

# The Cambridge Handbook of Physics Formulas

Graham Woan

CAMBRIDGE

CAMBRIDGE

[www.cambridge.org/9780521573498](http://www.cambridge.org/9780521573498)

This page intentionally left blank

## **The Cambridge Handbook of Physics Formulas**

*The Cambridge Handbook of Physics Formulas* is a quick-reference aid for students and professionals in the physical sciences and engineering. It contains more than 2000 of the most useful formulas and equations found in undergraduate physics courses, covering mathematics, dynamics and mechanics, quantum physics, thermodynamics, solid state physics, electromagnetism, optics, and astrophysics. An exhaustive index allows the required formulas to be located swiftly and simply, and the unique tabular format crisply identifies all the variables involved.

*The Cambridge Handbook of Physics Formulas* comprehensively covers the major topics explored in undergraduate physics courses. It is designed to be a compact, portable, reference book suitable for everyday work, problem solving, or exam revision. All students and professionals in physics, applied mathematics, engineering, and other physical sciences will want to have this essential reference book within easy reach.

Graham Woan is a senior lecturer in the Department of Physics and Astronomy at the University of Glasgow. Prior to this he taught physics at the University of Cambridge where he also received his degree in Natural Sciences, specialising in physics, and his PhD, in radio astronomy. His research interests range widely with a special focus on low-frequency radio astronomy. His publications span journals as diverse as *Astronomy & Astrophysics*, *Geophysical Research Letters*, *Advances in Space Science*, the *Journal of Navigation* and *Emergency Prehospital Medicine*. He was co-developer of the revolutionary CURSOR radio positioning system, which uses existing broadcast transmitters to determine position, and he is the designer of the Glasgow Millennium Sundial.



# The Cambridge Handbook of Physics Formulas

2003 Edition

GRAHAM WOAN

*Department of Physics & Astronomy  
University of Glasgow*

CAMBRIDGE UNIVERSITY PRESS

Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press

The Edinburgh Building, Cambridge CB2 2RU, UK

Published in the United States of America by Cambridge University Press, New York

[www.cambridge.org](http://www.cambridge.org)

Information on this title: [www.cambridge.org/9780521573498](http://www.cambridge.org/9780521573498)

© Cambridge University Press 2000

This publication is in copyright. Subject to statutory exception and to the provision of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press.

First published in print format 2000

ISBN-13 978-0-511-07589-6 eBook (EBL)

ISBN-10 0-511-07589-8 eBook (EBL)

ISBN-13 978-0-521-57349-8 hardback

ISBN-10 0-521-57349-1 hardback

ISBN-13 978-0-521-57507-2 paperback

ISBN-10 0-521-57507-9 paperback

Cambridge University Press has no responsibility for the persistence or accuracy of URLs for external or third-party internet websites referred to in this publication, and does not guarantee that any content on such websites is, or will remain, accurate or appropriate.

# Contents

Preface	page vii
How to use this book	1
<b><u>1 Units, constants, and conversions</u></b>	<b>3</b>
1.1 Introduction, 3 • 1.2 SI units, 4 • 1.3 Physical constants, 6 • 1.4 Converting between units, 10 • 1.5 Dimensions, 16 • 1.6 Miscellaneous, 18	
<b><u>2 Mathematics</u></b>	<b>19</b>
2.1 Notation, 19 • 2.2 Vectors and matrices, 20 • 2.3 Series, summations, and progressions, 27 • 2.4 Complex variables, 30 • 2.5 Trigonometric and hyperbolic formulas, 32 • 2.6 Mensuration, 35 • 2.7 Differentiation, 40 • 2.8 Integration, 44 • 2.9 Special functions and polynomials, 46 • 2.10 Roots of quadratic and cubic equations, 50 • 2.11 Fourier series and transforms, 52 • 2.12 Laplace transforms, 55 • 2.13 Probability and statistics, 57 • 2.14 Numerical methods, 60	
<b><u>3 Dynamics and mechanics</u></b>	<b>63</b>
3.1 Introduction, 63 • 3.2 Frames of reference, 64 • 3.3 Gravitation, 66 • 3.4 Particle motion, 68 • 3.5 Rigid body dynamics, 74 • 3.6 Oscillating systems, 78 • 3.7 Generalised dynamics, 79 • 3.8 Elasticity, 80 • 3.9 Fluid dynamics, 84	
<b><u>4 Quantum physics</u></b>	<b>89</b>
4.1 Introduction, 89 • 4.2 Quantum definitions, 90 • 4.3 Wave mechanics, 92 • 4.4 Hydrogenic atoms, 95 • 4.5 Angular momentum, 98 • 4.6 Perturbation theory, 102 • 4.7 High energy and nuclear physics, 103	
<b><u>5 Thermodynamics</u></b>	<b>105</b>
5.1 Introduction, 105 • 5.2 Classical thermodynamics, 106 • 5.3 Gas laws, 110 • 5.4 Kinetic theory, 112 • 5.5 Statistical thermodynamics, 114 • 5.6 Fluctuations and noise, 116 • 5.7 Radiation processes, 118	

<b>6</b>	<b><u>Solid state physics</u></b>	123
	6.1 Introduction, 123 • 6.2 Periodic table, 124 • 6.3 Crystalline structure, 126 • 6.4 Lattice dynamics, 129 • 6.5 Electrons in solids, 132	
<b>7</b>	<b><u>Electromagnetism</u></b>	135
	7.1 Introduction, 135 • 7.2 Static fields, 136 • 7.3 Electromagnetic fields (general), 139 • 7.4 Fields associated with media, 142 • 7.5 Force, torque, and energy, 145 • 7.6 LCR circuits, 147 • 7.7 Transmission lines and waveguides, 150 • 7.8 Waves in and out of media, 152 • 7.9 Plasma physics, 156	
<b>8</b>	<b><u>Optics</u></b>	161
	8.1 Introduction, 161 • 8.2 Interference, 162 • 8.3 Fraunhofer diffraction, 164 • 8.4 Fresnel diffraction, 166 • 8.5 Geometrical optics, 168 • 8.6 Polarisation, 170 • 8.7 Coherence (scalar theory), 172 • 8.8 Line radiation, 173	
<b>9</b>	<b><u>Astrophysics</u></b>	175
	9.1 Introduction, 175 • 9.2 Solar system data, 176 • 9.3 Coordinate transformations (astronomical), 177 • 9.4 Observational astrophysics, 179 • 9.5 Stellar evolution, 181 • 9.6 Cosmology, 184	
	Index	187



# Preface

In *A Brief History of Time*, Stephen Hawking relates that he was warned against including equations in the book because “each equation... would halve the sales.” Despite this dire prediction there is, for a scientific audience, some attraction in doing the exact opposite.

The reader should not be misled by this exercise. Although the equations and formulas contained here underpin a good deal of physical science they are useless unless the reader *understands* them. Learning physics is not about remembering equations, it is about appreciating the natural structures they express. Although its format should help make some topics clearer, this book is not designed to teach new physics; there are many excellent textbooks to help with that. It is intended to be useful rather than pedagogically complete, so that students can use it for revision and for structuring their knowledge *once they understand the physics*. More advanced users will benefit from having a compact, internally consistent, source of equations that can quickly deliver the relationship they require in a format that avoids the need to sift through pages of rubric.

Some difficult decisions have had to be made to achieve this. First, to be short the book only includes ideas that can be expressed succinctly in equations, without resorting to lengthy explanation. A small number of important topics are therefore absent. For example, Liouville’s theorem can be algebraically succinct ( $\dot{q} = 0$ ) but is meaningless unless  $\dot{q}$  is thoroughly (and carefully) explained. Anyone who already understands what  $\dot{q}$  represents will probably not need reminding that it equals zero. Second, empirical equations with numerical coefficients have been largely omitted, as have topics significantly more advanced than are found at undergraduate level. There are simply too many of these to be sensibly and confidently edited into a short handbook. Third, physical data are largely absent, although a periodic table, tables of physical constants, and data on the solar system are all included. Just a sighting of the marvellous (but dimensionally misnamed) *CRC Handbook of Chemistry and Physics* should be enough to convince the reader that a good science data book is thick.

Inevitably there is personal choice in what should or should not be included, and you may feel that an equation that meets the above criteria is missing. If this is the case, I would be delighted to hear from you so it can be considered for a subsequent edition. Contact details are at the end of this preface. Likewise, if you spot an error or an inconsistency then please let me know and I will post an erratum on the web page.

**Acknowledgments** This venture is founded on the generosity of colleagues in Glasgow and Cambridge whose inputs have strongly influenced the final product. The expertise of Dave Clarke, Declan Diver, Peter Duffett-Smith, Wolf-Gerrit Früh, Martin Hendry, Rico Ignace, David Ireland, John Simmons, and Harry Ward have been central to its production, as have the linguistic skills of Katie Lowe. I would also like to thank Richard Barrett, Matthew Cartmell, Steve Gull, Martin Hendry, Jim Hough, Darren McDonald, and Ken Riley who all agreed to field-test the book and gave invaluable feedback.

My greatest thanks though are to John Shakeshaft who, with remarkable knowledge and skill, worked through the entire manuscript more than once during its production and whose legendary red pen hovered over (or descended upon) every equation in the book. What errors remain are, of course, my own, but I take comfort from the fact that without John they would be much more numerous.

**Contact information** A website containing up-to-date information on this handbook and contact details can be found through the Cambridge University Press web pages at [us.cambridge.org](http://us.cambridge.org) (North America) or [uk.cambridge.org](http://uk.cambridge.org) (United Kingdom), or directly at [radio.astro.gla.ac.uk/hbhome.html](http://radio.astro.gla.ac.uk/hbhome.html).

**Production notes** This book was typeset by the author in  $\text{\LaTeX} 2_{\epsilon}$  using the CUP Times fonts. The software packages used were *WinEdt*, *MiKTeX*, *Mayura Draw*, *Gnuplot*, *Ghostscript*, *Ghostview*, and *Maple V*.

**Comments on the 2002 edition** I am grateful to all those who have suggested improvements, in particular Martin Hendry, Wolfgang Jitschin, and Joseph Katz. Although this edition contains only minor revisions to the original its production was also an opportunity to update the physical constants and periodic table entries and to reflect recent developments in cosmology.

# How to use this book

The format is largely self-explanatory, but a few comments may be helpful. Although it is very tempting to flick through the pages to find what you are looking for, the best starting point is the index. I have tried to make this as extensive as possible, and many equations are indexed more than once. Equations are listed both with their equation number (in square brackets) and the page on which they can be found. The equations themselves are grouped into self-contained and boxed “panels” on the pages. Each panel represents a separate topic, and you will find descriptions of all the variables used at the right-hand side of the panel, usually adjacent to the first equation in which they are used. You should therefore not need to stray outside the panel to understand the notation. Both the panel as a whole and its individual entries may have footnotes, shown below the panel. Be aware of these, as they contain important additional information and conditions relevant to the topic.

Although the panels are self-contained they may use concepts defined elsewhere in the handbook. Often these are cross-referenced, but again the index will help you to locate them if necessary. Notations and definitions are uniform over subject areas unless stated otherwise.



# Chapter 1 Units, constants, and conversions

## 1.1 Introduction

The determination of physical constants and the definition of the units with which they are measured is a specialised and, to many, hidden branch of science.

A quantity with dimensions is one whose value must be expressed relative to one or more standard units. In the spirit of the rest of the book, this section is based around the International System of units (SI). This system uses seven base units<sup>1</sup> (the number is somewhat arbitrary), such as the kilogram and the second, and defines their magnitudes in terms of physical laws or, in the case of the kilogram, an object called the “international prototype of the kilogram” kept in Paris. For convenience there are also a number of derived standards, such as the volt, which are defined as set combinations of the basic seven. Most of the physical observables we regard as being in some sense fundamental, such as the charge on an electron, are now known to a relative standard uncertainty,<sup>2</sup>  $u_r$ , of less than  $10^{-7}$ . The least well determined is the Newtonian constant of gravitation, presently standing at a rather lamentable  $u_r$  of  $1.5 \times 10^{-3}$ , and the best is the Rydberg constant ( $u_r = 7.6 \times 10^{-12}$ ). The dimensionless electron  $g$ -factor, representing twice the magnetic moment of an electron measured in Bohr magnetons, is now known to a relative uncertainty of only  $4.1 \times 10^{-12}$ .

No matter which base units are used, physical quantities are expressed as the product of a numerical value and a unit. These two components have more-or-less equal standing and can be manipulated by following the usual rules of algebra. So, if  $1 \cdot \text{eV} = 160.218 \times 10^{-21} \cdot \text{J}$  then  $1 \cdot \text{J} = [1/(160.218 \times 10^{-21})] \cdot \text{eV}$ . A measurement of energy,  $U$ , with joule as the unit has a numerical value of  $U/\text{J}$ . The same measurement with electron volt as the unit has a numerical value of  $U/\text{eV} = (U/\text{J}) \cdot (\text{J}/\text{eV})$  and so on.

<sup>1</sup>The **metre** is the length of the path travelled by light in vacuum during a time interval of  $1/299\,792\,458$  of a second. The **kilogram** is the unit of mass; it is equal to the mass of the international prototype of the kilogram. The **second** is the duration of  $9\,192\,631\,770$  periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium 133 atom. The **ampere** is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 metre apart in vacuum, would produce between these conductors a force equal to  $2 \times 10^{-7}$  newton per metre of length. The **kelvin**, unit of thermodynamic temperature, is the fraction  $1/273.16$  of the thermodynamic temperature of the triple point of water. The **mole** is the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon 12; its symbol is “mol.” When the mole is used, the elementary entities must be specified and may be atoms, molecules, ions, electrons, other particles, or specified groups of such particles. The **candela** is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency  $540 \times 10^{12}$  hertz and that has a radiant intensity in that direction of  $1/683$  watt per steradian.

<sup>2</sup>The relative standard uncertainty in  $x$  is defined as the estimated standard deviation in  $x$  divided by the modulus of  $x$  ( $x \neq 0$ ).

## 1.2 SI units

### SI base units

<i>physical quantity</i>	<i>name</i>	<i>symbol</i>
length	metre <sup>a</sup>	m
mass	kilogram	kg
time interval	second	s
electric current	ampere	A
thermodynamic temperature	kelvin	K
amount of substance	mole	mol
luminous intensity	candela	cd

<sup>a</sup>Or “meter”.

### SI derived units

<i>physical quantity</i>	<i>name</i>	<i>symbol</i>	<i>equivalent units</i>
catalytic activity	katal	kat	mol s <sup>-1</sup>
electric capacitance	farad	F	C V <sup>-1</sup>
electric charge	coulomb	C	A s
electric conductance	siemens	S	Ω <sup>-1</sup>
electric potential difference	volt	V	J C <sup>-1</sup>
electric resistance	ohm	Ω	V A <sup>-1</sup>
energy, work, heat	joule	J	N m
force	newton	N	m kg s <sup>-2</sup>
frequency	hertz	Hz	s <sup>-1</sup>
illuminance	lux	lx	cd sr m <sup>-2</sup>
inductance	henry	H	V A <sup>-1</sup> s
luminous flux	lumen	lm	cd sr
magnetic flux	weber	Wb	V s
magnetic flux density	tesla	T	V s m <sup>-2</sup>
plane angle	radian	rad	m m <sup>-1</sup>
power, radiant flux	watt	W	J s <sup>-1</sup>
pressure, stress	pascal	Pa	N m <sup>-2</sup>
radiation absorbed dose	gray	Gy	J kg <sup>-1</sup>
radiation dose equivalent <sup>a</sup>	sievert	Sv	[J kg <sup>-1</sup> ]
radioactive activity	becquerel	Bq	s <sup>-1</sup>
solid angle	steradian	sr	m <sup>2</sup> m <sup>-2</sup>
temperature <sup>b</sup>	degree Celsius	°C	K

<sup>a</sup>To distinguish it from the gray, units of J kg<sup>-1</sup> should not be used for the sievert in practice.

<sup>b</sup>The Celsius temperature,  $T_C$ , is defined from the temperature in kelvin,  $T_K$ , by  $T_C = T_K - 273.15$ .

**SI prefixes<sup>a</sup>**

<i>factor</i>	<i>prefix</i>	<i>symbol</i>	<i>factor</i>	<i>prefix</i>	<i>symbol</i>
10 <sup>24</sup>	yotta	Y	10 <sup>-24</sup>	yocto	y
10 <sup>21</sup>	zetta	Z	10 <sup>-21</sup>	zepto	z
10 <sup>18</sup>	exa	E	10 <sup>-18</sup>	atto	a
10 <sup>15</sup>	peta	P	10 <sup>-15</sup>	femto	f
10 <sup>12</sup>	tera	T	10 <sup>-12</sup>	pico	p
10 <sup>9</sup>	giga	G	10 <sup>-9</sup>	nano	n
10 <sup>6</sup>	mega	M	10 <sup>-6</sup>	micro	μ
10 <sup>3</sup>	kilo	k	10 <sup>-3</sup>	milli	m
10 <sup>2</sup>	hecto	h	10 <sup>-2</sup>	centi	c
10 <sup>1</sup>	deca <sup>b</sup>	da	10 <sup>-1</sup>	deci	d

<sup>a</sup>The kilogram is the only SI unit with a prefix embedded in its name and symbol. For mass, the unit name “gram” and unit symbol “g” should be used with these prefixes, hence 10<sup>-6</sup> kg can be written as 1 mg. Otherwise, any prefix can be applied to any SI unit.

<sup>b</sup>Or “deka”.

**Recognised non-SI units**

<i>physical quantity</i>	<i>name</i>	<i>symbol</i>	<i>SI value</i>
area	barn	b	10 <sup>-28</sup> m <sup>2</sup>
energy	electron volt	eV	≈ 1.602 18 × 10 <sup>-19</sup> J
length	ångström	Å	10 <sup>-10</sup> m
	fermi <sup>a</sup>	fm	10 <sup>-15</sup> m
	micron <sup>a</sup>	μm	10 <sup>-6</sup> m
plane angle	degree	°	(π/180) rad
	arcminute	'	(π/10 800) rad
	arcsecond	"	(π/648 000) rad
pressure	bar	bar	10 <sup>5</sup> N m <sup>-2</sup>
time	minute	min	60 s
	hour	h	3 600 s
	day	d	86 400 s
mass	unified atomic mass unit	u	≈ 1.660 54 × 10 <sup>-27</sup> kg
	tonne <sup>a,b</sup>	t	10 <sup>3</sup> kg
volume	litre <sup>c</sup>	l, L	10 <sup>-3</sup> m <sup>3</sup>

<sup>a</sup>These are non-SI names for SI quantities.

<sup>b</sup>Or “metric ton.”

<sup>c</sup>Or “liter”. The symbol “l” should be avoided.

### 1.3 Physical constants

The following 1998 CODATA recommended values for the fundamental physical constants can also be found on the Web at [physics.nist.gov/constants](http://physics.nist.gov/constants). Detailed background information is available in *Reviews of Modern Physics*, Vol. 72, No. 2, pp. 351–495, April 2000.

The digits in parentheses represent the  $1\sigma$  uncertainty in the previous two quoted digits. For example,  $G = (6.673 \pm 0.010) \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$ . It is important to note that the uncertainties for many of the listed quantities are correlated, so that the uncertainty in any expression using them in combination cannot necessarily be computed from the data presented. Suitable covariance values are available in the above references.

#### Summary of physical constants

speed of light in vacuum <sup>a</sup>	$c$	2.997 924 58	$\times 10^8 \text{ m s}^{-1}$
permeability of vacuum <sup>b</sup>	$\mu_0$	$4\pi$	$\times 10^{-7} \text{ H m}^{-1}$
		$=12.566 370 614 \dots$	$\times 10^{-7} \text{ H m}^{-1}$
permittivity of vacuum	$\epsilon_0$	$1/(\mu_0 c^2)$	$\text{F m}^{-1}$
		$=8.854 187 817 \dots$	$\times 10^{-12} \text{ F m}^{-1}$
constant of gravitation <sup>c</sup>	$G$	6.673(10)	$\times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$
Planck constant	$h$	6.626 068 76(52)	$\times 10^{-34} \text{ J s}$
$h/(2\pi)$	$\hbar$	1.054 571 596(82)	$\times 10^{-34} \text{ J s}$
elementary charge	$e$	1.602 176 462(63)	$\times 10^{-19} \text{ C}$
magnetic flux quantum, $h/(2e)$	$\Phi_0$	2.067 833 636(81)	$\times 10^{-15} \text{ Wb}$
electron volt	eV	1.602 176 462(63)	$\times 10^{-19} \text{ J}$
electron mass	$m_e$	9.109 381 88(72)	$\times 10^{-31} \text{ kg}$
proton mass	$m_p$	1.672 621 58(13)	$\times 10^{-27} \text{ kg}$
proton/electron mass ratio	$m_p/m_e$	1 836.152 667 5(39)	
unified atomic mass unit	u	1.660 538 73(13)	$\times 10^{-27} \text{ kg}$
fine-structure constant, $\mu_0 c e^2/(2h)$	$\alpha$	7.297 352 533(27)	$\times 10^{-3}$
		inverse	$1/\alpha$
Rydberg constant, $m_e c \alpha^2/(2h)$	$R_\infty$	1.097 373 156 854 9(83)	$\times 10^7 \text{ m}^{-1}$
Avogadro constant	$N_A$	6.022 141 99(47)	$\times 10^{23} \text{ mol}^{-1}$
Faraday constant, $N_A e$	$F$	9.648 534 15(39)	$\times 10^4 \text{ C mol}^{-1}$
molar gas constant	$R$	8.314 472(15)	$\text{J mol}^{-1} \text{ K}^{-1}$
Boltzmann constant, $R/N_A$	$k$	1.380 650 3(24)	$\times 10^{-23} \text{ J K}^{-1}$
Stefan–Boltzmann constant, $\pi^2 k^4/(60 \hbar^3 c^2)$	$\sigma$	5.670 400(40)	$\times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$
Bohr magneton, $e\hbar/(2m_e)$	$\mu_B$	9.274 008 99(37)	$\times 10^{-24} \text{ J T}^{-1}$

<sup>a</sup>By definition, the speed of light is exact.

<sup>b</sup>Also exact, by definition. Alternative units are  $\text{N A}^{-2}$ .

<sup>c</sup>The standard acceleration due to gravity,  $g$ , is defined as exactly  $9.806 65 \text{ m s}^{-2}$ .



**General constants**

speed of light in vacuum	$c$	2.997 924 58	$\times 10^8 \text{ m s}^{-1}$
permeability of vacuum	$\mu_0$	$4\pi$ $=12.566 370 614 \dots$	$\times 10^{-7} \text{ H m}^{-1}$ $\times 10^{-7} \text{ H m}^{-1}$
permittivity of vacuum	$\epsilon_0$	$1/(\mu_0 c^2)$ $=8.854 187 817 \dots$	$\text{F m}^{-1}$ $\times 10^{-12} \text{ F m}^{-1}$
impedance of free space	$Z_0$	$\mu_0 c$ $=376.730 313 461 \dots$	$\Omega$ $\Omega$
constant of gravitation	$G$	6.673(10)	$\times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$
Planck constant	$h$	6.626 068 76(52)	$\times 10^{-34} \text{ J s}$
in eV s		4.135 667 27(16)	$\times 10^{-15} \text{ eV s}$
$h/(2\pi)$	$\hbar$	1.054 571 596(82)	$\times 10^{-34} \text{ J s}$
in eV s		6.582 118 89(26)	$\times 10^{-16} \text{ eV s}$
Planck mass, $(\hbar c/G)^{1/2}$	$m_{\text{Pl}}$	2.176 7(16)	$\times 10^{-8} \text{ kg}$
Planck length, $\hbar/(m_{\text{Pl}}c) = (\hbar G/c^3)^{1/2}$	$l_{\text{Pl}}$	1.616 0(12)	$\times 10^{-35} \text{ m}$
Planck time, $l_{\text{Pl}}/c = (\hbar G/c^5)^{1/2}$	$t_{\text{Pl}}$	5.390 6(40)	$\times 10^{-44} \text{ s}$
elementary charge	$e$	1.602 176 462(63)	$\times 10^{-19} \text{ C}$
magnetic flux quantum, $h/(2e)$	$\Phi_0$	2.067 833 636(81)	$\times 10^{-15} \text{ Wb}$
Josephson frequency/voltage ratio	$2e/h$	4.835 978 98(19)	$\times 10^{14} \text{ Hz V}^{-1}$
Bohr magneton, $e\hbar/(2m_e)$	$\mu_{\text{B}}$	9.274 008 99(37)	$\times 10^{-24} \text{ J T}^{-1}$
in eV $\text{T}^{-1}$		5.788 381 749(43)	$\times 10^{-5} \text{ eV T}^{-1}$
$\mu_{\text{B}}/k$		0.671 713 1(12)	$\text{K T}^{-1}$
nuclear magneton, $e\hbar/(2m_{\text{p}})$	$\mu_{\text{N}}$	5.050 783 17(20)	$\times 10^{-27} \text{ J T}^{-1}$
in eV $\text{T}^{-1}$		3.152 451 238(24)	$\times 10^{-8} \text{ eV T}^{-1}$
$\mu_{\text{N}}/k$		3.658 263 8(64)	$\times 10^{-4} \text{ K T}^{-1}$
Zeeman splitting constant	$\mu_{\text{B}}/(hc)$	46.686 452 1(19)	$\text{m}^{-1} \text{ T}^{-1}$

**Atomic constants<sup>a</sup>**

fine-structure constant, $\mu_0 c e^2/(2h)$	$\alpha$	7.297 352 533(27)	$\times 10^{-3}$
inverse	$1/\alpha$	137.035 999 76(50)	
Rydberg constant, $m_e c \alpha^2/(2h)$	$R_{\infty}$	1.097 373 156 854 9(83)	$\times 10^7 \text{ m}^{-1}$
$R_{\infty} c$		3.289 841 960 368(25)	$\times 10^{15} \text{ Hz}$
$R_{\infty} hc$		2.179 871 90(17)	$\times 10^{-18} \text{ J}$
$R_{\infty} hc/e$		13.605 691 72(53)	eV
Bohr radius <sup>b</sup> , $\alpha/(4\pi R_{\infty})$	$a_0$	5.291 772 083(19)	$\times 10^{-11} \text{ m}$

<sup>a</sup>See also the Bohr model on page 95.<sup>b</sup>Fixed nucleus.

**Electron constants**

electron mass	$m_e$	9.109 381 88(72)	$\times 10^{-31}$ kg
in MeV		0.510 998 902(21)	MeV
electron/proton mass ratio	$m_e/m_p$	5.446 170 232(12)	$\times 10^{-4}$
electron charge	$-e$	-1.602 176 462(63)	$\times 10^{-19}$ C
electron specific charge	$-e/m_e$	-1.758 820 174(71)	$\times 10^{11}$ C kg <sup>-1</sup>
electron molar mass, $N_A m_e$	$M_e$	5.485 799 110(12)	$\times 10^{-7}$ kg mol <sup>-1</sup>
Compton wavelength, $h/(m_e c)$	$\lambda_C$	2.426 310 215(18)	$\times 10^{-12}$ m
classical electron radius, $\alpha^2 a_0$	$r_e$	2.817 940 285(31)	$\times 10^{-15}$ m
Thomson cross section, $(8\pi/3)r_e^2$	$\sigma_T$	6.652 458 54(15)	$\times 10^{-29}$ m <sup>2</sup>
electron magnetic moment	$\mu_e$	-9.284 763 62(37)	$\times 10^{-24}$ J T <sup>-1</sup>
in Bohr magnetons, $\mu_e/\mu_B$		-1.001 159 652 186 9(41)	
in nuclear magnetons, $\mu_e/\mu_N$		-1 838.281 966 0(39)	
electron gyromagnetic ratio, $2 \mu_e /\hbar$	$\gamma_e$	1.760 859 794(71)	$\times 10^{11}$ s <sup>-1</sup> T <sup>-1</sup>
electron g-factor, $2\mu_e/\mu_B$	$g_e$	-2.002 319 304 3737(82)	

**Proton constants**

proton mass	$m_p$	1.672 621 58(13)	$\times 10^{-27}$ kg
in MeV		938.271 998(38)	MeV
proton/electron mass ratio	$m_p/m_e$	1 836.152 667 5(39)	
proton charge	$e$	1.602 176 462(63)	$\times 10^{-19}$ C
proton specific charge	$e/m_p$	9.578 834 08(38)	$\times 10^7$ C kg <sup>-1</sup>
proton molar mass, $N_A m_p$	$M_p$	1.007 276 466 88(13)	$\times 10^{-3}$ kg mol <sup>-1</sup>
proton Compton wavelength, $h/(m_p c)$	$\lambda_{C,p}$	1.321 409 847(10)	$\times 10^{-15}$ m
proton magnetic moment	$\mu_p$	1.410 606 633(58)	$\times 10^{-26}$ J T <sup>-1</sup>
in Bohr magnetons, $\mu_p/\mu_B$		1.521 032 203(15)	$\times 10^{-3}$
in nuclear magnetons, $\mu_p/\mu_N$		2.792 847 337(29)	
proton gyromagnetic ratio, $2\mu_p/\hbar$	$\gamma_p$	2.675 222 12(11)	$\times 10^8$ s <sup>-1</sup> T <sup>-1</sup>

**Neutron constants**

neutron mass	$m_n$	1.674 927 16(13)	$\times 10^{-27}$ kg
in MeV		939.565 330(38)	MeV
neutron/electron mass ratio	$m_n/m_e$	1 838.683 655 0(40)	
neutron/proton mass ratio	$m_n/m_p$	1.001 378 418 87(58)	
neutron molar mass, $N_A m_n$	$M_n$	1.008 664 915 78(55)	$\times 10^{-3}$ kg mol <sup>-1</sup>
neutron Compton wavelength, $h/(m_n c)$	$\lambda_{C,n}$	1.319 590 898(10)	$\times 10^{-15}$ m
neutron magnetic moment	$\mu_n$	-9.662 364 0(23)	$\times 10^{-27}$ J T <sup>-1</sup>
in Bohr magnetons	$\mu_n/\mu_B$	-1.041 875 63(25)	$\times 10^{-3}$
in nuclear magnetons	$\mu_n/\mu_N$	-1.913 042 72(45)	
neutron gyromagnetic ratio, $2 \mu_n /\hbar$	$\gamma_n$	1.832 471 88(44)	$\times 10^8$ s <sup>-1</sup> T <sup>-1</sup>

**Muon and tau constants**

muon mass	$m_\mu$	1.883 531 09(16)	$\times 10^{-28}$ kg
in MeV		105.658 356 8(52)	MeV
tau mass	$m_\tau$	3.167 88(52)	$\times 10^{-27}$ kg
in MeV		1.777 05(29)	$\times 10^3$ MeV
muon/electron mass ratio	$m_\mu/m_e$	206.768 262(30)	
muon charge	$-e$	-1.602 176 462(63)	$\times 10^{-19}$ C
muon magnetic moment	$\mu_\mu$	-4.490 448 13(22)	$\times 10^{-26}$ J T <sup>-1</sup>
in Bohr magnetons, $\mu_\mu/\mu_B$		4.841 970 85(15)	$\times 10^{-3}$
in nuclear magnetons, $\mu_\mu/\mu_N$		8.890 597 70(27)	
muon g-factor	$g_\mu$	-2.002 331 832 0(13)	

**Bulk physical constants**

Avogadro constant	$N_A$	6.022 141 99(47)	$\times 10^{23}$ mol <sup>-1</sup>
atomic mass constant <sup>a</sup>	$m_u$	1.660 538 73(13)	$\times 10^{-27}$ kg
in MeV		931.494 013(37)	MeV
Faraday constant	$F$	9.648 534 15(39)	$\times 10^4$ C mol <sup>-1</sup>
molar gas constant	$R$	8.314 472(15)	J mol <sup>-1</sup> K <sup>-1</sup>
Boltzmann constant, $R/N_A$	$k$	1.380 650 3(24)	$\times 10^{-23}$ J K <sup>-1</sup>
in eV K <sup>-1</sup>		8.617 342(15)	$\times 10^{-5}$ eV K <sup>-1</sup>
molar volume (ideal gas at stp) <sup>b</sup>	$V_m$	22.413 996(39)	$\times 10^{-3}$ m <sup>3</sup> mol <sup>-1</sup>
Stefan–Boltzmann constant, $\pi^2 k^4/(60\hbar^3 c^2)$	$\sigma$	5.670 400(40)	$\times 10^{-8}$ W m <sup>-2</sup> K <sup>-4</sup>
Wien's displacement law constant, <sup>c</sup> $b = \lambda_m T$	$b$	2.897 768 6(51)	$\times 10^{-3}$ m K

<sup>a</sup>= mass of <sup>12</sup>C/12. Alternative nomenclature for the unified atomic mass unit, u.

<sup>b</sup>Standard temperature and pressure (stp) are  $T = 273.15$  K (0°C) and  $P = 101\,325$  Pa (1 standard atmosphere).

<sup>c</sup>See also page 121.

**Mathematical constants**

pi ( $\pi$ )	3.141 592 653 589 793 238 462 643 383 279 ...
exponential constant (e)	2.718 281 828 459 045 235 360 287 471 352 ...
Catalan's constant	0.915 965 594 177 219 015 054 603 514 932 ...
Euler's constant <sup>a</sup> ( $\gamma$ )	0.577 215 664 901 532 860 606 512 090 082 ...
Feigenbaum's constant ( $\alpha$ )	2.502 907 875 095 892 822 283 902 873 218 ...
Feigenbaum's constant ( $\delta$ )	4.669 201 609 102 990 671 853 203 820 466 ...
Gibbs constant	1.851 937 051 982 466 170 361 053 370 157 ...
golden mean	1.618 033 988 749 894 848 204 586 834 370 ...
Madelung constant <sup>b</sup>	1.747 564 594 633 182 190 636 212 035 544 ...

<sup>a</sup>See also Equation (2.119).

<sup>b</sup>NaCl structure.

## 1.4 Converting between units

The following table lists common (and not so common) measures of physical quantities. The numerical values given are the SI equivalent of one unit measure of the non-SI unit. Hence 1 astronomical unit equals  $149.5979 \times 10^9$  m. Those entries identified with a “\*” in the second column represent exact conversions; so 1 abampere equals exactly 10.0 A. Note that individual entries in this list are not recorded in the index, and that values are “international” unless otherwise stated.

There is a separate section on temperature conversions after this table.

<i>unit name</i>	<i>value in SI units</i>	
abampere	10.0*	A
abcoulomb	10.0*	C
abfarad	1.0*	$\times 10^9$ F
abhenry	1.0*	$\times 10^{-9}$ H
abmho	1.0*	$\times 10^9$ S
abohm	1.0*	$\times 10^{-9}$ $\Omega$
abvolt	10.0*	$\times 10^{-9}$ V
acre	4.046 856	$\times 10^3$ m <sup>2</sup>
amagat (at stp)	44.614 774	mol m <sup>-3</sup>
ampere hour	3.6*	$\times 10^3$ C
ångström	100.0*	$\times 10^{-12}$ m
apostilb	1.0*	lm m <sup>-2</sup>
arcminute	290.888 2	$\times 10^{-6}$ rad
arcsecond	4.848 137	$\times 10^{-6}$ rad
are	100.0*	m <sup>2</sup>
astronomical unit	149.5979	$\times 10^9$ m
atmosphere (standard)	101.325 0*	$\times 10^3$ Pa
atomic mass unit	1.660 540	$\times 10^{-27}$ kg
bar	100.0*	$\times 10^3$ Pa
barn	100.0*	$\times 10^{-30}$ m <sup>2</sup>
baromil	750.1	$\times 10^{-6}$ m
barrel (UK)	163.659 2	$\times 10^{-3}$ m <sup>3</sup>
barrel (US dry)	115.627 1	$\times 10^{-3}$ m <sup>3</sup>
barrel (US liquid)	119.240 5	$\times 10^{-3}$ m <sup>3</sup>
barrel (US oil)	158.987 3	$\times 10^{-3}$ m <sup>3</sup>
baud	1.0*	s <sup>-1</sup>
bayre	100.0*	$\times 10^{-3}$ Pa
biot	10.0	A
bolt (US)	36.576*	m
brewster	1.0*	$\times 10^{-12}$ m <sup>2</sup> N <sup>-1</sup>
British thermal unit	1.055 056	$\times 10^3$ J
bushel (UK)	36.36 872	$\times 10^{-3}$ m <sup>3</sup>
bushel (US)	35.23 907	$\times 10^{-3}$ m <sup>3</sup>
butt (UK)	477.339 4	$\times 10^{-3}$ m <sup>3</sup>
cable (US)	219.456*	m
calorie	4.186 8*	J

*continued on next page ...*

<i>unit name</i>	<i>value in SI units</i>	
candle power (spherical)	$4\pi$	lm
carat (metric)	200.0*	$\times 10^{-6}$ kg
cental	45.359 237	kg
centare	1.0*	m <sup>2</sup>
centimetre of Hg (0 °C)	1.333 222	$\times 10^3$ Pa
centimetre of H <sub>2</sub> O (4 °C)	98.060 616	Pa
chain (engineers')	30.48*	m
chain (US)	20.116 8*	m
Chu	1.899 101	$\times 10^3$ J
clusec	1.333 224	$\times 10^{-6}$ W
cord	3.624 556	m <sup>3</sup>
cubit	457.2*	$\times 10^{-3}$ m
cumec	1.0*	m <sup>3</sup> s <sup>-1</sup>
cup (US)	236.588 2	$\times 10^{-6}$ m <sup>3</sup>
curie	37.0*	$\times 10^9$ Bq
darcy	986.923 3	$\times 10^{-15}$ m <sup>2</sup>
day	86.4*	$\times 10^3$ s
day (sidereal)	86.164 09	$\times 10^3$ s
debye	3.335 641	$\times 10^{-30}$ C m
degree (angle)	17.453 29	$\times 10^{-3}$ rad
denier	111.111 1	$\times 10^{-9}$ kg m <sup>-1</sup>
digit	19.05*	$\times 10^{-3}$ m
dioptré	1.0*	m <sup>-1</sup>
Dobson unit	10.0*	$\times 10^{-6}$ m
dram (avoirdupois)	1.771 845	$\times 10^{-3}$ kg
dyne	10.0*	$\times 10^{-6}$ N
dyne centimetres	100.0*	$\times 10^{-9}$ J
electron volt	160.217 7	$\times 10^{-21}$ J
ell	1.143*	m
em	4.233 333	$\times 10^{-3}$ m
emu of capacitance	1.0*	$\times 10^9$ F
emu of current	10.0*	A
emu of electric potential	10.0*	$\times 10^{-9}$ V
emu of inductance	1.0*	$\times 10^{-9}$ H
emu of resistance	1.0*	$\times 10^{-9}$ $\Omega$
Eötvös unit	1.0*	$\times 10^{-9}$ m s <sup>-2</sup> m <sup>-1</sup>
esu of capacitance	1.112 650	$\times 10^{-12}$ F
esu of current	333.564 1	$\times 10^{-12}$ A
esu of electric potential	299.792 5	V
esu of inductance	898.755 2	$\times 10^9$ H
esu of resistance	898.755 2	$\times 10^9$ $\Omega$
erg	100.0*	$\times 10^{-9}$ J
faraday	96.485 3	$\times 10^3$ C
fathom	1.828 804	m
fermi	1.0*	$\times 10^{-15}$ m
Finsen unit	10.0*	$\times 10^{-6}$ W m <sup>-2</sup>
firkin (UK)	40.914 81	$\times 10^{-3}$ m <sup>3</sup>

*continued on next page ...*

<i>unit name</i>	<i>value in SI units</i>	
firkin (US)	34.068 71	$\times 10^{-3} \text{ m}^3$
fluid ounce (UK)	28.413 08	$\times 10^{-6} \text{ m}^3$
fluid ounce (US)	29.573 53	$\times 10^{-6} \text{ m}^3$
foot	304.8*	$\times 10^{-3} \text{ m}$
foot (US survey)	304.800 6	$\times 10^{-3} \text{ m}$
foot of water (4 °C)	2.988 887	$\times 10^3 \text{ Pa}$
footcandle	10.763 91	lx
footlambert	3.426 259	$\text{cd m}^{-2}$
footpoundal	42.140 11	$\times 10^{-3} \text{ J}$
footpounds (force)	1.355 818	J
fresnel	1.0*	$\times 10^{12} \text{ Hz}$
funal	1.0*	$\times 10^3 \text{ N}$
furlong	201.168*	m
g (standard acceleration)	9.806 65*	$\text{m s}^{-2}$
gal	10.0*	$\times 10^{-3} \text{ m s}^{-2}$
gallon (UK)	4.546 09*	$\times 10^{-3} \text{ m}^3$
gallon (US liquid)	3.785 412	$\times 10^{-3} \text{ m}^3$
gamma	1.0*	$\times 10^{-9} \text{ T}$
gauss	100.0*	$\times 10^{-6} \text{ T}$
gilbert	795.774 7	$\times 10^{-3} \text{ A turn}$
gill (UK)	142.065 4	$\times 10^{-6} \text{ m}^3$
gill (US)	118.294 1	$\times 10^{-6} \text{ m}^3$
gon	$\pi/200^*$	rad
grade	15.707 96	$\times 10^{-3} \text{ rad}$
grain	64.798 91*	$\times 10^{-6} \text{ kg}$
gram	1.0*	$\times 10^{-3} \text{ kg}$
gram-rad	100.0*	$\text{J kg}^{-1}$
gray	1.0*	$\text{J kg}^{-1}$
hand	101.6*	$\times 10^{-3} \text{ m}$
hartree	4.359 748	$\times 10^{-18} \text{ J}$
hectare	10.0*	$\times 10^3 \text{ m}^2$
hefner	902	$\times 10^{-3} \text{ cd}$
hogshead	238.669 7	$\times 10^{-3} \text{ m}^3$
horsepower (boiler)	9.809 50	$\times 10^3 \text{ W}$
horsepower (electric)	746*	W
horsepower (metric)	735.498 8	W
horsepower (UK)	745.699 9	W
hour	3.6*	$\times 10^3 \text{ s}$
hour (sidereal)	3.590 170	$\times 10^3 \text{ s}$
Hubble time	440	$\times 10^{15} \text{ s}$
Hubble distance	130	$\times 10^{24} \text{ m}$
hundredweight (UK long)	50.802 35	kg
hundredweight (US short)	45.359 24	kg
inch	25.4*	$\times 10^{-3} \text{ m}$
inch of mercury (0 °C)	3.386 389	$\times 10^3 \text{ Pa}$
inch of water (4 °C)	249.074 0	Pa
jansky	10.0*	$\times 10^{-27} \text{ W m}^{-2} \text{ Hz}^{-1}$

*continued on next page ...*

<i>unit name</i>	<i>value in SI units</i>	
jar	10/9*	$\times 10^{-9}$ F
kayser	100.0*	$\text{m}^{-1}$
kilocalorie	4.186 8*	$\times 10^3$ J
kilogram-force	9.806 65*	N
kilowatt hour	3.6*	$\times 10^6$ J
knot (international)	514.444 4	$\times 10^{-3}$ $\text{m s}^{-1}$
lambert	10/ $\pi$ *	$\times 10^3$ $\text{cd m}^{-2}$
langley	41.84*	$\times 10^3$ $\text{J m}^{-2}$
langmuir	133.322 4	$\times 10^{-6}$ Pa s
league (nautical, int.)	5.556*	$\times 10^3$ m
league (nautical, UK)	5.559 552	$\times 10^3$ m
league (statute)	4.828 032	$\times 10^3$ m
light year	9.460 73*	$\times 10^{15}$ m
ligne	2.256*	$\times 10^{-3}$ m
line	2.116 667	$\times 10^{-3}$ m
line (magnetic flux)	10.0*	$\times 10^{-9}$ Wb
link (engineers')	304.8*	$\times 10^{-3}$ m
link (US)	201.168 0	$\times 10^{-3}$ m
litre	1.0*	$\times 10^{-3}$ $\text{m}^3$
lumen (at 555 nm)	1.470 588	$\times 10^{-3}$ W
maxwell	10.0*	$\times 10^{-9}$ Wb
mho	1.0*	S
micron	1.0*	$\times 10^{-6}$ m
mil (length)	25.4*	$\times 10^{-6}$ m
mil (volume)	1.0*	$\times 10^{-6}$ $\text{m}^3$
mile (international)	1.609 344*	$\times 10^3$ m
mile (nautical, int.)	1.852*	$\times 10^3$ m
mile (nautical, UK)	1.853 184*	$\times 10^3$ m
mile per hour	447.04*	$\times 10^{-3}$ $\text{m s}^{-1}$
milliard	1.0*	$\times 10^9$ $\text{m}^3$
millibar	100.0*	Pa
millimetre of Hg (0 °C)	133.322 4	Pa
minim (UK)	59.193 90	$\times 10^{-9}$ $\text{m}^3$
minim (US)	61.611 51	$\times 10^{-9}$ $\text{m}^3$
minute (angle)	290.888 2	$\times 10^{-6}$ rad
minute	60.0*	s
minute (sidereal)	59.836 17	s
month (lunar)	2.551 444	$\times 10^6$ s
nit	1.0*	$\text{cd m}^{-2}$
noggin (UK)	142.065 4	$\times 10^{-6}$ $\text{m}^3$
oersted	1000/(4 $\pi$ )*	$\text{A m}^{-1}$
ounce (avoirdupois)	28.349 52	$\times 10^{-3}$ kg
ounce (UK fluid)	28.413 07	$\times 10^{-6}$ $\text{m}^3$
ounce (US fluid)	29.573 53	$\times 10^{-6}$ $\text{m}^3$
pace	762.0*	$\times 10^{-3}$ m
parsec	30.856 78	$\times 10^{15}$ m

*continued on next page ...*

<i>unit name</i>	<i>value in SI units</i>	
peck (UK)	9.092 18*	$\times 10^{-3} \text{ m}^3$
peck (US)	8.809 768	$\times 10^{-3} \text{ m}^3$
pennyweight (troy)	1.555 174	$\times 10^{-3} \text{ kg}$
perch	5.029 2*	m
phot	10.0*	$\times 10^3 \text{ lx}$
pica (printers')	4.217 518	$\times 10^{-3} \text{ m}$
pint (UK)	568.261 2	$\times 10^{-6} \text{ m}^3$
pint (US dry)	550.610 5	$\times 10^{-6} \text{ m}^3$
pint (US liquid)	473.176 5	$\times 10^{-6} \text{ m}^3$
point (printers')	351.459 8*	$\times 10^{-6} \text{ m}$
poise	100.0*	$\times 10^{-3} \text{ Pa s}$
pole	5.029 2*	m
poncelot	980.665*	W
pottle	2.273 045	$\times 10^{-3} \text{ m}^3$
pound (avoirdupois)	453.592 4	$\times 10^{-3} \text{ kg}$
poundal	138.255 0	$\times 10^{-3} \text{ N}$
pound-force	4.448 222	N
promaxwell	1.0*	Wb
psi	6.894 757	$\times 10^3 \text{ Pa}$
puncheon (UK)	317.974 6	$\times 10^{-3} \text{ m}^3$
quad	1.055 056	$\times 10^{18} \text{ J}$
quart (UK)	1.136 522	$\times 10^{-3} \text{ m}^3$
quart (US dry)	1.101 221	$\times 10^{-3} \text{ m}^3$
quart (US liquid)	946.352 9	$\times 10^{-6} \text{ m}^3$
quintal (metric)	100.0*	kg
rad	10.0*	$\times 10^{-3} \text{ Gy}$
rayleigh	10/(4 $\pi$ )	$\times 10^9 \text{ s}^{-1} \text{ m}^{-2} \text{ sr}^{-1}$
rem	10.0*	$\times 10^{-3} \text{ Sv}$
REN	1/4 000*	S
reyn	689.5	$\times 10^3 \text{ Pa s}$
rhe	10.0*	$\text{Pa}^{-1} \text{ s}^{-1}$
rod	5.029 2*	m
roentgen	258.0	$\times 10^{-6} \text{ C kg}^{-1}$
rood (UK)	1.011 714	$\times 10^3 \text{ m}^2$
rope (UK)	6.096*	m
rutherford	1.0*	$\times 10^6 \text{ Bq}$
rydberg	2.179 874	$\times 10^{-18} \text{ J}$
scruple	1.295 978	$\times 10^{-3} \text{ kg}$
seam	290.949 8	$\times 10^{-3} \text{ m}^3$
second (angle)	4.848 137	$\times 10^{-6} \text{ rad}$
second (sidereal)	997.269 6	$\times 10^{-3} \text{ s}$
shake	100.0*	$\times 10^{-10} \text{ s}$
shed	100.0*	$\times 10^{-54} \text{ m}^2$
slug	14.593 90	kg
square degree	( $\pi/180$ ) <sup>2</sup> *	sr
statampere	333.564 1	$\times 10^{-12} \text{ A}$
statcoulomb	333.564 1	$\times 10^{-12} \text{ C}$

*continued on next page ...*



<i>unit name</i>	<i>value in SI units</i>	
statfarad	1.112 650	$\times 10^{-12}$ F
stathenry	898.755 2	$\times 10^9$ H
statmho	1.112 650	$\times 10^{-12}$ S
statohm	898.755 2	$\times 10^9 \Omega$
statvolt	299.792 5	V
stere	1.0*	m <sup>3</sup>
sthéne	1.0*	$\times 10^3$ N
stilb	10.0*	$\times 10^3$ cd m <sup>-2</sup>
stokes	100.0*	$\times 10^{-6}$ m <sup>2</sup> s <sup>-1</sup>
stone	6.350 293	kg
tablespoon (UK)	14.206 53	$\times 10^{-6}$ m <sup>3</sup>
tablespoon (US)	14.786 76	$\times 10^{-6}$ m <sup>3</sup>
teaspoon (UK)	4.735 513	$\times 10^{-6}$ m <sup>3</sup>
teaspoon (US)	4.928 922	$\times 10^{-6}$ m <sup>3</sup>
tex	1.0*	$\times 10^{-6}$ kg m <sup>-1</sup>
therm (EEC)	105.506*	$\times 10^6$ J
therm (US)	105.480 4*	$\times 10^6$ J
thermie	4.185 407	$\times 10^6$ J
thou	25.4*	$\times 10^{-6}$ m
tog	100.0*	$\times 10^{-3}$ W <sup>-1</sup> m <sup>2</sup> K
ton (of TNT)	4.184*	$\times 10^9$ J
ton (UK long)	1.016 047	$\times 10^3$ kg
ton (US short)	907.184 7	kg
tonne (metric ton)	1.0*	$\times 10^3$ kg
torr	133.322 4	Pa
townsend	1.0*	$\times 10^{-21}$ V m <sup>2</sup>
troy dram	3.887 935	$\times 10^{-3}$ kg
troy ounce	31.103 48	$\times 10^{-3}$ kg
troy pound	373.241 7	$\times 10^{-3}$ kg
tun	954.678 9	$\times 10^{-3}$ m <sup>3</sup>
XU	100.209	$\times 10^{-15}$ m
yard	914.4*	$\times 10^{-3}$ m
year (365 days)	31.536*	$\times 10^6$ s
year (sidereal)	31.558 15	$\times 10^6$ s
year (tropical)	31.556 93	$\times 10^6$ s

### Temperature conversions

From degrees Celsius <sup>a</sup>	$T_K = T_C + 273.15$	(1.1)	$T_K$ temperature in kelvin $T_C$ temperature in °Celsius
From degrees Fahrenheit	$T_K = \frac{T_F - 32}{1.8} + 273.15$	(1.2)	$T_F$ temperature in °Fahrenheit
From degrees Rankine	$T_K = \frac{T_R}{1.8}$	(1.3)	$T_R$ temperature in °Rankine

<sup>a</sup>The term “centigrade” is not used in SI, to avoid confusion with “10<sup>-2</sup> of a degree”.

## 1.5 Dimensions

The following table lists the dimensions of common physical quantities, together with their conventional symbols and the SI units in which they are usually quoted. The dimensional basis used is length (L), mass (M), time (T), electric current (I), temperature ( $\Theta$ ), amount of substance (N), and luminous intensity (J).

<i>physical quantity</i>	<i>symbol</i>	<i>dimensions</i>	<i>SI units</i>
acceleration	$a$	$L T^{-2}$	$m s^{-2}$
action	$S$	$L^2 M T^{-1}$	J s
angular momentum	$L, J$	$L^2 M T^{-1}$	$m^2 kg s^{-1}$
angular speed	$\omega$	$T^{-1}$	$rad s^{-1}$
area	$A, S$	$L^2$	$m^2$
Avogadro constant	$N_A$	$N^{-1}$	$mol^{-1}$
bending moment	$G_b$	$L^2 M T^{-2}$	N m
Bohr magneton	$\mu_B$	$L^2 I$	$J T^{-1}$
Boltzmann constant	$k, k_B$	$L^2 M T^{-2} \Theta^{-1}$	$J K^{-1}$
bulk modulus	$K$	$L^{-1} M T^{-2}$	Pa
capacitance	$C$	$L^{-2} M^{-1} T^4 I^2$	F
charge (electric)	$q$	$T I$	C
charge density	$\rho$	$L^{-3} T I$	$C m^{-3}$
conductance	$G$	$L^{-2} M^{-1} T^3 I^2$	S
conductivity	$\sigma$	$L^{-3} M^{-1} T^3 I^2$	$S m^{-1}$
couple	$G, T$	$L^2 M T^{-2}$	N m
current	$I, i$	I	A
current density	$J, j$	$L^{-2} I$	$A m^{-2}$
density	$\rho$	$L^{-3} M$	$kg m^{-3}$
electric displacement	$D$	$L^{-2} T I$	$C m^{-2}$
electric field strength	$E$	$L M T^{-3} I^{-1}$	$V m^{-1}$
electric polarisability	$\alpha$	$M^{-1} T^4 I^2$	$C m^2 V^{-1}$
electric polarisation	$P$	$L^{-2} T I$	$C m^{-2}$
electric potential difference	$V$	$L^2 M T^{-3} I^{-1}$	V
energy	$E, U$	$L^2 M T^{-2}$	J
energy density	$u$	$L^{-1} M T^{-2}$	$J m^{-3}$
entropy	$S$	$L^2 M T^{-2} \Theta^{-1}$	$J K^{-1}$
Faraday constant	$F$	$T I N^{-1}$	$C mol^{-1}$
force	$F$	$L M T^{-2}$	N
frequency	$\nu, f$	$T^{-1}$	Hz
gravitational constant	$G$	$L^3 M^{-1} T^{-2}$	$m^3 kg^{-1} s^{-2}$
Hall coefficient	$R_H$	$L^3 T^{-1} I^{-1}$	$m^3 C^{-1}$
Hamiltonian	$H$	$L^2 M T^{-2}$	J
heat capacity	$C$	$L^2 M T^{-2} \Theta^{-1}$	$J K^{-1}$
Hubble constant <sup>1</sup>	$H$	$T^{-1}$	$s^{-1}$
illuminance	$E_v$	$L^{-2} J$	lx
impedance	$Z$	$L^2 M T^{-3} I^{-2}$	$\Omega$

*continued on next page ...*

<sup>1</sup>The Hubble constant is almost universally quoted in units of  $km s^{-1} Mpc^{-1}$ . There are about  $3.1 \times 10^{19}$  kilometres in a megaparsec.

<i>physical quantity</i>	<i>symbol</i>	<i>dimensions</i>	<i>SI units</i>
impulse	$I$	$L M T^{-1}$	N s
inductance	$L$	$L^2 M T^{-2} I^{-2}$	H
irradiance	$E_e$	$M T^{-3}$	$W m^{-2}$
Lagrangian	$L$	$L^2 M T^{-2}$	J
length	$L, l$	L	m
luminous intensity	$I_v$	J	cd
magnetic dipole moment	$m, \mu$	$L^2 I$	$A m^2$
magnetic field strength	$H$	$L^{-1} I$	$A m^{-1}$
magnetic flux	$\Phi$	$L^2 M T^{-2} I^{-1}$	Wb
magnetic flux density	$B$	$M T^{-2} I^{-1}$	T
magnetic vector potential	$A$	$L M T^{-2} I^{-1}$	$Wb m^{-1}$
magnetisation	$M$	$L^{-1} I$	$A m^{-1}$
mass	$m, M$	M	kg
mobility	$\mu$	$M^{-1} T^2 I$	$m^2 V^{-1} s^{-1}$
molar gas constant	$R$	$L^2 M T^{-2} \Theta^{-1} N^{-1}$	$J mol^{-1} K^{-1}$
moment of inertia	$I$	$L^2 M$	$kg m^2$
momentum	$p$	$L M T^{-1}$	$kg m s^{-1}$
number density	$n$	$L^{-3}$	$m^{-3}$
permeability	$\mu$	$L M T^{-2} I^{-2}$	$H m^{-1}$
permittivity	$\epsilon$	$L^{-3} M^{-1} T^4 I^2$	$F m^{-1}$
Planck constant	$h$	$L^2 M T^{-1}$	J s
power	$P$	$L^2 M T^{-3}$	W
Poynting vector	$S$	$M T^{-3}$	$W m^{-2}$
pressure	$p, P$	$L^{-1} M T^{-2}$	Pa
radiant intensity	$I_e$	$L^2 M T^{-3}$	$W sr^{-1}$
resistance	$R$	$L^2 M T^{-3} I^{-2}$	$\Omega$
Rydberg constant	$R_\infty$	$L^{-1}$	$m^{-1}$
shear modulus	$\mu, G$	$L^{-1} M T^{-2}$	Pa
specific heat capacity	$c$	$L^2 T^{-2} \Theta^{-1}$	$J kg^{-1} K^{-1}$
speed	$u, v, c$	$L T^{-1}$	$m s^{-1}$
Stefan–Boltzmann constant	$\sigma$	$M T^{-3} \Theta^{-4}$	$W m^{-2} K^{-4}$
stress	$\sigma, \tau$	$L^{-1} M T^{-2}$	Pa
surface tension	$\sigma, \gamma$	$M T^{-2}$	$N m^{-1}$
temperature	$T$	$\Theta$	K
thermal conductivity	$\lambda$	$L M T^{-3} \Theta^{-1}$	$W m^{-1} K^{-1}$
time	$t$	T	s
velocity	$v, u$	$L T^{-1}$	$m s^{-1}$
viscosity (dynamic)	$\eta, \mu$	$L^{-1} M T^{-1}$	Pa s
viscosity (kinematic)	$\nu$	$L^2 T^{-1}$	$m^2 s^{-1}$
volume	$V, v$	$L^3$	$m^3$
wavevector	$k$	$L^{-1}$	$m^{-1}$
weight	$W$	$L M T^{-2}$	N
work	$W$	$L^2 M T^{-2}$	J
Young modulus	$E$	$L^{-1} M T^{-2}$	Pa

## 1.6 Miscellaneous

### Greek alphabet

<i>A</i>	$\alpha$	alpha	<i>N</i>	$\nu$	nu
<i>B</i>	$\beta$	beta	$\Xi$	$\xi$	xi
$\Gamma$	$\gamma$	gamma	<i>O</i>	$o$	omicron
$\Delta$	$\delta$	delta	$\Pi$	$\pi$	$\varpi$ pi
<i>E</i>	$\epsilon$	$\varepsilon$ epsilon	<i>P</i>	$\rho$	$\varrho$ rho
<i>Z</i>	$\zeta$	zeta	$\Sigma$	$\sigma$	$\varsigma$ sigma
<i>H</i>	$\eta$	eta	<i>T</i>	$\tau$	tau
$\Theta$	$\theta$	$\vartheta$ theta	$\Upsilon$	$\upsilon$	upsilon
<i>I</i>	$\iota$	iota	$\Phi$	$\phi$	$\varphi$ phi
<i>K</i>	$\kappa$	kappa	<i>X</i>	$\chi$	chi
$\Lambda$	$\lambda$	lambda	$\Psi$	$\psi$	psi
<i>M</i>	$\mu$	mu	$\Omega$	$\omega$	omega

### Pi ( $\pi$ ) to 1 000 decimal places

3.1415926535 8979323846 2643383279 5028841971 6939937510 5820974944 5923078164 0628620899 8628034825 3421170679  
 8214808651 3282306647 0938446095 5058223172 5359408128 4811174502 8410270193 8521105559 6446229489 5493038196  
 4428810975 6659334461 2847564823 3786783165 2712019091 4564856692 3460348610 4543266482 1339360726 0249141273  
 7245870066 0631558817 4881520920 9628292540 9171536436 7892590360 0113305305 4882046652 1384146951 9415116094  
 3305727036 5759591953 0921861173 8193261179 3105118548 0744623799 6274956735 1885752724 8912279381 8301194912  
 9833673362 4406566430 8602139494 6395224737 1907021798 6094370277 0539217176 2931767523 8467481846 7669405132  
 0005681271 4526356082 7785771342 7577896091 7363717872 1468440901 2249534301 4654958537 1050792279 6892589235  
 4201995611 2129021960 8640344181 5981362977 4771309960 5187072113 4999999837 2978049951 0597317328 1609631859  
 5024459455 3469083026 4252230825 3344685035 2619311881 7101000313 7838752886 5875332083 8142061717 7669147303  
 5982534904 2875546873 1159562863 8823537875 9375195778 1857780532 1712268066 1300192787 6611195909 2164201989

### e to 1 000 decimal places

2.7182818284 5904523536 0287471352 6624977572 4709369995 9574966967 6277240766 3035354759 4571382178 5251664274  
 2746639193 2003059921 8174135966 2904357290 0334295260 5956307381 3232862794 3490763233 8298807531 9525101901  
 1573834187 9307021540 8914993488 4167509244 7614606680 8226480016 8477411853 7423454424 3710753907 7744992069  
 5517027618 3860626133 1384583000 7520449338 2656029760 6737113200 7093287091 2744374704 7230696977 2093101416  
 9283681902 5515108657 4637721112 5238978442 5056953696 7707854499 6996794686 4454905987 9316368892 3009879312  
 7736178215 4249992295 7635148220 8269895193 6680331825 2886939849 6465105820 9392398294 8879332036 2509443117  
 3012381970 6841614039 7019837679 3206832823 7646480429 5311802328 7825098194 5581530175 6717361332 0698112509  
 9618188159 3041690351 5988885193 4580727386 6738589422 8792284998 9208680582 5749279610 4841984443 6346324496  
 8487560233 6248270419 7862320900 2160990235 3043699418 4914631409 3431738143 6405462531 5209618369 0888707016  
 7683964243 7814059271 4563549061 3031072085 1038375051 0115747704 1718986106 8739696552 1267154688 9570350354

# Chapter 2 Mathematics

## 2.1 Notation

Mathematics is, of course, a vast subject, and so here we concentrate on those mathematical methods and relationships that are most often applied in the physical sciences and engineering.

Although there is a high degree of consistency in accepted mathematical notation, there is some variation. For example the spherical harmonics,  $Y_l^m$ , can be written  $Y_{lm}$ , and there is some freedom with their signs. In general, the conventions chosen here follow common practice as closely as possible, whilst maintaining consistency with the rest of the handbook.

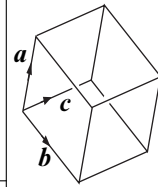
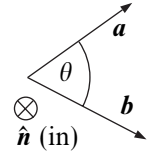
In particular:

scalars	$a$	general vectors	$\mathbf{a}$
unit vectors	$\hat{\mathbf{a}}$	scalar product	$\mathbf{a} \cdot \mathbf{b}$
vector cross-product	$\mathbf{a} \times \mathbf{b}$	gradient operator	$\nabla$
Laplacian operator	$\nabla^2$	derivative	$\frac{df}{dx}$ etc.
partial derivatives	$\frac{\partial f}{\partial x}$ etc.	derivative of $r$ with respect to $t$	$\dot{r}$
$n$ th derivative	$\frac{d^n f}{dx^n}$	closed loop integral	$\oint_L dl$
closed surface integral	$\oint_S ds$	matrix	$\mathbf{A}$ or $a_{ij}$
mean value (of $x$ )	$\langle x \rangle$	binomial coefficient	$\binom{n}{r}$
factorial	$!$	unit imaginary ( $\mathbf{i}^2 = -1$ )	$\mathbf{i}$
exponential constant	$e$	modulus (of $x$ )	$ x $
natural logarithm	$\ln$	log to base 10	$\log_{10}$

## 2.2 Vectors and matrices

### Vector algebra

Scalar product <sup>a</sup>	$\mathbf{a} \cdot \mathbf{b} =  \mathbf{a}  \mathbf{b} \cos\theta$	(2.1)
Vector product <sup>b</sup>	$\mathbf{a} \times \mathbf{b} =  \mathbf{a}  \mathbf{b} \sin\theta \hat{\mathbf{n}} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$	(2.2)
Product rules	$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$	(2.3)
	$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$	(2.4)
	$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b}) + (\mathbf{a} \cdot \mathbf{c})$	(2.5)
	$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$	(2.6)
Lagrange's identity	$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$	(2.7)
Scalar triple product	$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$	(2.8)
	$= (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}$	(2.9)
	$= \text{volume of parallelepiped}$	(2.10)
Vector triple product	$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$	(2.11)
	$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$	(2.12)
Reciprocal vectors	$\mathbf{a}' = (\mathbf{b} \times \mathbf{c}) / [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]$	(2.13)
	$\mathbf{b}' = (\mathbf{c} \times \mathbf{a}) / [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]$	(2.14)
	$\mathbf{c}' = (\mathbf{a} \times \mathbf{b}) / [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]$	(2.15)
	$(\mathbf{a}' \cdot \mathbf{a}) = (\mathbf{b}' \cdot \mathbf{b}) = (\mathbf{c}' \cdot \mathbf{c}) = 1$	(2.16)
Vector $\mathbf{a}$ with respect to a nonorthogonal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ <sup>c</sup>	$\mathbf{a} = (\mathbf{e}'_1 \cdot \mathbf{a})\mathbf{e}_1 + (\mathbf{e}'_2 \cdot \mathbf{a})\mathbf{e}_2 + (\mathbf{e}'_3 \cdot \mathbf{a})\mathbf{e}_3$	(2.17)

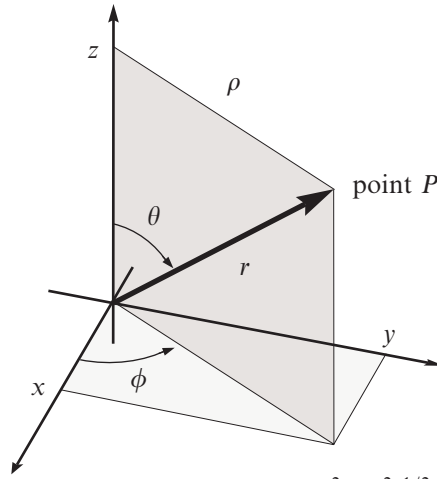


<sup>a</sup>Also known as the “dot product” or the “inner product.”

<sup>b</sup>Also known as the “cross-product.”  $\hat{\mathbf{n}}$  is a unit vector making a right-handed set with  $\mathbf{a}$  and  $\mathbf{b}$ .

<sup>c</sup>The prime (') denotes a reciprocal vector.

**Common three-dimensional coordinate systems**



$$x = \rho \cos \phi = r \sin \theta \cos \phi \quad (2.18) \qquad \rho = (x^2 + y^2)^{1/2} \quad (2.21)$$

$$y = \rho \sin \phi = r \sin \theta \sin \phi \quad (2.19) \qquad r = (x^2 + y^2 + z^2)^{1/2} \quad (2.22)$$

$$z = r \cos \theta \quad (2.20) \qquad \theta = \arccos(z/r) \quad (2.23)$$

$$\qquad \qquad \qquad \phi = \arctan(y/x) \quad (2.24)$$

coordinate system:	rectangular	spherical polar	cylindrical polar
coordinates of P:	(x, y, z)	(r, theta, phi)	(rho, phi, z)
volume element:	dx dy dz	r^2 sin theta dr dtheta dphi	rho d rho dz d phi
metric elements <sup>a</sup> (h1, h2, h3):	(1, 1, 1)	(1, r, r sin theta)	(1, rho, 1)

<sup>a</sup>In an orthogonal coordinate system (parameterised by coordinates q1, q2, q3), the differential line element dl is obtained from (dl)<sup>2</sup> = (h1 dq1)<sup>2</sup> + (h2 dq2)<sup>2</sup> + (h3 dq3)<sup>2</sup>.

**Gradient**

Rectangular coordinates	$\nabla f = \frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} + \frac{\partial f}{\partial z} \hat{z}$	(2.25)	f ^ scalar field unit vector
Cylindrical coordinates	$\nabla f = \frac{\partial f}{\partial \rho} \hat{\rho} + \frac{1}{r} \frac{\partial f}{\partial \phi} \hat{\phi} + \frac{\partial f}{\partial z} \hat{z}$	(2.26)	rho distance from the z axis
Spherical polar coordinates	$\nabla f = \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\phi}$	(2.27)	
General orthogonal coordinates	$\nabla f = \frac{\hat{q}_1}{h_1} \frac{\partial f}{\partial q_1} + \frac{\hat{q}_2}{h_2} \frac{\partial f}{\partial q_2} + \frac{\hat{q}_3}{h_3} \frac{\partial f}{\partial q_3}$	(2.28)	qi basis hi metric elements

## Divergence

Rectangular coordinates	$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad (2.29)$	$\mathbf{A}$ vector field $A_i$ $i$ th component of $\mathbf{A}$ $\rho$ distance from the $z$ axis
Cylindrical coordinates	$\nabla \cdot \mathbf{A} = \frac{1}{\rho} \frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \quad (2.30)$	
Spherical polar coordinates	$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(A_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi} \quad (2.31)$	
General orthogonal coordinates	$\nabla \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial q_1} (A_1 h_2 h_3) + \frac{\partial}{\partial q_2} (A_2 h_3 h_1) + \frac{\partial}{\partial q_3} (A_3 h_1 h_2) \right] \quad (2.32)$	$q_i$ basis $h_i$ metric elements

## Curl

Rectangular coordinates	$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix} \quad (2.33)$	$\hat{\phantom{x}}$ unit vector $\mathbf{A}$ vector field $A_i$ $i$ th component of $\mathbf{A}$ $\rho$ distance from the $z$ axis $q_i$ basis $h_i$ metric elements
Cylindrical coordinates	$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\rho}/\rho & \hat{\phi} & \hat{z}/\rho \\ \partial/\partial \rho & \partial/\partial \phi & \partial/\partial z \\ A_\rho & \rho A_\phi & A_z \end{vmatrix} \quad (2.34)$	
Spherical polar coordinates	$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{r}/(r^2 \sin \theta) & \hat{\theta}/(r \sin \theta) & \hat{\phi}/r \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial \phi \\ A_r & r A_\theta & r A_\phi \sin \theta \end{vmatrix} \quad (2.35)$	
General orthogonal coordinates	$\nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{q}_1 h_1 & \hat{q}_2 h_2 & \hat{q}_3 h_3 \\ \partial/\partial q_1 & \partial/\partial q_2 & \partial/\partial q_3 \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix} \quad (2.36)$	

## Radial forms<sup>a</sup>

$\nabla r = \frac{\mathbf{r}}{r} \quad (2.37)$	$\nabla(1/r) = \frac{-\mathbf{r}}{r^3} \quad (2.41)$
$\nabla \cdot \mathbf{r} = 3 \quad (2.38)$	$\nabla \cdot (\mathbf{r}/r^2) = \frac{1}{r^2} \quad (2.42)$
$\nabla r^2 = 2\mathbf{r} \quad (2.39)$	$\nabla(1/r^2) = \frac{-2\mathbf{r}}{r^4} \quad (2.43)$
$\nabla \cdot (r\mathbf{r}) = 4r \quad (2.40)$	$\nabla \cdot (\mathbf{r}/r^3) = 4\pi\delta(\mathbf{r}) \quad (2.44)$

<sup>a</sup>Note that the curl of any purely radial function is zero.  $\delta(\mathbf{r})$  is the Dirac delta function.





**Laplacian (scalar)**

Rectangular coordinates	$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$	(2.45)	$f$ scalar field
Cylindrical coordinates	$\nabla^2 f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$	(2.46)	$\rho$ distance from the $z$ axis
Spherical polar coordinates	$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}$	(2.47)	
General orthogonal coordinates	$\nabla^2 f = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial q_1} \left( \frac{h_2 h_3}{h_1} \frac{\partial f}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left( \frac{h_3 h_1}{h_2} \frac{\partial f}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left( \frac{h_1 h_2}{h_3} \frac{\partial f}{\partial q_3} \right) \right]$	(2.48)	$q_i$ basis $h_i$ metric elements

**Differential operator identities**

$\nabla(fg) \equiv f\nabla g + g\nabla f$	(2.49)	
$\nabla \cdot (fA) \equiv f\nabla \cdot A + A \cdot \nabla f$	(2.50)	
$\nabla \times (fA) \equiv f\nabla \times A + (\nabla f) \times A$	(2.51)	
$\nabla(A \cdot B) \equiv A \times (\nabla \times B) + (A \cdot \nabla)B + B \times (\nabla \times A) + (B \cdot \nabla)A$	(2.52)	
$\nabla \cdot (A \times B) \equiv B \cdot (\nabla \times A) - A \cdot (\nabla \times B)$	(2.53)	$f, g$ scalar fields
$\nabla \times (A \times B) \equiv A(\nabla \cdot B) - B(\nabla \cdot A) + (B \cdot \nabla)A - (A \cdot \nabla)B$	(2.54)	$A, B$ vector fields
$\nabla \cdot (\nabla f) \equiv \nabla^2 f \equiv \Delta f$	(2.55)	
$\nabla \times (\nabla f) \equiv \mathbf{0}$	(2.56)	
$\nabla \cdot (\nabla \times A) \equiv 0$	(2.57)	
$\nabla \times (\nabla \times A) \equiv \nabla(\nabla \cdot A) - \nabla^2 A$	(2.58)	

**Vector integral transformations**

Gauss's (Divergence) theorem	$\int_V (\nabla \cdot A) dV = \oint_{S_c} A \cdot ds$	(2.59)	$A$ vector field $dV$ volume element $S_c$ closed surface $V$ volume enclosed
Stokes's theorem	$\int_S (\nabla \times A) \cdot ds = \oint_L A \cdot dl$	(2.60)	$S$ surface $ds$ surface element $L$ loop bounding $S$ $dl$ line element
Green's first theorem	$\oint_S (f\nabla g) \cdot ds = \int_V \nabla \cdot (f\nabla g) dV$	(2.61)	$f, g$ scalar fields
	$= \int_V [f\nabla^2 g + (\nabla f) \cdot (\nabla g)] dV$	(2.62)	
Green's second theorem	$\oint_S [f(\nabla g) - g(\nabla f)] \cdot ds = \int_V (f\nabla^2 g - g\nabla^2 f) dV$	(2.63)	

**Matrix algebra<sup>a</sup>**

Matrix definition	$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad (2.64)$	$\mathbf{A}$ $m$ by $n$ matrix $a_{ij}$ matrix elements
Matrix addition	$\mathbf{C} = \mathbf{A} + \mathbf{B} \quad \text{if } c_{ij} = a_{ij} + b_{ij} \quad (2.65)$	
Matrix multiplication	$\mathbf{C} = \mathbf{AB} \quad \text{if } c_{ij} = a_{ik}b_{kj} \quad (2.66)$	
	$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) \quad (2.67)$	
	$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC} \quad (2.68)$	
Transpose matrix <sup>b</sup>	$\tilde{a}_{ij} = a_{ji} \quad (2.69)$	$\tilde{a}_{ij}$ transpose matrix (sometimes $a_{ij}^T$ , or $a'_{ij}$ )
	$(\mathbf{AB}\dots\mathbf{N}) = \tilde{\mathbf{N}}\dots\tilde{\mathbf{B}}\tilde{\mathbf{A}} \quad (2.70)$	
Adjoint matrix (definition 1) <sup>c</sup>	$\mathbf{A}^\dagger = \tilde{\mathbf{A}}^* \quad (2.71)$	* complex conjugate (of each component) † adjoint (or Hermitian conjugate)
	$(\mathbf{AB}\dots\mathbf{N})^\dagger = \mathbf{N}^\dagger \dots \mathbf{B}^\dagger \mathbf{A}^\dagger \quad (2.72)$	
Hermitian matrix <sup>d</sup>	$\mathbf{H}^\dagger = \mathbf{H} \quad (2.73)$	$\mathbf{H}$ Hermitian (or self-adjoint) matrix
examples:		
$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$	$\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$	
$\tilde{\mathbf{A}} = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$	$\mathbf{A} + \mathbf{B} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & a_{13} + b_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \\ a_{31} + b_{31} & a_{32} + b_{32} & a_{33} + b_{33} \end{pmatrix}$	
$\mathbf{AB} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{pmatrix}$		

<sup>a</sup>Terms are implicitly summed over repeated suffices; hence  $a_{ik}b_{kj}$  equals  $\sum_k a_{ik}b_{kj}$ .

<sup>b</sup>See also Equation (2.85).

<sup>c</sup>Or “Hermitian conjugate matrix.” The term “adjoint” is used in quantum physics for the transpose conjugate of a matrix and in linear algebra for the transpose matrix of its cofactors. These definitions are not compatible, but both are widely used [cf. Equation (2.80)].

<sup>d</sup>Hermitian matrices must also be square (see next table).



Square matrices<sup>a</sup>

Trace	$\text{tr} \mathbf{A} = a_{ii}$ (2.74)	$\mathbf{A}$ square matrix $a_{ij}$ matrix elements $a_{ii}$ implicitly = $\sum_i a_{ii}$
	$\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA})$ (2.75)	
Determinant <sup>b</sup>	$\det \mathbf{A} = \epsilon_{ijk\dots} a_{1i} a_{2j} a_{3k} \dots$ (2.76)	tr trace det determinant (or $ \mathbf{A} $ ) $M_{ij}$ minor of element $a_{ij}$ $C_{ij}$ cofactor of the element $a_{ij}$
	$= (-1)^{i+1} a_{i1} M_{i1}$ (2.77)	
	$= a_{i1} C_{i1}$ (2.78)	
	$\det(\mathbf{AB} \dots \mathbf{N}) = \det \mathbf{A} \det \mathbf{B} \dots \det \mathbf{N}$ (2.79)	
Adjoint matrix (definition 2) <sup>c</sup>	$\text{adj} \mathbf{A} = \tilde{C}_{ij} = C_{ji}$ (2.80)	adj adjoint (sometimes written $\hat{\mathbf{A}}$ ) ~ transpose
Inverse matrix ( $\det \mathbf{A} \neq 0$ )	$a_{ij}^{-1} = \frac{C_{ji}}{\det \mathbf{A}} = \frac{\text{adj} \mathbf{A}}{\det \mathbf{A}}$ (2.81)	$\mathbf{1}$ unit matrix
	$\mathbf{AA}^{-1} = \mathbf{1}$ (2.82)	
	$(\mathbf{AB} \dots \mathbf{N})^{-1} = \mathbf{N}^{-1} \dots \mathbf{B}^{-1} \mathbf{A}^{-1}$ (2.83)	
Orthogonality condition	$a_{ij} a_{ik} = \delta_{jk}$ (2.84)	$\delta_{jk}$ Kronecker delta (= 1 if $i = j$ , = 0 otherwise)
	i.e., $\tilde{\mathbf{A}} = \mathbf{A}^{-1}$ (2.85)	
Symmetry	If $\mathbf{A} = \tilde{\mathbf{A}}$ , $\mathbf{A}$ is symmetric (2.86)	
	If $\mathbf{A} = -\tilde{\mathbf{A}}$ , $\mathbf{A}$ is antisymmetric (2.87)	
Unitary matrix	$\mathbf{U}^\dagger = \mathbf{U}^{-1}$ (2.88)	$\mathbf{U}$ unitary matrix † Hermitian conjugate
examples:		
$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$	$\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$	
$\text{tr} \mathbf{A} = a_{11} + a_{22} + a_{33}$	$\text{tr} \mathbf{B} = b_{11} + b_{22}$	
$\det \mathbf{A} = a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{21} a_{12} a_{33} + a_{21} a_{13} a_{32} + a_{31} a_{12} a_{23} - a_{31} a_{13} a_{22}$		
$\det \mathbf{B} = b_{11} b_{22} - b_{12} b_{21}$		
$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} a_{22} a_{33} - a_{23} a_{32} & -a_{12} a_{33} + a_{13} a_{32} & a_{12} a_{23} - a_{13} a_{22} \\ -a_{21} a_{33} + a_{23} a_{31} & a_{11} a_{33} - a_{13} a_{31} & -a_{11} a_{23} + a_{13} a_{21} \\ a_{21} a_{32} - a_{22} a_{31} & -a_{11} a_{32} + a_{12} a_{31} & a_{11} a_{22} - a_{12} a_{21} \end{pmatrix}$		
$\mathbf{B}^{-1} = \frac{1}{\det \mathbf{B}} \begin{pmatrix} b_{22} & -b_{12} \\ -b_{21} & b_{11} \end{pmatrix}$		

<sup>a</sup>Terms are implicitly summed over repeated suffices; hence  $a_{ik} b_{kj}$  equals  $\sum_k a_{ik} b_{kj}$ .  
<sup>b</sup> $\epsilon_{ijk\dots}$  is defined as the natural extension of Equation (2.443) to  $n$ -dimensions (see page 50).  $M_{ij}$  is the determinant of the matrix  $\mathbf{A}$  with the  $i$ th row and the  $j$ th column deleted. The cofactor  $C_{ij} = (-1)^{i+j} M_{ij}$ .  
<sup>c</sup>Or “adjugate matrix.” See the footnote to Equation (2.71) for a discussion of the term “adjoint.”

**Commutators**

Commutator definition	$[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA} = -[\mathbf{B}, \mathbf{A}]$	(2.89)	$[\cdot, \cdot]$ commutator
Adjoint	$[\mathbf{A}, \mathbf{B}]^\dagger = [\mathbf{B}^\dagger, \mathbf{A}^\dagger]$	(2.90)	$\dagger$ adjoint
Distribution	$[\mathbf{A} + \mathbf{B}, \mathbf{C}] = [\mathbf{A}, \mathbf{C}] + [\mathbf{B}, \mathbf{C}]$	(2.91)	
Association	$[\mathbf{AB}, \mathbf{C}] = \mathbf{A}[\mathbf{B}, \mathbf{C}] + [\mathbf{A}, \mathbf{C}]\mathbf{B}$	(2.92)	
Jacobi identity	$[\mathbf{A}, [\mathbf{B}, \mathbf{C}]] = [\mathbf{B}, [\mathbf{A}, \mathbf{C}]] - [\mathbf{C}, [\mathbf{A}, \mathbf{B}]]$	(2.93)	

**Pauli matrices**

Pauli matrices	$\begin{aligned} \sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \sigma_2 &= \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix} \\ \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \mathbf{1} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$	(2.94)	$\sigma_i$ Pauli spin matrices $\mathbf{1}$ $2 \times 2$ unit matrix $\mathbf{i}$ $\mathbf{i}^2 = -1$
Anticommutation	$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij} \mathbf{1}$	(2.95)	$\delta_{ij}$ Kronecker delta
Cyclic permutation	$\sigma_i \sigma_j = \mathbf{i} \sigma_k$	(2.96)	
	$(\sigma_i)^2 = \mathbf{1}$	(2.97)	

**Rotation matrices<sup>a</sup>**

Rotation about $x_1$	$\mathbf{R}_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix}$	(2.98)	$\mathbf{R}_i(\theta)$ matrix for rotation about the $i$ th axis $\theta$ rotation angle
Rotation about $x_2$	$\mathbf{R}_2(\theta) = \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix}$	(2.99)	
Rotation about $x_3$	$\mathbf{R}_3(\theta) = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$	(2.100)	$\alpha$ rotation about $x_3$ $\beta$ rotation about $x'_2$ $\gamma$ rotation about $x''_3$
Euler angles			$\mathbf{R}$ rotation matrix
	$\mathbf{R}(\alpha, \beta, \gamma) = \begin{pmatrix} \cos \gamma \cos \beta \cos \alpha - \sin \gamma \sin \alpha & \cos \gamma \cos \beta \sin \alpha + \sin \gamma \cos \alpha & -\cos \gamma \sin \beta \\ -\sin \gamma \cos \beta \cos \alpha - \cos \gamma \sin \alpha & -\sin \gamma \cos \beta \sin \alpha + \cos \gamma \cos \alpha & \sin \gamma \sin \beta \\ \sin \beta \cos \alpha & \sin \beta \sin \alpha & \cos \beta \end{pmatrix}$		(2.101)

<sup>a</sup>Angles are in the right-handed sense for rotation of axes, or the left-handed sense for rotation of vectors. i.e., a vector  $\mathbf{v}$  is given a right-handed rotation of  $\theta$  about the  $x_3$ -axis using  $\mathbf{R}_3(-\theta)\mathbf{v} \mapsto \mathbf{v}'$ . Conventionally,  $x_1 \equiv x$ ,  $x_2 \equiv y$ , and  $x_3 \equiv z$ .

## 2.3 Series, summations, and progressions

### Progressions and summations

Arithmetic progression	$S_n = a + (a + d) + (a + 2d) + \dots$	(2.102)	$n$ number of terms $S_n$ sum of $n$ successive terms $a$ first term $d$ common difference $l$ last term	
	$+ [a + (n - 1)d]$			
	$= \frac{n}{2}[2a + (n - 1)d]$			(2.103)
	$= \frac{n}{2}(a + l)$			(2.104)
Geometric progression	$S_n = a + ar + ar^2 + \dots + ar^{n-1}$	(2.105)	$r$ common ratio	
	$= a \frac{1 - r^n}{1 - r}$	(2.106)		
	$S_\infty = \frac{a}{1 - r} \quad ( r  < 1)$	(2.107)		
Arithmetic mean	$\langle x \rangle_a = \frac{1}{n}(x_1 + x_2 + \dots + x_n)$	(2.108)	$\langle \cdot \rangle_a$ arithmetic mean	
Geometric mean	$\langle x \rangle_g = (x_1 x_2 x_3 \dots x_n)^{1/n}$	(2.109)	$\langle \cdot \rangle_g$ geometric mean	
Harmonic mean	$\langle x \rangle_h = n \left( \frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} \right)^{-1}$	(2.110)	$\langle \cdot \rangle_h$ harmonic mean	
Relative mean magnitudes	$\langle x \rangle_a \geq \langle x \rangle_g \geq \langle x \rangle_h \quad \text{if } x_i > 0 \text{ for all } i$	(2.111)		
Summation formulas	$\sum_{i=1}^n i = \frac{n}{2}(n + 1)$	(2.112)	$i$ dummy integer	
	$\sum_{i=1}^n i^2 = \frac{n}{6}(n + 1)(2n + 1)$	(2.113)		
	$\sum_{i=1}^n i^3 = \frac{n^2}{4}(n + 1)^2$	(2.114)		
	$\sum_{i=1}^n i^4 = \frac{n}{30}(n + 1)(2n + 1)(3n^2 + 3n - 1)$	(2.115)		
	$\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \ln 2$	(2.116)		
	$\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{2i - 1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$	(2.117)		
	$\sum_{i=1}^{\infty} \frac{1}{i^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots = \frac{\pi^2}{6}$	(2.118)		
	Euler's constant <sup>a</sup>	$\gamma = \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n \right)$		(2.119)

<sup>a</sup> $\gamma \simeq 0.577215664\dots$



**Power series**

Binomial series <sup>a</sup>	$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$	(2.120)
Binomial coefficient <sup>b</sup>	${}^nC_r \equiv \binom{n}{r} \equiv \frac{n!}{r!(n-r)!}$	(2.121)
Binomial theorem	$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$	(2.122)
Taylor series (about $a$ ) <sup>c</sup>	$f(a+x) = f(a) + xf^{(1)}(a) + \frac{x^2}{2!}f^{(2)}(a) + \dots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(a) + \dots$	(2.123)
Taylor series (3-D)	$f(\mathbf{a} + \mathbf{x}) = f(\mathbf{a}) + (\mathbf{x} \cdot \nabla)f _a + \frac{(\mathbf{x} \cdot \nabla)^2}{2!}f _a + \frac{(\mathbf{x} \cdot \nabla)^3}{3!}f _a + \dots$	(2.124)
Maclaurin series	$f(x) = f(0) + xf^{(1)}(0) + \frac{x^2}{2!}f^{(2)}(0) + \dots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \dots$	(2.125)

<sup>a</sup>If  $n$  is a positive integer the series terminates and is valid for all  $x$ . Otherwise the (infinite) series is convergent for  $|x| < 1$ .

<sup>b</sup>The coefficient of  $x^r$  in the binomial series.

<sup>c</sup> $f^{(n)}(a)$  is  $x$  times the  $n$ th derivative of the function  $f(x)$  with respect to  $x$  evaluated at  $a$ , taken as well behaved around  $a$ .  $(\mathbf{x} \cdot \nabla)^n f|_a$  is its extension to three dimensions.

**Limits**

$n^c x^n \rightarrow 0$ as $n \rightarrow \infty$ if $ x  < 1$ (for any fixed $c$ )	(2.126)
$x^n/n! \rightarrow 0$ as $n \rightarrow \infty$ (for any fixed $x$ )	(2.127)
$(1+x/n)^n \rightarrow e^x$ as $n \rightarrow \infty$	(2.128)
$x \ln x \rightarrow 0$ as $x \rightarrow 0$	(2.129)
$\frac{\sin x}{x} \rightarrow 1$ as $x \rightarrow 0$	(2.130)
If $f(a) = g(a) = 0$ or $\infty$ then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f^{(1)}(a)}{g^{(1)}(a)}$ (l'Hôpital's rule)	(2.131)

**Series expansions**

$\exp(x)$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$	(2.132)	(for all $x$ )
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$	(2.133)	$(-1 < x \leq 1)$
$\ln\left(\frac{1+x}{1-x}\right)$	$2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots\right)$	(2.134)	$( x  < 1)$
$\cos(x)$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$	(2.135)	(for all $x$ )
$\sin(x)$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$	(2.136)	(for all $x$ )
$\tan(x)$	$x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} \dots$	(2.137)	$( x  < \pi/2)$
$\sec(x)$	$1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \dots$	(2.138)	$( x  < \pi/2)$
$\csc(x)$	$\frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \frac{31x^5}{15120} + \dots$	(2.139)	$( x  < \pi)$
$\cot(x)$	$\frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \dots$	(2.140)	$( x  < \pi)$
$\arcsin(x)^a$	$x + \frac{1}{2} \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{x^7}{7} \dots$	(2.141)	$( x  < 1)$
$\arctan(x)^b$	$\begin{cases} x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots & ( x  \leq 1) \\ \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \dots & (x > 1) \\ -\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \dots & (x < -1) \end{cases}$	(2.142)	
$\cosh(x)$	$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$	(2.143)	(for all $x$ )
$\sinh(x)$	$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$	(2.144)	(for all $x$ )
$\tanh(x)$	$x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \dots$	(2.145)	$( x  < \pi/2)$

<sup>a</sup> $\arccos(x) = \pi/2 - \arcsin(x)$ . Note that  $\arcsin(x) \equiv \sin^{-1}(x)$  etc.

<sup>b</sup> $\operatorname{arccot}(x) = \pi/2 - \arctan(x)$ .

### Inequalities

Triangle inequality	$ a_1  -  a_2  \leq  a_1 + a_2  \leq  a_1  +  a_2 ;$	(2.146)
	$\left  \sum_{i=1}^n a_i \right  \leq \sum_{i=1}^n  a_i $	(2.147)
Chebyshev inequality	if $a_1 \geq a_2 \geq a_3 \geq \dots \geq a_n$	(2.148)
	and $b_1 \geq b_2 \geq b_3 \geq \dots \geq b_n$	(2.149)
	then $n \sum_{i=1}^n a_i b_i \geq \left( \sum_{i=1}^n a_i \right) \left( \sum_{i=1}^n b_i \right)$	(2.150)
Cauchy inequality	$\left( \sum_{i=1}^n a_i b_i \right)^2 \leq \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2$	(2.151)
Schwarz inequality	$\left[ \int_a^b f(x)g(x) dx \right]^2 \leq \int_a^b [f(x)]^2 dx \int_a^b [g(x)]^2 dx$	(2.152)

## 2.4 Complex variables

### Complex numbers

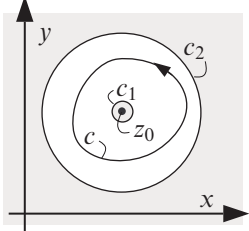
Cartesian form	$z = x + iy$	(2.153)	$z$ complex variable $i$ $i^2 = -1$ $x, y$ real variables
Polar form	$z = re^{i\theta} = r(\cos \theta + i \sin \theta)$	(2.154)	$r$ amplitude (real) $\theta$ phase (real)
Modulus <sup>a</sup>	$ z  = r = (x^2 + y^2)^{1/2}$	(2.155)	$ z $ modulus of $z$
	$ z_1 \cdot z_2  =  z_1  \cdot  z_2 $	(2.156)	
Argument	$\theta = \arg z = \arctan \frac{y}{x}$	(2.157)	$\arg z$ argument of $z$
	$\arg(z_1 z_2) = \arg z_1 + \arg z_2$	(2.158)	
Complex conjugate	$z^* = x - iy = re^{-i\theta}$	(2.159)	$z^*$ complex conjugate of $z = re^{i\theta}$
	$\arg(z^*) = -\arg z$	(2.160)	
	$z \cdot z^* =  z ^2$	(2.161)	
Logarithm <sup>b</sup>	$\ln z = \ln r + i(\theta + 2\pi n)$	(2.162)	$n$ integer

<sup>a</sup>Or "magnitude."

<sup>b</sup>The principal value of  $\ln z$  is given by  $n=0$  and  $-\pi < \theta \leq \pi$ .



**Complex analysis<sup>a</sup>**

Cauchy–Riemann equations <sup>b</sup>	if $f(z) = u(x, y) + iv(x, y)$ then $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ (2.163) $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (2.164)	$z$ complex variable $i$ $i^2 = -1$ $x, y$ real variables $f(z)$ function of $z$ $u, v$ real functions
Cauchy–Goursat theorem <sup>c</sup>	$\oint_c f(z) dz = 0$ (2.165)	
Cauchy integral formula <sup>d</sup>	$f(z_0) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z - z_0} dz$ (2.166) $f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_c \frac{f(z)}{(z - z_0)^{n+1}} dz$ (2.167)	<sup>(n)</sup> $n$ th derivative $a_n$ Laurent coefficients $a_{-1}$ residue of $f(z)$ at $z_0$ $z'$ dummy variable
Laurent expansion <sup>e</sup>	$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ (2.168) where $a_n = \frac{1}{2\pi i} \oint_c \frac{f(z')}{(z' - z_0)^{n+1}} dz'$ (2.169)	
Residue theorem	$\oint_c f(z) dz = 2\pi i \sum$ enclosed residues (2.170)	

<sup>a</sup>Closed contour integrals are taken in the counterclockwise sense, once.

<sup>b</sup>Necessary condition for  $f(z)$  to be analytic at a given point.

<sup>c</sup>If  $f(z)$  is analytic within and on a simple closed curve  $c$ . Sometimes called “Cauchy’s theorem.”

<sup>d</sup>If  $f(z)$  is analytic within and on a simple closed curve  $c$ , encircling  $z_0$ .

<sup>e</sup>Of  $f(z)$ , (analytic) in the annular region between concentric circles,  $c_1$  and  $c_2$ , centred on  $z_0$ .  $c$  is any closed curve in this region encircling  $z_0$ .

## 2.5 Trigonometric and hyperbolic formulas

### Trigonometric relationships

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \quad (2.171)$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \quad (2.172)$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \quad (2.173)$$

$$\cos A \cos B = \frac{1}{2} [\cos(A+B) + \cos(A-B)] \quad (2.174)$$

$$\sin A \cos B = \frac{1}{2} [\sin(A+B) + \sin(A-B)] \quad (2.175)$$

$$\sin A \sin B = \frac{1}{2} [\cos(A-B) - \cos(A+B)] \quad (2.176)$$

$$\cos^2 A + \sin^2 A = 1 \quad (2.177)$$

$$\sec^2 A - \tan^2 A = 1 \quad (2.178)$$

$$\csc^2 A - \cot^2 A = 1 \quad (2.179)$$

$$\sin 2A = 2 \sin A \cos A \quad (2.180)$$

$$\cos 2A = \cos^2 A - \sin^2 A \quad (2.181)$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A} \quad (2.182)$$

$$\sin 3A = 3 \sin A - 4 \sin^3 A \quad (2.183)$$

$$\cos 3A = 4 \cos^3 A - 3 \cos A \quad (2.184)$$

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} \quad (2.185)$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2} \quad (2.186)$$

$$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} \quad (2.187)$$

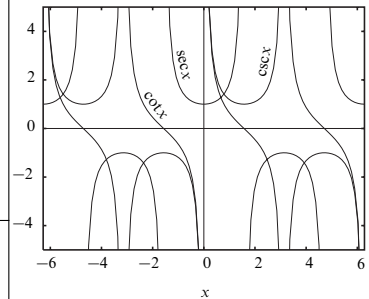
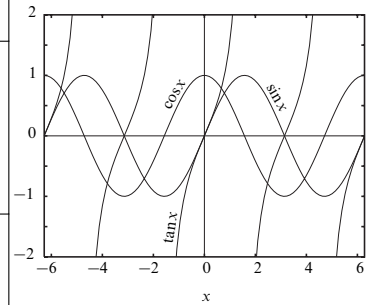
$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2} \quad (2.188)$$

$$\cos^2 A = \frac{1}{2} (1 + \cos 2A) \quad (2.189)$$

$$\sin^2 A = \frac{1}{2} (1 - \cos 2A) \quad (2.190)$$

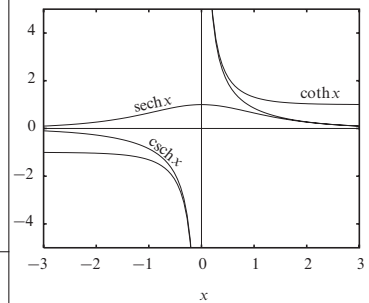
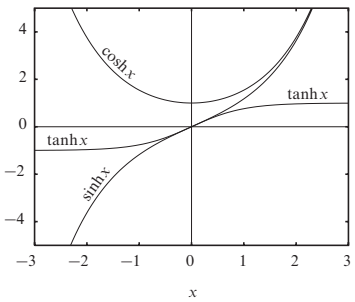
$$\cos^3 A = \frac{1}{4} (3 \cos A + \cos 3A) \quad (2.191)$$

$$\sin^3 A = \frac{1}{4} (3 \sin A - \sin 3A) \quad (2.192)$$



**Hyperbolic relationships<sup>a</sup>**

$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$	(2.193)
$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$	(2.194)
$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$	(2.195)
$\cosh x \cosh y = \frac{1}{2} [\cosh(x + y) + \cosh(x - y)]$	(2.196)
$\sinh x \cosh y = \frac{1}{2} [\sinh(x + y) + \sinh(x - y)]$	(2.197)
$\sinh x \sinh y = \frac{1}{2} [\cosh(x + y) - \cosh(x - y)]$	(2.198)
$\cosh^2 x - \sinh^2 x = 1$	(2.199)
$\operatorname{sech}^2 x + \tanh^2 x = 1$	(2.200)
$\operatorname{coth}^2 x - \operatorname{csch}^2 x = 1$	(2.201)
$\sinh 2x = 2 \sinh x \cosh x$	(2.202)
$\cosh 2x = \cosh^2 x + \sinh^2 x$	(2.203)
$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$	(2.204)
$\sinh 3x = 3 \sinh x + 4 \sinh^3 x$	(2.205)
$\cosh 3x = 4 \cosh^3 x - 3 \cosh x$	(2.206)
$\sinh x + \sinh y = 2 \sinh \frac{x+y}{2} \cosh \frac{x-y}{2}$	(2.207)
$\sinh x - \sinh y = 2 \cosh \frac{x+y}{2} \sinh \frac{x-y}{2}$	(2.208)
$\cosh x + \cosh y = 2 \cosh \frac{x+y}{2} \cosh \frac{x-y}{2}$	(2.209)
$\cosh x - \cosh y = 2 \sinh \frac{x+y}{2} \sinh \frac{x-y}{2}$	(2.210)
$\cosh^2 x = \frac{1}{2} (\cosh 2x + 1)$	(2.211)
$\sinh^2 x = \frac{1}{2} (\cosh 2x - 1)$	(2.212)
$\cosh^3 x = \frac{1}{4} (3 \cosh x + \cosh 3x)$	(2.213)
$\sinh^3 x = \frac{1}{4} (\sinh 3x - 3 \sinh x)$	(2.214)



<sup>a</sup>These can be derived from trigonometric relationships by using the substitutions  $\cos x \mapsto \cosh x$  and  $\sin x \mapsto i \sinh x$ .

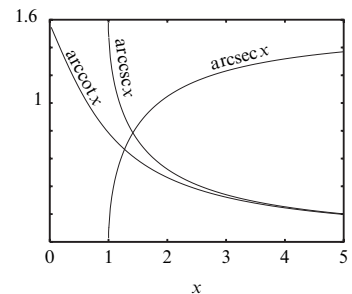
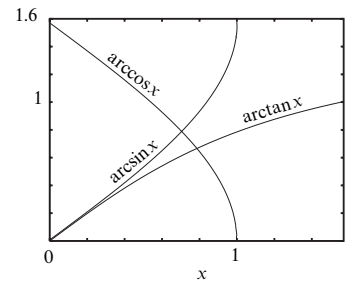
### Trigonometric and hyperbolic definitions

de Moivre's theorem	$(\cos x + i \sin x)^n = e^{inx} = \cos nx + i \sin nx$	(2.215)	
$\cos x = \frac{1}{2} (e^{ix} + e^{-ix})$	(2.216)	$\cosh x = \frac{1}{2} (e^x + e^{-x})$	(2.217)
$\sin x = \frac{1}{2i} (e^{ix} - e^{-ix})$	(2.218)	$\sinh x = \frac{1}{2} (e^x - e^{-x})$	(2.219)
$\tan x = \frac{\sin x}{\cos x}$	(2.220)	$\tanh x = \frac{\sinh x}{\cosh x}$	(2.221)
$\cos ix = \cosh x$	(2.222)	$\cosh ix = \cos x$	(2.223)
$\sin ix = i \sinh x$	(2.224)	$\sinh ix = i \sin x$	(2.225)
$\cot x = (\tan x)^{-1}$	(2.226)	$\coth x = (\tanh x)^{-1}$	(2.227)
$\sec x = (\cos x)^{-1}$	(2.228)	$\operatorname{sech} x = (\cosh x)^{-1}$	(2.229)
$\csc x = (\sin x)^{-1}$	(2.230)	$\operatorname{csch} x = (\sinh x)^{-1}$	(2.231)

### Inverse trigonometric functions<sup>a</sup>

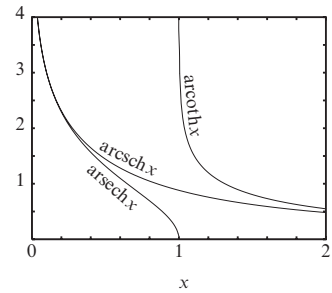
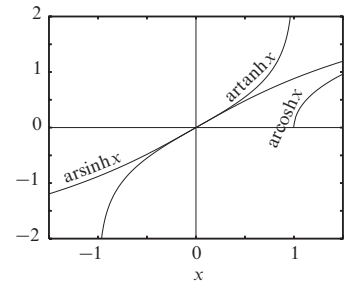
$\arcsin x = \arctan \left[ \frac{x}{(1-x^2)^{1/2}} \right]$	(2.232)
$\arccos x = \arctan \left[ \frac{(1-x^2)^{1/2}}{x} \right]$	(2.233)
$\operatorname{arccsc} x = \arctan \left[ \frac{1}{(x^2-1)^{1/2}} \right]$	(2.234)
$\operatorname{arcsec} x = \arctan \left[ (x^2-1)^{1/2} \right]$	(2.235)
$\operatorname{arccot} x = \arctan \left( \frac{1}{x} \right)$	(2.236)
$\arccos x = \frac{\pi}{2} - \arcsin x$	(2.237)

<sup>a</sup>Valid in the angle range  $0 \leq \theta \leq \pi/2$ . Note that  $\arcsin x \equiv \sin^{-1} x$  etc.



**Inverse hyperbolic functions**

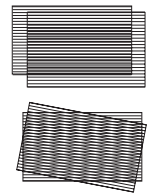
$\operatorname{arsinh} x \equiv \sinh^{-1} x = \ln \left[ x + (x^2 + 1)^{1/2} \right]$ (2.238)	for all $x$
$\operatorname{arcosh} x \equiv \cosh^{-1} x = \ln \left[ x + (x^2 - 1)^{1/2} \right]$ (2.239)	$x \geq 1$
$\operatorname{artanh} x \equiv \tanh^{-1} x = \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right)$ (2.240)	$ x  < 1$
$\operatorname{arcoth} x \equiv \coth^{-1} x = \frac{1}{2} \ln \left( \frac{x+1}{x-1} \right)$ (2.241)	$ x  > 1$
$\operatorname{arsech} x \equiv \operatorname{sech}^{-1} x = \ln \left[ \frac{1}{x} + \frac{(1-x^2)^{1/2}}{x} \right]$ (2.242)	$0 < x \leq 1$
$\operatorname{arcsch} x \equiv \operatorname{csch}^{-1} x = \ln \left[ \frac{1}{x} + \frac{(1+x^2)^{1/2}}{x} \right]$ (2.243)	$x \neq 0$



**2.6 Mensuration**

**Moiré fringes<sup>a</sup>**

Parallel pattern	$d_M = \left  \frac{1}{d_1} - \frac{1}{d_2} \right ^{-1}$ (2.244)	$d_M$ Moiré fringe spacing $d_{1,2}$ grating spacings
Rotational pattern <sup>b</sup>	$d_M = \frac{d}{2 \sin(\theta/2) }$ (2.245)	$d$ common grating spacing $\theta$ relative rotation angle ( $ \theta  \leq \pi/2$ )

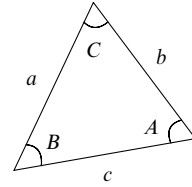


<sup>a</sup>From overlapping linear gratings.

<sup>b</sup>From identical gratings, spacing  $d$ , with a relative rotation  $\theta$ .

**Plane triangles**

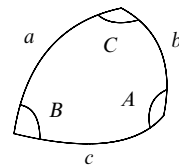
Sine formula <sup>a</sup>	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$	(2.246)
Cosine formulas	$a^2 = b^2 + c^2 - 2bc \cos A$	(2.247)
	$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$	(2.248)
	$a = b \cos C + c \cos B$	(2.249)
Tangent formula	$\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}$	(2.250)
Area	area = $\frac{1}{2} ab \sin C$	(2.251)
	= $\frac{a^2 \sin B \sin C}{2 \sin A}$	(2.252)
	= $[s(s-a)(s-b)(s-c)]^{1/2}$	(2.253)
	where $s = \frac{1}{2}(a+b+c)$	(2.254)



<sup>a</sup>The diameter of the circumscribed circle equals  $a/\sin A$ .

**Spherical triangles<sup>a</sup>**

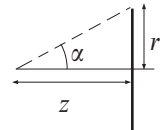
Sine formula	$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$	(2.255)
Cosine formulas	$\cos a = \cos b \cos c + \sin b \sin c \cos A$	(2.256)
	$\cos A = -\cos B \cos C + \sin B \sin C \cos a$	(2.257)
Analogue formula	$\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A$	(2.258)
Four-parts formula	$\cos a \cos C = \sin a \cot b - \sin C \cot B$	(2.259)
Area <sup>b</sup>	$E = A + B + C - \pi$	(2.260)



<sup>a</sup>On a unit sphere.

<sup>b</sup>Also called the "spherical excess."

**Perimeter, area, and volume**

Perimeter of circle	$P = 2\pi r$	(2.261)	$P$ perimeter $r$ radius
Area of circle	$A = \pi r^2$	(2.262)	$A$ area
Surface area of sphere <sup>a</sup>	$A = 4\pi R^2$	(2.263)	$R$ sphere radius
Volume of sphere	$V = \frac{4}{3}\pi R^3$	(2.264)	$V$ volume
Perimeter of ellipse <sup>b</sup>	$P = 4aE(\pi/2, e)$	(2.265)	$a$ semi-major axis $b$ semi-minor axis $E$ elliptic integral of the second kind (p. 45) $e$ eccentricity ( $= 1 - b^2/a^2$ )
	$\simeq 2\pi \left( \frac{a^2 + b^2}{2} \right)^{1/2}$	(2.266)	
Area of ellipse	$A = \pi ab$	(2.267)	
Volume of ellipsoid <sup>c</sup>	$V = 4\pi \frac{abc}{3}$	(2.268)	$c$ third semi-axis
Surface area of cylinder	$A = 2\pi r(h + r)$	(2.269)	$h$ height
Volume of cylinder	$V = \pi r^2 h$	(2.270)	
Area of circular cone <sup>d</sup>	$A = \pi rl$	(2.271)	$l$ slant height
Volume of cone or pyramid	$V = A_b h/3$	(2.272)	$A_b$ base area
Surface area of torus	$A = \pi^2(r_1 + r_2)(r_2 - r_1)$	(2.273)	$r_1$ inner radius $r_2$ outer radius
Volume of torus	$V = \frac{\pi^2}{4}(r_2^2 - r_1^2)(r_2 - r_1)$	(2.274)	
Area <sup>d</sup> of spherical cap, depth $d$	$A = 2\pi R d$	(2.275)	$d$ cap depth
Volume of spherical cap, depth $d$	$V = \pi d^2 \left( R - \frac{d}{3} \right)$	(2.276)	$\Omega$ solid angle $z$ distance from centre $\alpha$ half-angle subtended
Solid angle of a circle from a point on its axis, $z$ from centre	$\Omega = 2\pi \left[ 1 - \frac{z}{(z^2 + r^2)^{1/2}} \right]$	(2.277)	
	$= 2\pi(1 - \cos \alpha)$	(2.278)	

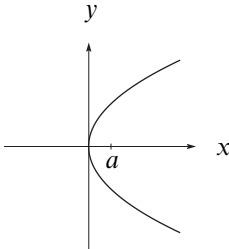
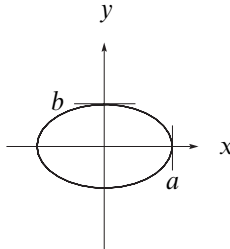
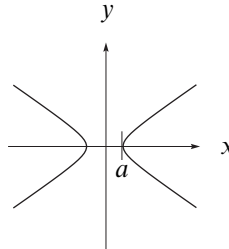
<sup>a</sup>Sphere defined by  $x^2 + y^2 + z^2 = R^2$ .

<sup>b</sup>The approximation is exact when  $e=0$  and  $e \simeq 0.91$ , giving a maximum error of 11% at  $e = 1$ .

<sup>c</sup>Ellipsoid defined by  $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ .

<sup>d</sup>Curved surface only.

## Conic sections

			
	<i>parabola</i>	<i>ellipse</i>	<i>hyperbola</i>
equation	$y^2 = 4ax$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
parametric form	$x = t^2/(4a)$ $y = t$	$x = a \cos t$ $y = b \sin t$	$x = \pm a \cosh t$ $y = b \sinh t$
foci	$(a, 0)$	$(\pm\sqrt{a^2 - b^2}, 0)$	$(\pm\sqrt{a^2 + b^2}, 0)$
eccentricity	$e = 1$	$e = \frac{\sqrt{a^2 - b^2}}{a}$	$e = \frac{\sqrt{a^2 + b^2}}{a}$
directrices	$x = -a$	$x = \pm \frac{a}{e}$	$x = \pm \frac{a}{e}$

Platonic solids<sup>a</sup>

<i>solid</i> <i>(faces, edges, vertices)</i>	<i>volume</i>	<i>surface area</i>	<i>circumradius</i>	<i>inradius</i>
tetrahedron (4, 6, 4)	$\frac{a^3 \sqrt{2}}{12}$	$a^2 \sqrt{3}$	$\frac{a\sqrt{6}}{4}$	$\frac{a\sqrt{6}}{12}$
cube (6, 12, 8)	$a^3$	$6a^2$	$\frac{a\sqrt{3}}{2}$	$\frac{a}{2}$
octahedron (8, 12, 6)	$\frac{a^3 \sqrt{2}}{3}$	$2a^2 \sqrt{3}$	$\frac{a}{\sqrt{2}}$	$\frac{a}{\sqrt{6}}$
dodecahedron (12, 30, 20)	$\frac{a^3(15 + 7\sqrt{5})}{4}$	$3a^2 \sqrt{5(5 + 2\sqrt{5})}$	$\frac{a}{4} \sqrt{3}(1 + \sqrt{5})$	$\frac{a}{4} \sqrt{\frac{50 + 22\sqrt{5}}{5}}$
icosahedron (20, 30, 12)	$\frac{5a^3(3 + \sqrt{5})}{12}$	$5a^2 \sqrt{3}$	$\frac{a}{4} \sqrt{2(5 + \sqrt{5})}$	$\frac{a}{4} \left( \sqrt{3} + \sqrt{\frac{5}{3}} \right)$

<sup>a</sup>Of side  $a$ . Both regular and irregular polyhedra follow the Euler relation, faces – edges + vertices = 2.



**Curve measure**

Length of plane curve	$l = \int_a^b \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{1/2} dx \quad (2.279)$	$a$ start point $b$ end point $y(x)$ plane curve $l$ length
Surface of revolution	$A = 2\pi \int_a^b y \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{1/2} dx \quad (2.280)$	$A$ surface area
Volume of revolution	$V = \pi \int_a^b y^2 dx \quad (2.281)$	$V$ volume
Radius of curvature	$\rho = \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{3/2} \left( \frac{d^2y}{dx^2} \right)^{-1} \quad (2.282)$	$\rho$ radius of curvature

**Differential geometry<sup>a</sup>**

Unit tangent	$\hat{\tau} = \frac{\dot{\mathbf{r}}}{ \dot{\mathbf{r}} } = \frac{\dot{\mathbf{r}}}{v} \quad (2.283)$	$\tau$ tangent $r$ curve parameterised by $r(t)$ $v$ $ \dot{\mathbf{r}}(t) $
Unit principal normal	$\hat{\mathbf{n}} = \frac{\ddot{\mathbf{r}} - \dot{v}\hat{\tau}}{ \ddot{\mathbf{r}} - \dot{v}\hat{\tau} } \quad (2.284)$	$\mathbf{n}$ principal normal
Unit binormal	$\hat{\mathbf{b}} = \hat{\tau} \times \hat{\mathbf{n}} \quad (2.285)$	$\mathbf{b}$ binormal
Curvature	$\kappa = \frac{ \dot{\mathbf{r}} \times \ddot{\mathbf{r}} }{ \dot{\mathbf{r}} ^3} \quad (2.286)$	$\kappa$ curvature
Radius of curvature	$\rho = \frac{1}{\kappa} \quad (2.287)$	$\rho$ radius of curvature
Torsion	$\lambda = \frac{\dot{\mathbf{r}} \cdot (\ddot{\mathbf{r}} \times \ddot{\ddot{\mathbf{r}}})}{ \dot{\mathbf{r}} \times \ddot{\mathbf{r}} ^2} \quad (2.288)$	$\lambda$ torsion
Frenet's formulas	$\dot{\hat{\tau}} = \kappa v \hat{\mathbf{n}} \quad (2.289)$	
	$\dot{\hat{\mathbf{n}}} = -\kappa v \hat{\tau} + \lambda v \hat{\mathbf{b}} \quad (2.290)$	
	$\dot{\hat{\mathbf{b}}} = -\lambda v \hat{\mathbf{n}} \quad (2.291)$	

<sup>a</sup>For a continuous curve in three dimensions, traced by the position vector  $r(t)$ .

## 2.7 Differentiation

### Derivatives (general)

Power	$\frac{d}{dx}(u^n) = nu^{n-1} \frac{du}{dx}$	(2.292)	$n$	power index
Product	$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$	(2.293)	$u, v$	functions of $x$
Quotient	$\frac{d}{dx} \left( \frac{u}{v} \right) = \frac{1}{v} \frac{du}{dx} - \frac{u}{v^2} \frac{dv}{dx}$	(2.294)		
Function of a function <sup>a</sup>	$\frac{d}{dx}[f(u)] = \frac{d}{du}[f(u)] \cdot \frac{du}{dx}$	(2.295)	$f(u)$	function of $u(x)$
Leibniz theorem	$\frac{d^n}{dx^n}[uv] = \binom{n}{0} v \frac{d^n u}{dx^n} + \binom{n}{1} \frac{dv}{dx} \frac{d^{n-1} u}{dx^{n-1}} + \cdots + \binom{n}{k} \frac{d^k v}{dx^k} \frac{d^{n-k} u}{dx^{n-k}} + \cdots + \binom{n}{n} u \frac{d^n v}{dx^n}$	(2.296)	$\binom{n}{k}$	binomial coefficient
Differentiation under the integral sign	$\frac{d}{dq} \left[ \int_p^q f(x) dx \right] = f(q) \quad (p \text{ constant})$	(2.297)		
	$\frac{d}{dp} \left[ \int_p^q f(x) dx \right] = -f(p) \quad (q \text{ constant})$	(2.298)		
General integral	$\frac{d}{dx} \left[ \int_{u(x)}^{v(x)} f(t) dt \right] = f(v) \frac{dv}{dx} - f(u) \frac{du}{dx}$	(2.299)		
Logarithm	$\frac{d}{dx}(\log_b  ax ) = (x \ln b)^{-1}$	(2.300)	$b$ $a$	log base constant
Exponential	$\frac{d}{dx}(e^{ax}) = ae^{ax}$	(2.301)		
Inverse functions	$\frac{dx}{dy} = \left( \frac{dy}{dx} \right)^{-1}$	(2.302)		
	$\frac{d^2 x}{dy^2} = -\frac{d^2 y}{dx^2} \left( \frac{dy}{dx} \right)^{-3}$	(2.303)		
	$\frac{d^3 x}{dy^3} = \left[ 3 \left( \frac{d^2 y}{dx^2} \right)^2 - \frac{dy}{dx} \frac{d^3 y}{dx^3} \right] \left( \frac{dy}{dx} \right)^{-5}$	(2.304)		

<sup>a</sup>The “chain rule.”

**Trigonometric derivatives<sup>a</sup>**

$\frac{d}{dx}(\sin ax) = a \cos ax$	(2.305)	$\frac{d}{dx}(\cos ax) = -a \sin ax$	(2.306)
$\frac{d}{dx}(\tan ax) = a \sec^2 ax$	(2.307)	$\frac{d}{dx}(\csc ax) = -a \csc ax \cdot \cot ax$	(2.308)
$\frac{d}{dx}(\sec ax) = a \sec ax \cdot \tan ax$	(2.309)	$\frac{d}{dx}(\cot ax) = -a \csc^2 ax$	(2.310)
$\frac{d}{dx}(\arcsin ax) = a(1 - a^2 x^2)^{-1/2}$	(2.311)	$\frac{d}{dx}(\arccos ax) = -a(1 - a^2 x^2)^{-1/2}$	(2.312)
$\frac{d}{dx}(\arctan ax) = a(1 + a^2 x^2)^{-1}$	(2.313)	$\frac{d}{dx}(\operatorname{arccsc} ax) = -\frac{a}{ ax }(a^2 x^2 - 1)^{-1/2}$	(2.314)
$\frac{d}{dx}(\operatorname{arcsec} ax) = \frac{a}{ ax }(a^2 x^2 - 1)^{-1/2}$	(2.315)	$\frac{d}{dx}(\operatorname{arccot} ax) = -a(a^2 x^2 + 1)^{-1}$	(2.316)

<sup>a</sup>*a* is a constant.**Hyperbolic derivatives<sup>a</sup>**

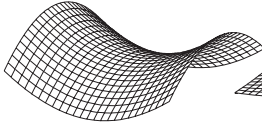
$\frac{d}{dx}(\sinh ax) = a \cosh ax$	(2.317)	$\frac{d}{dx}(\cosh ax) = a \sinh ax$	(2.318)
$\frac{d}{dx}(\tanh ax) = a \operatorname{sech}^2 ax$	(2.319)	$\frac{d}{dx}(\operatorname{csch} ax) = -a \operatorname{csch} ax \cdot \operatorname{coth} ax$	(2.320)
$\frac{d}{dx}(\operatorname{sech} ax) = -a \operatorname{sech} ax \cdot \tanh ax$	(2.321)	$\frac{d}{dx}(\operatorname{coth} ax) = -a \operatorname{csch}^2 ax$	(2.322)
$\frac{d}{dx}(\operatorname{arsinh} ax) = a(a^2 x^2 + 1)^{-1/2}$	(2.323)	$\frac{d}{dx}(\operatorname{arcosh} ax) = a(a^2 x^2 - 1)^{-1/2}$	(2.324)
$\frac{d}{dx}(\operatorname{artanh} ax) = a(1 - a^2 x^2)^{-1}$	(2.325)	$\frac{d}{dx}(\operatorname{arcsch} ax) = -\frac{a}{ ax }(1 + a^2 x^2)^{-1/2}$	(2.326)
$\frac{d}{dx}(\operatorname{arsech} ax) = -\frac{a}{ ax }(1 - a^2 x^2)^{-1/2}$	(2.327)	$\frac{d}{dx}(\operatorname{arcoth} ax) = a(1 - a^2 x^2)^{-1}$	(2.328)

<sup>a</sup>*a* is a constant.

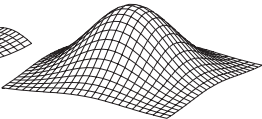
**Partial derivatives**

Total differential	$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$	(2.329)	$f = f(x,y,z)$
Reciprocity	$\left. \frac{\partial g}{\partial x} \right _y \left. \frac{\partial x}{\partial y} \right _g \left. \frac{\partial y}{\partial g} \right _x = -1$	(2.330)	$g = g(x,y)$
Chain rule	$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial u}$	(2.331)	
Jacobian	$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$	(2.332)	$J$ Jacobian $u = u(x,y,z)$ $v = v(x,y,z)$ $w = w(x,y,z)$
Change of variable	$\int_V f(x,y,z) dx dy dz = \int_{V'} f(u,v,w) J du dv dw$	(2.333)	$V$ volume in $(x,y,z)$ $V'$ volume in $(u,v,w)$ mapped to by $V$
Euler–Lagrange equation	if $I = \int_a^b F(x,y,y') dx$ then $\delta I = 0$ when $\frac{\partial F}{\partial y} = \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right)$	(2.334)	$y'$ $dy/dx$ $a,b$ fixed end points

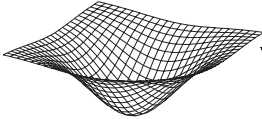
**Stationary points<sup>a</sup>**



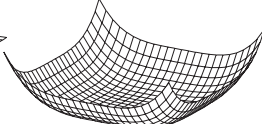
saddle point



maximum



minimum



quartic minimum

Stationary point if  $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$  at  $(x_0, y_0)$  (necessary condition) (2.335)

**Additional sufficient conditions**

for maximum  $\frac{\partial^2 f}{\partial x^2} < 0$ , and  $\frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} > \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2$  (2.336)

for minimum  $\frac{\partial^2 f}{\partial x^2} > 0$ , and  $\frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} > \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2$  (2.337)

for saddle point  $\frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} < \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2$  (2.338)

<sup>a</sup>Of a function  $f(x,y)$  at the point  $(x_0, y_0)$ . Note that at, for example, a *quartic minimum*  $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = 0$ .

**Differential equations**

Laplace	$\nabla^2 f = 0$	(2.339)	$f$	$f(x, y, z)$
Diffusion <sup>a</sup>	$\frac{\partial f}{\partial t} = D \nabla^2 f$	(2.340)	$D$	diffusion coefficient
Helmholtz	$\nabla^2 f + \alpha^2 f = 0$	(2.341)	$\alpha$	constant
Wave	$\nabla^2 f = \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2}$	(2.342)	$c$	wave speed
Legendre	$\frac{d}{dx} \left[ (1-x^2) \frac{dy}{dx} \right] + l(l+1)y = 0$	(2.343)	$l$	integer
Associated Legendre	$\frac{d}{dx} \left[ (1-x^2) \frac{dy}{dx} \right] + \left[ l(l+1) - \frac{m^2}{1-x^2} \right] y = 0$	(2.344)	$m$	integer
Bessel	$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - m^2)y = 0$	(2.345)		
Hermite	$\frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2\alpha y = 0$	(2.346)		
Laguerre	$x \frac{d^2 y}{dx^2} + (1-x) \frac{dy}{dx} + \alpha y = 0$	(2.347)		
Associated Laguerre	$x \frac{d^2 y}{dx^2} + (1+k-x) \frac{dy}{dx} + \alpha y = 0$	(2.348)	$k$	integer
Chebyshev	$(1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + n^2 y = 0$	(2.349)	$n$	integer
Euler (or Cauchy)	$x^2 \frac{d^2 y}{dx^2} + ax \frac{dy}{dx} + by = f(x)$	(2.350)	$a, b$	constants
Bernoulli	$\frac{dy}{dx} + p(x)y = q(x)y^a$	(2.351)	$p, q$	functions of $x$
Airy	$\frac{d^2 y}{dx^2} = xy$	(2.352)		

<sup>a</sup>Also known as the “conduction equation.” For thermal conduction,  $f \equiv T$  and  $D$ , the thermal diffusivity,  $\equiv \kappa \equiv \lambda / (\rho c_p)$ , where  $T$  is the temperature distribution,  $\lambda$  the thermal conductivity,  $\rho$  the density, and  $c_p$  the specific heat capacity of the material.

## 2.8 Integration

### Standard forms<sup>a</sup>

$$\int u \, dv = [uv] - \int v \, du \quad (2.353) \quad \int uv \, dx = v \int u \, dx - \int \left( \int u \, dx \right) \frac{dv}{dx} \, dx \quad (2.354)$$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1) \quad (2.355) \quad \int \frac{1}{x} \, dx = \ln|x| \quad (2.356)$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} \quad (2.357) \quad \int x e^{ax} \, dx = e^{ax} \left( \frac{x}{a} - \frac{1}{a^2} \right) \quad (2.358)$$

$$\int \ln ax \, dx = x(\ln ax - 1) \quad (2.359) \quad \int \frac{f'(x)}{f(x)} \, dx = \ln f(x) \quad (2.360)$$

$$\int x \ln ax \, dx = \frac{x^2}{2} \left( \ln ax - \frac{1}{2} \right) \quad (2.361) \quad \int b^{ax} \, dx = \frac{b^{ax}}{a \ln b} \quad (b > 0) \quad (2.362)$$

$$\int \frac{1}{a+bx} \, dx = \frac{1}{b} \ln(a+bx) \quad (2.363) \quad \int \frac{1}{x(a+bx)} \, dx = -\frac{1}{a} \ln \frac{a+bx}{x} \quad (2.364)$$

$$\int \frac{1}{(a+bx)^2} \, dx = \frac{-1}{b(a+bx)} \quad (2.365) \quad \int \frac{1}{a^2+b^2x^2} \, dx = \frac{1}{ab} \arctan \left( \frac{bx}{a} \right) \quad (2.366)$$

$$\int \frac{1}{x(x^n+a)} \, dx = \frac{1}{an} \ln \left| \frac{x^n}{x^n+a} \right| \quad (2.367) \quad \int \frac{1}{x^2-a^2} \, dx = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| \quad (2.368)$$

$$\int \frac{x}{x^2 \pm a^2} \, dx = \frac{1}{2} \ln|x^2 \pm a^2| \quad (2.369) \quad \int \frac{x}{(x^2 \pm a^2)^n} \, dx = \frac{-1}{2(n-1)(x^2 \pm a^2)^{n-1}} \quad (2.370)$$

$$\int \frac{1}{(a^2-x^2)^{1/2}} \, dx = \arcsin \left( \frac{x}{a} \right) \quad (2.371) \quad \int \frac{1}{(x^2 \pm a^2)^{1/2}} \, dx = \ln|x + (x^2 \pm a^2)^{1/2}| \quad (2.372)$$

$$\int \frac{x}{(x^2 \pm a^2)^{1/2}} \, dx = (x^2 \pm a^2)^{1/2} \quad (2.373) \quad \int \frac{1}{x(x^2-a^2)^{1/2}} \, dx = \frac{1}{a} \operatorname{arcsec} \left( \frac{x}{a} \right) \quad (2.374)$$

<sup>a</sup> $a$  and  $b$  are non-zero constants.

**Trigonometric and hyperbolic integrals**

$\int \sin x \, dx = -\cos x$	(2.375)	$\int \sinh x \, dx = \cosh x$	(2.376)
$\int \cos x \, dx = \sin x$	(2.377)	$\int \cosh x \, dx = \sinh x$	(2.378)
$\int \tan x \, dx = -\ln \cos x $	(2.379)	$\int \tanh x \, dx = \ln(\cosh x)$	(2.380)
$\int \csc x \, dx = \ln \left  \tan \frac{x}{2} \right $	(2.381)	$\int \operatorname{csch} x \, dx = \ln \left  \tanh \frac{x}{2} \right $	(2.382)
$\int \sec x \, dx = \ln  \sec x + \tan x $	(2.383)	$\int \operatorname{sech} x \, dx = 2 \arctan(e^x)$	(2.384)
$\int \cot x \, dx = \ln  \sin x $	(2.385)	$\int \operatorname{coth} x \, dx = \ln  \sinh x $	(2.386)
$\int \sin mx \cdot \sin nx \, dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} \quad (m^2 \neq n^2) \quad (2.387)$			
$\int \sin mx \cdot \cos nx \, dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)} \quad (m^2 \neq n^2) \quad (2.388)$			
$\int \cos mx \cdot \cos nx \, dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)} \quad (m^2 \neq n^2) \quad (2.389)$			

**Named integrals**

Error function	$\operatorname{erf}(x) = \frac{2}{\pi^{1/2}} \int_0^x \exp(-t^2) \, dt$	(2.390)
Complementary error function	$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\pi^{1/2}} \int_x^\infty \exp(-t^2) \, dt$	(2.391)
Fresnel integrals <sup>a</sup>	$C(x) = \int_0^x \cos \frac{\pi t^2}{2} \, dt; \quad S(x) = \int_0^x \sin \frac{\pi t^2}{2} \, dt$	(2.392)
	$C(x) + iS(x) = \frac{1+i}{2} \operatorname{erf} \left[ \frac{\pi^{1/2}}{2} (1-i)x \right]$	(2.393)
Exponential integral	$Ei(x) = \int_{-\infty}^x \frac{e^t}{t} \, dt \quad (x > 0)$	(2.394)
Gamma function	$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt \quad (x > 0)$	(2.395)
Elliptic integrals (trigonometric form)	$F(\phi, k) = \int_0^\phi \frac{1}{(1-k^2 \sin^2 \theta)^{1/2}} \, d\theta \quad (\text{first kind})$	(2.396)
	$E(\phi, k) = \int_0^\phi (1-k^2 \sin^2 \theta)^{1/2} \, d\theta \quad (\text{second kind})$	(2.397)

<sup>a</sup>See also page 167.

**Definite integrals**

$\int_0^\infty e^{-ax^2} dx = \frac{1}{2} \left(\frac{\pi}{a}\right)^{1/2} \quad (a > 0)$	(2.398)
$\int_0^\infty xe^{-ax^2} dx = \frac{1}{2a} \quad (a > 0)$	(2.399)
$\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}} \quad (a > 0; n = 0, 1, 2, \dots)$	(2.400)
$\int_{-\infty}^\infty \exp(2bx - ax^2) dx = \left(\frac{\pi}{a}\right)^{1/2} \exp\left(\frac{b^2}{a}\right) \quad (a > 0)$	(2.401)
$\int_0^\infty x^n e^{-ax^2} dx = \begin{cases} 1 \cdot 3 \cdot 5 \cdot \dots \cdot (n-1)(2a)^{-(n+1)/2}(\pi/2)^{1/2} & n > 0 \text{ and even} \\ 2 \cdot 4 \cdot 6 \cdot \dots \cdot (n-1)(2a)^{-(n+1)/2} & n > 1 \text{ and odd} \end{cases}$	(2.402)
$\int_0^1 x^p(1-x)^q dx = \frac{p!q!}{(p+q+1)!} \quad (p, q \text{ integers } > 0)$	(2.403)
$\int_0^\infty \cos(ax^2) dx = \int_0^\infty \sin(ax^2) dx = \frac{1}{2} \left(\frac{\pi}{2a}\right)^{1/2} \quad (a > 0)$	(2.404)
$\int_0^\infty \frac{\sin x}{x} dx = \int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}$	(2.405)
$\int_0^\infty \frac{1}{(1+x)x^a} dx = \frac{\pi}{\sin a\pi} \quad (0 < a < 1)$	(2.406)

**2.9 Special functions and polynomials**

**Gamma function**

Definition	$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \quad [\Re(z) > 0]$	(2.407)
Relations	$n! = \Gamma(n+1) = n\Gamma(n) \quad (n = 0, 1, 2, \dots)$	(2.408)
	$\Gamma(1/2) = \pi^{1/2}$	(2.409)
	$\binom{z}{w} = \frac{z!}{w!(z-w)!} = \frac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}$	(2.410)
Stirling's formulas (for $ z , n \gg 1$ )	$\Gamma(z) \simeq e^{-z} z^{z-(1/2)} (2\pi)^{1/2} \left(1 + \frac{1}{12z} + \frac{1}{288z^2} - \dots\right)$	(2.411)
	$n! \simeq n^{n+(1/2)} e^{-n} (2\pi)^{1/2}$	(2.412)
	$\ln(n!) \simeq n \ln n - n$	(2.413)





**Bessel functions**

Series expansion	$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-x^2/4)^k}{k! \Gamma(\nu+k+1)} \quad (2.414)$ $Y_\nu(x) = \frac{J_\nu(x) \cos(\pi\nu) - J_{-\nu}(x)}{\sin(\pi\nu)} \quad (2.415)$	$J_\nu(x)$ Bessel function of the first kind $Y_\nu(x)$ Bessel function of the second kind $\Gamma(\nu)$ Gamma function $\nu$ order ( $\nu \geq 0$ )
Approximations	$J_\nu(x) \simeq \begin{cases} \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu & (0 \leq x \ll \nu) \\ \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \frac{1}{2}\nu\pi - \frac{\pi}{4}\right) & (x \gg \nu) \end{cases} \quad (2.416)$ $Y_\nu(x) \simeq \begin{cases} \frac{-\Gamma(\nu)}{\pi} \left(\frac{x}{2}\right)^{-\nu} & (0 < x \ll \nu) \\ \left(\frac{2}{\pi x}\right)^{1/2} \sin\left(x - \frac{1}{2}\nu\pi - \frac{\pi}{4}\right) & (x \gg \nu) \end{cases} \quad (2.417)$	
Modified Bessel functions	$I_\nu(x) = (-i)^\nu J_\nu(ix) \quad (2.418)$ $K_\nu(x) = \frac{\pi}{2} i^{\nu+1} [J_\nu(ix) + iY_\nu(ix)] \quad (2.419)$	$I_\nu(x)$ modified Bessel function of the first kind $K_\nu(x)$ modified Bessel function of the second kind
Spherical Bessel function	$j_\nu(x) = \left(\frac{\pi}{2x}\right)^{1/2} J_{\nu+1/2}(x) \quad (2.420)$	$j_\nu(x)$ spherical Bessel function of the first kind [similarly for $y_\nu(x)$ ]

**Legendre polynomials<sup>a</sup>**

Legendre equation	$(1-x^2) \frac{d^2 P_l(x)}{dx^2} - 2x \frac{dP_l(x)}{dx} + l(l+1)P_l(x) = 0 \quad (2.421)$	$P_l$ Legendre polynomials $l$ order ( $l \geq 0$ )
Rodrigues' formula	$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2-1)^l \quad (2.422)$	
Recurrence relation	$(l+1)P_{l+1}(x) = (2l+1)xP_l(x) - lP_{l-1}(x) \quad (2.423)$	
Orthogonality	$\int_{-1}^1 P_l(x)P_{l'}(x) dx = \frac{2}{2l+1} \delta_{ll'} \quad (2.424)$	$\delta_{ll'}$ Kronecker delta
Explicit form	$P_l(x) = 2^{-l} \sum_{m=0}^{l/2} (-1)^m \binom{l}{m} \binom{2l-2m}{l} x^{l-2m} \quad (2.425)$	$\binom{l}{m}$ binomial coefficients
Expansion of plane wave	$\exp(ikz) = \exp(ikr \cos \theta) \quad (2.426)$	$k$ wavenumber $z$ propagation axis $z = r \cos \theta$
	$= \sum_{l=0}^{\infty} (2l+1) i^l j_l(kr) P_l(\cos \theta) \quad (2.427)$	$j_l$ spherical Bessel function of the first kind (order $l$ )
$P_0(x) = 1$ $P_1(x) = x$	$P_2(x) = (3x^2 - 1)/2$ $P_3(x) = (5x^3 - 3x)/2$	$P_4(x) = (35x^4 - 30x^2 + 3)/8$ $P_5(x) = (63x^5 - 70x^3 + 15x)/8$

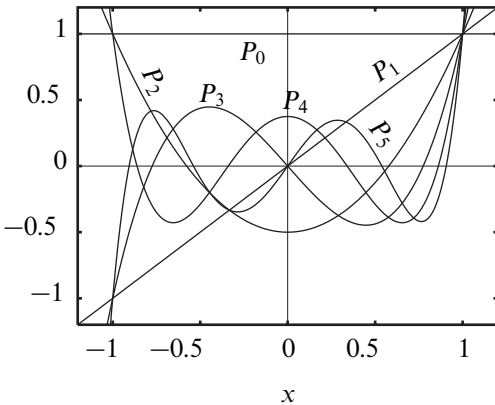
<sup>a</sup>Of the first kind.

**Associated Legendre functions<sup>a</sup>**

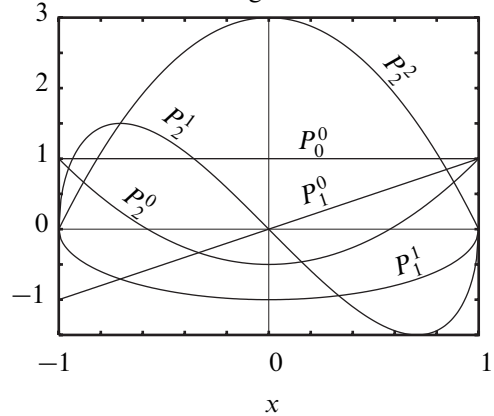
Associated Legendre equation	$\frac{d}{dx} \left[ (1-x^2) \frac{dP_l^m(x)}{dx} \right] + \left[ l(l+1) - \frac{m^2}{1-x^2} \right] P_l^m(x) = 0$ (2.428)	$P_l^m$ associated Legendre functions
From Legendre polynomials	$P_l^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x), \quad 0 \leq m \leq l$ (2.429) $P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x)$ (2.430)	$P_l$ Legendre polynomials
Recurrence relations	$P_{m+1}^m(x) = x(2m+1)P_m^m(x)$ (2.431) $P_m^m(x) = (-1)^m (2m-1)!! (1-x^2)^{m/2}$ (2.432) $(l-m+1)P_{l+1}^m(x) = (2l+1)xP_l^m(x) - (l+m)P_{l-1}^m(x)$ (2.433)	!! $5!! = 5 \cdot 3 \cdot 1$ etc.
Orthogonality	$\int_{-1}^1 P_l^m(x) P_{l'}^m(x) dx = \frac{(l+m)!}{(l-m)!} \frac{2}{2l+1} \delta_{ll'}$ (2.434)	$\delta_{ll'}$ Kronecker delta
	$P_0^0(x) = 1$ $P_2^0(x) = (3x^2 - 1)/2$	$P_1^0(x) = x$ $P_2^1(x) = -3x(1-x^2)^{1/2}$
		$P_1^1(x) = -(1-x^2)^{1/2}$ $P_2^2(x) = 3(1-x^2)$

<sup>a</sup>Of the first kind.  $P_l^m(x)$  can be defined with a  $(-1)^m$  factor in Equation (2.429) as well as Equation (2.430).

Legendre polynomials



associated Legendre functions





**Spherical harmonics**

Differential equation	$\left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] Y_l^m + l(l+1)Y_l^m = 0$ (2.435)	$Y_l^m$ spherical harmonics
Definition <sup>a</sup>	$Y_l^m(\theta, \phi) = (-1)^m \left[ \frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} \right]^{1/2} P_l^m(\cos \theta) e^{im\phi}$ (2.436)	$P_l^m$ associated Legendre functions
Orthogonality	$\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} Y_l^{m*}(\theta, \phi) Y_{l'}^{m'}(\theta, \phi) \sin \theta \, d\theta \, d\phi = \delta_{mm'} \delta_{ll'}$ (2.437)	$Y^*$ complex conjugate $\delta_{ll'}$ Kronecker delta
Laplace series	$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l a_{lm} Y_l^m(\theta, \phi)$ (2.438) where $a_{lm} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} Y_l^{m*}(\theta, \phi) f(\theta, \phi) \sin \theta \, d\theta \, d\phi$ (2.439)	$f$ continuous function
Solution to Laplace equation	if $\nabla^2 \psi(r, \theta, \phi) = 0$ , then $\psi(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l Y_l^m(\theta, \phi) \cdot [a_{lm} r^l + b_{lm} r^{-(l+1)}]$ (2.440)	$\psi$ continuous function $a, b$ constants
$Y_0^0(\theta, \phi) = \sqrt{\frac{1}{4\pi}} \qquad Y_1^0(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos \theta$ $Y_1^{\pm 1}(\theta, \phi) = \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi} \qquad Y_2^0(\theta, \phi) = \sqrt{\frac{5}{4\pi}} \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$ $Y_2^{\pm 1}(\theta, \phi) = \mp \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{\pm i\phi} \qquad Y_2^{\pm 2}(\theta, \phi) = \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{\pm 2i\phi}$ $Y_3^0(\theta, \phi) = \frac{1}{2} \sqrt{\frac{7}{4\pi}} (5 \cos^2 \theta - 3) \cos \theta \qquad Y_3^{\pm 1}(\theta, \phi) = \mp \frac{1}{4} \sqrt{\frac{21}{4\pi}} \sin \theta (5 \cos^2 \theta - 1) e^{\pm i\phi}$ $Y_3^{\pm 2}(\theta, \phi) = \frac{1}{4} \sqrt{\frac{105}{2\pi}} \sin^2 \theta \cos \theta e^{\pm 2i\phi} \qquad Y_3^{\pm 3}(\theta, \phi) = \mp \frac{1}{4} \sqrt{\frac{35}{4\pi}} \sin^3 \theta e^{\pm 3i\phi}$		

<sup>a</sup>Defined for  $-l \leq m \leq l$ , using the sign convention of the Condon–Shortley phase. Other sign conventions are possible.

**Delta functions**

Kronecker delta	$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$	(2.441)	$\delta_{ij}$ Kronecker delta $i, j, k, \dots$ indices (= 1, 2 or 3)
	$\delta_{ii} = 3$	(2.442)	
Three-dimensional Levi-Civita symbol (permutation tensor) <sup>a</sup>	$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$	(2.443)	$\epsilon_{ijk}$ Levi-Civita symbol (see also page 25)
	$\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$		
	all other $\epsilon_{ijk} = 0$		
	$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$	(2.444)	
	$\delta_{ij}\epsilon_{ijk} = 0$	(2.445)	
	$\epsilon_{ilm}\epsilon_{jlm} = 2\delta_{ij}$	(2.446)	
	$\epsilon_{ijk}\epsilon_{ijk} = 6$	(2.447)	
Dirac delta function	$\int_a^b \delta(x) dx = \begin{cases} 1 & \text{if } a < 0 < b \\ 0 & \text{otherwise} \end{cases}$	(2.448)	$\delta(x)$ Dirac delta function $f(x)$ smooth function of $x$ $a, b$ constants
	$\int_a^b f(x)\delta(x - x_0) dx = f(x_0)$	(2.449)	
	$\delta(x - x_0)f(x) = \delta(x - x_0)f(x_0)$	(2.450)	
	$\delta(-x) = \delta(x)$	(2.451)	
	$\delta(ax) =  a ^{-1}\delta(x) \quad (a \neq 0)$	(2.452)	
	$\delta(x) \simeq n\pi^{-1/2}e^{-n^2x^2} \quad (n \gg 1)$	(2.453)	

<sup>a</sup>The general symbol  $\epsilon_{ijk\dots}$  is defined to be +1 for even permutations of the suffices, -1 for odd permutations, and 0 if a suffix is repeated. The sequence (1, 2, 3, ...,  $n$ ) is taken to be even. Swapping adjacent suffices an odd (or even) number of times gives an odd (or even) permutation.

**2.10 Roots of quadratic and cubic equations**

**Quadratic equations**

Equation	$ax^2 + bx + c = 0 \quad (a \neq 0)$	(2.454)	$x$ variable $a, b, c$ real constants
Solutions	$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	(2.455)	$x_1, x_2$ quadratic roots
	$= \frac{-2c}{b \pm \sqrt{b^2 - 4ac}}$	(2.456)	
Solution combinations	$x_1 + x_2 = -b/a$	(2.457)	
	$x_1 x_2 = c/a$	(2.458)	



**Cubic equations**

Equation	$ax^3 + bx^2 + cx + d = 0 \quad (a \neq 0)$	(2.459)	$x$ $a, b, c, d$	variable real constants
Intermediate definitions	$p = \frac{1}{3} \left( \frac{3c}{a} - \frac{b^2}{a^2} \right)$	(2.460)	$D$	discriminant
	$q = \frac{1}{27} \left( \frac{2b^3}{a^3} - \frac{9bc}{a^2} + \frac{27d}{a} \right)$	(2.461)		
	$D = \left( \frac{p}{3} \right)^3 + \left( \frac{q}{2} \right)^2$	(2.462)		
If $D \geq 0$ , also define:		If $D < 0$ , also define:		
$u = \left( \frac{-q}{2} + D^{1/2} \right)^{1/3}$	(2.463)	$\phi = \arccos \left[ \frac{-q}{2} \left( \frac{ p }{3} \right)^{-3/2} \right]$	(2.467)	
$v = \left( \frac{-q}{2} - D^{1/2} \right)^{1/3}$	(2.464)	$y_1 = 2 \left( \frac{ p }{3} \right)^{1/2} \cos \frac{\phi}{3}$	(2.468)	
$y_1 = u + v$	(2.465)	$y_{2,3} = -2 \left( \frac{ p }{3} \right)^{1/2} \cos \frac{\phi \pm \pi}{3}$	(2.469)	
$y_{2,3} = \frac{-(u+v)}{2} \pm i \frac{u-v}{2} 3^{1/2}$	(2.466)	1 real, 2 complex roots (if $D = 0$ : 3 real roots, at least 2 equal)		
3 distinct real roots				
Solutions <sup>a</sup>	$x_n = y_n - \frac{b}{3a}$	(2.470)	$x_n$	cubic roots ( $n = 1, 2, 3$ )
Solution combinations	$x_1 + x_2 + x_3 = -b/a$	(2.471)		
	$x_1x_2 + x_1x_3 + x_2x_3 = c/a$	(2.472)		
	$x_1x_2x_3 = -d/a$	(2.473)		

<sup>a</sup> $y_n$  are solutions to the reduced equation  $y^3 + py + q = 0$ .

## 2.11 Fourier series and transforms

### Fourier series

Real form	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$	(2.474)	$f(x)$ periodic function, period $2L$ $a_n, b_n$ Fourier coefficients
	$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx$	(2.475)	
	$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx$	(2.476)	
Complex form	$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp \left( \frac{i n \pi x}{L} \right)$	(2.477)	$c_n$ complex Fourier coefficient
	$c_n = \frac{1}{2L} \int_{-L}^L f(x) \exp \left( \frac{-i n \pi x}{L} \right) dx$	(2.478)	
Parseval's theorem	$\frac{1}{2L} \int_{-L}^L  f(x) ^2 dx = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$	(2.479)	modulus
	$= \sum_{n=-\infty}^{\infty}  c_n ^2$	(2.480)	

### Fourier transform<sup>a</sup>

Definition 1	$F(s) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x s} dx$	(2.481)	$f(x)$ function of $x$ $F(s)$ Fourier transform of $f(x)$
	$f(x) = \int_{-\infty}^{\infty} F(s) e^{2\pi i x s} ds$	(2.482)	
Definition 2	$F(s) = \int_{-\infty}^{\infty} f(x) e^{-i x s} dx$	(2.483)	
	$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s) e^{i x s} ds$	(2.484)	
Definition 3	$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i x s} dx$	(2.485)	
	$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{i x s} ds$	(2.486)	

<sup>a</sup>All three (and more) definitions are used, but definition 1 is probably the best.



**Fourier transform theorems<sup>a</sup>**

Convolution	$f(x) * g(x) = \int_{-\infty}^{\infty} f(u)g(x-u) du$	(2.487)	$f, g$ general functions * convolution
Convolution rules	$f * g = g * f$ $f * (g * h) = (f * g) * h$	(2.488) (2.489)	$f$ $f(x) \rightleftharpoons F(s)$ $g$ $g(x) \rightleftharpoons G(s)$
Convolution theorem	$f(x)g(x) \rightleftharpoons F(s) * G(s)$	(2.490)	$\rightleftharpoons$ Fourier transform relation
Autocorrelation	$f^*(x) \star f(x) = \int_{-\infty}^{\infty} f^*(u-x)f(u) du$	(2.491)	* correlation $f^*$ complex conjugate of $f$
Wiener-Khintchine theorem	$f^*(x) \star f(x) \rightleftharpoons  F(s) ^2$	(2.492)	
Cross-correlation	$f^*(x) \star g(x) = \int_{-\infty}^{\infty} f^*(u-x)g(u) du$	(2.493)	
Correlation theorem	$h(x) \star j(x) \rightleftharpoons H(s)J^*(s)$	(2.494)	$h, j$ real functions $H$ $H(s) \rightleftharpoons h(x)$ $J$ $J(s) \rightleftharpoons j(x)$
Parseval's relation <sup>b</sup>	$\int_{-\infty}^{\infty} f(x)g^*(x) dx = \int_{-\infty}^{\infty} F(s)G^*(s) ds$	(2.495)	
Parseval's theorem <sup>c</sup>	$\int_{-\infty}^{\infty}  f(x) ^2 dx = \int_{-\infty}^{\infty}  F(s) ^2 ds$	(2.496)	
Derivatives	$\frac{df(x)}{dx} \rightleftharpoons 2\pi isF(s)$	(2.497)	
	$\frac{d}{dx}[f(x) * g(x)] = \frac{df(x)}{dx} * g(x) = \frac{dg(x)}{dx} * f(x)$	(2.498)	

<sup>a</sup>Defining the Fourier transform as  $F(s) = \int_{-\infty}^{\infty} f(x)e^{-2\pi isx} dx$ .

<sup>b</sup>Also called the "power theorem."

<sup>c</sup>Also called "Rayleigh's theorem."

**Fourier symmetry relationships**

$f(x)$	$\rightleftharpoons$	$F(s)$	definitions
even	$\rightleftharpoons$	even	real: $f(x) = f^*(x)$
odd	$\rightleftharpoons$	odd	imaginary: $f(x) = -f^*(x)$
real, even	$\rightleftharpoons$	real, even	even: $f(x) = f(-x)$
real, odd	$\rightleftharpoons$	imaginary, odd	odd: $f(x) = -f(-x)$
imaginary, even	$\rightleftharpoons$	imaginary, even	Hermitian: $f(x) = f^*(-x)$
complex, even	$\rightleftharpoons$	complex, even	anti-Hermitian: $f(x) = -f^*(-x)$
complex, odd	$\rightleftharpoons$	complex, odd	
real, asymmetric	$\rightleftharpoons$	complex, Hermitian	
imaginary, asymmetric	$\rightleftharpoons$	complex, anti-Hermitian	

Fourier transform pairs<sup>a</sup>

$$f(x) \Leftrightarrow F(s) = \int_{-\infty}^{\infty} f(x)e^{-2\pi isx} dx \quad (2.499)$$

$$f(ax) \Leftrightarrow \frac{1}{|a|} F(s/a) \quad (a \neq 0, \text{ real}) \quad (2.500)$$

$$f(x-a) \Leftrightarrow e^{-2\pi ias} F(s) \quad (a \text{ real}) \quad (2.501)$$

$$\frac{d^n}{dx^n} f(x) \Leftrightarrow (2\pi is)^n F(s) \quad (2.502)$$

$$\delta(x) \Leftrightarrow 1 \quad (2.503)$$

$$\delta(x-a) \Leftrightarrow e^{-2\pi ias} \quad (2.504)$$

$$e^{-a|x|} \Leftrightarrow \frac{2a}{a^2 + 4\pi^2 s^2} \quad (a > 0) \quad (2.505)$$

$$xe^{-a|x|} \Leftrightarrow \frac{8i\pi as}{(a^2 + 4\pi^2 s^2)^2} \quad (a > 0) \quad (2.506)$$

$$e^{-x^2/a^2} \Leftrightarrow a\sqrt{\pi}e^{-\pi^2 a^2 s^2} \quad (2.507)$$

$$\sin ax \Leftrightarrow \frac{1}{2i} \left[ \delta\left(s - \frac{a}{2\pi}\right) - \delta\left(s + \frac{a}{2\pi}\right) \right] \quad (2.508)$$

$$\cos ax \Leftrightarrow \frac{1}{2} \left[ \delta\left(s - \frac{a}{2\pi}\right) + \delta\left(s + \frac{a}{2\pi}\right) \right] \quad (2.509)$$

$$\sum_{m=-\infty}^{\infty} \delta(x-ma) \Leftrightarrow \frac{1}{a} \sum_{n=-\infty}^{\infty} \delta\left(s - \frac{n}{a}\right) \quad (2.510)$$

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \quad (\text{“step”}) \Leftrightarrow \frac{1}{2}\delta(s) - \frac{i}{2\pi s} \quad (2.511)$$

$$f(x) = \begin{cases} 1 & |x| \leq a \\ 0 & |x| > a \end{cases} \quad (\text{“top hat”}) \Leftrightarrow \frac{\sin 2\pi as}{\pi s} = 2a \operatorname{sinc} 2as \quad (2.512)$$

$$f(x) = \begin{cases} \left(1 - \frac{|x|}{a}\right) & |x| \leq a \\ 0 & |x| > a \end{cases} \quad (\text{“triangle”}) \Leftrightarrow \frac{1}{2\pi^2 a s^2} (1 - \cos 2\pi as) = a \operatorname{sinc}^2 as \quad (2.513)$$

<sup>a</sup>Equation (2.499) defines the Fourier transform used for these pairs. Note that  $\operatorname{sinc} x \equiv (\sin \pi x)/(\pi x)$ .





## 2.12 Laplace transforms

### Laplace transform theorems

Definition <sup>a</sup>	$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st} dt$	(2.514)	$\mathcal{L}\{\}$	Laplace transform
Convolution <sup>b</sup>	$F(s) \cdot G(s) = \mathcal{L}\left\{\int_0^{\infty} f(t-z)g(z) dz\right\}$	(2.515)	$F(s)$	$\mathcal{L}\{f(t)\}$
	$= \mathcal{L}\{f(t) * g(t)\}$	(2.516)	$G(s)$	$\mathcal{L}\{g(t)\}$
Inverse <sup>c</sup>	$f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} F(s) ds$	(2.517)	$\gamma$	constant
	$= \sum \text{residues (for } t > 0)$	(2.518)		
Transform of derivative	$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n \mathcal{L}\{f(t)\} - \sum_{r=0}^{n-1} s^{n-r-1} \left.\frac{d^r f(t)}{dt^r}\right _{t=0}$	(2.519)	$n$	integer > 0
Derivative of transform	$\frac{d^n F(s)}{ds^n} = \mathcal{L}\{(-t)^n f(t)\}$	(2.520)		
Substitution	$F(s-a) = \mathcal{L}\{e^{at} f(t)\}$	(2.521)	$a$	constant
Translation	$e^{-as} F(s) = \mathcal{L}\{u(t-a)f(t-a)\}$	(2.522)	$u(t)$	unit step function
	where $u(t) = \begin{cases} 0 & (t < 0) \\ 1 & (t > 0) \end{cases}$	(2.523)		

<sup>a</sup>If  $|e^{-s_0 t} f(t)|$  is finite for sufficiently large  $t$ , the Laplace transform exists for  $s > s_0$ .

<sup>b</sup>Also known as the “faltung (or folding) theorem.”

<sup>c</sup>Also known as the “Bromwich integral.”  $\gamma$  is chosen so that the singularities in  $F(s)$  are left of the integral line.

## Laplace transform pairs

$$f(t) \implies F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st} dt \quad (2.524)$$

$$\delta(t) \implies 1 \quad (2.525)$$

$$1 \implies 1/s \quad (s > 0) \quad (2.526)$$

$$t^n \implies \frac{n!}{s^{n+1}} \quad (s > 0, n > -1) \quad (2.527)$$

$$t^{1/2} \implies \sqrt{\frac{\pi}{4s^3}} \quad (2.528)$$

$$t^{-1/2} \implies \sqrt{\frac{\pi}{s}} \quad (2.529)$$

$$e^{at} \implies \frac{1}{s-a} \quad (s > a) \quad (2.530)$$

$$te^{at} \implies \frac{1}{(s-a)^2} \quad (s > a) \quad (2.531)$$

$$(1-at)e^{-at} \implies \frac{s}{(s+a)^2} \quad (2.532)$$

$$t^2e^{-at} \implies \frac{2}{(s+a)^3} \quad (2.533)$$

$$\sin at \implies \frac{a}{s^2+a^2} \quad (s > 0) \quad (2.534)$$

$$\cos at \implies \frac{s}{s^2+a^2} \quad (s > 0) \quad (2.535)$$

$$\sinh at \implies \frac{a}{s^2-a^2} \quad (s > a) \quad (2.536)$$

$$\cosh at \implies \frac{s}{s^2-a^2} \quad (s > a) \quad (2.537)$$

$$e^{-bt} \sin at \implies \frac{a}{(s+b)^2+a^2} \quad (2.538)$$

$$e^{-bt} \cos at \implies \frac{s+b}{(s+b)^2+a^2} \quad (2.539)$$

$$e^{-at} f(t) \implies F(s+a) \quad (2.540)$$

2.13 Probability and statistics

Discrete statistics

Mean	$\langle x \rangle = \frac{1}{N} \sum_{i=1}^N x_i$	(2.541)	$x_i$	data series
			$N$	series length
			$\langle \cdot \rangle$	mean value
Variance <sup>a</sup>	$\text{var}[x] = \frac{1}{N-1} \sum_{i=1}^N (x_i - \langle x \rangle)^2$	(2.542)	$\text{var}[\cdot]$	unbiased variance
Standard deviation	$\sigma[x] = (\text{var}[x])^{1/2}$	(2.543)	$\sigma$	standard deviation
Skewness	$\text{skew}[x] = \frac{N}{(N-1)(N-2)} \sum_{i=1}^N \left( \frac{x_i - \langle x \rangle}{\sigma} \right)^3$	(2.544)		
Kurtosis	$\text{kurt}[x] \simeq \left[ \frac{1}{N} \sum_{i=1}^N \left( \frac{x_i - \langle x \rangle}{\sigma} \right)^4 \right] - 3$	(2.545)		
Correlation coefficient <sup>b</sup>	$r = \frac{\sum_{i=1}^N (x_i - \langle x \rangle)(y_i - \langle y \rangle)}{\sqrt{\sum_{i=1}^N (x_i - \langle x \rangle)^2} \sqrt{\sum_{i=1}^N (y_i - \langle y \rangle)^2}}$	(2.546)	$x, y$	data series to correlate
			$r$	correlation coefficient

<sup>a</sup>If  $\langle x \rangle$  is derived from the data,  $\{x_i\}$ , the relation is as shown. If  $\langle x \rangle$  is known independently, then an unbiased estimate is obtained by dividing the right-hand side by  $N$  rather than  $N - 1$ .

<sup>b</sup>Also known as ‘‘Pearson’s  $r$ .’’

Discrete probability distributions

distribution	$\text{pr}(x)$	mean	variance	domain		
Binomial	$\binom{n}{x} p^x (1-p)^{n-x}$	$np$	$np(1-p)$	$(x = 0, 1, \dots, n)$	(2.547)	$\binom{n}{x}$ binomial coefficient
Geometric	$(1-p)^{x-1} p$	$1/p$	$(1-p)/p^2$	$(x = 1, 2, 3, \dots)$	(2.548)	
Poisson	$\lambda^x \exp(-\lambda)/x!$	$\lambda$	$\lambda$	$(x = 0, 1, 2, \dots)$	(2.549)	

**Continuous probability distributions**

distribution	pr(x)	mean	variance	domain	
Uniform	$\frac{1}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$(a \leq x \leq b)$	(2.550)
Exponential	$\lambda \exp(-\lambda x)$	$1/\lambda$	$1/\lambda^2$	$(x \geq 0)$	(2.551)
Normal/ Gaussian	$\frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right]$	$\mu$	$\sigma^2$	$(-\infty < x < \infty)$	(2.552)
Chi-squared <sup>a</sup>	$\frac{e^{-x/2} x^{(r/2)-1}}{2^{r/2} \Gamma(r/2)}$	$r$	$2r$	$(x \geq 0)$	(2.553)
Rayleigh	$\frac{x}{\sigma^2} \exp\left(\frac{-x^2}{2\sigma^2}\right)$	$\sigma\sqrt{\pi/2}$	$2\sigma^2\left(1 - \frac{\pi}{4}\right)$	$(x \geq 0)$	(2.554)
Cauchy/ Lorentzian	$\frac{a}{\pi(a^2 + x^2)}$	(none)	(none)	$(-\infty < x < \infty)$	(2.555)

<sup>a</sup>With *r* degrees of freedom.  $\Gamma$  is the gamma function.

**Multivariate normal distribution**

Density function	$\text{pr}(\mathbf{x}) = \frac{\exp\left[-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})\mathbf{C}^{-1}(\mathbf{x} - \boldsymbol{\mu})^T\right]}{(2\pi)^{k/2} [\det(\mathbf{C})]^{1/2}}$	(2.556)	pr probability density <i>k</i> number of dimensions <b>C</b> covariance matrix <i>x</i> variable ( <i>k</i> dimensional) <b>μ</b> vector of means <sup>T</sup> transpose
Mean	$\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_k)$	(2.557)	det determinant $\mu_i$ mean of <i>i</i> th variable
Covariance	$\mathbf{C} = \sigma_{ij} = \langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle$	(2.558)	$\sigma_{ij}$ components of <b>C</b>
Correlation coefficient	$r = \frac{\sigma_{ij}}{\sigma_i \sigma_j}$	(2.559)	<i>r</i> correlation coefficient
Box–Muller transformation	$x_1 = (-2 \ln y_1)^{1/2} \cos 2\pi y_2$	(2.560)	$x_i$ normally distributed deviates $y_i$ deviates distributed uniformly between 0 and 1
	$x_2 = (-2 \ln y_1)^{1/2} \sin 2\pi y_2$	(2.561)	



**Random walk**

One-dimensional	$\text{pr}(x) = \frac{1}{(2\pi Nl^2)^{1/2}} \exp\left(\frac{-x^2}{2Nl^2}\right)$ (2.562)	<p><math>x</math> displacement after <math>N</math> steps (can be positive or negative)</p> <p><math>\text{pr}(x)</math> probability density of <math>x</math> (<math>\int_{-\infty}^{\infty} \text{pr}(x) dx = 1</math>)</p> <p><math>N</math> number of steps</p> <p><math>l</math> step length (all equal)</p>
rms displacement	$x_{\text{rms}} = N^{1/2}l$ (2.563)	$x_{\text{rms}}$ root-mean-squared displacement from start point
Three-dimensional	$\text{pr}(r) = \left(\frac{a}{\pi^{1/2}}\right)^3 \exp(-a^2r^2)$ (2.564) where $a = \left(\frac{3}{2Nl^2}\right)^{1/2}$	<p><math>r</math> radial distance from start point</p> <p><math>\text{pr}(r)</math> probability density of <math>r</math> (<math>\int_0^{\infty} 4\pi r^2 \text{pr}(r) dr = 1</math>)</p> <p><math>a</math> (most probable distance)<sup>-1</sup></p>
Mean distance	$\langle r \rangle = \left(\frac{8}{3\pi}\right)^{1/2} N^{1/2}l$ (2.565)	$\langle r \rangle$ mean distance from start point
rms distance	$r_{\text{rms}} = N^{1/2}l$ (2.566)	$r_{\text{rms}}$ root-mean-squared distance from start point

**Bayesian inference**

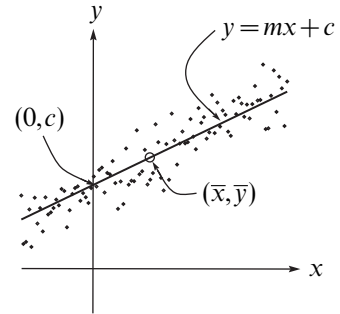
Conditional probability	$\text{pr}(x) = \int \text{pr}(x y') \text{pr}(y') dy'$ (2.567)	<p><math>\text{pr}(x)</math> probability (density) of <math>x</math></p> <p><math>\text{pr}(x y')</math> conditional probability of <math>x</math> given <math>y'</math></p>
Joint probability	$\text{pr}(x, y) = \text{pr}(x) \text{pr}(y x)$ (2.568)	$\text{pr}(x, y)$ joint probability of $x$ and $y$
Bayes' theorem <sup>a</sup>	$\text{pr}(y x) = \frac{\text{pr}(x y) \text{pr}(y)}{\text{pr}(x)}$ (2.569)	

<sup>a</sup>In this expression,  $\text{pr}(y|x)$  is known as the posterior probability,  $\text{pr}(x|y)$  the likelihood, and  $\text{pr}(y)$  the prior probability.

## 2.14 Numerical methods

### Straight-line fitting<sup>a</sup>

Data	$(\{x_i\}, \{y_i\})$ $n$ points	(2.570)
Weights <sup>b</sup>	$\{w_i\}$	(2.571)
Model	$y = mx + c$	(2.572)
Residuals	$d_i = y_i - mx_i - c$	(2.573)
Weighted centre	$(\bar{x}, \bar{y}) = \frac{1}{\sum w_i} \left( \sum w_i x_i, \sum w_i y_i \right)$	(2.574)
Weighted moment	$D = \sum w_i (x_i - \bar{x})^2$	(2.575)
Gradient	$m = \frac{1}{D} \sum w_i (x_i - \bar{x}) y_i$	(2.576)
	$\text{var}[m] \simeq \frac{1}{D} \frac{\sum w_i d_i^2}{n-2}$	(2.577)
Intercept	$c = \bar{y} - m\bar{x}$	(2.578)
	$\text{var}[c] \simeq \left( \frac{1}{\sum w_i} + \frac{\bar{x}^2}{D} \right) \frac{\sum w_i d_i^2}{n-2}$	(2.579)



<sup>a</sup>Least-squares fit of data to  $y = mx + c$ . Errors on  $y$ -values only.

<sup>b</sup>If the errors on  $y_i$  are uncorrelated, then  $w_i = 1/\text{var}[y_i]$ .

### Time series analysis<sup>a</sup>

Discrete convolution	$(r \star s)_j = \sum_{k=-(M/2)+1}^{M/2} s_{j-k} r_k$	(2.580)	$r_i$ response function $s_i$ time series $M$ response function duration
Bartlett (triangular) window	$w_j = 1 - \left  \frac{j - N/2}{N/2} \right $	(2.581)	$w_j$ windowing function $N$ length of time series 
Welch (quadratic) window	$w_j = 1 - \left[ \frac{j - N/2}{N/2} \right]^2$	(2.582)	
Hanning window	$w_j = \frac{1}{2} \left[ 1 - \cos \left( \frac{2\pi j}{N} \right) \right]$	(2.583)	
Hamming window	$w_j = 0.54 - 0.46 \cos \left( \frac{2\pi j}{N} \right)$	(2.584)	

<sup>a</sup>The time series runs from  $j = 0 \dots (N - 1)$ , and the windowing functions peak at  $j = N/2$ .

**Numerical integration**

Trapezoidal rule	$\int_{x_0}^{x_N} f(x) dx \simeq \frac{h}{2}(f_0 + 2f_1 + 2f_2 + \dots + 2f_{N-1} + f_N) \quad (2.585)$	$h = (x_N - x_0)/N$ (subinterval width) $f_i = f(x_i)$ $N$ = number of subintervals
Simpson's rule <sup>a</sup>	$\int_{x_0}^{x_N} f(x) dx \simeq \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + \dots + 4f_{N-1} + f_N) \quad (2.586)$	

<sup>a</sup> $N$  must be even. Simpson's rule is exact for quadratics and cubics.

**Numerical differentiation<sup>a</sup>**

$\frac{df}{dx} \simeq \frac{1}{12h} [-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)] \quad (2.587)$
$\sim \frac{1}{2h} [f(x+h) - f(x-h)] \quad (2.588)$
$\frac{d^2f}{dx^2} \simeq \frac{1}{12h^2} [-f(x+2h) + 16f(x+h) - 30f(x) + 16f(x-h) - f(x-2h)] \quad (2.589)$
$\sim \frac{1}{h^2} [f(x+h) - 2f(x) + f(x-h)] \quad (2.590)$
$\frac{d^3f}{dx^3} \sim \frac{1}{2h^3} [f(x+2h) - 2f(x+h) + 2f(x-h) - f(x-2h)] \quad (2.591)$

<sup>a</sup>Derivatives of  $f(x)$  at  $x$ .  $h$  is a small interval in  $x$ . Relations containing " $\simeq$ " are  $O(h^4)$ ; those containing " $\sim$ " are  $O(h^2)$ .

**Numerical solutions to  $f(x) = 0$**

Secant method	$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n) \quad (2.592)$	$f$ = function of $x$ $x_n = f(x_n) = 0$
Newton-Raphson method	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (2.593)$	$f'$ = $df/dx$

### Numerical solutions to ordinary differential equations<sup>a</sup>

Euler's method	if	$\frac{dy}{dx} = f(x, y)$	(2.594)
	and	$h = x_{n+1} - x_n$	(2.595)
	then	$y_{n+1} = y_n + hf(x_n, y_n) + O(h^2)$	(2.596)
Runge–Kutta method (fourth-order)	if	$\frac{dy}{dx} = f(x, y)$	(2.597)
	and	$h = x_{n+1} - x_n$	(2.598)
		$k_1 = hf(x_n, y_n)$	(2.599)
		$k_2 = hf(x_n + h/2, y_n + k_1/2)$	(2.600)
		$k_3 = hf(x_n + h/2, y_n + k_2/2)$	(2.601)
		$k_4 = hf(x_n + h, y_n + k_3)$	(2.602)
	then	$y_{n+1} = y_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} + O(h^5)$	(2.603)

<sup>a</sup>Ordinary differential equations (ODEs) of the form  $\frac{dy}{dx} = f(x, y)$ . Higher order equations should be reduced to a set of coupled first-order equations and solved in parallel.



# Chapter 3 Dynamics and mechanics

## 3.1 Introduction

Unusually in physics, there is no pithy phrase that sums up the study of *dynamics* (the way in which forces produce motion), *kinematics* (the motion of matter), *mechanics* (the study of the forces and the motion they produce), and *statics* (the way forces combine to produce equilibrium). We will take the phrase *dynamics and mechanics* to encompass all the above, although it clearly does not!

To some extent this is because the equations governing the motion of matter include some of our oldest insights into the physical world and are consequentially steeped in tradition. One of the more delightful, or for some annoying, facets of this is the occasional use of arcane vocabulary in the description of motion. The epitome must be what Goldstein<sup>1</sup> calls “the jabberwockian sounding statement” *the polhode rolls without slipping on the herpolhode lying in the invariable plane*, describing “Poinsot’s construction” – a method of visualising the free motion of a spinning rigid body. Despite this, dynamics and mechanics, including fluid mechanics, is arguably the most practically applicable of all the branches of physics.

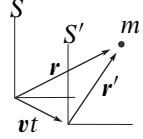
Moreover, and in common with electromagnetism, the study of dynamics and mechanics has spawned a good deal of mathematical apparatus that has found uses in other fields. Most notably, the ideas behind the generalised dynamics of Lagrange and Hamilton lie behind much of quantum mechanics.

<sup>1</sup>H. Goldstein, *Classical Mechanics*, 2nd ed., 1980, Addison-Wesley.

### 3.2 Frames of reference

#### Galilean transformations

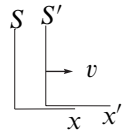
Time and position <sup>a</sup>	$\mathbf{r} = \mathbf{r}' + \mathbf{v}t$ $t = t'$	(3.1) (3.2)	$\mathbf{r}, \mathbf{r}'$ $\mathbf{v}$ $t, t'$	position in frames $S$ and $S'$ velocity of $S'$ in $S$ time in $S$ and $S'$
Velocity	$\mathbf{u} = \mathbf{u}' + \mathbf{v}$	(3.3)	$\mathbf{u}, \mathbf{u}'$	velocity in frames $S$ and $S'$
Momentum	$\mathbf{p} = \mathbf{p}' + m\mathbf{v}$	(3.4)	$\mathbf{p}, \mathbf{p}'$ $m$	particle momentum in frames $S$ and $S'$ particle mass
Angular momentum	$\mathbf{J} = \mathbf{J}' + m\mathbf{r}' \times \mathbf{v} + \mathbf{v} \times \mathbf{p}'t$	(3.5)	$\mathbf{J}, \mathbf{J}'$	angular momentum in frames $S$ and $S'$
Kinetic energy	$T = T' + m\mathbf{u}' \cdot \mathbf{v} + \frac{1}{2}m\mathbf{v}^2$	(3.6)	$T, T'$	kinetic energy in frames $S$ and $S'$



<sup>a</sup>Frames coincide at  $t=0$ .

#### Lorentz (spacetime) transformations<sup>a</sup>

Lorentz factor	$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$	(3.7)	$\gamma$ $v$ $c$	Lorentz factor velocity of $S'$ in $S$ speed of light
Time and position	$x = \gamma(x' + vt')$ ; $x' = \gamma(x - vt)$ $y = y'$ ; $y' = y$ $z = z'$ ; $z' = z$ $t = \gamma\left(t' + \frac{v}{c^2}x'\right)$ ; $t' = \gamma\left(t - \frac{v}{c^2}x\right)$	(3.8) (3.9) (3.10) (3.11)	$x, x'$ $t, t'$	$x$ -position in frames $S$ and $S'$ (similarly for $y$ and $z$ ) time in frames $S$ and $S'$
Differential four-vector <sup>b</sup>	$d\mathbf{X} = (cdt, -dx, -dy, -dz)$	(3.12)	$\mathbf{X}$	spacetime four-vector

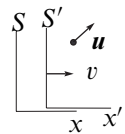


<sup>a</sup>For frames  $S$  and  $S'$  coincident at  $t=0$  in relative motion along  $x$ . See page 141 for the transformations of electromagnetic quantities.

<sup>b</sup>Covariant components, using the  $(1, -1, -1, -1)$  signature.

#### Velocity transformations<sup>a</sup>

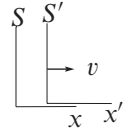
Velocity	$u_x = \frac{u'_x + v}{1 + u'_x v/c^2}$ ; $u'_x = \frac{u_x - v}{1 - u_x v/c^2}$	(3.13)	$\gamma$ $v$ $c$ $u_i, u'_i$	Lorentz factor $= [1 - (v/c)^2]^{-1/2}$ velocity of $S'$ in $S$ speed of light particle velocity components in frames $S$ and $S'$
	$u_y = \frac{u'_y}{\gamma(1 + u'_x v/c^2)}$ ; $u'_y = \frac{u_y}{\gamma(1 - u_x v/c^2)}$	(3.14)		
	$u_z = \frac{u'_z}{\gamma(1 + u'_x v/c^2)}$ ; $u'_z = \frac{u_z}{\gamma(1 - u_x v/c^2)}$	(3.15)		



<sup>a</sup>For frames  $S$  and  $S'$  coincident at  $t=0$  in relative motion along  $x$ .

**Momentum and energy transformations<sup>a</sup>**

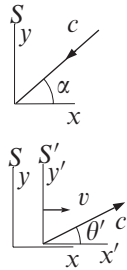
<p>Momentum and energy</p> $p_x = \gamma(p'_x + vE'/c^2); \quad p'_x = \gamma(p_x - vE/c^2) \quad (3.16)$ $p_y = p'_y; \quad p'_y = p_y \quad (3.17)$ $p_z = p'_z; \quad p'_z = p_z \quad (3.18)$ $E = \gamma(E' + vp'_x); \quad E' = \gamma(E - vp_x) \quad (3.19)$ $E^2 - p^2c^2 = E'^2 - p'^2c^2 = m_0^2c^4 \quad (3.20)$	<p><math>\gamma</math> Lorentz factor  <math>= [1 - (v/c)^2]^{-1/2}</math></p> <p><math>v</math> velocity of <math>S'</math> in <math>S</math></p> <p><math>c</math> speed of light</p> <p><math>p_x, p'_x</math> x components of momentum in <math>S</math> and <math>S'</math> (sim. for <math>y</math> and <math>z</math>)</p> <p><math>E, E'</math> energy in <math>S</math> and <math>S'</math></p> <p><math>m_0</math> (rest) mass</p> <p><math>p</math> total momentum in <math>S</math></p> <p><math>\mathbf{P}</math> momentum four-vector</p>
<p>Four-vector<sup>b</sup> <math>\mathbf{P} = (E/c, -p_x, -p_y, -p_z) \quad (3.21)</math></p>	



<sup>a</sup>For frames  $S$  and  $S'$  coincident at  $t=0$  in relative motion along  $x$ .  
<sup>b</sup>Covariant components, using the  $(1, -1, -1, -1)$  signature.

**Propagation of light<sup>a</sup>**

<p>Doppler effect</p> $\frac{v'}{v} = \gamma \left( 1 + \frac{v}{c} \cos \alpha \right) \quad (3.22)$	<p><math>v</math> frequency received in <math>S</math></p> <p><math>v'</math> frequency emitted in <math>S'</math></p> <p><math>\alpha</math> arrival angle in <math>S</math></p>
<p>Aberration<sup>b</sup></p> $\cos \theta = \frac{\cos \theta' + v/c}{1 + (v/c) \cos \theta'} \quad (3.23)$ $\cos \theta' = \frac{\cos \theta - v/c}{1 - (v/c) \cos \theta} \quad (3.24)$	<p><math>\gamma</math> Lorentz factor  <math>= [1 - (v/c)^2]^{-1/2}</math></p> <p><math>v</math> velocity of <math>S'</math> in <math>S</math></p> <p><math>c</math> speed of light</p> <p><math>\theta, \theta'</math> emission angle of light in <math>S</math> and <math>S'</math></p>
<p>Relativistic beaming<sup>c</sup></p> $P(\theta) = \frac{\sin \theta}{2\gamma^2 [1 - (v/c) \cos \theta]^2} \quad (3.25)$	<p><math>P(\theta)</math> angular distribution of photons in <math>S</math></p>



<sup>a</sup>For frames  $S$  and  $S'$  coincident at  $t=0$  in relative motion along  $x$ .  
<sup>b</sup>Light travelling in the opposite sense has a propagation angle of  $\pi + \theta$  radians.  
<sup>c</sup>Angular distribution of photons from a source, isotropic and stationary in  $S'$ .  $\int_0^\pi P(\theta) d\theta = 1$ .

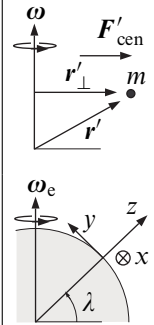
**Four-vectors<sup>a</sup>**

<p>Covariant and contravariant components</p> $x_0 = x^0 \quad x_1 = -x^1$ $x_2 = -x^2 \quad x_3 = -x^3 \quad (3.26)$	<p><math>x_i</math> covariant vector components</p> <p><math>x^i</math> contravariant components</p>
<p>Scalar product</p> $x^i y_i = x^0 y_0 + x^1 y_1 + x^2 y_2 + x^3 y_3 \quad (3.27)$	
<p>Lorentz transformations</p> $x^0 = \gamma[x'^0 + (v/c)x'^1]; \quad x'^0 = \gamma[x^0 - (v/c)x^1] \quad (3.28)$ $x^1 = \gamma[x'^1 + (v/c)x'^0]; \quad x'^1 = \gamma[x^1 - (v/c)x^0] \quad (3.29)$ $x^2 = x'^2; \quad x'^2 = x^2 \quad (3.30)$	<p><math>x^i, x'^i</math> four-vector components in frames <math>S</math> and <math>S'</math></p> <p><math>\gamma</math> Lorentz factor  <math>= [1 - (v/c)^2]^{-1/2}</math></p> <p><math>v</math> velocity of <math>S'</math> in <math>S</math></p> <p><math>c</math> speed of light</p>

<sup>a</sup>For frames  $S$  and  $S'$ , coincident at  $t=0$  in relative motion along the (1) direction. Note that the  $(1, -1, -1, -1)$  signature used here is common in special relativity, whereas  $(-1, 1, 1, 1)$  is often used in connection with general relativity (page 67).

### Rotating frames

Vector transformation	$\left[ \frac{dA}{dt} \right]_S = \left[ \frac{dA}{dt} \right]_{S'} + \omega \times A \quad (3.31)$	$A$ any vector $S$ stationary frame $S'$ rotating frame $\omega$ angular velocity of $S'$ in $S$
Acceleration	$\dot{v} = \dot{v}' + 2\omega \times v' + \omega \times (\omega \times r') \quad (3.32)$	$\dot{v}, \dot{v}'$ accelerations in $S$ and $S'$ $v'$ velocity in $S'$ $r'$ position in $S'$
Coriolis force	$F'_{\text{cor}} = -2m\omega \times v' \quad (3.33)$	$F'_{\text{cor}}$ coriolis force $m$ particle mass
Centrifugal force	$F'_{\text{cen}} = -m\omega \times (\omega \times r') \quad (3.34)$	$F'_{\text{cen}}$ centrifugal force $r'_\perp$ perpendicular to particle from rotation axis
	$= +m\omega^2 r'_\perp \quad (3.35)$	
Motion relative to Earth	$m\ddot{x} = F_x + 2m\omega_e(\dot{y} \sin \lambda - \dot{z} \cos \lambda) \quad (3.36)$	$F_i$ nongravitational force
	$m\ddot{y} = F_y - 2m\omega_e \dot{x} \sin \lambda \quad (3.37)$	$\lambda$ latitude
	$m\ddot{z} = F_z - mg + 2m\omega_e \dot{x} \cos \lambda \quad (3.38)$	$z$ local vertical axis $y$ northerly axis $x$ easterly axis
Foucault's pendulum <sup>a</sup>	$\Omega_f = -\omega_e \sin \lambda \quad (3.39)$	$\Omega_f$ pendulum's rate of turn $\omega_e$ Earth's spin rate

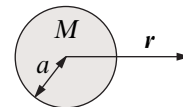


<sup>a</sup>The sign is such as to make the rotation clockwise in the northern hemisphere.

### 3.3 Gravitation

#### Newtonian gravitation

Newton's law of gravitation	$F_1 = \frac{Gm_1m_2}{r_{12}^2} \hat{r}_{12} \quad (3.40)$	$m_{1,2}$ masses $F_1$ force on $m_1$ ( $= -F_2$ ) $r_{12}$ vector from $m_1$ to $m_2$ $\hat{\phantom{r}}$ unit vector
Newtonian field equations <sup>a</sup>	$g = -\nabla\phi \quad (3.41)$	$G$ constant of gravitation
	$\nabla^2\phi = -\nabla \cdot g = 4\pi G\rho \quad (3.42)$	$g$ gravitational field strength $\phi$ gravitational potential $\rho$ mass density
Fields from an isolated uniform sphere, mass $M$ , $r$ from the centre	$g(r) = \begin{cases} -\frac{GM}{r^2} \hat{r} & (r > a) \\ -\frac{GM}{a^3} r \hat{r} & (r < a) \end{cases} \quad (3.43)$	$r$ vector from sphere centre $M$ mass of sphere $a$ radius of sphere
	$\phi(r) = \begin{cases} -\frac{GM}{r} & (r > a) \\ \frac{GM}{2a^3}(r^2 - 3a^2) & (r < a) \end{cases} \quad (3.44)$	



<sup>a</sup>The gravitational force on a mass  $m$  is  $mg$ .

**General relativity<sup>a</sup>**

Line element	$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -d\tau^2$	(3.45)	ds dτ $g_{\mu\nu}$ $dx^\mu$	invariant interval proper time interval metric tensor differential of $x^\mu$
Christoffel symbols and covariant differentiation	$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} (g_{\delta\beta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta})$	(3.46)	$\Gamma^\alpha_{\beta\gamma}$	Christoffel symbols
	$\phi_{;\gamma} = \phi_{,\gamma} \equiv \partial\phi/\partial x^\gamma$	(3.47)	$_{,\alpha}$	partial diff. w.r.t. $x^\alpha$
	$A^\alpha_{;\gamma} = A^\alpha_{,\gamma} + \Gamma^\alpha_{\beta\gamma} A^\beta$	(3.48)	$_{;\alpha}$	covariant diff. w.r.t. $x^\alpha$
	$B_{\alpha;\gamma} = B_{\alpha,\gamma} - \Gamma^\beta_{\alpha\gamma} B_\beta$	(3.49)	$\phi$ $A^\alpha$ $B_\alpha$	scalar contravariant vector covariant vector
Riemann tensor	$R^\alpha_{\beta\gamma\delta} = \Gamma^\alpha_{\mu\gamma} \Gamma^\mu_{\beta\delta} - \Gamma^\alpha_{\mu\delta} \Gamma^\mu_{\beta\gamma} + \Gamma^\alpha_{\beta\delta,\gamma} - \Gamma^\alpha_{\beta\gamma,\delta}$	(3.50)	$R^\alpha_{\beta\gamma\delta}$	Riemann tensor
	$B_{\mu;\alpha;\beta} - B_{\mu;\beta;\alpha} = R^\gamma_{\mu\alpha\beta} B_\gamma$	(3.51)		
	$R_{\alpha\beta\gamma\delta} = -R_{\alpha\delta\beta\gamma} ; R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\gamma\delta}$	(3.52)		
	$R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0$	(3.53)		
Geodesic equation	$\frac{Dv^\mu}{D\lambda} = 0$	(3.54)	$v^\mu$	tangent vector (= $dx^\mu/d\lambda$ )
	where $\frac{DA^\mu}{D\lambda} \equiv \frac{dA^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta} A^\alpha v^\beta$	(3.55)	$\lambda$	affine parameter (e.g., $\tau$ for material particles)
Geodesic deviation	$\frac{D^2 \xi^\mu}{D\lambda^2} = -R^\mu_{\alpha\beta\gamma} v^\alpha \xi^\beta v^\gamma$	(3.56)	$\xi^\mu$	geodesic deviation
Ricci tensor	$R_{\alpha\beta} \equiv R^\sigma_{\alpha\sigma\beta} = g^{\sigma\delta} R_{\delta\alpha\sigma\beta} = R_{\beta\alpha}$	(3.57)	$R_{\alpha\beta}$	Ricci tensor
Einstein tensor	$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$	(3.58)	$G^{\mu\nu}$ $R$	Einstein tensor Ricci scalar (= $g^{\mu\nu} R_{\mu\nu}$ )
Einstein's field equations	$G^{\mu\nu} = 8\pi T^{\mu\nu}$	(3.59)	$T^{\mu\nu}$ $p$	stress-energy tensor pressure (in rest frame)
Perfect fluid	$T^{\mu\nu} = (p + \rho)u^\mu u^\nu + pg^{\mu\nu}$	(3.60)	$\rho$ $u^\nu$	density (in rest frame) fluid four-velocity
Schwarzschild solution (exterior)	$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$	(3.61)	$M$ $(r, \theta, \phi)$ $t$	spherically symmetric mass (see page 183) spherical polar coords. time
Kerr solution (outside a spinning black hole)	$ds^2 = -\frac{\Delta - a^2 \sin^2\theta}{\rho^2} dt^2 - 2a \frac{2Mr \sin^2\theta}{\rho^2} dt d\phi + \frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2\theta}{\rho^2} \sin^2\theta d\phi^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2$	(3.62)	$J$ $a$ $\Delta$ $\rho^2$	angular momentum (along $z$ ) $\equiv J/M$ $\equiv r^2 - 2Mr + a^2$ $\equiv r^2 + a^2 \cos^2\theta$

<sup>a</sup>General relativity conventionally uses the (-1,1,1,1) metric signature and “geometrized units” in which  $G=1$  and  $c=1$ . Thus,  $1\text{ kg} = 7.425 \times 10^{-28}\text{ m}$  etc. Contravariant indices are written as superscripts and covariant indices as subscripts. Note also that  $ds^2$  means  $(ds)^2$  etc.



### 3.4 Particle motion

#### Dynamics definitions<sup>a</sup>

Newtonian force	$\mathbf{F} = m\ddot{\mathbf{r}} = \dot{\mathbf{p}}$	(3.63)	$\mathbf{F}$ force $m$ mass of particle $\mathbf{r}$ particle position vector
Momentum	$\mathbf{p} = m\dot{\mathbf{r}}$	(3.64)	$\mathbf{p}$ momentum
Kinetic energy	$T = \frac{1}{2}mv^2$	(3.65)	$T$ kinetic energy $v$ particle velocity
Angular momentum	$\mathbf{J} = \mathbf{r} \times \mathbf{p}$	(3.66)	$\mathbf{J}$ angular momentum
Couple (or torque)	$\mathbf{G} = \mathbf{r} \times \mathbf{F}$	(3.67)	$\mathbf{G}$ couple
Centre of mass (ensemble of $N$ particles)	$\mathbf{R}_0 = \frac{\sum_{i=1}^N m_i \mathbf{r}_i}{\sum_{i=1}^N m_i}$	(3.68)	$\mathbf{R}_0$ position vector of centre of mass $m_i$ mass of $i$ th particle $\mathbf{r}_i$ position vector of $i$ th particle

<sup>a</sup>In the Newtonian limit,  $v \ll c$ , assuming  $m$  is constant.

#### Relativistic dynamics<sup>a</sup>

Lorentz factor	$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$	(3.69)	$\gamma$ Lorentz factor $v$ particle velocity $c$ speed of light
Momentum	$\mathbf{p} = \gamma m_0 \mathbf{v}$	(3.70)	$\mathbf{p}$ relativistic momentum $m_0$ particle (rest) mass
Force	$\mathbf{F} = \frac{d\mathbf{p}}{dt}$	(3.71)	$\mathbf{F}$ force on particle $t$ time
Rest energy	$E_r = m_0 c^2$	(3.72)	$E_r$ particle rest energy
Kinetic energy	$T = m_0 c^2 (\gamma - 1)$	(3.73)	$T$ relativistic kinetic energy
Total energy	$E = \gamma m_0 c^2$	(3.74)	$E$ total energy ( $= E_r + T$ )
	$= (p^2 c^2 + m_0^2 c^4)^{1/2}$	(3.75)	

<sup>a</sup>It is now common to regard mass as a Lorentz invariant property and to drop the term “rest mass.” The symbol  $m_0$  is used here to avoid confusion with the idea of “relativistic mass” ( $= \gamma m_0$ ) used by some authors.

#### Constant acceleration

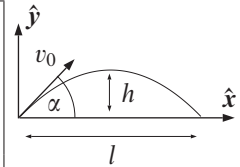
$v = u + at$	(3.76)	$u$ initial velocity $v$ final velocity $t$ time $s$ distance travelled $a$ acceleration
$v^2 = u^2 + 2as$	(3.77)	
$s = ut + \frac{1}{2}at^2$	(3.78)	
$s = \frac{u+v}{2}t$	(3.79)	

**Reduced mass (of two interacting bodies)**

Reduced mass	$\mu = \frac{m_1 m_2}{m_1 + m_2}$ (3.80)	$\mu$ reduced mass $m_i$ interacting masses
Distances from centre of mass	$r_1 = \frac{m_2}{m_1 + m_2} r$ (3.81)	$r_i$ position vectors from centre of mass
	$r_2 = \frac{-m_1}{m_1 + m_2} r$ (3.82)	$r$ $r = r_1 - r_2$ $ r $ distance between masses
Moment of inertia	$I = \mu  r ^2$ (3.83)	$I$ moment of inertia
Total angular momentum	$J = \mu r \times \dot{r}$ (3.84)	$J$ angular momentum
Lagrangian	$L = \frac{1}{2} \mu  \dot{r} ^2 - U( r )$ (3.85)	$L$ Lagrangian $U$ potential energy of interaction

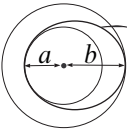
**Ballistics<sup>a</sup>**

Velocity	$v = v_0 \cos \alpha \hat{x} + (v_0 \sin \alpha - gt) \hat{y}$ (3.86)	$v_0$ initial velocity $v$ velocity at t $\alpha$ elevation angle $g$ gravitational acceleration
	$v^2 = v_0^2 - 2gy$ (3.87)	
Trajectory	$y = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha}$ (3.88)	$\hat{\phantom{x}}$ unit vector $t$ time
Maximum height	$h = \frac{v_0^2}{2g} \sin^2 \alpha$ (3.89)	$h$ maximum height
Horizontal range	$l = \frac{v_0^2}{g} \sin 2\alpha$ (3.90)	$l$ range



<sup>a</sup>Ignoring the curvature and rotation of the Earth and frictional losses,  $g$  is assumed constant.

## Rocketry

Escape velocity <sup>a</sup>	$v_{\text{esc}} = \left( \frac{2GM}{r} \right)^{1/2} \quad (3.91)$	$v_{\text{esc}}$ escape velocity $G$ constant of gravitation $M$ mass of central body $r$ central body radius
Specific impulse	$I_{\text{sp}} = \frac{u}{g} \quad (3.92)$	$I_{\text{sp}}$ specific impulse $u$ effective exhaust velocity $g$ acceleration due to gravity
Exhaust velocity (into a vacuum)	$u = \left[ \frac{2\gamma RT_c}{(\gamma - 1)\mu} \right]^{1/2} \quad (3.93)$	$R$ molar gas constant $\gamma$ ratio of heat capacities $T_c$ combustion temperature $\mu$ effective molecular mass of exhaust gas
Rocket equation ( $g = 0$ )	$\Delta v = u \ln \left( \frac{M_i}{M_f} \right) \equiv u \ln \mathcal{M} \quad (3.94)$	$\Delta v$ rocket velocity increment $M_i$ pre-burn rocket mass $M_f$ post-burn rocket mass $\mathcal{M}$ mass ratio
Multistage rocket	$\Delta v = \sum_{i=1}^N u_i \ln \mathcal{M}_i \quad (3.95)$	$N$ number of stages $\mathcal{M}_i$ mass ratio for $i$ th burn $u_i$ exhaust velocity of $i$ th burn
In a constant gravitational field	$\Delta v = u \ln \mathcal{M} - gt \cos \theta \quad (3.96)$	$t$ burn time $\theta$ rocket zenith angle
Hohmann cotangential transfer <sup>b</sup>	$\Delta v_{ah} = \left( \frac{GM}{r_a} \right)^{1/2} \left[ \left( \frac{2r_b}{r_a + r_b} \right)^{1/2} - 1 \right] \quad (3.97)$ $\Delta v_{hb} = \left( \frac{GM}{r_b} \right)^{1/2} \left[ 1 - \left( \frac{2r_a}{r_a + r_b} \right)^{1/2} \right] \quad (3.98)$	$\Delta v_{ah}$ velocity increment, $a$ to $h$ $\Delta v_{hb}$ velocity increment, $h$ to $b$ $r_a$ radius of inner orbit $r_b$ radius of outer orbit 

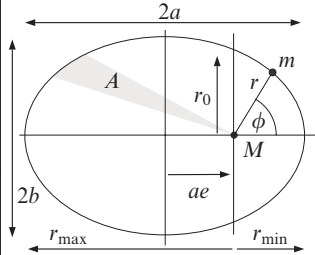
<sup>a</sup>From the surface of a spherically symmetric, nonrotating body, mass  $M$ .

<sup>b</sup>Transfer between coplanar, circular orbits  $a$  and  $b$ , via ellipse  $h$  with a minimal expenditure of energy.



**Gravitationally bound orbital motion<sup>a</sup>**

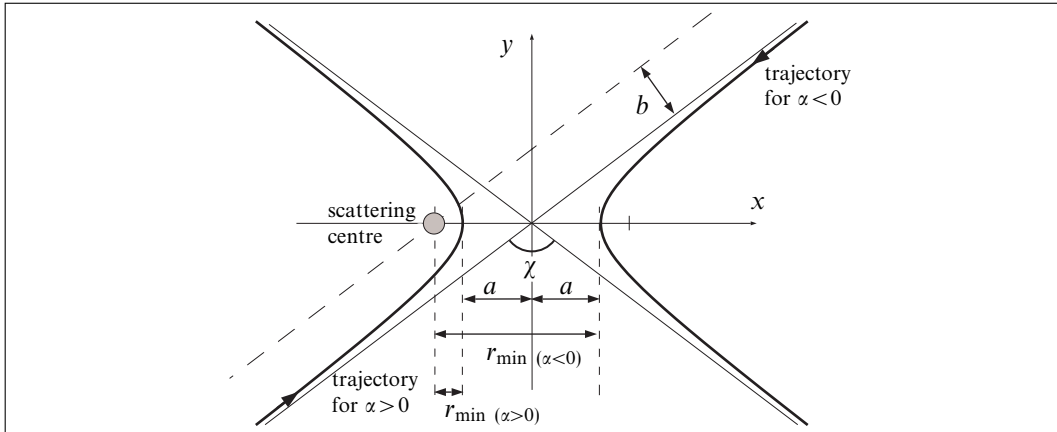
Potential energy of interaction	$U(r) = -\frac{GMm}{r} \equiv -\frac{\alpha}{r}$	(3.99)	$U(r)$ potential energy $G$ constant of gravitation $M$ central mass $m$ orbiting mass ( $\ll M$ ) $\alpha$ $GMm$ (for gravitation)
Total energy	$E = -\frac{\alpha}{r} + \frac{J^2}{2mr^2} = -\frac{\alpha}{2a}$	(3.100)	$E$ total energy (constant) $J$ total angular momentum (constant)
Virial theorem (1/r potential)	$E = \langle U \rangle / 2 = -\langle T \rangle$	(3.101)	$T$ kinetic energy
	$\langle U \rangle = -2\langle T \rangle$	(3.102)	$\langle \cdot \rangle$ mean value
Orbital equation (Kepler's 1st law)	$\frac{r_0}{r} = 1 + e \cos \phi, \quad \text{or}$	(3.103)	$r_0$ semi-latus-rectum $r$ distance of $m$ from $M$ $e$ eccentricity $\phi$ phase (true anomaly)
	$r = \frac{a(1 - e^2)}{1 + e \cos \phi}$	(3.104)	
Rate of sweeping area (Kepler's 2nd law)	$\frac{dA}{dt} = \frac{J}{2m} = \text{constant}$	(3.105)	$A$ area swept out by radius vector (total area = $\pi ab$ )
Semi-major axis	$a = \frac{r_0}{1 - e^2} = \frac{\alpha}{2 E }$	(3.106)	$a$ semi-major axis $b$ semi-minor axis
Semi-minor axis	$b = \frac{r_0}{(1 - e^2)^{1/2}} = \frac{J}{(2m E )^{1/2}}$	(3.107)	
Eccentricity <sup>b</sup>	$e = \left(1 + \frac{2EJ^2}{m\alpha^2}\right)^{1/2} = \left(1 - \frac{b^2}{a^2}\right)^{1/2}$	(3.108)	
Semi-latus-rectum	$r_0 = \frac{J^2}{m\alpha} = \frac{b^2}{a} = a(1 - e^2)$	(3.109)	
Pericentre	$r_{\min} = \frac{r_0}{1 + e} = a(1 - e)$	(3.110)	$r_{\min}$ pericentre distance
Apocentre	$r_{\max} = \frac{r_0}{1 - e} = a(1 + e)$	(3.111)	$r_{\max}$ apocentre distance
Speed	$v^2 = GM \left( \frac{2}{r} - \frac{1}{a} \right)$	(3.112)	$v$ orbital speed
Period (Kepler's 3rd law)	$P = \pi\alpha \left( \frac{m}{2 E ^3} \right)^{1/2} = 2\pi a^{3/2} \left( \frac{m}{\alpha} \right)^{1/2}$	(3.113)	$P$ orbital period



<sup>a</sup>For an inverse-square law of attraction between two isolated bodies in the nonrelativistic limit. If  $m$  is not  $\ll M$ , then the equations are valid with the substitutions  $m \rightarrow \mu = Mm/(M+m)$  and  $M \rightarrow (M+m)$  and with  $r$  taken as the body separation. The distance of mass  $m$  from the centre of mass is then  $r\mu/m$  (see earlier table on *Reduced mass*). Other orbital dimensions scale similarly, and the two orbits have the same eccentricity.

<sup>b</sup>Note that if the total energy,  $E$ , is  $< 0$  then  $e < 1$  and the orbit is an ellipse (a circle if  $e = 0$ ). If  $E = 0$ , then  $e = 1$  and the orbit is a parabola. If  $E > 0$  then  $e > 1$  and the orbit becomes a hyperbola (see *Rutherford scattering* on next page).

Rutherford scattering<sup>a</sup>



Scattering potential energy	$U(r) = -\frac{\alpha}{r}$	(3.114)	$U(r)$ potential energy
	$\alpha \begin{cases} < 0 & \text{repulsive} \\ > 0 & \text{attractive} \end{cases}$	(3.115)	$r$ particle separation $\alpha$ constant
Scattering angle	$\tan \frac{\chi}{2} = \frac{ \alpha }{2Eb}$	(3.116)	$\chi$ scattering angle $E$ total energy ( $> 0$ ) $b$ impact parameter
Closest approach	$r_{\min} = \frac{ \alpha }{2E} \left( \csc \frac{\chi}{2} - \frac{\alpha}{ \alpha } \right)$	(3.117)	$r_{\min}$ closest approach $a$ hyperbola semi-axis $e$ eccentricity
	$= a(e \pm 1)$	(3.118)	
Semi-axis	$a = \frac{ \alpha }{2E}$	(3.119)	
Eccentricity	$e = \left( \frac{4E^2 b^2}{\alpha^2} + 1 \right)^{1/2} = \csc \frac{\chi}{2}$	(3.120)	
Motion trajectory <sup>b</sup>	$\frac{4E^2}{\alpha^2} x^2 - \frac{y^2}{b^2} = 1$	(3.121)	$x, y$ position with respect to hyperbola centre
Scattering centre <sup>c</sup>	$x = \pm \left( \frac{\alpha^2}{4E^2} + b^2 \right)^{1/2}$	(3.122)	
Rutherford scattering formula <sup>d</sup>	$\frac{d\sigma}{d\Omega} = \frac{1}{n} \frac{dN}{d\Omega}$	(3.123)	$\frac{d\sigma}{d\Omega}$ differential scattering cross section
	$= \left( \frac{\alpha}{4E} \right)^2 \csc^4 \frac{\chi}{2}$	(3.124)	$n$ beam flux density $dN$ number of particles scattered into $d\Omega$ $\Omega$ solid angle

<sup>a</sup>Nonrelativistic treatment for an inverse-square force law and a fixed scattering centre. Similar scattering results from either an attractive or repulsive force. See also *Conic sections* on page 38.

<sup>b</sup>The correct branch can be chosen by inspection.

<sup>c</sup>Also the focal points of the hyperbola.

<sup>d</sup> $n$  is the number of particles per second passing through unit area perpendicular to the beam.

**Inelastic collisions<sup>a</sup>**

Coefficient of restitution	$v'_2 - v'_1 = \epsilon(v_1 - v_2)$ (3.125) $\epsilon = 1$ if perfectly elastic (3.126) $\epsilon = 0$ if perfectly inelastic (3.127)		$\epsilon$ coefficient of restitution $v_i$ pre-collision velocities $v'_i$ post-collision velocities
Loss of kinetic energy <sup>b</sup>	$\frac{T - T'}{T} = 1 - \epsilon^2$ (3.128)		$T, T'$ total KE in zero momentum frame before and after collision
Final velocities	$v'_1 = \frac{m_1 - \epsilon m_2}{m_1 + m_2} v_1 + \frac{(1 + \epsilon)m_2}{m_1 + m_2} v_2$ (3.129) $v'_2 = \frac{m_2 - \epsilon m_1}{m_1 + m_2} v_2 + \frac{(1 + \epsilon)m_1}{m_1 + m_2} v_1$ (3.130)		$m_i$ particle masses

<sup>a</sup>Along the line of centres,  $v_1, v_2 \ll c$ .

<sup>b</sup>In zero momentum frame.

**Oblique elastic collisions<sup>a</sup>**

Directions of motion	$\tan \theta'_1 = \frac{m_2 \sin 2\theta}{m_1 - m_2 \cos 2\theta}$ (3.131) $\theta'_2 = \theta$ (3.132)		$\theta$ angle between centre line and incident velocity $\theta'_i$ final trajectories $m_i$ sphere masses
Relative separation angle	$\theta'_1 + \theta'_2 \begin{cases} > \pi/2 & \text{if } m_1 < m_2 \\ = \pi/2 & \text{if } m_1 = m_2 \\ < \pi/2 & \text{if } m_1 > m_2 \end{cases}$ (3.133)		
Final velocities	$v'_1 = \frac{(m_1^2 + m_2^2 - 2m_1 m_2 \cos 2\theta)^{1/2}}{m_1 + m_2} v$ (3.134) $v'_2 = \frac{2m_1 v}{m_1 + m_2} \cos \theta$ (3.135)		$v$ incident velocity of $m_1$ $v'_i$ final velocities

<sup>a</sup>Collision between two perfectly elastic spheres:  $m_2$  initially at rest, velocities  $\ll c$ .

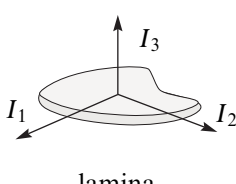
### 3.5 Rigid body dynamics

#### Moment of inertia tensor

Moment of inertia tensor <sup>a</sup>	$I_{ij} = \int (r^2 \delta_{ij} - x_i x_j) dm \quad (3.136)$	$r^2 = x^2 + y^2 + z^2$ $\delta_{ij}$ Kronecker delta $\mathbf{I}$ moment of inertia tensor $dm$ mass element $x_i$ position vector of $dm$ $I_{ij}$ components of $\mathbf{I}$
	$\mathbf{I} = \begin{pmatrix} \int (y^2 + z^2) dm & -\int xy dm & -\int xz dm \\ -\int xy dm & \int (x^2 + z^2) dm & -\int yz dm \\ -\int xz dm & -\int yz dm & \int (x^2 + y^2) dm \end{pmatrix} \quad (3.137)$	
Parallel axis theorem	$I_{12} = I_{12}^* - ma_1 a_2 \quad (3.138)$	$I_{ij}^*$ tensor with respect to centre of mass
	$I_{11} = I_{11}^* + m(a_2^2 + a_3^2) \quad (3.139)$	$a_i, \mathbf{a}$ position vector of centre of mass
	$I_{ij} = I_{ij}^* + m( \mathbf{a} ^2 \delta_{ij} - a_i a_j) \quad (3.140)$	$m$ mass of body
Angular momentum	$\mathbf{J} = \mathbf{I}\boldsymbol{\omega} \quad (3.141)$	$\mathbf{J}$ angular momentum $\boldsymbol{\omega}$ angular velocity
Rotational kinetic energy	$T = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{J} = \frac{1}{2} I_{ij} \omega_i \omega_j \quad (3.142)$	$T$ kinetic energy

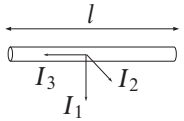
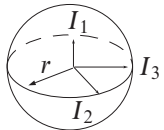
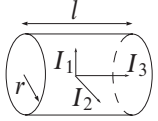
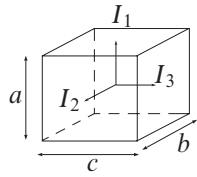
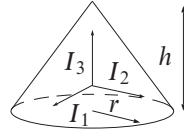
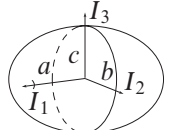
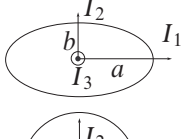
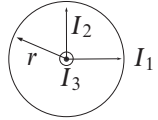
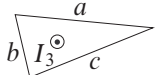
<sup>a</sup> $I_{ii}$  are the moments of inertia of the body.  $I_{ij}$  ( $i \neq j$ ) are its products of inertia. The integrals are over the body volume.

#### Principal axes

Principal moment of inertia tensor	$\mathbf{I}' = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \quad (3.143)$	$\mathbf{I}'$ principal moment of inertia tensor $I_i$ principal moments of inertia
Angular momentum	$\mathbf{J} = (I_1 \omega_1, I_2 \omega_2, I_3 \omega_3) \quad (3.144)$	$\mathbf{J}$ angular momentum $\omega_i$ components of $\boldsymbol{\omega}$ along principal axes
Rotational kinetic energy	$T = \frac{1}{2} (I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2) \quad (3.145)$	$T$ kinetic energy
Moment of inertia ellipsoid <sup>a</sup>	$T = T(\omega_1, \omega_2, \omega_3) \quad (3.146)$ $J_i = \frac{\partial T}{\partial \omega_i} \quad (\mathbf{J} \text{ is } \perp \text{ ellipsoid surface}) \quad (3.147)$	
Perpendicular axis theorem	$I_1 + I_2 \begin{cases} \geq I_3 & \text{generally} \\ = I_3 & \text{flat lamina } \perp \text{ to 3-axis} \end{cases} \quad (3.148)$	 <p style="text-align: center;">lamina</p>
Symmetries	$I_1 \neq I_2 \neq I_3 \quad \text{asymmetric top} \quad (3.149)$ $I_1 = I_2 \neq I_3 \quad \text{symmetric top}$ $I_1 = I_2 = I_3 \quad \text{spherical top}$	

<sup>a</sup>The ellipsoid is defined by the surface of constant  $T$ .

**Moments of inertia<sup>a</sup>**

Thin rod, length $l$	$I_1 = I_2 = \frac{ml^2}{12}$ (3.150)	
	$I_3 \simeq 0$ (3.151)	
Solid sphere, radius $r$	$I_1 = I_2 = I_3 = \frac{2}{5}mr^2$ (3.152)	
Spherical shell, radius $r$	$I_1 = I_2 = I_3 = \frac{2}{3}mr^2$ (3.153)	
Solid cylinder, radius $r$ , length $l$	$I_1 = I_2 = \frac{m}{4} \left( r^2 + \frac{l^2}{3} \right)$ (3.154)	
	$I_3 = \frac{1}{2}mr^2$ (3.155)	
Solid cuboid, sides $a, b, c$	$I_1 = m(b^2 + c^2)/12$ (3.156)	
	$I_2 = m(c^2 + a^2)/12$ (3.157)	
	$I_3 = m(a^2 + b^2)/12$ (3.158)	
Solid circular cone, base radius $r$ , height $h$ <sup>b</sup>	$I_1 = I_2 = \frac{3}{20}m \left( r^2 + \frac{h^2}{4} \right)$ (3.159)	
	$I_3 = \frac{3}{10}mr^2$ (3.160)	
Solid ellipsoid, semi-axes $a, b, c$	$I_1 = m(b^2 + c^2)/5$ (3.161)	
	$I_2 = m(c^2 + a^2)/5$ (3.162)	
	$I_3 = m(a^2 + b^2)/5$ (3.163)	
Elliptical lamina, semi-axes $a, b$	$I_1 = mb^2/4$ (3.164)	
	$I_2 = ma^2/4$ (3.165)	
	$I_3 = m(a^2 + b^2)/4$ (3.166)	
Disk, radius $r$	$I_1 = I_2 = mr^2/4$ (3.167)	
	$I_3 = mr^2/2$ (3.168)	
Triangular plate <sup>c</sup>	$I_3 = \frac{m}{36}(a^2 + b^2 + c^2)$ (3.169)	



<sup>a</sup>With respect to principal axes for bodies of mass  $m$  and uniform density. The radius of gyration is defined as  $k = (I/m)^{1/2}$ .

<sup>b</sup>Origin of axes is at the centre of mass ( $h/4$  above the base).

<sup>c</sup>Around an axis through the centre of mass and perpendicular to the plane of the plate.

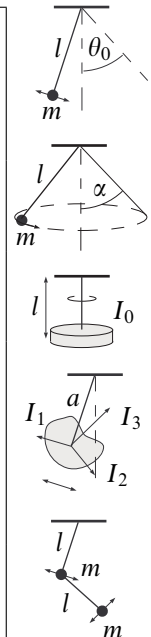
**Centres of mass**

Solid hemisphere, radius $r$	$d = 3r/8$ from sphere centre	(3.170)
Hemispherical shell, radius $r$	$d = r/2$ from sphere centre	(3.171)
Sector of disk, radius $r$ , angle $2\theta$	$d = \frac{2}{3}r \frac{\sin\theta}{\theta}$ from disk centre	(3.172)
Arc of circle, radius $r$ , angle $2\theta$	$d = r \frac{\sin\theta}{\theta}$ from circle centre	(3.173)
Arbitrary triangular lamina, height $h^a$	$d = h/3$ perpendicular from base	(3.174)
Solid cone or pyramid, height $h$	$d = h/4$ perpendicular from base	(3.175)
Spherical cap, height $h$ , sphere radius $r$	solid: $d = \frac{3}{4} \frac{(2r-h)^2}{3r-h}$ from sphere centre	(3.176)
	shell: $d = r - h/2$ from sphere centre	(3.177)
Semi-elliptical lamina, height $h$	$d = \frac{4h}{3\pi}$ from base	(3.178)

<sup>a</sup> $h$  is the perpendicular distance between the base and apex of the triangle.

**Pendulums**

Simple pendulum	$P = 2\pi \sqrt{\frac{l}{g} \left( 1 + \frac{\theta_0^2}{16} + \dots \right)}$ (3.179)	<p><math>P</math> period</p> <p><math>g</math> gravitational acceleration</p> <p><math>l</math> length</p> <p><math>\theta_0</math> maximum angular displacement</p>
Conical pendulum	$P = 2\pi \left( \frac{l \cos\alpha}{g} \right)^{1/2}$ (3.180)	$\alpha$ cone half-angle
Torsional pendulum <sup>a</sup>	$P = 2\pi \left( \frac{I I_0}{C} \right)^{1/2}$ (3.181)	<p><math>I_0</math> moment of inertia of bob</p> <p><math>C</math> torsional rigidity of wire (see page 81)</p>
Compound pendulum <sup>b</sup>	$P \simeq 2\pi \left[ \frac{1}{mg a} (ma^2 + I_1 \cos^2 \gamma_1 + I_2 \cos^2 \gamma_2 + I_3 \cos^2 \gamma_3) \right]^{1/2}$ (3.182)	<p><math>a</math> distance of rotation axis from centre of mass</p> <p><math>m</math> mass of body</p> <p><math>I_i</math> principal moments of inertia</p> <p><math>\gamma_i</math> angles between rotation axis and principal axes</p>
Equal double pendulum <sup>c</sup>	$P \simeq 2\pi \left[ \frac{l}{(2 \pm \sqrt{2})g} \right]^{1/2}$ (3.183)	

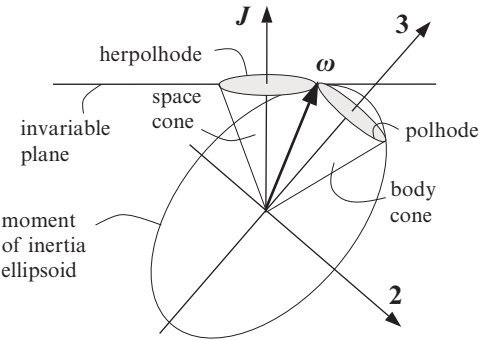
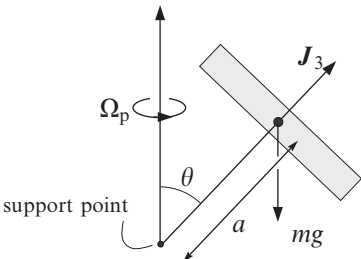


<sup>a</sup>Assuming the bob is supported parallel to a principal rotation axis.

<sup>b</sup>I.e., an arbitrary triaxial rigid body.

<sup>c</sup>For very small oscillations (two eigenmodes).

Tops and gyroscopes

			
prolate symmetric top		gyroscope	
Euler's equations <sup>a</sup>	$G_1 = I_1 \dot{\omega}_1 + (I_3 - I_2) \omega_2 \omega_3$	(3.184)	$G_i$ external couple (=0 for free rotation)
	$G_2 = I_2 \dot{\omega}_2 + (I_1 - I_3) \omega_3 \omega_1$	(3.185)	$I_i$ principal moments of inertia
	$G_3 = I_3 \dot{\omega}_3 + (I_2 - I_1) \omega_1 \omega_2$	(3.186)	$\omega_i$ angular velocity of rotation
Free symmetric top <sup>b</sup> ( $I_3 < I_2 = I_1$ )	$\Omega_b = \frac{I_1 - I_3}{I_1} \omega_3$	(3.187)	$\Omega_b$ body frequency
	$\Omega_s = \frac{J}{I_1}$	(3.188)	$\Omega_s$ space frequency
Free asymmetric top <sup>c</sup>	$\Omega_b^2 = \frac{(I_1 - I_3)(I_2 - I_3)}{I_1 I_2} \omega_3^2$	(3.189)	$J$ total angular momentum
Steady gyroscopic precession	$\Omega_p^2 I_1' \cos \theta - \Omega_p J_3 + m g a = 0$	(3.190)	$\Omega_p$ precession angular velocity
	$\Omega_p \simeq \begin{cases} M g a / J_3 & \text{(slow)} \\ J_3 / (I_1' \cos \theta) & \text{(fast)} \end{cases}$	(3.191)	$\theta$ angle from vertical
Gyroscopic stability	$J_3^2 \geq 4 I_1' m g a \cos \theta$	(3.192)	$J_3$ angular momentum around symmetry axis
Gyroscopic limit ("sleeping top")	$J_3^2 \gg I_1' m g a$	(3.193)	$m$ mass
Nutation rate	$\Omega_n = J_3 / I_1'$	(3.194)	$g$ gravitational acceleration
Gyroscope released from rest	$\Omega_p = \frac{m g a}{J_3} (1 - \cos \Omega_n t)$	(3.195)	$a$ distance of centre of mass from support point
			$I_1'$ moment of inertia about support point
			$\Omega_n$ nutation angular velocity
			$t$ time

<sup>a</sup>Components are with respect to the principal axes, rotating with the body.

<sup>b</sup>The body frequency is the angular velocity (with respect to principal axes) of  $\omega$  around the 3-axis. The space frequency is the angular velocity of the 3-axis around  $J$ , i.e., the angular velocity at which the body cone moves around the space cone.

<sup>c</sup> $J$  close to 3-axis. If  $\Omega_b^2 < 0$ , the body tumbles.



### 3.6 Oscillating systems

#### Free oscillations

Differential equation	$\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = 0 \quad (3.196)$	$x$ oscillating variable $t$ time $\gamma$ damping factor (per unit mass) $\omega_0$ undamped angular frequency
Underdamped solution ( $\gamma < \omega_0$ )	$x = Ae^{-\gamma t} \cos(\omega t + \phi) \quad (3.197)$ where $\omega = (\omega_0^2 - \gamma^2)^{1/2} \quad (3.198)$	$A$ amplitude constant $\phi$ phase constant $\omega$ angular eigenfrequency
Critically damped solution ( $\gamma = \omega_0$ )	$x = e^{-\gamma t} (A_1 + A_2 t) \quad (3.199)$	$A_i$ amplitude constants
Overdamped solution ( $\gamma > \omega_0$ )	$x = e^{-\gamma t} (A_1 e^{qt} + A_2 e^{-qt}) \quad (3.200)$ where $q = (\gamma^2 - \omega_0^2)^{1/2} \quad (3.201)$	
Logarithmic decrement <sup>a</sup>	$\Delta = \ln \frac{a_n}{a_{n+1}} = \frac{2\pi\gamma}{\omega} \quad (3.202)$	$\Delta$ logarithmic decrement $a_n$ $n$ th displacement maximum
Quality factor	$Q = \frac{\omega_0}{2\gamma} \left[ \simeq \frac{\pi}{\Delta} \text{ if } Q \gg 1 \right] \quad (3.203)$	$Q$ quality factor

<sup>a</sup>The *decrement* is usually the ratio of successive displacement *maxima* but is sometimes taken as the ratio of successive displacement *extrema*, reducing  $\Delta$  by a factor of 2. Logarithms are sometimes taken to base 10, introducing a further factor of  $\log_{10} e$ .

#### Forced oscillations

Differential equation	$\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = F_0 e^{i\omega_f t} \quad (3.204)$	$x$ oscillating variable $t$ time $\gamma$ damping factor (per unit mass)
Steady-state solution <sup>a</sup>	$x = Ae^{i(\omega_f t - \phi)}, \text{ where} \quad (3.205)$	$\omega_0$ undamped angular frequency
	$A = F_0 [(\omega_0^2 - \omega_f^2)^2 + (2\gamma\omega_f)^2]^{-1/2} \quad (3.206)$	$F_0$ force amplitude (per unit mass)
	$\simeq \frac{F_0/(2\omega_0)}{[(\omega_0 - \omega_f)^2 + \gamma^2]^{1/2}} \quad (\gamma \ll \omega_f) \quad (3.207)$	$\omega_f$ forcing angular frequency
	$\tan \phi = \frac{2\gamma\omega_f}{\omega_0^2 - \omega_f^2} \quad (3.208)$	$A$ amplitude $\phi$ phase lag of response behind driving force
Amplitude resonance <sup>b</sup>	$\omega_{ar}^2 = \omega_0^2 - 2\gamma^2 \quad (3.209)$	$\omega_{ar}$ amplitude resonant forcing angular frequency
Velocity resonance <sup>c</sup>	$\omega_{vr} = \omega_0 \quad (3.210)$	$\omega_{vr}$ velocity resonant forcing angular frequency
Quality factor	$Q = \frac{\omega_0}{2\gamma} \quad (3.211)$	$Q$ quality factor
Impedance	$Z = 2\gamma + i \frac{\omega_f^2 - \omega_0^2}{\omega_f} \quad (3.212)$	$Z$ impedance (per unit mass)

<sup>a</sup>Excluding the free oscillation terms.

<sup>b</sup>Forcing frequency for maximum displacement.

<sup>c</sup>Forcing frequency for maximum velocity. Note  $\phi = \pi/2$  at this frequency.





### 3.7 Generalised dynamics

#### Lagrangian dynamics

Action	$S = \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}}, t) dt \quad (3.213)$	$S$ action ( $\delta S = 0$ for the motion)
Euler–Lagrange equation	$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \quad (3.214)$	$\mathbf{q}$ generalised coordinates $\dot{\mathbf{q}}$ generalised velocities $L$ Lagrangian $t$ time $m$ mass
Lagrangian of particle in external field	$L = \frac{1}{2}mv^2 - U(\mathbf{r}, t) \quad (3.215)$ $= T - U \quad (3.216)$	$\mathbf{v}$ velocity $\mathbf{r}$ position vector $U$ potential energy $T$ kinetic energy
Relativistic Lagrangian of a charged particle	$L = -\frac{m_0 c^2}{\gamma} - e(\phi - \mathbf{A} \cdot \mathbf{v}) \quad (3.217)$	$m_0$ (rest) mass $\gamma$ Lorentz factor $+e$ positive charge $\phi$ electric potential $\mathbf{A}$ magnetic vector potential
Generalised momenta	$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad (3.218)$	$p_i$ generalised momenta

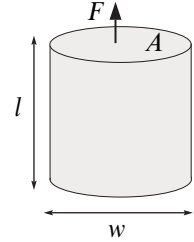
#### Hamiltonian dynamics

Hamiltonian	$H = \sum_i p_i \dot{q}_i - L \quad (3.219)$	$L$ Lagrangian $p_i$ generalised momenta $\dot{q}_i$ generalised velocities
Hamilton's equations	$\dot{q}_i = \frac{\partial H}{\partial p_i}; \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad (3.220)$	$H$ Hamiltonian $q_i$ generalised coordinates
Hamiltonian of particle in external field	$H = \frac{1}{2}mv^2 + U(\mathbf{r}, t) \quad (3.221)$ $= T + U \quad (3.222)$	$v$ particle speed $\mathbf{r}$ position vector $U$ potential energy $T$ kinetic energy
Relativistic Hamiltonian of a charged particle	$H = (m_0^2 c^4 +  \mathbf{p} - e\mathbf{A} ^2 c^2)^{1/2} + e\phi \quad (3.223)$	$m_0$ (rest) mass $c$ speed of light $+e$ positive charge $\phi$ electric potential $\mathbf{A}$ vector potential
Poisson brackets	$[f, g] = \sum_i \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \quad (3.224)$ $[q_i, g] = \frac{\partial g}{\partial p_i}, \quad [p_i, g] = -\frac{\partial g}{\partial q_i} \quad (3.225)$ $[H, g] = 0 \quad \text{if} \quad \frac{\partial g}{\partial t} = 0, \quad \frac{dg}{dt} = 0 \quad (3.226)$	$\mathbf{p}$ particle momentum $t$ time $f, g$ arbitrary functions $[\cdot, \cdot]$ Poisson bracket (also see <i>Commutators</i> on page 26)
Hamilton–Jacobi equation	$\frac{\partial S}{\partial t} + H \left( q_i, \frac{\partial S}{\partial q_i}, t \right) = 0 \quad (3.227)$	$S$ action

### 3.8 Elasticity

#### Elasticity definitions (simple)<sup>a</sup>

Stress	$\tau = F/A$	(3.228)	$\tau$ stress $F$ applied force $A$ cross-sectional area
Strain	$e = \delta l/l$	(3.229)	$e$ strain $\delta l$ change in length $l$ length
Young modulus (Hooke's law)	$E = \tau/e = \text{constant}$	(3.230)	$E$ Young modulus
Poisson ratio <sup>b</sup>	$\sigma = -\frac{\delta w/w}{\delta l/l}$	(3.231)	$\sigma$ Poisson ratio $\delta w$ change in width $w$ width



<sup>a</sup>These apply to a thin wire under longitudinal stress.

<sup>b</sup>Solids obeying Hooke's law are restricted by thermodynamics to  $-1 \leq \sigma \leq 1/2$ , but none are known with  $\sigma < 0$ . Non-Hookean materials can show  $\sigma > 1/2$ .

#### Elasticity definitions (general)

Stress tensor <sup>a</sup>	$\tau_{ij} = \frac{\text{force } \parallel i \text{ direction}}{\text{area } \perp j \text{ direction}}$	(3.232)	$\tau_{ij}$ stress tensor ( $\tau_{ij} = \tau_{ji}$ )
Strain tensor	$e_{kl} = \frac{1}{2} \left( \frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right)$	(3.233)	$e_{kl}$ strain tensor ( $e_{kl} = e_{lk}$ ) $u_k$ displacement $\parallel$ to $x_k$ $x_k$ coordinate system
Elastic modulus	$\tau_{ij} = \lambda_{ijkl} e_{kl}$	(3.234)	$\lambda_{ijkl}$ elastic modulus
Elastic energy <sup>b</sup>	$U = \frac{1}{2} \lambda_{ijkl} e_{ij} e_{kl}$	(3.235)	$U$ potential energy
Volume strain (dilatation)	$e_v = \frac{\delta V}{V} = e_{11} + e_{22} + e_{33}$	(3.236)	$e_v$ volume strain $\delta V$ change in volume $V$ volume
Shear strain	$e_{kl} = \underbrace{\left( e_{kl} - \frac{1}{3} e_v \delta_{kl} \right)}_{\text{pure shear}} + \underbrace{\frac{1}{3} e_v \delta_{kl}}_{\text{dilatation}}$	(3.237)	$\delta_{kl}$ Kronecker delta
Hydrostatic compression	$\tau_{ij} = -p \delta_{ij}$	(3.238)	$p$ hydrostatic pressure

<sup>a</sup> $\tau_{ii}$  are normal stresses,  $\tau_{ij}$  ( $i \neq j$ ) are torsional stresses.

<sup>b</sup>As usual, products are implicitly summed over repeated indices.

**Isotropic elastic solids**

Lamé coefficients	$\mu = \frac{E}{2(1+\sigma)}$	(3.239)	$\mu, \lambda$ Lamé coefficients
	$\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}$	(3.240)	$E$ Young modulus $\sigma$ Poisson ratio
Longitudinal modulus <sup>a</sup>	$M_l = \frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)} = \lambda + 2\mu$	(3.241)	$M_l$ longitudinal elastic modulus
Diagonalised equations <sup>b</sup>	$e_{ii} = \frac{1}{E} [\tau_{ii} - \sigma(\tau_{jj} + \tau_{kk})]$	(3.242)	$e_{ii}$ strain in $i$ direction $\tau_{ii}$ stress in $i$ direction
	$\tau_{ii} = M_l \left[ e_{ii} + \frac{\sigma}{1-\sigma} (e_{jj} + e_{kk}) \right]$	(3.243)	$\mathbf{e}$ strain tensor $\mathbf{t}$ stress tensor
	$\mathbf{t} = 2\mu\mathbf{e} + \lambda\mathbf{1}\text{tr}(\mathbf{e})$	(3.244)	$\mathbf{1}$ unit matrix $\text{tr}(\cdot)$ trace
Bulk modulus (compression modulus)	$K = \frac{E}{3(1-2\sigma)} = \lambda + \frac{2}{3}\mu$	(3.245)	$K$ bulk modulus $K_T$ isothermal bulk modulus
	$\frac{1}{K_T} = -\frac{1}{V} \frac{\partial V}{\partial p} \Big _T$	(3.246)	$V$ volume
	$-p = K e_v$	(3.247)	$p$ pressure $T$ temperature
Shear modulus (rigidity modulus)	$\mu = \frac{E}{2(1+\sigma)}$	(3.248)	$e_v$ volume strain $\mu$ shear modulus
	$\tau_T = \mu\theta_{sh}$	(3.249)	$\tau_T$ transverse stress $\theta_{sh}$ shear strain
Young modulus	$E = \frac{9\mu K}{\mu + 3K}$	(3.250)	
Poisson ratio	$\sigma = \frac{3K - 2\mu}{2(3K + \mu)}$	(3.251)	

<sup>a</sup>In an extended medium.

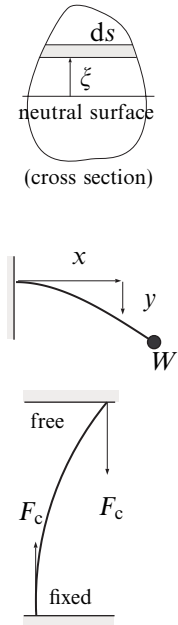
<sup>b</sup>Axes aligned along eigenvectors of the stress and strain tensors.

**Torsion**

Torsional rigidity (for a homogeneous rod)	$G = C \frac{\phi}{l}$	(3.252)	$G$ twisting couple $C$ torsional rigidity $l$ rod length $\phi$ twist angle in length $l$	
Thin circular cylinder	$C = 2\pi a^3 \mu t$	(3.253)	$a$ radius $t$ wall thickness $\mu$ shear modulus	
Thick circular cylinder	$C = \frac{1}{2} \mu \pi (a_2^4 - a_1^4)$	(3.254)	$a_1$ inner radius $a_2$ outer radius	
Arbitrary thin-walled tube	$C = \frac{4A^2 \mu t}{P}$	(3.255)	$A$ cross-sectional area $P$ perimeter	
Long flat ribbon	$C = \frac{1}{3} \mu w t^3$	(3.256)	$w$ cross-sectional width	

**Bending beams<sup>a</sup>**

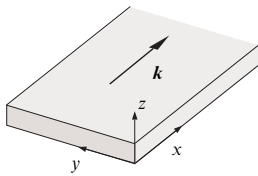
Bending moment	$G_b = \frac{E}{R_c} \int \xi^2 ds$ (3.257)	$G_b$ bending moment
	$= \frac{EI}{R_c}$ (3.258)	$E$ Young modulus $R_c$ radius of curvature $ds$ area element $\xi$ distance to neutral surface from $ds$
Light beam, horizontal at $x=0$ , weight at $x=l$	$y = \frac{W}{2EI} \left(l - \frac{x}{3}\right)x^2$ (3.259)	$l$ moment of area $y$ displacement from horizontal $W$ end-weight $l$ beam length $x$ distance along beam
	Heavy beam	$EI \frac{d^4y}{dx^4} = w(x)$ (3.260)
Euler strut failure	$F_c = \begin{cases} \pi^2 EI / l^2 & \text{(free ends)} \\ 4\pi^2 EI / l^2 & \text{(fixed ends)} \\ \pi^2 EI / (4l^2) & \text{(1 free end)} \end{cases}$ (3.261)	$F_c$ critical compression force $l$ strut length



<sup>a</sup>The radius of curvature is approximated by  $1/R_c \approx d^2y/dx^2$ .

**Elastic wave velocities<sup>a</sup>**

In an infinite isotropic solid <sup>b</sup>	$v_t = (\mu/\rho)^{1/2}$ (3.262)	$v_t$ speed of transverse wave
	$v_l = (M_1/\rho)^{1/2}$ (3.263)	$v_l$ speed of longitudinal wave
	$\frac{v_l}{v_t} = \left(\frac{2-2\sigma}{1-2\sigma}\right)^{1/2}$ (3.264)	$\mu$ shear modulus $\rho$ density $M_1$ longitudinal modulus ( $= \frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)}$ )
In a fluid	$v_l = (K/\rho)^{1/2}$ (3.265)	$K$ bulk modulus
On a thin plate (wave travelling along $x$ , plate thin in $z$ )	$v_l^{(x)} = \left[\frac{E}{\rho(1-\sigma^2)}\right]^{1/2}$ (3.266)	$v_l^{(i)}$ speed of longitudinal wave (displacement $\parallel i$ )
	$v_t^{(y)} = (\mu/\rho)^{1/2}$ (3.267)	$v_t^{(i)}$ speed of transverse wave (displacement $\parallel i$ )
	$v_t^{(z)} = k \left[\frac{Et^2}{12\rho(1-\sigma^2)}\right]^{1/2}$ (3.268)	$E$ Young modulus $\sigma$ Poisson ratio $k$ wavenumber ( $= 2\pi/\lambda$ ) $t$ plate thickness (in $z$ , $t \ll \lambda$ )
In a thin circular rod	$v_l = (E/\rho)^{1/2}$ (3.269)	
	$v_\phi = (\mu/\rho)^{1/2}$ (3.270)	$v_\phi$ torsional wave velocity
	$v_t = \frac{ka}{2} \left(\frac{E}{\rho}\right)^{1/2}$ (3.271)	$a$ rod radius ( $\ll \lambda$ )



<sup>a</sup>Waves that produce “bending” are generally dispersive. Wave (phase) speeds are quoted throughout.

<sup>b</sup>Transverse waves are also known as shear waves, or S-waves. Longitudinal waves are also known as pressure waves, or P-waves.

**Waves in strings and springs<sup>a</sup>**

In a spring	$v_l = (\kappa l / \rho_l)^{1/2}$	(3.272)	$v_l$ speed of longitudinal wave $\kappa$ spring constant <sup>b</sup> $l$ spring length $\rho_l$ mass per unit length <sup>c</sup>
On a stretched string	$v_t = (T / \rho_l)^{1/2}$	(3.273)	$v_t$ speed of transverse wave $T$ tension
On a stretched sheet	$v_t = (\tau / \rho_A)^{1/2}$	(3.274)	$\tau$ tension per unit width $\rho_A$ mass per unit area

<sup>a</sup>Wave amplitude assumed  $\ll$  wavelength.

<sup>b</sup>In the sense  $\kappa = \text{force}/\text{extension}$ .

<sup>c</sup>Measured along the axis of the spring.

**Propagation of elastic waves**

Acoustic impedance	$Z = \frac{\text{force}}{\text{response velocity}} = \frac{F}{\dot{u}}$	(3.275)	$Z$ impedance $F$ stress force $u$ strain displacement
	$= (E' \rho)^{1/2}$	(3.276)	
Wave velocity/impedance relation	if $v = \left(\frac{E'}{\rho}\right)^{1/2}$	(3.277)	$E'$ elastic modulus $\rho$ density $v$ wave phase velocity
	then $Z = (E' \rho)^{1/2} = \rho v$	(3.278)	
Mean energy density (nondispersive waves)	$\mathcal{U} = \frac{1}{2} E' k^2 u_0^2$	(3.279)	$\mathcal{U}$ energy density $k$ wavenumber
	$= \frac{1}{2} \rho \omega^2 u_0^2$	(3.280)	$\omega$ angular frequency $u_0$ maximum displacement
	$P = \mathcal{U} v$	(3.281)	$P$ mean energy flux
Normal coefficients <sup>a</sup>	$r = \frac{u_r}{u_i} = -\frac{\tau_r}{\tau_i} = \frac{Z_1 - Z_2}{Z_1 + Z_2}$	(3.282)	$r$ reflection coefficient $t$ transmission coefficient $\tau$ stress
	$t = \frac{2Z_1}{Z_1 + Z_2}$	(3.283)	
Snell's law <sup>b</sup>	$\frac{\sin \theta_i}{v_i} = \frac{\sin \theta_r}{v_r} = \frac{\sin \theta_t}{v_t}$	(3.284)	$\theta_i$ angle of incidence $\theta_r$ angle of reflection $\theta_t$ angle of refraction

<sup>a</sup>For stress and strain amplitudes. Because these reflection and transmission coefficients are usually defined in terms of displacement,  $u$ , rather than stress, there are differences between these coefficients and their equivalents defined in electromagnetism [see Equation (7.179) and page 154].

<sup>b</sup>Angles defined from the normal to the interface. An incident plane pressure wave will generally excite both shear and pressure waves in reflection and transmission. Use the velocity appropriate for the wave type.

### 3.9 Fluid dynamics

#### Ideal fluids<sup>a</sup>

Continuity <sup>b</sup>	$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$	(3.285)	$\rho$ density $\mathbf{v}$ fluid velocity field $t$ time
Kelvin circulation	$\Gamma = \oint \mathbf{v} \cdot d\mathbf{l} = \text{constant}$	(3.286)	$\Gamma$ circulation $d\mathbf{l}$ loop element
	$= \int_S \boldsymbol{\omega} \cdot d\mathbf{s}$	(3.287)	$d\mathbf{s}$ element of surface bounded by loop $\boldsymbol{\omega}$ vorticity ( $= \nabla \times \mathbf{v}$ )
Euler's equation <sup>c</sup>	$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\nabla p}{\rho} + \mathbf{g}$	(3.288)	$p$ pressure $\mathbf{g}$ gravitational field strength
	or $\frac{\partial}{\partial t}(\nabla \times \mathbf{v}) = \nabla \times [\mathbf{v} \times (\nabla \times \mathbf{v})]$	(3.289)	$(\mathbf{v} \cdot \nabla)$ advective operator
Bernoulli's equation (incompressible flow)	$\frac{1}{2} \rho v^2 + p + \rho g z = \text{constant}$	(3.290)	$z$ altitude
Bernoulli's equation (compressible adiabatic flow) <sup>d</sup>	$\frac{1}{2} v^2 + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} + g z = \text{constant}$	(3.291)	$\gamma$ ratio of specific heat capacities ( $c_p/c_v$ )
	$= \frac{1}{2} v^2 + c_p T + g z$	(3.292)	$c_p$ specific heat capacity at constant pressure $T$ temperature
Hydrostatics	$\nabla p = \rho \mathbf{g}$	(3.293)	
Adiabatic lapse rate (ideal gas)	$\frac{dT}{dz} = -\frac{g}{c_p}$	(3.294)	

<sup>a</sup>No thermal conductivity or viscosity.

<sup>b</sup>True generally.

<sup>c</sup>The second form of Euler's equation applies to incompressible flow only.

<sup>d</sup>Equation (3.292) is true only for an ideal gas.

#### Potential flow<sup>a</sup>

Velocity potential	$\mathbf{v} = \nabla \phi$	(3.295)	$\mathbf{v}$ velocity
	$\nabla^2 \phi = 0$	(3.296)	$\phi$ velocity potential
Vorticity condition	$\boldsymbol{\omega} = \nabla \times \mathbf{v} = 0$	(3.297)	$\boldsymbol{\omega}$ vorticity $\mathbf{F}$ drag force on moving sphere
Drag force on a sphere <sup>b</sup>	$\mathbf{F} = -\frac{2}{3} \pi \rho a^3 \dot{\mathbf{u}} = -\frac{1}{2} M_d \dot{\mathbf{u}}$	(3.298)	$a$ sphere radius $\dot{\mathbf{u}}$ sphere acceleration $\rho$ fluid density $M_d$ displaced fluid mass

<sup>a</sup>For incompressible fluids.

<sup>b</sup>The effect of this drag force is to give the sphere an additional effective mass equal to half the mass of fluid displaced.

**Viscous flow (incompressible)<sup>a</sup>**

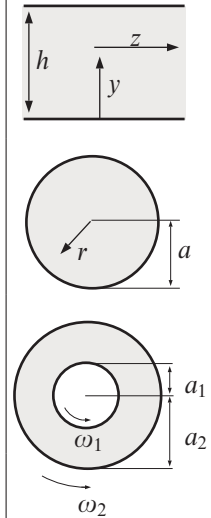
Fluid stress	$\tau_{ij} = -p\delta_{ij} + \eta \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$ (3.299)	$\tau_{ij}$ fluid stress tensor $p$ hydrostatic pressure $\eta$ shear viscosity $v_i$ velocity along $i$ axis $\delta_{ij}$ Kronecker delta
Navier–Stokes equation <sup>b</sup>	$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla p}{\rho} - \frac{\eta}{\rho} \nabla \times \boldsymbol{\omega} + \mathbf{g}$ (3.300) $= -\frac{\nabla p}{\rho} + \frac{\eta}{\rho} \nabla^2 \mathbf{v} + \mathbf{g}$ (3.301)	$\mathbf{v}$ fluid velocity field $\boldsymbol{\omega}$ vorticity $\mathbf{g}$ gravitational acceleration $\rho$ density
Kinematic viscosity	$\nu = \eta/\rho$ (3.302)	$\nu$ kinematic viscosity

<sup>a</sup>I.e.,  $\nabla \cdot \mathbf{v} = 0, \eta \neq 0$ .

<sup>b</sup>Neglecting bulk (second) viscosity.

**Laminar viscous flow**

Between parallel plates	$v_z(y) = \frac{1}{2\eta} y(h-y) \frac{\partial p}{\partial z}$ (3.303)	$v_z$ flow velocity $z$ direction of flow $y$ distance from plate $\eta$ shear viscosity $p$ pressure
Along a circular pipe <sup>a</sup>	$v_z(r) = \frac{1}{4\eta} (a^2 - r^2) \frac{\partial p}{\partial z}$ (3.304) $Q = \frac{dV}{dt} = \frac{\pi a^4}{8\eta} \frac{\partial p}{\partial z}$ (3.305)	$r$ distance from pipe axis $a$ pipe radius $V$ volume
Circulating between concentric rotating cylinders <sup>b</sup>	$G_z = \frac{4\pi\eta a_1^2 a_2^2}{a_2^2 - a_1^2} (\omega_2 - \omega_1)$ (3.306)	$G_z$ axial couple between cylinders per unit length $\omega_i$ angular velocity of $i$ th cylinder
Along an annular pipe	$Q = \frac{\pi}{8\eta} \frac{\partial p}{\partial z} \left[ a_2^4 - a_1^4 - \frac{(a_2^2 - a_1^2)^2}{\ln(a_2/a_1)} \right]$ (3.307)	$a_1$ inner radius $a_2$ outer radius $Q$ volume discharge rate



<sup>a</sup>Poiseuille flow.

<sup>b</sup>Couette flow.

**Drag<sup>a</sup>**

On a sphere (Stokes’s law)	$F = 6\pi a \eta v$ (3.308)	$F$ drag force $a$ radius
On a disk, broadside to flow	$F = 16a \eta v$ (3.309)	$v$ velocity $\eta$ shear viscosity
On a disk, edge on to flow	$F = 32a \eta v/3$ (3.310)	

<sup>a</sup>For Reynolds numbers  $\ll 1$ .

## Characteristic numbers

Reynolds number	$\text{Re} = \frac{\rho UL}{\eta} = \frac{\text{inertial force}}{\text{viscous force}} \quad (3.311)$	Re Reynolds number $\rho$ density $U$ characteristic velocity $L$ characteristic scale-length $\eta$ shear viscosity
Froude number <sup>a</sup>	$\text{F} = \frac{U^2}{Lg} = \frac{\text{inertial force}}{\text{gravitational force}} \quad (3.312)$	F Froude number $g$ gravitational acceleration
Strouhal number <sup>b</sup>	$\text{S} = \frac{U\tau}{L} = \frac{\text{evolution scale}}{\text{physical scale}} \quad (3.313)$	S Strouhal number $\tau$ characteristic timescale
Prandtl number	$\text{P} = \frac{\eta c_p}{\lambda} = \frac{\text{momentum transport}}{\text{heat transport}} \quad (3.314)$	P Prandtl number $c_p$ Specific heat capacity at constant pressure $\lambda$ thermal conductivity
Mach number	$\text{M} = \frac{U}{c} = \frac{\text{speed}}{\text{sound speed}} \quad (3.315)$	M Mach number $c$ sound speed
Rossby number	$\text{Ro} = \frac{U}{\Omega L} = \frac{\text{inertial force}}{\text{Coriolis force}} \quad (3.316)$	Ro Rossby number $\Omega$ angular velocity

<sup>a</sup>Sometimes the square root of this expression.  $L$  is usually the fluid depth.

<sup>b</sup>Sometimes the reciprocal of this expression.

## Fluid waves

Sound waves	$v_p = \left(\frac{K}{\rho}\right)^{1/2} = \left(\frac{dp}{d\rho}\right)^{1/2} \quad (3.317)$	$v_p$ wave (phase) speed $K$ bulk modulus $p$ pressure $\rho$ density
In an ideal gas (adiabatic conditions) <sup>a</sup>	$v_p = \left(\frac{\gamma RT}{\mu}\right)^{1/2} = \left(\frac{\gamma p}{\rho}\right)^{1/2} \quad (3.318)$	$\gamma$ ratio of heat capacities $R$ molar gas constant $T$ (absolute) temperature $\mu$ mean molecular mass
Gravity waves on a liquid surface <sup>b</sup>	$\omega^2 = gk \tanh kh \quad (3.319)$ $v_g \simeq \begin{cases} \frac{1}{2} \left(\frac{g}{k}\right)^{1/2} & (h \gg \lambda) \\ (gh)^{1/2} & (h \ll \lambda) \end{cases} \quad (3.320)$	$v_g$ group speed of wave $h$ liquid depth $\lambda$ wavelength $k$ wavenumber $g$ gravitational acceleration $\omega$ angular frequency
Capillary waves (ripples) <sup>c</sup>	$\omega^2 = \frac{\sigma k^3}{\rho} \quad (3.321)$	$\sigma$ surface tension
Capillary-gravity waves ( $h \gg \lambda$ )	$\omega^2 = gk + \frac{\sigma k^3}{\rho} \quad (3.322)$	

<sup>a</sup>If the waves are isothermal rather than adiabatic then  $v_p = (p/\rho)^{1/2}$ .

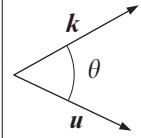
<sup>b</sup>Amplitude  $\ll$  wavelength.

<sup>c</sup>In the limit  $k^2 \gg g\rho/\sigma$ .



**Doppler effect<sup>a</sup>**

Source at rest, observer moving at $u$	$\frac{v'}{v} = 1 - \frac{ u }{v_p} \cos \theta$	(3.323)	$v', v''$ observed frequency $v$ emitted frequency $v_p$ wave (phase) speed in fluid
Observer at rest, source moving at $u$	$\frac{v''}{v} = \frac{1}{1 - \frac{ u }{v_p} \cos \theta}$	(3.324)	$u$ velocity $\theta$ angle between wavevector, $k$ , and $u$



<sup>a</sup>For plane waves in a stationary fluid.

**Wave speeds**

Phase speed	$v_p = \frac{\omega}{k} = v\lambda$	(3.325)	$v_p$ phase speed $v$ frequency $\omega$ angular frequency ( $= 2\pi v$ ) $\lambda$ wavelength $k$ wavenumber ( $= 2\pi/\lambda$ )
Group speed	$v_g = \frac{d\omega}{dk}$	(3.326)	$v_g$ group speed
	$= v_p - \lambda \frac{dv_p}{d\lambda}$	(3.327)	

**Shocks**

Mach wedge <sup>a</sup>	$\sin \theta_w = \frac{v_p}{v_b}$	(3.328)	$\theta_w$ wedge semi-angle $v_p$ wave (phase) speed $v_b$ body speed
Kelvin wedge <sup>b</sup>	$\lambda_K = \frac{4\pi v_b^2}{3g}$	(3.329)	$\lambda_K$ characteristic wavelength
	$\theta_w = \arcsin(1/3) = 19^\circ.5$	(3.330)	$g$ gravitational acceleration
Spherical adiabatic shock <sup>c</sup>	$r \simeq \left( \frac{Et^2}{\rho_0} \right)^{1/5}$	(3.331)	$r$ shock radius $E$ energy release $t$ time $\rho_0$ density of undisturbed medium
Rankine–Hugoniot shock relations <sup>d</sup>	$\frac{p_2}{p_1} = \frac{2\gamma M_1^2 - (\gamma - 1)}{\gamma + 1}$	(3.332)	1 upstream values 2 downstream values
	$\frac{v_1}{v_2} = \frac{\rho_2}{\rho_1} = \frac{\gamma + 1}{(\gamma - 1) + 2/M_1^2}$	(3.333)	$p$ pressure $v$ velocity
	$\frac{T_2}{T_1} = \frac{[2\gamma M_1^2 - (\gamma - 1)][2 + (\gamma - 1)M_1^2]}{(\gamma + 1)^2 M_1^2}$	(3.334)	$T$ temperature $\rho$ density $\gamma$ ratio of specific heats $M$ Mach number

<sup>a</sup>Approximating the wake generated by supersonic motion of a body in a nondispersive medium.

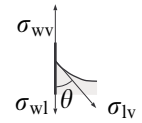
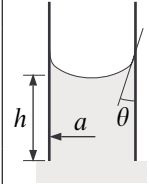
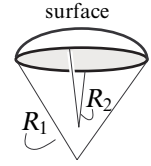
<sup>b</sup>For gravity waves, e.g., in the wake of a boat. Note that the wedge semi-angle is independent of  $v_b$ .

<sup>c</sup>Sedov–Taylor relation.

<sup>d</sup>Solutions for a steady, normal shock, in the frame moving with the shock front. If  $\gamma = 5/3$  then  $v_1/v_2 \leq 4$ .

## Surface tension

Definition	$\sigma_{lv} = \frac{\text{surface energy}}{\text{area}}$	(3.335)	$\sigma_{lv}$ surface tension (liquid/vapour interface)
	$= \frac{\text{surface tension}}{\text{length}}$	(3.336)	
Laplace's formula <sup>a</sup>	$\Delta p = \sigma_{lv} \left( \frac{1}{R_1} + \frac{1}{R_2} \right)$	(3.337)	$\Delta p$ pressure difference over surface $R_i$ principal radii of curvature
Capillary constant	$c_c = \left( \frac{2\sigma_{lv}}{g\rho} \right)^{1/2}$	(3.338)	$c_c$ capillary constant $\rho$ liquid density $g$ gravitational acceleration
Capillary rise (circular tube)	$h = \frac{2\sigma_{lv} \cos \theta}{\rho g a}$	(3.339)	$h$ rise height $\theta$ contact angle $a$ tube radius
Contact angle	$\cos \theta = \frac{\sigma_{wv} - \sigma_{wl}}{\sigma_{lv}}$	(3.340)	$\sigma_{wv}$ wall/vapour surface tension $\sigma_{wl}$ wall/liquid surface tension



<sup>a</sup>For a spherical bubble in a liquid  $\Delta p = 2\sigma_{lv}/R$ . For a soap bubble (two surfaces)  $\Delta p = 4\sigma_{lv}/R$ .

# Chapter 4 Quantum physics

## 4.1 Introduction

Quantum ideas occupy such a pivotal position in physics that different notations and algebras appropriate to each field have been developed. In the spirit of this book, only those formulas that are commonly present in undergraduate courses and that can be simply presented in tabular form are included here. For example, much of the detail of atomic spectroscopy and of specific perturbation analyses has been omitted, as have ideas from the somewhat specialised field of quantum electrodynamics. Traditionally, quantum physics is understood through standard “toy” problems, such as the potential step and the one-dimensional harmonic oscillator, and these are reproduced here. Operators are distinguished from observables using the “hat” notation, so that the momentum observable,  $p_x$ , has the operator  $\hat{p}_x = -i\hbar\partial/\partial x$ .

For clarity, many relations that can be generalised to three dimensions in an obvious way have been stated in their one-dimensional form, and wavefunctions are implicitly taken as normalised functions of space and time unless otherwise stated. With the exception of the last panel, all equations should be taken as nonrelativistic, so that “total energy” is the sum of potential and kinetic energies, excluding the rest mass energy.

## 4.2 Quantum definitions

### Quantum uncertainty relations

De Broglie relation	$p = \frac{h}{\lambda}$	(4.1)	$p, \mathbf{p}$	particle momentum
	$\mathbf{p} = \hbar \mathbf{k}$	(4.2)	$h$	Planck constant
Planck–Einstein relation	$E = h\nu = \hbar\omega$	(4.3)	$\hbar$	$h/(2\pi)$
			$\lambda$	de Broglie wavelength
			$\mathbf{k}$	de Broglie wavevector
Dispersion <sup>a</sup>	$(\Delta a)^2 = \langle (a - \langle a \rangle)^2 \rangle$	(4.4)	$E$	energy
			$\nu$	frequency
	$= \langle a^2 \rangle - \langle a \rangle^2$	(4.5)	$\omega$	angular frequency (= $2\pi\nu$ )
General uncertainty relation	$(\Delta a)^2(\Delta b)^2 \geq \frac{1}{4} \langle \mathbf{i}[\hat{a}, \hat{b}] \rangle^2$	(4.6)	$a, b$	observables <sup>b</sup>
			$\langle \cdot \rangle$	expectation value
Momentum–position uncertainty relation <sup>c</sup>	$\Delta p \Delta x \geq \frac{\hbar}{2}$	(4.7)	$(\Delta a)^2$	dispersion of $a$
			$\hat{a}$	operator for observable $a$
Energy–time uncertainty relation	$\Delta E \Delta t \geq \frac{\hbar}{2}$	(4.8)	$[\cdot, \cdot]$	commutator (see page 26)
			$x$	particle position
Number–phase uncertainty relation	$\Delta n \Delta \phi \geq \frac{1}{2}$	(4.9)	$t$	time
			$n$	number of photons
			$\phi$	wave phase

<sup>a</sup>Dispersion in quantum physics corresponds to variance in statistics.

<sup>b</sup>An observable is a directly measurable parameter of a system.

<sup>c</sup>Also known as the “Heisenberg uncertainty relation.”

### Wavefunctions

Probability density	$\text{pr}(x, t) dx =  \psi(x, t) ^2 dx$	(4.10)	$\text{pr}$	probability density
Probability density current <sup>a</sup>	$\mathbf{j}(x) = \frac{\hbar}{2im} \left( \psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x} \right)$	(4.11)	$\psi$	wavefunction
			$\mathbf{j}, j$	probability density current
			$\hbar$	(Planck constant)/( $2\pi$ )
	$\mathbf{j} = \frac{\hbar}{2im} [\psi^*(\mathbf{r}) \nabla \psi(\mathbf{r}) - \psi(\mathbf{r}) \nabla \psi^*(\mathbf{r})]$	(4.12)	$x$	position coordinate
			$\hat{\mathbf{p}}$	momentum operator
	$= \frac{1}{m} \Re(\psi^* \hat{\mathbf{p}} \psi)$	(4.13)	$m$	particle mass
			$\Re$	real part of
			$t$	time
Continuity equation	$\nabla \cdot \mathbf{j} = -\frac{\partial}{\partial t}(\psi \psi^*)$	(4.14)		
Schrödinger equation	$\hat{H}\psi = i\hbar \frac{\partial \psi}{\partial t}$	(4.15)	$H$	Hamiltonian
Particle stationary states <sup>b</sup>	$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} + V(x)\psi(x) = E\psi(x)$	(4.16)	$V$	potential energy
			$E$	total energy

<sup>a</sup>For particles. In three dimensions, suitable units would be particles  $\text{m}^{-2}\text{s}^{-1}$ .

<sup>b</sup>Time-independent Schrödinger equation for a particle, in one dimension.

### Operators

Hermitian conjugate operator	$\int (\hat{a}\phi)^* \psi \, dx = \int \phi^* \hat{a}\psi \, dx$	(4.17)	$\hat{a}$ Hermitian conjugate operator $\psi, \phi$ normalisable functions
Position operator	$\hat{x}^n = x^n$	(4.18)	* complex conjugate $x, y$ position coordinates
Momentum operator	$\hat{p}_x^n = \frac{\hbar^n}{i^n} \frac{\partial^n}{\partial x^n}$	(4.19)	$n$ arbitrary integer $\geq 1$ $p_x$ momentum coordinate
Kinetic energy operator	$\hat{T} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$	(4.20)	$T$ kinetic energy $\hbar$ (Planck constant)/( $2\pi$ ) $m$ particle mass
Hamiltonian operator	$\hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)$	(4.21)	$H$ Hamiltonian $V$ potential energy
Angular momentum operators	$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$	(4.22)	$L_z$ angular momentum along $z$ axis (sim. $x$ and $y$ )
	$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$	(4.23)	$L$ total angular momentum
Parity operator	$\hat{P}\psi(\mathbf{r}) = \psi(-\mathbf{r})$	(4.24)	$\hat{P}$ parity operator $\mathbf{r}$ position vector



### Expectation value

Expectation value <sup>a</sup>	$\langle a \rangle = \langle \hat{a} \rangle = \int \Psi^* \hat{a} \Psi \, dx$	(4.25)	$\langle a \rangle$ expectation value of $a$
	$= \langle \Psi   \hat{a}   \Psi \rangle$	(4.26)	$\hat{a}$ operator for $a$ $\Psi$ (spatial) wavefunction $x$ (spatial) coordinate
Time dependence	$\frac{d}{dt} \langle \hat{a} \rangle = \frac{i}{\hbar} \langle [\hat{H}, \hat{a}] \rangle + \left\langle \frac{\partial \hat{a}}{\partial t} \right\rangle$	(4.27)	$t$ time $\hbar$ (Planck constant)/( $2\pi$ )
Relation to eigenfunctions	if $\hat{a}\psi_n = a_n\psi_n$ and $\Psi = \sum c_n\psi_n$ then $\langle a \rangle = \sum  c_n ^2 a_n$	(4.28)	$\psi_n$ eigenfunctions of $\hat{a}$ $a_n$ eigenvalues $n$ dummy index $c_n$ probability amplitudes
Ehrenfest's theorem	$m \frac{d}{dt} \langle \mathbf{r} \rangle = \langle \mathbf{p} \rangle$	(4.29)	$m$ particle mass $\mathbf{r}$ position vector
	$\frac{d}{dt} \langle \mathbf{p} \rangle = -\langle \nabla V \rangle$	(4.30)	$\mathbf{p}$ momentum $V$ potential energy

<sup>a</sup>Equation (4.26) uses the Dirac “bra-ket” notation for integrals involving operators. The presence of vertical bars distinguishes this use of angled brackets from that on the left-hand side of the equations. Note that  $\langle a \rangle$  and  $\langle \hat{a} \rangle$  are taken as equivalent.

**Dirac notation**

Matrix element <sup>a</sup>	$a_{nm} = \int \psi_n^* \hat{a} \psi_m dx \quad (4.31)$ $= \langle n   \hat{a}   m \rangle \quad (4.32)$	$n, m$ eigenvector indices $a_{nm}$ matrix element $\psi_n$ basis states $\hat{a}$ operator $x$ spatial coordinate
Bra vector	bra state vector = $\langle n  $ <span style="float:right">(4.33)</span>	$\langle \cdot  $ bra
Ket vector	ket state vector = $  m \rangle$ <span style="float:right">(4.34)</span>	$  \cdot \rangle$ ket
Scalar product	$\langle n   m \rangle = \int \psi_n^* \psi_m dx$ <span style="float:right">(4.35)</span>	
Expectation	if $\Psi = \sum_n c_n \psi_n$ <span style="float:right">(4.36)</span>	$\Psi$ wavefunction
	then $\langle a \rangle = \sum_m \sum_n c_n^* c_m a_{nm}$ <span style="float:right">(4.37)</span>	$c_n$ probability amplitudes

<sup>a</sup>The Dirac bracket,  $\langle n | \hat{a} | m \rangle$ , can also be written  $\langle \psi_n | \hat{a} | \psi_m \rangle$ .

**4.3 Wave mechanics**

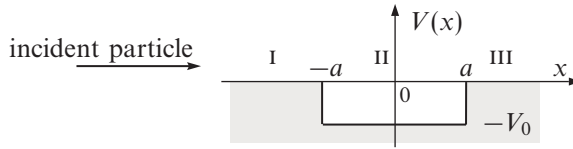
**Potential step<sup>a</sup>**

Potential function	$V(x) = \begin{cases} 0 & (x < 0) \\ V_0 & (x \geq 0) \end{cases} \quad (4.38)$	$V$ particle potential energy $V_0$ step height $\hbar$ (Planck constant)/( $2\pi$ )
Wavenumbers	$\hbar^2 k^2 = 2mE \quad (x < 0)$ <span style="float:right">(4.39)</span>	$k, q$ particle wavenumbers
	$\hbar^2 q^2 = 2m(E - V_0) \quad (x > 0)$ <span style="float:right">(4.40)</span>	$m$ particle mass $E$ total particle energy
Amplitude reflection coefficient	$r = \frac{k - q}{k + q}$ <span style="float:right">(4.41)</span>	$r$ amplitude reflection coefficient
Amplitude transmission coefficient	$t = \frac{2k}{k + q}$ <span style="float:right">(4.42)</span>	$t$ amplitude transmission coefficient
Probability currents <sup>b</sup>	$j_I = \frac{\hbar k}{m} (1 -  r ^2)$ <span style="float:right">(4.43)</span>	$j_I$ particle flux in zone I
	$j_{II} = \frac{\hbar q}{m}  t ^2$ <span style="float:right">(4.44)</span>	$j_{II}$ particle flux in zone II

<sup>a</sup>One-dimensional interaction with an incident particle of total energy  $E = KE + V$ . If  $E < V_0$  then  $q$  is imaginary and  $|r|^2 = 1$ .  $1/|q|$  is then a measure of the tunnelling depth.

<sup>b</sup>Particle flux with the sign of increasing  $x$ .

**Potential well<sup>a</sup>**



Potential function	$V(x) = \begin{cases} 0 & ( x  > a) \\ -V_0 & ( x  \leq a) \end{cases} \quad (4.45)$	$V$ particle potential energy $V_0$ well depth $\hbar$ (Planck constant)/( $2\pi$ ) $2a$ well width
Wavenumbers	$\hbar^2 k^2 = 2mE \quad ( x  > a) \quad (4.46)$ $\hbar^2 q^2 = 2m(E + V_0) \quad ( x  < a) \quad (4.47)$	$k, q$ particle wavenumbers $m$ particle mass $E$ total particle energy
Amplitude reflection coefficient	$r = \frac{\mathbf{i}e^{-2ika}(q^2 - k^2)\sin 2qa}{2kq \cos 2qa - \mathbf{i}(q^2 + k^2)\sin 2qa} \quad (4.48)$	$r$ amplitude reflection coefficient
Amplitude transmission coefficient	$t = \frac{2kqe^{-2ika}}{2kq \cos 2qa - \mathbf{i}(q^2 + k^2)\sin 2qa} \quad (4.49)$	$t$ amplitude transmission coefficient
Probability currents <sup>b</sup>	$j_I = \frac{\hbar k}{m}(1 -  r ^2) \quad (4.50)$ $j_{III} = \frac{\hbar k}{m} t ^2 \quad (4.51)$	$j_I$ particle flux in zone I $j_{III}$ particle flux in zone III
Ramsauer effect <sup>c</sup>	$E_n = -V_0 + \frac{n^2 \hbar^2 \pi^2}{8ma^2} \quad (4.52)$	$n$ integer $> 0$ $E_n$ Ramsauer energy
Bound states ( $V_0 < E < 0$ ) <sup>d</sup>	$\tan qa = \begin{cases}  k /q & \text{even parity} \\ -q/ k  & \text{odd parity} \end{cases} \quad (4.53)$ $q^2 -  k ^2 = 2mV_0/\hbar^2 \quad (4.54)$	

<sup>a</sup>One-dimensional interaction with an incident particle of total energy  $E = KE + V > 0$ .

<sup>b</sup>Particle flux in the sense of increasing  $x$ .

<sup>c</sup>Incident energy for which  $2qa = n\pi$ ,  $|r| = 0$ , and  $|t| = 1$ .

<sup>d</sup>When  $E < 0$ ,  $k$  is purely imaginary.  $|k|$  and  $q$  are obtained by solving these implicit equations.



**Barrier tunnelling<sup>a</sup>**

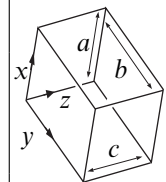
Potential function	$V(x) = \begin{cases} 0 & ( x  > a) \\ V_0 & ( x  \leq a) \end{cases} \quad (4.55)$	$V$ particle potential energy $V_0$ well depth $\hbar$ (Planck constant)/( $2\pi$ ) $2a$ barrier width
Wavenumber and tunnelling constant	$\hbar^2 k^2 = 2mE \quad ( x  > a) \quad (4.56)$	$k$ incident wavenumber $\kappa$ tunnelling constant
	$\hbar^2 \kappa^2 = 2m(V_0 - E) \quad ( x  < a) \quad (4.57)$	$m$ particle mass $E$ total energy ( $< V_0$ )
Amplitude reflection coefficient	$r = \frac{-ie^{-2ika}(k^2 + \kappa^2) \sinh 2\kappa a}{2k\kappa \cosh 2\kappa a - i(k^2 - \kappa^2) \sinh 2\kappa a} \quad (4.58)$	$r$ amplitude reflection coefficient
Amplitude transmission coefficient	$t = \frac{2k\kappa e^{-2ika}}{2k\kappa \cosh 2\kappa a - i(k^2 - \kappa^2) \sinh 2\kappa a} \quad (4.59)$	$t$ amplitude transmission coefficient
Tunnelling probability	$ t ^2 = \frac{4k^2 \kappa^2}{(k^2 + \kappa^2)^2 \sinh^2 2\kappa a + 4k^2 \kappa^2} \quad (4.60)$	$ t ^2$ tunnelling probability
	$\simeq \frac{16k^2 \kappa^2}{(k^2 + \kappa^2)^2} \exp(-4\kappa a) \quad ( t ^2 \ll 1) \quad (4.61)$	
Probability currents <sup>b</sup>	$j_I = \frac{\hbar k}{m} (1 -  r ^2) \quad (4.62)$	$j_I$ particle flux in zone I
	$j_{III} = \frac{\hbar k}{m}  t ^2 \quad (4.63)$	$j_{III}$ particle flux in zone III

<sup>a</sup>By a particle of total energy  $E = KE + V$ , through a one-dimensional rectangular potential barrier height  $V_0 > E$ .

<sup>b</sup>Particle flux in the sense of increasing  $x$ .

**Particle in a rectangular box<sup>a</sup>**

Eigenfunctions	$\Psi_{lmn} = \left(\frac{8}{abc}\right)^{1/2} \sin \frac{l\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c} \quad (4.64)$	$\Psi_{lmn}$ eigenfunctions $a, b, c$ box dimensions $l, m, n$ integers $\geq 1$
Energy levels	$E_{lmn} = \frac{h^2}{8M} \left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right) \quad (4.65)$	$E_{lmn}$ energy $h$ Planck constant $M$ particle mass
Density of states	$\rho(E) dE = \frac{4\pi}{h^3} (2M^3 E)^{1/2} dE \quad (4.66)$	$\rho(E)$ density of states (per unit volume)



<sup>a</sup>Spinless particle in a rectangular box bounded by the planes  $x=0, y=0, z=0, x=a, y=b,$  and  $z=c$ . The potential is zero inside and infinite outside the box.



**Harmonic oscillator**

Schrödinger equation	$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi_n}{\partial x^2} + \frac{1}{2} m \omega^2 x^2 \psi_n = E_n \psi_n \quad (4.67)$	$\hbar$ (Planck constant)/( $2\pi$ ) $m$ mass $\psi_n$ $n$ th eigenfunction $x$ displacement
Energy levels <sup>a</sup>	$E_n = \left(n + \frac{1}{2}\right) \hbar \omega \quad (4.68)$	$n$ integer $\geq 0$ $\omega$ angular frequency $E_n$ total energy in $n$ th state
Eigenfunctions	$\psi_n = \frac{H_n(x/a) \exp[-x^2/(2a^2)]}{(n! 2^n a \pi^{1/2})^{1/2}} \quad (4.69)$ where $a = \left(\frac{\hbar}{m\omega}\right)^{1/2}$	$H_n$ Hermite polynomials
Hermite polynomials	$H_0(y) = 1, \quad H_1(y) = 2y, \quad H_2(y) = 4y^2 - 2$ $H_{n+1}(y) = 2yH_n(y) - 2nH_{n-1}(y) \quad (4.70)$	$y$ dummy variable

<sup>a</sup> $E_0$  is the zero-point energy of the oscillator.

**4.4 Hydrogenic atoms**

**Bohr model<sup>a</sup>**

Quantisation condition	$\mu r_n^2 \Omega = n \hbar \quad (4.71)$	$r_n$ $n$ th orbit radius $\Omega$ orbital angular speed $n$ principal quantum number ( $>0$ )
Bohr radius	$a_0 = \frac{\epsilon_0 \hbar^2}{\pi m_e e^2} = \frac{\alpha}{4\pi R_\infty} \simeq 52.9 \text{ pm} \quad (4.72)$	$a_0$ Bohr radius $\mu$ reduced mass ( $\simeq m_e$ ) $-e$ electronic charge
Orbit radius	$r_n = \frac{n^2}{Z} a_0 \frac{m_e}{\mu} \quad (4.73)$	$Z$ atomic number $h$ Planck constant $\hbar$ $h/(2\pi)$
Total energy	$E_n = -\frac{\mu e^4 Z^2}{8\epsilon_0^2 \hbar^2 n^2} = -R_\infty hc \frac{\mu}{m_e} \frac{Z^2}{n^2} \quad (4.74)$	$E_n$ total energy of $n$ th orbit $\epsilon_0$ permittivity of free space $m_e$ electron mass
Fine structure constant	$\alpha = \frac{\mu_0 c e^2}{2h} = \frac{e^2}{4\pi \epsilon_0 \hbar c} \simeq \frac{1}{137} \quad (4.75)$	$\alpha$ fine structure constant $\mu_0$ permeability of free space
Hartree energy	$E_H = \frac{\hbar^2}{m_e a_0^2} \simeq 4.36 \times 10^{-18} \text{ J} \quad (4.76)$	$E_H$ Hartree energy
Rydberg constant	$R_\infty = \frac{m_e c \alpha^2}{2h} = \frac{m_e e^4}{8h^3 \epsilon_0^2 c} = \frac{E_H}{2hc} \quad (4.77)$	$R_\infty$ Rydberg constant $c$ speed of light
Rydberg's formula <sup>b</sup>	$\frac{1}{\lambda_{mn}} = R_\infty \frac{\mu}{m_e} Z^2 \left( \frac{1}{n^2} - \frac{1}{m^2} \right) \quad (4.78)$	$\lambda_{mn}$ photon wavelength $m$ integer $> n$

<sup>a</sup>Because the Bohr model is strictly a two-body problem, the equations use reduced mass,  $\mu = m_e m_{\text{nuc}} / (m_e + m_{\text{nuc}}) \simeq m_e$ , where  $m_{\text{nuc}}$  is the nuclear mass, throughout. The orbit radius is therefore the electron–nucleus distance.

<sup>b</sup>Wavelength of the spectral line corresponding to electron transitions between orbits  $m$  and  $n$ .

## Hydrogenlike atoms – Schrödinger solution<sup>a</sup>

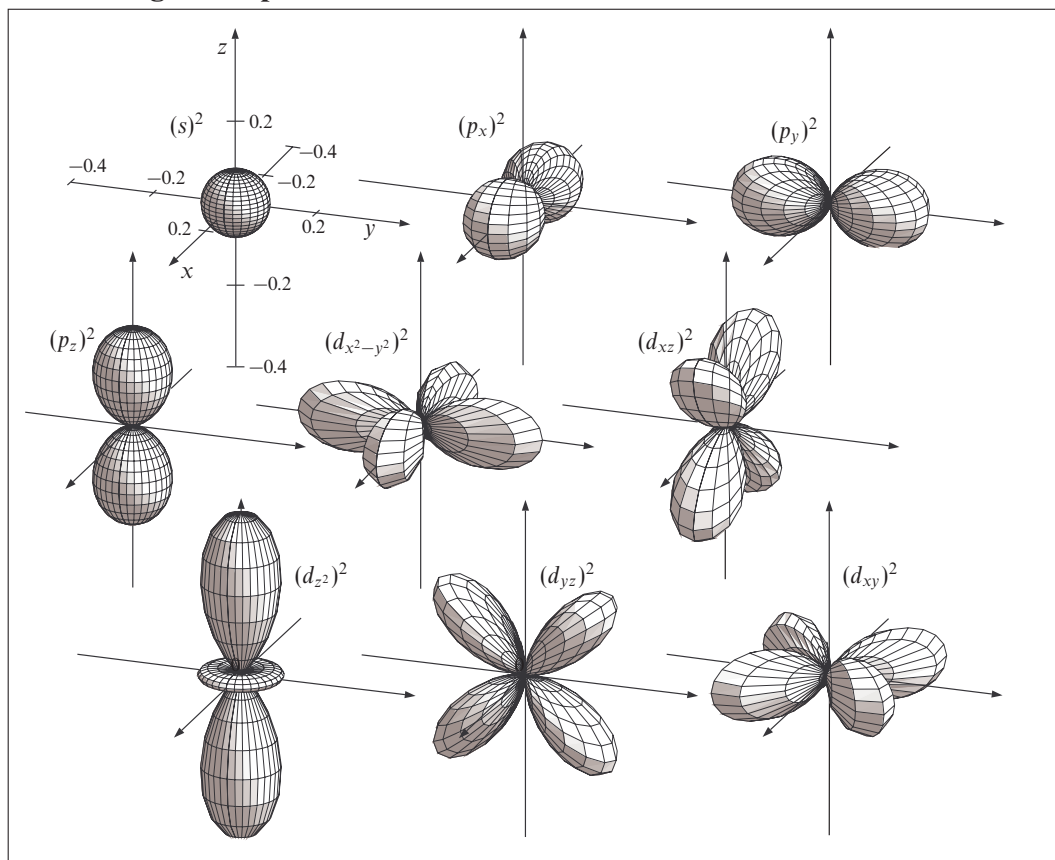
Schrödinger equation	
$-\frac{\hbar^2}{2\mu}\nabla^2\Psi_{nlm}-\frac{Ze^2}{4\pi\epsilon_0r}\Psi_{nlm}=E_n\Psi_{nlm} \quad \text{with} \quad \mu=\frac{m_em_{\text{nuc}}}{m_e+m_{\text{nuc}}} \quad (4.79)$	
Eigenfunctions	
$\Psi_{nlm}(r,\theta,\phi)=\left[\frac{(n-l-1)!}{2n(n+l)!}\right]^{1/2}\left(\frac{2}{an}\right)^{3/2}x^le^{-x/2}L_{n-l-1}^{2l+1}(x)Y_l^m(\theta,\phi) \quad (4.80)$	
with $a=\frac{m_e a_0}{\mu Z}$ , $x=\frac{2r}{an}$ , and $L_{n-l-1}^{2l+1}(x)=\sum_{k=0}^{n-l-1}\frac{(l+n)!(-x)^k}{(2l+1+k)!(n-l-1-k)!k!}$	
Total energy	$E_n=-\frac{\mu e^4 Z^2}{8\epsilon_0^2 h^2 n^2} \quad (4.81)$
	$E_n$ total energy $\epsilon_0$ permittivity of free space
Radial expectation values	$\langle r \rangle = \frac{a}{2}[3n^2 - l(l+1)] \quad (4.82)$
	$\langle r^2 \rangle = \frac{a^2 n^2}{2}[5n^2 + 1 - 3l(l+1)] \quad (4.83)$
	$\langle 1/r \rangle = \frac{1}{an^2} \quad (4.84)$
	$\langle 1/r^2 \rangle = \frac{2}{(2l+1)n^3 a^2} \quad (4.85)$
Allowed quantum numbers and selection rules <sup>b</sup>	$n = 1, 2, 3, \dots \quad (4.86)$
	$l = 0, 1, 2, \dots, (n-1) \quad (4.87)$
	$m = 0, \pm 1, \pm 2, \dots, \pm l \quad (4.88)$
	$\Delta n \neq 0 \quad (4.89)$
	$\Delta l = \pm 1 \quad (4.90)$
	$\Delta m = 0 \quad \text{or} \quad \pm 1 \quad (4.91)$
	$L_p^q$ associated Laguerre polynomials <sup>c</sup> $a$ classical orbit radius, $n=1$ $r$ electron–nucleus separation $Y_l^m$ spherical harmonics $a_0$ Bohr radius $=\frac{\epsilon_0 \hbar^2}{\pi m_e e^2}$
$\Psi_{100} = \frac{a^{-3/2}}{\pi^{1/2}} e^{-r/a}$	$\Psi_{200} = \frac{a^{-3/2}}{4(2\pi)^{1/2}} \left(2 - \frac{r}{a}\right) e^{-r/2a}$
$\Psi_{210} = \frac{a^{-3/2}}{4(2\pi)^{1/2}} \frac{r}{a} e^{-r/2a} \cos\theta$	$\Psi_{21\pm 1} = \mp \frac{a^{-3/2}}{8\pi^{1/2}} \frac{r}{a} e^{-r/2a} \sin\theta e^{\pm i\phi}$
$\Psi_{300} = \frac{a^{-3/2}}{81(3\pi)^{1/2}} \left(27 - 18\frac{r}{a} + 2\frac{r^2}{a^2}\right) e^{-r/3a}$	$\Psi_{310} = \frac{2^{1/2} a^{-3/2}}{81\pi^{1/2}} \left(6 - \frac{r}{a}\right) \frac{r}{a} e^{-r/3a} \cos\theta$
$\Psi_{31\pm 1} = \mp \frac{a^{-3/2}}{81\pi^{1/2}} \left(6 - \frac{r}{a}\right) \frac{r}{a} e^{-r/3a} \sin\theta e^{\pm i\phi}$	$\Psi_{320} = \frac{a^{-3/2}}{81(6\pi)^{1/2}} \frac{r^2}{a^2} e^{-r/3a} (3\cos^2\theta - 1)$
$\Psi_{32\pm 1} = \mp \frac{a^{-3/2}}{81\pi^{1/2}} \frac{r^2}{a^2} e^{-r/3a} \sin\theta \cos\theta e^{\pm i\phi}$	$\Psi_{32\pm 2} = \frac{a^{-3/2}}{162\pi^{1/2}} \frac{r^2}{a^2} e^{-r/3a} \sin^2\theta e^{\pm 2i\phi}$

<sup>a</sup>For a single bound electron in a perfect nuclear Coulomb potential (nonrelativistic and spin-free).

<sup>b</sup>For dipole transitions between orbitals.

<sup>c</sup>The sign and indexing definitions for this function vary. This form is appropriate to Equation (4.80).

## Orbital angular dependence



4

$s$ orbital ( $l=0$ )	$s = Y_0^0 = \text{constant}$	(4.92)	$Y_l^m$ spherical harmonics <sup>a</sup>
$p$ orbitals ( $l=1$ )	$p_x = \frac{-1}{2^{1/2}}(Y_1^1 - Y_1^{-1}) \propto \cos\phi \sin\theta$	(4.93)	$\theta, \phi$ spherical polar coordinates
	$p_y = \frac{i}{2^{1/2}}(Y_1^1 + Y_1^{-1}) \propto \sin\phi \sin\theta$	(4.94)	
	$p_z = Y_1^0 \propto \cos\theta$	(4.95)	
$d$ orbitals ( $l=2$ )	$d_{x^2-y^2} = \frac{1}{2^{1/2}}(Y_2^2 + Y_2^{-2}) \propto \sin^2\theta \cos 2\phi$	(4.96)	
	$d_{xz} = \frac{-1}{2^{1/2}}(Y_2^1 - Y_2^{-1}) \propto \sin\theta \cos\theta \cos\phi$	(4.97)	
	$d_{z^2} = Y_2^0 \propto (3\cos^2\theta - 1)$	(4.98)	
	$d_{yz} = \frac{i}{2^{1/2}}(Y_2^1 + Y_2^{-1}) \propto \sin\theta \cos\theta \sin\phi$	(4.99)	
	$d_{xy} = \frac{-i}{2^{1/2}}(Y_2^2 - Y_2^{-2}) \propto \sin^2\theta \sin 2\phi$	(4.100)	

<sup>a</sup>See page 49 for the definition of spherical harmonics.

## 4.5 Angular momentum

### Orbital angular momentum

Angular momentum operators	$\hat{L} = \mathbf{r} \times \hat{\mathbf{p}}$ (4.101)	$L$ angular momentum $\mathbf{p}$ linear momentum $\mathbf{r}$ position vector $xyz$ Cartesian coordinates $r\theta\phi$ spherical polar coordinates $\hbar$ (Planck constant)/(2 $\pi$ )
	$\hat{L}_z = \frac{\hbar}{i} \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)$ (4.102)	
	$= \frac{\hbar}{i} \frac{\partial}{\partial \phi}$ (4.103)	
	$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$ (4.104)	
	$= -\hbar^2 \left[ \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left( \sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} \right]$ (4.105)	
Ladder operators	$\hat{L}_{\pm} = \hat{L}_x \pm i\hat{L}_y$ (4.106)	$\hat{L}_{\pm}$ ladder operators $Y_l^{m_l}$ spherical harmonics $l, m_l$ integers
	$= \hbar e^{\pm i\phi} \left( i \cot\theta \frac{\partial}{\partial \phi} \pm \frac{\partial}{\partial \theta} \right)$ (4.107)	
	$\hat{L}_{\pm} Y_l^{m_l} = \hbar [l(l+1) - m_l(m_l \pm 1)]^{1/2} Y_l^{m_l \pm 1}$ (4.108)	
Eigenfunctions and eigenvalues	$\hat{L}^2 Y_l^{m_l} = l(l+1)\hbar^2 Y_l^{m_l} \quad (l \geq 0)$ (4.109)	
	$\hat{L}_z Y_l^{m_l} = m_l \hbar Y_l^{m_l} \quad ( m_l  \leq l)$ (4.110)	
	$\hat{L}_z [\hat{L}_{\pm} Y_l^{m_l}(\theta, \phi)] = (m_l \pm 1) \hbar \hat{L}_{\pm} Y_l^{m_l}(\theta, \phi)$ (4.111)	
	$l$ -multiplicity = $(2l+1)$ (4.112)	

### Angular momentum commutation relations<sup>a</sup>

Conservation of angular momentum <sup>b</sup>	$[\hat{H}, \hat{L}_z] = 0$ (4.113)	$L$ angular momentum $p$ momentum $H$ Hamiltonian $\hat{L}_{\pm}$ ladder operators
$[\hat{L}_z, x] = i\hbar y$ (4.114)	$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$ (4.120)	
$[\hat{L}_z, y] = -i\hbar x$ (4.115)	$[\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y$ (4.121)	
$[\hat{L}_z, z] = 0$ (4.116)	$[\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x$ (4.122)	
$[\hat{L}_z, \hat{p}_x] = i\hbar \hat{p}_y$ (4.117)	$[\hat{L}_+, \hat{L}_z] = -\hbar \hat{L}_+$ (4.123)	
$[\hat{L}_z, \hat{p}_y] = -i\hbar \hat{p}_x$ (4.118)	$[\hat{L}_-, \hat{L}_z] = \hbar \hat{L}_-$ (4.124)	
$[\hat{L}_z, \hat{p}_z] = 0$ (4.119)	$[\hat{L}_+, \hat{L}_-] = 2\hbar \hat{L}_z$ (4.125)	
	$[\hat{L}^2, \hat{L}_{\pm}] = 0$ (4.126)	
	$[\hat{L}^2, \hat{L}_x] = [\hat{L}^2, \hat{L}_y] = [\hat{L}^2, \hat{L}_z] = 0$ (4.127)	

<sup>a</sup>The commutation of  $a$  and  $b$  is defined as  $[a, b] = ab - ba$  (see page 26). Similar expressions hold for  $S$  and  $J$ .

<sup>b</sup>For motion under a central force.

Clebsch–Gordan coefficients<sup>a</sup>

$\langle j, -m_j | l_1, -m_1; l_2, -m_2 \rangle = (-1)^{l_1+l_2-j} \langle j, m_j | l_1, m_1; l_2, m_2 \rangle$

$1/2 \times 1/2$		$1$	$0$
$+1/2$	$+1/2$	$1$	$0$
$+1/2$	$-1/2$	$1/2$	$1/2$
$-1/2$	$+1/2$	$1/2$	$-1/2$

$1 \times 1/2$		$3/2$	$+1/2$
$+1$	$+1/2$	$1$	$3/2$
$+1$	$-1/2$	$1/3$	$2/3$
$0$	$+1/2$	$2/3$	$-1/3$

$l_1 \times l_2$		$m_j$
$m_1$	$m_2$	$j$
$m_1$	$m_2$	$j$
$\vdots$	$\vdots$	$\vdots$
		coefficients
		$\langle j, m_j   l_1, m_1; l_2, m_2 \rangle$

$3/2 \times 1/2$		$2$	$+1$
$+3/2$	$+1/2$	$1$	$2$
$+3/2$	$-1/2$	$1/4$	$3/4$
$+1/2$	$+1/2$	$3/4$	$-1/4$
$+1/2$	$-1/2$	$1/2$	$1/2$
$-1/2$	$+1/2$	$1/2$	$-1/2$

$2 \times 1/2$		$5/2$	$+3/2$
$+2$	$+1/2$	$1$	$5/2$
$+2$	$-1/2$	$1/5$	$4/5$
$+1$	$+1/2$	$4/5$	$-1/5$
$+1$	$-1/2$	$2/5$	$3/5$
$0$	$+1/2$	$3/5$	$-2/5$

$1 \times 1$		$2$	$+1$
$+1$	$1$	$2$	$1$
$+1$	$0$	$1/2$	$1/2$
$0$	$+1$	$1/2$	$-1/2$
$+1$	$-1$	$1/6$	$1/2$
$0$	$0$	$2/3$	$0$
$-1$	$+1$	$1/6$	$-1/2$

$3/2 \times 1$		$5/2$	$+3/2$
$+3/2$	$+1$	$1$	$5/2$
$+3/2$	$0$	$2/5$	$3/5$
$+1/2$	$+1$	$3/5$	$-2/5$
$+3/2$	$-1$	$1/10$	$2/5$
$1/2$	$0$	$3/5$	$1/5$
$-1/2$	$+1$	$3/10$	$-8/15$

$2 \times 1$		$3$	$+2$
$+2$	$1$	$3$	$2$
$+2$	$0$	$1/3$	$2/3$
$+1$	$+1$	$2/3$	$-1/3$
$+2$	$-1$	$1/15$	$1/3$
$+1$	$0$	$8/15$	$1/6$
$0$	$+1$	$6/15$	$-1/2$

$3/2 \times 3/2$		$3$	$+2$
$+3/2$	$+3/2$	$1$	$3$
$+3/2$	$+1/2$	$1/2$	$1/2$
$+1/2$	$+3/2$	$1/2$	$-1/2$
$+3/2$	$-1/2$	$1/5$	$1/2$
$+1/2$	$+1/2$	$3/5$	$0$
$-1/2$	$+3/2$	$1/5$	$-1/2$

$2 \times 3/2$		$7/2$	$+5/2$
$+2$	$+3/2$	$1$	$7/2$
$+2$	$+1/2$	$3/7$	$4/7$
$+1$	$+3/2$	$4/7$	$-3/7$
$+2$	$-1/2$	$1/7$	$16/35$
$+1$	$+1/2$	$4/7$	$1/35$
$0$	$+3/2$	$2/7$	$-18/35$

$2 \times 2$		$4$	$+3$
$+2$	$2$	$4$	$3$
$+2$	$+1$	$1/2$	$1/2$
$+1$	$+1$	$1/2$	$-1/2$
$+2$	$0$	$3/14$	$1/2$
$+1$	$+1$	$4/7$	$0$
$0$	$+2$	$3/14$	$-1/2$

$4$		$3$	$2$	$1$	$0$
$+2$	$-1$	$1/14$	$3/10$	$3/7$	$1/5$
$+1$	$0$	$3/7$	$1/5$	$-1/14$	$-3/10$
$0$	$+1$	$3/7$	$-1/5$	$-1/14$	$3/10$
$-1$	$+2$	$1/14$	$-3/10$	$3/7$	$-1/5$

$4$		$3$	$2$	$1$	$0$
$+2$	$-2$	$1/70$	$1/10$	$2/7$	$2/5$
$+1$	$-1$	$8/35$	$2/5$	$1/14$	$-1/10$
$0$	$0$	$18/35$	$0$	$-2/7$	$0$
$-1$	$+1$	$8/35$	$-2/5$	$1/14$	$1/10$
$-2$	$+2$	$1/70$	$-1/10$	$2/7$	$-2/5$



<sup>a</sup>Or “Wigner coefficients,” using the Condon–Shortley sign convention. Note that a square root is assumed over all coefficient digits, so that “ $-3/10$ ” corresponds to  $-\sqrt{3/10}$ . Also for clarity, only values of  $m_j \geq 0$  are listed here. The coefficients for  $m_j < 0$  can be obtained from the symmetry relation  $\langle j, -m_j | l_1, -m_1; l_2, -m_2 \rangle = (-1)^{l_1+l_2-j} \langle j, m_j | l_1, m_1; l_2, m_2 \rangle$ .

**Angular momentum addition<sup>a</sup>**

Total angular momentum	$\mathbf{J} = \mathbf{L} + \mathbf{S}$ (4.128)	$\mathbf{J}, \mathbf{J}$ total angular momentum $\mathbf{L}, \mathbf{L}$ orbital angular momentum $\mathbf{S}, \mathbf{S}$ spin angular momentum $\psi$ eigenfunctions $m_j$ magnetic quantum number $ m_j  \leq j$ $j$ $(l+s) \geq j \geq  l-s $
	$\hat{J}_z = \hat{L}_z + \hat{S}_z$ (4.129)	
	$\hat{J}^2 = \hat{L}^2 + \hat{S}^2 + 2\hat{\mathbf{L}} \cdot \hat{\mathbf{S}}$ (4.130)	
	$\hat{J}_z \psi_{j,m_j} = m_j \hbar \psi_{j,m_j}$ (4.131)	
	$\hat{J}^2 \psi_{j,m_j} = j(j+1) \hbar^2 \psi_{j,m_j}$ (4.132)	
$j$ -multiplicity $= (2l+1)(2s+1)$ (4.133)		
Mutually commuting sets	$\{L^2, S^2, J^2, J_z, \mathbf{L} \cdot \mathbf{S}\}$ (4.134)	{} set of mutually commuting observables
	$\{L^2, S^2, L_z, S_z, J_z\}$ (4.135)	
Clebsch–Gordan coefficients <sup>b</sup>	$ j, m_j\rangle = \sum_{\substack{m_l, m_s \\ m_l + m_s = m_j}} \langle j, m_j   l, m_l; s, m_s \rangle  l, m_l\rangle  s, m_s\rangle$ (4.136)	$  \cdot \rangle$ eigenstates $\langle \cdot   \cdot \rangle$ Clebsch–Gordan coefficients

<sup>a</sup>Summing spin and orbital angular momenta as examples, eigenstates  $|s, m_s\rangle$  and  $|l, m_l\rangle$ .

<sup>b</sup>Or “Wigner coefficients.” Assuming no  $L$ – $S$  interaction.

**Magnetic moments**

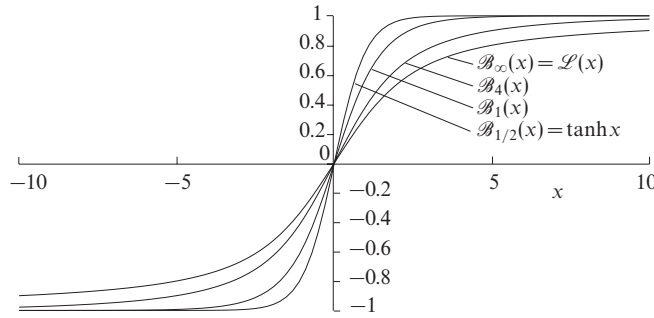
Bohr magneton	$\mu_B = \frac{e\hbar}{2m_e}$ (4.137)	$\mu_B$ Bohr magneton $-e$ electronic charge $\hbar$ (Planck constant)/( $2\pi$ ) $m_e$ electron mass
Gyromagnetic ratio <sup>a</sup>	$\gamma = \frac{\text{orbital magnetic moment}}{\text{orbital angular momentum}}$ (4.138)	$\gamma$ gyromagnetic ratio
Electron orbital gyromagnetic ratio	$\gamma_e = \frac{-\mu_B}{\hbar}$ (4.139)	$\gamma_e$ electron gyromagnetic ratio
	$= \frac{-e}{2m_e}$ (4.140)	
Spin magnetic moment of an electron <sup>b</sup>	$\mu_{e,z} = -g_e \mu_B m_s$ (4.141)	$\mu_{e,z}$ $z$ component of spin magnetic moment $g_e$ electron $g$ -factor ( $\approx 2.002$ ) $m_s$ spin quantum number ( $\pm 1/2$ )
	$= \pm g_e \gamma_e \frac{\hbar}{2}$ (4.142)	
	$= \pm \frac{g_e e \hbar}{4m_e}$ (4.143)	
Landé $g$ -factor <sup>c</sup>	$\mu_J = g_J \sqrt{J(J+1)} \mu_B$ (4.144)	$\mu_J$ total magnetic moment $\mu_{J,z}$ $z$ component of $\mu_J$ $m_J$ magnetic quantum number $J, L, S$ total, orbital, and spin quantum numbers $g_J$ Landé $g$ -factor
	$\mu_{J,z} = -g_J \mu_B m_J$ (4.145)	
	$g_J = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}$ (4.146)	

<sup>a</sup>Or “magnetogyric ratio.”

<sup>b</sup>The electron  $g$ -factor equals exactly 2 in Dirac theory. The modification  $g_e = 2 + \alpha/\pi + \dots$ , where  $\alpha$  is the fine structure constant, comes from quantum electrodynamics.

<sup>c</sup>Relating the spin + orbital angular momenta of an electron to its total magnetic moment, assuming  $g_e = 2$ .

Quantum paramagnetism



$$B_J(x) = \frac{2J+1}{2J} \coth \left[ \frac{(2J+1)x}{2J} \right] - \frac{1}{2J} \coth \frac{x}{2J} \tag{4.147}$$

Brillouin function

$$B_J(x) \simeq \begin{cases} \frac{J+1}{3J} x & (x \ll 1) \\ \mathcal{L}(x) & (J \gg 1) \end{cases} \tag{4.148}$$

$$B_{1/2}(x) = \tanh x \tag{4.149}$$

Mean magnetisation<sup>a</sup>

$$\langle M \rangle = n \mu_B J g_J B_J \left( J g_J \frac{\mu_B B}{kT} \right) \tag{4.150}$$

$\langle M \rangle$  for isolated spins ( $J = 1/2$ )

$$\langle M \rangle_{1/2} = n \mu_B \tanh \left( \frac{\mu_B B}{kT} \right) \tag{4.151}$$

- $B_J(x)$  Brillouin function
- $J$  total angular momentum quantum number
- $\mathcal{L}(x)$  Langevin function =  $\coth x - 1/x$  (see page 144)
- $\langle M \rangle$  mean magnetisation
- $n$  number density of atoms
- $g_J$  Landé g-factor
- $\mu_B$  Bohr magneton
- $B$  magnetic flux density
- $k$  Boltzmann constant
- $T$  temperature
- $\langle M \rangle_{1/2}$  mean magnetisation for  $J = 1/2$  (and  $g_J = 2$ )

<sup>a</sup>Of an ensemble of atoms in thermal equilibrium at temperature  $T$ , each with total angular momentum quantum number  $J$ .

### 4.6 Perturbation theory

#### Time-independent perturbation theory

Unperturbed states	$\hat{H}_0\psi_n = E_n\psi_n$ ( $\psi_n$ nondegenerate)	(4.152)	$\hat{H}_0$ unperturbed Hamiltonian $\psi_n$ eigenfunctions of $\hat{H}_0$ $E_n$ eigenvalues of $\hat{H}_0$ $n$ integer $\geq 0$
Perturbed Hamiltonian	$\hat{H} = \hat{H}_0 + \hat{H}'$	(4.153)	$\hat{H}$ perturbed Hamiltonian $\hat{H}'$ perturbation ( $\ll \hat{H}_0$ )
Perturbed eigenvalues <sup>a</sup>	$E'_k = E_k + \langle \psi_k   \hat{H}'   \psi_k \rangle$ $+ \sum_{n \neq k} \frac{ \langle \psi_k   \hat{H}'   \psi_n \rangle ^2}{E_k - E_n} + \dots$	(4.154)	$E'_k$ perturbed eigenvalue ( $\simeq E_k$ ) $\langle    \rangle$ Dirac bracket
Perturbed eigenfunctions <sup>b</sup>	$\psi'_k = \psi_k + \sum_{n \neq k} \frac{\langle \psi_k   \hat{H}'   \psi_n \rangle}{E_k - E_n} \psi_n + \dots$	(4.155)	$\psi'_k$ perturbed eigenfunction ( $\simeq \psi_k$ )

<sup>a</sup>To second order.

<sup>b</sup>To first order.

#### Time-dependent perturbation theory

Unperturbed stationary states	$\hat{H}_0\psi_n = E_n\psi_n$	(4.156)	$\hat{H}_0$ unperturbed Hamiltonian $\psi_n$ eigenfunctions of $\hat{H}_0$ $E_n$ eigenvalues of $\hat{H}_0$ $n$ integer $\geq 0$
Perturbed Hamiltonian	$\hat{H}(t) = \hat{H}_0 + \hat{H}'(t)$	(4.157)	$\hat{H}$ perturbed Hamiltonian $\hat{H}'(t)$ perturbation ( $\ll \hat{H}_0$ ) $t$ time
Schrödinger equation	$[\hat{H}_0 + \hat{H}'(t)]\Psi(t) = i\hbar \frac{\partial \Psi(t)}{\partial t}$ $\Psi(t=0) = \psi_0$	(4.158) (4.159)	$\Psi$ wavefunction $\psi_0$ initial state $\hbar$ (Planck constant)/( $2\pi$ )
Perturbed wave-function <sup>a</sup>	$\Psi(t) = \sum_n c_n(t)\psi_n \exp(-iE_n t/\hbar)$ where	(4.160)	$c_n$ probability amplitudes
	$c_n = \frac{-i}{\hbar} \int_0^t \langle \psi_n   \hat{H}'(t')   \psi_0 \rangle \exp[i(E_n - E_0)t'/\hbar] dt'$	(4.161)	
Fermi's golden rule	$\Gamma_{i \rightarrow f} = \frac{2\pi}{\hbar}  \langle \psi_f   \hat{H}'   \psi_i \rangle ^2 \rho(E_f)$	(4.162)	$\Gamma_{i \rightarrow f}$ transition probability per unit time from state $i$ to state $f$ $\rho(E_f)$ density of final states

<sup>a</sup>To first order.



## 4.7 High energy and nuclear physics

### Nuclear decay

Nuclear decay law	$N(t) = N(0)e^{-\lambda t}$	(4.163)	$N(t)$ number of nuclei remaining after time $t$ $t$ time
Half-life and mean life	$T_{1/2} = \frac{\ln 2}{\lambda}$	(4.164)	$\lambda$ decay constant
	$\langle T \rangle = 1/\lambda$	(4.165)	$T_{1/2}$ half-life $\langle T \rangle$ mean lifetime
Successive decays $1 \rightarrow 2 \rightarrow 3$ (species 3 stable)			
	$N_1(t) = N_1(0)e^{-\lambda_1 t}$	(4.166)	$N_1$ population of species 1
	$N_2(t) = N_2(0)e^{-\lambda_2 t} + \frac{N_1(0)\lambda_1(e^{-\lambda_1 t} - e^{-\lambda_2 t})}{\lambda_2 - \lambda_1}$	(4.167)	$N_2$ population of species 2
	$N_3(t) = N_3(0) + N_2(0)(1 - e^{-\lambda_2 t}) + N_1(0) \left( 1 + \frac{\lambda_1 e^{-\lambda_2 t} - \lambda_2 e^{-\lambda_1 t}}{\lambda_2 - \lambda_1} \right)$	(4.168)	$N_3$ population of species 3 $\lambda_1$ decay constant $1 \rightarrow 2$ $\lambda_2$ decay constant $2 \rightarrow 3$
Geiger's law <sup>a</sup>	$v^3 = a(R - x)$	(4.169)	$v$ velocity of $\alpha$ particle $x$ distance from source $a$ constant
Geiger-Nuttall rule	$\log \lambda = b + c \log R$	(4.170)	$R$ range $b, c$ constants for each series $\alpha, \beta,$ and $\gamma$

<sup>a</sup>For  $\alpha$  particles in air (empirical).

### Nuclear binding energy

Liquid drop model <sup>a</sup>	$B = a_v A - a_s A^{2/3} - a_c \frac{Z^2}{A^{1/3}} - a_a \frac{(N - Z)^2}{A} + \delta(A)$	(4.171)	$N$ number of neutrons $A$ mass number ( $= N + Z$ ) $B$ semi-empirical binding energy $Z$ number of protons $a_v$ volume term ( $\sim 15.8 \text{ MeV}$ ) $a_s$ surface term ( $\sim 18.0 \text{ MeV}$ ) $a_c$ Coulomb term ( $\sim 0.72 \text{ MeV}$ ) $a_a$ asymmetry term ( $\sim 23.5 \text{ MeV}$ ) $a_p$ pairing term ( $\sim 33.5 \text{ MeV}$ )
	$\delta(A) \simeq \begin{cases} +a_p A^{-3/4} & Z, N \text{ both even} \\ -a_p A^{-3/4} & Z, N \text{ both odd} \\ 0 & \text{otherwise} \end{cases}$	(4.172)	
Semi-empirical mass formula	$M(Z, A) = Z M_H + N m_n - B$	(4.173)	$M(Z, A)$ atomic mass $M_H$ mass of hydrogen atom $m_n$ neutron mass

<sup>a</sup>Coefficient values are empirical and approximate.

**Nuclear collisions**

Breit–Wigner formula <sup>a</sup>	$\sigma(E) = \frac{\pi}{k^2} g \frac{\Gamma_{ab}\Gamma_c}{(E - E_0)^2 + \Gamma^2/4} \quad (4.174)$	$\sigma(E)$ cross-section for $a + b \rightarrow c$ $k$ incoming wavenumber $g$ spin factor $E$ total energy (PE + KE) $E_0$ resonant energy $\Gamma$ width of resonant state $R$ $\Gamma_{ab}$ partial width into $a + b$ $\Gamma_c$ partial width into $c$ $\tau$ resonance lifetime $J$ total angular momentum quantum number of $R$ $s_{a,b}$ spins of $a$ and $b$
	$g = \frac{2J + 1}{(2s_a + 1)(2s_b + 1)} \quad (4.175)$	
Total width	$\Gamma = \Gamma_{ab} + \Gamma_c \quad (4.176)$	$\frac{d\sigma}{d\Omega}$ differential collision cross-section $\mu$ reduced mass $K =  \mathbf{k}_{in} - \mathbf{k}_{out} $ (see footnote) $r$ radial distance $V(r)$ potential energy of interaction
Resonance lifetime	$\tau = \frac{\hbar}{\Gamma} \quad (4.177)$	
Born scattering formula <sup>b</sup>	$\frac{d\sigma}{d\Omega} = \left  \frac{2\mu}{\hbar^2} \int_0^\infty \frac{\sin Kr}{Kr} V(r)r^2 dr \right ^2 \quad (4.178)$	$\hbar$ (Planck constant)/ $2\pi$ $\alpha/r$ scattering potential energy $\chi$ scattering angle $v$ closing velocity $A = 2$ for spin-zero particles, $= -1$ for spin-half particles
Mott scattering formula <sup>c</sup>	$\frac{d\sigma}{d\Omega} = \left( \frac{\alpha}{4E} \right)^2 \left[ \csc^4 \frac{\chi}{2} + \sec^4 \frac{\chi}{2} + \frac{A \cos \left( \frac{\alpha}{\hbar v} \ln \tan^2 \frac{\chi}{2} \right)}{\sin^2 \frac{\chi}{2} \cos \frac{\chi}{2}} \right] \quad (4.179)$ $\frac{d\sigma}{d\Omega} \simeq \left( \frac{\alpha}{2E} \right)^2 \frac{4 - 3 \sin^2 \chi}{\sin^4 \chi} \quad (A = -1, \alpha \ll v\hbar) \quad (4.180)$	

<sup>a</sup>For the reaction  $a + b \leftrightarrow R \rightarrow c$  in the centre of mass frame.

<sup>b</sup>For a central field. The Born approximation holds when the potential energy of scattering,  $V$ , is much less than the total kinetic energy.  $K$  is the magnitude of the change in the particle's wavevector due to scattering.

<sup>c</sup>For identical particles undergoing Coulomb scattering in the centre of mass frame. Nonidentical particles obey the Rutherford scattering formula (page 72).

**Relativistic wave equations<sup>a</sup>**

Klein–Gordon equation (massive, spin zero particles)	$(\nabla^2 - m^2)\psi = \frac{\partial^2 \psi}{\partial t^2} \quad (4.181)$	$\psi$ wavefunction $m$ particle mass $t$ time
Weyl equations (massless, spin 1/2 particles)	$\frac{\partial \psi}{\partial t} = \pm \left( \sigma_x \frac{\partial \psi}{\partial x} + \sigma_y \frac{\partial \psi}{\partial y} + \sigma_z \frac{\partial \psi}{\partial z} \right) \quad (4.182)$	$\psi$ spinor wavefunction $\sigma_i$ Pauli spin matrices (see page 26)
Dirac equation (massive, spin 1/2 particles)	$(\mathbf{i}\gamma^\mu \partial_\mu - m)\psi = 0 \quad (4.183)$	$\mathbf{i}$ $i^2 = -1$ $\gamma^\mu$ Dirac matrices: $\gamma^0 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix}$ $\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$ $\mathbf{1}_n$ $n \times n$ unit matrix
	where $\partial_\mu = \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \quad (4.184)$	
	$(\gamma^0)^2 = \mathbf{1}_4; \quad (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -\mathbf{1}_4 \quad (4.185)$	

<sup>a</sup>Written in natural units, with  $c = \hbar = 1$ .

# Chapter 5 Thermodynamics

## 5.1 Introduction

The term *thermodynamics* is used here loosely and includes classical thermodynamics, statistical thermodynamics, thermal physics, and radiation processes. Notation in these subjects can be confusing and the conventions used here are those found in the majority of modern treatments. In particular:

- The internal energy of a system is defined in terms of the heat supplied *to* the system plus the work done *on* the system, that is,  $dU = \delta Q + \delta W$ .
- The lowercase symbol  $p$  is used for pressure. Probability density functions are denoted by  $\text{pr}(x)$  and microstate probabilities by  $p_i$ .
- With the exception of *specific intensity*, quantities are taken as specific if they refer to unit mass and are distinguished from the extensive equivalent by using lowercase. Hence *specific volume*,  $v$ , equals  $V/m$ , where  $V$  is the volume of gas and  $m$  its mass. Also, the *specific heat capacity* of a gas at constant pressure is  $c_p = C_p/m$ , where  $C_p$  is the heat capacity of mass  $m$  of gas. Molar values take a subscript “m” (e.g.,  $V_m$  for molar volume) and remain in upper case.
- The component held constant during a partial differentiation is shown after a vertical bar; hence  $\left. \frac{\partial V}{\partial p} \right|_T$  is the partial differential of volume with respect to pressure, holding temperature constant.

The thermal properties of solids are dealt with more explicitly in the section on solid state physics (page 123). Note that in solid state literature *specific heat capacity* is often taken to mean heat capacity per unit volume.

## 5.2 Classical thermodynamics

### Thermodynamic laws

Thermodynamic temperature <sup>a</sup>	$T \propto \lim_{p \rightarrow 0} (pV)$	(5.1)	$T$ thermodynamic temperature $V$ volume of a fixed mass of gas $p$ gas pressure
Kelvin temperature scale	$T / \text{K} = 273.16 \frac{\lim_{p \rightarrow 0} (pV)_T}{\lim_{p \rightarrow 0} (pV)_{\text{tr}}}$	(5.2)	K kelvin unit tr temperature of the triple point of water
First law <sup>b</sup>	$dU = \delta Q + \delta W$	(5.3)	$dU$ change in internal energy $\delta W$ work done on system $\delta Q$ heat supplied to system
Entropy <sup>c</sup>	$dS = \frac{\delta Q_{\text{rev}}}{T} \geq \frac{\delta Q}{T}$	(5.4)	$S$ experimental entropy $T$ temperature rev reversible change

<sup>a</sup>As determined with a gas thermometer. The idea of temperature is associated with the zeroth law of thermodynamics: *If two systems are in thermal equilibrium with a third, they are also in thermal equilibrium with each other.*

<sup>b</sup>The  $\delta$  notation represents a differential change in a quantity that is not a function of state of the system.

<sup>c</sup>Associated with the second law of thermodynamics: *No process is possible with the sole effect of completely converting heat into work* (Kelvin statement).

### Thermodynamic work<sup>a</sup>

Hydrostatic pressure	$\delta W = -p dV$	(5.5)	$p$ (hydrostatic) pressure $dV$ volume change
Surface tension	$\delta W = \gamma dA$	(5.6)	$\delta W$ work done on the system $\gamma$ surface tension $dA$ change in area
Electric field	$\delta W = \mathbf{E} \cdot d\mathbf{p}$	(5.7)	$\mathbf{E}$ electric field $d\mathbf{p}$ induced electric dipole moment
Magnetic field	$\delta W = \mathbf{B} \cdot d\mathbf{m}$	(5.8)	$\mathbf{B}$ magnetic flux density $d\mathbf{m}$ induced magnetic dipole moment
Electric current	$\delta W = \Delta\phi dq$	(5.9)	$\Delta\phi$ potential difference $dq$ charge moved

<sup>a</sup>The sources of electric and magnetic fields are taken as being outside the thermodynamic system on which they are working.

**Cycle efficiencies (thermodynamic)<sup>a</sup>**

Heat engine	$\eta = \frac{\text{work extracted}}{\text{heat input}} \leq \frac{T_h - T_l}{T_h}$	(5.10)	$\eta$ efficiency $T_h$ higher temperature $T_l$ lower temperature
Refrigerator	$\eta = \frac{\text{heat extracted}}{\text{work done}} \leq \frac{T_l}{T_h - T_l}$	(5.11)	
Heat pump	$\eta = \frac{\text{heat supplied}}{\text{work done}} \leq \frac{T_h}{T_h - T_l}$	(5.12)	
Otto cycle <sup>b</sup>	$\eta = \frac{\text{work extracted}}{\text{heat input}} = 1 - \left(\frac{V_2}{V_1}\right)^{\gamma-1}$	(5.13)	$\frac{V_1}{V_2}$ compression ratio $\gamma$ ratio of heat capacities (assumed constant)

<sup>a</sup>The equalities are for reversible cycles, such as Carnot cycles, operating between temperatures  $T_h$  and  $T_l$ .

<sup>b</sup>Idealised reversible “petrol” (heat) engine.

**Heat capacities**

Constant volume	$C_V = \left.\frac{\delta Q}{dT}\right _V = \left.\frac{\partial U}{\partial T}\right _V = T \left.\frac{\partial S}{\partial T}\right _V$	(5.14)	$C_V$ heat capacity, $V$ constant $Q$ heat $T$ temperature $V$ volume $U$ internal energy $S$ entropy
Constant pressure	$C_p = \left.\frac{\delta Q}{dT}\right _p = \left.\frac{\partial H}{\partial T}\right _p = T \left.\frac{\partial S}{\partial T}\right _p$	(5.15)	$C_p$ heat capacity, $p$ constant $p$ pressure $H$ enthalpy
Difference in heat capacities	$C_p - C_V = \left(\left.\frac{\partial U}{\partial V}\right _T + p\right) \left.\frac{\partial V}{\partial T}\right _p$	(5.16)	$\beta_p$ isobaric expansivity $\kappa_T$ isothermal compressibility
	$= \frac{VT\beta_p^2}{\kappa_T}$	(5.17)	
Ratio of heat capacities	$\gamma = \frac{C_p}{C_V} = \frac{\kappa_T}{\kappa_S}$	(5.18)	$\gamma$ ratio of heat capacities $\kappa_S$ adiabatic compressibility

**Thermodynamic coefficients**

Isobaric expansivity <sup>a</sup>	$\beta_p = \frac{1}{V} \left.\frac{\partial V}{\partial T}\right _p$	(5.19)	$\beta_p$ isobaric expansivity $V$ volume $T$ temperature
Isothermal compressibility	$\kappa_T = -\frac{1}{V} \left.\frac{\partial V}{\partial p}\right _T$	(5.20)	$\kappa_T$ isothermal compressibility $p$ pressure
Adiabatic compressibility	$\kappa_S = -\frac{1}{V} \left.\frac{\partial V}{\partial p}\right _S$	(5.21)	$\kappa_S$ adiabatic compressibility
Isothermal bulk modulus	$K_T = \frac{1}{\kappa_T} = -V \left.\frac{\partial p}{\partial V}\right _T$	(5.22)	$K_T$ isothermal bulk modulus
Adiabatic bulk modulus	$K_S = \frac{1}{\kappa_S} = -V \left.\frac{\partial p}{\partial V}\right _S$	(5.23)	$K_S$ adiabatic bulk modulus

<sup>a</sup>Also called “cubic expansivity” or “volume expansivity.” The linear expansivity is  $\alpha_p = \beta_p/3$ .



## Expansion processes

Joule expansion <sup>a</sup>	$\eta = \left. \frac{\partial T}{\partial V} \right _U = - \frac{T^2}{C_V} \left. \frac{\partial(p/T)}{\partial T} \right _V \quad (5.24)$	$\eta$ Joule coefficient $T$ temperature $p$ pressure $U$ internal energy $C_V$ heat capacity, $V$ constant
	$= - \frac{1}{C_V} \left( T \left. \frac{\partial p}{\partial T} \right _V - p \right) \quad (5.25)$	
Joule–Kelvin expansion <sup>b</sup>	$\mu = \left. \frac{\partial T}{\partial p} \right _H = \frac{T^2}{C_p} \left. \frac{\partial(V/T)}{\partial T} \right _p \quad (5.26)$	$\mu$ Joule–Kelvin coefficient $V$ volume $H$ enthalpy $C_p$ heat capacity, $p$ constant
	$= \frac{1}{C_p} \left( T \left. \frac{\partial V}{\partial T} \right _p - V \right) \quad (5.27)$	

<sup>a</sup>Expansion with no change in internal energy.

<sup>b</sup>Expansion with no change in enthalpy. Also known as a “Joule–Thomson expansion” or “throttling” process.

Thermodynamic potentials<sup>a</sup>

Internal energy	$dU = T dS - p dV + \mu dN \quad (5.28)$	$U$ internal energy $T$ temperature $S$ entropy $\mu$ chemical potential $N$ number of particles
Enthalpy	$H = U + pV \quad (5.29)$	$H$ enthalpy $p$ pressure $V$ volume
	$dH = T dS + V dp + \mu dN \quad (5.30)$	
Helmholtz free energy <sup>b</sup>	$F = U - TS \quad (5.31)$	$F$ Helmholtz free energy
	$dF = -S dT - p dV + \mu dN \quad (5.32)$	
Gibbs free energy <sup>c</sup>	$G = U - TS + pV \quad (5.33)$	$G$ Gibbs free energy
	$= F + pV = H - TS \quad (5.34)$	
	$dG = -S dT + V dp + \mu dN \quad (5.35)$	
Grand potential	$\Phi = F - \mu N \quad (5.36)$	$\Phi$ grand potential
	$d\Phi = -S dT - p dV - N d\mu \quad (5.37)$	
Gibbs–Duhem relation	$-S dT + V dp - N d\mu = 0 \quad (5.38)$	
Availability	$A = U - T_0 S + p_0 V \quad (5.39)$	$A$ availability $T_0$ temperature of surroundings $p_0$ pressure of surroundings
	$dA = (T - T_0) dS - (p - p_0) dV \quad (5.40)$	

<sup>a</sup> $dN=0$  for a closed system.

<sup>b</sup>Sometimes called the “work function.”

<sup>c</sup>Sometimes called the “thermodynamic potential.”

## Maxwell's relations

Maxwell 1	$\left. \frac{\partial T}{\partial V} \right _S = - \left. \frac{\partial p}{\partial S} \right _V \quad \left( = \frac{\partial^2 U}{\partial S \partial V} \right)$	(5.41)	$U$ internal energy $T$ temperature $V$ volume
Maxwell 2	$\left. \frac{\partial T}{\partial p} \right _S = \left. \frac{\partial V}{\partial S} \right _p \quad \left( = \frac{\partial^2 H}{\partial p \partial S} \right)$	(5.42)	$H$ enthalpy $S$ entropy $p$ pressure
Maxwell 3	$\left. \frac{\partial p}{\partial T} \right _V = \left. \frac{\partial S}{\partial V} \right _T \quad \left( = \frac{\partial^2 F}{\partial T \partial V} \right)$	(5.43)	$F$ Helmholtz free energy
Maxwell 4	$\left. \frac{\partial V}{\partial T} \right _p = - \left. \frac{\partial S}{\partial p} \right _T \quad \left( = \frac{\partial^2 G}{\partial p \partial T} \right)$	(5.44)	$G$ Gibbs free energy

## Gibbs–Helmholtz equations

$U = -T^2 \left. \frac{\partial(F/T)}{\partial T} \right _V$	(5.45)	$F$ Helmholtz free energy $U$ internal energy
$G = -V^2 \left. \frac{\partial(F/V)}{\partial V} \right _T$	(5.46)	$G$ Gibbs free energy $H$ enthalpy
$H = -T^2 \left. \frac{\partial(G/T)}{\partial T} \right _p$	(5.47)	$T$ temperature $p$ pressure $V$ volume

## Phase transitions

Heat absorbed	$L = T(S_2 - S_1)$	(5.48)	$L$ (latent) heat absorbed (1 → 2) $T$ temperature of phase change $S$ entropy
Clausius–Clapeyron equation <sup>a</sup>	$\frac{dp}{dT} = \frac{S_2 - S_1}{V_2 - V_1}$	(5.49)	$p$ pressure $V$ volume 1,2 phase states
	$= \frac{L}{T(V_2 - V_1)}$	(5.50)	
Coexistence curve <sup>b</sup>	$p(T) \propto \exp\left(\frac{-L}{RT}\right)$	(5.51)	$R$ molar gas constant
Ehrenfest's equation <sup>c</sup>	$\frac{dp}{dT} = \frac{\beta_{p2} - \beta_{p1}}{\kappa_{T2} - \kappa_{T1}}$	(5.52)	$\beta_p$ isobaric expansivity $\kappa_T$ isothermal compressibility $C_p$ heat capacity ( $p$ constant)
	$= \frac{1}{VT} \frac{C_{p2} - C_{p1}}{\beta_{p2} - \beta_{p1}}$	(5.53)	
Gibbs's phase rule	$P + F = C + 2$	(5.54)	$P$ number of phases in equilibrium $F$ number of degrees of freedom $C$ number of components

<sup>a</sup>Phase boundary gradient for a first-order transition. Equation (5.50) is sometimes called the "Clapeyron equation."<sup>b</sup>For  $V_2 \gg V_1$ , e.g., if phase 1 is a liquid and phase 2 a vapour.<sup>c</sup>For a second-order phase transition.

### 5.3 Gas laws

#### Ideal gas

Joule's law	$U = U(T)$	(5.55)	$U$ internal energy $T$ temperature
Boyle's law	$pV _T = \text{constant}$	(5.56)	$p$ pressure $V$ volume
Equation of state (Ideal gas law)	$pV = nRT$	(5.57)	$n$ number of moles $R$ molar gas constant
Adiabatic equations	$pV^\gamma = \text{constant}$	(5.58)	$\gamma$ ratio of heat capacities ( $C_p/C_V$ ) $\Delta W$ work done on system
	$TV^{(\gamma-1)} = \text{constant}$	(5.59)	
	$T^\gamma p^{(1-\gamma)} = \text{constant}$	(5.60)	
	$\Delta W = \frac{1}{\gamma-1}(p_2V_2 - p_1V_1)$	(5.61)	
Internal energy	$U = \frac{nRT}{\gamma-1}$	(5.62)	
Reversible isothermal expansion	$\Delta Q = nRT \ln(V_2/V_1)$	(5.63)	$\Delta Q$ heat supplied to system 1,2 initial and final states
Joule expansion <sup>a</sup>	$\Delta S = nR \ln(V_2/V_1)$	(5.64)	$\Delta S$ change in entropy of the system

<sup>a</sup>Since  $\Delta Q = 0$  for a Joule expansion,  $\Delta S$  is due entirely to irreversibility. Because entropy is a function of state it has the same value as for the reversible isothermal expansion, where  $\Delta S = \Delta Q/T$ .

#### Virial expansion

Virial expansion	$pV = RT \left( 1 + \frac{B_2(T)}{V} + \frac{B_3(T)}{V^2} + \dots \right)$	(5.65)	$p$ pressure $V$ volume $R$ molar gas constant $T$ temperature $B_i$ virial coefficients
Boyle temperature	$B_2(T_B) = 0$	(5.66)	$T_B$ Boyle temperature

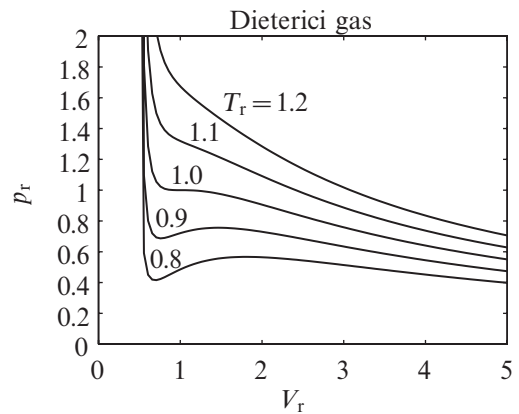
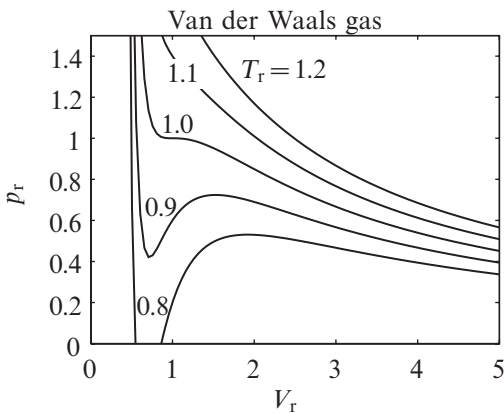


**Van der Waals gas**

Equation of state	$\left(p + \frac{a}{V_m^2}\right)(V_m - b) = RT \quad (5.67)$	<p><math>p</math> pressure</p> <p><math>V_m</math> molar volume</p> <p><math>R</math> molar gas constant</p> <p><math>T</math> temperature</p> <p><math>a, b</math> van der Waals' constants</p>
Critical point	$T_c = 8a/(27Rb) \quad (5.68)$	$T_c$ critical temperature
	$p_c = a/(27b^2) \quad (5.69)$	$p_c$ critical pressure
	$V_{mc} = 3b \quad (5.70)$	$V_{mc}$ critical molar volume
Reduced equation of state	$\left(p_r + \frac{3}{V_r^2}\right)(3V_r - 1) = 8T_r \quad (5.71)$	<p><math>p_r = p/p_c</math></p> <p><math>V_r = V_m/V_{mc}</math></p> <p><math>T_r = T/T_c</math></p>

**Dieterici gas**

Equation of state	$p = \frac{RT}{V_m - b'} \exp\left(\frac{-a'}{RTV_m}\right) \quad (5.72)$	<p><math>p</math> pressure</p> <p><math>V_m</math> molar volume</p> <p><math>R</math> molar gas constant</p> <p><math>T</math> temperature</p> <p><math>a', b'</math> Dieterici's constants</p>
Critical point	$T_c = a'/(4Rb') \quad (5.73)$	$T_c$ critical temperature
	$p_c = a'/(4b'^2e^2) \quad (5.74)$	$p_c$ critical pressure
	$V_{mc} = 2b' \quad (5.75)$	$V_{mc}$ critical molar volume
Reduced equation of state	$p_r = \frac{T_r}{2V_r - 1} \exp\left(2 - \frac{2}{V_r T_r}\right) \quad (5.76)$	<p><math>p_r = p/p_c</math></p> <p><math>V_r = V_m/V_{mc}</math></p> <p><math>T_r = T/T_c</math></p>



### 5.4 Kinetic theory

#### Monatomic gas

Pressure	$p = \frac{1}{3}nm\langle c^2 \rangle$	(5.77)	$p$ pressure $n$ number density = $N/V$ $m$ particle mass $\langle c^2 \rangle$ mean squared particle velocity
Equation of state of an ideal gas	$pV = NkT$	(5.78)	$V$ volume $k$ Boltzmann constant $N$ number of particles $T$ temperature
Internal energy	$U = \frac{3}{2}NkT = \frac{N}{2}m\langle c^2 \rangle$	(5.79)	$U$ internal energy
Heat capacities	$C_V = \frac{3}{2}Nk$	(5.80)	$C_V$ heat capacity, constant $V$ $C_p$ heat capacity, constant $p$ $\gamma$ ratio of heat capacities
	$C_p = C_V + Nk = \frac{5}{2}Nk$	(5.81)	
	$\gamma = \frac{C_p}{C_V} = \frac{5}{3}$	(5.82)	
Entropy (Sackur–Tetrode equation) <sup>a</sup>	$S = Nk \ln \left[ \left( \frac{mkT}{2\pi\hbar^2} \right)^{3/2} e^{5/2} \frac{V}{N} \right]$	(5.83)	$S$ entropy $\hbar$ = (Planck constant)/( $2\pi$ ) $e$ = 2.71828...

<sup>a</sup>For the uncondensed gas. The factor  $\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2}$  is the quantum concentration of the particles,  $n_Q$ . Their thermal de Broglie wavelength,  $\lambda_T$ , approximately equals  $n_Q^{-1/3}$ .

#### Maxwell–Boltzmann distribution<sup>a</sup>

Particle speed distribution	$pr(c) dc = \left( \frac{m}{2\pi kT} \right)^{3/2} \exp\left( \frac{-mc^2}{2kT} \right) 4\pi c^2 dc$	(5.84)	$pr$ probability density $m$ particle mass $k$ Boltzmann constant $T$ temperature $c$ particle speed
Particle energy distribution	$pr(E) dE = \frac{2E^{1/2}}{\pi^{1/2}(kT)^{3/2}} \exp\left( \frac{-E}{kT} \right) dE$	(5.85)	$E$ particle kinetic energy ( $=mc^2/2$ )
Mean speed	$\langle c \rangle = \left( \frac{8kT}{\pi m} \right)^{1/2}$	(5.86)	$\langle c \rangle$ mean speed
rms speed	$c_{rms} = \left( \frac{3kT}{m} \right)^{1/2} = \left( \frac{3\pi}{8} \right)^{1/2} \langle c \rangle$	(5.87)	$c_{rms}$ root mean squared speed
Most probable speed	$\hat{c} = \left( \frac{2kT}{m} \right)^{1/2} = \left( \frac{\pi}{4} \right)^{1/2} \langle c \rangle$	(5.88)	$\hat{c}$ most probable speed

<sup>a</sup>Probability density functions normalised so that  $\int_0^\infty pr(x) dx = 1$ .

**Transport properties**

Mean free path <sup>a</sup>	$l = \frac{1}{\sqrt{2}\pi d^2 n}$	(5.89)	<i>l</i> mean free path <i>d</i> molecular diameter <i>n</i> particle number density
Survival equation <sup>b</sup>	$\text{pr}(x) = \exp(-x/l)$	(5.90)	<i>pr</i> probability <i>x</i> linear distance
Flux through a plane <sup>c</sup>	$J = \frac{1}{4}n\langle c \rangle$	(5.91)	<i>J</i> molecular flux $\langle c \rangle$ mean molecular speed
Self-diffusion (Fick's law of diffusion) <sup>d</sup>	$J = -D\nabla n$	(5.92)	<i>D</i> diffusion coefficient
	where $D \simeq \frac{2}{3}l\langle c \rangle$	(5.93)	
Thermal conductivity <sup>d</sup>	$H = -\lambda\nabla T$	(5.94)	<i>H</i> heat flux per unit area
	$\nabla^2 T = \frac{1}{D} \frac{\partial T}{\partial t}$	(5.95)	$\lambda$ thermal conductivity <i>T</i> temperature
	for monatomic gas $\lambda \simeq \frac{5}{4}\rho l\langle c \rangle c_V$	(5.96)	$\rho$ density <i>c<sub>V</sub></i> specific heat capacity, <i>V</i> constant
Viscosity <sup>d</sup>	$\eta \simeq \frac{1}{2}\rho l\langle c \rangle$	(5.97)	$\eta$ dynamic viscosity <i>x</i> displacement of sphere in <i>x</i> direction after time <i>t</i>
Brownian motion (of a sphere)	$\langle x^2 \rangle = \frac{kTt}{3\pi\eta a}$	(5.98)	<i>k</i> Boltzmann constant <i>t</i> time interval <i>a</i> sphere radius
Free molecular flow (Knudsen flow) <sup>e</sup>	$\frac{dM}{dt} = \frac{4R_p^3}{3L} \left( \frac{2\pi m}{k} \right)^{1/2} \left( \frac{p_1}{T_1^{1/2}} - \frac{p_2}{T_2^{1/2}} \right)$	(5.99)	$\frac{dM}{dt}$ mass flow rate <i>R<sub>p</sub></i> pipe radius <i>L</i> pipe length <i>m</i> particle mass <i>p</i> pressure

<sup>a</sup>For a perfect gas of hard, spherical particles with a Maxwell–Boltzmann speed distribution.

<sup>b</sup>Probability of travelling distance *x* without a collision.

<sup>c</sup>From the side where the number density is *n*, assuming an isotropic velocity distribution. Also known as “collision number.”

<sup>d</sup>Simplistic kinetic theory yields numerical coefficients of 1/3 for *D*,  $\lambda$  and  $\eta$ .

<sup>e</sup>Through a pipe from end 1 to end 2, assuming  $R_p \ll l$  (i.e., at very low pressure).

**Gas equipartition**

Classical equipartition <sup>a</sup>	$E_q = \frac{1}{2}kT$	(5.100)	<i>E<sub>q</sub></i> energy per quadratic degree of freedom <i>k</i> Boltzmann constant <i>T</i> temperature
Ideal gas heat capacities	$C_V = \frac{1}{2}fNk = \frac{1}{2}fnR$	(5.101)	<i>C<sub>V</sub></i> heat capacity, <i>V</i> constant <i>C<sub>p</sub></i> heat capacity, <i>p</i> constant
	$C_p = Nk \left( 1 + \frac{f}{2} \right)$	(5.102)	<i>N</i> number of molecules <i>f</i> number of degrees of freedom
	$\gamma = \frac{C_p}{C_V} = 1 + \frac{2}{f}$	(5.103)	<i>n</i> number of moles <i>R</i> molar gas constant $\gamma$ ratio of heat capacities

<sup>a</sup>System in thermal equilibrium at temperature *T*.



## 5.5 Statistical thermodynamics

### Statistical entropy

Boltzmann formula <sup>a</sup>	$S = k \ln W$ (5.104)	$\simeq k \ln g(E)$ (5.105)	$S$ entropy $k$ Boltzmann constant $W$ number of accessible microstates $g(E)$ density of microstates with energy $E$
Gibbs entropy <sup>b</sup>	$S = -k \sum_i p_i \ln p_i$ (5.106)		$\sum_i$ sum over microstates $p_i$ probability that the system is in microstate $i$
$N$ two-level systems	$W = \frac{N!}{(N-n)!n!}$ (5.107)		$N$ number of systems $n$ number in upper state
$N$ harmonic oscillators	$W = \frac{(Q+N-1)!}{Q!(N-1)!}$ (5.108)		$Q$ total number of energy quanta available

<sup>a</sup>Sometimes called “configurational entropy.” Equation (5.105) is true only for large systems.

<sup>b</sup>Sometimes called “canonical entropy.”

### Ensemble probabilities

Microcanonical ensemble <sup>a</sup>	$p_i = \frac{1}{W}$ (5.109)	$p_i$ probability that the system is in microstate $i$ $W$ number of accessible microstates
Partition function <sup>b</sup>	$Z = \sum_i e^{-\beta E_i}$ (5.110)	$Z$ partition function $\sum_i$ sum over microstates $\beta = 1/(kT)$ $E_i$ energy of microstate $i$
Canonical ensemble (Boltzmann distribution) <sup>c</sup>	$p_i = \frac{1}{Z} e^{-\beta E_i}$ (5.111)	$k$ Boltzmann constant $T$ temperature
Grand partition function	$\Xi = \sum_i e^{-\beta(E_i - \mu N_i)}$ (5.112)	$\Xi$ grand partition function $\mu$ chemical potential $N_i$ number of particles in microstate $i$
Grand canonical ensemble (Gibbs distribution) <sup>d</sup>	$p_i = \frac{1}{\Xi} e^{-\beta(E_i - \mu N_i)}$ (5.113)	

<sup>a</sup>Energy fixed.

<sup>b</sup>Also called “sum over states.”

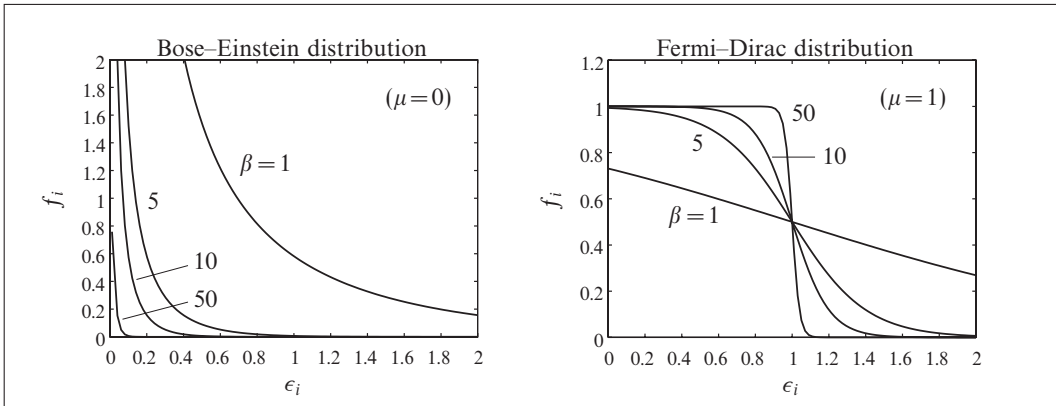
<sup>c</sup>Temperature fixed.

<sup>d</sup>Temperature fixed. Exchange of both heat and particles with a reservoir.

**Macroscopic thermodynamic variables**

Helmholtz free energy	$F = -kT \ln Z$	(5.114)	$F$ Helmholtz free energy $k$ Boltzmann constant $T$ temperature $Z$ partition function
Grand potential	$\Phi = -kT \ln \Xi$	(5.115)	$\Phi$ grand potential $\Xi$ grand partition function
Internal energy	$U = F + TS = -\left. \frac{\partial \ln Z}{\partial \beta} \right _{V,N}$	(5.116)	$U$ internal energy $\beta = 1/(kT)$
Entropy	$S = -\left. \frac{\partial F}{\partial T} \right _{V,N} = \left. \frac{\partial (kT \ln Z)}{\partial T} \right _{V,N}$	(5.117)	$S$ entropy $N$ number of particles
Pressure	$p = -\left. \frac{\partial F}{\partial V} \right _{T,N} = \left. \frac{\partial (kT \ln Z)}{\partial V} \right _{T,N}$	(5.118)	$p$ pressure
Chemical potential	$\mu = \left. \frac{\partial F}{\partial N} \right _{V,T} = -\left. \frac{\partial (kT \ln Z)}{\partial N} \right _{V,T}$	(5.119)	$\mu$ chemical potential

**Identical particles**



Bose-Einstein distribution <sup>a</sup>	$f_i = \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1}$	(5.120)	$f_i$ mean occupation number of $i$ th state $\beta = 1/(kT)$
Fermi-Dirac distribution <sup>b</sup>	$f_i = \frac{1}{e^{\beta(\epsilon_i - \mu)} + 1}$	(5.121)	$\epsilon_i$ energy quantum for $i$ th state $\mu$ chemical potential
Fermi energy <sup>c</sup>	$\epsilon_F = \frac{\hbar^2}{2m} \left( \frac{6\pi^2 n}{g} \right)^{2/3}$	(5.122)	$\epsilon_F$ Fermi energy $\hbar$ (Planck constant)/( $2\pi$ ) $n$ particle number density $m$ particle mass $g$ spin degeneracy ( $=2s+1$ )
Bose condensation temperature	$T_c = \frac{2\pi\hbar^2}{mk} \left[ \frac{n}{g\zeta(3/2)} \right]^{2/3}$	(5.123)	$\zeta$ Riemann zeta function $\zeta(3/2) \approx 2.612$ $T_c$ Bose condensation temperature

<sup>a</sup>For bosons.  $f_i \geq 0$ .

<sup>b</sup>For fermions.  $0 \leq f_i \leq 1$ .

<sup>c</sup>For noninteracting particles. At low temperatures,  $\mu \approx \epsilon_F$ .



Population densities<sup>a</sup>

Boltzmann excitation equation	$\frac{n_{mj}}{n_{lj}} = \frac{g_{mj}}{g_{lj}} \exp\left[\frac{-(\chi_{mj} - \chi_{lj})}{kT}\right]$	(5.124)	$n_{ij}$ number density of atoms in excitation level $i$ of ionisation state $j$ ( $j=0$ if not ionised)
	$= \frac{g_{mj}}{g_{lj}} \exp\left(\frac{-h\nu_{lm}}{kT}\right)$	(5.125)	$g_{ij}$ level degeneracy $\chi_{ij}$ excitation energy relative to the ground state
Partition function	$Z_j(T) = \sum_i g_{ij} \exp\left(\frac{-\chi_{ij}}{kT}\right)$	(5.126)	$\nu_{ij}$ photon transition frequency $h$ Planck constant
	$\frac{n_{ij}}{N_j} = \frac{g_{ij}}{Z_j(T)} \exp\left(\frac{-\chi_{ij}}{kT}\right)$	(5.127)	$k$ Boltzmann constant $T$ temperature
Saha equation (general)	$n_{ij} = n_{0,j+1} n_e \frac{g_{ij}}{g_{0,j+1}} \frac{h^3}{2} (2\pi m_e kT)^{-3/2} \exp\left(\frac{\chi_{lj} - \chi_{ij}}{kT}\right)$	(5.128)	$Z_j$ partition function for ionisation state $j$ $N_j$ total number density in ionisation state $j$
Saha equation (ion populations)	$\frac{N_j}{N_{j+1}} = n_e \frac{Z_j(T)}{Z_{j+1}(T)} \frac{h^3}{2} (2\pi m_e kT)^{-3/2} \exp\left(\frac{\chi_{lj}}{kT}\right)$	(5.129)	$n_e$ electron number density $m_e$ electron mass $\chi_{lj}$ ionisation energy of atom in ionisation state $j$

<sup>a</sup>All equations apply only under conditions of local thermodynamic equilibrium (LTE). In atoms with no magnetic splitting, the degeneracy of a level with total angular momentum quantum number  $J$  is  $g_{ij} = 2J + 1$ .

## 5.6 Fluctuations and noise

Thermodynamic fluctuations<sup>a</sup>

Fluctuation probability	$\text{pr}(x) \propto \exp[S(x)/k]$	(5.130)	$\text{pr}$ probability density $x$ unconstrained variable
	$\propto \exp\left[\frac{-A(x)}{kT}\right]$	(5.131)	$S$ entropy $A$ availability
General variance	$\text{var}[x] = kT \left[ \frac{\partial^2 A(x)}{\partial x^2} \right]^{-1}$	(5.132)	$\text{var}[\cdot]$ mean square deviation $k$ Boltzmann constant $T$ temperature
Temperature fluctuations	$\text{var}[T] = kT \left. \frac{\partial T}{\partial S} \right _V = \frac{kT^2}{C_V}$	(5.133)	$V$ volume $C_V$ heat capacity, $V$ constant
Volume fluctuations	$\text{var}[V] = -kT \left. \frac{\partial V}{\partial p} \right _T = \kappa_T V kT$	(5.134)	$p$ pressure $\kappa_T$ isothermal compressibility
Entropy fluctuations	$\text{var}[S] = kT \left. \frac{\partial S}{\partial T} \right _p = kC_p$	(5.135)	$C_p$ heat capacity, $p$ constant
Pressure fluctuations	$\text{var}[p] = -kT \left. \frac{\partial p}{\partial V} \right _S = \frac{K_S kT}{V}$	(5.136)	$K_S$ adiabatic bulk modulus
Density fluctuations	$\text{var}[n] = \frac{n^2}{V^2} \text{var}[V] = \frac{n^2}{V} \kappa_T kT$	(5.137)	$n$ number density

<sup>a</sup>In part of a large system, whose mean temperature is fixed. Quantum effects are assumed negligible.

**Noise**

Nyquist's noise theorem	$dw = kT \cdot \beta \epsilon (e^{\beta \epsilon} - 1)^{-1} dv$	(5.138)	$w$	exchangeable noise power
	$= kT_N dv$	(5.139)	$k$	Boltzmann constant
	$\simeq kT dv \quad (hv \ll kT)$	(5.140)	$T$	temperature
Johnson (thermal) noise voltage <sup>a</sup>	$v_{\text{rms}} = (4kT_N R \Delta v)^{1/2}$	(5.141)	$T_N$	noise temperature
			$\beta \epsilon$	$= hv / (kT)$
			$v$	frequency
			$h$	Planck constant
			$v_{\text{rms}}$	rms noise voltage
Shot noise (electrical)	$I_{\text{rms}} = (2eI_0 \Delta v)^{1/2}$	(5.142)	$R$	resistance
			$\Delta v$	bandwidth
			$I_{\text{rms}}$	rms noise current
Noise figure <sup>b</sup>	$f_{\text{dB}} = 10 \log_{10} \left( 1 + \frac{T_N}{T_0} \right)$	(5.143)	$-e$	electronic charge
			$I_0$	mean current
Relative power	$G = 10 \log_{10} \left( \frac{P_2}{P_1} \right)$	(5.144)	$f_{\text{dB}}$	noise figure (decibels)
			$T_0$	ambient temperature (usually taken as 290 K)
			$G$	decibel gain of $P_2$ over $P_1$
			$P_1, P_2$	power levels

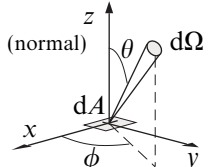
<sup>a</sup>Thermal voltage over an open-circuit resistance.

<sup>b</sup>Noise figure can also be defined as  $f = 1 + T_N/T_0$ , when it is also called "noise factor."



### 5.7 Radiation processes

#### Radiometry<sup>a</sup>

Radiant energy <sup>b</sup>	$Q_e = \iiint L_e \cos \theta \, dA \, d\Omega \, dt \quad \text{J} \quad (5.145)$	$Q_e$ radiant energy $L_e$ radiance (generally a function of position and direction) $\theta$ angle between dir. of $d\Omega$ and normal to $dA$ $\Omega$ solid angle
Radiant flux (“radiant power”)	$\Phi_e = \frac{\partial Q_e}{\partial t} \quad \text{W} \quad (5.146)$	$A$ area $t$ time $\Phi_e$ radiant flux
Radiant energy density <sup>c</sup>	$W_e = \frac{\partial Q_e}{\partial V} \quad \text{J m}^{-3} \quad (5.148)$	$W_e$ radiant energy density $dV$ differential volume of propagation medium
Radiant exitance <sup>d</sup>	$M_e = \frac{\partial \Phi_e}{\partial A} \quad \text{W m}^{-2} \quad (5.149)$	$M_e$ radiant exitance
Irradiance <sup>e</sup>	$E_e = \frac{\partial \Phi_e}{\partial A} \quad \text{W m}^{-2} \quad (5.151)$	
Radiant intensity	$I_e = \frac{\partial \Phi_e}{\partial \Omega} \quad \text{W sr}^{-1} \quad (5.153)$	$E_e$ irradiance $I_e$ radiant intensity
Radiance	$L_e = \frac{1}{\cos \theta} \frac{\partial^2 \Phi_e}{\partial A \, \partial \Omega} \quad \text{W m}^{-2} \text{sr}^{-1} \quad (5.155)$	
	$= \frac{1}{\cos \theta} \frac{\partial I_e}{\partial A} \quad (5.156)$	

<sup>a</sup>Radiometry is concerned with the treatment of light as energy.

<sup>b</sup>Sometimes called “total energy.” Note that we assume opaque radiant surfaces, so that  $0 \leq \theta \leq \pi/2$ .

<sup>c</sup>The instantaneous amount of radiant energy contained in a unit volume of propagation medium.

<sup>d</sup>Power per unit area leaving a surface. For a perfectly diffusing surface,  $M_e = \pi L_e$ .

<sup>e</sup>Power per unit area incident on a surface.



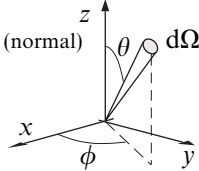
**Photometry<sup>a</sup>**

<p>Luminous energy ("total light")</p>	$Q_v = \iiint L_v \cos \theta \, dA \, d\Omega \, dt \quad \text{lms} \quad (5.157)$	<p><math>Q_v</math> luminous energy <math>L_v</math> luminance (generally a function of position and direction) <math>\theta</math> angle between dir. of <math>d\Omega</math> and normal to <math>dA</math> <math>\Omega</math> solid angle</p>
<p>Luminous flux</p>	$\Phi_v = \frac{\partial Q_v}{\partial t} \quad \text{lumen (lm)} \quad (5.158)$	<p><math>A</math> area <math>t</math> time <math>\Phi_v</math> luminous flux</p>
<p>Luminous density<sup>b</sup></p>	$W_v = \frac{\partial Q_v}{\partial V} \quad \text{lm m}^{-3} \quad (5.160)$	<p><math>W_v</math> luminous density <math>V</math> volume</p>
<p>Luminous exitance<sup>c</sup></p>	$M_v = \frac{\partial \Phi_v}{\partial A} \quad \text{lx} \quad (\text{lm m}^{-2}) \quad (5.161)$	<p><math>M_v</math> luminous exitance</p>
<p>Illuminance ("illumination")<sup>d</sup></p>	$E_v = \frac{\partial \Phi_v}{\partial A} \quad \text{lm m}^{-2} \quad (5.163)$	<p>(normal)</p>
<p>Luminous intensity<sup>e</sup></p>	$I_v = \frac{\partial \Phi_v}{\partial \Omega} \quad \text{cd} \quad (5.165)$	<p><math>E_v</math> illuminance <math>I_v</math> luminous intensity</p>
<p>Luminance ("photometric brightness")</p>	$L_v = \frac{1}{\cos \theta} \frac{\partial^2 \Phi_v}{\partial A \, \partial \Omega} \quad \text{cd m}^{-2} \quad (5.167)$	<p><math>\theta</math> <math>d\Omega</math> <math>dA</math> <math>\phi</math> <math>x</math> <math>y</math> <math>z</math></p>
<p>Luminous efficacy</p>	$K = \frac{\Phi_v}{\Phi_e} = \frac{L_v}{L_e} = \frac{I_v}{I_e} \quad \text{lm W}^{-1} \quad (5.169)$	<p><math>K</math> luminous efficacy <math>L_e</math> radiance <math>\Phi_e</math> radiant flux <math>I_e</math> radiant intensity</p>
<p>Luminous efficiency</p>	$V(\lambda) = \frac{K(\lambda)}{K_{\max}} \quad (5.170)$	<p><math>V</math> luminous efficiency <math>\lambda</math> wavelength <math>K_{\max}</math> spectral maximum of <math>K(\lambda)</math></p>



<sup>a</sup>Photometry is concerned with the treatment of light as seen by the human eye.  
<sup>b</sup>The instantaneous amount of luminous energy contained in a unit volume of propagating medium.  
<sup>c</sup>Luminous emitted flux per unit area.  
<sup>d</sup>Luminous incident flux per unit area. The derived SI unit is the lux (lx). 1lx = 1lm m<sup>-2</sup>.  
<sup>e</sup>The SI unit of luminous intensity is the candela (cd). 1cd = 1lm sr<sup>-1</sup>.

Radiative transfer<sup>a</sup>

Flux density (through a plane)	$F_v = \int I_v(\theta, \phi) \cos \theta \, d\Omega \quad \text{W m}^{-2} \text{Hz}^{-1} \quad (5.171)$	 <p>(normal)</p> <p><math>F_v</math> flux density  <math>I_v</math> specific intensity  <math>(\text{W m}^{-2} \text{Hz}^{-1} \text{sr}^{-1})</math>  <math>J_v</math> mean intensity  <math>u_v</math> spectral energy density  <math>\Omega</math> solid angle  <math>\theta</math> angle between normal  and direction of <math>\Omega</math>  <math>j_v</math> specific emission  coefficient  <math>\epsilon_v</math> emission coefficient  <math>(\text{W m}^{-3} \text{Hz}^{-1} \text{sr}^{-1})</math>  <math>\rho</math> density  <math>\alpha_v</math> linear absorption  coefficient  <math>n</math> particle number density  <math>\sigma_v</math> particle cross section  <math>l_v</math> mean free path  <math>\kappa_v</math> opacity  <math>\tau_v</math> optical depth, or  optical thickness  <math>ds</math> line element  <math>S_v</math> source function</p>
Mean intensity <sup>b</sup>	$J_v = \frac{1}{4\pi} \int I_v(\theta, \phi) \, d\Omega \quad \text{W m}^{-2} \text{Hz}^{-1} \quad (5.172)$	
Spectral energy density <sup>c</sup>	$u_v = \frac{1}{c} \int I_v(\theta, \phi) \, d\Omega \quad \text{J m}^{-3} \text{Hz}^{-1} \quad (5.173)$	
Specific emission coefficient	$j_v = \frac{\epsilon_v}{\rho} \quad \text{W kg}^{-1} \text{Hz}^{-1} \text{sr}^{-1} \quad (5.174)$	
Gas linear absorption coefficient ( $\alpha_v \ll 1$ )	$\alpha_v = n\sigma_v = \frac{1}{l_v} \quad \text{m}^{-1} \quad (5.175)$	
Opacity <sup>d</sup>	$\kappa_v = \frac{\alpha_v}{\rho} \quad \text{kg}^{-1} \text{m}^2 \quad (5.176)$	
Optical depth	$\tau_v = \int \kappa_v \rho \, ds \quad (5.177)$	
Transfer equation <sup>e</sup>	$\frac{1}{\rho} \frac{dI_v}{ds} = -\kappa_v I_v + j_v \quad (5.178)$ <p>or</p> $\frac{dI_v}{ds} = -\alpha_v I_v + \epsilon_v \quad (5.179)$	
Kirchhoff's law <sup>f</sup>	$S_v \equiv \frac{j_v}{\kappa_v} = \frac{\epsilon_v}{\alpha_v} \quad (5.180)$	
Emission from a homogeneous medium	$I_v = S_v(1 - e^{-\tau_v}) \quad (5.181)$	

<sup>a</sup>The definitions of these quantities vary in the literature. Those presented here are common in meteorology and astrophysics. Note particularly that the ambiguous term *specific* is taken to mean "per unit frequency interval" in the case of specific intensity and "per unit mass per unit frequency interval" in the case of specific emission coefficient.

<sup>b</sup>In radio astronomy, flux density is usually taken as  $S = 4\pi J_v$ .

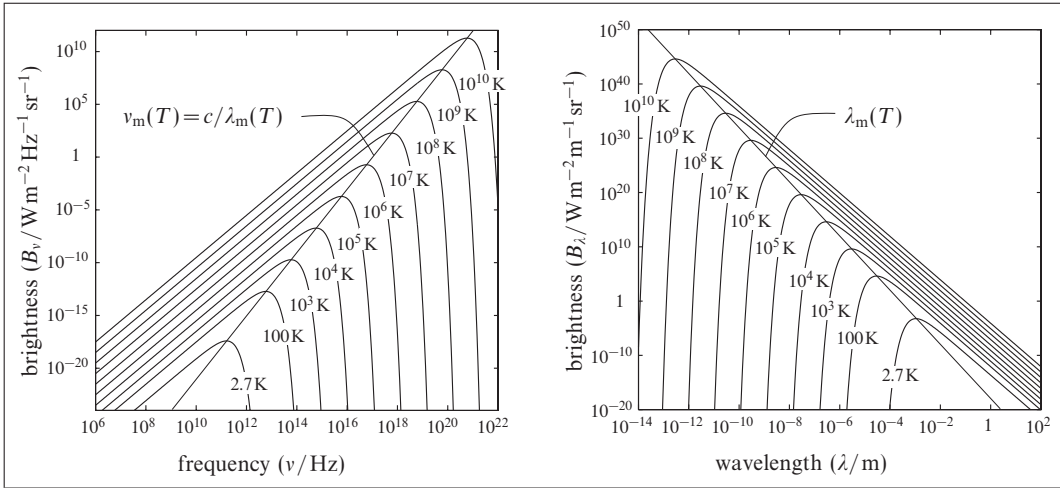
<sup>c</sup>Assuming a refractive index of 1.

<sup>d</sup>Or "mass absorption coefficient."

<sup>e</sup>Or "Schwarzschild's equation."

<sup>f</sup>Under conditions of local thermal equilibrium (LTE), the source function,  $S_v$ , equals the Planck function,  $B_v(T)$  [see Equation (5.182)].

**Blackbody radiation**



Planck function <sup>a</sup>	$B_\nu(T) = \frac{2h\nu^3}{c^2} \left[ \exp\left(\frac{h\nu}{kT}\right) - 1 \right]^{-1} \quad (5.182)$	$B_\nu$ surface brightness per unit frequency ( $\text{W m}^{-2} \text{Hz}^{-1} \text{sr}^{-1}$ )
	$B_\lambda(T) = B_\nu(T) \frac{d\nu}{d\lambda} \quad (5.183)$	$B_\lambda$ surface brightness per unit wavelength ( $\text{W m}^{-2} \text{m}^{-1} \text{sr}^{-1}$ )
	$= \frac{2hc^2}{\lambda^5} \left[ \exp\left(\frac{hc}{\lambda kT}\right) - 1 \right]^{-1} \quad (5.184)$	$h$ Planck constant
Spectral energy density	$u_\nu(T) = \frac{4\pi}{c} B_\nu(T) \quad \text{J m}^{-3} \text{Hz}^{-1} \quad (5.185)$	$c$ speed of light
	$u_\lambda(T) = \frac{4\pi}{c} B_\lambda(T) \quad \text{J m}^{-3} \text{m}^{-1} \quad (5.186)$	$k$ Boltzmann constant $T$ temperature $u_{\nu,\lambda}$ spectral energy density
Rayleigh–Jeans law ( $h\nu \ll kT$ )	$B_\nu(T) = \frac{2kT}{c^2} \nu^2 = \frac{2kT}{\lambda^2} \quad (5.187)$	
Wien’s law ( $h\nu \gg kT$ )	$B_\nu(T) = \frac{2h\nu^3}{c^2} \exp\left(\frac{-h\nu}{kT}\right) \quad (5.188)$	
Wien’s displacement law	$\lambda_m T = \begin{cases} 5.1 \times 10^{-3} \text{ mK} & \text{for } B_\nu \\ 2.9 \times 10^{-3} \text{ mK} & \text{for } B_\lambda \end{cases} \quad (5.189)$	$\lambda_m$ wavelength of maximum brightness
Stefan–Boltzmann law <sup>b</sup>	$M = \pi \int_0^\infty B_\nu(T) d\nu \quad (5.190)$	$M$ exitance
	$= \frac{2\pi^5 k^4}{15c^2 h^3} T^4 = \sigma T^4 \quad \text{W m}^{-2} \quad (5.191)$	$\sigma$ Stefan–Boltzmann constant ( $\approx 5.67 \times 10^{-8} \text{ W m}^{-2} \text{K}^{-4}$ )
Energy density	$u(T) = \frac{4}{c} \sigma T^4 \quad \text{J m}^{-3} \quad (5.192)$	$u$ energy density
Greybody	$M = \epsilon \sigma T^4 = (1 - A) \sigma T^4 \quad (5.193)$	$\epsilon$ mean emissivity $A$ albedo

<sup>a</sup>With respect to the projected area of the surface. Surface brightness is also known simply as “brightness.” “Specific intensity” is used for reception.

<sup>b</sup>Sometimes “Stefan’s law.” Exitance is the total radiated energy from unit area of the body per unit time.





# Chapter 6 Solid state physics

## 6.1 Introduction

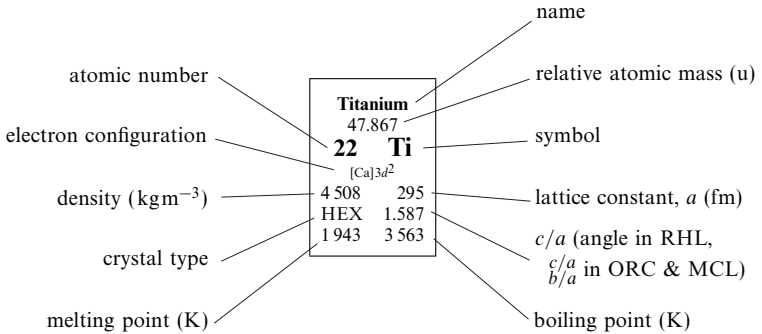
This section covers a few selected topics in solid state physics. There is no attempt to do more than scratch the surface of this vast field, although the basics of many undergraduate texts on the subject are covered. In addition a period table of elements, together with some of their physical properties, is displayed on the next two pages.

**Periodic table (overleaf)** Data for the periodic table of elements are taken from *Pure Appl. Chem.*, **71**, 1593–1607 (1999), from the 16th edition of Kaye and Laby *Tables of Physical and Chemical Constants* (Longman, 1995) and from the 74th edition of the CRC *Handbook of Chemistry and Physics* (CRC Press, 1993). Note that melting and boiling points have been converted to kelvins by adding 273.15 to the Celsius values listed in Kaye and Laby. The standard atomic masses reflect the relative isotopic abundances in samples found naturally on Earth, and the number of significant figures reflect the variations between samples. Elements with atomic masses shown in square brackets have no stable nuclides, and the values reflect the mass numbers of the longest-lived isotopes. Crystallographic data are based on the most common forms of the elements (the  $\alpha$ -form, unless stated otherwise) stable under standard conditions. Densities are for the solid state. For full details and footnotes for each element, the reader is advised to consult the original texts.

Elements 110, 111, 112 and 114 are known to exist but their names are not yet permanent.

6.2 Periodic table

1									9
1 <b>Hydrogen</b> 1.007 94 <b>1 H</b> $1s^1$ 89 ( $\beta$ ) 378 HEX 1.632 13.80 20.28									2
2 <b>Lithium</b> 6.941 <b>3 Li</b> $[He]2s^1$ 533 ( $\beta$ ) 351 BCC 453.65 1613	<b>Beryllium</b> 9.012 182 <b>4 Be</b> $[He]2s^2$ 1846 229 HEX 1.568 1560 2745								
3 <b>Sodium</b> 22.989 770 <b>11 Na</b> $[Ne]3s^1$ 966 429 BCC 370.8 1153	<b>Magnesium</b> 24.305 0 <b>12 Mg</b> $[Ne]3s^2$ 1738 321 HEX 1.624 923 1363								
4 <b>Potassium</b> 39.098 3 <b>19 K</b> $[Ar]4s^1$ 862 532 BCC 336.5 1033	<b>Calcium</b> 40.078 <b>20 Ca</b> $[Ar]4s^2$ 1530 559 FCC 1113 1757	5 <b>Scandium</b> 44.955 910 <b>21 Sc</b> $[Ca]3d^1$ 2992 331 HEX 1.592 1813 3103	6 <b>Titanium</b> 47.867 <b>22 Ti</b> $[Ca]3d^2$ 4508 295 HEX 1.587 1943 3563	7 <b>Vanadium</b> 50.941 5 <b>23 V</b> $[Ca]3d^3$ 6090 302 BCC 2193 3673	8 <b>Chromium</b> 51.996 1 <b>24 Cr</b> $[Ar]3d^5 4s^1$ 7194 388 BCC 2180 2943	9 <b>Manganese</b> 54.938 049 <b>25 Mn</b> $[Ca]3d^5$ 7473 891 FCC 1523 2333	10 <b>Iron</b> 55.845 <b>26 Fe</b> $[Ca]3d^6$ 7873 287 BCC 1813 3133	11 <b>Cobalt</b> 58.933 200 <b>27 Co</b> $[Ca]3d^7$ 8800 ( $\epsilon$ ) 251 HEX 1.623 1768 3203	
5 <b>Rubidium</b> 85.467 8 <b>37 Rb</b> $[Kr]5s^1$ 1533 571 BCC 312.4 963.1	<b>Strontium</b> 87.62 <b>38 Sr</b> $[Kr]5s^2$ 2583 608 FCC 1050 1653	<b>Yttrium</b> 88.905 85 <b>39 Y</b> $[Sr]4d^1$ 4475 365 HEX 1.571 1798 3613	<b>Zirconium</b> 91.224 <b>40 Zr</b> $[Sr]4d^2$ 6507 323 HEX 1.593 2123 4673	<b>Niobium</b> 92.906 38 <b>41 Nb</b> $[Kr]4d^4 5s^1$ 8578 330 BCC 2750 4973	<b>Molybdenum</b> 95.94 <b>42 Mo</b> $[Kr]4d^5 5s^1$ 10222 315 BCC 2896 4913	<b>Technetium</b> [98] <b>43 Tc</b> $[Sr]4d^5$ 11496 274 HEX 1.604 2433 4533	<b>Ruthenium</b> 101.07 <b>44 Ru</b> $[Kr]4d^7 5s^1$ 12360 270 HEX 1.582 2603 4423	<b>Rhodium</b> 102.905 50 <b>45 Rh</b> $[Kr]4d^8 5s^1$ 12420 380 FCC 2236 3973	
6 <b>Cesium</b> 132.905 45 <b>55 Cs</b> $[Xe]6s^1$ 1900 614 BCC 301.6 943.2	<b>Barium</b> 137.327 <b>56 Ba</b> $[Xe]6s^2$ 3594 502 BCC 1001 2173	7 <b>Lanthanides</b> <b>57 – 71</b>	<b>Hafnium</b> 178.49 <b>72 Hf</b> $[Yb]5d^2$ 13276 319 HEX 1.581 2503 4873	<b>Tantalum</b> 180.947 9 <b>73 Ta</b> $[Yb]5d^3$ 16670 330 BCC 3293 5833	<b>Tungsten</b> 183.84 <b>74 W</b> $[Yb]5d^4$ 19254 316 BCC 3695 5823	<b>Rhenium</b> 186.207 <b>75 Re</b> $[Yb]5d^5$ 21023 276 HEX 1.615 3459 5873	<b>Osmium</b> 190.23 <b>76 Os</b> $[Yb]5d^6$ 22580 273 HEX 1.606 3303 5273	<b>Iridium</b> 192.217 <b>77 Ir</b> $[Yb]5d^7$ 22550 384 FCC 2720 4703	
7 <b>Francium</b> [223] <b>87 Fr</b> $[Rn]7s^1$ 300 923	<b>Radium</b> [226] <b>88 Ra</b> $[Rn]7s^2$ 5000 515 BCC 973 1773	<b>Actinides</b> <b>89 – 103</b>	<b>Rutherfordium</b> [261] <b>104 Rf</b> $[Ra]5f^{14} 6d^2$	<b>Dubnium</b> [262] <b>105 Db</b> $[Ra]5f^{14} 6d^3 ?$	<b>Seaborgium</b> [263] <b>106 Sg</b> $[Ra]5f^{14} 6d^4 ?$	<b>Bohrium</b> [264] <b>107 Bh</b> $[Ra]5f^{14} 6d^5 ?$	<b>Hassium</b> [265] <b>108 Hs</b> $[Ra]5f^{14} 6d^6 ?$	<b>Meitnerium</b> [268] <b>109 Mt</b> $[Ra]5f^{14} 6d^7 ?$	



Lanthanides	<b>Lanthanum</b> 138.905 5 <b>57 La</b> $[Ba]5d^1$ 6174 377 HEX 3.23 1193 3733	<b>Cerium</b> 140.116 <b>58 Ce</b> $[Ba]4f^1 5d^1$ 6711 ( $\gamma$ ) 516 FCC 1073 3693	<b>Praseodymium</b> 140.907 65 <b>59 Pr</b> $[Ba]4f^3$ 6779 367 HEX 3.222 1204 3783	<b>Neodymium</b> 144.24 <b>60 Nd</b> $[Ba]4f^4$ 7000 366 HEX 3.225 1289 3343	<b>Promethium</b> [145] <b>61 Pm</b> $[Ba]4f^5$ 7220 365 HEX 3.19 1415 3573	<b>Samarium</b> 150.36 <b>62 Sm</b> $[Ba]4f^6$ 7536 363 HEX 7.221 1443 2063
Actinides	<b>Actinium</b> [227] <b>89 Ac</b> $[Ra]6d^1$ 10060 531 FCC 1323 3473	<b>Thorium</b> 232.038 1 <b>90 Th</b> $[Ra]6d^2$ 11725 508 FCC 2023 5063	<b>Protactinium</b> 231.035 88 <b>91 Pa</b> $[Rn]5f^2 6d^1 7s^2$ 15370 392 TET 0.825 1843 4273	<b>Uranium</b> 238.028 9 <b>92 U</b> $[Rn]5f^3 6d^1 7s^2$ 19050 285 ORC 1.736 2.056 1405.3 4403	<b>Neptunium</b> [237] <b>93 Np</b> $[Rn]5f^4 6d^1 7s^2$ 20450 666 ORC 0.733 0.709 913 4173	<b>Plutonium</b> [244] <b>94 Pu</b> $[Rn]5f^6 7s^2$ 19816 618 MCL 1.773 0.780 913 3503

								18
								<b>Helium</b> 4.002 602 <b>2 He</b> $1s^2$ 120 356 HEX 1.631 3-5 4.22
BCC body-centred cubic CUB simple cubic DIA diamond FCC face-centred cubic HEX hexagonal MCL monoclinic ORC orthorhombic RHL rhombohedral TET tetragonal (t-pt) triple point			13	14	15	16	17	
			<b>Boron</b> 10.811 <b>5 B</b> $[Be]2p^1$ 2.466 1017 RHL $65^\circ 7'$ 2.348 4.273	<b>Carbon</b> 12.0107 <b>6 C</b> $[Be]2p^2$ 2.266 357 DIA 4.763 (t-pt)	<b>Nitrogen</b> 14.006 74 <b>7 N</b> $[Be]2p^3$ 1.035 ( $\beta$ ) 405 HEX 1.631 63 77.35	<b>Oxygen</b> 15.999 4 <b>8 O</b> $[Be]2p^4$ 1.460 ( $\gamma$ ) 683 CUB 54.36 90.19	<b>Fluorine</b> 18.998 403 2 <b>9 F</b> $[Be]2p^5$ 1.140 550 MCL $1.32$ $0.61$ 53.55 85.05	<b>Neon</b> 20.179 7 <b>10 Ne</b> $[Be]2p^6$ 1.442 446 FCC 24.56 27.07
			<b>Aluminium</b> 26.981 538 <b>13 Al</b> $[Mg]3p^1$ 2.698 405 FCC 933.47 2.793	<b>Silicon</b> 28.085 5 <b>14 Si</b> $[Mg]3p^2$ 2.329 543 DIA 1.683 3.533	<b>Phosphorus</b> 30.973 761 <b>15 P</b> $[Mg]3p^3$ 1.820 331 ORC $1.320$ $3.162$ 317.3 550	<b>Sulfur</b> 32.066 <b>16 S</b> $[Mg]3p^4$ 2.086 1.046 ORC $1.340$ $1.229$ 388.47 717.82	<b>Chlorine</b> 35.452 7 <b>17 Cl</b> $[Mg]3p^5$ 2.030 624 ORC $1.324$ $0.718$ 172 239.1	<b>Argon</b> 39.948 <b>18 Ar</b> $[Mg]3p^6$ 1.656 532 FCC 83.81 87.30
10	11	12						
<b>Nickel</b> 58.693 4 <b>28 Ni</b> $[Ca]3d^8$ 8.907 352 FCC 1.728 3.263	<b>Copper</b> 63.546 <b>29 Cu</b> $[Ar]3d^{10}4s^1$ 8.933 361 FCC 1.357.8 2.833	<b>Zinc</b> 65.39 <b>30 Zn</b> $[Ca]3d^{10}$ 7.135 266 HEX 1.856 692.68 1.183	<b>Gallium</b> 69.723 <b>31 Ga</b> $[Zn]4p^1$ 5.905 452 ORC $1.991$ $1.695$ 302.9 2.473	<b>Germanium</b> 72.61 <b>32 Ge</b> $[Zn]4p^2$ 5.323 566 DIA 1.211 3.103	<b>Arsenic</b> 74.921 60 <b>33 As</b> $[Zn]4p^3$ 5.776 413 RHL $54^\circ 7'$ 883 (t-pt)	<b>Selenium</b> 78.96 <b>34 Se</b> $[Zn]4p^4$ 4.808 ( $\gamma$ ) 436 HEX 1.135 493 958	<b>Bromine</b> 79.904 <b>35 Br</b> $[Zn]4p^5$ 3.120 668 ORC $1.398$ $0.672$ 265.90 332.0	<b>Krypton</b> 83.80 <b>36 Kr</b> $[Zn]4p^6$ 3.000 581 FCC 115.8 119.9
<b>Palladium</b> 106.42 <b>46 Pd</b> $[Kr]4d^{10}$ 11.995 389 FCC 1.828 3.233	<b>Silver</b> 107.868 2 <b>47 Ag</b> $[Pd]5s^1$ 10.500 409 FCC 1.235 2.433	<b>Cadmium</b> 112.411 <b>48 Cd</b> $[Pd]5s^2$ 8.647 298 HEX 1.886 594.2 1.043	<b>Indium</b> 114.818 <b>49 In</b> $[Cd]5p^1$ 7.290 325 TET 1.521 429.75 2.343	<b>Tin</b> 118.710 <b>50 Sn</b> $[Cd]5p^2$ 7.285 ( $\beta$ ) 583 TET 0.546 505.08 2.893	<b>Antimony</b> 121.760 <b>51 Sb</b> $[Cd]5p^3$ 6.692 451 RHL $57^\circ 7'$ 903.8 1.860	<b>Tellurium</b> 127.60 <b>52 Te</b> $[Cd]5p^4$ 6.247 446 HEX 1.33 723 1.263	<b>Iodine</b> 126.904 47 <b>53 I</b> $[Cd]5p^5$ 4.953 727 ORC $1.347$ $0.659$ 386.7 457	<b>Xenon</b> 131.29 <b>54 Xe</b> $[Cd]5p^6$ 3.560 635 FCC 161.3 165.0
<b>Platinum</b> 195.078 <b>78 Pt</b> $[Xe]4f^{14}5d^96s^1$ 21.450 392 FCC 2.041 4.093	<b>Gold</b> 196.966 55 <b>79 Au</b> $[Xe]4f^{14}5d^{10}6s^1$ 19.281 408 FCC 1.337.3 3.123	<b>Mercury</b> 200.59 <b>80 Hg</b> $[Yb]5d^{10}$ 13.546 300 RHL $70^\circ 32'$ 234.32 629.9	<b>Thallium</b> 204.383 3 <b>81 Tl</b> $[Hg]6p^1$ 11.871 346 HEX 1.598 577 1.743	<b>Lead</b> 207.2 <b>82 Pb</b> $[Hg]6p^2$ 11.343 495 FCC 600.7 2.023	<b>Bismuth</b> 208.980 38 <b>83 Bi</b> $[Hg]6p^3$ 9.803 475 RHL $57^\circ 14'$ 544.59 1.833	<b>Polonium</b> [209] <b>84 Po</b> $[Hg]6p^4$ 9.400 337 CUB 527 1.233	<b>Astatine</b> [210] <b>85 At</b> $[Hg]6p^5$ 573 623	<b>Radon</b> [222] <b>86 Rn</b> $[Hg]6p^6$ 440 202 211
<b>Ununnilium</b> [271] <b>110 Uun</b>	<b>Unununium</b> [272] <b>111 Uuu</b>	<b>Ununbium</b> [285] <b>112 Uub</b>	<b>Ununquadium</b> [289] <b>114 Uuq</b>					
<b>Europium</b> 151.964 <b>63 Eu</b> $[Ba]4f^7$ 5.248 458 BCC 1.095 1.873	<b>Gadolinium</b> 157.25 <b>64 Gd</b> $[Ba]4f^75d^1$ 7.870 363 HEX 1.591 1.587 3.533	<b>Terbium</b> 158.925 34 <b>65 Tb</b> $[Ba]4f^9$ 8.267 361 HEX 1.580 1.633 3.493	<b>Dysprosium</b> 162.50 <b>66 Dy</b> $[Ba]4f^{10}$ 8.531 359 HEX 1.573 1.683 2.833	<b>Holmium</b> 164.930 32 <b>67 Ho</b> $[Ba]4f^{11}$ 8.797 358 HEX 1.570 1.743 2.973	<b>Erbium</b> 167.26 <b>68 Er</b> $[Ba]4f^{12}$ 9.044 356 HEX 1.570 1.803 3.133	<b>Thulium</b> 168.934 21 <b>69 Tm</b> $[Ba]4f^{13}$ 9.325 354 HEX 1.570 1.823 2.223	<b>Ytterbium</b> 173.04 <b>70 Yb</b> $[Ba]4f^{14}$ 6.966 ( $\beta$ ) 549 FCC 1.097 1.473	<b>Lutetium</b> 174.967 <b>71 Lu</b> $[Yb]5d^1$ 9.842 351 HEX 1.583 1.933 3.663
<b>Americium</b> [243] <b>95 Am</b> $[Ra]5f^7$ 13.670 347 HEX 3.24 1.449 2.873	<b>Curium</b> [247] <b>96 Cm</b> $[Ra]5f^76d^17s^2$ 13.510 350 HEX 3.24 1.618 3.383	<b>Berkelium</b> [247] <b>97 Bk</b> $[Ra]5f^9$ 14.780 342 HEX 3.24 1.323	<b>Californium</b> [251] <b>98 Cf</b> $[Ra]5f^{10}$ 15.100 338 HEX 3.24 1.173	<b>Einsteinium</b> [252] <b>99 Es</b> $[Ra]5f^{11}$ HEX 1.133	<b>Fermium</b> [257] <b>100 Fm</b> $[Ra]5f^{12}$ 1.803	<b>Mendelevium</b> [258] <b>101 Md</b> $[Ra]5f^{13}$ 1.103	<b>Nobelium</b> [259] <b>102 No</b> $[Ra]5f^{14}$ 1.103	<b>Lawrencium</b> [262] <b>103 Lr</b> $[Ra]5f^{14}7p^1$ 1.903



### 6.3 Crystalline structure

#### Bravais lattices

Volume of primitive cell	$V = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$	(6.1)	$\mathbf{a}, \mathbf{b}, \mathbf{c}$ $V$	primitive base vectors volume of primitive cell
Reciprocal primitive base vectors <sup>a</sup>	$\mathbf{a}^* = 2\pi \mathbf{b} \times \mathbf{c} / [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]$	(6.2)	$\mathbf{a}^*, \mathbf{b}^*, \mathbf{c}^*$	reciprocal primitive base vectors
	$\mathbf{b}^* = 2\pi \mathbf{c} \times \mathbf{a} / [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]$	(6.3)		
	$\mathbf{c}^* = 2\pi \mathbf{a} \times \mathbf{b} / [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]$	(6.4)		
	$\mathbf{a} \cdot \mathbf{a}^* = \mathbf{b} \cdot \mathbf{b}^* = \mathbf{c} \cdot \mathbf{c}^* = 2\pi$	(6.5)		
	$\mathbf{a} \cdot \mathbf{b}^* = \mathbf{a} \cdot \mathbf{c}^* = 0$ (etc.)	(6.6)		
Lattice vector	$\mathbf{R}_{uvw} = u\mathbf{a} + v\mathbf{b} + w\mathbf{c}$	(6.7)	$\mathbf{R}_{uvw}$ $u, v, w$	lattice vector $[uvw]$ integers
Reciprocal lattice vector	$\mathbf{G}_{hkl} = h\mathbf{a}^* + k\mathbf{b}^* + l\mathbf{c}^*$	(6.8)	$\mathbf{G}_{hkl}$ $\mathbf{i}$	reciprocal lattice vector $[hkl]$ $\mathbf{i}^2 = -1$
	$\exp(i\mathbf{G}_{hkl} \cdot \mathbf{R}_{uvw}) = 1$	(6.9)		
Weiss zone equation <sup>b</sup>	$hu + kv + lw = 0$	(6.10)	$(hkl)$	Miller indices of plane <sup>c</sup>
Interplanar spacing (general)	$d_{hkl} = \frac{2\pi}{G_{hkl}}$	(6.11)	$d_{hkl}$	distance between $(hkl)$ planes
Interplanar spacing (orthogonal basis)	$\frac{1}{d_{hkl}^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2} + \frac{l^2}{c^2}$	(6.12)		

<sup>a</sup>Note that this is  $2\pi$  times the usual definition of a “reciprocal vector” (see page 20).

<sup>b</sup>Condition for lattice vector  $[uvw]$  to be parallel to lattice plane  $(hkl)$  in an arbitrary Bravais lattice.

<sup>c</sup>Miller indices are defined so that  $G_{hkl}$  is the shortest reciprocal lattice vector normal to the  $(hkl)$  planes.

#### Weber symbols

Converting $[uvw]$ to $[UVTW]$	$U = \frac{1}{3}(2u - v)$	(6.13)	$U, V, T, W$ $u, v, w$ $[UVTW]$ $[uvw]$	Weber indices zone axis indices Weber symbol zone axis symbol
	$V = \frac{1}{3}(2v - u)$	(6.14)		
	$T = -\frac{1}{3}(u + v)$	(6.15)		
	$W = w$	(6.16)		
Converting $[UVTW]$ to $[uvw]$	$u = (U - T)$	(6.17)		
	$v = (V - T)$	(6.18)		
	$w = W$	(6.19)		
Zone law <sup>a</sup>	$hU + kV + iT + lW = 0$	(6.20)	$(hkil)$	Miller–Bravais indices

<sup>a</sup>For trigonal and hexagonal systems.



**Cubic lattices**

lattice	primitive (P)	body-centred (I)	face-centred (F)
lattice parameter	$a$	$a$	$a$
volume of conventional cell	$a^3$	$a^3$	$a^3$
lattice points per cell	1	2	4
1st nearest neighbours <sup>a</sup>	6	8	12
1st n.n. distance	$a$	$a\sqrt{3}/2$	$a/\sqrt{2}$
2nd nearest neighbours	12	6	6
2nd n.n. distance	$a\sqrt{2}$	$a$	$a$
packing fraction <sup>b</sup>	$\pi/6$	$\sqrt{3}\pi/8$	$\sqrt{2}\pi/6$
reciprocal lattice <sup>c</sup>	P	F	I
primitive base vectors <sup>d</sup>	$\mathbf{a}_1 = a\hat{x}$ $\mathbf{a}_2 = a\hat{y}$ $\mathbf{a}_3 = a\hat{z}$	$\mathbf{a}_1 = \frac{a}{2}(\hat{y} + \hat{z} - \hat{x})$ $\mathbf{a}_2 = \frac{a}{2}(\hat{z} + \hat{x} - \hat{y})$ $\mathbf{a}_3 = \frac{a}{2}(\hat{x} + \hat{y} - \hat{z})$	$\mathbf{a}_1 = \frac{a}{2}(\hat{y} + \hat{z})$ $\mathbf{a}_2 = \frac{a}{2}(\hat{z} + \hat{x})$ $\mathbf{a}_3 = \frac{a}{2}(\hat{x} + \hat{y})$

<sup>a</sup>Or "coordination number."

<sup>b</sup>For close-packed spheres. The maximum possible packing fraction for spheres is  $\sqrt{2}\pi/6$ .

<sup>c</sup>The lattice parameters for the reciprocal lattices of P, I, and F are  $2\pi/a$ ,  $4\pi/a$ , and  $4\pi/a$  respectively.

<sup>d</sup> $\hat{x}$ ,  $\hat{y}$ , and  $\hat{z}$  are unit vectors.

**Crystal systems<sup>a</sup>**

system	symmetry	unit cell <sup>b</sup>	lattices <sup>c</sup>
triclinic	none	$a \neq b \neq c$ ; $\alpha \neq \beta \neq \gamma \neq 90^\circ$	P
monoclinic	one diad $\parallel [010]$	$a \neq b \neq c$ ; $\alpha = \gamma = 90^\circ, \beta \neq 90^\circ$	P, C
orthorhombic	three orthogonal diads	$a \neq b \neq c$ ; $\alpha = \beta = \gamma = 90^\circ$	P, C, I, F
tetragonal	one tetrad $\parallel [001]$	$a = b \neq c$ ; $\alpha = \beta = \gamma = 90^\circ$	P, I
trigonal <sup>d</sup>	one triad $\parallel [111]$	$a = b = c$ ; $\alpha = \beta = \gamma < 120^\circ \neq 90^\circ$	P, R
hexagonal	one hexad $\parallel [001]$	$a = b \neq c$ ; $\alpha = \beta = 90^\circ, \gamma = 120^\circ$	P
cubic	four triads $\parallel \langle 111 \rangle$	$a = b = c$ ; $\alpha = \beta = \gamma = 90^\circ$	P, F, I

<sup>a</sup>The symbol " $\neq$ " implies that equality is not required by the symmetry, but neither is it forbidden.

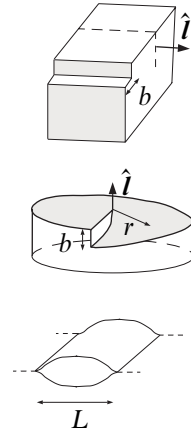
<sup>b</sup>The cell axes are  $a$ ,  $b$ , and  $c$  with  $\alpha$ ,  $\beta$ , and  $\gamma$  the angles between  $b:c$ ,  $c:a$ , and  $a:b$  respectively.

<sup>c</sup>The lattice types are primitive (P), body-centred (I), all face-centred (F), side-centred (C), and rhombohedral primitive (R).

<sup>d</sup>A primitive hexagonal unit cell, with a triad  $\parallel [001]$ , is generally preferred over this rhombohedral unit cell.

**Dislocations and cracks**

Edge dislocation	$\hat{l} \cdot \mathbf{b} = 0$	(6.21)	$\hat{l}$ unit vector $\parallel$ line of dislocation
Screw dislocation	$\hat{l} \cdot \mathbf{b} = b$	(6.22)	$\mathbf{b}, b$ Burgers vector <sup>a</sup>
Screw dislocation energy per unit length <sup>b</sup>	$U = \frac{\mu b^2}{4\pi} \ln \frac{R}{r_0}$	(6.23)	$U$ dislocation energy per unit length
	$\sim \mu b^2$	(6.24)	$\mu$ shear modulus
Critical crack length <sup>c</sup>	$L = \frac{4\alpha E}{\pi(1-\sigma^2)p_0^2}$	(6.25)	$R$ outer cutoff for $r$
			$r_0$ inner cutoff for $r$
			$L$ critical crack length
			$\alpha$ surface energy per unit area
			$E$ Young modulus
			$\sigma$ Poisson ratio
			$p_0$ applied widening stress



<sup>a</sup>The Burgers vector is a Bravais lattice vector characterising the total relative slip were the dislocation to travel throughout the crystal.

<sup>b</sup>Or “tension.” The energy per unit length of an edge dislocation is also  $\sim \mu b^2$ .

<sup>c</sup>For a crack cavity (long  $\perp L$ ) within an isotropic medium. Under uniform stress  $p_0$ , cracks  $\geq L$  will grow and smaller cracks will shrink.

**Crystal diffraction**

Laue equations	$a(\cos \alpha_1 - \cos \alpha_2) = h\lambda$	(6.26)	$a, b, c$ lattice parameters
	$b(\cos \beta_1 - \cos \beta_2) = k\lambda$	(6.27)	$\alpha_1, \beta_1, \gamma_1$ angles between lattice base vectors and input wavevector
	$c(\cos \gamma_1 - \cos \gamma_2) = l\lambda$	(6.28)	$\alpha_2, \beta_2, \gamma_2$ angles between lattice base vectors and output wavevector
Bragg’s law <sup>a</sup>	$2\mathbf{k}_{in} \cdot \mathbf{G} +  \mathbf{G} ^2 = 0$	(6.29)	$h, k, l$ integers (Laue indices)
			$\lambda$ wavelength
			$\mathbf{k}_{in}$ input wavevector
Atomic form factor	$f(\mathbf{G}) = \int_{vol} e^{-i\mathbf{G} \cdot \mathbf{r}} \rho(\mathbf{r}) d^3r$	(6.30)	$\mathbf{G}$ reciprocal lattice vector
			$f(\mathbf{G})$ atomic form factor
			$\mathbf{r}$ position vector
Structure factor <sup>b</sup>	$S(\mathbf{G}) = \sum_{j=1}^n f_j(\mathbf{G}) e^{-i\mathbf{G} \cdot \mathbf{d}_j}$	(6.31)	$\rho(\mathbf{r})$ atomic electron density
			$S(\mathbf{G})$ structure factor
			$n$ number of atoms in basis
Scattered intensity <sup>c</sup>	$I(\mathbf{K}) \propto N^2  S(\mathbf{K}) ^2$	(6.32)	$\mathbf{d}_j$ position of $j$ th atom within basis
			$\mathbf{K}$ change in wavevector ( $= \mathbf{k}_{out} - \mathbf{k}_{in}$ )
			$I(\mathbf{K})$ scattered intensity
Debye–Waller factor <sup>d</sup>	$I_T = I_0 \exp \left[ -\frac{1}{3} \langle u^2 \rangle  \mathbf{G} ^2 \right]$	(6.33)	$N$ number of lattice points illuminated
			$I_T$ intensity at temperature $T$
			$I_0$ intensity from a lattice with no motion
			$\langle u^2 \rangle$ mean-squared thermal displacement of atoms

<sup>a</sup>Alternatively, see Equation (8.32).

<sup>b</sup>The summation is over the atoms in the basis, i.e., the atomic motif repeating with the Bravais lattice.

<sup>c</sup>The Bragg condition makes  $\mathbf{K}$  a reciprocal lattice vector, with  $|\mathbf{k}_{in}| = |\mathbf{k}_{out}|$ .

<sup>d</sup>Effect of thermal vibrations.

### 6.4 Lattice dynamics

#### Phonon dispersion relations<sup>a</sup>

<p style="text-align: center;">monatomic chain</p>	<p style="text-align: center;">diatomic chain</p>
<p>Monatomic linear chain</p> $\omega^2 = 4 \frac{\alpha}{m} \sin^2 \left( \frac{ka}{2} \right) \quad (6.34)$ $v_p = \frac{\omega}{k} = a \left( \frac{\alpha}{m} \right)^{1/2} \text{sinc} \left( \frac{a}{\lambda} \right) \quad (6.35)$ $v_g = \frac{\partial \omega}{\partial k} = a \left( \frac{\alpha}{m} \right)^{1/2} \cos \left( \frac{ka}{2} \right) \quad (6.36)$	<p><math>\omega</math> phonon angular frequency  <math>\alpha</math> spring constant<sup>b</sup>  <math>m</math> atomic mass  <math>v_p</math> phase speed (<math>\text{sinc } x \equiv \frac{\sin \pi x}{\pi x}</math>)  <math>v_g</math> group speed  <math>\lambda</math> phonon wavelength</p>
<p>Diatomic linear chain<sup>c</sup></p> $\omega^2 = \frac{\alpha}{\mu} \pm \alpha \left[ \frac{1}{\mu^2} - \frac{4}{m_1 m_2} \sin^2(ka) \right]^{1/2} \quad (6.37)$	<p><math>k</math> wavenumber (<math>= 2\pi/\lambda</math>)  <math>a</math> atomic separation  <math>m_i</math> atomic masses (<math>m_2 &gt; m_1</math>)  <math>\mu</math> reduced mass  <math>[= m_1 m_2 / (m_1 + m_2)]</math></p>
<p>Identical masses, alternating spring constants</p> $\omega^2 = \frac{\alpha_1 + \alpha_2}{m} \pm \frac{1}{m} (\alpha_1^2 + \alpha_2^2 + 2\alpha_1 \alpha_2 \cos ka)^{1/2} \quad (6.38)$ $= \begin{cases} 0, & 2(\alpha_1 + \alpha_2)/m & \text{if } k=0 \\ 2\alpha_1/m, & 2\alpha_2/m & \text{if } k=\pi/a \end{cases} \quad (6.39)$	<p><math>\alpha_i</math> alternating spring constants</p>

<sup>a</sup>Along infinite linear atomic chains, considering simple harmonic nearest-neighbour interactions only. The shaded region of the dispersion relation is outside the first Brillouin zone of the reciprocal lattice.

<sup>b</sup>In the sense  $\alpha = \text{restoring force/relative displacement}$ .

<sup>c</sup>Note that the repeat distance for this chain is  $2a$ , so that the first Brillouin zone extends to  $|k| < \pi/(2a)$ . The optic and acoustic branches are the + and - solutions respectively.



## Debye theory

Mean energy per phonon mode <sup>a</sup>	$\langle E \rangle = \frac{1}{2} \hbar \omega + \frac{\hbar \omega}{\exp[\hbar \omega / (k_B T)] - 1} \quad (6.40)$	<p><math>\langle E \rangle</math> mean energy in a mode at <math>\omega</math></p> <p><math>\hbar</math> (Planck constant)/(<math>2\pi</math>)</p> <p><math>\omega</math> phonon angular frequency</p> <p><math>k_B</math> Boltzmann constant</p> <p><math>T</math> temperature</p>
Debye frequency	$\omega_D = v_s (6\pi^2 N/V)^{1/3} \quad (6.41)$ <p>where <math display="block">\frac{3}{v_s^3} = \frac{1}{v_l^3} + \frac{2}{v_t^3} \quad (6.42)</math></p>	<p><math>\omega_D</math> Debye (angular) frequency</p> <p><math>v_s</math> effective sound speed</p> <p><math>v_l</math> longitudinal phase speed</p> <p><math>v_t</math> transverse phase speed</p>
Debye temperature	$\theta_D = \hbar \omega_D / k_B \quad (6.43)$	<p><math>N</math> number of atoms in crystal</p> <p><math>V</math> crystal volume</p> <p><math>\theta_D</math> Debye temperature</p>
Phonon density of states	$g(\omega) d\omega = \frac{3V\omega^2}{2\pi^2 v_s^3} d\omega \quad (6.44)$ <p>(for <math>0 &lt; \omega &lt; \omega_D</math>, <math>g = 0</math> otherwise)</p>	<p><math>g(\omega)</math> density of states at <math>\omega</math></p> <p><math>C_V</math> heat capacity, <math>V</math> constant</p> <p><math>U</math> thermal phonon energy within crystal</p>
Debye heat capacity	$C_V = 9Nk_B \frac{T^3}{\theta_D^3} \int_0^{\theta_D/T} \frac{x^4 e^x}{(e^x - 1)^2} dx \quad (6.45)$	
Dulong and Petit's law	$\simeq 3Nk_B \quad (T \gg \theta_D) \quad (6.46)$	
Debye $T^3$ law	$\simeq \frac{12\pi^4}{5} Nk_B \frac{T^3}{\theta_D^3} \quad (T \ll \theta_D) \quad (6.47)$	
Internal thermal energy <sup>b</sup>	$U(T) = \frac{9N}{\omega_D^3} \int_0^{\omega_D} \frac{\hbar \omega^3}{\exp[\hbar \omega / (k_B T)] - 1} d\omega \equiv 3Nk_B T D(\theta_D/T) \quad (6.48)$ <p>where <math display="block">D(x) = \frac{3}{x^3} \int_0^x \frac{y^3}{e^y - 1} dy \quad (6.49)</math></p>	

<sup>a</sup>Or any simple harmonic oscillator in thermal equilibrium at temperature  $T$ .<sup>b</sup>Neglecting zero-point energy.

**Lattice forces (simple)**

Van der Waals interaction <sup>a</sup>	$\phi(r) = -\frac{3}{4} \frac{\alpha_p^2 \hbar \omega}{(4\pi\epsilon_0)^2 r^6} \quad (6.50)$	$\phi(r)$ two-particle potential energy $r$ particle separation $\alpha_p$ particle polarisability
Lennard–Jones 6-12 potential (molecular crystals)	$\phi(r) = -\frac{A}{r^6} + \frac{B}{r^{12}} \quad (6.51)$	$\hbar$ (Planck constant)/(2 $\pi$ ) $\epsilon_0$ permittivity of free space
	$= 4\epsilon \left[ \left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^6 \right] \quad (6.52)$	$\omega$ angular frequency of polarised orbital
	$\sigma = (B/A)^{1/6}; \quad \epsilon = A^2/(4B) \quad (6.53)$	$A, B$ constants $\epsilon, \sigma$ Lennard–Jones parameters
De Boer parameter	$\Lambda = \frac{h}{\sigma(m\epsilon)^{1/2}} \quad (6.54)$	$\Lambda$ de Boer parameter $h$ Planck constant $m$ particle mass
Coulomb interaction (ionic crystals)	$U_C = -\alpha_M \frac{e^2}{4\pi\epsilon_0 r_0} \quad (6.55)$	$U_C$ lattice Coulomb energy per ion pair $\alpha_M$ Madelung constant $-e$ electronic charge $r_0$ nearest neighbour separation

<sup>a</sup>London’s formula for fluctuating dipole interactions, neglecting the propagation time between particles.



**Lattice thermal expansion and conduction**

Grüneisen parameter <sup>a</sup>	$\gamma = -\frac{\partial \ln \omega}{\partial \ln V} \quad (6.56)$	$\gamma$ Grüneisen parameter $\omega$ normal mode frequency $V$ volume
Linear expansivity <sup>b</sup>	$\alpha = \frac{1}{3K_T} \left. \frac{\partial p}{\partial T} \right _V = \frac{\gamma C_V}{3K_T V} \quad (6.57)$	$\alpha$ linear expansivity $K_T$ isothermal bulk modulus $p$ pressure $T$ temperature $C_V$ lattice heat capacity, constant $V$
Thermal conductivity of a phonon gas	$\lambda = \frac{1}{3} \frac{C_V}{V} v_s l \quad (6.58)$	$\lambda$ thermal conductivity $v_s$ effective sound speed $l$ phonon mean free path
Umklapp mean free path <sup>c</sup>	$l_u \propto \exp(\theta_u/T) \quad (6.59)$	$l_u$ umklapp mean free path $\theta_u$ umklapp temperature ( $\sim \theta_D/2$ )

<sup>a</sup>Strictly, the Grüneisen parameter is the mean of  $\gamma$  over all normal modes, weighted by the mode’s contribution to  $C_V$ .

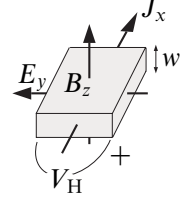
<sup>b</sup>Or “coefficient of thermal expansion,” for an isotropically expanding crystal.

<sup>c</sup>Mean free path determined solely by “umklapp processes” – the scattering of phonons outside the first Brillouin zone.

### 6.5 Electrons in solids

#### Free electron transport properties

Current density	$J = -nev_d$	(6.60)	$J$ current density
Mean electron drift velocity	$v_d = -\frac{e\tau}{m_e}E$	(6.61)	$n$ free electron number density $-e$ electronic charge $v_d$ mean electron drift velocity $\tau$ mean time between collisions (relaxation time) $m_e$ electronic mass
d.c. electrical conductivity	$\sigma_0 = \frac{ne^2\tau}{m_e}$	(6.62)	$E$ applied electric field $\sigma_0$ d.c. conductivity ( $J = \sigma E$ )
a.c. electrical conductivity <sup>a</sup>	$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}$	(6.63)	$\omega$ a.c. angular frequency $\sigma(\omega)$ a.c. conductivity
Thermal conductivity	$\lambda = \frac{1}{3} \frac{C_V}{V} \langle c^2 \rangle \tau$	(6.64)	$C_V$ total electron heat capacity, $V$ constant
	$= \frac{\pi^2 nk_B^2 T}{3m_e} \quad (T \ll T_F)$	(6.65)	$V$ volume $\langle c^2 \rangle$ mean square electron speed $k_B$ Boltzmann constant $T$ temperature $T_F$ Fermi temperature
Wiedemann-Franz law <sup>b</sup>	$\frac{\lambda}{\sigma T} = L = \frac{\pi^2 k_B^2}{3e^2}$	(6.66)	$L$ Lorenz constant ( $\simeq 2.45 \times 10^{-8} \text{ W}\Omega\text{K}^{-2}$ ) $\lambda$ thermal conductivity
Hall coefficient <sup>c</sup>	$R_H = -\frac{1}{ne} = \frac{E_y}{J_x B_z}$	(6.67)	$R_H$ Hall coefficient $E_y$ Hall electric field $J_x$ applied current density $B_z$ magnetic flux density
Hall voltage (rectangular strip)	$V_H = R_H \frac{B_z I_x}{w}$	(6.68)	$V_H$ Hall voltage $I_x$ applied current ( $= J_x \times$ cross-sectional area) $w$ strip thickness in $z$



<sup>a</sup>For an electric field varying as  $e^{-i\omega t}$ .

<sup>b</sup>Holds for an arbitrary band structure.

<sup>c</sup>The charge on an electron is  $-e$ , where  $e$  is the elementary charge (approximately  $+1.6 \times 10^{-19} \text{ C}$ ). The Hall coefficient is therefore a negative number when the dominant charge carriers are electrons.

**Fermi gas**

Electron density of states <sup>a</sup>	$g(E) = \frac{V}{2\pi^2} \left( \frac{2m_e}{\hbar^2} \right)^{3/2} E^{1/2} \quad (6.69)$ $g(E_F) = \frac{3nV}{2E_F} \quad (6.70)$	<i>E</i> electron energy (>0) <i>g(E)</i> density of states <i>V</i> “gas” volume <i>m<sub>e</sub></i> electronic mass $\hbar$ (Planck constant)/(2π)
Fermi wavenumber	$k_F = (3\pi^2 n)^{1/3} \quad (6.71)$	<i>k<sub>F</sub></i> Fermi wavenumber <i>n</i> number of electrons per unit volume
Fermi velocity	$v_F = \hbar k_F / m_e \quad (6.72)$	<i>v<sub>F</sub></i> Fermi velocity
Fermi energy ( <i>T</i> = 0)	$E_F = \frac{\hbar^2 k_F^2}{2m_e} = \frac{\hbar^2}{2m_e} (3\pi^2 n)^{2/3} \quad (6.73)$	<i>E<sub>F</sub></i> Fermi energy
Fermi temperature	$T_F = \frac{E_F}{k_B} \quad (6.74)$	<i>T<sub>F</sub></i> Fermi temperature <i>k<sub>B</sub></i> Boltzmann constant
Electron heat capacity <sup>b</sup> ( <i>T</i> ≪ <i>T<sub>F</sub></i> )	$C_{Ve} = \frac{\pi^2}{3} g(E_F) k_B^2 T \quad (6.75)$ $= \frac{\pi^2 k_B^2}{2E_F} T \quad (6.76)$	<i>C<sub>Ve</sub></i> heat capacity per electron <i>T</i> temperature
Total kinetic energy ( <i>T</i> = 0)	$U_0 = \frac{3}{5} n V E_F \quad (6.77)$	<i>U<sub>0</sub></i> total kinetic energy
Pauli paramagnetism	$\mathbf{M} = \chi_{HP} \mathbf{H} \quad (6.78)$	$\chi_{HP}$ Pauli magnetic susceptibility $\mathbf{H}$ magnetic field strength $\mathbf{M}$ magnetisation
	$= \frac{3n}{2E_F} \mu_0 \mu_B^2 \mathbf{H} \quad (6.79)$	$\mu_0$ permeability of free space $\mu_B$ Bohr magneton
Landau diamagnetism	$\chi_{HL} = -\frac{1}{3} \chi_{HP} \quad (6.80)$	$\chi_{HL}$ Landau magnetic susceptibility

<sup>a</sup>The density of states is often quoted per unit volume in real space (i.e., *g(E)/V* here).

<sup>b</sup>Equation (6.75) holds for any density of states.

**Thermoelectricity**

Thermopower <sup>a</sup>	$\mathcal{E} = \frac{\mathbf{J}}{\sigma} + S_T \nabla T \quad (6.81)$	$\mathcal{E}$ electrochemical field <sup>b</sup> $\mathbf{J}$ current density $\sigma$ electrical conductivity $S_T$ thermopower
Peltier effect	$\mathbf{H} = \Pi \mathbf{J} - \lambda \nabla T \quad (6.82)$	<i>T</i> temperature $\mathbf{H}$ heat flux per unit area
Kelvin relation	$\Pi = T S_T \quad (6.83)$	$\Pi$ Peltier coefficient $\lambda$ thermal conductivity

<sup>a</sup>Or “absolute thermoelectric power.”

<sup>b</sup>The electrochemical field is the gradient of  $(\mu/e) - \phi$ , where  $\mu$  is the chemical potential,  $-e$  the electronic charge, and  $\phi$  the electrical potential.



## Band theory and semiconductors

Bloch's theorem	$\Psi(\mathbf{r} + \mathbf{R}) = \exp(i\mathbf{k} \cdot \mathbf{R})\Psi(\mathbf{r})$ (6.84)	$\Psi$ electron eigenstate $\mathbf{k}$ Bloch wavevector $\mathbf{R}$ lattice vector $\mathbf{r}$ position vector
Electron velocity	$\mathbf{v}_b(\mathbf{k}) = \frac{1}{\hbar} \nabla_{\mathbf{k}} E_b(\mathbf{k})$ (6.85)	$\mathbf{v}_b$ electron velocity (for wavevector $\mathbf{k}$ ) $\hbar$ (Planck constant)/ $2\pi$ $b$ band index $E_b(\mathbf{k})$ energy band
Effective mass tensor	$m_{ij} = \hbar^2 \left[ \frac{\partial^2 E_b(\mathbf{k})}{\partial k_i \partial k_j} \right]^{-1}$ (6.86)	$m_{ij}$ effective mass tensor $k_i$ components of $\mathbf{k}$
Scalar effective mass <sup>a</sup>	$m^* = \hbar^2 \left[ \frac{\partial^2 E_b(k)}{\partial k^2} \right]^{-1}$ (6.87)	$m^*$ scalar effective mass $k =  \mathbf{k} $
Mobility	$\mu = \frac{ \mathbf{v}_d }{ \mathbf{E} } = \frac{eD}{k_B T}$ (6.88)	$\mu$ particle mobility $v_d$ mean drift velocity $\mathbf{E}$ applied electric field $-e$ electronic charge $D$ diffusion coefficient $T$ temperature
Net current density	$\mathbf{J} = (n_e \mu_e + n_h \mu_h) e \mathbf{E}$ (6.89)	$\mathbf{J}$ current density $n_{e,h}$ electron, hole, number densities $\mu_{e,h}$ electron, hole, mobilities
Semiconductor equation	$n_e n_h = \frac{(k_B T)^3}{2(\pi \hbar^2)^3} (m_e^* m_h^*)^{3/2} e^{-E_g/(k_B T)}$ (6.90)	$k_B$ Boltzmann constant $E_g$ band gap $m_{e,h}^*$ electron, hole, effective masses
p-n junction	$I = I_0 \left[ \exp\left(\frac{eV}{k_B T}\right) - 1 \right]$ (6.91)	$I$ current $I_0$ saturation current
	$I_0 = e n_i^2 A \left( \frac{D_e}{L_e N_a} + \frac{D_h}{L_h N_d} \right)$ (6.92)	$V$ bias voltage (+ for forward) $n_i$ intrinsic carrier concentration $A$ area of junction
	$L_e = (D_e \tau_e)^{1/2}$ (6.93)	$D_{e,h}$ electron, hole, diffusion coefficients
	$L_h = (D_h \tau_h)^{1/2}$ (6.94)	$L_{e,h}$ electron, hole, diffusion lengths $\tau_{e,h}$ electron, hole, recombination times
		$N_{a,d}$ acceptor, donor, concentrations

<sup>a</sup>Valid for regions of  $k$ -space in which  $E_b(\mathbf{k})$  can be taken as independent of the direction of  $\mathbf{k}$ .



# Chapter 7 Electromagnetism

## 7.1 Introduction

The electromagnetic force is central to nearly every physical process around us and is a major component of classical physics. In fact, the development of electromagnetic theory in the nineteenth century gave us much mathematical machinery that we now apply quite generally in other fields, including potential theory, vector calculus, and the ideas of divergence and curl.

It is therefore not surprising that this section deals with a large array of physical quantities and their relationships. As usual, SI units are assumed throughout. In the past electromagnetism has suffered from the use of a variety of systems of units, including the cgs system in both its electrostatic (esu) and electromagnetic (emu) forms. The fog has now all but cleared, but some specialised areas of research still cling to these historical measures. Readers are advised to consult the section on unit conversion if they come across such exotica in the literature.

Equations cast in the rationalised units of SI can be readily converted to the once common Gaussian (unrationalised) units by using the following symbol transformations:

### Equation conversion: SI to Gaussian units

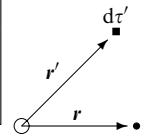
$\epsilon_0 \mapsto 1/(4\pi)$	$\mu_0 \mapsto 4\pi/c^2$	$\mathbf{B} \mapsto \mathbf{B}/c$
$\chi_E \mapsto 4\pi\chi_E$	$\chi_H \mapsto 4\pi\chi_H$	$\mathbf{H} \mapsto c\mathbf{H}/(4\pi)$
$\mathbf{A} \mapsto \mathbf{A}/c$	$\mathbf{M} \mapsto c\mathbf{M}$	$\mathbf{D} \mapsto \mathbf{D}/(4\pi)$

The quantities  $\rho$ ,  $\mathbf{J}$ ,  $\mathbf{E}$ ,  $\phi$ ,  $\sigma$ ,  $\mathbf{P}$ ,  $\epsilon_r$ , and  $\mu_r$  are all unchanged.

## 7.2 Static fields

### Electrostatics

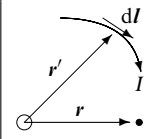
Electrostatic potential	$E = -\nabla\phi$ (7.1)	$E$ electric field $\phi$ electrostatic potential
Potential difference <sup>a</sup>	$\phi_a - \phi_b = \int_a^b E \cdot dI = - \int_b^a E \cdot dI$ (7.2)	$\phi_a$ potential at $a$ $\phi_b$ potential at $b$ $dI$ line element
Poisson's Equation (free space)	$\nabla^2\phi = -\frac{\rho}{\epsilon_0}$ (7.3)	$\rho$ charge density $\epsilon_0$ permittivity of free space
Point charge at $r'$	$\phi(r) = \frac{q}{4\pi\epsilon_0 r-r' }$ (7.4) $E(r) = \frac{q(r-r')}{4\pi\epsilon_0 r-r' ^3}$ (7.5)	$q$ point charge
Field from a charge distribution (free space)	$E(r) = \frac{1}{4\pi\epsilon_0} \int_{\text{volume}} \frac{\rho(r')(r-r')}{ r-r' ^3} d\tau'$ (7.6)	$d\tau'$ volume element $r'$ position vector of $d\tau'$



<sup>a</sup>Between points  $a$  and  $b$  along a path  $I$ .

### Magnetostatics<sup>a</sup>

Magnetic scalar potential	$B = -\mu_0\nabla\phi_m$ (7.7)	$\phi_m$ magnetic scalar potential $B$ magnetic flux density
$\phi_m$ in terms of the solid angle of a generating current loop	$\phi_m = \frac{I\Omega}{4\pi}$ (7.8)	$\Omega$ loop solid angle $I$ current
Biot-Savart law (the field from a line current)	$B(r) = \frac{\mu_0 I}{4\pi} \int_{\text{line}} \frac{dI \times (r-r')}{ r-r' ^3}$ (7.9)	$dI$ line element in the direction of the current $r'$ position vector of $dI$
Ampère's law (differential form)	$\nabla \times B = \mu_0 J$ (7.10)	$J$ current density $\mu_0$ permeability of free space
Ampère's law (integral form)	$\oint B \cdot dI = \mu_0 I_{\text{tot}}$ (7.11)	$I_{\text{tot}}$ total current through loop



<sup>a</sup>In free space.

**Capacitance<sup>a</sup>**

Of sphere, radius $a$	$C = 4\pi\epsilon_0\epsilon_r a$	(7.12)
Of circular disk, radius $a$	$C = 8\epsilon_0\epsilon_r a$	(7.13)
Of two spheres, radius $a$ , in contact	$C = 8\pi\epsilon_0\epsilon_r a \ln 2$	(7.14)
Of circular solid cylinder, radius $a$ , length $l$	$C \simeq [8 + 4.1(l/a)^{0.76}] \epsilon_0\epsilon_r a$	(7.15)
Of nearly spherical surface, area $S$	$C \simeq 3.139 \times 10^{-11} \epsilon_r S^{1/2}$	(7.16)
Of cube, side $a$	$C \simeq 7.283 \times 10^{-11} \epsilon_r a$	(7.17)
Between concentric spheres, radii $a < b$	$C = 4\pi\epsilon_0\epsilon_r ab(b-a)^{-1}$	(7.18)
Between coaxial cylinders, radii $a < b$	$C = \frac{2\pi\epsilon_0\epsilon_r}{\ln(b/a)}$ per unit length	(7.19)
Between parallel cylinders, separation $2d$ , radii $a$	$C = \frac{\pi\epsilon_0\epsilon_r}{\operatorname{arcosh}(d/a)}$ per unit length	(7.20)
	$\simeq \frac{\pi\epsilon_0\epsilon_r}{\ln(2d/a)} \quad (d \gg a)$	(7.21)
Between parallel, coaxial circular disks, separation $d$ , radii $a$	$C \simeq \frac{\epsilon_0\epsilon_r\pi a^2}{d} + \epsilon_0\epsilon_r a [\ln(16\pi a/d) - 1]$	(7.22)

<sup>a</sup>For conductors, in an embedding medium of relative permittivity  $\epsilon_r$ .

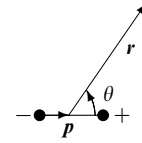
**Inductance<sup>a</sup>**

Of $N$ -turn solenoid (straight or toroidal), length $l$ , area $A$ ( $\ll l^2$ )	$L = \mu_0 N^2 A / l$	(7.23)
Of coaxial cylindrical tubes, radii $a, b$ ( $a < b$ )	$L = \frac{\mu_0}{2\pi} \ln \frac{b}{a}$ per unit length	(7.24)
Of parallel wires, radii $a$ , separation $2d$	$L \simeq \frac{\mu_0}{\pi} \ln \frac{2d}{a}$ per unit length, ( $2d \gg a$ )	(7.25)
Of wire of radius $a$ bent in a loop of radius $b \gg a$	$L \simeq \mu_0 b \left( \ln \frac{8b}{a} - 2 \right)$	(7.26)

<sup>a</sup>For currents confined to the surfaces of perfect conductors in free space.

**Electric fields<sup>a</sup>**

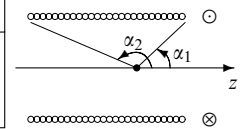
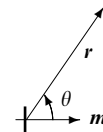
Uniformly charged sphere, radius $a$ , charge $q$	$E(\mathbf{r}) = \begin{cases} \frac{q}{4\pi\epsilon_0 a^3} \mathbf{r} & (r < a) \\ \frac{q}{4\pi\epsilon_0 r^3} \mathbf{r} & (r \geq a) \end{cases} \quad (7.27)$
Uniformly charged disk, radius $a$ , charge $q$ (on axis, $z$ )	$E(z) = \frac{q}{2\pi\epsilon_0 a^2 z} \left( \frac{1}{ z } - \frac{1}{\sqrt{z^2 + a^2}} \right) \quad (7.28)$
Line charge, charge density $\lambda$ per unit length	$E(\mathbf{r}) = \frac{\lambda}{2\pi\epsilon_0 r^2} \mathbf{r} \quad (7.29)$
Electric dipole, moment $\mathbf{p}$ (spherical polar coordinates, $\theta$ angle between $\mathbf{p}$ and $\mathbf{r}$ )	$E_r = \frac{p \cos \theta}{2\pi\epsilon_0 r^3} \quad (7.30)$
	$E_\theta = \frac{p \sin \theta}{4\pi\epsilon_0 r^3} \quad (7.31)$
Charge sheet, surface density $\sigma$	$E = \frac{\sigma}{2\epsilon_0} \quad (7.32)$



<sup>a</sup>For  $\epsilon_r = 1$  in the surrounding medium.

**Magnetic fields<sup>a</sup>**

Uniform infinite solenoid, current $I$ , $n$ turns per unit length	$B = \begin{cases} \mu_0 n I & \text{inside (axial)} \\ 0 & \text{outside} \end{cases} \quad (7.33)$
Uniform cylinder of current $I$ , radius $a$	$B(r) = \begin{cases} \mu_0 I r / (2\pi a^2) & r < a \\ \mu_0 I / (2\pi r) & r \geq a \end{cases} \quad (7.34)$
Magnetic dipole, moment $\mathbf{m}$ ( $\theta$ angle between $\mathbf{m}$ and $\mathbf{r}$ )	$B_r = \mu_0 \frac{m \cos \theta}{2\pi r^3} \quad (7.35)$
	$B_\theta = \frac{\mu_0 m \sin \theta}{4\pi r^3} \quad (7.36)$
Circular current loop of $N$ turns, radius $a$ , along axis, $z$	$B(z) = \frac{\mu_0 N I}{2} \frac{a^2}{(a^2 + z^2)^{3/2}} \quad (7.37)$
The axis, $z$ , of a straight solenoid, $n$ turns per unit length, current $I$	$B_{\text{axis}} = \frac{\mu_0 n I}{2} (\cos \alpha_1 - \cos \alpha_2) \quad (7.38)$



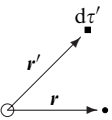
<sup>a</sup>For  $\mu_r = 1$  in the surrounding medium.

**Image charges**

Real charge, $+q$ , at a distance:	image point	image charge
$b$ from a conducting plane	$-b$	$-q$
$b$ from a conducting sphere, radius $a$	$a^2/b$	$-qa/b$
$b$ from a plane dielectric boundary:		
seen from free space	$-b$	$-q(\epsilon_r - 1)/(\epsilon_r + 1)$
seen from the dielectric	$b$	$+2q/(\epsilon_r + 1)$

### 7.3 Electromagnetic fields (general)


#### Field relationships

Conservation of charge	$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$	(7.39)	$\mathbf{J}$ current density $\rho$ charge density $t$ time
Magnetic vector potential	$\mathbf{B} = \nabla \times \mathbf{A}$	(7.40)	$\mathbf{A}$ vector potential
Electric field from potentials	$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi$	(7.41)	$\phi$ electrical potential
Coulomb gauge condition	$\nabla \cdot \mathbf{A} = 0$	(7.42)	
Lorenz gauge condition	$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0$	(7.43)	$c$ speed of light
Potential field equations <sup>a</sup>	$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = \frac{\rho}{\epsilon_0}$	(7.44)	
	$\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$	(7.45)	
Expression for $\phi$ in terms of $\rho^a$	$\phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int_{\text{volume}} \frac{\rho(\mathbf{r}', t -  \mathbf{r} - \mathbf{r}' /c)}{ \mathbf{r} - \mathbf{r}' } d\tau'$	(7.46)	$d\tau'$ volume element $\mathbf{r}'$ position vector of $d\tau'$
Expression for $\mathbf{A}$ in terms of $\mathbf{J}^a$	$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_{\text{volume}} \frac{\mathbf{J}(\mathbf{r}', t -  \mathbf{r} - \mathbf{r}' /c)}{ \mathbf{r} - \mathbf{r}' } d\tau'$	(7.47)	$\mu_0$ permeability of free space

<sup>a</sup>Assumes the Lorenz gauge.



#### Liénard–Wiechert potentials<sup>a</sup>

Electrical potential of a moving point charge	$\phi = \frac{q}{4\pi\epsilon_0( \mathbf{r}  - \mathbf{v} \cdot \mathbf{r}/c)}$	(7.48)	$q$ charge $\mathbf{r}$ vector from charge to point of observation $\mathbf{v}$ particle velocity
Magnetic vector potential of a moving point charge	$\mathbf{A} = \frac{\mu_0 q \mathbf{v}}{4\pi( \mathbf{r}  - \mathbf{v} \cdot \mathbf{r}/c)}$	(7.49)	

<sup>a</sup>In free space. The right-hand sides of these equations are evaluated at retarded times, i.e., at  $t' = t - |\mathbf{r}'|/c$ , where  $\mathbf{r}'$  is the vector from the charge to the observation point at time  $t'$ .

**Maxwell's equations**

Differential form:	Integral form:
$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (7.50)$	$\oint_{\text{closed surface}} \mathbf{E} \cdot d\mathbf{s} = \frac{1}{\epsilon_0} \int_{\text{volume}} \rho \, d\tau \quad (7.51)$
$\nabla \cdot \mathbf{B} = 0 \quad (7.52)$	$\oint_{\text{closed surface}} \mathbf{B} \cdot d\mathbf{s} = 0 \quad (7.53)$
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (7.54)$	$\oint_{\text{loop}} \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} \quad (7.55)$
$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (7.56)$	$\oint_{\text{loop}} \mathbf{B} \cdot d\mathbf{l} = \mu_0 I + \mu_0 \epsilon_0 \int_{\text{surface}} \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{s} \quad (7.57)$
Equation (7.51) is "Gauss's law" Equation (7.55) is "Faraday's law" <i>E</i> electric field <i>B</i> magnetic flux density <i>J</i> current density <i>ρ</i> charge density	<i>ds</i> surface element <i>dτ</i> volume element <i>dl</i> line element $\Phi$ linked magnetic flux ( $= \int \mathbf{B} \cdot d\mathbf{s}$ ) <i>I</i> linked current ( $= \int \mathbf{J} \cdot d\mathbf{s}$ ) <i>t</i> time

**Maxwell's equations (using *D* and *H*)**

Differential form:	Integral form:
$\nabla \cdot \mathbf{D} = \rho_{\text{free}} \quad (7.58)$	$\oint_{\text{closed surface}} \mathbf{D} \cdot d\mathbf{s} = \int_{\text{volume}} \rho_{\text{free}} \, d\tau \quad (7.59)$
$\nabla \cdot \mathbf{B} = 0 \quad (7.60)$	$\oint_{\text{closed surface}} \mathbf{B} \cdot d\mathbf{s} = 0 \quad (7.61)$
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (7.62)$	$\oint_{\text{loop}} \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} \quad (7.63)$
$\nabla \times \mathbf{H} = \mathbf{J}_{\text{free}} + \frac{\partial \mathbf{D}}{\partial t} \quad (7.64)$	$\oint_{\text{loop}} \mathbf{H} \cdot d\mathbf{l} = I_{\text{free}} + \int_{\text{surface}} \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s} \quad (7.65)$
<i>D</i> displacement field $\rho_{\text{free}}$ free charge density (in the sense of $\rho = \rho_{\text{induced}} + \rho_{\text{free}}$ ) <i>B</i> magnetic flux density <i>H</i> magnetic field strength $\mathbf{J}_{\text{free}}$ free current density (in the sense of $\mathbf{J} = \mathbf{J}_{\text{induced}} + \mathbf{J}_{\text{free}}$ )	<i>E</i> electric field <i>ds</i> surface element <i>dτ</i> volume element <i>dl</i> line element $\Phi$ linked magnetic flux ( $= \int \mathbf{B} \cdot d\mathbf{s}$ ) $I_{\text{free}}$ linked free current ( $= \int \mathbf{J}_{\text{free}} \cdot d\mathbf{s}$ ) <i>t</i> time

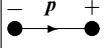
## Relativistic electrodynamics

Lorentz transformation of electric and magnetic fields	$E'_{\parallel} = E_{\parallel}$	(7.66)	$E$ electric field
	$E'_{\perp} = \gamma(\mathbf{E} + \mathbf{v} \times \mathbf{B})_{\perp}$	(7.67)	$B$ magnetic flux density
	$B'_{\parallel} = B_{\parallel}$	(7.68)	' measured in frame moving at relative velocity $\mathbf{v}$
	$B'_{\perp} = \gamma(\mathbf{B} - \mathbf{v} \times \mathbf{E}/c^2)_{\perp}$	(7.69)	$\gamma$ Lorentz factor = $[1 - (v/c)^2]^{-1/2}$
Lorentz transformation of current and charge densities	$\rho' = \gamma(\rho - vJ_{\parallel}/c^2)$	(7.70)	$\parallel$ parallel to $\mathbf{v}$
	$J'_{\perp} = J_{\perp}$	(7.71)	$\perp$ perpendicular to $\mathbf{v}$
	$J'_{\parallel} = \gamma(J_{\parallel} - v\rho)$	(7.72)	$J$ current density
Lorentz transformation of potential fields	$\phi' = \gamma(\phi - vA_{\parallel})$	(7.73)	$\rho$ charge density
	$A'_{\perp} = A_{\perp}$	(7.74)	$\phi$ electric potential
	$A'_{\parallel} = \gamma(A_{\parallel} - v\phi/c^2)$	(7.75)	$A$ magnetic vector potential
Four-vector fields <sup>a</sup>	$\tilde{\mathbf{J}} = (\rho c, \mathbf{J})$	(7.76)	
	$\tilde{\mathbf{A}} = \left( \frac{\phi}{c}, \mathbf{A} \right)$	(7.77)	$\tilde{\mathbf{J}}$ current density four-vector
	$\square^2 = \left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2}, -\nabla^2 \right)$	(7.78)	$\tilde{\mathbf{A}}$ potential four-vector
	$\square^2 \tilde{\mathbf{A}} = \mu_0 \tilde{\mathbf{J}}$	(7.79)	$\square^2$ D'Alembertian operator

<sup>a</sup>Other sign conventions are common here. See page 65 for a general definition of four-vectors.

### 7.4 Fields associated with media

#### Polarisation

Definition of electric dipole moment	$\mathbf{p} = q\mathbf{a}$	(7.80)	$\pm q$ end charges $\mathbf{a}$ charge separation vector (from $-$ to $+$ )	
Generalised electric dipole moment	$\mathbf{p} = \int_{\text{volume}} \mathbf{r}' \rho \, d\tau'$	(7.81)	$\mathbf{p}$ dipole moment $\rho$ charge density $d\tau'$ volume element $\mathbf{r}'$ vector to $d\tau'$	
Electric dipole potential	$\phi(\mathbf{r}) = \frac{\mathbf{p} \cdot \mathbf{r}}{4\pi\epsilon_0 r^3}$	(7.82)	$\phi$ dipole potential $\mathbf{r}$ vector from dipole $\epsilon_0$ permittivity of free space	
Dipole moment per unit volume (polarisation) <sup>a</sup>	$\mathbf{P} = n\mathbf{p}$	(7.83)	$\mathbf{P}$ polarisation $n$ number of dipoles per unit volume	
Induced volume charge density	$\nabla \cdot \mathbf{P} = -\rho_{\text{ind}}$	(7.84)	$\rho_{\text{ind}}$ volume charge density	
Induced surface charge density	$\sigma_{\text{ind}} = \mathbf{P} \cdot \hat{\mathbf{s}}$	(7.85)	$\sigma_{\text{ind}}$ surface charge density $\hat{\mathbf{s}}$ unit normal to surface	
Definition of electric displacement	$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$	(7.86)	$\mathbf{D}$ electric displacement $\mathbf{E}$ electric field	
Definition of electric susceptibility	$\mathbf{P} = \epsilon_0 \chi_E \mathbf{E}$	(7.87)	$\chi_E$ electrical susceptibility (may be a tensor)	
Definition of relative permittivity <sup>b</sup>	$\epsilon_r = 1 + \chi_E$	(7.88)	$\epsilon_r$ relative permittivity $\epsilon$ permittivity	
	$\mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E}$ $= \epsilon \mathbf{E}$	(7.89) (7.90)		
Atomic polarisability <sup>c</sup>	$\mathbf{p} = \alpha \mathbf{E}_{\text{loc}}$	(7.91)	$\alpha$ polarisability $\mathbf{E}_{\text{loc}}$ local electric field	
Depolarising fields	$\mathbf{E}_d = -\frac{N_d \mathbf{P}}{\epsilon_0}$	(7.92)	$\mathbf{E}_d$ depolarising field $N_d$ depolarising factor =1/3 (sphere) =1 (thin slab $\perp$ to $\mathbf{P}$ ) =0 (thin slab $\parallel$ to $\mathbf{P}$ ) =1/2 (long circular cylinder, axis $\perp$ to $\mathbf{P}$ )	
Clausius–Mossotti equation <sup>d</sup>	$\frac{n\alpha}{3\epsilon_0} = \frac{\epsilon_r - 1}{\epsilon_r + 2}$	(7.93)		

<sup>a</sup>Assuming dipoles are parallel. The equivalent of Equation (7.112) holds for a hot gas of electric dipoles.

<sup>b</sup>Relative permittivity as defined here is for a linear isotropic medium.

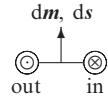
<sup>c</sup>The polarisability of a conducting sphere radius  $a$  is  $\alpha = 4\pi\epsilon_0 a^3$ . The definition  $\mathbf{p} = \alpha\epsilon_0 \mathbf{E}_{\text{loc}}$  is also used.

<sup>d</sup>With the substitution  $\eta^2 = \epsilon_r$  [cf. Equation (7.195) with  $\mu_r = 1$ ] this is also known as the ‘‘Lorentz–Lorenz formula.’’



**Magnetisation**

Definition of magnetic dipole moment	$d\mathbf{m} = I ds \quad (7.94)$	<p><math>d\mathbf{m}</math> dipole moment</p> <p><math>I</math> loop current</p> <p><math>ds</math> loop area (right-hand sense with respect to loop current)</p>
Generalised magnetic dipole moment	$\mathbf{m} = \frac{1}{2} \int_{\text{volume}} \mathbf{r}' \times \mathbf{J} d\tau' \quad (7.95)$	<p><math>\mathbf{m}</math> dipole moment</p> <p><math>\mathbf{J}</math> current density</p> <p><math>d\tau'</math> volume element</p> <p><math>\mathbf{r}'</math> vector to <math>d\tau'</math></p>
Magnetic dipole (scalar) potential	$\phi_m(\mathbf{r}) = \frac{\mu_0 \mathbf{m} \cdot \mathbf{r}}{4\pi r^3} \quad (7.96)$	<p><math>\phi_m</math> magnetic scalar potential</p> <p><math>\mathbf{r}</math> vector from dipole</p> <p><math>\mu_0</math> permeability of free space</p>
Dipole moment per unit volume (magnetisation) <sup>a</sup>	$\mathbf{M} = n\mathbf{m} \quad (7.97)$	<p><math>\mathbf{M}</math> magnetisation</p> <p><math>n</math> number of dipoles per unit volume</p>
Induced volume current density	$\mathbf{J}_{\text{ind}} = \nabla \times \mathbf{M} \quad (7.98)$	<p><math>\mathbf{J}_{\text{ind}}</math> volume current density (i.e., <math>A \text{ m}^{-2}</math>)</p>
Induced surface current density	$\mathbf{j}_{\text{ind}} = \mathbf{M} \times \hat{\mathbf{s}} \quad (7.99)$	<p><math>\mathbf{j}_{\text{ind}}</math> surface current density (i.e., <math>A \text{ m}^{-1}</math>)</p> <p><math>\hat{\mathbf{s}}</math> unit normal to surface</p>
Definition of magnetic field strength, $\mathbf{H}$	$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \quad (7.100)$	<p><math>\mathbf{B}</math> magnetic flux density</p> <p><math>\mathbf{H}</math> magnetic field strength</p>
Definition of magnetic susceptibility	$\mathbf{M} = \chi_H \mathbf{H} \quad (7.101)$	<p><math>\chi_H</math> magnetic susceptibility. <math>\chi_B</math> is also used (both may be tensors)</p>
	$= \frac{\chi_B \mathbf{B}}{\mu_0} \quad (7.102)$	
	$\chi_B = \frac{\chi_H}{1 + \chi_H} \quad (7.103)$	
Definition of relative permeability <sup>b</sup>	$\mathbf{B} = \mu_0 \mu_r \mathbf{H} \quad (7.104)$	<p><math>\mu_r</math> relative permeability</p> <p><math>\mu</math> permeability</p>
	$= \mu \mathbf{H} \quad (7.105)$	
	$\mu_r = 1 + \chi_H \quad (7.106)$	
	$= \frac{1}{1 - \chi_B} \quad (7.107)$	



<sup>a</sup>Assuming all the dipoles are parallel. See Equation (7.112) for a classical paramagnetic gas and page 101 for the quantum generalisation.

<sup>b</sup>Relative permeability as defined here is for a linear isotropic medium.

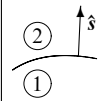
**Paramagnetism and diamagnetism**

Diamagnetic moment of an atom	$m = -\frac{e^2}{6m_e} Z \langle r^2 \rangle \mathbf{B}$	(7.108)	$m$ magnetic moment $\langle r^2 \rangle$ mean squared orbital radius (of all electrons) $Z$ atomic number $\mathbf{B}$ magnetic flux density $m_e$ electron mass $-e$ electronic charge
Intrinsic electron magnetic moment <sup>a</sup>	$m \simeq -\frac{e}{2m_e} g \mathbf{J}$	(7.109)	$\mathbf{J}$ total angular momentum $g$ Landé $g$ -factor (=2 for spin, =1 for orbital momentum)
Langevin function	$\mathcal{L}(x) = \coth x - \frac{1}{x}$	(7.110)	$\mathcal{L}(x)$ Langevin function
	$\simeq x/3 \quad (x \lesssim 1)$	(7.111)	
Classical gas paramagnetism ( $ \mathbf{J}  \gg \hbar$ )	$\langle M \rangle = nm_0 \mathcal{L} \left( \frac{m_0 B}{kT} \right)$	(7.112)	$\langle M \rangle$ apparent magnetisation $m_0$ magnitude of magnetic dipole moment $n$ dipole number density
Curie's law	$\chi_H = \frac{\mu_0 n m_0^2}{3kT}$	(7.113)	$T$ temperature $k$ Boltzmann constant $\chi_H$ magnetic susceptibility
Curie-Weiss law	$\chi_H = \frac{\mu_0 n m_0^2}{3k(T - T_c)}$	(7.114)	$\mu_0$ permeability of free space $T_c$ Curie temperature

<sup>a</sup>See also page 100.

**Boundary conditions for  $E$ ,  $D$ ,  $B$ , and  $H^a$**

Parallel component of the electric field	$E_{\parallel}$ continuous	(7.115)	$\parallel$ component parallel to interface
Perpendicular component of the magnetic flux density	$B_{\perp}$ continuous	(7.116)	$\perp$ component perpendicular to interface
Electric displacement <sup>b</sup>	$\hat{s} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma$	(7.117)	$\mathbf{D}_{1,2}$ electrical displacements in media 1 & 2 $\hat{s}$ unit normal to surface, directed 1 $\rightarrow$ 2 $\sigma$ surface density of free charge
Magnetic field strength <sup>c</sup>	$\hat{s} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{j}_s$	(7.118)	$\mathbf{H}_{1,2}$ magnetic field strengths in media 1 & 2 $\mathbf{j}_s$ surface current per unit width



<sup>a</sup>At the plane surface between two uniform media.

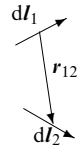
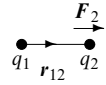
<sup>b</sup>If  $\sigma = 0$ , then  $D_{\perp}$  is continuous.

<sup>c</sup>If  $\mathbf{j}_s = \mathbf{0}$  then  $H_{\parallel}$  is continuous.

### 7.5 Force, torque, and energy

#### Electromagnetic force and torque

Force between two static charges: Coulomb's law	$F_2 = \frac{q_1 q_2}{4\pi\epsilon_0 r_{12}^2} \hat{r}_{12} \quad (7.119)$	<p><math>F_2</math> force on <math>q_2</math></p> <p><math>q_{1,2}</math> charges</p> <p><math>r_{12}</math> vector from 1 to 2</p> <p><math>\hat{\phantom{r}}</math> unit vector</p> <p><math>\epsilon_0</math> permittivity of free space</p>
Force between two current-carrying elements	$dF_2 = \frac{\mu_0 I_1 I_2}{4\pi r_{12}^2} [dI_2 \times (dI_1 \times \hat{r}_{12})] \quad (7.120)$	<p><math>dI_{1,2}</math> line elements</p> <p><math>I_{1,2}</math> currents flowing along <math>dI_1</math> and <math>dI_2</math></p> <p><math>dF_2</math> force on <math>dI_2</math></p> <p><math>\mu_0</math> permeability of free space</p>
Force on a current-carrying element in a magnetic field	$dF = I dI \times B \quad (7.121)$	<p><math>dI</math> line element</p> <p><math>F</math> force</p> <p><math>I</math> current flowing along <math>dI</math></p> <p><math>B</math> magnetic flux density</p>
Force on a charge (Lorentz force)	$F = q(E + v \times B) \quad (7.122)$	<p><math>E</math> electric field</p> <p><math>v</math> charge velocity</p>
Force on an electric dipole <sup>a</sup>	$F = (p \cdot \nabla) E \quad (7.123)$	<p><math>p</math> electric dipole moment</p>
Force on a magnetic dipole <sup>b</sup>	$F = (m \cdot \nabla) B \quad (7.124)$	<p><math>m</math> magnetic dipole moment</p>
Torque on an electric dipole	$G = p \times E \quad (7.125)$	<p><math>G</math> torque</p>
Torque on a magnetic dipole	$G = m \times B \quad (7.126)$	
Torque on a current loop	$G = I_L \oint_{\text{loop}} r \times (dI_L \times B) \quad (7.127)$	<p><math>dI_L</math> line-element (of loop)</p> <p><math>r</math> position vector of <math>dI_L</math></p> <p><math>I_L</math> current around loop</p>



<sup>a</sup> $F$  simplifies to  $\nabla(p \cdot E)$  if  $p$  is intrinsic,  $\nabla(pE/2)$  if  $p$  is induced by  $E$  and the medium is isotropic.

<sup>b</sup> $F$  simplifies to  $\nabla(m \cdot B)$  if  $m$  is intrinsic,  $\nabla(mB/2)$  if  $m$  is induced by  $B$  and the medium is isotropic.

## Electromagnetic energy

Electromagnetic field energy density (in free space)	$u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\frac{B^2}{\mu_0} \quad (7.128)$	$u$ energy density $E$ electric field $B$ magnetic flux density
Energy density in media	$u = \frac{1}{2}(\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H}) \quad (7.129)$	$\epsilon_0$ permittivity of free space $\mu_0$ permeability of free space $\mathbf{D}$ electric displacement $\mathbf{H}$ magnetic field strength
Energy flow (Poynting) vector	$\mathbf{N} = \mathbf{E} \times \mathbf{H} \quad (7.130)$	$c$ speed of light $N$ energy flow rate per unit area $\perp$ to the flow direction
Mean flux density at a distance $r$ from a short oscillating dipole	$\langle N \rangle = \frac{\omega^4 p_0^2 \sin^2 \theta}{32\pi^2 \epsilon_0 c^3 r^3} \mathbf{r} \quad (7.131)$	$p_0$ amplitude of dipole moment $\mathbf{r}$ vector from dipole ( $\gg$ wavelength) $\theta$ angle between $\mathbf{p}$ and $\mathbf{r}$ $\omega$ oscillation frequency
Total mean power from oscillating dipole <sup>a</sup>	$W = \frac{\omega^4 p_0^2}{6\pi\epsilon_0 c^3} \quad (7.132)$	$W$ total mean radiated power
Self-energy of a charge distribution	$U_{\text{tot}} = \frac{1}{2} \int_{\text{volume}} \phi(\mathbf{r}) \rho(\mathbf{r}) d\tau \quad (7.133)$	$U_{\text{tot}}$ total energy $d\tau$ volume element $\mathbf{r}$ position vector of $d\tau$ $\phi$ electrical potential $\rho$ charge density
Energy of an assembly of capacitors <sup>b</sup>	$U_{\text{tot}} = \frac{1}{2} \sum_i \sum_j C_{ij} V_i V_j \quad (7.134)$	$V_i$ potential of $i$ th capacitor $C_{ij}$ mutual capacitance between capacitors $i$ and $j$
Energy of an assembly of inductors <sup>c</sup>	$U_{\text{tot}} = \frac{1}{2} \sum_i \sum_j L_{ij} I_i I_j \quad (7.135)$	$L_{ij}$ mutual inductance between inductors $i$ and $j$
Intrinsic dipole in an electric field	$U_{\text{dip}} = -\mathbf{p} \cdot \mathbf{E} \quad (7.136)$	$U_{\text{dip}}$ energy of dipole $\mathbf{p}$ electric dipole moment
Intrinsic dipole in a magnetic field	$U_{\text{dip}} = -\mathbf{m} \cdot \mathbf{B} \quad (7.137)$	$\mathbf{m}$ magnetic dipole moment
Hamiltonian of a charged particle in an EM field <sup>d</sup>	$H = \frac{ \mathbf{p}_m - q\mathbf{A} ^2}{2m} + q\phi \quad (7.138)$	$H$ Hamiltonian $\mathbf{p}_m$ particle momentum $q$ particle charge $m$ particle mass $\mathbf{A}$ magnetic vector potential

<sup>a</sup>Sometimes called ‘‘Larmor’s formula.’’

<sup>b</sup> $C_{ii}$  is the self-capacitance of the  $i$ th body. Note that  $C_{ij} = C_{ji}$ .

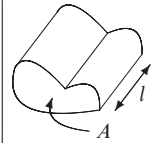
<sup>c</sup> $L_{ii}$  is the self-inductance of the  $i$ th body. Note that  $L_{ij} = L_{ji}$ .

<sup>d</sup>Newtonian limit, i.e., velocity  $\ll c$ .

## 7.6 LCR circuits

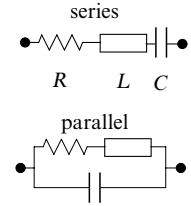
### LCR definitions

Current	$I = \frac{dQ}{dt}$	(7.139)	$I$ current $Q$ charge
Ohm's law	$V = IR$	(7.140)	$R$ resistance $V$ potential difference over $R$ $I$ current through $R$
Ohm's law (field form)	$\mathbf{J} = \sigma \mathbf{E}$	(7.141)	$\mathbf{J}$ current density $\mathbf{E}$ electric field $\sigma$ conductivity
Resistivity	$\rho = \frac{1}{\sigma} = \frac{RA}{l}$	(7.142)	$\rho$ resistivity $A$ area of face ( $I$ is normal to face) $l$ length
Capacitance	$C = \frac{Q}{V}$	(7.143)	$C$ capacitance $V$ potential difference across $C$
Current through capacitor	$I = C \frac{dV}{dt}$	(7.144)	$I$ current through $C$ $t$ time
Self-inductance	$L = \frac{\Phi}{I}$	(7.145)	$\Phi$ total linked flux $I$ current through inductor
Voltage across inductor	$V = -L \frac{dI}{dt}$	(7.146)	$V$ potential difference over $L$
Mutual inductance	$L_{12} = \frac{\Phi_1}{I_2} = L_{21}$	(7.147)	$\Phi_1$ total flux from loop 2 linked by loop 1 $L_{12}$ mutual inductance $I_2$ current through loop 2
Coefficient of coupling	$ L_{12}  = k \sqrt{L_1 L_2}$	(7.148)	$k$ coupling coefficient between $L_1$ and $L_2$ ( $\leq 1$ )
Linked magnetic flux through a coil	$\Phi = N\phi$	(7.149)	$\Phi$ linked flux $N$ number of turns around $\phi$ $\phi$ flux through area of turns



**Resonant LCR circuits**

Phase resonant frequency <sup>a</sup>	$\omega_0^2 = \begin{cases} 1/LC & \text{(series)} \\ 1/LC - R^2/L^2 & \text{(parallel)} \end{cases}$ (7.150)	$\omega_0$ resonant angular frequency $L$ inductance $C$ capacitance $R$ resistance
Tuning <sup>b</sup>	$\frac{\delta\omega}{\omega_0} = \frac{1}{Q} = \frac{R}{\omega_0 L}$ (7.151)	$\delta\omega$ half-power bandwidth $Q$ quality factor
Quality factor	$Q = 2\pi \frac{\text{stored energy}}{\text{energy lost per cycle}}$ (7.152)	



<sup>a</sup>At which the impedance is purely real.

<sup>b</sup>Assuming the capacitor is purely reactive. If  $L$  and  $R$  are parallel, then  $1/Q = \omega_0 L/R$ .

**Energy in capacitors, inductors, and resistors**

Energy stored in a capacitor	$U = \frac{1}{2} CV^2 = \frac{1}{2} QV = \frac{1}{2} \frac{Q^2}{C}$ (7.153)	$U$ stored energy $C$ capacitance $Q$ charge $V$ potential difference
Energy stored in an inductor	$U = \frac{1}{2} LI^2 = \frac{1}{2} \Phi I = \frac{1}{2} \frac{\Phi^2}{L}$ (7.154)	$L$ inductance $\Phi$ linked magnetic flux $I$ current
Power dissipated in a resistor <sup>a</sup> (Joule's law)	$W = IV = I^2 R = \frac{V^2}{R}$ (7.155)	$W$ power dissipated $R$ resistance
Relaxation time	$\tau = \frac{\epsilon_0 \epsilon_r}{\sigma}$ (7.156)	$\tau$ relaxation time $\epsilon_r$ relative permittivity $\sigma$ conductivity

<sup>a</sup>This is d.c., or instantaneous a.c., power.

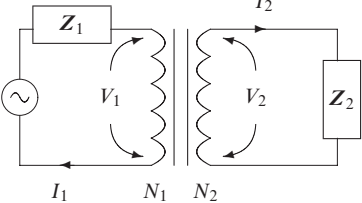
**Electrical impedance**

Impedances in series	$Z_{\text{tot}} = \sum_n Z_n$ (7.157)
Impedances in parallel	$Z_{\text{tot}} = \left( \sum_n Z_n^{-1} \right)^{-1}$ (7.158)
Impedance of capacitance	$Z_C = -\frac{\mathbf{i}}{\omega C}$ (7.159)
Impedance of inductance	$Z_L = \mathbf{i}\omega L$ (7.160)
Impedance: $Z$ Inductance: $L$ Conductance: $G = 1/R$ Admittance: $Y = 1/Z$	Capacitance: $C$ Resistance: $R = \text{Re}[Z]$ Reactance: $X = \text{Im}[Z]$ Susceptance: $S = 1/X$

**Kirchhoff's laws**

Current law	$\sum_{\text{node}} I_i = 0$	(7.161)	$I_i$ currents impinging on node
Voltage law	$\sum_{\text{loop}} V_i = 0$	(7.162)	$V_i$ potential differences around loop

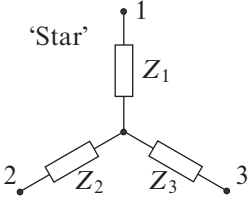
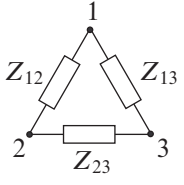
**Transformers<sup>a</sup>**

	<p><math>n</math> turns ratio</p> <p><math>N_1</math> number of primary turns</p> <p><math>N_2</math> number of secondary turns</p> <p><math>V_1</math> primary voltage</p> <p><math>V_2</math> secondary voltage</p> <p><math>I_1</math> primary current</p> <p><math>I_2</math> secondary current</p> <p><math>Z_{\text{out}}</math> output impedance</p> <p><math>Z_{\text{in}}</math> input impedance</p> <p><math>Z_1</math> source impedance</p> <p><math>Z_2</math> load impedance</p>	
Turns ratio	$n = N_2/N_1$	(7.163)
Transformation of voltage and current	$V_2 = nV_1$	(7.164)
	$I_2 = I_1/n$	(7.165)
Output impedance (seen by $Z_2$ )	$Z_{\text{out}} = n^2 Z_1$	(7.166)
Input impedance (seen by $Z_1$ )	$Z_{\text{in}} = Z_2/n^2$	(7.167)

<sup>a</sup>Ideal, with a coupling constant of 1 between loss-free windings.



**Star-delta transformation**

 	<p><math>i, j, k</math> node indices (1, 2, or 3)</p> <p><math>Z_i</math> impedance on node <math>i</math></p> <p><math>Z_{ij}</math> impedance connecting nodes <math>i</math> and <math>j</math></p>	
Star impedances	$Z_i = \frac{Z_{ij}Z_{ik}}{Z_{ij} + Z_{ik} + Z_{jk}}$	(7.168)
Delta impedances	$Z_{ij} = Z_i Z_j \left( \frac{1}{Z_i} + \frac{1}{Z_j} + \frac{1}{Z_k} \right)$	(7.169)

### 7.7 Transmission lines and waveguides

#### Transmission line relations

Loss-free transmission line equations	$\frac{\partial V}{\partial x} = -L \frac{\partial I}{\partial t} \quad (7.170)$ $\frac{\partial I}{\partial x} = -C \frac{\partial V}{\partial t} \quad (7.171)$	<p><math>V</math> potential difference across line</p> <p><math>I</math> current in line</p> <p><math>L</math> inductance per unit length</p> <p><math>C</math> capacitance per unit length</p>
Wave equation for a lossless transmission line	$\frac{1}{LC} \frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial t^2} \quad (7.172)$ $\frac{1}{LC} \frac{\partial^2 I}{\partial x^2} = \frac{\partial^2 I}{\partial t^2} \quad (7.173)$	<p><math>x</math> distance along line</p> <p><math>t</math> time</p>
Characteristic impedance of lossless line	$Z_c = \sqrt{\frac{L}{C}} \quad (7.174)$	<p><math>Z_c</math> characteristic impedance</p>
Characteristic impedance of lossy line	$Z_c = \sqrt{\frac{R + i\omega L}{G + i\omega C}} \quad (7.175)$	<p><math>R</math> resistance per unit length of conductor</p> <p><math>G</math> conductance per unit length of insulator</p> <p><math>\omega</math> angular frequency</p>
Wave speed along a lossless line	$v_p = v_g = \frac{1}{\sqrt{LC}} \quad (7.176)$	<p><math>v_p</math> phase speed</p> <p><math>v_g</math> group speed</p>
Input impedance of a terminated lossless line	$Z_{in} = Z_c \frac{Z_t \cos kl - iZ_c \sin kl}{Z_c \cos kl - iZ_t \sin kl} \quad (7.177)$ $= Z_c^2 / Z_t \quad \text{if } l = \lambda/4 \quad (7.178)$	<p><math>Z_{in}</math> (complex) input impedance</p> <p><math>Z_t</math> (complex) terminating impedance</p> <p><math>k</math> wavenumber (<math>= 2\pi/\lambda</math>)</p>
Reflection coefficient from a terminated line	$r = \frac{Z_t - Z_c}{Z_t + Z_c} \quad (7.179)$	<p><math>l</math> distance from termination</p> <p><math>r</math> (complex) voltage reflection coefficient</p>
Line voltage standing wave ratio	$V_{SWR} = \frac{1 +  r }{1 -  r } \quad (7.180)$	

#### Transmission line impedances<sup>a</sup>

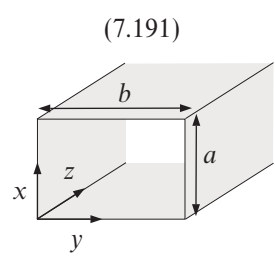
Coaxial line	$Z_c = \sqrt{\frac{\mu}{4\pi^2 \epsilon}} \ln \frac{b}{a} \approx \frac{60}{\sqrt{\epsilon_r}} \ln \frac{b}{a} \quad (7.181)$	<p><math>Z_c</math> characteristic impedance (<math>\Omega</math>)</p> <p><math>a</math> radius of inner conductor</p> <p><math>b</math> radius of outer conductor</p> <p><math>\epsilon</math> permittivity (<math>= \epsilon_0 \epsilon_r</math>)</p>
Open wire feeder	$Z_c = \sqrt{\frac{\mu}{\pi^2 \epsilon}} \ln \frac{l}{r} \approx \frac{120}{\sqrt{\epsilon_r}} \ln \frac{l}{r} \quad (7.182)$	<p><math>\mu</math> permeability (<math>= \mu_0 \mu_r</math>)</p> <p><math>r</math> radius of wires</p> <p><math>l</math> distance between wires (<math>\gg r</math>)</p>
Paired strip	$Z_c = \sqrt{\frac{\mu}{\epsilon}} \frac{d}{w} \approx \frac{377}{\sqrt{\epsilon_r}} \frac{d}{w} \quad (7.183)$	<p><math>d</math> strip separation</p> <p><math>w</math> strip width (<math>\gg d</math>)</p>
Microstrip line	$Z_c \approx \frac{377}{\sqrt{\epsilon_r} [(w/h) + 2]} \quad (7.184)$	<p><math>h</math> height above earth plane (<math>\ll w</math>)</p>

<sup>a</sup>For lossless lines.



**Waveguides<sup>a</sup>**

Waveguide equation	$k_g^2 = \frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2} \quad (7.185)$	$k_g$ wavenumber in guide $\omega$ angular frequency $a$ guide height $b$ guide width $m, n$ mode indices with respect to $a$ and $b$ (integers) $c$ speed of light
Guide cutoff frequency	$v_c = c\sqrt{\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2} \quad (7.186)$	$v_c$ cutoff frequency $\omega_c = 2\pi v_c$
Phase velocity above cutoff	$v_p = \frac{c}{\sqrt{1 - (v_c/v)^2}} \quad (7.187)$	$v_p$ phase velocity $v$ frequency
Group velocity above cutoff	$v_g = c^2/v_p = c\sqrt{1 - (v_c/v)^2} \quad (7.188)$	$v_g$ group velocity
Wave impedances <sup>b</sup>	$Z_{TM} = Z_0\sqrt{1 - (v_c/v)^2} \quad (7.189)$ $Z_{TE} = Z_0/\sqrt{1 - (v_c/v)^2} \quad (7.190)$	$Z_{TM}$ wave impedance for transverse magnetic modes $Z_{TE}$ wave impedance for transverse electric modes $Z_0$ impedance of free space ( $= \sqrt{\mu_0/\epsilon_0}$ )
<p>Field solutions for TE<sub>mn</sub> modes<sup>c</sup></p> $B_x = \frac{\mathbf{i}k_g c^2}{\omega_c^2} \frac{\partial B_z}{\partial x} \quad E_x = \frac{\mathbf{i}\omega c^2}{\omega_c^2} \frac{\partial B_z}{\partial y} \quad (7.191)$ $B_y = \frac{\mathbf{i}k_g c^2}{\omega_c^2} \frac{\partial B_z}{\partial y} \quad E_y = \frac{-\mathbf{i}\omega c^2}{\omega_c^2} \frac{\partial B_z}{\partial x}$ $B_z = B_0 \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \quad E_z = 0$ <p>Field solutions for TM<sub>mn</sub> modes<sup>c</sup></p> $E_x = \frac{\mathbf{i}k_g c^2}{\omega_c^2} \frac{\partial E_z}{\partial x} \quad B_x = \frac{-\mathbf{i}\omega}{\omega_c^2} \frac{\partial E_z}{\partial y} \quad (7.192)$ $E_y = \frac{\mathbf{i}k_g c^2}{\omega_c^2} \frac{\partial E_z}{\partial y} \quad B_y = \frac{\mathbf{i}\omega}{\omega_c^2} \frac{\partial E_z}{\partial x}$ $E_z = E_0 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad B_z = 0$		



<sup>a</sup>Equations are for lossless waveguides with rectangular cross sections and no dielectric.

<sup>b</sup>The ratio of the electric field to the magnetic field strength in the  $xy$  plane.

<sup>c</sup>Both TE and TM modes propagate in the  $z$  direction with a further factor of  $\exp[\mathbf{i}(k_g z - \omega t)]$  on all components.  $B_0$  and  $E_0$  are the amplitudes of the  $z$  components of magnetic flux density and electric field respectively.

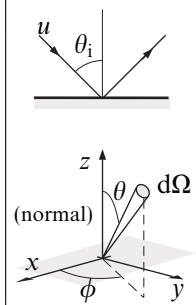
## 7.8 Waves in and out of media

### Waves in lossless media

Electric field	$\nabla^2 \mathbf{E} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$	(7.193)	$\mathbf{E}$ electric field $\mu$ permeability ( $=\mu_0\mu_r$ ) $\epsilon$ permittivity ( $=\epsilon_0\epsilon_r$ )
Magnetic field	$\nabla^2 \mathbf{B} = \mu\epsilon \frac{\partial^2 \mathbf{B}}{\partial t^2}$	(7.194)	$\mathbf{B}$ magnetic flux density $t$ time
Refractive index	$\eta = \sqrt{\epsilon_r\mu_r}$	(7.195)	
Wave speed	$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{\eta}$	(7.196)	$v$ wave phase speed $\eta$ refractive index $c$ speed of light
Impedance of free space	$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \simeq 376.7 \Omega$	(7.197)	$Z_0$ impedance of free space
Wave impedance	$Z = \frac{E}{H} = Z_0 \sqrt{\frac{\mu_r}{\epsilon_r}}$	(7.198)	$Z$ wave impedance $H$ magnetic field strength

### Radiation pressure<sup>a</sup>

Radiation momentum density	$\mathbf{G} = \frac{\mathbf{N}}{c^2}$	(7.199)	$\mathbf{G}$ momentum density $\mathbf{N}$ Poynting vector $c$ speed of light
Isotropic radiation	$p_n = \frac{1}{3}u(1+R)$	(7.200)	$p_n$ normal pressure $u$ incident radiation energy density $R$ (power) reflectance coefficient
Specular reflection	$p_n = u(1+R)\cos^2\theta_i$	(7.201)	$p_t$ tangential pressure $\theta_i$ angle of incidence
	$p_t = u(1-R)\sin\theta_i\cos\theta_i$	(7.202)	
From an extended source <sup>b</sup>	$p_n = \frac{1+R}{c} \iint I_v(\theta, \phi)\cos^2\theta \, d\Omega \, dv$	(7.203)	$I_v$ specific intensity $v$ frequency $\Omega$ solid angle $\theta$ angle between $d\Omega$ and normal to plane
From a point source, <sup>c</sup> luminosity $L$	$p_n = \frac{L(1+R)}{4\pi r^2 c}$	(7.204)	$L$ source luminosity (i.e., radiant power) $r$ distance from source



<sup>a</sup>On an opaque surface.

<sup>b</sup>In spherical polar coordinates. See page 120 for the meaning of specific intensity.

<sup>c</sup>Normal to the plane.

**Antennas**

Spherical polar geometry:			
Field from a short ( $l \ll \lambda$ ) electric dipole in free space <sup>a</sup>	$E_r = \frac{1}{2\pi\epsilon_0} \left( \frac{[\dot{p}]}{r^2c} + \frac{[p]}{r^3} \right) \cos\theta \quad (7.205)$ $E_\theta = \frac{1}{4\pi\epsilon_0} \left( \frac{[\ddot{p}]}{rc^2} + \frac{[\dot{p}]}{r^2c} + \frac{[p]}{r^3} \right) \sin\theta \quad (7.206)$ $B_\phi = \frac{\mu_0}{4\pi} \left( \frac{[\ddot{p}]}{rc} + \frac{[\dot{p}]}{r^2} \right) \sin\theta \quad (7.207)$	$r$ distance from dipole $\theta$ angle between $r$ and $P$ $[p]$ retarded dipole moment $[p] = p(t-r/c)$ $c$ speed of light	
Radiation resistance of a short electric dipole in free space	$R = \frac{\omega^2 l^2}{6\pi\epsilon_0 c^3} = \frac{2\pi Z_0}{3} \left( \frac{l}{\lambda} \right)^2 \quad (7.208)$ $\simeq 789 \left( \frac{l}{\lambda} \right)^2 \text{ ohm} \quad (7.209)$	$l$ dipole length ( $\ll \lambda$ ) $\omega$ angular frequency $\lambda$ wavelength $Z_0$ impedance of free space	
Beam solid angle	$\Omega_A = \int_{4\pi} P_n(\theta, \phi) d\Omega \quad (7.210)$	$\Omega_A$ beam solid angle $P_n$ normalised antenna power pattern $P_n(0,0) = 1$ $d\Omega$ differential solid angle	
Forward power gain	$G(0) = \frac{4\pi}{\Omega_A} \quad (7.211)$	$G$ antenna gain	
Antenna effective area	$A_e = \frac{\lambda^2}{\Omega_A} \quad (7.212)$	$A_e$ effective area	
Power gain of a short dipole	$G(\theta) = \frac{3}{2} \sin^2\theta \quad (7.213)$		
Beam efficiency	$\text{efficiency} = \frac{\Omega_M}{\Omega_A} \quad (7.214)$	$\Omega_M$ main lobe solid angle	
Antenna temperature <sup>b</sup>	$T_A = \frac{1}{\Omega_A} \int_{4\pi} T_b(\theta, \phi) P_n(\theta, \phi) d\Omega \quad (7.215)$	$T_A$ antenna temperature $T_b$ sky brightness temperature	

<sup>a</sup>All field components propagate with a further phase factor equal to  $\exp i(kr - \omega t)$ , where  $k = 2\pi/\lambda$ .

<sup>b</sup>The brightness temperature of a source of specific intensity  $I_\nu$  is  $T_b = \lambda^2 I_\nu / (2k_B)$ .



Reflection, refraction, and transmission<sup>a</sup>

<div style="display: flex; justify-content: space-around;"> <div style="text-align: center;"> <p>parallel incidence</p> </div> <div style="text-align: center;"> <p>perpendicular incidence</p> </div> </div>		<p><i>E</i> electric field  <i>B</i> magnetic flux density  <math>\eta_i</math> refractive index on incident side  <math>\eta_t</math> refractive index on transmitted side  <math>\theta_i</math> angle of incidence  <math>\theta_r</math> angle of reflection  <math>\theta_t</math> angle of refraction</p>
Law of reflection	$\theta_i = \theta_r$ (7.216)	
Snell's law <sup>b</sup>	$\eta_i \sin \theta_i = \eta_t \sin \theta_t$ (7.217)	
Brewster's law	$\tan \theta_B = \eta_t / \eta_i$ (7.218)	$\theta_B$ Brewster's angle of incidence for plane-polarised reflection ( $r_{\parallel} = 0$ )
Fresnel equations of reflection and refraction		
$r_{\parallel} = \frac{\sin 2\theta_i - \sin 2\theta_t}{\sin 2\theta_i + \sin 2\theta_t}$ (7.219)	$r_{\perp} = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$ (7.223)	
$t_{\parallel} = \frac{4 \cos \theta_i \sin \theta_t}{\sin 2\theta_i + \sin 2\theta_t}$ (7.220)	$t_{\perp} = \frac{2 \cos \theta_i \sin \theta_t}{\sin(\theta_i + \theta_t)}$ (7.224)	
$R_{\parallel} = r_{\parallel}^2$ (7.221)	$R_{\perp} = r_{\perp}^2$ (7.225)	
$T_{\parallel} = \frac{\eta_t \cos \theta_t}{\eta_i \cos \theta_i} t_{\parallel}^2$ (7.222)	$T_{\perp} = \frac{\eta_t \cos \theta_t}{\eta_i \cos \theta_i} t_{\perp}^2$ (7.226)	
Coefficients for normal incidence <sup>c</sup>		
$R = \frac{(\eta_i - \eta_t)^2}{(\eta_i + \eta_t)^2}$ (7.227)	$r = \frac{\eta_i - \eta_t}{\eta_i + \eta_t}$ (7.230)	
$T = \frac{4\eta_i \eta_t}{(\eta_i + \eta_t)^2}$ (7.228)	$t = \frac{2\eta_i}{\eta_i + \eta_t}$ (7.231)	
$R + T = 1$ (7.229)	$t - r = 1$ (7.232)	
$\parallel$ electric field parallel to the plane of incidence	$\perp$ electric field perpendicular to the plane of incidence	
<i>R</i> (power) reflectance coefficient	<i>r</i> amplitude reflection coefficient	
<i>T</i> (power) transmittance coefficient	<i>t</i> amplitude transmission coefficient	

<sup>a</sup>For the plane boundary between lossless dielectric media. All coefficients refer to the electric field component and whether it is parallel or perpendicular to the plane of incidence. Perpendicular components are out of the paper.

<sup>b</sup>The incident wave suffers total internal reflection if  $\frac{\eta_t}{\eta_i} \sin \theta_i > 1$ .

<sup>c</sup>I.e.,  $\theta_i = 0$ . Use the diagram labelled "perpendicular incidence" for correct phases.

**Propagation in conducting media<sup>a</sup>**

Electrical conductivity ( $B = 0$ )	$\sigma = n_e e \mu = \frac{n_e e^2}{m_e} \tau_c \quad (7.233)$	$\sigma$ electrical conductivity $n_e$ electron number density $\tau_c$ electron relaxation time $\mu$ electron mobility $B$ magnetic flux density
Refractive index of an ohmic conductor <sup>b</sup>	$\eta = (1 + \mathbf{i}) \left( \frac{\sigma}{4\pi\nu\epsilon_0} \right)^{1/2} \quad (7.234)$	$m_e$ electron mass $-e$ electronic charge $\eta$ refractive index $\epsilon_0$ permittivity of free space
Skin depth in an ohmic conductor	$\delta = (\mu_0 \sigma \pi \nu)^{-1/2} \quad (7.235)$	$\nu$ frequency $\delta$ skin depth $\mu_0$ permeability of free space

<sup>a</sup>Assuming a relative permeability,  $\mu_r$ , of 1.

<sup>b</sup>Taking the wave to have an  $e^{-i\omega t}$  time dependence, and the low-frequency limit ( $\sigma \gg 2\pi\nu\epsilon_0$ ).

**Electron scattering processes<sup>a</sup>**

Rayleigh scattering cross section <sup>b</sup>	$\sigma_R = \frac{\omega^4 \alpha^2}{6\pi\epsilon_0 c^4} \quad (7.236)$	$\sigma_R$ Rayleigh cross section $\omega$ radiation angular frequency $\alpha$ particle polarisability $\epsilon_0$ permittivity of free space
Thomson scattering cross section <sup>c</sup>	$\sigma_T = \frac{8\pi}{3} \left( \frac{e^2}{4\pi\epsilon_0 m_e c^2} \right)^2 \quad (7.237)$ $= \frac{8\pi}{3} r_e^2 \simeq 6.652 \times 10^{-29} \text{ m}^2 \quad (7.238)$	$\sigma_T$ Thomson cross section $m_e$ electron (rest) mass $r_e$ classical electron radius $c$ speed of light
Inverse Compton scattering <sup>d</sup>	$P_{\text{tot}} = \frac{4}{3} \sigma_T c u_{\text{rad}} \gamma^2 \left( \frac{v^2}{c^2} \right) \quad (7.239)$	$P_{\text{tot}}$ electron energy loss rate $u_{\text{rad}}$ radiation energy density $\gamma$ Lorentz factor $= [1 - (v/c)^2]^{-1/2}$ $v$ electron speed
Compton scattering <sup>e</sup>	$\lambda' - \lambda = \frac{h}{m_e c} (1 - \cos\theta) \quad (7.240)$ $h\nu' = \frac{m_e c^2}{1 - \cos\theta + (1/\epsilon)} \quad (7.241)$ $\cot\phi = (1 + \epsilon) \tan \frac{\theta}{2} \quad (7.242)$	$\lambda, \lambda'$ incident & scattered wavelengths $\nu, \nu'$ incident & scattered frequencies $\theta$ photon scattering angle $\frac{h}{m_e c}$ electron Compton wavelength $\epsilon = h\nu / (m_e c^2)$
Klein–Nishina cross section (for a free electron)	$\sigma_{\text{KN}} = \frac{\pi r_e^2}{\epsilon} \left\{ \left[ 1 - \frac{2(\epsilon + 1)}{\epsilon^2} \right] \ln(2\epsilon + 1) + \frac{1}{2} + \frac{4}{\epsilon} - \frac{1}{2(2\epsilon + 1)^2} \right\} \quad (7.243)$ $\simeq \sigma_T \quad (\epsilon \ll 1) \quad (7.244)$ $\simeq \frac{\pi r_e^2}{\epsilon} \left( \ln 2\epsilon + \frac{1}{2} \right) \quad (\epsilon \gg 1) \quad (7.245)$	$\sigma_{\text{KN}}$ Klein–Nishina cross section

<sup>a</sup>For Rutherford scattering see page 72.

<sup>b</sup>Scattering by bound electrons.

<sup>c</sup>Scattering from free electrons,  $\epsilon \ll 1$ .

<sup>d</sup>Electron energy loss rate due to photon scattering in the Thomson limit ( $\gamma h\nu \ll m_e c^2$ ).

<sup>e</sup>From an electron at rest.



**Cherenkov radiation**

Cherenkov cone angle	$\sin \theta = \frac{c}{\eta v}$ (7.246)	$\theta$ cone semi-angle $c$ (vacuum) speed of light $\eta(\omega)$ refractive index $v$ particle velocity
Radiated power <sup>a</sup>	$P_{\text{tot}} = \frac{e^2 \mu_0}{4\pi} v \int_0^{\omega_c} \left[ 1 - \frac{c^2}{v^2 \eta^2(\omega)} \right] \omega \, d\omega$ (7.247) where $\eta(\omega) \geq \frac{c}{v}$ for $0 < \omega < \omega_c$	$P_{\text{tot}}$ total radiated power $-e$ electronic charge $\mu_0$ free space permeability $\omega$ angular frequency $\omega_c$ cutoff frequency

<sup>a</sup>From a point charge,  $e$ , travelling at speed  $v$  through a medium of refractive index  $\eta(\omega)$ .

**7.9 Plasma physics**

**Warm plasmas**

Landau length	$l_L = \frac{e^2}{4\pi\epsilon_0 k_B T_e}$ (7.248) $\simeq 1.67 \times 10^{-5} T_e^{-1} \text{ m}$ (7.249)	$l_L$ Landau length $-e$ electronic charge $\epsilon_0$ permittivity of free space $k_B$ Boltzmann constant $T_e$ electron temperature (K)
Electron Debye length	$\lambda_{De} = \left( \frac{\epsilon_0 k_B T_e}{n_e e^2} \right)^{1/2}$ (7.250) $\simeq 69 (T_e/n_e)^{1/2} \text{ m}$ (7.251)	$\lambda_{De}$ electron Debye length $n_e$ electron number density ( $\text{m}^{-3}$ )
Debye screening <sup>a</sup>	$\phi(r) = \frac{q \exp(-2^{1/2} r/\lambda_{De})}{4\pi\epsilon_0 r}$ (7.252)	$\phi$ effective potential $q$ point charge $r$ distance from $q$
Debye number	$N_{De} = \frac{4}{3} \pi n_e \lambda_{De}^3$ (7.253)	$N_{De}$ electron Debye number
Relaxation times ( $B = 0$ ) <sup>b</sup>	$\tau_e = 3.44 \times 10^5 \frac{T_e^{3/2}}{n_e \ln \Lambda} \text{ s}$ (7.254) $\tau_i = 2.09 \times 10^7 \frac{T_i^{3/2}}{n_e \ln \Lambda} \left( \frac{m_i}{m_p} \right)^{1/2} \text{ s}$ (7.255)	$\tau_e$ electron relaxation time $\tau_i$ ion relaxation time $T_i$ ion temperature (K) $\ln \Lambda$ Coulomb logarithm (typically 10 to 20) $B$ magnetic flux density
Characteristic electron thermal speed <sup>c</sup>	$v_{te} = \left( \frac{2k_B T_e}{m_e} \right)^{1/2}$ (7.256) $\simeq 5.51 \times 10^3 T_e^{1/2} \text{ ms}^{-1}$ (7.257)	$v_{te}$ electron thermal speed $m_e$ electron mass

<sup>a</sup>Effective (Yukawa) potential from a point charge  $q$  immersed in a plasma.

<sup>b</sup>Collision times for electrons and singly ionised ions with Maxwellian speed distributions,  $T_i \lesssim T_e$ . The Spitzer conductivity can be calculated from Equation (7.233).

<sup>c</sup>Defined so that the Maxwellian velocity distribution  $\propto \exp(-v^2/v_{te}^2)$ . There are other definitions (see *Maxwell-Boltzmann distribution* on page 112).

Electromagnetic propagation in cold plasmas<sup>a</sup>

Plasma frequency	$(2\pi\nu_p)^2 = \frac{n_e e^2}{\epsilon_0 m_e} = \omega_p^2$ (7.258)	$\nu_p$ plasma frequency
	$\nu_p \simeq 8.98 n_e^{1/2}$ Hz (7.259)	$\omega_p$ plasma angular frequency
		$n_e$ electron number density ( $\text{m}^{-3}$ )
		$m_e$ electron mass
Plasma refractive index ( $B=0$ )	$\eta = [1 - (\nu_p/\nu)^2]^{1/2}$ (7.260)	$-e$ electronic charge
		$\epsilon_0$ permittivity of free space
		$\eta$ refractive index
		$\nu$ frequency
Plasma dispersion relation ( $B=0$ )	$c^2 k^2 = \omega^2 - \omega_p^2$ (7.261)	$k$ wavenumber ( $=2\pi/\lambda$ )
		$\omega$ angular frequency ( $=2\pi/\nu$ )
		$c$ speed of light
Plasma phase velocity ( $B=0$ )	$v_\phi = c/\eta$ (7.262)	$v_\phi$ phase velocity
Plasma group velocity ( $B=0$ )	$v_g = c\eta$ (7.263)	
	$v_\phi v_g = c^2$ (7.264)	$v_g$ group velocity
Cyclotron (Larmor, or gyro-) frequency	$2\pi\nu_C = \frac{qB}{m} = \omega_C$ (7.265)	$\nu_C$ cyclotron frequency
	$\nu_{Ce} \simeq 28 \times 10^9 B$ Hz (7.266)	$\omega_C$ cyclotron angular frequency
	$\nu_{Cp} \simeq 15.2 \times 10^6 B$ Hz (7.267)	$\nu_{Ce}$ electron $\nu_C$
		$\nu_{Cp}$ proton $\nu_C$
		$q$ particle charge
		$B$ magnetic flux density (T)
Larmor (cyclotron, or gyro-) radius	$r_L = \frac{v_\perp}{\omega_C} = v_\perp \frac{m}{qB}$ (7.268)	$m$ particle mass ( $\gamma m$ if relativistic)
	$r_{Le} = 5.69 \times 10^{-12} \left(\frac{v_\perp}{B}\right)$ m (7.269)	$r_L$ Larmor radius
	$r_{Lp} = 10.4 \times 10^{-9} \left(\frac{v_\perp}{B}\right)$ m (7.270)	$r_{Le}$ electron $r_L$
		$r_{Lp}$ proton $r_L$
		$v_\perp$ speed $\perp$ to $\mathbf{B}$ ( $\text{ms}^{-1}$ )
Mixed propagation modes <sup>b</sup>	$\eta^2 = 1 - \frac{X(1-X)}{(1-X) - \frac{1}{2}Y^2 \sin^2 \theta_B \pm S}$ , (7.271)	$\theta_B$ angle between wavefront normal ( $\hat{\mathbf{k}}$ ) and $\mathbf{B}$
	where $X = (\omega_p/\omega)^2$ , $Y = \omega_{Ce}/\omega$ ,	
	and $S^2 = \frac{1}{4}Y^4 \sin^4 \theta_B + Y^2(1-X)^2 \cos^2 \theta_B$	
Faraday rotation <sup>c</sup>	$\Delta\psi = \frac{\mu_0 e^3}{8\pi^2 m_e^2 c} \lambda^2 \int_{\text{line}} n_e \mathbf{B} \cdot d\mathbf{l}$ (7.272)	$\Delta\psi$ rotation angle
	$= R\lambda^2$ (7.273)	$\lambda$ wavelength ( $=2\pi/k$ )
		$d\mathbf{l}$ line element in direction of wave propagation
		$R$ rotation measure

<sup>a</sup>I.e., plasmas in which electromagnetic force terms dominate over thermal pressure terms. Also taking  $\mu_r = 1$ .<sup>b</sup>In a collisionless electron plasma. The ordinary and extraordinary modes are the + and - roots of  $S^2$  when  $\theta_B = \pi/2$ . When  $\theta_B = 0$ , these roots are the right and left circularly polarised modes respectively, using the optical convention for handedness.<sup>c</sup>In a tenuous plasma, SI units throughout.  $\Delta\psi$  is taken positive if  $\mathbf{B}$  is directed towards the observer.

## Magnetohydrodynamics<sup>a</sup>

Sound speed	$v_s = \left(\frac{\gamma p}{\rho}\right)^{1/2} = \left(\frac{2\gamma k_B T}{m_p}\right)^{1/2} \quad (7.274)$ $\simeq 166 T^{1/2} \text{ms}^{-1} \quad (7.275)$	$v_s$ sound (wave) speed $\gamma$ ratio of heat capacities $p$ hydrostatic pressure $\rho$ plasma mass density $k_B$ Boltzmann constant $T$ temperature (K)
Alfvén speed	$v_A = \frac{B}{(\mu_0 \rho)^{1/2}} \quad (7.276)$ $\simeq 2.18 \times 10^{16} B n_e^{-1/2} \text{ms}^{-1} \quad (7.277)$	$m_p$ proton mass $v_A$ Alfvén speed $B$ magnetic flux density (T) $\mu_0$ permeability of free space $n_e$ electron number density ( $\text{m}^{-3}$ )
Plasma beta	$\beta = \frac{2\mu_0 p}{B^2} = \frac{4\mu_0 n_e k_B T}{B^2} = \frac{2v_s^2}{\gamma v_A^2} \quad (7.278)$	$\beta$ plasma beta (ratio of hydrostatic to magnetic pressure)
Direct electrical conductivity	$\sigma_d = \frac{n_e^2 e^2 \sigma}{n_e^2 e^2 + \sigma^2 B^2} \quad (7.279)$	$-e$ electronic charge $\sigma_d$ direct conductivity $\sigma$ conductivity ( $B=0$ )
Hall electrical conductivity	$\sigma_H = \frac{\sigma B}{n_e e} \sigma_d \quad (7.280)$	$\sigma_H$ Hall conductivity
Generalised Ohm's law	$\mathbf{J} = \sigma_d (\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \sigma_H \hat{\mathbf{B}} \times (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (7.281)$	$\mathbf{J}$ current density $\mathbf{E}$ electric field $\mathbf{v}$ plasma velocity field $\hat{\mathbf{B}} = \mathbf{B}/ \mathbf{B} $
Resistive MHD equations (single-fluid model) <sup>b</sup>	$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} \quad (7.282)$	$\mu_0$ permeability of free space $\eta$ magnetic diffusivity [ $= 1/(\mu_0 \sigma)$ ] $\nu$ kinematic viscosity $\mathbf{g}$ gravitational field strength
	$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\nabla p}{\rho} + \frac{1}{\mu_0 \rho} (\nabla \times \mathbf{B}) \times \mathbf{B} + \nu \nabla^2 \mathbf{v} + \frac{1}{3} \nu \nabla (\nabla \cdot \mathbf{v}) + \mathbf{g} \quad (7.283)$	
Shear Alfvénic dispersion relation <sup>c</sup>	$\omega = k v_A \cos \theta_B \quad (7.284)$	$\omega$ angular frequency ( $= 2\pi\nu$ ) $\mathbf{k}$ wavevector ( $k = 2\pi/\lambda$ ) $\theta_B$ angle between $\mathbf{k}$ and $\mathbf{B}$
Magnetosonic dispersion relation <sup>d</sup>	$\omega^2 k^2 (v_s^2 + v_A^2) - \omega^4 = v_s^2 v_A^2 k^4 \cos^2 \theta_B \quad (7.285)$	

<sup>a</sup>For a warm, fully ionised, electrically neutral  $p^+/e^-$  plasma,  $\mu_r = 1$ . Relativistic and displacement current effects are assumed to be negligible and all oscillations are taken as being well below all resonance frequencies.

<sup>b</sup>Neglecting bulk (second) viscosity.

<sup>c</sup>Nonresistive, inviscid flow.

<sup>d</sup>Nonresistive, inviscid flow. The greater and lesser solutions for  $\omega^2$  are the fast and slow magnetosonic waves respectively.



**Synchrotron radiation**

Power radiated by a single electron <sup>a</sup>	$P_{\text{tot}} = 2\sigma_{\text{T}}cu_{\text{mag}}\gamma^2 \left(\frac{v}{c}\right)^2 \sin^2\theta \quad (7.286)$	$P_{\text{tot}}$ total radiated power $\sigma_{\text{T}}$ Thomson cross section
	$\simeq 1.59 \times 10^{-14} B^2 \gamma^2 \left(\frac{v}{c}\right)^2 \sin^2\theta \quad \text{W} \quad (7.287)$	$u_{\text{mag}}$ magnetic energy density = $B^2/(2\mu_0)$ $v$ electron velocity ( $\sim c$ )
... averaged over pitch angles	$P_{\text{tot}} = \frac{4}{3}\sigma_{\text{T}}cu_{\text{mag}}\gamma^2 \left(\frac{v}{c}\right)^2 \quad (7.288)$	$\gamma$ Lorentz factor = $[1 - (v/c)^2]^{-1/2}$ $\theta$ pitch angle (angle between $v$ and $B$ )
	$\simeq 1.06 \times 10^{-14} B^2 \gamma^2 \left(\frac{v}{c}\right)^2 \quad \text{W} \quad (7.289)$	$B$ magnetic flux density $c$ speed of light
Single electron emission spectrum <sup>b</sup>	$P(\nu) = \frac{3^{1/2}e^3B \sin\theta}{4\pi\epsilon_0cm_e} F(\nu/\nu_{\text{ch}}) \quad (7.290)$	$P(\nu)$ emission spectrum $\nu$ frequency
	$\simeq 2.34 \times 10^{-25} B \sin\theta F(\nu/\nu_{\text{ch}}) \quad \text{W Hz}^{-1} \quad (7.291)$	$\nu_{\text{ch}}$ characteristic frequency $-e$ electronic charge $\epsilon_0$ free space permittivity $m_e$ electronic (rest) mass
Characteristic frequency	$\nu_{\text{ch}} = \frac{3}{2}\gamma^2 \frac{eB}{2\pi m_e} \sin\theta \quad (7.292)$	$F$ spectral function
	$\simeq 4.2 \times 10^{10} \gamma^2 B \sin\theta \quad \text{Hz} \quad (7.293)$	$K_{5/3}$ modified Bessel fn. of the 2nd kind, order 5/3
Spectral function	$F(x) = x \int_x^\infty K_{5/3}(y) dy \quad (7.294)$	
	$\simeq \begin{cases} 2.15x^{1/3} & (x \ll 1) \\ 1.25x^{1/2}e^{-x} & (x \gg 1) \end{cases} \quad (7.295)$	

<sup>a</sup>This expression also holds for cyclotron radiation ( $v \ll c$ ).

<sup>b</sup>I.e., total radiated power per unit frequency interval.

**Bremsstrahlung<sup>a</sup>**Single electron and ion<sup>b</sup>

$$\frac{dW}{d\omega} = \frac{Z^2 e^6}{24\pi^4 \epsilon_0^3 c^3 m_e^2} \frac{\omega^2}{\gamma^2 v^4} \left[ \frac{1}{\gamma^2} K_0^2 \left( \frac{\omega b}{\gamma v} \right) + K_1^2 \left( \frac{\omega b}{\gamma v} \right) \right] \quad (7.296)$$

$$\simeq \frac{Z^2 e^6}{24\pi^4 \epsilon_0^3 c^3 m_e^2 b^2 v^2} \quad (\omega b \ll \gamma v) \quad (7.297)$$

Thermal bremsstrahlung radiation ( $v \ll c$ ; Maxwellian distribution)

$$\frac{dP}{dV dv} = 6.8 \times 10^{-51} Z^2 T^{-1/2} n_i n_e g(v, T) \exp\left(\frac{-hv}{kT}\right) \quad \text{W m}^{-3} \text{Hz}^{-1} \quad (7.298)$$

$$\text{where } g(v, T) \simeq \begin{cases} 0.28 [\ln(4.4 \times 10^{16} T^3 v^{-2} Z^{-2}) - 0.76] & (hv \ll kT \lesssim 10^5 kZ^2) \\ 0.55 \ln(2.1 \times 10^{10} T v^{-1}) & (hv \ll 10^5 kZ^2 \lesssim kT) \\ (2.1 \times 10^{10} T v^{-1})^{-1/2} & (hv \gg kT) \end{cases} \quad (7.299)$$

$$\frac{dP}{dV} \simeq 1.7 \times 10^{-40} Z^2 T^{1/2} n_i n_e \quad \text{W m}^{-3} \quad (7.300)$$

$\omega$  angular frequency ( $= 2\pi\nu$ )  
 $Ze$  ionic charge  
 $-e$  electronic charge  
 $\epsilon_0$  permittivity of free space  
 $c$  speed of light  
 $m_e$  electronic mass  
 $b$  collision parameter<sup>c</sup>

$v$  electron velocity  
 $K_i$  modified Bessel functions of order  $i$  (see page 47)  
 $\gamma$  Lorentz factor  
 $= [1 - (v/c)^2]^{-1/2}$   
 $P$  power radiated  
 $V$  volume  
 $\nu$  frequency (Hz)

$W$  energy radiated  
 $T$  electron temperature (K)  
 $n_i$  ion number density ( $\text{m}^{-3}$ )  
 $n_e$  electron number density ( $\text{m}^{-3}$ )  
 $k$  Boltzmann constant  
 $h$  Planck constant  
 $g$  Gaunt factor

<sup>a</sup>Classical treatment. The ions are at rest, and all frequencies are above the plasma frequency.<sup>b</sup>The spectrum is approximately flat at low frequencies and drops exponentially at frequencies  $\gtrsim \gamma v/b$ .<sup>c</sup>Distance of closest approach.

# Chapter 8 Optics

## 8.1 Introduction

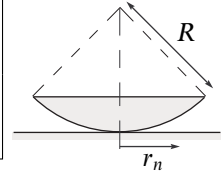
Any attempt to unify the notations and terminology of optics is doomed to failure. This is partly due to the long and illustrious history of the subject (a pedigree shared only with mechanics), which has allowed a variety of approaches to develop, and partly due to the disparate fields of physics to which its basic principles have been applied. Optical ideas find their way into most wave-based branches of physics, from quantum mechanics to radio propagation.

Nowhere is the lack of convention more apparent than in the study of polarisation, and so a cautionary note follows. The conventions used here can be taken largely from context, but the reader should be aware that alternative sign and handedness conventions do exist and are widely used. In particular we will take a circularly polarised wave as being right-handed if, for an observer looking *towards* the source, the electric field vector in a plane perpendicular to the line of sight rotates clockwise. This convention is often used in optics textbooks and has the conceptual advantage that the electric field orientation describes a right-hand corkscrew in space, with the direction of energy flow defining the screw direction. It is however opposite to the system widely used in radio engineering, where the handedness of a helical antenna generating or receiving the wave defines the handedness and is also in the opposite sense to the wave's own angular momentum vector.

## 8.2 Interference

### Newton's rings<sup>a</sup>

nth dark ring	$r_n^2 = nR\lambda_0$	(8.1)	$r_n$ radius of nth ring
			$n$ integer ( $\geq 0$ )
			$R$ lens radius of curvature
nth bright ring	$r_n^2 = \left(n + \frac{1}{2}\right) R\lambda_0$	(8.2)	$\lambda_0$ wavelength in external medium



<sup>a</sup>Viewed in reflection.

### Dielectric layers<sup>a</sup>

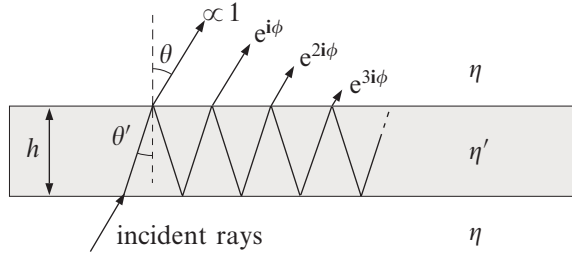
<p>The diagram shows a single layer of thickness <math>a</math> with refractive index <math>\eta_2</math> between media <math>\eta_1</math> and <math>\eta_3</math>. The reflection coefficient is <math>R</math> and the transmission coefficient is <math>1 - R</math>. To the right, a multilayer stack of <math>N</math> pairs of layers with refractive indices <math>\eta_a</math> and <math>\eta_b</math> is shown between media <math>\eta_1</math> and <math>\eta_3</math>. The overall reflection coefficient is <math>R_N</math> and the transmission coefficient is <math>1 - R_N</math>.</p>			
Quarter-wave condition	$a = \frac{m \lambda_0}{\eta_2 4}$	(8.3)	$a$ film thickness
			$m$ thickness integer ( $m \geq 0$ )
			$\eta_2$ film refractive index
			$\lambda_0$ free-space wavelength
			$R$ power reflectance coefficient
Single-layer reflectance <sup>b</sup>	$R = \begin{cases} \left( \frac{\eta_1 \eta_3 - \eta_2^2}{\eta_1 \eta_3 + \eta_2^2} \right)^2 & (m \text{ odd}) \\ \left( \frac{\eta_1 - \eta_3}{\eta_1 + \eta_3} \right)^2 & (m \text{ even}) \end{cases}$	(8.4)	$\eta_1$ entry-side refractive index
			$\eta_3$ exit-side refractive index
Dependence of $R$ on layer thickness, $m$	max if $(-1)^m (\eta_1 - \eta_2)(\eta_2 - \eta_3) > 0$	(8.5)	
	min if $(-1)^m (\eta_1 - \eta_2)(\eta_2 - \eta_3) < 0$	(8.6)	
	$R = 0$ if $\eta_2 = (\eta_1 \eta_3)^{1/2}$ and $m$ odd	(8.7)	
Multilayer reflectance <sup>c</sup>	$R_N = \left[ \frac{\eta_1 - \eta_3 (\eta_a / \eta_b)^{2N}}{\eta_1 + \eta_3 (\eta_a / \eta_b)^{2N}} \right]^2$	(8.8)	$R_N$ multilayer reflectance
			$N$ number of layer pairs
			$\eta_a$ refractive index of top layer
			$\eta_b$ refractive index of bottom layer

<sup>a</sup>For normal incidence, assuming the quarter-wave condition. The media are also assumed lossless, with  $\mu_r = 1$ .

<sup>b</sup>See page 154 for the definition of  $R$ .

<sup>c</sup>For a stack of  $N$  layer pairs, giving an overall refractive index sequence  $\eta_1 \eta_a \eta_b \eta_a \dots \eta_a \eta_b \eta_3$  (see right-hand diagram). Each layer in the stack meets the quarter-wave condition with  $m = 1$ .

**Fabry-Perot etalon<sup>a</sup>**



Incremental phase difference <sup>b</sup>	$\phi = 2k_0 h \eta' \cos \theta' \quad (8.9)$	$\phi$ incremental phase difference
	$= 2k_0 h \eta' \left[ 1 - \left( \frac{\eta \sin \theta}{\eta'} \right)^2 \right]^{1/2} \quad (8.10)$	$k_0$ free-space wavenumber ( $= 2\pi/\lambda_0$ )
	$= 2\pi n \quad \text{for a maximum} \quad (8.11)$	$h$ cavity width
Coefficient of finesse	$F = \frac{4R}{(1-R)^2} \quad (8.12)$	$\theta$ fringe inclination (usually $\ll 1$ )
		$\theta'$ internal angle of refraction
		$\eta'$ cavity refractive index
Finesse	$\mathcal{F} = \frac{\pi}{2} F^{1/2} \quad (8.13)$	$\eta$ external refractive index
	$= \frac{\lambda_0}{\eta' h} Q \quad (8.14)$	$n$ fringe order (integer)
		$F$ coefficient of finesse
Transmitted intensity	$I(\theta) = \frac{I_0(1-R)^2}{1+R^2-2R\cos\phi} \quad (8.15)$	$R$ interface power reflectance
	$= \frac{I_0}{1+F\sin^2(\phi/2)} \quad (8.16)$	$\mathcal{F}$ finesse
	$= I_0 A(\theta) \quad (8.17)$	$\lambda_0$ free-space wavelength
Fringe intensity profile	$\Delta\phi = 2\arcsin(F^{-1/2}) \quad (8.18)$	$Q$ cavity quality factor
	$\simeq 2F^{-1/2} \quad (8.19)$	$I$ transmitted intensity
Chromatic resolving power	$\frac{\lambda_0}{\delta\lambda} \simeq \frac{R^{1/2}\pi n}{1-R} = n\mathcal{F} \quad (8.20)$	$I_0$ incident intensity
	$\simeq \frac{2\mathcal{F} h \eta'}{\lambda_0} \quad (\theta \ll 1) \quad (8.21)$	$A$ Airy function
Free spectral range <sup>c</sup>	$\delta\lambda_f = \mathcal{F} \delta\lambda \quad (8.22)$	$\Delta\phi$ phase difference at half intensity point
	$\delta\nu_f = \frac{c}{2\eta' h} \quad (8.23)$	$\delta\lambda$ minimum resolvable wavelength difference

<sup>a</sup>Neglecting any effects due to surface coatings on the etalon. See also *Lasers* on page 174.

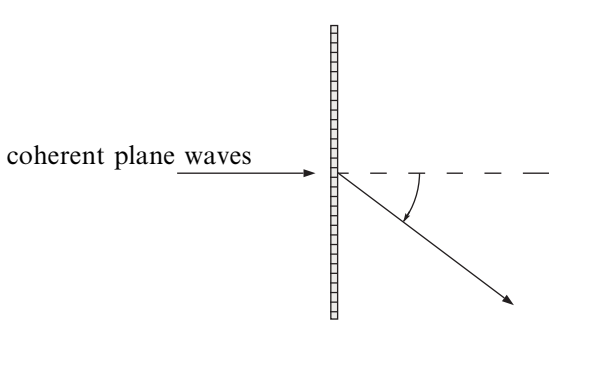
<sup>b</sup>Between adjacent rays. Highest order fringes are near the centre of the pattern.

<sup>c</sup>At near-normal incidence ( $\theta \simeq 0$ ), the orders of two spectral components separated by  $< \delta\lambda_f$  will not overlap.

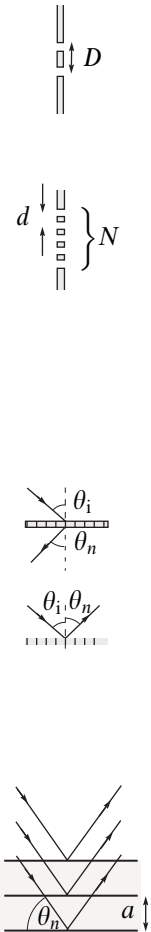


### 8.3 Fraunhofer diffraction

#### Gratings<sup>a</sup>

		
Young's double slits <sup>b</sup>	$I(s) = I_0 \cos^2 \frac{kDs}{2}$	(8.24)
N equally spaced narrow slits	$I(s) = I_0 \left[ \frac{\sin(Nkds/2)}{N \sin(kds/2)} \right]^2$	(8.25)
Infinite grating	$I(s) = I_0 \sum_{n=-\infty}^{\infty} \delta \left( s - \frac{n\lambda}{d} \right)$	(8.26)
Normal incidence	$\sin \theta_n = \frac{n\lambda}{d}$	(8.27)
Oblique incidence	$\sin \theta_n + \sin \theta_i = \frac{n\lambda}{d}$	(8.28)
Reflection grating	$\sin \theta_n - \sin \theta_i = \frac{n\lambda}{d}$	(8.29)
Chromatic resolving power	$\frac{\lambda}{\delta\lambda} = Nn$	(8.30)
Grating dispersion	$\frac{\partial\theta}{\partial\lambda} = \frac{n}{d \cos\theta}$	(8.31)
Bragg's law <sup>c</sup>	$2a \sin \theta_n = n\lambda$	(8.32)

$I(s)$  diffracted intensity  
 $I_0$  peak intensity  
 $\theta$  diffraction angle  
 $s = \sin\theta$   
 $D$  slit separation  
 $\lambda$  wavelength  
 $N$  number of slits  
 $k$  wavenumber ( $= 2\pi/\lambda$ )  
 $d$  slit spacing  
 $n$  diffraction order  
 $\delta$  Dirac delta function  
 $\theta_n$  angle of diffracted maximum  
 $\theta_i$  angle of incident illumination  
 $\delta\lambda$  diffraction peak width  
 $a$  atomic plane spacing



<sup>a</sup>Unless stated otherwise, the illumination is normal to the grating.  
<sup>b</sup>Two narrow slits separated by  $D$ .  
<sup>c</sup>The condition is for Bragg reflection, with  $\theta_n = \theta_i$ .

**Aperture diffraction**

<p>coherent plane-wave illumination, normal to the <math>xy</math> plane</p>		
General 1-D aperture <sup>a</sup>	$\psi(s) \propto \int_{-\infty}^{\infty} f(x)e^{-ik_sx} dx \quad (8.33)$ $I(s) \propto \psi\psi^*(s) \quad (8.34)$	$\psi$ diffracted wavefunction $I$ diffracted intensity $\theta$ diffraction angle $s = \sin\theta$
General 2-D aperture in $(x,y)$ plane (small angles)	$\psi(s_x, s_y) \propto \iint_{\infty} f(x,y)e^{-ik(s_x x + s_y y)} dx dy \quad (8.35)$	$f$ aperture amplitude transmission function $x,y$ distance across aperture $k$ wavenumber ( $= 2\pi/\lambda$ ) $s_x$ deflection $\parallel$ $xz$ plane $s_y$ deflection $\perp$ $xz$ plane
Broad 1-D slit <sup>b</sup>	$I(s) = I_0 \frac{\sin^2(kas/2)}{(kas/2)^2} \quad (8.36)$ $\equiv I_0 \text{sinc}^2(as/\lambda) \quad (8.37)$	$I_0$ peak intensity $a$ slit width (in $x$ ) $\lambda$ wavelength
Sidelobe intensity	$\frac{I_n}{I_0} = \left(\frac{2}{\pi}\right)^2 \frac{1}{(2n+1)^2} \quad (n > 0) \quad (8.38)$	$I_n$ $n$ th sidelobe intensity
Rectangular aperture (small angles)	$I(s_x, s_y) = I_0 \text{sinc}^2 \frac{as_x}{\lambda} \text{sinc}^2 \frac{bs_y}{\lambda} \quad (8.39)$	$a$ aperture width in $x$ $b$ aperture width in $y$
Circular aperture <sup>c</sup>	$I(s) = I_0 \left[ \frac{2J_1(kDs/2)}{kDs/2} \right]^2 \quad (8.40)$	$J_1$ first-order Bessel function $D$ aperture diameter
First minimum <sup>d</sup>	$s = 1.22 \frac{\lambda}{D} \quad (8.41)$	$\lambda$ wavelength
First subside. maximum	$s = 1.64 \frac{\lambda}{D} \quad (8.42)$	
Weak 1-D phase object	$f(x) = \exp[i\phi(x)] \simeq 1 + i\phi(x) \quad (8.43)$	$\phi(x)$ phase distribution $i^2 = -1$
Fraunhofer limit <sup>e</sup>	$L \gg \frac{(\Delta x)^2}{\lambda} \quad (8.44)$	$L$ distance of aperture from observation point $\Delta x$ aperture size

<sup>a</sup>The Fraunhofer integral.

<sup>b</sup>Note that  $\text{sinc } x = (\sin \pi x)/(\pi x)$ .

<sup>c</sup>The central maximum is known as the ‘‘Airy disk.’’

<sup>d</sup>The ‘‘Rayleigh resolution criterion’’ states that two point sources of equal intensity can just be resolved with diffraction-limited optics if separated in angle by  $1.22\lambda/D$ .

<sup>e</sup>Plane-wave illumination.



8.4 Fresnel diffraction

Kirchhoff's diffraction formula<sup>a</sup>

(source at infinity)			
Source at infinity	$\psi_P = -\frac{i}{\lambda} \psi_0 \int_{\text{plane}} K(\theta) \frac{e^{ikr}}{r} dA \quad (8.45)$	$\psi_P$ complex amplitude at $P$ $\lambda$ wavelength $k$ wavenumber ( $= 2\pi/\lambda$ ) $\psi_0$ incident amplitude $\theta$ obliquity angle $r$ distance of $dA$ from $P$ ( $\gg \lambda$ ) $dA$ area element on incident wavefront $K$ obliquity factor $dS$ element of closed surface $\hat{s}$ unit vector $s$ vector normal to $dS$ $r$ vector from $P$ to $dS$ $\rho$ vector from source to $dS$ $E_0$ amplitude (see footnote)	
where:			
Obliquity factor (cardioid)	$K(\theta) = \frac{1}{2}(1 + \cos\theta) \quad (8.46)$		
Source at finite distance <sup>b</sup>	$\psi_P = -\frac{iE_0}{\lambda} \oint_{\text{closed surface}} \frac{e^{ik(\rho+r)}}{2\rho r} [\cos(\hat{s} \cdot \hat{r}) - \cos(\hat{s} \cdot \hat{\rho})] dS \quad (8.47)$		

<sup>a</sup>Also known as the ‘‘Fresnel–Kirchhoff formula.’’ Diffraction by an obstacle coincident with the integration surface can be approximated by omitting that part of the surface from the integral.

<sup>b</sup>The source amplitude at  $\rho$  is  $\psi(\rho) = E_0 e^{ik\rho}/\rho$ . The integral is taken over a surface enclosing the point  $P$ .

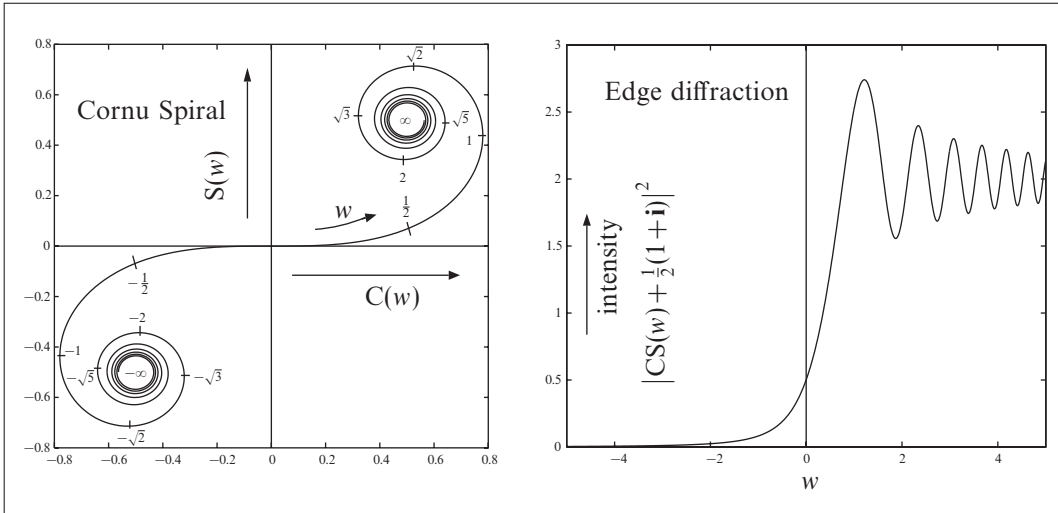
Fresnel zones

Effective aperture distance <sup>a</sup>	$\frac{1}{z} = \frac{1}{z_1} + \frac{1}{z_2} \quad (8.48)$	$z$ effective distance $z_1$ source–aperture distance $z_2$ aperture–observer distance
Half-period zone radius	$y_n = (n\lambda z)^{1/2} \quad (8.49)$	$n$ half-period zone number $\lambda$ wavelength $y_n$ $n$ th half-period zone radius
Axial zeros (circular aperture)	$z_m = \frac{R^2}{2m\lambda} \quad (8.50)$	$z_m$ distance of $m$ th zero from aperture $R$ aperture radius

<sup>a</sup>I.e., the aperture–observer distance to be employed when the source is not at infinity.



**Cornu spiral**



<p>Fresnel integrals<sup>a</sup></p>	$C(w) = \int_0^w \cos \frac{\pi t^2}{2} dt \quad (8.51)$ $S(w) = \int_0^w \sin \frac{\pi t^2}{2} dt \quad (8.52)$	<p>C Fresnel cosine integral S Fresnel sine integral</p>
<p>Cornu spiral</p>	$\text{CS}(w) = C(w) + iS(w) \quad (8.53)$ $\text{CS}(\pm\infty) = \pm \frac{1}{2}(1+i) \quad (8.54)$	<p>CS Cornu spiral <math>v, w</math> length along spiral</p>
<p>Edge diffraction</p>	$\psi_P = \frac{\psi_0}{2^{1/2}} [\text{CS}(w) + \frac{1}{2}(1+i)] \quad (8.55)$ <p>where <math>w = y \left( \frac{2}{\lambda z} \right)^{1/2} \quad (8.56)</math></p>	<p><math>\psi_P</math> complex amplitude at <math>P</math> <math>\psi_0</math> unobstructed amplitude <math>\lambda</math> wavelength <math>z</math> distance of <math>P</math> from aperture plane [see (8.48)] <math>y</math> position of edge</p>
<p>Diffraction from a long slit<sup>b</sup></p>	$\psi_P = \frac{\psi_0}{2^{1/2}} [\text{CS}(w_2) - \text{CS}(w_1)] \quad (8.57)$ <p>where <math>w_i = y_i \left( \frac{2}{\lambda z} \right)^{1/2} \quad (8.58)</math></p>	<p>coherent plane waves</p>
<p>Diffraction from a rectangular aperture</p>	$\psi_P = \frac{\psi_0}{2} [\text{CS}(v_2) - \text{CS}(v_1)] \times [\text{CS}(w_2) - \text{CS}(w_1)] \quad (8.59)$ <p>where <math>v_i = x_i \left( \frac{2}{\lambda z} \right)^{1/2} \quad (8.61)</math></p> <p>and <math>w_i = y_i \left( \frac{2}{\lambda z} \right)^{1/2} \quad (8.62)</math></p>	<p><math>x_i</math> positions of slit sides <math>y_i</math> positions of slit top/bottom</p>

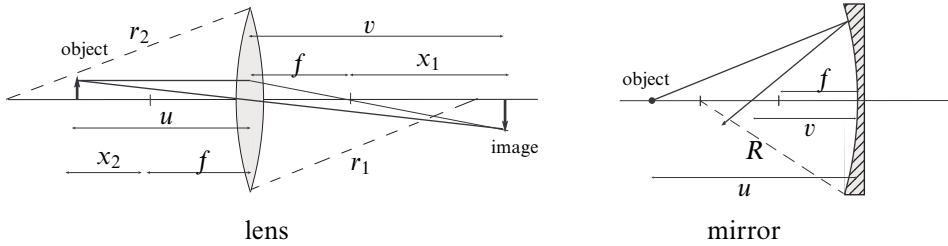
<sup>a</sup>See also Equation (2.393) on page 45.

<sup>b</sup>Slit long in  $x$ .



### 8.5 Geometrical optics

#### Lenses and mirrors<sup>a</sup>



sign convention		
	+	-
$r$	centred to right	centred to left
$u$	real object	virtual object
$v$	real image	virtual image
$f$	converging lens/ concave mirror	diverging lens/ convex mirror
$M_T$	erect image	inverted image

Fermat's principle <sup>b</sup>	$L = \int \eta dl$ is stationary	(8.63)	$L$ optical path length $\eta$ refractive index $dl$ ray path element
Gauss's lens formula	$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$	(8.64)	$u$ object distance $v$ image distance $f$ focal length
Newton's lens formula	$x_1 x_2 = f^2$	(8.65)	$x_1 = v - f$ $x_2 = u - f$
Lensmaker's formula	$\frac{1}{u} + \frac{1}{v} = (\eta - 1) \left( \frac{1}{r_1} - \frac{1}{r_2} \right)$	(8.66)	$r_i$ radii of curvature of lens surfaces
Mirror formula <sup>c</sup>	$\frac{1}{u} + \frac{1}{v} = -\frac{2}{R} = \frac{1}{f}$	(8.67)	$R$ mirror radius of curvature
Dioptre number	$D = \frac{1}{f}$ m <sup>-1</sup>	(8.68)	$D$ dioptre number ( $f$ in metres)
Focal ratio <sup>d</sup>	$n = \frac{f}{d}$	(8.69)	$n$ focal ratio $d$ lens or mirror diameter
Transverse linear magnification	$M_T = -\frac{v}{u}$	(8.70)	$M_T$ transverse magnification
Longitudinal linear magnification	$M_L = -M_T^2$	(8.71)	$M_L$ longitudinal magnification

<sup>a</sup>Formulas assume "Gaussian optics," i.e., all lenses are thin and all angles small. Light enters from the left.

<sup>b</sup>A stationary optical path length has, to first order, a length identical to that of adjacent paths.

<sup>c</sup>The mirror is concave if  $R < 0$ , convex if  $R > 0$ .

<sup>d</sup>Or "f-number," written  $f/2$  if  $n = 2$  etc.

**Prisms (dispersing)**

Transmission angle	$\sin \theta_t = (\eta^2 - \sin^2 \theta_i)^{1/2} \sin \alpha - \sin \theta_i \cos \alpha \quad (8.72)$	$\theta_i$ angle of incidence $\theta_t$ angle of transmission $\alpha$ apex angle $\eta$ refractive index
Deviation	$\delta = \theta_i + \theta_t - \alpha \quad (8.73)$	$\delta$ angle of deviation
Minimum deviation condition	$\sin \theta_i = \sin \theta_t = \eta \sin \frac{\alpha}{2} \quad (8.74)$	
Refractive index	$\eta = \frac{\sin[(\delta_m + \alpha)/2]}{\sin(\alpha/2)} \quad (8.75)$	$\delta_m$ minimum deviation
Angular dispersion <sup>a</sup>	$D = \frac{d\delta}{d\lambda} = \frac{2 \sin(\alpha/2)}{\cos[(\delta_m + \alpha)/2]} \frac{d\eta}{d\lambda} \quad (8.76)$	$D$ dispersion $\lambda$ wavelength

<sup>a</sup>At minimum deviation.

**Optical fibres**

Acceptance angle	$\sin \theta_m = \frac{1}{\eta_0} (\eta_f^2 - \eta_c^2)^{1/2} \quad (8.77)$	$\theta_m$ maximum angle of incidence $\eta_0$ exterior refractive index $\eta_f$ fibre refractive index $\eta_c$ cladding refractive index
Numerical aperture	$N = \eta_0 \sin \theta_m \quad (8.78)$	$N$ numerical aperture
Multimode dispersion <sup>a</sup>	$\frac{\Delta t}{L} = \frac{\eta_f}{c} \left( \frac{\eta_f}{\eta_c} - 1 \right) \quad (8.79)$	$\Delta t$ temporal dispersion $L$ fibre length $c$ speed of light

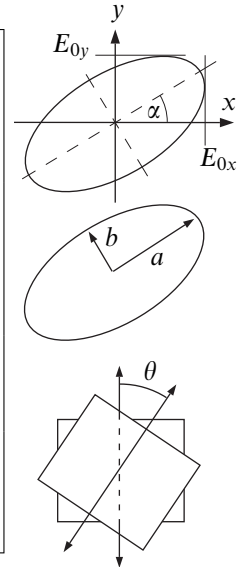
<sup>a</sup>Of a pulse with a given wavelength, caused by the range of incident angles up to  $\theta_m$ . Sometimes called “intermodal dispersion” or “modal dispersion.”



### 8.6 Polarisation

#### Elliptical polarisation<sup>a</sup>

Elliptical polarisation	$E = (E_{0x}, E_{0y}e^{i\delta})e^{i(kz-\omega t)}$ (8.80)	$E$ electric field $k$ wavevector $z$ propagation axis $\omega t$ angular frequency $\times$ time
Polarisation angle <sup>b</sup>	$\tan 2\alpha = \frac{2E_{0x}E_{0y}}{E_{0x}^2 - E_{0y}^2} \cos \delta$ (8.81)	$E_{0x}$ x amplitude of $E$ $E_{0y}$ y amplitude of $E$ $\delta$ relative phase of $E_y$ with respect to $E_x$ $\alpha$ polarisation angle
Ellipticity <sup>c</sup>	$e = \frac{a-b}{a}$ (8.82)	$e$ ellipticity $a$ semi-major axis $b$ semi-minor axis
Malus's law <sup>d</sup>	$I(\theta) = I_0 \cos^2 \theta$ (8.83)	$I(\theta)$ transmitted intensity $I_0$ incident intensity $\theta$ polariser-analyser angle



<sup>a</sup>See the introduction (page 161) for a discussion of sign and handedness conventions.

<sup>b</sup>Angle between ellipse major axis and x axis. Sometimes the polarisation angle is defined as  $\pi/2 - \alpha$ .

<sup>c</sup>This is one of several definitions for ellipticity.

<sup>d</sup>Transmission through skewed polarisers for unpolarised incident light.

#### Jones vectors and matrices

Normalised electric field <sup>a</sup>	$E = \begin{pmatrix} E_x \\ E_y \end{pmatrix};  E  = 1$ (8.84)	$E$ electric field $E_x$ x component of $E$ $E_y$ y component of $E$	
Example vectors:	$E_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $E_r = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$	$E_{45} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $E_l = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$ $E_{45}$ 45° to x axis $E_r$ right-hand circular $E_l$ left-hand circular	
Jones matrix	$E_t = \mathbf{A}E_i$ (8.85)	$E_t$ transmitted vector $E_i$ incident vector $\mathbf{A}$ Jones matrix	
Example matrices:			
Linear polariser $\parallel x$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	Linear polariser $\parallel y$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
Linear polariser at 45°	$\frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	Linear polariser at -45°	$\frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$
Right circular polariser	$\frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$	Left circular polariser	$\frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$
$\lambda/4$ plate (fast $\parallel x$ )	$e^{i\pi/4} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$	$\lambda/4$ plate (fast $\perp x$ )	$e^{i\pi/4} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$

<sup>a</sup>Known as the “normalised Jones vector.”

Stokes parameters<sup>a</sup>

Electric fields	$E_x = E_{0x}e^{i(kz-\omega t)} \quad (8.86)$ $E_y = E_{0y}e^{i(kz-\omega t+\delta)} \quad (8.87)$	$k$ wavevector $\omega t$ angular frequency $\times$ time $\delta$ relative phase of $E_y$ with respect to $E_x$																																								
Axial ratio <sup>b</sup>	$\tan \chi = \pm r = \pm \frac{b}{a} \quad (8.88)$	$\chi$ (see diagram) $r$ axial ratio																																								
Stokes parameters	$I = \langle E_x^2 \rangle + \langle E_y^2 \rangle \quad (8.89)$	$E_x$ electric field component $\parallel$ x $E_y$ electric field component $\parallel$ y $E_{0x}$ field amplitude in x direction $E_{0y}$ field amplitude in y direction $\alpha$ polarisation angle $p$ degree of polarisation $\langle \cdot \rangle$ mean over time																																								
	$Q = \langle E_x^2 \rangle - \langle E_y^2 \rangle \quad (8.90)$																																									
	$= pI \cos 2\chi \cos 2\alpha \quad (8.91)$																																									
	$U = 2\langle E_x E_y \rangle \cos \delta \quad (8.92)$																																									
	$= pI \cos 2\chi \sin 2\alpha \quad (8.93)$																																									
	$V = 2\langle E_x E_y \rangle \sin \delta \quad (8.94)$																																									
	$= pI \sin 2\chi \quad (8.95)$																																									
Degree of polarisation	$p = \frac{(Q^2 + U^2 + V^2)^{1/2}}{I} \leq 1 \quad (8.96)$																																									
	<table border="1"> <thead> <tr> <th></th> <th><math>Q/I</math></th> <th><math>U/I</math></th> <th><math>V/I</math></th> <th></th> <th><math>Q/I</math></th> <th><math>U/I</math></th> <th><math>V/I</math></th> </tr> </thead> <tbody> <tr> <td>left circular</td> <td>0</td> <td>0</td> <td>-1</td> <td>right circular</td> <td>0</td> <td>0</td> <td>1</td> </tr> <tr> <td>linear <math>\parallel</math> x</td> <td>1</td> <td>0</td> <td>0</td> <td>linear <math>\parallel</math> y</td> <td>-1</td> <td>0</td> <td>0</td> </tr> <tr> <td>linear <math>45^\circ</math> to x</td> <td>0</td> <td>1</td> <td>0</td> <td>linear <math>-45^\circ</math> to x</td> <td>0</td> <td>-1</td> <td>0</td> </tr> <tr> <td>unpolarised</td> <td>0</td> <td>0</td> <td>0</td> <td></td> <td></td> <td></td> <td></td> </tr> </tbody> </table>		$Q/I$	$U/I$	$V/I$		$Q/I$	$U/I$	$V/I$	left circular	0	0	-1	right circular	0	0	1	linear $\parallel$ x	1	0	0	linear $\parallel$ y	-1	0	0	linear $45^\circ$ to x	0	1	0	linear $-45^\circ$ to x	0	-1	0	unpolarised	0	0	0					
	$Q/I$	$U/I$	$V/I$		$Q/I$	$U/I$	$V/I$																																			
left circular	0	0	-1	right circular	0	0	1																																			
linear $\parallel$ x	1	0	0	linear $\parallel$ y	-1	0	0																																			
linear $45^\circ$ to x	0	1	0	linear $-45^\circ$ to x	0	-1	0																																			
unpolarised	0	0	0																																							

<sup>a</sup>Using the convention that right-handed circular polarisation corresponds to a clockwise rotation of the electric field in a given plane when looking towards the source. The propagation direction in the diagram is out of the plane. The parameters  $I$ ,  $Q$ ,  $U$ , and  $V$  are sometimes denoted  $s_0$ ,  $s_1$ ,  $s_2$ , and  $s_3$ , and other nomenclatures exist. There is no generally accepted definition – often the parameters are scaled to be dimensionless, with  $s_0 = 1$ , or to represent power flux through a plane  $\perp$  the beam, i.e.,  $I = (\langle E_x^2 \rangle + \langle E_y^2 \rangle)/Z_0$  etc., where  $Z_0$  is the impedance of free space.

<sup>b</sup>The axial ratio is positive for right-handed polarisation and negative for left-handed polarisation using our definitions.



### 8.7 Coherence (scalar theory)

Mutual coherence function	$\Gamma_{12}(\tau) = \langle \psi_1(t)\psi_2^*(t+\tau) \rangle$ (8.97)	$\Gamma_{ij}$ mutual coherence function $\tau$ temporal interval $\psi_i$ (complex) wave disturbance at spatial point $i$
Complex degree of coherence	$\gamma_{12}(\tau) = \frac{\langle \psi_1(t)\psi_2^*(t+\tau) \rangle}{[\langle  \psi_1 ^2 \rangle \langle  \psi_2 ^2 \rangle]^{1/2}}$ (8.98)	$t$ time $\langle \cdot \rangle$ mean over time
	$= \frac{\Gamma_{12}(\tau)}{[\Gamma_{11}(0)\Gamma_{22}(0)]^{1/2}}$ (8.99)	$\gamma_{ij}$ complex degree of coherence * complex conjugate
Combined intensity <sup>a</sup>	$I_{tot} = I_1 + I_2 + 2(I_1 I_2)^{1/2} \Re[\gamma_{12}(\tau)]$ (8.100)	$I_{tot}$ combined intensity $I_i$ intensity of disturbance at point $i$ $\Re$ real part of
Fringe visibility	$V(\tau) = \frac{2(I_1 I_2)^{1/2}}{I_1 + I_2}  \gamma_{12}(\tau) $ (8.101)	
if $ \gamma_{12}(\tau) $ is a constant:	$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}}$ (8.102)	$I_{max}$ max. combined intensity $I_{min}$ min. combined intensity
if $I_1 = I_2$ :	$V(\tau) =  \gamma_{12}(\tau) $ (8.103)	
Complex degree of temporal coherence <sup>b</sup>	$\gamma(\tau) = \frac{\langle \psi_1(t)\psi_1^*(t+\tau) \rangle}{\langle  \psi_1(t) ^2 \rangle}$ (8.104)	$\gamma(\tau)$ degree of temporal coherence
	$= \frac{\int I(\omega) e^{-i\omega\tau} d\omega}{\int I(\omega) d\omega}$ (8.105)	$I(\omega)$ specific intensity $\omega$ radiation angular frequency $c$ speed of light
Coherence time and length	$\Delta\tau_c = \frac{\Delta l_c}{c} \sim \frac{1}{\Delta\nu}$ (8.106)	$\Delta\tau_c$ coherence time $\Delta l_c$ coherence length $\Delta\nu$ spectral bandwidth
Complex degree of spatial coherence <sup>c</sup>	$\gamma(\mathbf{D}) = \frac{\langle \psi_1 \psi_2^* \rangle}{[\langle  \psi_1 ^2 \rangle \langle  \psi_2 ^2 \rangle]^{1/2}}$ (8.107)	$\gamma(\mathbf{D})$ degree of spatial coherence $\mathbf{D}$ spatial separation of points 1 and 2
	$= \frac{\int I(\hat{s}) e^{ik\mathbf{D}\cdot\hat{s}} d\Omega}{\int I(\hat{s}) d\Omega}$ (8.108)	$I(\hat{s})$ specific intensity of distant extended source in direction $\hat{s}$ $d\Omega$ differential solid angle $\hat{s}$ unit vector in the direction of $d\Omega$
Intensity correlation <sup>d</sup>	$\frac{\langle I_1 I_2 \rangle}{[\langle I_1 \rangle^2 \langle I_2 \rangle^2]^{1/2}} = 1 + \gamma^2(\mathbf{D})$ (8.109)	$k$ wavenumber
Speckle intensity distribution <sup>e</sup>	$\text{pr}(I) = \frac{1}{\langle I \rangle} e^{-I/\langle I \rangle}$ (8.110)	$\text{pr}$ probability density
Speckle size (coherence width)	$\Delta w_c \simeq \frac{\lambda}{\alpha}$ (8.111)	$\Delta w_c$ characteristic speckle size $\lambda$ wavelength $\alpha$ source angular size as seen from the screen

<sup>a</sup>From interfering the disturbances at points 1 and 2 with a relative delay  $\tau$ .

<sup>b</sup>Or "autocorrelation function."

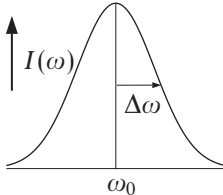
<sup>c</sup>Between two points on a wavefront, separated by  $\mathbf{D}$ . The integral is over the entire extended source.

<sup>d</sup>For wave disturbances that have a Gaussian probability distribution in amplitude. This is "Gaussian light" such as from a thermal source.

<sup>e</sup>Also for Gaussian light.

## 8.8 Line radiation

### Spectral line broadening

Natural broadening <sup>a</sup>	$I(\omega) = \frac{(2\pi\tau)^{-1}}{(2\tau)^{-2} + (\omega - \omega_0)^2} \quad (8.112)$	$I(\omega)$ normalised intensity <sup>b</sup> $\tau$ lifetime of excited state $\omega$ angular frequency ( $= 2\pi\nu$ )
Natural half-width	$\Delta\omega = \frac{1}{2\tau} \quad (8.113)$	$\Delta\omega$ half-width at half-power $\omega_0$ centre frequency
Collision broadening	$I(\omega) = \frac{(\pi\tau_c)^{-1}}{(\tau_c)^{-2} + (\omega - \omega_0)^2} \quad (8.114)$	$\tau_c$ mean time between collisions $p$ pressure $d$ effective atomic diameter
Collision and pressure half-width <sup>c</sup>	$\Delta\omega = \frac{1}{\tau_c} = p\pi d^2 \left( \frac{\pi mkT}{16} \right)^{-1/2} \quad (8.115)$	$m$ gas particle mass $k$ Boltzmann constant $T$ temperature $c$ speed of light
Doppler broadening	$I(\omega) = \left( \frac{mc^2}{2kT\omega_0^2\pi} \right)^{1/2} \exp \left[ -\frac{mc^2}{2kT} \frac{(\omega - \omega_0)^2}{\omega_0^2} \right] \quad (8.116)$	
Doppler half-width	$\Delta\omega = \omega_0 \left( \frac{2kT \ln 2}{mc^2} \right)^{1/2} \quad (8.117)$	

<sup>a</sup>The transition probability per unit time for the state is  $= 1/\tau$ . In the classical limit of a damped oscillator, the e-folding time of the electric field is  $2\tau$ . Both the natural and collision profiles described here are Lorentzian.

<sup>b</sup>The intensity spectra are normalised so that  $\int I(\omega) d\omega = 1$ , assuming  $\Delta\omega/\omega_0 \ll 1$ .

<sup>c</sup>The pressure-broadening relation combines Equations (5.78), (5.86) and (5.89) and assumes an otherwise perfect gas of finite-sized atoms. More accurate expressions are considerably more complicated.

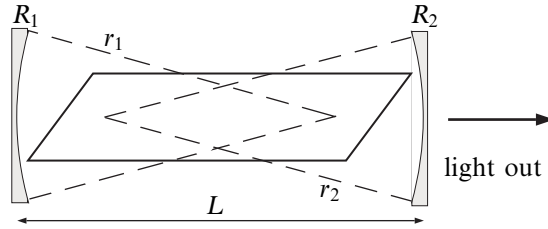
### Einstein coefficients<sup>a</sup>

Absorption	$R_{12} = B_{12}I_\nu n_1 \quad (8.118)$	$R_{ij}$ transition rate, level $i \rightarrow j$ ( $m^{-3} s^{-1}$ ) $B_{ij}$ Einstein $B$ coefficients $I_\nu$ specific intensity of radiation field
Spontaneous emission	$R_{21} = A_{21}n_2 \quad (8.119)$	$A_{21}$ Einstein $A$ coefficient $n_i$ number density of atoms in quantum level $i$ ( $m^{-3}$ )
Stimulated emission	$R'_{21} = B_{21}I_\nu n_2 \quad (8.120)$	
Coefficient ratios	$\frac{A_{21}}{B_{12}} = \frac{2h\nu^3}{c^2} \frac{g_1}{g_2} \quad (8.121)$	$h$ Planck constant $\nu$ frequency
	$\frac{B_{21}}{B_{12}} = \frac{g_1}{g_2} \quad (8.122)$	$c$ speed of light $g_i$ degeneracy of $i$ th level

<sup>a</sup>Note that the coefficients can also be defined in terms of spectral energy density,  $u_\nu = 4\pi I_\nu/c$  rather than  $I_\nu$ . In this case  $\frac{A_{21}}{B_{12}} = \frac{8\pi h\nu^3}{c^3} \frac{g_1}{g_2}$ . See also *Population densities* on page 116.



**Lasers<sup>a</sup>**



Cavity stability condition	$0 \leq \left(1 - \frac{L}{r_1}\right) \left(1 - \frac{L}{r_2}\right) \leq 1$ (8.123)	$r_{1,2}$ radii of curvature of end-mirrors $L$ distance between mirror centres
Longitudinal cavity modes <sup>b</sup>	$\nu_n = \frac{c}{2L}n$ (8.124)	$\nu_n$ mode frequency $n$ integer $c$ speed of light
Cavity $Q$	$Q = \frac{2\pi L(R_1 R_2)^{1/4}}{\lambda[1 - (R_1 R_2)^{1/2}]}$ (8.125)	$Q$ quality factor $R_{1,2}$ mirror (power) reflectances $\lambda$ wavelength
	$\simeq \frac{4\pi L}{\lambda(1 - R_1 R_2)}$ (8.126)	
Cavity line width	$\Delta\nu_c = \frac{\nu_n}{Q} = 1/(2\pi\tau_c)$ (8.127)	$\Delta\nu_c$ cavity line width (FWHP) $\tau_c$ cavity photon lifetime
Schawlow–Townes line width	$\frac{\Delta\nu}{\nu_n} = \frac{2\pi h(\Delta\nu_c)^2}{P} \left( \frac{g_u N_u}{g_l N_l - g_u N_l} \right)$ (8.128)	$\Delta\nu$ line width (FWHP) $P$ laser power $g_{u,l}$ degeneracy of upper/lower levels $N_{u,l}$ number density of upper/lower levels
Threshold lasing condition	$R_1 R_2 \exp[2(\alpha - \beta)L] > 1$ (8.129)	$\alpha$ gain per unit length of medium $\beta$ loss per unit length of medium

<sup>a</sup>Also see the *Fabry-Perot etalon* on page 163. Note that “cavity” refers to the empty cavity, with no lasing medium present.

<sup>b</sup>The mode spacing equals the cavity free spectral range.



# Chapter 9 Astrophysics

## 9.1 Introduction

Many of the formulas associated with astronomy and astrophysics are either too specialised for a general work such as this or are common to other fields and can therefore be found elsewhere in this book. The following section includes many of the relationships that fall into neither of these categories, including equations to convert between various astronomical coordinate systems and some basic formulas associated with cosmology.

Exceptionally, this section also includes data on the Sun, Earth, Moon, and planets. Observational astrophysics remains a largely inexact science, and parameters of these (and other) bodies are often used as approximate base units in measurements. For example, the masses of stars and galaxies are frequently quoted as multiples of the mass of the Sun ( $1M_{\odot} = 1.989 \times 10^{30}$  kg), extra-solar system planets in terms of the mass of Jupiter, and so on. Astronomers seem to find it particularly difficult to drop arcane units and conventions, resulting in a profusion of measures and nomenclatures throughout the subject. However, the convention of using suitable astronomical objects in this way is both useful and widely accepted.

## 9.2 Solar system data

### Solar data

equatorial radius	$R_{\odot}$	=	$6.960 \times 10^8 \text{ m}$	=	$109.1 R_{\oplus}$
mass	$M_{\odot}$	=	$1.9891 \times 10^{30} \text{ kg}$	=	$3.32946 \times 10^5 M_{\oplus}$
polar moment of inertia	$I_{\odot}$	=	$5.7 \times 10^{46} \text{ kg m}^2$	=	$7.09 \times 10^8 I_{\oplus}$
bolometric luminosity	$L_{\odot}$	=	$3.826 \times 10^{26} \text{ W}$		
effective surface temperature	$T_{\odot}$	=	5770 K		
solar constant <sup>a</sup>			$1.368 \times 10^3 \text{ W m}^{-2}$		
absolute magnitude	$M_V$	=	+4.83;	$M_{\text{bol}}$	= +4.75
apparent magnitude	$m_V$	=	-26.74;	$m_{\text{bol}}$	= -26.82

<sup>a</sup>Bolometric flux at a distance of 1 astronomical unit (AU).

### Earth data

equatorial radius	$R_{\oplus}$	=	$6.37814 \times 10^6 \text{ m}$	=	$9.166 \times 10^{-3} R_{\odot}$
flattening <sup>a</sup>	$f$	=	0.00335364	=	1/298.183
mass	$M_{\oplus}$	=	$5.9742 \times 10^{24} \text{ kg}$	=	$3.0035 \times 10^{-6} M_{\odot}$
polar moment of inertia	$I_{\oplus}$	=	$8.037 \times 10^{37} \text{ kg m}^2$	=	$1.41 \times 10^{-9} I_{\odot}$
orbital semi-major axis <sup>b</sup>	1 AU	=	$1.495979 \times 10^{11} \text{ m}$	=	$214.9 R_{\odot}$
mean orbital velocity			$2.979 \times 10^4 \text{ m s}^{-1}$		
equatorial surface gravity	$g_e$	=	$9.780327 \text{ m s}^{-2}$	(includes rotation)	
polar surface gravity	$g_p$	=	$9.832186 \text{ m s}^{-2}$		
rotational angular velocity	$\omega_e$	=	$7.292115 \times 10^{-5} \text{ rad s}^{-1}$		

<sup>a</sup> $f$  equals  $(R_{\oplus} - R_{\text{polar}})/R_{\oplus}$ . The mean radius of the Earth is  $6.3710 \times 10^6 \text{ m}$ .

<sup>b</sup>About the Sun.

### Moon data

equatorial radius	$R_m$	=	$1.7374 \times 10^6 \text{ m}$	=	$0.27240 R_{\oplus}$
mass	$M_m$	=	$7.3483 \times 10^{22} \text{ kg}$	=	$1.230 \times 10^{-2} M_{\oplus}$
mean orbital radius <sup>a</sup>	$a_m$	=	$3.84400 \times 10^8 \text{ m}$	=	$60.27 R_{\oplus}$
mean orbital velocity			$1.03 \times 10^3 \text{ m s}^{-1}$		
orbital period (sidereal)			27.32166 d		
equatorial surface gravity			$1.62 \text{ m s}^{-2}$	=	$0.166 g_e$

<sup>a</sup>About the Earth.

### Planetary data<sup>a</sup>

	$M/M_{\oplus}$	$R/R_{\oplus}$	$T(\text{d})$	$P(\text{yr})$	$a(\text{AU})$	$M$	mass
Mercury	0.055 274	0.382 51	58.646	0.240 85	0.387 10	$R$	equatorial radius
Venus <sup>b</sup>	0.815 00	0.948 83	243.018	0.615 228	0.723 35	$T$	rotational period
Earth	1	1	0.997 27	1.000 04	1.000 00	$P$	orbital period
Mars	0.107 45	0.532 60	1.025 96	1.880 93	1.523 71	$a$	mean distance
Jupiter	317.85	11.209	0.413 54	11.861 3	5.202 53	$M_{\oplus}$	$5.9742 \times 10^{24} \text{ kg}$
Saturn	95.159	9.449 1	0.444 01	29.628 2	9.575 60	$R_{\oplus}$	$6.37814 \times 10^6 \text{ m}$
Uranus <sup>b</sup>	14.500	4.007 3	0.718 33	84.746 6	19.293 4	1d	86400 s
Neptune	17.204	3.882 6	0.671 25	166.344	30.245 9	1yr	$3.15569 \times 10^7 \text{ s}$
Pluto <sup>b</sup>	0.002 51	0.187 36	6.387 2	248.348	39.509 0	1AU	$1.495979 \times 10^{11} \text{ m}$

<sup>a</sup>Using the osculating orbital elements for 1998. Note that  $P$  is the instantaneous orbital period, calculated from the planet's daily motion. The radii of gas giants are taken at 1 atmosphere pressure.

<sup>b</sup>Retrograde rotation.

### 9.3 Coordinate transformations (astronomical)

#### Time in astronomy

Julian day number <sup>a</sup> $JD = D - 32075 + 1461 * (Y + 4800 + (M - 14)/12)/4 + 367 * (M - 2 - (M - 14)/12 * 12)/12 - 3 * ((Y + 4900 + (M - 14)/12)/100)/4$ (9.1)	<i>JD</i> Julian day number <i>D</i> day of month number <i>Y</i> calendar year, e.g., 1963 <i>M</i> calendar month (Jan=1) * integer multiply / integer divide <i>MJD</i> modified Julian day number
Modified Julian day number $MJD = JD - 2400000.5$ (9.2)	
Day of week $W = (JD + 1) \text{ mod } 7$ (9.3)	<i>W</i> day of week (0=Sunday, 1=Monday, ... )
Local civil time $LCT = UTC + TZC + DSC$ (9.4)	LCT local civil time UTC coordinated universal time TZC time zone correction DSC daylight saving correction
Julian centuries $T = \frac{JD - 2451545.5}{36525}$ (9.5)	<i>T</i> Julian centuries between 12 <sup>h</sup> UTC 1 Jan 2000 and 0 <sup>h</sup> UTC <i>D/M/Y</i>
Greenwich sidereal time $GMST = 6^h 41^m 50^s.54841 + 8640184^s.812866T + 0^s.093104T^2 - 0^s.0000062T^3$ (9.6)	GMST Greenwich mean sidereal time at 0 <sup>h</sup> UTC <i>D/M/Y</i> (for later times use 1s=1.002738 sidereal seconds)
Local sidereal time $LST = GMST + \frac{\lambda^\circ}{15^\circ}$ (9.7)	LST local sidereal time $\lambda^\circ$ geographic longitude, degrees east of Greenwich

<sup>a</sup>For the Julian day starting at noon on the calendar day in question. The routine is designed around integer arithmetic with “truncation towards zero” (so that  $-5/3 = -1$ ) and is valid for dates from the onset of the Gregorian calendar, 15 October 1582. *JD* represents the number of days since Greenwich mean noon 1 Jan 4713 BC. For reference, noon, 1 Jan 2000 = *JD*2451545 and was a Saturday (*W* = 6).

#### Horizon coordinates<sup>a</sup>

Hour angle $H = LST - \alpha$ (9.8)	LST local sidereal time <i>H</i> (local) hour angle
Equatorial to horizon $\sin a = \sin \delta \sin \phi + \cos \delta \cos \phi \cos H$ (9.9)	$\alpha$ right ascension $\delta$ declination $a$ altitude $A$ azimuth (E from N) $\phi$ observer's latitude
$\tan A \equiv \frac{-\cos \delta \sin H}{\sin \delta \cos \phi - \sin \phi \cos \delta \cos H}$ (9.10)	
Horizon to equatorial $\sin \delta = \sin a \sin \phi + \cos a \cos \phi \cos A$ (9.11)	
$\tan H \equiv \frac{-\cos a \sin A}{\sin a \cos \phi - \sin \phi \cos a \cos A}$ (9.12)	

<sup>a</sup>Conversions between horizon or alt-azimuth coordinates, (*a, A*), and celestial equatorial coordinates, ( $\delta, \alpha$ ). There are a number of conventions for defining azimuth. For example, it is sometimes taken as the angle west from south rather than east from north. The quadrants for *A* and *H* can be obtained from the signs of the numerators and denominators in Equations (9.10) and (9.12) (see diagram).



**Ecliptic coordinates<sup>a</sup>**

Obliquity of the ecliptic	$\varepsilon = 23^\circ 26' 21''.45 - 46''.815 T$ $- 0''.0006 T^2$ $+ 0''.00181 T^3$	(9.13)	$\varepsilon$ mean ecliptic obliquity $T$ Julian centuries since J2000.0 <sup>b</sup>
Equatorial to ecliptic	$\sin \beta = \sin \delta \cos \varepsilon - \cos \delta \sin \varepsilon \sin \alpha$ $\tan \lambda \equiv \frac{\sin \alpha \cos \varepsilon + \tan \delta \sin \varepsilon}{\cos \alpha}$	(9.14) (9.15)	$\alpha$ right ascension $\delta$ declination $\lambda$ ecliptic longitude $\beta$ ecliptic latitude
Ecliptic to equatorial	$\sin \delta = \sin \beta \cos \varepsilon + \cos \beta \sin \varepsilon \sin \lambda$ $\tan \alpha \equiv \frac{\sin \lambda \cos \varepsilon - \tan \beta \sin \varepsilon}{\cos \lambda}$	(9.16) (9.17)	

<sup>a</sup>Conversions between ecliptic, ( $\beta, \lambda$ ), and celestial equatorial, ( $\delta, \alpha$ ), coordinates.  $\beta$  is positive above the ecliptic and  $\lambda$  increases eastwards. The quadrants for  $\lambda$  and  $\alpha$  can be obtained from the signs of the numerators and denominators in Equations (9.15) and (9.17) (see diagram).

<sup>b</sup>See Equation (9.5).

**Galactic coordinates<sup>a</sup>**

Galactic frame	$\alpha_g = 192^\circ 15'$ $\delta_g = 27^\circ 24'$ $l_g = 33^\circ$	(9.18) (9.19) (9.20)	$\alpha_g$ right ascension of north galactic pole $\delta_g$ declination of north galactic pole
Equatorial to galactic	$\sin b = \cos \delta \cos \delta_g \cos(\alpha - \alpha_g) + \sin \delta \sin \delta_g$ $\tan(l - l_g) \equiv \frac{\tan \delta \cos \delta_g - \cos(\alpha - \alpha_g) \sin \delta_g}{\sin(\alpha - \alpha_g)}$	(9.21) (9.22)	$l_g$ ascending node of galactic plane on equator
Galactic to equatorial	$\sin \delta = \cos b \cos \delta_g \sin(l - l_g) + \sin b \sin \delta_g$ $\tan(\alpha - \alpha_g) \equiv \frac{\cos(l - l_g)}{\tan b \cos \delta_g - \sin \delta_g \sin(l - l_g)}$	(9.23) (9.24)	$\delta$ declination $\alpha$ right ascension $b$ galactic latitude $l$ galactic longitude

<sup>a</sup>Conversions between galactic, ( $b, l$ ), and celestial equatorial, ( $\delta, \alpha$ ), coordinates. The galactic frame is defined at epoch B1950.0. The quadrants of  $l$  and  $\alpha$  can be obtained from the signs of the numerators and denominators in Equations (9.22) and (9.24).

**Precession of equinoxes<sup>a</sup>**

In right ascension	$\alpha \simeq \alpha_0 + (3^s.075 + 1^s.336 \sin \alpha_0 \tan \delta_0) N$	(9.25)	$\alpha$ right ascension of date $\alpha_0$ right ascension at J2000.0 $N$ number of years since J2000.0
In declination	$\delta \simeq \delta_0 + (20''.043 \cos \alpha_0) N$	(9.26)	$\delta$ declination of date $\delta_0$ declination at J2000.0

<sup>a</sup>Right ascension in hours, minutes, and seconds; declination in degrees, arcminutes, and arcseconds. These equations are valid for several hundred years each side of J2000.0.

## 9.4 Observational astrophysics

### Astronomical magnitudes

Apparent magnitude	$m_1 - m_2 = -2.5 \log_{10} \frac{F_1}{F_2}$ (9.27)	$m_i$ apparent magnitude of object $i$ $F_i$ energy flux from object $i$
Distance modulus <sup>a</sup>	$m - M = 5 \log_{10} D - 5$ (9.28)	$M$ absolute magnitude
	$= -5 \log_{10} p - 5$ (9.29)	$m - M$ distance modulus $D$ distance to object (parsec) $p$ annual parallax (arcsec)
Luminosity–magnitude relation	$M_{\text{bol}} = 4.75 - 2.5 \log_{10} \frac{L}{L_{\odot}}$ (9.30)	$M_{\text{bol}}$ bolometric absolute magnitude
	$L \simeq 3.04 \times 10^{(28 - 0.4 M_{\text{bol}})}$ (9.31)	$L$ luminosity (W) $L_{\odot}$ solar luminosity ( $3.826 \times 10^{26}$ W)
Flux–magnitude relation	$F_{\text{bol}} \simeq 2.559 \times 10^{-(8 + 0.4 m_{\text{bol}})}$ (9.32)	$F_{\text{bol}}$ bolometric flux ( $\text{W m}^{-2}$ ) $m_{\text{bol}}$ bolometric apparent magnitude
Bolometric correction	$BC = m_{\text{bol}} - m_V$ (9.33)	$BC$ bolometric correction
	$= M_{\text{bol}} - M_V$ (9.34)	$m_V$ $V$ -band apparent magnitude $M_V$ $V$ -band absolute magnitude
Colour index <sup>b</sup>	$B - V = m_B - m_V$ (9.35)	$B - V$ observed $B - V$ colour index
	$U - B = m_U - m_B$ (9.36)	$U - B$ observed $U - B$ colour index
Colour excess <sup>c</sup>	$E = (B - V) - (B - V)_0$ (9.37)	$E$ $B - V$ colour excess $(B - V)_0$ intrinsic $B - V$ colour index

<sup>a</sup>Neglecting extinction.

<sup>b</sup>Using the  $UBV$  magnitude system. The bands are centred around 365 nm ( $U$ ), 440 nm ( $B$ ), and 550 nm ( $V$ ).

<sup>c</sup>The  $U - B$  colour excess is defined similarly.

### Photometric wavelengths

Mean wavelength	$\lambda_0 = \frac{\int \lambda R(\lambda) d\lambda}{\int R(\lambda) d\lambda}$ (9.38)	$\lambda_0$ mean wavelength $\lambda$ wavelength $R$ system spectral response
Isophotal wavelength	$F(\lambda_i) = \frac{\int F(\lambda) R(\lambda) d\lambda}{\int R(\lambda) d\lambda}$ (9.39)	$F(\lambda)$ flux density of source (in terms of wavelength) $\lambda_i$ isophotal wavelength
Effective wavelength	$\lambda_{\text{eff}} = \frac{\int \lambda F(\lambda) R(\lambda) d\lambda}{\int F(\lambda) R(\lambda) d\lambda}$ (9.40)	$\lambda_{\text{eff}}$ effective wavelength



## Planetary bodies

Bode's law <sup>a</sup>	$D_{\text{AU}} = \frac{4 + 3 \times 2^n}{10}$	(9.41)	$D_{\text{AU}}$	planetary orbital radius (AU)
			$n$	index: Mercury = $-\infty$ , Venus = 0, Earth = 1, Mars = 2, Ceres = 3, Jupiter = 4, ...
Roche limit	$R \gtrsim \left( \frac{100M}{9\pi\rho} \right)^{1/3}$	(9.42)	$R$	satellite orbital radius
	$\gtrsim 2.46R_0$ (if densities equal)	(9.43)	$M$	central mass
			$\rho$	satellite density
			$R_0$	central body radius
Synodic period <sup>b</sup>	$\frac{1}{S} = \left  \frac{1}{P} - \frac{1}{P_{\oplus}} \right $	(9.44)	$S$	synodic period
			$P$	planetary orbital period
			$P_{\oplus}$	Earth's orbital period

<sup>a</sup>Also known as the "Titius–Bode rule." Note that the asteroid Ceres is counted as a planet in this scheme. The relationship breaks down for Neptune and Pluto.

<sup>b</sup>Of a planet.

## Distance indicators

Hubble law	$v = H_0 d$	(9.45)	$v$	cosmological recession velocity
			$H_0$	Hubble parameter (present epoch)
			$d$	(proper) distance
Annual parallax	$D_{\text{pc}} = p^{-1}$	(9.46)	$D_{\text{pc}}$	distance (parsec)
			$p$	annual parallax ( $\pm p$ arcsec from mean)
Cepheid variables <sup>a</sup>	$\log_{10} \frac{\langle L \rangle}{L_{\odot}} \simeq 1.15 \log_{10} P_d + 2.47$	(9.47)	$\langle L \rangle$	mean cepheid luminosity
	$M_V \simeq -2.76 \log_{10} P_d - 1.40$	(9.48)	$L_{\odot}$	Solar luminosity
			$P_d$	pulsation period (days)
			$M_V$	absolute visual magnitude
Tully–Fisher relation <sup>b</sup>	$M_I \simeq -7.68 \log_{10} \left( \frac{2v_{\text{rot}}}{\sin i} \right) - 2.58$	(9.49)	$M_I$	$I$ -band absolute magnitude
			$v_{\text{rot}}$	observed maximum rotation velocity ( $\text{kms}^{-1}$ )
			$i$	galactic inclination ( $90^\circ$ when edge-on)
Einstein rings	$\theta^2 = \frac{4GM}{c^2} \left( \frac{d_s - d_l}{d_s d_l} \right)$	(9.50)	$\theta$	ring angular radius
			$M$	lens mass
			$d_s$	distance from observer to source
			$d_l$	distance from observer to lens
Sunyaev–Zel'dovich effect <sup>c</sup>	$\frac{\Delta T}{T} = -2 \int \frac{n_e k T_e \sigma_T}{m_e c^2} dl$	(9.51)	$T$	apparent CMBR temperature
			$dl$	path element through cloud
			$R$	cloud radius
			$n_e$	electron number density
			$k$	Boltzmann constant
... for a homogeneous sphere	$\frac{\Delta T}{T} = -\frac{4R n_e k T_e \sigma_T}{m_e c^2}$	(9.52)	$T_e$	electron temperature
			$\sigma_T$	Thomson cross section
			$m_e$	electron mass
			$c$	speed of light

<sup>a</sup>Period–luminosity relation for classical Cepheids. Uncertainty in  $M_V$  is  $\pm 0.27$  (Madore & Freedman, 1991, Publications of the Astronomical Society of the Pacific, **103**, 933).

<sup>b</sup>Galaxy rotation velocity–magnitude relation in the infrared  $I$  waveband, centred at  $0.90 \mu\text{m}$ . The coefficients depend on waveband and galaxy type (see Giovanelli *et al.*, 1997, The Astronomical Journal, **113**, 1).

<sup>c</sup>Scattering of the cosmic microwave background radiation (CMBR) by a cloud of electrons, seen as a temperature decrement,  $\Delta T$ , in the Rayleigh–Jeans limit ( $\lambda \gg 1 \text{mm}$ ).

## 9.5 Stellar evolution

### Evolutionary timescales

Free-fall timescale <sup>a</sup>	$\tau_{\text{ff}} = \left( \frac{3\pi}{32G\rho_0} \right)^{1/2} \quad (9.53)$	$\tau_{\text{ff}}$ free-fall timescale $G$ constant of gravitation $\rho_0$ initial mass density
Kelvin–Helmholtz timescale	$\tau_{\text{KH}} = \frac{-U_g}{L} \quad (9.54)$	$\tau_{\text{KH}}$ Kelvin–Helmholtz timescale $U_g$ gravitational potential energy
	$\simeq \frac{GM^2}{R_0L} \quad (9.55)$	$M$ body's mass $R_0$ body's initial radius $L$ body's luminosity

<sup>a</sup>For the gravitational collapse of a uniform sphere.

### Star formation

Jeans length <sup>a</sup>	$\lambda_J = \left( \frac{\pi}{G\rho} \frac{dp}{d\rho} \right)^{1/2} \quad (9.56)$	$\lambda_J$ Jeans length $G$ constant of gravitation $\rho$ cloud mass density $p$ pressure
Jeans mass	$M_J = \frac{\pi}{6} \rho \lambda_J^3 \quad (9.57)$	$M_J$ (spherical) Jeans mass
Eddington limiting luminosity <sup>b</sup>	$L_E = \frac{4\pi GMm_p c}{\sigma_T} \quad (9.58)$	$L_E$ Eddington luminosity $M$ stellar mass $M_\odot$ solar mass
	$\simeq 1.26 \times 10^{31} \frac{M}{M_\odot} \text{ W} \quad (9.59)$	$m_p$ proton mass $c$ speed of light $\sigma_T$ Thomson cross section

<sup>a</sup>Note that  $(dp/d\rho)^{1/2}$  is the sound speed in the cloud.

<sup>b</sup>Assuming the opacity is mostly from Thomson scattering.

### Stellar theory<sup>a</sup>

Conservation of mass	$\frac{dM_r}{dr} = 4\pi\rho r^2 \quad (9.60)$	$r$ radial distance $M_r$ mass interior to $r$ $\rho$ mass density
Hydrostatic equilibrium	$\frac{dp}{dr} = \frac{-G\rho M_r}{r^2} \quad (9.61)$	$p$ pressure $G$ constant of gravitation
Energy release	$\frac{dL_r}{dr} = 4\pi\rho r^2 \epsilon \quad (9.62)$	$L_r$ luminosity interior to $r$ $\epsilon$ power generated per unit mass
Radiative transport	$\frac{dT}{dr} = \frac{-3}{16\sigma} \frac{\langle\kappa\rangle\rho}{T^3} \frac{L_r}{4\pi r^2} \quad (9.63)$	$T$ temperature $\sigma$ Stefan–Boltzmann constant $\langle\kappa\rangle$ mean opacity
Convective transport	$\frac{dT}{dr} = \frac{\gamma-1}{\gamma} \frac{T}{p} \frac{dp}{dr} \quad (9.64)$	$\gamma$ ratio of heat capacities, $c_p/c_V$

<sup>a</sup>For stars in static equilibrium with adiabatic convection. Note that  $\rho$  is a function of  $r$ .  $\kappa$  and  $\epsilon$  are functions of temperature and composition.

Stellar fusion processes<sup>a</sup>

PP I chain	PP II chain	PP III chain
$p^+ + p^+ \rightarrow {}^2_1\text{H} + e^+ + \nu_e$ ${}^2_1\text{H} + p^+ \rightarrow {}^3_2\text{He} + \gamma$ ${}^3_2\text{He} + {}^3_2\text{He} \rightarrow {}^4_2\text{He} + 2p^+$	$p^+ + p^+ \rightarrow {}^2_1\text{H} + e^+ + \nu_e$ ${}^2_1\text{H} + p^+ \rightarrow {}^3_2\text{He} + \gamma$ ${}^3_2\text{He} + {}^4_2\text{He} \rightarrow {}^7_4\text{Be} + \gamma$ ${}^7_4\text{Be} + e^- \rightarrow {}^7_3\text{Li} + \nu_e$ ${}^7_3\text{Li} + p^+ \rightarrow 2{}^4_2\text{He}$	$p^+ + p^+ \rightarrow {}^2_1\text{H} + e^+ + \nu_e$ ${}^2_1\text{H} + p^+ \rightarrow {}^3_2\text{He} + \gamma$ ${}^3_2\text{He} + {}^4_2\text{He} \rightarrow {}^7_4\text{Be} + \gamma$ ${}^7_4\text{Be} + p^+ \rightarrow {}^8_5\text{B} + \gamma$ ${}^8_5\text{B} \rightarrow {}^8_4\text{Be} + e^+ + \nu_e$ ${}^8_4\text{Be} \rightarrow 2{}^4_2\text{He}$
CNO cycle	triple- $\alpha$ process	
${}^{12}_6\text{C} + p^+ \rightarrow {}^{13}_7\text{N} + \gamma$ ${}^{13}_7\text{N} \rightarrow {}^{13}_6\text{C} + e^+ + \nu_e$ ${}^{13}_6\text{C} + p^+ \rightarrow {}^{14}_7\text{N} + \gamma$ ${}^{14}_7\text{N} + p^+ \rightarrow {}^{15}_8\text{O} + \gamma$ ${}^{15}_8\text{O} \rightarrow {}^{15}_7\text{N} + e^+ + \nu_e$ ${}^{15}_7\text{N} + p^+ \rightarrow {}^{12}_6\text{C} + {}^4_2\text{He}$	$4{}^4_2\text{He} + {}^4_2\text{He} \rightleftharpoons {}^8_4\text{Be} + \gamma$ ${}^8_4\text{Be} + {}^4_2\text{He} \rightleftharpoons {}^{12}_6\text{C}^*$ ${}^{12}_6\text{C}^* \rightarrow {}^{12}_6\text{C} + \gamma$	$\gamma$ photon $p^+$ proton $e^+$ positron $e^-$ electron $\nu_e$ electron neutrino

<sup>a</sup>All species are taken as fully ionised.

## Pulsars

Braking index	$\dot{\omega} \propto -\omega^n$ (9.65)	$\omega$ rotational angular velocity
	$n = 2 - \frac{P\ddot{P}}{\dot{P}^2}$ (9.66)	$P$ rotational period ( $= 2\pi/\omega$ )
		$n$ braking index
Characteristic age <sup>a</sup>	$T = \frac{1}{n-1} \frac{P}{\dot{P}}$ (9.67)	$T$ characteristic age
		$L$ luminosity
		$\mu_0$ permeability of free space
		$c$ speed of light
Magnetic dipole radiation	$L = \frac{\mu_0  \ddot{m} ^2 \sin^2 \theta}{6\pi c^3}$ (9.68)	$m$ pulsar magnetic dipole moment
	$= \frac{2\pi R^6 B_p^2 \omega^4 \sin^2 \theta}{3c^3 \mu_0}$ (9.69)	$R$ pulsar radius
		$B_p$ magnetic flux density at magnetic pole
		$\theta$ angle between magnetic and rotational axes
Dispersion measure	$\text{DM} = \int_0^D n_e dl$ (9.70)	DM dispersion measure
		$D$ path length to pulsar
		$dl$ path element
		$n_e$ electron number density
Dispersion <sup>b</sup>	$\frac{d\tau}{dv} = \frac{-e^2}{4\pi^2 \epsilon_0 m_e c v^3} \text{DM}$ (9.71)	$\tau$ pulse arrival time
	$\Delta\tau = \frac{e^2}{8\pi^2 \epsilon_0 m_e c} \left( \frac{1}{v_1^2} - \frac{1}{v_2^2} \right) \text{DM}$ (9.72)	$\Delta\tau$ difference in pulse arrival time
		$v_i$ observing frequencies
		$m_e$ electron mass

<sup>a</sup>Assuming  $n \neq 1$  and that the pulsar has already slowed significantly. Usually  $n$  is assumed to be 3 (magnetic dipole radiation), giving  $T = P/(2\dot{P})$ .

<sup>b</sup>The pulse arrives first at the higher observing frequency.



**Compact objects and black holes**

Schwarzschild radius	$r_s = \frac{2GM}{c^2} \simeq 3 \frac{M}{M_\odot} \text{ km}$	(9.73)	$r_s$ Schwarzschild radius $G$ constant of gravitation $M$ mass of body $c$ speed of light $M_\odot$ solar mass
Gravitational redshift	$\frac{v_\infty}{v_r} = \left(1 - \frac{2GM}{rc^2}\right)^{1/2}$	(9.74)	$r$ distance from mass centre $v_\infty$ frequency at infinity $v_r$ frequency at $r$
Gravitational wave radiation <sup>a</sup>	$L_g = \frac{32 G^4 m_1^2 m_2^2 (m_1 + m_2)}{5 c^5 a^5}$	(9.75)	$m_i$ orbiting masses $a$ mass separation $L_g$ gravitational luminosity
Rate of change of orbital period	$\dot{P} = -\frac{96}{5} (4\pi^2)^{4/3} \frac{G^{5/3} m_1 m_2 P^{-5/3}}{c^5 (m_1 + m_2)^{1/3}}$	(9.76)	$P$ orbital period
Neutron star degeneracy pressure (nonrelativistic)	$p = \frac{(3\pi^2)^{2/3} \hbar^2}{5 m_n} \left(\frac{\rho}{m_n}\right)^{5/3} = \frac{2}{3} u$	(9.77)	$p$ pressure $\hbar$ (Planck constant)/(2π) $m_n$ neutron mass $\rho$ density
Relativistic <sup>b</sup>	$p = \frac{\hbar c (3\pi^2)^{1/3}}{4} \left(\frac{\rho}{m_n}\right)^{4/3} = \frac{1}{3} u$	(9.78)	$u$ energy density
Chandrasekhar mass <sup>c</sup>	$M_{\text{Ch}} \simeq 1.46 M_\odot$	(9.79)	$M_{\text{Ch}}$ Chandrasekhar mass
Maximum black hole angular momentum	$J_m = \frac{GM^2}{c}$	(9.80)	$J_m$ maximum angular momentum
Black hole evaporation time	$\tau_e \sim \frac{M^3}{M_\odot^3} \times 10^{66} \text{ yr}$	(9.81)	$\tau_e$ evaporation time
Black hole temperature	$T = \frac{\hbar c^3}{8\pi G M k} \simeq 10^{-7} \frac{M_\odot}{M} \text{ K}$	(9.82)	$T$ temperature $k$ Boltzmann constant

<sup>a</sup>From two bodies,  $m_1$  and  $m_2$ , in circular orbits about their centre of mass. Note that the frequency of the radiation is twice the orbital frequency.

<sup>b</sup>Particle velocities  $\sim c$ .

<sup>c</sup>Upper limit to mass of a white dwarf.

## 9.6 Cosmology

### Cosmological model parameters

Hubble law	$v_r = Hd$	(9.83)	$v_r$ radial velocity $H$ Hubble parameter $d$ proper distance
Hubble parameter <sup>a</sup>	$H(t) = \frac{\dot{R}(t)}{R(t)}$	(9.84)	$0$ present epoch $R$ cosmic scale factor $t$ cosmic time $z$ redshift
	$H(z) = H_0 [\Omega_{m0}(1+z)^3 + \Omega_{\Lambda 0} + (1 - \Omega_{m0} - \Omega_{\Lambda 0})(1+z)^2]^{1/2}$	(9.85)	
Redshift	$z = \frac{\lambda_{\text{obs}} - \lambda_{\text{em}}}{\lambda_{\text{em}}} = \frac{R_0}{R(t_{\text{em}})} - 1$	(9.86)	$\lambda_{\text{obs}}$ observed wavelength $\lambda_{\text{em}}$ emitted wavelength $t_{\text{em}}$ epoch of emission
Robertson–Walker metric <sup>b</sup>	$ds^2 = c^2 dt^2 - R^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right]$	(9.87)	$ds$ interval $c$ speed of light $r, \theta, \phi$ comoving spherical polar coordinates
Friedmann equations <sup>c</sup>	$\ddot{R} = -\frac{4\pi}{3}GR \left( \rho + 3\frac{p}{c^2} \right) + \frac{\Lambda R}{3}$	(9.88)	$k$ curvature parameter $G$ constant of gravitation
	$\dot{R}^2 = \frac{8\pi}{3}G\rho R^2 - kc^2 + \frac{\Lambda R^2}{3}$	(9.89)	$p$ pressure $\Lambda$ cosmological constant
Critical density	$\rho_{\text{crit}} = \frac{3H^2}{8\pi G}$	(9.90)	$\rho$ (mass) density $\rho_{\text{crit}}$ critical density
Density parameters	$\Omega_m = \frac{\rho}{\rho_{\text{crit}}} = \frac{8\pi G\rho}{3H^2}$	(9.91)	
	$\Omega_\Lambda = \frac{\Lambda}{3H^2}$	(9.92)	$\Omega_m$ matter density parameter $\Omega_\Lambda$ lambda density parameter
	$\Omega_k = -\frac{kc^2}{R^2H^2}$	(9.93)	$\Omega_k$ curvature density parameter
	$\Omega_m + \Omega_\Lambda + \Omega_k = 1$	(9.94)	
Deceleration parameter	$q_0 = -\frac{R_0\ddot{R}_0}{\dot{R}_0^2} = \frac{\Omega_{m0}}{2} - \Omega_{\Lambda 0}$	(9.95)	$q_0$ deceleration parameter

<sup>a</sup>Often called the Hubble “constant.” At the present epoch,  $60 \lesssim H_0 \lesssim 80 \text{ km s}^{-1} \text{ Mpc}^{-1} \equiv 100 h \text{ km s}^{-1} \text{ Mpc}^{-1}$ , where  $h$  is a dimensionless scaling parameter. The Hubble time is  $t_H = 1/H_0$ . Equation (9.85) assumes a matter dominated universe and mass conservation.

<sup>b</sup>For a homogeneous, isotropic universe, using the  $(-1, 1, 1, 1)$  metric signature.  $r$  is scaled so that  $k = 0, \pm 1$ . Note that  $ds^2 \equiv (ds)^2$  etc.

<sup>c</sup> $\Lambda = 0$  in a Friedmann universe. Note that the cosmological constant is sometimes defined as equalling the value used here divided by  $c^2$ .

**Cosmological distance measures**

Look-back time	$t_{\text{lb}}(z) = t_0 - t(z)$	(9.96)	$t_{\text{lb}}(z)$ light travel time from an object at redshift $z$ $t_0$ present cosmic time $t(z)$ cosmic time at $z$
Proper distance	$d_p = R_0 \int_0^r \frac{dr}{(1-kr^2)^{1/2}} = cR_0 \int_t^{t_0} \frac{dt}{R(t)}$	(9.97)	$d_p$ proper distance $R$ cosmic scale factor $c$ speed of light $t_0$ present epoch
Luminosity distance <sup>a</sup>	$d_L = d_p(1+z) = c(1+z) \int_0^z \frac{dz}{H(z)}$	(9.98)	$d_L$ luminosity distance $z$ redshift $H$ Hubble parameter <sup>b</sup>
Flux density–redshift relation	$F(\nu) = \frac{L(\nu')}{4\pi d_L^2(z)}$ where $\nu' = (1+z)\nu$	(9.99)	$F$ spectral flux density $\nu$ frequency $L(\nu)$ spectral luminosity <sup>c</sup>
Angular diameter distance <sup>d</sup>	$d_a = d_L(1+z)^{-2}$	(9.100)	$d_a$ angular diameter distance $k$ curvature parameter

<sup>a</sup>Assuming a flat universe ( $k=0$ ). The apparent flux density of a source varies as  $d_L^{-2}$ .

<sup>b</sup>See Equation (9.85).

<sup>c</sup>Defined as the output power of the body per unit frequency interval.

<sup>d</sup>True for all  $k$ . The angular diameter of a source varies as  $d_a^{-1}$ .

**Cosmological models<sup>a</sup>**

	$d_p = \frac{2c}{H_0} [1 - (1+z)^{-1/2}]$	(9.101)	$d_p$ proper distance
Einstein – de Sitter model ( $\Omega_k = 0$ , $\Lambda = 0$ , $p = 0$ and $\Omega_{m0} = 1$ )	$H(z) = H_0(1+z)^{3/2}$	(9.102)	$H$ Hubble parameter
	$q_0 = 1/2$	(9.103)	$t_0$ present epoch
	$t(z) = \frac{2}{3H(z)}$	(9.104)	$z$ redshift
	$\rho = (6\pi Gt^2)^{-1}$	(9.105)	$c$ speed of light
	$R(t) = R_0(t/t_0)^{2/3}$	(9.106)	$q$ deceleration parameter $t(z)$ time at redshift $z$
Concordance model ( $\Omega_k = 0$ , $\Lambda = 3(1 - \Omega_{m0})H_0^2$ , $p = 0$ and $\Omega_{m0} < 1$ )	$d_p = \frac{c}{H_0} \int_0^z \frac{\Omega_{m0}^{-1/2} dz'}{[(1+z')^3 - 1 + \Omega_{m0}^{-1}]^{1/2}}$	(9.107)	$R$ cosmic scale factor
	$H(z) = H_0[\Omega_{m0}(1+z)^3 + (1 - \Omega_{m0})]$	(9.108)	$\Omega_{m0}$ present mass density parameter
	$q_0 = 3\Omega_{m0}/2 - 1$	(9.109)	$G$ constant of gravitation
	$t(z) = \frac{2}{3H_0} (1 - \Omega_{m0})^{-1/2} \operatorname{arsinh} \left[ \frac{(1 - \Omega_{m0})^{1/2}}{(1+z)^{3/2}} \right]$	(9.110)	$\rho$ mass density

<sup>a</sup>Currently popular.





# Index

Section headings are shown in boldface and panel labels in small caps. Equation numbers are contained within square brackets.

## A

- aberration (relativistic) [3.24], 65
- absolute magnitude [9.29], 179
- absorption (Einstein coefficient) [8.118], 173
- absorption coefficient (linear) [5.175], 120
- accelerated point charge
  - bremsstrahlung, 160
  - Liénard–Wiechert potentials, 139
  - oscillating [7.132], 146
  - synchrotron, 159
- acceleration
  - constant, 68
  - dimensions, 16
  - due to gravity (value on Earth), 176
  - in a rotating frame [3.32], 66
- acceptance angle (optical fibre) [8.77], 169
- acoustic branch (phonon) [6.37], 129
- acoustic impedance [3.276], 83
- action (definition) [3.213], 79
- action (dimensions), 16
- addition of velocities
  - Galilean [3.3], 64
  - relativistic [3.15], 64
- adiabatic
  - bulk modulus [5.23], 107
  - compressibility [5.21], 107
  - expansion (ideal gas) [5.58], 110
  - lapse rate [3.294], 84
- adjoint matrix
  - definition 1 [2.71], 24
  - definition 2 [2.80], 25
- adjugate matrix [2.80], 25
- admittance (definition), 148
- advective operator [3.289], 84
- Airy
  - disk [8.40], 165
  - function [8.17], 163
  - resolution criterion [8.41], 165
- Airy's differential equation [2.352], 43
- albedo [5.193], 121
- Alfvén speed [7.277], 158
- Alfvén waves [7.284], 158
- alt-azimuth coordinates, 177
- alternating tensor ( $\epsilon_{ijk}$ ) [2.443], 50
- altitude coordinate [9.9], 177
- Ampère's law [7.10], 136
- ampere (SI definition), 3
- ampere (unit), 4
- analogue formula [2.258], 36
- angle
  - aberration [3.24], 65
  - acceptance [8.77], 169
  - beam solid [7.210], 153
  - Brewster's [7.218], 154
  - Compton scattering [7.240], 155
  - contact (surface tension) [3.340], 88
  - deviation [8.73], 169
  - Euler [2.101], 26
  - Faraday rotation [7.273], 157
  - hour (coordinate) [9.8], 177
  - Kelvin wedge [3.330], 87
  - Mach wedge [3.328], 87
  - polarisation [8.81], 170
  - principal range (inverse trig.), 34
  - refraction, 154
  - rotation, 26
  - Rutherford scattering [3.116], 72
  - separation [3.133], 73
  - spherical excess [2.260], 36
  - units, 4, 5
- ångström (unit), 5

- angular diameter distance [9.100], 185
- Angular momentum**, 98
- angular momentum
- conservation [4.113], 98
  - definition [3.66], 68
  - dimensions, 16
  - eigenvalues [4.109] [4.109], 98
  - ladder operators [4.108], 98
  - operators
    - and other operators [4.23], 91
    - definitions [4.105], 98
  - rigid body [3.141], 74
- ANGULAR MOMENTUM ADDITION, 100
- ANGULAR MOMENTUM COMMUTATION RELATIONS, 98
- angular speed (dimensions), 16
- anomaly (true) [3.104], 71
- antenna
- beam efficiency [7.214], 153
  - effective area [7.212], 153
  - power gain [7.211], 153
  - temperature [7.215], 153
- ANTENNAS, 153
- anticommutation [2.95], 26
- antihermitian symmetry, 53
- antisymmetric matrix [2.87], 25
- APERTURE DIFFRACTION, 165
- aperture function [8.34], 165
- apocentre (of an orbit) [3.111], 71
- apparent magnitude [9.27], 179
- Appleton-Hartree formula [7.271], 157
- arc length [2.279], 39
- arccos  $x$
- from arctan [2.233], 34
  - series expansion [2.141], 29
- arcosh  $x$  (definition) [2.239], 35
- arccot  $x$  (from arctan) [2.236], 34
- arcoth  $x$  (definition) [2.241], 35
- arccsc  $x$  (from arctan) [2.234], 34
- arsch  $x$  (definition) [2.243], 35
- arcminute (unit), 5
- arcsec  $x$  (from arctan) [2.235], 34
- arsech  $x$  (definition) [2.242], 35
- arcsecond (unit), 5
- arcsin  $x$
- from arctan [2.232], 34
  - series expansion [2.141], 29
- arsinh  $x$  (definition) [2.238], 35
- arctan  $x$  (series expansion) [2.142], 29
- artanh  $x$  (definition) [2.240], 35
- area
- of circle [2.262], 37
  - of cone [2.271], 37
  - of cylinder [2.269], 37
  - of ellipse [2.267], 37
  - of plane triangle [2.254], 36
  - of sphere [2.263], 37
  - of spherical cap [2.275], 37
  - of torus [2.273], 37
- area (dimensions), 16
- argument (of a complex number) [2.157], 30
- arithmetic mean [2.108], 27
- arithmetic progression [2.104], 27
- associated Laguerre equation [2.348], 43
- associated Laguerre polynomials, 96
- associated Legendre equation
- and polynomial solutions [2.428], 48
  - differential equation [2.344], 43
- ASSOCIATED LEGENDRE FUNCTIONS, 48
- astronomical constants, 176
- ASTRONOMICAL MAGNITUDES, 179
- Astrophysics, 175–185
- asymmetric top [3.189], 77
- atomic
- form factor [6.30], 128
  - mass unit, 6, 9
  - numbers of elements, 124
  - polarisability [7.91], 142
  - weights of elements, 124
- ATOMIC CONSTANTS, 7
- atto, 5
- autocorrelation (Fourier) [2.491], 53
- autocorrelation function [8.104], 172
- availability
- and fluctuation probability [5.131], 116
  - definition [5.40], 108
- Avogadro constant, 6, 9
- Avogadro constant (dimensions), 16
- azimuth coordinate [9.10], 177
- B**
- BALLISTICS, 69
- band index [6.85], 134
- BAND THEORY AND SEMICONDUCTORS, 134
- bandwidth
- and coherence time [8.106], 172

- and Johnson noise [5.141], 117
- Doppler [8.117], 173
- natural [8.113], 173
- of a diffraction grating [8.30], 164
- of an LCR circuit [7.151], 148
- of laser cavity [8.127], 174
- Schawlow-Townes [8.128], 174
- bar (unit), 5
- barn (unit), 5
- BARRIER TUNNELLING, 94
- Bartlett window [2.581], 60
- base vectors (crystallographic), 126
- basis vectors [2.17], 20
- Bayes' theorem [2.569], 59
- BAYESIAN INFERENCE, 59
- bcc structure, 127
- beam bowing under its own weight [3.260], 82
- beam efficiency [7.214], 153
- beam solid angle [7.210], 153
- beam with end-weight [3.259], 82
- beaming (relativistic) [3.25], 65
- becquerel (unit), 4
- BENDING BEAMS, 82
- bending moment (dimensions), 16
- bending moment [3.258], 82
- bending waves [3.268], 82
- Bernoulli's differential equation [2.351], 43
- Bernoulli's equation
  - compressible flow [3.292], 84
  - incompressible flow [3.290], 84
- Bessel equation [2.345], 43
- BESSEL FUNCTIONS, 47
- beta (in plasmas) [7.278], 158
- binomial
  - coefficient [2.121], 28
  - distribution [2.547], 57
  - series [2.120], 28
  - theorem [2.122], 28
- binormal [2.285], 39
- Biot-Savart law [7.9], 136
- Biot-Fourier equation [5.95], 113
- black hole
  - evaporation time [9.81], 183
  - Kerr solution [3.62], 67
  - maximum angular momentum [9.80], 183
  - Schwarzschild radius [9.73], 183
  - Schwarzschild solution [3.61], 67
  - temperature [9.82], 183
- blackbody
  - energy density [5.192], 121
  - spectral energy density [5.186], 121
  - spectrum [5.184], 121
- BLACKBODY RADIATION, 121
- Bloch's theorem [6.84], 134
- Bode's law [9.41], 180
- body cone, 77
- body frequency [3.187], 77
- body-centred cubic structure, 127
- Bohr
  - energy [4.74], 95
  - magneton (equation) [4.137], 100
  - magneton (value), 6, 7
  - quantisation [4.71], 95
  - radius (equation) [4.72], 95
  - radius (value), 7
- Bohr magneton (dimensions), 16
- BOHR MODEL, 95
- boiling points of elements, 124
- bolometric correction [9.34], 179
- Boltzmann
  - constant, 6, 9
  - constant (dimensions), 16
  - distribution [5.111], 114
  - entropy [5.105], 114
  - excitation equation [5.125], 116
- Born collision formula [4.178], 104
- Bose condensation [5.123], 115
- Bose-Einstein distribution [5.120], 115
- boson statistics [5.120], 115
- BOUNDARY CONDITIONS FOR  $E$ ,  $D$ ,  $B$ , AND  $H$ , 144
- box (particle in a) [4.64], 94
- Box Muller transformation [2.561], 58
- Boyle temperature [5.66], 110
- Boyle's law [5.56], 110
- bra vector [4.33], 92
- bra-ket notation, 91, 92
- Bragg's reflection law
  - in crystals [6.29], 128
  - in optics [8.32], 164
- braking index (pulsar) [9.66], 182
- BRAVAIS LATTICES, 126
- Breit-Wigner formula [4.174], 104
- BREMSSTRAHLUNG, 160
- bremsstrahlung

- single electron and ion [7.297], 160
  - thermal [7.300], 160
  - Brewster's law [7.218], 154
  - brightness (blackbody) [5.184], 121
  - Brillouin function [4.147], 101
  - Bromwich integral [2.518], 55
  - Brownian motion [5.98], 113
  - bubbles [3.337], 88
  - bulk modulus
    - adiabatic [5.23], 107
    - general [3.245], 81
    - isothermal [5.22], 107
  - bulk modulus (dimensions), 16
  - BULK PHYSICAL CONSTANTS, 9
  - Burgers vector [6.21], 128
- C**
- calculus of variations [2.334], 42
  - candela, 119
  - candela (SI definition), 3
  - candela (unit), 4
  - canonical
    - ensemble [5.111], 114
    - entropy [5.106], 114
    - equations [3.220], 79
    - momenta [3.218], 79
  - cap, *see* spherical cap
  - CAPACITANCE, 137
  - capacitance
    - current through [7.144], 147
    - definition [7.143], 147
    - dimensions, 16
    - energy [7.153], 148
    - energy of an assembly [7.134], 146
    - impedance [7.159], 148
    - mutual [7.134], 146
  - capacitance of
    - cube [7.17], 137
    - cylinder [7.15], 137
    - cylinders (adjacent) [7.21], 137
    - cylinders (coaxial) [7.19], 137
    - disk [7.13], 137
    - disks (coaxial) [7.22], 137
    - nearly spherical surface [7.16], 137
    - sphere [7.12], 137
    - spheres (adjacent) [7.14], 137
    - spheres (concentric) [7.18], 137
  - capacitor, *see* capacitance
  - capillary
    - constant [3.338], 88
    - contact angle [3.340], 88
    - rise [3.339], 88
    - waves [3.321], 86
  - capillary-gravity waves [3.322], 86
  - cardioid [8.46], 166
  - Carnot cycles, 107
  - Cartesian coordinates, 21
  - Catalan's constant (value), 9
  - Cauchy
    - differential equation [2.350], 43
    - distribution [2.555], 58
    - inequality [2.151], 30
    - integral formula [2.167], 31
  - Cauchy-Goursat theorem [2.165], 31
  - Cauchy-Riemann conditions [2.164], 31
  - cavity modes (laser) [8.124], 174
  - Celsius (unit), 4
  - Celsius conversion [1.1], 15
  - centi, 5
  - centigrade (avoidance of), 15
  - centre of mass
    - circular arc [3.173], 76
    - cone [3.175], 76
    - definition [3.68], 68
    - disk sector [3.172], 76
    - hemisphere [3.170], 76
    - hemispherical shell [3.171], 76
    - pyramid [3.175], 76
    - semi-ellipse [3.178], 76
    - spherical cap [3.177], 76
    - triangular lamina [3.174], 76
  - CENTRES OF MASS, 76
  - centrifugal force [3.35], 66
  - centripetal acceleration [3.32], 66
  - cepheid variables [9.48], 180
  - Cerenkov, *see* Cherenkov
  - chain rule
    - function of a function [2.295], 40
    - partial derivatives [2.331], 42
  - Chandrasekhar mass [9.79], 183
  - change of variable [2.333], 42
  - CHARACTERISTIC NUMBERS, 86
  - charge
    - conservation [7.39], 139
    - dimensions, 16
    - elementary, 6, 7
    - force between two [7.119], 145
    - Hamiltonian [7.138], 146



- to mass ratio of electron, 8
- charge density
  - dimensions, 16
  - free [7.57], 140
  - induced [7.84], 142
  - Lorentz transformation, 141
- charge distribution
  - electric field from [7.6], 136
  - energy of [7.133], 146
- charge-sheet (electric field) [7.32], 138
- Chebyshev equation [2.349], 43
- Chebyshev inequality [2.150], 30
- chemical potential
  - definition [5.28], 108
  - from partition function [5.119], 115
- Cherenkov cone angle [7.246], 156
- CHERENKOV RADIATION, 156
- $\chi_E$  (electric susceptibility) [7.87], 142
- $\chi_H$ ,  $\chi_B$  (magnetic susceptibility) [7.103], 143
- chi-squared ( $\chi^2$ ) distribution [2.553], 58
- Christoffel symbols [3.49], 67
- circle
  - (arc of) centre of mass [3.173], 76
  - area [2.262], 37
  - perimeter [2.261], 37
- circular aperture
  - Fraunhofer diffraction [8.40], 165
  - Fresnel diffraction [8.50], 166
- circular polarisation, 170
- circulation [3.287], 84
- civil time [9.4], 177
- Clapeyron equation [5.50], 109
- classical electron radius, 8
- Classical thermodynamics**, 106
- Clausius–Mossotti equation [7.93], 142
- Clausius–Clapeyron equation [5.49], 109
- CLEBSCH–GORDAN COEFFICIENTS, 99
- Clebsch–Gordan coefficients (spin-orbit) [4.136], 100
- close-packed spheres, 127
- closure density (of the universe) [9.90], 184
- CNO cycle, 182
- coaxial cable
  - capacitance [7.19], 137
  - inductance [7.24], 137
- coaxial transmission line [7.181], 150
- coefficient of
  - coupling [7.148], 147
  - fineness [8.12], 163
  - reflectance [7.227], 154
  - reflection [7.230], 154
  - restitution [3.127], 73
  - transmission [7.232], 154
  - transmittance [7.229], 154
- coexistence curve [5.51], 109
- coherence
  - length [8.106], 172
  - mutual [8.97], 172
  - temporal [8.105], 172
  - time [8.106], 172
  - width [8.111], 172
- Coherence (scalar theory)**, 172
- cold plasmas, 157
- collision
  - broadening [8.114], 173
  - elastic, 73
  - inelastic, 73
  - number [5.91], 113
  - time (electron drift) [6.61], 132
- colour excess [9.37], 179
- colour index [9.36], 179
- COMMON THREE-DIMENSIONAL COORDINATE SYSTEMS, 21
- commutator (in uncertainty relation) [4.6], 90
- COMMUTATORS, 26
- COMPACT OBJECTS AND BLACK HOLES, 183
- complementary error function [2.391], 45
- COMPLEX ANALYSIS, 31
- complex conjugate [2.159], 30
- COMPLEX NUMBERS, 30
- complex numbers
  - argument [2.157], 30
  - cartesian form [2.153], 30
  - conjugate [2.159], 30
  - logarithm [2.162], 30
  - modulus [2.155], 30
  - polar form [2.154], 30
- Complex variables**, 30
- compound pendulum [3.182], 76
- compressibility
  - adiabatic [5.21], 107
  - isothermal [5.20], 107
- compression modulus, *see* bulk modulus
- compression ratio [5.13], 107
- Compton

- scattering [7.240], 155
- wavelength (value), 8
- wavelength [7.240], 155
- Concordance model, 185
- conditional probability [2.567], 59
- conductance (definition), 148
- conductance (dimensions), 16
- conduction equation (and transport) [5.96], 113
- conduction equation [2.340], 43
- conductivity
  - and resistivity [7.142], 147
  - dimensions, 16
  - direct [7.279], 158
  - electrical, of a plasma [7.233], 155
  - free electron a.c. [6.63], 132
  - free electron d.c. [6.62], 132
  - Hall [7.280], 158
- conductor refractive index [7.234], 155
- cone
  - centre of mass [3.175], 76
  - moment of inertia [3.160], 75
  - surface area [2.271], 37
  - volume [2.272], 37
- configurational entropy [5.105], 114
- CONIC SECTIONS, 38
- conical pendulum [3.180], 76
- conservation of
  - angular momentum [4.113], 98
  - charge [7.39], 139
  - mass [3.285], 84
- CONSTANT ACCELERATION, 68
- constant of gravitation, 7
- contact angle (surface tension) [3.340], 88
- continuity equation (quantum physics) [4.14], 90
- continuity in fluids [3.285], 84
- CONTINUOUS PROBABILITY DISTRIBUTIONS, 58
- contravariant components
  - in general relativity, 67
  - in special relativity [3.26], 65
- convection (in a star) [9.64], 181
- convergence and limits, 28
- CONVERSION FACTORS, 10
- Converting between units**, 10
- convolution
  - definition [2.487], 53
  - derivative [2.498], 53
  - discrete [2.580], 60
  - Laplace transform [2.516], 55
  - rules [2.489], 53
  - theorem [2.490], 53
- coordinate systems, 21
- coordinate transformations
  - astronomical, 177
  - Galilean, 64
  - relativistic, 64
  - rotating frames [3.31], 66
- Coordinate transformations (astronomical)**, 177
- coordinates (generalised ) [3.213], 79
- coordination number (cubic lattices), 127
- Coriolis force [3.33], 66
- CORNU SPIRAL, 167
- Cornu spiral and Fresnel integrals [8.54], 167
- correlation coefficient
  - multinormal [2.559], 58
  - Pearson's  $r$  [2.546], 57
- correlation intensity [8.109], 172
- correlation theorem [2.494], 53
- $\cos x$ 
  - and Euler's formula [2.216], 34
  - series expansion [2.135], 29
- cosec, *see* csc
- csch  $x$  [2.231], 34
- cosh  $x$ 
  - definition [2.217], 34
  - series expansion [2.143], 29
- cosine formula
  - planar triangles [2.249], 36
  - spherical triangles [2.257], 36
- cosmic scale factor [9.87], 184
- cosmological constant [9.89], 184
- COSMOLOGICAL DISTANCE MEASURES, 185
- COSMOLOGICAL MODEL PARAMETERS, 184
- COSMOLOGICAL MODELS, 185
- Cosmology**, 184
- $\cos^{-1} x$ , *see* arccos  $x$
- cot  $x$ 
  - definition [2.226], 34
  - series expansion [2.140], 29
- coth  $x$  [2.227], 34
- Couette flow [3.306], 85
- coulomb (unit), 4
- Coulomb gauge condition [7.42], 139

- Coulomb logarithm [7.254], 156  
 Coulomb's law [7.119], 145  
 couple  
   definition [3.67], 68  
   dimensions, 16  
   electromagnetic, 145  
   for Couette flow [3.306], 85  
   on a current-loop [7.127], 145  
   on a magnetic dipole [7.126], 145  
   on a rigid body, 77  
   on an electric dipole [7.125], 145  
   twisting [3.252], 81  
 coupling coefficient [7.148], 147  
 covariance [2.558], 58  
 covariant components [3.26], 65  
 cracks (critical length) [6.25], 128  
 critical damping [3.199], 78  
 critical density (of the universe) [9.90], 184  
 critical frequency (synchrotron) [7.293], 159  
 critical point  
   Dieterici gas [5.75], 111  
   van der Waals gas [5.70], 111  
 cross section  
   absorption [5.175], 120  
 cross-correlation [2.493], 53  
 cross-product [2.2], 20  
 cross-section  
   Breit-Wigner [4.174], 104  
   Mott scattering [4.180], 104  
   Rayleigh scattering [7.236], 155  
   Rutherford scattering [3.124], 72  
   Thomson scattering [7.238], 155  
 CRYSTAL DIFFRACTION, 128  
 CRYSTAL SYSTEMS, 127  
**Crystalline structure**, 126  
 csc  $x$   
   definition [2.230], 34  
   series expansion [2.139], 29  
 csch  $x$  [2.231], 34  
 cube  
   electrical capacitance [7.17], 137  
   mensuration, 38  
 CUBIC EQUATIONS, 51  
 cubic expansivity [5.19], 107  
 CUBIC LATTICES, 127  
 cubic system (crystallographic), 127  
 Curie temperature [7.114], 144  
 Curie's law [7.113], 144  
 Curie-Weiss law [7.114], 144  
 CURL, 22  
 curl  
   cylindrical coordinates [2.34], 22  
   general coordinates [2.36], 22  
   of curl [2.57], 23  
   rectangular coordinates [2.33], 22  
   spherical coordinates [2.35], 22  
 current  
   dimensions, 16  
   electric [7.139], 147  
   law (Kirchhoff's) [7.161], 149  
   magnetic flux density from [7.11], 136  
   probability density [4.13], 90  
   thermodynamic work [5.9], 106  
   transformation [7.165], 149  
 current density  
   dimensions, 16  
   four-vector [7.76], 141  
   free [7.63], 140  
   free electron [6.60], 132  
   hole [6.89], 134  
   Lorentz transformation, 141  
   magnetic flux density [7.10], 136  
 curvature  
   in differential geometry [2.286], 39  
   parameter (cosmic) [9.87], 184  
   radius of  
   and curvature [2.287], 39  
   plane curve [2.282], 39  
 curve length (plane curve) [2.279], 39  
 CURVE MEASURE, 39  
 CYCLE EFFICIENCIES (THERMODYNAMIC), 107  
 cyclic permutation [2.97], 26  
 cyclotron frequency [7.265], 157  
 cylinder  
   area [2.269], 37  
   capacitance [7.15], 137  
   moment of inertia [3.155], 75  
   torsional rigidity [3.253], 81  
   volume [2.270], 37  
 cylinders (adjacent)  
   capacitance [7.21], 137  
   inductance [7.25], 137  
 cylinders (coaxial)  
   capacitance [7.19], 137  
   inductance [7.24], 137

cylindrical polar coordinates, 21

## D

*d* orbitals [4.100], 97  
 D'Alembertian [7.78], 141  
 damped harmonic oscillator [3.196], 78  
 damping profile [8.112], 173  
 day (unit), 5  
 day of week [9.3], 177  
 daylight saving time [9.4], 177  
 de Boer parameter [6.54], 131  
 de Broglie relation [4.2], 90  
 de Broglie wavelength (thermal) [5.83], 112  
 de Moivre's theorem [2.214], 34  
 Debye  
    $T^3$  law [6.47], 130  
   frequency [6.41], 130  
   function [6.49], 130  
   heat capacity [6.45], 130  
   length [7.251], 156  
   number [7.253], 156  
   screening [7.252], 156  
   temperature [6.43], 130  
 DEBYE THEORY, 130  
 Debye-Waller factor [6.33], 128  
 deca, 5  
 decay constant [4.163], 103  
 decay law [4.163], 103  
 deceleration parameter [9.95], 184  
 deci, 5  
 decibel [5.144], 117  
 declination coordinate [9.11], 177  
 decrement (oscillating systems) [3.202], 78  
 DEFINITE INTEGRALS, 46  
 degeneracy pressure [9.77], 183  
 degree (unit), 5  
 degree Celsius (unit), 4  
 degree kelvin [5.2], 106  
 degree of freedom (and equipartition), 113  
 degree of mutual coherence [8.99], 172  
 degree of polarisation [8.96], 171  
 degree of temporal coherence, 172  
 deka, 5  
 del operator, 21  
 del-squared operator, 23  
 del-squared operator [2.55], 23  
 DELTA FUNCTIONS, 50

delta-star transformation, 149  
 densities of elements, 124  
 density (dimensions), 16  
 density of states  
   electron [6.70], 133  
   particle [4.66], 94  
   phonon [6.44], 130  
 density parameters [9.94], 184  
 depolarising factors [7.92], 142  
 DERIVATIVES (GENERAL), 40  
 determinant [2.79], 25  
 deviation (of a prism) [8.73], 169  
 diamagnetic moment (electron) [7.108], 144  
 diamagnetic susceptibility (Landau) [6.80], 133  
 DIAMAGNETISM, 144  
 DIELECTRIC LAYERS, 162  
 DIETERICI GAS, 111  
 Dieterici gas law [5.72], 111  
 DIFFERENTIAL EQUATIONS, 43  
 differential equations (numerical solutions), 62  
 DIFFERENTIAL GEOMETRY, 39  
 DIFFERENTIAL OPERATOR IDENTITIES, 23  
 differential scattering cross-section [3.124], 72  
**Differentiation**, 40  
 differentiation  
   hyperbolic functions, 41  
   numerical, 61  
   of a function of a function [2.295], 40  
   of a log [2.300], 40  
   of a power [2.292], 40  
   of a product [2.293], 40  
   of a quotient [2.294], 40  
   of exponential [2.301], 40  
   of integral [2.299], 40  
   of inverse functions [2.304], 40  
   trigonometric functions, 41  
   under integral sign [2.298], 40  
 diffraction from  
   *N* slits [8.25], 164  
   1 slit [8.37], 165  
   2 slits [8.24], 164  
   circular aperture [8.40], 165  
   crystals, 128  
   infinite grating [8.26], 164

- rectangular aperture [8.39], 165
- diffraction grating
  - finite [8.25], 164
  - general, 164
  - infinite [8.26], 164
- diffusion coefficient (semiconductor) [6.88], 134
- diffusion equation
  - differential equation [2.340], 43
  - Fick's first law [5.93], 113
- diffusion length (semiconductor) [6.94], 134
- diffusivity (magnetic) [7.282], 158
- dilatation (volume strain) [3.236], 80
- Dimensions**, 16
- diode (semiconductor) [6.92], 134
- diopetre number [8.68], 168
- dipole
  - antenna power
    - flux [7.131], 146
    - gain [7.213], 153
    - total [7.132], 146
  - electric field [7.31], 138
  - energy of
    - electric [7.136], 146
    - magnetic [7.137], 146
  - field from
    - magnetic [7.36], 138
  - moment (dimensions), 17
  - moment of
    - electric [7.80], 142
    - magnetic [7.94], 143
  - potential
    - electric [7.82], 142
    - magnetic [7.95], 143
  - radiation
    - field [7.207], 153
    - magnetic [9.69], 182
  - radiation resistance [7.209], 153
- dipole moment per unit volume
  - electric [7.83], 142
  - magnetic [7.97], 143
- Dirac bracket, 92
- Dirac delta function [2.448], 50
- Dirac equation [4.183], 104
- Dirac matrices [4.185], 104
- DIRAC NOTATION, 92
- direct conductivity [7.279], 158
- directrix (of conic section), 38
- disc, *see* disk
- discrete convolution, 60
- DISCRETE PROBABILITY DISTRIBUTIONS, 57
- DISCRETE STATISTICS, 57
- disk
  - Airy [8.40], 165
  - capacitance [7.13], 137
  - centre of mass of sector [3.172], 76
  - coaxial capacitance [7.22], 137
  - drag in a fluid, 85
  - electric field [7.28], 138
  - moment of inertia [3.168], 75
- DISLOCATIONS AND CRACKS, 128
- dispersion
  - diffraction grating [8.31], 164
  - in a plasma [7.261], 157
  - in fluid waves, 86
  - in quantum physics [4.5], 90
  - in waveguides [7.188], 151
  - intermodal (optical fibre) [8.79], 169
  - measure [9.70], 182
  - of a prism [8.76], 169
  - phonon (alternating springs) [6.39], 129
  - phonon (diatomic chain) [6.37], 129
  - phonon (monatomic chain) [6.34], 129
  - pulsar [9.72], 182
- displacement, **D** [7.86], 142
- DISTANCE INDICATORS, 180
- DIVERGENCE, 22
- divergence
  - cylindrical coordinates [2.30], 22
  - general coordinates [2.32], 22
  - rectangular coordinates [2.29], 22
  - spherical coordinates [2.31], 22
  - theorem [2.59], 23
- dodecahedron, 38
- Doppler
  - beaming [3.25], 65
  - effect (non-relativistic), 87
  - effect (relativistic) [3.22], 65
  - line broadening [8.116], 173
  - width [8.117], 173
- DOPPLER EFFECT, 87
- dot product [2.1], 20
- double factorial, 48
- double pendulum [3.183], 76
- DRAG, 85

- drag  
 on a disk  $\parallel$  to flow [3.310], 85  
 on a disk  $\perp$  to flow [3.309], 85  
 on a sphere [3.308], 85
- drift velocity (electron) [6.61], 132
- Dulong and Petit's law [6.46], 130
- Dynamics and Mechanics, 63–88
- DYNAMICS DEFINITIONS, 68
- E**
- e (exponential constant), 9
- e TO 1 000 DECIMAL PLACES, 18
- Earth (motion relative to) [3.38], 66
- EARTH DATA, 176
- eccentricity  
 of conic section, 38  
 of orbit [3.108], 71  
 of scattering hyperbola [3.120], 72
- ECLIPTIC COORDINATES, 178
- ecliptic latitude [9.14], 178
- ecliptic longitude [9.15], 178
- Eddington limit [9.59], 181
- edge dislocation [6.21], 128
- effective  
 area (antenna) [7.212], 153  
 distance (Fresnel diffraction) [8.48],  
 166  
 mass (in solids) [6.86], 134  
 wavelength [9.40], 179
- efficiency  
 heat engine [5.10], 107  
 heat pump [5.12], 107  
 Otto cycle [5.13], 107  
 refrigerator [5.11], 107
- Ehrenfest's equations [5.53], 109
- Ehrenfest's theorem [4.30], 91
- eigenfunctions (quantum) [4.28], 91
- Einstein  
 $A$  coefficient [8.119], 173  
 $B$  coefficients [8.118], 173  
 diffusion equation [5.98], 113  
 field equation [3.59], 67  
 lens (rings) [9.50], 180  
 tensor [3.58], 67
- Einstein - de Sitter model, 185
- EINSTEIN COEFFICIENTS, 173
- elastic  
 collisions, 73  
 media (isotropic), 81  
 modulus (longitudinal) [3.241], 81  
 modulus [3.234], 80  
 potential energy [3.235], 80
- elastic scattering, 72
- ELASTIC WAVE VELOCITIES, 82
- Elasticity**, 80
- ELASTICITY DEFINITIONS (GENERAL), 80
- ELASTICITY DEFINITIONS (SIMPLE), 80
- electric current [7.139], 147
- electric dipole, *see* dipole
- electric displacement (dimensions), 16
- electric displacement,  $D$  [7.86], 142
- electric field  
 around objects, 138  
 energy density [7.128], 146  
 static, 136  
 thermodynamic work [5.7], 106  
 wave equation [7.193], 152
- electric field from  
 $A$  and  $\phi$  [7.41], 139  
 charge distribution [7.6], 136  
 charge-sheet [7.32], 138  
 dipole [7.31], 138  
 disk [7.28], 138  
 line charge [7.29], 138  
 point charge [7.5], 136  
 sphere [7.27], 138  
 waveguide [7.190], 151  
 wire [7.29], 138
- electric field strength (dimensions), 16
- ELECTRIC FIELDS, 138
- electric polarisability (dimensions), 16
- electric polarisation (dimensions), 16
- electric potential  
 from a charge density [7.46], 139  
 Lorentz transformation [7.75], 141  
 of a moving charge [7.48], 139  
 short dipole [7.82], 142
- electric potential difference (dimensions),  
 16
- electric susceptibility,  $\chi_E$  [7.87], 142
- electrical conductivity, *see* conductivity
- ELECTRICAL IMPEDANCE, 148
- electrical permittivity,  $\epsilon$ ,  $\epsilon_r$  [7.90], 142
- electromagnet (magnetic flux density) [7.38],  
 138
- electromagnetic  
 boundary conditions, 144  
 constants, 7

- fields, 139
- wave speed [7.196], 152
- waves in media, 152
- electromagnetic coupling constant, *see* fine structure constant
- ELECTROMAGNETIC ENERGY, 146
- Electromagnetic fields (general)**, 139
- ELECTROMAGNETIC FORCE AND TORQUE, 145
- ELECTROMAGNETIC PROPAGATION IN COLD PLASMAS, 157
- Electromagnetism, 135–160
- electron
  - charge, 6, 7
  - density of states [6.70], 133
  - diamagnetic moment [7.108], 144
  - drift velocity [6.61], 132
  - g*-factor [4.143], 100
  - gyromagnetic ratio (value), 8
  - gyromagnetic ratio [4.140], 100
  - heat capacity [6.76], 133
  - intrinsic magnetic moment [7.109], 144
  - mass, 6
  - radius (equation) [7.238], 155
  - radius (value), 8
  - scattering cross-section [7.238], 155
  - spin magnetic moment [4.143], 100
  - thermal velocity [7.257], 156
  - velocity in conductors [6.85], 134
- ELECTRON CONSTANTS, 8
- ELECTRON SCATTERING PROCESSES, 155
- electron volt (unit), 5
- electron volt (value), 6
- Electrons in solids**, 132
- electrostatic potential [7.1], 136
- ELECTROSTATICS, 136
- elementary charge, 6, 7
- elements (periodic table of), 124
- ellipse, 38
  - (semi) centre of mass [3.178], 76
  - area [2.267], 37
  - moment of inertia [3.166], 75
  - perimeter [2.266], 37
  - semi-latus-rectum [3.109], 71
  - semi-major axis [3.106], 71
  - semi-minor axis [3.107], 71
- ellipsoid
  - moment of inertia of solid [3.163], 75
  - the moment of inertia [3.147], 74
  - volume [2.268], 37
- elliptic integrals [2.397], 45
- elliptical orbit [3.104], 71
- ELLIPTICAL POLARISATION, 170
- elliptical polarisation [8.80], 170
- ellipticity [8.82], 170
- $E = mc^2$  [3.72], 68
- emission coefficient [5.174], 120
- emission spectrum [7.291], 159
- emissivity [5.193], 121
- energy
  - density
    - blackbody [5.192], 121
    - dimensions, 16
    - elastic wave [3.281], 83
    - electromagnetic [7.128], 146
    - radiant [5.148], 118
    - spectral [5.173], 120
  - dimensions, 16
  - dissipated in resistor [7.155], 148
  - distribution (Maxwellian) [5.85], 112
  - elastic [3.235], 80
  - electromagnetic, 146
  - equipartition [5.100], 113
  - Fermi [5.122], 115
  - first law of thermodynamics [5.3], 106
  - Galilean transformation [3.6], 64
  - kinetic, *see* kinetic energy
  - Lorentz transformation [3.19], 65
  - loss after collision [3.128], 73
  - mass relation [3.20], 65
  - of capacitive assembly [7.134], 146
  - of capacitor [7.153], 148
  - of charge distribution [7.133], 146
  - of electric dipole [7.136], 146
  - of inductive assembly [7.135], 146
  - of inductor [7.154], 148
  - of magnetic dipole [7.137], 146
  - of orbit [3.100], 71
  - potential, *see* potential energy
  - relativistic rest [3.72], 68
  - rotational kinetic
    - rigid body [3.142], 74
    - w.r.t. principal axes [3.145], 74
  - thermodynamic work, 106
- ENERGY IN CAPACITORS, INDUCTORS, AND RESISTORS, 148

- energy-time uncertainty relation [4.8], 90  
 ENSEMBLE PROBABILITIES, 114  
 enthalpy  
   definition [5.30], 108  
   Joule-Kelvin expansion [5.27], 108  
 entropy  
   Boltzmann formula [5.105], 114  
   change in Joule expansion [5.64], 110  
   experimental [5.4], 106  
   fluctuations [5.135], 116  
   from partition function [5.117], 115  
   Gibbs formula [5.106], 114  
   of a monatomic gas [5.83], 112  
 entropy (dimensions), 16  
 $\epsilon$ ,  $\epsilon_r$  (electrical permittivity) [7.90], 142  
 EQUATION CONVERSION: SI TO GAUSSIAN UNITS, 135  
 equation of state  
   Dieterici gas [5.72], 111  
   ideal gas [5.57], 110  
   monatomic gas [5.78], 112  
   van der Waals gas [5.67], 111  
 equipartition theorem [5.100], 113  
 error function [2.390], 45  
 errors, 60  
 escape velocity [3.91], 70  
 estimator  
   kurtosis [2.545], 57  
   mean [2.541], 57  
   skewness [2.544], 57  
   standard deviation [2.543], 57  
   variance [2.542], 57  
 Euler  
   angles [2.101], 26  
   constant  
     expression [2.119], 27  
     value, 9  
   differential equation [2.350], 43  
   formula [2.216], 34  
   relation, 38  
   strut [3.261], 82  
 Euler's equation (fluids) [3.289], 84  
 Euler's equations (rigid bodies) [3.186], 77  
 Euler's method (for ordinary differential equations) [2.596], 62  
 Euler-Lagrange equation  
   and Lagrangians [3.214], 79  
     calculus of variations [2.334], 42  
   even functions, 53  
 EVOLUTIONARY TIMESCALES, 181  
 exa, 5  
 exhaust velocity (of a rocket) [3.93], 70  
 exitance  
   blackbody [5.191], 121  
   luminous [5.162], 119  
   radiant [5.150], 118  
 $\exp(x)$  [2.132], 29  
 expansion coefficient [5.19], 107  
 EXPANSION PROCESSES, 108  
 expansivity [5.19], 107  
 EXPECTATION VALUE, 91  
 expectation value  
   Dirac notation [4.37], 92  
   from a wavefunction [4.25], 91  
 explosions [3.331], 87  
 exponential  
   distribution [2.551], 58  
   integral [2.394], 45  
   series expansion [2.132], 29  
 exponential constant (e), 9  
 extraordinary modes [7.271], 157  
 extrema [2.335], 42
- ## F
- $f$ -number [8.69], 168  
 Fabry-Perot etalon  
   chromatic resolving power [8.21], 163  
   free spectral range [8.23], 163  
   fringe width [8.19], 163  
   transmitted intensity [8.17], 163  
 FABRY-PEROT ETALON, 163  
 face-centred cubic structure, 127  
 factorial [2.409], 46  
 factorial (double), 48  
 Fahrenheit conversion [1.2], 15  
 faltung theorem [2.516], 55  
 farad (unit), 4  
 Faraday constant, 6, 9  
 Faraday constant (dimensions), 16  
 Faraday rotation [7.273], 157  
 Faraday's law [7.55], 140  
 fcc structure, 127  
 Feigenbaum's constants, 9  
 femto, 5  
 Fermat's principle [8.63], 168  
 Fermi



- energy [6.73], 133
- temperature [6.74], 133
- velocity [6.72], 133
- wavenumber [6.71], 133
- fermi (unit), 5
- Fermi energy [5.122], 115
- FERMI GAS, 133
- Fermi's golden rule [4.162], 102
- Fermi–Dirac distribution [5.121], 115
- fermion statistics [5.121], 115
- fibre optic
  - acceptance angle [8.77], 169
  - dispersion [8.79], 169
  - numerical aperture [8.78], 169
- Fick's first law [5.92], 113
- Fick's second law [5.95], 113
- field equations (gravitational) [3.42], 66
- FIELD RELATIONSHIPS, 139
- fields
  - depolarising [7.92], 142
  - electrochemical [6.81], 133
  - electromagnetic, 139
  - gravitational, 66
  - static  $E$  and  $B$ , 136
  - velocity [3.285], 84
- Fields associated with media**, 142
- film reflectance [8.4], 162
- fine-structure constant
  - expression [4.75], 95
  - value, 6, 7
- finesse (coefficient of) [8.12], 163
- finesse (Fabry-Perot etalon) [8.14], 163
- first law of thermodynamics [5.3], 106
- fitting straight-lines, 60
- fluctuating dipole interaction [6.50], 131
- fluctuation
  - of density [5.137], 116
  - of entropy [5.135], 116
  - of pressure [5.136], 116
  - of temperature [5.133], 116
  - of volume [5.134], 116
  - probability (thermodynamic) [5.131], 116
  - variance (general) [5.132], 116
- Fluctuations and noise**, 116
- Fluid dynamics**, 84
- fluid stress [3.299], 85
- FLUID WAVES, 86
- flux density [5.171], 120
- flux density–redshift relation [9.99], 185
- flux linked [7.149], 147
- flux of molecules through a plane [5.91], 113
- flux–magnitude relation [9.32], 179
- focal length [8.64], 168
- focus (of conic section), 38
- force
  - and acoustic impedance [3.276], 83
  - and stress [3.228], 80
  - between two charges [7.119], 145
  - between two currents [7.120], 145
  - between two masses [3.40], 66
  - central [4.113], 98
  - centrifugal [3.35], 66
  - Coriolis [3.33], 66
  - critical compression [3.261], 82
  - definition [3.63], 68
  - dimensions, 16
  - electromagnetic, 145
  - Newtonian [3.63], 68
  - on
    - charge in a field [7.122], 145
    - current in a field [7.121], 145
    - electric dipole [7.123], 145
    - magnetic dipole [7.124], 145
    - sphere (potential flow) [3.298], 84
    - sphere (viscous drag) [3.308], 85
  - relativistic [3.71], 68
  - unit, 4
- Force, torque, and energy**, 145
- FORCED OSCILLATIONS, 78
- form factor [6.30], 128
- formula (the) [2.455], 50
- Foucault's pendulum [3.39], 66
- four-parts formula [2.259], 36
- four-scalar product [3.27], 65
- four-vector
  - electromagnetic [7.79], 141
  - momentum [3.21], 65
  - spacetime [3.12], 64
- FOUR-VECTORS, 65
- Fourier series
  - complex form [2.478], 52
  - real form [2.476], 52
- FOURIER SERIES, 52
- Fourier series and transforms**, 52
- FOURIER SYMMETRY RELATIONSHIPS, 53
- Fourier transform

- cosine [2.509], 54
- definition [2.482], 52
- derivatives
  - and inverse [2.502], 54
  - general [2.498], 53
- Gaussian [2.507], 54
- Lorentzian [2.505], 54
- shah function [2.510], 54
- shift theorem [2.501], 54
- similarity theorem [2.500], 54
- sine [2.508], 54
- step [2.511], 54
- top hat [2.512], 54
- triangle function [2.513], 54
- FOURIER TRANSFORM, 52
- FOURIER TRANSFORM PAIRS, 54
- FOURIER TRANSFORM THEOREMS, 53
- Fourier's law [5.94], 113
- Frames of reference**, 64
- Fraunhofer diffraction**, 164
- Fraunhofer integral [8.34], 165
- Fraunhofer limit [8.44], 165
- free charge density [7.57], 140
- free current density [7.63], 140
- FREE ELECTRON TRANSPORT PROPERTIES, 132
- free energy [5.32], 108
- free molecular flow [5.99], 113
- FREE OSCILLATIONS, 78
- free space impedance [7.197], 152
- free spectral range
  - Fabry Perot etalon [8.23], 163
  - laser cavity [8.124], 174
- free-fall timescale [9.53], 181
- Frenet's formulas [2.291], 39
- frequency (dimensions), 16
- Fresnel diffraction
  - Cornu spiral [8.54], 167
  - edge [8.56], 167
  - long slit [8.58], 167
  - rectangular aperture [8.62], 167
- Fresnel diffraction**, 166
- Fresnel Equations, 154
- Fresnel half-period zones [8.49], 166
- Fresnel integrals
  - and the Cornu spiral [8.52], 167
  - definition [2.392], 45
  - in diffraction [8.54], 167
- FRESNEL ZONES, 166
- Fresnel-Kirchhoff formula
  - plane waves [8.45], 166
  - spherical waves [8.47], 166
- Friedmann equations [9.89], 184
- fringe visibility [8.101], 172
- fringes (Moiré), 35
- Froude number [3.312], 86
- G**
- g-factor
  - electron, 8
  - Landé [4.146], 100
  - muon, 9
- gain in decibels [5.144], 117
- galactic
  - coordinates [9.20], 178
  - latitude [9.21], 178
  - longitude [9.22], 178
- GALACTIC COORDINATES, 178
- Galilean transformation
  - of angular momentum [3.5], 64
  - of kinetic energy [3.6], 64
  - of momentum [3.4], 64
  - of time and position [3.2], 64
  - of velocity [3.3], 64
- GALILEAN TRANSFORMATIONS, 64
- GAMMA FUNCTION, 46
- gamma function
  - and other integrals [2.395], 45
  - definition [2.407], 46
- gas
  - adiabatic expansion [5.58], 110
  - adiabatic lapse rate [3.294], 84
  - constant, 6, 9, 86, 110
  - Dieterici, 111
  - Doppler broadened [8.116], 173
  - flow [3.292], 84
  - giant (astronomical data), 176
  - ideal equation of state [5.57], 110
  - ideal heat capacities, 113
  - ideal, or perfect, 110
  - internal energy (ideal) [5.62], 110
  - isothermal expansion [5.63], 110
  - linear absorption coefficient [5.175], 120
  - molecular flow [5.99], 113
  - monatomic, 112
  - paramagnetism [7.112], 144
  - pressure broadened [8.115], 173
  - speed of sound [3.318], 86

- temperature scale [5.1], 106
- Van der Waals, 111
- GAS EQUIPARTITION, 113
- Gas laws**, 110
- gauge condition
  - Coulomb [7.42], 139
  - Lorenz [7.43], 139
- Gaunt factor [7.299], 160
- Gauss's
  - law [7.51], 140
  - lens formula [8.64], 168
  - theorem [2.59], 23
- Gaussian
  - electromagnetism, 135
  - Fourier transform of [2.507], 54
  - integral [2.398], 46
  - light [8.110], 172
  - optics, 168
  - probability distribution
    - $k$ -dimensional [2.556], 58
    - 1-dimensional [2.552], 58
- Geiger's law [4.169], 103
- Geiger-Nuttall rule [4.170], 103
- GENERAL CONSTANTS, 7
- GENERAL RELATIVITY, 67
- generalised coordinates [3.213], 79
- Generalised dynamics**, 79
- generalised momentum [3.218], 79
- geodesic deviation [3.56], 67
- geodesic equation [3.54], 67
- geometric
  - distribution [2.548], 57
  - mean [2.109], 27
  - progression [2.107], 27
- Geometrical optics**, 168
- Gibbs
  - constant (value), 9
  - distribution [5.113], 114
  - entropy [5.106], 114
  - free energy [5.35], 108
- Gibbs's phase rule [5.54], 109
- GIBBS–HELMHOLTZ EQUATIONS, 109
- Gibbs-Duhem relation [5.38], 108
- giga, 5
- golden mean (value), 9
- golden rule (Fermi's) [4.162], 102
- GRADIENT, 21
- gradient
  - cylindrical coordinates [2.26], 21
  - general coordinates [2.28], 21
  - rectangular coordinates [2.25], 21
  - spherical coordinates [2.27], 21
- gram (use in SI), 5
- grand canonical ensemble [5.113], 114
- grand partition function [5.112], 114
- grand potential
  - definition [5.37], 108
  - from grand partition function [5.115], 115
- grating
  - dispersion [8.31], 164
  - formula [8.27], 164
  - resolving power [8.30], 164
- GRATINGS, 164
- Gravitation**, 66
- gravitation
  - field from a sphere [3.44], 66
  - general relativity, 67
  - Newton's law [3.40], 66
  - Newtonian, 71
  - Newtonian field equations [3.42], 66
- gravitational
  - collapse [9.53], 181
  - constant, 6, 7, 16
  - lens [9.50], 180
  - potential [3.42], 66
  - redshift [9.74], 183
  - wave radiation [9.75], 183
- GRAVITATIONALLY BOUND ORBITAL MOTION, 71
- gravity
  - and motion on Earth [3.38], 66
  - waves (on a fluid surface) [3.320], 86
- gray (unit), 4
- GREEK ALPHABET, 18
- Green's first theorem [2.62], 23
- Green's second theorem [2.63], 23
- Greenwich sidereal time [9.6], 177
- Gregory's series [2.141], 29
- greybody [5.193], 121
- group speed (wave) [3.327], 87
- Grüneisen parameter [6.56], 131
- gyro-frequency [7.265], 157
- gyro-radius [7.268], 157
- gyromagnetic ratio
  - definition [4.138], 100
  - electron [4.140], 100

proton (value), 8  
 gyroscopes, 77  
 gyroscopic  
   limit [3.193], 77  
   nutation [3.194], 77  
   precession [3.191], 77  
   stability [3.192], 77

**H**  
***H*** (magnetic field strength) [7.100], 143  
 half-life (nuclear decay) [4.164], 103  
 half-period zones (Fresnel) [8.49], 166  
 Hall  
   coefficient (dimensions), 16  
   conductivity [7.280], 158  
   effect and coefficient [6.67], 132  
   voltage [6.68], 132  
 Hamilton's equations [3.220], 79  
 Hamilton's principal function [3.213], 79  
 Hamilton-Jacobi equation [3.227], 79  
 Hamiltonian  
   charged particle (Newtonian) [7.138],  
     146  
   charged particle [3.223], 79  
   definition [3.219], 79  
   of a particle [3.222], 79  
   quantum mechanical [4.21], 91  
 Hamiltonian (dimensions), 16  
 HAMILTONIAN DYNAMICS, 79  
 Hamming window [2.584], 60  
 Hanbury Brown and Twiss interferometry,  
   172  
 Hanning window [2.583], 60  
 harmonic mean [2.110], 27  
 HARMONIC OSCILLATOR, 95  
 harmonic oscillator  
   damped [3.196], 78  
   energy levels [4.68], 95  
   entropy [5.108], 114  
   forced [3.204], 78  
   mean energy [6.40], 130  
 Hartree energy [4.76], 95  
 HEAT CAPACITIES, 107  
 heat capacity (dimensions), 16  
 heat capacity in solids  
   Debye [6.45], 130  
   free electron [6.76], 133  
 heat capacity of a gas  
    $C_p - C_V$  [5.17], 107

constant pressure [5.15], 107  
 constant volume [5.14], 107  
 for  $f$  degrees of freedom, 113  
 ratio ( $\gamma$ ) [5.18], 107  
 heat conduction/diffusion equation  
   differential equation [2.340], 43  
   Fick's second law [5.96], 113  
 heat engine efficiency [5.10], 107  
 heat pump efficiency [5.12], 107  
 heavy beam [3.260], 82  
 hectare, 12  
 hecto, 5  
 Heisenberg uncertainty relation [4.7], 90  
 Helmholtz equation [2.341], 43  
 Helmholtz free energy  
   definition [5.32], 108  
   from partition function [5.114], 115  
 hemisphere (centre of mass) [3.170], 76  
 hemispherical shell (centre of mass) [3.171],  
   76  
 henry (unit), 4  
 Hermite equation [2.346], 43  
 Hermite polynomials [4.70], 95  
 Hermitian  
   conjugate operator [4.17], 91  
   matrix [2.73], 24  
   symmetry, 53  
 Heron's formula [2.253], 36  
 herpolhode, 63, 77  
 hertz (unit), 4  
 Hertzian dipole [7.207], 153  
 hexagonal system (crystallographic), 127  
**High energy and nuclear physics**, 103  
 Hohmann cotangential transfer [3.98],  
   70  
 hole current density [6.89], 134  
 Hooke's law [3.230], 80  
 l'Hôpital's rule [2.131], 28  
 HORIZON COORDINATES, 177  
 hour (unit), 5  
 hour angle [9.8], 177  
 Hubble constant (dimensions), 16  
 Hubble constant [9.85], 184  
 Hubble law  
   as a distance indicator [9.45], 180  
   in cosmology [9.83], 184  
 hydrogen atom  
   eigenfunctions [4.80], 96  
   energy [4.81], 96

- Schrödinger equation [4.79], 96
- Hydrogenic atoms**, 95
- HYDROGENLIKE ATOMS – SCHRÖDINGER SOLUTION, 96
- hydrostatic
- compression [3.238], 80
  - condition [3.293], 84
  - equilibrium (of a star) [9.61], 181
- hyperbola, 38
- HYPERBOLIC DERIVATIVES, 41
- hyperbolic motion, 72
- HYPERBOLIC RELATIONSHIPS, 33
- I**
- I* (Stokes parameter) [8.89], 171
- icosahedron, 38
- IDEAL FLUIDS, 84
- IDEAL GAS, 110
- ideal gas
- adiabatic equations [5.58], 110
  - internal energy [5.62], 110
  - isothermal reversible expansion [5.63], 110
  - law [5.57], 110
  - speed of sound [3.318], 86
- IDENTICAL PARTICLES, 115
- illuminance (definition) [5.164], 119
- illuminance (dimensions), 16
- IMAGE CHARGES, 138
- impedance
- acoustic [3.276], 83
  - dimensions, 17
  - electrical, 148
  - transformation [7.166], 149
- impedance of
- capacitor [7.159], 148
  - coaxial transmission line [7.181], 150
  - electromagnetic wave [7.198], 152
  - forced harmonic oscillator [3.212], 78
  - free space
    - definition [7.197], 152
    - value, 7
  - inductor [7.160], 148
  - lossless transmission line [7.174], 150
  - lossy transmission line [7.175], 150
  - microstrip line [7.184], 150
  - open-wire transmission line [7.182], 150
  - paired strip transmission line [7.183], 150
  - terminated transmission line [7.178], 150
  - waveguide
    - TE modes [7.189], 151
    - TM modes [7.188], 151
- impedances
- in parallel [7.158], 148
  - in series [7.157], 148
- impulse (dimensions), 17
- impulse (specific) [3.92], 70
- incompressible flow, 84, 85
- indefinite integrals, 44
- induced charge density [7.84], 142
- INDUCTANCE, 137
- inductance
- dimensions, 17
  - energy [7.154], 148
  - energy of an assembly [7.135], 146
  - impedance [7.160], 148
  - mutual
    - definition [7.147], 147
    - energy [7.135], 146
  - self [7.145], 147
  - voltage across [7.146], 147
- inductance of
- cylinders (coaxial) [7.24], 137
  - solenoid [7.23], 137
  - wire loop [7.26], 137
  - wires (parallel) [7.25], 137
- induction equation (MHD) [7.282], 158
- inductor, *see* inductance
- INELASTIC COLLISIONS, 73
- INEQUALITIES, 30
- inertia tensor [3.136], 74
- inner product [2.1], 20
- Integration**, 44
- integration (numerical), 61
- integration by parts [2.354], 44
- intensity
- correlation [8.109], 172
  - luminous [5.166], 119
  - of interfering beams [8.100], 172
  - radiant [5.154], 118
  - specific [5.171], 120
- Interference**, 162
- interference and coherence [8.100], 172
- intermodal dispersion (optical fibre) [8.79],

169  
 internal energy  
   definition [5.28], 108  
   from partition function [5.116], 115  
   ideal gas [5.62], 110  
   Joule's law [5.55], 110  
   monatomic gas [5.79], 112  
 interval (in general relativity) [3.45], 67  
 invariable plane, 63, 77  
 inverse Compton scattering [7.239], 155  
 INVERSE HYPERBOLIC FUNCTIONS, 35  
 inverse Laplace transform [2.518], 55  
 inverse matrix [2.83], 25  
 inverse square law [3.99], 71  
 INVERSE TRIGONOMETRIC FUNCTIONS, 34  
 ionic bonding [6.55], 131  
 irradiance (definition) [5.152], 118  
 irradiance (dimensions), 17  
 isobaric expansivity [5.19], 107  
 isophotal wavelength [9.39], 179  
 isothermal bulk modulus [5.22], 107  
 isothermal compressibility [5.20], 107  
 ISOTROPIC ELASTIC SOLIDS, 81

**J**  
 Jacobi identity [2.93], 26  
 Jacobian  
   definition [2.332], 42  
   in change of variable [2.333], 42  
 Jeans length [9.56], 181  
 Jeans mass [9.57], 181  
 Johnson noise [5.141], 117  
 joint probability [2.568], 59  
 Jones matrix [8.85], 170  
 Jones vectors  
   definition [8.84], 170  
   examples [8.84], 170  
 JONES VECTORS AND MATRICES, 170  
 Josephson frequency-voltage ratio, 7  
 joule (unit), 4  
 Joule expansion (and Joule coefficient) [5.25],  
   108  
 Joule expansion (entropy change) [5.64],  
   110  
 Joule's law (of internal energy) [5.55],  
   110  
 Joule's law (of power dissipation) [7.155],  
   148  
 Joule-Kelvin coefficient [5.27], 108

Julian centuries [9.5], 177  
 Julian day number [9.1], 177  
 Jupiter data, 176

## K

katal (unit), 4  
 Kelvin  
   circulation theorem [3.287], 84  
   relation [6.83], 133  
   temperature conversion, 15  
   temperature scale [5.2], 106  
   wedge [3.330], 87  
 kelvin (SI definition), 3  
 kelvin (unit), 4  
 Kelvin-Helmholtz timescale [9.55], 181  
 Kepler's laws, 71  
 Kepler's problem, 71  
 Kerr solution (in general relativity) [3.62],  
   67  
 ket vector [4.34], 92  
 kilo, 5  
 kilogram (SI definition), 3  
 kilogram (unit), 4  
 kinematic viscosity [3.302], 85  
 kinematics, 63  
 kinetic energy  
   definition [3.65], 68  
   for a rotating body [3.142], 74  
   Galilean transformation [3.6], 64  
   in the virial theorem [3.102], 71  
   loss after collision [3.128], 73  
   of a particle [3.216], 79  
   of monatomic gas [5.79], 112  
   operator (quantum) [4.20], 91  
   relativistic [3.73], 68  
   w.r.t. principal axes [3.145], 74  
**Kinetic theory**, 112  
 Kirchhoff's (radiation) law [5.180], 120  
 KIRCHHOFF'S DIFFRACTION FORMULA, 166  
 KIRCHHOFF'S LAWS, 149  
 Klein-Nishina cross section [7.243], 155  
 Klein-Gordon equation [4.181], 104  
 Knudsen flow [5.99], 113  
 Kronecker delta [2.442], 50  
 kurtosis estimator [2.545], 57

**L**  
 ladder operators (angular momentum) [4.108],  
   98

- Lagrange's identity [2.7], 20
- Lagrangian (dimensions), 17
- LAGRANGIAN DYNAMICS, 79
- Lagrangian of
- charged particle [3.217], 79
  - particle [3.216], 79
  - two mutually attracting bodies [3.85], 69
- Laguerre equation [2.347], 43
- Laguerre polynomials (associated), 96
- Lamé coefficients [3.240], 81
- LAMINAR VISCOUS FLOW, 85
- Landé  $g$ -factor [4.146], 100
- Landau diamagnetic susceptibility [6.80], 133
- Landau length [7.249], 156
- Langevin function (from Brillouin fn) [4.147], 101
- Langevin function [7.111], 144
- Laplace equation
- definition [2.339], 43
  - solution in spherical harmonics [2.440], 49
- Laplace series [2.439], 49
- Laplace transform
- convolution [2.516], 55
  - definition [2.514], 55
  - derivative of transform [2.520], 55
  - inverse [2.518], 55
  - of derivative [2.519], 55
  - substitution [2.521], 55
  - translation [2.523], 55
- LAPLACE TRANSFORM PAIRS, 56
- LAPLACE TRANSFORM THEOREMS, 55
- Laplace transforms**, 55
- Laplace's formula (surface tension) [3.337], 88
- Laplacian
- cylindrical coordinates [2.46], 23
  - general coordinates [2.48], 23
  - rectangular coordinates [2.45], 23
  - spherical coordinates [2.47], 23
- LAPLACIAN (SCALAR), 23
- lapse rate (adiabatic) [3.294], 84
- Larmor frequency [7.265], 157
- Larmor radius [7.268], 157
- Larmor's formula [7.132], 146
- laser
- cavity  $Q$  [8.126], 174
  - cavity line width [8.127], 174
  - cavity modes [8.124], 174
  - cavity stability [8.123], 174
  - threshold condition [8.129], 174
- LASERS, 174
- latent heat [5.48], 109
- lattice constants of elements, 124
- Lattice dynamics**, 129
- LATTICE FORCES (SIMPLE), 131
- lattice plane spacing [6.11], 126
- LATTICE THERMAL EXPANSION AND CONDUCTION, 131
- lattice vector [6.7], 126
- latus-rectum [3.109], 71
- Laue equations [6.28], 128
- Laurent series [2.168], 31
- LCR circuits**, 147
- LCR DEFINITIONS, 147
- least-squares fitting, 60
- Legendre equation
- and polynomials [2.421], 47
  - definition [2.343], 43
- LEGENDRE POLYNOMIALS, 47
- Leibniz theorem [2.296], 40
- length (dimensions), 17
- Lennard-Jones 6-12 potential [6.52], 131
- lens blooming [8.7], 162
- LENSES AND MIRRORS, 168
- lensmaker's formula [8.66], 168
- Levi-Civita symbol (3-D) [2.443], 50
- l'Hôpital's rule [2.131], 28
- LIÉNARD-WIECHERT POTENTIALS, 139
- light (speed of), 6, 7
- LIMITS, 28
- line charge (electric field from) [7.29], 138
- line fitting, 60
- Line radiation**, 173
- line shape
- collisional [8.114], 173
  - Doppler [8.116], 173
  - natural [8.112], 173
- line width
- collisional/pressure [8.115], 173
  - Doppler broadened [8.117], 173
  - laser cavity [8.127], 174
  - natural [8.113], 173
  - Schawlow-Townes [8.128], 174
- linear absorption coefficient [5.175], 120
- linear expansivity (definition) [5.19], 107

- linear expansivity (of a crystal) [6.57], 131
- linear regression, 60
- linked flux [7.149], 147
- liquid drop model [4.172], 103
- litre (unit), 5
- local civil time [9.4], 177
- local sidereal time [9.7], 177
- local thermodynamic equilibrium (LTE), 116, 120
- $\ln(1+x)$  (series expansion) [2.133], 29
- logarithm of complex numbers [2.162], 30
- logarithmic decrement [3.202], 78
- London's formula (interacting dipoles) [6.50], 131
- longitudinal elastic modulus [3.241], 81
- look-back time [9.96], 185
- Lorentz
  - broadening [8.112], 173
  - contraction [3.8], 64
  - factor ( $\gamma$ ) [3.7], 64
  - force [7.122], 145
- LORENTZ (SPACETIME) TRANSFORMATIONS, 64
- Lorentz factor (dynamical) [3.69], 68
- Lorentz transformation
  - in electrodynamics, 141
  - of four-vectors, 65
  - of momentum and energy, 65
  - of time and position, 64
  - of velocity, 64
- Lorentz-Lorenz formula [7.93], 142
- Lorentzian distribution [2.555], 58
- Lorentzian (Fourier transform of) [2.505], 54
- Lorenz
  - constant [6.66], 132
  - gauge condition [7.43], 139
- lumen (unit), 4
- luminance [5.168], 119
- luminosity distance [9.98], 185
- luminosity–magnitude relation [9.31], 179
- luminous
  - density [5.160], 119
  - efficacy [5.169], 119
  - efficiency [5.170], 119
  - energy [5.157], 119
  - exitance [5.162], 119
  - flux [5.159], 119
  - intensity (dimensions), 17
  - intensity [5.166], 119
- lux (unit), 4
- ## M
- Mach number [3.315], 86
- Mach wedge [3.328], 87
- Maclaurin series [2.125], 28
- MACROSCOPIC THERMODYNAMIC VARIABLES, 115
- Madelung constant (value), 9
- Madelung constant [6.55], 131
- magnetic
  - diffusivity [7.282], 158
  - flux quantum, 6, 7
  - monopoles (none) [7.52], 140
  - permeability,  $\mu$ ,  $\mu_r$  [7.107], 143
  - quantum number [4.131], 100
  - scalar potential [7.7], 136
  - susceptibility,  $\chi_H$ ,  $\chi_B$  [7.103], 143
  - vector potential
    - definition [7.40], 139
    - from  $\mathbf{J}$  [7.47], 139
    - of a moving charge [7.49], 139
- magnetic dipole, *see* dipole
- magnetic field
  - around objects, 138
  - dimensions, 17
  - energy density [7.128], 146
  - Lorentz transformation, 141
  - static, 136
  - strength ( $\mathbf{H}$ ) [7.100], 143
  - thermodynamic work [5.8], 106
  - wave equation [7.194], 152
- MAGNETIC FIELDS, 138
- magnetic flux (dimensions), 17
- magnetic flux density (dimensions), 17
- magnetic flux density from
  - current [7.11], 136
  - current density [7.10], 136
  - dipole [7.36], 138
  - electromagnet [7.38], 138
  - line current (Biot–Savart law) [7.9], 136
  - solenoid (finite) [7.38], 138
  - solenoid (infinite) [7.33], 138
  - uniform cylindrical current [7.34], 138
  - waveguide [7.190], 151



- wire [7.34], 138  
 wire loop [7.37], 138
- MAGNETIC MOMENTS, 100
- magnetic vector potential (dimensions), 17
- MAGNETISATION, 143
- magnetisation  
   definition [7.97], 143  
   dimensions, 17  
   isolated spins [4.151], 101  
   quantum paramagnetic [4.150], 101
- magnetogyric ratio [4.138], 100
- MAGNETOHYDRODYNAMICS, 158
- magnetosonic waves [7.285], 158
- MAGNETOSTATICS, 136
- magnification (longitudinal) [8.71], 168
- magnification (transverse) [8.70], 168
- magnitude (astronomical)  
   –flux relation [9.32], 179  
   –luminosity relation [9.31], 179  
   absolute [9.29], 179  
   apparent [9.27], 179
- major axis [3.106], 71
- Malus's law [8.83], 170
- Mars data, 176
- mass (dimensions), 17
- mass absorption coefficient [5.176], 120
- mass ratio (of a rocket) [3.94], 70
- MATHEMATICAL CONSTANTS, 9
- Mathematics, 19–62
- matrices (square), 25
- MATRIX ALGEBRA, 24
- matrix element (quantum) [4.32], 92
- maxima [2.336], 42
- MAXWELL'S EQUATIONS, 140
- MAXWELL'S EQUATIONS (USING  $D$  AND  $H$ ), 140
- MAXWELL'S RELATIONS, 109
- MAXWELL–BOLTZMANN DISTRIBUTION, 112
- Maxwell-Boltzmann distribution  
   mean speed [5.86], 112  
   most probable speed [5.88], 112  
   rms speed [5.87], 112  
   speed distribution [5.84], 112
- mean  
   arithmetic [2.108], 27  
   geometric [2.109], 27  
   harmonic [2.110], 27
- mean estimator [2.541], 57
- mean free path  
   and absorption coefficient [5.175], 120  
   Maxwell-Boltzmann [5.89], 113
- mean intensity [5.172], 120
- mean-life (nuclear decay) [4.165], 103
- mega, 5
- melting points of elements, 124
- meniscus [3.339], 88
- Mensuration**, 35
- Mercury data, 176
- method of images, 138
- metre (SI definition), 3
- metre (unit), 4
- metric elements and coordinate systems, 21
- MHD equations [7.283], 158
- micro, 5
- microcanonical ensemble [5.109], 114
- micron (unit), 5
- microstrip line (impedance) [7.184], 150
- Miller-Bravais indices [6.20], 126
- milli, 5
- minima [2.337], 42
- minimum deviation (of a prism) [8.74], 169
- minor axis [3.107], 71
- minute (unit), 5
- mirror formula [8.67], 168
- Miscellaneous**, 18
- mobility (dimensions), 17
- mobility (in conductors) [6.88], 134
- modal dispersion (optical fibre) [8.79], 169
- modified Bessel functions [2.419], 47
- modified Julian day number [9.2], 177
- modulus (of a complex number) [2.155], 30
- MOIRÉ FRINGES, 35
- molar gas constant (dimensions), 17
- molar volume, 9
- mole (SI definition), 3
- mole (unit), 4
- molecular flow [5.99], 113
- moment  
   electric dipole [7.81], 142  
   magnetic dipole [7.94], 143  
   magnetic dipole [7.95], 143
- moment of area [3.258], 82
- moment of inertia

- cone [3.160], 75
- cylinder [3.155], 75
- dimensions, 17
- disk [3.168], 75
- ellipsoid [3.163], 75
- elliptical lamina [3.166], 75
- rectangular cuboid [3.158], 75
- sphere [3.152], 75
- spherical shell [3.153], 75
- thin rod [3.150], 75
- triangular plate [3.169], 75
- two-body system [3.83], 69
- moment of inertia ellipsoid [3.147], 74
- MOMENT OF INERTIA TENSOR, 74
- moment of inertia tensor [3.136], 74
- MOMENTS OF INERTIA, 75
- momentum
  - definition [3.64], 68
  - dimensions, 17
  - generalised [3.218], 79
  - relativistic [3.70], 68
- MOMENTUM AND ENERGY TRANSFORMATIONS, 65
- MONATOMIC GAS, 112
- monatomic gas
  - entropy [5.83], 112
  - equation of state [5.78], 112
  - heat capacity [5.82], 112
  - internal energy [5.79], 112
  - pressure [5.77], 112
- monoclinic system (crystallographic), 127
- MOON DATA, 176
- motif [6.31], 128
- motion under constant acceleration, 68
- Mott scattering formula [4.180], 104
- $\mu$ ,  $\mu_r$  (magnetic permeability) [7.107], 143
- multilayer films (in optics) [8.8], 162
- multimode dispersion (optical fibre) [8.79], 169
- multiplicity (quantum)
  - $j$  [4.133], 100
  - $l$  [4.112], 98
- multistage rocket [3.95], 70
- MULTIVARIATE NORMAL DISTRIBUTION, 58
- MUON AND TAU CONSTANTS, 9
- muon physical constants, 9
- mutual
  - capacitance [7.134], 146
  - inductance (definition) [7.147], 147
  - inductance (energy) [7.135], 146
  - mutual coherence function [8.97], 172
- N**
- nabla, 21
- NAMED INTEGRALS, 45
- nano, 5
- natural broadening profile [8.112], 173
- natural line width [8.113], 173
- Navier-Stokes equation [3.301], 85
- nearest neighbour distances, 127
- Neptune data, 176
- neutron
  - Compton wavelength, 8
  - gyromagnetic ratio, 8
  - magnetic moment, 8
  - mass, 8
  - molar mass, 8
- NEUTRON CONSTANTS, 8
- neutron star degeneracy pressure [9.77], 183
- newton (unit), 4
- Newton's law of Gravitation [3.40], 66
- Newton's lens formula [8.65], 168
- NEWTON'S RINGS, 162
- Newton's rings [8.1], 162
- Newton-Raphson method [2.593], 61
- NEWTONIAN GRAVITATION, 66
- noggin, 13
- NOISE, 117
- noise
  - figure [5.143], 117
  - Johnson [5.141], 117
  - Nyquist's theorem [5.140], 117
  - shot [5.142], 117
  - temperature [5.140], 117
- normal (unit principal) [2.284], 39
- normal distribution [2.552], 58
- normal plane, 39
- NUCLEAR BINDING ENERGY, 103
- NUCLEAR COLLISIONS, 104
- NUCLEAR DECAY, 103
- nuclear decay law [4.163], 103
- nuclear magneton, 7
- number density (dimensions), 17
- numerical aperture (optical fibre) [8.78], 169
- NUMERICAL DIFFERENTIATION, 61
- NUMERICAL INTEGRATION, 61

**Numerical methods**, 60NUMERICAL SOLUTIONS TO  $f(x)=0$ , 61

NUMERICAL SOLUTIONS TO ORDINARY DIFFERENTIAL EQUATIONS, 62

notation [3.194], 77

Nyquist's theorem [5.140], 117

**O**

OBLIQUE ELASTIC COLLISIONS, 73

obliquity factor (diffraction) [8.46], 166

obliquity of the ecliptic [9.13], 178

observable (quantum physics) [4.5], 90

**Observational astrophysics**, 179

octahedron, 38

odd functions, 53

ODEs (numerical solutions), 62

ohm (unit), 4

Ohm's law (in MHD) [7.281], 158

Ohm's law [7.140], 147

opacity [5.176], 120

open-wire transmission line [7.182], 150

operator

angular momentum

and other operators [4.23], 91

definitions [4.105], 98

Hamiltonian [4.21], 91

kinetic energy [4.20], 91

momentum [4.19], 91

parity [4.24], 91

position [4.18], 91

time dependence [4.27], 91

OPERATORS, 91

optic branch (phonon) [6.37], 129

optical coating [8.8], 162

optical depth [5.177], 120

OPTICAL FIBRES, 169

optical path length [8.63], 168

Optics, 161–174

ORBITAL ANGULAR DEPENDENCE, 97

ORBITAL ANGULAR MOMENTUM, 98

orbital motion, 71

orbital radius (Bohr atom) [4.73], 95

order (in diffraction) [8.26], 164

ordinary modes [7.271], 157

orthogonal matrix [2.85], 25

orthogonality

associated Legendre functions [2.434],  
48

Legendre polynomials [2.424], 47

orthorhombic system (crystallographic), 127

**Oscillating systems**, 78

osculating plane, 39

Otto cycle efficiency [5.13], 107

overdamping [3.201], 78

**P** $p$  orbitals [4.95], 97

P-waves [3.263], 82

packing fraction (of spheres), 127

paired strip (impedance of) [7.183], 150

parabola, 38

parabolic motion [3.88], 69

parallax (astronomical) [9.46], 180

parallel axis theorem [3.140], 74

parallel impedances [7.158], 148

parallel wire feeder (inductance) [7.25],  
137paramagnetic susceptibility (Pauli) [6.79],  
133

paramagnetism (quantum), 101

PARAMAGNETISM AND DIAMAGNETISM, 144

parity operator [4.24], 91

Parseval's relation [2.495], 53

Parseval's theorem

integral form [2.496], 53

series form [2.480], 52

PARTIAL DERIVATIVES, 42

partial widths (and total width) [4.176],  
104

PARTICLE IN A RECTANGULAR BOX, 94

**Particle motion**, 68

partition function

atomic [5.126], 116

definition [5.110], 114

macroscopic variables from, 115

pascal (unit), 4

PAULI MATRICES, 26

Pauli matrices [2.94], 26

Pauli paramagnetic susceptibility [6.79],  
133Pauli spin matrices (and Weyl eqn.) [4.182]  
104Pearson's  $r$  [2.546], 57

Peltier effect [6.82], 133

pendulum

compound [3.182], 76

conical [3.180], 76

double [3.183], 76

- simple [3.179], 76
- torsional [3.181], 76
- PENDULUMS, 76
- perfect gas, 110
- pericentre (of an orbit) [3.110], 71
- perimeter
  - of circle [2.261], 37
  - of ellipse [2.266], 37
- PERIMETER, AREA, AND VOLUME, 37
- period (of an orbit) [3.113], 71
- Periodic table**, 124
- permeability
  - dimensions, 17
  - magnetic [7.107], 143
  - of vacuum, 6, 7
- permittivity
  - dimensions, 17
  - electrical [7.90], 142
  - of vacuum, 6, 7
- permutation tensor ( $\epsilon_{ijk}$ ) [2.443], 50
- perpendicular axis theorem [3.148], 74
- Perturbation theory**, 102
- peta, 5
- petrol engine efficiency [5.13], 107
- phase object (diffraction by weak) [8.43], 165
- phase rule (Gibbs's) [5.54], 109
- phase speed (wave) [3.325], 87
- PHASE TRANSITIONS, 109
- PHONON DISPERSION RELATIONS, 129
- phonon modes (mean energy) [6.40], 130
- PHOTOMETRIC WAVELENGTHS, 179
- PHOTOMETRY, 119
- photon energy [4.3], 90
- Physical constants**, 6
- PI ( $\pi$ ) TO 1 000 DECIMAL PLACES, 18
- PI ( $\pi$ ), 9
- pico, 5
- pipe (flow of fluid along) [3.305], 85
- pipe (twisting of) [3.255], 81
- pitch angle, 159
- Planck
  - constant, 6, 7
  - constant (dimensions), 17
  - function [5.184], 121
  - length, 7
  - mass, 7
  - time, 7
- Planck-Einstein relation [4.3], 90
- plane polarisation, 170
- PLANE TRIANGLES, 36
- plane wave expansion [2.427], 47
- PLANETARY BODIES, 180
- PLANETARY DATA, 176
- plasma
  - beta [7.278], 158
  - dispersion relation [7.261], 157
  - frequency [7.259], 157
  - group velocity [7.264], 157
  - phase velocity [7.262], 157
  - refractive index [7.260], 157
- Plasma physics**, 156
- PLATONIC SOLIDS, 38
- Pluto data, 176
- p-n junction [6.92], 134
- Poincaré sphere, 171
- point charge (electric field from) [7.5], 136
- Poiseuille flow [3.305], 85
- Poisson brackets [3.224], 79
- Poisson distribution [2.549], 57
- Poisson ratio
  - and elastic constants [3.251], 81
  - simple definition [3.231], 80
- Poisson's equation [7.3], 136
- polarisability [7.91], 142
- Polarisation**, 170
- POLARISATION, 142
- polarisation (electrical, per unit volume) [7.83], 142
- polarisation (of radiation)
  - angle [8.81], 170
  - axial ratio [8.88], 171
  - degree of [8.96], 171
  - elliptical [8.80], 170
  - ellipticity [8.82], 170
  - reflection law [7.218], 154
- polarisers [8.85], 170
- polhode, 63, 77
- POPULATION DENSITIES, 116
- potential
  - chemical [5.28], 108
  - difference (and work) [5.9], 106
  - difference (between points) [7.2], 136
  - electrical [7.46], 139
  - electrostatic [7.1], 136
  - energy (elastic) [3.235], 80
  - energy in Hamiltonian [3.222], 79

- energy in Lagrangian [3.216], 79
  - field equations [7.45], 139
  - four-vector [7.77], 141
  - grand [5.37], 108
  - Liénard–Wiechert, 139
  - Lorentz transformation [7.75], 141
  - magnetic scalar [7.7], 136
  - magnetic vector [7.40], 139
  - Rutherford scattering [3.114], 72
  - thermodynamic [5.35], 108
  - velocity [3.296], 84
  - POTENTIAL FLOW, 84
  - POTENTIAL STEP, 92
  - POTENTIAL WELL, 93
  - power (dimensions), 17
  - power gain
    - antenna [7.211], 153
    - short dipole [7.213], 153
  - POWER SERIES, 28
  - Power theorem [2.495], 53
  - Poynting vector (dimensions), 17
  - Poynting vector [7.130], 146
  - pp (proton-proton) chain, 182
  - Prandtl number [3.314], 86
  - precession (gyroscopic) [3.191], 77
  - PRECESSION OF EQUINOXES, 178
  - pressure
    - broadening [8.115], 173
    - critical [5.75], 111
    - degeneracy [9.77], 183
    - dimensions, 17
    - fluctuations [5.136], 116
    - from partition function [5.118], 115
    - hydrostatic [3.238], 80
    - in a monatomic gas [5.77], 112
    - radiation, 152
    - thermodynamic work [5.5], 106
    - waves [3.263], 82
  - primitive cell [6.1], 126
  - primitive vectors (and lattice vectors) [6.7], 126
  - primitive vectors (of cubic lattices), 127
  - PRINCIPAL AXES, 74
  - principal moments of inertia [3.143], 74
  - principal quantum number [4.71], 95
  - principle of least action [3.213], 79
  - prism
    - determining refractive index [8.75], 169
    - deviation [8.73], 169
    - dispersion [8.76], 169
    - minimum deviation [8.74], 169
    - transmission angle [8.72], 169
  - PRISMS (DISPERSING), 169
  - probability
    - conditional [2.567], 59
    - density current [4.13], 90
    - distributions
      - continuous, 58
      - discrete, 57
    - joint [2.568], 59
  - Probability and statistics**, 57
  - product (derivative of) [2.293], 40
  - product (integral of) [2.354], 44
  - product of inertia [3.136], 74
  - progression (arithmetic) [2.104], 27
  - progression (geometric) [2.107], 27
  - PROGRESSIONS AND SUMMATIONS, 27
  - projectiles, 69
  - propagation in cold plasmas, 157
  - PROPAGATION IN CONDUCTING MEDIA, 155
  - PROPAGATION OF ELASTIC WAVES, 83
  - PROPAGATION OF LIGHT, 65
  - proper distance [9.97], 185
  - PROTON CONSTANTS, 8
  - proton mass, 6
  - proton-proton chain, 182
  - pulsar
    - braking index [9.66], 182
    - characteristic age [9.67], 182
    - dispersion [9.72], 182
    - magnetic dipole radiation [9.69], 182
  - PULSARS, 182
  - pyramid (centre of mass) [3.175], 76
  - pyramid (volume) [2.272], 37
- ## Q
- Q*, *see* quality factor
  - Q* (Stokes parameter) [8.90], 171
  - QUADRATIC EQUATIONS, 50
  - quadrature, 61
  - quadrature (integration), 44
  - quality factor
    - Fabry-Perot etalon [8.14], 163
    - forced harmonic oscillator [3.211], 78
    - free harmonic oscillator [3.203], 78
    - laser cavity [8.126], 174

- LCR circuits [7.152], 148  
 quantum concentration [5.83], 112  
**Quantum definitions**, 90  
 QUANTUM PARAMAGNETISM, 101  
 Quantum physics, 89–104  
 QUANTUM UNCERTAINTY RELATIONS, 90  
 quarter-wave condition [8.3], 162  
 quarter-wave plate [8.85], 170  
 quartic minimum, 42
- R**
- RADIAL FORMS, 22  
 radian (unit), 4  
 radiance [5.156], 118  
 radiant  
   energy [5.145], 118  
   energy density [5.148], 118  
   exitance [5.150], 118  
   flux [5.147], 118  
   intensity (dimensions), 17  
   intensity [5.154], 118  
 radiation  
   blackbody [5.184], 121  
   bremsstrahlung [7.297], 160  
   Cherenkov [7.247], 156  
   field of a dipole [7.207], 153  
   flux from dipole [7.131], 146  
   resistance [7.209], 153  
   synchrotron [7.287], 159  
 RADIATION PRESSURE, 152  
 radiation pressure  
   extended source [7.203], 152  
   isotropic [7.200], 152  
   momentum density [7.199], 152  
   point source [7.204], 152  
   specular reflection [7.202], 152  
**Radiation processes**, 118  
 RADIATIVE TRANSFER, 120  
 radiative transfer equation [5.179], 120  
 radiative transport (in stars) [9.63], 181  
 radioactivity, 103  
 RADIOMETRY, 118  
 radius of curvature  
   definition [2.282], 39  
   in bending [3.258], 82  
   relation to curvature [2.287], 39  
 radius of gyration (see footnote), 75  
 Ramsauer effect [4.52], 93  
 RANDOM WALK, 59  
 random walk  
   Brownian motion [5.98], 113  
   one-dimensional [2.562], 59  
   three-dimensional [2.564], 59  
 range (of projectile) [3.90], 69  
 Rankine conversion [1.3], 15  
 Rankine-Hugoniot shock relations [3.334], 87  
 Rayleigh  
   distribution [2.554], 58  
   resolution criterion [8.41], 165  
   scattering [7.236], 155  
   theorem [2.496], 53  
 Rayleigh-Jeans law [5.187], 121  
 reactance (definition), 148  
 reciprocal  
   lattice vector [6.8], 126  
   matrix [2.83], 25  
   vectors [2.16], 20  
 reciprocity [2.330], 42  
 RECOGNISED NON-SI UNITS, 5  
 rectangular aperture diffraction [8.39], 165  
 rectangular coordinates, 21  
 rectangular cuboid moment of inertia [3.158], 75  
 rectifying plane, 39  
 recurrence relation  
   associated Legendre functions [2.433], 48  
   Legendre polynomials [2.423], 47  
 redshift  
   –flux density relation [9.99], 185  
   cosmological [9.86], 184  
   gravitational [9.74], 183  
 REDUCED MASS (OF TWO INTERACTING BODIES), 69  
 reduced units (thermodynamics) [5.71], 111  
 reflectance coefficient  
   and Fresnel equations [7.227], 154  
   dielectric film [8.4], 162  
   dielectric multilayer [8.8], 162  
 reflection coefficient  
   acoustic [3.283], 83  
   dielectric boundary [7.230], 154  
   potential barrier [4.58], 94  
   potential step [4.41], 92  
   potential well [4.48], 93

- transmission line [7.179], 150
  - reflection grating [8.29], 164
  - reflection law [7.216], 154
  - REFLECTION, REFRACTION, AND TRANSMISSION, 154
  - refraction law (Snell's) [7.217], 154
  - refractive index of
    - dielectric medium [7.195], 152
    - ohmic conductor [7.234], 155
    - plasma [7.260], 157
  - refrigerator efficiency [5.11], 107
  - regression (linear), 60
  - relativistic beaming [3.25], 65
  - relativistic doppler effect [3.22], 65
  - RELATIVISTIC DYNAMICS, 68
  - RELATIVISTIC ELECTRODYNAMICS, 141
  - RELATIVISTIC WAVE EQUATIONS, 104
  - relativity (general), 67
  - relativity (special), 64
  - relaxation time
    - and electron drift [6.61], 132
    - in a conductor [7.156], 148
    - in plasmas, 156
  - residuals [2.572], 60
  - Residue theorem [2.170], 31
  - residues (in complex analysis), 31
  - resistance
    - and impedance, 148
    - dimensions, 17
    - energy dissipated in [7.155], 148
    - radiation [7.209], 153
  - resistivity [7.142], 147
  - resistor, *see* resistance
  - resolving power
    - chromatic (of an etalon) [8.21], 163
    - of a diffraction grating [8.30], 164
    - Rayleigh resolution criterion [8.41], 165
  - resonance
    - forced oscillator [3.209], 78
  - resonance lifetime [4.177], 104
  - resonant frequency (LCR) [7.150], 148
  - RESONANT LCR CIRCUITS, 148
  - restitution (coefficient of) [3.127], 73
  - retarded time, 139
  - revolution (volume and surface of), 39
  - Reynolds number [3.311], 86
  - ribbon (twisting of) [3.256], 81
  - Ricci tensor [3.57], 67
  - Riemann tensor [3.50], 67
  - right ascension [9.8], 177
  - rigid body
    - angular momentum [3.141], 74
    - kinetic energy [3.142], 74
  - Rigid body dynamics**, 74
  - rigidity modulus [3.249], 81
  - ripples [3.321], 86
  - rms (standard deviation) [2.543], 57
  - Robertson-Walker metric [9.87], 184
  - Roche limit [9.43], 180
  - rocket equation [3.94], 70
  - ROCKETRY, 70
  - rod
    - bending, 82
    - moment of inertia [3.150], 75
    - stretching [3.230], 80
    - waves in [3.271], 82
  - Rodrigues' formula [2.422], 47
  - Roots of quadratic and cubic equations**, 50
  - Rossby number [3.316], 86
  - rot (curl), 22
  - ROTATING FRAMES, 66
  - ROTATION MATRICES, 26
  - rotation measure [7.273], 157
  - Runge Kutta method [2.603], 62
  - RUTHERFORD SCATTERING, 72
  - Rutherford scattering formula [3.124], 72
  - Rydberg constant, 6, 7
    - and Bohr atom [4.77], 95
    - dimensions, 17
  - Rydberg's formula [4.78], 95
- ## S
- s orbitals [4.92], 97
  - S-waves [3.262], 82
  - Sackur-Tetrode equation [5.83], 112
  - saddle point [2.338], 42
  - Saha equation (general) [5.128], 116
  - Saha equation (ionisation) [5.129], 116
  - Saturn data, 176
  - scalar effective mass [6.87], 134
  - scalar product [2.1], 20
  - scalar triple product [2.10], 20
  - scale factor (cosmic) [9.87], 184
  - scattering
    - angle (Rutherford) [3.116], 72
    - Born approximation [4.178], 104
    - Compton [7.240], 155

- crystal [6.32], 128
- inverse Compton [7.239], 155
- Klein-Nishina [7.243], 155
- Mott (identical particles) [4.180], 104
- potential (Rutherford) [3.114], 72
- processes (electron), 155
- Rayleigh [7.236], 155
- Rutherford [3.124], 72
- Thomson [7.238], 155
- scattering cross-section, *see* cross-section
- Schawlow-Townes line width [8.128], 174
- Schrödinger equation [4.15], 90
- Schwarz inequality [2.152], 30
- Schwarzschild geometry (in GR) [3.61], 67
- Schwarzschild radius [9.73], 183
- Schwarzschild's equation [5.179], 120
- screw dislocation [6.22], 128
- sec  $x$ 
  - definition [2.228], 34
  - series expansion [2.138], 29
- secant method (of root-finding) [2.592], 61
- sech  $x$  [2.229], 34
- second (SI definition), 3
- second (time interval), 4
- second moment of area [3.258], 82
- Sedov-Taylor shock relation [3.331], 87
- selection rules (dipole transition) [4.91], 96
- self-diffusion [5.93], 113
- self-inductance [7.145], 147
- semi-ellipse (centre of mass) [3.178], 76
- semi-empirical mass formula [4.173], 103
- semi-latus-rectum [3.109], 71
- semi-major axis [3.106], 71
- semi-minor axis [3.107], 71
- semiconductor diode [6.92], 134
- semiconductor equation [6.90], 134
- SERIES EXPANSIONS, 29
- series impedances [7.157], 148
- Series, summations, and progressions**, 27
- shah function (Fourier transform of) [2.510], 54
- shear
  - modulus [3.249], 81
  - strain [3.237], 80
  - viscosity [3.299], 85
  - waves [3.262], 82
- shear modulus (dimensions), 17
- sheet of charge (electric field) [7.32], 138
- shift theorem (Fourier transform) [2.501], 54
- shock
  - Rankine-Hugoniot conditions [3.334], 87
  - spherical [3.331], 87
- SHOCKS, 87
- shot noise [5.142], 117
- SI base unit definitions, 3
- SI BASE UNITS, 4
- SI DERIVED UNITS, 4
- SI PREFIXES, 5
- SI units**, 4
- sidelobes (diffraction by 1-D slit) [8.38], 165
- sidereal time [9.7], 177
- siemens (unit), 4
- sievert (unit), 4
- similarity theorem (Fourier transform) [2.500], 54
- simple cubic structure, 127
- simple harmonic oscillator, *see* harmonic oscillator
- simple pendulum [3.179], 76
- Simpson's rule [2.586], 61
- sin  $x$ 
  - and Euler's formula [2.218], 34
  - series expansion [2.136], 29
- sinc function [2.512], 54
- sine formula
  - planar triangles [2.246], 36
  - spherical triangles [2.255], 36
- sinh  $x$ 
  - definition [2.219], 34
  - series expansion [2.144], 29
- $\sin^{-1}x$ , *see* arccos  $x$
- skew-symmetric matrix [2.87], 25
- skewness estimator [2.544], 57
- skin depth [7.235], 155
- slit diffraction (broad slit) [8.37], 165
- slit diffraction (Young's) [8.24], 164
- Snell's law (acoustics) [3.284], 83
- Snell's law (electromagnetism) [7.217], 154
- soap bubbles [3.337], 88
- solar constant, 176
- SOLAR DATA, 176
- Solar system data**, 176



- solenoid  
 finite [7.38], 138  
 infinite [7.33], 138  
 self inductance [7.23], 137
- solid angle (subtended by a circle) [2.278], 37
- Solid state physics, 123–134
- sound speed (in a plasma) [7.275], 158
- sound, speed of [3.317], 86
- space cone, 77
- space frequency [3.188], 77
- space impedance [7.197], 152
- spatial coherence [8.108], 172
- Special functions and polynomials**, 46
- special relativity, 64
- specific  
 charge on electron, 8  
 emission coefficient [5.174], 120  
 heat capacity, *see* heat capacity  
 definition, 105  
 dimensions, 17  
 intensity (blackbody) [5.184], 121  
 intensity [5.171], 120
- specific impulse [3.92], 70
- speckle intensity distribution [8.110], 172
- speckle size [8.111], 172
- spectral energy density  
 blackbody [5.186], 121  
 definition [5.173], 120
- spectral function (synchrotron) [7.295], 159
- SPECTRAL LINE BROADENING, 173
- speed (dimensions), 17
- speed distribution (Maxwell-Boltzmann) [5.84], 112
- speed of light (equation) [7.196], 152
- speed of light (value), 6
- speed of sound [3.317], 86
- sphere  
 area [2.263], 37  
 Brownian motion [5.98], 113  
 capacitance [7.12], 137  
 capacitance of adjacent [7.14], 137  
 capacitance of concentric [7.18], 137  
 close-packed, 127  
 collisions of, 73  
 electric field [7.27], 138  
 geometry on a, 36  
 gravitation field from a [3.44], 66  
 in a viscous fluid [3.308], 85  
 in potential flow [3.298], 84  
 moment of inertia [3.152], 75  
 Poincaré, 171  
 polarisability, 142  
 volume [2.264], 37
- spherical Bessel function [2.420], 47
- spherical cap  
 area [2.275], 37  
 centre of mass [3.177], 76  
 volume [2.276], 37
- spherical excess [2.260], 36
- SPHERICAL HARMONICS, 49
- spherical harmonics  
 definition [2.436], 49  
 Laplace equation [2.440], 49  
 orthogonality [2.437], 49
- spherical polar coordinates, 21
- spherical shell (moment of inertia) [3.153], 75
- spherical surface (capacitance of near) [7.16], 137
- SPHERICAL TRIANGLES, 36
- spin  
 and total angular momentum [4.128], 100  
 degeneracy, 115  
 electron magnetic moment [4.141], 100  
 Pauli matrices, 26
- spinning bodies, 77
- spinors [4.182], 104
- Spitzer conductivity [7.254], 156
- spontaneous emission [8.119], 173
- spring constant and wave velocity [3.272], 83
- SQUARE MATRICES, 25
- standard deviation estimator [2.543], 57
- STANDARD FORMS, 44
- STAR FORMATION, 181
- STAR-DELTA TRANSFORMATION, 149
- Static fields**, 136
- statics, 63
- STATIONARY POINTS, 42
- STATISTICAL ENTROPY, 114
- Statistical thermodynamics**, 114
- Stefan-Boltzmann constant, 9
- Stefan-Boltzmann constant (dimensions), 17

- Stefan-Boltzmann constant, 121  
 Stefan-Boltzmann law [5.191], 121  
 stellar aberration [3.24], 65  
**Stellar evolution**, 181  
 STELLAR FUSION PROCESSES, 182  
 STELLAR THEORY, 181  
 step function (Fourier transform of) [2.511], 54  
 steradian (unit), 4  
 stimulated emission [8.120], 173  
 Stirling's formula [2.411], 46  
 STOKES PARAMETERS, 171  
 Stokes parameters [8.95], 171  
 Stokes's law [3.308], 85  
 Stokes's theorem [2.60], 23  
 STRAIGHT-LINE FITTING, 60  
 strain  
   simple [3.229], 80  
   tensor [3.233], 80  
   volume [3.236], 80  
 stress  
   dimensions, 17  
   in fluids [3.299], 85  
   simple [3.228], 80  
   tensor [3.232], 80  
 stress-energy tensor  
   and field equations [3.59], 67  
   perfect fluid [3.60], 67  
 string (waves along a stretched) [3.273], 83  
 Strouhal number [3.313], 86  
 structure factor [6.31], 128  
 sum over states [5.110], 114  
 SUMMARY OF PHYSICAL CONSTANTS, 6  
 summation formulas [2.118], 27  
 Sun data, 176  
 Sunyaev-Zel'dovich effect [9.51], 180  
 surface brightness (blackbody) [5.184], 121  
 surface of revolution [2.280], 39  
 SURFACE TENSION, 88  
 surface tension  
   Laplace's formula [3.337], 88  
   work done [5.6], 106  
 surface tension (dimensions), 17  
 surface waves [3.320], 86  
 survival equation (for mean free path) [5.90], 113  
 susceptance (definition), 148  
 susceptibility  
   electric [7.87], 142  
   Landau diamagnetic [6.80], 133  
   magnetic [7.103], 143  
   Pauli paramagnetic [6.79], 133  
 symmetric matrix [2.86], 25  
 symmetric top [3.188], 77  
 SYNCHROTRON RADIATION, 159  
 synodic period [9.44], 180
- T**
- tan  $x$   
   definition [2.220], 34  
   series expansion [2.137], 29  
 tangent [2.283], 39  
 tangent formula [2.250], 36  
 tanh  $x$   
   definition [2.221], 34  
   series expansion [2.145], 29  
 $\tan^{-1} x$ , *see* arctan  $x$   
 tau physical constants, 9  
 Taylor series  
   one-dimensional [2.123], 28  
   three-dimensional [2.124], 28  
 telegraphist's equations [7.171], 150  
 temperature  
   antenna [7.215], 153  
   Celsius, 4  
   dimensions, 17  
   Kelvin scale [5.2], 106  
   thermodynamic [5.1], 106  
 TEMPERATURE CONVERSIONS, 15  
 temporal coherence [8.105], 172  
 tensor  
   Einstein [3.58], 67  
   electric susceptibility [7.87], 142  
    $\epsilon_{ijk}$  [2.443], 50  
   fluid stress [3.299], 85  
   magnetic susceptibility [7.103], 143  
   moment of inertia [3.136], 74  
   Ricci [3.57], 67  
   Riemann [3.50], 67  
   strain [3.233], 80  
   stress [3.232], 80  
 tera, 5  
 tesla (unit), 4  
 tetragonal system (crystallographic), 127  
 tetrahedron, 38  
 thermal conductivity

- diffusion equation [2.340], 43
- dimensions, 17
- free electron [6.65], 132
- phonon gas [6.58], 131
- transport property [5.96], 113
- thermal de Broglie wavelength [5.83], 112
- thermal diffusion [5.93], 113
- thermal diffusivity [2.340], 43
- thermal noise [5.141], 117
- thermal velocity (electron) [7.257], 156
- THERMODYNAMIC COEFFICIENTS, 107
- THERMODYNAMIC FLUCTUATIONS, 116
- THERMODYNAMIC LAWS, 106
- THERMODYNAMIC POTENTIALS, 108
- thermodynamic temperature [5.1], 106
- THERMODYNAMIC WORK, 106
- Thermodynamics, 105–121
- THERMOELECTRICITY, 133
- thermopower [6.81], 133
- Thomson cross section, 8
- Thomson scattering [7.238], 155
- throttling process [5.27], 108
- time (dimensions), 17
- time dilation [3.11], 64
- TIME IN ASTRONOMY, 177
- TIME SERIES ANALYSIS, 60
- TIME-DEPENDENT PERTURBATION THEORY, 102
- TIME-INDEPENDENT PERTURBATION THEORY, 102
- timescale
  - free-fall [9.53], 181
  - Kelvin-Helmholtz [9.55], 181
- Titius-Bode rule [9.41], 180
- tonne (unit), 5
- top
  - asymmetric [3.189], 77
  - symmetric [3.188], 77
  - symmetries [3.149], 74
- top hat function (Fourier transform of) [2.512], 54
- TOPS AND GYROSCOPES, 77
- torque, *see* couple
- TORSION, 81
- torsion
  - in a thick cylinder [3.254], 81
  - in a thin cylinder [3.253], 81
  - in an arbitrary ribbon [3.256], 81
  - in an arbitrary tube [3.255], 81
  - in differential geometry [2.288], 39
- torsional pendulum [3.181], 76
- torsional rigidity [3.252], 81
- torus (surface area) [2.273], 37
- torus (volume) [2.274], 37
- total differential [2.329], 42
- total internal reflection [7.217], 154
- total width (and partial widths) [4.176], 104
- trace [2.75], 25
- trajectory (of projectile) [3.88], 69
- transfer equation [5.179], 120
- TRANSFORMERS, 149
- transmission coefficient
  - Fresnel [7.232], 154
  - potential barrier [4.59], 94
  - potential step [4.42], 92
  - potential well [4.49], 93
- transmission grating [8.27], 164
- transmission line, 150
  - coaxial [7.181], 150
  - equations [7.171], 150
  - impedance
    - lossless [7.174], 150
    - lossy [7.175], 150
  - input impedance [7.178], 150
  - open-wire [7.182], 150
  - paired strip [7.183], 150
  - reflection coefficient [7.179], 150
  - vswr [7.180], 150
  - wave speed [7.176], 150
  - waves [7.173], 150
- TRANSMISSION LINE IMPEDANCES, 150
- TRANSMISSION LINE RELATIONS, 150
- Transmission lines and waveguides**, 150
- transmittance coefficient [7.229], 154
- TRANSPORT PROPERTIES, 113
- transpose matrix [2.70], 24
- trapezoidal rule [2.585], 61
- triangle
  - area [2.254], 36
  - centre of mass [3.174], 76
  - inequality [2.147], 30
  - plane, 36
  - spherical, 36
- triangle function (Fourier transform of) [2.513], 54
- triclinic system (crystallographic), 127
- trigonal system (crystallographic), 127
- TRIGONOMETRIC AND HYPERBOLIC DEFINI-

- TIONS, 34  
**Trigonometric and hyperbolic formulas**, 32  
 TRIGONOMETRIC AND HYPERBOLIC INTEGRALS, 45  
 TRIGONOMETRIC DERIVATIVES, 41  
 TRIGONOMETRIC RELATIONSHIPS, 32  
 triple- $\alpha$  process, 182  
 true anomaly [3.104], 71  
 tube, *see* pipe  
 Tully-Fisher relation [9.49], 180  
 tunnelling (quantum mechanical), 94  
 tunnelling probability [4.61], 94  
 turns ratio (of transformer) [7.163], 149  
 two-level system (microstates of) [5.107], 114
- U**  
*U* (Stokes parameter) [8.92], 171  
*UBV* magnitude system [9.36], 179  
 umklapp processes [6.59], 131  
 uncertainty relation
  - energy-time [4.8], 90
  - general [4.6], 90
  - momentum-position [4.7], 90
  - number-phase [4.9], 90
 underdamping [3.198], 78  
 unified atomic mass unit, 5, 6  
 uniform distribution [2.550], 58  
 uniform to normal distribution transformation, 58  
 unitary matrix [2.88], 25  
 units (conversion of SI to Gaussian), 135  
 Units, constants and conversions, 3–18  
 universal time [9.4], 177  
 Uranus data, 176  
 UTC [9.4], 177
- V**  
*V* (Stokes parameter) [8.94], 171  
 van der Waals equation [5.67], 111  
 VAN DER WAALS GAS, 111  
 van der Waals interaction [6.50], 131  
 Van-Cittert Zernicke theorem [8.108], 172  
 variance estimator [2.542], 57  
 variations, calculus of [2.334], 42  
 VECTOR ALGEBRA, 20  
 VECTOR INTEGRAL TRANSFORMATIONS, 23  
 vector product [2.2], 20  
 vector triple product [2.12], 20  
**Vectors and matrices**, 20  
 velocity (dimensions), 17  
 velocity distribution (Maxwell-Boltzmann) [5.84], 112  
 velocity potential [3.296], 84  
 VELOCITY TRANSFORMATIONS, 64  
 Venus data, 176  
 virial coefficients [5.65], 110  
 VIRIAL EXPANSION, 110  
 virial theorem [3.102], 71  
 vis-viva equation [3.112], 71  
 viscosity
  - dimensions, 17
  - from kinetic theory [5.97], 113
  - kinematic [3.302], 85
  - shear [3.299], 85
 viscous flow
  - between cylinders [3.306], 85
  - between plates [3.303], 85
  - through a circular pipe [3.305], 85
  - through an annular pipe [3.307], 85
 VISCOUS FLOW (INCOMPRESSIBLE), 85  
 volt (unit), 4  
 voltage
  - across an inductor [7.146], 147
  - bias [6.92], 134
  - Hall [6.68], 132
  - law (Kirchhoff's) [7.162], 149
  - standing wave ratio [7.180], 150
  - thermal noise [5.141], 117
  - transformation [7.164], 149
 volume
  - dimensions, 17
  - of cone [2.272], 37
  - of cube, 38
  - of cylinder [2.270], 37
  - of dodecahedron, 38
  - of ellipsoid [2.268], 37
  - of icosahedron, 38
  - of octahedron, 38
  - of parallelepiped [2.10], 20
  - of pyramid [2.272], 37
  - of revolution [2.281], 39
  - of sphere [2.264], 37
  - of spherical cap [2.276], 37
  - of tetrahedron, 38
  - of torus [2.274], 37
 volume expansivity [5.19], 107  
 volume strain [3.236], 80  
 vorticity and Kelvin circulation [3.287],

84  
 vorticity and potential flow [3.297], 84  
 vswr [7.180], 150

## W

wakes [3.330], 87  
 WARM PLASMAS, 156  
 watt (unit), 4  
 wave equation [2.342], 43  
 wave impedance  
   acoustic [3.276], 83  
   electromagnetic [7.198], 152  
   in a waveguide [7.189], 151  
**Wave mechanics**, 92  
 WAVE SPEEDS, 87  
 wavefunction  
   and expectation value [4.25], 91  
   and probability density [4.10], 90  
   diffracted in 1-D [8.34], 165  
   hydrogenic atom [4.91], 96  
   perturbed [4.160], 102  
 WAVEFUNCTIONS, 90  
 waveguide  
   cut-off frequency [7.186], 151  
   equation [7.185], 151  
   impedance  
     TE modes [7.189], 151  
     TM modes [7.188], 151  
   TE<sub>*mn*</sub> modes [7.190], 151  
   TM<sub>*mn*</sub> modes [7.192], 151  
   velocity  
     group [7.188], 151  
     phase [7.187], 151  
 WAVEGUIDES, 151  
 wavelength  
   Compton [7.240], 155  
   de Broglie [4.2], 90  
   photometric, 179  
   redshift [9.86], 184  
   thermal de Broglie [5.83], 112  
 waves  
   capillary [3.321], 86  
   electromagnetic, 152  
   in a spring [3.272], 83  
   in a thin rod [3.271], 82  
   in bulk fluids [3.265], 82  
   in fluids, 86  
   in infinite isotropic solids [3.264], 82  
   magnetosonic [7.285], 158

on a stretched sheet [3.274], 83  
 on a stretched string [3.273], 83  
 on a thin plate [3.268], 82  
 sound [3.317], 86  
 surface (gravity) [3.320], 86  
 transverse (shear) Alfvén [7.284], 158

**Waves in and out of media**, 152  
 WAVES IN LOSSLESS MEDIA, 152  
 WAVES IN STRINGS AND SPRINGS, 83  
 wavevector (dimensions), 17  
 weber (unit), 4  
 WEBER SYMBOLS, 126  
 weight (dimensions), 17  
 Weiss constant [7.114], 144  
 Weiss zone equation [6.10], 126  
 Welch window [2.582], 60  
 Weyl equation [4.182], 104  
 Wiedemann-Franz law [6.66], 132  
 Wien's displacement law [5.189], 121  
 Wien's displacement law constant, 9  
 Wien's radiation law [5.188], 121  
 Wiener-Khintchine theorem  
   in Fourier transforms [2.492], 53  
   in temporal coherence [8.105], 172  
 Wigner coefficients (spin-orbit) [4.136], 100  
 Wigner coefficients (table of), 99  
 windowing  
   Bartlett [2.581], 60  
   Hamming [2.584], 60  
   Hanning [2.583], 60  
   Welch [2.582], 60  
 wire  
   electric field [7.29], 138  
   magnetic flux density [7.34], 138  
 wire loop (inductance) [7.26], 137  
 wire loop (magnetic flux density) [7.37], 138  
 wires (inductance of parallel) [7.25], 137  
 work (dimensions), 17

## X

X-ray diffraction, 128

## Y

yocto, 5  
 yotta, 5  
 Young modulus  
   and Lamé coefficients [3.240], 81



and other elastic constants [3.250],  
81  
Hooke's law [3.230], 80  
Young modulus (dimensions), 17  
Young's slits [8.24], 164  
Yukawa potential [7.252], 156

## **Z**

Zeeman splitting constant, 7  
zepto, 5  
zero-point energy [4.68], 95  
zetta, 5  
zone law [6.20], 126