{T}_1} \leq \left(\left|\frac{X}{\hat{T}(X)}\right|+\left|\frac{Y}{\hat{T}(Y)}\right|\right)\sqrt{\hat{T}_1} = |\hat{X}\hat{|}+|\hat{Y}\hat{|}.$$

Also,

$$|\hat{X}+\hat{Y}\hat{|} = |\hat{X}\hat{|}+|\hat{Y}\hat{|} \qquad \Longleftrightarrow$$

$$\left|\frac{X}{\hat{T}(X)}+\frac{Y}{\hat{T}(Y)}\right| = \left|\frac{X}{\hat{T}(X)}\right|+\left|\frac{Y}{\hat{T}(Y)}\right|$$

if and only if there exists $t \geq 0$ such that

$$\frac{X}{\hat{T}(X)} = t\frac{Y}{\hat{T}(Y)} \qquad \Longleftrightarrow$$

$$\frac{X}{\hat{T}(X)} = \frac{t}{\hat{T}_1}\hat{T}_1\frac{Y}{\hat{T}(Y)} \qquad \Longleftrightarrow$$

$$\hat{X} = \hat{t}\hat{\times}\hat{Y}.$$

We note that $\hat{t} \in \hat{F}_{\mathbb{R}}$ and $\hat{t} \geq 0$. $\square$

Exercise 1.1.24. *(Iso-Triangle Inequality) If $\hat{X}$ and $\hat{Y}$ are in $\hat{F}_{\mathbb{R}^n}$, then*

$$|\hat{X}+\hat{Y}| \leq |\hat{X}|+|\hat{Y}|,$$

with equality if and only if one of the iso-vectors is a nonegative iso-scalar iso-multiple of the other.

Corollary 1.1.25. *If $\hat{X}$, $\hat{Y}$ and $\hat{Z}$ are in $\hat{F}_{\mathbb{R}^n}$, then*

$$|\hat{X}-\hat{Z}\hat{|} \leq |\hat{X}-\hat{Y}\hat{|}+|\hat{Y}-\hat{Z}\hat{|},$$

$$|\hat{X}-\hat{Z}| \leq |\hat{X}-\hat{Y}|+|\hat{Y}-\hat{Z}|.$$

Corollary 1.1.26. *If $\hat{X}$ and $\hat{Y}$ are in $\hat{F}_{\mathbb{R}^n}$, then*

$$|\hat{X}-\hat{Y}\hat{|} \geq ||\hat{X}\hat{|}-|\hat{Y}\hat{|}|,$$

$$|\hat{X}-\hat{Y}| \geq ||\hat{X}|-|\hat{Y}||.$$

The next theorem lists some of the properties of the iso-length, the iso-distance, and the inner iso-product that follow directly from their definitions. We leave its proof to the reader.

Theorem 1.1.27. *If $\hat{X}$, $\hat{Y}$ and $\hat{Z}$ are members of $\hat{F}_{\mathbb{R}^n}$, and $\hat{a} \in \hat{F}_{|R}$, then*

1. $|\hat{a}\hat{\times}\hat{X}\hat{|} = |\hat{a}|\hat{\times}|\hat{X}\hat{|} = |\hat{a}\hat{|}\hat{\times}|\hat{X}|$,

2. $|\hat{a}\hat{\times}\hat{X}| = |\hat{a}|\hat{\times}|\hat{X}|$,

3. $|\hat{a}\hat{X}\hat{|} = |\hat{a}||\hat{X}\hat{|} = |\hat{a}\hat{|}|\hat{X}|$,

4. $|\hat{a}\hat{X}| = |\hat{a}||\hat{X}|$,

5. $|a\hat{\times}\hat{X}\hat{|} = |a|\hat{\times}|\hat{X}\hat{|}$,

6. $|a\hat{\times}\hat{X}| = |a|\hat{\times}|\hat{X}|$,

7. $|a\hat{X}\hat{|} = |a||\hat{X}\hat{|}$,

8. $|a\hat{X}| = |a||\hat{X}|$,

9. $|\hat{X}\hat{|} \geq \hat{0}$, *with equality if and only if* $\hat{X} = \hat{0}$,

10. $|\hat{X}| \geq 0$, *with equality if and only if* $\hat{X} = \hat{0}$,

11. $|\hat{X}-\hat{Y}\hat{|} \geq 0$, *with equality if and only if* $\hat{X} = \hat{Y}$,

12. $|\hat{X}-\hat{Y} \geq 0$, *with equality if and only if* $\hat{X} = \hat{Y}$,

13. $\hat{X}\hat{\cdot}\hat{Y} = \hat{Y}\hat{\cdot}\hat{X}$,

14. $\hat{X}\cdot\hat{Y} = \hat{Y}\cdot\hat{X}$,

15. $\hat{X}\hat{\cdot}(\hat{Y}+\hat{Z}) = \hat{X}\hat{\cdot}\hat{Y}+\hat{X}\hat{\cdot}\hat{Z}$,

16. $\hat{X}\cdot(\hat{Y}+\hat{Z}) = \hat{X}\cdot\hat{Y}+\hat{X}\cdot\hat{Z}$,

17. $(\hat{a}\hat{\times}\hat{X})\hat{\cdot}\hat{Y} = \hat{X}\hat{\cdot}(\hat{a}\hat{\times}\hat{Y}) = \hat{a}\hat{\times}(\hat{X}\hat{\cdot}\hat{Y})$,

18. $(\hat{a}\hat{X})\hat{\cdot}\hat{Y} = \hat{X}\hat{\cdot}(\hat{a}\hat{Y}) = \hat{a}(\hat{X}\hat{\cdot}\hat{Y})$,

19. $(a\hat{\times}\hat{X})\hat{\cdot}\hat{Y} = \hat{X}\hat{\cdot}(a\hat{\times}\hat{Y}) = a\hat{\times}(\hat{X}\hat{\cdot}\hat{Y})$,

20. $(a\hat{X})\hat{\cdot}\hat{Y} = \hat{X}\hat{\cdot}(a\hat{Y}) = a(\hat{X}\hat{\cdot}\hat{Y})$,

21. $(\hat{a}\hat{\times}\hat{X})\cdot\hat{Y} = \hat{X}\cdot(\hat{a}\hat{\times}\hat{Y}) = \hat{a}\hat{\times}(\hat{X}\cdot\hat{Y})$,

22. $(\hat{a}\hat{X})\cdot\hat{Y} = \hat{X}\cdot(\hat{a}\hat{Y}) = \hat{a}(\hat{X}\cdot\hat{Y})$,

23. $(a\hat{\times}\hat{X})\cdot\hat{Y} = \hat{X}\cdot(a\hat{\times}\hat{Y}) = a\hat{\times}(\hat{X}\cdot\hat{Y})$,

24. $(a\hat{X})\cdot\hat{Y} = \hat{X}\cdot(a\hat{Y}) = a(\hat{X}\cdot\hat{Y})$.

Iso-Line Segments in $\hat{F}_{\mathbb{R}^n}$

The equation of an iso-line through an iso-point $\hat{X}_0 = (\hat{x}_0, \hat{y}_0, \hat{z}_0)$ in $\hat{F}_{\mathbb{R}^3}$ can be written parametrically as

$(A6)\hat{X} = \hat{X}_0 + \hat{t}\hat{\times}\hat{U}, \qquad \hat{t} \in \hat{F}_{\mathbb{R}},$

where $\hat{U} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$ and $\hat{u}_1$, $\hat{u}_2$, and $\hat{u}_3$ are not all zero. We will write this in the iso-coordinate form

$$\hat{x}_1 = \hat{x}_0 + \hat{t}\hat{\times}\hat{u}_1,$$

$$\hat{x}_2 = \hat{y}_0 + \hat{t}\hat{\times}\hat{u}_2,$$

$$\hat{x}_3 = \hat{z}_0 + \hat{t}\hat{\times}\hat{u}_3,$$

or

$$\frac{x_1}{\hat{T}(x_1,x_2,x_3)} = \frac{x_0}{\hat{T}(x_0,y_0,z_0)} + t\frac{u_1}{\hat{T}(u_1,u_2,u_3)},$$

$$\frac{x_2}{\hat{T}(x_1,x_2,x_3)} = \frac{y_0}{\hat{T}(x_0,y_0,z_0)} + t\frac{u_2}{\hat{T}(u_1,u_2,u_3)},$$

$$\frac{x_3}{\hat{T}(x_1,x_2,x_3)} = \frac{z_0}{\hat{T}(x_0,y_0,z_0)} + t\frac{u_3}{\hat{T}(u_1,u_2,u_3)}.$$

We say that the iso-line is through $\hat{X}_0$ in the direction $\hat{U}$.

There are many ways to represent a given iso-line parameterically. For example,

$$\hat{X} = \hat{X}_0 + \hat{s}\hat{\times}\hat{V}, \qquad \hat{s} \in \hat{F}_{\mathbb{R}},$$

represents the same iso-line as (A6) if and only if $\hat{V} = \hat{a}\hat{\times}\hat{U}$ for some nonzero iso-real iso-number $\hat{a}$.

To write the parametric equation of an iso-line through two iso-points $\hat{X}_0$ and $\hat{X}_1$ in $\hat{F}_{\mathbb{R}^3}$, we take $\hat{U} = \hat{X}_1 - \hat{X}_0$, which yields

$$\hat{X} = \hat{X}_0 + \hat{t}\hat{\times}(\hat{X}_1 - \hat{X}_0) = \hat{t}\hat{\times}\hat{X}_1 + (\hat{I}_1 - \hat{t})\hat{\times}\hat{X}_0, \qquad \hat{t} \in \hat{F}_{\mathbb{R}}.$$

The iso-line segment consists of those iso-points for which $\hat{0} \le \hat{t} \le \hat{I}_1$.

Example 1.1.28. *Let* $\hat{T}(x) = |x_1| + 1$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\hat{T}_1 = 2$, $X_0 = (-1, 3, 1)$, $U = (2, -4, 0)$. *Then*

$$\hat{X}_0 = \Big(-\frac{1}{2}, \frac{3}{2}, -\frac{1}{2}\Big), \qquad \hat{U} = \Big(\frac{2}{3}, -\frac{4}{3}, 0\Big).$$

The iso-line segment is

$$\hat{X} = \hat{X}_0 + \hat{t}\hat{\times}\hat{U},$$

which we can rewrite in the form

$$\frac{x_1}{|x_1|+1} = -\frac{1}{2} + \frac{2}{3}t,$$

$$\frac{x_2}{|x_1|+1} = \frac{3}{2} - \frac{4}{3}t,$$

$$\frac{x_3}{|x_1|+1} = -\frac{1}{2}.$$

1. case $x_1 \geq 0$. *Then we have*

$$x_1 = \left(-\frac{1}{2} + \frac{2}{3}t\right)(x_1 + 1),$$

$$x_2 = \left(\frac{3}{2} - \frac{4}{3}t\right)(x_1 + 1),$$

$$x_3 = -\frac{1}{2}(x_1 + 1),$$

from where

$$x_1 = \frac{-3+4t}{9-4t},$$

$$x_2 = \frac{(9-8t)(15-4t)}{36},$$

$$x_3 = \frac{4t-15}{12}, \qquad t \in \left[\frac{3}{4}, \frac{9}{4}\right).$$

2. case $x_1 \leq 0$. *Then we have*

$$x_1 = \left(-\frac{1}{2} + \frac{2}{3}t\right)(-x_1 + 1),$$

$$x_2 = \left(\frac{3}{2} - \frac{4}{3}t\right)(-x_1 + 1),$$

$$x_3 = -\frac{1}{2}(-x_1 + 1),$$

whereupon we get

$$x_1 = \frac{4t-3}{4t+3},$$

$$x_2 = \frac{9-8t}{4t+3},$$

$$x_3 = -\frac{3}{4t+3}, \qquad t \in \left(-\frac{3}{4}, \frac{3}{4}\right].$$

Definition 1.1.29. *Suppose that $\hat{X}_0$ and $\hat{U}$ are in $\hat{F}_{\mathbb{R}^n}$ and $\hat{U} \neq \hat{0}$. Then the iso-line through $\hat{X}_0$ in the direction of $\hat{U}$ is the set of all iso-points in $\hat{F}_{\mathbb{R}^n}$ of the form*

$$\hat{X} = \hat{X}_0 + \hat{t}\hat{\times}\hat{U}, \qquad \hat{t} \in \hat{F}_{\mathbb{R}}.$$

A set of iso-points of the form

$$\hat{X} = \hat{X}_0 + \hat{t}\hat{\times}\hat{U}, \qquad \hat{t}_1 \leq \hat{t} \leq \hat{t}_2,$$

is called an iso-line segment. In particular, the iso-line segment from $\hat{X}_0$ to $\hat{X}_1$ is the set of iso-points of the form

$$\hat{X} = \hat{X}_0 + \hat{t}\hat{\times}(\hat{X}_1 - \hat{X}_0) = \hat{t}\hat{\times}\hat{X}_1 + (\hat{I}_1 - \hat{t})\hat{\times}\hat{X}_0, \qquad \hat{0} \leq \hat{t} \leq \hat{I}_1.$$

Iso-neighborhood and Iso-open sets in $\hat{F}_{\mathbb{R}^n}$

Having defined iso-distance in $\hat{F}_{\mathbb{R}^n}$, we are now able to say what we mean by an iso-neighborhood of an iso-point in $\hat{F}_{\mathbb{R}^n}$.

Definition 1.1.30. *If $\hat{\varepsilon} > 0$, $\hat{\varepsilon}$-iso-neighborhood of an iso-point $\hat{X}_0$ in $\hat{F}_{\mathbb{R}^n}$ is the set*

$$\hat{N}_{\varepsilon}(\hat{X}_0) = \{\hat{X} \in \hat{F}_{\mathbb{R}^n} : |\hat{X} - \hat{X}_0\hat{|} < \hat{\varepsilon}\}.$$

In $\hat{F}_{\mathbb{R}^3}$ it is the inside, but not the surface of the iso-sphere of iso-radius $\hat{\varepsilon}$ about $\hat{X}_0$.

Example 1.1.31. *In $\hat{F}_{\mathbb{R}^3}$, let $\hat{T}(x) = |x_1| + |x_2| + |x_3| + 2$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $X_0 = (-1, 2, 3)$, $\hat{T}_1 = 4$, $\varepsilon = 3$. Then*

$$\hat{X}_0 = \left(-\tfrac{1}{8}, \tfrac{1}{4}, \tfrac{3}{8}\right),$$

$$\hat{X} = \left(\tfrac{x_1}{|x_1|+|x_2|+|x_3|+2}, \tfrac{x_2}{|x_1|+|x_2|+|x_3|+2}, \tfrac{x_3}{|x_1|+|x_2|+|x_3|+2}\right),$$

$$\hat{\varepsilon} = \tfrac{3}{4}.$$

From here

$$|\hat{X} - \hat{X}_0\hat{|} < \hat{\varepsilon} \qquad \Longleftrightarrow \qquad |\hat{X} - \hat{X}_0|2 < \tfrac{3}{4} \qquad \Longleftrightarrow$$

$$\left(\tfrac{x_1}{|x_1|+|x_2|+|x_3|+2} + \tfrac{1}{8}\right)^2 + \left(\tfrac{x_2}{|x_1|+|x_2|+|x_3|+2} - \tfrac{1}{4}\right)^2 + \left(\tfrac{x_3}{|x_1|+|x_2|+|x_3|+2} - \tfrac{3}{8}\right)^2 < \tfrac{9}{64},$$

or

$$\hat{N}_{\varepsilon}(\hat{X}_0) = \Big\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \left(\tfrac{x_1}{|x_1|+|x_2|+|x_3|+2} + \tfrac{1}{8}\right)^2$$

$$+\left(\tfrac{x_2}{|x_1|+|x_2|+|x_3|+2} - \tfrac{1}{4}\right)^2 + \left(\tfrac{x_3}{|x_1|+|x_2|+|x_3|+2} - \tfrac{3}{8}\right)^2 < \tfrac{9}{64}\Big\}.$$

Definition 1.1.32. *The iso-open n - ball of radius $\hat{r}$ about $\hat{X}_0$ is the set*

$$\hat{B}_{\hat{r}}(\hat{X}_0) = \{\hat{X} : |\hat{X} - \hat{X}_0\hat{|} < \hat{r}\}.$$

The iso-sphere $\hat{S}_{\hat{r}}(\hat{X}_0)$ of radius $\hat{r}$ and iso-centre $\hat{X}_0$ is the set

$$\hat{S}_{\hat{r}}(\hat{X}_0) = \{\hat{X} : |\hat{X} - \hat{X}_0| = \hat{r}\}.$$

Lemma 1.1.33. *If $\hat{X}_1$ and $\hat{X}_2$ are in $\hat{B}_{\hat{r}}(\hat{X}_0)$ for some $\hat{r} > 0$, then so is every iso-point on the iso-line segment from $\hat{X}_1$ to $\hat{X}_2$.*

Proof. From $\hat{X}_1, \hat{X}_2 \in \hat{B}_{\hat{r}}(\hat{X}_0)$ it follows that

$$|\hat{X}_1 - \hat{X}_0\hat{|} < \hat{r}, \qquad |\hat{X}_2 - \hat{X}_0\hat{|} < \hat{r},$$

or

$$\Big|\frac{X_1}{\hat{T}(X_1)} - \frac{X_0}{\hat{T}(X_0)}\Big|\sqrt{\hat{T}_1} < \frac{r}{\hat{T}_1}, \qquad \Big|\frac{X_2}{\hat{T}(X_2)} - \frac{X_0}{\hat{T}(X_0)}\Big|\sqrt{\hat{T}_1} < \frac{r}{\hat{T}_1}.$$

The iso-line segment is given by

$$\hat{X} = \hat{t}\hat{\times}\hat{X}_2 + (\hat{I}_1 - \hat{t})\hat{\times}\hat{X}_1, \qquad \hat{0} < \hat{t} < \hat{T}_1$$

or

$$\frac{X}{\hat{T}(X)} = t\frac{X_2}{\hat{T}(X_2)} + (1-t)\frac{X_1}{\hat{T}(X_1)},$$

and from here

$$\begin{aligned}
|\hat{X} - \hat{X}_0\hat{|} &= \left|\tfrac{X}{\hat{T}(X)} - \tfrac{X_0}{\hat{T}(X_0)}\right|\sqrt{\hat{T}_1} \\
&= \left|t\tfrac{X_2}{\hat{T}(X_2)} + (1-t)\tfrac{X_1}{\hat{T}(X_1)} - t\tfrac{X_0}{\hat{T}(X_0)} - (1-t)\tfrac{X_0}{\hat{T}(X_0)}\right|\sqrt{\hat{T}_1} \\
&= \left|t\left(\tfrac{X_2}{\hat{T}(X_2)} - \tfrac{X_0}{\hat{T}(X_0)}\right) + (1-t)\left(\tfrac{X_2}{\hat{T}(X_2)} - \tfrac{X_0}{\hat{T}(X_0)}\right)\right|\sqrt{\hat{T}_1} \\
&\le t\left|\tfrac{X_2}{\hat{T}(X_2)} - \tfrac{X_0}{\hat{T}(X_0)}\right|\sqrt{\hat{T}_1} + (1-t)\left|\tfrac{X_2}{\hat{T}(X_2)} - \tfrac{X_0}{\hat{T}(X_0)}\right|\sqrt{\hat{T}_1} \\
&< t\tfrac{r}{\hat{T}_1} + (1-t)\tfrac{r}{\hat{T}_1} = \tfrac{r}{\hat{T}_1} = \hat{r}.
\end{aligned}$$

□

Definition 1.1.34. *A sequence of iso-points* $\{\hat{X}_l\}_{l=1}^{\infty}$ *in* $\hat{F}_{\mathbb{R}^n}$ *converges to the limit* $\hat{X}$ *if*

$$\lim_{l\longrightarrow\infty} |\hat{X}_l - \hat{X}\hat{|} = \hat{0}.$$

In this case we will write

$$\lim_{l\longrightarrow\infty} \hat{X}_l = \hat{X}.$$

Remark 1.1.35. *Let* $X_l = (l+1, l+2, \ldots, l+n)$*. Then the sequence* $\{X_l\}_{l=1}^{\infty}$ *is not convergent in* $\mathbb{R}^n$*. Also, if* $\hat{T}(x) = x_1^2 + x_2^2 + \cdots + x_n^2 + 2$*,* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$*, then*

$$\hat{X}_l = \Big(\tfrac{l+1}{(l+1)^2+(l+2)^2+\cdots+(l+n)^2+2}, \tfrac{l+2}{(l+1)^2+(l+2)^2+\cdots+(l+n)^2+2},$$

$$\ldots, \tfrac{l+n}{(l+1)^2+(l+2)^2+\cdots+(l+n)^2+2}\Big),$$

and the sequence $\{\hat{X}_l\}_{l=1}^{\infty}$ *is convergent to* $(0,0,\ldots,0)$*.*

If $X_l = \left(\frac{1}{l}, \frac{1}{l}, \ldots, \frac{1}{l}\right)$*, then the sequence* $\{X_l\}_{l=1}^{\infty}$ *is a convergent sequence in* $\mathbb{R}^n$ *to* $(0,0,\ldots,0)$*. Also, if* $\hat{T}(x) = \frac{x_1^4}{1+x_2^2}$*,* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \backslash \{(0,0,\ldots,0)\}$*, then*

$$\hat{X}_l = \Big(l(l^2+1), l(l^2+1), \ldots, l(l^2+1)\Big)$$

which is not a convergent sequence.

Theorem 1.1.36. *Let* $\hat{X}_l = (x_{1l}, x_{2l}, \ldots, x_{nl})$, $\hat{X} = (x_1, x_2, \ldots, x_n)$. *Then*

$$\lim_{l \longrightarrow \infty} \hat{X}_l = \hat{X}$$

if and only if

$$\lim_{l \longrightarrow \infty} \hat{x}_{il} = \hat{x}_i, \qquad i = 1, 2, \ldots, n.$$

Proof. 1. Let

$$\lim_{l \longrightarrow \infty} \hat{X}_l = \hat{X}$$

and $\hat{\varepsilon} = \frac{\varepsilon}{\hat{T}_1} > 0$ be fixed. Then there exists $L = L(\hat{\varepsilon})$ such that for every $l > L$ we have

$$|\hat{X}_l - \hat{X}\hat{|} < \hat{\varepsilon} \qquad \text{or}$$

$$\left| \frac{X_l}{\hat{T}(X_l)} - \frac{X}{\hat{T}(X)} \right| \sqrt{\hat{T}_1} < \frac{\varepsilon}{\hat{T}_1} \qquad \text{or}$$

$$\sqrt{\sum_{i=1}^{n} \left(\frac{x_{il}}{\hat{T}(X_l)} - \frac{x_i}{\hat{T}(X)} \right)^2} \sqrt{\hat{T}_1} < \frac{\varepsilon}{\hat{T}_1},$$

whereupon

$$\left| \frac{x_{il}}{\hat{T}(X_l)} - \frac{x_i}{\hat{T}(X)} \right| \sqrt{\hat{T}_1}, \qquad i = 1, 2, \ldots, n, \qquad \text{or}$$

$$|\hat{x}_{il} - \hat{x}_i\hat{|} < \hat{\varepsilon}, \qquad i = 1, 2, \ldots, n.$$

2. Let now

$$\lim_{l \longrightarrow \infty} \hat{x}_{il} = \hat{x}_i, \qquad i = 1, 2, \ldots, n.$$

Let also, $\hat{\varepsilon} = \frac{\varepsilon}{\hat{T}_1} > 0$ be arbitrarily chosen. Then there exists $L = L(\hat{\varepsilon}) > 0$ such that for every $l > L$ we have

$$|\hat{x}_{il} - \hat{x}_i\hat{|} < \frac{\hat{\varepsilon}}{\sqrt{n}}, \qquad i = 1, 2, \ldots, n, \qquad \text{or}$$

$$\left| \frac{x_{il}}{\hat{T}(X_l)} - \frac{x_i}{\hat{T}(X)} \right| \sqrt{\hat{T}_1} < \frac{\varepsilon}{\sqrt{n}\hat{T}_1}, \qquad i = 1, 2, \ldots, n,$$

$$\left(\frac{x_{il}}{\hat{T}(X_l)} - \frac{x_i}{\hat{T}(X)} \right)^2 \hat{T}_1 < \frac{\varepsilon^2}{n\hat{T}_1^2}, \qquad i = 1, 2, \ldots, n.$$

From here we obtain the following inequality

$$\sum_{i=1}^{n} \left(\frac{x_{il}}{\hat{T}(X_l)} - \frac{x_i}{\hat{T}(X)} \right)^2 \hat{T}_1 < \frac{\varepsilon^2}{\hat{T}_1^2},$$

therefore

$$|\hat{X}_l - \hat{X}\hat{|} < \hat{\varepsilon}.$$

$\square$

Theorem 1.1.37. *Let $\{X_l\}_{l=1}^{\infty}$ be a convergent sequence in $\mathbb{R}^n$ to the point Y, $Y \neq 0$, $X_l \neq 0$ for every $l = 1, 2, \ldots$. Let also, the sequence $\{\hat{T}(X_l)\}_{l=1}^{\infty}$ be a convergent sequence in $\mathbb{R}^n$ to the origin. Then the sequence $\{\hat{X}_l\}_{l=1}^{\infty}$ is not a convergent sequence in $\hat{F}_{\mathbb{R}^n}$.*

Proof. Let us suppose that the sequence $\{\hat{X}_l\}_{l=1}^{\infty}$ is a convergent sequence in $\hat{F}_{\mathbb{R}^n}$ to the element $\hat{X}$. We fix $\hat{\varepsilon} > 0$, $\hat{\varepsilon} \in \hat{F}_{\mathbb{R}}$. Then there exists $L = L(\hat{\varepsilon}) > 0$ such that

$$|\hat{X}_l - \hat{X}\hat{|} < \hat{\varepsilon} \qquad \text{for} \qquad \forall l > L \qquad \text{or}$$

$$\left|\tfrac{X_l}{\hat{T}(X_l)} - \tfrac{X}{\hat{T}(X)}\right|\sqrt{\hat{T}_1} < \tfrac{\varepsilon}{\hat{T}_1} \qquad \text{for} \qquad \forall l > L,$$

whereupon

$$\left|\frac{X_l}{\hat{T}(X_l)}\right| - \left|\frac{X}{\hat{T}(X)}\right| < \frac{\varepsilon}{\hat{T}_1^{\frac{3}{2}}} \qquad \text{for} \qquad \forall l > L,$$

consequently

$$\hat{T}(X_l) > \frac{|X_l|}{\frac{\varepsilon}{\hat{T}_1^{\frac{3}{2}}} + \left|\frac{X}{\hat{T}(X)}\right|} \qquad \text{for} \qquad \forall l > L,$$

which is a contradiction because

$$\lim_{l \longrightarrow \infty} \hat{T}(X_l) = 0$$

and

$$\frac{|X_l|}{\frac{\varepsilon}{\hat{T}_1^{\frac{3}{2}}} + \left|\frac{X}{\hat{T}(X)}\right|} > 0 \qquad \text{for} \qquad \forall l > L.$$

Therefore the sequence $\{\hat{X}_l\}_{l=1}^{\infty}$ is not a convergent sequence in $\hat{F}_{\mathbb{R}^n}$. □

Theorem 1.1.38. *Let $\{X_l\}_{l=1}^{\infty}$ be a convergent sequence in $\mathbb{R}^n$ to the point $X \in \mathbb{R}^n$, let also $\{\hat{T}(X_l)\}_{l=1}^{\infty}$ be a convergent sequence to $B \in \mathbb{R}^n$, $B \neq 0$. Then the sequence $\{\hat{X}_l\}_{l=1}^{\infty}$ is a convergent sequence and*

$$\lim_{l \longrightarrow \infty} \hat{X}_l = \frac{X}{B}.$$

Proof. We have

$$\lim_{l \longrightarrow \infty} \hat{X}_l = \lim_{l \longrightarrow \infty} \frac{X_l}{\hat{T}(X_l)} = \frac{\lim_{l \longrightarrow \infty} X_l}{\lim_{l \longrightarrow \infty} \hat{T}(X_l)} = \frac{X}{B}.$$

□

Corollary 1.1.39. *In addition, if $\hat{T} : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a continuous function, then $B = \hat{T}(X)$ and*

$$\lim_{l \longrightarrow \infty} \hat{X}_l = \hat{X}.$$

Next theorem lists some of the properties of the convergent sequences that follow directly from the definition for convergent sequences.

Theorem 1.1.40. *Let* $\lim_{l \longrightarrow \infty} \hat{X}_l = \hat{X}_0$, $\lim_{l \longrightarrow \infty} \hat{Y}_l = \hat{Y}_0$, *then*

1. $\lim_{l\longrightarrow\infty}(\hat{X}_l \pm \hat{Y}_l) = \hat{X}_0 + \hat{Y}_0$,

2. $\lim_{l\longrightarrow\infty}\hat{\alpha}\hat{\times}\hat{X}_l = \hat{\alpha}\hat{\times}\hat{X}_0$,

3. $\lim_{l\longrightarrow\infty}\alpha\hat{\times}\hat{X}_l = \alpha\hat{\times}\hat{X}_0$,

4. $\lim_{l\longrightarrow\infty}\alpha\hat{X}_l = \alpha\hat{X}_0$,

5. $\lim_{l\longrightarrow\infty}\hat{\alpha}\hat{X}_l = \alpha\hat{X}_0$.

Exercise 1.1.41. *In $\hat{F}_{\mathbb{R}^n}$, let $\hat{T}(x) = \sum_{i=1}^{n}|x_i|^3 + 4$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Investigate for convergence the sequence $\{\hat{X}_l\}_{l=1}^{\infty}$, where*

1. $X_l = (l, l-1, l-2, \ldots, l-n)$,

2. $X_l = (\sqrt{l}, \sqrt{l+1}, \sqrt{l+2}, \ldots, \sqrt{l+n})$,

3. $X_l = (\sqrt{l+1}-\sqrt{l}, 2(\sqrt{l+1}-\sqrt{l}), 3(\sqrt{l+1}-\sqrt{l}), \ldots, n(\sqrt{l+1}-\sqrt{l}))$,

4. $X_l = (\sqrt{l^2+1}-l, 2(\sqrt{l^2+1}-l), 3(\sqrt{l^2+1}-l), \ldots, n(\sqrt{l^2+1}-l))$,

5. $X_l = \left(\frac{1}{2n}\sqrt[3]{1-l^3}, \frac{1}{2n-1}\sqrt[3]{1-l^3}, \frac{1}{2n-2}\sqrt[3]{1-l^3}, \ldots, \frac{1}{n+1}\sqrt[3]{1-l^3}\right)$.

Definition 1.1.42. *A sequence $\{\hat{X}_l\}_{l=1}^{\infty}$ of elements of $\hat{F}_{\mathbb{R}^n}$ will be called a bounded sequence if there exists an iso-real iso-number $\hat{M} \in \hat{F}_{\mathbb{R}}$ such that*

$$|\hat{X}_l\hat{|} \leq \hat{M} \qquad \text{for} \qquad \forall l \in \mathbb{N}.$$

Theorem 1.1.43. *Let $\{X_l\}_{l=1}^{\infty}$ be a bounded sequence in $\mathbb{R}^n$, let also the sequence $\{\hat{T}(X_l)\}_{l=1}^{\infty}$ is a bounded below sequence in $\mathbb{R}^n$ by the positive constant P. Then the sequence $\{\hat{X}_l\}_{l=1}^{\infty}$ is a bounded sequence.*

Proof. There exists a positive constant M such that

$$|X_l| \leq M.$$

Then

$$\frac{1}{\hat{T}(X_l)} \leq \frac{1}{P}$$

and

$$|\hat{X}_l\hat{|} = \left|\frac{X_l}{\hat{T}(X_l)}\right|\sqrt{\hat{T}_1} = \frac{|X_l|}{\hat{T}(X_l)}\sqrt{\hat{T}_1} \leq \frac{M}{P}\sqrt{\hat{T}_1}.$$

Consequently, the sequence $\{\hat{X}_l\}_{l=1}^{\infty}$ is a bounded sequence in $\hat{F}_{\mathbb{R}^n}$. □

Theorem 1.1.44. *Let $\{X_l\}_{l=1}^{\infty}$ be a bounded below sequence in $\mathbb{R}^n$ by a positive constant, let also the sequence $\{\hat{T}(X_l)\}_{l=1}^{\infty}$ is a sequence in $\mathbb{R}^n$ such that*

$$\lim_{l\longrightarrow\infty}\hat{T}(X_l) = 0.$$

Then the sequence $\{\hat{X}_l\}_{l=1}^{\infty}$ is not a bounded sequence.

Proof. Let us suppose that the sequence $\{\hat{X}_l\}_{l=1}^{\infty}$ is a bounded sequence in $\hat{F}_{\mathbb{R}^n}$. There exist a positive constant $M \in \mathbb{R}$ and a positive iso-real iso-number $\hat{P} \in \hat{F}_{\mathbb{R}}$ such that

$$|X_l| \geq M \qquad \text{and} \qquad |\hat{X}_l\hat{|} \leq \hat{P}$$

Then

$$\hat{P} = \frac{P}{\hat{T}_1} \geq |\hat{X}_l\hat{|} = \frac{|X_l|}{\hat{T}(X_l)}\sqrt{\hat{T}_1} \geq \frac{M\sqrt{\hat{T}_1}}{\hat{T}(X_l)},$$

whereupon

$$\hat{T}(X_l) \geq \frac{M\hat{T}_1^{\frac{3}{2}}}{P},$$

which is a contradiction because the sequence $\{\hat{T}(X_l)\}_{l=1}^{\infty}$ is a convergent sequence to the origin. Consequently, the sequence $\{\hat{X}_l\}_{l=1}^{\infty}$ is not a bounded sequence in $\hat{F}_{\mathbb{R}^n}$. $\square$

Theorem 1.1.45. *(Iso-Cauchy's Convergence Criterion) A sequence $\{\hat{X}_l\}_{l=1}^{\infty}$ is convergent if and only if for each $\hat{\varepsilon} > 0$, $\hat{\varepsilon} \in \hat{F}_{\mathbb{R}}$, there exists $L = L(\hat{\varepsilon}) > 0$ such that*

$$|\hat{X}|_l - \hat{X}_s\hat{|} < \hat{\varepsilon} \qquad \text{for} \qquad \forall s, l > L.$$

Proof. We observe that

$$|\hat{X}_l - \hat{X}_s\hat{|} < \hat{\varepsilon} \qquad \Longleftrightarrow$$

$$\left|\frac{X_l}{\hat{T}(X_l)} - \frac{X_s}{\hat{T}(X_s)}\right| < \frac{\varepsilon}{\hat{T}_1^{\frac{3}{2}}}.$$

Therefore the criterion follows immediately from the classical Cauchy's convergence criterion applied for the sequence $\left\{\frac{X_l}{\hat{T}(X_l)}\right\}_{l=1}^{\infty}$. $\square$

Definition 1.1.46. *If $\hat{S}$ is a nonempty subset of $\hat{F}_{\mathbb{R}^n}$, then*

$$\hat{d}(\hat{S}) = \sup\{|\hat{X} - \hat{Y}\hat{|} : \hat{X}, \hat{Y} \in \hat{S}\}$$

will be called the iso-diameter of $\hat{S}$. If $\hat{d}(\hat{S}) < \infty$, then $\hat{S}$ will be called bounded, if $\hat{d}(\hat{S}) = \infty$, $\hat{S}$ will be called unbounded.

Definition 1.1.47. *A nonempty subset $\hat{A}$ of $\hat{F}_{\mathbb{R}^n}$ will be called closed if every limit of every sequence of elements of $\hat{S}$ is an element of $\hat{S}$.*

Remark 1.1.48. *Since, if $\{X_l\}_{l=1}^{\infty}$ is a convergent sequence in $\mathbb{R}^n$, there are cases such that the corresponding lift $\{\hat{X}_l\}_{l=1}^{\infty}$ is not a convergent sequence in $\hat{F}_{\mathbb{R}^n}$ and the conversely. Therefore, if S is a closed set in $\mathbb{R}^n$, there are cases such that $\hat{S}$ is not a closed set in $\hat{F}_{\mathbb{R}^n}$ and the conversely.*

1.2. Iso-real Iso-valued Iso-Functions of n Variables

Let $D \subset \mathbb{R}^n$ and $\hat{T}, f : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$ for every $x \in D$.

For $x \in D$ we introduce the following notations

$$\frac{x}{\hat{T}(x)} = \Big(\frac{x_1}{\hat{T}(x)}, \frac{x_2}{\hat{T}(x)}, \ldots, \frac{x_n}{\hat{T}(x)}\Big),$$

and

$$x\hat{T}(x) = \hat{T}(x)x = (x_1\hat{T}(x), x_2\hat{T}(x), \ldots, x_n\hat{T}(x)).$$

Definition 1.2.1. *We will say that in the set D is defined the iso-function of the first kind or the iso-map of the first kind $\hat{f}^{\wedge\wedge}$ if*

$$\hat{y} := \hat{f}^{\wedge}(\hat{x}) = \frac{f(x)}{\hat{T}(x)}, \qquad x \in D,$$

is a function(map) in the set D.

The element $x \in D$ will be called the argument or the iso-independent variable of the iso-function of the first kind, and its iso-image $\hat{y} = \hat{f}^{\wedge}(\hat{x})$ will be called the iso-dependent iso-variable or the iso-value of the iso-function of the first kind at the point x. The set

$$\{\hat{f}^{\wedge}(\hat{x}) : x \in D\}$$

will be called the iso-codomain of the iso-values of the iso-function of the first kind. The set D will be called the domain of the iso-function of the first kind. The function $\frac{f(x)}{\hat{T}(x)}$ will be called the iso-original of the iso-function of the first kind .

Example 1.2.2. *Let $D = \mathbb{R}^2$, $f(x) = x_1x_2$, $\hat{T}(x) = x_1^2 + x_2^2 + 1$, $x \in D$. Then*

$$\hat{f}^{\wedge}(\hat{x}) = \frac{f(x)}{\hat{T}(x)} = \frac{x_1x_2}{x_1^2 + x_2^2 + 1}.$$

Remark 1.2.3. *We will note that if f is not a function in D, then there is a possibility $\hat{f}^{\wedge\wedge}$ to be a function in D and the conversely.*

Example 1.2.4. *Let $D = \mathbb{R}^2$,*

$$f(x) = \begin{cases} x_1 \qquad x_1 \geq 1, x_2 \leq 1, \\ \\ x_1(x_2+1) \qquad x_1 \geq 1, x_2 \geq 1, \\ \\ x_1(x_2+1) \qquad x_1 \leq 1, x_2 \leq 1, \\ \\ (x_1+1)(x_2+1) \qquad x_1 \geq 1, x_2 \geq 1. \end{cases}$$

Then f is not a function because

$$f(1,1) = 1, \qquad f(1,1) = 2, \qquad f(1,1) = 3, \qquad f(1,1) = 4.$$

Let

$$\hat{T}(x) = \begin{cases} 1 & x_1 \geq 1, x_2 \leq 1, \\ 2 & x_1 \geq 1, x_2 \geq 1, \\ 3 & x_1 \leq 1, x_2 \leq 1, \\ 4 & x_1 \leq 1, x_2 \geq 1. \end{cases}$$

We have that $\hat{T}$ *is not a function in D since*

$$\hat{T}(1,1) = 1, \qquad \hat{T}(1,1) = 2, \qquad \hat{T}(1,1) = 3, \qquad \hat{T}(1,1) = 4.$$

On the other hand,

$$\hat{f}^{\wedge}(\hat{x}) = \frac{f(x)}{\hat{T}(x)} = \begin{cases} x_1 & x_1 \geq 1, x_2 \leq 1, \\ \frac{x_1(x_2+1)}{2} & x_1 \geq 1, x_2 \geq 1, \\ \frac{x_1(x_2+2)}{3} & x_1 \leq 1, x_2 \leq 1, \\ \frac{(x_1+1)(x_2+1)}{4} & x_1 \geq 1, x_2 \leq 1. \end{cases}$$

We have that

$$\frac{f(1,1)}{\hat{T}(1,1)} = 1$$

and

$$\frac{f(x_1,1)}{\hat{T}(x_1,1)} = \begin{cases} x_1 & x_1 \geq 1, \\ x_1 & \geq 1, \\ x_1 & x_1 \leq 1, \\ \frac{x_1+1}{2} & x_1 \geq 1, \end{cases}$$

$$\frac{f(1,x_2)}{\hat{T}(1,x_2)} = \begin{cases} 1 & x_2 \leq 1, \\ \frac{x_2+1}{2} & x_2 \geq 1, \\ \frac{x_2+2}{3} & x_2 \leq 1, \\ \frac{x_2+1}{2} & x_2 \leq 1. \end{cases}$$

Therefore $\hat{f}^{\wedge\wedge}$ *is a function.*

Let now

$$f(x) = x_1^2 + x_2^2 + 1, \qquad x \in D,$$

$$\hat{T}(x) = \begin{cases} x_1^2 + 1 & \text{for} \quad x_2 \le 1, \quad x_1 \in \mathbb{R}, \\ x_1^2 + x_2^2 + 1 & \text{for} \quad x_2 \ge 1, \quad x_1 \in \mathbb{R}. \end{cases}$$

Then $f : D \longrightarrow \mathbb{R}$ *is a function. On other hand,*

$$\hat{f}^{\wedge}(\hat{x}) = \begin{cases} \frac{x_1^2+x_2^2+1}{x_1^2+1} & \text{for} \quad x_2 \le 1, \quad x_1 \in \mathbb{R}, \\ \frac{x_1^2+x_2^2+1}{x_1^2+x_2^2+2} & \text{for} \quad x_2 \ge 1, \quad x_1 \in \mathbb{R}. \end{cases}$$

Since

$$\hat{f}^{\wedge}(\hat{x})\Big|_{x_2=1-} = \frac{x_1^2+2}{x_1^2+1}, \qquad \hat{f}^{\wedge}(\hat{x})\Big|_{x_2=1+} = \frac{x_1^2+2}{x_1^2+3}, \qquad x_1 \in \mathbb{R},$$

then $\hat{f}^{\wedge\wedge} : D \longrightarrow \mathbb{R}$ *is not a function.*

Exercise 1.2.5. *Let* $D = \mathbb{R}^3$, $\hat{T}(x) = x_1^2 + x_2^2 + |x_3| + 2$, $f(x) = x_1^2 - 2x_1x_2 + x_3^2$, $x = (x_1, x_2, x_3) \in D$. *Find* $\hat{f}^{\wedge}(\hat{x})$.

Answer. $\frac{x_1^2 - 2x_1x_2 + x_3^2}{x_1^2 + x_2^2 + |x_3| + 2}$.

Exercise 1.2.6. *Let* $D = \mathbb{R}^3$, $f(x) = |x_1| - 2|x_2| + 3x_3^2 - 4$,

$$\hat{T}(x) = \begin{cases} |x_1 - x_2| + 4 & x_1 \le 2, \quad x_2 \le 1, \quad x_3 \in \mathbb{R}, \\ |x_1| + 3|x_3| + 4 & x_1 \le 2, \quad x_2 \ge 1, \quad x_3 \in \mathbb{R}, \\ |x_1| + 5 & x_1 \ge 2, \quad x_2 \le 1, \quad x_3 \in \mathbb{R}, \\ x_1^2 + 2x_2 - 3x_3^2 + 5 & x_1 \ge 2, \quad x_2 \ge 1, \quad x_3 \in \mathbb{R}. \end{cases}$$

Check if $\hat{f}^{\wedge}(\hat{x})$ *is a function.*

Answer. No.

Definition 1.2.7. *We will tell that in the set D is defined the iso-function of the second kind or the iso-map of the second kind* $\hat{f}^{\wedge}$ *if* $x\hat{T}(x) \in D$ *for every* $x \in D$ *and*

$$\hat{y} := \hat{f}^{\wedge}(x) = \frac{f(x\hat{T}(x))}{\hat{T}(x)}, \qquad x \in D,$$

is a function(map) in D.

The element x will be called the argument of the iso-function of the second kind or the independent variable, and its iso-image $\hat{y} = \hat{f}^{\wedge}(x)$ will be called the iso-dependent iso-variable or the iso-value of the iso-function of the second kind. The set

$$\{\hat{f}^{\wedge}(x) : x \in D\}$$

will be called the iso-codomain of the iso-values of the iso-function of the second kind. The set D will be called the domain of the iso-function of the second kind. The function $\frac{f(x\hat{T}(x))}{\hat{T}(x)}$ will be called the iso-original of the iso-function of the second kind.

Example 1.2.8. *Let $D = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$, $f(x) = x_1 + x_2$, $\hat{T}(x) = \frac{x_1^2+x_2^2+2}{10}$, $x \in D$. Then*

$$x\hat{T}(x) = (x_1\hat{T}(x), x_2\hat{T}(x)) = \Big(\frac{x_1(x_1^2+x_2^2+2)}{10}, \frac{x_2(x_1^2+x_2^2+2)}{10}\Big), \quad x \in D.$$

Then

$$x_1^2\frac{(x_1^2+x_2^2+2)^2}{100} + x_2^2\frac{(x_1^2+x_2^2+2)^2}{100} = \frac{(x_1^2+x_2^2+2)^2}{100}(x_1^2+x_2^2) \leq \frac{9}{100}.$$

Consequently $x\hat{T}(x) \in D$ and

$$\begin{aligned}
\hat{f}^{\wedge}(x) &= \tfrac{f(x\hat{T}(x))}{\hat{T}(x)} \\
&= \frac{f\left(\frac{10x_1}{x_1^2+x_2^2+2}, \frac{10x_2}{x_1^2+x_2^2+2}\right)}{\frac{x_1^2+x_2^2+2}{10}} \\
&= \tfrac{10}{x_1^2+x_2^2+2}\left(\tfrac{10x_1}{x_1^2+x_2^2+2} + \tfrac{10x_2}{x_1^2+x_2^2+2}\right) \\
&= \tfrac{100(x_1+x_2)}{(x_1^2+x_2^2+2)^2}.
\end{aligned}$$

Example 1.2.9. *Let $D = \mathbb{R}^2$,*

$$f(x) = \begin{cases} x_1 + x_2^2 + 2 & x_1 \leq 1, \quad x_2 \in \mathbb{R}, \\ x_1 + 2x_2^2 + 1 & x_1 \geq 1, \quad x_2 \in \mathbb{R}, \end{cases}$$

Then $f : D \longrightarrow \mathbb{R}$ is not a function. Let us take

$$\hat{T}(x) = \begin{cases} 2 & x_1 \leq 1, \quad x_2 \in \mathbb{R}, \\ 1 & x_1 \geq 1, \quad x_2 \in \mathbb{R}. \end{cases}$$

For $\hat{f}^{\wedge}(x)$ we have the representation

$$\hat{f}^{\wedge}(x) = \frac{f(x\hat{T}(x))}{\hat{T}(x)} = \frac{f(x_1\hat{T}(x), x_2\hat{T}(x))}{\hat{T}(x)} = \begin{cases} \frac{x_1\hat{T}(x)+x_2^2\hat{T}^2(x)+2}{\hat{T}(x)} & x_1 \leq 1, \quad x_2 \in \mathbb{R}, \\ \frac{x_1+2x_2^2\hat{T}^2(x)+1}{\hat{T}(x)} & x_1 \geq 1, \quad x_2 \in \mathbb{R}, \end{cases}$$

$$= \begin{cases} x_1 + 2x_2^2 + 1 & x_1 \leq 1, \quad x_2 \in \mathbb{R}, \\ x_1 + 2x_2^2 + 1 & x_1 \geq 1, \quad x_2 \in \mathbb{R}. \end{cases}$$

We have that $\hat{f}^{\wedge} : D \longrightarrow \mathbb{R}$ is a function.

Example 1.2.10. *Let $D = \mathbb{R}^2$, $f(x) = x_1 + x_2 + 1$, $x = (x_1, x_2) \in D$. We have that $f : D \longrightarrow \mathbb{R}$ is a function. Let us take*

$$\hat{T}(x) = \begin{cases} x_1^2 + 1 & x_1 \leq 1, \quad x_2 \in \mathbb{R}, \\ x_1^2 + 2 & x_1 \geq 1, \quad x_2 \in \mathbb{R}. \end{cases}$$

Then

$$\hat{f}^{\wedge}(x) = \frac{f(x\hat{T}(x))}{\hat{T}(x)} = \frac{f(x_1\hat{T}(x), x_2\hat{T}(x))}{\hat{T}(x)}$$

$$= \begin{cases} \frac{x_1\hat{T}(x)+x_2\hat{T}(x)+1}{\hat{T}(x)} & x_1 \leq 1, \quad x_2 \in \mathbb{R}, \\ \frac{x_1\hat{T}(x)+x_2\hat{T}(x)+2}{\hat{T}(x)} & x_1 \geq 1, \quad x_2 \in \mathbb{R}, \end{cases}$$

$$= \begin{cases} x_1 + x_2 + \frac{1}{x_1^2+1} & x_1 \leq 1, \quad x_2 \in \mathbb{R}, \\ x_1 + x_2 + \frac{2}{x_1^2+x_2^2} & x_1 \geq 1, \quad x_2 \in \mathbb{R}. \end{cases}$$

We note that $\hat{f}^{\wedge} : D \longrightarrow \mathbb{R}$ is not a function.

Exercise 1.2.11. *Let $D = \mathbb{R}^2$, $f(x) = x_1 - 2x_2 + 3$, $\hat{T}(x) = x_1^2 + x_2^2 + 2$, $x = (x_1, x_2) \in D$. Find $\hat{f}^{\wedge}(x)$.*

Answer. $\hat{f}^{\wedge}(x) = x_1 - 2x_2 + \frac{3}{x_1^2+x_2^2+2}$.

Definition 1.2.12. *We will tell that in the set D is defined the iso-function of the third kind or the iso-map of the third kind $\hat{\hat{f}}$ if $\frac{x}{\hat{T}(x)} \in D$ for every $x \in D$ and*

$$\hat{y} := \hat{f}(\hat{x}) = \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)}, \quad x \in D,$$

is a function(map) in D.

The element x will be called the argument of the iso-function of the third kind or the independent variable, and its iso-image $\hat{y} = \hat{f}(\hat{x})$ will be called the iso-dependent iso-variable or the iso-value of the iso-function of the third kind . The set

$$\{\hat{f}(\hat{x}) : x \in D\}$$

will be called the iso-codomain of the iso-values of the iso-function of the third kind. The set D will be called the domain of the iso-function of the third kind. The function $\frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)}$ will be called the iso-original of the iso-function of the third kind .

Example 1.2.13. *Let $D = \mathbb{R}^2$, $\hat{T}(x) = x_1^2 + x_2^2 + 1$, $f(x) = x_1^3 + x_2$. Then*

$$\hat{f}(\hat{x}) = \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)} = \frac{f\left(\frac{x_1}{\hat{T}(x)}, \frac{x_2}{\hat{T}(x)}\right)}{\hat{T}(x)}$$

$$= \frac{\frac{x_1^3}{\hat{T}^3(x)} + \frac{x_2}{\hat{T}(x)}}{\hat{T}(x)}$$

$$= \frac{x_1^3 + x_2\hat{T}^2(x)}{\hat{T}^4(x)}$$

$$= \frac{x_1^3 + x_2(x_1^2 + x_2^2 + 1)^2}{(x_1^2 + x_2^2 + 1)^4}.$$

Example 1.2.14. *Let $D = \mathbb{R}^3$,*

$$f(x) = \begin{cases} x_1^2 + x_2^2 + x_3^2 \qquad (x_1, x_2) \in \mathbb{R}^2, \qquad x_3 \leq 1, \\ \\ 1 \qquad (x_1, x_2) \in \mathbb{R}^2, \qquad x_3 \geq 1. \end{cases}$$

Then $f : D \longrightarrow \mathbb{R}$ is not a function. If we take

$$\hat{T}(x) = \begin{cases} \sqrt{x_1^2 + x_2^2 + x_3^2} \qquad (x_1, x_2) \in \mathbb{R}^2, \qquad x_3 \leq 1, 1 \qquad (x_1, x_2) \in R^2, x_3 \geq 1, \end{cases}$$

then

$$\hat{f}(\hat{x}) = \frac{f\left(\frac{x_1}{\hat{T}(x)}, \frac{x_2}{\hat{T}(x)}, \frac{x_3}{\hat{T}(x)}\right)}{\hat{T}(x)}$$

$$= \begin{cases} \frac{x_1^2}{x_1^2+x_2^2+x_3^2} + \frac{x_2^2}{x_1^2+x_2^2+x_3^2} + \frac{x_1^2}{x_1^2+x_2^2+x_3^2} \qquad (x_1, x_2) \in \mathbb{R}^2, \qquad x_3 \leq 1, \\ \\ 1 \qquad (x_1, x_2) \in \mathbb{R}^2, \qquad x_3 \geq 1, \end{cases}$$

whereupon $\hat{f}(\hat{x}) = 1$ for every $(x_1, x_2, x_3) \in D$ and therefore $\hat{\hat{f}} : D \longrightarrow \mathbb{R}$ is a function.

Exercise 1.2.15. *Let $D = \mathbb{R}^2$, $f(x) = x_1^2 - 2x_1x_2$, $\hat{T}(x) = x_1^2 + 1$, $x = (x_1, x_2) \in D$. Find $\hat{f}(\hat{x})$.*

Answer. $\frac{x_1^2 - 2x_1x_2}{(x_1^2+1)^3}$.

Exercise 1.2.16. *Let $D = \mathbb{R}^3$, $f(x) = x_1^3 - 3x_1x_2 + 4x_1^4 + x_2^5$, $x = (x_1, x_2, x_3) \in D$,*

$$\hat{T}(x) = \begin{cases} \sqrt{x_1^2 + x_2^2 + x_3^2} + 4 \qquad (x_1, x_2) \in \mathbb{R}^2, \qquad x_3 \leq 2, \\ \\ 2 + \frac{1}{x_1^2+1} \qquad (x_1, x_2) \in \mathbb{R}^2, \qquad x_3 \geq 2. \end{cases}$$

Check if $\hat{\hat{f}}$ is a function.

Answer. No.

Definition 1.2.17. *We will tell that in the set D is defined the iso-function of the fourth kind or the iso-map of the fourth kind $f^{\wedge}$ if $x\hat{T}(x) \in D$ for every $x \in D$ and*

$$\hat{y} := f^{\wedge}(x) = f\Big(x\hat{T}(x)\Big), \quad x \in D,$$

is a function(map) in D.

The element x will be called the argument of the iso-function of the fourth kind or the independent variable, and its iso-image $\hat{y} = f^{\wedge}(x)$ will be called the iso-dependent iso-variable or the iso-value of the iso-function of the fourth kind. The set

$$\{f^{\wedge}(x) : x \in D\}$$

will be called the iso-codomain of the iso-values of the iso-function of the fourth kind. The set D will be called the domain of the iso-function of the fourth kind. The function $f\Big(x\hat{T}(x)\Big)$ will be called the iso-original of the iso-function of the fourth kind.

Example 1.2.18. *Let $D = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$, $f(x) = x_1 + x_2^2$, $\hat{T}(x) = \frac{1}{10}(x_1^2 + x_2^2 + 1)$, $x \in D$. Then for $x \in D$*

$$x\hat{T}(x) = (x_1\hat{T}(x), x_2\hat{T}(x)),$$

$$x_1^2\hat{T}^2(x) + x_2^2\hat{T}^2(x) = (x_1^2 + x_2^2)\hat{T}^2(x) = \frac{1}{10}(x_1^2 + x_2^2 + 1)(x_1^2 + x_2^2) \leq \frac{1}{5},$$

i.e. $x\hat{T}(x) \in D$. Therefore the function $f^{\wedge}$ is well defined on D and

$$\begin{aligned} f^{\wedge}(x) &= x_1\hat{T}(x) + x_2^2\hat{T}^2(x) \\ &= (x_1 + x_2^2\hat{T}(x))\hat{T}(x) \\ &= \Big(x_1 + \tfrac{x_2^2}{10}(x_1^2 + x_2^2 + 1)\Big)\tfrac{1}{10}(x_1^2 + x_2^2 + 1) \\ &= \tfrac{x_1^3x_2^2 + x_1x_2^4 + x_1^2x_2^4 + 11x_1x_2^2 + x_2^6 + x_2^4 + x_1^2x_2^2 + x_2^4 + x_2^2 + 10x_1^3 + 10x_1}{100}. \end{aligned}$$

Example 1.2.19. *Let $D = \mathbb{R}^2$, $f(x) = x_1^2 - x_2$, $x = (x_1, x_2) \in D$,*

$$\hat{T}(x) = \begin{cases} |x_1| + 1 \qquad x_1 \in \mathbb{R}, \qquad x_2 \leq 1, \\ \\ x_1^2 + x_2^2 + 1 \qquad x_1 \in \mathbb{R}, \qquad x_2 \geq 1. \end{cases}$$

Then $f : D \longrightarrow \mathbb{R}$ is a function. Also,

$$f^{\wedge}(x) = f(x\hat{T}(x)) = f(x_1\hat{T}(x), x_2\hat{T}(x)) = f(x_1\hat{T}(x), x_2\hat{T}(x))$$

$$= \begin{cases} x_1^2(|x_1|+1)^2 - x_2(|x_1|+1) & x_1 \in \mathbb{R}, \quad x_2 \leq 1, \\ \\ x_1^2(x_1^2+x_2^2+1) - x_2(x_1^2+x_2^2+1) & x_1 \in \mathbb{R}, \quad x_2 \geq 1. \end{cases}$$

Consequently $f^{\wedge} : D \longrightarrow \mathbb{R}$ is not a function.

Exercise 1.2.20. *Let $D = \mathbb{R}^2$, $f(x) = x_1 - x_2$, $\hat{T}(x) = |x_1| + |x_2| + 1$, $x = (x_1, x_2) \in D$. Find $f^{\wedge}(x)$.*

Answer. $(x_1 - x_2)(|x_1| + |x_2| + 1)$.

Exercise 1.2.21. *Let $D = \mathbb{R}^2$, $\hat{T}(x) = x_1^2 + x_2^2 + 4$, $x = (x_1, x_2) \in D$,*

$$f(x) = \begin{cases} x_1 - 2x_2 + 3x_1^2 & x_1 \in \mathbb{R}, \quad x_2 \leq 1, \\ \\ x_1 + 4x_2^2 & x_1 \in \mathbb{R}, \quad x_2 \geq 1. \end{cases}$$

Check if f and $f^{\wedge}$ are functions.

Answer. No, No.

Definition 1.2.22. *We will tell that in the set D is defined the iso-function of the fifth kind or the iso-map of the fifth kind $f^{\vee}$ if $x\hat{T}(x) \in D$ for every $x \in D$ and*

$$\hat{y} := f^{\vee}(x) = f(\hat{x}) = f\left(\frac{x}{\hat{T}(x)}\right), \quad x \in D,$$

is a function(map) in D. We will use the notation $f^{\vee}$.

The element x will be called the argument or the independent variable of the iso-function of the fifth kind , and its iso-image $\hat{y} = f^{\vee}(x)$ will be called the iso-dependent iso-variable or the iso-value of the iso-function of the fifth kind. The set

$$\{f^{\vee}(x) : x \in D\}$$

will be called the iso-codomain of the iso-values of the iso-function of the fifth kind . The set D will be called the domain of the iso-function of the fifth kind

. The function $f\left(\frac{x}{\hat{T}(x)}\right)$ will be called the iso-original of the iso-function of the fifth kind

.

Example 1.2.23. *Let $D = \mathbb{R}^2$,*

$$f(x) = \begin{cases} x_1 + x_2 & x_1 \leq 1, \quad x_2 \leq 2, \\ \\ 2x_2 + 1 & x_1 \leq 1, \quad x_2 \geq 2, \\ \\ 3x_1 + x_2^2 & x_1 \geq 1, \quad x_2 \leq 2, \\ \\ x_1^2 + 2x_1x_2 & x_1 \geq 1, \quad x_2 \geq 2, \end{cases}$$

$\hat{T}(x) = x_1^2 + x_2^2 + 1$, $x(x_1, x_2) \in D$. *Then*

$$f^{\vee}(x) = f(\hat{x}) = f\left(\frac{x}{\hat{T}(x)}\right) = f\left(\frac{x_1}{\hat{T}(x)}, \frac{x_2}{\hat{T}(x)}\right)$$

$$= \begin{cases} \frac{x_1}{\hat{T}(x)} + \frac{x_2}{\hat{T}(x)} & x_1 \le 1, \quad x_2 \le 2, \\ 2\frac{x_2}{\hat{T}(x)} + 1 & x_1 \le 1, \quad x_2 \ge 2, \\ 3\frac{x_1}{\hat{T}(x)} + \frac{x_2^2}{\hat{T}^2(x)} & x_1 \ge 1, \quad x_2 \le 2, \\ \frac{x_1^2}{\hat{T}^2(x)} + 2\frac{x_1}{\hat{T}(x)}\frac{x_2}{\hat{T}(x)} & x_1 \ge 1, \quad x_2 \le 2, \end{cases}$$

$$= \begin{cases} \frac{x_1+x_2}{x_1^2+x_2^2+1} & x_1 \le 1, \quad x_2 \le 2, \\ 2\frac{x_2}{x_1^2+x_2^2+1} + 1 & x_1 \le 1, \quad x_2 \ge 2, \\ 3\frac{x_1}{x_1^2+x_2^2+1} + \frac{x_2^2}{(x_1^2+x_2^2+1)^2} & x_1 \ge 1, \quad x_2 \le 2, \\ \frac{x_1^2+2x_1x_2}{x_1^2+x_2^2+1} & x_1 \ge 1, \quad x_2 \ge 2. \end{cases}$$

We have that f and $f^{\vee}$ are not functions.

Example 1.2.24. *Let $D = \mathbb{R}^2$,*

$$f(x) = \begin{cases} x_1^3 + 2x_1^2 + 3x_2^2 + 6x_1x_2 & x_1 \le 1, \quad x_2 \in \mathbb{R}, \\ 4x_1^5 - 3x_1^4 + 2x_2^6 + 7x_1^2x_2^3 & x_1 \ge 1, \quad x_2 \in \mathbb{R}, \end{cases}$$

$$\hat{T}(x) = \begin{cases} 6x_1^2 + 6x_1x_2 & x_1 \le 1, \quad x_2 \in \mathbb{R}, \\ x_1^5 + 2x_2^6 + 7x_2^3 & x_1 \ge 1, \quad x_2 \in \mathbb{R}. \end{cases}$$

We have that f and $\hat{T}$ are not functions. On the other hand,

$$f^{\vee}(x) = f(\hat{x}) = f\left(\frac{x_1}{\hat{T}(x)}, \frac{x_2}{\hat{T}(x)}\right)$$

$$= \begin{cases} \frac{x_1^3}{\hat{T}^3(x)} - 2\frac{x_1^2}{\hat{T}^2(x)} + 3\frac{x_2^2}{\hat{T}^2(x)} + 6\frac{x_1}{\hat{T}(x)}\frac{x_2}{\hat{T}(x)} & x_1 \le 1, \quad x_2 \in \mathbb{R}, \\ 4\frac{x_1^5}{\hat{T}^5(x)} - 3\frac{x_1^4}{\hat{T}^4(x)} + 2\frac{x_2^6}{\hat{T}^6(x)} + 7\frac{x_1^2}{\hat{T}^2(x)}\frac{x_2^3}{\hat{T}^3(x)}, & x_1 \ge 1, \quad x_2 \in \mathbb{R}, \end{cases}$$

$$= \begin{cases} \frac{x_1^3 + (2x_1^2 + 3x_2^2 + 6x_1x_2)\hat{T}(x)}{\hat{T}^3(x)} & x_1 \le 1, \quad x_2 \in \mathbb{R}, \\ \frac{2x_2^6 + (4x_1^5 + 7x_1^2x_2^3)\hat{T}(x) - 3x_1^4\hat{T}^2(x)}{\hat{T}^6(x)} & x_1 \ge 1, \quad x_2 \in \mathbb{R}. \end{cases}$$

We have that $f^{\vee}$ is a function.

Exercise 1.2.25. *Let $D = \{(x_1, x_2) \in \mathbb{R} : x_1 \geq 1, x_2 \geq 1\}$,*

$$f(x) = \begin{cases} x_1^4 + 3x_1^2 x_2 + x_1 x_2^7 + x_1^2 x_2^2 & 1 \leq x_1 \leq 2, \quad x_2 \in \mathbb{R}, \\ \\ x_1^7 - 7x_1^2 x_2 + x_1 x_2^3 & x_1 \geq 2, \quad x_2 \in \mathbb{R}, \end{cases}$$

$\hat{T}(x) = x_1$, $x = (x_1, x_2) \in D$. Find $f^{\vee}(x)$, $x \in D$.

Answer.

$$f^{\vee}(x) = \begin{cases} 1 + 3\frac{x_2}{x_1} + \frac{x_2^7}{x_1^7} + \frac{x_2^2}{x_1^2} & 1 \leq x_1 \leq 2, \quad x_2 \in \mathbb{R}, \\ \\ 1 - 7\frac{x_2}{x_1} + \frac{x_2^3}{x_1^3} & x_1 \geq 2, \quad x_2 \in \mathbb{R}. \end{cases}$$

Exercise 1.2.26. *Let $D = \mathbb{R}$, $f(x) = 2x_1 x_2$, $\hat{T}(x) = x_1^2 + x_2^2 + 1$, $x = (x_1, x_2) \in D$. Find $f^{\vee}(x)$, $x \in D$.*

Answer. $2\frac{x_1 x_2}{(x_1^2 + x_2^2 + 1)^2}$.

Exercise 1.2.27. *Let $D = \mathbb{R}^2$,*

$$f(x) = \begin{cases} x_1^2 + 3x_1 x_2^3 + 6x_2^4 - 4x_1 x_2 - 5x_1 x_2^4 & x_1 \leq 1, \quad x_2 \in \mathbb{R}, \\ \\ x_2^3 + 3x_1^2 x_2 + 4x_2^3 & x_1 \geq 1, \quad x_2 \in \mathbb{R}, \end{cases}$$

$$\hat{T}(x) = \begin{cases} x_1^4 + x_2^4 + x_1^6 + x_1^2 x_2^2 + 2 & x_1 \leq 1, \quad x_2 \in \mathbb{R}, \\ \\ x_1^8 + 7x_1^2 x_2^2 + 6x_1^4 x_2^4 + 5x_1^{10} + 9 & x_1 \geq 1, \quad x_2 \in \mathbb{R}. \end{cases}$$

Check if $f^{\vee}$ is a function.

Answer. No.

Exercise 1.2.28. *Let $D = \mathbb{R}^3$, $f(x) = x_1^3 + x_2 + 3x_1 x_2 x_3 + x_3^4$, $\hat{T}(x) = x_1^2 + x_2^2 + x_3^2$, $x = (x_1, x_2, x_3) \in D$. Check if $f^{\vee}$ is a function.*

Answer. Yes.

Definition 1.2.29. *Let $\hat{f}$ and $\hat{g}$ are iso-functions of the first, the second, the third, the fourth or the fifth kind, $\tilde{f}$ and $\tilde{g}$ are their iso-originals, respectively. Let also, $a \in \mathbb{R}$ and $\hat{a} = \frac{a}{\hat{T}_1}$. We define*

1. $\hat{a} \hat{\times} \hat{f} = a\tilde{f}$,

2. $\hat{a}\hat{f} = \frac{a}{\hat{T}_1}\tilde{f}$,

3. $a \hat{\times} \hat{f} = a\hat{T}_1 \tilde{f}$,

4. $\hat{f} \pm \hat{g} = \tilde{f} \pm \tilde{g}$,

5. $\hat{f} \hat{\times} \hat{g} = \tilde{f}\hat{T}_1\tilde{g}$,

6. $\hat{f}\hat{g} = \tilde{f}\tilde{g}$,

7. $\hat{f} \nearrow \hat{g} = \frac{1}{\hat{T}_1}\frac{\hat{f}}{\hat{g}}$.

Example 1.2.30. *Let* $D + \mathbb{R}^2$, $f(x) = x_1^2 + 2x_1x_2$, $g(x) = x_1 - x_2$, $\hat{T}(x) = x_1^2 + x_2^2 + 1$, $x = (x_1, x_2) \in D$,, $\hat{T}_1 = 4$. *We will find*

$$A = f^{\vee}(x)\hat{\times}(2\hat{\times}\hat{f}^{\wedge}(\hat{x}) - \hat{3}\hat{\times}\hat{g}(\hat{x})).$$

We have

$$f^{\vee}(x) = f(\hat{x}) = f\left(\frac{x_1}{\hat{T}(x)}, \frac{x_2}{\hat{T}(x)}\right)$$

$$= \frac{x_1^2}{\hat{T}^2(x)} + 2\frac{x_1}{\hat{T}(x)}\frac{x_2}{\hat{T}(x)}$$

$$= \frac{x_1^2+2x_1x_2}{\hat{T}^2(x)}$$

$$= \frac{x_1^2+2x_1x_2}{(x_1^2+x_2^2+1)^2},$$

$$\hat{f}^{\wedge}(\hat{x}) = \frac{f(x)}{\hat{T}(x)}$$

$$= \frac{x_1^2+2x_1x_2}{x_1^2+x_2^2+1},$$

$$\hat{g}(\hat{x}) = \frac{g\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)} = \frac{g\left(\frac{x_1}{\hat{T}(x)}, \frac{x_2}{\hat{T}(x)}\right)}{\hat{T}(x)}$$

$$= \frac{\frac{x_1}{\hat{T}(x)} - \frac{x_2}{\hat{T}(x)}}{\hat{T}(x)}$$

$$= \frac{x_1-x_2}{\hat{T}^2(x)}$$

$$= \frac{x_1-x_2}{(x_1^2+x_2^2+1)^2},$$

$$\hat{3}\hat{\times}\hat{g}(\hat{x}) = \frac{3x_1 - 3x_2}{(x_1^2+x_2^2+1)^2},$$

$$2\hat{\times}\hat{f}^{\wedge}(\hat{x}) = 2 \cdot 4 \cdot \frac{x_1^2+2x_1x_2}{(x_1^2+x_2^2+1)^2}$$

$$= \frac{8x_1^2+16x_1x_2}{(x_1^2+x_2^2+1)^2},$$

$$2\hat{\times}\hat{f}^{\wedge}(\hat{x}) - \hat{3}\hat{\times}\hat{g}(\hat{x}) = \frac{8x_1^2+16x_1x_2}{(x_1^2+x_2^2+1)^2} - \frac{3x_1-3x_2}{(x_1^2+x_2^2+1)^2}$$

$$= \frac{8x_1^2+16x_1x_2-3x_1+3x_2}{(x_1^2+x_2^2+1)^2},$$

$$
\begin{aligned}
A &= \tfrac{x_1^2+2x_1x_2}{(x_1^2+x_2^2+1)^2} \cdot 4 \cdot \tfrac{8x_1^2+16x_1x_2-3x_1+3x_2}{(x_1^2+x_2^2+1)^2} \\
&= \tfrac{32x_1^4+128x_1^3x_2-12x_1^3-12x_1^2x_2+24x_1x_2^2}{(x_1^2+x_2^2+1)^4}.
\end{aligned}
$$

Exercise 1.2.31. *Let $D=\mathbb{R}^2$, $f(x)=x_1+x_2$, $\hat{T}(x)=x_1^2+x_2^2+1$, $x=(x_1,x_2)\in D$. Find*

$$\hat{f}^{\wedge}(\hat{x})+\hat{2}\hat{\times}f^{\wedge}(x), \qquad x\in D.$$

Solution. We have

$$
\begin{aligned}
\hat{f}^{\wedge}(\hat{x}) &= \tfrac{f(x)}{\hat{T}(x)} \\
&= \tfrac{x_1+x_2}{x_1^2+x_2^2+1}, \\
f^{\wedge}(x) &= f(x\hat{T}(x)) = f(x_1\hat{T}(x), x_2\hat{T}(x)) \\
&= x_1(x_1^2+x_2^2+1)+x_2(x_1^2+x_2^2+1) \\
&= (x_1+x_2)(x_1^2+x_2^2+1), \\
\hat{2}\hat{\times}f^{\wedge}(x) &= 2(x_1+x_2)(x_1^2+x_2^2+1), \\
\hat{f}^{\wedge}(\hat{x})+\hat{2}\hat{\times}f^{\wedge}(x) &= \tfrac{x_1+x_2}{x_1^2+x_2^2+1}+2(x_1+x_2)(x_1^2+x_2^2+1) \\
&= \tfrac{(x_1+x_2)(1+2(x_1^2+x_2+1)^2)}{x_1^2+x_2^2+1}.
\end{aligned}
$$

Exercise 1.2.32. *Let $D=\mathbb{R}^2$, $\hat{T}_1=4$, $\hat{T}(x)=|x_1|+2$, $f(x)=x_1^2+x_2^4$, $x=(x_1,x_2)\in D$. Find*

$$\hat{3}\hat{f}^{\wedge}(\hat{x})-\hat{2}\hat{\times}f^{\wedge}(x).$$

Answer.

$$\tfrac{3x_1^2+3x_2^4}{4(|x_1|+2)}-2x_1^2-8|x_1|x_1^2-8x_1^2-4x_1^4x_2^4-48x_1^2x_2^4-16x_1^2|x_1|x_2^4-32x_2^4-32|x_1|x_2^4.$$

Definition 1.2.33. *Let $\hat{f}$ is an iso-function of the first, the second, the third, the fourth or* *the fifth kind, $\tilde{f}$ is its iso-original. Then*

$$\hat{f}^{\hat{2}} = \hat{f}\hat{\times}\hat{f} = \tilde{f}\hat{T}_1\tilde{f},$$

$$\hat{f}^{\hat{3}} = \hat{f}^{\hat{2}}\hat{\times}\hat{f} = \tilde{f}\hat{T}_1\tilde{f}\hat{T}_1\tilde{f},$$

$$\dots$$

$$\hat{f}^{n\hat{+}1} = \hat{f}^{\hat{n}}\hat{\times}\hat{f},$$

$$\hat{f}^2 = \hat{f}\hat{f} = \tilde{f}\tilde{f} = \tilde{f}^2,$$

$$\hat{f}^3 = \hat{f}^2\hat{f} = \tilde{f}^2\tilde{f} = \tilde{f}^3,$$

$$\dots$$

$$\tilde{f}^{n+1} = \hat{f}^n\hat{f} = \tilde{f}^n\tilde{f} = \tilde{f}^{n+1}, \qquad n \in \mathbb{N}.$$

Exercise 1.2.34. *Let $D = \mathbb{R}^2$, $\hat{T}_1 = 2$, $\hat{T}(x) = 1 + |x_2|$, $f(x) = x_1 - x_2$, $x = (x_1, x_2) \in \mathbb{R}$. Find*

$$A = \left(\hat{f}^\wedge(\hat{x})\right)^{\hat{2}} - \hat{2}\hat{\times}\left(f^\wedge(x)\right)^2.$$

Solution. We have

$$\hat{f}^\wedge(\hat{x}) = \tfrac{f(x)}{\hat{T}(x)}$$

$$= \tfrac{x_1-x_2}{1+|x_2|},$$

$$\left(\hat{f}^\wedge(\hat{x})\right)^{\hat{2}} = \hat{f}^\wedge(\hat{x})\hat{\times}\hat{f}^\wedge(\hat{x})$$

$$= \tfrac{x_1-x_2}{1+|x_2|}2\tfrac{x_1-x_2}{1+|x_2|}$$

$$= 2\tfrac{(x_1-x_2)^2}{(1+|x_2|)^2},$$

$$f^\wedge(x) = f(x\hat{T}(x)) = f(x_1\hat{T}(x), x_2\hat{T}(x))$$

$$= x_1\hat{T}(x) - x_2\hat{T}(x)$$

$$= (x_1 - x_2)\hat{T}(x) = (1 + |x_2|)(x_1 - x_2),$$

$$\hat{2}\hat{\times}\left(f^\wedge(x)\right)^2 = 2f^\wedge(x)f^\wedge(x)$$

$$= 2(1 + |x_2|)^2(x_1 - x_2)^2.$$

Consequently

$$A = 2\frac{(x_1 - x_2)^2}{(1+|x_2|)^2} - 2(1+|x_2|)^2(x_1 - x_2)^2.$$

Exercise 1.2.35. *Let $D = \mathbb{R}^2$, $\hat{T}_1 = 3$, $f(x) = x_1 + 2x_2$, $\hat{T}(x) = x_1^2 + 1$. Find*

$$\left(f^{\wedge}(x)\right)^{\hat{2}} - \left(f^{\wedge}(x)\right)^{2}.$$

Answer. $2(x_1 + 2x_2)^2(1+x_1^2)^2$.

Definition 1.2.36. *An iso-function $\hat{h}$ of the first, the second, the third, the fourth or the fifth kind will be called an iso-injection, an iso-surjection or an iso-bijection if its iso-original $\tilde{h}$ is an injection, a surjection or a bijection, respectively.*

1.3. Limits of Iso-Real Iso-Valued Iso-Functions of n Variables

Let $D \subset \mathbb{R}^n$ and $\hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$ for every $x \in D$, $\hat{f} : D \longrightarrow \mathbb{R}$ is an iso-function of the first, the second, the third, the fourth or the fifth kind and let $\tilde{f}$ be its iso-original.

Definition 1.3.1. *The real number a will be called the left limit of $\hat{f}$ at $x_0 \in D$ if it is the left limit of $\tilde{f}$ at x_0.*

Definition 1.3.2. *The real number a will be called the right limit of $\hat{f}$ at $x_0 \in D$ if it is the right limit of $\tilde{f}$ at x_0.*

Definition 1.3.3. *The real number a will be called the limit of $\hat{f}$ at $x_0 \in D$ if it is the limit of $\tilde{f}$ at x_0.*

Example 1.3.4. *Let* $D = \mathbb{R}^2$, $f(x) = 1 - x_1^2 - 2x_2^2$, $\hat{T}(x) = 1 + x_1^2 + x_2^2$, $x = (x_1, x_2) \in D$. *Then*

$$\hat{f}^{\wedge}(\hat{x}) = \frac{f(x)}{\hat{T}(x)}$$

$$= \frac{1-x_1^2-2x_2^2}{1+x_1^2+x_2^2},$$

$$\hat{f}^{\wedge}(x) = \frac{f(x\hat{T}(x))}{\hat{T}(x)} = \frac{f(x_1\hat{T}(x), x_2\hat{T}(x))}{\hat{T}(x)}$$

$$= \frac{1-x_1^2(1+x_1^2+x_2^2)^2-2x_2^2(1+x_1^2+x_2^2)^2}{1+x_1^2+x_2^2},$$

$$\hat{f}(\hat{x}) = \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)}$$

$$= \frac{x_1^4+x_2^4+1+x_1^2+2x_1^2x_2^2}{(1+x_1^2+x_2^2)^4},$$

$$f^{\wedge}(x) = f(x\hat{T}(x))$$

$$= f(x_1\hat{T}(x), x_2\hat{T}(x))$$

$$= 1 - x_1^2(1+x_1^2+x_2^2)^2 - 2x_2^2(1+x_1^2+x_2^2)^2.$$

Then

$$\lim_{x\longrightarrow(1,1)} \hat{f}^{\wedge}(\hat{x}) = \lim_{x\longrightarrow(1,1)} \frac{1-x_1^2-2x_2^2}{1+x_1^2+x_2^2}$$

$$= -\frac{2}{3},$$

$$\lim_{x\longrightarrow(1,1)} \hat{f}^{\wedge}(x) = \frac{f(x\hat{T}(x))}{\hat{T}(x)}$$

$$= \lim_{x\longrightarrow(1,1)} \frac{1-x_1^2(1+x_1^2+x_2^2)^2-2x_2^2(1+x_1^2+x_2^2)^2}{1+x_1^2+x_2^2}$$

$$= -\frac{26}{3},$$

$$\lim_{x\longrightarrow(1,1)} \hat{f}(\hat{x}) = \lim_{x\longrightarrow(1,1)} \frac{x_1^4+x_2^4+1+x_1^2+2x_1^2x_2^2}{(1+x_1^2+x_2^2)^4}$$

$$= \frac{2}{27},$$

$$\lim_{x\longrightarrow(1,1)} f^{\wedge}(x) = \lim_{x\longrightarrow(1,1)}(1 - x_1^2(1+x_1^2+x_2^2)^2 - 2x_2^2(1+x_1^2+x_2^2)^2)$$

$$= -26.$$

Exercise 1.3.5. *Let* $D = \mathbb{R}^2$, $f(x) = x_1^4 + x_2^2 + 7$, $\hat{T}(x) = 1 + 2x_1^2 + x_2^2$, $x = (x_1, x_2) \in D$. *Find*

$$\lim_{x\longrightarrow(2,-1)} \hat{f}^{\wedge}(\hat{x}).$$

Answer. $\frac{12}{5}$.

The theorem below follows immediately from the definition for the limit of an iso-function, therefore we leave its proof to the reader.

Theorem 1.3.6. *Let $x_0 \in D$. Then there exists*

$$\lim_{x \longrightarrow x_0, x \in D} \hat{f}(x) = a$$

if and only if there exist $\hat{f}(x_0+0)$, $\hat{f}(x_0-0)$ and

$$\hat{f}(x_0-0) = \hat{f}(x_0+0) = a.$$

Definition 1.3.7. *We will say that the number $b \in \mathbb{R}$ is the limit of the iso-function $\hat{f}$ when $x \longrightarrow \pm\infty$ if it is the limit of its iso-original $\tilde{f}$ when $x \longrightarrow \pm\infty$.*

The proof of the following theorems repeats the steps of the proof in the case $n = 1$(see [1]).

Theorem 1.3.8. *Let the iso-function $\hat{f}$ has a limit a at the point $x_0 \in D$. Then there exist a neighbourhood $U(x_0)$ and a number $b > 0$ such that for every $x \in U(x_0) \cap D$, $x \neq x_0$, we have*

$$|\hat{f}(x)| \leq b.$$

Theorem 1.3.9. *Let $\lim_{x \longrightarrow x_0} \hat{f}(x) = b$, $b \neq 0$.*

1. *There exists a neighbourhood $U(x_0)$ such that for every $x \in U(x_0) \cap D$, $x \neq x_0$, we have*
$$|\hat{f}(x)| > \frac{|b|}{2},$$
2. *If $b > 0$ then there exists a neighbourhood $U(x_0)$ such that for every $x \in U(x_0) \cap D$, $x \neq x_0$, we have*
$$\hat{f}(x) > \frac{b}{2},$$
3. *If $b < 0$ then there exists a neighbourhood $U(x_0)$ such that for every $x \in U(x_0) \cap D$, $x \neq x_0$, we have*
$$\hat{f}(x) < \frac{b}{2}.$$

Theorem 1.3.10. *Let $\hat{\phi} : D \longrightarrow \hat{\phi}(D)$ and $\lim_{x \longrightarrow x_0} \hat{f}(x) = a$, $\lim_{x \longrightarrow x_0} \hat{\phi}(x) = b$ and $\hat{f}(x) \leq \hat{\phi}(x)$ for every $x \in D$. Then $a \leq b$.*

Theorem 1.3.11. *Let $\hat{\phi} : D \longrightarrow \hat{\phi}(D)$, $\hat{g} : D \longrightarrow \hat{g}(D)$ and*

$$\lim_{x \longrightarrow x_0} \hat{f}(x) = \lim_{x \longrightarrow x_0} \hat{\phi}(x) = a,$$

and

$$\hat{f}(x) \leq \hat{g}(x) \leq \hat{\phi}(x) \quad \forall x \in D.$$

Then

$$\lim_{x \longrightarrow x_0} \hat{g}(x) = a.$$

The following theorem lists some of the properties of the limit of an iso-function of the first, the second, the third, the fourth or the fifth kind. Its proof follows from the definition for the limit of an iso-function of the first, the second, the third, the fourth or the fifth kind .

Theorem 1.3.12. *Let $\hat{g} : D \longrightarrow \hat{g}(D)$ and $\hat{f}$ has a limit at the iso-point $x_0 \in D$. Then*

1. $\lim_{x\longrightarrow x_0}(\hat{f}(x) \pm \hat{g}(x)) = \lim_{x\longrightarrow x_0} \hat{f}(x) \pm \lim_{x\longrightarrow x_0} \hat{f}(x)$,
2. $\lim_{x\longrightarrow x_0}(\hat{f}(x) \hat{\times} \hat{g}(x)) = \lim_{x\longrightarrow x_0} \hat{f}(x) \hat{\times} \lim_{x\longrightarrow x_0} \hat{g}(x)$,
3. $\lim_{x\longrightarrow x_0}(\hat{f}(x)\hat{g}(x)) = \lim_{x\longrightarrow x_0} \hat{f}(x) \lim_{x\longrightarrow x_0} \hat{g}(x)$,
4. $\lim_{x\longrightarrow x_0}(\hat{f}(x) \diagup \hat{g}(x)) = \lim_{x\longrightarrow x_0} \hat{f}(x) \diagup \lim_{x\longrightarrow x_0} \hat{g}(x)$, *if* $\lim_{x\longrightarrow x_0} \hat{g}(x) \neq 0$,
5. $\lim_{x\longrightarrow x_0} \frac{\hat{f}(x)}{\hat{g}(x)} = \frac{\lim_{x\longrightarrow x_0} \hat{f}(x)}{\lim_{x\longrightarrow x_0} \hat{g}(x)}$, *if* $\lim_{x\longrightarrow x_0} \hat{g}(x) \neq 0$,
6. *if* $|\hat{f}(x)|$ *is bounded below and* $\lim_{x\longrightarrow x_0} \hat{g}(x) = 0$, *we have that* $\lim_{x\longrightarrow x_0}(\hat{f}(x) \diagup \hat{g}(x)) = \infty$,
7. *if* $\lim_{x\longrightarrow x_0} \hat{f}(x) = a$ *and* $\lim_{x\longrightarrow x_0} \hat{g}(x) = \infty$, *we have that* $\lim_{x\longrightarrow x_0}(\hat{f}(x) \diagup \hat{g}(x)) = 0$.

Theorem 1.3.13. *The limit* $\lim_{x\longrightarrow x_0} \hat{f}(x) = a$ *exists if and only if for every* $\varepsilon > 0$ *there exists a neighbourhood* $U(x_0)$ *such that for every* $x_1, x_2 \in U(x_0)$, $x_1 \neq x_0$, $x_2 \neq x_0$, *we have*

$$|\hat{f}(x_1) - \hat{f}(x_2)| < \varepsilon.$$

Definition 1.3.14. *We say that the iso-function $\hat{f}$ approaches $\pm\infty$ as x approaches x_0 if its iso-original approaches $\pm\infty$ as x approaches x_0.*

Exercise 1.3.15. *Find* $\lim_{x\longrightarrow(0,0)} \hat{f}^{\wedge}(\hat{x})$, *where*

$$f(x) = \frac{1-\cos(x_1x_2)}{x_1^2x_2^2}, \qquad \hat{T}(x) = x_1^2 + x_2^2 + 1, \qquad x = (x_1, x_2) \in D = \mathbb{R}^2.$$

Answer. $\frac{1}{2}$.

Exercise 1.3.16. *Find* $\lim_{x\longrightarrow(0,0)} \hat{f}^{\wedge}(\hat{x})$, *where*

$$f(x) = \frac{\log(1+x_1x_2)}{x_1x_2}, \qquad \hat{T}(x) = x_1^2 + x_2^2 + 1, \qquad x = (x_1, x_2) \in D = \mathbb{R}^2.$$

Answer. 1.

Exercise 1.3.17. *Find* $\lim_{x\longrightarrow(\infty,0)} \hat{f}^{\wedge}(\hat{x})$, *where*

$$f(x) = \frac{1}{x_1^2 + x_2^2}, \qquad \hat{T}(x) = x_1^2 + x_2^2 + 1, \qquad x = (x_1, x_2) \in D = \mathbb{R}^2.$$

Answer. 0.

Exercise 1.3.18. *Find* $\lim_{x\longrightarrow\infty}\hat{f}^{\wedge}(\hat{x})$, *where*

$$f(x)=\frac{x_1\sin x_1}{x_1^4+x_2^4},\qquad \hat{T}(x)=x_1^2+x_2^2+1,\qquad x=(x_1,x_2)\in D=\mathbb{R}^2.$$

Answer. 0.

Exercise 1.3.19. *Find* $\lim_{x\longrightarrow(\infty,-\infty)}\hat{f}^{\wedge}(\hat{x})$, *where*

$$f(x)=x_1x_2,\qquad \hat{T}(x)=\frac{1}{x_1^2+x_2^2+1},\qquad x=(x_1,x_2)\in D=\mathbb{R}^2.$$

1.4. Continuous Iso-Real Iso-Valued Iso-Functions of n Variables

Let $D\subset\mathbb{R}^n$ and $\hat{T}:D\longrightarrow\mathbb{R}$, $\hat{T}(x)>0$ for every $x\in D$, $\hat{f}:D\longrightarrow\mathbb{R}$ is an iso-function of the first, the second, the third, the fourth or the fifth kind and let $\tilde{f}$ be its iso-original.

Definition 1.4.1. *The iso-function $\hat{f}$ will be called continuous at the point $x_0\in D$ if its iso-original is a continuous function at x_0.*

Definition 1.4.2. *The iso-function $\hat{f}$ will be called continuous function in D if it is a continuous function at every point of D.*

Example 1.4.3. *Let $D=\mathbb{R}^2$, $f(x)=x_1^2+x_2^2$, $\hat{T}(x)=1+x_1^2+x_2^2$. Then*

$$\hat{f}^{\wedge}(\hat{x})=\frac{f(x)}{\hat{T}(x)}$$

$$=\frac{x_1^2+x_2^2}{1+x_1^2+x_2^2},$$

$$\hat{f}^{\wedge}(x)=\frac{f(x\hat{T}(x))}{\hat{T}(x)}=\frac{f(x_1\hat{T}(x),x_2\hat{T}(x))}{\hat{T}(x)}$$

$$=(x_1^2+x_2^2)(1+x_1^2+x_2^2),$$

$$\hat{f}(\hat{x})=\frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)}=\frac{f\left(\frac{x_1}{\hat{T}(x)},\frac{x_2}{\hat{T}(x)}\right)}{\hat{T}(x)}$$

$$=\frac{x_1^2+x_2^2}{(1+x_1^2+x_2^2)^3},$$

$$f^{\wedge}(x)=f(x\hat{T}(x))=f(x_1\hat{T}(x),x_2\hat{T}(x))$$

$$=(x_1^2+x_2^2)(1+x_1^2+x_2^2)^3.$$

The iso-functions $\hat{f}^{\wedge\wedge}$, $\hat{f}^{\wedge}$, $\hat{\hat{f}}$ and $f^{\wedge}$ are continuous in D.

Example 1.4.4. *Let $D = \mathbb{R}^2$, $f(x) = x_1^3 + x_2^3 + 3$, $x = (x_1, x_2) \in D$, and*

$$\hat{T}(x) = \begin{cases} 1 & x_1 \in \mathbb{R}, \quad x_2 \leq 1, \\ \\ 2 & x_1 \in \mathbb{R}, \quad x_2 \geq 1. \end{cases}$$

Then f is a continuous function in D. Then

$$\hat{f}^{\wedge}(\hat{x}) = \frac{f(x)}{\hat{T}(x)} = \begin{cases} x_1^3 + x_2^3 + 3 & x_1 \in \mathbb{R}, \quad x_2 \leq 1, \\ \\ \frac{x_1^3 + x_2^3 + 3}{2} & x_1 \in \mathbb{R}, \quad x_2 \geq 1, \end{cases}$$

$$\hat{f}^{\wedge}(x) = \frac{f(x\hat{T}(x))}{\hat{T}(x)} = \begin{cases} x_1^3 + x_2^3 + 3 & x_1 \in \mathbb{R}, \quad x_2 \leq 1, \\ \\ \frac{8x_1^3 + 8x_2^3 + 3}{2} & x_1 \in \mathbb{R}, \quad x_2 \geq 1, \end{cases}$$

$$\hat{f}(\hat{x}) = \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)} = \begin{cases} x_1^3 + x_2^3 + 3 & x_1 \in \mathbb{R}, \quad x_2 \leq 1, \\ \\ \frac{x_1^3 + x_2^3 + 24}{16} & x_1 \in \mathbb{R}, \quad x_2 \geq 1, \end{cases}$$

$$f^{\wedge}(x) = f(x\hat{T}(x)) = \begin{cases} x_1^3 + x_2^3 + 3 & x_1 \in \mathbb{R}, \quad x_2 \leq 1, \\ \\ 8x_1^3 + 8x_2^3 + 3 & x_1 \in \mathbb{R}, x_2 \geq 1. \end{cases}$$

Then the iso-functions $\hat{f}^{\wedge\wedge}$, $\hat{f}^{\wedge}$, $\hat{\hat{f}}$ and $f^{\wedge}$ are not continuous functions at $(x_1, 1)$, $x_1 \in \mathbb{R}$.

Exercise 1.4.5. *Let $D = \mathbb{R}^2$, $f(x_1, x_2) = \frac{x_1 + x_2}{x_1^2 + x_2^2}$ for $(x_1, x_2) \neq (0, 0)$, $f(0, 0) = 0$, $\hat{T}(x) = 1 + x_1^2 + x_2^2$, $x = (x_1, x_2) \in D$. Check if $\hat{f}^{\wedge}(\hat{x})$ is a continuous function in D.*

Answer. Yes.

Exercise 1.4.6. *Let $D = \mathbb{R}^2$, $f(x) = (x_1 + x_2) \sin \frac{1}{x_1} \sin \frac{1}{x_2}$ for $x_1 x_2 \neq 0$, $f(0, x_2) = f(x_1, 0) = 0$, x_1, $x_2 \in \mathbb{R}$, $\hat{T}(x) = 2 + |x_1| + x_2^2$, $x = (x_1, x_2) \in D$. Check if $\hat{f}^{\wedge}(\hat{x})$ is a continuous function in D.*

Answer. $\hat{f}^{\wedge\wedge}$ is not a continuous function at every point (x_1, x_2) for which $x_1 x_2 = 0$ and $x_1^2 + x_2^2 \neq 0$.

Below we list some of the properties of the continuous iso-functions of the first, the second, the third, the fourth or the fifth kind. Their proofs repeat the proofs in the case $n = 1$.

Theorem 1.4.7. *Let $\hat{g} : D \longrightarrow \hat{g}(D)$ and $\hat{f}$ are continuous at x_0, $x_0 \in D$. Then*

1. *$\hat{f} \pm \hat{g}$ is continuous at x_0,*
2. *$\hat{f} \hat{\times} \hat{g}$ is continuous at x_0,*
3. *$\hat{f} \hat{g}$ is continuous at x_0,*

4. $\hat{f} \leftthreetimes \hat{g}$ *is continuous at* x_0 *if* $\hat{g}(x_0) \neq 0$,

5. $\frac{\hat{f}}{\hat{g}}$ *is continuous at* x_0 *if* $\hat{g}(x_0) \neq 0$

Theorem 1.4.8. *Let* $\hat{f}$ *is continuous at* $x_0 \in D$. *Then*

1. *there exists a neighbourhood* $U(x_0)$ *such that for every* $x \in U(x_0) \cap D$, $x \neq x_0$, *we have*

$$|\hat{f}(x)| > \frac{|\hat{f}(x_0)|}{2},$$

2. *if* $\hat{f}(x_0) > 0$, *there exists a neighbourhood* $U(x_0)$ *such that for every* $x \in U(x_0) \cap D$, $x \neq x_0$, *we have*

$$\hat{f}(x) > \frac{\hat{f}(x_0)}{2},$$

3. *if* $\hat{f}(x_0) < 0$, *there exists a neighbourhood* $U(x_0)$ *such that for every* $x \in U(x_0) \cap D$, $x \neq x_0$, *we have*

$$\hat{f}(x) < \frac{\hat{f}(x_0)}{2}.$$

Definition 1.4.9. *The iso-function* $\hat{f}$ *of the first, the second, the third, the fourth or the fifth kind will be called discontinuous at* $x_0 \in D$ *of the first kind if there exist*

$$\hat{f}(x_0 - 0), \quad \hat{f}(x_0 + 0),$$

and

$$\hat{f}(x_0 - 0) \neq \hat{f}(x_0 + 0).$$

Definition 1.4.10. *The iso-function* $\hat{f}$ *will be called discontinuous of the second kind at* $x_0 \in D$ *if one of the both of the limits*

$$\hat{f}(x_0 - 0), \quad \hat{f}(x_0 + 0)$$

does not exist. Here are included the cases

$$\hat{f}(x_0 - 0) = \pm\infty, \quad \hat{f}(x_0 + 0) = \pm\infty.$$

Theorem 1.4.11. *Let* K *be a compact set in* $\mathbb{R}^n$ *and* $\hat{f} : K \longrightarrow D$ *be a continuous function in* K. *Then* $\hat{f}$ *is bounded.*

Definition 1.4.12. *We will say that the iso-function* $\hat{f} : D \longrightarrow \mathbb{R}$ *is uniformly continuous in* D *if for every* $\hat{\varepsilon} \in \hat{F}_{\mathbb{R}}$, $\hat{\varepsilon} > 0$, *there exists* $\hat{\delta} = \hat{\delta}(\hat{\varepsilon}) > 0$, $\hat{\delta} \in \hat{F}_{\mathbb{R}}$, *such that*

$$|\hat{f}(x) - \hat{f}(x')\hat{|} < \hat{\varepsilon}$$

whenever $|x - x'\hat{|} < \hat{\delta}$, $x, x' \in D$.

Theorem 1.4.13. *If* $\hat{f}$ *is a continuous function on a compact set* $D \subset \mathbb{R}^n$ *then* $\hat{f}$ *is an uniformly continuous function on* D.

1.5. Iso-Partial Derivatives of Iso-Real Iso-Valued Iso-Functions of n Variables

Let $D \subset \mathbb{R}^n$ and $\hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$ for every $x \in D$. Here we suppose that $f, \hat{T} : D \longrightarrow \mathbb{R}$ are enough times differentiable functions with respect to their variables.

Definition 1.5.1. *Let $i \in \{1, 2, \ldots, n\}$ be fixed. Then the iso-differential with respect to x_i we define as follows*

$$\hat{\partial}_{x_i}(\cdot) = \hat{T}(x)\partial_{x_i}(\cdot)dx_i.$$

Using the above definition we can deduct the following iso-differentials.

1. The iso-differential with respect to x_i of x_i, $i = 1, 2, \ldots, n$,

$$\hat{\partial}_{x_i}(\hat{x}_i) = \hat{T}(x)\partial_{x_i}\tfrac{x_i}{\hat{T}(x)}dx_i$$

$$= \hat{T}(x)\tfrac{\hat{T}(x)dx_i - x_i\partial_{x_i}\hat{T}(x)dx_i}{\hat{T}^2(x)}$$

$$= \tfrac{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)}{\hat{T}(x)}dx_i.$$

2. The iso-differential with respect to x_i of x_j, for $i \neq j$, $i, j = 1, 2, \ldots, n$,

$$\hat{\partial}_{x_i}(\hat{x}_j) = \hat{T}(x)\partial_{x_i}\tfrac{x_j}{\hat{T}(x)}dx_i$$

$$= \hat{T}(x)\tfrac{-x_j\partial_{x_i}\hat{T}(x)dx_i}{\hat{T}^2(x)}$$

$$= \tfrac{-x_j\partial_{x_i}\hat{T}(x)}{\hat{T}(x)}dx_i.$$

3. The iso-differential with respect to x_i of an iso-function of the first kind

$$\hat{\partial}_{x_i}\hat{f}^{\wedge}(\hat{x}) = \hat{T}(x)\partial_{x_i}\tfrac{f(x)}{\hat{T}(x)}dx_i$$

$$= \hat{T}(x)\tfrac{\partial_{x_i}f(x)\hat{T}(x)dx_i - f(x)\partial_{x_i}\hat{T}(x)dx_i}{\hat{T}^2(x)}$$

$$= \tfrac{\partial_{x_i}f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x)}dx_i.$$

4. The iso-differential with respect to x_i of an iso-function of the second kind

$$\hat{\partial}_{x_i}\hat{f}^{\wedge}(x) = \hat{T}(x)\partial_{x_i}\tfrac{f(x\hat{T}(x))}{\hat{T}(x)}dx_i$$

$$= \hat{T}(x)\tfrac{\partial_{x_i}f(x\hat{T}(x))\partial_{x_i}(x_i\hat{T}(x))\hat{T}(x)dx_i + \sum_{j=1, j\neq i}^{n}\partial_{x_j}f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)dx_i - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)dx_i}{\hat{T}^2(x)}$$

$$= \tfrac{\partial_{x_i}f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x))\hat{T}(x) + \sum_{j=1, j\neq i}^{n}\partial_{x_j}f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)}{\hat{T}(x)}dx_i.$$

5. The iso-differential with respect to x_i of an iso-function of the third kind

$$\hat{\partial}_{x_i}\hat{f}(\hat{x}) = \hat{T}(x)\partial_{x_i}\frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)}dx_i$$

$$= \hat{T}(x)\frac{\partial_{x_i}f\left(\frac{x}{\hat{T}(x)}\right)\partial_{x_i}\left(\frac{x_i}{\hat{T}(x)}\right)\hat{T}(x)dx_i+\sum_{j=1,j\neq i}^{n}\partial_{x_j}f\left(\frac{x}{\hat{T}(x)}\right)\partial_{x_i}\frac{x_j}{\hat{T}(x)}\hat{T}(x)dx_i-f\left(\frac{x}{\hat{T}(x)}\right)\partial_{x_i}\hat{T}(x)dx_i}{\hat{T}^2(x)}$$

$$= \frac{\partial_{x_i}f\left(\frac{x}{\hat{T}(x)}\right)\frac{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}{\hat{T}(x)}-\sum_{j=1,j\neq i}^{n}\partial_{x_j}f\left(\frac{x}{\hat{T}(x)}\right)x_j\frac{\partial_{x_i}\hat{T}(x)}{\hat{T}^2(x)}\hat{T}(x)-f\left(\frac{x}{\hat{T}(x)}\right)\partial_{x_i}\hat{T}(x)}{\hat{T}(x)}dx_i$$

$$= \frac{\partial_{x_i}f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x))-\sum_{j=1,j\neq i}^{n}\partial_{x_j}f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)-f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}^2(x)}dx_i.$$

6. The iso-differential with respect to x_i of an iso-function of the fourth kind

$$\hat{\partial}_{x_i}f^{\wedge}(x) = \hat{T}(x)\partial_{x_i}(f(x\hat{T}(x)))dx_i$$

$$= \hat{T}(x)\partial_{x_i}f(x\hat{T}(x))\partial_{x_i}(x_i\hat{T}(x))dx_i + \hat{T}(x) + \hat{T}(x)\sum_{j=1,j\neq i}\partial_{x_j}f(x\hat{T}(x))dx_i$$

$$= \hat{T}(x)\partial_{x_i}f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))dx_i + \hat{T}(x)\sum_{j=1,j\neq i}^{n}\partial_{x_j}f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)dx_i.$$

7. The iso-differential with respect to x_i of an iso-function of the fifth kind

$$\hat{\partial}_{x_i}f^{\vee}(x) = \hat{T}(x)\partial_{x_i}f(\hat{x})$$

$$= \hat{T}(x)\partial_{x_i}f\left(\frac{x}{\hat{T}(x)}\right)$$

$$= \hat{T}(x)\partial_{x_i}f\left(\frac{x}{\hat{T}(x)}\right)\partial_{x_i}\frac{x_i}{\hat{T}(x)}dx_i + \hat{T}(x)\sum_{j=1,j\neq i}^{n}\partial_{x_j}f\left(\frac{x}{\hat{T}(x)}\right)\partial_{x_i}\frac{x_j}{\hat{T}(x)}dx_i$$

$$= \hat{T}(x)\partial_{x_i}f\left(\frac{x}{\hat{T}(x)}\right)\frac{\hat{T}(x)-x_i\hat{T}_{x_i}(x)}{\hat{T}^2(x)}dx_i - \hat{T}(x)\sum_{j=1,j\neq i}^{n}\partial_{x_j}f\left(\frac{x}{\hat{T}(x)}\right)x_j\frac{\hat{T}_{x_i}(x)}{\hat{T}^2(x)}dx_i$$

$$= \partial_{x_i}f\left(\frac{x}{\hat{T}(x)}\right)\frac{\hat{T}(x)-x_i\hat{T}_{x_i}(x)}{\hat{T}(x)}dx_i - \sum_{j=1,j\neq i}^{n}\partial_{x_j}f\left(\frac{x}{\hat{T}(x)}\right)x_j\frac{\hat{T}_{x_i}(x)}{\hat{T}(x)}dx_i$$

Definition 1.5.2. *The first order iso-partial derivative of the first kind with respect to x_i will be defined as follows*

$$(\cdot)_{x_i}^{1\circledast} = \hat{\partial}_{x_i}(\cdot) \nearrow \hat{\partial}_{x_i}(\hat{x}_i).$$

Using the above definition we have

1. The first order iso-partial derivative of the first kind with respect to x_i of an iso-function of the first kind

$$(\hat{f}^{\wedge}(\hat{x}))_{x_i}^{1\circledast} = \frac{1}{\hat{T}(x)}\frac{\partial_{x_i}f(x)\hat{T}(x)-f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}.$$

2. The first order iso-partial derivative of the first kind with respect x_i of an iso-function of the second kind

$$(\hat{f}^{\wedge}(x))^{1\circledast}_{x_i} = \frac{1}{\hat{T}(x)} \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))\hat{T}(x)+\sum_{j=1,j\neq i}^{n}\partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\hat{T}(x)-f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}.$$

3. The first order iso-partial derivative of the first kind with respect to x_i of an iso-function of the third kind

$$(\hat{f}(\hat{x}))^{1\circledast}_{x_i} = \frac{1}{\hat{T}^2(x)} \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x))-\sum_{j=1,j\neq i}^{n}\partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)-f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}.$$

4. The first order iso-partial derivative of the first kind with respect to x_i of an iso-function of the fourth kind

$$(f^{\wedge}(x))^{1\circledast}_{x_i} = \hat{T}(x) \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))+\sum_{j=1,j\neq i}^{n}\partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}.$$

5. The first order iso-partial derivative of the first kind with respect to x_i of an iso-function of the fifth kind

$$(f^{\vee}(x))^{1\circledast}(x) = \frac{1}{\hat{T}(x)} \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\hat{T}_{x_i}(x))-\sum_{j=1,j\neq i}^{n}\partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}.$$

Example 1.5.3. *Let $D=\mathbb{R}^2$, $f(x)=x_1+x_2$, $\hat{T}(x)=x_1^2+x_2^2+1$, $x=(x_1,x_2)\in D$. Then*

$$\hat{f}^{\wedge}(\hat{x}) = \frac{f(x)}{\hat{T}(x)} = \frac{x_1+x_2}{x_1^2+x_2^2+1},$$

$$\hat{f}^{\wedge}(x) = \frac{f(x\hat{T}(x)}{\hat{T}(x)} = \frac{f(x_1\hat{T}(x),x_2\hat{T}(x))}{\hat{T}(x)} = x_1+x_2,$$

$$\hat{f}(\hat{x}) = \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)} = \frac{f\left(\frac{x_1}{\hat{T}(x)},\frac{x_2}{\hat{T}(x)}\right)}{\hat{T}(x)} = \frac{x_1+x_2}{(x_1^2+x_2^2+1)^2},$$

$$f^{\wedge}(x) = f(x\hat{T}(x)) = f(x_1\hat{T}(x),x_2\hat{T}(x)) = (x_1+x_2)(x_1^2+x_2^2+1)$$

$$= x_1^3+x_1x_2^2+x_1+x_2x_1^2+x_2^3+x_2.$$

From here,

$$\hat{\partial}_{x_1}(\hat{x}_1) = \hat{T}(x)\partial_{x_1}\hat{x}_1 dx_1 = \hat{T}(x)\partial_{x_1}\tfrac{x_1}{\hat{T}(x)}dx_1 = (x_1+x_2)\partial_{x_1}\tfrac{x_1}{x_1^2+x_2^2+1}dx_1 = \tfrac{-x_1^2+x_2^2+1}{x_1^2+x_2^2+1}dx_1,$$

$$\hat{\partial}_{x_1}\hat{f}^\wedge(\hat{x}) = \hat{T}(x)\partial_{x_1}\hat{f}^\wedge(\hat{x})dx_1 = (x_1^2+x_2^2+1)\partial_{x_1}\tfrac{x_1+x_2}{x_1^2+x_2^2+1}dx_1 = \tfrac{-x_1^2+x_2^2-2x_1x_2+1}{x_1^2+x_2^2+1}dx_1,$$

$$\hat{\partial}_{x_1}\hat{f}^\wedge(x) = \hat{T}(x)\partial_{x_1}\hat{f}^\wedge(x)dx_1 = (x_1^2+x_2^2+1)\partial_{x_1}(x_1+x_2)dx_1 = (x_1^2+x_2^2+1)dx_1,$$

$$\hat{\partial}_{x_1}\hat{f}(\hat{x}) = \hat{T}(x)\partial_{x_1}\hat{f}(\hat{x})dx_1$$

$$= (x_1^2+x_2^+1)\partial_{x_1}\left(\tfrac{x_1+x_2}{(x_1^2+x_2^2+1)^2}\right)dx_1 = \tfrac{-3x_1^2+x_2^2-4x_1x_2+1}{(x_1^2+x_2^2+1)^2}dx_1,$$

$$\hat{\partial}_{x_1}f^\wedge(x) = \hat{T}(x)\partial_{x_1}f^\wedge(x)dx_1 = (x_1^2+x_2^2+1)\partial_{x_1}(x_1^3+x_1x_2^2+x_1+x_2x_1^2+x_2^3+x_2)dx_1$$

$$= (x_1^2+x_2^2+1)(3x_1^2+x_2^2+2x_1x_2+1)dx_1.$$

Using the above computations we get

$$\left(\hat{f}^\wedge(\hat{x})\right)_{x_1}^{1\circledast} = \hat{\partial}_{x_1}\hat{f}^\wedge(\hat{x}) \nearrow \hat{\partial}_{x_1}(\hat{x}_1) = \tfrac{1}{\hat{T}(x)}\tfrac{\hat{\partial}_{x_1}\hat{f}^\wedge(\hat{x})}{\hat{\partial}_{x_1}(\hat{x})} = \tfrac{-x_1^2+x_2^2-2x_1x_2+1}{x_2^4+2x_2^2-x_1^4+1},$$

$$\left(\hat{f}^\wedge(x)\right)_{x_1}^{1\circledast} = \hat{\partial}_{x_1}\hat{f}^\wedge(x) \nearrow \hat{\partial}_{x_1}(\hat{x}_1) = \tfrac{1}{\hat{T}(x)}\tfrac{\hat{\partial}_{x_1}\hat{f}^\wedge(x)}{\hat{\partial}_{x_1}(\hat{x}_1)} = \tfrac{x_1^2+x_2^+1}{-x_1^2+x_2^2+1},$$

$$\left(\hat{f}(\hat{x})\right)_{x_1}^{1\circledast} = \hat{\partial}_{x_1}\hat{f}(\hat{x} \nearrow \hat{\partial}_{x_1}(\hat{x}_1) = \tfrac{1}{\hat{T}(x)}\tfrac{\hat{\partial}_{x_1}\hat{f}(\hat{x})}{\hat{\partial}_{x_1}(\hat{x}_1)} = \tfrac{-3x_1^2+x_2^2-4x_1x_2+1}{(-x_1^2+x_2^2+1)(x_1^2+x_2^2+1)^2},$$

$$\left(f^\wedge(x)\right)_{x_1}^{1\circledast} = \hat{\partial}_{x_1}f^\wedge(x) \nearrow \hat{\partial}_{x_1}(\hat{x}_1) = \tfrac{1}{\hat{T}(x)}\tfrac{\hat{\partial}_{x_1}f^\wedge(x)}{\hat{\partial}_{x_1}(\hat{x}_1)} = \tfrac{(x_1^2+x_2^2+1)(3x_1^2+x_2^2+2x_1x_2+1)}{-x_1^2+x_2^2+1}.$$

Exercise 1.5.4. *Let $D=\mathbb{R}^2$, $f(x)=x_1^2+x_2$, $\hat{T}(x)=x_1^4+x_2^4+3$, $x=(x_1,x_2)\in D$. Find $\left(\hat{f}^\wedge(\hat{x})\right)_{x_2}^{1\circledast}$.*

Answer. $\frac{x_1^4-3x_2^4-4x_1^2x_2^3+3}{(x_1^4+x_2^4+3)(x_1^4-3x_2^4+3)}$.

Definition 1.5.5. *The second order iso-partial derivative of the first kind we define as follows*

$$\left((\cdot)_{x_i}^{1\circledast}\right)_{x_l}^{1\circledast}, \qquad i,l=1,2,\ldots,n.$$

The third order iso-partial derivative of the first kind we define as follows

$$\left(\left((\cdot)_{x_i}^{1\circledast}\right)_{x_l}^{1\circledast}\right)_{x_k}^{1\circledast}, \qquad i,l,k=1,2,\ldots,n,$$

and so on.

Definition 1.5.6. *The first order iso-partial derivative of the second kind with respect to x_i will be defined as follows*

$$(\cdot)^{2\circledast}_{x_i} = \hat{\partial}_{x_i}(\cdot) \nearrow dx_i.$$

Using the above definition we have

1. The first order iso-partial derivative of the second kind with respect to x_i of an iso-function of the first kind

$$(\hat{f}^{\wedge}(\hat{x}))^{2\circledast}_{x_i} = \frac{\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}^2(x)}.$$

2. The first order iso-partial derivative of the second kind with respect to x_i of an iso-function of the second kind

$$(\hat{f}^{\wedge}(x))^{2\circledast}_{x_i} = \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))\hat{T}(x)+\sum_{j=1,j\neq i}^{n}\partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\hat{T}(x)-f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)}{\hat{T}^2(x)}.$$

3. The first order iso-partial derivative of the second kind with respect to x_i of an iso-function of the third kind

$$(\hat{f}(\hat{x}))^{2\circledast}_{x_i} = \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x))-\sum_{j=1,j\neq i}^{n}\partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)-f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}^3(x)}.$$

4. The first order iso-partial derivative of the second kind with respect to x_i of an iso-function of the fourth kind

$$(f^{\wedge}(x))^{2\circledast}_{x_i} = \partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))+\sum_{j=1,j\neq i}^{n}\partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x).$$

5. The first order iso-partial derivative of the second kind with respect to x_i of an iso-function of the fifth kind

$$(f^{\vee}(x))^{2\circledast}_{x_i} = \partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)\frac{\hat{T}(x)-x_i\hat{T}_{x_i}(x)}{\hat{T}^2(x)} - \sum_{j=1,j\neq i}^{n}\partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right)x_j\frac{\hat{T}_{x_i}(x)}{\hat{T}^2(x)}$$

Remark 1.5.7. *In fact,*

$$\hat{\partial}_{x_i}(\hat{f}^{\wedge}(\hat{x})) \nearrow dx_i = (\hat{f}^{\wedge}(\hat{x}))_{x_i}, \qquad \hat{\partial}_{x_i}(\hat{f}^{\wedge}(x)) \nearrow dx_i = (\hat{f}^{\wedge}(x))_{x_i},$$

$$\hat{\partial}_{x_i}(\hat{f}(\hat{x})) \nearrow dx_i = (\hat{f}(\hat{x}))_{x_i}, \qquad \hat{\partial}_{x_i}(f^{\wedge}(x)) \nearrow dx_i = (f^{\wedge}(x))_{x_i}.$$

Example 1.5.8. *Let $D=\mathbb{R}^2$, $f(x)=2x_1x_2$, $\hat{T}(x)=1+x_1^2+x_2^2$, $x=(x_1,x_2)\in D$. Then*

$$\hat{f}^{\wedge}(\hat{x}) = \frac{f(x)}{\hat{T}(x)} = 2\frac{x_1x_2}{1+x_1^2+x_2^2},$$

$$\hat{f}^{\wedge}(x) = \frac{f(x\hat{T}(x))}{\hat{T}(x)} = 2x_1x_2(1+x_1^2+x_2^2),$$

$$\hat{f}(\hat{x}) = \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)} = 2\frac{x_1x_2}{(x_1^2+x_2^2+1)^3},$$

$$f^{\wedge}(x) = f(x\hat{T}(x)) = 2x_1x_2(1+x_1^2+x_2^2).$$

From here

$$\hat{\partial}_{x_2}(\hat{f}^\wedge(\hat{x}) \nearrow dx_2 = 2\frac{\partial}{\partial x_2}\left(\frac{x_1x_2}{1+x_1^2+x_2^2}\right) = 2\frac{x_1^3-3x_1x_2^2+x_1}{(x_1^2+x_2^2+1)^2},$$

$$\hat{\partial}_{x_2}(\hat{f}^\wedge(x)) \nearrow dx_i = 2\frac{\partial}{\partial x_2}(x_1x_2(1+x_1^2+x_2^2)) = 2x_1^3+6x_1x_2^2+2x_1,$$

$$\hat{\partial}_{x_2}(\hat{f}(\hat{x})) \nearrow dx_2 = 2\frac{\partial}{\partial x_2}\left(\frac{x_1x_2}{(x_1^2+x_2^2+1)^3}\right) = 2\frac{x_1^3-5x_1x_2^2+x_1}{(1+x_1^2+x_2^2)^4},$$

$$\hat{\partial}_{x_2}(f^\wedge(x)) \nearrow dx_2 = 2\frac{\partial}{\partial x_2}(x_1x_2(1+x_1^2+x_2^2)) = 2(x_1^2+x_2^2+1)(x_1^3+5x_1x_2^2+x_1).$$

Exercise 1.5.9. *Let $D=\mathbb{R}^2$, $\hat{T}(x)=x_1^2+x_2^2+3$, $f(x)=x_1-2x_2$, $x=(x_1,x_2)\in D$. Find $(f^\wedge(x))^{2\circledast}_{x_1}$.*

Answer. $-2x_1^2x_2-2x_2^3-6x_2+2x_1^2-4x_1x_2$.

Definition 1.5.10. *The second order iso-partial derivative of the second kind we define as follows*

$$\left((\cdot)^{2\circledast}_{x_i}\right)^{2\circledast}_{x_l}, \qquad i,l=1,2,\ldots,n.$$

The third order iso-partial derivative of the second kind we define as follows

$$\left(\left((\cdot)^{2\circledast}_{x_i}\right)^{2\circledast}_{x_l}\right)^{2\circledast}_{x_k}, \qquad i,l,k=1,2,\ldots,n,$$

and so on.

Definition 1.5.11. *The first order iso-partial derivative of the third kind with respect to x_i will be defined as follows*

$$(\cdot)^{3\circledast}_{x_i} = \partial_{x_i}(\cdot)dx_i \nearrow \hat{\partial}_{x_i}(\hat{x}_i).$$

Using the above definition we have

1. The first order iso-partial derivative of the third kind with respect to x_i of an iso-function of the first kind

$$(\hat{f}^\wedge(\hat{x}))^{3\circledast}_{x_i} = \frac{1}{\hat{T}^2(x)}\frac{\partial_{x_i}f(x)\hat{T}(x)-f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}.$$

2. The first order iso-partial derivative of the third kind with respect to x_i of an iso-function of the second kind

$$(\hat{f}^\wedge(x))^{3\circledast}_{x_i} = \frac{1}{\hat{T}^2(x)}\frac{\partial_{x_i}f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))\hat{T}(x)+\sum_{j=1,j\neq i}^n\partial_{x_j}f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\hat{T}(x)-f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}.$$

3. The first order iso-partial derivative of the third kind with respect to x_i of an iso-function of the third kind

$$(\hat{f}(\hat{x}))^{3\circledast}_{x_i} = \frac{1}{\hat{T}^3(x)}\frac{\partial_{x_i}f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x))-\sum_{j=1,j\neq i}^n\partial_{x_j}f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)-f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}.$$

4. The first order iso-partial derivative of the third kind with respect to x_i of an iso-function of the fourth kind

$$(f^\wedge(x))^{3\circledast}_{x_i} = \hat{T}(x)\frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))+\sum_{j=1,j\neq i}^n \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}.$$

5. The first order iso-partial derivative of the third kind with respect to x_i of an iso-function of the fifth kind

$$(f^\vee(x))^{3\circledast}_{x_i} = \frac{1}{\hat{T}^2(x)}\frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\hat{T}_{x_i}(x))-\sum_{j=1,j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)}{\hat{T}(x)-x_i\hat{T}_{x_i}(x)}.$$

Example 1.5.12. *Let $D = \{(x_1,x_2)\in\mathbb{R}^2 : x_1\geq 1, x_2\geq 1\}$, $f(x) = x_1-x_2$, $\hat{T}(x) = x_1+x_2$, $x=(x_1,x_2)\in D$. Then*

$$\hat{f}^\wedge(\hat{x}) = \frac{f(x)}{\hat{T}(x)} = \frac{x_1-x_2}{x_1+x_2},$$

$$\hat{f}^\wedge(x) = \frac{f(x\hat{T}(x))}{\hat{T}(x)} = \frac{f(x_1\hat{T}(x),x_2\hat{T}(x))}{\hat{T}(x)} = \frac{x_1\hat{T}(x)-x_2\hat{T}(x)}{\hat{T}(x)} = x_1-x_2,$$

$$\hat{f}(\hat{x}) = \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)} = \frac{f\left(\frac{x_1}{\hat{T}(x)},\frac{x_2}{\hat{T}(x)}\right)}{\hat{T}(x)} = \frac{\frac{x_1}{\hat{T}(x)}-\frac{x_2}{\hat{T}(x)}}{\hat{T}(x)} = \frac{x_1-x_2}{(x_1+x_2)^2},$$

$$f^\wedge(x) = f(x\hat{T}(x)) = f(x_1\hat{T}(x),x_2\hat{T}(x)) = x_1\hat{T}(x)-x_2\hat{T}(x) = x_1^2-x_2^2,$$

$$\hat{x}_1 = \frac{x_1}{\hat{T}(x)} = \frac{x_1}{x_1+x_2}.$$

From here

$$\hat{\partial}_{x_1}(\hat{x}_1) = \hat{T}(x)\partial_{x_1}\left(\frac{x_1}{x_1+x_2}\right)dx_1 = (x_1+x_2)\frac{x_2}{(x_1+x_2)^2}dx_1 = \frac{x_2}{x_1+x_2}dx_1,$$

$$\partial_{x_1}(\hat{f}^\wedge(\hat{x})) = \partial_{x_1}\left(\frac{x_1-x_2}{x_1+x_2}\right) = \frac{2x_2}{(x_1+x_2)^2},$$

$$\partial_{x_1}(\hat{f}^\wedge(x)) = \partial_{x_1}(x_1-x_2) = 1,$$

$$\partial_{x_1}(\hat{f}(\hat{x})) = \partial_{x_1}\left(\frac{x_1-x_2}{(x_1+x_2)^2}\right) = \frac{-x_1+3x_2}{(x_1+x_2)^3},$$

$$\partial_{x_1}(f^\wedge(x)) = \partial_{x_1}(x_1^2-x_2^2) = 2x_1.$$

Using the above computations we get

$$(\hat{f}^\wedge(\hat{x}))^{3\circledast}_{x_1} = \partial_{x_1}(\hat{f}^\wedge(\hat{x})) \nearrow \hat{\partial}_{x_1}(\hat{x}_1) = \frac{1}{\hat{T}(x)} \frac{\partial_{x_1}(\hat{f}^\wedge(x))}{\hat{\partial}_{x_1}(\hat{x}_1)} = \frac{2}{(x_1+x_2)^2},$$

$$(\hat{f}^\wedge(x))^{3\circledast}_{x_1} = \partial_{x_1}(\hat{f}^\wedge(x)) \nearrow \hat{\partial}_{x_1}(\hat{x}_1) = \frac{1}{\hat{T}(x)} \frac{\partial_{x_1}(\hat{f}^\wedge(x))}{\hat{\partial}_{x_1}(\hat{x}_1)} = \frac{1}{x_2},$$

$$(\hat{f}(\hat{x}))^{3\circledast}_{x_1} = \partial_{x_1}(\hat{f}(\hat{x})) \nearrow \hat{\partial}_{x_1}(\hat{x}_1) = \frac{1}{\hat{T}(x)} \frac{\partial_{x_1}(\hat{f}(\hat{x}))}{\hat{\partial}_{x_1}(\hat{x}_1)} = \frac{-x_1+3x_2}{x_2(x_1+x_2)^3},$$

$$(f^\wedge(x))^{3\circledast}_{x_1} = \partial_{x_1}(f^\wedge(x)) \nearrow \hat{\partial}_{x_1}(\hat{x}_1) = \frac{1}{\hat{T}(x)} \frac{\partial_{x_1}(f^\wedge(x)}{)} \hat{\partial}_{x_1}(\hat{x}_1) = 2\frac{x_1}{x_2}.$$

Exercise 1.5.13. *Let $D = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 \geq 1, x_2 \geq 1\}$, $f(x) = x_1^2 + x_2^2$, $\hat{T}(x) = x_1 + x_2$, $x = (x_1,x_2) \in D$. Find $(f^\wedge(x))^{3\circledast}_{x_2}$.*

Answer. $\frac{x_1}{(x_1+x_2)^2(2x_1^2+4x_2^2+2x_1x_2)}$.

Definition 1.5.14. *The second order iso-partial derivative of the third kind we define as follows*

$$\left((\cdot)^{3\circledast}_{x_i}\right)^{3\circledast}_{x_l}, \qquad i,l = 1,2,\ldots,n.$$

The third order iso-partial derivative of the third kind we define as follows

$$\left(\left((\cdot)^{3\circledast}_{x_i}\right)^{3\circledast}_{x_l}\right)^{3\circledast}_{x_k}, \qquad i,l,k = 1,2,\ldots,n,$$

and so on.

Definition 1.5.15. *The first order iso-partial derivative of the fourth kind with respect to x_i will be defined as follows*

$$(\cdot)^{4\circledast}_{x_i} = \frac{1}{\hat{T}(x)} \partial_{x_i}(\cdot).$$

Using the above definition we have

1. The first order iso-partial derivative of the fourth kind with respect to x_i of an iso-function of the first kind

$$(\hat{f}^\wedge(\hat{x}))^{4\circledast}_{x_i} = \frac{\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}^3(x)}.$$

2. The first order iso-partial derivative of the fourth kind with respect to x_i of an iso-function of the second kind

$$(\hat{f}^\wedge(x))^{4\circledast}_{x_i} = \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))\hat{T}(x) + \sum_{j=1, j\neq i}^{n} \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)}{\hat{T}^3(x)}.$$

3. The first order iso-partial derivative of the fourth kind with respect to x_i of an iso-function of the third kind

$$(\hat{f}(\hat{x}))^{4\circledast}_{x_i} = \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)) - \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x) - f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}^4(x)}.$$

4. The first order iso-partial derivative of the fourth kind with respect to x_i of an iso-function of the fourth kind

$$(f^\wedge(x))^{4\circledast}_{x_i} = \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))+\sum_{j=1, j\neq i}^{n}\partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)}{\hat{T}(x)}.$$

5. The first order iso-partial derivative of the fourth kind with respect to x_i of an iso-function of fifth kind

$$(f^\vee(x))^{4\circledast}_{x_i} = \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\hat{T}_{x_i}(x))-\sum_{j=1, j\neq i}^{n}\partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)}{\hat{T}^3(x)}.$$

Example 1.5.16. *Let* $D = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 \geq 1, x_2 \geq 1\}$, $f(x) = 2x_1 - x_2$, $\hat{T}(x) = x_1 + x_2$, $x = (x_1,x_2) \in D$. *Then*

$$\hat{f}^\wedge(\hat{x}) = \frac{f(x)}{\hat{T}(x)} = \frac{2x_1-x_2}{x_1+x_2},$$

$$\hat{f}^\wedge(x) = \frac{f(x\hat{T}(x))}{\hat{T}(x)} = \frac{f(x_1\hat{T}(x),x_2\hat{T}(x))}{\hat{T}(x)} = \frac{2x_1\hat{T}(x)-x_2\hat{T}(x)}{\hat{T}(x)} = 2x_1 - x_2,$$

$$\hat{f}(\hat{x}) = \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)} = \frac{f\left(\frac{x_1}{\hat{T}(x)},\frac{x_2}{\hat{T}(x)}\right)}{\hat{T}(x)} = \frac{\frac{2x_1}{\hat{T}(x)}-\frac{x_2}{\hat{T}(x)}}{\hat{T}(x)} = \frac{2x_1-x_2}{(x_1+x_2)^2},$$

$$f^\wedge(x) = f(x\hat{T}(x)) = f(x_1\hat{T}(x),x_2\hat{T}(x)) = 2x_1\hat{T}(x) - x_2\hat{T}(x) = (2x_1-x_2)(x_1+x_2),$$

from here

$$\partial_{x_1}(\hat{f}^\wedge(\hat{x})) = \partial_{x_1}\left(\frac{2x_1-x_2}{x_1+x_2}\right) = \frac{3x_2}{(x_1+x_2)^2},$$

$$\partial_{x_1}(\hat{f}^\wedge(x)) = \partial_{x_1}(2x_1-x_2) = 2,$$

$$\partial_{x_1}(\hat{f}(\hat{x})) = \partial_{x_1}\left(\frac{2x_1-x_2}{(x_1+x_2)^2}\right) = \frac{-2x_1+4x_2}{(x_1+x_2)^3},$$

$$\partial_{x_1}(f^\wedge(x)) = \partial_{x_1}((2x_1-x_2)(x_1+x_2)) = 4x_1 + x_2.$$

Using the above computations we get

$$(\hat{f}^\wedge(\hat{x}))^{4\circledast}_{x_1} = \frac{1}{\hat{T}(x)}\partial_{x_1}(\hat{f}^\wedge(\hat{x})) = \frac{3x_2}{(x_1+x_2)^3},$$

$$(\hat{f}^\wedge(x))^{4\circledast}_{x_1} = \frac{1}{\hat{T}(x)}\partial_{x_1}(\hat{f}^\wedge(x)) = \frac{2}{x_1+x_2},$$

$$(\hat{f}(\hat{x})^{4\circledast}_{x_1} = \frac{1}{\hat{T}(x)}\partial_{x_1}\hat{f}(\hat{x}) = \frac{-2x_1+4x_2}{(x_1+x_2)^4},$$

$$(f^\wedge(x)^{4\circledast}_{x_1} = \frac{1}{\hat{T}(x)}\partial_{x_1}(f^\wedge(x)) = \frac{4x_1+x_2}{x_1+x_2}.$$

Exercise 1.5.17. *Let* $D = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 \geq 1, x_2 \geq 1\}$, $f(x) = x_1 - x_2$, $\hat{T}(x) = x_1 + x_2$, $x = (x_1,x_2) \in D$. *Find* $(f^\wedge(x))^{4\circledast}_{x_1}$.

Answer. $\frac{2x_1}{x_1+x_2}$.

Definition 1.5.18. *The second order iso-partial derivative of the fourth kind we define as follows*

$$\left((\cdot)^{4\circledast}_{x_i}\right)^{4\circledast}_{x_l}, \qquad i,l=1,2,\ldots,n.$$

The third order iso-partial derivative of the fourth kind we define as follows

$$\left(\left((\cdot)^{4\circledast}_{x_i}\right)^{4\circledast}_{x_l}\right)^{4\circledast}_{x_k}, \qquad i,l,k=1,2,\ldots,n$$

and so on.

Definition 1.5.19. *The first order iso-partial derivative of the fifth kind with respect to x_i will be defined as follows*

$$(\cdot)^{5\circledast}_{x_i}=\frac{\hat{\partial}_{x_i}(\cdot)}{\hat{\partial}_{x_i}(\hat{x}_i)}.$$

From the definition it follows

$$(\cdot)^{5\circledast}_{x_i}=\frac{\hat{\partial}_{x_i}(\cdot)}{\hat{\partial}_{x_i}(\hat{x}_i)}=\frac{\hat{T}(x)\partial_{x_i}(\cdot)}{\hat{T}(x)\partial_{x_i}(\hat{x}_i)}=\frac{\partial_{x_i}(\cdot)}{\partial_{x_i}(\hat{x}_i)}.$$

Using the above definition we have

1. The first order iso-partial derivative of the first kind with respect to x_i of an iso-function of the first kind

$$(\hat{f}^\wedge(\hat{x}))^{5\circledast}_{x_i}=\frac{\partial_{x_i}f(x)\hat{T}(x)-f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}.$$

2. The first order iso-partial derivative of the fifth kind with respect to x_i of an iso-function of the second kind

$$(\hat{f}^\wedge(x))^{5\circledast}_{x_i}=\frac{\partial_{x_i}f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))\hat{T}(x)+\sum_{j=1,j\neq i}^{n}\partial_{x_j}f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\hat{T}(x)-f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}.$$

3. The first order iso-partial derivative of the fifth kind with respect to x_i of an iso-function of the third kind

$$(\hat{f}(\hat{x}))^{5\circledast}_{x_i}=\frac{1}{\hat{T}(x)}\frac{\partial_{x_i}f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x))-\sum_{j=1,j\neq i}^{n}\partial_{x_j}f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)-f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}.$$

4. The first order iso-partial derivative of the fifth kind with respect to x_i of an iso-function of the fourth kind

$$(f^\wedge(x))^{5\circledast}_{x_i}=\hat{T}^2(x)\frac{\partial_{x_i}f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))+\sum_{j=1,j\neq i}^{n}\partial_{x_j}f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}.$$

5. The first order iso-partial derivative of the fifth kind with respect to x_i of an iso-function of the fifth kind

$$(f^\vee)^{5\circledast}_{x_i}=\frac{\partial_{x_i}f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\hat{T}_{x_i}(x))-\sum_{j=1,j\neq i}^{n}\partial_{x_j}f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)}{\hat{T}(x)-x_i\hat{T}_{x_i}(x)}.$$

Example 1.5.20. *Let $D = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 \geq 1, x_2 \geq 1\}$, $f(x) = x_1 - x_2$, $\hat{T}(x) = 3x_1 + x_2$, $x = (x_1,x_2) \in D$. Then we have*

$$\hat{f}^{\wedge}(\hat{x}) = \frac{f(x)}{\hat{T}(x)} = \frac{x_1-x_2}{3x_1+x_2},$$

$$\hat{f}^{\wedge}(x) = \frac{f(x\hat{T}(x))}{\hat{T}(x)} = \frac{f(x_1\hat{T}(x),x_2\hat{T}(x))}{\hat{T}(x)} = \frac{x_1\hat{T}(x)-x_2\hat{T}(x)}{\hat{T}(x)} = x_1 - x_2,$$

$$\hat{f}(\hat{x}) = \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)} = \frac{f\left(\frac{x_1}{\hat{T}(x)},\frac{x_2}{\hat{T}(x)}\right)}{\hat{T}(x)} = \frac{\frac{x_1}{\hat{T}(x)} - \frac{x_2}{\hat{T}(x)}}{\hat{T}(x)} = \frac{x_1-x_2}{(3x_1+x_2)^2},$$

$$f^{\wedge}(x) = f(x\hat{T}(x)) = f(x_1\hat{T}(x), x_2\hat{T}(x)) = x_1\hat{T}(x) - x_2\hat{T}(x) = (x_1-x_2)(3x_1+x_2),$$

$$\hat{x}_1 = \frac{x_1}{3x_1+x_2},$$

and from here

$$\partial_{x_1}(\hat{f}^{\wedge}(\hat{x})) = \partial_{x_1}\left(\frac{x_1-x_2}{3x_1+x_2}\right) = \frac{4x_2}{(3x_1+x_2)^2},$$

$$\partial_{x_1}(\hat{f}^{\wedge}(x)) = \partial_{x_1}(x_1-x_2) = 1,$$

$$\partial_{x_1}(\hat{f}(\hat{x})) = \partial_{x_1}\left(\frac{x_1-x_2}{(3x_1+x_2)^2}\right) = \frac{-3x_1+7x_2}{(3x_1+x_2)^3},$$

$$\partial_{x_1}(f^{\wedge}(x)) = \partial_{x_1}((x_1-x_2)(3x_1+x_2)) = 6x_1 - 2x_2,$$

$$\partial_{x_1}(\hat{x}_1) = \partial_{x_1}\left(\frac{x_1}{3x_1+x_2}\right)dx_1 = \frac{x_2}{(3x_1+x_2)^2}dx_1.$$

Using the above computations we get

$$(\hat{f}^{\wedge}(\hat{x}))^{5\circledast}_{x_1} = \frac{\partial_{x_1}\hat{f}^{\wedge}(\hat{x})}{\partial_{x_1}(\hat{x}_1)} = 4,$$

$$(\hat{f}^{\wedge}(x))^{5\circledast}_{x_1} = \frac{\partial_{x_1}\hat{f}^{\wedge}(x)}{\partial_{x_1}(\hat{x}_1)} = \frac{(3x_1+x_2)^2}{x_2},$$

$$(\hat{f}(\hat{x}))^{5\circledast}_{x_1} = \frac{\partial_{x_1}\hat{f}(\hat{x})}{\partial_{x_1}(\hat{x}_1)} = \frac{-3x_1+7x_2}{x_2(3x_1+x_2)},$$

$$(f^{\wedge}(x))^{5\circledast}_{x_1} = \frac{\partial_{x_1}f^{\wedge}(x)}{\partial_{x_1}(\hat{x}_1)} = \frac{2(9x_1^2-x_2^2)(3x_1+x_2)}{x_2}.$$

Exercise 1.5.21. *Let $D = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 \geq 1, x_2 \geq 1\}$, $f(x) = x_1 - 5x_2$, $\hat{T}(x) = x_1 + x_2$, $x = (x_1,x_2) \in D$. Find $(\hat{f}^{\wedge}(\hat{x}))^{5\circledast}_{x_1}$.*

Answer. 6.

Definition 1.5.22. *The second order iso-partial derivative of the fifth kind we define as follows*

$$\left((\cdot)^{5\circledast}_{x_5}\right)^{5\circledast}_{x_l}, \qquad i,l = 1,2,\ldots,n.$$

The third order iso-partial derivative of the fifth kind we define as follows

$$\left(\left((\cdot)_{x_i}^{5\circledast}\right)_{x_l}^{5\circledast}\right)_{x_k}^{5\circledast}, \qquad i,l,k=1,2,\ldots,n$$

and so on.

Definition 1.5.23. *The first order iso-partial derivative of the sixth kind with respect to x_i will be defined as follows*

$$(\cdot)_{x_i}^{6\circledast}=\frac{\hat{\partial}_{x_i}(\cdot)}{dx_i}.$$

We can rewrite it in the form

$$(\cdot)_{x_i}^{6\circledast}=\hat{T}(x)\partial_{x_i}(\cdot).$$

Using the above definition we have

1. The first order iso-partial derivative of the sixth kind with respect to x_i of an iso-function of the first kind

$$(\hat{f}^{\wedge}(\hat{x}))_{x_i}^{6\circledast}=\frac{\partial_{x_i}f(x)\hat{T}(x)-f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x)}.$$

2. The first order iso-partial derivative of the sixth kind with respect to x_i of an iso-function of the second kind

$$(\hat{f}^{\wedge}(x))_{x_i}^{6\circledast}=\frac{\partial_{x_i}f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))\hat{T}(x)+\sum_{j=1,j\neq i}^{n}\partial_{x_j}f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\hat{T}(x)-f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)}{\hat{T}(x)}.$$

3. The first order iso-partial derivative of the sixth kind with respect to x_i of an iso-function of the third kind

$$(\hat{f}(\hat{x}))_{x_i}^{6\circledast}=\frac{\partial_{x_i}f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x))-\sum_{j=1,j\neq i}^{n}\partial_{x_j}f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)-f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}^2(x)}.$$

4. The first order iso-partial derivative of the sixth kind with respect to x_i of an iso-function of the fourth kind

$$(f^{\wedge}(x))_{x_i}^{6\circledast}=\hat{T}(x)\partial_{x_i}f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))$$

$$+\hat{T}(x)\sum_{j=1,j\neq i}^{n}\partial_{x_j}f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x).$$

5. The first order iso-partial derivative of the sixth kind with respect to x_i of an iso-function of the fifth kind

$$(f^{\vee}(x))_{x_i}^{6\circledast}=\frac{\partial_{x_i}f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\hat{T}_{x_i}(x))-\sum_{j=1,j\neq i}^{n}\partial_{x_j}f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)}{\hat{T}(x)}.$$

Example 1.5.24. *Let $D = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}$, $f(x) = x_1^2 + 2x_2$, $\hat{T}(x) = x_1 + x_2 + 1$, $x = (x_1, x_2) \in D$. Then*

$$\hat{f}^\wedge(\hat{x}) = \frac{f(x)}{\hat{T}(x)} = \frac{x_1^2+2x_2}{x_1+x_2+1},$$

$$\hat{f}^\wedge(x) = \frac{f(x\hat{T}(x))}{\hat{T}(x)} = \frac{f(x_1\hat{T}(x), x_2\hat{T}(x))}{\hat{T}(x)} = \frac{x_1^2\hat{T}^2(x)+2x_2\hat{T}(x)}{\hat{T}(x)} = x_1^3 + x_1^2x_2 + x_1^2 + 2x_2,$$

$$\hat{f}(\hat{x}) = \frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)} = \frac{f\left(\frac{x_1}{\hat{T}(x)}, \frac{x_2}{\hat{T}(x)}\right)}{\hat{T}(x)} = \frac{\frac{x_1^2}{\hat{T}^2(x)}+2\frac{x_2}{\hat{T}(x)}}{\hat{T}(x)} = \frac{x_1^2+2x_1x_2+2x_2^2+2x_2}{(x_1+x_2+1)^3},$$

$$f^\wedge(x) = f(x\hat{T}(x)) = f(x_1\hat{T}(x), x_2\hat{T}(x)) = x_1^2\hat{T}(x) + 2x_2\hat{T}(x)$$

$$= (x_1^3 + x_1^2x_2 + x_1^2 + 2x_2)(x_1 + x_2 + 1),$$

whereupon

$$(\hat{f}^\wedge(\hat{x}))^{6\circledast}_{x_1} = \hat{T}(x)\partial_{x_1}(\hat{f}^\wedge(\hat{x})) = (x_1 + x_2 + 1)\partial_{x_1}\left(\frac{x_1^2+2x_2}{x_1+x_2+1}\right) = \frac{x_1^2+2x_1x_2+2x_1-2x_2}{x_1+x_2+1}$$

$$(\hat{f}^\wedge(x))^{6\circledast}_{x_1} = \hat{T}(x)\partial_{x_1}(\hat{T}^\wedge(x)) = (x_1 + x_2 + 1)\partial_{x_1}(x_1^3 + x_1^2x_2 + x_1^2 + 2x_2)$$

$$= (x_1 + x_2 + 1)(3x_1^2 + 2x_1x_2 + 2x_1)$$

$$(\hat{f}(\hat{x}))^{6\circledast}_{x_1} = \hat{T}(x)\partial_{x_1}(\hat{f}(\hat{x})) = (x_1 + x_2 + 1)\partial_{x_1}\left(\frac{x_1^2+2x_1x_2+2x_2^2+2x_2}{(x_1+x_2+1)^3}\right)$$

$$= \frac{-x_1^2-4x_2^2-2x_1x_2+2x_1-4x_2}{(x_1+x_2+1)^3},$$

$$(f^\wedge(x))^{6\circledast}_{x_1} = \hat{T}(x)\partial_{x_1}(f^\wedge(x)) = (x_1 + x_2 + 1)\partial_{x_1}((x_1^3 + x_1^2x_2 + x_1^2 + 2x_2)(x_1 + x_2 + 1))$$

$$= (x_1 + x_2 + 1)(4x_1^3 + 6x_1^2x_2 + 6x_1^2 + 2x_1x_2^2 + 4x_1x_2 + 2x_1).$$

Exercise 1.5.25. *Let $D = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0$, $f(x) = x_1^3 + 2x_1x_2$, $\hat{T}(x) = x_1^2 + x_2 + 1$, $x = (x_1, x_2) \in D$. Find $(\hat{f}^\wedge(\hat{x}))^{6\circledast}_{x_2}$.*

Answer. $\frac{x_1^3+2x_1}{x_1^2+x_2+1}$.

Definition 1.5.26. *The second order iso-partial derivative of the sixth kind we define as follows*

$$\left((\cdot)^{6\circledast}_{x_i}\right)^{6\circledast}_{x_l}, \qquad i, l = 1, 2, \ldots, n.$$

The third order iso-partial derivative of the sixth kind we define as follows

$$\left(\left((\cdot)^{6\circledast}_{x_i}\right)^{6\circledast}_{x_l}\right)^{6\circledast}_{x_k}, \qquad i, l, k = 1, 2, \ldots, n,$$

and so on.

Definition 1.5.27. *The first order iso-partial derivative of the seventh kind with respect to x_i will be defined as follows*

$$(\cdot)^{7\circledast}_{x_i} = \frac{\hat{\partial}_{x_i}(\cdot)}{\partial_{x_i}(\hat{x}_i)}.$$

Using the above definition we have

1. The first order iso-partial derivative of the seventh kind with respect to x_i of an iso-function of the first kind

$$(\hat{f}^{\wedge}(\hat{x}))^{7\circledast}_{x_i} = \hat{T}(x)\frac{\partial_{x_i}f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)}.$$

2. The first order iso-partial derivative of the seventh kind with respect to x_i of an iso-function of the second kind

$$(\hat{f}^{\wedge}(x))^{7\circledast}_{x_i} = \hat{T}(x)\frac{\partial_{x_i}f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))\hat{T}(x) + \sum_{j=1,j\neq i}^{n}\partial_{x_j}f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)}.$$

3. The first order iso-partial derivative of the seventh kind with respect to x_i of an iso-function of the third kind

$$(\hat{f}(\hat{x}))^{7\circledast}_{x_i} = \hat{T}^2(x)\frac{\partial_{x_i}f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)) - \sum_{j=1,j\neq i}^{n}\partial_{x_j}f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x) - f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)}.$$

4. The first order iso-partial derivative of the seventh kind with respect to x_i of an iso-function of the fourth kind

$$(f^{\wedge}(x))^{7\circledast}_{x_i} = \hat{T}^3(x)\frac{\partial_{x_i}f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x)) + \sum_{j=1,j\neq i}^{n}\partial_{x_j}f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)}.$$

5. The first order iso-partial derivative of the seventh kind with respect to x_i of an iso-function of the fifth kind

$$(f^{\vee}(x))^{7\circledast}_{x_i} = \hat{T}(x)\frac{\partial_{x_i}f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i\hat{T}_{x_i}(x)) - \sum_{j=1,j\neq i}^{n}\partial_{x_j}f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)}{\hat{T}(x) - x_i\hat{T}_{x_i}(x)}.$$

Example 1.5.28. *Let* $D = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$, $f(x) = x_1 - x_2^2$, $\hat{T}(x) = 1 + x_1 + x_2$,

$x=(x_1,x_2)\in D$. Then

$$\hat{f}^{\wedge}(\hat{x})=\frac{f(x)}{\hat{T}(x)}=\frac{x_1-x_2^2}{1+x_1+x_2},$$

$$\hat{f}^{\wedge}(x)=\frac{f(x\hat{T}(x))}{\hat{T}(x)}=\frac{f(x_1\hat{T}(x),x_2\hat{T}(x))}{\hat{T}(x)}=\frac{x_1\hat{T}(x)-x_2^2\hat{T}^2(x)}{\hat{T}(x)}=x_1-x_1x_2^2-x_2^3-x_2^2,$$

$$\hat{f}(\hat{x})=\frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)}=\frac{f\left(\frac{x_1}{\hat{T}(x)},\frac{x_2}{\hat{T}(x)}\right)}{\hat{T}(x)}=\frac{\frac{x_1}{\hat{T}(x)}-\frac{x_2^2}{\hat{T}^2(x)}}{\hat{T}(x)}=\frac{x_1^2-x_2^2+x_1x_2+1}{(x_1+x_2+1)^3},$$

$$f^{\wedge}(x)=f(x\hat{T}(x))=f(x_1\hat{T}(x),x_2\hat{T}(x))=x_1\hat{T}(x)-x_2^2\hat{T}^2(x)$$

$$=(x_1+x_2+1)(x_1-x_1x_2^2-x_2^2-x_2^3),$$

$$\hat{x}_1=\frac{x_1}{\hat{T}(x)}=\frac{x_1}{1+x_1+x_2},$$

whereupon

$$\partial_{x_1}(\hat{f}^{\wedge}(\hat{x}))=\partial_{x_1}\left(\frac{x_1-x_2^2}{1+x_1+x_2}\right)=\frac{1+x_2+x_2^2}{1+x_1+x_2},$$

$$\partial_{x_1}(\hat{f}^{\wedge}(x))=\partial_{x_1}(x_1-x_1x_2^2-x_2^2-x_2^3)=1-x_2^2,$$

$$\partial_{x_1}(\hat{f}(\hat{x}))=\partial_{x_1}\left(\frac{x_1^2-x_2^2+x_1x_2+1}{(x_1+x_2+1)^3}\right)=\frac{-x_1^2+4x_2^2+2x_1+x_2-3}{(1+x_1+x_2)^4},$$

$$\partial_{x_1}(f^{\wedge}(x))=\partial_{x_1}((1+x_1+x_2)(x_1-x_1x_2^2-x_2^2-x_2^3))$$

$$=-2x_2^3-2x_1x_2^2-2x_2^2+2x_1+x_2+1,$$

$$\partial_{x_1}(\hat{x}_1)=\partial_{x_1}(\hat{x}_1)dx_1=\frac{1+x_2}{(1+x_1+x_2)^2}dx_1.$$

Using the above computations we get

$$(\hat{f}^{\wedge}(\hat{x})^{7\circledast}_{x_1}=\frac{(1+x_1+x_2)^2(1+x_2+x_2^2)}{1+x_2},$$

$$(\hat{f}^{\wedge}(x))^{7\circledast}_{x_1}=(1-x_2)(1+x_1+x_2)^3,$$

$$(\hat{f}(\hat{x}))^{7\circledast}_{x_1}=\frac{-x_1^2+4x_2^2+2x_1+x_2-3}{(1+x_2)(1+x_1+x_2)},$$

$$(f^{\wedge}(x))^{7\circledast}_{x_1}=\frac{(1+x_1+x_2)^3(-2x_2^3-2x_1x_2^2-2x_2^2+2x_1+x_2+1)}{1+x_2}.$$

Exercise 1.5.29. *Let $D=\{(x_1,x_2):x_1\geq 0,x_2\geq 0\}$, $f(x)=2x_1+x_2^2$, $\hat{T}(x)=1+x_1+x_2$, $x=(x_1,x_2)\in D$. Find $(\hat{f}^{\wedge}(\hat{x}))^{7\circledast}_{x_1}$.*

Answer. $\frac{(2+2x_2-x_2^2)(1+x_1+x_2)}{1+x_2}$.

Definition 1.5.30. *The second order iso-partial derivative of the seventh kind we define as follows*

$$\left((\cdot)^{7\circledast}_{x_i}\right)^{7\circledast}_{x_l}, \qquad i,l=1,2,\ldots,n.$$

The third order iso-partial derivative of the seventh kind we define as follows

$$\left(\left((\cdot)^{7\circledast}_{x_i}\right)^{7\circledast}_{x_l}\right)^{7\circledast}_{x_k}, \qquad i,l,k=1,2,\ldots,n,$$

and so on.

Remark 1.5.31. *In the general case there is no equality between the mixed iso-partial derivatives. We will consider the following example.*

Example 1.5.32. *Let $D=\{(x_1,x_2\in\mathbb{R}^2 : x_1\geq\frac{1}{3}, x_2\geq\frac{1}{3}, x_1+x_2\geq 1\}$, $f(x)=x_1$, $\hat{T}(x)=x_1+x_2$. Then*

$$(\hat{f}^{\wedge}(\hat{x}))^{1\circledast}_{x_1}=\frac{1}{x_1+x_2}\frac{x_1+x_2-x_1}{x_1+x_2-x_1}=\frac{1}{x_1+x_2}=\frac{1}{x_1+x_2},$$

$$(\hat{f}^{\wedge}(\hat{x}))^{1\circledast}_{x_2}=\frac{1}{x_1+x_2}\frac{-x_1}{x_1+x_2-x_1}=-\frac{1}{x_1+x_2},$$

$$(\hat{f}^{\wedge}(\hat{x}))^{1\circledast}_{x_1})^{1\circledast}_{x_2}=\frac{1}{x_1+x_2}\frac{(x_1+x_2)\partial_{x_2}\frac{1}{x_1+x_2}}{\frac{x_1+x_2-x_2}{x_1+x_2}}=-\frac{1}{x_1(x_1+x_2)},$$

$$((\hat{f}^{\wedge}(\hat{x}))^{1\circledast}_{x_2})^{1\circledast}_{x_1}=\frac{1}{x_1+x_2}\frac{(x_1+x_2)\frac{1}{(x_1+x_2)^2}}{(x_1+x_2)\frac{x_1+x_2-x_2}{x_1+x_2}}=\frac{1}{x_2(x_1+x_2)}.$$

Consequently

$$(\hat{f}^{\wedge}(\hat{x}))^{1\circledast}_{x_1})^{1\circledast}_{x_2}\neq((\hat{f}^{\wedge}(\hat{x}))^{1\circledast}_{x_2})^{1\circledast}_{x_1}.$$

We note that the function $\hat{f}(\hat{x})$ is a continuously-differentiable function on D.

Definition 1.5.33. *An iso-function of the first, the second, the third, the fourth or the fifth kind will be called iso-differentiable of the first, third, fifth or seventh kind at the point $x^0\in D$ if its iso-original is differentiable at the same point and*

$$\hat{T}(x^0)-x^0_i\partial_{x_i}\hat{T}(x^0)\neq 0, \quad \text{for} \quad \forall i=1,2,\ldots,n.$$

Definition 1.5.34. *An iso-function of the first, the second, the third, the fourth or the fifth kind will be called iso-differentiable of the second, fourth or sixth kind at the point $x^0\in D$ if its iso-original is differentiable at the same point.*

Definition 1.5.35. *An iso-function of the first, the second, the third, the fourth or the fifth kind will be called iso-differentiable of the first, the second, the third, the fourth, the fifth, the sixth or the seventh kind on D if it is - iso-differentiable of the first, the second, the third, the fourth, the fifth, the sixth or the seventh kind at every point of D.*

Exercise 1.5.36. *Let $\hat{f},\hat{g}:D\longrightarrow\mathbb{R}$ be iso-functions of the first, the second, the third, the fourth or the fifth kind, which are iso-differentiable of the first, the second, the third, the fourth, the fifth, the sixth or the seventh kind at $x\in D$. Let also, $a\in\mathbb{R}$, $\hat{a}\in\hat{F}_{\mathbb{R}}$. Prove*

1. $\left(\hat{f}(x) \pm \hat{g}(x)\right)_{x_i}^{j\circledast} = (\hat{f}(x))_{x_i}^{j\circledast} \pm (\hat{g}(x))_{x_i}^{j\circledast}$.

2. $(\hat{a}\hat{\times}\hat{f}(x))_{x_i}^{j\circledast} = \hat{a}\hat{\times}(\hat{f}(x))_{x_i}^{j\circledast}$.

3. $(\hat{a}\hat{f}(x))_{x_i}^{j\circledast} = \hat{a}(\hat{f}(x))_{x_i}^{j\circledast}$.

4. $(a\hat{\times}\hat{f}(x))_{x_i}^{j\circledast} = a\hat{\times}(\hat{f}(x))_{x_i}^{j\circledast}$.

5. $(a\hat{f}(x))_{x_i}^{j\circledast} = a(\hat{f}(x))_{x_i}^{j\circledast}$.

6. $(\hat{f}(x)\hat{\times}\hat{g}(x))_{x_i}^{j\circledast} = (\hat{f}(x))_{x_i}^{j\circledast}\hat{\times}\hat{g}(x) + \hat{f}(x)\hat{\times}(\hat{g}(x))_{x_i}^{j\circledast}$.

7. $(\hat{f}(x)\hat{g}(x))_{x_i}^{j\circledast} = (\hat{f}(x))_{x_i}^{j\circledast}\hat{g}(x) + \hat{f}(x)(\hat{g}(x))_{x_i}^{j\circledast}$.

8. $\left(\hat{f}(x) \nearrow \hat{g}(x)\right)_{x_i}^{j\circledast} = \left((\hat{f}(x))_{x_i}^{j\circledast}\hat{g}(x) - \hat{f}(x)(\hat{g}(x))_{x_i}^{j\circledast}\right) \nearrow \hat{g}^2(x)$.

9. $\left(\frac{\hat{f}(x)}{\hat{g}(x)}\right)_{x_i}^{j\circledast} = \frac{(\hat{f}(x))_{x_i}^{j\circledast}\hat{g}(x)-\hat{f}(x)(\hat{g}(x))_{x_i}^{j\circledast}}{\hat{g}^2(x)}$, *j=1, ..., 7, i=1,..., n.*

Exercise 1.5.37. *Let $\hat{f} : D \longrightarrow \mathbb{R}$ be an iso-differentiable of the first, the second, the third, the fourth, the fifth, the sixth or the seventh kind at $x \in D$ iso-function of the first, the second, the third, the fourth or the fifth kind. Prove that it is continuous at x.*

Definition 1.5.38. *Let $\hat{f}$ be an iso-function of the first, the second, the third, the fourth or the fifth kind, which is iso-differentiable of the first, the second, the third, the fourth, the fifth, the sixth or the seventh kind at the point $x^0 \in D$. With $\hat{f}_{x_i}^{\circledast}(x^0)$ will be denoted the iso-partial derivative of the first, the second, the third, the fourth, the fifth, the sixth or the seventh kind of $\hat{f}$ at the point x^0.*

1. *The total iso-differential of the first kind for an iso-differentiable of the i-th kind iso-function of the j-th kind, $i = 1,2,3,4,5,6,7$, $j = 1,2,3,4,5$, is*

$$d_1\hat{f}(x^0) = \sum_{i=1}^{n} \hat{f}_{x_i}^{i\circledast}(x^0)\hat{\times}\hat{d}\hat{x}_i,$$

2. *The total iso-differential of the second kind for an iso-differentiable of the i-th kind iso-function of the j-th kind, $i = 1,2,3,4,5,6,7$, $j = 1,2,3,4,5$, is*

$$d_2\hat{f}(x^0) = \textstyle\sum_{i=1}^{n} \hat{f}_{x_i}^{i\circledast}(x^0)\hat{\times}d\hat{x}_i,$$

3. *The total iso-differential of the third kind for an iso-differentiable of the i-th kind iso-function of the j-th, $i = 1,2,3,4,5,6,7$, $j = 1,2,3,4,5$, is*

$$d_3\hat{f}(x^0) = \sum_{i=1}^{n} \hat{f}_{x_i}^{i\circledast}(x^0)d\hat{x}_i,$$

4. *The total iso-differential of the fourth kind for an iso-differentiable of the i-th kind iso-function of the j-th kind, $i = 1,2,3,4,5,6,7$, $j = 1,2,3,4,5$, is*

$$d_4\hat{f}(x^0) = \sum_{i=1}^{n} \hat{f}_{x_i}^{i\circledast}(x^0)\hat{d}\hat{x}_i,$$

5. *The total iso-differential of the fifth kind for an iso-differentiable of the i-th kind iso-function of the j-th ,* $i = 1,2,3,4,5,6,7,\ j = 1,2,3,4,5,$ *is*

$$d_5\hat{f}(x^0) = \sum_{i=1}^{n} \hat{f}_{x_i}^{i\circledast}(x^0)\hat{\times}\hat{d}x_i,$$

6. *The total iso-differential of the sixth kind for an iso-differentiable of the i-th kind iso-function of the j-th kind,* $i = 1,2,3,4,5,6,7,\ j = 1,2,3,4,5,$ *is*

$$d_6\hat{f}(x^0) = \sum_{i=1}^{n} \hat{f}_{x_i}^{i\circledast}(x^0)\hat{\times}dx_i,$$

7. *The total iso-differential of the seventh kind for an iso-differentiable of the i-th kind iso-function of the j-th kind,* $i = 1,2,3,4,5,6,7,\ j = 1,2,3,4,5,$ *is*

$$d_7\hat{f}(x^0) = \sum_{i=1}^{n} \hat{f}_{x_i}^{i\circledast}(x^0)dx_i,$$

8. *The total iso-differential of the eighth kind for an iso-differentiable of the i-th kind iso-function of the j-th kind,* $i = 1,2,3,4,5,6,7,\ j = 1,2,3,4,5,$ *is*

$$d_8\hat{f}(x^0) = \sum_{i=1}^{n} \hat{f}_{x_i}^{i\circledast}(x^0)\hat{d}x_i.$$

Now we will give the explicit expressions of the iso-differentials of the iso-functions.

1. The total iso-differential of the first kind of the iso-differentiable of the first kind iso-functions of the first kind is

$$d_1(\hat{f}^\wedge(\hat{x})) = \hat{T}_1\tfrac{1}{\hat{T}^2(x)}\textstyle\sum_{i=1}^{n}\Big(\partial_{x_i}f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)\Big)dx_i.$$

2. The total iso-differential of the first kind of the iso-differentiable of the first kind iso-functions of the second kind is

$$d_1(\hat{f}^\wedge(x)) = \hat{T}_1\tfrac{1}{\hat{T}^2(x)}\textstyle\sum_{i=1}^{n}\Big(\partial_{x_i}f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))\hat{T}(x)$$

$$+\textstyle\sum_{j=1,j\neq i}^{n}\partial_{x_j}f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)\Big)dx_i.$$

3. The total iso-differential of the first kind of the iso-differentiable of the first kind iso-functions of the third kind is

$$d_1(\hat{f}(\hat{x})) = \hat{T}_1\tfrac{1}{\hat{T}^3(x)}\textstyle\sum_{i=1}^{n}\Big(\partial_{x_i}f\Big(\tfrac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x))$$

$$-\textstyle\sum_{j=1,j\neq i}^{n}\partial_{x_j}f\Big(\tfrac{x}{\hat{T}(x)}\Big)x_j\hat{T}_{x_i}(x) - f\Big(\tfrac{x}{\hat{T}(x)}\Big)\hat{T}(x)\partial_{x_i}\hat{T}(x)\Big)dx_i.$$

4. The total iso-differential of the first kind of the iso-differentiable of the first kind iso-functions of the fourth kind is

$$d_1(f^{\wedge}(x)) = \hat{T}_1 \sum_{i=1}^{n} \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i} \hat{T}(x))$$

$$+ \sum_{j=1, j\neq i}^{n} \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i} \hat{T}(x)\Big) dx_i.$$

5. The total iso-differential of the first kind of the iso-differentiable of the first kind iso-functions of the fifth kind is

$$d_1(f^{\vee}(x)) = \hat{T}_1 \frac{1}{\hat{T}(x)} \sum_{i=1}^{n} \left(\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \hat{T}_{x_i}(x))\right.$$

$$\left. - \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x)\right) dx_i$$

6. The total iso-differential of the first kind of the iso-differentiable of the second kind iso-functions of the first kind is

$$d_1(\hat{f}^{\wedge}(\hat{x})) = \hat{T}_1 \sum_{i=1}^{n} \frac{(\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x))(\hat{T}(x) - x_i \partial_{x_i}\hat{T}(x))}{\hat{T}^3(x)} dx_i.$$

7. The total iso-differential of the first kind of the iso-differentiable of the second kind iso-functions of the second kind is

$$d_1(\hat{f}^{\wedge}(x))$$

$$= \hat{T}_1 \frac{1}{\hat{T}^3(x)} \sum_{i=1}^{n} \Big((\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i}\hat{T}(x))\hat{T}(x)$$

$$+ \sum_{j=1, j\neq i}^{n} \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x))(\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x))\Big) dx_i.$$

8. The total iso-differential of the first kind of the iso-differentiable of the second kind iso-functions of the third kind is

$$d_1(\hat{f}(\hat{x})) = \hat{T}_1 \frac{1}{\hat{T}^4(x)} \sum_{i=1}^{n} \left(\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \partial_{x_i}\hat{T}(x))\right.$$

$$\left. - \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x) - f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)\right)(\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)) dx_i.$$

9. The total iso-differential of the first kind of the iso-differentiable of the second kind iso-functions of the fourth kind is

$$d_1(f^{\wedge}(x)) = \hat{T}_1 \frac{1}{\hat{T}(x)} \sum_{i=1}^{n} \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x))$$

$$+ \sum_{j=1, j\neq i}^{n} \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\Big)(\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)) dx_i.$$

10. The total iso-differential of the first kind of the iso-differentiable of the second kind iso-functions of the fifth kind is

$$d_1(f^{\vee}(x)) = \hat{T}_1 \frac{1}{\hat{T}^3(x)} \sum_{i=1}^{n} \left(\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \hat{T}_{x_i}(x)) \right.$$

$$\left. - \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x) \right)(\hat{T}(x) - x_i \hat{T}_{x_i}(x)) dx_i.$$

11. The total iso-differential of the first kind of the iso-differentiable of the third kind iso-functions of the first kind is

$$d_1(\hat{f}^{\wedge}(\hat{x})) = \hat{T}_1 \frac{1}{\hat{T}^3(x)} \sum_{i=1}^{n} \left(\partial_{x_i} f(x) \hat{T}(x) - f(x) \partial_{x_i} \hat{T}(x) \right) dx_i.$$

12. The total iso-differential of the first kind of the iso-differentiable of the third kind iso-functions of the second kind is

$$d_1(\hat{f}^{\wedge}(x)) = \hat{T}_1 \frac{1}{\hat{T}^4(x)} \sum_{i=1}^{n} \left(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i} \hat{T}(x)) \hat{T}(x) \right.$$

$$\left. + \sum_{j=1, j\neq i}^{n} \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i} \hat{T}(x) \hat{T}(x) - f(x\hat{T}(x)) \partial_{x_i} \hat{T}(x) \right) dx_i.$$

13. The total iso-differential of the first kind of the iso-differentiable of the third kind iso-functions of the third kind is

$$d_1(\hat{f}(\hat{x})) = \hat{T}_1 \frac{1}{\hat{T}^4(x)} \sum_{i=1}^{n} \left(\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \partial_{x_i} \hat{T}(x)) \right.$$

$$\left. - \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x) - f\left(\frac{x}{\hat{T}(x)}\right) \hat{T}(x) \partial_{x_i} \hat{T}(x) \right) dx_i.$$

14. The total iso-differential of the first kind of the iso-differentiable of the third kind iso-functions of the fourth kind is

$$d_1(f^{\wedge}(x)) = \hat{T}_1 \sum_{i=1}^{n} \left(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i} \hat{T}(x)) \right.$$

$$\left. + \sum_{j=1, j\neq i}^{n} \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i} \hat{T}(x) \right) dx_i.$$

15. The total iso-differential of the first kind of the iso-differentiable of the third kind iso-functions of the fifth kind is

$$d_1(f^{\vee}(x)) = \hat{T}_1 \frac{1}{\hat{T}^3(x)} \sum_{i=1}^{n} \left(\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \hat{T}_{x_i}(x)) \right.$$

$$\left. - \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x) \right) dx_i$$

16. The total iso-differential of the first kind of the iso-differentiable of the fourth kind iso-functions of the first kind is

$$d_1(\hat{f}^{\wedge}(\hat{x})) = \hat{T}_1 \sum_{i=1}^{n} \frac{\left(\partial_{x_i} f(x) \hat{T}(x) - f(x) \partial_{x_i} \hat{T}(x) \right)(\hat{T}(x) - x_1 \partial_{x_i} \hat{T}(x))}{\hat{T}^4(x)} dx_i.$$

17. The total iso-differential of the first kind of the iso-differentiable of the fourth kind iso-functions of the second kind is

$$d_1(\hat{f}^\wedge(x)) = \hat{T}_1 \frac{1}{\hat{T}^4(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i} \hat{T}(x))\hat{T}(x)$$

$$\sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i} \hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)\Big)(\hat{T}(x) - x_1 \partial_{x_i}\hat{T}(x))dx_i.$$

18. The total iso-differential of the first kind of the iso-differentiable of the fourth kind iso-functions of the third kind is

$$d_1(\hat{f}(\hat{x})) = \hat{T}_1 \frac{1}{\hat{T}^5(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i \partial_{x_i}\hat{T}(x))$$

$$-\sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big) x_j \hat{T}_{x_i}(x) - f\Big(\frac{x}{\hat{T}(x)}\Big)\hat{T}(x)\partial_{x_i}\hat{T}(x)\Big)dx_i.$$

19. The total iso-differential of the first kind of the iso-differentiable of the fourth kind iso-functions of the fourth kind is

$$d_1(f^\wedge(x)) = \hat{T}_1 \frac{1}{\hat{T}^2(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i}\hat{T}(x))$$

$$+\sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\Big)(\hat{T}(x) - x_1 \partial_{x_i}\hat{T}(x))dx_i.$$

20. The total iso-differential of the first kind of the iso-differentiable of the fourth kind iso-functions of the fifth kind is

$$d_1(f^\vee(x)) = \hat{T}_1 \frac{1}{\hat{T}^4(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i \hat{T}_{x_i}(x))$$

$$-\sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big) x_j \hat{T}_{x_i}(x)\Big)(\hat{T}(x) - x_i \hat{T}_{x_i}(x))dx_i.$$

21. The total iso-differential of the first kind of the iso-differentiable of the fifth kind iso-functions of the first kind is

$$d_1(\hat{f}^\wedge(\hat{x})) = \hat{T}_1 \frac{1}{\hat{T}(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)\Big)dx_i.$$

22. The total iso-differential of the first kind of the iso-differentiable of the fifth kind iso-functions of the second kind is

$$d_1(\hat{f}^\wedge(x)) = \hat{T}_1 \frac{1}{\hat{T}(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i}\hat{T}(x))\hat{T}(x)$$

$$+\sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)\Big)dx_i.$$

23. The total iso-differential of the first kind of the iso-differentiable of the fifth kind iso-functions of the third kind is

$$d_1(\hat{f}(\hat{x})) = \hat{T}_1 \frac{1}{\hat{T}^2(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i \partial_{x_i}\hat{T}(x))$$

$$-\sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big) x_j \hat{T}_{x_i}(x) - f\Big(\frac{x}{\hat{T}(x)}\Big)\hat{T}(x)\partial_{x_i}\hat{T}(x)\Big)dx_i.$$

24. The total iso-differential of the first kind of the iso-differentiable of the fifth kind iso-functions of the fourth kind is

$$d_1(f^{\wedge}(x)) = \hat{T}_1\hat{T}(x)\sum_{i=1}^{n}\Big(\partial_{x_i}f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))$$

$$+\sum_{j=1,j\neq i}^{n}\partial_{x_j}f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\Big)dx_i.$$

25. The total iso-differential of the first kind of the iso-differentiable of the fifth kind iso-functions of the fifth kind is

$$d_1(f^{\vee}(x)) = \hat{T}_1\frac{1}{\hat{T}(x)}\sum_{i=1}^{n}\Big(\partial_{x_i}f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x)-x_i\hat{T}_{x_i}(x))$$

$$-\sum_{j=1,j\neq i}^{n}\partial_{x_j}f\Big(\frac{x}{\hat{T}(x)}\Big)x_j\hat{T}_{x_i}(x)\Big)dx_i.$$

26. The total iso-differential of the first kind of the iso-differentiable of the sixth kind iso-functions of the first kind is

$$d_1(\hat{f}^{\wedge}(\hat{x})) = \hat{T}_1\sum_{i=1}^{n}\frac{\Big(\partial_{x_i}f(x)\hat{T}(x)-f(x)\partial_{x_i}\hat{T}(x)\Big)(\hat{T}(x)-x_1\partial_{x_i}\hat{T}(x))}{\hat{T}^2(x)}dx_i.$$

27. The total iso-differential of the first kind of the iso-differentiable of the fifth kind iso-functions of the second kind is

$$d_1(\hat{f}^{\wedge}(x)) = \hat{T}_1\frac{1}{\hat{T}^2(x)}\sum_{i=1}^{n}\Big(\partial_{x_i}f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))\hat{T}(x)$$

$$+\sum_{j=1,j\neq i}^{n}\partial_{x_j}f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\hat{T}(x)-f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)\Big)(\hat{T}(x)-x_1\partial_{x_i}\hat{T}(x))dx_i.$$

28. The total iso-differential of the first kind of the iso-differentiable of the fifth kind iso-functions of the third kind is

$$d_1(\hat{f}(\hat{x})) = \hat{T}_1\frac{1}{\hat{T}^3(x)}\sum_{i=1}^{n}\Big(\partial_{x_i}f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x))$$

$$-\sum_{j=1,j\neq i}^{n}\partial_{x_j}f\Big(\frac{x}{\hat{T}(x)}\Big)x_j\hat{T}_{x_i}(x)-f\Big(\frac{x}{\hat{T}(x)}\Big)\hat{T}(x)\partial_{x_i}\hat{T}(x)\Big)(\hat{T}(x)-x_1\partial_{x_i}\hat{T}(x))dx_i.$$

29. The total iso-differential of the first kind of the iso-differentiable of the fifth kind iso-functions of the fourth kind is

$$d_1(f^{\wedge}(x)) = \hat{T}_1\sum_{i=1}^{n}\Big(\partial_{x_i}f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))$$

$$+\sum_{j=1,j\neq i}^{n}\partial_{x_j}f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\Big)(\hat{T}(x)-x_1\partial_{x_i}\hat{T}(x))dx_i.$$

30. The total iso-differential of the first kind of the iso-differentiable of the sixth kind iso-functions of the fourth kind is

$$d_1(f^{\vee}(x)) = \hat{T}_1 \frac{1}{\hat{T}^2(x)} \sum_{i=1}^{n} \left(\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \hat{T}_{x_i}(x)) \right.$$

$$\left. - \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x) \right)(\hat{T}(x) - x_i \hat{T}_{x_i}(x)).$$

31. The total iso-differential of the first kind of the iso-differentiable of the seventh kind iso-functions of the first kind is

$$d_1(\hat{f}^{\wedge}(\hat{x})) = \hat{T}_1 \sum_{i=1}^{n} \frac{\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x)} dx_i.$$

32. The total iso-differential of the first kind of the iso-differentiable of the seventh kind iso-functions of the second kind is

$$d_1(\hat{f}^{\wedge}(x)) = \hat{T}_1 \frac{1}{\hat{T}(x)} \sum_{i=1}^{n} \left(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i}\hat{T}(x))\hat{T}(x) \right.$$

$$\left. + \sum_{j=1, j\neq i}^{n} \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x) \right) dx_i.$$

33. The total iso-differential of the first kind of the iso-differentiable of the seventh kind iso-functions of the third kind is

$$d_1(\hat{f}(\hat{x})) = \hat{T}_1 \hat{T}(x) \sum_{i=1}^{n} \partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \partial_{x_i}\hat{T}(x))$$

$$- \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x) - f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x) dx_i.$$

34. The total iso-differential of the first kind of the iso-differentiable of the seventh kind iso-functions of the fourth kind is

$$d_1(f^{\wedge}(x)) = \hat{T}_1 \hat{T}^2(x) \sum_{i=1}^{n} \left(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i}\hat{T}(x)) \right.$$

$$\left. + \sum_{j=1, j\neq i}^{n} \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x) \right) dx_i.$$

35. The total iso-differential of the first kind of the iso-differentiable of the seventh kind iso-functions of the fifth kind is

$$d_1(f^{\vee}(x)) = \hat{T}_1 \sum_{i=1}^{n} \left(\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \hat{T}_{x_i}(x)) \right.$$

$$\left. - \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x) \right) dx_i.$$

36. The total iso-differential of the second kind of the iso-differentiable of the first kind iso-functions of the first kind is

$$d_2(\hat{f}^{\wedge}(\hat{x})) = \hat{T}_1 \frac{1}{\hat{T}^3(x)} \sum_{i=1}^{n} \left(\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x) \right) dx_i.$$

37. The total iso-differential of the second kind of the iso-differentiable of the first kind iso-functions of the second kind is

$$d_2(\hat{f}^\wedge(x)) = \hat{T}_1 \frac{1}{\hat{T}^3(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i}\hat{T}(x))\hat{T}(x)$$

$$+ \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x) \Big) dx_i.$$

38. The total iso-differential of the second kind of the iso-differentiable of the first kind iso-functions of the third kind is

$$d_2(\hat{f}(\hat{x})) = \hat{T}_1 \frac{1}{\hat{T}^4(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i \partial_{x_i}\hat{T}(x))$$

$$- \sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big) x_j \hat{T}_{x_i}(x) - f\Big(\frac{x}{\hat{T}(x)}\Big)\hat{T}(x)\partial_{x_i}\hat{T}(x) \Big) dx_i.$$

39. The total iso-differential of the second kind of the iso-differentiable of the first kind iso-functions of the fourth kind is

$$d_2(f^\wedge(x)) = \hat{T}_1 \frac{1}{\hat{T}(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i}\hat{T}(x))$$

$$+ \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x) \Big) dx_i.$$

40. The total iso-differential of the second kind of the iso-differentiable of the first kind iso-functions of the fifth kind is

$$d_2(f^\vee(x)) = \hat{T}_1 \frac{1}{\hat{T}^4(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i \hat{T}_{x_i}(x))$$

$$- \sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big) x_j \hat{T}_{x_i}(x) \Big) dx_i.$$

41. The total iso-differential of the second kind of the iso-differentiable of the second kind iso-functions of the first kind is

$$d_2(\hat{f}^\wedge(\hat{x})) = \hat{T}_1 \frac{1}{\hat{T}^4(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x) \Big)(\hat{T}(x) - x_1 \partial_{x_i}\hat{T}(x)) dx_i.$$

42. The total iso-differential of the second kind of the iso-differentiable of the second kind iso-functions of the second kind is

$$d_2(\hat{f}^\wedge(x)) = \hat{T}_1 \frac{1}{\hat{T}^4(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i}\hat{T}(x))\hat{T}(x)$$

$$+ \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x) \Big)(\hat{T}(x) - x_1 \partial_{x_i}\hat{T}(x)) dx_i.$$

43. The total iso-differential of the second kind of the iso-differentiable of the second kind iso-functions of the third kind is

$$d_2(\hat{f}(\hat{x})) = \hat{T}_1 \frac{1}{\hat{T}^5(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i \partial_{x_i}\hat{T}(x))$$

$$- \sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big) x_j \hat{T}_{x_i}(x) - f\Big(\frac{x}{\hat{T}(x)}\Big)\hat{T}(x)\partial_{x_i}\hat{T}(x) \Big)(\hat{T}(x) - x_1 \partial_{x_i}\hat{T}(x)) dx_i.$$

44. The total iso-differential of the second kind of the iso-differentiable of the second kind iso-functions of the fourth kind is

$$d_2(f^\wedge(x)) = \hat{T}_1 \tfrac{1}{\hat{T}^2(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i}\hat{T}(x))$$

$$+\sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\Big)(\hat{T}(x) - x_i\hat{T}_{x_i})dx_i.$$

45. The total iso-differential of the second kind of the iso-differentiable of the second kind iso-functions of the fourth kind is

$$d_2(f^\wedge(x)) = \hat{T}_1 \tfrac{1}{\hat{T}^4(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\Big(\tfrac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i\hat{T}_{x_i}(x))$$

$$-\sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\tfrac{x}{\hat{T}(x)}\Big) x_j \hat{T}_{x_i}(x)\Big)(\hat{T}(x) - x_i\hat{T}_{x_i})dx_i.$$

46. The total iso-differential of the second kind of the iso-differentiable of the third kind iso-functions of the first kind is

$$d_2(\hat{f}^\wedge(\hat{x})) = \hat{T}_1 \tfrac{1}{\hat{T}^4(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)\Big)dx_i.$$

47. The total iso-differential of the second kind of the iso-differentiable of the third kind iso-functions of the second kind is

$$d_2(\hat{f}^\wedge(x)) = \hat{T}_1 \tfrac{1}{\hat{T}^4(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x))\hat{T}(x)$$

$$+\sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)\Big)dx_i.$$

48. The total iso-differential of the second kind of the iso-differentiable of the third kind iso-functions of the third kind is

$$d_2(\hat{f}(\hat{x})) = \hat{T}_1 \tfrac{1}{\hat{T}^5(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\Big(\tfrac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x))$$

$$-\sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\tfrac{x}{\hat{T}(x)}\Big) x_j \hat{T}_{x_i}(x) - f\Big(\tfrac{x}{\hat{T}(x)}\Big)\hat{T}(x)\partial_{x_i}\hat{T}(x)\Big)dx_i.$$

49. The total iso-differential of the second kind of the iso-differentiable of the third kind iso-functions of the fourth kind is

$$d_2(f^\wedge(x)) = \hat{T}_1 \tfrac{1}{\hat{T}(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x))$$

$$+\sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\Big)dx_i.$$

50. The total iso-differential of the second kind of the iso-differentiable of the third kind iso-functions of the fifth kind is

$$d_2(f^\wedge(x)) = \hat{T}_1 \tfrac{1}{\hat{T}^4(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\Big(\tfrac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i\hat{T}_{x_i}(x))$$

$$-\sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\tfrac{x}{\hat{T}(x)}\Big) x_j \hat{T}_{x_i}(x)\Big)dx_i.$$

51. The total iso-differential of the second kind of the iso-differentiable of the fourth kind iso-functions of the first kind is

$$d_2(\hat{f}^\wedge(\hat{x})) = \hat{T}_1 \frac{1}{\hat{T}^5(x)} \sum_{i=1}^n \left(\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)\right)(\hat{T}(x) - x_1\partial_{x_i}\hat{T}(x))dx_i.$$

52. The total iso-differential of the second kind of the iso-differentiable of the fourth kind iso-functions of the second kind is

$$(\hat{f}^\wedge(x))^{4\circledast}_{x_i} = \hat{T}_1 \frac{1}{\hat{T}^5(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x))\hat{T}(x)$$

$$+ \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)\Big)(\hat{T}(x) - x_1\partial_{x_i}\hat{T}(x))dx_i.$$

53. The total iso-differential of the second kind of the iso-differentiable of the fourth kind iso-functions of the third kind is

$$d_2(\hat{f}(\hat{x})) = \hat{T}_1 \frac{1}{\hat{T}^6(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x))$$

$$- \sum_{j=1, j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x) - f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)\Big)(\hat{T}(x) - x_1\partial_{x_i}\hat{T}(x))dx_i.$$

54. The total iso-differential of the second kind of the iso-differentiable of the fourth kind iso-functions of the fourth kind is

$$d_2(f^\wedge(x)) = \hat{T}_1 \frac{1}{\hat{T}^3(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x))$$

$$+ \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\Big)(\hat{T}(x) - x_1\partial_{x_i}\hat{T}(x))dx_i.$$

55. The total iso-differential of the second kind of the iso-differentiable of the fourth kind iso-functions of the fifth kind is

$$d_2(f^\vee(x)) = \hat{T}_1 \frac{1}{\hat{T}^3(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i\hat{T}_{x_i}(x))$$

$$- \sum_{j=1, j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x)\Big)(\hat{T}(x) - x_i\hat{T}_{x_i}(x))dx_i.$$

56. The total iso-differential of the second kind of the iso-differentiable of the fifth kind iso-functions of the first kind is

$$d_2(\hat{f}^\wedge(\hat{x})) = \hat{T}_1 \frac{1}{\hat{T}^2(x)} \sum_{i=1}^n \left(\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)\right)dx_i.$$

57. The total iso-differential of the second kind of the iso-differentiable of the fifth kind iso-functions of the second kind is

$$d_2(\hat{f}^\wedge(x)) = \hat{T}_1 \frac{1}{\hat{T}^2(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x))\hat{T}(x)$$

$$+ \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)\Big)dx_i.$$

58. The total iso-differential of the second kind of the iso-differentiable of the fifth kind iso-functions of the third kind is

$$d_2(\hat{f}(\hat{x})) = \hat{T}_1 \frac{1}{\hat{T}^3(x)} \sum_{i=1}^n \left(\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \partial_{x_i} \hat{T}(x))\right.$$

$$\left. - \sum_{j=1, j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x) - f\left(\frac{x}{\hat{T}(x)}\right) \hat{T}(x) \partial_{x_i} \hat{T}(x)\right) dx_i.$$

59. The total iso-differential of the second kind of the iso-differentiable of the fifth kind iso-functions of the fourth kind is

$$d_2(f^\wedge(x)) = \hat{T}_1 \sum_{i=1}^n \left(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i} \hat{T}(x))\right.$$

$$\left. + \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i} \hat{T}(x)\right) dx_i.$$

60. The total iso-differential of the second kind of the iso-differentiable of the fifth kind iso-functions of the fifth kind is

$$d_2(f^\vee(x)) = \hat{T}_1 \frac{1}{\hat{T}^2(x)} \sum_{i=1}^n \left(\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \hat{T}_{x_i}(x))\right.$$

$$- \sum_{j=1, j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x) dx_i.$$

61. The total iso-differential of the second kind of the iso-differentiable of the sixth kind iso-functions of the first kind is

$$d_2(\hat{f}^\wedge(\hat{x})) = \hat{T}_1 \frac{1}{\hat{T}^3(x)} \sum_{i=1}^n \left(\partial_{x_i} f(x)\hat{T}(x) - f(x) \partial_{x_i} \hat{T}(x)\right)(\hat{T}(x) - x_1 \partial_{x_i} \hat{T}(x)) dx_i.$$

62. The total iso-differential of the second kind of the iso-differentiable of the sixth kind iso-functions of the second kind is

$$d_2(\hat{f}^\wedge(x)) = \hat{T}_1 \frac{1}{\hat{T}^3(x)} \sum_{i=1}^n \left(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i} \hat{T}(x))\hat{T}(x) - f(x\hat{T}(x)) \partial_{x_i} \hat{T}(x)\right.$$

$$\left. + \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i} \hat{T}(x) \hat{T}(x)\right)(\hat{T}(x) - x_1 \partial_{x_i} \hat{T}(x)) dx_i.$$

63. The total iso-differential of the second kind of the iso-differentiable of the sixth kind iso-functions of the third kind is

$$d_2(\hat{f}(\hat{x})) = \hat{T}_1 \frac{1}{\hat{T}^4(x)} \sum_{i=1}^n \left(\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \partial_{x_i} \hat{T}(x))\right.$$

$$\left. - \sum_{j=1, j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x) - f\left(\frac{x}{\hat{T}(x)}\right) \hat{T}(x) \partial_{x_i} \hat{T}(x)\right)(\hat{T}(x) - x_1 \partial_{x_i} \hat{T}(x)) dx_i.$$

64. The total iso-differential of the second kind of the iso-differentiable of the sixth kind iso-functions of the fourth kind is

$$d_2(f^\wedge(x)) = \hat{T}_1 \frac{1}{\hat{T}(x)} \sum_{i=1}^n \left(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i} \hat{T}(x))\right.$$

$$\left. + \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i} \hat{T}(x)\right)(\hat{T}(x) - x_1 \partial_{x_i} \hat{T}(x)) dx_i.$$

65. The total iso-differential of the second kind of the iso-differentiable of the sixth kind iso-functions of the fourth kind is

$$d_2(f^{\vee}(x)) = \hat{T}_1 \frac{1}{\hat{T}^3(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i \hat{T}_{x_i}(x))$$

$$-\sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big) x_j \hat{T}_{x_i}(x)\Big)(\hat{T}(x) - x_i \hat{T}_{x_i}(x)) dx_i.$$

66. The total iso-differential of the second kind of the iso-differentiable of the seventh kind iso-functions of the first kind is

$$d_2(\hat{f}^{\wedge}(\hat{x})) = \hat{T}_1 \frac{1}{\hat{T}^2(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)\Big) dx_i.$$

67. The total iso-differential of the second kind of the iso-differentiable of the seventh kind iso-functions of the second kind is

$$d_2(\hat{f}^{\wedge}(x)) = \hat{T}_1 \frac{1}{\hat{T}^2(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i}\hat{T}(x))\hat{T}(x)$$

$$\sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)|bigr) dx_i.$$

68. The total iso-differential of the second kind of the iso-differentiable of the seventh kind iso-functions of the third kind is

$$d_2(\hat{f}(\hat{x})) = \hat{T}_1 \frac{1}{\hat{T}^3(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i \partial_{x_i}\hat{T}(x))$$

$$-\sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big) x_j \hat{T}_{x_i}(x) - f\Big(\frac{x}{\hat{T}(x)}\Big)\hat{T}(x)\partial_{x_i}\hat{T}(x)\Big) dx_i.$$

69. The total iso-differential of the second kind of the iso-differentiable of the seventh kind iso-functions of the fourth kind is

$$d_2(f^{\wedge}(x)) = \hat{T}_1 \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i}\hat{T}(x))$$

$$+\sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\Big) dx_i.$$

70. The total iso-differential of the second kind of the iso-differentiable of the seventh kind iso-functions of the fifth kind is

$$d_2(f^{\vee}(x)) = \hat{T}_1 \frac{1}{\hat{T}(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i \hat{T}_{x_i}(x))$$

$$-\sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big) x_j \hat{T}_{x_i}(x)\Big) dx_i.$$

71. The total iso-differential of the third kind of the iso-differentiable of the first kind iso-functions of the first kind is

$$d_3(\hat{f}^{\wedge}(\hat{x})) = \frac{1}{\hat{T}^3(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)\Big) dx_i.$$

72. The total iso-differential of the third kind of the iso-differentiable of the first kind iso-functions of the second kind is

$$d_3(\hat{f}^{\wedge}(x)) = \frac{1}{\hat{T}^3(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i} \hat{T}(x))\hat{T}(x)$$

$$+ \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i} \hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x) \Big) dx_i.$$

73. The total iso-differential of the third kind of the iso-differentiable of the first kind iso-functions of the third kind is

$$d_3(\hat{f}(\hat{x})) = \frac{1}{\hat{T}^4(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i \partial_{x_i} \hat{T}(x))$$

$$- \sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big) x_j \hat{T}_{x_i}(x) - f\Big(\frac{x}{\hat{T}(x)}\Big)\hat{T}(x)\partial_{x_i}\hat{T}(x) \Big) dx_i.$$

74. The total iso-differential of the third kind of the iso-differentiable of the first kind iso-functions of the fourth kind is

$$d_3(f^{\wedge}(x)) = \frac{1}{\hat{T}(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i} \hat{T}(x))$$

$$+ \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i} \hat{T}(x) \Big) dx_i.$$

75. The total iso-differential of the third kind of the iso-differentiable of the first kind iso-functions of the fifth kind is

$$d_3(f^{\vee}(x)) = \frac{1}{\hat{T}^4(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i \hat{T}_{x_i}(x))$$

$$- \sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big) x_j \hat{T}_{x_i}(x) \Big) dx_i.$$

76. The total iso-differential of the third kind of the iso-differentiable of the second kind iso-functions of the first kind is

$$d_3(\hat{f}^{\wedge}(\hat{x})) = \frac{1}{\hat{T}^4(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x) \Big)(\hat{T}(x) - x_1 \partial_{x_i}\hat{T}(x)) dx_i.$$

77. The total iso-differential of the third kind of the iso-differentiable of the second kind iso-functions of the second kind is

$$d_3(\hat{f}^{\wedge}(x)) = \frac{1}{\hat{T}^4(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i} \hat{T}(x))\hat{T}(x)$$

$$+ \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i} \hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x) \Big)(\hat{T}(x) - x_1 \partial_{x_i}\hat{T}(x)) dx_i.$$

78. The total iso-differential of the third kind of the iso-differentiable of the second kind iso-functions of the third kind is

$$d_3(\hat{f}(\hat{x})) = \frac{1}{\hat{T}^5(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i \partial_{x_i} \hat{T}(x))$$

$$- \sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big) x_j \hat{T}_{x_i}(x) - f\Big(\frac{x}{\hat{T}(x)}\Big)\hat{T}(x)\partial_{x_i}\hat{T}(x) \Big)(\hat{T}(x) - x_1 \partial_{x_i}\hat{T}(x)) dx_i.$$

79. The total iso-differential of the third kind of the iso-differentiable of the second kind iso-functions of the fourth kind is

$$d_3(f^\wedge(x)) = \frac{1}{\hat{T}^2(x)}\sum_{i=1}^n\Big(\partial_{x_i}f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))$$

$$+\sum_{j=1,j\neq i}^n\partial_{x_j}f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\Big)(\hat{T}(x)-x_i\hat{T}_{x_i})dx_i.$$

80. The total iso-differential of the third kind of the iso-differentiable of the second kind iso-functions of the fourth kind is

$$d_3(f^\wedge(x)) = \frac{1}{\hat{T}^4(x)}\sum_{i=1}^n\Big(\partial_{x_i}f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x)-x_i\hat{T}_{x_i}(x))$$

$$-\sum_{j=1,j\neq i}^n\partial_{x_j}f\Big(\frac{x}{\hat{T}(x)}\Big)x_j\hat{T}_{x_i}(x)\Big)(\hat{T}(x)-x_i\hat{T}_{x_i})dx_i.$$

81. The total iso-differential of the third kind of the iso-differentiable of the third kind iso-functions of the first kind is

$$d_3(\hat{f}^\wedge(\hat{x})) = \frac{1}{\hat{T}^4(x)}\sum_{i=1}^n\Big(\partial_{x_i}f(x)\hat{T}(x)-f(x)\partial_{x_i}\hat{T}(x)\Big)dx_i.$$

82. The total iso-differential of the third kind of the iso-differentiable of the third kind iso-functions of the second kind is

$$d_3(\hat{f}^\wedge(x)) = \frac{1}{\hat{T}^4(x)}\sum_{i=1}^n\Big(\partial_{x_i}f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))\hat{T}(x)$$

$$+\sum_{j=1,j\neq i}^n\partial_{x_j}f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\hat{T}(x)-f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)\Big)dx_i.$$

83. The total iso-differential of the third kind of the iso-differentiable of the third kind iso-functions of the third kind is

$$d_3(\hat{f}(\hat{x})) = \frac{1}{\hat{T}^5(x)}\sum_{i=1}^n\Big(\partial_{x_i}f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x))$$

$$-\sum_{j=1,j\neq i}^n\partial_{x_j}f\Big(\frac{x}{\hat{T}(x)}\Big)x_j\hat{T}_{x_i}(x)-f\Big(\frac{x}{\hat{T}(x)}\Big)\hat{T}(x)\partial_{x_i}\hat{T}(x)\Big)dx_i.$$

84. The total iso-differential of the third kind of the iso-differentiable of the third kind iso-functions of the fourth kind is

$$d_3(f^\wedge(x)) = \frac{1}{\hat{T}(x)}\sum_{i=1}^n\Big(\partial_{x_i}f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))$$

$$+\sum_{j=1,j\neq i}^n\partial_{x_j}f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\Big)dx_i.$$

85. The total iso-differential of the third kind of the iso-differentiable of the third kind iso-functions of the fifth kind is

$$d_3(f^\wedge(x)) = \frac{1}{\hat{T}^4(x)}\sum_{i=1}^n\Big(\partial_{x_i}f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x)-x_i\hat{T}_{x_i}(x))$$

$$-\sum_{j=1,j\neq i}^n\partial_{x_j}f\Big(\frac{x}{\hat{T}(x)}\Big)x_j\hat{T}_{x_i}(x)\Big)dx_i.$$

86. The total iso-differential of the third kind of the iso-differentiable of the fourth kind iso-functions of the first kind is

$$d_3(\hat{f}^{\wedge}(\hat{x})) = \frac{1}{\hat{T}^5(x)} \sum_{i=1}^{n} \Big(\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x) \Big)(\hat{T}(x) - x_1\partial_{x_i}\hat{T}(x))dx_i.$$

87. The total iso-differential of the third kind of the iso-differentiable of the fourth kind iso-functions of the second kind is

$$d_3(\hat{f}^{\wedge}(x))_{x_i}^{4\circledast} = \frac{1}{\hat{T}^5(x)} \sum_{i=1}^{n} \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x))\hat{T}(x)$$

$$+ \sum_{j=1, j\neq i}^{n} \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x) \Big)(\hat{T}(x) - x_1\partial_{x_i}\hat{T}(x))dx_i.$$

88. The total iso-differential of the third kind of the iso-differentiable of the fourth kind iso-functions of the third kind is

$$d_3(\hat{f}(\hat{x})) = \frac{1}{\hat{T}^6(x)} \sum_{i=1}^{n} \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x))$$

$$- \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big)x_j\hat{T}_{x_i}(x) - f\Big(\frac{x}{\hat{T}(x)}\Big)\hat{T}(x)\partial_{x_i}\hat{T}(x) \Big)(\hat{T}(x) - x_1\partial_{x_i}\hat{T}(x))dx_i.$$

89. The total iso-differential of the third kind of the iso-differentiable of the fourth kind iso-functions of the fourth kind is

$$d_3(f^{\wedge}(x)) = \frac{1}{\hat{T}^3(x)} \sum_{i=1}^{n} \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x))$$

$$+ \sum_{j=1, j\neq i}^{n} \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x) \Big)(\hat{T}(x) - x_1\partial_{x_i}\hat{T}(x))dx_i.$$

90. The total iso-differential of the third kind of the iso-differentiable of the fourth kind iso-functions of the fifth kind is

$$d_3(f^{\vee}(x)) = \frac{1}{\hat{T}^3(x)} \sum_{i=1}^{n} \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i\hat{T}_{x_i}(x))$$

$$- \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big)x_j\hat{T}_{x_i}(x) \Big)(\hat{T}(x) - x_i\hat{T}_{x_i}(x))dx_i.$$

91. The total iso-differential of the third kind of the iso-differentiable of the fifth kind iso-functions of the first kind is

$$d_3(\hat{f}^{\wedge}(\hat{x})) = \frac{1}{\hat{T}^2(x)} \sum_{i=1}^{n} \Big(\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x) \Big)dx_i.$$

92. The total iso-differential of the third kind of the iso-differentiable of the fifth kind iso-functions of the second kind is

$$d_3(\hat{f}^{\wedge}(x)) = \frac{1}{\hat{T}^2(x)} \sum_{i=1}^{n} \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x))\hat{T}(x)$$

$$+ \sum_{j=1, j\neq i}^{n} \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x) \Big)dx_i.$$

93. The total iso-differential of the third kind of the iso-differentiable of the fifth kind iso-functions of the third kind is

$$d_3(\hat{f}(\hat{x})) = \frac{1}{\hat{T}^3(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i \partial_{x_i} \hat{T}(x))$$

$$- \sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big) x_j \hat{T}_{x_i}(x) - f\Big(\frac{x}{\hat{T}(x)}\Big)\hat{T}(x)\partial_{x_i}\hat{T}(x)\Big) dx_i.$$

94. The total iso-differential of the third kind of the iso-differentiable of the fifth kind iso-functions of the fourth kind is

$$d_3(f^\wedge(x)) = \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i} \hat{T}(x))$$

$$+ \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i} \hat{T}(x)\Big) dx_i.$$

95. The total iso-differential of the third kind of the iso-differentiable of the fifth kind iso-functions of the fifth kind is

$$d_3(f^\vee(x)) = \frac{1}{\hat{T}^2(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i \hat{T}_{x_i}(x))$$

$$- \sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big) x_j \hat{T}_{x_i}(x) dx_i.$$

96. The total iso-differential of the third kind of the iso-differentiable of the sixth kind iso-functions of the first kind is

$$d_3(\hat{f}^\wedge(\hat{x})) = \frac{1}{\hat{T}^3(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)\Big)(\hat{T}(x) - x_1 \partial_{x_i} \hat{T}(x)) dx_i.$$

97. The total iso-differential of the third kind of the iso-differentiable of the sixth kind iso-functions of the second kind is

$$d_3(\hat{f}^\wedge(x)) = \frac{1}{\hat{T}^3(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i}\hat{T}(x))\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)$$

$$+ \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\hat{T}(x)\Big)(\hat{T}(x) - x_1\partial_{x_i}\hat{T}(x)) dx_i.$$

98. The total iso-differential of the third kind of the iso-differentiable of the sixth kind iso-functions of the third kind is

$$d_3(\hat{f}(\hat{x})) = \frac{1}{\hat{T}^4(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i \partial_{x_i} \hat{T}(x))$$

$$- \sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big) x_j \hat{T}_{x_i}(x) - f\Big(\frac{x}{\hat{T}(x)}\Big)\hat{T}(x)\partial_{x_i}\hat{T}(x)\Big)(\hat{T}(x) - x_1\partial_{x_i}\hat{T}(x)) dx_i.$$

99. The total iso-differential of the third kind of the iso-differentiable of the sixth kind iso-functions of the fourth kind is

$$d_3(f^\wedge(x)) = \frac{1}{\hat{T}(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i} \hat{T}(x))$$

$$+ \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i} \hat{T}(x)\Big)(\hat{T}(x) - x_1\partial_{x_i}\hat{T}(x)) dx_i.$$

100. The total iso-differential of the third kind of the iso-differentiable of the sixth kind iso-functions of the fourth kind is

$$d_3(f^\vee(x)) = \frac{1}{\hat{T}^3(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i \hat{T}_{x_i}(x))$$

$$-\sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big) x_j \hat{T}_{x_i}(x)\Big)(\hat{T}(x) - x_i \hat{T}_{x_i}(x)) dx_i.$$

101. The total iso-differential of the third kind of the iso-differentiable of the seventh kind iso-functions of the first kind is

$$d_3(\hat{f}^\wedge(\hat{x})) = \frac{1}{\hat{T}^2(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)\Big) dx_i.$$

102. The total iso-differential of the third kind of the iso-differentiable of the seventh kind iso-functions of the second kind is

$$d_3(\hat{f}^\wedge(x)) = \frac{1}{\hat{T}^2(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i}\hat{T}(x))\hat{T}(x)$$

$$\sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)\Big) dx_i.$$

103. The total iso-differential of the third kind of the iso-differentiable of the seventh kind iso-functions of the third kind is

$$d_3(\hat{f}(\hat{x})) = \frac{1}{\hat{T}^3(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i \partial_{x_i}\hat{T}(x))$$

$$-\sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big) x_j \hat{T}_{x_i}(x) - f\Big(\frac{x}{\hat{T}(x)}\Big)\hat{T}(x)\partial_{x_i}\hat{T}(x)\Big) dx_i.$$

104. The total iso-differential of the third kind of the iso-differentiable of the seventh kind iso-functions of the fourth kind is

$$d_3(f^\wedge(x)) = \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i}\hat{T}(x))$$

$$+\sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\Big) dx_i.$$

105. The total iso-differential of the third kind of the iso-differentiable of the seventh kind iso-functions of the fifth kind is

$$d_3(f^\vee(x)) = \frac{1}{\hat{T}(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i \hat{T}_{x_i}(x))$$

$$-\sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big) x_j \hat{T}_{x_i}(x)\Big) dx_i.$$

106. The total iso-differential of the fourth kind of the iso-differentiable of the first kind iso-functions of the first kind is

$$d_4(\hat{f}^\wedge(\hat{x})) = \frac{1}{\hat{T}^2(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)\Big) dx_i.$$

107. The total iso-differential of the fourth kind of the iso-differentiable of the first kind iso-functions of the second kind is

$$d_4(\hat{f}^{\wedge}(x)) = \frac{1}{\hat{T}^2(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i} \hat{T}(x))\hat{T}(x)$$

$$+\sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i} \hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)\Big) dx_i.$$

108. The total iso-differential of the fourth kind of the iso-differentiable of the first kind iso-functions of the third kind is

$$d_4(\hat{f}(\hat{x})) = \frac{1}{\hat{T}^3(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i \partial_{x_i} \hat{T}(x))$$

$$-\sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big) x_j \hat{T}_{x_i}(x) - f\Big(\frac{x}{\hat{T}(x)}\Big)\hat{T}(x)\partial_{x_i}\hat{T}(x)\Big) dx_i.$$

109. The total iso-differential of the fourth kind of the iso-differentiable of the first kind iso-functions of the fourth kind is

$$d_4(f^{\wedge}(x)) = \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i} \hat{T}(x))$$

$$+\sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i} \hat{T}(x)\Big) dx_i.$$

110. The total iso-differential of the fourth kind of the iso-differentiable of the first kind iso-functions of the fifth kind is

$$d_4(f^{\vee}(x)) = \frac{1}{\hat{T}(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i \hat{T}_{x_i}(x))$$

$$-\sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big) x_j \hat{T}_{x_i}(x)\Big) dx_i$$

111. The total iso-differential of the fourth kind of the iso-differentiable of the second kind iso-functions of the first kind is

$$d_4(\hat{f}^{\wedge}(\hat{x})) = \sum_{i=1}^n \frac{(\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x))(\hat{T}(x) - x_i \partial_{x_i}\hat{T}(x))}{\hat{T}^3(x)} dx_i.$$

112. The total iso-differential of the fourth kind of the iso-differentiable of the second kind iso-functions of the second kind is

$$d_4(\hat{f}^{\wedge}(x)) = \frac{1}{\hat{T}^3(x)} \sum_{i=1}^n \Big((\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i} \hat{T}(x))\hat{T}(x)$$

$$+\sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i} \hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x))(\hat{T}(x) - x_i \partial_{x_i}\hat{T}(x))\Big) dx_i.$$

113. The total iso-differential of the fourth kind of the iso-differentiable of the second kind iso-functions of the third kind is

$$d_4(\hat{f}(\hat{x})) = \frac{1}{\hat{T}^4(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i \partial_{x_i} \hat{T}(x))$$

$$-\sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big) x_j \hat{T}_{x_i}(x) - f\Big(\frac{x}{\hat{T}(x)}\Big)\hat{T}(x)\partial_{x_i}\hat{T}(x)\Big)(\hat{T}(x) - x_i \partial_{x_i}\hat{T}(x)) dx_i.$$

114. The total iso-differential of the fourth kind of the iso-differentiable of the second kind iso-functions of the fourth kind is

$$d_4(f^\wedge(x)) = \frac{1}{\hat{T}(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i} \hat{T}(x))$$

$$+ \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i} \hat{T}(x)\Big)(\hat{T}(x) - x_i \partial_{x_i} \hat{T}(x)) dx_i.$$

115. The total iso-differential of the fourth kind of the iso-differentiable of the second kind iso-functions of the fifth kind is

$$d_4(f^\vee(x)) = \frac{1}{\hat{T}^3(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i \hat{T}_{x_i}(x))$$

$$- \sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big) x_j \hat{T}_{x_i}(x)\Big)(\hat{T}(x) - x_i \hat{T}_{x_i}(x)) dx_i.$$

116. The total iso-differential of the fourth kind of the iso-differentiable of the third kind iso-functions of the first kind is

$$d_4(\hat{f}^\wedge(\hat{x})) = \frac{1}{\hat{T}^3(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i} \hat{T}(x)\Big) dx_i.$$

117. The total iso-differential of the fourth kind of the iso-differentiable of the third kind iso-functions of the second kind is

$$d_4(\hat{f}^\wedge(x)) = \frac{1}{\hat{T}^4(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i} \hat{T}(x))\hat{T}(x)$$

$$+ \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i} \hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i} \hat{T}(x)\Big) dx_i.$$

118. The total iso-differential of the fourth kind of the iso-differentiable of the third kind iso-functions of the third kind is

$$d_4(\hat{f}(\hat{x})) = \frac{1}{\hat{T}^4(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i \partial_{x_i} \hat{T}(x))$$

$$- \sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big) x_j \hat{T}_{x_i}(x) - f\Big(\frac{x}{\hat{T}(x)}\Big)\hat{T}(x)\partial_{x_i} \hat{T}(x)\Big) dx_i.$$

119. The total iso-differential of the fourth kind of the iso-differentiable of the third kind iso-functions of the fourth kind is

$$d_4(f^\wedge(x)) = \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i} \hat{T}(x))$$

$$+ \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i} \hat{T}(x)\Big) dx_i.$$

120. The total iso-differential of the fourth kind of the iso-differentiable of the third kind iso-functions of the fifth kind is

$$d_4(f^\vee(x)) = \frac{1}{\hat{T}^3(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i \hat{T}_{x_i}(x))$$

$$- \sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big) x_j \hat{T}_{x_i}(x)\Big) dx_i.$$

121. The total iso-differential of the fourth kind of the iso-differentiable of the fourth kind iso-functions of the first kind is

$$d_4(\hat{f}^\wedge(\hat{x})) = \sum_{i=1}^n \frac{\left(\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)\right)(\hat{T}(x) - x_1\partial_{x_i}\hat{T}(x))}{\hat{T}^4(x)} dx_i.$$

122. The total iso-differential of the fourth kind of the iso-differentiable of the fourth kind iso-functions of the second kind is

$$d_4(\hat{f}^\wedge(x)) = \frac{1}{\hat{T}^4(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x))\hat{T}(x)$$

$$+\sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)\Big)(\hat{T}(x) - x_1\partial_{x_i}\hat{T}(x)) dx_i.$$

123. The total iso-differential of the fourth kind of the iso-differentiable of the fourth kind iso-functions of the third kind is

$$d_4(\hat{f}(\hat{x})) = \frac{1}{\hat{T}^5(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x))$$

$$-\sum_{j=1, j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x) - f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)\Big) dx_i.$$

124. The total iso-differential of the fourth kind of the iso-differentiable of the fourth kind iso-functions of the fourth kind is

$$d_4(f^\wedge(x)) = \frac{1}{\hat{T}^2(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x))$$

$$+\sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\Big)(\hat{T}(x) - x_1\partial_{x_i}\hat{T}(x)) dx_i.$$

125. The total iso-differential of the fourth kind of the iso-differentiable of the fourth kind iso-functions of the fifth kind is

$$d_4(f^\vee(x)) = \frac{1}{\hat{T}^4(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i\hat{T}_{x_i}(x))$$

$$-\sum_{j=1, j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x)\Big)(\hat{T}(x) - x_i\hat{T}_{x_i}(x)) dx_i.$$

126. The total iso-differential of the fourth kind of the iso-differentiable of the fifth kind iso-functions of the first kind is

$$d_4(\hat{f}^\wedge(\hat{x})) = \frac{1}{\hat{T}(x)} \sum_{i=1}^n \left(\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)\right) dx_i.$$

127. The total iso-differential of the fourth kind of the iso-differentiable of the fifth kind iso-functions of the second kind is

$$d_4(\hat{f}^\wedge(x)) = \frac{1}{\hat{T}(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x))\hat{T}(x)$$

$$+\sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)\Big) dx_i.$$

128. The total iso-differential of the fourth kind of the iso-differentiable of the fifth kind iso-functions of the third kind is

$$d_4(\hat{f}(\hat{x})) = \frac{1}{\hat{T}^2(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x))$$

$$-\sum_{j=1, j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x) - f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)\Big) dx_i.$$

129. The total iso-differential of the fourth kind of the iso-differentiable of the fifth kind iso-functions of the fourth kind is

$$d_4(f^\wedge(x)) = \hat{T}(x) \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x))$$

$$+\sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\Big) dx_i.$$

130. The total iso-differential of the fourth kind of the iso-differentiable of the fifth kind iso-functions of the fifth kind is

$$d_4(f^\vee(x)) = \frac{1}{\hat{T}(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i\hat{T}_{x_i}(x))$$

$$-\sum_{j=1, j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x)\Big) dx_i.$$

131. The total iso-differential of the fourth kind of the iso-differentiable of the sixth kind iso-functions of the first kind is

$$d_4(\hat{f}^\wedge(\hat{x})) = \sum_{i=1}^n \frac{\Big(\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)\Big)(\hat{T}(x) - x_1\partial_{x_i}\hat{T}(x))}{\hat{T}^2(x)} dx_i.$$

132. The total iso-differential of the fourth kind of the iso-differentiable of the fifth kind iso-functions of the second kind is

$$d_4(\hat{f}^\wedge(x)) = \frac{1}{\hat{T}^2(x)} \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x))\hat{T}(x)$$

$$+\sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)\Big)(\hat{T}(x) - x_1\partial_{x_i}\hat{T}(x)) dx_i.$$

133. The total iso-differential of the fourth kind of the iso-differentiable of the fifth kind iso-functions of the third kind is

$$d_4(\hat{f}(\hat{x})) = \frac{1}{\hat{T}^3(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x))$$

$$-\sum_{j=1, j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x) - f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)\Big)(\hat{T}(x) - x_1\partial_{x_i}\hat{T}(x)) dx_i.$$

134. The total iso-differential of the fourth kind of the iso-differentiable of the fifth kind iso-functions of the fourth kind is

$$d_4(f^\wedge(x)) = \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x))$$

$$+\sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\Big)(\hat{T}(x) - x_1\partial_{x_i}\hat{T}(x))dx_i.$$

135. The total iso-differential of the fourth kind of the iso-differentiable of the sixth kind iso-functions of the fourth kind is

$$d_4(f^\vee(x)) = \frac{1}{\hat{T}^2(x)}\sum_{i=1}^n \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i\hat{T}_{x_i}(x))$$

$$-\sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big) x_j \hat{T}_{x_i}(x)\Big)(\hat{T}(x) - x_i\hat{T}_{x_i}(x)).$$

136. The total iso-differential of the fourth kind of the iso-differentiable of the seventh kind iso-functions of the first kind is

$$d_4(\hat{f}^\wedge(\hat{x})) = \sum_{i=1}^n \frac{\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x)} dx_i.$$

137. The total iso-differential of the fourth kind of the iso-differentiable of the seventh kind iso-functions of the second kind is

$$d_4(\hat{f}^\wedge(x)) = \frac{1}{\hat{T}(x)}\sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x))\hat{T}(x)$$

$$+\sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)\Big)dx_i.$$

138. The total iso-differential of the fourth kind of the iso-differentiable of the seventh kind iso-functions of the third kind is

$$d_4(\hat{f}(\hat{x})) = \hat{T}(x)\sum_{i=1}^n \partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x))$$

$$-\sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big) x_j \hat{T}_{x_i}(x) - f\Big(\frac{x}{\hat{T}(x)}\Big)\hat{T}(x)\partial_{x_i}\hat{T}(x)dx_i.$$

139. The total iso-differential of the fourth kind of the iso-differentiable of the seventh kind iso-functions of the fourth kind is

$$d_4(f^\wedge(x)) = \hat{T}^2(x)\sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x))$$

$$+\sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\Big)dx_i.$$

140. The total iso-differential of the fourth kind of the iso-differentiable of the seventh kind iso-functions of the fifth kind is

$$d_4(f^\vee(x)) = \sum_{i=1}^n \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i\hat{T}_{x_i}(x))$$

$$-\sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big) x_j \hat{T}_{x_i}(x)\Big)dx_i.$$

141. The total iso-differential of the fifth kind of the iso-differentiable of the first kind iso-functions of the first kind is

$$d_5(\hat{f}^{\wedge}(\hat{x})) = \hat{T}_1 \sum_{i=1}^n \frac{\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)} dx_i.$$

142. The total iso-differential of the fifth kind of the iso-differentiable of the first kind iso-functions of the second kind is

$$d_5(\hat{f}^{\wedge}(x))$$

$$= \hat{T}_1 \sum_{i=1}^n \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))\hat{T}(x) + \sum_{j=1,j\neq i}^n \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)} dx_i.$$

143. The total iso-differential of the fifth kind of the iso-differentiable of the first kind iso-functions of the third kind is

$$d_5(\hat{f}(\hat{x}))$$

$$= \hat{T}_1 \sum_{i=1}^n \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)) - \sum_{j=1,j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x) - f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)} dx_i.$$

144. The total iso-differential of the fifth kind of the iso-differentiable of the first kind iso-functions of the fourth kind is

$$d_5(f^{\wedge}(x))$$

$$= \hat{T}_1\hat{T}^2(x) \sum_{i=1}^n \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x)) + \sum_{j=1,j\neq i}^n \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)} dx_i.$$

145. The total iso-differential of the fifth kind of the iso-differentiable of the first kind iso-functions of the fifth kind is

$$d_5(f^{\wedge}(x))$$

$$= \hat{T}_1 \sum_{i=1}^n \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i\hat{T}_{x_i}(x)) - \sum_{j=1,j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)} dx_i.$$

146. The total iso-differential of the fifth kind of the iso-differentiable of the second kind iso-functions of the first kind is

$$d_5(\hat{f}^{\wedge}(\hat{x})) = \hat{T}_1 \sum_{i=1}^n \frac{\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x)} dx_i.$$

147. The total iso-differential of the fifth kind of the iso-differentiable of the second kind iso-functions of the second kind is

$$d_5(\hat{f}^{\wedge}(x))$$

$$= \hat{T}_1 \sum_{i=1}^n \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))\hat{T}(x) + \sum_{j=1,j\neq i}^n \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)}{\hat{T}(x)} dx_i.$$

148. The total iso-differential of the fifth kind of the iso-differentiable of the second kind iso-functions of the third kind is

$$d_5(\hat{f}(\hat{x}))$$

$$= \hat{T}_1 \sum_{i=1}^{n} \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \partial_{x_i} \hat{T}(x)) - \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x) - f\left(\frac{x}{\hat{T}(x)}\right) \hat{T}(x) \partial_{x_i} \hat{T}(x)}{\hat{T}^2(x)} dx_i.$$

149. The total iso-differential of the fifth kind of the iso-differentiable of the second kind iso-functions of the fourth kind is

$$d_5(f^{\wedge}(x))$$

$$= \hat{T}_1 \hat{T}(x) \sum_{i=1}^{n} \partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i} \hat{T}(x)) + \sum_{j=1, j\neq i}^{n} \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i} \hat{T}(x) dx_i.$$

150. The total iso-differential of the fifth kind of the iso-differentiable of the second kind iso-functions of the fifth kind is

$$d_5(f^{\wedge}(x)) = \hat{T}_1 \frac{1}{\hat{T}(x)} \sum_{i=1}^{n} \left(\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \hat{T}_{x_i}(x)) \right.$$

$$\left. - \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x) \right) dx_i.$$

151. The total iso-differential of the fifth kind of the iso-differentiable of the third kind iso-functions of the first kind is

$$d_5(\hat{f}^{\wedge}(\hat{x})) = \hat{T}_1 \frac{1}{\hat{T}(x)} \sum_{i=1}^{n} \frac{\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i} \hat{T}(x)}{\hat{T}(x) - x_i \partial_{x_i} \hat{T}(x)} dx_i.$$

152. The total iso-differential of the fifth kind of the iso-differentiable of the third kind iso-functions of the second kind is

$$d_5(\hat{f}^{\wedge}(x))$$

$$= \hat{T}_1 \sum_{i=1}^{n} \frac{1}{\hat{T}(x)} \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i} \hat{T}(x))\hat{T}(x) + \sum_{j=1, j\neq i}^{n} \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i} \hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i} \hat{T}(x)}{\hat{T}(x) - x_i \partial_{x_i} \hat{T}(x)} dx_i.$$

153. The total iso-differential of the fifth kind of the iso-differentiable of the third kind iso-functions of the third kind is

$$d_5(\hat{f}(\hat{x}))$$

$$= \hat{T}_1 \frac{1}{\hat{T}^2(x)} \sum_{i=1}^{n} \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \partial_{x_i} \hat{T}(x)) - \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x) - f\left(\frac{x}{\hat{T}(x)}\right) \hat{T}(x) \partial_{x_i} \hat{T}(x)}{\hat{T}(x) - x_i \partial_{x_i} \hat{T}(x)} dx_i.$$

154. The total iso-differential of the fifth kind of the iso-differentiable of the third kind iso-functions of the fourth kind is

$$d_5(f^{\wedge}(x))$$

$$= \hat{T}_1 \hat{T}^2(x) \sum_{i=1}^{n} \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i} \hat{T}(x)) + \sum_{j=1, j\neq i}^{n} \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i} \hat{T}(x)}{\hat{T}(x) - x_i \partial_{x_i} \hat{T}(x)} dx_i.$$

155. The total iso-differential of the fifth kind of the iso-differentiable of the third kind iso-functions of the fifth kind is

$$d_5(f^\wedge(x))$$

$$= \hat{T}_1 \frac{1}{\hat{T}(x)} \sum_{i=1}^n \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \hat{T}_{x_i}(x)) - \sum_{j=1, j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x)}{\hat{T}(x) - x_i \partial_{x_i} \hat{T}(x)} dx_i.$$

156. The total iso-differential of the fifth kind of the iso-differentiable of the fourth kind iso-functions of the first kind is

$$d_5(\hat{f}^\wedge(\hat{x})) = \hat{T}_1 \sum_{i=1}^n \frac{\partial_{x_i} f(x) \hat{T}(x) - f(x) \partial_{x_i} \hat{T}(x)}{\hat{T}^2(x)} dx_i.$$

157. The total iso-differential of the fifth kind of the iso-differentiable of the fourth kind iso-functions of the second kind is

$$d_5(\hat{f}^\wedge(x))$$

$$= \hat{T}_1 \sum_{i=1}^n \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i} \hat{T}(x)) \hat{T}(x) + \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i} \hat{T}(x) \hat{T}(x) - f(x\hat{T}(x)) \partial_{x_i} \hat{T}(x)}{\hat{T}^2(x)} dx_i.$$

158. The total iso-differential of the fifth kind of the iso-differentiable of the fourth kind iso-functions of the third kind is

$$d_5(\hat{f}(\hat{x}))$$

$$= \hat{T}_1 \sum_{i=1}^n \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \partial_{x_i} \hat{T}(x)) - \sum_{j=1, j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x) - f\left(\frac{x}{\hat{T}(x)}\right) \hat{T}(x) \partial_{x_i} \hat{T}(x)}{\hat{T}^3(x)} dx_i.$$

159. The total iso-differential of the fifth kind of the iso-differentiable of the fourth kind iso-functions of the fourth kind is

$$d_5(f^\wedge(x))$$

$$= \hat{T}_1 \sum_{i=1}^n \left(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i} \hat{T}(x)) + \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i} \hat{T}(x)\right) dx_i.$$

160. The total iso-differential of the fifth kind of the iso-differentiable of the fourth kind iso-functions of the fifth kind is

$$d_5(f^\wedge(x))$$

$$= \hat{T}_1 \frac{1}{\hat{T}^2(x)} \sum_{i=1}^n \left(\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \hat{T}_{x_i}(x)) - \sum_{j=1, j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x)\right) dx_i.$$

161. The total iso-differential of fifth kind of the iso-differentiable of the fifth kind iso-functions of the first kind is

$$d_5(\hat{f}^\wedge(\hat{x})) = \hat{T}_1 \hat{T}(x) \sum_{i=1}^n \frac{\partial_{x_i} f(x) \hat{T}(x) - f(x) \partial_{x_i} \hat{T}(x)}{\hat{T}(x) - x_i \partial_{x_i} \hat{T}(x)} dx_i.$$

162. The total iso-differential of the fifth kind of the iso-differentiable of the fifth kind iso-functions of the second kind is

$$d_5(\hat{f}^\wedge(x))$$

$$=\hat{T}_1\hat{T}(x)\sum_{i=1}^n \frac{\partial_{x_i}f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))\hat{T}(x)+\sum_{j=1,j\neq i}^n\partial_{x_j}f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)-f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

163. The total iso-differential of the fifth kind of the iso-differentiable of the fifth kind iso-functions of the third kind is

$$d_5(\hat{f}(\hat{x}))$$

$$=\hat{T}_1\sum_{i=1}^n \frac{\partial_{x_i}f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x))-\sum_{j=1,j\neq i}^n\partial_{x_j}f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)-f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

164. The total iso-differential of the fifth kind of the iso-differentiable of the fifth kind iso-functions of the fourth kind is

$$d_5(f^\wedge(x))$$

$$=\hat{T}_1\hat{T}^3(x)\sum_{i=1}^n \frac{\partial_{x_i}f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))+\sum_{j=1,j\neq i}^n\partial_{x_j}f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

165. The total iso-differential of the fifth kind of the iso-differentiable of the fifth kind iso-functions of the fifth kind is

$$d_5(f^\wedge(x))$$

$$=\hat{T}_1\hat{T}(x)\sum_{i=1}^n \frac{\partial_{x_i}f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\hat{T}_{x_i}(x))-\sum_{j=1,j\neq i}^n\partial_{x_j}f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)}{\hat{T}(x)-x_1\partial_{x_i}\hat{T}(x)}dx_i.$$

166. The total iso-differential of the fifth kind of the iso-differentiable of the sixth kind iso-functions of the first kind is

$$d_5(\hat{f}^\wedge(\hat{x}))=\hat{T}_1\sum_{i=1}^n\Big(\partial_{x_i}f(x)\hat{T}(x)-f(x)\partial_{x_i}\hat{T}(x)\Big)dx_i.$$

167. The total iso-differential of fifth kind of the iso-differentiable of the sixth kind iso-functions of the second kind is

$$d_5(\hat{f}^\wedge(x))=\hat{T}_1\sum_{i=1}^n\Big(\partial_{x_i}f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))\hat{T}(x)$$

$$+\sum_{j=1,j\neq i}^n\partial_{x_j}f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\hat{T}(x)-f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)\Big)dx_i.$$

168. The total iso-differential of the fifth kind of the iso-differentiable of the sixth kind iso-functions of the third kind is

$$d_5(\hat{f}(\hat{x}))$$

$$=\hat{T}_1\sum_{i=1}^n \frac{\partial_{x_i}f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x))-\sum_{j=1,j\neq i}^n\partial_{x_j}f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)-f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x)}dx_i.$$

169. The total iso-differential of the fifth kind of the iso-differentiable of the sixth kind iso-functions of the fourth kind is

$$d_5(f^\wedge(x)) = \hat{T}_1\hat{T}^2(x)\sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x))$$

$$+\sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\Big)dx_i.$$

170. The total iso-differential of fifth kind of the iso-differentiable of the sixth kind iso-functions of the fifth kind is

$$d_5(f^\wedge(x))$$

$$= \hat{T}_1\sum_{i=1}^n \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i\hat{T}_{x_i}(x)) - \sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big)x_j\hat{T}_{x_i}(x)\Big)dx_i.$$

171. The total iso-differential of the fifth kind of the iso-differentiable of the seventh kind iso-functions of the first kind is

$$d_5(\hat{f}^\wedge(\hat{x})) = \hat{T}_1\hat{T}(x)\sum_{i=1}^n \frac{\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

172. The total iso-differential of fifth kind of the iso-differentiable of the seventh kind iso-functions of the second kind is

$$d_5(\hat{f}^\wedge(x))$$

$$= \hat{T}_1\hat{T}(x)\sum_{i=1}^n \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x) + \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\hat{T}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

173. The total iso-differential of the fifth kind of the iso-differentiable of the seventh kind iso-functions of the third kind is

$$d_5(\hat{f}(\hat{x}))$$

$$= \hat{T}_1\sum_{i=1}^n \frac{\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)) - \sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big)x_j\hat{T}_{x_i}(x) - f\Big(\frac{x}{\hat{T}(x)}\Big)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{(\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x))}dx_i.$$

174. The total iso-differential of fifth kind of the iso-differentiable of the seventh kind iso-functions of the fourth kind is

$$d_5(f^\wedge(x))$$

$$= \hat{T}_1\hat{T}^3(x)\sum_{i=1}^n \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x)) + \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

175. The total iso-differential of the fifth kind of the iso-differentiable of the seventh kind iso-functions of the fifth kind is

$$d_5(f^\wedge(x))$$

$$= \hat{T}_1\hat{T}^2(x)\sum_{i=1}^n \frac{\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i\hat{T}_{x_i}(x)) - \sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big)x_j\hat{T}_{x_i}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

176. The total iso-differential of the sixth kind of the iso-differentiable of the first kind iso-functions of the first kind is

$$d_6(\hat{f}^{\wedge}(\hat{x})) = \hat{T}_1 \frac{1}{\hat{T}(x)} \sum_{i=1}^{n} \frac{\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)} dx_i.$$

177. The total iso-differential of the sixth kind of the iso-differentiable of the first kind iso-functions of the second kind is

$$d_6(\hat{f}^{\wedge}(x))$$

$$= \frac{1}{\hat{T}(x)}\hat{T}_1 \sum_{i=1}^{n} \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x) + \sum_{j=1, j\neq i}^{n} \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\hat{T}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)} dx_i.$$

178. The total iso-differential of the sixth kind of the iso-differentiable of the first kind iso-functions of the third kind is

$$d_6(\hat{f}(\hat{x}))$$

$$= \frac{1}{\hat{T}^2(x)}\hat{T}_1 \sum_{i=1}^{n} \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)) - \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j\hat{T}_{x_i}(x) - f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)} dx_i.$$

179. The total iso-differential of the sixth kind of the iso-differentiable of the first kind iso-functions of the fourth kind is

$$d_6(f^{\wedge}(x))$$

$$= \hat{T}(x)\hat{T}_1 \sum_{i=1}^{n} \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x)) + \sum_{j=1, j\neq i}^{n} \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)} dx_i.$$

180. The total iso-differential of the sixth kind of the iso-differentiable of the first kind iso-functions of the fifth kind is

$$d_6(f^{\vee}(x))$$

$$= \hat{T}_1 \frac{1}{\hat{T}(x)} \sum_{i=1}^{n} \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i\hat{T}_{x_i}(x)) - \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j\hat{T}_{x_i}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)} dx_i.$$

181. The total iso-differential of the sixth kind of the iso-differentiable of the second kind iso-functions of the first kind is

$$d_6(\hat{f}^{\wedge}(\hat{x})) = \hat{T}_1 \sum_{i=1}^{n} \frac{\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}^2(x)} dx_i.$$

182. The total iso-differential of the sixth kind of the iso-differentiable of the second kind iso-functions of the second kind is

$$d_6(\hat{f}^{\wedge}(x))$$

$$= \hat{T}_1 \sum_{i=1}^{n} \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))\hat{T}(x) \sum_{j=1, j\neq i}^{n} \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)}{\hat{T}^2(x)} dx_i.$$

183. The total iso-differential of the sixth kind of the iso-differentiable of the second kind iso-functions of the third kind is

$$d_6(\hat{f}(\hat{x}))$$

$$= \hat{T}_1 \sum_{i=1}^n \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)) - \sum_{j=1, j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x) - f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}^3(x)} dx_i.$$

184. The total iso-differential of the sixth kind of the iso-differentiable of the second kind iso-functions of the fourth kind is

$$d_6(f^{\wedge}(x))$$

$$= \hat{T}_1 \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x)) + \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x) dx_i.$$

185. The total iso-differential of the sixth kind of the iso-differentiable of the second kind iso-functions of the fourth kind is

$$d_6(f^{\vee}(x))$$

$$= \hat{T}_1 \frac{1}{\hat{T}^2(x)} \sum_{i=1}^n \left(\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i\hat{T}_{x_i}(x)) - \sum_{j=1, j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x)\right) dx_i.$$

186. The total iso-differential of the sixth kind of the iso-differentiable of the third kind iso-functions of the first kind is

$$d_6(\hat{f}^{\wedge}(\hat{x})) = \frac{1}{\hat{T}^2(x)} \hat{T}_1 \sum_{i=1}^n \frac{\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)} dx_i.$$

187. The total iso-differential of the sixth kind of the iso-differentiable of the third kind iso-functions of the second kind is

$$d_6(\hat{f}^{\wedge}(x))$$

$$= \frac{1}{\hat{T}^2(x)} \hat{T}_1 \sum_{i=1}^n \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x))\hat{T}(x) + \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)} dx_i.$$

188. The total iso-differential of the sixth kind of the iso-differentiable of the third kind iso-functions of the third kind is

$$d_6(\hat{f}(\hat{x}))$$

$$= \frac{1}{\hat{T}^3(x)} \hat{T}_1 \sum_{i=1}^n \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)) - \sum_{j=1, j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x) - f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)} dx_i.$$

189. The total iso-differential of the sixth kind of the iso-differentiable of the third kind iso-functions of the fourth kind is

$$d_6(f^{\wedge}(x))$$

$$= \hat{T}(x)\hat{T}_1 \sum_{i=1}^n \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x)) + \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)} dx_i.$$

190. The total iso-differential of the sixth kind of the iso-differentiable of the third kind iso-functions of the fifth kind is

$$d_6(f^{\vee}(x))$$

$$= \hat{T}_1 \frac{1}{\hat{T}^2(x)} \sum_{i=1}^{n} \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \hat{T}_{x_i}(x)) - \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x)}{\hat{T}(x) - x_i \partial_{x_i} \hat{T}(x)} dx_i.$$

191. The total iso-differential of the sixth kind of the iso-differentiable of the fourth kind iso-functions of the first kind is

$$d_6(\hat{f}^{\wedge}(\hat{x})) = \hat{T}_1 \sum_{i=1}^{n} \frac{\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}^3(x)} dx_i.$$

192. The total iso-differential of the sixth kind of the iso-differentiable of the fourth kind iso-functions of the second kind is

$$d_6(\hat{f}^{\wedge}(x))$$

$$= \hat{T}_1 \sum_{i=1}^{n} \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i}\hat{T}(x))\hat{T}(x) + \sum_{j=1, j\neq i}^{n} \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)}{\hat{T}^3(x)} dx_i.$$

193. The total iso-differential of the sixth kind of the iso-differentiable of the fourth kind iso-functions of the third kind is

$$d_6(\hat{f}(\hat{x}))$$

$$= \hat{T}_1 \sum_{i=1}^{n} \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \partial_{x_i}\hat{T}(x)) - \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x - f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}^4(x)} dx_i.$$

194. The total iso-differential of the sixth kind of the iso-differentiable of the fourth kind iso-functions of the fourth kind is

$$d_6(f^{\wedge}(x))$$

$$= \hat{T}_1 \sum_{i=1}^{n} \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i}\hat{T}(x)) + \sum_{j=1, j\neq i}^{n} \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)}{\hat{T}(x)} dx_i.$$

195. The total iso-differential of the sixth kind of the iso-differentiable of the fourth kind iso-functions of the fifth kind is

$$d_6(f^{\vee}(x))$$

$$= \hat{T}_1 \frac{1}{\hat{T}^3(x)} \sum_{i=1}^{n} \left(\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \hat{T}_{x_i}(x)) - \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x)\right) dx_i.$$

196. The total iso-differential of the sixth kind of the iso-differentiable of the fifth kind iso-functions of the first kind is

$$d_6(\hat{f}^{\wedge}(\hat{x})) = \hat{T}_1 \sum_{i=1}^{n} \frac{\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x) - x_i \partial_{x_i}\hat{T}(x)} dx_i.$$

197. The total iso-differential of the sixth kind of the iso-differentiable of the fifth kind iso-functions of the second kind is

$$d_6(\hat{f}^\wedge(x))$$

$$= \hat{T}_1 \sum_{i=1}^n \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))\hat{T}(x)+\sum_{j=1,j\neq i}^n \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\hat{T}(x)-f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)} dx_i.$$

198. The total iso-differential of the sixth kind of the iso-differentiable of the fifth kind iso-functions of the third kind is

$$d_6(\hat{f}(\hat{x}))$$

$$= \frac{1}{\hat{T}(x)} \hat{T}_1 \sum_{i=1}^n \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x))-\sum_{j=1,j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)-f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)} dx_i.$$

199. The total iso-differential of the sixth kind of the iso-differentiable of the fifth kind iso-functions of the fourth kind is

$$d_6(f^\wedge(x))$$

$$= \hat{T}^2(x)\hat{T}_1 \sum_{i=1}^n \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))+\sum_{j=1,j\neq i}^n \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)} dx_i.$$

200. The total iso-differential of the sixth kind of the iso-differentiable of the fifth kind iso-functions of the fifth kind is

$$d_6(f^\vee(x))$$

$$= \hat{T}_1 \sum_{i=1}^n \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\hat{T}_{x_i}(x))-\sum_{j=1,j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)} dx_i.$$

201. The total iso-differential of the sixth kind of the iso-differentiable of the sixth kind iso-functions of the first kind is

$$d_6(\hat{f}^\wedge(\hat{x})) = \hat{T}_1 \sum_{i=1}^n \frac{\partial_{x_i} f(x)\hat{T}(x)-f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x)} dx_i.$$

202. The total iso-differential of the sixth kind of the iso-differentiable of the sixth kind iso-functions of the second kind is

$$d_6(\hat{f}^\wedge(x))$$

$$= \hat{T}_1 \sum_{i=1}^n \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))\hat{T}(x)+\sum_{j=1,j\neq i}^n \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\hat{T}(x)-f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)}{\hat{T}(x)} dx_i.$$

203. The total iso-differential of the sixth kind of the iso-differentiable of the sixth kind iso-functions of the third kind is

$$d_6(\hat{f}(\hat{x}))$$

$$= \hat{T}_1 \sum_{i=1}^n \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x))-\sum_{j=1,j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)-f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}^2(x)} dx_i.$$

204. The total iso-differential of the sixth kind of the iso-differentiable of the sixth kind iso-functions of the fourth kind is

$$d_6(f^\wedge(x))$$

$$= \hat{T}(x)\hat{T}_1 \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x)) + \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\Big)dx_i.$$

205. The total iso-differential of the sixth kind of the iso-differentiable of the sixth kind iso-functions of the fifth kind is

$$d_6(f^\vee(x))$$

$$= \hat{T}_1 \frac{1}{\hat{T}(x)} \sum_{i=1}^n \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i\hat{T}_{x_i}(x)) - \sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big)x_j\hat{T}_{x_i}(x)\Big)dx_i.$$

206. The total iso-differential of the sixth kind of the iso-differentiable of the seventh kind iso-functions of the first kind is

$$d_6(\hat{f}^\wedge(\hat{x})) = \hat{T}_1 \sum_{i=1}^n \frac{\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

207. The total iso-differential of the sixth kind of the iso-differentiable of the seventh kind iso-functions of the second kind is

$$d_6(\hat{f}^\wedge(x))$$

$$= \hat{T}_1 \sum_{i=1}^n \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x))\hat{T}(x) + \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

208. The total iso-differential of the sixth kind of the iso-differentiable of the seventh kind iso-functions of the third kind is

$$d_6(\hat{f}(\hat{x}))$$

$$= \hat{T}_1 \sum_{i=1}^n \frac{\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)) - \sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big)x_j\hat{T}_{x_i}(x) - f\Big(\frac{x}{\hat{T}(x)}\Big)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x)(\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x))}dx_i.$$

209. The total iso-differential of the sixth kind of the iso-differentiable of the seventh kind iso-functions of the fourth kind is

$$d_6(f^\wedge(x))$$

$$= \hat{T}^2(x)\hat{T}_1 \sum_{i=1}^n \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x)) + \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

210. The total iso-differential of the sixth kind of the iso-differentiable of the seventh kind iso-functions of the fourth kind is

$$d_6(f^\vee(x))$$

$$= \hat{T}_1\hat{T}(x) \sum_{i=1}^n \frac{\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i\hat{T}_{x_i}(x)) - \sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big)x_j\hat{T}_{x_i}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

211. The total iso-differential of the seventh kind of the iso-differentiable of the first kind iso-functions of the first kind is

$$d_7(\hat{f}^\wedge(\hat{x})) = \frac{1}{\hat{T}(x)}\sum_{i=1}^n \frac{\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)} dx_i.$$

212. The total iso-differential of the seventh kind of the iso-differentiable of the first kind iso-functions of the second kind is

$$d_7(\hat{f}^\wedge(x))$$

$$= \frac{1}{\hat{T}(x)}\sum_{i=1}^n \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x))\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x) + \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\hat{T}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)} dx_i.$$

213. The total iso-differential of seventh kind of the iso-differentiable of the first kind iso-functions of the third kind is

$$d_7(\hat{f}(\hat{x}))$$

$$= \frac{1}{\hat{T}^2(x)}\sum_{i=1}^n \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)) - \sum_{j=1, j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x) - f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)} dx_i.$$

214. The total iso-differential of the seventh kind of the iso-differentiable of the first kind iso-functions of the fourth kind is

$$d_7(f^\wedge(x))$$

$$= \hat{T}(x)\sum_{i=1}^n \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x)) + \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)} dx_i.$$

215. The total iso-differential of the seventh kind of the iso-differentiable of the first kind iso-functions of the fifth kind is

$$d_7(f^\vee(x))$$

$$= \frac{1}{\hat{T}(x)}\sum_{i=1}^n \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i\hat{T}_{x_i}(x)) - \sum_{j=1, j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)} dx_i.$$

216. The total iso-differential of the seventh kind of the iso-differentiable of the second kind iso-functions of the first kind is

$$d_7(\hat{f}^\wedge(\hat{x})) = \sum_{i=1}^n \frac{\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}^2(x)} dx_i.$$

217. The total iso-differential of the seventh kind of the iso-differentiable of the second kind iso-functions of the second kind is

$$d_7(\hat{f}^\wedge(x))$$

$$= \sum_{i=1}^n \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x))\hat{T}(x)\sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)}{\hat{T}^2(x)} dx_i.$$

218. The total iso-differential of the seventh kind of the iso-differentiable of the second kind iso-functions of the third kind is

$$d_7(\hat{f}(\hat{x}))$$

$$= \sum_{i=1}^n \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x))-\sum_{j=1,j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)-f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}^3(x)}dx_i.$$

219. The total iso-differential of the seventh kind of the iso-differentiable of the second kind iso-functions of the fourth kind is

$$d_7(f^\wedge(x))$$

$$= \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))+\sum_{j=1,j\neq i}^n \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)dx_i.$$

220. The total iso-differential of the seventh kind of the iso-differentiable of the second kind iso-functions of the fourth kind is

$$d_7(f^\vee(x))$$

$$= \frac{1}{\hat{T}^2(x)}\sum_{i=1}^n \left(\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\hat{T}_{x_i}(x))-\sum_{j=1,j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)\right)dx_i.$$

221. The total iso-differential of the seventh kind of the iso-differentiable of the third kind iso-functions of the first kind is

$$d_7(\hat{f}^\wedge(\hat{x})) = \frac{1}{\hat{T}^2(x)}\sum_{i=1}^n \frac{\partial_{x_i} f(x)\hat{T}(x)-f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

222. The total iso-differential of the seventh kind of the iso-differentiable of the third kind iso-functions of the second kind is

$$d_7(\hat{f}^\wedge(x))$$

$$= \frac{1}{\hat{T}^2(x)}\sum_{i=1}^n \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))\hat{T}(x)+\sum_{j=1,j\neq i}^n \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\hat{T}(x)-f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

223. The total iso-differential of the seventh kind of the iso-differentiable of the third kind iso-functions of the third kind is

$$d_7(\hat{f}(\hat{x}))$$

$$= \frac{1}{\hat{T}^3(x)}\sum_{i=1}^n \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x))-\sum_{j=1,j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)-f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

224. The total iso-differential of the seventh kind of the iso-differentiable of the third kind iso-functions of the fourth kind is

$$d_7(f^\wedge(x))$$

$$= \hat{T}(x)\sum_{i=1}^n \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))+\sum_{j=1,j\neq i}^n \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

225. The total iso-differential of the seventh kind of the iso-differentiable of the third kind iso-functions of the fifth kind is

$$d_7(f^{\vee}(x))$$

$$= \frac{1}{\hat{T}^2(x)} \sum_{i=1}^{n} \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \hat{T}_{x_i}(x)) - \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x)}{\hat{T}(x) - x_i \partial_{x_i} \hat{T}(x)} dx_i.$$

226. The total iso-differential of the seventh kind of the iso-differentiable of the fourth kind iso-functions of the first kind is

$$d_7(\hat{f}^{\wedge}(\hat{x})) = \sum_{i=1}^{n} \frac{\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}^3(x)} dx_i.$$

227. The total iso-differential of the seventh kind of the iso-differentiable of the fourth kind iso-functions of the second kind is

$$d_7(\hat{f}^{\wedge}(x))$$

$$= \sum_{i=1}^{n} \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i}\hat{T}(x))\hat{T}(x) + \sum_{j=1, j\neq i}^{n} \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)}{\hat{T}^3(x)} dx_i.$$

228. The total iso-differential of the seventh kind of the iso-differentiable of the fourth kind iso-functions of the third kind is

$$d_7(\hat{f}(\hat{x}))$$

$$= \sum_{i=1}^{n} \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \partial_{x_i}\hat{T}(x)) - \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x - f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}^4(x)} dx_i.$$

229. The total iso-differential of the seventh kind of the iso-differentiable of the fourth kind iso-functions of the fourth kind is

$$d_7(f^{\wedge}(x))$$

$$= \sum_{i=1}^{n} \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i}\hat{T}(x)) + \sum_{j=1, j\neq i}^{n} \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)}{\hat{T}(x)} dx_i.$$

230. The total iso-differential of the seventh kind of the iso-differentiable of the fourth kind iso-functions of the fifth kind is

$$d_7(f^{\vee}(x))$$

$$= \frac{1}{\hat{T}^3(x)} \sum_{i=1}^{n} \left(\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \hat{T}_{x_i}(x)) - \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x)\right) dx_i.$$

231. The total iso-differential of the seventh kind of the iso-differentiable of the fifth kind iso-functions of the first kind is

$$d_7(\hat{f}^{\wedge}(\hat{x})) = \sum_{i=1}^{n} \frac{\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x) - x_i \partial_{x_i}\hat{T}(x)} dx_i.$$

232. The total iso-differential of the seventh kind of the iso-differentiable of the fifth kind iso-functions of the second kind is

$$d_7(\hat{f}^\wedge(x))$$

$$= \sum_{i=1}^n \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))\hat{T}(x)+\sum_{j=1,j\neq i}^n \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\hat{T}(x)-f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

233. The total iso-differential of the seventh kind of the iso-differentiable of the fifth kind iso-functions of the third kind is

$$d_7(\hat{f}(\hat{x}))$$

$$= \frac{1}{\hat{T}(x)}\sum_{i=1}^n \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x))-\sum_{j=1,j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)-f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

234. The total iso-differential of the seventh kind of the iso-differentiable of the fifth kind iso-functions of the fourth kind is

$$d_7(f^\wedge(x))$$

$$= \hat{T}^2(x)\sum_{i=1}^n \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))+\sum_{j=1,j\neq i}^n \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

235. The total iso-differential of the seventh kind of the iso-differentiable of the fifth kind iso-functions of the fifth kind is

$$d_7(f^\vee(x))$$

$$= \sum_{i=1}^n \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\hat{T}_{x_i}(x))-\sum_{j=1,j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

236. The total iso-differential of the seventh kind of the iso-differentiable of the sixth kind iso-functions of the first kind is

$$d_7(\hat{f}^\wedge(\hat{x})) = \sum_{i=1}^n \frac{\partial_{x_i} f(x)\hat{T}(x)-f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x)}dx_i.$$

237. The total iso-differential of the seventh kind of the iso-differentiable of the sixth kind iso-functions of the second kind is

$$d_7(\hat{f}^\wedge(x))$$

$$= \sum_{i=1}^n \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))\hat{T}(x)+\sum_{j=1,j\neq i}^n \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\hat{T}(x)-f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)}{\hat{T}(x)}dx_i.$$

238. The total iso-differential of the seventh kind of the iso-differentiable of the sixth kind iso-functions of the third kind is

$$d_7(\hat{f}(\hat{x}))$$

$$= \sum_{i=1}^n \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x))-\sum_{j=1,j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)-f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}^2(x)}dx_i.$$

239. The total iso-differential of the seventh kind of the iso-differentiable of the sixth kind iso-functions of the fourth kind is

$$d_7(f^\wedge(x))$$

$$= \hat{T}(x)\sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x)) + \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\Big)dx_i.$$

240. The total iso-differential of the seventh kind of the iso-differentiable of the sixth kind iso-functions of the fifth kind is

$$d_7(f^\vee(x))$$

$$= \frac{1}{\hat{T}(x)}\sum_{i=1}^n \Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x)-x_i\hat{T}_{x_i}(x)) - \sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big)x_j\hat{T}_{x_i}(x)\Big)dx_i.$$

241. The total iso-differential of the seventh kind of the iso-differentiable of the seventh kind iso-functions of the first kind is

$$d_7(\hat{f}^\wedge(\hat{x})) = \sum_{i=1}^n \frac{\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

242. The total iso-differential of the seventh kind of the iso-differentiable of the seventh kind iso-functions of the second kind is

$$d_7(\hat{f}^\wedge(x))$$

$$= \sum_{i=1}^n \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))\hat{T}(x) + \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

243. The total iso-differential of the seventh kind of the iso-differentiable of the seventh kind iso-functions of the third kind is

$$d_7(\hat{f}(\hat{x}))$$

$$= \sum_{i=1}^n \frac{\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)) - \sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big)x_j\hat{T}_{x_i}(x) - f\Big(\frac{x}{\hat{T}(x)}\Big)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x)(\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x))}dx_i.$$

244. The total iso-differential of the seventh kind of the iso-differentiable of the seventh kind iso-functions of the fourth kind is

$$d_7(f^\wedge(x))$$

$$= \hat{T}^2(x)\sum_{i=1}^n \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x)) + \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

245. The total iso-differential of the seventh kind of the iso-differentiable of the seventh kind iso-functions of the fourth kind is

$$d_7(f^\vee(x))$$

$$= \hat{T}(x)\sum_{i=1}^n \frac{\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x)-x_i\hat{T}_{x_i}(x)) - \sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big)x_j\hat{T}_{x_i}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

246. The total iso-differential of the eighth kind of the iso-differentiable of the first kind iso-functions of the first kind is

$$d_8(\hat{f}^\wedge(\hat{x})) = \sum_{i=1}^n \frac{\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)} dx_i.$$

247. The total iso-differential of the eighth kind of the iso-differentiable of the first kind iso-functions of the second kind is

$$d_8(\hat{f}^\wedge(x))$$

$$= \sum_{i=1}^n \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))\hat{T}(x) + \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)} dx_i.$$

248. The total iso-differential of the the eighth kind of the iso-differentiable of the first kind iso-functions of the third kind is

$$d_8(\hat{f}(\hat{x}))$$

$$= \sum_{i=1}^n \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)) - \sum_{j=1, j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x) - f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)} dx_i.$$

249. The total iso-differential of the eighth kind of the iso-differentiable of the first kind iso-functions of the fourth kind is

$$d_8(f^\wedge(x))$$

$$= \hat{T}^2(x)\sum_{i=1}^n \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x)) + \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)} dx_i.$$

250. The total iso-differential of the eighth kind of the iso-differentiable of the first kind iso-functions of the fifth kind is

$$d_8(f^\wedge(x))$$

$$= \sum_{i=1}^n \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i\hat{T}_{x_i}(x)) - \sum_{j=1, j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)}{\hat{T}(x) - x_i\partial_{x_i}\hat{T}(x)} dx_i.$$

251. The total iso-differential of the eighth kind of the iso-differentiable of the second kind iso-functions of the first kind is

$$d_8(\hat{f}^\wedge(\hat{x})) = \sum_{i=1}^n \frac{\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x)} dx_i.$$

252. The total iso-differential of the eighth kind of the iso-differentiable of the second kind iso-functions of the second kind is

$$d_8(\hat{f}^\wedge(x))$$

$$= \sum_{i=1}^n \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))\hat{T}(x) + \sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)}{\hat{T}(x)} dx_i.$$

253. The total iso-differential of the eighth kind of the iso-differentiable of the second kind iso-functions of the third kind is

$$d_8(\hat{f}(\hat{x})) = \sum_{i=1}^{n} \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)) - \sum_{j=1,j\neq i}^{n}\partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x) - f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}^2(x)} dx_i.$$

254. The total iso-differential of the eighth kind of the iso-differentiable of the second kind iso-functions of the fourth kind is

$$d_8(f^\wedge(x))$$

$$= \hat{T}(x)\sum_{i=1}^{n}\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x)) + \sum_{j=1,j\neq i}^{n}\partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)dx_i.$$

255. The total iso-differential of the eighth kind of the iso-differentiable of the second kind iso-functions of the fifth kind is

$$d_8(f^\wedge(x)) = \frac{1}{\hat{T}(x)}\sum_{i=1}^{n}\left(\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\hat{T}_{x_i}(x))\right.$$

$$\left.-\sum_{j=1,j\neq i}^{n}\partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)\right)dx_i.$$

256. The total iso-differential of the eighth kind of the iso-differentiable of the third kind iso-functions of the first kind is

$$d_8(\hat{f}^\wedge(\hat{x})) = \frac{1}{\hat{T}(x)}\sum_{i=1}^{n}\frac{\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

257. The total iso-differential of the eighth kind of the iso-differentiable of the third kind iso-functions of the second kind is

$$d_8(\hat{f}^\wedge(x))$$

$$= \sum_{i=1}^{n}\frac{1}{\hat{T}(x)}\frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))\hat{T}(x) + \sum_{j=1,j\neq i}^{n}\partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

258. The total iso-differential of the eighth kind of the iso-differentiable of the third kind iso-functions of the third kind is

$$d_8(\hat{f}(\hat{x}))$$

$$= \frac{1}{\hat{T}^2(x)}\sum_{i=1}^{n}\frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)) - \sum_{j=1,j\neq i}^{n}\partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x) - f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

259. The total iso-differential of the eighth kind of the iso-differentiable of the third kind iso-functions of the fourth kind is

$$d_8(f^\wedge(x))$$

$$= \hat{T}^2(x)\sum_{i=1}^{n}\frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x)) + \sum_{j=1,j\neq i}^{n}\partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

260. The total iso-differential of the eighth kind of the iso-differentiable of the third kind iso-functions of the fifth kind is

$$d_8(f^{\wedge}(x))$$

$$= \frac{1}{\hat{T}(x)} \sum_{i=1}^{n} \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \hat{T}_{x_i}(x)) - \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x)}{\hat{T}(x) - x_i \partial_{x_i} \hat{T}(x)} dx_i.$$

261. The total iso-differential of the eighth kind of the iso-differentiable of the fourth kind iso-functions of the first kind is

$$d_8(\hat{f}^{\wedge}(\hat{x})) = \sum_{i=1}^{n} \frac{\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}^2(x)} dx_i.$$

262. The total iso-differential of the eighth kind of the iso-differentiable of the fourth kind iso-functions of the second kind is

$$d_8(\hat{f}^{\wedge}(x))$$

$$= \sum_{i=1}^{n} \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i}\hat{T}(x))\hat{T}(x) + \sum_{j=1, j\neq i}^{n} \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)}{\hat{T}^2(x)} dx_i.$$

263. The total iso-differential of the eighth kind of the iso-differentiable of the fourth kind iso-functions of the third kind is

$$d_8(\hat{f}(\hat{x}))$$

$$= \sum_{i=1}^{n} \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \partial_{x_i}\hat{T}(x)) - \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x) - f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}^3(x)} dx_i.$$

264. The total iso-differential of the eighth kind of the iso-differentiable of the fourth kind iso-functions of the fourth kind is

$$d_8(f^{\wedge}(x))$$

$$= \sum_{i=1}^{n} \left(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i}\hat{T}(x)) + \sum_{j=1, j\neq i}^{n} \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x)\right) dx_i.$$

265. The total iso-differential of the eighth kind of the iso-differentiable of the fourth kind iso-functions of the fifth kind is

$$d_8(f^{\wedge}(x))$$

$$= \frac{1}{\hat{T}^2(x)} \sum_{i=1}^{n} \left(\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \hat{T}_{x_i}(x)) - \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x)\right) dx_i.$$

266. The total iso-differential of the eighth kind of the iso-differentiable of the fifth kind iso-functions of the first kind is

$$d_8(\hat{f}^{\wedge}(\hat{x})) = \hat{T}(x) \sum_{i=1}^{n} \frac{\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x) - x_i \partial_{x_i}\hat{T}(x)} dx_i.$$

267. The total iso-differential of the eighth kind of the iso-differentiable of the fifth kind iso-functions of the second kind is

$$d_8(\hat{f}^\wedge(x))$$

$$= \hat{T}(x)\sum_{i=1}^n \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))\hat{T}(x)+\sum_{j=1,j\neq i}^n \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)-f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

268. The total iso-differential of the eighth kind of the iso-differentiable of the fifth kind iso-functions of the third kind is

$$d_8(\hat{f}(\hat{x}))$$

$$= \sum_{i=1}^n \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x))-\sum_{j=1,j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)-f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

269. The total iso-differential of the eighth kind of the iso-differentiable of the fifth kind iso-functions of the fourth kind is

$$d_8(f^\wedge(x))$$

$$= \hat{T}^3(x)\sum_{i=1}^n \frac{\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))+\sum_{j=1,j\neq i}^n \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

270. The total iso-differential of the eighth kind of the iso-differentiable of the fifth kind iso-functions of the fifth kind is

$$d_8(f^\wedge(x))$$

$$= \hat{T}(x)\sum_{i=1}^n \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\hat{T}_{x_i}(x))-\sum_{j=1,j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)}{\hat{T}(x)-x_1\partial_{x_i}\hat{T}(x)}dx_i.$$

271. The total iso-differential of the eighth kind of the iso-differentiable of the sixth kind iso-functions of the first kind is

$$d_8(\hat{f}^\wedge(\hat{x})) = \sum_{i=1}^n \Big(\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)\Big)dx_i.$$

272. The total iso-differential of the eighth kind of the iso-differentiable of the sixth kind iso-functions of the second kind is

$$d_8(\hat{f}^\wedge(x)) = \sum_{i=1}^n \Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))\hat{T}(x)$$

$$+\sum_{j=1,j\neq i}^n \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\hat{T}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)\Big)dx_i.$$

273. The total iso-differential of the eighth kind of the iso-differentiable of the sixth kind iso-functions of the third kind is

$$d_8(\hat{f}(\hat{x}))$$

$$= \sum_{i=1}^n \frac{\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x))-\sum_{j=1,j\neq i}^n \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)-f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x)}dx_i.$$

274. The total iso-differential of the eighth kind of the iso-differentiable of the sixth kind iso-functions of the fourth kind is

$$d_8(f^\wedge(x)) = \hat{T}^2(x)\sum_{i=1}^n\Big(\partial_{x_i}f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))$$

$$+\sum_{j=1,j\neq i}^n\partial_{x_j}f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\Big)dx_i.$$

275. The total iso-differential of the eighth kind of the iso-differentiable of the sixth kind iso-functions of the fifth kind is

$$d_8(f^\wedge(x))$$

$$=\sum_{i=1}^n\left(\partial_{x_i}f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\hat{T}_{x_i}(x))-\sum_{j=1,j\neq i}^n\partial_{x_j}f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)\right)dx_i.$$

276. The total iso-differential of the eighth kind of the iso-differentiable of the seventh kind iso-functions of the first kind is

$$d_8(\hat{f}^\wedge(\hat{x})) = \hat{T}(x)\sum_{i=1}^n\frac{\partial_{x_i}f(x)\hat{T}(x)-f(x)\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

277. The total iso-differential of the eighth kind of the iso-differentiable of the seventh kind iso-functions of the second kind is

$$d_8(\hat{f}^\wedge(x))$$

$$=\hat{T}(x)\sum_{i=1}^n\frac{\partial_{x_i}f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))\hat{T}(x)-f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)+\sum_{j=1,j\neq i}^n\partial_{x_j}f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

278. The total iso-differential of the eighth kind of the iso-differentiable of the seventh kind iso-functions of the third kind is

$$d_8(\hat{f}(\hat{x}))$$

$$=\sum_{i=1}^n\frac{\partial_{x_i}f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x))-\sum_{j=1,j\neq i}^n\partial_{x_j}f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)-f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}(x)\partial_{x_i}\hat{T}(x)}{(\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x))}dx_i.$$

279. The total iso-differential of the eighth kind of the iso-differentiable of the seventh kind iso-functions of the fourth kind is

$$d_8(f^\wedge(x))$$

$$=\hat{T}^3(x)\sum_{i=1}^n\frac{\partial_{x_i}f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))+\sum_{j=1,j\neq i}^n\partial_{x_j}f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

280. The total iso-differential of the eighth kind of the iso-differentiable of the seventh kind iso-functions of the fifth kind is

$$d_8(f^\wedge(x))$$

$$=\hat{T}^2(x)\sum_{i=1}^n\frac{\partial_{x_i}f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\hat{T}_{x_i}(x))-\sum_{j=1,j\neq i}^n\partial_{x_j}f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)}{\hat{T}(x)-x_i\partial_{x_i}\hat{T}(x)}dx_i.$$

Definition 1.5.39. *The second order total iso-differential of the (i,j)-kind of an iso-function $\hat{f}$ is defined as follows*

$$d_i^{\wedge}(d_j(\hat{f})) = d_i(\hat{T}d_j(\hat{f})), \qquad i,j = 1,2,\ldots,8.$$

The third order total iso-differential of the (l,i,j)-kind of an iso-function $\hat{f}$ is defined as follows

$$d_l^{\wedge}(d_i^{\wedge}(d_j(\hat{f}))) = d_l(\hat{T}d_i(\hat{T}d_j(\hat{f}))), \qquad l,i,j = 1,2,\ldots,8$$

and so on.

Exercise 1.5.40. *Let $\hat{f},\hat{g} : D \longrightarrow \mathbb{R}$ be iso-functions of the first, the second, the third, the fourth or the fifth kind, which are iso-differentiable at $x \in D$ of the first, the second, the third, the fourth, the fifth, the sixth or the seventh kind. Let also, $a \in \mathbb{R}$, $\hat{a} \in \hat{F}_{\mathbb{R}}$. Prove*

1. $d_j(\hat{f}(x) \pm \hat{g}(x)\Big) = d_j(\hat{f}(x)) \pm d_j(\hat{g}(x))_{x_i}^{j\circledast}$.

2. $d_j(\hat{a}\hat{\times}\hat{f}(x)) = \hat{a}\hat{\times}d_j(\hat{f}(x))$.

3. $d_j(\hat{a}\hat{f}(x)) = \hat{a}d_j(\hat{f}(x))$.

4. $d_j(a\hat{\times}\hat{f}(x)) = a\hat{\times}d_j(\hat{f}(x))$.

5. $d_j(a\hat{f}(x)) = ad_j(\hat{f}(x))$.

6. $d_j(\hat{f}(x)\hat{\times}\hat{g}(x)) = d_j(\hat{f}(x))\hat{\times}\hat{g}(x) + \hat{f}(x)\hat{\times}d_j(\hat{g}(x))$.

7. $d_j(\hat{f}(x)\hat{g}(x)) = d_j(\hat{f}(x))\hat{g}(x) + \hat{f}(x)d_j(\hat{g}(x))$.

8 $d_j\Big(\hat{f}(x) \nearrow \hat{g}(x)\Big) = \Big(d_j(\hat{f}(x))\hat{g}(x) - \hat{f}(x)d_j(\hat{g}(x))\Big) \nearrow \hat{g}^2(x)$.

9. $d_j\left(\frac{\hat{f}(x)}{\hat{g}(x)}\right) = \frac{d_j(\hat{f}(x))\hat{g}(x) - \hat{f}(x)d_j(\hat{g}(x))}{\hat{g}^2(x)}, \qquad j = 1,\ldots,8, i = 1,\ldots,n.$

Remark 1.5.41. *The iso- derivatives of the first, the second, the third, the fourth, the fifth, the sixth or the seventh kind of the iso-composite iso-functions of the first, the second, the third, the fourth or the fifth kind can be computed using the definition of the iso-composite iso-functions, the iso-derivatives and the rules for computation of the derivatives of composite functions.*

Definition 1.5.42. *Let $\hat{f} : D \longrightarrow \mathbb{R}$ be an iso-function of the first, the second, the third, the fourth or the fifth kind, which is iso-differentiable of the first, the second, the third, the fourth, the fifth, the sixth or the seventh kind at $x \in D$. Let also, $\hat{Y} = (\hat{y}_1,\hat{y}_2,\ldots,\hat{y}_n) \in \hat{F}_{\mathbb{R}^n}$. Then the directional iso-derivative of $\hat{f}$ is defined as follows*

$$\hat{\partial}\hat{f}(x) \nearrow \hat{\partial}\hat{Y} = \sum_{i=1}^n (\hat{f}(x))_{x_i}^{j\circledast}\hat{\times}\hat{y}_i \qquad \text{or}$$

$$\frac{\hat{\partial}\hat{f}(x)}{\hat{\partial}\hat{Y}} = \sum_{i=1}^n (\hat{f}(x))_{x_i}^{j\circledast}\hat{y}_i, \qquad j = 1,2,\ldots,7.$$

Definition 1.5.43. *Let $\hat{f} : D \longrightarrow \mathbb{R}$ be an iso-function of the first, the second, the third, the fourth or the fifth kind, which is iso-differentiable of the first, the second, the third, the fourth, the fifth, the sixth or the seventh kind at $x \in D$. Then the iso-gradient of $\hat{f}$ of the j-th kind, $j = 1, 2, \dots 7$, is defined as follows*

$$\hat{\nabla}^j \hat{f}(x) = \Big((\hat{f})^{j\circledast}_{x_1}, (\hat{f})^{j\circledast}_{x_2}, \dots, (\hat{f})^{j\circledast}_{x_n} \Big).$$

Homogeneous iso-functions

Let $D \subset \mathbb{R}^n$ and $\hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$ for every $x \in D$.

Definition 1.5.44. *An iso-function of the first, the second, the third, the fourth or the fifth kind, defined on $D \subset \mathbb{R}^n$, will be called a homogeneous iso-function of degree n at the point $x^0 \in D$ if its iso-original is a homogeneous function of degree n at the point x^0.*

Theorem 1.5.45. *Let $\hat{f}^{\wedge\wedge}$ is defined on D, f is homogeneous of degree n at the point $x \in D$, $\hat{T}$ is homogeneous of degree m at the point $x \in D$. Then $\hat{f}^{\wedge\wedge}$ is homogeneous of degree $n - m$.*

Proof. Let t belongs to an enough small neighborhood of 1. Then

$$\frac{f(tx)}{\hat{T}(tx)} = \frac{t^n f(x)}{t^m \hat{T}(x)} = t^{n-m} \frac{f(x)}{\hat{T}(x)}.$$

$\square$

Corollary 1.5.46. *In addition, if f and $\hat{T}$ are differentiable at x, then we have the following iso- Euler equality*

$$\sum_{i=1}^{n} x_i \partial_{x_i} \Big(\frac{f(x)}{\hat{T}(x)} \Big) = (n - m) \frac{f(x)}{\hat{T}(x)}.$$

Theorem 1.5.47. *Let $\hat{f}^{\wedge}$ is defined on D, f is homogeneous of degree n at the point $x \in D$, $\hat{T}$ is homogeneous of degree m at the point $x \in D$. Then $\hat{f}^{\wedge}$ is homogeneous of degree $m(n-1)+n$.*

Proof. Let t belongs to an enough small neighbourhood of 1. Then

$$\frac{f(tx\hat{T}(tx))}{\hat{T}(tx)} = \frac{f(t^{m+1}x\hat{T}(x)}{t^m \hat{T}(x)} = \frac{t^{n(m+1)} f(x\hat{T}(x))}{t^m \hat{T}(x)} = t^{m(n-1)+n} \frac{f(x\hat{T}(x))}{\hat{T}(x)}.$$

$\square$

Corollary 1.5.48. *In addition, if f and $\hat{T}$ are differentiable at x, then we have the following iso- Euler equality*

$$\sum_{i=1}^{n} x_i \partial_{x_i} \Big(\frac{f(x\hat{T}(x))}{\hat{T}(x)} \Big) = (m(n-1)+n) \frac{f\Big(\frac{x}{\hat{T}(x)}\Big)}{\hat{T}(x)}.$$

Theorem 1.5.49. *Let $\hat{\hat{f}}$ is defined on D, f is homogeneous of degree n at the point $x \in D$, $\hat{T}$ is homogeneous of degree m at the point $x \in D$. Then $\hat{\hat{f}}$ is homogeneous of degree $-(n+1)m+n$.*

Proof. Let t belongs to an enough small neighbourhood of 1. Then

$$\frac{f\left(\frac{tx}{\hat{T}(tx)}\right)}{\hat{T}(tx)} = \frac{f\left(\frac{tx}{t^m\hat{T}(x)}\right)}{t^m\hat{T}(x)} = \frac{f\left(t^{1-m}\frac{x}{\hat{T}(x)}\right)}{t^m\hat{T}(x)} = \frac{t^{n(1-m)}}{t^m}\frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)} = t^{-(n+1)m+n}\frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)}.$$

□

Corollary 1.5.50. *In addition, if f and $\hat{T}$ are differentiable at x, then we have the following iso- Euler equality*

$$\sum_{i=1}^{n} x_i\partial_{x_i}\left(\frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)}\right) = (-(n+1)m+n)\frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)}.$$

Theorem 1.5.51. *Let $f^{\wedge}$ is defined on D, f is homogeneous of degree n at the point $x \in D$, $\hat{T}$ is homogeneous of degree m at the point $x \in D$. Then $f^{\wedge}$ is homogeneous of degree $(m+1)n$.*

Proof. Let t belongs to an enough small neighbourhood of 1. Then

$$f(tx\hat{T}(tx)) = f(t^{m+1}x\hat{T}(x)) = t^{(m+1)n}f(x\hat{T}(x)).$$

□

Corollary 1.5.52. *In addition, if f and $\hat{T}$ are differentiable at x, then we have the following iso- Euler equality*

$$\sum_{i=1}^{n} x_i\partial_{x_i}(f(x\hat{T}(x))) = (m+1)nf(x\hat{T}(x)).$$

Theorem 1.5.53. *Let $f^{\vee}$ is defined on D, f is homogeneous of degree n at the point $x \in D$, $\hat{T}$ is homogeneous of degree m at the point $x \in D$. Then $f^{\vee}$ is homogeneous of degree $n(1-m)$.*

Proof. Let t belongs to an enough small neighbourhood of 1. Then

$$f\left(\frac{tx}{\hat{T}(tx)}\right) = f\left(\frac{tx}{t^m\hat{T}(x)}\right) = f\left(t^{1-m}\frac{x}{\hat{T}(x)}\right) = t^{n(1-m)}f\left(\frac{x}{\hat{T}(x)}\right).$$

□

Corollary 1.5.54. *In addition, if f and $\hat{T}$ are differentiable at x, then we have the following iso- Euler equality*

$$\sum_{i=1}^{n} x_i\partial_{x_i}\left(f\left(\frac{x}{\hat{T}(x)}\right)\right) = n(1-m)f\left(\frac{x}{\hat{T}(x)}\right).$$

1.6. Minima and Maxima of Iso-Functions of n Iso-Variables

Let $D \subset \mathbb{R}^n$ and $\hat{T} : D \longrightarrow \mathbb{R}$, $\hat{T}(x) > 0$ for every $x \in D$, and $x^0 \in D$.

Definition 1.6.1. *We will say that the iso-point $\hat{x} \in \hat{F}_{\mathbb{R}^n}$ is a local extreme iso-point of the iso-function $\hat{f}$ of the first, the second, the third, the fourth or the fifth kind if the point x is a local extreme point of its iso-original $\tilde{f}$.*

For $x \in D$ we introduce the following quantities.

$$A_i(x) = \frac{1}{\hat{T}^2(x)}\Big(\partial_{x_i} f(x)\hat{T}(x) - f(x)\partial_{x_i}\hat{T}(x)\Big),$$

$$B_i(x) = \frac{1}{\hat{T}^2(x)}\Big(\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x))\hat{T}(x)$$

$$+\sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x))x_j\hat{T}_{x_i}(x) - f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)\Big),$$

$$C_i(x) = \frac{1}{\hat{T}^3(x)}\Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i\hat{T}_{x_i}(x))$$

$$-\sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big)x_j\hat{T}_{x_i}(x) - f\Big(\frac{x}{\hat{T}(x)}\Big)\partial_{x_i}\hat{T}(x)\hat{T}(x)\Big),$$

$$D_i(x) = \partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\hat{T}_{x_i}(x))$$

$$+\sum_{j=1, j\neq i}^n \partial_{x_j} f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x),$$

$$E_i(x) = \frac{1}{\hat{T}^2(x)}\Big(\partial_{x_i} f\Big(\frac{x}{\hat{T}(x)}\Big)(\hat{T}(x) - x_i\hat{T}_{x_i}(x))$$

$$-\sum_{j=1, j\neq i}^n \partial_{x_j} f\Big(\frac{x}{\hat{T}(x)}\Big)x_j\hat{T}_{x_i}(x)\Big), \qquad i = 1, 2, \ldots, n.$$

In fact, we have

$$A_i(x) = \partial_{x_i}\hat{f}^{\wedge}(\hat{x}), \qquad B_i(x) = \partial_{x_i}\hat{f}^{\wedge}(x), \qquad C_i(x) = \partial_{x_i}\hat{f}(\hat{x}),$$

$$D_i(x) = \partial_{x_i} f^{\wedge}(x), \qquad E_i(x) = \partial_{x_i} f^{\vee}(x), \qquad x \in D, \qquad i = 1, 2, \ldots, n.$$

Theorem 1.6.2. *Let $f, \hat{T} : D \longrightarrow \mathbb{R}$ be differentiable functions at $x^0 \in D$ and x^0 is a local extreme point of $\hat{f}^{\wedge\wedge}$. Then*

$$f_{x_i}(x^0)\hat{T}(x^0) = f(x^0)\hat{T}_{x_i}(x^0), \qquad i = 1, 2, \ldots, n.$$

Proof. Since x^0 is a local extreme point of $\hat{f}^{\wedge\wedge}$ then x^0 is a local extreme point of the function $\frac{f(x)}{\hat{T}(x)}$. Because f and $\hat{T}$ are differentiable at x_0 and $\hat{T}(x) > 0$ for every $x \in D$, then $\frac{f(x)}{\hat{T}(x)}$ is a differentiable function at x_0. From here, using that x^0 is a local extreme point of

$\frac{f(x)}{\hat{T}(x)}$, we get

$$A_i(x^0) = 0 \qquad \Longleftrightarrow \frac{f_{x_i}(x^0)\hat{T}(x^0) - f(x^0)\hat{T}_{x_i}(x^0)}{\hat{T}^2(x^0)} = 0 \qquad \Longleftrightarrow$$

$$f_{x_i}(x^0)\hat{T}(x^0) = f(x^0)\hat{T}_{x_i}(x^0), \qquad i = 1, 2, \ldots, n.$$

$\square$

Theorem 1.6.3. *Let $f, \hat{T} : D \longrightarrow \mathbb{R}$ be differentiable functions at x^0 and $x^0\hat{T}(x_0) \in D$, respectively. Let also, x^0 is a local extreme point of $\hat{f}^{\wedge}$. Then*

$$\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x))\hat{T}(x) + \sum_{j=1, j\neq i}^{n} \partial_{x_j} f(x\hat{T}(x))x_j\hat{T}_{x_i}(x)$$

$$= f(x\hat{T}(x))\partial_{x_i}\hat{T}(x), \qquad i = 1, 2, \ldots, n.$$

Proof. Since x^0 is a local extreme point of $\hat{f}^{\wedge}$ then x^0 is a local extreme point of $\frac{f(x\hat{T}(x))}{\hat{T}(x)}$. Because f is differentiable at $x^0\hat{T}(x^0)$ and $\hat{T}$ is differentiable at x^0, and $\hat{T}(x) > 0$ for every $x \in D$, then $\frac{f(x\hat{T}(x))}{\hat{T}(x)}$ is differentiable at x^0. From here, using that x^0 is a local extreme point of $\frac{f(x\hat{T}(x))}{\hat{T}(x)}$, we get

$$B_i(x^0) = 0 \qquad \Longleftrightarrow$$

$$\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i\partial_{x_i}\hat{T}(x))\hat{T}(x) + \sum_{j=1, j\neq i}^{n} \partial_{x_j} f(x\hat{T}(x))x_j\hat{T}_{x_i}(x)$$

$$-f(x\hat{T}(x))\partial_{x_i}\hat{T}(x) = 0, \qquad i = 1, 2, \ldots, n.$$

$\square$

Theorem 1.6.4. *Let $f, \hat{T} : D \longrightarrow \mathbb{R}$ be differentiable at x^0 and $\frac{x^0}{\hat{T}(x^0)}$, respectively. Let also, x^0 is a local extreme point of $\hat{\hat{f}}$. Then*

$$\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i\hat{T}_{x_i}(x)) - \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)$$

$$= f\left(\frac{x}{\hat{T}(x)}\right)\partial_{x_i}\hat{T}(x)\hat{T}(x), \qquad i = 1, 2, \ldots, n.$$

Proof. Since x^0 is a local extreme point of $\hat{\hat{f}}$ then x^0 is a local extreme point of $\frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)}$. Because f is a differentiable function at $\frac{x^0}{\hat{T}(x^0)}$ and $\hat{T}$ is a differentiable function at x^0, and $\hat{T}(x^0) > 0$, then the function $\frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)}$ is a differentiable function at x_0. Using that x^0 is a local extreme point of $\frac{f\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)}$, we get

$$C_i(x^0) = 0 \qquad \Longleftrightarrow$$

$$\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \hat{T}_{x_i}(x)) - \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x)$$

$$-f\left(\frac{x}{\hat{T}(x)}\right)\partial_{x_i}\hat{T}(x)\hat{T}(x) = 0, \qquad i = 1, 2, \ldots, n.$$

□

Theorem 1.6.5. *Let $f, \hat{T} : D \longrightarrow \mathbb{R}$ be differentiable functions at x^0 and $x^0\hat{T}(x^0)$, respectively. Let also x^0 is a local extreme point of $f^{\wedge}$. Then*

$$\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \hat{T}_{x_i}(x)) = -\sum_{j=1, j\neq i}^{n} \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x),$$

$$i = 1, 2, \ldots, n.$$

Proof. Since x^0 is a local extreme point of $f^{\wedge}$ then x^0 is a local extreme point of $f(x\hat{T}(x))$. Therefore

$$D_i(x^0) = 0 \qquad \Longleftrightarrow$$

$$\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \hat{T}_{x_i}(x)) + \sum_{j=1, j\neq i}^{n} \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x) = 0, \qquad i = 1, 2, \ldots, n.$$

□

Theorem 1.6.6. *Let $f, \hat{T} : D \longrightarrow \mathbb{R}$ be differentiable functions at x^0 and $\frac{x^0}{\hat{T}(x^0)}$, respectively. Let also x^0 is a local extreme point of $f^{\vee}$. Then*

$$\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \hat{T}_{x_i}(x)) = \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x), \qquad i = 1, 2, \ldots, n.$$

Proof. Since x^0 is a local extreme point of $f^{\vee}$ then x^0 is a local extreme point of $f\left(\frac{x}{\hat{T}(x)}\right)$. Therefore

$$E_i(x^0) = 0 \qquad \Longleftrightarrow$$

$$\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \hat{T}_{x_i}(x)) - \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x) = 0, \qquad i = 1, 2, \ldots, n.$$

□

Remark 1.6.7. *If $f, \hat{T} : D \longrightarrow \mathbb{R}$ are twice differentiable function at $x \in D$, we introduce the following quantities*

$$A_{ij}(x) = \partial_{x_j} A_i(x), \qquad B_{ij}(x) = \partial_{x_j} B_i(x), \qquad C_{ij}(x) = \partial_{x_j} C_i(x),$$

$$D_{ij}(x) = \partial_{x_j} D_i(x), \qquad E_{ij}(x) = \partial_{x_j} E_i(x), \qquad i, j = 1, 2, \ldots, n.$$

Using some basic facts concerning the local extreme of the real-valued functions one can prove the following theorems.

Theorem 1.6.8. *Let $f, \hat{T} : D \longrightarrow \mathbb{R}$ be twice continuously-differentiable functions at $x^0 \in D$ and*

$$f_{x_i}(x^0)\hat{T}(x^0) = f(x^0)\hat{T}_{x_i}(x^0), \qquad i = 1, 2, \ldots, n.$$

If

$$\sum_{ij=1}^{n} A_{ij}(x^0) dx_i dx_j$$

is a positive(negative) definite quadratic form, then x^0 is a local minimum(maximum) point of $\hat{f}^{\wedge\wedge}$.

Theorem 1.6.9. *Let $f, \hat{T} : D \longrightarrow \mathbb{R}$ be twice continuously-differentiable functions at x^0 and $x^0\hat{T}(x_0) \in D$, respectively. Let also,*

$$\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \partial_{x_i}\hat{T}(x))\hat{T}(x) + \sum_{j=1, j\neq i}^{n} \partial_{x_j} f(x\hat{T}(x)) x_j \hat{T}_{x_i}(x)$$

$$= f(x\hat{T}(x))\partial_{x_i}\hat{T}(x), \qquad i = 1, 2, \ldots, n.$$

If

$$\sum_{i,j=1}^{n} B_{ij}(x^0) dx_i dx_j$$

is a positive(negative) definite quadratic form, then x^0 is a local minimum(maximum) point of $\hat{f}^{\wedge}$.

Theorem 1.6.10. *Let $f, \hat{T} : D \longrightarrow \mathbb{R}$ be twice continuously-differentiable at x^0 and $\frac{x^0}{\hat{T}(x^0)}$, respectively. Let also,*

$$\partial_{x_i} f\left(\tfrac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \hat{T}_{x_i}(x)) - \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\left(\tfrac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x)$$

$$= f\left(\tfrac{x}{\hat{T}(x)}\right)\partial_{x_i}\hat{T}(x)\hat{T}(x), \qquad i = 1, 2, \ldots, n.$$

If

$$\sum_{ij=1}^{n} C_{ij}(x^0) dx_i dx_j$$

is a positive(negative) definite quadratic form, then then x^0 is a local minimum(maximum) point of $\hat{\hat{f}}$.

Theorem 1.6.11. *Let $f, \hat{T} : D \longrightarrow \mathbb{R}$ be twice continuously-differentiable functions at x^0 and $x^0\hat{T}(x^0)$, respectively. Let also,*

$$\partial_{x_i} f(x\hat{T}(x))(\hat{T}(x) + x_i \hat{T}_{x_i}(x)) = -\sum_{j=1, j\neq i}^{n} \partial_{x_j} f(x\hat{T}(x)) x_j \partial_{x_i}\hat{T}(x),$$

$$i = 1, 2, \ldots, n.$$

If

$$\sum_{ij=1}^{n} D_{ij}(x^0)dx_i dx_j$$

is a positive(negative) definite quadratic form, then then x^0 is a local minimum(maximum) point of $\hat{f}^{\wedge}$.

Theorem 1.6.12. *Let $f, \hat{T} : D \longrightarrow \mathbb{R}$ be twice continuously-differentiable functions at x^0 and $\frac{x^0}{\hat{T}(x^0)}$, respectively. Let also,*

$$\partial_{x_i} f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x) - x_i \hat{T}_{x_i}(x)) = \sum_{j=1, j\neq i}^{n} \partial_{x_j} f\left(\frac{x}{\hat{T}(x)}\right) x_j \hat{T}_{x_i}(x), \qquad i = 1, 2, \ldots, n.$$

If

$$\sum_{ij=1}^{n} E_{ij}(x^0)dx_i dx_j$$

is a positive(negative) definite quadratic form, then then x^0 is a local minimum(maximum) point of $\hat{f}^{\vee}$.

Now we will use the following notations $x = (x_1, x_2, \ldots, x_l)$, $y = (y_1, y_2, \ldots, y_m) = (x_{m+1}, x_{m+2}, \ldots, x_n)$, $l + m = n$. We fix a point $(x^0, y^0) \in D$. We put

$$\tilde{D}_1 = \{(x,y) \in D : x_i^0 - a_i \leq x_i \leq x_i^0 + a_i,$$

$$y_j^0 - b_j \leq y_j \leq y_j^0 + b_j, \quad i = 1, 2, \ldots, l, j = 1, 2, \ldots, m\},$$

where a_i, b_j, $i = 1, 2, \ldots, l$, $j = 1, 2, \ldots, m$ are enough small positive constants.

We suppose that the iso-function $\hat{f}$ of the first, the second, the third, the fourth or the fifth kind, and the functions G_i, $i = 1, 2, \ldots, m$, are defined and continuously-differentiable on $\tilde{D}_1$. We introduce the set

$$\tilde{D}_2 = \{(x,y) \in D_1 : G_i(x,y) = 0, \quad i = 1, 2, \ldots, m\}.$$

We assume that (x^0, y^0), $(x^0\hat{T}(x^0,y^0), y^0\hat{T}(x^0, hy^0))$, $\left(\frac{x^0}{\hat{T}(x^0,y^0)}, \frac{y^0}{\hat{T}(x^0,hy^0)}\right) \in \tilde{D}_2$, and

$$G_i(x^0, y^0) = 0, \qquad i = 1, 2, \ldots, m,$$

$$(A7) \quad \frac{\partial(G_1, G_2, \ldots, G_m)}{\partial(y_1, y_2, \ldots, y_m)} = \det \begin{pmatrix} \frac{\partial G_1}{\partial y_1} & \frac{\partial G_1}{\partial y_2} & \cdots & \frac{\partial G_1}{\partial y_m} \\ \frac{\partial G_2}{\partial y_1} & \frac{\partial G_2}{\partial y_2} & \cdots & \frac{\partial G_2}{\partial y_m} \\ \cdots & & & \\ \frac{\partial G_m}{\partial y_1} & \frac{\partial G_m}{\partial y_2} & \cdots & \frac{\partial G_m}{\partial y_m} \end{pmatrix} (x^0, y^0) \neq 0.$$

Let (x^0, y^0) is a local extreme point of $\hat{f}$. We define the function

$$\phi(x,y) = \hat{f}(x,y) + \sum_{i=1}^{n} \lambda_i G_i(x,y).$$

The iso- Lagrange multipliers are the constants λ_i, $i = 1, 2, \ldots, m$. Then the iso-Lagrange multipliers can be determined by the following system

1. (in the case when $\hat{f}$ is an iso-function of the first kind)

$$\begin{cases} \frac{f_{x_i}(x^0,y^0)\hat{T}(x^0,y^0)-f(x^0,y^0)\hat{T}_{x_i}(x^0,y^0)}{\hat{T}^2(x^0,y^0)}+\sum_{k=1}^m \lambda_k G_{kx_i}(x^0,y^0)=0, \qquad i=1,2,\ldots,l, \\ \frac{f_{y_j}(x^0,y^0)\hat{T}(x^0,y^0)-f(x^0,y^0)\hat{T}_{y_j}(x^0,y^0)}{\hat{T}^2(x^0,y^0)}+\sum_{k=1}^m \lambda_k G_{ky_j}(x^0,y^0)=0, \qquad j=1,2,\ldots,m, \\ G_i(x^0,y^0)=0 \qquad i=1,2,\ldots,m. \end{cases}$$

2. (in the case when $\hat{f}$ is an iso-function of the second kind)

$$\begin{cases} \frac{1}{\hat{T}^2(x^0,y^0)}\Big(f_{x_i}(x^0\hat{T}(x^0,y^0),y^0\hat{T}(x^0,y^0))(\hat{T}(x^0,y^0)+x_i^0\hat{T}_{x_i}(x^0,y^0)) \\ +\sum_{j=1,j\neq i}^n f_{x_j}(x^0\hat{T}(x^0,y^0),y^0\hat{T}(x^0,y^0))x_j^0\hat{T}_{x_i}(x^0,y^0)\hat{T}(x^0,y^0) \\ -f(x^0\hat{T}(x^0,y^0),y^0\hat{T}(x^0,y^0))\hat{T}_{x_i}(x^0,y^0)\Big) \\ +\sum_{k=1}^m \lambda_k G_{kx_i}(x^0,y^0)=0, i=1,2,\ldots,l, \\ \frac{1}{\hat{T}^2(x^0,y^0)}\Big(f_{y_j}(x^0\hat{T}(x^0,y^0),y^0\hat{T}(x^0,y^0))(\hat{T}(x^0,y^0)+y_j^0\hat{T}_{x_i}(x^0,y^0)) \\ +\sum_{j=1,j\neq i}^n f_{y_j}(x^0\hat{T}(x^0,y^0),y^0\hat{T}(x^0,y^0))y_j^0\hat{T}_{y_i}(x^0,y^0)\hat{T}(x^0,y^0) \\ -f(x^0\hat{T}(x^0,y^0),y^0\hat{T}(x^0,y^0))\hat{T}_{y_j}(x^0,y^0)\Big) \\ +\sum_{k=1}^m \lambda_k G_{ky_j}(x^0,y^0)=0, j=1,2,\ldots,m, \\ G_i(x^0,y^0)=0, \qquad i=1,2,\ldots,m, \end{cases}$$

3. (in the case when $\hat{f}$ is an iso-function of the third kind)

$$\begin{cases} \frac{1}{\hat{T}^3(x^0,y^0)}\Big(f_{x_i}\Big(\frac{x^0}{\hat{T}(x^0,y^0)},\frac{y^0}{\hat{T}(x^0,y^0)}\Big)(\hat{T}(x^0,y^0)-x_i^0\hat{T}_{x_i}(x^0,y^0)) \\ -f\Big(\frac{x^0}{\hat{T}(x^0,y^0)},\frac{y^0}{\hat{T}(x^0,y^0)}\Big)\hat{T}(x^0,y^0)\hat{T}_{x_i}(x^0,y^0) \\ -\sum_{j=1,j\neq i}^{n} f_{x_j}\Big(\frac{x^0}{\hat{T}(x^0,y^0)},\frac{y^0}{\hat{T}(x^0,y^0)}\Big)x_j^0\hat{T}_{x_i}(x^0,y^0)\Big) \\ +\sum_{k=1}^{m}\lambda_k G_{kx_i}(x^0,y^0)=0, i=1,2,\ldots,l, \\ \frac{1}{\hat{T}^3(x^0,y^0)}\Big(f_{y_j}\Big(\frac{x^0}{\hat{T}(x^0,y^0)},\frac{y^0}{\hat{T}(x^0,y^0)}\Big)(\hat{T}(x^0,y^0)-y_j^0\hat{T}_{x_i}(x^0,y^0)) \\ -f\Big(\frac{x^0}{\hat{T}(x^0,y^0)},\frac{y^0}{\hat{T}(x^0,y^0)}\Big)\hat{T}_{y_j}(x^0,y^0)\hat{T}(x^0,y^0) \\ -\sum_{j=1,j\neq i}^{n} f_{y_j}\Big(\frac{x^0}{\hat{T}(x^0,y^0)},\frac{y^0}{\hat{T}(x^0,y^0)}\Big)y_j^0\hat{T}_{y_i}(x^0,y^0)\Big) \\ +\sum_{k=1}^{m}\lambda_k G_{ky_j}(x^0,y^0)=0, j=1,2,\ldots,m, \\ G_i(x^0,y^0)=0, \qquad i=1,2,\ldots,m, \end{cases}$$

4. (in the case when $\hat{f}$ is an iso-function of the fourth kind)

$$\begin{cases} f_{x_i}(x^0\hat{T}(x^0,y^0),y^0\hat{T}(x^0,y^0))(\hat{T}(x^0,y^0)+x_i^0\hat{T}_{x_i}(x^0,y^0)) \\ +\sum_{j=1,j\neq i}^{n} f_{x_j}(x^0\hat{T}(x^0,y^0),y^0\hat{T}(x^0,y^0))x_j^0\hat{T}_{x_i}(x^0,y^0) \\ +\sum_{k=1}^{m}\lambda_k G_{kx_i}(x^0,y^0)=0, \\ i=1,2,\ldots,l, \\ f_{y_j}(x^0\hat{T}(x^0,y^0),y^0\hat{T}(x^0,y^0))(\hat{T}(x^0,y^0)+y_j^0\hat{T}_{y_j}(x^0,y^0)) \\ +\sum_{j=1,j\neq i}^{n} f_{y_j}(x^0\hat{T}(x^0,y^0),y^0\hat{T}(x^0,y^0))y_j^0\hat{T}_{y_i}(x^0,y^0) \\ +\sum_{k=1}^{m}\lambda_k G_{ky_j}(x^0,y^0)=0, \\ j=1,2,\ldots,m, \\ G_i(x^0,y^0)=0, \qquad i=1,2,\ldots,m. \end{cases}$$

5. (in the case when $\hat{f}$ is an iso-function of the fifth kind)

$$\begin{cases} \frac{1}{\hat{T}^2(x^0,y^0)}\Big(f_{x_i}\Big(\frac{x^0}{\hat{T}(x^0,y^0)},\frac{y^0}{\hat{T}(x^0,y^0)}\Big)(\hat{T}(x^0,y^0)-x_i^0\hat{T}_{x_i}(x^0,y^0)) \\ -\sum_{j=1,j\neq i}^n f_{x_j}\Big(\frac{x^0}{\hat{T}(x^0,y^0)},\frac{y^0}{\hat{T}(x^0,y^0)}\Big)x_j^0\hat{T}_{x_i}(x^0,y^0)\Big) \\ +\sum_{k=1}^m \lambda_k G_{kx_i}(x^0,y^0)=0, i=1,2,\dots,l, \\ \frac{1}{\hat{T}^2(x^0,y^0)}\Big(f_{y_j}\Big(\frac{x^0}{\hat{T}(x^0,y^0)},\frac{y^0}{\hat{T}(x^0,y^0)}\Big)(\hat{T}(x^0,y^0)-y_j^0\hat{T}_{x_i}(x^0,y^0)) \\ -\sum_{j=1,j\neq i}^n f_{y_j}\Big(\frac{x^0}{\hat{T}(x^0,y^0)},\frac{y^0}{\hat{T}(x^0,y^0)}\Big)y_j^0\hat{T}_{y_i}(x^0,y^0)\Big) \\ +\sum_{k=1}^m \lambda_k G_{ky_j}(x^0,y^0)=0, j=1,2,\dots,m, \\ G_i(x^0,y^0)=0, \qquad i=1,2,\dots,m. \end{cases}$$

Now we will give some conditions for the existence of the constrained extreme values. In addition, we suppose that $f, G_k : D_1 \longrightarrow \mathbb{R}$, $k = 1,2,\dots,m$, are twice continuously-differentiable functions. Since (A7), the system

$$\sum_{i=1}^{l} G_{jx_i}dx_i + \sum_{k=1}^{m} G_{jy_k}dy_k = 0$$

has an unique solution

$$dy_k = \sum_{i=1}^{m} \alpha_{ik}dx_i, \qquad k = 1,2,\dots,m.$$

Then

1. for the iso-functions of the first kind

$$d^2\phi(x^0,y^0) = \sum_{i,j=1}^{m} A_{ij}(x^0,y^0)dx_dx_j + \sum_{i,j=m+1}^{n} A_{ij}(x^0,y^0)\Big(\sum_{l=1}^{m}\alpha_{li}dx_l\Big)\Big(\sum_{l=1}^{m}\alpha_{lj}dx_l\Big),$$

2. for the iso-functions of the second kind

$$d^2\phi(x^0,y^0) = \sum_{i,j=1}^{m} B_{ij}(x^0,y^0)dx_dx_j + \sum_{i,j=m+1}^{n} B_{ij}(x^0,y^0)\Big(\sum_{l=1}^{m}\alpha_{li}dx_l\Big)\Big(\sum_{l=1}^{m}\alpha_{lj}dx_l\Big),$$

3. for the iso-functions of the third kind

$$d^2\phi(x^0,y^0) = \sum_{i,j=1}^{m} C_{ij}(x^0,y^0)dx_dx_j + \sum_{i,j=m+1}^{n} C_{ij}(x^0,y^0)\Big(\sum_{l=1}^{m}\alpha_{li}dx_l\Big)\Big(\sum_{l=1}^{m}\alpha_{lj}dx_l\Big),$$

4. for the iso-functions of the fourth kind

$$d^2\phi(x^0,y^0) = \sum_{i,j=1}^{m} D_{ij}(x^0,y^0)dx_d x_j + \sum_{i,j=m+1}^{n} D_{ij}(x^0,y^0)\Big(\sum_{l=1}^{m} \alpha_{li}dx_l\Big)\Big(\sum_{l=1}^{m} \alpha_{lj}dx_l\Big).$$

5. for the iso-functions of the fifth kind

$$d^2\phi(x^0,y^0) = \sum_{i,j=1}^{m} E_{ij}(x^0,y^0)dx_d x_j + \sum_{i,j=m+1}^{n} E_{ij}(x^0,y^0)\Big(\sum_{l=1}^{m} \alpha_{li}dx_l\Big)\Big(\sum_{l=1}^{m} \alpha_{lj}dx_l\Big).$$

If $d^2\phi(x^0,y^0)$ is a positive(negative) definite quadratic form, then (x^0,y^0) is a minimum(maximum) point of the iso-function $\hat{f}$.

Exercise 1.6.13. *Let* $D=\mathbb{R}^2$, $f(x)=2-2x_1^2-x_2^2$, $\hat{T}(x)=x_1^2+2$, $x=(x_1,x_2)\in D$. *Find the minima and the maxima of* $\hat{f}^{\wedge\wedge}$ *on the ellipse*

$$2(x_1-1)^2+(x_2-1)^2=1.$$

Now we will formulate the mean value theorems for the iso-functions of n variables.

1. The mean value theorem for the iso-functions of the first kind

$$\hat{f}^{\wedge}(\hat{x}^1)-\hat{f}^{\wedge}(\hat{x}^2)=\sum_{i=1}^{n}\frac{f_{x_i}(x^0)\hat{T}(x^0)-f(x^0)\hat{T}_{x_i}(x^0)}{\hat{T}^2(x^0)}(x_i^1-x_i^0),$$

2. The mean value theorem for the iso-functions of the second kind

$$\hat{f}^{\wedge}(x^1)-\hat{f}^{\wedge}(x^2)=\textstyle\sum_{i=1}^{n}\frac{1}{\hat{T}^2(x^0)}\Big(\partial_{x_i}f(x^0\hat{T}(x^0))(\hat{T}(x^0)+x_i^0\partial_{x_i^0}\hat{T}(x^0))\hat{T}(x^0)$$

$$+\textstyle\sum_{j=1,j\neq i}^{n}\partial_{x_j}f(x^0\hat{T}(x^0))x_j\hat{T}_{x_i}(x^0)-f(x^0\hat{T}(x^0))\partial_{x_i}\hat{T}(x^0)\Big)(x_i^1-x_i^2).$$

3. The mean value theorem for the iso-functions of the third kind

$$\hat{f}(\hat{x}^1)-\hat{f}(\hat{x}^2)$$

$$=\textstyle\sum_{i=1}^{n}\frac{1}{\hat{T}^3(x^0)}\Big(\partial_{x_i}f\Big(\frac{x^0}{\hat{T}(x^0)}\Big)(\hat{T}(x^0)-x_i^0\hat{T}_{x_i}(x^0))$$

$$-\textstyle\sum_{j=1,j\neq i}^{n}\partial_{x_j}f\Big(\frac{x^0}{\hat{T}(x^0)}\Big)x_j^0\hat{T}_{x_i}(x^0)-f\Big(\frac{x^0}{\hat{T}(x^0)}\Big)\partial_{x_i}\hat{T}(x^0)\hat{T}(x^0)\Big)(x_i^1-x_i^2).$$

4. The mean value theorem for the iso-functions of the fourth kind

$$f^{\wedge}(x^1)-f^{\wedge}(x^2)=\textstyle\sum_{i=1}^{n}\Big(\partial_{x_i}f(x^0\hat{T}(x^0))(\hat{T}(x^0)+x_i^0\hat{T}_{x_i}(x^0))$$

$$+\textstyle\sum_{j=1,j\neq i}^{n}\partial_{x_j}f(x^0\hat{T}(x^0))x_j^0\partial_{x_i}\hat{T}(x^0)\Big)(x_i^1-x_i^2),$$

5. The mean value theorem for the iso-functions of the fifth kind

$$f^{\vee}(x^1)-f^{\vee}(x^2)$$
$$=\sum_{i=1}^{n}\frac{1}{\hat{T}^2(x^0)}\left(\partial_{x_i}f\left(\frac{x^0}{\hat{T}(x^0)}\right)(\hat{T}(x^0)-x_i^0\hat{T}_{x_i}(x^0))\right.$$
$$\left.-\sum_{j=1,j\neq i}^{n}\partial_{x_j}f\left(\frac{x^0}{\hat{T}(x^0)}\right)x_j^0\hat{T}_{x_i}(x^0)\right)(x_i^1-x_i^2).$$

where x^0 belongs to the line from x^1 to x^2 and $x^0\neq x^1,x^2$. Here $x^1,x^2\in D$ are arbitrarily chosen.

Corollary 1.6.14. *If $f,\hat{T}:D\longrightarrow\mathbb{R}$ are differentiable functions and*

$$f_{x_i}(x)\hat{T}(x)-f(x)\hat{T}_{x_i}(x)=0 \qquad \text{for} \qquad \forall x\in D, \qquad i=1,2,\ldots,n,$$

then $\hat{f}^{\wedge\wedge}$ is a constant in D.

Corollary 1.6.15. *If $f,\hat{T}:D\longrightarrow\mathbb{R}$ are differentiable functions, $x\hat{T}(x)\in D$ for every $x\in D$, and*

$$\partial_{x_i}f(x\hat{T}(x))(\hat{T}(x)+x_i\partial_{x_i}\hat{T}(x))\hat{T}(x)$$
$$+\sum_{j=1,j\neq i}^{n}\partial_{x_j}f(x\hat{T}(x))x_j\hat{T}_{x_i}(x)-f(x\hat{T}(x))\partial_{x_i}\hat{T}(x)=0 \qquad \text{for} \qquad \forall x\in D, \qquad i=1,2,\ldots,n,$$

then $\hat{f}^{\wedge}$ is a constant in D.

Corollary 1.6.16. *If $f,\hat{T}:D\longrightarrow\mathbb{R}$ are differentiable functions, $\frac{x}{\hat{T}(x)}\in D$ for every $x\in D$, and*

$$\partial_{x_i}f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\hat{T}_{x_i}(x))$$
$$-\sum_{j=1,j\neq i}^{n}\partial_{x_j}f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)-f\left(\frac{x}{\hat{T}(x)}\right)\partial_{x_i}\hat{T}(x)\hat{T}(x)=0 \qquad \text{for} \qquad \forall x\in D, \qquad i=1,2,\ldots,n,$$

then $\hat{\hat{f}}$ is a constant in D.

Corollary 1.6.17. *If $f,\hat{T}:D\longrightarrow\mathbb{R}$ are differentiable functions, $x\hat{T}(x)\in D$ for every $x\in D$, and*

$$\partial_{x_i}f(x\hat{T}(x))(\hat{T}(x)+x_i\hat{T}_{x_i}(x))$$
$$+\sum_{j=1,j\neq i}^{n}\partial_{x_j}f(x\hat{T}(x))x_j\partial_{x_i}\hat{T}(x)=0 \qquad \text{for} \qquad \forall x\in D, \qquad i=1,2,\ldots,n,$$

then f is a constant in D.

Corollary 1.6.18. *If $f,\hat{T}:D\longrightarrow\mathbb{R}$ are differentiable functions, $\frac{x}{\hat{T}(x)}\in D$ for every $x\in D$, and*

$$\partial_{x_i}f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x)-x_i\hat{T}_{x_i}(x))-\sum_{j=1,j\neq i}^{n}\partial_{x_j}f\left(\frac{x}{\hat{T}(x)}\right)x_j\hat{T}_{x_i}(x)=0$$
$$\text{for} \qquad \forall x\in D, \qquad i=1,2,\ldots,n,$$

then $f^{\vee}$ is a constant in D.

Now we suppose that $\hat{f}$ is an iso-function of the first, the second, the third, the fourth or the fifth kind, which iso-original is an enough times differentiable function in D. Then we can formulate the iso-Taylor series for $\hat{f}$ as follows.

1. The iso-Taylor series of the first kind

$$\hat{f}(x) = \hat{f}(x^0) + \widehat{\tfrac{1}{1!}} \hat{\times} \left(\sum_{i=1}^{n} (\hat{\partial}_{x_i} \nearrow \hat{\partial}_{x_i} \hat{x}_i \hat{\times} \Delta \hat{x}_i) \right) \hat{f}(x^0)$$

$$+ \widehat{\tfrac{1}{2!}} \hat{\times} \left(\sum_{i=1}^{n} (\hat{\partial}_{x_i} \nearrow \hat{\partial}_{x_i} \hat{x}_i \hat{\times} \Delta \hat{x}_i) \right)^{\hat{2}} \hat{f}(x^0)$$

$$+ \cdots$$

$$+ \widehat{\tfrac{1}{m!}} \hat{\times} \left(\sum_{i=1}^{n} (\hat{\partial}_{x_i} \nearrow \hat{\partial}_{x_i} \hat{x}_i \hat{\times} \Delta \hat{x}_i) \right)^{\hat{n}} \hat{f}(x^0)$$

$$+ \widehat{\tfrac{1}{(m+1)!}} \hat{\times} \left(\sum_{i=1}^{n} (\hat{\partial}_{x_i} \nearrow \hat{\partial}_{x_i} \hat{x}_i \hat{\times} \Delta \hat{x}_i) \right)^{\widehat{n+1}} \hat{f}(x^0 + \xi \Delta x), \qquad \xi \in (0,1),$$

$$\Delta \hat{x}_i = \hat{x}_i - \hat{x}_i^0, \qquad \Delta \hat{x} = (\Delta \hat{x}_1, \Delta \hat{x}_2, \ldots, \Delta \hat{x}_n).$$

2. The iso-Taylor series of the second kind

$$\hat{f}(x) = \hat{f}(x^0) + \widehat{\tfrac{1}{1!}} \hat{\times} \left(\sum_{i=1}^{n} (\hat{\partial}_{x_i} \nearrow dx_i \hat{\times} \Delta \hat{x}_i) \right) \hat{f}(x^0)$$

$$+ \widehat{\tfrac{1}{2!}} \hat{\times} \left(\sum_{i=1}^{n} (\hat{\partial}_{x_i} \nearrow dx_i \hat{\times} \Delta \hat{x}_i) \right)^{\hat{2}} \hat{f}(x^0)$$

$$+ \cdots$$

$$+ \widehat{\tfrac{1}{m!}} \hat{\times} \left(\sum_{i=1}^{n} (\hat{\partial}_{x_i} \nearrow dx_i \hat{\times} \Delta \hat{x}_i) \right)^{\hat{n}} \hat{f}(x^0)$$

$$+ \widehat{\tfrac{1}{(m+1)!}} \hat{\times} \left(\sum_{i=1}^{n} (\hat{\partial}_{x_i} \nearrow dx_i \hat{\times} \Delta \hat{x}_i) \right)^{\widehat{n+1}} \hat{f}(x^0 + \xi \Delta x), \qquad \xi \in (0,1),$$

$$\Delta \hat{x}_i = \hat{x}_i - \hat{x}_i^0, \qquad \Delta \hat{x} = (\Delta \hat{x}_1, \Delta \hat{x}_2, \ldots, \Delta \hat{x}_n).$$

3. The iso-Taylor series of the third kind

$$\hat{f}(x)=\hat{f}(x^0)+\widehat{\tfrac{1}{1!}}\hat{\times}\Big(\textstyle\sum_{i=1}^n(dx_i\partial_{x_i}\nearrow\hat{\partial}_{x_i}\hat{x}_i\hat{\times}\Delta\hat{x}_i)\Big)\hat{f}(x^0)$$

$$+\widehat{\tfrac{1}{2!}}\hat{\times}\Big(\textstyle\sum_{i=1}^n(dx_i\partial_{x_i}\nearrow\hat{\partial}_{x_i}\hat{x}_i\hat{\times}\Delta\hat{x}_i)\Big)^{\hat{2}}\hat{f}(x^0)$$

$$+\cdots$$

$$+\widehat{\tfrac{1}{m!}}\hat{\times}\Big(\textstyle\sum_{i=1}^n(dx_i\partial_{x_i}\nearrow\hat{\partial}_{x_i}\hat{x}_i\hat{\times}\Delta\hat{x}_i)\Big)^{\hat{n}}\hat{f}(x^0)$$

$$+\widehat{\tfrac{1}{(m+1)!}}\hat{\times}\Big(\textstyle\sum_{i=1}^n(dx_i\partial_{x_i}\nearrow\hat{\partial}_{x_i}\hat{x}_i\hat{\times}\Delta\hat{x}_i)\Big)^{\widehat{n+1}}\hat{f}(x^0+\xi\Delta x),\qquad \xi\in(0,1),$$

$$\Delta\hat{x}_i=\hat{x}_i-\hat{x}_i^0,\qquad \Delta\hat{x}=(\Delta\hat{x}_1,\Delta\hat{x}_2,\ldots,\Delta\hat{x}_n).$$

4. The iso-Taylor series of the fourth kind

$$\hat{f}(x)=\hat{f}(x^0)+\widehat{\tfrac{1}{1!}}\hat{\times}\Big(\textstyle\sum_{i=1}^n\Big(\tfrac{1}{\hat{T}(x)}\partial_{x_i}\hat{\times}\Delta\hat{x}_i\Big)\Big)\hat{f}(x^0)$$

$$+\widehat{\tfrac{1}{2!}}\hat{\times}\Big(\textstyle\sum_{i=1}^n\Big(\tfrac{1}{\hat{T}(x)}\partial_{x_i}\hat{\times}\Delta\hat{x}_i\Big)\Big)^{\hat{2}}\hat{f}(x^0)$$

$$+\cdots$$

$$+\widehat{\tfrac{1}{m!}}\hat{\times}\Big(\textstyle\sum_{i=1}^n\Big(\tfrac{1}{\hat{T}(x)}\partial_{x_i}\hat{\times}\Delta\hat{x}_i\Big)\Big)^{\hat{n}}\hat{f}(x^0)$$

$$+\widehat{\tfrac{1}{(m+1)!}}\hat{\times}\Big(\textstyle\sum_{i=1}^n\Big(\partial_{x_i}\hat{\times}\Delta\hat{x}_i\Big)\Big)^{\widehat{n+1}}\hat{f}(x^0+\xi\Delta x),\qquad \xi\in(0,1),$$

$$\Delta\hat{x}_i=\hat{x}_i-\hat{x}_i^0,\qquad \Delta\hat{x}=(\Delta\hat{x}_1,\Delta\hat{x}_2,\ldots,\Delta\hat{x}_n).$$

5. The iso-Taylor series of the fifth kind

$$\hat{f}(x) = \hat{f}(x^0) + \widehat{\frac{1}{1!}} \hat{\times} \left(\sum_{i=1}^{n} \left(\frac{\hat{\partial}_{x_i}}{\hat{\partial}_{x_i} \hat{x}_i} \hat{\times} \Delta\hat{x}_i \right) \right) \hat{f}(x^0)$$

$$+ \widehat{\frac{1}{2!}} \hat{\times} \left(\sum_{i=1}^{n} \left(\frac{\hat{\partial}_{x_i}}{\hat{\partial}_{x_i} \hat{x}_i} \hat{\times} \Delta\hat{x}_i \right) \right)^{\hat{2}} \hat{f}(x^0)$$

$$+ \cdots$$

$$+ \widehat{\frac{1}{m!}} \hat{\times} \left(\sum_{i=1}^{n} \left(\frac{\hat{\partial}_{x_i}}{\hat{\partial}_{x_i} \hat{x}_i} \hat{\times} \Delta\hat{x}_i \right) \right)^{\hat{n}} \hat{f}(x^0)$$

$$+ \widehat{\frac{1}{(m+1)!}} \hat{\times} \left(\sum_{i=1}^{n} \left(\frac{\hat{\partial}_{x_i}}{\hat{\partial}_{x_i} \hat{x}_i} \hat{\times} \Delta\hat{x}_i \right) \right)^{\widehat{n+1}} \hat{f}(x^0 + \xi \Delta x), \qquad \xi \in (0,1),$$

$$\Delta\hat{x}_i = \hat{x}_i - \hat{x}_i^0, \qquad \Delta\hat{x} = (\Delta\hat{x}_1, \Delta\hat{x}_2, \ldots, \Delta\hat{x}_n).$$

6. The iso-Taylor series of the sixth kind

$$\hat{f}(x) = \hat{f}(x^0) + \widehat{\frac{1}{1!}} \hat{\times} \left(\sum_{i=1}^{n} \left(\frac{\hat{\partial}_{x_i}}{dx_i} \hat{\times} \Delta\hat{x}_i \right) \right) \hat{f}(x^0)$$

$$+ \widehat{\frac{1}{2!}} \hat{\times} \left(\sum_{i=1}^{n} \left(\frac{\hat{\partial}_{x_i}}{dx_i} \hat{\times} \Delta\hat{x}_i \right) \right)^{\hat{2}} \hat{f}(x^0)$$

$$+ \cdots$$

$$+ \widehat{\frac{1}{m!}} \hat{\times} \left(\sum_{i=1}^{n} \left(\frac{\hat{\partial}_{x_i}}{dx_i} \hat{\times} \Delta\hat{x}_i \right) \right)^{\hat{n}} \hat{f}(x^0)$$

$$+ \widehat{\frac{1}{(m+1)!}} \hat{\times} \left(\sum_{i=1}^{n} \left(\frac{\hat{\partial}_{x_i}}{dx_i} \hat{\times} \Delta\hat{x}_i \right) \right)^{\widehat{n+1}} \hat{f}(x^0 + \xi \Delta x), \qquad \xi \in (0,1),$$

$$\Delta\hat{x}_i = \hat{x}_i - \hat{x}_i^0, \qquad \Delta\hat{x} = (\Delta\hat{x}_1, \Delta\hat{x}_2, \ldots, \Delta\hat{x}_n).$$

7. The iso-Taylor series of the seventh kind

$$\hat{f}(x) = \hat{f}(x^0) + \widehat{\frac{1}{1!}} \hat{\times} \left(\sum_{i=1}^{n} \left(\frac{\hat{\partial}_{x_i}}{\partial_{x_i} \hat{x}_i} \hat{\times} \Delta \hat{x}_i \right) \right) \hat{f}(x^0)$$

$$+ \widehat{\frac{1}{2!}} \hat{\times} \left(\sum_{i=1}^{n} \left(\frac{\hat{\partial}_{x_i}}{\partial_{x_i} \hat{x}_i} \hat{\times} \Delta \hat{x}_i \right) \right)^{\hat{2}} \hat{f}(x^0)$$

$$+ \cdots$$

$$+ \widehat{\frac{1}{m!}} \hat{\times} \left(\sum_{i=1}^{n} \left(\frac{\hat{\partial}_{x_i}}{\partial_{x_i} \hat{x}_i} \hat{\times} \Delta \hat{x}_i \right) \right)^{\hat{n}} \hat{f}(x^0)$$

$$+ \widehat{\frac{1}{(m+1)!}} \hat{\times} \left(\sum_{i=1}^{n} \left(\frac{\hat{\partial}_{x_i}}{\partial_{x_i} \hat{x}_i} \hat{\times} \Delta \hat{x}_i \right) \right)^{\widehat{n+1}} \hat{f}(x^0 + \xi \Delta x), \qquad \xi \in (0,1),$$

$$\Delta \hat{x}_i = \hat{x}_i - \hat{x}_i^0, \qquad \Delta \hat{x} = (\Delta \hat{x}_1, \Delta \hat{x}_2, \ldots, \Delta \hat{x}_n).$$

Definition 1.6.19. *If $\hat{f}_1$, $\hat{f}_2$, …, $\hat{f}_m$ are iso-functions of the first, the second, the third, the fourth or the fifth kind, then a vector iso-function is*

$$(\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_m).$$

Example 1.6.20. *Let $D = \mathbb{R}^2$, $\hat{T}(x) = 1 + x_1^2 + x_2^2$, $f_1(x) = x_1^2$, $f_2(x) = x_1 + x_2$, $f_3(x) = x_2$, $x = (x_1, x_2) \in D$. Then*

$$(\hat{f}_1^{\wedge}(\hat{x}), \hat{f}_2^{\wedge}(x), \hat{f}_3^{\wedge}(\hat{x})) = \left(\frac{x_1^2}{1 + x_1^2 + x_2^2}, (x_1 + x_2)(1 + x_1^2 + x_2^2), \frac{x_2}{1 + x_1^2 + x_2^2} \right)$$

is a vector iso-function.

1.7. Advanced practical exercises

Problem 1.7.1. *In $\hat{F}_{\mathbb{R}^4}$, let $\hat{T}(x) = x_1^2 + x_4^2$, $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, $\hat{T}_1(y) = y^4 + 1$, $y \in \mathbb{R}$, $X = (1,0,0,1)$, $Y = (1,-2,-3,1)$. Find*

$$\hat{X}, \qquad \hat{Y}, \qquad \hat{X} + \hat{Y}, \qquad \hat{3} \hat{\times} \hat{X}, \qquad \hat{2} \hat{\times} \hat{X} + \hat{Y}.$$

Answer. $\hat{X} = \left(\frac{1}{2}, 0, 0, \frac{1}{2} \right)$, $\hat{Y} = \left(\frac{1}{2}, -1, -\frac{3}{2}, \frac{1}{2} \right)$, $\hat{X} + \hat{Y} = \left(1, -1, -\frac{3}{2}, 1 \right)$, $\hat{3} \hat{\times} \hat{Y} = \left(\frac{3}{2}, -3, -\frac{9}{2}, \frac{3}{2} \right)$, $\hat{2} \hat{\times} \hat{X} + \hat{Y} = \left(\frac{3}{2}, -1, -\frac{3}{2}, \frac{3}{2} \right)$.

Problem 1.7.2. *Let $D = \mathbb{R}$,*

$$f(x) = \begin{cases} x_1^4 + 2x_1^5 x_2 + 7x_2^3 & x_1 \le 1, \quad x_2 \in \mathbb{R}, \\ \\ x_1^7 - 7x_1^2 x_2 + 6x_2^3 & x_1 \ge 1, \quad x_2 \in \mathbb{R}, \end{cases}$$

$\hat{T}(x) = x_2$, $x = (x_1, x_2) \in \mathbb{R}$. Find $f^{\vee}(x)$, $x \in D$.

Answer.

$$f^{\vee}(x) = \begin{cases} \frac{x_1^4}{x_2^4} + 2\frac{x_1^5}{x_2^5} + 7 & x_1 \leq 1, \qquad x_2 \in \mathbb{R}, \\ \\ \frac{x_1^7}{x_2^7} - 7\frac{x_1^2}{x_2^2} + 6 & x_1 \geq 1, \qquad x_2 \in \mathbb{R}. \end{cases}$$

Problem 1.7.3. *In $\hat{F}_{\mathbb{R}^2}$, let $\hat{T}(x) = x_1^4 + x_2^4 + 2$, $x = (x_1, x_2) \in \mathbb{R}^2$, $\hat{T}(y) = 1 + y^6$, $y \in \mathbb{R}$, $X = (1,1)$, $Y = (-1,-1)$. Find*

$$\hat{2} \hat{\times} \hat{X} + 3 \hat{\times} \hat{Y}.$$

Answer. $\left(-\frac{2185}{4}, -\frac{2185}{4}\right)$.

Problem 1.7.4. *In $\hat{F}_{\mathbb{R}^2}$, let $\hat{T}(x) = |x_1| + |x_2| + 2$, $x = (x_1, x_2) \in \mathbb{R}^2$, $\hat{T}_1(y) = 2 + |y|$, $y \in \mathbb{R}$, $X = (2,-1)$, $Y = (-1,2)$. Find*

$$\hat{3} \hat{\times} (\hat{2}\hat{X} + 2 \hat{\times} \hat{Y}).$$

Answer. $\left(-\frac{81}{20}, \frac{369}{40}\right)$.

Problem 1.7.5. *In $\hat{F}_{\mathbb{R}^2}$, let $\hat{T}(x) = x_1^2 + x_2^2 + 2$, $x = (x_1, x_2) \in \mathbb{R}^2$, $\hat{T}_1(y) = 2 + |y|$, $y \in \mathbb{R}$, $X = (1,-1)$, $Y = (-1,1)$. Find*

$$\hat{2} \hat{\times} (3\hat{X} + \hat{2}\hat{Y}) - 3 \hat{\times} (\hat{X} - \hat{Y}).$$

Answer. $\left(-\frac{25}{4}, \frac{25}{4}\right)$.

Problem 1.7.6. *In $\hat{F}_{\mathbb{R}^2}$, let $\hat{T}(x) = |x_1| + 2$, $x = (x_1, x_2) \in \mathbb{R}^2$, $\hat{T}_1 = 4$, $X = (-2,3)$, $Y = (3,4)$. Find*

$$|\hat{X}\hat{|}, \qquad |\hat{Y}\hat{|}, \qquad |\hat{X} - \hat{Y}\hat{|}.$$

Answer. $\frac{\sqrt{13}}{2}$, 2, $2\frac{\sqrt{26}}{7}$.

Problem 1.7.7. *In $\hat{F}_{\mathbb{R}^3}$, let $\hat{T}(x) = x_1^2 + x_2^2 + 3$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\hat{T}_1 = 4$, $X(1,-1,2)$, $Y = (2,-1,3)$. Find*

$$\hat{X} \hat{\cdot} \hat{Y}.$$

Answer. $\frac{9}{10}$.

Problem 1.7.8. *In $\hat{F}_{\mathbb{R}^n}$, let $\hat{T}(x) = \sum_{i=1}^{n} |x_i|^5 + 1$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Investigate for convergence the sequence $\{\hat{X}_l\}_{l=1}^{\infty}$, where*

1. $X_l = \left(\frac{l}{2}, \frac{l-1}{2}, \frac{l-2}{2}, \ldots, \frac{l-n}{2}\right)$,

2. $X_l = (\sqrt{l}, \sqrt{l^2+1}, \sqrt{l^2+2}, \ldots, \sqrt{l^2+n})$,

3. $X_l = (\sqrt{l+1} + \sqrt{l}, 2(\sqrt{l+1} + \sqrt{l}), 3(\sqrt{l+1} + \sqrt{l}), \ldots, n(\sqrt{l+1} + \sqrt{l}))$,

4. $X_l = (\sqrt[5]{l^2+1} - l, 2(\sqrt[5]{l^2+1} - l), 3(\sqrt[5]{l^2+1} - l), \ldots, n(\sqrt[5]{l^2+1} - l))$,

5. $X_l = \left(\frac{1}{3n}\sqrt[4]{1+l^3}, \frac{1}{3n-1}\sqrt[4]{1+l^3}, \frac{1}{3n-2}\sqrt[4]{1+l^3}, \ldots, \frac{1}{2n+1}\sqrt[4]{1+l^3}\right)$.

Problem 1.7.9. *Let $D=\mathbb{R}^2$, $\hat{T}(x)=|x_1|+4$, $f(x)=x_1-x_2$, $x=(x_1,x_2)\in\mathbb{R}^2$. Find $\hat{f}^{\wedge}(\hat{x})$.*

Answer. $\frac{x_1-x_2}{|x_1|+4}$.

Problem 1.7.10. *Let $D=\mathbb{R}^3$, $\hat{T}(x)=|x_1|+|x_2|+|x_3|+3$,*

$$f(x)=\begin{cases} x_1-x_2 & x_1\le 1, \quad x_2\le 1, \quad x_3\in\mathbb{R}, \\ x_1^2+x_3^2+4 & x_1\le 1, \quad x_2\le 1, \quad x_3\in\mathbb{R}, \\ x_1^2+2x_3 & x_1\ge 1, \quad x_2\le 1, \quad x_3\in\mathbb{R}, \\ x_1^2-x_3^2 & x_1\ge 1, \quad x_2\ge 1, \quad x_3\in\mathbb{R}. \end{cases}$$

Check if $\hat{f}^{\wedge}(\hat{x})$ is a function.

Answer. No.

Problem 1.7.11. *Let $D=\mathbb{R}^2$, $f(x)=x_1^2+x_2$, $\hat{T}(x)=x_2^2+1$, $x=(x_1,x_2)\in D$. Find $\hat{f}^{\wedge}(x)$.*

Answer. $\hat{f}^{\wedge}(x)=x_1^2x_2^2+2x_1^2+x_2$.

Problem 1.7.12. *Let $D=\mathbb{R}^2$, $f(x)=x_1^3+2x_2-3x_1x_2$,*

$$\hat{T}(x)=\begin{cases} x_1^2+x_2^2+4 & x_1\in\mathbb{R}, \quad x_2\le 3, \\ |x_1|+5|x_2|+4 & x_1\in\mathbb{R}, \quad x_2\ge 3. \end{cases}$$

Check if $\hat{f}^{\wedge}(x)$ is a function.

Answer. No.

Problem 1.7.13. *Let $D=\mathbb{R}^2$, $f(x)=x_1^2-x_2$, $\hat{T}(x)=x_1^2+2$, $x=(x_1,x_2)\in D$. Find $\hat{f}(\hat{x})$.*

Answer. $\frac{x_1^2-x_1^2x_2-2x_2}{(x_1^2+2)^3}$.

Problem 1.7.14. *Let $D=\mathbb{R}^3$, $f(x)=x_1-2x_2+3x_2^2$, $x=(x_1,x_2)\in D$,*

$$\hat{T}(x)=\begin{cases} x_1^2+2x_2^2+3x_3^2+4 & (x_1,x_2)\in\mathbb{R}^2, \quad x_3\le 1, \\ \sqrt{x_1t+x_2^4+x_3^4}+5 & (x_1,x_2)\in\mathbb{R}^2, \quad x_3\ge 1. \end{cases}$$

Check if $\hat{\hat{f}}$ is a function.

Answer. No.

Problem 1.7.15. *Let $D=\mathbb{R}^2$, $\hat{T}(x)=x_1^2+x_2^2+3$, $f(x)=x_1^2+x_2$, $x=(x_1,x_2)\in D$. Find $f^{\wedge}(x)$.*

Answer. $(x_1^2+x_2^2+3)(x_1^4+x_1^2x_2^2+3x_1^2+x_2)$.

Problem 1.7.16. *Let $D = \mathbb{R}^2$, $f(x) = x_1^3 - x_2^2$, $x = (x_1, x_2) \in D$. Let also*

$$\hat{T}(x) = \begin{cases} x_1^2 + x_2^2 + 3 & x_1 \in \mathbb{R}, \quad x_2 \leq 1, \\ \\ |x_1| + |x_2| + 2 & x_1 \in \mathbb{R}, \quad x_2 \geq 1. \end{cases}$$

Check if $f^\wedge$ is a function.

Answer. No.

Problem 1.7.17. *Let $D = \mathbb{R}^2$,*

$$f(x) = \begin{cases} x_1^4 + x_1x_2 + x_1x_2^3 - x_2^4 - x_1x_2 - 5x_1x_2^4 & x_1 \leq 1, \quad x_2 \in \mathbb{R}, \\ \\ x_2 + x_1^2x_2 + 4x_2^4 & x_1 \geq 1, \quad x_2 \in \mathbb{R}, \end{cases}$$

$$\hat{T}(x) = \begin{cases} x_1^2 + x_2^2 + x_1^4 + 2x_1^2x_2^2 + 2 & x_1 \leq 1, \quad x_2 \in \mathbb{R}, \\ \\ x_1^6 + x_1^2x_2^2 + x_1^4x_2^4 + x_1^8 + 9 & x_1 \geq 1, \quad x_2 \in \mathbb{R}. \end{cases}$$

Check if $f^\vee$ is a function.

Answer. No.

Problem 1.7.18. *Let $D = \mathbb{R}^3$, $f(x) = x_1^3x_2^3x_3^3 + x_2 + x_1x_2x_3 + x_3^7$, $\hat{T}(x) = x_1^2 + x_2^2 + x_3^2 + 5$, $x = (x_1, x_2, x_3) \in D$. Check if $f^\vee$ is a function.*

Answer. Yes.

Problem 1.7.19. *Let $D = \mathbb{R}^2$, $\hat{T}_1 = 4$, $\hat{T}(x) = 2 + x_2^2$, $f(x) = x_1^2 - 2x_2$, $x = (x_1, x_2) \in D$. Find*

$$\hat{2} \hat{\times} \hat{f}^\wedge(\hat{x}) - 4 \hat{\times} f^\wedge(x).$$

Answer.

$$\frac{2x_1^2 - 4x_2}{x_2^2 + 2} + 64x_1^2 + 64x_1^2x_2^2 + 16x_1^2x_2^4 - 64x_2 - 32x_2^3.$$

Problem 1.7.20. *Let $D = \mathbb{R}^2$, $\hat{T}_1 = 2$, $f(x) = x_1 - 2x_2$, $\hat{T}(x) = x_1^2 + x_2^2 + 4$, $x = (x_1, x_2) \in D$. Find*

$$\Big(f^\wedge(x)\Big)^{\hat{3}} - \Big(f^\wedge(x)\Big)^3.$$

Answer. $3(x_1 - 2x_2)^3(x_1^2 + x_2^2 + 4)^3$.

Problem 1.7.21. *Let $D = \mathbb{R}^2$, $\hat{T}(x) = 1 + x_1^2 + x_2^2$, $\hat{T}(x) = x_1^3 + x_2^4 + 3$, $x = (x_1, x_2) \in D$. Find*

$$\lim_{x \longrightarrow (1,1)} \hat{f}^\wedge(\hat{x}).$$

Answer. $\frac{5}{3}$.

Problem 1.7.22. *Find* $\lim_{x\longrightarrow(0,0)} \hat{f}^{\wedge}(\hat{x})$*, where*

$$f(x) = x_1^2\log(x_1^2+x_2^2), \qquad \hat{T}(x) = x_1^2+x_2^2+1, \qquad x=(x_1,x_2)\in D=\mathbb{R}^2.$$

Answer. 0.

Problem 1.7.23. *Find* $\lim_{x\longrightarrow(1,1)} \hat{f}^{\wedge}(\hat{x})$*, where*

$$f(x) = \frac{e^{x_1x_2}-1}{x_1^2x_2^2-1}, \qquad \hat{T}(x) = \frac{1}{3}(x_1^2+x_2^2+1), \qquad x=(x_1,x_2)\in D=\mathbb{R}^2.$$

Answer. e.

Problem 1.7.24. *Find* $\lim_{x\longrightarrow\infty} \hat{f}^{\wedge}(\hat{x})$*, where*

$$f(x) = \frac{x_1^2+x_2^2}{x_1^4+x_2^4}, \qquad \hat{T}(x) = x_1^2+x_2^2+1, \qquad x=(x_1,x_2)\in D=\mathbb{R}^2.$$

Answer. 0.

Problem 1.7.25. *Find* $\lim_{x\longrightarrow\infty} \hat{f}^{\wedge}(\hat{x})$*, where*

$$f(x) = (x_1^2+x_2^2)e^{-(x_1+x_2)}, \qquad \hat{T}(x) = x_1^2+x_2^2+1, \qquad x=(x_1,x_2)\in D=\mathbb{R}^2.$$

Answer.0.

Problem 1.7.26. *Let* $D=\{(x_1,x_2)\in\mathbb{R}^2 : x_1\geq 0, \quad x_1^2+x_2^2\neq 0\}$*,* $\hat{T}(x)=3+x_1^2+x_2^4$*,*

$$f(x) = \frac{\log\Big(x_1+e^{x_2}\Big)}{\sqrt{x_1^2+x_2^2}}, \qquad x=(x_1,x_2)\in D.$$

Check if $\hat{f}^{\wedge\wedge}$ *is a continuous function in D.*

Answer. Yes.

Problem 1.7.27. *Let* $D=\mathbb{R}^2$*,* $f(x)=x_1-x_2$*,* $\hat{T}(x)=x_1^2+x_2^2+1$*,* $x=(x_1,x_2)\in D$*. Find* $\Big(f^{\wedge}(x)\Big)_{x_2}^{1\circledast}$.

Answer. $\frac{(x_1^2+x_2^2+1)(x_2^2-x_1^2+2x_1x_2+1)}{x_1^2-x_2^2+1}$.

Problem 1.7.28. *Let* $D=\mathbb{R}^2$*,* $f(x)=2x_1x_2$*,* $\hat{T}(x)=x_1^2+x_2^2+1$*,* $x=(x_1,x_2)\in D$*. Find* $(\hat{f}^{\wedge}(\hat{x}))_{x_1}^{2\circledast}$.

Answer. $2\frac{x_2^3-3x_1^2x_2+x_2}{(x_1^2+x_2^2+1)^2}$.

Problem 1.7.29. *Let* $D=\{(x_1,x_2)\in\mathbb{R}^2 : x_1\geq 1, x_2\geq 1\}$*,* $f(x)=x_1^2+x_2^2$*,* $\hat{T}(x)=x_1+x_2$*,* $x=(x_1,x_2)\in\mathbb{R}^2$*. Find* $(\hat{f}^{\wedge}(\hat{x}))_{x_2}^{3\circledast}$.

Answer. $\frac{-x_1^2+x_2^2+2x_1x_2}{x_1(x_1+x_2)^2}$.

Problem 1.7.30. *Let* $D = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 \geq 1, x_2 \geq 1\}$, $f(x) = x_1 - x_2$, $\hat{T}(x) = x_1 + x_2$, $x = (x_1,x_2) \in D$. *Find* $(f^{\wedge}(x))^{4\circledast}_{x_2}$.

Answer. $\frac{-2x_2}{x_1+x_2}$.

Problem 1.7.31. *Let* $D = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 \geq 2, x_2 \geq 3\}$, $f(x) = x_1 - 5x_2$, $\hat{T}(x) = x_1 + x_2$, $x = (x_1,x_2) \in D$. *Find* $(\hat{f}^{\wedge}(\hat{x}))^{5\circledast}_{x_2}$.

Answer. -6.

Problem 1.7.32. *Let* $D = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$, $f(x) = x_1 + 2x_2^2$, $\hat{T}(x) = x_1^2 + x_2^2 + 1$, $x = (x_1,x_2) \in D$. *Find* $(f^{\wedge}(x))^{6\circledast}$.

Answer.

$$(x_1^2+x_2^2+1)(8x_1^3x_2^2+8x_1x_2^4+8x_1x_2^2+3x_1^2+x_2^2+1).$$

Problem 1.7.33. *Let* $D = \{(x_1,x_2) : x_1 \geq 0, x_2 \geq 0\}$, $f(x) = x_1 - x_2$, $\hat{T}(x) = 1 + x_1 + x_2$, $x = (x_1,x_2) \in D$. *Find* $(\hat{f}^{\wedge}(\hat{x}))^{7\circledast}_{x_1}$.

Answer. $\frac{(1+2x_2)(1+x_1+x_2)}{1+x_2}$.

Problem 1.7.34. *Let* $D = \mathbb{R}^3$, $\hat{T}(x) = x_1^2 + x_2^2 + x_3^2 + 1$, $f(x) = x_1x_2x_3$. *Find minima and maxima of* $\hat{f}^{\wedge\wedge}$, $\hat{f}^{\wedge}$, $\hat{\hat{f}}$, $f^{\wedge}$ *on the sphere* $x_1^2 + x_2^2 + x_3^2 = 1$.

Chapter 2

Multiple Iso-Integrals

Let $D \subset \mathbb{R}^n$ be a bounded set, $f : D \longrightarrow$ be an integrable on D function, $\hat{T} : D \longrightarrow \mathbb{R}$ be a positive continuously-differentiable function such that

$$(A8)\quad \begin{array}{l} M_1 \le \hat{T}(x) \le M_2, \qquad M_1 \le |\hat{T}(x) - x_i \hat{T}_{x_i}(x)| \le M_2 \qquad \text{for} \qquad \forall x \in D, \\ x\hat{T}(x) \in D, \qquad \frac{x}{\hat{T}(x)} \in D \qquad \text{for} \qquad \forall x \in D, \qquad i = 1, 2, \dots, n, \end{array}$$

for some positive constants M_1 and M_2.

2.1. Definition of Multiple Iso-Integrals

We suppose that $\hat{f}$ is an iso-function of the first, the second, the third, the fourth or the fifth kind.

Definition 2.1.1. *The multiple iso-integral of the first kind of the iso-function $\hat{f}$ over D is defined as follows*

$$\hat{\int}^1_D \hat{f}(x) \hat{\times} \hat{d}\hat{x},$$

where

$$\hat{d}\hat{x} = \hat{d}\hat{x}_1 \hat{d}\hat{x}_2 \dots \hat{d}\hat{x}_n,$$

$$\hat{d}\hat{x}_i = \hat{T}(x) d\hat{x}_i = \hat{T}(x) d\left(\frac{x_i}{\hat{T}(x)}\right) = \frac{\hat{T}(x) - x_i \hat{T}_{x_i}}{\hat{T}(x)} dx_i, \qquad i = 1, 2, \dots, n.$$

We can rewrite the multiple iso-integral of the first kind in the following manner

$$\hat{\int}^1_D \hat{T}(x) \hat{\times} \hat{d}\hat{x} = \int_D \frac{1}{\hat{T}(x)} \hat{f}(x) \hat{T}_1 \prod_{i=1}^n \frac{\hat{T}(x) - x_i \hat{T}_{x_i}(x)}{\hat{T}(x)} dx_i$$

$$= \hat{T}_1 \int_D \hat{f}(x) \frac{1}{\hat{T}^{(n+1)}(x)} \prod_{i=1}^n (\hat{T}(x) - x_i \hat{T}_{x_i}(x)) dx, \qquad dx = dx_1 dx_2 \dots dx_n.$$

Since f is an integrable function and (A8) holds we have that every iso-functions $\hat{f}^{\wedge\wedge}$, $\hat{f}^{\wedge}$, $\hat{\hat{f}}$, $f^{\wedge}$ and $f^{\vee}$ are integrable functions. From here, using that $\hat{T}$ satisfies (A8), we conclude that the multiple iso-integral of the first kind of $\hat{f}$ over D exists.

Example 2.1.2. *Let $D=\{(x_1,x_2)\in R^2 : 0\le x_1\le 2, 0\le x_2\le 2-x_1\}$, $\hat{T}_1=3$, $f(x)=x_1+x_2$, $\hat{T}(x)=e^{x_1+x_2}$, $x=(x_1,x_2)\in D$. Then*

$$\hat{f}^{\wedge}(\hat{x})=\tfrac{f(x)}{\hat{T}(x)}=\tfrac{x_1+x_2}{e^{x_1+x_2}}=(x_1+x_2)e^{-(x_1+x_2)},$$

$$\hat{T}(x)-x_1\hat{T}_{x_1}(x)=e^{x_1+x_2}-x_1e^{x_1+x_2}=(1-x_1)e^{x_1+x_2},$$

$$\hat{T}(x)-x_2\hat{T}_{x_2}(x)=e^{x_1+x_2}-x_2e^{x_1+x_2}=(1-x_2)e^{x_1+x_2}.$$

From here

$$I=\hat{\int}_D^1\hat{f}^{\wedge}(\hat{x})\hat{\times}\hat{d}\hat{x}=3\int_0^2\int_0^{2-x_1}(x_1+x_2)(1-x_1)(1-x_2)e^{-2(x_1+x_2)}dx_2dx_1$$

$$=3\int_0^2\int_0^{2-x_1}(x_1+x_2-x_1^2-x_2^2+x_1^2x_2+x_1x_2^2-2x_1x_2)e^{-2(x_1+x_2)}dx_2dx_1$$

$$=-\tfrac{3}{2}\int_0^2(x_1+x_2-x_1^2-x_2^2+x_1^2x_2+x_1x_2^2-2x_1x_2)e^{-2(x_1+x_2)}\Big|_{x_2=0}^{x_2=1}dx_1$$

$$+\tfrac{3}{2}\int_0^2\int_0^{2-x_1}(1-2x_2-2x_1+x_1^3+2x_1x_2)e^{-2(x_1+x_2)}dx_2dx_1 \qquad l$$

$$=-\tfrac{3}{2}e^{-4}\int_0^2(-2+4x_1-2x_1^2)dx_1+\tfrac{3}{2}\int_0^2(x_1-x_1^2)e^{-2x_1}dx_1$$

$$+\tfrac{3}{2}\int_0^2\int_0^{2-x_1}(1-2x_2-2x_1+x_1^3+2x_1x_2)e^{-2(x_1+x_2)}dx_2dx_1$$

$$=3\int_0^2(x_1-1)^2e^{-4}dx_1+\tfrac{3}{2}\int_0^2(x_1-x_1^2)e^{-2x_1}dx_1$$

$$+\tfrac{3}{2}\int_0^2\int_0^{2-x_1}(1-2x_2-2x_1+x_1^3+2x_1x_2)e^{-2(x_1+x_2)}dx_2dx_1.$$

Let

$$I_1=3\int_0^2(x_1-1)^2e^{-4}dx_1+\tfrac{3}{2}\int_0^2(x_1-x_1^2)e^{-2x_1}dx_1,$$

$$J_1=\tfrac{3}{2}\int_0^2\int_0^{2-x_1}(1-2x_2-2x_1+x_1^3+2x_1x_2)e^{-2(x_1+x_2)}dx_2dx_1.$$

Then

$$I_1=e^{-4}(x_1-1)^3\Big|_{x_1=0}^{x_1=2}-\tfrac{3}{4}(x_1-x_1^2)e^{-2x_1}\Big|_{x_1=0}^{x_1=2}+\tfrac{3}{4}\int_0^2(1-2x_1)e^{-2x_1}dx_1$$

$$=\tfrac{7}{2}e^{-4}+\tfrac{3}{4}\int_0^2(1-2x_1)e^{-2x_1}dx_1$$

$$=\tfrac{7}{2}e^{-4}-\tfrac{3}{8}(1-2x_1)e^{-2x_1}\Big|_{x_1=0}^{x_1=2}+\tfrac{3}{4}\int_0^2e^{-2x_1}dx_1$$

$$=\tfrac{37}{8}e^{-4}+\tfrac{3}{8}-\tfrac{3}{8}e^{-2x_1}\Big|_{x_1=0}^{x_1=2}$$

$$=\tfrac{17}{4}e^{-4}+\tfrac{3}{4}.$$

Now we consider J_1. For it we have

$$J_1 = -\tfrac{3}{2}\int_0^2 (1-2x_2-2x_1+x_1^2+2x_1x_2)e^{-2(x_1+x_2)}\Big|_{x_2=0}^{x_2=2-x_1} dx_1$$

$$+\tfrac{3}{4}\int_0^2\int_0^{2-x_1}(-2+2x_1)e^{-2(x_1+x_2)}dx_2dx_1$$

$$= -\tfrac{3}{4}\int_0^2(-3+4x_1-x_1^2)e^{-4}dx_1+\tfrac{3}{4}\int_0^2(x_1-1)^2e^{-2x_1}dx_1$$

$$-\tfrac{3}{4}\int_0^2(-1+x_1)e^{-2(x_1+x_2)}\Big|_{x_2=0}^{x_2=2-x_1}dx_1$$

$$= -\tfrac{3}{4}\int_0^2(-3+4x_1-x_1^2)e^{-4}dx_1+\tfrac{3}{4}\int_0^2(x_1-1)^2e^{-2x_1}dx_1$$

$$-\tfrac{3}{4}\int_0^2(-1+x_1)e^{-4}dx_1+\tfrac{3}{4}\int_0^2(-1+x_1)e^{-2x_1}dx_1$$

$$= -\tfrac{3}{4}\int_0^2(-4+5x_1-x_1^2)e^{-4}dx_1+\tfrac{3}{4}\int_0^2(x_1^2-x_1)e^{-2x_1}dx_1$$

$$= -\tfrac{3}{4}e^{-4}\left(-4x_1+\tfrac{5}{2}x_1^2-\tfrac{x_1^3}{3}\right)\Big|_{x_1=0}^{x_1=2}-\tfrac{3}{8}(x_1^2-x_1)e^{-2x_1}\Big|_{x_1=0}^{x_1=2}$$

$$+\tfrac{3}{8}\int_0^2(2x_1-1)e^{-2x_1}dx_1$$

$$= -\tfrac{1}{4}e^{-4}-\tfrac{3}{16}(2x_1-1)e^{-2x_1}\Big|_{x_1=0}^{x_1=2}+\tfrac{3}{8}\int_0^2 e^{-2x_1}dx_1$$

$$= -\tfrac{13}{16}e^{-4}-\tfrac{3}{16}-\tfrac{3}{16}e^{-2x_1}\Big|_{x_1=0}^{x_1=2}$$

$$= -e^{-4}.$$

Consequently,

$$I = I_1+J_1 = \frac{13}{4}e^{-4}+\frac{3}{4}.$$

Exercise 2.1.3. *Let $D=\{(x_1,x_2)\in\mathbb{R}^3 : 0\le x_1\le 3-2x_2, 0\le x_2\le 1\}$, $\hat{T}_1=3$, $f(x)=x_1^2-x_2$, $\hat{T}(x)=e^{x_1}$, $x=(x_1,x_2)\in D$. Compute*

$$\hat{\int}_D^1 \hat{f}^\wedge(x)\hat{\times}d\hat{x}, \qquad \hat{\int}_D^1 f^\wedge(x)\hat{\times}d\hat{x}.$$

Definition 2.1.4. *The multiple iso-integral of the second kind of the iso-function $\hat{f}$ over D is defined as follows*

$$\hat{\int}_D^2 \hat{f}(x)\hat{\times}d\hat{x},$$

where

$$d\hat{x} = d\hat{x}_1 d\hat{x}_2\dots d\hat{x}_n,$$

$$d\hat{x}_i = d\left(\tfrac{x_i}{\hat{T}(x)}\right) = \tfrac{\hat{T}(x)-x_i\hat{T}_{x_i}}{\hat{T}^2(x)}dx_i, \qquad i=1,2,\dots,n.$$

We can rewrite the multiple iso-integral of the second kind in the following manner

$$\hat{\int}^{1}_{D}\hat{T}(x)\hat{\times}\hat{d}\hat{x} = \int_D \frac{1}{\hat{T}(x)}\hat{f}(x)\hat{T}_1\prod_{i=1}^{n}\frac{\hat{T}(x)-x_i\hat{T}_{x_i}(x)}{\hat{T}^2(x)}dx_i$$

$$= \hat{T}_1\int_D \hat{f}(x)\frac{1}{\hat{T}^{2n+1}(x)}\prod_{i=1}^{n}(\hat{T}(x)-x_i\hat{T}_{x_i}(x))dx, \qquad dx = dx_1dx_2\dots dx_n.$$

Since f is an integrable function and (A8) holds we have that every iso-functions $\hat{f}^{\wedge\wedge}$, $\hat{f}^{\wedge}$, $\hat{\hat{f}}$, $f^{\wedge}$ and $f^{\vee}$ are integrable functions. From here, using that $\hat{T}$ satisfies (A8), we conclude that the multiple iso-integral of the second kind of $\hat{f}$ over D exists.

Example 2.1.5. *Let $D = \{(x_1,x_2) : 0 \le x_1 \le 1, 0 \le x_2 \le 2-x_1\}$, $f(x) = x_1+x_2$, $\hat{T}(x) = e^{x_1}$, $\hat{T}_1 = 3$, $x = (x_1,x_2) \in D$. Then*

$$f^{\wedge}(x) = f(x\hat{T}(x)) = f(x_1\hat{T}(x), x_2\hat{T}(x)) = (x_1+x_2)e^{x_1},$$

$$\hat{T}(x)-x_1\hat{T}_{x_1}(x) = e^{x_1}-x_1e^{x_1} = (1-x_1)e^{x_1},$$

$$\hat{T}(x)-x_2\hat{T}_{x_2}(x) = e^{x_1}.$$

From here and from the definition for the multiple iso-integral of the second kind we get

$$\hat{\int}^{2} f^{\wedge}(x)\hat{\times}\hat{d}\hat{x} = 3\int_0^1\int_0^{2-x_1}(x_1+x_2)e^{-x_1}\frac{(1-x_1)e^{x_1}}{e^{2x_1}}\frac{e^{x_1}}{e^{2x_1}}dx_2dx_1$$

$$= 3\int_0^1\int_0^{2-x_1}(x_1+x_2)(1-x_1)e^{-3x_1}dx_2dx_1$$

$$= 3\int_0^1 x_1(1-x_1)(2-x_1)e^{-3x_1}dx_1 + 3\int_0^1(1-x_1)e^{-3x_1}\int_0^{2-x_1}x_2dx_2dx_1$$

$$= 3\int_0^1 x_1(1-x_1)(2-x_1)e^{-3x_1}dx_1 + \frac{3}{2}\int_0^1(1-x_1)e^{-3x_1}x_2^2\Big|_{x_2=0}^{x_2=2-x_1}dx_1$$

$$= \frac{3}{2}\int_0^1(x_1^3-x_1^2-4x_1+4)e^{-3x_1}dx_1$$

$$= -\frac{1}{2}(x_1^3-x_1^2-4x_1+4)e^{-3x_1}\Big|_{x_1=0}^{x_1=1} + \frac{1}{2}\int_0^1(3x_1^2-2x_1-4)e^{-3x_1}dx_1$$

$$= 2-\frac{1}{6}(3x_1^2-2x_1-4)e^{-3x_1}\Big|_{x_1=0}^{x_1=1} + \frac{1}{3}\int_0^1(3x_1-1)e^{-3x_1}dx_1$$

$$= \frac{4}{3}+\frac{1}{2}e^{-3}-\frac{1}{9}(3x_1-1)e^{-3x_1}\Big|_{x_1=0}^{x_1=1} + \frac{1}{3}\int_0^1 e^{-3x_1}dx_1$$

$$= \frac{11}{9}+\frac{5}{18}e^{-3}-\frac{1}{9}e^{-3x_1}\Big|_{x_1=0}^{x_1=1}$$

$$= \frac{4}{3}+\frac{1}{6}e^{-3}.$$

Exercise 2.1.6. *Let* $D = \{(x_1, x_2) : 0 \le x_1 \le 3 - x_2, 0 \le x_2 \le 2\}$, $f(x) = x_1^2 + x_2^2$, $\hat{T}(x) = e^{x_2}$, $x = (x_1, x_2) \in D$, $\hat{T}_1 = 3$. *Compute*

$$\hat{\int}^2_D \hat{f}^\wedge(\hat{x})\hat{\times}d\hat{x}, \qquad \hat{\int}^2_D \hat{f}^\wedge(x)\hat{\times}d\hat{x}, \qquad \hat{\int}^2_D \hat{f}(\hat{x})\hat{\times}d\hat{x}, \qquad \hat{\int}^2_D f^\wedge(x)\hat{\times}d\hat{x}.$$

Definition 2.1.7. *The multiple iso-integral of the third kind of the iso-function* $\hat{f}$ *over D is defined as follows*

$$\hat{\int}^3_D \hat{f}(x)\hat{\times}\hat{d}x,$$

where

$$\hat{d}x = \hat{d}x_1\hat{d}x_2\ldots\hat{d}x_n,$$

$$\hat{d}x_i = \hat{T}(x)dx_i, \qquad i = 1, 2, \ldots, n.$$

We can rewrite the multiple iso-integral of the third kind in the following manner

$$\hat{\int}^3_D \hat{f}(x)\hat{\times}\hat{d}x = \int_D \tfrac{1}{\hat{T}(x)}\hat{f}(x)\hat{T}_1\hat{T}^n(x)dx = \hat{T}_1\int_D \hat{f}(x)\hat{T}^{n-1}(x)dx.$$

Since f is an integrable function and (A8) holds we have that every iso-functions $\hat{f}^{\wedge\wedge}$, $\hat{f}^\wedge$, $\hat{\hat{f}}$, $f^\wedge$ and $f^\vee$ are integrable functions. From here, using that $\hat{T}$ satisfies (A8), we conclude that the multiple iso-integral of the third kind of $\hat{f}$ over D exists.

Example 2.1.8. *Let* $D = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \le x_1 \le 2, 0 \le x_2 \le 2 - x_1\}$, $f(x) = \sqrt{x_1^2 + 2x_2}$, $\hat{T}(x) = e^{x_2}$, $x = (x_1, x_2) \in D$, $\hat{T}_1 = 4$. *Then*

$$\hat{f}^\wedge(\hat{x}) = \frac{f(x)}{\hat{T}(x)} = \frac{x_1^2 + 2x_2}{e^{x_2}} = \sqrt{x_1^2 + 2x_2}e^{-x_2}.$$

We will compute the iso-integral

$$I = \hat{\int}^3_D (\hat{f}^\wedge(\hat{x}))^2\hat{\times}\hat{d}x.$$

For it we have

$$I = 4\int_0^2\int_0^{2-x_1}(x_1^2+2x_2)e^{-x_2}dx_1dx_2$$

$$= 4\int_0^2 x_1^2\int_0^{2-x_1} e^{-x_2}dx_2dx_1 + 8\int_0^2\int_0^{2-x} x_2e^{-x_2}dx_2dx_1$$

$$= 4\int_0^2 x_1^2 e^{-x_2}\Big|_{x_2=0}^{x_2=2-x_1} dx_1 + 8\int_0^2\left(-x_2e^{-x_2}\Big|_{x_2=0}^{x_2=2-x_1}\right)dx_1$$

$$+8\int_0^2\int_0^{2-x_1} e^{-x_2}dx_2dx_1$$

$$= 4\int_0^2 x_1^2\left(e^{x_1-2}-1\right)dx_1 + 8\int_0^2(x_1-2)e^{x_1-2}dx_1$$

$$+8\int_0^2 e^{-x_2}\Big|_{x_2=0}^{x_2=2-x_1} dx_1$$

$$= 4x_1^2e^{x_1-2}\Big|_{x_1=0}^{x_1=2} - \tfrac{4}{3}x_1^3\Big|_{x_1=0}^{x_1=2} - 8\int_0^2 x_1e^{x_1-2}dx_1$$

$$+8(x_1-2)e^{x_1-2}\Big|_{x_1=0}^{x_1=2} - 8\int_0^2 e^{x_1-2}dx_1 - 8\int_0^2 e^{x_1-2}dx_1 + 8\int_0^2 e^{x_1-2}dx_1 - 16$$

$$= \tfrac{16}{3} + 16e^{-2} - 8x_1e^{x_1-2}\Big|_{x_1=0}^{x_1=2} + 8\int_0^2 e^{x_1-2}dx_1$$

$$= -\tfrac{32}{3} + 16e^{-2} + 8e^{x_1-2}\Big|_{x_1=0}^{x_1=2}$$

$$= -\tfrac{8}{3} + 8e^{-2}.$$

Exercise 2.1.9. *Let* $D = \{(x_1,x_2) : 0 \le x_1 \le x_2^2+1, 0 \le x_2 \le 2\}$, $f(x) = x_1^2+x_2$, $\hat{T}(x) = x_1^2+x_2^2+1$, $x = (x_1,x_2) \in D$, $\hat{T}_1 = 3$. *Compute*

$$\hat{\int}_D^3 f(\hat{x})\hat{\times}\hat{d}x.$$

Definition 2.1.10. *The multiple iso-integral of the fourth kind of the iso-function* $\hat{f}$ *over D is defined as follows*

$$\hat{\int}_D^4 \hat{f}(x)\hat{d}\hat{x}.$$

We can rewrite the multiple iso-integral of the fourth kind in the following manner

$$\hat{\int}_D^4 \hat{f}(x)\hat{\times}\hat{d}x = \int_D \tfrac{1}{\hat{T}(x)}\hat{f}(x)\prod_{i=1}^n \tfrac{\hat{T}(x)-x_i\hat{T}_{x_i}(x)}{\hat{T}(x)}dx$$

$$= \int_D f(x)\tfrac{1}{\hat{T}^{n+1}(x)}\prod_{i=1}^n(\hat{T}(x)-x_i\hat{T}_{x_i}(x))dx.$$

Remark 2.1.11. *In fact, we have*

$$\hat{\int}_D^1 \hat{f}(x)\hat{\times}\hat{d}\hat{x} = \hat{T}_1\hat{\int}_D^4 \hat{f}(x)\hat{d}\hat{x}.$$

Therefore the multiple iso-integral of the fourth kind exists for every kind iso-functions $\hat{f}^{\wedge\wedge}$, $\hat{f}^{\wedge}$, $\hat{\hat{f}}$, $f^{\wedge}$ and $f^{\vee}$.

Definition 2.1.12. *The multiple iso-integral of the fifth kind of the iso-function $\hat{f}$ over D is defined as follows*

$$\hat{\int}_D^5 \hat{f}(x)d\hat{x}.$$

We can rewrite the multiple iso-integral of the fifth kind in the following manner

$$\hat{\int}_D^5 \hat{f}(x)d\hat{x} = \int_D \frac{1}{\hat{T}(x)}\hat{f}(x)\prod_{i=1}^n \frac{\hat{T}(x)-x_i\hat{T}_{x_i}^2(x)}{\hat{T}^2(x)}dx$$

$$= \int_D f(x)\frac{1}{\hat{T}^{2n+1}(x)}\prod_{i=1}^n(\hat{T}(x)-x_i\hat{T}_{x_i}(x))dx.$$

Remark 2.1.13. *In fact, we have*

$$\hat{\int}_D^2 \hat{f}(x)\hat{\times}d\hat{x} = \hat{T}_1\hat{\int}_D^5 \hat{f}(x)d\hat{x}.$$

Therefore the multiple iso-integral of the fifth kind exists for every iso-functions $\hat{f}^{\wedge\wedge}$, $\hat{f}^{\wedge}$, $\hat{\hat{f}}$, $f^{\wedge}$ and $f^{\vee}$

Definition 2.1.14. *The multiple iso-integral of the sixth kind of the iso-function $\hat{f}$ over D we define as follows*

$$\hat{\int}_D^6 \hat{f}(x)\hat{\times}dx.$$

We can rewrite the multiple iso-integral of the sixth kind as follows

$$\hat{\int}_D^6 \hat{f}(x)\hat{\times}dx = \hat{T}_1\int_D \frac{1}{\hat{T}(x)}\hat{f}(x)dx.$$

Since f is an integrable function over D and $\hat{T}$ satisfies $(A8)$ we conclude that the multiple iso-integral of the sixth kind exists for every iso-functions $\hat{f}^{\wedge\wedge}$, $\hat{f}^{\wedge}$, $\hat{\hat{f}}$, $f^{\wedge}$ and $f^{\vee}$.

Example 2.1.15. *Let $D = \{(x_1,x_2) : 0 \le x_1 \le 4-x_2, 0 \le x_2 \le 2\}$, $f(x) = x_1+4x_2$, $\hat{T}(x) = e^{-x_1}$, $x = (x_1,x_2) \in D$, $\hat{T}_1 = 2$. We will compute*

$$I = \hat{\int}_D^6 f(\hat{x})\hat{\times}dx.$$

We have

$$f(\hat{x}) = f\Big(\frac{x}{\hat{T}(x)}\Big) = f\Big(\frac{x_1}{\hat{T}(x)}, \frac{x_2}{\hat{T}(x)}\Big) = \frac{x_1}{\hat{T}(x)} + 4\frac{x_2}{\hat{T}(x)} = (x_1+4x_2)e^{x_1},$$

and

$$I = 2\int_0^2 \int_0^{4-x_2} e^{x_1}(x_1+4x_2)e^{x_1}dx_1dx_2$$

$$= 2\int_0^2 \int_0^{4-x_2} x_1e^{2x_1}dx_1dx_2 + 8\int_0^2 x_2\int_0^{4-x_2} e^{2x_1}dx_1dx_2$$

$$= \int_0^2 \left(x_1e^{2x_1}\Big|_{x_1=0}^{x_1=4-x_2}\right)dx_2 - \int_0^2\int_0^{4-x_2} e^{2x_1}dx_1dx_2$$

$$+4\int_0^2 x_2e^{2x_1}\Big|_{x_1=0}^{x_1=4-x_2}dx_2$$

$$= \int_0^2(4-x_2)e^{8-2x_2}dx_2 - \frac{1}{2}\int_0^2 e^{2x_1}\Big|_{x_1=0}^{x_1=4-x_2}dx_2$$

$$+4\int_0^2 x_2e^{8-x_2}dx_2 - 4\int_0^2 x_2dx_2$$

$$= -\frac{1}{2}(4-x_2)e^{8-2x_2}\Big|_{x_2=0}^{x_2=2} - \frac{1}{2}\int_0^2 e^{8-2x_2}dx_2$$

$$-\frac{1}{2}\int_0^2 e^{8-2x_2}dx_2 + 1 - 2x_2e^{8-2x_2}\Big|_{x_2=0}^{x_2=4}$$

$$+2\int_0^2 e^{8-2x_2}dx_2 - 2x_2^2\Big|_{x_2=0}^{x_2=2}$$

$$= -e^4 + 2e^8 - 15 - \frac{1}{2}e^{8-2x_2}\Big|_{x_2=0}^{x_2=2}$$

$$= -\frac{3}{2}e^4 + \frac{5}{2}e^8 - 15.$$

Exercise 2.1.16. *Let $D = \{(x_1,x_2) \in \mathbb{R}^2 : 0 \le x_1 \le 3, 0 \le x_2 \le 3-2x_1\}$, $f(x) = x_1^2 + 2x_1x_2$, $\hat{T}(x) = x_1 + x_2$, $x = (x_1,x_2) \in D$, $\hat{T}_1 = 4$. Compute*

$$\hat{\int}_D^6 \hat{f}^\wedge(x)\hat{\times}dx.$$

Definition 2.1.17. *The multiple iso-integral of the seventh kind of the iso-function $\hat{f}$ over D is defined as follows*

$$\hat{\int}_D \hat{f}(x)\hat{d}x.$$

We can represent the multiple iso-integral of the seventh kind in the following way.

$$\hat{\int}_D \hat{f}(x)\hat{d}x = \int_D \frac{1}{\hat{T}(x)}\hat{f}(x)\hat{T}^n(x)dx = \int_D \hat{T}^{n-1}(x)\hat{f}(x)dx.$$

Remark 2.1.18. *In fact, we have*

$$\hat{\int}_D^3 \hat{f}(x)\hat{\times}\hat{d}x = \hat{T}_1\hat{\int}_D^7 \hat{f}(x)\hat{d}x.$$

Consequently the multiple iso-integral of the seventh kind exists for every iso-functions $\hat{f}^{\wedge\wedge}$, $\hat{f}^{\wedge}$, $\hat{\hat{f}}$, $f^{\wedge}$ and $f^{\vee}$.

Definition 2.1.19. *The multiple iso-integral of the eighth kind of the iso-function $\hat{f}$ over D is defined as follows*

$$\hat{\int}^{8}_{D}\hat{f}(x)dx.$$

We can represent the multiple iso-integral of the eighth kind in the following way

$$\hat{\int}^{8}_{D}\hat{f}(x)dx = \int_{D}\frac{1}{\hat{T}(x)}\hat{f}(x)dx.$$

Because f is an integrable function and $\hat{T}$ satisfies (A8) we have that the multiple iso-integral of the eighth kind exists for every iso-functions $\hat{f}^{\wedge\wedge}$, $\hat{f}^{\wedge}$, $\hat{\hat{f}}$, $f^{\wedge}$ and $f^{\vee}$.

Example 2.1.20. *Let $D = \{(x_1,x_2) \in \mathbb{R}^2 : 0 \le x_1 \le 1, 0 \le x_2 \le 1-x_1\}$, $f(x) = 2x_1^2 + x_2$, $\hat{T}(x) = e^{x_2}$, $x = (x_1,x_2) \in D$, $\hat{T}_1 = 2$. Then*

$$f^{\wedge}(x) = f(x\hat{T}(x)) = f(x_1\hat{T}(x), x_2\hat{T}(x) = 2x_1^2\hat{T}^2(x) + x_2\hat{T}(x) = 2x_1^2e^{2x_2} + x_2e^{x_2}.$$

From here

$$\hat{\int}^{8}_{D}\hat{f}(x)dx = \int_0^1\int_0^{1-x_1}\tfrac{1}{e^{x_2}}(2x_1^2e^{2x_2} + x_2e^{x_2})dx_2dx_1$$

$$= \int_0^1\int_0^{1-x_1}(2x_1^2e^{x_2} + x_2)dx_2dx_1$$

$$= \int_0^1\left(2x_1^2e^{x_2}\Big|_{x_2=0}^{x_2=1-x_1} + \tfrac{x_2^2}{2}\Big|_{x_2=0}^{x_2=1-x_1}\right)dx_1$$

$$= 2\int_0^1 x_1^2e^{1-x_1}dx_1 + \tfrac{1}{2}\int_0^1(1-x_1)^2dx_1$$

$$= -2x_1^2e^{1-x_1}\Big|_{x_1=0}^{x_1=1} - 4\int_0^1 x_1e^{1-x_1}dx_1 - \tfrac{(1-x_1)^3}{6}\Big|_{x_1=0}^{x_1=1}$$

$$= -\tfrac{11}{6} + 4x_1e^{1-x_1}\Big|_{x_1=0}^{x_1=1} - 4\int_0^1 e^{1-x_1}dx_1$$

$$= \tfrac{13}{6} + 4e^{1-x_1}\Big|_{x_1=0}^{x_1=1}$$

$$= \tfrac{37}{6} - 4e.$$

Exercise 2.1.21. *Let $D = \{(x_1,x_2) \in \mathbb{R}^2 : 0 \le x_1 \le 3, 0 \le x_2 \le 4 - x_1^2\}$, $f(x) = x_1 + x_2^2$, $\hat{T}(x) = x_1 + x_2$, $\hat{T}_1 = 3$. Compute*

$$\hat{\int}^{8}_{D}\hat{f}^{\wedge}(x)dx.$$

-

Definition 2.1.22. *The multiple iso-integral of the ninth kind of the iso-function $\hat{f}$ over D is defined in the following manner*

$$\int_D^9 \hat{f}(x) \hat{\times} \hat{d}\hat{x}.$$

The multiple iso-integral of the ninth kind of the iso-function $\hat{f}$ over D can be represented as follows

$$\int_D^9 \hat{f}(x) \hat{\times} \hat{d}\hat{x} = \int_D \hat{f}(x) \hat{T}_1 \prod_{i=1}^n \frac{\hat{T}(x) - x_i \hat{T}_{x_i}(x)}{\hat{T}(x)} dx$$

$$= \hat{T}_1 \int_D \hat{f}(x) \frac{1}{\hat{T}^n(x)} \prod_{i=1}^n (\hat{T}(x) - x_i \hat{T}_{x_i}(x)) dx.$$

Because f is an integrable function over D and $\hat{T}$ satisfies (A8) then the multiple iso-integral of the ninth kind exists for every iso-functions $\hat{f}^{\wedge\wedge}$, $\hat{f}^{\wedge}$, $\hat{\hat{f}}$, $f^{\wedge}$ and $f^{\vee}$.

Example 2.1.23. *Let $D = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \le x_1 \le 1, 0 \le x_2 \le 2x_1\}$, $f(x) = 2x_1 + 3x_2$, $\hat{T}(x) = e^{x_1}$, $x = (x_1, x_2) \in D$, $\hat{T}_1 = 3$. Then*

$$f^{\wedge}(x) = f(x\hat{T}(x)) = f(x_1\hat{T}(x), x_2\hat{T}(x)) = 2x_1\hat{T}(x) + 3x_2\hat{T}(x) = (2x_1 + 3x_2)e^{x_1},$$

$$\hat{T}(x) - x_1\hat{T}_{x_1}(x) = e^{x_1} - x_1 e^{x_1} = (1 - x_1)e^{x_1},$$

$$\hat{T}(x) - x_2\hat{T}_{x_2}(x) = e^{x_1}.$$

From here

$$\int_D^9 f^{\wedge}(x) \hat{\times} \hat{d}\hat{x} = 3\int_0^1 \int_0^{2x_1} (2x_1 + 3x_2)e^{x_1} \frac{1}{e^{2x_1}} (1 - x_1)e^{2x_1} dx_2 dx_1$$

$$= 3\int_0^1 \int_0^{2x_1} \Big((2x_1 - 2x_1^2) + 3x_2(1 - x_1)\Big) e^{x_1} dx_2 dx_1$$

$$= 12\int_0^1 (x_1^2 - x_1^3)e^{x_1} dx_1 + 9\int_0^1 (1 - x_1)e^{x_1} \frac{x_2^2}{2}\Big|_{x_2=0}^{x_2=2x_1} dx_1$$

$$= 12(x_1^2 - x_1^3)e^{x_1}\Big|_{x_1=0}^{x_1=1} + 6\int_0^1 (-x_1 + 3x_1^2)e^{x_1} dx_1$$

$$= 6(-x_1 + 3x_1^2)e^{x_1}\Big|_{x_1=0}^{x_1=1} - 6\int_0^1 (-1 + 6x_1)e^{x_1} dx_1$$

$$= 12e - 6(-1 + 6x_1)e^{x_1}\Big|_{x_1=0}^{x_1=1} + 36\int_0^1 e^{x_1} dx_1$$

$$= -18e - 6 + 36e^{x_1}\Big|_{x_1=0}^{x_1=1}$$

$$= 18e - 42.$$

Exercise 2.1.24. *Let* $D = \{(x_1,x_2) \in \mathbb{R}^{\cdot} 0 \le x_1 \le 2x_2+1, 0 \le x_2 \le 1\}$*,* $f(x) = x_1 + x_2$*,* $\hat{T}(x) = x_1 + x_2 + 1$*,* $x = (x_1,x_2) \in D$*,* $\hat{T}_1 = 4$*. Compute*

$$\int_D^9 \hat{f}(\hat{x}) \hat{\times} \hat{d}\hat{x}.$$

Definition 2.1.25. *The multiple iso-integral of the tenth kind of the iso-function* $\hat{f}$ *is defined as follows*

$$\int_D^{10} \hat{f}(x) \hat{\times} d\hat{x}.$$

The multiple iso-integral of the tenth kind can be represented in the form

$$\int_D^{10} \hat{f}(x) \hat{\times} d\hat{x} = \int_D \hat{f}(x) \hat{T}_1 \prod_{i=1}^n \frac{\hat{T}(x) - x_i \hat{T}_{x_1}(x)}{\hat{T}^2(x)} dx$$

$$= \hat{T}_1 \int_D \hat{f}(x) \frac{1}{\hat{T}^{2n}(x)} \prod_{i=1}^n (\hat{T}(x) - x_i \hat{T}_{x_i}(x)) dx.$$

Since f is an integrable function over D and $\hat{T}$ satisfies (A8) then the multiple iso-integral of the tenth kind exists for all iso-functions $\hat{f}^{\wedge\wedge}$, $\hat{f}^{\wedge}$, $\hat{\hat{f}}$, $f^{\wedge}$ and $f^{\vee}$.

Example 2.1.26. *Let* $D = \{(x_1,x_2) \in \mathbb{R}^2 : 0 \le x_1 \le 1, 0 \le x_2 \le x_1\}$*,* $f(x) = x_1$*,* $\hat{T}(x) = \frac{1}{1+x_1+x_2}$*,* $x = (x_1,x_2) \in D$*,* $\hat{T}_1 = 2$*. Then*

$$\hat{f}^{\wedge}(\hat{x}) = \frac{f(x)}{\hat{T}(x)} = x_1(1+x_1+x_2),$$

$$\hat{T}(x) - x_1 \hat{T}_{x_1}(x) = \frac{1}{1+x_1+x_2} + \frac{x_1}{(1+x_1+x_2)^2} = \frac{1+2x_1+x_2}{(1+x_1+x_2)^2},$$

$$\hat{T}(x) - x_2 \hat{T}_{x_2}(x) = \frac{1}{1+x_1+x_2} + \frac{x_2}{(1+x_1+x_2)^2} = \frac{1+x_1+2x_2}{(1+x_1+x_2)^2}.$$

From here,

$$\int_D^{10} \hat{f}^\wedge(\hat{x})\hat{\times}d\hat{x} = 2\int_0^1\int_0^{x_1} x_1(1+x_1+x_2)(1+x_1+x_2)^4\frac{(1+2x_1+x_2)(1+x_1+2x_2)}{(1+x_1+x_2)^4}dx_2dx_1$$

$$= 2\int_0^1\int_0^{x_1} x_1(1+x_1+x_2)(1+2x_1+x_2)(1+x_1+2x_2)dx_2dx_1$$

$$= 2\int_0^1\int_0^{x_1}(x_1+4x_1^2+5x_1^3+2x_1^4)dx_2dx_1$$

$$+2\int_0^1\int_0^{x_1}(4x_1x_2+11x_1^2x_2+5x_1x_2^2+7x_1^3x_2+2x_1x_2^3+7x_1^2x_2^2)dx_2dx_1$$

$$= 2\int_0^1(x_1^2+4x_1^3+5x_1^4+2x_1^5)dx_1$$

$$+\int_0^1\left(4x_1x_2^2+11x_1^2x_2^2+\frac{10}{3}x_1x_2^3+7x_1^3x_2^2+x_1x_2^4+\frac{14}{3}x_1^2x_2^3\right)\Big|_{x_2=0}^{x_2=x_1}dx_1$$

$$= \left(\frac{2}{3}x_1^3+2x_1^4+2x_1^5+\frac{2}{3}x_1^6\right)\Big|_{x_1=0}^{x_1=1}+\int_0^1\left(4x_1^3+\frac{43}{3}x_1^4+\frac{38}{3}x_1^5\right)dx_1$$

$$= \frac{16}{3}+\left(x_1^4+\frac{43}{15}x_1^5+\frac{19}{9}x_1^6\right)\Big|_{x_1=0}^{x_1=1}$$

$$= \frac{509}{45}.$$

Exercise 2.1.27. *Let* $D = \{(x_1,x_2) \in \mathbb{R}^2 : 0 \le x_1 \le 2, 0 \le x_2 \le 3x_1+1\}$, $f(x) = x_1+x_2$, $\hat{T}(x) = x_1^2+x_2^2+1$, $x = (x_1,x_2) \in D$, $\hat{T}_1 = 4$. *Compute*

$$\int_D^{10} \hat{f}^\wedge(\hat{x})\hat{\times}d\hat{x}, \qquad \int_D^{10} f^\wedge(x)\hat{\times}d\hat{x}.$$

Definition 2.1.28. *The multiple iso-integral of the eleventh kind of the iso-function* $\hat{f}$ *is defined as follows*

$$\int_D^{11} \hat{f}(x)\hat{\times}\hat{d}x.$$

The multiple iso-integral of the eleventh kind can be represented in the following manner

$$\int^{11} \hat{f}(x)\hat{\times}\hat{d}x = \int_D \hat{f}(x)\hat{T}_1\hat{T}^n(x)dx = \hat{T}_1\int_D \hat{f}(x)\hat{T}^n(x)dx.$$

Because f is an integrable function over D and $\hat{T}$ satisfies (A8) then the multiple iso-integral of the eleventh kind exists for all iso-functions $\hat{f}^{\wedge\wedge}$, $\hat{f}^\wedge$, $\hat{\hat{f}}$, $f^\wedge$ and $f^\vee$.

Example 2.1.29. *Let* $D = \{(x_1,x_2) \in \mathbb{R}^2 : 0 \le x_1 \le 1, 0 \le x_2 \le 2x_1\}$, $f(x) = x_1+7x_2$, $\hat{T}(x) = e^{x_1+x_2}$, $x = (x_1,x_2) \in D$, $\hat{T}_1 = 2$. *Then*

$$f^\wedge(x) = f(x\hat{T}(x)) = f(x_1\hat{T}(x), x_2\hat{T}(x)) = x_1\hat{T}(x)+7x_2\hat{T}(x) = (x_1+7x_2)e^{x_1+x_2}.$$

From here,

$$\int_D^{11} f^{\wedge}(x)\hat{\times}\hat{d}x = 2\int_0^1\int_0^{2x_2}(x_1+7x_2)e^{x_1+x_2}e^{2(x_1+x_2)}dx_2dx_1$$

$$= 2\int_0^1\int_0^{2x_1}(x_1+7x_2)e^{3(x_1+x_2)}dx_2dx_1$$

$$= 2\int_0^1 x_1e^{3x_1}\int_0^{2x_1}e^{3x_2}dx_2dx_1 + 14\int_0^1 e^{3x_1}\int_0^{2x_1}x_2e^{3x_2}dx_2dx_1$$

$$= \frac{2}{3}\int_0^1 x_1e^{3x_1}e^{3x_2}\Big|_{x_2=0}^{x_2=2x_1}dx_1 + \frac{14}{3}\int_0^1 e^{3x_1}x_2e^{3x_2}\Big|_{x_2=0}^{x_2=2x_1}dx_1$$

$$-\frac{14}{3}\int_0^1 e^{3x_1}\int_0^{2x_1}e^{3x_2}dx_2dx_1$$

$$= 10\int_0^1 x_1e^{9x_1}dx_1 - \frac{2}{3}\int_0^1 x_1e^{3x_1}dx_1 - \frac{14}{9}\int_0^1 e^{3x_1}e^{3x_2}\Big|_{x_2=0}^{x_2=2x_1}dx_1$$

$$= \frac{10}{9}x_1e^{9x_1}\Big|_{x_1=0}^{x_1=1} - \frac{8}{3}\int_0^1 e^{9x_1}dx_1 - \frac{2}{9}x_1e^{3x_1}\Big|_{x_1=0}^{x_1=1} + \frac{16}{9}\int_0^1 e^{3x_1}dx_1$$

$$= \frac{10}{9}e^9 - \frac{2}{9}e^3 - \frac{8}{27}e^{9x_1}\Big|_{x_1=0}^{x_1=1} + \frac{16}{27}e^{3x_1}\Big|_{x_1=0}^{x_1=1}$$

$$= \frac{22}{27}e^9 + \frac{10}{27}e^3 - \frac{8}{27}.$$

Exercise 2.1.30. *Let $D = \{(x_1,x_2) \in \mathbb{R}^2 : 0 \le x_1 \le 2x_2+1, 0 \le x_2 \le 1\}$, $f(x) = x_1^2+x_2+1$, $\hat{T}(x) = x_1^2+x_2^2+1$, $x = (x_1,x_2) \in D$, $\hat{T}_1 = 12$. Compute*

$$\int_D^{11} \hat{f}(\hat{x})\hat{\times}\hat{d}x.$$

Definition 2.1.31. *The multiple iso-integral of the twelfth kind of the iso-function $\hat{f}$ over D is defined in the following manner*

$$\int_D^{12}\hat{f}(x)\hat{d}\hat{x}.$$

The multiple iso-integral of the twelfth kind of the iso-function $\hat{f}$ over D can be represented as follows

$$\int_D^{12}\hat{f}(x)\hat{d}\hat{x} = \int_D \hat{f}(x)\prod_{i=1}^n \frac{\hat{T}(x)-x_i\hat{T}_{x_i}(x)}{\hat{T}(x)}dx$$

$$= \int_D \hat{f}(x)\frac{1}{\hat{T}^n(x)}\prod_{i=1}^n(\hat{T}(x)-x_i\hat{T}_{x_i}(x))dx.$$

Remark 2.1.32. *In fact, we have*

$$\int_D^9 \hat{f}(x)\hat{\times}\hat{d}\hat{x} = \hat{T}_1\int_D^{12}\hat{f}(x)\hat{d}x,$$

therefore the multiple iso-integral of the twelfth kind exists for all iso-functions $\hat{f}^{\wedge\wedge}$, $\hat{f}^{\wedge}$, $\hat{\hat{f}}$, $f^{\wedge}$ and $f^{\vee}$.

Definition 2.1.33. *The multiple iso-integral of the thirteenth kind of the iso-function $\hat{f}$ is defined as follows*

$$\int_D^{13} \hat{f}(x) d\hat{x}.$$

The multiple iso-integral of the thirteenth kind can be represented in the form

$$\int_D^{13} \hat{f}(x) d\hat{x} = \int_D \hat{f}(x) \prod_{i=1}^n \frac{\hat{T}(x) - x_i \hat{T}_{x_1}(x)}{\hat{T}^2(x)} dx = \int_D \hat{f}(x) \frac{1}{\hat{T}^{2n}(x)} \prod_{i=1}^n (\hat{T}(x) - x_i \hat{T}_{x_i}(x)) dx.$$

Remark 2.1.34. *In fact, we have*

$$\int_D^{10} \hat{f}(x) \hat{\times} d\hat{x} = \hat{T}_1 \int_D^{13} \hat{f}(x) d\hat{x}.$$

therefore the multiple iso-integral of the thirteenth kind exists for all iso-functions $\hat{f}^{\wedge\wedge}$, $\hat{f}^{\wedge}$, $\hat{\hat{f}}$, $f^{\wedge}$ and $f^{\vee}$.

Definition 2.1.35. *The multiple iso-integral of the fourteenth kind of the iso-function $\hat{f}$ over D is defined as follows*

$$\int_D^{14} \hat{f}(x) \hat{\times} dx.$$

For the multiple iso-integral of the fourteenth kind we have the following representation

$$\int_D^{14} \hat{f}(x) \hat{\times} dx = \hat{T}_1 \int_D \hat{f}(x) dx.$$

Because f is an integrable function over D and $\hat{T}$ satisfies (A8) the multiple iso-integral of the fourteenth kind exists for all iso-functions $\hat{f}^{\wedge\wedge}$, $\hat{f}^{\wedge}$, $\hat{\hat{f}}$, $f^{\wedge}$ and $f^{\vee}$.

Definition 2.1.36. *The multiple iso-integral of the fifteenth kind of the iso-function $\hat{f}$ is defined as follows*

$$\int_D^{15} \hat{f}(x) \hat{d}x.$$

The multiple iso-integral of the fifteenth kind can be represented in the following manner

$$\int_D^{15} \hat{f}(x) \hat{d}x = \int_D \hat{f}(x) \hat{T}^n(x) dx.$$

Remark 2.1.37. *In fact, we have*

$$\int_D^{11} \hat{f}(x) \hat{\times} \hat{d}x = \hat{T}_1 \int_D^{15} \hat{f}(x) \hat{d}x.$$

Below we will use the following notation

$$P_i(x) = \hat{T}(x) - x_i \hat{T}_{x_i}(x), \qquad i = 1, 2, \ldots, n,$$

$$P(x) = \prod_{i=1}^n (\hat{T}(x) - x_i \hat{T}_{x_i}(x)),$$

and the following notation for the multiple iso-integral of the j-th kind, $j = 1, 2, \ldots, 15$, for the iso-function $\hat{f}$ over D

$$\int_D^i \hat{f}(x) \circledast^i x.$$

2.2. Properties of Multiple Iso-Integrals

Let g be an integrable function on D, h be a continuous function on D and $c \in \mathbb{R}$.

We now list some of the properties of the multiple iso-integrals.

1. $\int_D^i (\hat{f}(x) + \hat{g}(x)) \circledast^i x = \int_D^i \hat{f}(x) \circledast^i x + \int_D^i \hat{g}(x) \circledast^i x$, $i = 1, 2, \ldots, 15$.

2. $\int_D^i \hat{c} \hat{\times} \hat{f}(x) \circledast^i x = \hat{c} \hat{\times} \int_D^i \hat{f}(x) \circledast^i x$, $i = 1, 2, \ldots, 15$.

3. $\int_D^i \hat{c} \hat{f}(x) \circledast^i x = \hat{c} \int_D^i \hat{f}(x) \circledast^i x$, $i = 1, 2, \ldots, 15$.

4. $\int_D^i c \hat{\times} \hat{f}(x) \circledast^i x = c \hat{\times} \int_D^i \hat{f}(x) \circledast^i x$, $i = 1, 2, \ldots, 15$.

5. $\int_D^i c \hat{f}(x) \circledast^i x = c \int_D^i \hat{f}(x) \circledast^i x$, $i = 1, 2, \ldots, 15$.

6. If $f(x) \geq g(x)$ for every $x \in D$, then if $P(x) \geq 0$ for every $x \in D$, we have

$$\int_D^i \hat{f}(x) \circledast^i x \leq \int_D^i g(x) \circledast^i x, \qquad i = 1, 2, \ldots, 15.$$

7. If $f(x) \leq g(x)$ for every $x \in D$ and $P(x) \leq 0$ for every $x \in D$, then

$$\int_D^i \hat{f}(x) \circledast^i x \geq \int_D^i \hat{g}(x) \circledast^i x$$

for $i = 1, 2, 4, 5, 9, 10, 12, 13$, and

$$\int_D^i \hat{f}(x) \circledast^i x \leq \int_D^i \hat{g}(x) \circledast^i x$$

for $i = 3, 6, 7, 8, 11, 14, 15$.

8. $\left| \hat{\int}_D^1 \hat{f}(x) \hat{\times} \hat{d}\hat{x} \right| \leq \hat{T}_1 \int_D \frac{1}{\hat{T}^{n+1}(x)} |\hat{f}(x) P(x)| dx$.

9. $\left| \hat{\int}_D^2 \hat{f}(x) \hat{\times} d\hat{x} \right| \leq \hat{T}_1 \int_D \frac{1}{\hat{T}^{2n+1}(x)} |\hat{f}(x) P(x)| dx$.

10. $\left| \hat{\int}_D^3 \hat{f}(x) \hat{\times} \hat{d}x \right| \leq \hat{T}_1 \int_D \hat{T}^{n-1}(x) |\hat{f}(x)| dx$.

11. $\left| \hat{\int}_D^4 \hat{f}(x) \hat{d}\hat{x} \right| \leq \int_D \frac{1}{\hat{T}^{n+1}(x)} |\hat{f}(x) P(x)| dx$.

12. $\left| \hat{\int}_D^5 \hat{f}(x) d\hat{x} \right| \leq \int_D \frac{1}{\hat{T}^{2n+1}(x)} |\hat{f}(x) P(x)| dx$.

13. $\left| \hat{\int}_D^6 \hat{f}(x) \hat{\times} dx \right| \leq \hat{T}_1 \int_D \frac{1}{\hat{T}(x)} |\hat{f}(x)| dx$.

13. $\left| \hat{\int}_D^7 \hat{f}(x) \hat{d}x \right| \leq \int_D \hat{T}^{n-1}(x) |\hat{f}(x)| dx$.

14. $\left| \hat{\int}_D^8 \hat{f}(x) dx = \int_D \hat{T}(x) |\hat{f}(x)| dx$.

15. $\left|\int_D^9 \hat{f}(x)\hat{\times}\hat{d}\hat{x}\right| \leq \hat{T}_1 \int_D \frac{1}{\hat{T}^n(x)}|\hat{f}(x)P(x)|dx.$

16. $\left|\int_D^{10} \hat{f}(x)\hat{\times}\hat{d}x\right| \leq \hat{T}_1 \int_D \frac{1}{\hat{T}^{2n}(x)}|\hat{f}(x)P(x)|dx.$

17. $\left|\int_D^{11} \hat{f}(x)\hat{\times}\hat{d}x\right| \leq \hat{T}_1 \int_D \hat{T}^n(x)|\hat{f}(x)|dx.$

18. $\left|\int_D^{12} \hat{f}(x)d\hat{x}\right| \leq \int_D \frac{1}{\hat{T}^n(x)}|\hat{f}(x)P(x)|dx.$

19. $\left|\int_D^{13} \hat{f}(x)d\hat{x}\right| \leq \int_D \frac{1}{\hat{T}^{2n}(x)}|\hat{f}(x)P(x)|dx.$

20. $\left|\int_D^{14} \hat{f}(x)\hat{\times}dx\right| \leq \hat{T}_1 \int_D |\hat{f}(x)|dx.$

21. $\left|\int_D^{15} \hat{f}(x)\hat{d}x\right| \leq \int_D \hat{T}^n(x)|\hat{f}(x)|dx.$

22. (the iso-integral form of the mean value theorem)

$$\int_D^i \hat{h}(x)\hat{\times}\hat{f}(x) \circledast^i x = \hat{h}(x_0)\hat{\times}\int_D^i \hat{f}(x) \circledast^i x$$

for some $x_0 \in D$, $i = 1, 2, \ldots, 15$.

23. (the iso-integral form of the mean value theorem)

$$\int_D^i \hat{h}(x)\hat{f}(x) \circledast^i x = \hat{h}(x_0)\int_D^i \hat{f}(x) \circledast^i x$$

for some $x_0 \in D$, $i = 1, 2, \ldots, 15$.

24. (the iso-integral form of the mean value theorem)

$$\hat{\int}_D^{!} \hat{f}(x)\hat{\times}\hat{d}\hat{x} = \hat{T}_1 P_j(x_0)\int_D \hat{f}(x)\prod_{i=1, i\neq j}^{n} P_i(x)dx, \qquad j = 1, 2, \ldots, n,$$

for some $x_0 \in D$.

25. (the iso-integral form of the mean value theorem)

$$\hat{\int}_D^{1} \hat{f}(x)\hat{\times}\hat{d}\hat{x} = \hat{T}_1 P(x_0)\int_D \frac{1}{\hat{T}^{n+1}(x)}\hat{f}(x)dx.$$

for some $x_0 \in D$.

26. (the iso-integral form of the mean value theorem)

$$\hat{\int}_D^{1} \hat{f}(x)\hat{\times}\hat{d}\hat{x} = \hat{T}_1 \frac{1}{\hat{T}^l(x_0)}\int_D \frac{1}{\hat{T}^{n+1-l}(x)}\hat{f}(x)P(x)dx, \qquad l = 1, \ldots, n+1,$$

for some $x_0 \in D$.

27. (the iso-integral form of the mean value theorem)

$$\hat{\int}_D^2 \hat{f}(x)\hat{\times}d\hat{x} = \hat{T}_1 P_j(x_0)\int_D \hat{f}(x)\frac{1}{\hat{T}^{2n+1}(x)}\prod_{i=1,i\neq j}^{n} P_i(x)dx, \qquad j = 1,2,\ldots,n,$$

for some $x_0 \in D$.

28. (the iso-integral form of the mean value theorem)

$$\hat{\int}_D^2 \hat{f}(x)\hat{\times}d\hat{x} = \hat{T}_1 P(x_0)\int_D \hat{f}(x)\frac{1}{\hat{T}^{2n+1}(x)}dx,$$

for some $x_0 \in D$.

29. (the iso-integral form of the mean value theorem)

$$\hat{\int}_D^2 \hat{f}(x)\hat{\times}d\hat{x} = \hat{T}_1 \frac{1}{\hat{T}^l(x_0)}\int_D \hat{f}(x)\frac{1}{\hat{T}^{2n+1-l}(x)}P(x)dx, \qquad l = 1,2,\ldots,2n+1,$$

for some $x_0 \in D$.

30. (the iso-integral form of the mean value theorem)

$$\hat{\int}_D^3 \hat{f}(x)\hat{\times}\hat{d}x = \hat{T}_1 \hat{T}^l(x_0)\int_D \hat{f}(x)\hat{T}^{n-1-l}(x)dx, \qquad l = 1,2,\ldots,n-1,$$

for some $x_0 \in D$.

31. (the iso-integral form of the mean value theorem)

$$\hat{\int}_D^4 \hat{f}(x)\hat{d}\hat{x} = P_j(x_0)\int_D \hat{f}(x)\prod_{i=1,i\neq j}^{n} P_i(x)dx, \qquad j = 1,2,\ldots,n,$$

for some $x_0 \in D$.

32. (the iso-integral form of the mean value theorem)

$$\hat{\int}_D^4 \hat{f}(x)\hat{d}\hat{x} = P(x_0)\int_D \frac{1}{\hat{T}^{n+1}(x)}\hat{f}(x)dx.$$

for some $x_0 \in D$.

33. (the iso-integral form of the mean value theorem)

$$\hat{\int}_D^4 \hat{f}(x)\hat{d}\hat{x} = \frac{1}{\hat{T}^l(x_0)}\int_D \frac{1}{\hat{T}^{n+1-l}(x)}\hat{f}(x)P(x)dx, \qquad l = 1,\ldots,n+1,$$

for some $x_0 \in D$.

34. (the iso-integral form of the mean value theorem)

$$\hat{\int}_D^5 \hat{f}(x)d\hat{x} = P_j(x_0)\int_D \hat{f}(x)\frac{1}{\hat{T}^{2n+1}(x)}\prod_{i=1,i\neq j}^{n} P_i(x)dx, \qquad j=1,2,\ldots,n,$$

for some $x_0 \in D$.

35. (the iso-integral form of the mean value theorem)

$$\hat{\int}_D^5 \hat{f}(x)d\hat{x} = P(x_0)\int_D \hat{f}(x)\frac{1}{\hat{T}^{2n+1}(x)}dx,$$

for some $x_0 \in D$.

36. (the iso-integral form of the mean value theorem)

$$\hat{\int}_D^5 \hat{f}(x)d\hat{x} = \frac{1}{\hat{T}^l(x_0)}\int_D \hat{f}(x)\frac{1}{\hat{T}^{2n+1-l}(x)}P(x)dx, \qquad l=1,2,\ldots,2n+1,$$

for some $x_0 \in D$.

37. (the iso-integral form of the mean value theorem)

$$\hat{\int}_D^6 \hat{f}(x)\hat{\times}dx = \hat{T}_1\frac{1}{\hat{T}(x_0)}\int_D \hat{f}(x)dx,$$

for some $x_0 \in D$.

38. (the iso-integral form of the mean value theorem)

$$\hat{\int}_D^7 \hat{f}(x)\hat{d}x = \hat{T}^l(x_0)\int_D \hat{T}^{n-1}(x)\hat{f}(x)dx, \qquad l=1,2,\ldots,n-1,$$

for some $x_0 \in D$.

39. (the iso-integral form of the mean value theorem)

$$\hat{\int}_D^8 \hat{f}(x)dx = \frac{1}{\hat{T}(x_0)}\int_D \hat{f}(x)dx.$$

for some $x_0 \in D$.

40. (the iso-integral form of the mean value theorem)

$$\hat{\int}^9 \hat{f}(x)\hat{\times}\hat{d}\hat{x} = \hat{T}_1 P_j(x_0)\int_D \frac{1}{\hat{T}^n(x)}\hat{f}(x)\prod_{i=1,i\neq j}^{n} P_i(x)dx, \qquad j=1,2,\ldots,n,$$

for some $x_0 \in D$.

41. (the iso-integral form of the mean value theorem)

$$\hat{\int}^{9} \hat{f}(x)\hat{\times}\hat{d}\hat{x} = \hat{T}_1 P(x_0)\int_D \frac{1}{\hat{T}^n(x)}\hat{f}(x)dx,$$

for some $x_0 \in D$.

42. (the iso-integral form of the mean value theorem)

$$\hat{\int}^{9} \hat{f}(x)\hat{\times}\hat{d}\hat{x} = \hat{T}_1 \frac{1}{\hat{T}^l(x_0)}\int_D \frac{1}{\hat{T}^{n-l}(x)}\hat{f}(x)P(x)dx, \qquad l = 1,2,\ldots,n,$$

for some $x_0 \in D$.

43. (the iso-integral form of the mean value theorem)

$$\int^{10} \hat{f}(x)\hat{\times}d\hat{x} = \hat{T}_1 P_j(x_0)\int_D \hat{f}(x)\frac{1}{\hat{T}^{2n}(x)}\prod_{i=1,i\neq j}^{n} P_i(x)dx, \qquad j = 1,2,\ldots,n,$$

for some $x_0 \in D$.

44. (the iso-integral form of the mean value theorem)

$$\int^{10} \hat{f}(x)\hat{\times}d\hat{x} = \hat{T}_1 P(x_0)\int_D \hat{f}(x)\frac{1}{\hat{T}^{2n}(x)}dx,$$

for some $x_0 \in D$.

45. (the iso-integral form of the mean value theorem)

$$\int^{10} \hat{f}(x)\hat{\times}d\hat{x} = \hat{T}_1 \frac{1}{\hat{T}^l(x_0)}\int_D \hat{f}(x)\frac{1}{\hat{T}^{2n-l}(x)}P(x)dx, \qquad l = 1,2,\ldots,2n,$$

for some $x_0 \in D$.

46. (the iso-integral form of the mean value theorem)

$$\int_D^{11} \hat{f}(x)\hat{\times}\hat{d}x = \hat{T}_1\hat{T}^l(x_0)\int_D \hat{f}(x)\hat{T}^{n-l}(x)dx, \qquad l = 1,2,\ldots,n,$$

for some $x_0 \in D$.

47. (the iso-integral form of the mean value theorem)

$$\int_D^{12} \hat{f}(x)\hat{d}\hat{x} = P_j(x_0)\int_D \hat{f}(x)\frac{1}{\hat{T}^{2n}(x)}\prod_{i=1,i\neq j}^{n} P_i(x)dx, \qquad j = 1,2,\ldots,n,$$

for some $x_0 \in D$.

48. (the iso-integral form of the mean value theorem)

$$\int_D^{12} \hat{f}(x)\hat{d}\hat{x} = P(x_0)\int_D \hat{f}(x)\frac{1}{\hat{T}^n(x)}dx,$$

for some $x_0 \in D$.

49. (the iso-integral form of the mean value theorem)

$$\int_D^{12} \hat{f}(x)\hat{d}\hat{x} = \frac{1}{\hat{T}^l(x_0)} \int_D \hat{f}(x) \frac{1}{\hat{T}^{n-l}(x)} P(x)dx, \qquad l = 1, 2, \ldots, n,$$

for some $x_0 \in D$.

50. (the iso-integral form of the mean value theorem)

$$\int_D^{13} \hat{f}(x)d\hat{x} = P_j(x_0) \int_D \hat{f}(x) \frac{1}{\hat{T}^{2n}(x)} \prod_{i=1, i\neq j}^{n} P_i(x)dx, \qquad j = 1, 2, \ldots, n,$$

for some $x_0 \in D$.

51. (the iso-integral form of the mean value theorem)

$$\int_D^{13} \hat{f}(x)d\hat{x} = P(x_0) \int_D \hat{f}(x) \frac{1}{\hat{T}^{2n}(x)} dx,$$

for some $x_0 \in D$.

52. (the iso-integral form of the mean value theorem)

$$\int_D^{13} \hat{f}(x)d\hat{x} = \frac{1}{\hat{T}^l(x_0)} \int_D \hat{f}(x) \frac{1}{\hat{T}^{2n-l}(x)} P(x)dx, \qquad l = 1, 2, \ldots, 2n,$$

for some $x_0 \in D$.

53. (the iso-integral form of the mean value theorem)

$$\int_D^{15} \hat{f}(x)\hat{d}x = \hat{T}^l(x_0) \int_D \hat{f}(x)\hat{T}^{n-l}(x)dx, \qquad l = 1, 2, \ldots, n,$$

for some $x_0 \in D$.

54. If $D_1, D_2 \subset D$, $D_1 \bigcap D_2 = \emptyset$, $\hat{F}^{\wedge\wedge}$, $\hat{f}^{\wedge}$, $\hat{\hat{f}}$, $f^{\wedge}$, $f^{\vee}$ are defined on D_1 and D_2 then

$$\int_{D_1 \bigcup D_2}^{i} \hat{f}(x) \circledast^i x = \int_{D_1}^{i} \hat{f}(x) \circledast^i x + \int_{D_2}^{i} \hat{f}(x) \circledast^i x.$$

55. If the measure of D, $\mu(D)$, is equal to zero then

$$\int_D^i \hat{f}(x) \circledast^i x = 0.$$

56. Let $D_1 \subset D$ and $\hat{f}^{\wedge\wedge}$, $\hat{f}^{\wedge}$, $\hat{\hat{f}}$, $f^{\wedge}$, $f^{\vee}$ are defined on D_1. Then if $P(x) \geq 0$ for every $x \in D$ we have

$(A9)\ \int_{D_1}^i \hat{f}(x) \circledast^i x \leq \int_D^i \hat{f}(x) \circledast^i x, \qquad i = 1, 2, \ldots, 15,$

if $P(x) \leq 0$ for every $x \in D$, then we can not make the conclusion (A9) for $i = 1, 2, 4, 5, 9, 10, 12, 13$.

Definition 2.2.1. *The iso-volume of the first kind of D is defined as follows*

$$\hat{\int}_D^1 1 \hat{\times} \hat{d}\hat{x}.$$

Definition 2.2.2. *The iso-volume of the second kind of D is defined as follows*

$$\hat{\int}_D^2 1 \hat{\times} d\hat{x}.$$

Definition 2.2.3. *The iso-volume of the third kind of D is defined as follows*

$$\hat{\int}_D^3 1 \hat{\times} \hat{d}x.$$

Definition 2.2.4. *The iso-volume of the fourth kind of D is defined as follows*

$$\hat{\int}_D^4 1 \hat{d}\hat{x}.$$

Definition 2.2.5. *The iso-volume of the fifth kind of D is defined as follows*

$$\hat{\int}_D^5 1 d\hat{x}.$$

Definition 2.2.6. *The iso-volume of the sixth kind of D is defined as follows*

$$\hat{\int}_D^6 1 \hat{\times} dx.$$

Definition 2.2.7. *The iso-volume of the seventh kind of D is defined as follows*

$$\hat{\int}_D^7 1 \hat{d}x.$$

Definition 2.2.8. *The iso-volume of the eighth kind of D is defined as follows*

$$\hat{\int}_D^8 1 dx.$$

Definition 2.2.9. *The iso-volume of the ninth kind of D is defined as follows*

$$\int_D^9 1 \hat{\times} \hat{d}\hat{x}.$$

Definition 2.2.10. *The iso-volume of the tenth kind of D is defined as follows*

$$\int_D^{10} 1 \hat{\times} d\hat{x}.$$

Definition 2.2.11. *The iso-volume of the eleventh kind of D is defined as follows*

$$\int_D^{11} 1 \hat{\times} \hat{d}x.$$

Definition 2.2.12. *The iso-volume of the twelfth kind of D is defined as follows*

$$\int_{D}^{12} 1 \hat{d}\hat{x}.$$

Definition 2.2.13. *The iso-volume of the thirteenth kind of D is defined as follows*

$$\int_{D}^{13} 1 d\hat{x}.$$

Definition 2.2.14. *The iso-volume of the fourteenth kind of D is defined as follows*

$$\int_{D}^{14} 1 \hat{\times} dx.$$

Definition 2.2.15. *The iso-volume of the fifteenth kind of D is defined as follows*

$$\int_{D}^{15} 1 \hat{d}x.$$

Definition 2.2.16. *The iso-volume of the sixteenth kind of D is defined as follows*

$$\hat{\int}_{D}^{1} \hat{T}(x) \hat{\times} \hat{d}\hat{x}.$$

Definition 2.2.17. *The iso-volume of the seventeenth kind of D is defined as follows*

$$\hat{\int}_{D}^{2} \hat{T}(x) \hat{\times} d\hat{x}.$$

Definition 2.2.18. *The iso-volume of the eighteenth kind of D is defined as follows*

$$\hat{\int}_{D}^{3} \hat{T}(x) \hat{\times} \hat{d}x.$$

Definition 2.2.19. *The iso-volume of the nineteenth kind of D is defined as follows*

$$\hat{\int}_{D}^{4} \hat{T}(x) \hat{d}\hat{x}.$$

Definition 2.2.20. *The iso-volume of the twentieth kind of D is defined as follows*

$$\hat{\int}_{D}^{5} \hat{T}(x) d\hat{x}.$$

Definition 2.2.21. *The iso-volume of the twenty-first kind of D is defined as follows*

$$\hat{\int}_{D}^{6} \hat{T}(x) \hat{\times} dx.$$

Definition 2.2.22. *The iso-volume of the twenty-second kind of D is defined as follows*

$$\hat{\int}_{D}^{7} \hat{T}(x) \hat{d}x.$$

Definition 2.2.23. *The iso-volume of the twenty-third kind of D is defined as follows*

$$\hat{\int}_D^8 \hat{T}(x)dx.$$

Definition 2.2.24. *The iso-volume of the twenty-fourth kind of D is defined as follows*

$$\int_D^9 \hat{T}(x)\hat{\times}\hat{d}\hat{x}.$$

Definition 2.2.25. *The iso-volume of the twenty-fifth kind of D is defined as follows*

$$\int_D^{10} \hat{T}(x)\hat{\times}d\hat{x}.$$

Definition 2.2.26. *The iso-volume of the twenty-sixth kind of D is defined as follows*

$$\int_D^{11} \hat{T}(x)\hat{\times}\hat{d}x.$$

Definition 2.2.27. *The iso-volume of the twenty-seventh kind of D is defined as follows*

$$\int_D^{12} \hat{T}(x)\hat{d}\hat{x}.$$

Definition 2.2.28. *The iso-volume of the twenty-eighth kind of D is defined as follows*

$$\int_D^{13} \hat{T}(x)d\hat{x}.$$

Definition 2.2.29. *The iso-volume of the twenty-ninth kind of D is defined as follows*

$$\int_D^{14} \hat{T}(x)\hat{\times}dx.$$

Definition 2.2.30. *The iso-volume of the thirtieth kind of D is defined as follows*

$$\int_D^{15} \hat{T}(x)\hat{d}x.$$

Sometimes, after we reduce the multiple iso-integrals to the multiple integrals it is suitable to be made a change of the variables.

Example 2.2.31. *Let $D = \{(x_1,x_2) \in \mathbb{R}^2 : 1 \leq x_1^2 + x_2^2 \leq 4\}$, $\hat{T}(x) = \sqrt{x_1^2+x_2^2}$, $f(x) = x_1^2 + x_2^2$, $x = (x_1,x_2) \in D$, $\hat{T}_1 = 2$. Then*

$$f^\wedge(x) = f(x\hat{T}(x)) = f(x_1\hat{T}(x), x_2\hat{T}(x)) = x_1^2\hat{T}^2(x) + x_2^2\hat{T}^2(x) = (x_1^2+x_2^2)^2.$$

From here

$$I = \hat{\int}_D^3 f^\wedge(x)\hat{\times}\hat{d}x = 2\int_D (x_1^2+x_2^2)^2(x_1^2+x_2^2)^{\frac{1}{2}}dx = 2\int_D (x_1^2+x_2^2)^{\frac{5}{2}}dx.$$

Now we make the following change of the variables

$$x_1 = \rho\cos\phi, \quad x_2 = \rho\sin\phi, \quad 1 \le \rho \le 2, \quad 0 \le \phi \le 2\pi.$$

Then for I we have

$$I = 2\int_1^2\int_0^{2\pi}\rho^5 d\phi d\rho = 4\pi\int_1^2\rho^5 d\rho = 4\pi\frac{\rho^6}{6}\Big|_{\rho=1}^{\rho=2} = \frac{128}{3}\pi.$$

Exercise 2.2.32. *Let* $D = \{(x_1,x_2) \in \mathbb{R}^2 : 0 \le x_1^2 + 2x_1x_2 \le 8\}$, $f(x) = x_1^4 + x_2^2$, $\hat{T}(x) = x_1^2 + x_2^2 + 1$, $x = (x_1,x_2) \in D$, $\hat{T}_1 = 4$. *Compute*

$$\hat{\int}_D^4 \hat{f}(\hat{x})\hat{d}\hat{x}.$$

2.3. Advanced Practical Exercises

Problem 2.3.1. *Let* $D = \{x = (x_1,x_2) : x_1 + x_2 \le 1, -1 \le x_1 \le 1, 0 \le x_2\}$, $\hat{T}_1 = 4$, $f(x) = x_1^2 - 2x_1x_2$, $\hat{T}(x) = x_1^2 + x_2^2 + 1$, $x = (x_1,x_2) \in D$. *Compute*

$$\hat{\int}_D^1 \hat{f}^\wedge(\hat{x})\hat{\times}\hat{d}\hat{x}, \qquad \hat{\int}_D^1 \hat{f}^\wedge(x)\hat{\times}\hat{d}\hat{x}, \qquad \hat{\int}_D^1 \hat{f}(\hat{x})\hat{\times}\hat{d}\hat{x}, \qquad \hat{\int}_D^1 f^\wedge(x)\hat{\times}\hat{d}\hat{x}.$$

Problem 2.3.2. *Let* $D = \{x = (x_1,x_2) : 0 \le x_1 \le 3 - x_2, 0 \le x_2 \le 1, 0 \le x_2\}$, $\hat{T}_1 = 4$, $f(x) = x_1^2 - 2x_1x_2$, $\hat{T}(x) = x_1^2 + x_2^2 + 2$, $x = (x_1,x_2) \in D$. *Compute*

$$\hat{\int}_D^2 \hat{f}^\wedge(\hat{x})\hat{\times}d\hat{x}, \qquad \hat{\int}_D^2 \hat{f}^\wedge(x)\hat{\times}d\hat{x}, \qquad \hat{\int}_D^2 \hat{f}(\hat{x})\hat{\times}d\hat{x}, \qquad \hat{\int}_D^2 f^\wedge(x)\hat{\times}d\hat{x}.$$

Problem 2.3.3. *Let* $D = \{(x_1,x_2) : 0 \le x_1 \le x_2 + 1, 0 \le x_2 \le 2\}$, $f(x) = x_1^2 - x_2 + 2$, $\hat{T}(x) = x_1^2 + x_2^2 + 1$, $x = (x_1,x_2) \in D$, $\hat{T}_1 = 2$. *Compute*

$$\hat{\int}_D^3 f(\hat{x})\hat{\times}\hat{d}x, \qquad \hat{\int}_D^3 \hat{f}^\wedge(x)\hat{\times}\hat{d}x.$$

Problem 2.3.4. *Let* $D = \{(x_1,x_2) \in \mathbb{R}^2 : 0 \le x_1 \le 4 - x_2, 0 \le x_2 \le 3\}$, $f(x) = x_1 + 2x_1x_2$, $\hat{T}(x) = x_1 + x_2$, $x = (x_1,x_2) \in D$, $\hat{T}_1 = 3$. *Compute*

$$\hat{\int}_D^6 \hat{f}^\wedge(\hat{x})\hat{\times}dx.$$

Problem 2.3.5. *Let* $D = \{(x_1,x_2) \in \mathbb{R}^2 : 0 \le x_1 \le 3, 0 \le x_2 \le 16 - x_1^2\}$, $f(x) = x_1 + 3x_2$, $\hat{T}(x) = x_1 + x_2 + 1$, $x = (x_1,x_2) \in D$, $\hat{T}_1 = 2$. *Compute*

$$\hat{\int}_D^8 \hat{f}(\hat{x})dx.$$

Problem 2.3.6. *Let* $d = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1 - x_2^2, 0 \leq x_2 \leq 1\}$*,* $f(x) = x_1 - 2x_2^2 + x_1^2$*,* $\hat{T}(x) = 1 + x_1 + x_2$*,* $x = (x_1, x_2) \in D$*,* $\hat{T}_1 = 4$*. Compute*

$$\hat{\int}_D^8 \hat{f}^{\wedge}(x)dx.$$

Problem 2.3.7. *Let* $D = \{(x_1, x_2) \in \mathbb{R}^{\cdot}0 \leq x_1 \leq 2x_2 + x_2^2, 0 \leq x_2 \leq 1\}$*,* $f(x) = x_1^2 + x_2$*,* $\hat{T}(x) = x_1 + x_2 + 1$*,* $x = (x_1, x_2) \in D$*,* $\hat{T}_1 = 4$*. Compute*

$$\int_D^9 \hat{f}(\hat{x})\hat{\times}\hat{d}\hat{x}.$$

Problem 2.3.8. *Let* $D = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq x_1^2 + 1\}$*,* $f(x) = 2x_1 + x_2^2$*,* $\hat{T}(x) = x_1^2 + x_2^2 + 1$*,* $x = (x_1, x_2) \in D$*,* $\hat{T}_1 = 4$*. Compute*

$$\int_D^{10} \hat{f}(\hat{x})\hat{\times}d\hat{x}, \qquad \int_D^{10} f(\hat{x})\hat{\times}d\hat{x}.$$

Problem 2.3.9. *Let* $D = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 3, 0 \leq x_2 \leq 2x_1 + 5\}$*,* $f(x) = x_1 - 7x_2 - 12$*,* $\hat{T}(x) = x_1^2 + x_2^2 + 1$*,* $x = (x_1, x_2) \in D$*,* $\hat{T}_1 = 4$*. Compute*

$$\int_D^{11} \hat{f}^{\wedge}(x)\hat{\times}\hat{d}x.$$

Problem 2.3.10. *Let* $D = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1^4 + 2x_1x_2^2 \leq 8\}$*,* $f(x) = x_1^4 + x_2^2 + 2x_1x_2$*,* $\hat{T}(x) = x_1^2 + x_2^2 + 1$*,* $x = (x_1, x_2) \in D$*,* $\hat{T}_1 = 4$*. Compute*

$$\hat{\int}_D^6 \hat{f}(\hat{x})\hat{\times}d\hat{x}.$$

Problem 2.3.11. *Let* $D = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1^2 + 2x_1^4x_2^2 \leq 8\}$*,* $f(x) = x_1^2 + x_2^2 + 2x_1^2x_2$*,* $\hat{T}(x) = x_1^2 + x_2^2 + 1$*,* $x = (x_1, x_2) \in D$*,* $\hat{T}_1 = 4$*. Compute*

$$\int_D^9 \hat{f}^{\wedge}(\hat{x})\hat{\times}\hat{d}\hat{x}.$$

Chapter 3

Line and Surface Iso-Integrals

3.1. Definition of Line Iso-Integrals

Let $\hat{T} : \mathbb{R} \longrightarrow \mathbb{R}$ be a positive continuously-differentiable function, C be a curve in $\mathbb{R}^2$, parameterized by the equations

$$x_1 = x_1(t), \quad x_2 = x_2(t), \quad t \in [a,b].$$

Let also, $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be an integrable function and $\hat{f}$ be its iso-lift as an iso-function of the first, the second, the third, the fourth or the fifth kind. With s we will denote the arc length

$$s(t) = \int_a^t \sqrt{x_1'(t)^2 + x_2'(t)^2} dt.$$

Definition 3.1.1. *The line iso-integral of the first kind of $\hat{f}$ along the curve C is defined as follows*

$$\hat{\int}_C^1 \hat{f}(x_1, x_2) \hat{\times} \hat{d}\hat{s}^{\wedge\wedge} = \hat{\int}_a^{1b} \hat{f}(x_1(t), x_2(t)) \hat{\times} \hat{d}\hat{s}^{\wedge}(\hat{t}).$$

We can rewrite the line iso-integral of the first kind in the following way

$$\hat{\int}_a^{1b} \hat{f}(x) \hat{\times} \hat{d}\hat{s}^{\wedge}(\hat{t}) = \int_a^b \hat{f}(x_1(t), x_2(t)) \frac{s'(t)\hat{T}(t) - s(t)\hat{T}'(t)}{\hat{T}(t)} dt.$$

Example 3.1.2. *Let* $C : x_1(t) = r\cos t$, $x_2(t) = r\sin t$, $\hat{T}(x) = t+1$, $t \in [0, 2\pi]$, $r \equiv \text{const} > 0$,

$f(x_1,x_2) = x_1x_2$. *Then*

$$x'(t) = -r\sin t, \quad x_2'(t) = r\cos t,$$

$$s(t) = \int_0^t \sqrt{(-r\sin u)^2+(r\cos u)^2}du$$

$$= r\int_0^t du = rt,$$

$$s'(t) = r,$$

$$f^\wedge(x_1,x_2) = f(x_1(t)\hat{T}(t), x_2(t)\hat{T}(t)) = x_1(t)x_2(t)\hat{T}^2(t)$$

$$= (r\cos t)(r\sin t)(t+1)^2$$

$$= \frac{r^2}{2}(t+1)^2\sin(2t),$$

$$\frac{s'(t)\hat{T}(t)-s(t)\hat{T}'(t)}{\hat{T}(t)} = \frac{r(t+1)-rt}{t+1} = \frac{r}{t+1}.$$

From here,

$$\hat{\int}_C f^\wedge(x_1,x_2)\hat{\times}\hat{d}\hat{s}^{\wedge\wedge} = \int_0^{2\pi}\frac{r^2}{2}(t+1)^2\sin(2t)\frac{r}{t+1}dt$$

$$= \frac{r^3}{2}\int_0^{2\pi}(t+1)\sin(2t)dt$$

$$= -\frac{r^3}{4}\int_0^{2\pi}(t+1)d\cos(2t)$$

$$= -\frac{r^3}{4}(t+1)\cos(2t)\Big|_{t=0}^{t=2\pi} + \frac{r^3}{4}\int_0^{2\pi}\cos(2t)dt$$

$$= -\frac{r^3}{2}\pi + \frac{r^3}{8}\sin(2t)\Big|_{t=0}^{t=2\pi}$$

$$= -\frac{r^3}{2}\pi.$$

Exercise 3.1.3. *Let* $C : x_1(t) = r\sin t,\ x_2(t) = r\cos t,\ t \in [0,\pi],\ \hat{T}(t) = t^2+1,\ f(x_1,x_2) = x_1^2+x_2^2$. *Compute*

$$\hat{\int}_L \hat{f}^\wedge(x_1x_2)\hat{\times}\hat{d}\hat{s}^{\wedge\wedge}.$$

Definition 3.1.4. *The line iso-integral of the second kind of $\hat{f}$ along the curve C is defined as follows*

$$\hat{\int}_C^2 \hat{f}(x_1,x_2)\hat{\times}\hat{d}\hat{s}^\wedge = \hat{\int}_a^{2b}\hat{f}(x_1(t),x_2(t))\hat{\times}\hat{d}\hat{s}^\wedge(t).$$

We can rewrite the line iso-integral of the second kind in the following way

$$\hat{\int}_a^{2b}\hat{f}(x)\hat{\times}\hat{d}\hat{s}^\wedge(\hat{t}) = \int_a^b \hat{f}(x_1(t),x_2(t))\frac{s'(t\hat{T}(t))(\hat{T}(t)+t\hat{T}'(t))\hat{T}(t)-s(t\hat{T}(t))\hat{T}'(t)}{\hat{T}(t)}dt.$$

Example 3.1.5. *Let* $C : x_1(t) = \sqrt{2}t,\ x_2(t) = \sqrt{2}t+1,\ t \in [1,2],\ f(x_1,x_2) = x_1x_2,\ \hat{T}(t) = t+1$. *Then*

$$s(t) = \int_1^t \sqrt{x_1'(u)^2 + x_2'(u)^2}du = 2\int_1^t du = 2t-2,$$

$$s(t\hat{T}(t)) = 2t\hat{T}(t) - 2 = 2t(t+1) - 2 = 2t^2+2t-2,$$

$$s'(t) = 2,$$

$$\frac{s'(t\hat{T}(t))(\hat{T}(t)+t\hat{T}'(t)) - s(t\hat{T}(t))\hat{T}'(t)}{\hat{T}(t)} = \frac{2(t+1+t)-(2t^2+2t-2)}{t+1}$$

$$= \frac{-2t^2+2t+4}{t+1},$$

$$f^\wedge(x_1,x_2) = f(x_1(t)\hat{T}(t), x_2(t)\hat{T}(t)) = x_1(t)x_2(t)\hat{T}^2(t) = \sqrt{2}t(\sqrt{2}t+1)(t+1)^2.$$

From here

$$\hat{\int}_L^2 f^\wedge(x_1,x_2)\hat{\times}\hat{d}\hat{s}^\wedge = \int_1^2 \sqrt{2}(\sqrt{2}t+1)(t+1)^2\frac{-2t^2+2t+1}{t+1}dt$$

$$= \int_1^2 (2t^2+\sqrt{2}t)(t+1)(-2t^2+2t+1)dt$$

$$= \int_1^2 (-4t^5 - 2\sqrt{2}t^4 + 6t^3 + (2+3\sqrt{2})t^2 + \sqrt{2}t)dt$$

$$= \left(-\frac{2}{3}t^6 - \frac{2\sqrt{2}}{5}t^5 + \frac{3}{2}t^4 + \frac{2+3\sqrt{2}}{3}t^3 + \frac{\sqrt{2}}{2}\right)\Big|_{t=1}^{t=2}$$

$$= -\frac{89}{6} - \frac{39}{10}\sqrt{2}.$$

Exercise 3.1.6. *Let* $C : x_1(t) = 2\sqrt{2}t,\ x_2(t) = 2\sqrt{2}t+1,\ t \in [1,2],\ f(x_1,x_2) = x_1x_2,\ \hat{T}(t) = 2t+1$. *Compute*

$$\hat{\int}_C^2 \hat{f}^\wedge(\hat{x}_1,\hat{x}_2)\hat{\times}\hat{d}\hat{s}^\wedge.$$

Definition 3.1.7. *The line iso-integral of the third kind of* $\hat{f}$ *along the curve C is defined as follows*

$$\hat{\int}_C^3 \hat{f}(x_1,x_2)\hat{\times}\hat{d}\hat{\hat{s}} = \hat{\int}_a^{3b} \hat{f}(x_1(t),x_2(t))\hat{\times}\hat{d}\hat{s}(\hat{t}).$$

We can rewrite the line iso-integral of the third kind in the following way

$$\hat{\int}_a^{3b} \hat{f}(x)\hat{\times}\hat{d}\hat{s}(\hat{t}) = \int_a^b \hat{f}(x_1(t),x_2(t))\frac{s'\left(\frac{t}{\hat{T}(t)}\right)(\hat{T}(t)-t\hat{T}'(t))\hat{T}(t) - s\left(\frac{t}{\hat{T}(t)}\right)\hat{T}'(t)\hat{T}(t)}{\hat{T}^2(t)}dt.$$

Example 3.1.8. *Let* $C : x_1(t) = 2t+1,\ x_2(t) = 2t+2,\ t \in [1,2],\ f(x_1,x_2) = 2x_1 - x_2,\ \hat{T}(t) =$

$t+1$. *Then*

$$\hat{f}^\wedge(\hat{x}_1,\hat{x}_2) = \frac{f(x_1(t),x_2(t))}{\hat{T}(t)} = \frac{2x_1(t)-x_2(t)}{\hat{T}(t)} = \frac{2t}{t+1},$$

$$s(t) = \int_1^t \sqrt{x_1'(u)^2+x_2'(u)^2}du = 2\sqrt{2}\int_1^t du = 2\sqrt{2}(t-1),$$

$$s'(t) = 2\sqrt{2},$$

$$s\left(\frac{t}{\hat{T}(t)}\right) = 2\sqrt{2}\left(\frac{t}{\hat{T}(t)}-1\right) = 2\sqrt{2}\left(\frac{t}{t+1}-1\right) = -\frac{2\sqrt{2}}{t+1},$$

$$\frac{s'\left(\frac{t}{\hat{T}(t)}\right)(\hat{T}(t)-t\hat{T}'(t))\hat{T}(t)-s\left(\frac{t}{\hat{T}(t)}\right)\hat{T}'(t)\hat{T}(t)}{\hat{T}^2(t)} = \frac{2\sqrt{2}(t+1-t)+\frac{2\sqrt{2}}{t+1}(t+1)}{(t+1)^2}$$

$$= \frac{4\sqrt{2}}{(t+1)^2}.$$

From here,

$$\hat{\int}_C^3 \hat{f}^\wedge(\hat{x}c_1,\hat{x}_2)\hat{\times}\hat{d}\hat{s}(\hat{t}) = \int_1^2 \frac{2t}{t+1}\frac{4\sqrt{2}}{(t+1)^2}dt$$

$$= -4\sqrt{2}\int_1^2 td\frac{1}{(t+1)^2}$$

$$= -4\sqrt{2}\frac{t}{(t+1)^2}\Big|_{t=1}^{t=2} + 4\sqrt{2}\int_1^2 \frac{1}{(t+1)^2}dt$$

$$= \frac{\sqrt{2}}{9} - 4\sqrt{2}\frac{1}{t+1}\Big|_{t=1}^{t=2}$$

$$= \frac{7\sqrt{2}}{9}.$$

Exercise 3.1.9. *Let* $C: x_1(t) = t^2+1$, $x_2(t) = 2t+2$, $t \in [1,2]$, $f(x_1,x_2) = x_1^2 - x_2$, $\hat{T}(t) = t+1$. *Compute*

$$\hat{\int}_C^3 \hat{f}^\wedge(x_1,x_2)\hat{\times}\hat{d}\hat{s}(\hat{t}).$$

Definition 3.1.10. *The line iso-integral of the fourth kind of* $\hat{f}$ *along the curve C is defined as follows*

$$\hat{\int}_C^4 \hat{f}(x_1,x_2)\hat{\times}\hat{d}\hat{s}^\wedge = \hat{\int}_a^{4b} \hat{f}(x_1(t),x_2(t))\hat{\times}\hat{d}\hat{s}^\wedge(t).$$

We can rewrite the line iso-integral of the fourth kind in the following way

$$\hat{\int}_a^{4b} \hat{f}(x)\hat{\times}\hat{d}\hat{s}^\wedge(t) = \int_a^b \hat{f}(x_1(t),x_2(t))\hat{T}(t)s'(t\hat{T}(t))(\hat{T}(t)+t\hat{T}'(t))dt.$$

Example 3.1.11. *Let* $C: x_1(t) = t^2+1$, $x_2(t) = t^2+2$, $t \in [0,1]$, $f(x_1,x_2) = x_1^2+x_2^2$, $\hat{T}(t) =$

$t+1$. *Then*

$$\hat{f}^\wedge(\hat{x}_1(t),\hat{x}_2(t)) = \frac{f(x_1(t),x_2(t))}{\hat{T}(t)}$$

$$= \frac{x_1^2(t)+x_2^2(t)}{\hat{T}(t)}$$

$$= \frac{(t^2+1)^2+(t^2+2)^2}{t+1}$$

$$= \frac{2t^4+6t^2+5}{t+1},$$

$$s(t) = \int_0^t \sqrt{x_1'(u)^2+x_2'(u)^2}du$$

$$= \int_0^t \sqrt{4u^2+4u^2}du$$

$$= 2\sqrt{2}\int_0^t udu$$

$$= \sqrt{2}t^2,$$

$$s'(t) = 2\sqrt{2}t,$$

$$s'(t\hat{T}(t)) = 2\sqrt{2}t\hat{T}(t) = 2\sqrt{2}t(t+1).$$

From here

$$\hat{\int}_0^1 \hat{f}^\wedge(\hat{x}_1(t),\hat{x}_2(t))\hat{\times}\hat{d}s^\wedge(t) = \int_0^1 \frac{2t^4+6t^2+5}{t+1}(t+1)2\sqrt{2}t(t+1)(t+1+t)dt$$

$$= 2\sqrt{2}\int_0^1(2t^4+6t^2+5)t(t+1)(2t+1)dt$$

$$= 2\sqrt{2}\int_0^1(4t^7+6t^6+14t^5+18t^4+16t^3+15t^2+5t)dt$$

$$= 2\sqrt{2}\left(\frac{t^8}{2}+\frac{6}{7}t^7+\frac{7}{3}t^6+\frac{18}{5}t^5+4t^4+5t^3+\frac{5}{2}t^2\right)\Big|_{t=0}^{t=1}$$

$$= \frac{3946\sqrt{2}}{105}.$$

Exercise 3.1.12. *Let* $C: x_1(t) = t+1$, $x_2(t) = t^2+1$, $t \in [0,1]$, $f(x_1,x_2) = x_1+x_2^2$, $\hat{T}(t) = t+1$. *Compute*

$$\hat{\int}_C^4 \hat{f}^\wedge(x_1,x_2)\hat{\times}\hat{d}s^\wedge.$$

Definition 3.1.13. *The line iso-integral of the fifth kind of* $\hat{f}$ *along the curve* C *is defined as follows*

$$\hat{\int}_C^5 \hat{f}(x_1,x_2)\hat{\times}\hat{d}s = \hat{\int}_a^{4b} \hat{f}(x_1(t),x_2(t))\hat{\times}\hat{d}s(\hat{t}).$$

We can rewrite the line iso-integral of the fifth kind in the following way

$$\hat{\int}_a^{5b} \hat{f}(x)\hat{\times}\hat{d}s(\hat{t}) = \int_a^b \hat{f}(x_1(t),x_2(t))s'\left(\frac{t}{\hat{T}(t)}\right)\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)}dt.$$

Example 3.1.14. *Let* $C: x_1(t)=t^2+2,\ x_2(t)=t^2,\ t\in[0,1],\ f(x)=x_1^2+x_2^2,\ \hat{T}(t)=t+2.$ *Then*

$$f^\wedge(x_1,x_2)=f(x_1(t)\hat{T}(t),x_2(t)\hat{T}(t))$$

$$=x_1^2(t)\hat{T}^2(t)+x_2^2(t)\hat{T}^2(t)$$

$$=(t^2+2)^2(t+2)^2+t^4(t+2)^2$$

$$=2(t^4+2t^2+2)(t+2)^2,$$

$$s(t)=\int_0^t\sqrt{x_1'(u)^2+x_2'(u)^2}du$$

$$=\int_0^t\sqrt{4u^2+4u^2}du$$

$$=2\sqrt{2}\int_0^t udu$$

$$=\sqrt{2}t^2,$$

$$s'(t)=2\sqrt{2}t,$$

$$s'\left(\frac{t}{\hat{T}(t)}\right)=2\sqrt{2}\frac{t}{\hat{T}(t)}=2\sqrt{2}\frac{t}{t+2},$$

$$\frac{\hat{T}(t)-\hat{T}'(t)}{\hat{T}(t)}=\frac{t+2-t}{t+2}=\frac{2}{t+2}.$$

From here

$$\hat{\int}_C^5 f^\wedge(x_1,x_2)\hat{\times}\hat{d}s^\wedge=\int_0^1 2(t^4+2t^2+2)(t+2)^2 2\sqrt{2}\frac{t}{t+2}\frac{2}{t+2}dt$$

$$=8\sqrt{2}\int_0^1 t(t^4+2t^2+2)dt$$

$$=8\sqrt{2}\int_0^1(t^5+2t^3+2t)dt$$

$$=8\sqrt{2}\left(\frac{t^6}{6}+\frac{t^4}{2}+t^2\right)\Big|_{t=0}^{t=1}$$

$$=\frac{40\sqrt{2}}{3}.$$

Let now, $f_1,f_2:\mathbb{R}^2\longrightarrow\mathbb{R}$ be integrable functions and $\hat{f}_1$, $\hat{f}_2$ be their iso-lifts as iso-functions of the first, the second, the third, the fourth or the fifth kind . Let also, $e_1=(1,0)$, $e_2=(0,1)$.

Definition 3.1.15. *The line iso-integral of the first kind of* $\hat{f}_1e_1+\hat{f}_2e_2$ *along the curve* C *is defined as follows*

$$\hat{\int}_C^1\hat{f}_1(x_1,x_2)\hat{\times}\hat{d}\hat{x}_1^{\wedge\wedge}+\hat{\int}_C^1\hat{f}_2(x_1,x_2)\hat{\times}\hat{d}\hat{x}_2^{\wedge\wedge}$$

$$=\hat{\int}_a^{1b}\hat{f}_1(x_1(t),x_2(t))\hat{\times}\hat{d}\hat{x}_1^\wedge(\hat{t})+\hat{\int}_a^{1b}\hat{f}_2(x_1(t),x_2(t))\hat{\times}\hat{d}\hat{x}_2^\wedge(\hat{t}).$$

We can rewrite the line iso-integral of the first kind in the following way

$$\hat{\int}_a^{1b} \hat{f}_1(x) \hat{\times} \hat{d}\hat{x}_1^{\wedge}(\hat{t}) + \hat{\int}_a^{1b} \hat{f}_2(x) \hat{\times} \hat{d}\hat{x}_2^{\wedge}(\hat{t})$$

$$= \int_a^b \hat{f}_1(x_1(t), x_2(t)) \frac{x_1'(t)\hat{T}(t) - x_1(t)\hat{T}'(t)}{\hat{T}(t)} dt + \int_a^b \hat{f}_2(x_1(t), x_2(t)) \frac{x_2'(t)\hat{T}(t) - x_2(t)\hat{T}'(t)}{\hat{T}(t)} dt.$$

Exercise 3.1.16. *Let* $C : x_1(t) = t+1$, $x_2(t) = t$, $t \in [0,1]$, $f_1(x_1, x_2) = x_1 - x_2$, $f_2(x_1, x_2) = x_1 + x_2$, $\hat{T}(t) = t^2 + 1$. *Compute*

$$\hat{\int}_C^1 \hat{f}_1(x_1, x_2) \hat{\times} \hat{d}\hat{x}_1^{\wedge\wedge} + \hat{\int}_C^1 \hat{f}_2(x_1, x_2) \hat{\times} \hat{d}\hat{x}_2^{\wedge\wedge}.$$

Definition 3.1.17. *The line iso-integral of the second kind of* $\hat{f}_1 e_1 + \hat{f}_2 e_2$ *along the curve* C *is defined as follows*

$$\hat{\int}_C^2 \hat{f}_1(x_1, x_2) \hat{\times} \hat{d}\hat{x}_1^{\wedge} + \hat{\int}_C^2 \hat{f}_2(x_1, x_2) \hat{\times} \hat{d}\hat{x}_2^{\wedge}$$

$$= \hat{\int}_a^{2b} \hat{f}_1(x_1(t), x_2(t)) \hat{\times} \hat{d}\hat{x}_1^{\wedge}(t) + \hat{\int}_a^{2b} \hat{f}_2(x_1(t), x_2(t)) \hat{\times} \hat{d}\hat{x}_2^{\wedge}(t).$$

We can rewrite the line iso-integral of the second kind in the following way

$$\hat{\int}_a^{2b} \hat{f}_1(x) \hat{\times} \hat{d}\hat{x}_1^{\wedge}(\hat{t}) + \hat{\int}_a^{2b} \hat{f}_2(x) \hat{\times} \hat{d}\hat{x}_2^{\wedge}(\hat{t})$$

$$= \int_a^b \hat{f}_1(x_1(t), x_2(t)) \frac{x_1'(t\hat{T}(t))(\hat{T}(t) + t\hat{T}'(t))\hat{T}(t) - x_1(t\hat{T}(t))\hat{T}'(t)}{\hat{T}(t)} dt$$

$$+ \int_a^b \hat{f}_2(x_1(t), x_2(t)) \frac{x_2'(t\hat{T}(t))(\hat{T}(t) + t\hat{T}'(t))\hat{T}(t) - x_2(t\hat{T}(t))\hat{T}'(t)}{\hat{T}(t)} dt.$$

Exercise 3.1.18. *Let* $C : x_1(t) = t+1$, $x_2(t) = t$, $t \in [0,1]$, $f_1(x_1, x_2) = x_1 - x_2^2$, $f_2(x_1, x_2) = x_1^2 + x_2$, $\hat{T}(t) = t^2 + 1$. *Compute*

$$\hat{\int}_C^2 \hat{f}_1(x_1, x_2) \hat{\times} \hat{d}\hat{x}_1^{\wedge} + \hat{\int}_C^2 \hat{f}_2(x_1, x_2) \hat{\times} \hat{d}\hat{x}_2^{\wedge}.$$

Definition 3.1.19. *The line iso-integral of the third kind of* $\hat{f}_1 e_1 + \hat{f} e_2$ *along the curve* C *is defined as follows*

$$\hat{\int}_C^3 \hat{f}_1(x_1, x_2) \hat{\times} \hat{d}\hat{\hat{x}}_1 + \hat{\int}_C^3 \hat{f}_2(x_1, x_2) \hat{\times} \hat{d}\hat{\hat{x}}_2$$

$$= \hat{\int}_a^{3b} \hat{f}_1(x_1(t), x_2(t)) \hat{\times} \hat{d}\hat{x}_1(\hat{t}) + \hat{\int}_a^{3b} \hat{f}_2(x_1(t), x_2(t)) \hat{\times} \hat{d}\hat{x}_2(\hat{t}).$$

We can rewrite the line iso-integral of the third kind in the following way

$$\hat{\int}_a^{3b} \hat{f}_1(x) \hat{\times} \hat{d}\hat{x}_1(\hat{t}) + \hat{\int}_a^{3b} \hat{f}_2(x) \hat{\times} \hat{d}\hat{x}_2(\hat{t})$$

$$= \int_a^b \hat{f}_1(x_1(t), x_2(t)) \frac{x_1'\left(\frac{t}{\hat{T}(t)}\right)(\hat{T}(t) - t\hat{T}'(t))\hat{T}(t) - x_1\left(\frac{t}{\hat{T}(t)}\right)\hat{T}'(t)\hat{T}(t)}{\hat{T}^2(t)} dt$$

$$+ \int_a^b \hat{f}_2(x_1(t), x_2(t)) \frac{x_2'\left(\frac{t}{\hat{T}(t)}\right)(\hat{T}(t) - t\hat{T}'(t))\hat{T}(t) - x_2\left(\frac{t}{\hat{T}(t)}\right)\hat{T}'(t)\hat{T}(t)}{\hat{T}^2(t)} dt.$$

Exercise 3.1.20. *Let* $C : x_1(t) = t+1,\ x_2(t) = t+4,\ t \in [0,1],\ f_1(x_1,x_2) = 2x_1 + 3x_1x_2 - x_2^2,$ $f_2(x_1,x_2) = x_1^2 + x_2,\ \hat{T}(t) = t^2+1.$ *Compute*

$$\hat{\int}_C^3 \hat{f}_1(x_1,x_2)\hat{\times}\hat{d}\hat{\hat{x}}_1 + \hat{\int}_C^3 \hat{f}_2(x_1,x_2)\hat{\times}\hat{d}\hat{\hat{x}}_2.$$

Definition 3.1.21. *The line iso-integral of the fourth kind of* $\hat{f}_1 e_1 + \hat{f} e_2$ *along the curve C is defined as follows*

$$\hat{\int}_C^4 \hat{f}_1(x_1,x_2)\hat{\times}\hat{d}x_1{}^\wedge + \hat{\int}_C^4 \hat{f}_2(x_1,x_2)\hat{\times}\hat{d}x_2{}^\wedge$$

$$= \hat{\int}_a^{4b} \hat{f}_1(x_1(t),x_2(t))\hat{\times}\hat{d}x_1{}^\wedge(t) + \hat{\int}_a^{4b} \hat{f}_2(x_1(t),x_2(t))\hat{\times}\hat{d}x_2{}^\wedge(t).$$

We can rewrite the line iso-integral of the fourth kind in the following way

$$\hat{\int}_a^{4b} \hat{f}_1(x)\hat{\times}\hat{d}x_1^\wedge(t) + \hat{\int}_a^{4b} \hat{f}_2(x)\hat{\times}\hat{d}x_2^\wedge(t)$$

$$= \int_a^b \hat{f}_1(x_1(t),x_2(t))\hat{T}(t)x_1'(t\hat{T}(t))(\hat{T}(t)+t\hat{T}'(t))dt$$

$$+ \int_a^b \hat{f}_2(x_1(t),x_2(t))\hat{T}(t)x_2'(t\hat{T}(t))(\hat{T}(t)+t\hat{T}'(t))dt.$$

Exercise 3.1.22. *Let* $C : x_1(t) = t+1,\ x_2(t) = t+4,\ t \in [0,1],\ f_1(x_1,x_2) = x_1 + 3x_1x_2,$ $f_2(x_1,x_2) = x_1 + x_2,\ \hat{T}(t) = t^2+1.$ *Compute*

$$\hat{\int}_C^4 \hat{f}_1(x_1,x_2)\hat{\times}\hat{d}x_1{}^\wedge + \hat{\int}_C^4 \hat{f}_2(x_1,x_2)\hat{\times}\hat{d}x_2{}^\wedge.$$

Definition 3.1.23. *The line iso-integral of the fifth kind of* $\hat{f}$ *along the curve C is defined as follows*

$$\hat{\int}_C^5 \hat{f}_1(x_1,x_2)\hat{\times}\hat{d}x_1 + \hat{\int}_C^5 \hat{f}_2(x_1,x_2)\hat{\times}\hat{d}x_2$$

$$= \hat{\int}_a^{4b} \hat{f}_1(x_1(t),x_2(t))\hat{\times}\hat{d}x_1(\hat{t}) + \hat{\int}_a^{4b} \hat{f}_2(x_1(t),x_2(t))\hat{\times}\hat{d}x_2(\hat{t}).$$

We can rewrite the line iso-integral of the fifth kind in the following way

$$\hat{\int}_a^{5b} \hat{f}_1(x)\hat{\times}\hat{d}x_1(\hat{t}) + \hat{\int}_a^{5b} \hat{f}_2(x)\hat{\times}\hat{d}x_2(\hat{t})$$

$$= \int_a^b \hat{f}_1(x_1(t),x_2(t))x_1'\left(\frac{t}{\hat{T}(t)}\right)\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)}dt$$

$$+ \int_a^b \hat{f}_2(x_1(t),x_2(t))x_2'\left(\frac{t}{\hat{T}(t)}\right)\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)}dt.$$

Exercise 3.1.24. *Let* $C : x_1(t) = t+1,\ x_2(t) = t+4t^2,\ t \in [0,1],\ f_1(x_1,x_2) = x_1x_2,$ $f_2(x_1,x_2) = x_1 + x_2,\ \hat{T}(t) = t^2+1.$ *Compute*

$$\hat{\int}_C^5 \hat{f}_1^\wedge(x_1,x_2)\hat{\times}\hat{d}x_1 + \hat{\int}_C^5 \hat{f}_2^\wedge(x_1,x_2)\hat{\times}\hat{d}x_2.$$

3.2. Properties of Line Iso-Integrals

With $-C$ we will denote the curve $x_= x_1(a+b-t)$, $x_2 = x_2(a+b-t)$, $t \in [a,b]$. The curve $-C$ is traversed in the opposite direction.

Theorem 3.2.1.

$$\hat{\int}^1_{-C}\hat{f}_1(x_1,x_2)\hat{\times}\hat{d}\hat{x}_1^{\wedge\wedge}+\hat{\int}^1_{-C}\hat{f}_2(x_1,x_2)\hat{\times}\hat{d}\hat{x}_2^{\wedge\wedge}$$

$$=-\hat{\int}^1_{C}\hat{f}_1(x_1,x_2)\hat{\times}\hat{d}\hat{x}_1^{\wedge\wedge}-\hat{\int}^1_{C}\hat{f}_2(x_1,x_2)\hat{\times}\hat{d}\hat{x}_2^{\wedge\wedge}$$

Proof.

$$\hat{\int}^1_{-C}\hat{f}_1(x_1,x_2)\hat{\times}\hat{d}\hat{x}_1^{\wedge\wedge}+\hat{\int}^1_{-C}\hat{f}_2(x_1,x_2)\hat{\times}\hat{d}\hat{x}_2^{\wedge\wedge}$$

$$=\int_a^b \hat{f}_1(x_1(a+b-t),x_2(a+b-t))\frac{\frac{d}{dt}x_1(a+b-t)\hat{T}(a+b-t)-x_1(a+b-t)\frac{d}{dt}\hat{T}(a+b-t)}{\hat{T}(a+b-t)}dt$$

$$+\int_a^b \hat{f}_2(x_1(a+b-t),x_2(a+b-t))\frac{\frac{d}{dt}x_2(a+b-t)\hat{T}(a+b-t)-x_2(a+b-t)\frac{d}{dt}\hat{T}(a+b-t)}{\hat{T}(a+b-t)}dt$$

$$=\int_a^b \hat{f}_1(x_1(a+b-t),x_2(a+b-t))\frac{-x_1'(a+b-t)\hat{T}(a+b-t)+x_1(a+b-t)\hat{T}'(a+b-t)}{\hat{T}(a+b-t)}dt$$

$$+\int_a^b \hat{f}_2(x_1(a+b-t),x_2(a+b-t))\frac{-x_2'(a+b-t)\hat{T}(a+b-t)+x_2(a+b-t)\hat{T}'(a+b-t)}{\hat{T}(a+b-t)}dt$$

$$a+b-t=u$$

$$=-\int_b^a \hat{f}_1(x_1(u),x_2(u))\frac{-x_1'(u)\hat{T}(u)+x_1(u)\hat{T}'(u)}{\hat{T}(u)}du-\int_b^a \hat{f}_2(x_1(u),x_2(u))\frac{-x_2'(u)\hat{T}(u)+x_2(u)\hat{T}'(u)}{\hat{T}(u)}du$$

$$=\int_a^b \hat{f}_1(x_1(u),x_2(u))\frac{-x_1'(u)\hat{T}(u)+x_1(u)\hat{T}'(u)}{\hat{T}(u)}du+\int_a^b \hat{f}_2(x_1(u),x_2(u))\frac{-x_2'(u)\hat{T}(u)+x_2(u)\hat{T}'(u)}{\hat{T}(u)}du$$

$$=-\hat{\int}^1_{C}\hat{f}_1(x_1,x_2)\hat{\times}\hat{d}\hat{x}_1^{\wedge\wedge}-\hat{\int}^1_{C}\hat{f}_2(x_1,x_2)\hat{\times}\hat{d}\hat{x}_2^{\wedge\wedge}.$$

□

Theorem 3.2.2.

$$\hat{\int}^2_{-C}\hat{f}_1(x_1,x_2)\hat{\times}\hat{d}\hat{x}_1^{\wedge}+\hat{\int}^2_{-C}\hat{f}_2(x_1,x_2)\hat{\times}\hat{d}\hat{x}_2^{\wedge}$$

$$\neq-\hat{\int}^2_{C}\hat{f}_1(x_1,x_2)\hat{\times}\hat{d}\hat{x}_1^{\wedge}-\hat{\int}^2_{C}\hat{f}_2(x_1,x_2)\hat{\times}\hat{d}\hat{x}_2^{\wedge}$$

Proof.

$$\hat{\int}^{2}_{-C}\hat{f}_1(x_1,x_2)\hat{\times}\hat{d}\hat{x}_1^{\wedge}+\hat{\int}^{2}_{-C}\hat{f}_2(x_1,x_2)\hat{\times}\hat{d}\hat{x}_2^{\wedge}$$

$$=\int_a^b \hat{f}_1(x_1(a+b-t),x_2(a+b-t))$$

$$\frac{\frac{d}{dt}x_1((a+b-t)\hat{T}(a+b-t))(\hat{T}(a+b-t)+(a+b-t)\frac{d}{dt}\hat{T}(a+b-t))\hat{T}(a+b-t)-x_1((a+b-t)\hat{T}(a+b-t))\frac{d}{dt}\hat{T}(a+b-t)}{\hat{T}(a+b-t)}dt$$

$$+\int_a^b \hat{f}_2(x_1(a+b-t),x_2(a+b-t))$$

$$\frac{\frac{d}{dt}x_2((a+b-t)\hat{T}(a+b-t))(\hat{T}(a+b-t)+(a+b-t)\frac{d}{dt}\hat{T}(a+b-t))\hat{T}(a+b-t)-x_2((a+b-t)\hat{T}(a+b-t))\frac{d}{dt}\hat{T}(a+b-t)}{\hat{T}(a+b-t)}dt$$

$$=\int_a^b \hat{f}_1(x_1(a+b-t),x_2(a+b-t))$$

$$\frac{-x_1'((a+b-t)\hat{T}(a+b-t))(\hat{T}(a+b-t)-(a+b-t)\hat{T}'(a+b-t))\hat{T}(a+b-t)+x_1((a+b-t)\hat{T}(a+b-t))\hat{T}'(a+b-t)}{\hat{T}(a+b-t)}dt$$

$$+\int_a^b \hat{f}_2(x_1(a+b-t),x_2(a+b-t))$$

$$\frac{-x_2'((a+b-t)\hat{T}(a+b-t))(\hat{T}(a+b-t)-(a+b-t)\hat{T}'(a+b-t))\hat{T}(a+b-t)+x_2((a+b-t)\hat{T}(a+b-t))\hat{T}'(a+b-t)}{\hat{T}(a+b-t)}dt$$

$$a+b-t=u$$

$$=-\int_b^a \hat{f}_1(x_1(u),x_2(u))\frac{-x_1'(u\hat{T}(u))(\hat{T}(u)-u\hat{T}(u))\hat{T}(u)+x_1(u\hat{T}(u))\hat{T}'(u)}{\hat{T}(u)}dt$$

$$-\int_b^a \hat{f}_2(x_1(u),x_2(u))\frac{-x_2'(u\hat{T}(u))(\hat{T}(u)-u\hat{T}'(u))\hat{T}(u)+x_2(u\hat{T}(u))\hat{T}'(u)}{\hat{T}(u)}dt$$

$$=\int_a^b \hat{f}_1(x_1(u),x_2(u))\frac{-x_1'(u\hat{T}(u))(\hat{T}(u)-u\hat{T}(u))\hat{T}(u)+x_1(u\hat{T}(u))\hat{T}'(u)}{\hat{T}(u)}dt$$

$$+\int_a^b \hat{f}_2(x_1(u),x_2(u))\frac{-x_2'(u\hat{T}(u))(\hat{T}(u)-u\hat{T}'(u))\hat{T}(u)+x_2(u\hat{T}(u))\hat{T}'(u)}{\hat{T}(u)}dt$$

$$\neq-\hat{\int}^{2}_{C}\hat{f}_1(x_1,x_2)\hat{\times}\hat{d}\hat{x}_1^{\wedge}-\hat{\int}^{2}_{C}\hat{f}_2(x_1,x_2)\hat{\times}\hat{d}\hat{x}_2^{\wedge}$$

□

Theorem 3.2.3.

$$\hat{\int}^{3}_{-C}\hat{f}_1(x_1,x_2)\hat{\times}\hat{d}\hat{\hat{x}}_1+\hat{\int}^{3}_{-C}\hat{f}_2(x_1,x_2)\hat{\times}\hat{d}\hat{\hat{x}}_2$$

$$\neq-\hat{\int}^{3}_{C}\hat{f}_1(x_1,x_2)\hat{\times}\hat{d}\hat{\hat{x}}_1-\hat{\int}^{3}_{C}\hat{f}_2(x_1,x_2)\hat{\times}\hat{d}\hat{\hat{x}}_2$$

Proof.

$$\hat{\int}^3_{-C}\hat{f}_1(x_1,x_2)\hat{\times}\hat{d}\hat{\hat{x}}_1+\hat{\int}^3_{-C}\hat{f}_2(x_1,x_2)\hat{\times}\hat{d}\hat{\hat{x}}_2$$

$$=\int_a^b\hat{f}_1(x_1(a+b-t),x_2(a+b-t))$$

$$\frac{\frac{d}{dt}x_1((a+b-t)\hat{T}(a+b-t))(\hat{T}(a+b-t)+(a+b-t)\frac{d}{dt}\hat{T}(a+b-t))\hat{T}(a+b-t)-x_1((a+b-t)\hat{T}(a+b-t))\frac{d}{dt}\hat{T}(a+b-t)}{\hat{T}(a+b-t)}dt$$

$$+\int_a^b\hat{f}_2(x_1(a+b-t),x_2(a+b-t))$$

$$\frac{\frac{d}{dt}x_2((as+b-t)\hat{T}(a+b-t))(\hat{T}(a+b-t)+(a+b-t)\frac{d}{dt}\hat{T}(a+b-t))\hat{T}(a+b-t)-x_2((a+b-t)\hat{T}(a+b-t))\frac{d}{dt}\hat{T}(a+b-t)}{\hat{T}(a+b-t)}dt$$

$$=\int_a^b\hat{f}_1(x_1(a+b-t),x_2(a+b-t))$$

$$\frac{-x_1'((a+b-t)\hat{T}(a+b-t))(\hat{T}(a+b-t)-(a+b-t)\hat{T}'(a+b-t))\hat{T}(a+b-t)+x_1((a+b-t)\hat{T}(a+b-t))\hat{T}'(a+b-t)}{\hat{T}(a+b-t)}dt$$

$$+\int_a^b\hat{f}_2(x_1(a+b-t),x_2(a+b-t))$$

$$\frac{-x_2'((as+b-t)\hat{T}(a+b-t))(\hat{T}(a+b-t)-(a+b-t)\hat{T}'(a+b-t))\hat{T}(a+b-t)+x_2((a+b-t)\hat{T}(a+b-t))\hat{T}'(a+b-t)}{\hat{T}(a+b-t)}dt$$

$$a+b-t=u$$

$$=-\int_b^a\hat{f}_1(x_1(u),x_2(u))\frac{-x_1'(u\hat{T}(u))(\hat{T}(u)-u\hat{T}'(u))\hat{T}(u)+x_1(u\hat{T}(u))\hat{T}'(u)}{\hat{T}(u)}du$$

$$-\int_b^a\hat{f}_2(x_1(u),x_2(u))\frac{-x_2'(u\hat{T}(u))(\hat{T}(u)-u\hat{T}'(u))\hat{T}(u)+x_2(u\hat{T}(u))\hat{T}'(u)}{\hat{T}(u)}du$$

$$=\int_a^b\hat{f}_1(x_1(u),x_2(u))\frac{-x_1'(u\hat{T}(u))(\hat{T}(u)-u\hat{T}'(u))\hat{T}(u)+x_1(u\hat{T}(u))\hat{T}'(u)}{\hat{T}(u)}du$$

$$+\int_a^b\hat{f}_2(x_1(u),x_2(u))\frac{-x_2'(u\hat{T}(u))(\hat{T}(u)-u\hat{T}'(u))\hat{T}(u)+x_2(u\hat{T}(u))\hat{T}'(u)}{\hat{T}(u)}du$$

$$\neq-\hat{\int}^3_{C}\hat{f}_1(x_1,x_2)\hat{\times}\hat{d}\hat{\hat{x}}_1-\hat{\int}^3_{C}\hat{f}_2(x_1,x_2)\hat{\times}\hat{d}\hat{\hat{x}}_2.$$

□

Theorem 3.2.4.

$$\hat{\int}^4_{-C}\hat{f}_1(x_1,x_2)\hat{\times}\hat{d}x_1{}^\wedge+\hat{\int}^4_{-C}\hat{f}_2(x_1,x_2)\hat{\times}\hat{d}x_2{}^\wedge$$

$$\neq-\hat{\int}^4_{C}\hat{f}_1(x_1,x_2)\hat{\times}\hat{d}x_1{}^\wedge-\hat{\int}^4_{C}\hat{f}_2(x_1,x_2)\hat{\times}\hat{d}x_2{}^\wedge.$$

Proof.

$\hat{\int}^4_{-C}\hat{f}_1(x_1,x_2)\hat{\times}\hat{d}x_1{}^\wedge+\hat{\int}^4_{-C}\hat{f}_2(x_1,x_2)\hat{\times}\hat{d}x_2{}^\wedge$

$=\int_a^b\hat{f}_1(x_1(a+b-t),x_2(a+b-t))\hat{T}(a+b-t)\frac{d}{dt}x_1((a+b-t)\hat{T}(a+b-t))(\hat{T}(a+b-t)$

$+(a+b-t)\frac{d}{dt}\hat{T}(a+b-t))dt$

$+\int_a^b\hat{f}_2(x_1(a+b-t),x_2(a+b-t))\hat{T}(a+b-t)\frac{d}{dt}x_2((a+b-t)\hat{T}(a+b-t))(\hat{T}(a+b-t)$

$+(a+b-t)\frac{d}{dt}\hat{T}(a+b-t))dt$

$=\int_a^b\hat{f}_1(x_1(a+b-t),x_2(a+b-t))\hat{T}(a+b-t)(-x_1'((a+b-t)\hat{T}(a+b-t))(\hat{T}(a+b-t)$

$-(a+b-t)\hat{T}'(a+b-t))dt$

$+\int_a^b\hat{f}_2(x_1(a+b-t),x_2(a+b-t))\hat{T}(a+b-t)x_2'((a+b-t)\hat{T}(a+b-t))(\hat{T}(a+b-t)$

$-(a+b-t)\hat{T}'(a+b-t))dt$

$a+b-t=u$

$=-\int_b^a\hat{f}_1(x_1(u),x_2(u))\hat{T}(u)(-x_1'(u\hat{T}(u))(\hat{T}(u)-u\hat{T}'(u))dt$

$-\int_b^a\hat{f}_2(x_1(u),x_2(u))\hat{T}(u)x_2'(u\hat{T}(u))(\hat{T}(u)-u\hat{T}'(u))dt$

$=\int_a^b\hat{f}_1(x_1(u),x_2(u))\hat{T}(u)(-x_1'(u\hat{T}(u))(\hat{T}(u)-u\hat{T}'(u))dt$

$+\int_a^b\hat{f}_2(x_1(u),x_2(u))\hat{T}(u)x_2'(u\hat{T}(u))(\hat{T}(u)-u\hat{T}'(u))dt$

$\neq-\hat{\int}^4_{C}\hat{f}_1(x_1,x_2)\hat{\times}\hat{d}x_1{}^\wedge-\hat{\int}^4_{C}\hat{f}_2(x_1,x_2)\hat{\times}\hat{d}x_2{}^\wedge.$

$\square$

Theorem 3.2.5.

$$\hat{\int}^5_{-C}\hat{f}_1(x_1,x_2)\hat{\times}\hat{d}x_1+\hat{\int}^5_{-C}\hat{f}_2(x_1,x_2)\hat{\times}\hat{d}x_2$$

$$\neq-\hat{\int}^5_{C}\hat{f}_1(x_1,x_2)\hat{\times}\hat{d}x_1-\hat{\int}^5_{C}\hat{f}_2(x_1,x_2)\hat{\times}\hat{d}x_2.$$

Proof.

$$\hat{\int}^{5}_{-C}\hat{f}_1(x_1,x_2)\hat{\times}\hat{d}x_1+\hat{\int}^{5}_{-C}\hat{f}_2(x_1,x_2)\hat{\times}\hat{d}x_2$$

$$=\int_a^b \hat{f}_1(x_1(a+b-t),x_2(a+b-t))\frac{d}{dt}x_1\left(\frac{a+b-t}{\hat{T}(a+b-t)}\right)\frac{\hat{T}(a+b-t)-(a+b-t)\frac{d}{dt}\hat{T}(a+b-t)}{\hat{T}(a+b-t)}dt$$

$$+\int_a^b \hat{f}_2(x_1(a+b-t),x_2(a+b-t))\frac{d}{dt}x_2\left(\frac{a+b-t}{\hat{T}(a+b-t)}\right)\frac{\hat{T}(a+b-t)-(a+b-t)\frac{d}{dt}\hat{T}(a+b-t)}{\hat{T}(a+b-t)}dt$$

$$=\int_a^b \hat{f}_1(x_1(a+b-t),x_2(a+b-t))\left(-x_1'\left(\frac{a+b-t}{\hat{T}(a+b-t)}\right)\right)\frac{\hat{T}(a+b-t)+(a+b-t)\hat{T}'(a+b-t)}{\hat{T}(a+b-t)}dt$$

$$+\int_a^b \hat{f}_2(x_1(a+b-t),x_2(a+b-t))\left(-x_2'\left(\frac{a+b-t}{\hat{T}(a+b-t)}\right)\right)\frac{\hat{T}(a+b-t)+(a+b-t)\hat{T}'(a+b-t)}{\hat{T}(a+b-t)}dt$$

$$a+b-t=u$$

$$=-\int_b^a \hat{f}_1(x_1(u),x_2(u))\left(-x_1'\left(\frac{u}{\hat{T}(u)}\right)\right)\frac{\hat{T}(u)+u\hat{T}'(u)}{\hat{T}(u)}du$$

$$-\int_b^a \hat{f}_2(x_1(u),x_2(u))\left(-x_2'\left(\frac{u}{\hat{T}(u)}\right)\right)\frac{\hat{T}(u)+u\hat{T}'(u)}{\hat{T}(u)}du$$

$$=\int_a^b \hat{f}_1(x_1(u),x_2(u))\left(-x_1'\left(\frac{u}{\hat{T}(u)}\right)\right)\frac{\hat{T}(u)+u\hat{T}'(u)}{\hat{T}(u)}du$$

$$+\int_a^b \hat{f}_2(x_1(u),x_2(u))\left(-x_2'\left(\frac{u}{\hat{T}(u)}\right)\right)\frac{\hat{T}(u)+u\hat{T}'(u)}{\hat{T}(u)}du$$

$$\neq-\hat{\int}^{5}_{C}\hat{f}_1(x_1,x_2)\hat{\times}\hat{d}x_1-\hat{\int}^{5}_{C}\hat{f}_2(x_1,x_2)\hat{\times}\hat{d}x_2.$$

□

In the general case we can not formulate analogues of the Green's theorems connected with the line integrals because $\hat{d}\hat{x}_i^{\wedge\wedge}$, $\hat{d}\hat{x}_i^{\wedge}$, $\hat{d}x_i^{\wedge}$, $\hat{d}\hat{x}_i$ and $\hat{d}x_1$, $i=1,2$, depends on x_i and t and they are not invertible relation to t in the general case.

3.3. Surface Iso-Integrals

Let Σ be a surface in $\mathbb{R}^3$, parameterized by the equations

$$x_1=x_1(u,v),\quad x_2=x_2(u,v),\quad x_3=x_3(u,v),\quad (u,v)\in G,\quad G|subset\mathbb{R}^2.$$

We put the quantities

$$E=x_{1u}(u,v)^2+x_{2u}(u,v)^2+x_{3u}(u,v)^2,$$

$$F=x_{1u}(u,v)x_{1v}(u,v)+x_{2u}(u,v)x_{2v}(u,v)+x_{3u}(u,v)x_{3v}(u,v),$$

$$D=x_{1v}(u,v)^2+x_{2v}(u,v)^2+x_{3v}(u,v)^2.$$

Suppose that $f, P, Q, R : \Sigma \longrightarrow \mathbb{R}$ be continuous functions and $\hat{T} : \Sigma \longrightarrow \mathbb{R}$ be a positive continuously-differentiable function. Let also, the iso-lifts of the first, the second, the third, the fourth or the fifth kind of f, P, Q and R exist on Σ.

Definition 3.3.1. *The surface iso-integral of the first kind of $\hat{f}$ over Σ is defined as follows*

$$\hat{\int}\hat{\int}_{\Sigma} \hat{f}(x_1, x_2, x_3) \hat{d}\hat{\sigma} = \int\int_{G} \hat{f}(x_1(u,v), x_2(u,v), x_3(u,v)) \hat{\times} \sqrt{ED - F^2} \hat{\times} \hat{d}\hat{u}\hat{d}\hat{v}.$$

Exercise 3.3.2. *Let $\Sigma : \frac{x_1}{2} + \frac{x_2}{3} + \frac{x_3}{4} = 1$, $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $f(x_1, x_2, x_3) = x_1 + x_2 + x_3$, $\hat{T}(u,v) = u + v + 1$, $u \geq 0$, $v \geq 0$. Compute*

$$\hat{\int}\hat{\int}_{\Sigma} \hat{f}^{\wedge}(x_1, x_2, x_3) \hat{d}\hat{\sigma}.$$

Exercise 3.3.3. *Let $\Sigma : x_1{}^2 + x_2{}^2 + x_3{}^2 = 1$, $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $f(x_1, x_2, x_3) = 2x_1 - x_2 + x_3^2$, $\hat{T}(u,v) = u + v + 1$, $u \geq 0$, $v \geq 0$. Compute*

$$\hat{\int}\hat{\int}_{\Sigma} \hat{f}^{\wedge}(\hat{x}_1, \hat{x}_2, \hat{x}_3) \hat{d}\hat{\sigma}.$$

Now we will define the quantities

$$A = x_{2u}x_{3v} - x_{3u}x_{2v}, \quad B = x_{3u}x_{1v} - x_{3v}x_{1u}, \quad C = x_{1u}x_{2v} - x_{2u}x_{1v}.$$

Definition 3.3.4. *The surface iso-integral of the second kind is defined as follows*

$$\hat{\int}\hat{\int}_{\Sigma} \hat{P} \hat{\times} \hat{d}\hat{x}_2 \hat{d}\hat{x}_3 + \hat{Q} \hat{\times} \hat{d}\hat{x}_3 \hat{d}\hat{x}_1 + \hat{R} \hat{\times} \hat{d}\hat{x}_1 \hat{d}\hat{x}_2$$

$$= \hat{\int}\hat{\int}_{\Sigma} \left(\hat{P} \hat{\times} \frac{A}{\sqrt{A^2+B^2+C^2}} + \hat{Q} \hat{\times} \frac{B}{\sqrt{A^2+B^2+C^2}} + \hat{R} \hat{\times} \frac{C}{\sqrt{A^2+B^2+C^2}} \right) \hat{\times} \hat{d}\hat{x}_1 \hat{d}\hat{x}_2 \hat{d}\hat{x}_3.$$

Exercise 3.3.5. *Let $\Sigma : x_1{}^2 + x_2{}^2 + x_3{}^2 = 1$, $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $P(x_1, x_2, x_3) = 2x_1 - x_2 + x_3^2$, $Q(x_1, x_2, x_3) = x_2 + x_3^2$, $R(x_1, x_2, x_3) = 2x_1 + x_3$ $\hat{T}(u,v) = u + v + 1$, $u \geq 0$, $v \geq 0$. Compute*

$$\hat{\int}\hat{\int}_{\Sigma} \hat{P}(\hat{x}_1, \hat{x}_2, \hat{x}_3) \hat{\times} \hat{d}\hat{x}_2 \hat{d}\hat{x}_3 + \hat{Q}(\hat{x}_1, \hat{x}_2, \hat{x}_3) \hat{\times} \hat{d}\hat{x}_3 \hat{d}\hat{x}_1 + \hat{R}(\hat{x}_1, \hat{x}_2, \hat{x}_3) \hat{\times} \hat{d}\hat{x}_1 \hat{d}\hat{x}_2.$$

3.4. Advanced practical exercises

Problem 3.4.1. *Let $C : x_1(t) = r\sin t$, $x_2(t) = r\cos t$, $t \in [0, \pi]$, $\hat{T}(t) = t^4 + 1$, $f(x_1, x_2) = x_1^2 + 2x_1x_2$. Compute*

$$\hat{\int}_{L} \hat{f}^{\wedge}(\hat{x}_1\hat{x}_2) \hat{\times} \hat{d}\hat{s}^{\wedge\wedge}.$$

Problem 3.4.2. *Let $C : x_1(t) = 2\sqrt{2}t + 2$, $x_2(t) = 2\sqrt{2}t + 1$, $t \in [1, 2]$, $f(x_1, x_2) = x_1^2 x_2 + x_1$, $\hat{T}(t) = t + 2$. Compute*

$$\hat{\int}_{C}^{2} \hat{f}(\hat{x}_1, \hat{x}_2) \hat{\times} \hat{d}\hat{s}^{\wedge}.$$

Problem 3.4.3. *Let* $C : x_1(t) = t+2t^2,\ x_2(t) = t+2,\ t \in [1,2],\ f(x_1,x_2) = x_1^2 - x_2^2,\ \hat{T}(t) = t+1$. *Compute*

$$\hat{\int}_C^3 f^{\wedge}(x_1,x_2)\hat{\times}\hat{d}\hat{s}(\hat{t}).$$

Problem 3.4.4. *Let* $C : x_1(t) = t+1,\ x_2(t) = t^2+2t+1,\ t \in [0,1],\ f(x_1,x_2) = x_1 + x_2^2,\ \hat{T}(t) = t^2+1$. *Compute*

$$\hat{\int}_C^4 f^{\wedge}(x_1,x_2)\hat{\times}\hat{d}s^{\wedge}.$$

Problem 3.4.5. *Let* $C : x_1(t) = t^2+2t+3,\ x_2(t) = t^2+t+1,\ t \in [0,1],\ f(x_1,x_2) = x_1^4 + x_2^4 + 2x_1^2x_2^2,\ \hat{T}(t) = t^2+1$. *Compute*

$$\hat{\int}_C^5 \hat{f}^{\wedge}(\hat{x}_1,\hat{x}_2)\hat{\times}\hat{d}s^{\wedge}.$$

Problem 3.4.6. *Let* $C : x_1(t) = t^2+1,\ x_2(t) = t^2,\ t \in [0,1],\ f_1(x_1,x_2) = x_1^2 - 2x_2,\ f_2(x_1,x_2) = 2x_1 + x_2,\ \hat{T}(t) = t^2+1$. *Compute*

$$\hat{\int}_C^1 \hat{f}_1(x_1,x_2)\hat{\times}\hat{d}\hat{x}_1^{\wedge\wedge} + \hat{\int}_C^1 \hat{f}_2(x_1,x_2)\hat{\times}\hat{d}\hat{x}_2^{\wedge\wedge}.$$

Problem 3.4.7. *Let* $C : x_1(t) = t^2+t+1,\ x_2(t) = t^2,\ t \in [0,1],\ f_1(x_1,x_2) = 2x_1 + x_2^2,\ f_2(x_1,x_2) = x_1 + 3x_2,\ \hat{T}(t) = t^2+t+1$. *Compute*

$$\hat{\int}_C^2 \hat{f}_1(x_1,x_2)\hat{\times}\hat{d}\hat{x}_1^{\wedge} + \hat{\int}_C^2 \hat{f}_2(x_1,x_2)\hat{\times}\hat{d}\hat{x}_2^{\wedge}.$$

Problem 3.4.8. *Let* $C : x_1(t) = t+1,\ x_2(t) = t+4,\ t \in [0,1],\ f_1(x_1,x_2) = 2x_1^2 - x_2^2,\ f_2(x_1,x_2) = x_1^2 + x_2 + 2x_1x_2,\ \hat{T}(t) = t^2+1$. *Compute*

$$\hat{\int}_C^3 \hat{f}_1(x_1,x_2)\hat{\times}\hat{d}\hat{\hat{x}}_1 + \hat{\int}_C^3 \hat{f}_2(x_1,x_2)\hat{\times}\hat{d}\hat{\hat{x}}_2.$$

Problem 3.4.9. *Let* $C : x_1(t) = t+1,\ x_2(t) = t+4,\ t \in [0,1],\ f_1(x_1,x_2) = x_1^2 + x_2^2 + 3x_1x_2,\ f_2(x_1,x_2) = x_1^2 + x_2^2,\ \hat{T}(t) = t+1$. *Compute*

$$\hat{\int}_C^4 \hat{f}_1(x_1,x_2)\hat{\times}\hat{d}x_1^{\wedge} + \hat{\int}_C^4 \hat{f}_2(x_1,x_2)\hat{\times}\hat{d}x_2^{\wedge}.$$

Problem 3.4.10. *Let* $C : x_1(t) = t+1,\ x_2(t) = t^2,\ t \in [0,1],\ f_1(x_1,x_2) = x_1 - x_2,\ f_2(x_1,x_2) = x_1 - x_2^2,\ \hat{T}(t) = t+1$. *Compute*

$$\hat{\int}_C^5 \hat{f}_1(x_1,x_2)\hat{\times}\hat{d}x_1 + \hat{\int}_C^5 \hat{f}_2(x_1,x_2)\hat{\times}\hat{d}x_2.$$

Problem 3.4.11. *Let* $C : x_1(t) = t^2+1,\ x_2(t) = 2t^2+2,\ t \in [0,1],\ f_1(x_1,x_2) = 2x_1 - x_2,\ f_2(x_1,x_2) = x_1 - 2x_2^2,\ \hat{T}(t) = t+1$. *Compute*

$$\hat{\int}_C^4 \hat{f}_1(x_1,x_2)\hat{\times}\hat{d}x_1 + \hat{\int}_C^4 \hat{f}_2(x_1,x_2)\hat{\times}\hat{d}x_2.$$

Problem 3.4.12. *Let* $C : x_1(t) = t^2+1,\ x_2(t) = 2t,\ t \in [0,1],\ f_1(x_1,x_2) = 2x_1 - x_2,$ $f_2(x_1,x_2) = x_1 - 2x_2,\ \hat{T}(t) = t+1$. *Compute*

$$\hat{\int}_C^5 \hat{f}_1(x_1,x_2)\hat{\times}\hat{d}x_1 + \hat{\int}_C^5 \hat{f}_2(x_1,x_2)\hat{\times}\hat{d}x_2.$$

Problem 3.4.13. *Let* $C : x_1(t) = t+1,\ x_2(t) = t+2,\ t \in [0,1],\ f_1(x_1,x_2) = 2x_1 - 3x_2,$ $f_2(x_1,x_2) = 4x_1 - 5x_2,\ \hat{T}(t) = 2t+1$. *Compute*

$$\hat{\int}_C^2 \hat{f}_1(x_1,x_2)\hat{\times}\hat{d}x_1 + \hat{\int}_C^2 \hat{f}_2(x_1,x_2)\hat{\times}\hat{d}x_2.$$

Problem 3.4.14. *Let* $\Sigma : {x_1}^2 + 4{x_2}^2 + {x_3}^2 2 = 1,\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.\ \ f(x_1,x_2,x_3) =$ $2x_1^2 - x_2,\ \hat{T}(u,v) = u+v+1,\ u \geq 0,\ v \geq 0$. *Compute*

$$\hat{\int}\hat{\int}_{\Sigma} \hat{f}^{\wedge}(\hat{x}_1,\hat{x}_2,\hat{x}_3)\hat{d}\hat{\sigma}.$$

Problem 3.4.15. *Let* $\Sigma : x_1 + x_2 + {x_3}^2 = 1,\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ P(x_1,x_2,x_3) = x_1^2 - x_2 + x_3^2,$ $Q(x_1,x_2,x_3) = x_3^2,\ R(x_1,x_2,x_3) = 2x_1,\ \hat{T}(u,v) = u+v+1,\ u \geq 0,\ v \geq 0$. *Compute*

$$\hat{\int}\hat{\int}_{\Sigma} \hat{P}(\hat{x}_1,\hat{x}_2,\hat{x}_3)\hat{\times}\hat{d}\hat{x}_2\hat{d}\hat{x}_3 + \hat{Q}(\hat{x}_1,\hat{x}_2,\hat{x}_3)\hat{\times}\hat{d}\hat{x}_3\hat{d}\hat{x}_1 + \hat{R}(\hat{x}_1,\hat{x}_2,\hat{x}_3)\hat{\times}\hat{d}\hat{x}_1\hat{d}\hat{x}_2.$$

Chapter 4

The Iso-Fourier Iso- Integral

4.1. Definition of the Iso-Fourier Iso-Integral

We suppose that E is a measurable set in $\mathbb{R}$, $f : E \longrightarrow \mathbb{R}$ is defined and integrable on E. Let also, $\hat{T} : E \longrightarrow \mathbb{R}$ is a positive continuously-differentiable function.

Definition 4.1.1. *The iso-Fourier iso-integral is defined with*

$$\int_E f(x)\Big(1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx.$$

4.2. Properties of the Iso-Fourier Iso-Integral

Here we will study some of the properties of the iso-Fourier iso-integral.

Theorem 4.2.1. *Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of bounded and measurable functions on the measurable set E, which converges in measure to the measurable function $F(x)$ on E. Let also, $\hat{T}(x)$ is a measurable function on E, its derivative $\hat{T}'(x)$ exists on E and it is measurable on E, and $\left|1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\right| \le A$ for almost all $x \in E$, where A is a positive constant. If $|f_n(x)| \le K$ for every $x \in E$, every $n \in \mathbb{N}$ and for some positive constant K, then*

$$\lim_{n\longrightarrow\infty}\int_E f_n(x)\Big(1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx = \int_E F(x)\Big(1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx.$$

Proof. Since $f_n \longrightarrow_{n\longrightarrow\infty} F$ in measure, by the Riesz's Theorem it follows that there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of the sequence $\{f_n\}_{n=1}^{\infty}$ such that $f_{n_k} \longrightarrow_{k\longrightarrow\infty} F$ almost everywhere in E. From here, using that $|f_{n_k}(x)| \le K$ for every $x \in E$ and every $k \in \mathbb{N}$, we have that $|F(x)| \le K$ for almost all $x \in E$.

For $n \in \mathbb{N}$ and $\sigma > 0$, we define the sets

$$A_n(\sigma) = E(|f_n - F| \ge \sigma), \quad B_n(\sigma) = E(|f_n - F| < \sigma).$$

We have

$$A_n(\sigma) \cup B_n(\sigma) = E, \quad A_n(\sigma) \cap B_n(\sigma) = \emptyset.$$

Then

$$\int_E |f_n(x)-F(x)|\left|1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\right|dx \le A\int_E |f_n(x)-F(x)|dx$$

$$= A\int_{A_n(\sigma)\cup B_n(\sigma)} |f_n(x)-F(x)|dx$$

$$= A\int_{A_n(\sigma)} |f_n(x)-F(x)|dx + A\int_{B_n(\sigma)} |f_n(x)-F(x)|$$

$$\le A\int_{E(|f_n-F|\ge\sigma)} |f_n(x)-F(x)|dx + A\sigma\mu E(|f_n-F|<\sigma)$$

$$\le A\int_{E(|f_n-F|\ge\sigma)} |f_n(x)-F(x)|dx + A\sigma\mu E$$

$$\le A\int_{E(|f_n-F|\ge\sigma)} (|f_n(x)|+|F(x)|)dx + A\sigma\mu E,$$

now we use that $|f_n(x)|\le K$ for every $x\in E$ and for every $n\in\mathbb{N}$, and $|F(x)|\le K$ for almost all $x\in E$, therefore

$$(A10)\ \int_E |f_n(x)-F(x)|\left|1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\right|dx \le 2AK\mu E(|f_n-F|\ge\sigma)+A\sigma\mu E$$

Let $\varepsilon>0$ be arbitrarily chosen and fixed. From $f_n \longrightarrow_{n\longrightarrow\infty} F$ in measure, we have

$$\lim_{n\longrightarrow\infty}\mu E(|f_n-F|\ge\sigma)=0$$

for every $\sigma\ge 0$. Then there exists $N_1=N_1(\varepsilon)\in\mathbb{N}$ such that for every $n\ge N_1$ we have

$$\mu E(|f_n-F|\ge\frac{\varepsilon}{2A\mu E})<\frac{\varepsilon}{4AK}.$$

From here and from (A10), for $\sigma=\frac{\varepsilon}{2A\mu E}$,

$$\int_E |f_n(x)-F(x)|\left|1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\right|dx < 2AK\frac{\varepsilon}{4AK}+A\frac{\varepsilon}{2A\mu E}\mu E$$

$$=\frac{\varepsilon}{2}+\frac{\varepsilon}{2}$$

$$=\varepsilon.$$

Because $\varepsilon>0$ was arbitrarily chosen, we conclude that

$$\lim_{n\longrightarrow\infty}\int_E |f_n(x)-F(x)|\left|1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\right|dx=0.$$

Using the last limit, we obtain that

$$\lim_{n\longrightarrow\infty}\left|\int_E f_n(x)\left(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\right)dx-\int_E F(x)\left(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\right)dx\right|$$

$$=\lim_{n\longrightarrow\infty}\left|\int_E (f_n(x)-F(x))\left(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\right)dx\right|$$

$$\le\lim_{n\longrightarrow\infty}\int_E |f_n(x)-F(x)|\left|1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\right|dx=0,$$

consequently

$$\lim_{n\longrightarrow\infty}\int_E f_n(x)\Big(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx=\int_E F(x)\Big(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx.$$

$\square$

Corollary 4.2.2. *Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of bounded and measurable functions on the measurable set E, which converges in measure to the measurable function $F(x)$ on E. Let also, $\hat{T}(x)$ is a measurable function on E, its derivative $\hat{T}'(x)$ exists on E and it is measurable on E, and $\left|1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\right|\le\Psi(x)$ for almost all $x\in E$, where $\Psi(x)$ is a bounded and measurable function on E. If $|f_n(x)|\le K$ for every $x\in E$, every $n\in\mathbb{N}$ and for some positive constant K, then*

$$\lim_{n\longrightarrow\infty}\int_E f_n(x)\Big(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx=\int_E F(x)\Big(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx.$$

Proof. Since $\Psi(x)$ is a bounded and measurable function on E then there exists a positive constant A such that

$$\left|1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\right|\le A\quad\text{for}\quad\forall x\in E.$$

From here and by Theorem 4.2.1 it follows the assertion. $\square$

Corollary 4.2.3. *Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of bounded and measurable functions on the measurable set E which converges in measure to the measurable function $F(x)$ on E. Let also, $\hat{T}(x)$ is a measurable function on E, its derivative $\hat{T}'(x)$ exists on E and it is measurable on E, and $\left|1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\right|\le\Psi(x)$ for almost all $x\in E$, where $\Psi(x)$ is a summable and measurable function on E. If $|f_n(x)|\le K$ for every $x\in E$, every $n\in\mathbb{N}$ and for some positive constant K, then*

$$\lim_{n\longrightarrow\infty}\int_E f_n(x)\Big(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx=\int_E F(x)\Big(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx.$$

Proof. Since $\Psi(x)$ is a summable and measurable function on E then there exists a positive constant A such that

$$\left|1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\right|\le A$$

for almost all $x\in E$. From here and by Theorem 4.2.1 it follows the assertion. $\square$

Theorem 4.2.4. *Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of bounded and measurable functions on the measurable set E, which converges in measure to the measurable function $F(x)$ on E. Let also, $\hat{T}(x)$ is a measurable function on E, its derivative $\hat{T}'(x)$ exists on E and it is measurable on E, and $\left|1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\right|\le A$ for almost all $x\in E$, where A is a positive constant. If $|f_n(x)|\le\Phi(x)$ for every $x\in E$, every $n\in\mathbb{N}$, for some measurable and summable function $\Phi(x)$ on E, then*

$$\lim_{n\longrightarrow\infty}\int_E f_n(x)\Big(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx=\int_E F(x)\Big(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx.$$

Proof. Since $f_n \longrightarrow_{n\longrightarrow\infty} F$ in measure, from the Riesz's Theorem it follows that there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of the sequence $\{f_n\}_{n=1}^{\infty}$ such that $f_{n_k} \longrightarrow_{k\longrightarrow\infty} F$ almost everywhere in E. From here, using that $|f_{n_k}(x)| \le \Phi(x)$ for every $x \in E$, every $k \in \mathbb{N}$, we conclude that $|F(x)| \le \Phi(x)$ for almost all $x \in E$.

For $n \in \mathbb{N}$ and $\sigma > 0$, we define the sets

$$A_n(\sigma) = E(|f_n - F| \ge \sigma), \quad B_n(\sigma) = E(|f_n - F| < \sigma).$$

We have

$$A_n(\sigma) \cup B_n(\sigma) = E, \quad A_n(\sigma) \cap B_n(\sigma) = \emptyset.$$

Then

$$\int_E |f_n(x) - F(x)| \left|1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\right| dx \le A \int_E |f_n(x) - F(x)| dx$$

$$= A \int_{A_n(\sigma)\cup B_n(\sigma)} |f_n(x) - F(x)| dx$$

$$= A \int_{A_n(\sigma)} |f_n(x) - F(x)| dx + A \int_{B_n(\sigma)} |f_n(x) - F(x)|$$

$$\le A \int_{E(|f_n - F| \ge \sigma)} |f_n(x) - F(x)| dx + A\sigma\mu E(|f_n - F| < \sigma)$$

$$\le A \int_{E(|f_n - F| \ge \sigma)} |f_n(x) - F(x)| dx + A\sigma\mu E$$

$$\le A \int_{E(|f_n - F| \ge \sigma)} (|f_n(x)| + |F(x)|) dx + A\sigma\mu E,$$

now we use that $|f_n(x)| \le \Phi(x)$ for every $x \in E$, every $n \in \mathbb{N}$, and $|F(x)| \le \Phi(x)$ for almost all $x \in E$, therefore

$$(A11)\ \int_E |f_n(x) - F(x)| \left|1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\right| dx \le 2A \int_{E(|f_n - F| \ge \sigma)} \Phi(x) dx + A\sigma\mu E.$$

Let $\varepsilon > 0$ be arbitrarily chosen and fixed. From $f_n \longrightarrow_{n\longrightarrow\infty} F$ in measure, we have

$$\lim_{n\longrightarrow\infty} \mu E(|f_n - F| \ge \sigma) = 0$$

for every $\sigma \ge 0$. Then there exists $N_1 = N_1(\varepsilon) \in \mathbb{N}$ such that

$$\mu E(|f_n - F| \ge \frac{\varepsilon}{2A\mu E}) < \frac{\varepsilon}{4A}$$

and

$$\int_{E(|f_n - F| \ge \sigma)} \Phi(x) dx < \frac{\varepsilon}{4A}$$

for every $n \ge N_1$. From here and from (A11), for $\sigma = \frac{\varepsilon}{2A\mu E}$,

$$\int_E |f_n(x) - F(x)| \left|1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\right| dx < 2A\frac{\varepsilon}{4A} + A\frac{\varepsilon}{2A\mu E}\mu E$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon.$$

Because $\varepsilon > 0$ was arbitrarily chosen, we conclude that

$$\lim_{n\longrightarrow\infty}\int_E |f_n(x)-F(x)|\Big|1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big|dx = 0.$$

From here,

$$\lim_{n\longrightarrow\infty}\Big|\int_E f_n(x)\Big(1-x\tfrac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx - \int_E F(x)\Big(1-x\tfrac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx\Big|$$

$$= \lim_{n\longrightarrow\infty}\Big|\int_E (f_n(x)-F(x))\Big(1-x\tfrac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx\Big|$$

$$\le \lim_{n\longrightarrow\infty}\int_E |f_n(x)-F(x)|\Big|1-x\tfrac{\hat{T}'(x)}{\hat{T}(x)}\Big|dx = 0,$$

consequently

$$\lim_{n\longrightarrow\infty}\int_E f_n(x)\Big(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx = \int_E F(x)\Big(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx.$$

□

Corollary 4.2.5. *Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of bounded and measurable functions on the measurable set E, which converges in measure to the measurable function $F(x)$ on E. Let also, $\hat{T}(x)$ is a measurable function on E, its derivative $\hat{T}'(x)$ exists on E and it is measurable on E, and $\Big|1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big| \le \Psi(x)$ for almost all $x \in E$, where $\Psi(x)$ is a bounded and measurable function on E. If $|f_n(x)| \le \Phi(x)$ for every $x \in E$, every $n \in \mathbb{N}$, for some measurable and bounded function $\Phi(x)$ on E, then*

$$\lim_{n\longrightarrow\infty}\int_E f_n(x)\Big(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx = \int_E F(x)\Big(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx.$$

Proof. Since $\Psi(x)$ is a bounded and measurable function on E, we have that there exists a positive constant A such that

$$\Big|1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big| \le A \quad \text{for} \quad \forall x \in E.$$

From here and the above Theorem 4.2.4 it follows the assertion. □

Corollary 4.2.6. *Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of bounded and measurable functions on the measurable set E, which converges in measure to the measurable function $F(x)$ on E. Let also, $\hat{T}(x)$ is a measurable function on E, its derivative $\hat{T}'(x)$ exists on E and it is measurable on E, and $\Big|1-\frac{\hat{T}'(x)}{\hat{T}(x)}\Big| \le \Psi(x)$ for almost all $x \in E$, where $\Psi(x)$ is a summable and measurable function on E. If $|f_n(x)| \le \Phi(x)$ for every $x \in E$, every $n \in \mathbb{N}$, for some measurable and summable function $\Phi(x)$ on E, then*

$$\lim_{n\longrightarrow\infty}\int_E f_n(x)\Big(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx = \int_E F(x)\Big(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx.$$

Proof. Since $\Psi(x)$ is a summable and measurable function on E, we have that there exists a positive constant A such that

$$\Big|1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big| \leq A$$

for almost all $x \in E$. From here and by Theorem 4.2.4 it follows the assertion. $\square$

Corollary 4.2.7. *Under the hypotheses of the Theorem 4.2.4, if* ϕ *is a bounded measurable function on E, we have*

$$\lim_{n\longrightarrow\infty}\int_E f_n(x)\phi(x)\Big(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx = \int_E F(x)\phi(x)\Big(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx.$$

Proof. Since ϕ is a bounded measurable function on E, we have that there exists a positive constant A_1 such that

$$|\phi(x)| \leq A_1 \quad \text{for} \quad \forall x \in E.$$

Then, for every $\sigma \geq 0$, we obtain that

$(A12) E(|f_n\phi - F\phi| \geq \sigma) \subset E(|f_n - F| \geq \frac{\sigma}{A_1})$.

Because $f_n \longrightarrow_{n\longrightarrow\infty} F$ in measure, we have

$$\lim_{n\longrightarrow\infty}\mu E(|f_n - F| \geq \frac{\sigma}{A_1}) = 0.$$

From here and (A12), we conclude that

$$\lim_{n\longrightarrow\infty}\mu E(|f_n\phi - F\phi| \geq \sigma) = 0$$

for every $\sigma \geq 0$. Therefore $f_n\phi \longrightarrow_{n\longrightarrow\infty} F\phi$ in measure. From the last limit and by Theorem 4.2.4, we obtain that

$$\lim_{n\longrightarrow\infty}\int_E f_n(x)\phi(x)dx = \int_E F(x)\phi(x)dx.$$

$\square$

Corollary 4.2.8. *Under the hypotheses of Corollary 4.2.5, if* ϕ *is a bounded measurable function on E, we have*

$$\lim_{n\longrightarrow\infty}\int_E f_n(x)\phi(x)\Big(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx = \int_E F(x)\phi(x)\Big(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx.$$

Corollary 4.2.9. *Under the hypotheses of Corollary 4.2.6, if* ϕ *is a bounded measurable function on E, we have*

$$\lim_{n\longrightarrow\infty}\int_E f_n(x)\phi(x)\Big(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx = \int_E F(x)\phi(x)\Big(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx.$$

Chapter 5

Elements of the Theory of Iso-Hilbert Spaces

In this chapter we make a lift of the Hilbert spaces to the iso-Hilbert iso-spaces. They are given the main definitions and the main conceptions for such iso-spaces and they are made comparisons with the real or complex Hilbert spaces.

5.1. Definition of Iso-inner product and properties

Let H be a real or complex Hilbert space with an inner product $(\cdot,\cdot)$.

We suppose

(H1) $\hat{T} \in \mathcal{L}(H)$, $\hat{T}^{-1}$ exists and $\hat{T}^{-1} \in \mathcal{L}(H)$ and $\hat{T}^{-1}$ is positive, i.e. $\hat{T}^{-1} : H \longrightarrow H$ is a self-adjoint operator and $(\hat{T}^{-1}x, x) > 0$ for every $x \in H$.

We lift the Hilbert space H into the set

$$\hat{H} := \{\hat{T}^{-1}(x) := \hat{x} \quad \text{for} \quad x \in H\}$$

and we define an iso-inner product as follows

$$\widehat{(\hat{x},\hat{y})} := (\hat{T}^{-1}x, \hat{T}^{-1}y)\frac{1}{\hat{T}_1} \quad \text{for} \quad \hat{x},\hat{y} \in \hat{H}.$$

Since $\hat{T}^{-1} \in \mathcal{L}(H)$ then $\hat{H}$ is a linear space.

Remark 5.1.1. *We note that the initial space H should be some Hilbert space because we can define a positive definite operator only on some Hilbert space not on a linear space with an inner product. One of the main assumption for the isotopic element $\hat{T}$ is it to be a positive definite operator on H.*

Proposition 5.1.2. *We suppose $(H1)$. Then $\widehat{(\hat{\cdot},\hat{\cdot})}$ is an inner product.*

Proof. 1. Let $\hat{x} \in \hat{H}$ be arbitrarily chosen. We have

$$\widehat{(\hat{x},\hat{x})} = (\hat{T}^{-1}x, \hat{T}^{-1}x)\frac{1}{\hat{T}_1} \geq 0$$

because $\hat{T}_1 > 0$, $\hat{T}^{-1} : H \longrightarrow H$, $(\cdot,\cdot)$ is an inner product in the Hilbert space H. Also,

$$\widehat{(\hat{x},\hat{x})} = 0 \quad \Longleftrightarrow \quad (\hat{T}^{-1}x, \hat{T}^{-1}x)\tfrac{1}{\hat{T}_1} \quad \Longleftrightarrow$$

$$(\hat{T}^{-1}x, \hat{T}^{-1}x) = 0 \quad \Longleftrightarrow \hat{T}^{-1}x = 0 \quad \Longleftrightarrow x = 0$$

because $\hat{T}^{-1} \in \mathcal{L}(H)$.

2. Let $\hat{\lambda} \in \hat{F}_{\mathbb{C}}$. Then, for $\hat{x}, \hat{y} \in \hat{H}$, we have

$$\widehat{(\hat{\lambda}\hat{\times}\hat{x},\hat{y})} = \left(\lambda\tfrac{1}{\hat{T}_1}T_1\hat{T}^{-1}x, \hat{T}^{-1}y\right)\tfrac{1}{\hat{T}_1}$$

$$= \lambda\tfrac{1}{\hat{T}_1}\hat{T}_1(\hat{T}^{-1}x, \hat{T}^{1}y)\tfrac{1}{\hat{T}_1}$$

$$= \hat{\lambda}\hat{\times}\widehat{(\hat{x},\hat{y})},$$

and

$$\widehat{(\hat{x},\hat{\lambda}\hat{\times}\hat{y})} = \left(\hat{T}^{-1}x, \lambda\tfrac{1}{\hat{T}_1}\hat{T}_1\hat{T}^{-1}y\right)\tfrac{1}{\hat{T}_1}$$

$$= \overline{\lambda\tfrac{1}{\hat{T}_1}\hat{T}_1}(\hat{T}^{-1}x, \hat{T}^{-1}y)\tfrac{1}{\hat{T}_1}$$

$$= \overline{\lambda}(\hat{T}^{-1}x, \hat{T}^{-1}y)\tfrac{1}{\hat{T}_1}$$

$$= \overline{\lambda}\tfrac{1}{\hat{T}_1}T_1(\hat{T}^{-1}x, \hat{T}^{-1}y)\tfrac{1}{\hat{T}_1}$$

$$= \hat{\overline{\lambda}}\hat{\times}\widehat{(\hat{x},\hat{y})}.$$

3. For $\hat{x}, \hat{y}, \hat{z} \in \hat{H}$ we have

$$\widehat{(\hat{x}+\hat{y},\hat{z})} = (\hat{T}^{-1}x + \hat{T}^{-1}y, \hat{T}^{-1}z)\tfrac{1}{\hat{T}_1}$$

$$= [(\hat{T}^{-1}x, \hat{T}^{-1}z) + (\hat{T}^{-1}y, \hat{T}^{-1}z)]\tfrac{1}{\hat{T}_1}$$

$$= (\hat{T}^{-1}x, \hat{T}^{-1}z)\tfrac{1}{\hat{T}_1} + (\hat{T}^{-1}y, \hat{T}^{-1}z)\tfrac{1}{\hat{T}_1}$$

$$= \widehat{(\hat{x},\hat{z})} + \widehat{(\hat{y},\hat{z})}.$$

□

Remark 5.1.3. *We will note that not for any $\hat{T}$ we can make a lift of a space with an inner product in an iso-space with an iso - inner product.*

Really, let us consider $\mathcal{C}([1,2])$ in which is defined an inner product as follows

$$(f,g)=\int_1^2 f(x)g(x)dx \quad \text{for} \quad f,g\in\mathcal{C}([1,2]).$$

If we make the lift, then the corresponding iso-inner product is

$$\widehat{(\hat{f}^{\wedge\wedge},\hat{g}^{\wedge\wedge})}=\hat{\int}_1^{\hat{2}}\hat{f}^{\wedge}(\hat{x})\hat{\times}\hat{g}^{\wedge}(\hat{x})\hat{\times}\hat{d}\hat{x}$$

$$=\int_1^2 f(x)g(x)\Big(1-x\tfrac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx$$

and if $f=g$ we have

$$\widehat{(\hat{f}^{\wedge\wedge},\hat{f}^{\wedge\wedge})}=\int_1^2 f^2(x)\Big(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big)dx.$$

If $\hat{T}(x)=x+1$, then $\hat{T}(x)>0$ for every $x\in[1,2]$ and we have

$$\widehat{(\hat{f}^{\wedge\wedge},\hat{f}^{\wedge\wedge})}=\int_1^2 f^2(x)\Big(1-x\frac{1}{x+1}\Big)dx=\int_1^2 f^2(x)\frac{1}{x+1}dx\geq 0.$$

Also, if $\hat{T}(x)=e^x$, $x\in[1,2]$, then $\hat{T}>0$ for every $x\in[1,2]$, on the other hand,

$$\widehat{(\hat{f}^{\wedge\wedge},\hat{f}^{\wedge\wedge})}=\int_1^2 f^2(x)(1-x)dx\leq 0$$

because $1-x\leq 0$ for every $x\in[1,2]$.

Example 5.1.4. **1.** *Let us consider $\mathbb{R}^n$ and let $\hat{T}=(\hat{T}_1,\hat{T}_2,\dots,\hat{T}_n)$, where $\hat{T}_l$, $l=1,2,\dots,n$ are positive real numbers. Then we lift $\mathbb{R}^n$ into the space $\hat{R}^n$ in the following manner: for given $x=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$ we set*

$$\hat{x}=\Big(\frac{x_1}{\hat{T}_1},\frac{x_2}{\hat{T}_2},\dots,\frac{x_n}{\hat{T}_n}\Big),$$

which is the corresponding iso-lift of the element x, and for $\hat{x}$, $\hat{y}\in\hat{\mathbb{R}}^n$ we define an iso-inner iso-product as follows

$$\widehat{(\hat{x},\hat{y})}=\sum_{l=1}^n\hat{x}_l\hat{\times}\hat{y}_l=\sum_{l=1}^n x_l\tfrac{1}{\hat{T}_l}\hat{T}_l y_l\tfrac{1}{\hat{T}_l}=\sum_{l=1}^n x_l y_l\tfrac{1}{\hat{T}_l}.$$

2. *The space l_2 consists of all real sequences $\xi=\{\xi_l\}_{l=1}^{\infty}$ so that $\sum_{l=1}^{\infty}\xi_l^2<\infty$. Let also, $\hat{T}=\{\hat{T}_l\}_{l=1}^{\infty}$ to be a sequence of positive real numbers. We want to make a lift of l_2 into $\hat{l}_2$ as follows*

$$\xi\longrightarrow\hat{\xi}=\{\hat{\xi}_l\}_{l=1}^{\infty}=\Big\{\frac{\xi_l}{\hat{T}_l}\Big\}_{l=1}^{\infty}$$

therefore we have a need of an additional condition for $\hat{T}$, namely the sequence $\{\hat{T}_l\}_{l=1}^{\infty}$ to be bounded below by a positive real number a. In this way we have

$$\frac{1}{\hat{T}_l} \leq \frac{1}{a} \quad \text{for} \quad \forall l = 1, 2, \ldots$$

and from $\xi \in l_2$ it follows

$$\sum_{l=1}^{\infty} \frac{\xi_l^2}{\hat{T}_l^2} \leq \frac{1}{a^2} \sum_{l=1}^{\infty} \xi_l^2 < \infty.$$

In this case we define an iso-inner iso-product in $\hat{l}_2$ in the following manner: for $\hat{\xi}$, $\hat{\eta} \in \hat{l}_2$

$$\widehat{(\hat{\xi}, \hat{\eta})} = \sum_{l=1}^{\infty} \hat{\xi}_l \hat{\times} \hat{\eta}_l = \sum_{l=1}^{\infty} \xi_l \frac{1}{\hat{T}_l} \hat{T}_l \eta_l \frac{1}{\hat{T}_l} = \sum_{l=1}^{\infty} \xi_l \eta_l \frac{1}{\hat{T}_l}.$$

let now, $\xi = \{\frac{1}{l}\}_{l=1}^{\infty}$ and $\hat{T} = \{\frac{1}{\sqrt{l}}\}_{l=1}^{\infty}$. Then

$$\sum_{l=1}^{\infty} \xi_l^2 = \sum_{l=1}^{\infty} \frac{1}{l^2} < \infty,$$

the sequence $\hat{T}$ is not bounded below by a positive real number and

$$\sum_{l=1}^{\infty} \frac{\xi_l^2}{\hat{T}_l^2} = \sum_{l=1}^{\infty} \frac{1}{l^2} l = \sum_{l=1}^{\infty} \frac{1}{l} = \infty.$$

We obtain that if the positive sequence $\hat{T}$ is bounded below by a positive real number and $\xi \in l^2$ then $\hat{\xi} \in \hat{l}_2$.

*Let $\hat{T} = \{\hat{T}_l\}_{l=1}^{\infty} = \{l\}_{l=1}^{\infty}$. Then $\hat{T}$ is a bounded below sequence by*1*. Let also, $\xi = \{\xi_l\}_{l=1}^{\infty} = \left\{\frac{1}{\sqrt{l}}\right\}_{l=1}^{\infty}$. Then*

$$\sum_{l=1}^{\infty} \xi_l^2 = \sum_{l=1}^{\infty} \frac{1}{l} = \infty,$$

i.e. $\xi \notin l_2$. Also, for

$$\hat{\xi} = \left\{\frac{\xi_l}{\hat{T}_l}\right\}_{l=1}^{\infty} = \left\{\frac{1}{l^{\frac{3}{2}}}\right\}_{l=1}^{\infty}$$

we have

$$\sum_{l=1}^{\infty} \hat{\xi}_l^2 = \sum_{l=1}^{\infty} \frac{1}{l^3} < \infty.$$

This example shows that we have $\hat{\xi} \in \hat{l}_2$ and $\xi \notin l_2$.

Now we will give a condition for the positive sequence $\hat{T}$ such that from $\hat{\xi} \in \hat{l}_2$ it follows $\xi \in l^2$. We suppose that the positive sequence $\hat{T} = \{\hat{T}_l\}_{l=1}^{\infty}$ is bounded above by a positive real number b. For $\hat{\xi} = \left\{\frac{\xi_l}{\hat{T}_l}\right\}_{l=1}^{\infty} \in \hat{l}_2$ we have

$$\frac{1}{b^2} \sum_{l=1}^{\infty} \xi_l^2 = \sum_{l=1}^{\infty} \frac{\xi_l^2}{b^2} \leq \sum_{l=1}^{\infty} \frac{\xi_l^2}{\hat{T}_l^2} < \infty$$

and since $b > 0$ we conclude that

$$\sum_{l=1}^{\infty} \xi_l^2 < \infty,$$

i.e. $\xi = \{\xi_l\}_{l=1}^{\infty} \in l_2$.

If the positive sequence $\hat{T}$ is bounded above by a positive real number b then from $\xi \in l^2$ it does not follow that $\hat{\xi} \in \hat{l}_2$. Really, let $\xi = \left\{\frac{1}{l}\right\}_{l=1}^{\infty}$. Then

$$\sum_{l=1}^{\infty} \xi_l^2 = \sum_{l=1}^{\infty} \frac{1}{l^2} < \infty,$$

i.e. $\xi \in l_2$. *Let now $\hat{T} = \left\{\frac{1}{\sqrt{l}}\right\}_{l=1}^{\infty}$. Then $\hat{T}$ is a bounded above sequence and*

$$\sum_{l=1}^{\infty} \frac{\xi_l^2}{\hat{T}_l^2} = \sum_{l=1}^{\infty} \frac{\frac{1}{l^2}}{\frac{1}{l}} = \sum_{l=1}^{\infty} \frac{1}{l} = \infty,$$

consequently $\hat{\xi} = \left\{\frac{\xi_l}{\hat{T}_l}\right\}_{l=1}^{\infty} \notin \hat{l}_2$.

The sequence $\hat{T}$ to be bounded above is too important. Indeed, let $\hat{T} = \{l^2\}_{l=1}^{\infty}$ and $\xi = \{\sqrt{l}\}_{l=1}^{\infty}$. Then $\hat{T}$ is unbounded above and

$$\sum_{l=1}^{\infty} \frac{\xi_l^2}{\hat{T}_l^2} = \sum_{l=1}^{\infty} \frac{l}{l^4} = \sum_{l=1}^{\infty} \frac{1}{l^3} < \infty,$$

in other words $\hat{\xi} \in \hat{l}_2$, *and*

$$\sum_{l=1}^{\infty} \xi_l^2 = \sum_{l=1}^{\infty} l = \infty,$$

consequently $\xi \notin l_2$.

If we suppose that the positive sequence $\hat{T}$ is bounded below and above by some positive real numbers a and b, respectively, then from $\xi \in l^2$ it follows that $\hat{\xi} \in \hat{l}_2$ and from $\hat{\xi} \in \hat{l}_2$ it follows that $\xi \in l_2$, because

$$\frac{1}{b^2} \sum_{l=1}^{\infty} \xi_l^2 \leq \sum_{l=1}^{\infty} \frac{\xi_l^2}{\hat{T}_l^2} \leq \frac{1}{a^2} \sum_{l=1}^{\infty} \xi_l^2.$$

Remark 5.1.5. *In the above example we saw that if $\hat{T} = \{\hat{T}_l\}_{l=1}^{\infty}$ is a bounded below positive sequence by a positive real number then there exists $\hat{\xi} \in \hat{l}_2$ so that $\xi \notin l_2$. Now we will see that if $\hat{T} = \{\hat{T}_l\}_{l=1}^{\infty}$ is a positive bounded below sequence by a positive real number then there exists $\hat{\xi} \in \hat{l}_2$ such that $\xi \in l_2$. Let*

$$\hat{T} = \{\hat{T}_l\}_{l=1}^{\infty} = \{l\}_{l=1}^{\infty}$$

which is bounded below by 1, *and let*

$$\xi = \{\xi_l\}_{l=1}^{\infty} = \left\{\frac{1}{l^2}\right\}_{l=1}^{\infty}.$$

Then

$$\sum_{l=1}^{\infty}\frac{\xi_l^2}{\hat{T}_l^2}=\sum_{l=1}^{\infty}\frac{\frac{1}{l^4}}{l^2}=\sum_{l=1}^{\infty}\frac{1}{l^6}<\infty,\quad \text{i.e.}\quad \hat{\xi}\in\hat{l}_2,$$

and

$$\sum_{l=1}^{\infty}\xi_l^2=\sum_{l=1}^{\infty}\frac{1}{l^4}<\infty,\quad \text{i.e.}\quad \xi\in l_2.$$

Remark 5.1.6. *In the above example we saw that if $\hat{T}=\{\hat{T}_l\}_{l=1}^{\infty}$ is a positive bounded above sequence then there exists $\xi\in l_2$ so that $\hat{\xi}\notin\hat{l}_2$. Now we will see that if $\hat{T}=\{\hat{T}_l\}_{l=1}^{\infty}$ is a positive bounded above sequence then there exists $\xi\in l_2$ so that $\hat{\xi}\in\hat{l}_2$. Let*

$$\hat{T}=\{\hat{T}_l\}_{l=1}^{\infty}=\left\{\frac{1}{\sqrt{l}}\right\}_{l=1}^{\infty}$$

which is bounded above by 1*, and let*

$$\xi=\{\xi_l\}_{l=1}^{\infty}=\left\{\frac{1}{l^4}\right\}_{l=1}^{\infty}.$$

Then

$$\sum_{lk=1}^{\infty}\xi_l^2=\sum_{l=1}^{\infty}\frac{1}{l^8}<\infty,\quad \text{i.e.}\quad \xi\in l_2,$$

$$\sum_{l=1}^{\infty}\frac{\xi_l^2}{\hat{T}_l^2}=\sum_{l=1}^{\infty}\frac{\frac{1}{l^8}}{\frac{1}{l}}=\sum_{l=1}^{\infty}\frac{1}{l^7}<\infty,\quad \text{i.e.}\quad \hat{\xi}\in\hat{l}_2.$$

From the above examples and remarks it follows that there exists an iso-space which is a generalization of the iso-space $\hat{l}_2$. Now we will construct it.

Definition 5.1.7. *For given sequence $\hat{T}=\{\hat{T}_l\}_{l=1}^{\infty}$ of positive real numbers we define the iso-space*

$$\hat{l}l_{\hat{T}}=\left\{\frac{\xi_l}{\hat{T}_l}:\xi_l\in\mathbb{R}^+,\sum_{l=1}^{\infty}\frac{\xi_l^2}{\hat{T}_l^2}<\infty\right\}.$$

With $\hat{\mathcal{T}}$ we will denote the set of all sequences of positive real numbers.

Definition 5.1.8. *The set*

$$\hat{l}l_2=\cup_{\hat{T}\in\hat{\mathcal{T}}}\hat{l}l_{\hat{T}}$$

is called an iso-generalization of the iso-space $\hat{l}_2$.

From the above investigations we have the following inclusion.

Proposition 5.1.9. $\hat{l}_2\subset\hat{l}l_2$ *and* $\hat{l}_2\neq\hat{l}l_2$.

Proposition 5.1.10. *(iso-Cauchy inequality) We suppose $(H1)$. For $\hat{x}$, $\hat{y}\in\hat{H}$ we have*

$$\widehat{(\hat{x},\hat{y})}\,\hat{\times}\,\overline{\widehat{(\hat{x},\hat{y})}}\leq\widehat{(\hat{x},\hat{x})}\,\hat{\times}\,\widehat{(\hat{y},\hat{y})}.$$

Proof. Let $\hat{\lambda} \in \hat{F}_{\mathbb{C}}$. Then

$$A := (\widehat{\hat{x}+\hat{\lambda}\hat{\times}\hat{y}, \hat{x}+\hat{\lambda}\hat{\times}\hat{y}})$$

$$= (\widehat{\hat{x}, \hat{x}+\hat{\lambda}\hat{\times}\hat{y}}) + (\widehat{\hat{\lambda}\hat{\times}\hat{y}, \hat{x}+\hat{\lambda}\hat{\times}\hat{y}})$$

$$= \widehat{(\hat{x},\hat{x})} + \hat{\bar{\hat{\lambda}}}\hat{\times}\widehat{(\hat{x},\hat{y})} + \hat{\lambda}\hat{\times}\widehat{(\hat{y},\hat{x})} + \hat{\lambda}\hat{\times}\hat{\bar{\hat{\lambda}}}\hat{\times}\widehat{(\hat{y},\hat{y})} \geq 0.$$

Let

$$\hat{\lambda} = -\widehat{(\hat{x},\hat{y})} \nearrow \widehat{(\hat{y},\hat{y})}.$$

Then

$$A = \overline{\widehat{(\hat{x},\hat{x})}} - \overline{\widehat{(\hat{x},\hat{y})}}\hat{\times}\widehat{(\hat{x},\hat{y})} \nearrow \widehat{(\hat{y},\hat{y})} - \widehat{(\hat{x},\hat{y})}\hat{\times}\overline{\widehat{(\hat{x},\hat{y})}} \nearrow \widehat{(\hat{y},\hat{y})}$$

$$+\widehat{(\hat{x},\hat{y})}\hat{\times}\overline{\widehat{(\hat{x},\hat{y})}}\hat{\times}\widehat{(\hat{y},\hat{y})} \nearrow (\widehat{(\hat{y},\hat{y})}\hat{\times}\widehat{(\hat{y},\hat{y})})$$

and since $A \geq 0$ we get

$$\widehat{(\hat{x},\hat{y})} - \overline{\widehat{(\hat{x},\hat{y})}}\hat{\times}\widehat{(\hat{x},\hat{y})} \nearrow \widehat{(\hat{y},\hat{y})} \geq 0.$$

$\square$

Definition 5.1.11. *Two elements $\hat{x}$, $\hat{y} \in \hat{H}$ will be called iso-orthogonal if*

$$\widehat{(\hat{x},\hat{y})} = 0.$$

Remark 5.1.12. *We will note that if two elements of the Hilbert space H are orthogonal with respect to the inner product $(\cdot,\cdot)$ they are not iso-orthogonal iso-elements of the iso-space $\hat{H}$ with respect to the iso-inner iso-product $\widehat{(\hat{\cdot},\hat{\cdot})}$ and the conversely. We will consider an example for this.*

Let $H = \mathcal{C}([-1,1])$ with an inner product

$$(f,g) = \int_{-1}^{1} f(x)g(x)dx.$$

Then x, x^2, $x^3 \in H$ and

$$(x,x^2) = \int_{-1}^{1} x^3 dx = \frac{x^4}{4}\Big|_{x=-1}^{x=1} = \frac{1}{4} - \frac{1}{4} = 0,$$

$$(x,x^3) = \int_{-1}^{1} x^4 dx = \frac{x^5}{5}\Big|_{x=-1}^{x=1} = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}.$$

Consequently x and x^2 are orthogonal elements of $\mathcal{C}([-1,1])$ and x and x^3 are not orthogonal elements of $\mathcal{C}([-1,1])$.

Let now

$$\hat{T}(x) = e^{x+\frac{7}{10}x^2}, \quad x \in [-1,1].$$

Then

$$\hat{T}(x) \geq 0 \quad \text{for} \quad \forall x \in [-1,1]$$

and

$$\hat{T}'(x) = \left(1 + \tfrac{7}{5}x\right)e^{x+\frac{7}{10}x^2},$$

$$\frac{\hat{T}'(x)}{\hat{T}(x)} = \frac{\left(1+\frac{7}{5}x\right)e^{x+\frac{7}{5}x^2}}{e^{x+\frac{7}{5}x^2}} = 1 + \tfrac{7}{5}x,$$

$$1 - x\frac{\hat{T}'(x)}{\hat{T}(x)} = 1 - x\left(1 + \tfrac{7}{5}x\right) = 1 - x - \tfrac{7}{5}x^2.$$

Now we consider $\hat{\mathcal{C}}([-1,1])$ with the isotopic element $\hat{T}(x)$, then the corresponding iso-inner iso- product is

$$\widehat{(\hat{f},\hat{g})} = \int_{-1}^{1} f(x)g(x)\left(1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\right)dx.$$

From here,

$$\widehat{(\hat{x},\hat{x}^2)} = \int_{-1}^{1} x^3\left(1 - x - \tfrac{7}{5}x^2\right)dx$$

$$= \int_{-1}^{1} x^3dx - \int_{-1}^{1} x^4dx - \tfrac{7}{5}\int_{-1}^{1} x^5dx$$

$$= \frac{x^4}{4}\Big|_{x=-1}^{x=1} - \frac{x^5}{5}\Big|_{x=-1}^{x=1} - \frac{7}{5}\frac{x^6}{6}\Big|_{x=-1}^{x=1}$$

$$= -\tfrac{2}{5} \neq 0,$$

and

$$\widehat{(\hat{x},\hat{x}^3)} = \int_{-1}^{1} x^4\left(1 - x - \tfrac{7}{5}x^2\right)dx$$

$$= \int_{-1}^{1} x^4dx - \int_{-1}^{1} x^5dx - \tfrac{7}{5}\int_{-1}^{1} x^6dx$$

$$= \frac{x^5}{5}\Big|_{x=-1}^{x=1} - \frac{x^6}{6}\Big|_{x=-1}^{x=1} - \frac{7}{5}\frac{x^7}{7}\Big|_{x=-1}^{x=1}$$

$$= \tfrac{2}{5} - \tfrac{7}{5}\tfrac{2}{7} = 0.$$

Consequently $\hat{x}$ and $\hat{x}^2$ are not iso-orthogonal in $\hat{\mathcal{C}}([-1,1])$ and $\hat{x}$ and $\hat{x}^3$ are iso-orthogonal in $\hat{\mathcal{C}}([-1,1])$.

Below we will give a condition for the isotopic element $\hat{T}$ such that if $x,y \in H$ are orthogonal then $\hat{x}$, $\hat{y} \in \hat{H}$ are iso-orthogonal and the conversely.

Proposition 5.1.13. *We suppose $(H1)$ and $\hat{T} = \hat{T}^{-1*}$. If $x,y \in H$ are orthogonal then $\hat{x}$, $\hat{y} \in \hat{H}$ are iso-orthogonal and the conversely.*

Here with $\hat{T}^{-1*}$ we denote the adjoint operator of the operator $\hat{T}^{-1}$. Since $\hat{T}^{-1}$ is positive definite then $\hat{T}^{-1} = \hat{T}^{-1*}$ and from here $\hat{T} = \hat{T}^{-1*}$ is equivalent to $\hat{T}^2 = I$.

Proof. Below with I we will denote the identity operator in $\mathcal{L}(H)$.

1. Let $x, y \in H$ are orthogonal. Then

$(A13)(x,y) = 0.$

On the other hand, we have

$$= \widehat{(\hat{x},\hat{y})} = (\hat{T}^{-1}x, \hat{T}^{-1}y)\frac{1}{\hat{T}_1}$$

now we use that $\hat{T}^{-1*} = \hat{T}$

$$(\hat{T}^{-1*}\hat{T}^{-1}x, y)\frac{1}{\hat{T}_1} = (\hat{T}\hat{T}^{-1}x, y)\frac{1}{\hat{T}_1} = (x,y)\frac{1}{\hat{T}_1},$$

and using (A13) we conclude that

$$\widehat{(\hat{x},\hat{y})} = 0.$$

Consequently $\hat{x}$ and $\hat{y}$ are iso-orthogonal.

2. Let now $\hat{x}$, $\hat{y} \in \hat{H}$ are iso-orthogonal. Then

$(A14)\widehat{(\hat{x},\hat{y})} = 0.$

On the other hand, using (A14), we have

$$(x,y)\tfrac{1}{\hat{T}_1} = (Ix, y)\tfrac{1}{\hat{T}_1} = (\hat{T}\hat{T}^{-1}x, y)\tfrac{1}{\hat{T}_1}$$

$$= (\hat{T}^{-1*}\hat{T}^{-1}x, y)\tfrac{1}{\hat{T}_1} = (\hat{T}^{-1}x, \hat{T}^{-1}y)\tfrac{1}{\hat{T}_1} = 0.$$

Therefore x and y are orthogonal in H.

□

Proposition 5.1.14. *We suppose $(H1)$ and $\hat{T}^{-1*} = \hat{T}$. Then $\hat{x}_1$, $\hat{x}_2$, …, $\hat{x}_n \in \hat{H}$ is an iso-orthogonal system then it is an iso-linear independent system.*

Proof. Since $\hat{T}^{-1*} = \hat{T}$ from the previous Proposition it follows that the system x_1, x_2, …, $x_n \in H$ is an orthogonal system. From the properties of the linear spaces with an inner product we conclude that x_1, x_2, …, x_n is a linear independent system in H. From here and since $\hat{T}$ is a linear operator, we have

$$\hat{\lambda}_1 \hat{\times} \hat{x}_1 + \hat{\lambda}_2 \hat{\times} \hat{x}_2 + \cdots + \hat{\lambda}_n \hat{\times} \hat{x}_n = 0 \iff$$

$$\lambda_1 \hat{T}^{-1}x_1 + \lambda_2 \hat{T}^{-1}x_2 + \cdots + \lambda_n \hat{T}^{-1}x_n = 0 \iff$$

$$\hat{T}(\lambda_1 \hat{T}^{-1}x_1 + \lambda_2 \hat{T}^{-1}x_2 + \cdots + \lambda_n \hat{T}^{-1}x_n) = \hat{T}0 \iff$$

$$\hat{T}(\lambda_1 \hat{T}^{-1}x_1) + \hat{T}(\lambda_2 \hat{T}^{-1}x_2) + \cdots + \hat{T}(\lambda_n \hat{T}^{-1}x_n) = 0 \iff$$

$$\lambda_1 \hat{T}\hat{T}^{-1}x_1 + \lambda_2 \hat{T}\hat{T}^{-1}x_2 + \cdots + \lambda_n \hat{T}\hat{T}^{-1}x_n = 0 \iff$$

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = 0,$$

from where we conclude that the system $\hat{x}_1$, $\hat{x}_2$, ..., $\hat{x}_n$ is an iso-linear independent system.
$\square$

Proposition 5.1.15. *We suppose* $(H1)$*. Let* $\hat{x}$*,* $\hat{y} \in \hat{H}$ *are iso-orthogonal. Then*

$$(\hat{x}+\widehat{\hat{y},\hat{x}}+\hat{y}) = \widehat{(\hat{x},\hat{x})} + \widehat{(\hat{y},\hat{y})}.$$

Proof. Since $\hat{x}$ and $\hat{y}$ are iso-orthogonal then

$$\widehat{(\hat{x},\hat{y})} = \widehat{(\hat{y},\hat{x})} = 0.$$

From here

$$(\hat{x}+\widehat{\hat{y},\hat{x}}+\hat{y}) = (\widehat{\hat{x},\hat{x}+\hat{y}}) + (\widehat{\hat{y},\hat{x}+\hat{y}})$$

$$\widehat{(\hat{x},\hat{x})} + \widehat{(\hat{x},\hat{y})} + \widehat{(\hat{y},\hat{x})} + \widehat{(\hat{y},\hat{y})}$$

$$= \widehat{(\hat{x},\hat{x})} + \widehat{(\hat{y},\hat{y})}.$$

$\square$

Proposition 5.1.16. *We suppose* $(H1)$*. If* $\hat{x}$*,* $\hat{y} \in \hat{H}$ *then*

$$(\hat{x}+\widehat{\hat{y},\hat{x}}+\hat{y}) + (\hat{x}-\widehat{\hat{y},\hat{x}}-\hat{y}) = \hat{2}\hat{\times}\widehat{(\hat{x},\hat{x})} + \hat{2}\hat{\times}\widehat{(\hat{y},\hat{y})}.$$

Proof.

$$(\hat{x}+\widehat{\hat{y},\hat{x}}+\hat{y}) + (\hat{x}-\widehat{\hat{y},\hat{x}}-\hat{y})$$

$$= (\widehat{\hat{x},\hat{x}+\hat{y}}) + (\widehat{\hat{y},\hat{x}+\hat{y}}) + (\widehat{\hat{x},\hat{x}-\hat{y}}) - (\widehat{\hat{y},\hat{x}-\hat{y}})$$

$$= \widehat{(\hat{x},\hat{x})} + \widehat{(\hat{x},\hat{y})} + \widehat{(\hat{y},\hat{x})} + \widehat{(\hat{y},\hat{y})} + \widehat{(\hat{x},\hat{x})} - \widehat{(\hat{x},\hat{y})} - \widehat{(\hat{y},\hat{x})} + \widehat{(\hat{y},\hat{y})}$$

$$= 2\widehat{(\hat{x},\hat{x})} + 2\widehat{(\hat{y},\hat{y})}$$

$$= \hat{2}\hat{\times}\widehat{(\hat{x},\hat{x})} + \hat{2}\hat{\times}\widehat{(\hat{y},\hat{y})}.$$

$\square$

Definition 5.1.17. *We suppose* $(H1)$*. We will say that the iso-sequence* $\{\hat{x}_n\}_{n=1}^{\infty}$ *of iso-elements of* $\hat{H}$ *is convergent to* $\hat{x} \in \hat{H}$ *if*

$$\lim_{n\longrightarrow\infty} \widehat{(\hat{x}_n,\hat{x}_n)} = \lim_{n\longrightarrow\infty} (\hat{T}^{-1}x_n, \hat{T}^{-1}x_n)\frac{1}{\hat{T}_1}$$

$$= (\hat{T}^{-1}x, \hat{T}^{-1}x)\frac{1}{\hat{T}_1}$$

$$= \widehat{(\hat{x},\hat{x})}.$$

Remark 5.1.18. *We suppose* $(H1)$. *If* $x_n \in H$ *and* $\lim_{n\longrightarrow\infty} x_n = x$ *in* H, *since* $\hat{T}^{-1} \in \mathcal{L}(H)$, *we have that*

$$\lim_{n\longrightarrow\infty} \hat{T}^{-1}x_n = \hat{T}^{-1}x \quad \text{in} \quad H$$

or

$$\lim_{n\longrightarrow\infty} (\hat{T}^{-1}x_n, \hat{T}^{-1}x_n) = (\hat{T}^{-1}x, \hat{T}^{-1}x)$$

and from the definition above we obtain that

$$\lim_{n\longrightarrow\infty} \widehat{(\hat{x}_n, \hat{x}_n)} = \widehat{(\hat{x}, \hat{x})}.$$

If $\hat{x}_n \longrightarrow_{n\longrightarrow\infty} \hat{x}$, *then from the definition above we have*

$$\hat{T}^{-1}x_n \longrightarrow_{n\longrightarrow\infty} \hat{T}^{-1}x \quad \text{in} \quad H$$

and using that $\hat{T}^{-1} \in \mathcal{L}(H)$ *we conclude that* $x_n \longrightarrow_{n\longrightarrow\infty} x$ *in* H.

Consequently, under the assumption $(H1)$, *the convergence in* H *and* $\hat{H}$ *are equivalent.*

Proposition 5.1.19. *We suppose* $(H1)$. *If* $\hat{x}_n, y \in \hat{H}$, $\lim_{n\longrightarrow\infty} \hat{x}_n = \hat{x} \in \hat{H}$, *then*

$$\lim_{n\longrightarrow\infty} \widehat{(\hat{x}_n, \hat{y})} = \widehat{(\hat{x}, \hat{y})}.$$

Proof. Since the convergence in H and $\hat{H}$ are equivalent, then $\lim_{n\longrightarrow\infty} x_n = x$. From $\hat{T}^{-1} \in \mathcal{L}(H)$ we conclude that

$$\lim_{n\longrightarrow\infty} \hat{T}^{-1}x_n = \hat{T}^{-1}x.$$

From here and since the inner product in H is continuous, we get

$$\lim_{n\longrightarrow\infty} (\hat{T}^{-1}x_n, \hat{T}^{-1}y) = (\hat{T}^{-1}x, \hat{T}^{-1}y)$$

from where we conclude

$$\lim_{n\longrightarrow\infty} \widehat{(\hat{x}_n, y)} = \widehat{(\hat{x}, \hat{y})}.$$

□

Definition 5.1.20. *We suppose* $(H1)$. *Let* $\hat{x}_n \in \hat{H}$ *is a convergent sequence in* $\hat{H}$ *to the element* $\hat{x} \in \hat{H}$. *Then, since the convergence in* H *and in* $\hat{H}$ *are equivalent, we have that* x_n *is a convergent sequence in* H *and* $x \in H$, *because* H *is a Hilbert space. Because* $\hat{T}^{-1} \in \mathcal{L}(H)$ *we have that*

$$\hat{T}^{-1}x_n \longrightarrow_{n\longrightarrow\infty} \hat{T}^{-1}x$$

and $\hat{T}^{-1}x \in \hat{H}$. *Therefore* $\hat{H}$ *is a complete space which will be called an iso-Hilbert iso-space.*

Definition 5.1.21. *We suppose* $(H1)$. *Then if* $M \subset H$ *we will write* $\hat{M} \subset \hat{H}$.

Since $\hat{T}^{-1} \in \mathcal{L}(H)$, *then, if* M *is a closed subset of* H, *we have that* $\hat{M}$ *is a closed subset of* $\hat{H}$, *and if* M *is a convex subset of* H, *then* $\hat{M}$ *is a convex subset of* $\hat{H}$.

If $\hat{M}$ *is a subspace of the iso-Hilbert iso-space, then every* $\hat{x} \in \hat{H}$ *can be represented in the form*

$$\hat{x} = \hat{y} + \hat{z},$$

where $\hat{y} \in \hat{M}$ *and* $\widehat{(\hat{z}, \hat{p})} = 0$ *for every* $\hat{p} \in \hat{M}$.

If $\hat{M} \subset \hat{H}$ *is a linear manifold, then the iso-set of all iso-elements* $\hat{z}$ *of* $\hat{H}$ *such that* $\widehat{(\hat{z}, \hat{p})} = 0$ *for every* $\hat{p} \in \hat{M}$ *will be called the iso-orthogonal supplement of* $\hat{M}$ *and will be denoted by* $\hat{M}^{\perp}$.

5.2. Iso-operators in iso-Hilbert spaces

Definition 5.2.1. *We suppose* $(H1)$.

Let $A : H \longrightarrow H$ *is a linear operator. The corresponding lift of* A *will be defined as follows*

$$\hat{A} = A\hat{T}^{-1}$$

and will be called an iso-operator.

The act of the iso-operator $\hat{A}$ *on the iso-element* $\hat{x} \in \hat{H}$ *will be defined as follows*

$$\hat{A}\hat{x} := \hat{A}\hat{T}^{-1}\hat{T}\hat{T}^{-1}x = \hat{A}\hat{T}^{-1}x.$$

Because A *and* $\hat{T}^{-1}$ *are linear operators in* $\mathcal{L}(H)$ *and since the composition of two linear operators is a linear operator we have that* $\hat{A}$ *is a linear operator.*

The iso-norm of the iso-operator $\hat{A}$ *will be defined as follows*

$$\widehat{||\hat{A}||} := ||A\hat{T}^{-1}||\frac{1}{\hat{T}_1} = \frac{1}{\hat{T}_1}\sup_{x\in H:||x||\leq 1}||A\hat{T}^{-1}x||.$$

The iso-operator $\hat{A}$ *will be called iso-bounded if there exists* $\hat{c} \in \hat{F}_{\mathbb{R}}$, $\hat{c} \geq 0$, *such that*

$$\widehat{||\hat{A}||} \leq \hat{c}.$$

Remark 5.2.2. *If* $A : H \longrightarrow H$ *is a linear bounded operator, then* $\hat{A}$ *is an iso-bounded linear iso-operator. Really, since* $A, \hat{T}^{-1} \in \mathcal{L}(\mathcal{H})$, *then there exists a constant* $c > 0$ *such that*

$$||A|| \leq c, \quad ||T^{-1}|| \leq c.$$

From here, it follows

$$\widehat{||\hat{A}||} = ||A\hat{T}^{-1}||\frac{1}{\hat{T}_1} \leq ||A||||\hat{T}^{-1}||\frac{1}{\hat{T}_1} \leq \frac{c^2}{\hat{T}_1} < \infty.$$

Remark 5.2.3. *If* $\hat{A}$ *is an iso-bounded linear iso-operator, then there is a possibility* A *to be an unbounded operator. To see this we will consider the following example.*

Let $\mathcal{C}^{+}([0,1])$ *to be the space of all nonnegative continuous functions on* $[0,1]$, *endowed with the standard maximum norm, and let*

$$Af(t) = \int_0^1 \tfrac{1}{s} f(s)ds,$$

$$\hat{T}^{-1}f(t) = \int_0^t sf(s)ds, \quad t \in [0,1], f \in \mathcal{C}([0,1]).$$

Then A is an unbounded operator, because when $f \equiv 1$ we have

$$A1 = \int_0^1 \frac{1}{s} ds = -\ln 0,$$

$$|A1| = |\ln 0| = \infty.$$

from where, since $||1|| = 1$ and $||A|| = \sup_{x \in \mathcal{C}^+([0,1]), ||x|| \leq 1} ||Ax||$, we conclude that A is an unbounded operator.

Also, for some $f \in \mathcal{C}^+([0,1])$, we have

$$A\hat{T}^{-1} f(x) = \int_0^1 \frac{1}{s} \hat{T}^{-1} f(s) ds$$

$$= \int_0^1 \frac{1}{s} \int_0^s s_1 f(s_1) ds_1 ds.$$

From here,

$$|A\hat{T}^{-1} f(x)| = \left| \int_0^1 \frac{1}{s} \int_0^s s_1 f(s_1) ds_1 ds \right|$$

$$= \int_0^1 \frac{1}{s} \int_0^s s_1 f(s_1) ds_1 ds$$

$$\leq ||f|| \int_0^1 \frac{1}{s} \int_0^s s_1 ds_1 ds$$

$$= ||f|| \frac{1}{2} \int_0^1 \frac{1}{s} s^2 ds$$

$$= ||f|| \frac{1}{2} \int_0^1 s ds$$

$$= ||f|| \frac{1}{2} \frac{1}{2} = \frac{1}{4} ||f||,$$

from where,

$$||A\hat{T}^{-1} f|| \leq \frac{1}{4} ||f||,$$

and therefore

$$||A\hat{T}^{-1}|| \leq \frac{1}{4},$$

and

$$||A\hat{T}^{-1}|| \frac{1}{\hat{T}_1} \leq \frac{1}{4} \frac{1}{\hat{T}_1} \quad \Longleftrightarrow \widehat{||\hat{A}||} \leq \frac{\hat{1}}{4},$$

i.e. $\hat{A}$ is an iso-bounded iso-operator.

Definition 5.2.4. *The linear iso-operator $\hat{A} : \hat{H} \longrightarrow \hat{H}$ will be called iso-continuous iso-operator at $\hat{x_0} \in \hat{H}$ if whenever $\hat{x_n} \in \hat{H}$ and*

$$\widehat{||\hat{x_n} - \hat{x_0}|| \longrightarrow}_{n \longrightarrow \infty} 0,$$

we have

$$\widehat{||\hat{A}\hat{x_n} - \hat{A}\hat{x_0}|| \longrightarrow}_{n \longrightarrow \infty} 0.$$

The linear iso-operator $\hat{A} : \hat{H} \longrightarrow \hat{H}$ will be called iso-continuous in $\hat{H}$ if it is iso-continuous at every iso-point of $\hat{H}$.

Theorem 5.2.5. *If a linear iso-operator is bounded then it is continuous and the conversely.*

Proof. Let $\hat{A} : \hat{H} \longrightarrow \hat{H}$ be a linear iso-operator.

1. We suppose that $\hat{A}$ is an iso-bounded iso-operator. Then there exists $\hat{c} \in \hat{F}_R$ such that

$$\widehat{||\hat{A}\hat{x}||} \leq \hat{c} \hat{\times} \widehat{||\hat{x}||}.$$

Since $\hat{A}$ is a linear iso-operator we have

$$||\hat{A}\widehat{(\hat{x}_n - \hat{x})}|| = ||\widehat{\hat{A}\hat{x}_n - \hat{A}\hat{x}}|| \leq \hat{c} \hat{\times} ||\widehat{\hat{x}_n - \hat{x}}||,$$

therefore whenever

$$||\widehat{\hat{x}_n - \hat{x}|| \longrightarrow}_{n \longrightarrow \infty} 0$$

we have

$$||\widehat{\hat{A}\hat{x}_n - \hat{A}\hat{x}|| \longrightarrow}_{n \longrightarrow \infty} 0.$$

Consequently $\hat{A} : \hat{H} \longrightarrow \hat{H}$ is an iso-continuous iso-operator.

2. We suppose that $\hat{A} : \hat{H} \longrightarrow \hat{H}$ is iso-continuous and iso-unbounded. Therefore, from $\hat{x}_n \in \hat{H}$, $\widehat{||\hat{x}_n||} \leq \hat{I}$, we have that

$$\widehat{||\hat{A}\hat{x}_n||} \geq \hat{n}.$$

If we put $\hat{x}'_n = \hat{x}_n \hat{/} \hat{n}$, then

$$(A15) \widehat{||\hat{A}\hat{x}'_n||} = \hat{I} \hat{/} \hat{n} \hat{\times} \widehat{||\hat{A}\hat{x}_n||} \geq \hat{I}$$

and

$$\widehat{||\hat{x}'_n||} = \hat{I} \hat{/} \hat{n} \hat{\times} \widehat{||\hat{x}_n||} \leq \hat{I} \hat{/} \hat{n} \longrightarrow_{n \longrightarrow 0} 0$$

and since $\hat{A}$ is iso-continuous, then

$$\widehat{||\hat{A}\hat{x}'_n||} \longrightarrow_{n \longrightarrow \infty} 0,$$

which contradicts of (A15).

$\square$

Remark 5.2.6. *Since, as we saw that if $\hat{A}$ is an iso-bounded iso-operator, then in the general case we have not that A is a bounded operator, and because the last theorem we conclude that if $\hat{A}$ is an iso-continuous iso-operator, then in the general case it does not follow that A is a continuous operator.*

Definition 5.2.7. *The space of all linear iso-bounded iso-operators acting from $\hat{H}$ in $\hat{H}$ will be denoted with $\hat{\mathcal{L}}(\hat{H})$.*

Definition 5.2.8. *If $\hat{A}$, $\hat{B} \in \mathcal{L}(\hat{H})$, then their composition is defined as follows*

$$\hat{A}\hat{B} = A\hat{T}^{-1}\hat{T}\hat{B}\hat{T}^{-1} = AB\hat{T}^{-1},$$

and

$$\hat{A}^2 = \hat{A}\hat{A} = A\hat{T}^{-1}\hat{T}A\hat{T}^{-1} = A^2\hat{T}^{-1},$$

$$\hat{A}^3 = \hat{A}\hat{A}^2 = A\hat{T}^{-1}\hat{T}A\hat{T}^{-1}\hat{T}A\hat{T}^{-1} = A^3\hat{T}^{-1}$$

$$\text{and} \quad \text{so}a \quad \text{on.}$$

Definition 5.2.9. *Let $A \in \mathcal{L}(H)$ and there exists A^{-1}. The corresponding lift*

$$\hat{A}^{-1} = A^{-1}\hat{T}^{-1}$$

will be called an iso-inverse iso-operator of the iso-operator A.

From here it follows that, using the definition of composition of two iso-operators,

$$\hat{A}^{-1}\hat{A} = A^{-1}\hat{T}^{-1}\hat{T}A\hat{T}^{-1} = A^{-1}A\hat{T}^{-1} = \hat{T}^{-1} = \hat{I},$$

$$\hat{A}\hat{A}^{-1} = A\hat{T}^{-1}\hat{T}A^{-1}\hat{T}^{-1} = AA^{-1}\hat{T}^{-1} = \hat{T}^{-1} = \hat{I}.$$

Remark 5.2.10. *If we define $\hat{A}^{-1}$ as follows*

$$\hat{A}^{-1} = \left(A\hat{T}^{-1}\right)^{-1},$$

then

$$\hat{A}^{-1} = \hat{T}A^{-1}.$$

From here we obtain that

$$\hat{A}\hat{A}^{-1} = \hat{I} \quad \Longleftrightarrow$$

$$A\hat{T}^{-1}\hat{T}\hat{T}A^{-1} = \hat{T}^{-1} \quad \Longleftrightarrow$$

$$A\hat{T}A^{-1} = \hat{T}^{-1} \quad \Longleftrightarrow$$

$$\hat{T}A\hat{T}A^{-1} = I \quad \Longleftrightarrow$$

$$\hat{T}A\hat{T} = A,$$

and

$$\hat{A}^{-1}\hat{A} = \hat{I} \quad \Longleftrightarrow$$

$$\hat{T}A^{-1}\hat{T}A\hat{T}^{-1} = \hat{T}^{-1} \quad \Longleftrightarrow$$

$$\hat{T}A^{-1}\hat{T}A = I \quad \Longleftrightarrow$$

$$A^{-1}\hat{T}A = \hat{T}^{-1} \quad \Longleftrightarrow$$

$$\hat{T}A = A\hat{T}^{-1} \quad \Longleftrightarrow$$

$$\hat{T}A\hat{T} = A.$$

Therefore, to be built a conception for an iso-inverse iso-operator, we have to have the following relation between A and $\hat{T}$:

$$\hat{T}A\hat{T} = A.$$

Theorem 5.2.11. *Let A, $B \in \mathcal{L}(H)$ and there exist A^{-1}, B^{-1}. Then*

$$\hat{A}^{-1}\hat{B}^{-1} = \widehat{BA}^{-1}.$$

Proof.

$$\hat{A}^{-1}\hat{B}^{-1} = A^{-1}\hat{T}^{-1}\hat{T}B^{-1}\hat{T}^{-1} = A^{-1}B^{-1}\hat{T}^{-1} = (BA)^{-1}\hat{T}^{-1} = \widehat{BA}^{-1}.$$

□

Definition 5.2.12. *Let $A \in \mathcal{L}(H)$ and A^* is its adjoint operator. Then the lift*

$$\hat{A}^* = A^*\hat{T}^{-1}$$

will be called an iso-adjoint iso-operator of the iso-operator $\hat{A}$.

Here we use the word like "iso-adjoint" because in the general case we have

$$(A\hat{T}^{-1})^* \neq A^*\hat{T}^{-1}.$$

Theorem 5.2.13.

$$\hat{A}^*\hat{B}^* = \widehat{BA}^*.$$

Proof.

$$\begin{aligned}\hat{A}^*\hat{B}^* &= A^*\hat{T}^{-1}\hat{T}B^*\hat{T}^{-1}\\ &= A^*B^*\hat{T}^{-1}\\ &= (BA)^*\hat{T}^{-1}\\ &= \widehat{BA}^*.\end{aligned}$$

□

Definition 5.2.14. *Let $A \in \mathcal{L}(H)$ be an operator of orthogonal projection. Then the lift*

$$\hat{A} = A\hat{T}^{-1}$$

will be called an iso-operator of iso-orthogonal projection.

Theorem 5.2.15. *Let $\hat{A}$ be an iso-operator of iso-orthogonal projection. Then*

$$1) \quad \hat{A}^2 = \hat{A},$$

$$2) \quad \hat{A}^* = \hat{A}.$$

Proof.

$$1)\quad \hat{A}^2 = \hat{A}\hat{A} = A\hat{T}^{-1}\hat{T}A\hat{T}^{-1} = A^2\hat{T}^{-1} = A\hat{T}^{-1} = \hat{A},$$

$$2)\quad \hat{A}^* = A^*\hat{T}^{-1} = A\hat{T}^{-1} = \hat{A}.$$

In the last representations we use the definition for the iso-operator of iso-orthogonal projection and from it $A^* = A$ and $A^2 = A$. $\square$

Let $||\cdot||$ is the norm determined by the inner product $(\cdot,\cdot)$ in H. The lift of this norm is

$$\widehat{||\cdot||} = ||\cdot||\frac{1}{\hat{T}_1}.$$

For $\hat{x} \in \hat{H}$ we have

$$\widehat{||\hat{x}||}\hat{\times}\widehat{||\hat{x}||} = ||\hat{T}^{-1}x||\tfrac{1}{\hat{T}_1}\hat{T}_1||||\hat{T}^{-1}x||\tfrac{1}{\hat{T}_1}$$

$$(A16)\quad = ||\hat{T}^{-1}x||^2\tfrac{1}{\hat{T}_1}$$

$$= (\hat{T}^{-1}x, \hat{T}^{-1}x)\tfrac{1}{\hat{T}_1},$$

and

$$\widehat{(\hat{x},\hat{x})} = (\hat{T}^{-1}x, \hat{T}^{-1}x)\frac{1}{\hat{T}_1}.$$

From here and (A16) it follows that

$$\widehat{||\hat{x}||}\hat{\times}\widehat{||\hat{x}||} = \widehat{(\hat{x},\hat{x})}.$$

Definition 5.2.16. *Let $\{\hat{T}^n\}_{n=1}^{\infty}$ satisfy $(H1)$. We will say that the iso-sequence $\{\hat{A}_n\}_{n=1}^{\infty}$ of iso-elements of $\hat{L}(\hat{H})$ is uniformly convergent to $\hat{A} \in \hat{L}(\hat{H})$ if*

$$||\widehat{\hat{A}_n - \hat{A}}|| \longrightarrow_{n\longrightarrow\infty} 0.$$

Remark 5.2.17. *We will note that if the sequence $\{\hat{A}_n\}_{n=1}^{\infty}$ is uniformly convergent to $\hat{A}$ then it does not follow that the sequence $\{A_n\}_{n=1}^{\infty}$ is uniformly convergent to A and the conversely. We will see this in the next examples.*

Example 5.2.18. *Let $A, B, \hat{T}^{-1}, A_n, \hat{T}^{-1n} : H \longrightarrow H$ and for $x \in H$*

$$A_n x = \tfrac{n^2+1}{2n^2+3}x, \quad \hat{T}^{-1n}x = \tfrac{n+1}{2n+1}x,$$

$$\hat{T}^{-1}x = \tfrac{1}{4}x, \quad Ax = x, \quad Bx = \tfrac{1}{2}x.$$

Then, for $x \in H$,

$$||A_n x - Ax|| = \left|\left|\tfrac{n^2+1}{2n^2+3}x - x\right|\right| = \left|\left|\Big(\tfrac{n^2+1}{2n^2+3} - 1\Big)x\right|\right|,$$

$$||A_n - A|| = \sup_{x\in H, ||x||\leq 1}\left|1 - \tfrac{n^2+1}{2n^2+3}\right|||x||$$

$$= \left|1 - \tfrac{n^2+1}{2n^2+3}\right|,$$

from here

$$\lim_{n\longrightarrow\infty}||A_n - A|| = \lim_{n\longrightarrow\infty}\left|1 - \frac{n^2+1}{2n^2+3}\right| = 1 - \frac{1}{2} = \frac{1}{2},$$

consequently the sequence $\{A_n\}_{n=1}^{\infty}$ *is not uniformly convergent to* A.

Also, for $\hat{x} \in \hat{H}$, *we have*

$$\hat{A}_n\hat{x} = A_n\hat{T}^{-1n}x = A_n\left(\tfrac{n+1}{2n+3}x\right) = \tfrac{(n^2+1)(n+1)}{(2n^2+3)(2n+1)}x,$$

$$\hat{A}\hat{x} = A\hat{T}^{-1}x = A\left(\tfrac{1}{4}x\right) = \tfrac{1}{4}x,$$

from where

$$||\widehat{\hat{A}_n - \hat{A}}|| = ||A_n\hat{T}^{-1n}x - A\hat{T}^{-1}x||\tfrac{1}{T_1}$$

$$= \left|\left|\tfrac{(n^2+1)(n+1)}{(2n^2+1)(2n+1)}x - \tfrac{1}{4}x\right|\right|\tfrac{1}{T_1}$$

$$= \left|\left|\left(\tfrac{(n^2+1)(n+1)}{(2n^2+1)(2n+1)} - \tfrac{1}{4}\right)x\right|\right|\tfrac{1}{T_1}$$

$$= \left|\tfrac{(n^2+1)(n+1)}{(2n^2+1)(2n+1)} - \tfrac{1}{4}\right|||x||\tfrac{1}{T_1},$$

$$||\widehat{\hat{A}_n - \hat{A}}|| = ||A_n\hat{T}^{-1n} - A\hat{T}^{-1}||$$

$$= \sup_{x\in H,||x||\le 1}\left|\tfrac{(n^2+1)(n+1)}{(2n^2+1)(2n+1)} - \tfrac{1}{4}\right|||x||\tfrac{1}{T_1}$$

$$= \left|\tfrac{(n^2+1)(n+1)}{(2n^2+1)(2n+1)} - \tfrac{1}{4}\right|\tfrac{1}{T_1},$$

and then

$$\lim_{n\longrightarrow\infty}||\widehat{\hat{A}_n - \hat{A}}|| = \lim_{n\longrightarrow\infty}\left|\frac{(n^2+1)(n+1)}{(2n^2+1)(2n+1)} - \frac{1}{4}\right|\frac{1}{T_1} = 0,$$

consequently the iso-sequence $\{\hat{A}_n\}_{n=1}^{\infty}$ *is uniformly convergent to* $\hat{A}$.

For $\hat{x} \in \hat{H}$ *we have*

$$\hat{B}\hat{x} = B\hat{T}^{-1}x = B\left(\tfrac{1}{4}x\right) = \tfrac{1}{8}x,$$

from where

$$||\widehat{\hat{A}_n - \hat{B}}|| = ||A_n \hat{T}^{-1n} x - B\hat{T}^{-1}x||$$

$$= \left|\left|\tfrac{(n^2+1)(n+1)}{(2n^2+1)(2n+1)} x - \tfrac{1}{8} x\right|\right| \tfrac{1}{T_1}$$

$$= \left|\left|\left(\tfrac{(n^2+1)(n+1)}{(2n^2+1)(2n+1)} - \tfrac{1}{8}\right) x\right|\right| \tfrac{1}{T_1}$$

$$= \left|\tfrac{(n^2+1)(n+1)}{(2n^2+1)(2n+1)} - \tfrac{1}{8}\right| \tfrac{1}{T_1} ||x||,$$

$$||\widehat{\hat{A}_n - \hat{B}}|| = ||A_n \hat{T}^{-1n} - B\hat{T}^{-1}|| \tfrac{1}{T_1}$$

$$= \sup_{x \in H, ||x|| \leq 1} \left|\tfrac{(n^2+1)(n+1)}{(2n^2+1)(2n+1)} - \tfrac{1}{8}\right| \tfrac{1}{T_1} ||x||$$

$$= \left|\tfrac{(n^2+1)(n+1)}{(2n^2+1)(2n+1)} - \tfrac{1}{8}\right| \tfrac{1}{T_1},$$

and then

$$\lim_{n \longrightarrow \infty} ||\widehat{\hat{A}_n - \hat{A}}|| = \lim_{n \longrightarrow \infty} \left|\frac{(n^2+1)(n+1)}{(2n^2+1)(2n+1)} - \frac{1}{8}\right| \frac{1}{T_1} \neq 0,$$

consequently the iso-sequence $\{\hat{A}_n\}_{n=1}^{\infty}$ *is not uniformly convergent to* $\hat{B}$.

On the other hand, for $x \in H$,

$$||A_n x - Bx|| = \left|\left|\tfrac{n^2+1}{2n^2+3} x - \tfrac{1}{2} x\right|\right| = \left|\left|\left(\tfrac{n^2+1}{2n^2+3} - \tfrac{1}{2}\right) x\right|\right|,$$

$$||A_n - B|| = \sup_{x \in H, ||x|| \leq 1} \left|\tfrac{1}{2} - \tfrac{n^2+1}{2n^2+3}\right| ||x||$$

$$= \left|\tfrac{1}{2} - \tfrac{n^2+1}{2n^2+3}\right|$$

from here

$$\lim_{n \longrightarrow \infty} ||A_n - B|| = \lim_{n \longrightarrow \infty} \left|\frac{1}{2} - \frac{n^2+1}{2n^2+3}\right| = 0,$$

consequently the sequence $\{A_n\}_{n=1}^{\infty}$ *is uniformly convergent to* B.

Definition 5.2.19. *Let* $\{\hat{T}^n\}_{n=1}^{\infty}$ *satisfy* $(H1)$. *We will say that the iso-sequence* $\{\hat{A}_n\}_{n=1}^{\infty}$ *of iso-elements of* $\hat{L}(\hat{H})$ *is strongly convergent to* $\hat{A} \in \hat{L}(\hat{H})$ *if*

$$||\widehat{\hat{A}_n \hat{x} - \hat{A}\hat{x}}|| \longrightarrow_{n \longrightarrow \infty} 0$$

for every $\hat{x} \in \hat{H}$.

Remark 5.2.20. *From the above examples it follows that from the strongly convergence of* $\{\hat{A}_n\}_{n=1}^{\infty}$ *to* $\hat{A}$ *it does not follow the strongly convergence of* $\{A_n\}_{n=1}^{\infty}$ *to* A *and the conversely.*

Theorem 5.2.21. *Let* $\{\hat{T}^n\}_{n=1}^{\infty}$ *satisfy* $(H1)$ *and* $\{\hat{A}_n\}_{n=1}^{\infty}$ *is uniformly convergent to* $\hat{A} \in \hat{L}(\hat{H})$. *Then* $\{\hat{A}_n\}_{n=1}^{\infty}$ *is strongly convergent to* $\hat{A} \in \hat{L}(\hat{H})$.

Proof. The proof follows from the following iso-inequality

$$||\widehat{\hat{A}_n\hat{x} - \hat{A}\hat{x}}|| \leq ||\widehat{\hat{A}_n - \hat{A}}|| \hat{\times} \widehat{||\hat{x}||}.$$

$\square$

Definition 5.2.22. *Every linear iso-operator $\hat{L} : \hat{H} \longrightarrow \hat{F}_{\mathbb{R}}$ will be called a linear iso-functional.*

Definition 5.2.23. *The iso-sequence $\{\hat{x}_n\}_{n=1}^{\infty}$ of iso-elements of $\hat{H}$ will be called weakly convergent to $\hat{x} \in \hat{H}$ if*

$$\hat{L}(\hat{x}_n) \longrightarrow_{n \longrightarrow \infty} \hat{L}(\hat{x})$$

for every linear iso-functional $\hat{L}$ defined on $\hat{H}$.

If $\hat{L}$ is a linear iso-functional on $\hat{H}$, then for every $\hat{x} \in \hat{H}$ we have

$$|\hat{L}(\hat{x})| \leq \widehat{||\hat{L}||} \hat{\times} \widehat{||\hat{x}||}.$$

Theorem 5.2.24. *Let $\{\hat{x}_n\}_{n=1}^{\infty}$ be a sequence of iso-elements of $\hat{H}$ which is strongly convergent to $\hat{x} \in \hat{H}$. Then it is weakly convergent.*

Proof. Let $\hat{L}$ be arbitrarily chosen a linear iso-functional on $\hat{H}$. Then we have

$$(A16') \quad |\hat{L}(\hat{x}_n) - \hat{L}(\hat{x})| = |\hat{L}(\hat{x}_n - \hat{x})| \leq \widehat{||\hat{L}||} ||\widehat{\hat{x}_n - \hat{x}}||.$$

Since $\{\hat{x}_n\}_{n=1}^{\infty}$ is strongly convergent to $\hat{x}$ we have

$$\lim_{n \longrightarrow \infty} ||\widehat{\hat{x}_n - \hat{x}}|| = 0,$$

from here and $(\mathrm{A16}')$, we conclude that

$$\lim_{n \longrightarrow \infty} |\hat{L}(\hat{x}_n) - \hat{L}(\hat{x})| = 0,$$

because $\hat{L}$ was arbitrarily chosen, then $\{\hat{x}_n\}_{n=1}^{\infty}$ is weakly convergent to $\hat{x}$. $\square$

Chapter 6

Elements of Santilli-Lie-isotopic time evolution theory

6.1. Definition of Santilli's Lie isotopic power series

Let X and Y be complex Banach spaces. With $\mathcal{L}(X,Y)$ we will denote the space of all linear bounded operators $C : X \longrightarrow Y$.

Let A, T and $H \in \mathcal{L}(X,Y)$ and

$$\frac{dA}{dt} = -i(ATH - HTA).$$

Our aim here is to be investigated the series

$(A17) A(0) + \frac{dA}{dt}(0)w + \frac{1}{2!}\frac{d^2A}{dt^2}(0)w^2 + \cdots .$

Definition 6.1.1. *The series* (A17) *will be called the Santilli's Lie isotopic power series.*

Firstly, we will deduct the general term of (A17).

We have

$$\begin{aligned}
\tfrac{d^2A}{dt^2} &= \tfrac{d}{dt}\left(\tfrac{dA}{dt}\right) \\
&= -i\left(\tfrac{dA}{dt}TH - HT\tfrac{dA}{dt}\right) \\
&= -i\Big(-i(ATH - HTA)TH - HT(-i)(ATH - HTA)\Big) \\
&= (-i)^2((ATH - HTA)TH - HT(ATH - HTA)),
\end{aligned}$$

$$\frac{d^3A}{dt^3} = \frac{d}{dt}\left(\frac{d^2A}{dt^2}\right)$$

$$= -i\left(\frac{d^2A}{dt^2}TH - HT\frac{d^2A}{dt^2}\right)$$

$$= -i((-i)^2((ATH - HTA)TH - HT(ATH - HTA))TH$$

$$-HT(-i)^2((ATH - HTA)TH - HT(ATH - HTA)))$$

$$= (-i)^3((ATH - HTA)(TH)^2 - HT(ATH - HTA)TH$$

$$-HT(ATH - HTA)TH + (HT)^2(ATH - HTA))$$

$$= (-i)^3((ATH - HTA)(TH)^2 - 2HT(ATH - HTA)TH$$

$$+(HT)^2(ATH - HTA)),$$

$$\frac{d^4A}{dt^4} = \frac{d}{dt}\left(\frac{d^3A}{dt^3}\right)$$

$$= -i\left(\frac{d^3A}{dt^3}TH - HT\frac{d^3A}{dt^3}\right)$$

$$= -i((-i)^3((ATH - HTA)(TH)^2 - 2(HT)(ATH - HTA)TH$$

$$+(HT)^2(ATH - HTA))TH - (-i)^3(HT)((ATH - HTA)(TH)^2$$

$$-2(HT)(ATH - HTA)TH + (HT)^2(ATH - HTA)))$$

$$= (-i)^4((ATH - HTA)(TH)^3 - 2HT(ATH - HTA)(TH)^2$$

$$+(HT)^2(ATH - HTA)(TH) - (HT)(ATH - HTA)(TH)^2$$

$$+2(HT)^2(ATH - HTA)TH - (HT)^3(ATH - HTA))$$

$$= (-i)^4((ATH - HTA)(TH)^3 - 3(HT)(ATH - HTA)(TH)^2$$

$$+3(HT)^2(ATH - HTA)(TH) - (HT)^3(ATH - HTA))$$

$$= (-i)^4\sum_{k=0}^{3}\left(\begin{array}{c} 3 \\ 3-k \end{array}\right)(HT)^k(ATH - HTA)(TH)^{3-k}.$$

We suppose that for some natural number n we have

$$\frac{d^nA}{dt^n}$$

$$= (-i)^n\sum_{k=0}^{n-1}\left(\begin{array}{c} n-1 \\ n-1-k \end{array}\right)(-1)^k(HT)^k(ATH - HTA)(TH)^{n-1-k}.$$

We will prove that

$$\frac{d^{n+1}A}{dt^{n+1}}$$

$$= (-i)^{n+1}\sum_{k=0}^{n}\begin{pmatrix} n \\ n-k \end{pmatrix}(-1)^k(HT)^k(ATH-HTA)(TH)^{n-k}.$$

Really,

$$\frac{d^{n+1}A}{dt^{n+1}} = \frac{d}{dt}\left(\frac{d^nA}{dt^n}\right)$$

$$= -i\left(\frac{d^nA}{dt^n}TH - HT\frac{d^nA}{dt^n}\right)$$

$$= -i\Big((-i)^n\sum_{k=0}^{n-1}(-1)^k\begin{pmatrix} n-1 \\ n-1-k \end{pmatrix}(HT)^k(ATH-HTA)(TH)^{n-1-k}$$

$$-(-i)^n HT\sum_{k=0}^{n-1}(-1)^k\begin{pmatrix} n-1 \\ n-1-k \end{pmatrix}(HT)^k(ATH-HTA)(TH)^{n-1-k}\Big)$$

$$= (-i)^{n+1}\Big(((ATH-HTA)(TH)^{n-1} - (HT)(ATH-HTA)(TH)^{n-2}$$

$$+\cdots+(-1)^{n-1}(HT)^{n-1}(ATH-HTA))TH$$

$$-HT((ATH-HTA)(TH)^{n-1} - (HT)(ATH-HTA)(TH)^{n-2}+\cdots$$

$$+(-1)^{n-1}(HT)^{n-1}(ATH-HTA))\Big)$$

$$= (-i)^{n+1}\Big((ATH-HTA)(TH)^n - \begin{pmatrix} n \\ n-1 \end{pmatrix}(HT)(ATH-HTA)(TH)^{n-1}$$

$$+\cdots+(-1)^n(HT)^n(ATH-HTA)\Big)$$

$$= (-i)^n\sum_{k=0}^{n}(-1)^k\begin{pmatrix} n \\ n-k \end{pmatrix}(HT)^k(ATH-HTA)(TH)^{n-k}.$$

From here and the induction it follows that for every natural n we have

$$\frac{d^nA}{dt^n} = (-i)^n\sum_{k=0}^{n-1}(-1)^k\begin{pmatrix} n-1 \\ n-1-k \end{pmatrix}(HT)^k(ATH-HTA)(TH)^{n-1-k}.$$

Let

$$A^n = \frac{(-i)^n}{n!}\sum_{k=0}^{n-1}(-1)^k\begin{pmatrix} n-1 \\ n-1-k \end{pmatrix}(HT)^k(ATH-HTA)(TH)^{n-1-k}(0),$$

$$A^0 = A(0)..$$

We note that when we write $C(0)$ we have in mind that the operator C acts on the zero in X.

6.2. Properties of Santilli's Lie isotopic power series

Now we will investigate the series

$(A18) g(w) = \sum_{n=0}^{\infty} A^n w^n$,

where w is a complex variable. If $|w| > 1$ then we will make the change $w_1 = w - 1$ and therefore $|w_1| = |w-1| \geq |w| - 1 > 0$.

Let Ω be the set of all w for which the series (A18) is convergent. The set Ω is not empty because $0 \in \Omega$.

For $r > 0$ and $x_0 \in \Omega$ we will denote with $S_r(x_0)$ the ball

$$S_r(x_0) = \{x \in \mathbb{C} : |x - x_0| < r\}.$$

Theorem 6.2.1. *Let $w_0 \neq 0$ and $w_0 \in \Omega$. Then $S_{|w_0|}(0) \subset \Omega$ and in every ball $S_r(0)$, $0 < r < |w_0|$, the series* (A18) *is absolutely and uniformly convergent.*

Proof. Since $w_0 \in \Omega$ then the series $\sum_{n=0}^{\infty} A^n w_0^n$ is convergent. From the properties of the convergent series we have that $\lim_{n \longrightarrow \infty} A^n w_0^n = 0$. From here we conclude that the sequence $\{A^n w_0^n\}_{n=1}^{\infty}$ is bounded. Therefore there exists a constant $M > 0$ such that

$$||A^n w_0^n|| \leq M \quad \text{for} \quad \forall n \in \mathbb{N}.$$

Let $w \in S_{|w_0|}(0)$. Then $|w| < |w_0|$ and

$$||A^n w^n|| = \left|\left|A^n w_0^n \frac{w^n}{w_0^n}\right|\right|$$

$$= \left|\frac{w}{w_0}\right|^n ||A^n w_0^n|| \leq M \left|\frac{w}{w_0}\right|^n$$

and from here

$$\sum_{n=0}^{\infty} ||A^n w^n|| \leq M \sum_{n=0}^{\infty} \left|\frac{w}{w_0}\right|^n < \infty.$$

If $|w| < r$

$$||A^n w^n|| = \left|\left|A^n r^n \frac{w^n}{r^n}\right|\right|$$

$$= r^n ||A^n|| \left|\frac{w}{r}\right|^n$$

$$< |w_0|^n ||A^n|| \left|\frac{w}{r}\right|^n$$

$$= ||A^n w_0^n|| \left|\frac{w}{r}\right|^n$$

$$\leq M \left|\frac{w}{r}\right|^n.$$

Consequently

$$\sum_{n=0}^{\infty} ||A^n w^n|| \leq M \sum_{n=0}^{\infty} \left|\frac{w}{r}\right|^n < \infty,$$

i.e. the series (A18) is absolutely and uniformly convergent. □

With R we will denote the radius of the convergence of (A18).

From the definition of the radius of the convergence of a power series we have

$$R = \sup_{w \in \Omega} |w|.$$

Also,

1. If $R = 0$ then $\Omega = \{0\}$.

2. If $R = \infty$ then the series (A18) is convergent in all complex plane.

3. From the Cauchy - Hadamard formula we have

$$R = \frac{1}{\overline{\lim}_{n \longrightarrow \infty} ||A^n||^{\frac{1}{n}}}.$$

Theorem 6.2.2. *Let A, T, $H \in \mathcal{L}(X,y)$, $||T|| > 0$, $||H|| > 0$. Then*

$$R > \frac{1}{2||T||||H||}.$$

Proof. We have

$$||A^n|| = \left|\left|(-i)^n \sum_{k=0}^{n-1} \begin{pmatrix} n-1 \\ n-1-k \end{pmatrix} (-1)^k (HT)^k ATH - HTA)(TH)^{n-k-1}\right|\right|$$

$$\leq \sum_{k=0}^{\infty} \begin{pmatrix} n-1 \\ n-1-k \end{pmatrix} ||(HT)^k (ATH - HTA)(TH)^{n-1-k}||$$

$$\leq \sum_{k=0}^{\infty} \begin{pmatrix} n-1 \\ n-1-k \end{pmatrix} ||(HT)^k (ATH - HTA)||||(TH)^{n-1-k}||$$

$$\leq \sum_{k=0}^{\infty} \begin{pmatrix} n-1 \\ n-1-k \end{pmatrix} ||(HT)^k (ATH - HTA)||||TH||^{n-1-k}$$

$$\leq \sum_{k=0}^{\infty} \begin{pmatrix} n-1 \\ n-1-k \end{pmatrix} ||(HT)^k||||ATH - HTA||||TH||^{n-1-k}$$

$$\sum_{k=0}^{\infty} \begin{pmatrix} n-1 \\ n-1-k \end{pmatrix} ||HT||^k (||ATH|| + ||THA||)||T||^{n-1-k}||H||^{n-1-k}$$

$$\sum_{k=0}^{\infty} \begin{pmatrix} n-1 \\ n-1-k \end{pmatrix} ||H||^k||T||^k (||A||||T||||H|| + ||T||||H||||A||)||H||^{n-1-k}||T||^{n-1-k}$$

$$= 2||A||||T||^n||H||^n \sum_{k=0}^{\infty} \begin{pmatrix} n-1 \\ n-1-k \end{pmatrix}$$

$$= 2^n ||A||||T||^n||H||^n,$$

i.e.

$$||A^n|| \leq 2^n ||A|| ||T||^n ||H||^n,$$

from here

$$||A^n||^{\frac{1}{n}} \leq 2||T|| ||H|| ||A||^{\frac{1}{n}},$$

therefore

$$\overline{\lim}_{n\longrightarrow\infty} ||A^n||^{\frac{1}{n}}$$

$$\leq 2||T|| ||H|| \overline{\lim}_{n\longrightarrow\infty} ||A||^{\frac{1}{n}} = 2||T|| ||H||.$$

From here and the Cauchy - Hadamard formulae we conclude that

$$R = \frac{1}{\overline{\lim}_{n\longrightarrow\infty} ||A^n||^{\frac{1}{n}}} \geq \frac{1}{2||T|| ||H||}.$$

□

Theorem 6.2.3. *If there exist a positive constants M_1 and l such that*

$$||A^n|| \leq M_1 l^n$$

then $R \geq \frac{1}{l}$.

Proof. It is enough to be proved that the series (A18) is uniformly bounded for $|w| < \frac{1}{l}$.

Let

$$|w| l = q < 1.$$

Then

$$||A^n w^n|| = |w|^n ||A^n|| \leq |w|^n M_1 l^n = M_1 q^n,$$

therefore

$$\left|\left| \sum_{k=0}^{\infty} A^k w^k \right|\right| \leq \sum_{k=0}^{\infty} ||A^k w^k||$$

$$\leq M_1 \sum_{k=0}^{\infty} q^k = \frac{M_1}{1-q} < \infty.$$

□

Theorem 6.2.4. *Let*

$(A19) \sum_{n=0}^{\infty} A^n w^n = \sum_{n=0}^{\infty} \tilde{A}^n w^n \quad \text{in} \quad S_R(0).$

Then

$$A^n = \tilde{A}^n \quad \text{for} \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Proof. Since $w = 0 \in \Omega$ then, after we put $w = 0$ in (A19), we get

$$A^0 = \tilde{A}^0.$$

From here and (A19) we obtain

$$\sum_{k=1}^{\infty} A^k w^k = \sum_{k=1}^{\infty} \tilde{A}^k w^k \quad \text{in} \quad S_R(0),$$

from where

$(A20) \sum_{k=1}^{\infty} A^k w^{k-1} = \sum_{k=1}^{\infty} \tilde{A}^k w^{k-1} \quad \text{in} \quad S_R(0).$

We put $w = 0$ in the last equality and we have

$$A^1 = \tilde{A}^1.$$

From here and (A20)

$$\sum_{k=2}^{\infty} A^k w^{k-1} = \sum_{k=2}^{\infty} \tilde{A}^k w^{k-1} \quad \text{in} \quad S_R(0)$$

and etc. □

Theorem 6.2.5. *The function g is a continuous function in $S_R(0)$.*

Proof. Let $\rho \in (0,R)$ and $w, w_0 \in S_R(0)$. Then

$$g(w) - g(w_0) = \sum_{n=1}^{\infty} A^n w^n - \sum_{n=1}^{\infty} A^n w_0^n$$

$$= \sum_{n=1}^{\infty} A^n (w^n - w_0^n)$$

$$= \sum_{n=1}^{\infty} A^n (w - w_0)(w^{n-1} + w^{n-2} w_0 + \cdots + w_0^{n-1}),$$

therefore

$$||g(w) - g(w_0)|| = \left|\left|\sum_{n=1}^{\infty} A^n (w - w_0)(w^{n-1} + w^{n-2} w_0 + \cdots + w_0^{n-1})\right|\right|$$

$$\leq \sum_{n=1}^{\infty} ||A^n (w - w_0)(w^{n-1} + w^{n-2} w_0 + \cdots + w_0^{n-1})||$$

$$= \sum_{n=1}^{\infty} ||A^n|| |w - w_0| |w^{n-1} + w^{n-2} w_0 + \cdots + w_0^{n-1}|$$

$$\leq \sum_{n=1}^{\infty} ||A^n|| |w - w_0| (|w|^{n-1} + |w|^{n-2} |w_0| + \cdots + |w_0|^{n-1})$$

$$\leq \sum_{n=1}^{\infty} ||A^n|| |w - w_0| (\rho^{n-1} + \rho^{n-2} \rho + \cdots + \rho^{n-1})$$

$$= \sum_{n=1}^{\infty} n \rho^{n-1} ||A^n||,$$

i.e.

$(A21) ||g(w) - g(w_0)|| \leq \sum_{n=1}^{\infty} n \rho^{n-1} ||A^n|| |w - w_0|.$

Now we will prove that the series $\sum_{n=1}^{\infty} n \rho^{n-1} A^n$ is uniformly convergent for every $\rho \in (0,R)$. Let $\rho \in (0,R)$ is arbitrarily chosen and fixed. Let also $\tilde{\rho} \in (\rho, R)$. Then, since $\tilde{\rho} < R$ then the series $\sum_{n=1}^{\infty} A^n \tilde{\rho}^n$ is uniformly convergent, from where $\lim_{n \longrightarrow \infty} A^n \tilde{\rho}^n = 0$ and therefore the sequence $\{\tilde{\rho}^n A^n\}_{n=1}^{\infty}$ is a bounded sequence. Consequently there exists a positive constant M_2 such that

$$||A^n|| \tilde{\rho}^n \leq M_2 \quad \text{for} \quad \forall n \in \mathbb{N}.$$

From here

$$\sum_{n=1}^{\infty} n||A^n||\rho^{n-1} = \sum_{n=1}^{\infty} n||A^n||\tilde{\rho}^n \frac{1}{\tilde{\rho}}\left(\frac{\rho}{\tilde{\rho}}\right)^{n-1}$$

$$\leq \frac{M_2}{\tilde{\rho}} \sum_{n=1}^{\infty} n\left(\frac{\rho}{\tilde{\rho}}\right)^{n-1}.$$

Let $q_1 = \frac{\rho}{\tilde{\rho}}$. Then $q_1 < 1$ and

$$\left|\left|\sum_{n=1}^{\infty} nA^n\rho^{n-1}\right|\right| \leq \sum_{n=1}^{\infty} ||nA^n\rho^{n-1}||$$

$$\leq \sum_{n=1}^{\infty} n||A^n||\rho^{n-1}$$

$$\leq \frac{M_2}{\tilde{\rho}} \sum_{n=1}^{\infty} n\left(\frac{\rho}{\tilde{\rho}}\right)^{n-1}$$

$$= \frac{M_2}{\tilde{\rho}} \sum_{n=1}^{\infty} nq_1^{n-1}.$$

Let $b_k = kq_1^{k-1}$. Then

$$\lim_{k\longrightarrow\infty} \frac{b_{k+1}}{b_k} = \lim_{k\longrightarrow\infty} \frac{(k+1)q_1^k}{kq_1^{k-1}}$$

$$= \lim_{k\longrightarrow\infty} \frac{k+1}{k} q_1 = q_1 < 1.$$

Consequently the series $\sum_{n=1}^{\infty} nq_1^{n-1}$ is convergent and then

$$c(\rho) := \sum_{n=1}^{\infty} n||A^n||\rho^{n-1} < \infty.$$

Since $\rho \in S_R(0)$ was arbitrarily chosen then the series $\sum_{n=1}^{\infty} nA^n\rho^{n-1}$ is uniformly convergent for every $\rho \in S_R(0)$.

From (A21) we obtain

$(A22)||g(w) - g(w_0)|| \leq c(\rho)|w - w_0|.$

Let $\varepsilon > 0$ be arbitrarily chosen and fixed. Let also $\delta = \frac{\varepsilon}{1+c(\rho)}$. Then if $|w - w_0| < \delta$, from (A22), we get

$$||g(w) - g(w_0)|| \leq c(\rho)|w - w_0| < c(\rho)\delta = c(\rho)\frac{\varepsilon}{1+c(\rho)} < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrarily chosen and for it we find $\delta = \delta(\varepsilon) > 0$ such that whenever $|w - w_0| < \delta$ we have $||g(w) - g(w_0)|| < \varepsilon$, we conclude that g is a continuous function at w_0.

Because $w_0 \in S_R(0)$ was arbitrarily chosen then g is a continuous function in $S_R(0)$. $\square$

Corollary 6.2.6. *The series*

$$\sum_{n=1}^{\infty} nA^n w^{n-1}$$

is a convergent series in $S_R(0)$.

Proof. Let $|w| < R$ and $\rho \in (|w|, R)$. Then $\frac{|w|}{\rho} < 1$ and from here and the proof of the previous Theorem we have

$$\left|\left|\sum_{n=1}^{\infty} nA^n w^{n-1}\right|\right| \leq \sum_{n=1}^{\infty} ||nA^n w^{n-1}||$$

$$= \sum_{n=1}^{\infty} n||A^n|||w|^{n-1}$$

$$= \sum_{n=1}^{\infty} n||A^n||\rho^{n-1}\frac{|w|^{n-1}}{\rho^{n-1}}$$

$$\leq \sum_{n=1}^{\infty} n||A^n||\rho^{n-1} < \infty.$$

Because $w \in S_R(0)$ was arbitrarily chosen we conclude that $\sum_{n=1}^{\infty} nA^n w^{n-1}$ is convergent in $S_R(0)$. □

Theorem 6.2.7. *The function g is a differentiable function in $S_R(0)$.*

Proof. For $w \in S_R(0)$ we define the function

$$u(w) = \sum_{n=2}^{\infty} nA^n w^{n-1}.$$

For $w, w_1 \in S_R(0)$ we have

$$\frac{g(w)-g(w_1)}{w-w_1} - u(w_1)$$

$$= \frac{1}{w-w_1}\left(\sum_{n=2}^{\infty} A^n w^n - \sum_{n=2}^{\infty} A^n w^{n-1}\right) - \sum_{n=2}^{\infty} nA^n w_1^{n-1}$$

$$= \frac{1}{w-w_1}\sum_{n=2}^{\infty} A^n (w^n - w_1^n) - \sum_{n=2}^{\infty} nA^n w_1^{n-1}$$

$$= \sum_{n=2}^{\infty} A^n \frac{w^n - w_1^n}{w-w_1} - \sum_{n=2}^{\infty} nA^n w_1^{n-1}$$

$$= \sum_{n=2}^{\infty} A^n \left(\frac{w^n - w_1^n}{w-w_1} - nw_1^{n-1}\right),$$

i.e.

$$(A23)\ \frac{g(w)-g(w_1)}{w-w_1} - u(w_1) = \sum_{n=2}^{\infty} A^n \left(\frac{w^n - w_1^n}{w-w_1} - nw_1^{n-1}\right).$$

We will note that

$$(A24)\ \ \frac{w^n - w_1^n}{w-w_1} - nw_1^{n-1} = n(n-1)(w-w_1)\int_0^1 (1-\theta)((1-\theta)w_1 + \theta w)^{n-2} d\theta.$$

Really,

$$n(n-1)(w-w_1)\int_0^1(1-\theta)((1-\theta)w_1+\theta w)^{n-2}d\theta$$

$$=n(n-1)(w-w_1)\int_0^1(1-\theta)(w_1+\theta(w-w_1))^{n-2}d\theta$$

$$=n(n-1)\int_0^1(1-\theta)(w_1+\theta(w-w_1))^{n-2}d(w_1+\theta(w-w_1))$$

$$=n\int_0^1(1-\theta)d(w_1+\theta(w-w_1))^{n-1}$$

$$=n(1-\theta)(w_1+\theta(w-w_1))^{n-1}\Big|_{\theta=0}^{\theta=1}+n\int_0^1(w_1+\theta(w-w_1))^{n-1}d\theta$$

$$=-nw_1^{n-1}$$

$$+\frac{n}{w-w_1}\int_0^1(w_1+\theta(w-w_1))^{n-1}d(w_1+\theta(w-w_1))$$

$$=-nw_1^{n-1}$$

$$+\frac{1}{w-w_1}(w_1+\theta(w-w_1))^{n}\Big|_{\theta=0}^{\theta=1}$$

$$=\frac{w^n-w_1^n}{w-w_1}-nw_1^{n-1}.$$

Now we apply (A24) in (A23) and we obtain

$$\begin{aligned}&\frac{g(w)-g(w_1)}{w-w_1}-u(w_1)\\ (A25)\quad &=(w-w_1)\sum_{n=1}^{\infty}n(n-1)A^n\int_0^1(1-\theta)((1-\theta)w_1+\theta w)^{n-2}d\theta.\end{aligned}$$

Let $\rho\in(0,R)$ is arbitrarily chosen and fixed. Then for $w,w_1\in S_\rho(0)$ and from (A25) we have

$$
\begin{aligned}
&\left|\left|\frac{g(w)-g(w_1)}{w-w_1}-u(w_1)\right|\right| \\
&= \left|\left|(w-w_1)\sum_{n=2}^{\infty} n(n-1)A^n \int_0^1 (1-\theta)((1-\theta)w_1+\theta w)^{n-2} d\theta\right|\right| \\
&\le |w-w_1| \sum_{n=2}^{\infty} n(n-1)\left|\left|A^n \int_0^1 (1-\theta)((1-\theta)w_1+\theta w)^{n-2} d\theta\right|\right| \\
&= |w-w_1| \sum_{n=2}^{\infty} n(n-1)||A^n||\left|\int_0^1 (1-\theta)((1-\theta)w_1+\theta w)^{n-2} d\theta\right| \\
&\le |w-w_1| \sum_{n=2}^{\infty} n(n-1)||A^n|| \int_0^1 (1-\theta)|((1-\theta)w_1+\theta w)^{n-2}| d\theta \\
&= |w-w_1| \sum_{n=2}^{\infty} n(n-1)||A^n|| \int_0^1 (1-\theta)|(1-\theta)w_1+\theta w|^{n-2} d\theta \\
&\le |w-w_1| \sum_{n=2}^{\infty} n(n-1)||A^n|| \int_0^1 (1-\theta)((1-\theta)|w_1|+\theta|w|)^{n-2} d\theta \\
&\le |w-w_1| \sum_{n=2}^{\infty} n(n-1)||A^n|| \int_0^1 (1-\theta)((1-\theta)\rho+\theta\rho)^{n-2} d\theta \\
&= |w-w_1| \sum_{n=1}^{\infty} n(n-1)||A^n|| \int_0^1 (1-\theta)\rho^{n-2} d\theta \\
&\le |w-w_1| \sum_{n=1}^{\infty} n(n-1)||A^n||\rho^{n-2},
\end{aligned}
$$

i.e.

$$
\begin{aligned}
&\left|\left|\frac{g(w)-g(w_1)}{w-w_1}-u(w_1)\right|\right| \\
&\quad\le |w-w_1| \sum_{n=1}^{\infty} n(n-1)||A^n||\rho^{n-2}.
\end{aligned}
\tag{A26}
$$

Now we will prove that for every $\rho \in (0,R)$ the series $\sum_{n=2}^{\infty} n(n-1)A^n\rho^{n-2}$ is a convergent series. Really, let $\rho \in (0,R)$ is arbitrarily chosen and fixed. and let also $\tilde{\rho} \in (\rho,R)$. Since $0<\tilde{\rho}<R$ then the series $\sum_{n=2}^{\infty} A^n\tilde{\rho}^n$ is a convergent series. Therefore $\lim_{n\longrightarrow\infty} A^n\tilde{\rho}^n = 0$ and from here the sequence $\{A^n\tilde{\rho}^n\}_{n=1}^{\infty}$ is a convergent sequence. Consequently, there exists a positive constant M_3 so that

$$
||A^n||\tilde{\rho}^n \le M_3 \quad \text{for} \quad \forall n \in \mathbb{N}.
$$

Therefore

$$
\begin{aligned}
&\left|\left|\sum_{n=1}^{\infty} n(n-1)A^n\rho^{n-2}\right|\right| \\
&\le \sum_{n=2}^{\infty} ||n(n-1)A^n\rho^{n-2}|| \\
&= \sum_{n=2}^{\infty} n(n-1)||A^n|||\rho^{n-2} \\
&= \sum_{n=2}^{\infty} n(n-1)||A^n||\tilde{\rho}^n\left(\frac{\rho}{\tilde{\rho}}\right)^{n-2}\tilde{\rho}^2 \\
&\le M_3R^2\sum_{n=2}^{\infty} n(n-1)\left(\frac{\rho}{\tilde{\rho}}\right)^{n-2},
\end{aligned}
$$

i.e.

$$(A27)\quad \left|\left|\sum_{n=2}^{\infty} n(n-1)A^n\rho^{n-2}\right|\right| \le M_3R^2\sum_{n=2}^{\infty} n(n-1)\left(\frac{\rho}{\tilde{\rho}}\right)^{n-2}.$$

We put

$$q_2 = \frac{\rho}{\tilde{\rho}}.$$

Then, using (A27), we have $q_2 < 1$ and

$$\left|\left|\sum_{n=2}^{\infty} n(n-1)A^n\rho^{n-2}\right|\right|$$

$$\le M_3R^2\sum_{n=2}^{\infty} n(n-1)q_2^{n-2}.$$

Let

$$d_n = n(n-1)q_2^{n-2}.$$

Then

$$\lim_{n\longrightarrow\infty}\frac{d_{n+1}}{d_n} = \lim_{n\longrightarrow\infty}\frac{(n+1)nq_2^{n-1}}{n(n-1)q_2^{n-2}}$$

$$= q_2\lim_{n\longrightarrow\infty}\frac{n+1}{n-1} = q_2 < 1.$$

Consequently $\sum_{n=2}^{\infty} n(n-1)q_2^{n-1}$ is convergent and from (A27) it follows that the series $\sum_{n=2}^{\infty} n(n-1)A^n\rho^{n-2}$ is convergent. Because $\rho \in (0,R)$ was arbitrarily chosen then the series $\sum_{n=2}^{\infty} n(n-1)A^n\rho^{n-2}$ is convergent for every $\rho \in (0,R)$. Therefore

$$c_1(\rho) = \sum_{n=2}^{\infty} n(n-1)||A^n||\rho^{n-2} < \infty$$

for every $\rho \in (0,R)$. From (A26) it follows that

$$\left|\left|\frac{g(w)-g(w_1)}{w-w_1} - u(w_1)\right|\right| \le c_1(\rho)|w-w_1|$$

for every $\rho \in (0,R)$. Let $\varepsilon > 0$ be arbitrarily chosen and fixed. Let also $\delta = \frac{\varepsilon}{1+c_1(\rho)}$. Then from $|w-w_1| < \delta$ we have

$$\left|\left|\frac{g(w)-g(w_1)}{w-w_1} - u(w_1)\right|\right| \le c_1(\rho)|w-w_1| < c_1(\rho)\delta$$

$$= c_1(\rho)\frac{\varepsilon}{1+c_1(\rho)} < \varepsilon$$

for every $\rho \in (0,R)$. Because $\varepsilon > 0$ was arbitrarily chosen and for it we found $\delta = \delta(\varepsilon) > 0$ so that whenever $|w-w_1| < \delta$ we have $\left|\left|\frac{g(w)-g(w_1)}{w-w_1} - u(w_1)\right|\right| < \varepsilon$, then the function g is a differentiable function at w_1 and $g'(w_1) = u(w_1)$. Since $w_1 \in S_R(0)$ was arbitrarily chosen then the function g is a differentiable function in $S_R(0)$ and for every $w \in S_R(0)$ we have $g'(w) = u(w)$. $\square$

Using the induction one can prove

Corollary 6.2.8. $g \in \mathcal{C}^\infty(S_R(0))$.

Theorem 6.2.9. *Let $\sum_{n=0}^\infty A^n$ be an absolutely convergent series to* 0*. Then*

$$\lim_{w \longrightarrow 1, w: \left|\frac{|1-w|}{1-|w|}\right| < \infty} g(w) = 0.$$

Proof. Without loss of generality we will consider the case when $w \longrightarrow 1$ and $0 < \frac{|1-w|}{1-|w|} < \infty$.

Then $|w| < 1$ and there exists a positive constant M_4 such that

$(A28) 0 < \frac{|1-w|}{1-|w|} \leq M_4$.

Let

$$P^n = \sum_{k=0}^n A^k, \quad n = 0, 1, 2, \ldots.$$

Then the sequence $\{P^n\}_{n=1}^\infty$ is a convergent sequence.

$$A^0 = P^0, A^k = P^k - P^{k-1}, \quad k = 1, 2, \ldots.$$

We put

$$s_n(w) = \sum_{k=0}^n A^k w^k.$$

Then

$$\begin{aligned}
s_n(w) &= A^0 + A^1 w + A^2 w^2 + \cdots + A^n w^n \\
&= P^0 + (P^1 - P^0)w + (P^2 - P^1)w^2 + \cdots + (P^n - P^{n-1})w^n \\
&= P^0(1-w) + P^1(w - w^2) + P^2(w^2 - w^3) + \cdots + P^{n-1}(w^{n-1} - w^{n-2}) + P^n w^n \\
&= P^0(1-w) + P^1 w(1-w) + P^2 w^2(1-w) + \cdots + P^{n-1}w^{n-1}(1-w) + P^n w^n,
\end{aligned}$$

i.e.

$(A29) s_n(w) = (1-w)\sum_{k=0}^{n-1} P^k w^k + P^n w^n$.

Since $\sum_{k=0}^\infty A^n$ is an absolutely convergent series to 0 then for $|w| < 1$ we have $\lim_{n\longrightarrow\infty} P^n w^n = 0$ and from (A29)

$(A30) g(w) = \lim_{n\longrightarrow\infty} s_n(w) = (1-w)\sum_{k=0}^\infty P^k w^k$.

Let $\varepsilon > 0$. Then there exists $m \in \mathbb{N}$ such that $||P^n|| < \varepsilon$ for every $n \geq m$.

We choose w so that to satisfy (A28) and $|1-w| < \varepsilon$. From here

$$(A31) \quad \begin{aligned}
&\left|\left|\sum_{n=m}^\infty P^n w^n\right|\right| \leq \sum_{n=m}^\infty ||P^n|| |w|^n \\
&< \varepsilon \sum_{n=m}^\infty |w|^n = \varepsilon \frac{|w|^m}{1-|w|}.
\end{aligned}$$

From (A28) we have

$$|w|-1 \le |1-w| \le M_4(1-|w|),$$

from where

$$|w| \le M_4(1-|w|)+1.$$

Since $|w|<1$ then

$$|w|^m \le |w| \le M_4(1-|w|)+1$$

and from (A31) we obtain

$$\left|\left|\sum_{n=m}^{\infty} P^n w^n\right|\right| < \varepsilon \frac{M_4(1-|w|)+1}{1-|w|}$$

$$= \varepsilon M_4 + \frac{\varepsilon}{1-|w|}.$$

From the last inequality we get

$$\left|\left|(1-w)\sum_{n=m}^{\infty} P^n w^n\right|\right|$$

$$\le |1-w|\left(\varepsilon M_4 + \frac{\varepsilon}{1-|w|}\right)$$

$$= M_4\varepsilon|1-w| + \varepsilon\frac{|1-w|}{1-|w|}$$

$$\le M_4\varepsilon|1-w| + M_4\varepsilon$$

$$= M_4\varepsilon(1+|1-w|)$$

and from (A30)

$$||g(w)|| = \left|\left|(1-w)\sum_{n=0}^{\infty} P^n w^n\right|\right|$$

$$= \left|\left|(1-w)\sum_{n=0}^{m-1} P^n w^n + (1-w)\sum_{n=m}^{\infty} P6nw^n\right|\right|$$

$$\le |1-w|\left|\left|\sum_{n=0}^{m-1} P^n w^n\right|\right| + \left|\left|(1-w)\sum_{n=m}^{\infty} P^n w^n\right|\right|$$

$$\le |1-w|\left|\left|\sum_{n=0}^{m-1} P^n w^n\right|\right| + M_4\varepsilon(1+|1-w|).$$

Because $\varepsilon > 0$ was arbitrarily chosen

$$\lim_{w\longrightarrow 1, w:0<\frac{|1-w|}{1-|w|}<\infty} g(w) = 0.$$

$\square$

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Index