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AND

THE FOUNDATIONS OF MATHEMATICS

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JAN ŁUKASIEWICZ

SELECTED WORKS

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* Translated from the Polish by O. Wojtasiewicz.

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FOREWORD BY PROF. J. SŁUPECKI

Jan Łukasiewicz's selected papers on philosophy and logic appeared in Polish in 1961.[1]) The present publication differs considerably from the Polish edition. On the one hand, several of the articles in the Polish edition are not included here because they are now mainly of historical interest. On the other hand, the English-language version includes ten papers on mathematical logic, certainly the most valuable part of Łukasiewicz's contribution to science, which do not form part of the 1961 book. These papers have been selected so as to bring out the problem in which Łukasiewicz was most interested almost all his life and which he strove to solve with extraordinary effort and passion, namely the problem of determinism. It inspired him with his most brilliant idea, that of many-valued logics.

Łukasiewicz's scholarly activity may be divided into three periods, separated from one another by two world wars.[2])

Before World War I, Łukasiewicz's attention was most strongly attracted to problems in the methodology of the empirical sciences. He discussed those questions in two comprehensive papers: "On Induc-

[1]) Jan Łukasiewicz, *Z zagadnień logiki i filozofii. Pisma wybrane* (Problems of logic and philosophy. Selected writings). Polish Scientific Publishers, Warsaw, 1961.

[2]) Jan Łukasiewicz was born in Lwów on December 21, 1878. He took his Ph.D. degree at the University of Lwów in 1902, and in 1906 became a docent (roughly equivalent to an assistant professor). From 1915 to 1939 he was a professor at the University of Warsaw, of which he was also Rector in 1922/3 and 1931/2. After World War II he was Professor of Mathematical Logic at the Royal Irish Academy in Dublin, which conferred an honorary doctor's degree on him in 1955. (Before the war Łukasiewicz became an honorary doctor of Münster University.) He died on February 13, 1956. Łukasiewicz was undoubtedly one of the most eminent logicians of the first half of the 20th century. The study of many-valued logics and the methodological researches he initiated have developed into separate disciplines. The logical systems constructed by him are masterpieces of simplicity and formal elegance. He was also one of the best historians of logic, even though he wrote very little on that subject.

tion as the Inversion of Deduction" (1903) and "Analysis and Construction of the Concept of Cause" (1906).[3] Neither paper has been included in the present English-language edition, because in the passage of sixty years from the date of their original appearance both papers have lost much of their scientific value. Łukasiewicz's research on the methodology of the empirical sciences is represented by only one brief paper: "Creative Elements in Science" (1912). It includes Łukasiewicz's views on the tasks and value of science and also an extremely simple classification of methods of reasoning. His interest in the methodology of the empirical sciences is reflected in the paper "The Logical Foundations of Probability Theory" (1913), which is certainly one of Łukasiewicz's most valuable works. His ideas contained therein were many years later repeated by the most eminent founders of contemporary probability theory.

Even before World War I Łukasiewicz had become concerned with mathematical logic, which in his earliest papers he termed "algebraic logic". His first comprehensive and valuable work connected with the issues of formal logic was the article "On the Concept of Magnitude" (1916), reprinted here with the omission of those parts which are no longer of any interest, especially for the foreign reader (Łukasiewicz's criticism of Zaremba's book). The present publication also includes "The Farewell Lecture" delivered in the Warsaw University Lecture Hall on March 7, 1918. That lecture makes the earliest reference to three-valued logic.

After World War I mathematical logic came to dominate Łukasiewicz's research. The principal subject matter of his research was the propositional calculus and Aristotle's syllogistic. The results obtained by Łukasiewicz concerning the methodology of these systems are among the earliest works in this field.

His papers on mathematical logic, published between the two world wars and included in the present publication are: "On Three-Valued Logic" (1920), "Two-Valued Logic" (1921), "A Numerical Interpretation of the Theory of Propositions" (1922/3), "Investigations into the Sentential Calculus" (1930), "Comments on Nicod's Axiom and on 'Generalizing Deduction'", and "The Equivalential Calculus" (1939).

[3] A bibliography of Łukasiewicz's works is included in this publication (see p. 401).

The paper "On Three-Valued Logic" for the first time formulates the formal foundations of logical calculus other than classical logic. Some comments on non-classical logic were formulated by Łukasiewicz in his earlier works, for instance in the monograph *The Principle of Contradiction in Aristotle's Works* (1910), but they were rather intuitive than formal in nature.

The paper "A Numerical Interpretation of the Theory of Propositions" contains Łukasiewicz's earliest remarks on many-valued logics and on the applications they may have in the proofs of the independence of theses of the propositional calculus.

The paper "Two-Valued Logic" was intended by Łukasiewicz to be part of a more comprehensive study of three-valued logic, which, however, has never been published. Neither did Łukasiewicz ever revert to the method of constructing a system of propositional calculus which he used in that paper. The paper, however, has visibly influenced the works of other logicians. It is interesting to note that in "Two-Valued Logic" Łukasiewicz first used the concept of rejected proposition, a concept which later came to play an important role in his research on Aristotle's syllogistic.

"Investigations into the Sentential Calculus" was written jointly by Łukasiewicz and Alfred Tarski, and in addition to the results obtained by its authors it also includes results obtained by their disciples. This paper is to this day a classic and is probably the most important work on the methodology of the propositional calculus.

"Comments on Nicod's Axiom and on 'Generalizing Deduction'" discusses in detail Łukasiewicz's parenthesis-free notation and offers simple and elegant methods of proving logical theses and writing down such proofs. It also discusses a philosophically important property of some kinds of logical reasoning, which Łukasiewicz termed "generalizing deduction".

"The Equivalential Calculus" was to appear in Vol. 1 of *Collectanea Logica*, a periodical initiated by Łukasiewicz, but the publication was destroyed during the hostilities in 1939. Only a few off-prints, including Łukasiewicz's paper, have been saved.

The present publication includes the following five philosophical articles, published between the two world wars or during World War II: "On Determinism", "Philosophical Remarks on Many-Valued

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Systems of Propositional Logic" (1930), "Logistic and Philosophy" (1936), "In Defence of Logistics" (1937), "Logic and the Problem of the Foundations of Mathematics" (1941).

The first of these is a revised version of the speech delivered by Łukasiewicz as Rector of the University of Warsaw at the opening of the academic year 1922/1923. This paper discusses the intuitions which contributed to the formulation of three-valued logic as well as the significance of that logic in the analysis of the problem of determinism.

The second proves the thesis that modal logic cannot be based on two-valued logic but can be based on three-valued logic.

The third is the text of the paper read by Łukasiewicz in 1938 at the Zurich conference on the foundations and methods of the mathematical sciences. It outlines a modal three-valued propositional calculus different from the system discussed in "Philosophical Remarks on Many-Valued Systems of Propositional Logic".

The remaining two of these five papers are concerned with Łukasiewicz's defence of mathematical logic, which he sees as the modern form of the formal logic originated by Aristotle, against the objections of nominalism, formalism, conventionalism and relativism. This paper provides a very fine example of Łukasiewicz's polemic talent.

"On the History of the Logic of Propositions" (1934), concerned exclusively with the history of logic, is, according to H. Scholz, the most interesting thirty pages ever written on the history of logic. It demonstrates that Stoic dialectic, contrary to K. Prantl's opinion, is propositional logic, and not term logic. Many historical remarks are also included in "Philosophical Remarks on Many-Valued Systems of the Propositional Logic". One of Łukasiewicz's most important works is the monograph *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic* (1951), which is largely a historical study.

After World War II Łukasiewicz published twelve works, all of them on logic, seven of which have been included in the present publication: "The Shortest Axiom of the Implicational Calculus of Propositions" (1948), "On the System of Axioms of the Implicational Propositional Calculus" (1950), "On Variable Functors of Propositional Arguments" (1951), "On the Intuitionistic Theory of Deduction" (1952), "Formalization of Mathematical Theories" (1953, in French), "A System of Modal Logic" (1953), and "Arithmetic and Modal Logic" (1954).

The subject matter of the first two is explained by their titles. The third is concerned with a part of a system originating with Stanisław Leśniewski (eminent Polish logician, 1886–1939) and termed protothetics by him. The method of writing a definition as a single implication, which is made possible by the introduction into the propositional calculus of functor variables, is particularly interesting. "On the Intuitionistic Theory of Deduction" formulates the rather unexpected conclusion that the classical propositional calculus is a proper part of the intuitionistic calculus enriched by definitions of the terms of the classical propositional calculus. "Formalization of Mathematical Theories" is concerned with the arithmetic of natural numbers. In the sixth paper of those specified above Łukasiewicz constructed a four-valued modal calculus, thus reverting to the problems in which he was most interested for many years. He now offered a new solution of those problems, which both formally and intuitively differed essentially from his earlier solutions. That paper is to a certain extent supplemented by his "Arithmetic and Modal Logic".

The present publication does not include any of Łukasiewicz's works on Aristotle's syllogistic, in spite of the fact that research on syllogistics was for many years one of the principal subject matter of his studies. This is due to the fact that in his monograph on Aristotle's syllogistic, referred to above and easily accessible to English-speaking readers, Łukasiewicz formulated, in a fuller and more satisfactory form, all his results included in his earlier works.

*

* *

Łukasiewicz's papers are here arranged in chronological order, which makes it easier for the reader to follow the evolution of Łukasiewicz's views on many philosophical and logical issues. This is important because in some cases (cf. the problem of the relationship between logic and reality) Łukasiewicz changed his opinions completely.

The terminology used in this publication is based mainly, though not without some exceptions, on that used in Łukasiewicz's works published originally in English during his lifetime. All editorial notes are marked by asterisks or included in brackets.

The notation used here had not been unified as no consistent procedure could be used with reference to the various notations used by Łukasiewicz over a period of more than forty years.

The terminology used in the text reprinted from *Polish Logic*[4]) and *Logic, Semantics, Metamathematics*[5]) has remained unchanged.

JERZY SŁUPECKI

[4]) *Polish Logic 1920–1939*, ed. by Storrs McCall, Oxford, Clarendon Press, 1967. These texts are: "On Determinism", "Philosophical Remarks on Many-Valued Systems of Propositional Logic", "On the History of the Logic of Propositions" and "The Equivalential Calculus".

[5]) *Logic, Semantics, Metamathematics*, Papers from 1923 to 1938 by Alfred Tarski, Oxford, Clarendon Press, 1956. The text in question is "Investigations into the Sentential Calculus" by Łukasiewicz and Tarski.

CREATIVE ELEMENTS IN SCIENCE *)

Both scientists and those who are remote from science often deem that the goal of science is *truth*, and they understand truth as agreement between thought and existence. Hence they think that the scientist's work consists in *reproducing* facts in true judgements, similarly as a photographic plate reproduces light and shadow and a phonograph reproduces sound. The poet, the painter, and the composer work *creatively*; the scientist does not create anything, but merely *discovers* the truth.[1])

This knot of ideas makes many a scientist feel undue pride and makes many an artist treat science lightly. Such opinions have dug a chasm between science and art, and that chasm has engulfed the comprehension of the priceless quality that is *the creative element in science*.

Let us cut this knot of ideas with the sword of logical criticism.

*

* *

1. *Not all true judgements are scientific truths*. There are truths that are too *trifling* for science. Aristophanes says in *The Clouds*:[2])

[1]) After writing the introduction to the present paper I found the following formulations: "La science n'est pas une création de notre esprit, dans le genre de l'art... Elle n'est que la reproduction intellectuelle de l'univers", in a work by Xénopol, a well-known methodologist of the historical sciences (cf. *La théorie de l'histoire*, Paris, 1908, p. 30).

[2]) Aristophanes, *The Clouds*, Loeb Classical Library, London and Cambridge, 1960, p. 275.

*) First published as "O twórczości w nauce" in *Księga pamiątkowa ku uczczeniu 250 rocznicy założenia Uniwersytetu Lwowskiego*, Lwów, 1912, pp. 1–15. Also published by the Philosophical Library, Lwów, 1934, and reprinted, in an abridged version, as "O nauce" (On science), in *Poradnik dla samouków*, Vol. 1, Warsaw, 1915. Republished in the 1961 edition *Z zagadnień logiki i filozofii*.

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"Twas Socrates was asking Chaerephon,
How many feet of its own a flea could jump.
For one first bit the brow of Chaerephon,
Then bounded off to Socrates's head."

Socrates caught the flea and immersed its feet in molten wax; in this way he made shoes and took them off the flea's feet, then used them to measure the distance. There is a truth about a flea's jump which disturbed Socrates, but the proper place for such truths is in a comedy, not science.

The human mind, when producing science, does not strive for *omniscience*. If it were so, we would be concerned with even the most trifling truths. In fact, omniscience seems to be a religious, rather than scientific, ideal. God knows *all* the facts, for He is the Maker and the Providence of the world, and the Judge of human intentions and deeds. As the psalmist puts it,

"The LORD looketh from heaven; he beholdeth all the sons of men.
From the place of his habitation he looketh upon all the inhabitants of the earth.
He fashioneth their hearts alike; he considereth all their works."[3])

How different is Aristotle's idea of perfect knowledge! He, too, thinks that a sage knows everything; *yet he does not know detailed facts*, and has only a knowledge of *the general*. And as he knows the general, in a way he knows all the details falling under the general. Thus *potentially* he knows everything that can be known. But potentially only: actual omniscience is not the Stagirite's ideal. [4])

2. Since it is not so that all true judgements belong to science, then *besides their truth there must be some other value which gives some judgements the rank of scientific truths.*

Even Socrates and his great followers considered generality to be that additional value. Aristotle said that scientific knowledge is concerned not with incidental events (like the flea's jump from Chaerephon's brow), but with facts which recur *constantly* or at least *often*. Such

[3]) Psalm 33, *Exultate iusti in Domino*, verse 29–30. Cf. also Psalm 139.

[4]) *Metaphysics* A 2, 982 a 8 ff, 21 ff: ὑπολαμβάνομεν δὴ πρῶτον μὲν ἐπίστασθαι πάντα τὸν σοφὸν ὡς ἐνδέχεται, μὴ καθ' ἕκαδτον ἔχοντα ἐπιστήμην αὐτῶν [...] τὸ μὲν πάντα ἐπίστασθαι τῷ μάλιστ' ἔχοντι τὴν καθόλου ἐπισθήμην ἀναγκαῖον ὑπάρχειν· οὗτος γὰρ οἶδέ πως πάντα τά ὑποκείμενα.

facts are reflected in *general* judgements, and only such judgements belong to science [5]).

Yet generality is neither a necessary nor a sufficient characteristic of scientific truths. It is not *necessary*, for we may not eliminate *singular* judgements from science. The singular proposition "Władysław Jagiełło was the victor in the battle of Grunwald" refers to an important historical event; the singular judgement which, on the basis of computations, foresaw the existence of Neptune was one of the greatest triumphs of astronomy. Without singular judgements, history would cease to exist *qua* science, and natural science would be reduced to shreds of theory.

Generality is not a *sufficient* characteristic of scientific truths. The following four-line stanza by Mickiewicz:

"Na każdym miejscu i o każdej dobie,
gdziem z tobą płakał, gdziem się z tobą bawił,
wszędzie i zawsze będę ja przy tobie,
bom wszędzie cząstkę mej duszy zostawił."*)

can be the subject-matter of the following general judgements:

"Every line includes the letter s,"

"In every line which includes the letter m that letter occurs twice,"

"In every line the number of occurrences of the letter m is a function of the number of the occurrences of the letter s expressed by the formula: $m = s^2 - 5s + 6$." [6])

Such general truths can be turned out endlessly; shall we include them in science?

3. Aristotle, when adopting generality as the characteristic of scientific truth, was succumbing to the charm of *metaphysical* value. Behind

[5]) *Metaphysics* E 2, 1027 a 20, 21, 26: ὅτι δ' ἐπιστήμη οὐκ ἔστι τοῦ συμβεβηκότος φανερόν· ἐπιστήμη μὲν γὰρ πᾶσα ἤ τοῦ ἀεὶ ἤ τοῦ ὡς ἐπὶ τὸ πυλὺ [...] τὸ δὲ συμβεβηκός ἐστι παρὰ ταῦτα. B 6, 1003 a 15: καθὸλου γὰρ αἱ ἐπιστῆμαι πὰντων.

[6]) These four lines form the third stanza in the poem *Do M**** (*To M****), which begins with the words *Precz z moich oczu*, Adam Mickiewicz, *Dzieła* (Works). The Adam Mickiewicz Literary Society, Lwów, 1896, Vol. I, p. 179). It follows from the formula that $m = 2$ for $s = 1$ (lines one and two), $m = 0$ for $s = 2$ (line three), and $m = 2$ for $s = 4$ (line four).

*) The original example is left untranslated, since reference is made by the author not to the meaning of the poem, but to the occurrence of certain letters.

the constantly recurring facts he sensed a permanent existence different from the vanishing phenomena of the world of the senses. Today, scientists are more inclined to see in generality a *practical* value.

General judgements, by defining the conditions under which phenomena occur, make it possible to forecast the future, to bring about useful phenomena, and to prevent detrimental ones from taking place. Hence the view that scientific truths are *practically valuable* judgements, rules of effective action. [7])

But practical value, too, is neither a necessary nor a sufficient property of scientific truths. Gauss's theorem stating that every prime number of the form $4n+1$ is a product of two conjugate numbers has no practical value. [8]) On the other hand, the information supplied by the police that stolen things have been recovered from thieves is, for all practical purposes, very valuable for the owners of the stolen property. And how many phemomena can be foreseen, and how many accidents can effectively be averted, on the strength of the law in the formulation unknown to Galileo: "All pencils produced by Majewski & Co., Ltd., Warsaw, when neither suspended nor supported, fall with a velocity that increases in proportion to the period of fall!"

Those who would like to turn science into a servant of everyday needs hold a low opinion of science. More exalted, though not better, was Tolstoy's idea of condemning the experimental sciences and of

[7]) A. Comte (cf. *Cours de philosophie*, 2nd ed., Paris, 1864, Vol. I, p. 51) defined as follows the relationship between science and action: "Science, d'où prévoyance; prévoyance, d'où action." But Comte did not yet see the goal of science in prediction or action (cf. footnote 3 on p. 6). Today, *pragmatism* identifies truth with utility, and H. Bergson, by replacing, in *L'évolution créatrice* (5th ed., Paris, 1909, p. 151) the term *homo sapiens* by *homo faber* (which was done before him by Carlyle: Man is a tool-using animal, *Sartor Resarius*, Book 1, Chap. 5) wants the whole of man's mind to serve the purpose of practical activity. H. Poincaré in his book *La valeur de la science* (Paris, 1911, p. 218) quotes the following statement by Le Roy, one of Bergson's followers: "La science n'est qu'une règle d'action."

[8]) Gauss, *Theoria residuorum biquadraticorum, commentatio secunda*, § 33. Examples: $5 = (1+2i)(1-2i)$, $13 = (2+3i)(2-3i)$, etc. Gauss's theorem is equivalent to Fermat's theorem stating that every prime number of the form $4n+1$ can be represented as a sum of two square numbers; e.g., $5 = 1^2+2^2$, $13 = 2^2+3^2$, etc.

demanding of science instruction in *ethical* issues only. [9]) Science has immense importance in practical matters, it can elevate man ethically, and it happens to be a source of aesthetic satisfaction; but the essence of its value rests elsewhere.

4. Aristotle saw the origin of science in *astonishment*. The Greeks were astonished when they found out that the side and the diagonal of a square have no common measure. [10]) Astonishment is a psychological state which is both intellectual and emotional. There are other such states, such as *curiosity*, *fear* of the unknown, *incredulity*, *uncertainty*. They have not been thoroughly studied so far, but even a cursory analysis shows that they all include, along with emotional factors, an intellectual element which is a desire for knowledge. [11])

This desire is concerned with facts which are important for individuals or for all men. A man who is in love and who is tortured by doubts as to whether his beloved responds, would like to know the fact that is important to himself. But every man views death with fear and curiosity while he tries in vain to fathom its mystery. Science is not concerned with the desires of individuals; it investigates that which may arouse desire for knowledge in every man.

If the above statement is true, then *the additional value besides truth which every judgement ought to have in order to belong to science might be defined as the ability to arouse, or to satisfy, directly or indirectly, intellectual needs common to humanity, i.e., which may be felt by any man who has a certain level of mental development.*

5. The truth about the flea's jump from Chaerephon's brow does

[9]) L. Tolstoy included his remarks on the goals of science in the conclusion of his book against modern art. (I know that book only in a German translation: *Gegen die moderne Kunst*, deutsch von Wilhelm Thal, Berlin, 1898, pp. 171 ff.) Tolstoy is quoted by H. Poincaré in his article "Le choix des faits", included in his book *Science et méthode* (Paris, 1908, p. 7).

[10]) *Metaphysics* A 2, 982 b 11 ff: διὰ γὰρ τὸ θαυμάζειν οἱ ἄνθρωποι καὶ νῦν καὶ τὸ πρῶτον ἤρξαντο φιλοσοφεῖν [...] 983 a 16: θαυμαστὸν γὰρ εἶναι δοκεῖ πᾶσιν, εἴ τι (scil. ἡ διάμετρος) τῷ ἐλαχίστῳ μὴ μετρεῖται. Comte (*loc. cit.*) says that the cognition of the laws governing phenomena satisfies that urgent need of the human mind which is expressed in astonishment, *étonnement*.

[11]) States of uncertainty, as far as they occur in desires, have been analysed by W. Witwicki in *Analiza psychologiczna objawów woli* (A psychological analysis of the manifestations of will), Lwów, 1904, pp. 99 ff.

not belong to science, since it neither arouses nor satisfies any one's intellectual needs. The information supplied by the police about the recovery of stolen property may be of interest at the most to the persons concerned. Likewise, no one is interested in knowing how many times the letters *m* and *s* occur in a given poem, or what is the relationship between these two numbers. Even the judgement about the fall of pencils produced by Majewski & Co. will not find its way into a textbook of physics, since our desire for knowledge is *satisfied* by a general law about the fall of bodies.

Gauss's theorem on the factoring of prime numbers of the form $4n+1$ into complex numbers is known only to a few scientists. Yet it belongs to science because it reveals a strange regularity in the laws governing *numbers*, which, being powerful instrument of research, arouse curiosity in every thinking man. Not everyone need be concerned about the existence of Neptune, but that fact confirms Newton's synthetic theory about *the structure of the solar system*, and thus indirectly helps to satisfy the intellectual need which mankind has felt since the earliest times. The victory of Jagiełło as such may be of little interest to a Japanese, but that event was an important element in the history of the relations between two nations, and the history of a *nation* may not be a matter of indifference to any cultured individual.

While art developed from a longing for beauty, science was shaped by a striving for knowledge. To look for the goals of science outside the sphere of intellect is as grossly erroneous as to restrict art by considerations of utility. The slogans "science for science's sake" and "art for art's sake" are equally legitimate.

6. Every intellectual need that cannot be immediately satisfied in an empirical manner gives rise to *reasoning*. Whoever is astonished by the incommensurability of the side and the diagonal of a square wants to find an *explanation* of that fact; hence he looks for the reasons of which the judgement about incommensurability would be a consequence. Whoever is afraid of the Earth's passing through a comet's tail tries to *infer*, on the strength of the known laws of Nature, what might be the consequences of such an event. A mathematician who is not sure whether the equation $x^n+y^n=z^n$ has no solution in positive integers for $n>2$ looks for a proof, i.e., reliable judgements which would justify Fermat's well-known theorem. A person who is suffering from

hallucinations and at a given moment *does not believe* what he perceives, wants to *verify* the objective nature of what he perceives; hence he looks for the consequence of the assumption that he does not suffer from hallucinations. For instance, he asks others whether they see the same things he does. Explanation, inference, proof, and verification are kinds of reasoning. [12])

Every reasoning includes at least two judgements between which the relation of *consequence* holds. A set of judgements connected by such relations might be called a synthesis. Since any intellectual need common to humanity can be satisfied by reasoning only, and not by experience, which by its very nature is *individual* only, then *science includes not isolated judgements, but only syntheses of judgements.*

7. Every synthesis of judgements includes the formal relation of consequence as a necessary factor. The *syllogism*: "If every *S* is *M*, and every *M* is *P*, then every *S* is *P*," is the most common, though not the only, example of judgements connected by such a logical relation. The relation of consequence which holds between the premisses of a syllogism and its conclusion is called *formal*, because it holds regardless of the meanings of the terms *S*, *M*, and *P*, which form the "matter" of the syllogism.

The formal relation of consequence is *non-symmetrical*, i.e., it has the property that while the relation of consequence holds between a judgement or a set of judgements *A* and *B*, the same relation may, but need not, hold between *B* and *A*. The judgement *A*, of which *B* is a consequence, is the *reason*, and *B* is the *consequence*.*) The transition from reason to consequence determines the *direction* of the relation of consequence.

Reasoning which starts from reasons and looks for consequences is called *deduction*; that which starts from consequences and looks for reasons is called *reduction*. In the case of deduction the direction of

[12]) Professor K. Twardowski was the first to use the term "reasoning" as a general term covering "inference" and "proof" in *Zasadnicze pojęcia dydaktyki i logiki* (The fundamental concepts of teaching methods and logic), Lwów, 1901, p. 19, para. 97. As a continuation of his views I introduce the theory of reasoning outlined under 7 in the present paper.

*Unfortunately, two Polish terms have to be rendered by one English term "consequence" (the relation of consequence, and consequence as opposed to reason).

reasoning is in agreement with that of the relation of consequence; in reduction the two are *contrary* to one another.

Deductive reasoning can be either inference or verification, and reductive reasoning can be either explanation or proof. If from given reliable judgements we deduce a consequence, we *infer*; if we look for reasons for given reliable judgements, we *explain*. If we look for reliable judgements which are consequences of given *unreliable* judgements, we *verify*; if we look for reliable judgements of which given *unreliable* judgements are consequences, we *prove*.

8. There is a *creative* element in every reasoning; this is most strongly manifested in explanation.

Incomplete induction is one kind of explanation. It is a way of reasoning which for given reliable singular judgements: "S_1 is P, S_2 is P, S_3 is P, ..." looks for a reason in the form of a general judgement: "every S is P".

Like all reductive reasoning, incomplete induction does not justify the result of reasoning by its starting point. For S_1, S_2, S_3 do not exhaust the extension of the concept S, and inferring a general judgement from a few singular judgements is not formally permissible. That is why the result of an argument by incomplete induction as such is not a reliable judgement, but only a probable one.[13])

The generalization "every S is P" may be interpreted either as a set of singular descriptions or as the relationship "if something is S, then it is P". If a generalization is a set of singular judgements, it covers not only those cases which have been investigated, but unknown cases as well. By assuming that the unknown cases behave like the known ones *we do not reproduce* facts that are empirically given, but *we create* new judgements on the model of judgements about known cases.

If a generalization expresses a relationship, it introduces a factor that is alien to experience. Since Hume's time we have been permitted to say only that we perceive a coincidence or a sequence of events, but

[13]) This view on the essence of inductive inference is in agreement with what is called the inversion theory of induction, formulated by Jevons and Sigwart (cf. my paper "O indukcji jako inwersji dedukcji" (On induction as the inversion of deduction), in *Przegląd Filozoficzny* 6 (1903), p. 9).

not a relationship between them.[14]) Thus a judgement about a relationship *does not reproduce* facts that are empirically given, but again is a manifestation of man's *creative* thought.

This is still insignificant creative activity; we shall come to know a fuller one.

9. Consider Galileo's generalization: "All heavy bodies, if neither suspended nor supported, fall with a velocity that increases in proportion to the time of fall." This generalization includes a law that expresses the functional relationship between the velocity v and the time of fall t, given by the formula: $v = gt$.

The quantity t may take on values that are expressed by integers, fractions, irrational numbers, and transcendental numbers. This yields an infinite number of judgements about cases which no one has ever observed or will ever be able to observe. This is one element of creative thought which was already mentioned above.

The other is inherent in the form of the relationship. No measurement is exact. Hence it is impossible to state that the velocity is *exactly* proportional to the time of fall. Thus *neither* does the form of the relationship *reproduce* facts that are empirically given: the *entire* relationship is a product of the creative activity of the human mind.

Indeed, we know that the law governing the fall of heavy bodies can be true only in approximation, since it supposes such non-existent conditions as a constant gravitational acceleration or a lack of resistance offered by the air. Thus it does not reproduce reality, but only refers to a *fiction*.

That is why history tells us that the law did not emerge from the observation of phenomena, but was born *a priori* in Galileo's creative mind. It was only *after formulating* his law that Galileo verified its consequences with facts.[15]) Such is the role of experience in every theory of natural science: *to be a stimulus for creative ideas and to provide subjects for their verification.*

10. Another kind of explanation consists in the *formulation of hypotheses*. To formulate a hypothesis means to assume the existence of

[14]) Cf. David Hume, *Enquiry Concerning Human Understanding*, Leipzig, 1913, Felix Meisner, p. 64: "... we are never able, in a single instance, to discover any power of necessary connection."

[15]) Cf. E. Mach, *Die Mechanik in ihrer Entwickelung*, 6th ed., Leipzig, 1908, pp. 129 ff.

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a fact, not confirmed empirically, in order to deduce from a judgement about such a fact as its partial reason a given reliable judgement as a consequence. For instance, a person knows that some S is P, but does not known why. As he wants to find an explanation he assumes that the same S is M, although he does not verify it empirically. But he knows that all M are P; so if he assumes that S is M, then from these two judgements he may conclude that S is P.

The judgement about the existence of Neptune was a hypothesis before the fact was confirmed empirically. The judgement about the existence of Vulcan, a planet situated closer to the Sun than Mercury, is still a hypothesis. The views stating that atoms, electrons, and aether exist will always be hypotheses.[16]) All palaeontology is based on hypotheses; for instance, the statement that certain gray lumps of limestone found in Podolia are traces of the Brachiopoda which lived in the Silurian and the Lower Devonian periods pertains to phenomena which are not accessible to observation. History is an immense network of hypotheses which, by means of general judgements, in most cases drawn from experience, empirically explain given data, such as historical monuments, documents, institutions and customs that exist *now*.

All hypotheses are *products* of the human mind, for a person who assumes a fact that is not empirically confirmed creates something new. Hypotheses are *permanent* elements of knowledge and not temporary ideas that by verification can be changed into established truths. A judgement about a fact ceases to be a hypothesis only if that fact can be confirmed by *direct* experience. This happens only exceptionally. And to demonstrate that the *consequences* of a hypothesis are in agreement with facts does not mean turning a hypothesis into a truth, for the truth of the reason does not follow from the truth of the consequence.

[16]) Many examples pointing to creative elements in physics are quoted by Dr. Bronisław Biegeleisen in his paper "O twórczości w naukach ścisłych" (On creative elements in the exact sciences), *Przegląd Filozoficzny* 13 (1910), pp. 263, 387. Dr Biegeleisen draws attention to the visualization of physical theories by mechanical models (pp. 389 ff). Between a model that explains a theory and an *invention*, which certainly is a creative work, there is only a difference in the goals and the applications of two such objects. There are also models in logic: for instance, Jevons's logical abacus (see the drawing in his book *The Principles of Science*, London, 1883) or Marquand's logical machines (cf. *Studies in Logic by Members of the John Hopkins University*, Boston, 1883, pp. 12 ff).

11. Other kinds of reasoning do not contain *primary* creative elements, as explanation does. This is so because proving consists in looking for *known* reasons, and inference and verification develop the consequences already *contained* in the premisses in question. Yet in all reasoning there is inherent *formal* creative reasoning: a logical *principle of reasoning*.

A principle of reasoning is a judgement stating that the relation of consequence holds between certain forms of judgements. The syllogism "if S is M, and M is P, then S is P" is a principle of reasoning.[17])

Principles of reasoning *do not reproduce* facts that are empirically given; for neither is the non-symmetrical relation of consequence a subject-matter of experience, nor do the forms of judgements, such as "S is P", express phenomena.

Non-symmetrical relations never link real objects with one another. For we call non-symmetrical a relation which *may*, but need not, hold between B and A if holds between A and B. And if A and B really exist, then every relation either holds between them or *does not* hold. Actuality excludes possibility.

Possibility is inherent in the forms of judgements, too. The terms S and P are *variables* which do not denote anything definite, but which *may* denote anything. The element of possibility suffices to make us consider the principles of reasoning as *creations* of the human mind, and not as reproductions of real facts.

Logic is an *a priori* science. Its theorems are true on the strength of definitions and axioms derived from reason and not from experience. This science is a sphere of pure mental activity.

12. Logic gives rise to mathematics. Mathematics, according to Russell, is a set of judgements of the form "p implies q", where the judgements p and q may, in addition to the same variables, contain only logical constants.[18]) The logical constants include such concepts as the relation of consequence, the relation of membership that holds between

[17]) For the concept of the "principle of reasoning" I am indebted to Professor K. Twardowski (cf. *Zasadnicze pojęcia dydaktyki i logiki* (The fundamental concepts of teaching methods and logic)), Lwów, 1901, p. 30, para. 64).

[18]) B. Russell, *The Principles of Mathematics*, Cambridge, 1903, p. 3.

an individual and a class, etc.*) If all mathematics is reducible to logic, then it is also a *pure mental product*.

An analysis of the various mathematical disciplines leads to the same conclusion. The point, the straight line, the triangle, the cube, all the objects investigated by geometry have only an ideal existence; they are not empirically given. How much less are non-Euclidean figures and many-dimensional solids given empirically! Nor are there, in the world of phenomena, integral, irrational, imaginary, or conjugate numbers. Dedekind called numbers "free products of the human spirit".[19]) ~~And numbers are the foundation of all analysis.~~

Logic, with mathematics, might be compared to a fine net which is cast into the immense abyss of phenomena in order to catch the pearls that are scientific syntheses. It is a powerful *instrument* of research, but an instrument only. Logical and mathematical judgements are truths only in the world of ideal entities. We shall probably never know whether these entities have counterparts in any real objects.[20])

The a priori *mental constructions, which are contained in every synthesis, imbue the whole science with the ideal and creative element.*

13. The time has come now to consider the question: which scientific judgements are *pure reproductions* of facts? For if generalizations, laws, and hypotheses, and hence all the theories of the empirical sciences and the entire sphere of the *a priori* sciences are a result of the creative work of the human mind, then there are probably few judgements in science that are purely reproductive.

The answer to this question appears to be easy. *Only a singular statement about a fact which is directly given in experience can be a purely reproductive judgement*, for instance: "a pine grows here", "this magnetic needle now deviates (from its previous position)", "in this room there are two chairs". But whoever investigates these judgements more closely

*) It would seem that Łukasiewicz means here the symbol of implication and the symbol "ε" which denotes the membership relation that holds between an object and a set of which that object is an element.

[19]) R. Dedekind, *Was sind und was sollen die Zahlen*, Braunschweig, 1888, p. VII: "die Zahlen sind freie Schöpfungen des menschlichen Geistes."

[20]) In my book *O zasadzie sprzeczności u Arystotelesa* (On the principle of contradiction in Aristotle's works), Cracow, 1910, pp. 133 ff, I tried to demonstrate that we cannot even be sure that real objects are subject to the principle of contradiction.

will perhaps find creative elements even in them. The words "pine", "magnetic needle", and "two" stand for *concepts*, and hence concealed labour of the spirit works through them. All the facts formulated in words are, primitively it may be, interpreted by man. A "crude fact", untouched by the human mind, seems to be a limiting concept.

Whatever the actual situation may be, we feel that the creative ability ~~of the human mind is not unlimited. Idealistic systems of epistemology~~ fail to eliminate the feeling that some reality exists independent of man and that it is to be sought in the objects of *observation*, in *experience*. It has long since been the great task of philosophy to investigate which elements in that reality come from the human mind.[21])

14. Two kinds of judgements must be distinguished in science: some are supposed to *reproduce* facts given in experience, the others are *produced* by the human mind. The judgements of the first category are true, because truth consists in agreement between thought and existence. Are the judgements of the second category true as well?

We cannot state categorically that they are *false*. That which the human mind has produced need not necessarily be a fantasy. But neither are we entitled to consider them as *true*, for we usually do not know whether they have counterparts in real existence. Nevertheless we include them in science if they are linked by relations of consequence with judgements of the first category and if they do not lead to consequences that are at variance with the facts.

Hence it is erroneous to think that *truth* is the goal of science. The human mind does not work creatively for the sake of truth. *The goal of science is to construct syntheses that satisfy the intellectual needs common to humanity.*

Such syntheses include true judgements about facts; they are the ones which mainly arouse intellectual needs. They are *reconstructive* elements. But these syntheses also include creative judgements; they are the ones which satisfy intellectual needs. They are *constructive* elements.

[21]) The Copernican idea of Kant, who tried to prove that objects follow cognition rather than cognition follows objects, includes views that favour the thesis of creative elements in science. But I have tried to demonstrate that thesis not on the basis of any special theory of cognition, but on the basis of common realism, by means of *logical* research. For the same reason I have not taken into consideration James's pragmatism and Schiller's humanism.

Elements of the former and the latter combine into a whole by the logical relations of *consequence*. It is these relations which impart to the syntheses of judgements their *scientific* character.

Poetic creativity does not differ from scientific creativity by a greater amount of fantasy. Anyone who, like Copernicus, has moved the Earth from its position and sent it revolving around the Sun, or, like Darwin, has perceived in the mists of the past the genetic transformations of species, may vie with the greatest poet. But the scientist differs from the poet in that he *reasons* at all times and places. He need not and cannot justify everything, but whatever he states he must link with ties of logic into a coherent whole. The foundation of that whole consists of judgements about facts, and it supports the theory, which explains, orders and predicts facts.

This is how the *poem of science* is created. [22])

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* *

We are living in a period of a busy collecting of facts. We set up natural science museums and make herbaria. We list stars and draw maps of the Moon. We organize expeditions to the Poles of our globe and to the towering mountains of Tibet. We measure, we compute, and we collect statistical data. We accumulate artifacts from prehistoric civilizations and specimens of folk art. We search ancient tombs in quest of new papyri. We publish historical sources and list bibliographies. We would like to preserve from destruction every scrap of print-covered paper. All this is valuable and necessary work.

But a collection of facts is not yet science. He is a true scientist who knows how to link facts into syntheses. To do so it does not suffice to acquire the knowledge of facts; it is also necessary to contribute *creative thought*.

The more a person trains both his mind and his heart, and the closer he associates with the great creative minds of mankind, the more creative ideas he can form in his rich soul. And perhaps in a happy moment he will be illuminated by a spark of *inspiration* which will beget something great. For, as Adam Mickiewicz has said, [23]) "all great things in the world—nations, legislation, age-old institutions, all creeds before the coming of Christ, all sciences, inventions, discoveries, all masterpieces of poetry and art—have taken their origin from the *inspiration* of prophets, sages, heroes, and poets."

[22]) Ignacy Matuszewski in his paper "Cele sztuki" (The goals of art), included in the book *Twórczość i twórcy* (Creation and creators), Warsaw, 1904, offers similar views on creative elements in science. His studies, undertaken with different ends in view and from a different standpoint, have led him to the same results to which logical considerations have led me.

[23]) This formulation, drawn from Odyniec's letters, is quoted by W. Biegański in his paper "O filozofii Mickiewicza" (On Mickiewicz's philosophy), in *Przegląd Filozoficzny* 10 (1907), p. 205.

LOGICAL FOUNDATIONS OF PROBABILITY THEORY [1]*)

I

THE THEORY OF TRUTH VALUES

1. Indefinite propositions

I call indefinite those propositions which contain a variable. For instance, "x is an Englishman", "x is greater than 4".

I shall consider hereafter only those indefinite propositions in which the values of the variables range over a well-defined, *finite* class of individuals. For instance, it may be assumed that in the statement "x is greater than 4" x will stand only for integers from 1 to 6.

If in an indefinite proposition we substitute for the variable one of its values, we obtain a *definite* singular judgement which is either true or false. For instance, "5 is greater than 4", "3 is greater than 4".

Indefinite propositions are *true* if they yield true judgements for all the values of the variables. For instance, "x is greater than 0" for $x = 1, 2, \ldots, 6$.

Indefinite propositions are *false* if they yield false judgements for all the values of the variables. For instance, "x is greater than 6" for $x = 1, 2, \ldots, 6$.

[1]) I undertook the study of the subject described in this paper in Graz in 1909, where I studied as a fellow of the W. Osławski, Foundation, administered by the Cracow Academy of Learning.

*) First published in Cracow, 1913, as *Die logische Grundlagen der Wahrscheinlichkeitsrechnung*, reprinted in the 1961 edition *Z zagadnień logiki i filozofii*.

Indefinite propositions which yield true judgements for some values of the variables and false judgements for other values of the variables are *neither true nor false*. For instance, "x is greater than 4" for $x = 1, 2, \ldots, 6$.

2. Truth values

By the truth value of an indefinite proposition I mean the ratio between the number of values of the variables for which the proposition yields true judgements and the total number of values of the variables. For instance, the truth value of the proposition "x is greater than 4", for $x = 1, 2, \ldots, 6$, is $^2/_6 = {}^1/_3$, since out of the 6 values of x only 2 values, when substituted for the variable, yield a true judgement, i.e., "verify" it.

The truth value of a *true* indefinite proposition is 1, since that proposition yields true judgements for all the value of its variables.

The truth value of a *false* indefinite proposition is 0, since no value of its variables can verify such a proposition.

The truth values of indefinite propositions which are neither true nor false are *proper fractions*.

3. Implication

The relation of implication, or the relation between reason and consequence, holds between two indefinite propositions a and b if for every pair of values of the variables occurring in a and in b either the reason a yields a false judgement or the consequence b yields a true judgement.

The three following cases may be distinguished:

1. The reason a yields false judgements for all the values of its variables, i.e., a is a false indefinite proposition. Then the consequence may be arbitrary, since of the two conditions for the occurrence of implication formulated above, each of which is sufficient, the *first* is satisfied.

2. The consequence b yields true judgements for all the value of its variables, i.e., b is a *true* indefinite proposition. Then the reason may be arbitrary, since of the two conditions of the occurrence of implication the *second* is satisfied.

3. Neither the reason a yields false judgements for all values of its variables, nor the consequence b yields true judgements for all values

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of its variables. *Then the statements a and b must contain the same variable x, and all values of x which verify the reason a must verify the consequence b.* For if i is a value of the variable occurring in a, then for that value the statement a yields either a false or a true judgement. In the former case, the first condition of the occurrence of implication is satisfied. In the latter case, which by assumption must take place for some value of the variable occurring in a, the first condition is not satisfied, hence the second must be. Now, if b contains a variable *other* than that contained in a, and if the assumption that b does not yield true judgements for all values of its variable is valid, then we can always select a value j of the variable b for which b yields a false judgement. But then the relation of implication cannot hold between a and b, since for the pair (i, j) of values of the variables neither a yields a false judgement, nor b yields a true one. Therefore a and b must contain the *same* variable, and the same value i which yields a true judgement when substituted for the variable in a, must also yield a true judgement when substituted in b.

Examples. The reason: "x is greater than 4", the consequence: "x is greater than 3". The range of the values of x is arbitrary. All values of x which verify the reason, also verify the consequence, since as x is greater than 4, it must also be greater than 3. But it is obvious that for any value of x either the reason yields a false judgement or the consequence yields a true one, i.e., x must either be not greater than 4 or be greater than 3. On the other hand, the relation of implication does not hold between the following statements: "x is greater than 4" (a) and "x is greater than 5" (b), for $x = 1, 2, \ldots, 6$, because for $x = 5$ a yields a true judgement, and b yields a false one. Or: "x is greater than 4" (a) and "y is greater than 3" (b), for $x = 1, 2, \ldots, 6$; $y = 1, 2, \ldots, 6$. Since the variables x and y are different, we may substitute the value 5 for x and the value 2 for y, whereby a will yield a true judgement and b a false one.

Usually only the cases falling under 3 are classified as instances of the relation between reason and consequence, but in formal logic it has proved useful to extend that concept to 1 and 2 as well. The definition given at the beginning of this section covers *all* these cases.

4. Theorem on the truth value of reason

The following theorem holds: *The truth value of a reason cannot be greater than the truth value of the consequence.* For, if the reason is false, it has the *least* truth value 0; if the consequence is true, it has the *greatest* truth value 1; if neither the reason is false nor the consequence is true, then both the reason and the consequence must contain the same variable, and all values of the variable which verify the reason also verify the consequence. If their denominators are the same, the numerator of the fraction representing the truth value of the reason cannot be greater than the numerator of the fraction which represents the truth value of the consequence.

Here are examples of the third case: $x = 1, 2, \ldots, 6$.

Reason:	Consequences:	Truth values of the consequences:
	x is greater than 5	$^1/_6$
	x is greater than 4	$^2/_6$
$x = 6$	x is greater than 3	$^3/_6$
	x is greater than 2	$^4/_6$
	x is greater than 1	$^5/_6$

The truth value of the reason is $^1/_6$.

It can be seen that in the third case the numerator of the *) difference between the truth value of the consequence and that of the reason equals the number of values of the variable which verify the consequence but not the reason. For instance: Reason: "$x = 6$"; consequence: "x is greater than 3", for $x = 1, 2, \ldots, 6$. The truth value of the reason is $^1/_6$, and that of the consequence is $^3/_6$; hence the difference is $^2/_6$. In fact, there are only two values of the variable, 4 and 5, which verify the consequence but not the reason and thus change the proposition consisting of the negation of the reason and of the consequence: "x is different from 6 and greater than 3" into a true judgement. In algebraic logic, such propositions, connected by the word "and", are called

*) The words "the numerator of" are added in translation; they are omitted in the original, but it follows from the rest of the sentence that the numerator of the difference is meant, and not the difference itself. (Ed.)

logical products. Hence for the third case of implication we may formulate the following "theorem on the truth value of a reason":

The truth value of a reason, augmented by the truth value of the logical product of the negation of the reason and of the consequence, equals the truth value of the consequence.

The same theorem is also valid for the first two cases of implication. For, if the reason has the truth value 0, then the negation of the reason is true and the truth value of the product of the negation of the reason and of the consequence depends solely on the truth value of the consequence, and hence equals the truth value of the consequence. If, on the other hand, the consequence has the truth value 1, then the truth value of the product of the negation of the reason and of the consequence equals the truth value of the negation of the reason. But then it is obvious that the truth value of any indefinite proposition plus the truth value of its negation, equals 1, and hence in our case also equals the truth value of the consequence. Hence the theorem on the truth value of a reason, as formulated above, is universally valid.

5. The calculus of truth values

A special calculus, which abounds in formulae, can now be constructed on the basis of the foregoing explanations and the theorem formulated above, with the help of the algebra of logic.

Indefinite propositions are denoted by $a, b, c, \dots$, and their truth values by $w(a)$, $w(b)$, $w(c)$, ... The *logical product* ab denotes "a and b", while the *logical sum* $a+b$ denotes "a or b" (the word "or" being taken in its inclusive sense); a' is the *negation* (contradictory opposite) of a; $a < b$ stands for the relation of *implication*: "from a follows b"; the *equivalence* $a = b$ is identical with the logical product of $(a < b)\ (b < a)$ and means: "from a follows b and from b follows a".

Most formulae of the calculus consist of a logical and a mathematical part each. The logical part of a formula represents a relation between indefinite propositions, and the mathematical part is a numerical equation between the truth values of such propositions. The whole of a formula expresses a relationship between a logical relation and a numerical equation. In some cases the formulae are reduced to *pure* numerical equations; this occurs whenever a numerical equation is a consequence of a *universally valid* logical relation.

In the text that follows it is assumed that the elementary and obvious rules of the algebra of logic are known to the reader.[2])

Note: The formulae of the calculus contain both logical and mathematical operations and equations which are denoted by the same symbols (+; juxtaposition of the symbols in the case of multiplication; =). Misunderstandings, however, are excluded, since mathematical operations and equations occur only between expressions which stand for *numbers*, and such expressions can easily be recognized as they begin with w, for instance, $w(a)$, $w(a+b)$, etc.

6. Principles of the calculus

The calculus of truth values is based on the following three principles:

$$\text{I} \quad (a = 0) = [w(a) = 0].$$
$$\text{II} \quad (a = 1) = [w(a) = 1].$$
$$\text{III} \quad (a < b) < [w(a)+w(a'b) = w(b)].$$

The first two principles state: "If an indefinite proposition a is false (= 0) or true (= 1), then its truth value equals 0 or 1 respectively, and conversely, if the truth value of a proposition a equals 0 or 1, then a is false or true respectively". The figures 0 and 1 in these equivalences are not numbers but convenient symbols for false and true statements borrowed from the algebra of logic.

The third principle is that of the truth value of a reason: "If a is the reason for b, then the truth value of a, augmented by the truth value of the logical product $a'b$, equals the truth value of b."

In the first two principles the logical part is equivalent to the mathematical, and the third principle as a whole is only an implication.

These principles are based on the analyses carried out in the first four sections where they are explained by examples. Within the calculus of truth values they play the role of *axioms*.

[2]) Couturat's concise work *L'algèbre de la logique* (The Scientia series, division of mathematics and physics, No. 24, Paris, 1905) can serve as the best introduction to the algebra of logic.

7. Theorems

(1) $$w(0) = 0.$$

"The truth value of a false proposition is 0."

(2) $$w(1) = 1.$$

"The truth value of a true proposition is 1."

Proof: The theorems result from I and II following the substitution for a in I of the symbol of a false proposition, and, in II, of the symbol of a true proposition. Thus the equivalences:

$$a = 0 \quad \text{and} \quad a = 1$$

yield universally valid identities:

$$0 = 0 \quad \text{and} \quad 1 = 1,$$

so that their equivalent equalities:

$$w(0) = 0 \quad \text{and} \quad w(1) = 1$$

are proved.

(3) $$(a = b) < [w(a) = w(b)].$$

"If the propositions a and b are equivalent, then their truth values are equal."

Proof: $a = b$ means the same as $(a < b)(b < a)$. By III, $(a < b)$ yields:

(α) $$w(a)+w(a'b) = w(b).$$

On the other hand,

$$(b < a) = (a'b = 0),$$

which by I yields:

(β) $$w(a'b) = 0.$$

From the assumption that $a = b$ we obtain, by (α) and (β):

$$w(a) = w(b).$$

Theorem (3) is *not reversible*, i.e., it may not be asserted that propositions which have the same truth values are equivalent. For instance, for $x = 1, 2, \ldots, 6$, the statements "$x = 4$" and "$x = 5$" have the same

truth value, namely $^1/_6$, yet not only are they *not* equivalent, they are even mutually exclusive.

(4) $$w(a)+w(a') = 1.$$

"The sum of the truth values of two contradictory propositions equals 1."

Proof: If 1, the symbol of a true proposition, is substituted for b in III, this yields:

$$(a < 1) < [w(a)+w(a'1) = w(1)].$$

Now since

$$a'1 = a',$$

hence by (3)

$$w(a'1) = w(a').$$

Further, by (2)

$$w(1) = 1.$$

This yields

$$(a < 1) < [w(a)+w(a') = 1].$$

In this formula the antecedent $a < 1$ is a logical law of universal validity; hence the consequent, i.e., the thesis, also has universal validity.

8. The law of addition

Some auxiliary theorems will be proved first.

(5) $$w(ab)+w(a'b) = w(b).$$

Proof: If the logical product ab is substituted for a in III, this yields

$$(ab < b) < \{w(ab)+w[(ab)'b] = w(b)\}.$$

The following equivalences are valid in the algebra of logic:

$$(ab)'b = (a'+b')b = a'b+b'b = a'b.$$

By (3) we obtain:

$$w[(ab)'b] = w(a'b)$$

and

$$(ab < b) < [w(ab)+w(a'b) = w(b)].$$

Now, the antecedent $ab < b$ is a logical law of universal validity; hence the consequent, i.e., the thesis, is proved.

(6) $$w(a)+w(a'b) = w(a+b).$$

Proof: If the logical sum $a+b$ is substituted for b in III, this yields:

$$(a < a+b) < \{w(a)+w[a'(a+b)] = w(a+b)\}.$$

The following equivalences hold:

$$a'(a+b) = a'a+a'b = a'b.$$

By (3), this yields

$$w[a'(a+b)] = w(a'b)$$

and

$$(a < a+b) < [w(a)+w(a'b) = w(a+b)].$$

Now, since the antecedent $a < a+b$ is a logical law of universal validity, the consequent, i.e., the thesis, is proved.

(7) $$w(a+b) = w(a)+w(b)-w(ab).$$

Formula (7) is obtained by subtracting equation (5) from equation (6).

The law of addition:

(8) $$(ab = 0) < [w(a+b) = w(a)+w(b)].$$

"If the propositions a and b are mutually exclusive, then the truth value of their sum equals the sum of their truth values".

Proof: Formula (8) results from (7), since, by I, the assumption

$$ab = 0$$

yields

$$w(ab) = 0.$$

Example. Let a stand for "$x = 4$", and b, for "x is less than 3". For $x = 1, 2, \ldots, 6$, $w(a) = {}^1/_6$, and $w(b) = {}^2/_6$. The propositions a and b are mutually exclusive. The truth value of the logical sum: "$x = 4$ or x is less than 3" is ${}^3/_6$, hence $w(a+b) = w(a)+w(b)$.

The law of addition can, on the strength of mathematical induction, be extended so as to cover more than two propositions. The following holds:

(9) $$(\textstyle\sum_{ij} a_i a_j = 0) < [w(\sum_i a_i) = \sum_i w(a_i)];$$
$$i \neq j, \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, n.$$

"If n propositions are pairwise mutually exclusive, then the truth value of their sum equals the sum of their truth values".

9. Conclusions determined numerically

The law of addition is reversible:

(10) $$[w(a+b) = w(a)+w(b)] < (ab = 0).$$

"If the truth value of the sum of two propositions equals the sum of their truth values, then the propositions are mutually exclusive."

Proof: By (7), the equation

$$w(a+b) = w(a)+w(b)-w(ab)$$

is universally valid. If now

$$w(a+b) = w(a)+w(b),$$

then

$$w(ab) = 0.$$

But then, by I, also

$$ab = 0.$$

Consequently, the law of addition may be formulated as an equivalence:

(11) $$(ab = 0) = [w(a+b) = w(a)+w(b)].$$

If in (10) we put b' for b, we obtain

(α) $$[w(a+b') = w(a)+w(b')] < (ab' = 0).$$

But now, on the one hand,

$$(ab' = 0) = (a < b),$$

and on the other, by (4)

$$w(a+b') = 1-w[(a+b')'] = 1-w(a'b)$$
$$= w(b)+w(b')-w(a'b).$$

When these results are substituted in (α), then after easy transformations we obtain:

(12) $$[w(a)+w(a'b) = w(b)] < (a < b).$$

Theorem (12) is a reversion of Axiom III, i.e., the law of the truth

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value of reason. *Hence that axiom, too, may be formulated as an equivalence*:

(13) $$(a < b) = [w(a)+w(a'b) = w(b)].$$

Theorems (10) and (12) are most interesting from the logical point of view. *They make it possible, on the strength of numerical equations holding between the truth values of certain propositions, to determine the logical relationships between those propositions.* For instance, let us denote the indefinite proposition "x is A" by a, and the indefinite proposition "x is B" by b, and assume that computations yield the following numbers:

$$w(a) = \frac{m}{n},$$

$$w(b) = \frac{m+r}{n},$$

$$w(a'b) = \frac{r}{n}.$$

Since the following equation holds between the truth values of the propositions in question:

$$w(a)+w(a'b) = w(b),$$

we may conclude that from the proposition "x is A" follows the proposition "x is B". Such conclusions may be called *numerically determined*, since they can be presented in the following manner:

m individuals out of a given n are A,
$m+r$ individuals out of the same n are B,
r individuals out of the same n are B, but not A.
Conclusion: All A out of the given n individuals are B.

10. Relative truth values

Define:

Df(1) $$w_a(b) = \frac{w(ab)}{w(a)},$$

it being assumed that $w(a) \neq 0$.

This definition introduces an *abbreviated notation*, which is important since it points to a new concept, namely that of *relative truth value*. The quotient $\frac{w(ab)}{w(a)}$, symbolized by $w_a(b)$, indicates how many of those values of the variable which verify the proposition a also verify the proposition b, and hence also the product ab. In other words, *it indicates ~~how great is the truth value of~~ b, assuming that ~~a is true~~.* ~~If this assumption~~ is satisfied, so that $a = 1$, then the relative value of b equals its *absolute* value, since the following theorem can easily be proved:

(14) $$w_1(b) = w(b).$$

Proof:

$$w_1(b) = \frac{w(1b)}{w(1)} = \frac{w(b)}{1} = w(b).$$

Example. Let a stand for "x is divisible by 2", and b for "x is divisible by 3". For $x = 1, 2, \ldots, 9$ the absolute truth value of b is $^3/_9 = {}^1/_3$, so that $w(b) = {}^1/_3$. The relative truth value of b with respect to a is only $^1/_4$, so that $w_a(b) = {}^1/_4$. This is so because, assuming that a is true, for *four* values of x which verify the proposition a: "x is divisible by 2" there is only *one* value, namely 6, which also verifies the proposition b: "x is divisible by 3". The same result is obtained by the computation of the truth values for a and ab: $w(a) = {}^4/_9$; $w(ab) = {}^1/_9$; the quotient $\frac{w(ab)}{w(a)} = w_a(b) = {}^1/_4$.

Df(1) yields immediately:

(15) $$w(ab) = w(a)w_a(b) = w(b)w_b(a).$$

The truth value of a (logical) product equals the product of the (absolute) truth value of one factor and the relative value of the other factor with respect to the first factor.

11. Independence of indefinite propositions

Define:

Df(2) $$aUb = [w_a(b) = w_{a'}(b)].$$

While Df(1) introduces a new *mathematical* concept, or a concept from the theory of truth values, Df(2) introduces a new *logical* concept. aUb denotes a relation between the propositions a and b which holds if and only if the relative truth value of b with respect to a equals the relative truth value of b with respect to a'. Note that a can be neither true nor false, since for $a = 0$, $w_a(b)$ is meaningless, and for $a = 1$, $w_{a'}(b)$ is meaningless. Before the logical sense of aUb is clarified, certain formulae must first be proved.

(16) $$aUb = \left[\frac{w(ab)}{w(a)} = \frac{w(a'b)}{w(a')}\right].$$

Theorem (16) follows from Df(2) on the strength of Df(1).

(17) $$aUb = \left[\frac{w(ab)}{w(a)} = \frac{w(a'b)}{w(a')} = w(b)\right].$$

Proof: By (16), aUb is equivalent to the equation

$$\frac{w(ab)}{w(a)} = \frac{w(a'b)}{w(a')}.$$

On the strength of a well-known theorem from the theory of proportions we obtain:

$$\frac{w(ab)}{w(a)} = \frac{w(a'b)}{w(a')} = \frac{w(ab)+w(a'b)}{w(a)+w(a')}.$$

Now, by (4) and (5)

$$\frac{w(ab)+w(a'b)}{w(a)+w(a')} = \frac{w(b)}{1} = w(b).$$

(18) $$aUb = [w_a(b) = w_{a'}(b) = w(b)].$$

Theorem (18) follows from (17) on the strength of Df(1).

(19) $$aUb = \left[\frac{w(ab)}{w(b)} = \frac{w(ab)'}{w(b')}\right].$$

Proof: By (17) aUb is equivalent to the equation

$$\frac{w(ab)}{w(a)} = w(b).$$

This yields:

(α) $$\frac{w(ab)}{w(b)} = w(a).$$

Further, by (4) and (5) it follows that

$$\frac{w(ab')}{w(b')} = \frac{w(a)-w(ab)}{1-w(b)}.$$

If in this equation the product $w(a)w(b)$, obtained from (α), is substituted for $w(ab)$, we obtain

(β) $$\frac{w(ab')}{w(b')} = \frac{w(a)-w(a)w(b)}{1-w(b)} = w(a)\,\frac{1-w(b)}{1-w(b)} = w(a).$$

(α) and (β) yield:

$$\frac{w(ab)}{w(b)} = w(a) = \frac{w(ab')}{w(b')}.$$

(20) $$aUb = bUa.$$

Proof: If a and b are exchanged in (19), we obtain

$$bUa = \left[\frac{w(ab)}{w(a)} = \frac{w(a'b)}{w(a')}\right] = aUb.$$

On the basis of the formulae proved above the meaning of the relation U can now be defined.

U is a symmetrical relation (Theorem 20) *which holds between two propositions a and b if and only if both propositions are neither true nor false* (Df(1), Df(2)) *and the relative truth value of one proposition with respect to the other or with respect to its negation equals its absolute truth value* (Theorem 18).

If the relation U holds between the propositions a and b, then it is immaterial for the truth value of b whether we take it with respect to a or a', i.e., whether we assume that a is, or is not, verified; likewise, it is immaterial for the truth value of a whether we assume that b is, or is not, verified. We say that *the propositions a and b are independent of one another.*

Example. Let a stand for "x is divisible by 2", and let b stand for "x is divisible by 3". For $x = 1, 2, \ldots, 6$, the absolute value of a is $^1/_2$, and the absolute value of b is $^1/_3$. The relative value of b with respect to a is

$$w_a(b) = \frac{w(ab)}{w(a)} = {}^1/_6 : {}^1/_2 = {}^1/_3;$$

likewise,

$$w_{a'}(b) = \frac{w(a'b)}{w(a')} = {}^1/_6 : {}^1/_2 = {}^1/_3.$$

The propositions a and b are independent of one another.

Now, it is obvious that *propositions which are neither true nor false and contain different variables are always independent of one another*.

The concept of independence of indefinite propositions, as formulated above, is narrower than the usual concept of logical independence. Usually, propositions are treated as logically independent if neither implication nor exclusion holds between them or their negations. That characteristic does not suffice for *our* concept of independence, which in addition requires satisfaction of a certain *numerical relationship* formulated in Df(2).

12. The law of multiplication

(21) $$aUb = [w(ab) = w(a)w(b)].$$

"If the propositions a and b are independent of one another, then the truth value of their product equals the product of their truth values; and conversely, if the truth value of the product of two propositions equals the product of their truth values, then these propositions are independent of one another."

Theorem (21) follows directly from (17).

Example. Let a stand for "x is divisible by 2", and b for "x is divisible by 3". For $x = 1, 2, \ldots, 6$, a and b are independent of one another. The truth value of a is $^1/_2$, and that of b is $^1/_3$. The truth value of the product equals $^1/_6$, hence $w(ab) = w(a)w(b)$.

The propositions which are neither true nor false and contain different variables are always independent of one another, so that the law of multiplication is always valid for them. For instance, let a stand for "$x = 4$", and b for "$y = 4$", where x and y can take on the values of the integers from 1 to 6. The truth value of both a and b is $^1/_6$, and the truth value of the logical product ab is equal to the arithmetic product $^1/_6 \times {}^1/_6 = {}^1/_{36}$. Now out of the 36 pairs of values for which the indefinite propositions "$x = 4$" and "$y = 4$" yield definite judgements, only one pair of values, namely (4, 4), yields a true judgement.

Like the law of addition, the law of multiplication is reversible. Hence, if the product of the truth values of any propositions is compared with the truth value of their product, it can always be decided whether the propositions in question are independent of one another, or not.

The law of multiplication can also be expanded so as to cover more than two propositions, but the formula

$$w(\textstyle\prod_i a_i) = \prod_i w(a_i), \quad i = 1, 2, \ldots, n,$$

depends on fairly complicated conditions, and it would take us too far to lay them down here. One point, however, can be emphasized: if all n propositions contain *different* variables, then the formula certainly holds.

13. A special theorem

Finally, the following theorem can be proved:

(22) $$(\textstyle\sum_i x_i = 1)(\sum_{ij} x_i x_j = 0) < \left[w_a(x_m) = \dfrac{w(x_m)w_{x_m}(a)}{\sum_i w(x_i)w_{x_i}(a)}\right];$$

$$i \neq j, \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, n.$$

The first assumption means that the sum of the propositions $x_1 + x_2 + \ldots$ $\ldots + x_n$ is true, while the second indicates that all the propositions x are pairwise mutually exclusive. The consequent states that under these assumptions the relative truth value of any proposition x, for instance x_m, with respect to any proposition a equals the quotient, the numerator of which is the product of the absolute truth value of x_m and the relative truth value of a with respect to x_m, and the denominator of which is the sum of all the expressions formed for all x in the same way as the numerator is formed.

Proof: It follows from Df(1) that

(α) $$w_a(x_m) = \frac{w(ax_m)}{w(a)}.$$

By (15), the numerator of the quotient on the right yields

(β) $$w(ax_m) = w(x_m)w_{x_m}(a).$$

By (4), the denominator yields

$$w(a) = w(ax_j) + w(ax_j').$$

Now, on the strength of the first assumption stating that $\sum_i x_i = 1$:

$$x_j' = \overset{-j}{\textstyle\sum_i} x_i,$$

where $\overset{-j}{\sum}_i x_i$ means the sum of all the propositions x with the exception of x_j. Hence

$$w(a) = w(ax_j) + w(a\overset{-j}{\sum}_i x_i) = w(ax_j) + w(\overset{-j}{\sum}_i ax_i).$$

But in view of the second assumption, all the propositions x are pairwise mutually exclusive; consequently, all ax_i must also be mutually exclusive, and on the strength of the law of addition (9) we obtain

$$w(a) = w(ax_j) + \overset{-j}{\sum}_i w(ax_i) = \overset{-j}{\sum}_i w(ax_i).$$

Now by (15)

$$w(ax_i) = w(x_i)w_{x_i}(a),$$

hence:

$$w(a) = \sum_i w(x_i)w_{x_i}(a). \qquad (\gamma)$$

Now if we substitute the results obtained by (β) and (γ) for the numerator and the denominator of (α), we obtain the thesis.

II

THE CONCEPT OF PROBABILITY

14. The calculus of truth values versus the calculus of probability.—15. Two principal difficulties in probability theory.—16. Objective and subjective probability theory—17. Indefinite propositions and probabilistic propositions.—18. The principles of necessary and insufficient reason.—19. Truth values and probability fractions.—20. Interpretation of probabilistic propositions.—21. The listing of results.

14. The calculus of truth values versus the calculus of probability

The calculus of truth values outlined in the foregoing Chapter has the property that, without in any way assuming the concept of probability, without even *mentioning* it, its theorems are in agreement with the principles of probability theory. If the expressions $w(a)$, $w(b)$, etc., are interpreted not as *truth values* of statements but as *probabilities* of "events", then the theory of truth values becomes the theory of probability. In particular, the law of addition then becomes the rule of *complete*

probability; the law of multiplication, the rule of *compound probability*; the concept of relative truth value, the concept of *relative probability*; and the theorem proved in the preceding Section becomes a general interpretation of *Bayes' theorem.*

This remarkable agreement suggests that *probability propositions are nothing else than indefinite propositions, while probability fractions are their truth values.* This supposition becomes a certainty when we find that all the difficulties thus far attending the laying of the logical foundations of probability theory can be removed only on the basis of the interpretation of probability offered in this paper. The following Sections will be concerned with proving this claim.

15. Two principal difficulties in probability theory

According to Laplace, "the probability of an event is the ratio of the number of the cases favourable to that event to the number of all possible cases, if there is no reason for believing that one of the cases is more likely to occur than another, so that for us they are all equally possible".[3] This definition, which abounds in errors, is best suited to expose all the difficulties of probability theory.

If we at first disregard minor errors, for instance, that the definition refers only to the probability of *events, future events at that*, the principal shortcoming of Laplace's formulation seems to be that it is not a definition of *probability*, but at most only a definition of the probability *fraction.* It does not explain what probability is, but only tells us how to compute probabilities, in doing which it identifies probability with a numerical ratio. But probability is no more a numerical ratio than time, for example, although time is also measured in terms of a numerical ratio. Since Laplace's formulation is basically not an explanation of the concept of probability, it would not, perhaps, be correct to raise against it the objection that it is a vicious circle by pointing out that it explains probability by referring to "equally possible" cases, which cannot mean anything but "equiprobable" cases.[4] The principal error of the definition is the more strongly brought into relief: it explains an obscure

[3]) *Théorie analytique des probabilités*, 3rd ed., Paris, 1820, p. 179.

[4]) This objection has been raised, among others, by Poincaré (cf. *Calcul des probabilités*, Paris, 1896, pp. 5–6).

concept by means of another concept which is wrapped in a similar obscurity.

The mysterious essence of probability has not been fathomed by the great founders of probability calculus. Hence the first basic difficulty and the first basic problem of probability theory is: *what is probability?*

But even if we assume that Laplace's definition is not in any way intended as an explanation of the concept of probability, but merely indicates how to compute probability fractions, it still may not be absolved of error. According to Laplace, the computation of a probability fraction is based on the concept of "equally possible", i.e., "equiprobable", cases. But then the question arises: under what conditions can we consider two possibilities or two probabilities to be equal? Since Laplace answers this question by a reference to the subjective element of *belief*, he does so in a way which does not agree with the objective nature of the probability calculus and is the object of a controversy which has continued till the present day. Here lies the second basic difficulty and the second basic problem of probability theory: *how are probabilities computed?*

I shall now try to demonstrate that both difficulties vanish only when "probabilistic" propositions are interpreted as indefinite propositions.

16. Objective and subjective probability theory

The predicate "probable" is usually connected with "*events*", and *future* events at that. Without, for the time being, going into the problem as to whether it is at all possible to speak of a probability of *events*, we must state that in the same way in which we speak (falsely, as will be seen later) about the probability of *future* events, *present* and *past* events can also be probable; moreover, this refers not only to events, but also to many other states of things which do not fall under the concept of "event". It may be asked: what is the probability that a two-digit integer is divisible by 3; but divisibility of a number by 3 is not an event. Hence another subject must be sought for the predicate "probable". Now, it is quite certain that all states of things which are referred to as probable can be represented in the form of propositions. That is why it would be advisable to speak about the probability of propositions, the more so as in this way nothing is prejudged about the nature of probability.[5]

Propositions can express *subjective* conditions, on the one hand, and *objective* facts, on the other. When I am looking at a game of dice and state: "6 has turned up", then this proposition, on the one hand, expresses my subjective conviction, and on the other, denotes an objective state of affairs. Now, it is possible that propositions either are called probable because the *subjective* conditions which they express are not *convictions* but mere *suppositions*, or that they are considered probable because the *objective* facts which they denote are not *realities* but *possibilities*. A theory of probability which adopted the former standpoint might be called *subjective*, while the other could be called *objective*.

As far as I know, a purely objective theory of probability has not yet been formulated by anyone. This is so because it seems to be irreconcilable with two universally accepted principles: the principle of causality, and the principle of the excluded middle. By the former principle, we assume that everything in the world occurs of necessity, so that no room is left for *possibility*. Even such an insignificant event as a delicate movement of the hand which sets the dice rolling has been predetermined in all its details. The causes work in such a way that six either *must* turn up, or *cannot* turn up. In the former case, the proposition "6 has turned up" is necessarily true; in the latter, it is necessarily false; in neither case is it objectively probable.

However, should anyone believe that the principle of causality is not evidently true and that there are events which are not subject to the coercion of necessity, another principle can be referred to, namely the principle of the excluded middle, which, like the former, excludes all possibility. On the strength of that principle, of two contradictory propositions which pertain to definite individual objects, one must be true. Hence even under the assumption that an event, for instance, the drawing of a black ball out of an urn containing black and white balls, does not occur of necessity, one of the two must be true: a black ball either *is drawn* or *is not drawn*. If the former proposition is true,

[5]) Cf. pertinent remarks by Stumpf in "Über den Begriff der mathematischen Wahrscheinlichkeit", *Sitzungsberichte der philosophisch-philologischen und historischen Klasse der bayerischen Akademie der Wissenschaften*, No. 1, Munich, 1892, pp. 43 and 46.

then it is true at every time, hence also *before* the drawing of the ball; a black ball *will be* drawn, even if that event be not predetermined. If the latter proposition is true, then it is also true at every time, hence also *before* the drawing of the ball; a black ball *will not* be drawn, even if that event be not predetermined. In both cases we have to do with *realities*, even though not with *necessities*; again there is no room for objective *possibility*.

It follows from these considerations that it would be useless to try to explain the essence of probability by the study of objective facts. This was sensed by the founders of the probability calculus, such as Jacob Bernoulli or Laplace, and hence, more or less explicitly, they tried to solve the puzzle of probability by reference to *subjective* elements.

There is no consistent subjective theory of probability; it can immediately be seen that any such theory would be unsatisfactory. Endeavours to formulate such a theory are to be found, for instance, in the works of Jacob Bernoulli, who defined probability as a degree or part of certainty. If, says he, complete certainty, which is assumed to equal 1, consists of 5 parts or probabilities of which 3 favour a certain event and the rest oppose it, then the probability of that event is $^3/_5$.[6])

It is not difficult to uncover the shortcomings of such a formulation. Should probability be a part of certainty then, like certainty, it would have to be a property of psychic processes, namely convictions and suppositions. Now, *first*, so far no one has succeeded in measuring beliefs and suppositions and their properties. The computation of probability fractions would then be impossible, which is at variance with evident facts. *Secondly*, in the probability calculus we assume that under given objective conditions the probability of a proposition has *only one*, definite value. Now, should probability be a degree of certainty, then it would have to vary according to the psychic states of various individuals. The hope of winning and the fear of losing in gambling have a strong effect on the degree of certainty with which events are expected. *Finally*, there is no doubt that the probability calculus has nothing to do with the measurement of degrees of certainty and with psychic phenomena. Should the essence of probability consist in subjective processes, then the entire theory of probability would have to be relegated to a psychological laboratory and would then be an empirical discipline, like Fechner's psychophysics, but not an *a priori* branch of pure mathematics. The argument last adduced has a general validity and precludes any attempt to lay a subjective foundation for probability theory.

6) *Ars conjectandi*, Basel, 1713, p. 211.

Both paths, the objective and the subjective, are thus blocked; hence a third must be sought.

17. Indefinite propositions and probabilistic propositions

Although probability does not exist objectively, the probability calculus is not a science of subjective processes and has a thoroughly objective nature. Hence the essence of probability must be sought not in a relationship between propositions and psychic states, but in a relationship between propositions and objective facts. Now, such a relationship is not purely *positive*, but also *negative*. What is meant by this formulation can best be explained by reference to the concept of falsehood.

Falsehood, like truth, is a property of propositions which results from the relationship between the latter and objective facts. But while true propositions always have counterparts in certain facts, false propositions have no objective correlates. Thus falsehood is characterized by a *negative* relation to facts. Yet, although falsehood does not exist objectively, the concept of falsehood is free from subjective elements.

The same holds for probability, which also does not exist objectively and is only a property of *propositions*. A negative relation to facts is also included in the concept of probability as one of its characteristics. Probability lies midway between truth and falsehood, in the same way as a proper fraction lies between 0 and 1, and gray between black and white. But it is not possible to mix truth with falsehood in order to obtain probability as it is possible to obtain a gray colour from a mixture of white and black paint. No proposition can be both true and false. Something intermediate between truth and falsehood can be obtained only by forming a *group* consisting of true and false statements. This condition is satisfied precisely by indefinite propositions.

In my opinion, *probability is a property of those indefinite propositions which are neither true nor false*. Thus each probable proposition has its counterpart in a group of true and false judgements. The proposition "x is A" is probable if at least *one* value of the variable x verifies that proposition, i.e., yields a true judgement, and at least *one* other value does not verify that statement, i.e., yields a false judgement. This shows that *falsehood*, and with it the *negative* relation to facts, is necessarily contained in each probable proposition. If man were not in a position to formulate false propositions, then he would not know the concept of probability. This is not a subjective, but a purely *human* element, which should not astonish us, since it is man who is the creator of the concept of probability.

The interpretation of the essence of probability presented here might be called the *logical* theory of probability. According to this viewpoint, probability is only a property of propositions, i.e., of logical entities, and its explanation requires neither psychic processes nor the assumption of objective possibility. *Probability, as a purely logical concept, is a creative construction of the human mind, an instrument invented for the purpose of mastering those facts which cannot be interpreted by universally true judgements* (*laws of nature*).

The logical theory of probability seems to me to be only way out which avoids the reefs of both the objective and the subjective theory. In the light of that theory, the interpretation of those propositions which in probability calculus are considered probable must be subjected to a thorough criticism. It can no longer be asserted that such events or propositions as "this die will now turn up 6" or "the next drawing from this urn will yield a black ball" are probable. Such propositions, being *definite* judgements, are either *true* or *false*, even if *before* the event we can never know *which* of them are true and which are false. The fact that such judgements are to this day considered probable, although neither the objective nor the subjective theory of probability is tenable, and hence no real sense may be associated with the probability of such judgements, is, in my opinion, to be explained by the fact that people have so far been unable to cope with the concept of probability. Only those propositions which contain a variable can be *probable*, for instance: "the x-th throw of the die yields 6" or "the x-th drawing from the urn yields a black ball". But if a proposition has once been probable, it remains so forever; it can never become either true or false, and even an omniscient and omnipotent mind could not in the least change its degree of probability.

18. The principles of necessary and insufficient reason

Both basic problems of probability theory: What is probability? and How is probability computed?, are so closely intertwined that the latter cannot be solved if the former has not been solved. But since there are reasons to claim that there has been so far no satisfactory solution of the former problem, we cannot expect to find, in the existing literature on the subject, a definitive clarification of the latter problem.

All the attempts made so far to compute probabilities have been based on the concept of "equally possible" or "equiprobable" cases. There are two interpretations of this concept: an objective and a subjective one. The most eminent representatives of these two theories in modern times are von Kries and Stumpf.

As the basis of Kries's theory we have to consider the theorem that assumptions pertaining to equal and "indifferent" original ranges are equiprobable.[7] On the contrary, according to Stumpf, those cases are equally possible, and hence also equiprobable, with respect to which we are equally ignorant; two cases of ignorance can be held to be equal only if we know absolutely *nothing* as to which of the cases will occur.[8] The contrast between the two theories can best be explained by an example.

The probabilities that a geometrically and physically regular die will yield 1, or 2, or ... or 6, are assumed to be equal. According to Kries, they are equal because "here the geometrical and physical regularity of the die necessarily results in the fact that a definite interconnected complex of possible movements, which yields 6, is always accompanied by other complexes, which in every respect differ *very little* from the former, are contained within almost the same reach, and yield 1, 2, 3, 4, 5, and that these six kinds of movements, repeated regularly in turn,

[7]) *Die Prinzipien der Wahrscheinlichkeitsrechnung*, Freiburg and Br., 1886, p. 157.
[8]) Stumpf, *op. cit.*, p. 41.

fill the entire range of possible movements." [9]) The complexes of possible movements mentioned in the quotation are called ranges by Kries. Thus, in a game of dice the ranges of all throws are approximately *equal*; further, they are also *primary*, i.e., not deducible from any other ranges, and finally they are "*indifferent*", i.e., there is no reason to think that one of them is more probable than any other.[10]) These three conditions — equality, primacy, and indifference of ranges — are necessary but also sufficient to justify the equal probability of all throws of the dice.

Thus while Kries, as can be seen from the above example, requires the existence of *objective* equality, which would *necessarily* lead to the listing of equally possible cases, Stumpf holds that it is possible to disregard objective equality and, like Laplace, bases equal possibility or probability on *subjective* elements, namely on a *lack of knowledge*. According to Stumpf we could thus assume that the possibilities of a die turning up 1, or 2, ... or 6 are equal even if the die is geometrically or physically irregular, but we *know nothing about it*. The contrast between the two theories is now clear. Special terms have been coined to express that contrast tellingly: the assumptions underlying both theories are set in opposition to one another as the principles of *necessary* reason and of *insufficient* reason.

It is, however, clear that both principles must fail if, as is usual, they are applied to the computation of the probabilities of *definite* individual events. *Definite* events cannot be probable at all, since they are either *necessary* or *impossible*, either *real* or *unreal*. When, in a game of dice, we mean a definite individual throw of a die, then there is *only one* strictly defined "complex of possible movements", which necessarily yields a given number. It is true that a quite small change in the original position of the die would result in another complex of possible movements, which in every respect would differ very little from the previous one and would effect another throw, but such a change in fact does not take place. We cannot here compare any ranges at all, since only one range is given, which necessarily exists, and all others are excluded. Comparable ranges exist only when we consider not a *definite* throw, but *any arbitrary* throw. Then with each complex of possible movements which yields 6, we can compare other complexes, which differ very little from the former and which effect the throws of 1, 2, 3, 4, 5; but in this case we have to do with an *indefinite* proposition, and to compute the probabilities of such propositions we must follow another path.

[9]) Kries, *op. cit.*, p. 55.
[10]) *Ibid.*, p. 25.

Thus we can see anew that a purely objective theory of probability, based on the concept of objective possibility, cannot be realized. Kries seems to have sensed this, for he remarked that the totality of conditions which constitute the known range of a probable event does not suffice to predetermine the result. There is a remainder which evades our knowledge, *but we should not have any reason to assume* that that remainder favours any definite range.[11]) By introducing in this way the concept of indifference of range into his considerations Kries actually abandoned his original, purely objective, standpoint and came closer to those favouring the principle of *insufficient* reason.

The followers of that principle are in a still worse position when exposed to criticism than are the defenders of the principle of necessary reason. It is a subtle but untenable paradox to make lack of knowledge the basis of knowledge, be it only *probable*. Nothing results for the probability of an event from the fact that we know little or nothing about that event or have no reason to assume one event to be more probable than another. The probability of drawing just the *white* ball out of an urn which contains 999 black balls and one white ball is objectively very small, but it cannot be increased by the fact that a person does not know the ratio of the balls. If the principle of insufficient reason were true, then the computation of the probability would have to depend on subjective factors and would be different for different individuals, but the probability calculus as an objective discipline is not concerned with stating how probable an event is for an insufficiently informed observer, and strives to determine the probabilities of events or propositions without regard to any subjective factors.

Thus both principles have proved unable to solve the second basic problem of probability theory; hence a third principle must be sought.

[11]) *Ibid.*, p. 61.

19. Truth values and probability fractions

The principles of necessary and insufficient reason *cannot* be correct since they assume no correct concept, or rather no concept at all, of probability. But the probability *fraction*, if it is not to be a purely external and arbitrary appendage of the concept of probability, must find its justification and its legitimation in the *essence* of probability.

Now, if it is assumed that probability is a property of every indefinite proposition which is neither true nor false, then the probability fraction is in fact based on the essence of the concept of probability. The element of *plurality* and *number* is immediately contained in that concept. An indefinite proposition can be probable only if the variable which occurs in it can take on values, both which verify and which do not verify the proposition. The greater the number of the verifying values in proportion to all values, the greater the probability of a proposition, which in a limiting case can reach the degree of truth. It is thus self-evident that *the probability fraction is identical with the truth value of an indefinite proposition.*

If this assumption is made, then all the difficulties associated with the concept of "equally possible" or "equiprobable" cases vanish immediately. That concept, so obscure and so hotly disputed, is no longer needed as a basis for the computation of probabilities. Probabilities will be computed not by being *compared*, but by having the verifying and the non-verifying values of the variables *counted.* If the range of the values of the variables involved is well defined and finite, no difficulties can arise. There may also be cases in which the degree of probability of an indefinite proposition can be found *without* counting. Only when probability fractions have been *computed* in this way can they be *compared* with one another, and such a comparison immediately reveals which propositions are to be considered "equiprobable".

Now is perhaps the time to make use of the most important argument which the logical theory of probability has at its disposal and which forms the content of the entire first part of the present paper: *if probabilistic propositions are interpreted as indefinite propositions and probability fractions are interpreted as truth values, then all the principles of the probability calculus can be obtained from this assumption in a strictly deductive manner by means of the algebra of logic.* Moreover, *new* principles *can be formulated* and old concepts *can be made more precise* The *theorem on the truth value of a reason*, which can be considered an analogue of the law of addition, has, as far as I know, been unknown before; the relationship between the probability calculus and *numerically defined conclusions* also seems not to have been noticed previously; the important concept of *independence* of "events" or propositions has first been given a strictly scientific formulation in Section 11 of this paper. The logical theory of probability has thereby stood its test. It has proved workable from the very outset and, when supported by the algebra of logic, promises to remain a fertile working theory.

20. Interpretation of probabilistic propositions

It cannot be denied that the probability calculus, for all the previous uncertainty of its *logical* foundations, in its wider *mathematical* expansion has yielded numerous results that are in agreement with empirical data. Theoretically interesting and practically important disciplines, such as the theory of games of chance on the one hand, and mathematical statistics, insurance theory, descriptive statistics (*Kollektionmasslehre*) on the other, are based on the principles of the probability calculus. Now in all these disciplines reference usually is made to the probability of *definite individual* events. For instance, the question may be: what is the probability of throwing a given number with a given regular die at a given moment, or of drawing a ball of a given colour from a given urn; or else the problem may be how to compute the probability of the death within one year of a given person 40 years old. But in the light of the logical theory of probability, all such problem formulations and computations must be waived aside as meaningless. *Individual events* can never be probable, since probability is exclusively a property of *indefinite propositions*. Hence the problem arises how, in the light of the new theory as presented in this paper, we should interpret the propositions which in the probability calculus are interpreted as probable.

The question is not difficult to answer when it comes to *statistical* probability, that is, probability which is determined *a posteriori*. The following example will suffice: on the basis of the tables of survival of 23 German insurance companies it has been computed that out of the 85 020.5 insured persons who were 40 years old, 940 died before becoming 41. This yields the following empirical value of the probability of death of a German who is 40 years old:[12])

$$P_{40} = \frac{940}{85\,020.5} = 0.01106.$$

The probability fraction thus obtained by the statistical method, does not in the least mean that a given person who is 40 years old has to expect death with the probability of about eleven thousandths. It merely indicates that so far, on the average, out of 1000 persons 40 years of age, 11 have died before turning 41. It is supposed that this number of deaths depends on certain conditions, which are expected to continue to exist in their essentials in the near future and to have similar effects. Hence it is expected that also in the future out of 1000 persons who are 40 years old, on the average 11 persons will die before becoming 41 years old. But this means nothing else than that the indefinite proposition: "x, who is now 40 years old, will die before becoming 41 years old", has on the average the truth value of 11 thousandths. This means in turn that if we substitute individual values for x, we obtain on the average 11 true propositions out of 1000 definite propositions.

If the above probability fraction is interpreted in this way, then all the difficulties which inevitably accompany the usual interpretation disappear. In the light of the ordinary formulation, one had to assume that the probability of death within one year would be the same for a strong and healthy man of forty as for a seriously ill man of forty who may even be breathing his last. Such an assumption, however, is not admissible. But the new interpretation explains how it happens that an insurance company, which bases its calculations on the principles of probability theory, does not need to fear losses. It need not be concerned *which* of its customers are doomed to death in a given year if it can only hope that the number of deaths will correspond to the statistically obtained truth value of a given indefinite proposition.

All *empirically* determined probabilistic propositions can be interpreted in the same way without any difficulty. The analogous interpretation seems, however, to encounter greater difficulties when it comes

[12]) Cf. E. Czuber, *Wahrscheinlichkeitsrechnung*, Vol. II, Leipzig and Berlin, 1910, p. 154.

to *a priori* probability. Probability values obtained *a priori* usually occur in the theory of games of chance, that is, in that field of science which gave rise to the probability calculus. We again select an example, namely the oldest one, which led to the computation of probabilities.

Galileo Galilei was once asked by a careful observer of a game of dice (*passe-dix*), in which the point is to obtain more than 10 in three throws, why the sum of 11 occurred more often than the sum of 12, although in the opinion of the person who asked the question both sums could be obtained by the same number of combinations. Galileo solved the problem by proving that the sum of 11 could be obtained by 27 combinations, whereas that of 12 by only 25 combinations.[13]) The calculation made by Galileo can be very well presented as the computation of truth values of indefinite propositions. The task reduces to finding the truth values of the following propositions with three variables each: "$x+y+z = 11$" and "$x+y+z = 12$", for $x = 1, 2, \ldots, 6$; $y = 1, 2, \ldots, 6$; $z = 1, 2, \ldots, 6$. It can easily be seen that out of the $6^3 = 216$ combinations of values, 27 verify the former proposition, while 25 verify the latter.

So far so good. But it cannot be asserted that in the interpretation of the probability of obtaining the sum of 11 by three throws the definite judgement: "the throw of these three dice will now yield 11" may be replaced by the indefinite proposition: "$x+y+z = 11$". It is true that the judgement referred to above cannot be probable as it is a definite proposition, so that in its place we must look for something else, which would in fact be probable; yet the above indefinite proposition cannot be what we seek since it pertains to something else than the demand to obtain the sum of 11 in three throws. This becomes clear as soon as we assume that the dice in question are not regular. In this case the probability of throwing 11 cannot be assumed to be $^{27}/_{216}$, while the truth value of the indefinite proposition "$x+y+z = 11$" is $^{27}/_{216}$ as before, if it is assumed that the variables x, y and z can take on the values from 1 to 6, which corresponds to the condition that every die has six sides. Hence we must seek another interpretation of the probability in question.

[13]) Cf. Czuber, *op. cit.*, Vol. I, p. 27.

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In my opinion, the probable proposition which we seek in this case must be formulated as follows: "the x-th throw of three dice yields the sum of 11". To give the truth value of this proposition we would have to know the total number of throws and to count those of them which yielded 11. But it is impossible to make such a count in any empirical way. We must therefore resort to an *a priori* determination by assuming that in view of the geometrical and physical regularity of the dice, the frequency of the various sums which can be obtained by throwing the dice *depends only on the number of ways in which they can be obtained.* That is, for each die it is assumed that all the throws which can be made with it must be *equally* distributed over the six numbers. Should that not be the case, we would have to suppose a cause that would favour the throw of a given number, for instance 6, and such a cause would have to find its explanation in the *irregularity* of the die, which would contradict the assumption. It can thus be seen how the equality of the range, required by Kries in the computation of probabilities, finds its application here. It serves the purpose of *determining the ratio of the verifying values of a variable to all its values on the basis of an a priori fiction,* without *counting* the individual values and even without *knowing* their number.

This interpretation of *a priori* probabilistic propositions is not only in full agreement with the logical theory of probability, but also sheds light on the *a priori* element of probability. It can now be understood that here, as in all cases involving empirical data, *a priori* propositions have only the values of *hypotheses* which must later be checked in the light of the facts. It may be that the probability calculus, by becoming a sober logical theory, loses the charm of the mysterious, which has attracted so many eminent minds. But in exchange, it rises now as a clearly and sharply outlined structure to which we may not refuse a certain logical elegance.

21. The listing of results

To conclude let me list the most important results of the present paper, concisely and clearly, in the form of theses.

I. The concept of probability

1. Propositions are *indefinite* if they contain variables.

2. Indefinite propositions are *true* if they are verified by all the values of the variables.

3. Indefinite propositions are *false* if they are not verified by any values of the variables.

4. Indefinite propositions are *neither true nor false* if they are verified by some, but not by all, values of the variables.

5. *Probability* is a property of indefinite propositions which are neither true nor false.

6. *Definite* propositions can never be probable, but are either true or false.

7. *Probable* propositions can be neither true nor false, but are *always probable.*

8. Propositions which in the probability calculus are considered probable must be formulated not for any *definite* case, but for any *arbitrary* case x.

9. A *purely objective* theory of probability is impossible, since there is no objective possibility.

10. A *subjective* theory of probability is impossible, since the probability calculus has nothing to do with subjective processes.

11. The *logical* theory of probability, as presented here, is objective in so far as it interprets probability as a property of propositions which is characterized by its relationship to the *objective* world.

12. Nevertheless, probability is a concept invented by the *human mind* for the purpose of scientific treatment of those facts which cannot be interpreted by general judgements.

II. The principles of the probability calculus

13. The *truth value* of an indefinite proposition is the ratio of the number of those values of the variables which verify that proposition to the number of all the values of the variables.

14. The *degree of probability* of an indefinite proposition is identical with its truth value.

15. The relation of *implication*, or the relation between *reason* and *consequence*, holds between two indefinite propositions a and b if for each pair of the values of the variables occurring in a and b either the reason a yields a false judgement, or the consequence b yields a true one.

16. *Theorem on the truth value of the reason*: The truth value of the reason, augmented by the truth value of the logical product of the negation of the reason and of the consequence equals the truth value of the consequence.

17. All the *principles of the probability calculus* can be obtained from the foregoing explanations and theorems by means of the algebra of logic in a strictly deductive manner.

In particular:

a) the law of addition, or the rule of complete probability;

b) the law of multiplication, or the rule of compound probability;

c) Bayes's theorem.

18. The theorem on the truth value of the reason, the law of addition, and the law of multiplication are *conversible*, i.e., they make it possible, on the basis of the numerical equalities between the truth values of given propositions, to infer the logical relations between such propositions.

19. The *relative truth value* of a proposition b with respect to another proposition a is the ratio between the truth value of the logical product of both propositions and the truth value of the proposition a.

20. Relative truth values are identical with *relative degrees of probability*.

21. The concept of *independence*, as used in the probability calculus, denotes a symmetrical relation which holds between two probable propositions if and only if the relative truth value of one proposition with respect to the other proposition equals the absolute truth value of the former.

III

HISTORICAL AND CRITICAL REMARKS

22. The history of the emergence of the present theory

It is to the study of algebraic logic that I owe the first inspiration to treat probabilistic propositions as indefinite propositions. In that discipline, the concept of indefinite proposition plays a very important role, and eminent mathematicians and logicians, such as G. Frege, G. Peano, and B. Russell, have recently contributed much to its clarification.[14]) To Frege I am moreover indebted for the term "truth value", which, however, I use in a different sense, since I denote by it not only truth and falsehood, as Frege does, but also all degrees of probability. The idea of using the rules of algebraic logic as foundations for the principles of the probability calculus was first suggested to me by the form of the formulae in the principal work on the probability calculus by the Polish mathematician Gosiewski.[15])

The ideas, as developed in the present paper, were for the first time outlined by me at Professor Meinong's philosophical seminar in Graz in the summer semester of 1909. I treated the same subject more comprehensively in my lectures on probability theory which I gave in Lwów University in the winter semester of 1910/11. At the same time, the principal theses of my probability theory were presented in two public lectures at the meetings of the Polish Philosophical Society in Lwów, namely at the 100th meeting, held on November 4, 1910, in honour

[14]) Cf. Frege, *Funktion und Begriff*, Jena, 1891; Frege, *Grundgesetze der Arithmetik*, Jena, Vol. I, 1893, Vol. II, 1903; Peano, *Formulaire mathématique*, ed. 1902/3, Turin; Russell, *The Principles of Mathematics*, Cambridge, 1903.

[15]) *Zasady rachunku prawdopodobieństwa* (The principles of the probability calculus), Warsaw, 1906. The introduction to this work, which was of particular interest to me, was published earlier by the author as "O zasadach prawdopodobieństwa" (On the principles of probability) in the Polish philosophical journal *Przegląd Filozoficzny*, Warsaw, ed. Dr. Weryho (cf. Vol. VII for 1904, p. 270).

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of Professor Twardowski, the founder of the Society, and at the 101st meeting, held on November 12 of the same year.[16])

This brief history of the emergence of the probability theory presented in this paper would not be complete if I did not raise two issues. *First*, I wish to point to certain difficulties inherent in the concept of indefinite propositions as it is usually formulated by mathematical logicians, which for a long time disturbed my comprehending that concept, and, *second*, I should like to discuss two outstanding works which reveal tendencies similar to mine and which I came to know only at a later date.

These critical analyses, which will add more clarity to the problem of probability, will form the content of the following Sections.

23. Russell's propositional functions

Many authors, and Russell [17]) among them, divide those propositions which I call "indefinite" into two categories: sentences which contain a variable but are neither true nor false, for instance "x is a man", are called by Russell "propositional functions" and are not treated by him as propositions, since by "propositions" he means only *true* or *false* sentences. Accordingly Russell calls sentences which contain a variable but are true or false, for instance "x is a man implies that x is mortal", "genuine propositions", and following Peano [18]) he denotes the variables which they contain as "apparent variables" in opposition to "real variables", which occur in propositional functions.

It might be supposed that it is purely a matter of terminology whether indefinite sentences which are neither true nor false be called propositions or propositional functions. Yet it is otherwise: Russell's terminology artificially divides entities which by their very nature belong to the same category. There is only a *quantitative* difference between propositional functions and propositions which contain apparent variables. Propositions with apparent variables yield true or false judgements for *all* values of their variables, whereas propositional functions

[16]) A brief communiqué appeared in the philosophical periodical *Ruch Filozoficzny*, edited in Lwów by Prof. Twardowski (cf. 1 (1911), p. 52).

[17]) Cf. Russell, *op. cit.*, pp. 12–13 and Chap. VII.

[18]) Cf. Peano, *op. cit.*, p. 5.

are true only for *some* values of the variables and false for the others. For instance, the proposition "x is not a prime number" yields, for $x = 90, 91, \ldots, 96$, 7 true and 0 false judgements; the propositional function "x is not divisible by 7" yields, for the same range of the values of the variables, 6 true and 1 false judgements. The difference between these two indefinite sentences finds its only expression in the ratios 7 to 7 and 6 to 7. Although it is true that sentences with apparent variables can take on only the *limiting values* 0 and 1 out of the whole range of all truth values, yet these limiting values do not differ essentially from the remaining truth values. That is why I came to the conclusion that no essential difference can be drawn between propositional functions and propositions with apparent variables, and I accordingly covered these two categories of sentences by *one* concept, namely that of "indefinite proposition". Moreover, it seems to me that the terminology adopted by Peano and Russell can easily prove misleading. The variable which is contained in true or false proposition cannot be called *apparent*, since it is as *real*, i.e., *actual a variable*, as those which occur in propositional functions. There is no difference in the nature of the variables in the two cases.

The sharp demarcation between propositional functions and propositions perhaps contributed most to the fact that the idea of applying indefinite propositions to probability calculus could not develop easily. Probability is a property of propositions. But even now, after that idea has developed, the opposition to calling propositional functions, i.e., sentences which are neither true nor false, *propositions* or *judgements* and to treating them as *probable* does not easily vanish in view of its underlying venerable and widespread prejudice.

Aristotle was the first to formulate the fateful assertion that all propositions must be either true or false.[19]) He wanted thereby to characterize propositions as opposed to other kinds of sentences, which express *requests*, *questions*, and *commands*. He gave no other motive, much less *proof* of his assertion. But where there is no proof, there are also no counter-proofs, and thus the Aristotelian assertion has been uncritically repeated until the present day, although formal logic since Aristotle has always demonstrated its theorems by means of *indefinite* propositions

[19]) *De interpretatione*, c.4, 17 *a* 1–3.

such as "all S are P", and has always considered the latter as judgements or propositions, although they can be neither true nor false.

An end must be put to this prejudice once and for all. In order to characterize propositions as opposed to other categories of sentences it is not necessary to squeeze them into two drawers, those of truth and falsehood, but it suffices to accept that which is self-evident and to admit that propositions are just sentences which *predicate* something about something and hence *assert* something, i.e., state that something *is* or *is not*, that it is so or not so. Hence the question: "is x a man?" cannot be a proposition since it *does not assert* anything, but the indefinite proposition "x is a man" must be called a proposition in the same way as the definite judgement "Socrates is a man", because both sentences *assert* something. This not only leads to a better comprehension of probability, but also protects formal logic against inconsistencies.

Russell could not overcome the Aristotelian prejudice. Perhaps that was why it was impossible for him, as for the other inspired founders and promoters of mathematical logic, to interpret probabilistic statements as indefinite propositions, in spite of the fact that most of them well knew the concept of indefinite proposition.

24. Bolzano's concept of validity

The Aristotelian prejudice also influenced an earlier author whose works have at present acquired great importance, as they well deserve, and who developed opinions that come quite close to mine.[20] In the second volume of his *Wissenschaftslehre*[21] Bolzano introduced a new logical concept, which he called the "validity" of a proposition. The starting point of his analysis is Aristotle's claim, quoted above, which Bolzano often repeated[22], that every sentence (i.e., every proposition) is either *true*, and then it is always true, or *false*, and then it is always false, unless, he adds, we change something in it, so that we have to

[20] For the reference to Bolzano I am indebted to Professor Twardowski; although Bolzano's principal work was long known to me, I had previously paid no attention to his remarks on the concept of the "validity" of a sentence.

[21] Sulzbach, 1837, Vol. II, Sec. 147, p. 77 ff.

[22] Cf. *Ibid.*, Vol. II, Sec. 125, p. 7, and Vol. I, Sec. 23, p. 93, Para. 2, where the well-known quotations from Aristotle, mentioned in foregoing sections, are cited.

consider not that proposition, but some other proposition in its place. For instance, we say that the proposition: "this flower smells nice" can be both true and false, according as "this" refers to a rose or to a stapelia (a cactoid plant with fleshy leaves that smells of decayed meat). This, however, does not in the least contradict the principle formulated above, since here we no longer have to do with a *single* proposition but have to consider *two* essentially different propositions which are obtained by changing the idea denoted by the word "this". But when we, often without clearly realizing this fact, assume certain concepts in a proposition to be variable and then observe the relationship between the proposition and truth, then, Bolzano says, it pays to take the pains of doing so with full consciousness and with the definite purpose of acquiring the knowledge about the nature of a given sentence by observing its relationship to truth.

Bolzano's consideration can best be explained by examples. If in the proposition "the man Caius is mortal" the idea "Caius" is treated as one that can be changed at will and replaced by ever new ideas, e.g., "Sempronius", "Titus", "rose", "triangle", etc., then all the propositions obtained in this way are usually true, provided only that the subject of the proposition, and consequently the proposition itself, makes sense. But if the same idea is changed in the proposition "the man Caius is omniscient", only false propositions are obtained. Finally, from the proposition "the entity Caius is mortal", by changing the idea "Caius" we obtain propositions some of which are true, and some false, since in addition to mortal entities there are also immortal entities.

The quantity of the true and the false propositions obtained by changing an idea in a given proposition can in certain cases be *computed*. For instance, if in the proposition "the ball marked by the number 8 is among those which will be obtained in the next drawing" the idea 8 be taken as changeable and replaced, consecutively, by integers from 1 to 90, then under the assumption of the usual principles of a lottery (5 numbers are drawn out of 90 in each case) we obtain 5 true and 85 false propositions. Now the ratio of the number of the true propositions obtainable from a given proposition by exchanging certain ideas, considered changeable, with other ideas in accordance with a given rule, to the number of all propositions obtainable in this way, was called by Bolzano the "validity" of a proposition. The degree of validity is represented by a fraction whose numerator is in the same ratio to its denominator as the number of the true propositions to the number of all propositions. Thus, for instance, the degree of validity of the proposition discussed above is ${}^{5}/_{90} = {}^{1}/_{18}$.

It can be seen that Bolzano defined the concept of *validity* of a proposition in a manner quite similar to that in which I have formed the concept of *truth value*. There is, however, an essential difference between the two concepts: Bolzano's validity is a property of *definite* sentences or propositions, whereas truth values can be characteristic traits of *indefinite* propositions only. This primary difference, which is explained by the fact that *Bolzano did not know the concept of indefinite proposition* and could not accept it as long as he was influenced by the Aristotelian prejudice, leads to numerous secondary differences. One of the most important is expressed by the following remark by Bolzano: "It is self-evident that the validity of a sentence must depend on which ideas and how many are considered changeable".[23]) Bolzano clarifies this remark by the example: if in the proposition: "this triangle has three sides" only the idea "this" is changed and changed so that the proposition always is meaningful, then we always obtain true propositions and the degree of validity of this proposition is 1. But if, along with the idea "this" also the idea "triangle", or instead of those two the idea "side", is taken as changeable, then the degree of validity of the proposition turns out to be quite different, since in addition to true propositions we also obtain false ones.

The difference pointed up by this remark, between the concept of validity and that of truth value, is strongly borne out if we select examples constructed by analogy to those which I have used in the "theory of truth values". According to Bolzano, the true definite proposition: "6 is divisible by 3" (and likewise the false proposition: "5 is divisible by 3") must have a degree of validity of ${}^{2}/_{6}$, if the term "6" (or "5") changes and is replaced by integers from 1 to 6 (of these numbers only two, namely 3 and 6, are divisible by 3). But the degree of validity of the

[23]) *Ibid*, Vol. II, p. 81. A misprint seems to have crept into the text, which should read "mehrere" (many) instead of "wahre" (true). Bolzano, like Aristotle, considers truth and falsehood as exclusive properties of sentences, but not of ideas. (Cf. Vol. I, Sec. 55, p. 238). (In translation, Bolzano's text has been corrected as suggested by Łukasiewicz. (Ed.))

same true proposition "6 is divisible by 3" (and also of the false one: "6 is divisible by 5") is ${}^{4}/_{6}$ if the term "3" (or "5") takes on the range of integers from 1 to 6 (since 6 is divisible by 1, 2, 3, 6). In my opinion, in all these cases the concept of truth value is not applicable, since truth values are attributes of indefinite propositions only. Consequently, it can only by asserted that in the former case the truth value of the indefinite proposition "x is divisible by 3" is ${}^{2}/_{6}$, and in the latter case the truth value of the indefinite proposition "6 is divisible by x" is ${}^{4}/_{6}$, under the assumption that in both cases x stands only for integers from 1 to 6. The two propositions are different from one another, and hence it is not surprising that they have different truth values.

For all this, Bolzano's considerations so far are certainly free of error. Should it be possible to formulate and to solve problems occurring in the theory of truth values without the concept of *variable* and of *indefinite proposition*, then such a procedure would even have to be given methodological priority. *Entia non sunt multiplicanda praeter necessitatem.* It does seem to me, however, that in our case the formation of new logical concepts is *advisable* for many reasons.

The concept of indefinite proposition, and with it the concept of logical variable, play an important role not only in probability theory but also in logic in general. All the laws of formal logic are formulated and proved with the help of indefinite propositions. For instance, the law of the conversion of general negative propositions is: from the truth of the proposition "no A is B" follows the truth of the proposition "no B is A", and conversely. In these propositions occur the logical *variables* A and B, which may denote all possible objects; hence the propositions themselves are indefinite and can be neither true nor false, even though they have no truth values, which can be calculated, since the ranges of the values of A and B are not strictly outlined. If the same law of conversion were to be formulated without using variables, we would have to select an example like this: "no man is an angel" and hence "no angel is a man", and to add the following rule: if in these propositions "man" and "angel" are replaced by any other terms, then the propositions obtained in this way are always either both true or both false. It can thus be seen that it would be possible, though in a complicated form, to *formulate* logical laws without using variables, but I cannot conceive any way in which it would be possible to *prove* such laws in a general way without variables. If we refuse to accept variables, then we always have to do with *examples*, and instead of constructing strictly deductive proofs we must be satisfied with uncertain inductive generalizations. Mathematics has developed only when indefinite letters, that is variables, have been introduced instead of definite numbers, and the foundations of algebra have been laid in this way. It need not be mentioned that at present mathematics would not be possible without the concept of variable; but the concept of *mathematical* variable falls under that of logical *variable*. Hence at least *in mathematics* it is impossible to do without the concept of variable and that of indefinite propositions (all mathematical equations, such as "$2x+$ $+1 = x+2$", must be treated as indefinite propositions). Bolzano's procedure can be used as a striking testimony of the fact that the same is also valid in logic: without realizing the lack of consistency in his procedure, which he thus accepts, Bolzano formulates all logical laws with the help of indefinite propositions, which he denotes sometimes by single letters A, B, C, ..., M, and sometimes by the words "A has b", etc.

If the concepts of variable and of indefinite proposition even for these reasons can be eliminated neither from logic nor from science in general, they prove even more necessary when it comes to the explanation of *possibility* and *probability*. In my opinion, the essence of possibility cannot be grasped if it is not reduced to the concept of variable. Bolzano failed to fathom the essence of possibility, although to explain that concept he chose the path which could have led him to his goal. According to him, the existence of an object is called *possible* if it is not impossible. But an object A is *impossible* if the sentence "A does not exist" is a pure conceptual truth. For instance, we say that an almighty creature is impossible, since the sentence that there is no such being is a pure conceptual truth. On the other hand, it is possible that a man errs, since there is no conceptual truth which denies the existence of an erring man.[24] These are the most essential explanations to be found in Bolzano concerning the concept of possibility. Had he advanced his analysis of this concept somewhat further, he would undoubtedly have come across the concepts of variable and of indefinite

24) *Ibid.*, Vol. II, Sec. 182, p. 230.

proposition. That the conceptual truth, e.g., "A does not exist" does not hold, can be asserted *a priori*, i.e., without recourse to facts or examples, only if the proposition "A exists", which contradicts the former, can be proved conceptually. But then the existence of the object A is not only possible, but also necessary. Hence if we want to define pure possibility, which would be confused neither with necessity nor with actuality, we must assume that the second proposition, "A exists", is also not a conceptual truth. But it can never be proved *a priori* that out of the two contradictory propositions: "A exists" and "A does not exist" *neither* the former *nor* the latter is a conceptual truth; *examples* or *cases* must be found which show that among the objects falling under the concept A *some* verify the one proposition and *some* the other. If a person wants to prove, for instance, that neither of the two propositions: "man errs" and "man does not err" is a pure conceptual truth, he must find both human individuals *who err* and *who do not*. But then the existence of erring and non-erring men is not only *possible*, but also *real*. If we want to sort *pure possibility* out of that reality, we must select as the subject of such possibility not an individual erring man, but any man, the man x. It is possible, and only possible, that "the man x errs", but it is neither necessary nor real, since there are erring and non-erring men. It is obvious that Bolzano's definition of the concept of possibility does not lose its validity, since if there are men who err then the sentence "there are *no* erring men" is certainly not conceptual truth. But in addition to Bolzano's *negative* condition other, *positive*, conditions are necessary too in order to explain the concept of *pure* possibility. These positive conditions consist in the acceptance of logical *variables* and the assumption that there are *indefinite* propositions, which need not be either true or false.

What has been said concerning the concept of pure possibility is also valid for the concept of *probability*. Bolzano sensed clearly that there is close relationship between the concept of the validity of a proposition and that of probability. Since, however, the concepts of logical variable and of indefinite proposition were not known to him, he did not find the correct solution. Bolzano defined the concept of probability as follows: "We consider ... in a single proposition A or in several propositions $A, B, C, D, \ldots$ certain ideas $i, j, \ldots$ as changeable, although always agreeing among the propositions $A, B, C, D, \ldots$ then it is ex-

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ceptionally important to establish the ratio of the number of those cases in which the propositions $A, B, C, D, \ldots$ are all true to the number of those cases in which also another proposition M is true... I take the liberty ... to call this ratio between the given numbers the *comparative validity* of the proposition M with respect to the propositions $A, B, C, D, \ldots$, or the *probability* of the proposition M resulting from the assumptions $A, B, C, D, \ldots$"[25]) I shall try to explain this definition of probability by an example selected by myself, since Bolzano's definition is not followed by an example.

Although Bolzano denotes the changeable ideas occurring in the propositions $A, B, C, D, \ldots$ with the letters i, j, yet to be consistent he must mean by them not *variables* but only *definite* ideas, which are replaced by others in accordance with a certain rule. Let A and B stand for the following *definite* propositions: "18 is divisible by 2" and "18 is divisible by 3". The proposition M might be: "18 is divisible by 5". In all these propositions the idea "18" is treated as changeable, and we replace it in turn by two-digit integers. The comparative validity, or probability, of the proposition M with respect to the propositions A and B is $^3/_{15} = {}^1/_5$, since out of 15 two-digit numbers for which the propositions A and B are true (they are the numbers of the arithmetical progression: 12, 18, 24, ..., 96), there are only three numbers, namely 30, 60, 90, for which the proposition M is true too.[26]) In this way we come to the conclusion that the probability of a *false* proposition: "18 is divisible by 5" has the value $^1/_5$. Obviously, under the same assumptions, if the number 18 is replaced by the number 10, the comparative validity, or probability, of the *true* proposition: "10 is divisible by 5", also is $^3/_{15} = {}^1/_5$.

The example I have selected and the problem resulting therefrom have sense when the definite numbers 18 or 10 are replaced by the variable x. Then the problem is: to find the *relative truth value* of the indefinite proposition "x is divisible by 5" with respect to the indefinite propositions: "x is divisible by 2" and "x is divisible by 3", under the assumption that x ranges over all two-digit integers. This problem can

[25]) *Ibid.*, Vol. II, Sec. 161, p. 171 ff.

[26]) As a result of an inexact formulation or just by mistake, in this quotation Bolzano mentions in the *first* place that quantity which occurs in the denominator, and in the *second* that which occurs in the numerator.

be solved very easily on the basis of the explanations given in Section 10. But in Bolzano we obtain something nonsensical, not only because *absolute* probability is confused with *relative* probability, but also because propositions must be treated as probable which can never be probable, since they are either true or false. It might be objected that perhaps the examples I have chosen do not comply with Bolzano's ~~reasoning and intentions. To this~~ I can only answer that ~~these examples~~ have been selected with the strictest consistency with Bolzano's analysis. He asserts that 1) all propositions must be either true or false; 2) the degree of validity of one and the same proposition varies according to which idea occurring in the proposition is treated as changeable. These two assertions are incompatible with the assumption of the existence of indefinite propositions. Consequently, *definite* propositions must be treated as examples of probabilistic propositions. Moreover, I am convinced that Bolzano would have changed his mind in view of this consequence, if he had thought of the examples I have given. In his further considerations he makes use of non self-evident judgements such as "Caius draws a black ball from the urn", as examples of probabilistic propositions, without noticing the error inherent in his reasoning from the very outset.

Bolzano had sensed in the "validity" of a proposition an important concept and was moving toward the formulation of a new and original probability theory. But the force of the Aristotelian prejudice that every proposition must be either true or false nipped his ideas in the bud and resulted in an abortive concept which, not noticed by anyone before, can now claim only a historical importance.

25. Grelling's probability theory

More or less at the same time that I first presented my probability theory at the meetings of the Polish Philosophical Society in Lwów, the very interesting and valuable paper by Kurt Grelling was published under the title *Die philosophischen Grundlagen der Wahrscheinlichkeitsrechnung.*[27]) The author starts from the assumption that the logical

[27]) *Abhandlungen der Fries'schen Schule*, New Series, Vol. III, Göttingen, 1912, pp. 439–478. Grelling's work appeared in the third part of that volume at the end of 1910.

and philosophical foundations of probability theory as seen by philosophers and mathematicians are still controversial and obscure and endeavours to scrutinize the solutions of the problem obtained so far, especially those announced in recent decades in the German literature on the subject. For this purpose he selects three works which seem to him to exceed all others in importance: the works of von Kries and Stumpf, which I have quoted above, and A. Fick's *Philosophischer Versuch über die Wahrscheinlichkeiten.* [28]) We must now focus our attention for a while on the last-named item.

According to Fick, mathematical probability is a property of incompletely formulated hypothetical judgements: for instance, "when a coin falls upon a table, it turns heads up". Fick calls this sentence incomplete, because in a complete and hence universally valid judgement the consequent would have to be: "it turns heads up or tails up", or else the antecendent would have to be: "when a coin falls upon a table and its tail-side forms an angle of less than 90 degrees with the surface of the table".

Grelling's comment on this presentation of Fick's fundamental idea is that Fick in fact by incompletely formulated hypothetical judgements means *indefinite* judgements, but since he did not have at his disposal the concepts of modern mathematical logic, which only later were formulated and examined mainly by Frege and Russell, his analyses are handicapped by a certain obscurity and helplessness. In Grelling's opinion, Aristotelian terminology is very poorly suited to describe the concepts mentioned above.

Without engaging in a discussion of whether Fick in fact did know the concept of indefinite proposition but was unable to formulate it clearly, or whether terminological helplessness was in his case combined with a lack of knowledge of concepts, I shall now outline Grelling's opinions.

He recognizes with great clarity all the difficulties which mark both the objective and the subjective theory of probability. Grelling thinks that we could come to terms with the subjective interpretation if the only point were to justify probabilistic propositions which are concerned with every-day life and in fact are usually nothing but expressions

[28]) Würzburg, 1883.

of our defective knowledge. Yet the probabilistic propositions in science, for instance in statistical mechanics, theory of games of chance, theory of mass phenomena, also require an explanation. They are propositions which have been well confirmed by experience so that *objective validity* must be their property. Yet the judgements of the calculus of probability cannot be referred to objective definite events; hence they must refer to something else.

Grelling notices the resolution of this difficulty in the concept of indefinite judgement. He seems to understand by an *indefinite judgement* a sentence which contains a variable and is neither true nor false, for instance "$x = 8$". Sentences which contain variables but are either true or false, such as "$(x+1)^2 = x + 2x + 1$" are treated by Grelling as *definite judgements*. He presents the relationship which should hold between indefinite judgements and probabilistic propositions in a way which points to a lack of clarity of his basic idea. He states: "We may say that the state of things formulated in an indefinite judgement is certain if it can be confirmed, and is impossible if it can be denied. This is not far from saying that in all other cases it is more or less probable".[29]) Now, I find it difficult to understand how the state of things formulated in an indefinite judgement can be certain or impossible if it is assumed that indefinite judgements can be neither true nor false.

That this obscure point is not due to stylistic clumsiness, but has deeper reasons, can be seen quite clearly from the way in which Grelling tries to define mathematical probability. We find the following formulation: "The question is about the probability of the indefinite judgement: 'If the assumption A is satisfied, then the event B occurs'. Let the following definite judgements be given: 'If A occurs, then there occurs one and only one of N equiprobable cases, among which there are n such that if one of them occurs, then B occurs, too'. Then n/N is the measure of the probability sought." [30])

In this definition it is, first, unclear to me why Grelling calls the former judgement *indefinite* and the latter *definite*. In the former occur two indefinite terms, A and B, but the same terms reappear in the latter

[29]) *Ibid.*, p. 454.
[30]) *Ibid.*, p. 463.

judgement, which is to be treated as definite. Now if A and B are variables in the first judgement, they must also be so in the second. Or is, perhaps, the second judgement to be interpreted like an identity which contains variables but must nevertheless be true?

This obscure point is, however, far less important than the way in which Grelling defines the measure of probability. Although he is correct in noticing that probabilistic judgements must be interpreted as indefinite judgements, and although he does not hesitate to treat as judgements those indefinite sentences which are neither true nor false, he does not bring his idea to a conclusion. *He remains ignorant of the concept of truth value*, a concept which in a more penetrating analysis of indefinite propositions proves inevitable and which forms the natural measure of probability. Since Grelling does not possess that concept, he must revert to the old, obscure concept of "equiprobable cases", a concept that is ridden with difficulties. Finally, there is for him no other way out than to accept Kries's theory.

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The history of the formation of those concepts which have served as the building material of probability theory as presented in this paper is instructive and interesting. *Two* concepts are the corner-stones of that theory: the concept of *indefinite proposition* and that of *truth value*. Both concepts had been known previously, but they have not always been interpreted clearly, and never *associated* with one another. Representatives of modern mathematical logic, such as Frege and Russell, knew the concept of *indefinite proposition*, even if they did not always treat it as a judgement; yet none of them has tried to apply that concept to probabilistic propositions. Bolzano formulated the concept of *validity*, which corresponds to the concept of *truth value*, and used it in his own way, though not quite satisfactorily, to explain the concept of probability, but he did not know the concept of *indefinite proposition*. On the other hand, Grelling knew the last-named concept and applied it to probabilistic propositions, but the concept of *truth value* escaped his notice. It seems as if fate interfered enviously to prevent the lifting of the veil of mystery that surrounded the concept of probability.

Every correct scientific theory has its forerunners. It is only gradually that the human mind forces its way through to a clear comprehension of a difficult problem. The fact that the logical theory of probability also has its own forerunners, and such eminent ones at that, allows us to hope that the path toward the resolution of the problem of probability has ultimately been found.

ON THE CONCEPT OF MAGNITUDE *)

(In connection with Stanisław Zaremba's *Theoretical Arithmetic*)

1. Introduction.—2. Professor Zaremba's definition of magnitude.—3. On the formulation of the principles defining the formal properties of the relations "equal to", "less than" and "greater than".—4. On some logical relationships between those principles.—5. On theorems "devoid of content".—6. On the principle: "the inequalities $A < B$ and $B > A$ must hold simultaneously".—7. Other logical relationships between the properties defining the formal properties of the relations "equal to", "less than" and "greater than".—8. How the formal properties of those relationships are to be defined.—9. A criticism of Professor Zaremba's definition of magnitude.—10. How the concept of magnitude is to be defined.—11. How to present the theory of magnitude to make it easily comprehensible and precise.—12. The importance and the tasks of contemporary formal logic.

Stanisław Zaremba, Professor of the Jagellonian University, has written a comprehensive work of 859 pages, entitled *Arytmetyka teoretyczna* (*Theoretical Arithmetic*). His work, published in Cracow in 1912 by the Academy of Learning, met with the fullest approval from mathematicians.[1]) I have no doubt that approval is justified; as I am not a mathematician, I cannot appraise the book from the standpoint of mathematics and I rely completely on the opinion of the experts. But I can appraise that work from the standpoint of logic, the more so as some of the problems it raises are closely connected with logic.

The author himself, when referring, on page XV, to advances in theoretical arithmetic, says: "These advances consist partly in the introduction of new concepts and new kinds of numbers, but the development of theoretical arithmetic in recent times is primarily marked by the fact that certain quite elementary concepts that have existed in science for centuries have become well founded and interconnected in a way more satisfactory from the viewpoint of logic."

[1]) Cf. the review by A. Hoborski, docent of the Jagellonian University, published in *Wektor* (Vector), Vol. III, No. 9, Warsaw, April 1914, pp. 418–431.

*) First published in *Przegląd Filozoficzny* 19(1916) as *O pojęciu wielkości*. It is included here in a largely abbreviated form, which retains only those parts which are of general theoretical importance. The sections concerned purely with the criticism of Zaremba's book have been left out as being now of merely historical interest for the Polish reader, who can read Zaremba's book in the Polish original. The table of contents that follows the title of the article is quoted in full in order to give an idea about the full text.

I fully agree with the statement. In fact, both theoretical arithmetic and mathematics in general have recently gained much in logical precision. This logical improvement of mathematics is, in my opinion, to be ascribed first of all to the unprecedented development of formal logic. That development has been due to research by logicians who were also mathematicians. In 1854 Boole laid the foundations for the algebra of logic; he was followed by De Morgan, Peirce, and Schröder, who improved that algebra and formulated the theory of relations; finally, in recent years Russell, Frege, and Peano and his school applied the algebra of logic to mathematics and improved it considerably. Owing to the work of these scientists, traditional formal logic, which has had practically no effect on contemporary research, has been replaced by a new logic that will undoubtedly become a powerful but subtle instrument of cognition in all fields of knowledge.

This new logic, which is now flourishing, is as yet very little known. Only some of its concepts, often distorted, are penetrating the circles of those scientists who are not professional logicians. Much time will be needed before these new logical concepts and methods overcome all the obstacles of prejudice and become the property of all scientists. That is why I was not astonished when, in a book written by a learned professor of the Jagellonian University, I found none of the names mentioned above, but did find in many places opinions and methods which, from the standpoint of contemporary logic, are inexact or even erroneous.

To justify my last statement I shall analyse critically in the present paper the definition of the concept of *magnitude* as formulated by Professor Zaremba, discuss some logical and methodological issues connected with that subject, and offer my own tentative definition of that concept. [...]

Logic tells us that every deductive theory, hence also theoretical arithmetic, includes two kinds of propositions: some, called *principles*, are assumed in a given theory without proof, while the others, called *theorems*, are proved on the strength of the principles. All the principles must be *in agreement* with one another, and none may be a *consequence* of others; in other words, it may not *be derivable* from others. Should a principle be derivable from others, it could be proved on their basis, and as such would be not a principle, but a theorem. Every theorem is derivable either from *all* the principles taken together or from only *some* of them. It is always necessary to investigate most carefully which principles are necessary to the proof of a given theorem and which are not. It is also necessary to investigate which logical relations hold between the various theorems. For a loose set of propositions becomes a systematic scientific theory only when the logical relations between principles and theorems and between one theorem and another are established.

This interpretation of the essence of deductive theory, as outlined above, results in a number of methodological rules, which are not explicitly formulated in logic, but which are observed in the construction of logical theories. For instance, a combination of *several* propositions, which do not result from one another, into a *single* principle is avoided, because the formulation of such "compound" principles prevents us from establishing exactly the logical relations holding between principles and theorems, since it may happen that a given theorem is derivable from *only one* part of a principle and is not derivable from the *other* parts of that principle. That is why each logical proposition should be formulated as *one* principle in *one* grammatical sentence. [...]

When we speak about consequence in logic, we usually mean *formal* consequence. The expression: a proposition is "a formal consequence" of another proposition, is used to denote a certain relation which holds between indefinite propositions or propositional functions. These two terms are used with reference to propositions which contain some *variables*; for instance, "x is divisible by y" or "A is equal to B". Hence, if we want to examine whether certain *definite* propositions are, or are not, formal consequences of other propositions, we must first transform all these propositions into indefinite propositions, that is, we must replace definite terms by variables represented by certain letters.

The indefinite proposition Z is a consequence of the indefinite propositions $P, R, S, \ldots$, if there exist *no* values of the variables contained in the propositions $P, R, S, \ldots, Z$ which verify the propositions P, R $S, \ldots$, but do *not* verify the proposition Z. In other words, all the values of the variables which verify the propositions $P, R, S, \ldots$, must also verify the proposition Z. For instance, the proposition "A is equal to C" is a consequence of the propositions "A is equal to B" and "B is equal to C", for if we assume that the variables A, B, C range over the class of natural numbers, there are no values of these variables which verify the last two propositions without also verifying the first proposition, so that all the values of the variables A, B, C which verify the last two propositions also verify the first.

The indefinite proposition Z is *not* a consequence of the propositions $P, R, S, \ldots$, if there *do exist* values of the variables contained in the propositions $P, R, S, \ldots, Z$ which verify the propositions $P, R, S, \ldots$, but do not verify the proposition Z. For instance, the proposition "A is not equal to C" is not a consequence of the propositions "A is not equal to B" and "B is not equal to C", for if we assume that the range of the variables A, B, C is the class of natural numbers, there exist values of these variables which verify the last two propositions without verifying the first, for instance $A = 2$, $B = 3$, $C = 2$. [...]

As we know, every deductive theory includes certain propositions which are accepted without proof; they are *principles*. Such propositions must exist, because we cannot prove everything. With concepts it is the same as with propositions. Every deductive theory includes certain concepts which are adopted without definition; they are *primitive* concepts. Such concepts must exist, because we cannot define everything. We may formulate the following methodological rule, which pertains to both principles and primitive concepts: the principles and the primitive concepts of a given deductive theory should be selected so as to reduce their number as far as possible. In adopting this rule we are guided by two considerations: first, we want to have as few unproved propositions and undefined concepts as possible, because we treat both as a *malum necessarium*. Secondly, the fewer primitive concepts and principles we need to present a deductive theory, the more fundamental are the concepts and principles we have chosen, and the simpler is the theory. [...]

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Both straight line segments and real numbers are called magnitudes. But there is one essential difference between these two kinds of magnitudes, which is revealed in the fact that two different straight line segments *can* be equal to one another, whereas two different real numbers *cannot* be equal to one another.

On the basis of this fact I make a distinction between those objects which only *have* magnitude, and those which *are* magnitudes. I say that objects belonging to a set *are* magnitudes if any two different, that is, not identical, ones among them *cannot* be equal to one another, but are not magnitudes and only *have* magnitude if two different ones among them *can* be equal to one another.

Thus straight line segments are not magnitudes, but only *have* magnitude, since two different straight line segments, e.g., two sides of a triangle, can be equal to one another. The magnitude which straight line segments have is called their "length". The lengths of straight line segments *are* magnitudes, because two different, that is, not identical, lengths cannot be equal to one another. Hence straight line segments which have *different* lengths cannot be equal to one another. Likewise, real numbers *are* magnitudes, for two different real numbers cannot be equal to one another. That is why there is a perfect correspondence between lengths of straight line segments and real numbers which does not exist between real numbers and straight line segments.

The difference between objects which only *have* magnitude and those which *are* magnitudes is very common. A man is not a magnitude; but he *has* magnitude, for he has height, age, and body weight. The height of a man, his age, the weight of his body *are* magnitudes; this is why those properties can be expressed, whereas it is impossible to express *man* in terms of numbers.[2])

[2]) The distinction between those objects which only *have* magnitude and those which *are* magnitudes occurred to me under the influence of a paper by B. Russell, *L'idée d'ordre et la position absolue dans l'espace et le temps* (Bibliothèque du Congrès international de Philosophie. III. Logique et Histoire des Sciences. Paris, 1901, pp. 241–277). In that paper Russell (on p. 242) makes a distinction between those series which *are* positions and those which *have* positions. It will soon be seen that the concept of magnitude is closely associated with those of series and order. I regret that at the time of writing the present paper I did not have at my disposal Russell's principal work, *The Principles of Mathematics*, published in Cambridge in 1903.

When we want to define the concept of magnitude, we have to take into account not objects which merely *have* magnitude, but objects which *are* magnitudes. [...] In this Section I intend to present the results of my analysis in the fullest detail, by following the path of deduction from the final definition to its ultimate consequences, and I intend to present them in a way that is comprehensible even to those readers who do not grasp the preceding parts of my paper.

1. Real numbers are typical magnitudes. These numbers form an *ordered* set, that is, form a series, in which each number has its own *unique* and determined place. This is possible because of the three properties characteristic of the relation "less than" in the domain of real numbers:

First, if of any two real numbers, A and B, A is less than B, then B is not less than A; second, if of any three real numbers, A, B, and C, A is less than B and B is less than C, then A is less than C; third, if any two real numbers, A and B, are different from one another, then either A is less than B, or B less than A.

These three properties are necessary, and also sufficient, for real numbers to form an ordered set. For let us assume that A, B, and C are any real numbers different from one another. By the *third* property, either A is less than B, or B is less than A, and likewise either B is less than C, or C is less than B. Assume that A is less than B and B is less than C. By the *first* property, B is not less than A, nor is C less than B. Further, by the *second* property and the assumption that A is less than B and B is less than C, A is less than C and thus C is not less than A. Thus the relation "less than" connects the numbers A, B, C only in the direction from A to B, from B to C, and from A to C. Thus we can arrange these numbers in a series so that each of them bears the relation "less than" to every following one, but does not bear that relation to any preceding one. These conditions are satisfied only by the series: A-B-C, but they are not satisfied, for instance, by the series: B-C-A. Hence, with respect to the relation "less than", each number has a uniquely determined place in the series: A comes first, C comes last, and B comes between A and C. Likewise, any *other* real number has some uniquely determined place in a series arranged on the basis of the relation "less than", and all of them taken together form an *ordered* set.

Assume that A, B, C are any elements of a certain set. Every relation which has the property that if it holds between A and B then it does not hold in the opposite direction, between B and A, is called *asymmetric*. Every relation which has the property that if it holds between A and B and B and C, then it holds between A and C, is called *transitive*. The relation "less than" is asymmetric and transitive. The third property has no special name of its own. *) It can only be said that any two different real numbers, A and B, must be connected by the relation "less than" in a definite direction, hence either in the direction from A to B, or in the opposite direction from B to A. The same three properties are characteristic of the relation "greater than", which, being converse to the relation "less than", co-exists with it in the opposite direction, so that real numbers, which are ordered by the relation "less than", are ordered in the opposite direction by the relation "greater than".

2. By generalizing the foregoing considerations we may formulate the following definitions:

If every two different elements of a set are connected in a definite direction by an asymmetric and transitive relation r, then the set is termed *ordered*. The elements of an ordered set are termed *magnitudes*.

Thus the following principles are true for any magnitudes A, B, C that are members of a set:

I. If A bears the relation r to B, then B does not bear the relation r to A.

II. If A bears the relation r to B and B bears the relation r to C, then A bears the relation r to C.

III. If A is different from B, then either A bears the relation r to B, or B bears the relation r to A.

3. All properties of the relation r expressed by these principles are necessary for a set to be ordered and for its elements to be magnitudes. This will be seen from the following considerations:

Assume that the set of straight-line segments is given, and let the relation "less than" be represented by r. In the domain of straight-line segments, the relation "less than" satisfies the first two principles,

*) This property of relations is denoted by the terms "connexive" or "connected". The Polish term ("spójny") for this property was coined later.

but does not satisfy the third, for of two different straight-line segments one need not be less than the other, since they can be equal, as in the case of two sides of an equilateral triangle. Thus the set of straight-line segments is not ordered. In fact, the set of such segments also includes segments equal to one another, and equal segments cannot be arranged in a series so that each one has its uniquely determined place in the series. If the elements A, B, C are equal to one another, then there is no reason that they should be arranged in the order "A-B-C" rather than "B-C-A". Likewise, brothers cannot be uniquely placed in series according to the relation of brotherhood, while they can be so arranged on the basis of their age or height, if they differ in age and height.

Since straight-line segments do not form an ordered set, they are not magnitudes. This may seem strange, and yet it is true, because straight line segments *are not* magnitudes, but only *have* magnitudes, in the same way as a man *is not* a magnitude, but only *has* a magnitude, be it age or height. The length of a straight-line segment is its magnitude, and lengths *are* magnitudes, because of two different lengths one must be less than the other. Hence we must make a distinction between those objects which *are* magnitudes, and those objects which are not magnitudes but *have* magnitudes.

This shows that the first two properties of the relation r do not suffice for the elements of a set between which the relation r holds, to be magnitudes. Likewise, the last two properties do not suffice. This is proved by the following example drawn from the field of logic.

Those propositions which contain only definite terms are called *definite*, as opposed to indefinite propositions, which contain variables. For instance, the proposition "10 is divisible by 5" is definite, while the proposition "10 is divisible by x" is indefinite. Every definite proposition is either true or false. Assume that the set of definite propositions is given, and let r stand for the relation of *material consequence*, defined as follows: the relation of material consequence holds between two definite propositions, A and B, when either A is false or B is true. Since every definite proposition is either true or false, the relation of material consequence holds between every two such propositions in some direction. For if A is false, then every definite proposition B follows from it; if A is true, then, too, from A follows B, if B is true; and if B is false, then again from B follows A. Thus the third principle is satisfied. The second principle is satisfied as well, for if from A follows B and from B follows C, then from A follows C. On the other hand, the first principle is not satisfied, for if A and B are both true or both false, then both from A follows B and from B follows A. This is why the set of definite propositions is not ordered with respect to the relation of material consequence, and its elements are not magnitudes.

It can also be demonstrated that the first and third properties of the relation r do not suffice for a set to be ordered and for its elements to be magnitudes. Assume that the set of the following three months: "January, May, September" is given, and let r stand for the relation: *the fourth month following* a given month. Since May is the fourth month following January, and September is the fourth month following May, and January is the fourth month following September, then the third principle is satisfied under these assumptions. For let any of these months be symbolized by A or B: if A is other than B, then either the fourth month following A is B or the fourth month following B is A. The first principle is satisfied too, for if B is the fourth month following A, then A cannot be the fourth month following B. On the other hand, the second principle is not satisfied, for if out of these three months B is the fourth month following A, and C is the fourth month following B, then C is not the fourth month following A. For instance, the relation r holds between January and May, and between May and September, but is does not hold between January and September, but conversely, it holds between September and January. That is why these three elements cannot be ordered in a unique manner, because with respect to the relation r the series "January-May-September" and "May-September-January" and "September-January-May" are equally possible. These elements form not an ordered series, but a *cycle*. Accordingly they may not be treated as magnitudes.

This proves that all three properties of the relation r, formulated in principles I, II and III, are necessary for a set of elements to be ordered and for its elements to be magnitudes. It can also be seen that none of these principles follows from the remaining ones.

4. If a set of elements satisfies all three principles defining the formal properties of the relation r, then that relation, depending on certain other factual factors which are of no logical importance, may be called

either the relation "less than" or the relation "greater than". By using the former of these terms we may present the general principles I, II, and III as principles pertaining to the relation "less than" and give them first place in the *theory of magnitude*.

For any magnitudes A, B, C, belonging to a set of magnitudes, the following principles are true:

P1. *If A is less than B, then B is not less than A.*

P2. *If A is less than B and B is less than C, then A is less than C.*

P3. *If A is different from B, then either A is less than B or B is less than A.*

These principles make it possible to formulate the following definitions:

Df. 1. *"A is greater than B" means "B is less than A".*

Df. 2. *"A is equal to B" means "A is not less than B and B is not less than A".*

The relation "greater than" is *converse* to the relation "less than". Thus these relations are two aspects of one and the same connection. This is revealed even in the symbolism used in mathematics. The expression "$A < B$", when read from left to right means "A is less than B", and when read from right to left means "B is greater than A".

The relation of equality holds only between *identical* magnitudes. This follows from P1 and P3. If A is less than B or B is less than A, then A is different from B, for should A and B be identical, then, by P1 A would be both less and not less than itself. And if A is different from B, then, by P3, either A is less than B or B is less than A. Hence there is equivalence between the propositions "A is different from B" and "A is less than B or B is less than A". Likewise, there is equivalence between the negations of these propositions: "A is not different from B" or "A is identical with B", and "A is not less than B and B is not less than A" or "A is equal to B". Thus the relation of equality may be interpreted as the relation of identity. This is why we say that equal straight-line segments have *the same* length and that people who are equal to one another in height or age are of *the same* height or age.

5. The above principles and definitions yield a number of theorems, the most important of which are listed below together with their proofs

and with explicit references to the principles and definitions on which they are based.

The following theorems are true for any magnitudes A, B, and C, belonging to a certain set of magnitudes:

Th. 1. *If A is less than B, then B is greater than A.*

Th. 2. *If A is greater than B, then B is less than A.*

These theorems follow immediately from Df. 1.

Th. 3. *If A is equal to B, then A is not less than B.*

This theorem follows immediately from Df. 2.

Th. 4. *If A is equal to B, then A is not greater than B.*

Proof: It follows from Df. 2 that if A is equal to B then B is not less than A. And from the proposition: B is not less than A, we obtain, by contraposition of Th. 2, the proposition: A is not greater than B.

Th. 5. *If A is less than B, then A is not greater than B.*

Proof: It follows from P1 that when A is less than B, then B is not less than A. And from the proposition: B is not less than A, we obtain, by contraposition of Th. 2, the proposition: A is not greater than B.

Theorems 3, 4, and 5, taken together, state that only *one* of the relationships: A is equal to B, A is less than B, A is greater than B, holds between any two magnitudes A and B.

Th. 6. *If A is not less than B and A is not greater than B, then A is equal to B.*

Proof: From the proposition: A is not greater than B, we obtain, by the contraposition of Th. 1 and with a simultaneous exchange of the letters A and B, the proposition: B is not less than A. And from the propositions: A is not less than B, and B is not less than A, by Df. 2 follows the proposition: A is equal to B. It must be borne in mind that all the propositions of the present theory, and hence all the principles, definitions, and theorems, are valid for *any* magnitudes A, B, C, and hence the letters contained in those propositions may be replaced by other letters, provided only that the places occupied by like letters must, after the replacement, also be occupied by like letters.

Theorem 6 states that one of the relationships: A is equal to B, A is less than B, A is greater than B, *always* holds between any two magnitudes A and B.

Th. 7. *If A is not less than B and B is not less than C, then A is not less than C.*

Proof: Suppose that with the assumptions: A is not less than B, and B is not less than C, the proposition: A is less than C, is true. Then one of the following propositions must hold: either C is less than B, or C is not less than B. If C is less than B, then that proposition, together with the supposition that A is less than C, yields, by P2, the proposition: A is less than B. But this proposition is in contradiction to the assumption that A is not less than B. Hence, if we suppose that A is less than C, we must assume that C is not less than B. But from the proposition: C is not less than B, and the assumption: B is not less than C, taken together, it follows that B and C must be identical. For should they not be identical, but different from one another, then by P3 it would follow that either B is less than C or C is less than B. Hence, when assuming that A is less than C, we must assume that B is identical with C. But then the assumption: A is not less than B, is equivalent to the proposition: A is not less than C, and thus is in contradiction to the supposition that A is less than C. Consequently, we must assume that if A is not less than B and B is not less than C, then A is not less than C.

Theorem 7 shows that not only is the relation "less than" transitive, but that its negation, that is the relation "not less than", is transitive, too. It is worth while noticing that this theorem is based on P2 and P3.

Th. 8. *A is equal to A.*

Proof: By substituting A for B in P1 we obtain: If A is less than A, then A is not less than A. This is an example of a logical relationship in which a proposition implies its own negation. Hence the antecedent: A is less than A, must be false, for should it be true, then a contradiction would result, and the consequent: A is not less than A, must be true. Hence it follows, by the contraposition of Th. 2, that the proposition: A is not greater than A, must be true, too. By substituting in Th. 6 A for B we obtain: If A is not less than A and A is not greater than A, then A is equal to A. Since the propositions which together form the antecedent in this conditional proposition are true, the consequent must be true too. Hence: A is equal to A.

Every relation which has the property that it holds between any element of a set and that element itself is called *reflexive*. Theorem 8 states that the relation of equality is reflexive.

Th. 9. *If A is equal to B, then B is equal to A.*

Proof: By Df. 2 the proposition: A is equal to B, means: A is not less than B and B is not less than A; and the proposition: B is equal to A, means: B is not less than A and A is not less than B. Both propositions mean the same and one follows from the other.

Every relation which has the property that when it holds between any two elements of a set, A and B, then it also holds in the opposite direction between B and A, is called *symmetric*. Theorem 9 states that the relation of equality is symmetric.

Th. 10. *If A is equal to B and B is equal to C, then A is equal to C.*

Proof: By Df. 2 this theorem can be expressed in the following words: If A is not less than B and B is not less than A, and B is not less than C and C is not less than B, then A is not less than C and C is not less than A. The consequent in this theorem consists of two propositions: The first: A is not less than C, follows by Th. 7 from the propositions: A is not less than B and B is not less than C, contained in the antecedent. The second: C is not less than A, follows in the same way by Th. 7 from the propositions: C is not less than B and B is not less than A, contained in the antecedent. In this way Th. 10 is based on Th. 7, and hence also on P2 and P3.

Theorem 10 states that the relation of equality is transitive. Theorems 8, 9, and 10 can also be proved in a shorter way, by making use of the proof included in Section 4, based on P1 and P3 and demonstrating that elements equal to one another are identical.

Th. 11. *If A is equal to B, and B is less than C, then A is less than C.*

Proof: If A is equal to B, then A is identical with B by P1 and P3. Hence the proposition: B is less than C, implies the proposition: A is less than C.

Th. 12. *If A is less than B and B is equal to C, then A is less than C.*

Proof: If B is equal to C, then B is identical with C by P1 and P3. Hence the proposition: A is less than B, implies the proposition: A is less than C.

Many other theorems could be proved in a similar way. It could be demonstrated, for instance, that both the relation "greater than" and its negation, that is, the relation "not greater than", are transitive. Likewise, we could prove theorems analogous to Th. 11 and Th. 12 and differing from them only in that the word "greater" replaces the word "less". Those proofs, like all the preceding ones, except perhaps for the proof of Th. 7, are quite easy.

6. In this way the foregoing principles, definitions and theorems give rise to a deductive theory of magnitude. This theory combines in a logical whole all the laws defining the formal properties of the relations "equal to", "less than", and "greater than", by deducing these laws from three simple principles and two definitions. Its logical significance is, accordingly, quite considerable. It turns out, for instance, that the proposition "two magnitudes, equal to a third one, are equal to one another", which so far has been considered as axiom, is a logical consequence of more general principles. This is so because that proposition is equivalent to Th. 10 or to the theorem: if A is equal to B and C is equal to B, then A is equal to C, which can easily be proved on the basis of Th. 9 and Th. 10.

But the practical significance of this theory is quite considerable, too. If we want to make sure whether a set of elements is a set of magnitudes, it suffices to examine whether there is a relation r which holds between those elements and which is asymmetrical and transitive and such that it holds in a definite direction between any two elements of that set. Of course, in each particular case the relation r must be defined precisely so that we can see whether it in fact has the three properties specified above. We shall consider a number of examples for that purpose.

In the domain of natural numbers, the relation r is the relation "less than" interpreted in the ordinary way. It is obviously asymmetrical and transitive, and it also has the third property, since of any two different natural numbers one is always less than the other. This is why natural numbers form an ordered set, which we can call the *series* of natural numbers, and are magnitudes.

In the domain of the lengths of straight-line segments, the relation r can be defined as that holding between *a part and the whole*. Assume that a straight line is given, and let a point on that line be marked O. On that line we measure the lengths of straight-line segments by making a given segment coincide with some part of the straight line to the right of the fixed point O. It is obvious that of two such different lengths one is always a part of the other, and that the relation holding between a part and the whole is asymmetric and transitive. Hence, in view of the relation holding between a part and the whole, which may be called the relation "less than", the lengths of straight-line segments form an ordered set and are magnitudes. Since we assume that there is a perfect correspondence between the lengths of straight-line segments and real numbers, real numbers also form an ordered set and are magnitudes.

There are also sets of elements for which we can define several relations r satisfying the three principles of the theory of magnitude. For instance, the set of proper fractions is an ordered set with respect to the relation "less than" defined in the ordinary way. In the domain of proper fractions that relation satisfies principles P1, P2, and P3 and accounts for the fact that proper fractions are magnitudes. But set theory tells us that there is a perfect correspondence between the set of all proper fractions and the set of all natural numbers, so that all proper fractions can be numbered. When that is done, they are ordered on the basis of another relation, but one which satisfies principles P1, P2, and P3 in the same way as does the relation "less than". Thus, proper fractions can also be magnitudes in a sense different from the usual.

Finally, there are sets of elements for which we have not thus far succeeded in finding or defining any relation r that satisfies principles P1, P2, and P3, although we suspect that such a relation exists. We suppose, namely, that the infinite sets have magnitude as have the finite sets and that the "powers" of the infinite sets *are* magnitudes like finite numbers. But we cannot prove this, because so far we have failed to demonstrate the existence of an appropriate relation r. By adopting the non-self-evident "postulate of choice" formulated by Zermelo, we can prove that all infinite sets can be "well ordered", and from that we could deduce the statement that their powers form an ordered set. Hence, if Zermelo's postulate is correct, then infinite sets have the nature of magnitudes, and their powers are magnitudes.[3]) Nevertheless I

[3]) Cf. Artur Schoenflies, *Entwickelung der Mengenlehre und ihrer Anwendungen*, Leipzig and Berlin, 1913, Chap. III and Chap. X.

think that the proofs of those theorems will be truly precise only when we can demonstrate that any two different powers of infinite sets are connected in a definite direction by an asymmetric and transitive relation r, that is, that the set of those powers satisfies all the principles of the theory of magnitude, as formulated in this paper. [...]

The way in which Professor Zaremba uses the term "convention" I consider to be not a linguistic error, but a lack of verbal precision, which may result in grave logical errors. That term is very often encountered on the pages of Professor Zaremba's book. For instance, when he adopts the phrase "quantitative comparability" in order to simplify formulations, he says that he "adopts a convention". When referring to the principles defining the formal properties of the relations "equal to", "less than", and "greater than", he says that "no logical necessity forces us to accept these principles unconditionally, but these principles have the nature of conventions adopted of free will but certainly not incidentally".

In view of the above we have to say that the term "convention" always means the mutual agreement of at least *two* persons. The author who writes a book does not make an agreement with the reader, whom he does not meet and whom he does not know, but only expresses his ideas, of which the reader later takes cognizance and either approves or not. With whom then does the author make an agreement?

Further, it seems to me that in science only *terminological* issues may be the subject of a convention. Scientists representing a given discipline, having met at a scientific conference, may agree that they will use a certain term or sign always with a given definite meaning. But no conference may adopt conventions as to which *concepts* or *propositions* are to be accepted and which are to be rejected. For in science we accept noncontradictory concepts and true propositions, and we reject contradictory concepts and false propositions. We also accept propositions about which we do not know whether they are true or false, but only if such propositions imply other propositions that help us to explain, foresee, or order certain facts. Now the contradictoriness and noncontradictoriness of concepts, the truth and falsehood of propositions, and the following or non-following of some propositions from others do not depend on any conventions. That is why Professor Zaremba is inexact in his formulation when he says that the acceptance of the principles that define the properties of the relations "equal to", "less than", and "greater than" is not forced upon us by any logical necessity, and that those principles have the nature of conventions. No principle can have the nature of a convention, since none is accepted on the strength of a convention. Two or more scientists can only agree that they will use a *term* to denote objects that have properties formulated as certain principles. For instance, they may agree to use the term "equality" to denote a relation which satisfies the principles that express the properties of reflexivity, symmetry, and transitivity.

The fashion of using the term "convention", which has recently become widespread among mathematicians — perhaps under the influence of Poincaré, a great mathematician, but not a logician — is not satisfactory for another reason as well. It leads to the view that mathematics is largely based on "conventions" and hence is *conventional* in nature. Such an opinion is erroneous. This is why I think that it is better to avoid the term "convention" in mathematical works, or to use it only in such phrases as: "in accordance with the convention of such-and-such a scientific conference I use such-and-such a term in such-and-such a meaning." [...]

Contemporary formal logic, which is based on the algebra of propositions and relations, is as much superior to Aristotle's traditional logic as contemporary geometry is to Euclid's *Elements*. We may say without hesitation that today there is no form of reasoning in science that has not been analysed by contemporary logic, whereas traditional logic knew practically no forms of reasoning other than direct reasoning based on the square of contradiction and syllogistic. For instance, the form of reasoning stating that if from a proposition follows its own contradition, then that proposition is false — which reasoning has been used above in the demonstration that every magnitude is equal to itself — was not known in traditional logic. That logic also did not know the concepts of indefinite proposition or propositional function, the concept of logical variable, and formal and material consequence. Today, owing to these new concepts and forms of reasoning, we can check and appraise all methods of reasoning and find errors where they could not be found by traditional logic.

Moreover, contemporary logic formulates a number of *methodological* rules about which former logic knew nothing. We have become

acquainted with some of those rules, as far as they are concerned with deductive theories. We know that every such theory includes some *principles* and *theorems*, that all the principles ought to be in agreement with one another and ought not to follow from one another, that their number ought to be the least possible, and that it is to be investigated which principles yield given theorems and which do not yield them. Contemporary logic says that it does not suffice to be satisfied with a *proof* of a theorem; it is also necessary to find out all the logical relationships which hold between that theorem and the principles and even between various theorems. Former logic did not formulate such rules because, not knowing the concepts of indefinite proposition and logical variable, it could not even state what it means for a proposition not to be a formal consequence of another proposition.

If we realize all this, we can easily understand why present-day logic exceeds former logic in its requirements of scientific precision. Mathematics, which so far has been considered the most precise discipline, turns out to be full of defects and errors when gauged by this new standard of precision. And if *mathematics* cannot pass the test, what shall we say about other disciplines, which always have been less precise and less perfect than mathematics? What a fine target for logical criticism are such natural sciences as physics and chemistry, astronomy and crystallography. How much more imprecision must be inherent in those natural sciences which do not make use of mathematics, such as biology or geology. And how can contemporary philology, psychology, sociology and philosophy defend themselves against precise logical criticism?

Contemporary logic is faced with great and important tasks: to subject all scientific theories to criticism from the point of view of the new standard of logical *precision*, and to *systematize* those theories in accordance with the new methodological rules. The implementation of these tasks seems to me to be exceptionally important mainly for three reasons.

Firstly, *it will facilitate the understanding of scientific theories*. Because of the lack of logical precision, many scientific theorems have been obscure and not properly understood even by experts in a given discipline. We easily grasp any idea which is formulated precisely, even if it is false. It is equally clear to us that the proposition "two times two makes four" is true and that the proposition "two times two makes

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five" is false. But if a scientific theorem is formulated without adequate precision, then along with truth it always includes an admixture of falsehood. We do not realize the existence of that falsehood if a given theorem is accepted in that discipline, for we think that it is true *in toto*. And nothing is so obscure and difficult to understand as a proposition which includes falsehood and in which we want by all means to see only truth. For centuries differential calculus was inaccessible even to very gifted minds. When the student complained about the difficulties he encountered in the study of that calculus, he was answered in the words of d'Alembert: "Allez, Monsieur, allez, et la foi vous viendra." A person can become *accustomed* to inexact theorems and become used to them as something known, but he cannot *understand* them. Today we know what errors and inexactitudes underlay the foundations of differential calculus, and no one now believes in the existence of differentials as infinitesimally small quantities.

Secondly, compliance with these logical requirements *will facilitate the remembering of scientific theories.* The results of contemporary science are extremely rich. Although we accumulate them and list them in textbooks and encyclopaedias, scientific theories seem to be loose collections of propositions rather than systematized wholes. They are only *scientific data* which increase every day with terrifying speed, and which even today, even within the scope of a *single* discipline, cannot be grasped by the human mind and memory. If all the propositions combining to form a scientific theory are arranged according to the logical relationships between them, and if at the head of those propositions we place a few simple principles, of which the other propositions are consequences, a clear and coherent whole will be formed, which can more easily be grasped by memory than the loose collections of propositions in today's textbooks. This is so because we remember better a well constructed *poem* than a loose series of words, and we remember better a *tune* than the chaos of loose sounds. It is easier, I think, to remember that magnitudes are elements of an ordered set than to learn Professor Zaremba's definition of magnitude and to assimilate eight logically unanalysed principles joined to that definition.

Thirdly, the carrying out of these logical tasks *will help to distinguish, in scientific theories, between those things which are really important and subordinate details.* If a proposition contains other propositions as its consequences, then it is *more important* than those consequences. In science, a law of nature is more important than singular propositions about facts which are subsumed under that law. It is more important because whoever knows the law can deduce the consequences and knows more than he who knows only detailed facts. Hence, whoever has assimilated the *principles* of a scientific theory knows potentially the whole theory. The arrangement of the propositions belonging to a theory according to logical relationships between them and a study of such relationships will show which propositions ought to be adopted as principles, and which as theorems, and will also permit one to judge which theorems have more consequences than the others. In this way, the role of each proposition in a given scientific theory is strictly defined. It turns out then that the number of scientific laws which really deserve to be known is small in every discipline, and the rest are subordinate details which proliferate in science, merely covering it with weeds. Science must be cleared of such weeds, so that they should not stifle *great* scientific truths and great creative ideas developed by brilliant minds.

My desire is that in every discipline there should be scientists versed in contemporary formal logic, for they can most effectively improve the discipline in which they work in accordance with the requirements of logic. And above all, my desire is that such scientists be found in *our* nation. Science in the hands of man is a weapon not only against elemental forces, but also against man. Nations struggle with other nations for existence. The nation which is better equipped with the power of science has a greater chance of victory. To strive to improve and systematize science and thus to facilitate its assimilation means to work not only for the progress of human knowledge, but also for the good of one's own nation.

FAREWELL LECTURE BY PROFESSOR JAN ŁUKASIEWICZ, DELIVERED IN THE WARSAW UNIVERSITY LECTURE HALL ON MARCH 7, 1918

In this farewell lecture I wish to offer a synthesis of my research, based on autobiographical confessions. I wish to describe the emotional background against which my views have developed.

I have declared a spiritual war upon all coercion that restricts man's free creative activity.

There are two kinds of coercion. One of them is *physical*, which occurs either as an external force that fetters the freedom of movement, or as inner impotence that incapacitates all action.

We can free ourselves from that coercion. By straining our muscles we can break the fetters, and by exerting our will we can overcome the inertia of the body. And when all measures fail, there is still death as the great liberator.

The other kind of coercion is *logical*. We must accept self-evident principles and the theorems resulting therefrom. That coercion is much stronger than the physical; there is no hope for liberation. No physical or intellectual force can overcome the principles of logic and mathematics.

That coercion originated with the rise of Aristotelian logic and Euclidean geometry. The concept was born of *science* as a system of principles and theorems connected by logical relationships. The concept came from Greece and has reigned supreme. The universe was conceived after the pattern of a scientific system: all events and phenomena are interconnected by causal links and follow from one another as theorems in a scientific theory. All that exists is subject to necessary *laws*.

In the universe conceived in this way there is no place for a creative act resulting not from a law but from a spontaneous impulse. Impulses, too, are subject to laws, originate from necessity, and could be foreseen by an omniscient being. Before I came into this world, my actions had been predetermined in the minutest details.

This idea pervaded even practical life. It turned out that action subject to laws, both natural and social, and hence orderly and purposive, is always *effective*. If the whole nation can become a mechanism whose structure reproduces the scientific system, it gains such enormous strength that it can strive to become the master of the world.

The creative mind revolts against this concept of science, the universe, and life. A brave individual, conscious of his value, does not want to be just a link in the chain of cause and effect, but wants himself to affect the course of events.

This has always been the background of the opposition between science and art. But artists are remote from scientific issues and do not feel logical coercion. And what does a scientist to do?

He has two paths to choose from: either to submerge himself in scepticism and abandon research, or *to come to grips with the concept of science based on Aristotelian logic*.

I have chosen that second path. Slowly and gradually I have come to realize the final objective of the campaign I am conducting now. Yet even all my previous works also unconsciously served the same purpose.

In my striving to transform the concept of science based on Aristotelian logic I had to forge weapons stronger than that logic. It was symbolic logic that became such a weapon for me.

I examined the great *philosophical systems*, proclaiming the universal causality of phenomena, in the light of that logic. I made sure that all of them, Kant's criticism not excluded, fall into nothingness when subjected to logical criticism. They become a collection of loose ideas, sometimes brilliant, but devoid of scientific value. They are no threat to freedom at all.

The *empirical sciences* arrive at general laws by inductive reasoning. I examined the logical structure of inductive conclusions. I started from the research done by Jevons and Sigwart and strove to demonstrate that induction is a reductive reasoning that seeks reasons for given consequences. Such a reasoning never yields reliable results, but only yields hypotheses. Thus here, too, logical coercion ceases to work.

The laws and theories of natural science, by being hypotheses, are not reproductions of facts, but creative products of human thought.

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They should be compared not to a photograph, but to a picture painted by an artist. The same landscape can be interpreted in different ways in works by different artists; by analogy, different theories may serve to explain the same phenomena. In this I saw for the first time a proximity between scientific and artistic work.

Logical coercion is most strongly manifested *in a priori sciences*. Here the contest was to the strongest. In 1910 I published a book on the principle of contradiction in Aristotle's work, in which I strove to demonstrate that that principle is not so self-evident as it is believed to be. Even then I strove to construct non-Aristotelian logic, but in vain.

Now I believe to have succeeded in this. My path was indicated to me by *antinomies*, which prove that there is a gap in Aristotle's logic. Filling that gap led me to a transformation of the traditional principles of logic.

Examination of that issue was the subject-matter of my last lectures. I have proved that in addition to true and false propositions there are *possible* propositions, to which objective possibility corresponds as a third in addition to being and non-being.

This gave rise to a system of *three-valued logic*, which I worked out in detail last summer. That system is as coherent and self-consistent as Aristotle's logic, and is much richer in laws and formulae.

That new logic, by introducing the concept of objective possibility, destroys the former concept of science, based on necessity. Possible phenomena have no causes, although they themselves can be the beginning of a causal sequence. An act of a creative individual can be free and at the same fact affect the course of the world.

The possibility of constructing different logical systems shows that logic is not restricted to reproduction of facts but is a free product of man, like a work of art. Logical coercion vanishes at its very source.

Such was my research, its emotional background, and the objective by which it was guided.

And now I have to lay my work aside for some time and to subject myself to coercion and to observe laws and regulations and even become their guardian. I shall not be free, although I decided that of my own will. But when I feel free again, I shall revert to science. I shall revert to it and shall perhaps face you or your successors to continue that ideal struggle for the liberation of the human spirit.

ON THREE-VALUED LOGIC *)

Aristotelian logic, by assuming that every proposition is either true or false, distinguishes only two kinds of logical values, truth and falsehood. If truth is symbolized by 1, falsehood by 0, identity by $=$, and implication by $<$, we can deduce all the laws of Aristotelian logic from the following principles and definitions:

I. The principles of the identity of falsehood, identity of truth, and non-identity of truth and falsehood: $(0 = 0) = 1$, $(1 = 1) = 1$, $(0 = 1) = (1 = 0) = 0$.

II. The principles of implication: $(0 < 0) = (0 < 1) = (1 < 1) = 1$, $(1 < 0) = 0$.

III. The definitions of negation, addition and multiplication: $a' = (a < 0)$, $a+b = [(a < b) < b]$, $ab = (a'+b')'$.

In these definitions, a and b are variables which may take on only two values, 0 or 1. All logical laws, expressed by means of variables, can be verified by the substitution of 0 and 1 for the letters; e.g., $(a = 1) = a$ is true, for $(0 = 1) = 0$ and $(1 = 1) = 1$.

Three-valued logic is a system of non-Aristotelian logic, since it assumes that in addition to true and false propositions there also are propositions that are neither true nor false, and hence, that there exists a third logical value. That third logical value may be interpreted as "possibility" and may be symbolized by $\frac{1}{2}$.** If we want to formulate a system of three-valued logic, we have to supplement the principles concerning 0 and 1 by the principles concerning $\frac{1}{2}$. This can be done in various ways; the system adopted by the present author in the present stage of research, and which deviates least from "two-valued" logic, is as follows:

*) First published as "O logice trójwartościowej" in *Ruch Filozoficzny* 5 (1920), pp. 170–171.

**) In this paper Łukasiewicz used the symbol "2" to denote a third logical value; in his later papers he always used the symbol "$\frac{1}{2}$" in that sense.

I. The principles of identity: $(0=\frac{1}{2})=(\frac{1}{2}=0)=(1=\frac{1}{2})=(\frac{1}{2}=1)=\frac{1}{2}$, $(\frac{1}{2}=\frac{1}{2})=1$.

II. The principles of implication: $(0<\frac{1}{2})=(\frac{1}{2}<1)=(\frac{1}{2}<\frac{1}{2})=1$, $(\frac{1}{2}<0)=(1<\frac{1}{2})=\frac{1}{2}$.

The principles specified above concerning 0 and 1, and the definitions of negation, addition and multiplication remain the same in three-valued logic, with the only difference that the variables a and b may take on three values, 0, 1, and $\frac{1}{2}$.

The laws of three-valued logic differ partly from those of two-valued logic. Some laws of Aristotelian logic are only "possible" in three-valued logic, for instance, the principle of the syllogism in the ordinary formulation: $(a<b)(b<c)<(a<c)$ {but the principle of the syllogism in the formulation: $(a<b)<[(b<c)<(a<c)]$ is true}, the principle of contradiction $aa'=0$, the principle of the excluded middle $a+a'=1$, etc. Some laws of two-valued logic are false in three-valued logic, among them the law: $(a=a')=0$, since for $a=\frac{1}{2}$ the sentence $a=a'$ is true. This accounts for the fact that in three-valued logic there are no antinomies.

The present author is of the opinion that three-valued logic has above all theoretical importance as an endeavour to construct a system of non-Aristotelian logic. Whether that new system of logic has any practical importance will be seen only when logical phenomena, especially those in the deductive sciences, are thoroughly examined, and when the consequences of the indeterministic philosophy, which is the metaphysical substratum of the new logic, can be compared with empirical data.

TWO-VALUED LOGIC *)

The text that follows is an excerpt from a more comprehensive work on three-valued logic, which I am preparing for publication. I intend here to interpret two-valued logic in such a way that three-valued logic will prove a natural extension of it.

The present paper is a listing of truths and opinions already known. Let me mention briefly from which authors I have drawn most. The concepts of "truth", "falsehood", and "assertion" I owe to Frege. In adding "rejection" to "assertion" I have followed Brentano. The idea of deducing logical laws from the principles pertaining to 0 and 1 I have drawn from Schröder. For practical reasons I have adopted the symbolism developed by Boole and Schröder, as simplified by Couturat; from the symbolism used by Peano and Russell I have only taken the use of dots after the symbol of assertion or rejection and after quantifiers. The term and the symbols for "quantifiers" are due to Peirce. In accepting only apparent variables I have followed Professor Leśniewski.

The principles of three-valued logic have been summarized by me in a report published in *Ruch Filozoficzny* 5 (1920), p. 170.

Contents

*) First published as "Logika dwuwartościowa" in *Przegląd Filozoficzny* 13(1921), pp. 189–205.

1. Truth and falsehood

These terms are not defined; and by *truth* I mean not a true proposition, but the *object* denoted by a true proposition, and by *falsehood* I mean not a false proposition, but the *object* denoted by a false proposition. I say that "2 times 2 is 4" is a truth, because the proposition "2 times 2 is 4" denotes the same object as does the term "truth", in the same way as 2 times 2 is four, because the expression "2 times 2" denotes the same object as does the term "four".

Two different true propositions, for instance "2 times 2 is 4" and "Warsaw lies on the Vistula" differ only by their *contents*, but they denote *the same* object, that is truth, in the same way as the expressions "2 times 2" and "3 plus 1" differ only by their contents, but denote *the same* object, that is the number 4. All true propositions denote one and the same object, namely truth, and all false propositions denote one and the same object, namely falsehood. I consider truth and falsehood to be *singular* objects in the same sense as the number 2 or 4 is. There are as many different names of the one and only truth as there are true propositions, and as many different names of the one and only falsehood as there are false propositions. Ontologically, truth has its analogue in being, and falsehood, in non-being.

The objects denoted by propositions are called *logical values*. Truth is the positive, and falsehood is the negative logical value. Truth is represented by 1, falsehood by 0. These symbols are also read as propositions "truth is", "falsehood is".

2. Two-valued logic

By *logic* I mean the science of logical values. Conceived in this way, logic has its own subject-matter of research, with which no other discipline is concerned. Logic is not a science of propositions, since that belongs to grammar; it is not a science of judgements or convictions, since that belongs to psychology; it is not a science of contents expressed by propositions, since that, according to the content involved, is the concern of the various detailed disciplines; it is not a science of "objects in general", since that belongs to ontology. Logic is the science of objects of a specific kind, namely a science of *logical values*.

All systems of logic known so far, both Aristotelian logic and Stoic logic, both traditional formal logic and modern symbolic logic, have been based on the principle that every proposition is either true or false. That principle, which has served so far as the foundation of all logic, will be called the principle of *bivalence*, and the logic which assumes that there are two and only two logical values will be called *two-valued*.

3. Assertion and rejection

I do not define these terms, and by *assertion* and *rejection* I mean the ways of behaviour with respect to the logical values, the ways known to everyone from his own experience. I wish to assert truth and only truth, and to reject falsehood and only falsehood. The words "I assert" are denoted by U, and the words "I reject" by N. I consider the sentences:

$$U:1, \quad N:0,$$

which are read: "I assert truth" and "I reject falsehood", respectively, to be the fundamental principles of two-valued logic, although I do not quote them anywhere. These propositions are also read: "I assert that truth is" and "I reject that falsehood is".

When I say and write: "I assert that something is" and "I reject that something is" I mean that I assert or reject the object denoted by the *that*-clause, that is, that I assert or reject some logical value. Likewise, when saying or writing "I assert the proposition p" or "I reject the proposition r" I mean that I assert or reject the object denoted by the proposition p or the proposition r. I assert that the proposition "truth is" denotes truth, and the proposition "falsehood is" denotes falsehood.

4. Correct and erroneous procedure

Although I wish to assert truth and only truth and to reject falsehood and only falsehood, it may nevertheless happen that as a result of ignorance or carelessness I may assert falsehood or reject truth. Then I commit an error. I say that I *proceed correctly* when I assert truth or reject falsehood, and that I *proceed erroneously* when I assert falsehood or reject truth. An error is, obviously, something different from a falsehood.

If before an expression denoting truth, that is, before a true proposition or the symbol 1, I write U to indicate that I assert the object denoted by that expression, then I proceed correctly. I also proceed correctly if I write N before an expression which denotes falsehood. Should I write N in the former case or U in the latter, I would proceed erroneously.

In the present paper I write U and N only before symbolic expressions, that is expressions consisting exclusively of symbols, and not of words.

5. Abstention and indifferent procedure

If I do not know whether something is a truth or a falsehood, I usually neither assert nor reject it, but *abstain*. This is the third way of proceeding with respect to the logical values. For instance I do not know whether "the beginning of Cicero's work *De fato* will be found sometime" or "the beginning of Cicero's work *De fato* will never be found". I cannot write U or N before either of these two propositions, because neither proposition is asserted or rejected by me. By proceeding in this way I do not commit an error, but neither do I proceed correctly. I say that in such cases I proceed *indifferently*.

I also assume that I would proceed indifferently by asserting or rejecting something which is not any logical value at all.

Abstention resulting from ignorance has no logical, objective, justification, but is justified only psychologically, subjectively. For that reason this third way of proceeding with respect to the logical values has no significance in logic.

6. The principles of implication

The relation of implication is one of the 16 relations which can be distinguished in analysing the relationships between truth and falsehood. That relation I symbolize by $<$ and read "implies", and an expression of the form "$p < r$" I read: "p implies r" or "if p is, then r is". I do not

define the relation of implication, but I assume that it satisfies the following *principles*:

Z_1 $\quad U{:}\ 0 < 0,$

Z_2 $\quad U{:}\ 0 < 1,$

Z_3 $\quad N{:}\ 1 < 0,$

Z_4 $\quad U{:}\ 1 < 1.$

These expressions are read, respectively: "I assert that falsehood implies falsehood", "I assert that falsehood implies truth", "I reject that truth implies falsehood", "I assert that truth implies truth". They are also read: "I assert that if falsehood is, then falsehood is", "I assert that if falsehood is, then truth is", "I reject that if truth is, then falsehood is", "I assert that if truth is, then truth is".

I do not prove the principles of implication; they are in agreement with the meaning which in logic is ascribed to what is called the relation of *material* implication.

7. Logical variables and quantifiers

I introduce the symbols p, r, s, which I call *variables*, and which range over the logical values. Thus they are variables which may stand for *any* logical value, but *only* for a logical value. Such variables are called *logical variables*. In two-valued logic, logical variables range over two objects only, truth and falsehood. The symbols for these objects, 0 and 1, are the values of the logical variables and may replace them. In contradistinction to the variables, the symbols 0 and 1 are called *logical constants*.

Further, I introduce two kinds of other symbols, consisting of the Greek letters $\prod$ or $\sum$ and the subscripts p, r, or s, for instance, $\prod_p$, $\sum_p$. These symbols are called *quantifiers*, $\prod_p$ being the *universal*, and $\sum_p$ the *existential* quantifier. $\prod_p$ is read: "for any p", and $\sum_p$ is read: "for some p". For the time being I make no use of the existential quantifier.

A quantifier is inseparably connected with the expression that contains the variable indicated by the subscript. I place a quantifier before every expression that contains a variable. When a variable is replaced by one of its values, the quantifier is omitted. For the sake of brevity

I write $\prod_{pr}$ instead of $\prod_p \prod_r$, which is read "for any p and r", and instead of $\prod_p \prod_r \prod_s$ I write $\prod_{prs}$, which is read "for any p, r and s".

In contemporary logic the variables preceded by quantifiers are called *apparent* variables. I make use of apparent variables only.

8. An example of an expression containing a variable

An example will best show the meaning of the symbolic expressions containing variables preceded by universal quantifiers. I formulate the proposition:

T_3 $$U: \prod_p \cdot p < p.$$

This proposition is read: "I assert that for any p, p implies p"; I also read it: "I assert that for any p, if p is, then p is". $\prod_p$ followed by a dot pertains to the entire expression that comes after it. Since p may stand for any logical value, but only for a logical value, hence the sentence T_3 means the same as "every logical value implies itself". Hence falsehood implies falsehood and truth implies truth, in agreement with the principles of implication, accepted above:

Z_1 $$U: 0 < 0,$$
and
Z_4 $$U: 1 < 1.$$

If we compare these principles with the proposition T_3, we see that they are obtained from T_3 by replacing the variable p by its logical values 0 and 1. The principles Z_1 and Z_4 are thus contained in the proposition T_3, and the proposition T_3 is based on them as their *generalization*.

One of the important functions of the logical variables and quantifiers is that they make it possible to generalize principles.

9. Dual interpretation of logical variables

I have defined the logical variables as symbols which range over the logical values. In two-valued logic there are only two logical values, which are the objects truth and falsehood. Falsehood is symbolized by 0 and truth by 1, so that the logical variables can take on only two

values, 0 and 1. This is the first interpretation of the logical variables, which I term *objective*. However, a second interpretation is also possible, which I term *propositional*. According to the latter, the logical variables can take on not two, but infinitely many values, all of which are propositions. The latter interpretation is based on the former.

I have assumed that all true propositions denote one and the same object, namely truth, and all false propositions denote one and the same object, namely falsehood. Hence every true proposition, for instance, "2 times 2 is 4", is a *name* of truth, which is only more complicated than the symbol 1, and every false proposition, for instance, "2 times 2 is 5", is a *name* of falsehood, which is only more complicated than the symbol 0.

In the objective interpretation I do not consider different names of truth (and the same applies to falsehood) to be *different* values of the variable, in the same way as I do not consider "2×2" and "$3+1$" to be two different roots of the equation $x-4 = 0$. But in the propositional interpretation I do consider different names of truth (and the same applies to falsehood) to be different values of the variable, and hence they are infinitely many, although each of them denotes one of the two objects over which the variable ranges. Such an interpretation does not change the logical value of the theorems thus interpreted, since that value depends on the objects denoted by those names, and not on the names themselves.

In the objective interpretation the proposition:

T_3 $$U: \prod_p \cdot p < p$$

states that every *logical value* implies itself, and in the propositional interpretation, that every *proposition* implies itself.

10. The laws of implication

The following three laws of implication are based on the principles of implication:

T_1 $$U: \prod_p \cdot 0 < p,$$
T_2 $$U: \prod_p \cdot p < 1,$$
T_3 $$U: \prod_p \cdot p < p.$$

The first law states: "I assert that, for any p, falsehood implies p"; the second states: "I assert that, for any p, p implies truth". The first law generalizes the principles Z_1 and Z_2, while the second generalizes Z_2 and Z_4. The third law has been discussed previously; it generalizes the principles Z_1 and Z_4 and expresses the *reflexivity* of the relation of implication.

In the propositional interpretation, the laws T_1 and T_2 may be expressed thus: "falsehood implies any proposition" and "truth is implied by any proposition".

Thus, since "2 times 2 is 5" is a falsehood, hence "if 2 times 2 is 5, then London lies on the Thames" and "if 2 times 2 is 5, then Paris lies on the Thames", etc., are truths. And since "2 times 2 is 4" is a truth, hence "if London lies on the Thames, then 2 times 2 is 4" and "if Paris lies on the Thames, then 2 times 2 is 4", etc., are truths.

Any two laws of implication, together with the principle Z_3, can, in an axiomatic presentation of the principles of two-valued logic, replace the principles Z_1 to Z_4.

11. Definitions

I use the principles of implication, logical variables, and universal quantifiers to define certain logical operations and relations. At the beginning of every definition, after the symbol of assertion, I place the universal quantifier, which pertains to the entire definition, after it I write the expression to be defined. Next I place the symbol $\equiv$, which I read "means the same as", and finally on the other side of the symbol $\equiv$ I place the defining expression, which in addition to variables contains either symbols already known, such as 0 and $<$, or symbols defined previously. I adopt the following four definitions:

D_1 $\quad U\colon\ \prod_p \cdot p' \equiv (p < 0),$

$^*D_2$ $\quad U\colon\ \prod_{pr} \cdot p+r \equiv (p' < r),$

D_3 $\quad U\colon\ \prod_{pr} \cdot pr \equiv (p'+r')',$

D_4 $\quad U\colon\ \prod_{pr} \cdot (p = r) \equiv (p < r)(r < p).$

These are the definitions of negation, logical sum, logical product, and equivalence.

12. Negation

The first definition defines the expression p', which I term the *negation* of p and read "not-p" or "p is not". The definition states: "I assert that, for any p, 'not-p' means the same as 'p implies falsehood'"; or: "I assert that, for any p, 'p is not' means the same as 'if p is, then falsehood is'". The idea contained in this definition is this: "p is not" is a truth if and only if it is truth that p implies falsehood. Now p implies falsehood if and only if p is a falsehood. Hence we may say that "p is not" means the same as "p is a falsehood". With such an interpretation the definition agrees with the intuitive meaning of the word "not" and with Sigwart's and Bergson's views on negation.

The operation transforming the expression p into its negation by adding the apostrophe or the word "not" is also called *denying*. Denying, as opposed to other logical operations, is an operation of *one argument*, which means that one expression suffices for the performance of negation. If p is a proposition, for instance, "Peter is honest", then p' means: "Peter is not honest". The propositions p and p' are contradictory.

13. Addition

The second definition defines the expression $p+r$, which I term *logical sum* and read "p or r" or else "p is or r is". This definition states: "I assert that, for any p and r, 'p or r' means the same as 'not-p implies r'"; or "I assert that, for any p and r, 'p is or r is' means the same as 'if p is not, then r is'". This definition, especially in the second verbal formulation, is in agreement with the intuitive meaning of the word "or". Let it be added that, in conformity with the usage adopted in contemporary symbolic logic, I interpret the word "or" in its non-exclusive meaning, that is, when I say "p or r" I do not mean that p excludes r.

The operation which connects two symbols p and r by the symbol $+$ or the word "or" is termed *addition*. If p and r are propositions, for instance, "Peter is honest" and "Paul is honest", then the sum $p+r$ means: "Peter is honest or Paul is honest".

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14. Multiplication

The third definition defines the expression *pr*, which is termed *logical product*, and is read "*p* and *r*" or "*p* is and *r* is". This definition contains the expression $(p'+r')'$, which, if read literally, would be clumsy: "not-(not-*p* or not-*r*)" or "(*p* is not or *r* is not) is not". On the strength of the explanation of negation I say instead: "it is a falsehood that not-*p* or not-*r*" and "it is a falsehood that *p* is not or *r* is not". Thus this definition states: "I assert that, for any *p* and *r*, '*p* and *r*' means the same as 'it is a falsehood that not-*p* or not-*r*'"; or "I assert that, for any *p* and *r*, '*p* is and *r* is' means the same as 'it is a falsehood that *p* is not or *r* is not'". This definition, especially in its second verbal formulation, is in agreement with the intuitive meaning of the word "and".

The operation which connects two symbols *p* and *r* by their simple juxtaposition or by the word "and" is termed *multiplication*. If *p* and *r* are propositions, for instance, "Peter is honest" and "Paul is honest", then *pr* means: "Peter is honest and Paul is honest".

15. Equivalence

The fourth definition defines the expression $p = r$, which denotes the relation of *equivalence*, and which is read "*p* is equivalent to *r*". The simplest way of formulating this definition verbally is: "I assert that, for any *p* and *r*, '*p* is equivalent to *r*' means the same as '*p* implies *r* and *r* implies *p*'". Hence equivalence is *bilateral implication*. This definition, too, is in agreement with the intuitive meaning of equivalence and with the opinions prevailing in logic.

Equivalence is often interpreted as *identity*. The principles of equivalence, which I deduce below from the definition of equivalence, show that no objection is to be raised against such an interpretation.

16. Verbal rules

From the definitions I deduce, by inference, the principles of negation, addition, multiplication, and equivalence. In deducing principles from definitions, and later in verifying laws on the strength of principles, I make use of certain rules which I cannot formulate in symbols.

Accordingly, I term them *verbal rules*. I adopt four such verbal rules; the first two are needed for deducing principles from definitions, the fourth for verifying laws, and the third in both cases. These rules are as follows:

(a) I assert any expression which is obtained from an asserted expression containing variables with universal quantifiers by the replacement of the variables by the values 0 or 1.

(b) I assert any expression which means the same as some asserted expression; I reject any expression which means the same as some rejected expression.

(c) I assert any expression which becomes an asserted expression on the substitution of the symbol 1 for an asserted expression or the symbol 0 for a rejected expression; I reject any expression which becomes a rejected expression upon such substitutions.

(d) I assert any expression containing variables with universal quantifiers which yields only asserted expressions on the replacement of the variables by the values 0 and 1.

I consider all these rules to be obvious.

17. Examples of deducing principles from definitions

Example 1: The expression

$$\prod_p \cdot p' \equiv (p < 0)$$

is asserted on the strength of Definition D_1; hence on the strength of the verbal rule (a) I assert the expression obtained from D_1 by the replacement of the variable *p* by the value 0:

$$U : 0' \equiv (0 < 0).$$

Thus $0'$ means the same as $(0 < 0)$. The expression $(0 < 0)$ is asserted on the strength of principle Z_1; hence, on the strength of the verbal rule (b), I assert $0'$. In this way I obtain the principle of negation:

Z_5 $\qquad U: 0'.$

Example 2: The expression

$$\prod_{pr} \cdot p+r \equiv (p' < r)$$

is asserted on the strength of Definition $^*D_2$; hence, on the strength

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of the verbal rule (a), I assert the expression obtained from $^*D_2$ by the replacement of the variables p and r by the value 0:

$$U:\ 0+0 \equiv (0' < 0).$$

In the expression $(0' < 0)$ I substitute 1 for the asserted $0'$; I may do so, because I assert truth and only truth, unless I commit an error. The expression $(0' < 0)$ by the substitution of 1 for $0'$ becomes the rejected expression $(1 < 0)$. Hence, on the strength of the verbal rule (c) I reject $(0' < 0)$. Thus the expression $0+0$ means the same as a certain rejected expression, and accordingly I reject it on the strength of the verbal rule (b). In this way I obtain the principle of addition:

Z_7 $\quad N:\ 0+0.$

The examples suffice to explain how I have inferred from definitions the principles listed below.

18. The principles of logical operations and equivalence

The principles of negation:

Z_5 $\quad U:\ 0',$
Z_6 $\quad N:\ 1'.$

The principles of addition:

Z_7 $\quad N:\ 0+0,$
Z_8 $\quad U:\ 0+1,$
Z_9 $\quad U:\ 1+0,$
Z_{10} $\quad U:\ 1+1.$

The principles of multiplication:

Z_{11} $\quad N:\ 00,$
Z_{12} $\quad N:\ 01,$
Z_{13} $\quad N:\ 10,$
Z_{14} $\quad U:\ 11.$

The principles of equivalence:

Z_{15} $\quad U:\ 0 = 0,$
Z_{16} $\quad N:\ 0 = 1,$
Z_{17} $\quad N:\ 1 = 0,$
Z_{18} $\quad U:\ 1 = 1.$

I assert the negation of falsehood and I reject the negation of truth, which means that I assert that falsehood is not, and I reject that truth is not.

I assert the sum when at least one of its elements is a truth, and I reject it only when every element of it is a falsehood.

I reject the product when at least one its factor is a falsehood, and I assert it only when every factor of it is a truth.

I assert the equivalence of falsehood to falsehood and truth to truth, and I reject the equivalence of falsehood to truth and truth to falsehood.

19. The meaning of quantifiers

The universal quantifier is closely connected with multiplication, and the existential, with addition. When I place before an expression that contains the variable p, the universal quantifier $\prod_p$, I understand that in that expression I have to replace the variable p by the values 0 and 1 and to *multiply* the expressions obtained in this way. When I place before an expression that contains the variable p, the existential quantifier $\sum_p$, I understand that in that expression I have to replace the variable p by the values 0 and 1 and to *add* the expressions obtained in this way. I proceed in an analogous way with expressions containing more variables. Thus all the expressions containing variables with quantifiers could be defined as products or sums containing only logical constants, for instance:

$$U:\ [\textstyle\prod_p \cdot p < p] \equiv (0 < 0)(1 < 1),$$

$$U:\ [\textstyle\prod_{pr} \cdot p < (r < p)]$$
$$\equiv [0 < (0 < 0)]\,[0 < (1 < 0)]\,[1 < (0 < 1)]\,[1 < (1 < 1)],$$

$$U: (\textstyle\sum_p \cdot pp') \equiv 00'+11', \text{ etc.}$$

Since on the strength of the principles of multiplication I assert the product only if each of its factors is a truth, hence it is obvious that I assert the expressions preceded by universal quantifiers if and only if I assert all the expressions obtained from them by the replacement of the variables by the values 0 and 1. The foregoing sentence contains the verbal rules (a) and (d).

20. Theorems and laws

I term *theorems* those propositions, other than principles, which I can deduce from principles by means of the verbal rules. I divide the theorems of two-valued logic into theorems of the first kind, which do not contain variables, e.g.:

$$U\colon\ (0=1)=0,$$

and theorems of the second kind, which contain variables with quantifiers, e.g.:

$$U\colon\ \prod_p \cdot p<p.$$

Among the theorems of the second kind I consider those in which the symbol of assertion is followed by the universal quantifier to be particularly important and term them *laws*. The laws generalize either principles or theorems of the first kind, and in the propositional interpretation of the variables they present the principles of *reasoning*. Three such laws, generalizations of the principles of implication, were quoted in Section 10. The most important laws are listed in the last Section.

One of the merits of the logical system as presented in this paper is the ease of proving theorems. Theorems of the first kind are deduced from principles by means of the verbal rule (c). The proving of laws consists in *verifying* them. To verify a law it suffices to demonstrate that, in accordance with the verbal rule (d), the law yields only asserted expressions when the variables are replaced by the values 0 and 1. This procedure is explained by the examples given below.

21. Examples of verification of laws

Example 1: Verify the law:

T_5 $$U\colon\ \prod_p \cdot (p=1)=p.$$

I verify it by demonstrating that, in accordance with the verbal rule (d), the law yields only asserted expressions when the variable p is replaced by the values 0 and 1, namely:

(1) $$U\colon\ (0=1)=0,$$

(2) $$U\colon\ (1=1)=1.$$

After 0 is substituted for $(0=1)$, expression (1) yields the asserted expression $0=0$, and, after 1 is substituted for $(1=1)$, expression (2) yields the asserted expression $1=1$. Thus both (1) and (2) must be asserted on the strength of the verbal rule (c). The law has been verified.

Example 2: Verify the law:

T_{24} $$U\colon\ \prod_{pr} \cdot p<(r<p).$$

It turns out that, after the values 0 and 1 are substituted for the variables p and r, this law yields asserted expressions only, namely:

(1) $$U\colon\ 0<(0<0),$$

(2) $$U\colon\ 0<(1<0),$$

(3) $$U\colon\ 1<(0<1),$$

(4) $$U\colon\ 1<(1<1).$$

When 1 is substituted for $(0<0)$, expression (1) yields the asserted expression $0<1$; the substitution of 0 for $(1<0)$ in expression (2) yields the asserted expression $0<0$; the substitution of 1 for $(0<1)$ and $(1<1)$ in expressions (3) and (4), respectively, yields the asserted expression $1<1$. The law has been verified.

With some practice such verifications can be performed automatically.

22. An axiomatic system of two-valued logic

I base all the theorems of two-valued logic on the principles Z_1 to Z_{18} and the verbal rules (c) and (d). Hence, to construct an axiomatic system of that logic it suffices to give a set of axioms from which the principles Z_1 to Z_{18} can be deduced.

I construct such a system out of three axioms, namely the laws of implication T_1 and T_2 and the principle of implication Z_3, quoted in Section 10. To these axioms I join the definitions of negation, sum, product and equivalence, as well as the verbal rules (a), (b) and (c). The previous sections have provided proofs that all the principles can be deduced from the whole consisting of these axioms, definitions, and verbal rules.

23. The list of axioms and definitions of two-valued logic

In view of three-valued logic, instead of the definition of logical sum $*D_2$, given in Section 11, I adopt the following definition in the list below:

D_{2a} $$U\colon \prod_{pr} \cdot p+r \equiv [(p < r) < r].$$

This definition states: "I assert that, for any p and r, 'p is or r is' means the same as 'if p implies r, then r is'". This definition is less simple and obvious than $*D_2$, but it results in the same principles of addition. Since p' means the same as $(p < 0)$, Definition $*D_2$ can also be formulated as follows:

$*D_2$ $$U\colon \prod_{pr} \cdot p+r \equiv [(p < 0) < r].$$

The difference between $*D_2$ and D_{2a} thus consists only in the replacement of the symbol 0 in $*D_2$ by the variable r in D_{2a}. The expressions $[(p < 0) < r]$ and $[(p < r) < r]$ are equivalent in two-valued logic:

$$U\colon \prod_{pr} \cdot [(p < 0) < r] = [(p < r) < r].$$

When r is 0, both expressions become $[(p < 0) < 0]$, and when r is 1, both are true.

Here is the list of axioms and definitions:

T_1	$U\colon \prod_p \cdot 0 < p,$
T_2	$U\colon \prod_p \cdot p < 1,$
Z_3	$N\colon 1 < 0,$
D_1	$U\colon \prod_p \cdot p' \equiv (p < 0),$
D_{2a}	$U\colon \prod_{pr} \cdot p+r \equiv [(p < r) < r],$
D_3	$U\colon \prod_{pr} \cdot pr \equiv (p'+r')',$
D_4	$U\colon \prod_{pr} \cdot (p = r) \equiv (p < r)(r < p).$

24. The most important laws of two-valued logic

Out of the unlimited number of logical laws I specify here 40 which for various reasons seem to me important. Every law is accompanied with a short explanation. The asterisks which mark some laws are connected with three-valued logic: one asterisk means that the expression which follows the symbol of assertion is not true in three-valued logic, but is not false either; two asterisks mean that the expression which follows the symbol of assertion is false in three-valued logic.

T_1 $$U\colon \prod_p \cdot 0 < p.$$

T_2 $$U\colon \prod_p \cdot p < 1.$$

T_3 $$U\colon \prod_p \cdot p < p.$$

These are the already known laws of *implication*; I have chosen T_1 and T_2 as axioms, and T_3 states that the relation of implication is *reflexive*.

T_4 $$U\colon \prod_p \cdot p = p.$$

T_4 is the law of *identity*, if equivalence is interpreted as identity. This law also states that the relation of equivalence is *reflexive*.

T_5 $$U\colon \prod_p \cdot (p = 1) = p.$$

In *L'Algèbre de la logique* Couturat calls the law T_5 the principle of *assertion*. The proposition "$p = 1$", i.e., "p is equivalent to truth" I read briefly "p is a truth"; likewise, the proposition "$p = 0$", i.e., "p is equivalent to falsehood" I read briefly "p is a falsehood". T_5 states that the proposition "p is a truth" is, for any p, equivalent to the proposition "p is".

T_6 $$U\colon \prod_p \cdot (p = 0) = p'.$$

T_6 states that the proposition "p is a falsehood" is, for any p, equivalent to the proposition "p is not". On the strength of this law p' might be defined by $(p = 0)$.

T_7 $$U\colon \prod_p \cdot p'' = p.$$

By T_7, the expressions "not-(not-p)" and "p" are equivalent for any p. This is the law of *double negation*.

$*T_8$ $$U\colon \prod_p \cdot pp' = 0.$$

$*T_9$ $$U\colon \prod_p \cdot p+p' = 1.$$

$*T_8$ is the law of *contradiction*, and $*T_9$, the law of the *excluded middle*. Since a product is a falsehood only if at least one of its factors is a falsehood, and the sum is a truth only if at least one of its elements is a truth, it follows from these laws that of two contradictory propositions one is false $^*T_8$, and that of two contradictory propositions one is true $^*T_9$.

$^{**}T_{10}$ $$U:\ \prod_p \cdot (p = p') = 0.$$

For any p, the equivalence of the contradictory propositions p and p' is a falsehood. If the equivalence of contradictory propositions is called *absurdity*, it may be said that $^{**}T_{10}$ states the falsehood of absurdity.

$^*T_{11}$ $$U:\ \prod_p \cdot (p' < p) < p.$$

If a proposition is implied by its own negation, then it is true. This is a form of *apagogic proof*. Vailati has written a monograph on the history of this form of reasoning (cf. article CXV in *Scritti di G. Vailati*, Leipzig–Florence, 1911).

T_{12} $$U:\ \prod_p \cdot p+p = p.$$

T_{13} $$U:\ \prod_p \cdot pp = p.$$

T_{12} and T_{13} are the laws of *tautology*. The expressions "p or p" and "p and p" are, for any p, equivalent to "p".

T_{14} $$U:\ \prod_{pr} \cdot p+r = r+p.$$

T_{15} $$U:\ \prod_{pr} \cdot pr = rp.$$

T_{16} $$U:\ \prod_{pr} \cdot (p = r) = (r = p).$$

T_{14} is the law of *commutativity of addition*, T_{15} is the law of *commutativity of multiplication*, and T_{16} states that the relation of equivalence is *symmetrical*. Symmetry of relations corresponds to commutativity of operations.

T_{17} $$U:\ \prod_{prs} \cdot p(r+s) = pr+ps.$$

T_{17} is the law of *distributivity*.

T_{18} $$U:\ \prod_{pr} \cdot (p+r)' = p'r'.$$

T_{19} $$U:\ \prod_{pr} \cdot (pr)' = p'+r'.$$

These are *De Morgan's laws*. By T_{18}, the negation of a sum is equivalent to the product of the negated elements, and by T_{19}, the negation of a product is equivalent to the sum of the negated factors.

T_{20} $$U:\ \prod_{pr} \cdot pr < p.$$

T_{21} $$U:\ \prod_{pr} \cdot p < p+r.$$

T_{20} and T_{21} are called the laws of *simplification*. A product may be simplified in the consequent by deducing from it any factor as a consequence, and a sum may be simplified in the antecedent by deducing it from any element as a reason.

T_{22} $$U:\ \prod_{prs} \cdot (p < s)(r < s) = (p+r < s).$$

T_{23} $$U:\ \prod_{prs} \cdot (s < p)(s < r) = (s < pr).$$

T_{22} and T_{23} are called the laws of *combination*. The proposition "p implies s and r implies s" is, for any p, r and s, equivalent to the proposition "p or r implies s", and the proposition "s implies p and s implies r" is, for any p, r and s, equivalent to the proposition "s implies p and r".

T_{24} $$U:\ \prod_{pr} \cdot p < (r < p).$$

T_{24} is a law of implication that has no special name. Its content resembles Axiom T_2.

T_{25} $$U:\ \prod_{pr} \cdot (p < r)' < (r < p).$$

T_{25} states that, for any p and r, if p does not imply r, then r implies p. On the strength of this law the relation of implication might be termed *anti-asymmetric*, that is such that its negation is an asymmetric relation.

T_{26} $$U:\ \prod_{pr} \cdot (p < r) = (p = pr).$$

$^*T_{27}$ $$U:\ \prod_{pr} \cdot (p < r) = (pr' = 0).$$

T_{26} and $^*T_{27}$ give the methods of transforming implication into equivalence. The latter method was known even to Chrysippus (cf. Cicero, *De Fato* 15, and Diog. Läert. VII 73).

$^*T_{28}$ $$U: \prod_{pr} \cdot (p < r) = p'+r.$$

The proposition "if p is, then r is" is, for any p and r, equivalent to the proposition "p is not or r is". By $^*T_{28}$ the relation of implication could be defined in terms of sum and negation.

T_{29} $$U:\ \prod_{pr} \cdot (p < r) = (r' < p').$$

$^*T_{30}$ $$U:\ \prod_{prs} \cdot (pr < s) = (rs' < p').$$

These are the laws of *transposition* for two and three propositions. By T_{29} the proposition "if p is, then r is" is, for any p and r, equivalent to the proposition "if r is not, then p is not". $*T_{30}$ becomes T_{29} following the substitution of 1 for r and of r for s. $*T_{30}$ serves as the foundation of what is termed *reductio syllogismi*, i.e., the form of reasoning which traditional logic used in reducing the moods *Baroco* and *Bocardo* to the mood *Barbara*.

$*T_{31}$ $\quad U\colon \prod_{prs} \cdot (pr < s) = [p < (r < s)].$

$*T_{32}$ $\quad U\colon \prod_{pr} \cdot (p < r) = [p < (p < r)].$

$*T_{31}$ is called the law of *importation* and *exportation*. I *export* p when I reason: if p and r imply s, then, if p is, r implies s. I *import* p when I reason in the reverse direction: if, if p is, r implies s, then p and r imply s. $*T_{32}$ is a special case of $*T_{31}$, from which it is obtained by the substitution of p for r and of r for s.

$*T_{33}$ $\quad U\colon \prod_{prs} \cdot (p < r)(r < s) < (p < s).$

T_{34} $\quad U\colon \prod_{prs} \cdot (p < r) < [(r < s) < (p < s)].$

$*T_{33}$ is the law of the *syllogism*: it is true, for any p, r and s, that if p implies r and r implies s, then p implies s. This law also states that the relation of implication is *transitive*. T_{34} is another form of the law of the syllogism, obtained from $*T_{33}$ by the application of the law of exportation.

$*T_{35}$ $\quad U\colon \prod_{prs} \cdot (p = r)(r = s) < (p = s).$

T_{36} $\quad U\colon \prod_{prs} \cdot (p = r) < [(r = s) < (p = s)].$

The laws $*T_{35}$ and T_{36} are analogues of the laws $*T_{33}$ and T_{34} for the relation of equivalence. $*T_{35}$ states that the relation of equivalence is *transitive*.

$*T_{37}$ $\quad U\colon \prod_{pr} \cdot (p < r)p < r.$

$*T_{38}$ $\quad U\colon \prod_{pr} \cdot (p < r)r' < p'.$

These are the laws of reasoning known as *modus ponens* and *modus tollens*. $*T_{37}$ states that, for any p and r, if p implies r and p is, then r is. $*T_{38}$ states that, for any p and r, if p implies r and r is not, then p is not.

$*T_{39}$ $\quad U\colon \prod_{pr} \cdot (p' < rr') < p.$

$**T_{40}$ $\quad U\colon \prod_{pr} \cdot [p' < (r = r')] < p.$

$*T_{39}$ and $**T_{40}$ are certain forms of *apagogic proofs*, used especially in mathematics. On their basis we prove that if the negation of p, i.e., p', implies a contradiction $*T_{39}$ or an absurdity $**T_{40}$, then p is true.

ON DETERMINISM *)

This article is a revision of an address which I delivered as Rector of Warsaw University at the Inauguration of the academic year 1922/1923. As was my habit, I spoke without notes. I wrote down my address later on, but never published it.

In the course of the next twenty-four years I frequently returned to the editing of my lecture, improving its form and content. The main ideas, and in particular the critical examination of the arguments in favour of determinism, remained, however, unchanged.

At the time when I gave my address those facts and theories in the field of atomic physics which subsequently led to the undermining of determinism were still unknown. In order not to deviate too much from, and not to interfere with, the original content of the address, I have not amplified my article with arguments drawn from this branch of knowledge.

Dublin, November 1946

* * *

1. It is an old academic custom that the Rector should open a new session with an inaugural address. In such a lecture he should state his scientific creed and give a synthesis of his investigations.

A synthesis of philosophical investigations is expressed in a philosophical system, in a comprehensive view of the world and life. I am unable to give such a system, for I do not believe that today one can establish a philosophical system satisfying the requirements of scientific method.

*) [Editorial note from *Polish Logic 1920–1939*, ed. by Storrs McCall, Oxford, 1967, The Clarendon Press: This paper, entitled "O Determinizmie", was published for the first time in *Z zagadnień logiki i filozofii*, an anthology of Łukasiewicz's works edited by J. Słupecki, Warsaw, 1961. Translated by Z. Jordan.]

I belong, with a few fellow workers, to a still tiny group of philosophers and mathematicians who have chosen mathematical logic as the subject or the basis of their investigations. This discipline was initiated by Leibniz, the great mathematician and philosopher, but his efforts had fallen into oblivion when, about the middle of the nineteenth century, George Boole became its second founder. Gottlob Frege in Germany, Charles Peirce in the United States, and Bertrand Russell in England have been the most prominent representatives of mathematical logic in our own times.

In Poland the cultivation of mathematical logic has produced more plentiful and fruitful results than in many other countries. We have constructed logical systems which greatly surpass not only traditional logic but also the systems of mathematical logic formulated until now. We have understood, perhaps better than others, what a deductive system is and how such systems should be built. We have been the first to grasp the connexion of mathematical logic with the ancient systems of formal logic. Above all, we have achieved standards of scientific precision that are much superior to the requirements accepted so far.

Compared with these new standards of precision, the exactness of mathematics, previously regarded as an unequalled model, has not held its own. The degree of precision sufficient for the mathematician does not satisfy us any longer. We require that every branch of mathematics should be a correctly constructed deductive system. We want to know the axioms on which each system is based, and the rules of inference of which it makes use. We demand that proofs should be carried out in accordance with these rules of inference, that they should be complete and capable of being mechanically checked. We are no longer satisfied with ordinary mathematical deductions, which usually start somewhere "in the middle", reveal frequent gaps, and constantly appeal to intuition. If mathematics has not withstood the test of the new standard of precision, how are other disciplines, less exact than mathematics, to stand up to it? How is philosophy, in which fantastic speculations often stifle systematic investigations, to survive?

When we approach the great philosophical systems of Plato or Aristotle, Descartes or Spinoza, Kant or Hegel, with the criteria of precision set up by mathematical logic, these systems fall to pieces as if they were houses of cards. Their basic concepts are not clear, their most important theses are incomprehensible, their reasoning and proofs are inexact, and the logical theories which often underlie them are practically all erroneous. Philosophy must be reconstructed from its very foundations; it should take its inspiration from scientific method and be based on the new logic. No single individual can dream of accomplishing this task. This is a work for generations and for intellects much more powerful than those yet born.

2. This is my scientific creed. Since I cannot give a philosophical system, today I shall try to discuss a certain problem which no philosophical synthesis can ignore and which is closely connected with my logical investigations. I should like to confess in advance that I am unable to examine this problem, in all its details, with the scientific precision that I demand from myself. What I give is only a very imperfect essay, of which perhaps somebody will one day take advantage to establish, on the basis of these preliminary examinations, a more exact and mature synthesis.

I want to speak of determinism. I understand by determinism something more than that belief which rejects the freedom of the will. I shall first explain what I mean by an example.

John met Paul in the Old Town Square in Warsaw yesterday noon. The fact of yesterday's meeting no longer exists today. Yet that fact of yesterday is not a mere illusion today, but some part of the reality which both John and Paul have to take into account. They both remember their yesterday's meeting. The effects or traces of that meeting somehow exist in them today. Each of them could take an oath in a court of law that he saw the other in the Old Town Square in Warsaw yesterday noon.

On the basis of these data I say, "it is true at every instant of today that John met Paul in the Old Town Square in Warsaw yesterday noon". I do not intend to maintain by this that the *sentence* "John met Paul in the Old Town Square in Warsaw yesterday noon" is true at every instant of today, for such a sentence, if nobody utters it or thinks of it, may not exist at all. I make use of the expression "it is true at instant *t* that *p*"—in which "instant" means an unextended time point and "*p*" any statement of fact—as equivalent to "it is the case at instant *t* that *p*". For the present I am unable to give a further analysis of the latter expression.

We believe that what has happened cannot be undone, *facta infecta fieri non possunt*. What once was true remains true for ever. All truth is eternal. These sentences seem to be intuitively certain. We believe, therefore, that if an object *A* is *b* at instant *t*, it is true at any instant later than *t* that *A* is *b* at instant *t*. If John met Paul in the Old Town Square in Warsaw yesterday noon, it is true at any instant later than yesterday noon that John met Paul in the Old Town Square in Warsaw yesterday noon.

The question occurs whether it was also true at any instant earlier than yesterday noon that John would meet Paul in the Old Town Square in Warsaw yesterday noon? Was it true the day before yesterday and one year ago, at the moment of John's birth and at any instant preceding his birth? Is everything which will happen and be true at some future time true already today, and has it been true from all eternity? Is every truth eternal?

Intuition fails us in this case and the problem becomes controversial. The determinist answers the question in the affirmative and the indeterminist in the negative. By determinism I understand the belief that if *A* is *b* at instant *t* it is true at any instant earlier than *t* that *A* is *b* at instant *t*.

Nobody who adopts this belief can treat the future differently from the past. If everything that is to occur and become true at some future time is true already today, and has been true from all eternity, the future is as much determined as the past and differs from the past only in so far as it has not yet come to pass. The determinist looks at the events taking place in the world as if they were a film drama produced in some cinematographic studio in the universe. We are in the middle of the performance and do not know its ending, although each of us is not only a spectator but also an actor in the drama. But the ending is there, it exists from the beginning of the performance, for the whole picture is completed from eternity. In it all our parts, all our adventures and vicissitudes of life, all our decisions and deeds, both good and bad, are fixed in advance. Even the moment of our death, of yours and mine, is laid down beforehand. We are only puppets in the universal drama. There remains for us nothing else to do but watch the spectacle and patiently await its end.

This is a strange view and by no means obvious. However, there are two arguments of considerable persuasive power which have been known for a long time and which seem to support determinism. One of them, originating with Aristotle, is based on the logical principle of the excluded middle, and the other, which was known to the Stoics, on the physical principle of causality. I shall try to present these two arguments, however difficult and abstract they are, in a way as easy to understand as possible.

3. Two sentences of which one is the denial of the other are called *contradictory*. I shall illustrate this notion by an example taken from Aristotle. "There will be a sea battle tomorrow" and "There will not be a sea battle tomorrow" are contradictory sentences. Two famous principles derived from Aristotle, the principle of the excluded contradiction and the principle of the excluded middle, are concerned with contradictory sentences. The first of these states that two contradictory sentences are not true together, that is, that one of them must be false. In my subsequent inquiry I shall not deal with this important principle, which Aristotle and, following him, numerous other thinkers regarded as the deepest mainstay of our thinking. I am concerned here with the principle of the excluded middle. It lays down that two contradictory sentences are not false together, that is, that one of them must be true. Either there will be or there will not be a sea battle tomorrow. *Tertium non datur*. There is nothing in between the arguments of this alternative, no third thing that, being true, would invalidate both its arguments. It may sometimes happen that two disputants, of whom one regards as white what the other considers black, are both mistaken, and the truth lies somewhere in between these two assertions. There is no contradiction, however, between regarding something as white and considering the same thing as black. Only the sentences stating that the same thing is and is not white would be contradictory. In such cases truth cannot lie in between or outside of these sentences, but must inhere in one of them.

To return to our everyday example, if the principle of the excluded middle holds, and if Peter says today "John will be at home tomorrow noon" and Paul denies it by saying "John will not be at home tomorrow ~~noon", then one of them speaks the truth.~~ We may not know today which one of them does so, but we shall learn by visiting John tomorrow noon. If we find John at home, Peter made a true statement, and if John is away, Paul spoke the truth today.

Therefore, either it is already true today that John will be at home tomorrow noon or it is true today that John will not be at home tomorrow noon. If someone utters the sentence "p", and someone else utters its denial, "not-p", then one of them makes a true statement not-~~only today but at any instant t; for either "p" or "not-p" is true.~~ It does not matter at all whether anyone actually expresses these sentences or even thinks of them; it seems to be in the very nature of the case that either it is true at instant t that "p" or it is true at instant t that "not-p". This alternative seems to be intuitively true. As applied to our example, it takes the following form:

(a) *Either it is true at instant* t *that John will be at home tomorrow noon or it is true at instant* t *that John will not be at home tomorrow noon.*

Let us keep in mind this sentence as the first premiss of our reasoning.

The second premiss is not based on any logical principle and can be expressed in general form as the conditional "if it is true at instant t that p, then p". In this conditional, "p" stands for any sentence, either affirmative or negative. If we substitute for "p" the negative sentence "John will not be at home tomorrow noon" we obtain

(b) *If it is true at instant* t *that John will not be at home tomorrow noon, then John will not be at home tomorrow noon.*

This premiss also seems to be intuitively true. If it is true at an arbitrary instant, t, e.g. now, that John will not be at home tomorrow noon—for we know that he has just left for a distant destination and for a long time—there is no use calling upon John tomorrow noon. We are certain that we shall not find him at home.

We accept both premisses without proof as intuitively certain. The thesis of determinism is based upon these premisses. Its proof will be carried out rigorously in accordance with the so-called theory of deduction.

4. Thanks to mathematical logic we know today that the basic system of logic is not the small fragment of the logic of terms known as Aristotle's syllogistic, but the logic of propositions, incomparably more important than syllogistic. Aristotle made intuitive use of the

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logic of propositions, and only the Stoics, with Chrysippus at their head, formulated it systematically. In our own times the logic of propositions was constructed in an almost perfect axiomatic form by Gottlob Frege in 1879; it was discovered independently of Frege and enriched with new methods and theorems by Charles Peirce in 1895; and under the name of "the theory of deduction" it was made the basis of mathematics and logic by Bertrand Russell in 1910. It was also Bertrand Russell who extended knowledge of it to the scientific community at large.

The theory of deduction should become as universally well known as elementary arithmetic, for it comprises the most important rules of inference used in science and life. It teaches us how to use correctly such common words as "not", "and", "or", "if-then". In the course of the present exposition, which I begin with our second premiss, we shall become acquainted with three rules of inference included in the theory of deduction.

The second premiss is a conditional of the form "if α, then not-β" in which "α" stands for the sentence "it is true at instant t that John will not be at home tomorrow noon" and "β" for the sentence "John will be at home tomorrow noon". In the consequent of premiss (b) there occurs the denial of the sentence "β", that is, the sentence "not-β", "John will not be at home tomorrow noon". In accordance with the theory of deduction the premiss "if α, then not-β" implies the conclusion "if β, then not-α". For if "α" implies "not-β" then "α" and "β" exclude each other, and therefore "β" implies "not-α". According to this rule of inference, premiss (b) is transformed into the sentence

(c) *If John will be at home tomorrow noon, then it is not true at instant t that John will not be at home tomorrow noon.*

Let us now pass to the first premiss, to the alternation of the form "γ or α", in which "γ" signifies the sentence "it is true at instant t that John will be at home tomorrow noon", and "α" the same sentence as before, "it is true at instant t that John will not be at home tomorrow noon". It follows from the theory of deduction that the premiss "γ or α" implies the conclusion "if not-α, then γ". For an alternative is true if and only if at least one of its arguments is true. If the second argu-

ment is false, the first one must be true. In accordance with this rule of inference premiss (a) is transformed into the sentence

(d) *If it is not true at instant t that John will not be at home tomorrow noon, then it is true at instant t that John will be at home tomorrow noon.*

Let us now compare sentences (c) and (d). They are both conditionals, and the consequent in (c) is equiform with the antecedent in (d); these two sentences have the form "if β, then not-α" and "if not-α, then γ". According to the theory of deduction two such premisses imply the conclusion "if β, then γ". For if it is true that "if the first, then the second" and "if the second, then the third", then it is also true that "if the first, then the third". This is the law of the hypothetical syllogism, as known by Aristotle. If we remember that "β" stands for the sentence "John will be at home tomorrow noon", and "γ" for the sentence "it is true at instant t that John will be at home tomorrow noon", we obtain the conclusion

(e) *If John will be at home tomorrow noon, then it is true at instant t that John will be at home tomorrow noon.*

Instant t is an arbitrary instant; therefore, it is either earlier than or simultaneous with or later than tomorrow noon. It follows that if John will be at home tomorrow noon, then it is true at an arbitrary or at any instant that John will be at home tomorrow noon. To put it in general form, it has been proved on the basis of a particular example that if A is b at instant t, then it is true at any instant, and therefore at any instant earlier than t, that A is b at instant t. The thesis of determinism has been proved by deducing it from the principle of the excluded middle.

5. The second argument in favour of determinism is based on the principle of causality. It is not easy to present this argument in a comprehensible way, for neither the word "cause" nor the proposition known as the principle of causality have acquired an established meaning in science. They are only associated with a certain intuitive meaning which I should like to make explicit by giving a few explanations.

I say that the ringing of the bell at the entrance door to my apartment at this moment is a fact taking place now. I regard John's presence at home at instant t as a fact occurring at instant t. Every fact takes place somewhere at some time. Statements of fact are singular and include an indication of time and place.

Fact F occurring at instant s is called the *cause* of fact G occurring at instant t, and fact G the *effect* of fact F, if instant s is earlier than instant t, and if facts F and G are so connected with each other that by means of known laws obtaining between the respective states of affairs it is possible to infer the statement of fact G from the statement of fact F.*) For instance, I consider the pressing of the button of an electric bell the cause of its ringing, because the bell is pressed at an instant earlier than the instant of its ringing, and I can deduce the statement of the second fact from the statement of the first one by means of the known laws of physics on which the construction of an electric bell is based.

The definition of cause implies that the causal relation is transitive. This means that for any facts F, G, and H, if F is the cause of G and G is the cause of H, then F is the cause of H.

I understand by the principle of causality the proposition that every fact G occurring at instant t has its cause in some fact F occurring at instant s earlier than t, and that at every instant later than s and earlier than t there occur facts which are both effects of fact F and causes of fact G.

These explanations are intended to make explicit the following intuitions. The fact which is the cause takes place earlier than the fact which is the effect. I first press the button of the bell and the bell rings later, even if it appears to us that both facts happen simultaneously. If there occurs a fact which is the cause of some other fact, then the latter fact, which is the effect of the former, follows the cause inevitably. Thus if I press the button, then the bell rings. It is possible to infer the effect from the cause. As the conclusion is true provided that its premisses are true, in a similar way the effect has to occur provided that its causes exist. Nothing happens without cause. The bell does not

*) This definition of the concept of cause differs from the definition accepted in Łukasiewicz's paper "Analiza i konstrukcja pojęcia przyczyny" (The analysis and construction of the concept of cause), *Przegląd Filozoficzny* 9 (1906), pp. 105–179, reprinted in the 1961 edition *Z zagadnień logiki i filozofii*. Both definitions lay down, however, that the relation of causality is transitive, and this point is of paramount importance in Łukasiewicz's subsequent investigations.

ring of itself; this only happens because of some earlier facts. In the set of facts succeeding each other, ordered by the causal relation, there are neither gaps nor jumps. From the instant when the button is pressed to the instant when the bell rings there constantly occur facts each of which is simultaneously an effect of the pressing of the button and a cause of the ringing of the bell. Moreover, everyone of these facts occurring earlier is the cause of everyone occurring later.

6. The argument deducing the thesis of determinism from the principle of causality may become more intelligible after these explanations. Let us assume that a certain fact F occurs at instant t; for instance, that John is at home tomorrow noon. Fact F has its cause in some fact F_1, taking place at instant t_1 earlier than t. Again, fact F_1 has its cause in some fact F_2, taking place at instant t_2 earlier than t_1. Since according to the principle of causality every fact has its cause in some earlier fact, this procedure can be repeated over and over again. Therefore, we obtain an infinite sequence of facts which extends back indefinitely

$$\dots F_n, F_{n-1}, \dots, F_2, F_1, F,$$

because the facts take place at ever earlier instants

$$\dots t_n, t_{n-1}, \dots, t_2, t_1, t.$$

In this sequence every earlier fact is the cause of every later fact, for the causal relation is transitive. Moreover, if fact F_n occurring at instant t_n is the cause of fact F occurring at instant t, then, in accordance with the principle of causality, at every instant later than t_n and earlier than t there occur facts which are simultaneously effects of fact F_n and causes of fact F. Since these facts are infinitely many, we are unable to order all of them in the sequence and can designate only some, for instance F_{n-1}, F_2, or F_1.

While everything seems to be in order so far, the most important step in the determinist's argument comes only now. His reasoning would probably take the following course.

As the sequence of facts which occur earlier than and which are the causes of fact F is infinite, at every instant earlier than t, and therefore at every present and past instant, there occurs some fact that is the cause of F. If it is the case that John will be at home tomorrow noon, then the cause of this fact exists already today and also at every instant earlier than tomorrow noon. If the cause exists or existed, all the effects of this cause must inevitably exist. Therefore it is already true today and has been true from all eternity that John would be at home tomorrow noon. In general, if A is b at instant t, it is true at every instant earlier than t that A is b at instant t; for at every instant earlier than t there exist the causes of this fact. Thus the thesis of determinism may be proved by means of the principle of causality.

These are the two strongest arguments which can be used in support of determinism. Should we give up and accept them? Should we believe that everything in the world takes place of necessity and that every free and creative act is only an illusion? Or, on the contrary, should we reject the principle of causality along with the principle of the excluded middle?

7. Leibniz writes that there are two famous labyrinths in which our reason is often lost. One of them is the problem of freedom and necessity, and the other is concerned with continuity and infinity. While writing this Leibniz did not think it plausible that these two labyrinths should constitute one single whole and that freedom, if it exists at all, could be concealed in some nook of infinity.

Should the causes of all facts which could ever occur exist at every instant, there would be no freedom. Fortunately, the principle of causality does not compel us to accept this consequence. Infinity and continuity come to our rescue.

There is an error in the argument which derives the thesis of determinism from the principle of causality. For it is not the case that if John is at home tomorrow noon, then the infinite sequence of causes of this fact must reach the present and every past instant. This sequence may have its lower limit at an instant later than the present instant: one which, therefore, has not yet come to pass. This is clearly implied by the following considerations.

Let us consider time as a straight line and let us establish a one-to-one correspondence between a certain interval of time and the segment (0, 1) of that line. Let us assume that the present instant corresponds to point 0, that a certain future fact occurs at instant (corresponding to point 1), and that the causes of this fact occur at instants determined by real numbers greater than $\frac{1}{2}$. This sequence of causes is infinite and

has no beginning, that is, no first cause. For this first cause would have to take place at the instant corresponding to the smallest real number greater than $\frac{1}{2}$, and no such real number exists; not even the smallest rational number greater than $\frac{1}{2}$ exists. In the set of real numbers, and similarly in the ordered set of rational numbers, there are no two numbers succeeding each other immediately, that is, being the immediate predecessor and successor of each other; between any two numbers there is always another one, and consequently there are infinitely many numbers between any two of them. Similarly there are no two instants succeeding each other immediately, that is, being the immediate predecessor and successor of each other; between any two instants there is another one, and consequently there are infinitely many instants between any two of them. In accordance with the principle of causality, every fact of the sequence under consideration has its cause in some earlier fact. Although it has a lower limit at instant $\frac{1}{2}$, which is later than the present instant 0 and has not yet been attained, the sequence is infinite. Furthermore, this sequence cannot exceed its lower limit and therefore cannot reach back to the present instant.

This reasoning shows that there might exist infinite causal sequences which have not yet begun and which belong entirely to the future. This view is not only logically possible but also seems to be more prudent than the belief that each, even the smallest, future event has its causes acting from the beginning of the universe. I do not doubt at all that some future facts have their causes already in existence today and have had them from eternity. By means of observations and the laws of motion of the heavenly bodies astronomers predict eclipses of the moon and sun with great precision many years in advance. But nobody is able to predict today that a fly which does not yet exist will buzz into my ear at noon on 7 September of next year. The belief that this future behaviour of that future fly has its causes already today and has had them from all eternity seems to be a fantasy rather than a proposition supported by even a shadow of scientific validation.

Therefore the argument based on the principle of causality falls to the ground. One can be strongly convinced that nothing happens without cause, and that every fact has its cause in some earlier fact, without being a determinist. There remains to be considered the argument based on the principle of the excluded middle.

8. Although the argument based on the principle of the excluded middle is independent of that derived from the principle of causality, the former indeed becomes fully intelligible if every fact has its causes existing from all eternity. I shall explain what I mean by an example taken from ordinary life.*) Let us assume that John will be at home tomorrow noon. If the causes of all facts exist from all eternity, we should recognize that at the present instant there exists the cause of John's presence at home tomorrow noon. Therefore it is true, i.e. it is the case at the present instant, that John will be at home tomorrow noon. The somewhat confused expression "it is the case at instant t that p", in which "p" stands for sentences about future events, which I have previously been unable to elucidate, now becomes perfectly intelligible. It is the case at the present instant that "John will be at home tomorrow noon" implies first that at the present instant there exists a fact which is the cause of John's presence at home tomorrow noon, and secondly that this future effect is as much comprehended in that cause as a conclusion is included in its premisses. The cause of the future fact, which the sentence "p" states and which exists at instant t, is an actual correlate of the sentence "it is the case at instant t that p".

Should we assume that John will not be at home tomorrow noon, we can follow the same course of reasoning. If we recognize that the causes of every fact exist from all eternity, we must also accept the fact that the cause of John's absence from home tomorrow noon exists already at the present instant. Therefore the sentence "it is true, i.e. it is the case at the present instant, that John will not be at home tomorrow noon" has its actual correlate in the cause of the stated fact, and this cause exists at present.

As John will or will not be at home tomorrow noon, there exists either the cause of his presence at or of his absence from home tomorrow noon, provided that the causes of all facts exist from all eternity. Therefore, either it is true at the present instant that John will be at home tomorrow noon or it is true at the present instant that John will not be at home tomorrow noon. The argument based on the principle of the excluded middle has additional support in the argument derived from the principle of causality.

*) Łukasiewicz repeats this reasoning in his paper "Philosophical Remarks on Many-Valued Systems of Propositional Logic" (pp. 153–178 of this book).

9. However, the second of these arguments has proved itself to be invalid. In accordance with the preceding investigations, we may assume that at the present instant there exists as yet neither the cause of John's presence nor the cause of John's absence from home tomorrow noon. Thus it might happen that the infinite sequence of causes, which bring about John's presence or absence from home tomorrow noon, has not yet begun and lies entirely in the future. To put it colloquially, we can say that the question whether John will or will not be at home tomorrow noon is not yet decided either way. How should we argue in this case?

We might adopt the following course. The sentence "it is true at the present instant t that John will be at home tomorrow noon" has no actual correlate, for the cause of this fact does not exist at instant t; therefore nothing compels us to recognize this sentence as true. Thus it might happen that John would not be at home tomorrow noon. In the same way the sentence "it is true at the present instant t that John will not be at home tomorrow noon" has no real correlate, for the cause of this fact does not exist at instant t; again, nothing compels us to recognize this sentence as true. Thus it might happen that John would be at home tomorrow noon. We may, therefore, reject both these sentences as false and accept their denials "it is not true at instant t that John will be at home tomorrow noon", and "it is not true at instant t that John will not be at home tomorrow noon". The previously established conditional (e), "if John will be at home tomorrow noon, then it is true at instant t that John will be at home tomorrow noon" becomes invalid. For its antecedent turns out to be true if John is at home tomorrow noon, and its consequent becomes false if we choose an instant t, earlier than tomorrow noon, at which the cause of John's presence at home tomorrow noon does not yet exist. But with conditional (e) the thesis of determinism, "if A is b at instant t, it is true at every instant earlier than t that A is b at instant t" also becomes invalid; for we can substitute values for variables A, b, and t such that the antecedent of this thesis becomes true and the consequent false.

If on the assumption that a certain future fact is not yet decided either way the thesis of determinism becomes false, the deduction of this thesis from the principle of the excluded middle must involve an error. Indeed, if we reject as false the sentence "it is true at instant *t* that John will be at home tomorrow noon" as well as the sentence "it is true at instant *t* that John will not be at home tomorrow noon", we must also reject alternative (a) which is composed of these sentences as its arguments and which has been the starting-point of the deduction. An alternative both of whose arguments are false is itself false. So also conditional (d), obtained by transforming premiss (a), "if it is not true at instant *t* that John will not be at home tomorrow noon, then it is true at instant *t* that John will be at home tomorrow noon", turns out to be false, for we accept its antecedent and reject its consequent. It is no wonder that the inference produces a false conclusion if one of its premisses and one of its intervening theorems are false.

It should be pointed out that the rejection of alternative (a) is not a transgression of the principle of the excluded middle; for its arguments do not contradict each other. Only the sentences "John will be at home tomorrow noon" and "John will not be at home tomorrow noon" are contradictory, and the alternative composed of these sentences, "either John will be at home tomorrow noon or John will not be at home tomorrow noon", must be true in accordance with the principle of the excluded middle. But the sentences "it is true at instant *t* that John will be at home tomorrow noon" and "it is true at instant *t* that John will not be at home tomorrow noon" are not contradictory, for the one is not the denial of the other, and their presentation as alternatives need not be true. Premiss (a) has been deduced from the principle of the excluded middle on the basis of purely intuitive investigations and not by applying a logical principle. However, intuitive investigations may be fallacious, and they seem to have deceived us in this case.

10. Although this solution appears to be logically valid, I do not regard it as entirely satisfactory, for it does not satisfy all my intuitions. I believe that there is a difference between the non-acceptance of the sentence "it is true at the present instant that John will be at home tomorrow noon" because John's presence at or absence from home tomorrow is not yet decided, and the non-acceptance of this sentence because the cause of his absence tomorrow already exists at the present instant. I think that solely in the latter case have we the right to reject the sentence in question and say, "it is not true at the present instant that John will be at home tomorrow noon". In the former case we can neither accept nor reject the sentence but should suspend our judgement.

This attitude finds its justification both in life and in colloquial speech. If John's presence at or absence from home tomorrow noon is not yet decided, we say, "*it is possible* that John will be at home tomorrow noon, but also *it is possible* that John will not be at home tomorrow noon". On the other hand, if the cause of John's absence from home tomorrow noon exists already at the present instant, we say, provided that we know this cause, "*it is not possible* that John will be at home tomorrow noon". On the assumption of John's presence at or absence from home tomorrow noon not yet being decided, the sentence "it is true at the present instant that John will be at home tomorrow noon" can be neither accepted nor rejected, that is, we cannot consider it either true or false. Consequently also the denial of this sentence, "it is not true at the present instant that John will be at home tomorrow noon", can be neither accepted nor rejected, i.e. we cannot consider it as either true or false. The previous reasoning, which consisted in the rejection of the sentence under discussion and in the acceptance of its denial, is now inapplicable. In particular conditional (d), which was previously rejected, for its antecedent was accepted and its consequent rejected, need not now be rejected, for it is not true any longer that its antecedent is accepted and its consequent rejected. Furthermore, since conditional (d) together with premiss (c), which does not seem to involve any doubts whatsoever, suffice to validate the thesis of determinism, it appears as though Aristotle's argument regains its persuasive power.

11. However, this is not the case. I think that only now do we achieve a solution which is in agreement both with our intuitions and with the views of Aristotle himself. For Aristotle formulated his argument in support of determinism solely for the purpose of its subsequent rejection as invalid. In the famous chapter 9 of *De Interpretatione* Aristotle seems to have reached the conclusion that the alternative "either there will be a sea battle tomorrow or there will not be a sea battle tomorrow" is already true and necessary today, but it is neither true today that "there will be a sea battle tomorrow" nor that "there will not be a sea battle tomorrow". These sentences concern future contingent events and as such they are neither true nor false today. This was the interpretation of Aristotle given by the Stoics, who, being determinists, disputed his view, and by the Epicureans, who defended indeterminism and Aristotle.

Aristotle's reasoning does not undermine so much the principle of the excluded middle as one of the basic principles of our entire logic, which he himself was the first to state, namely, that *every proposition is either true or false*. That is, it can assume one and only one of two truth-values: truth or falsity. I call this principle *the principle of bivalence*. In ancient times this principle was emphatically defended by the Stoics and opposed by the Epicureans, both parties being fully aware of the issues involved. Because it lies at the very foundations of logic, the principle under discussion cannot be proved. One can only believe it, and he alone who considers it self-evident believes it. To me, personally, the principle of bivalence does not appear to be self-evident. Therefore I am entitled not to recognize it, and to accept the view that besides truth and falsehood there exist other truth-values, including at least one more, the third truth-value.

What is this third truth-value? I have no suitable name for it.*). But after the preceding explanations it should not be difficult to understand what I have in mind. I maintain that there are propositions which are neither true nor false but *indeterminate*. All sentences about future facts which are not yet decided belong to this category. Such sentences are neither true at the present moment, for they have no real correlate, nor are they false, for their denials too have no real correlate. If we make use of philosophical terminology which is not particularly clear, we could say that ontologically there corresponds to these sentences neither being nor non-being but possibility. Indeterminate sentences, which ontologically have possibility as their correlate, take the third truth-value.

If this third value is introduced into logic we change its very foundations. A trivalent system of logic, whose first outline I was able to give in 1920**), differs from ordinary bivalent logic, the only one known so far, as much as non-Euclidean systems of geometry differ from Euclidean geometry. In spite of this, trivalent logic is as consistent and free from contradictions as is bivalent logic. Whatever form, when worked out in detail, this new logic assumes, the thesis of determinism will be no part of it. For in the conditional in terms of which this thesis is expressed, "if A is b at instant t, then it is true at every instant earlier than t that A is b at instant t", we can assign such values to variables "A", "b", and "t" that its antecedent changes into a true sentence and its consequent into an indeterminate one, that is, into a sentence having the third truth-value. This always happens when the cause of the fact that A is b at a future instant t does not yet exist today. A conditional whose antecedent is true and consequent indeterminate cannot be accepted as true; for truth can imply only truth. The logical argument which seems to support determinism falls decisively.

*) In "Philosophical Remarks..." Łukasiewicz uses the term "possibility".

**) The first mention of the three-valued logic was made earlier, in the "Farewell Lecture..." of 1918 (pp. 84–86 of this book).

12. I am near the end of my investigations. In my view, the age-old arguments in support of determinism do not withstand the test of critical examination. This does not at all imply that determinism is a false view; the falsehood of the arguments does not demonstrate the falsehood of the thesis. Taking advantage of my preceding critical examination, I should like to state only one thing, namely that determinism is not a view better justified than indeterminism.

Therefore, without exposing myself to the charge of thoughtlessness, I may declare myself for indeterminism. I may assume that not the whole future is determined in advance. If there are causal chains commencing only in the future, then only some future facts and events, those nearest to the present time, are causally determined at the present instant. On the basis of present knowledge even an omniscient mind could predict fewer and fewer facts the deeper into the future it tried to reach: the only thing actually determined in the ever broader framework within which facts occur, and within which there is more and more room for possibility. The universal drama is not a picture completed from eternity; the further away we move from the parts of the film which are being shown just now, the more gaps and blanks the picture includes. It is well that it should be so. We may believe that we are not merely passive spectators of the drama but also its active participants. Among the contingencies that await us we can choose the better course and avoid the worse. We can ourselves somehow shape the future of the world in accordance with our designs. I do not know how this is possible, but I believe that it is.

We should not treat the past differently from the future. If the only part of the future that is now real is that which is causally determined by the present instant, and if causal chains commencing in the future belong to the realm of possibility, then only those parts of the past are at present real which still continue to act by their effects today. Facts whose effects have disappeared altogether, and which even an omniscient mind could not infer from those now occurring, belong to the realm of possibility. One cannot say about them that they took place, but only that they were *possible*. It is well that it should be so. There are hard moments of suffering and still harder ones of guilt in everyone's life. We should be glad to be able to erase them not only from our memory but also from existence. We may believe that when all the effects of those fateful moments are exhausted, even should that happen only *after* our death, then their causes too will be effaced from the world of actuality and pass into the realm of possibility. Time calms our cares and brings us forgiveness.

A NUMERICAL INTERPRETATION OF THE THEORY OF PROPOSITIONS *)

Refers to Chapters *1 to *5, Vol. 1. of *Principia Mathematica*. The authors of that work, Whitehead and Russell, present there the "theory of deduction"; they do so by introducing variables connected by symbols of operations and logical relations, and by using that notation to formulate 192 logical laws (axioms and theorems) marked with the symbol of assertion. Professor Łukasiewicz offered a numerical interpretation of those variables and their combinations, and of logical laws. The principles of that interpretation are as follows:

I. The variables $p, q, r, s,$ stand for any real numbers in the interval 0-1, including the limiting values of that interval.

II. The formula "$p \supset q$" equals the number 1 if $p \leqslant q$, i.e.,

$$p \supset q \,.\!=.\, 1 \qquad \text{for} \quad p \leqslant q.$$

III. $p \supset q \,.\!=.\, 1-p+q$ for $p \geqslant q$.

Thus the formula "$p \supset q$" always denotes a number in the interval 0-1. Further the following definitions are adopted

D1 $\sim p \,.\!=.\, p \supset 0$ (where 0 is a number).
D2 $p \vee q \,.\!=.\, p \supset q \,.\supset q.$
D3 $p \,.\, q \,.\!=.\, \sim(\sim p \vee \sim q).$
D4 $p \equiv q \,.\!=.\, p \supset q \,.\, q \supset p.$

It can easily be proved, on the strength of these definitions and the principles I–III above, that the number $\sim p$ equals the number $1-p$, and that the logical sum of two different numbers always equals the greater number of the two, and that the logical product of two different numbers always equals the lesser of the two. Thus every logical function

*) Report on Prof. Jan Łukasiewicz's lecture delivered at the 232nd meeting of the Polish Philosophical Society in Lwów on October 14, 1922. The report was published in *Ruch Filozoficzny* 7 (1923), pp. 92–93.

is a number in the interval 0-1. The logical laws marked by the symbol of assertion also are numbers in that interval on the strength of the following principle:

IV. Every logical law is equal to the least of the numbers obtained by substituting numerical values for the variables occurring in that law.

Professor Łukasiewicz stated that in this numerical interpretation out of the 192 laws included in the part of *Principia Mathematica* under consideration 60 laws take on the numerical value $^1/_2$, three take on the numerical value 0, and the rest take on the numerical value 1. The application of this interpretation is twofold: 1) It can be demonstrated that if those verbal rules, or directives, which are accepted by the authors of *Principia Mathematica* (the rule of deduction and the rule of substitution) are adopted, then no set of the logical laws that have the numerical value 1 can yield any law that would have a lesser numerical value. This would show that some logical laws are independent of the others. 2) If 0 is interpreted as falsehood, 1 as truth, and other numbers in the interval 0-1 as the degrees of probability corresponding to various possibilities, a many-valued logic is obtained, which is an expansion of three-valued logic and differs from the latter in certain details.

Speakers in the discussion were Messrs Ajdukiewicz, Bad, Ingarden, Kleiner, Łomnicki, Steinhaus, Weyberg and Łukasiewicz.

INVESTIGATIONS INTO THE SENTENTIAL CALCULUS *)

In the course of the years 1920–1930 investigations were carried out in Warsaw belonging to that part of metamathematics—or better metalogic—which has as its field of study the simplest deductive discipline, namely the sentential calculus. These investigations were initiated by Łukasiewicz; the first results originated both with him and with Tarski. In the seminar for mathematical logic which was conducted by Łukasiewicz in the University of Warsaw beginning in 1926, most of the results stated below of Lindenbaum, Sobociński, and Wajsberg were found and discussed. The systematization of all the results and the clarification of the concepts concerned was the work of Tarski.

In the present communication the most important results of these investigations—for the most part not previously published—are collected together.**)

1. General concepts

It is our intention to refer our considerations to the conceptual apparatus which was developed in the preceding article (see "On Some Fundamental Concepts of Mathematics" published as paper III in A. Tarski, *op. cit.*, pp. 30–37). For this purpose we wish first to define

*) [Note from A. Tarski, *Logic, Semantics, Metamathematics. Papers from 1923 to 1938*, Oxford, 1956, The Clarendon Press : *Bibliographical Note*. This joint communication of J. Łukasiewicz and A. Tarski was presented (by Łukasiewicz) to the Warsaw Scientific Society on 27 March 1930; it was published under the title "Untersuchungen über den Aussagenkalkül" in *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie* 23 (1930), cl. iii, pp. 39–50]. The following text has been reprinted in the 1961 edition *Z zagadnień logiki i filozofii*; in this book it is published as a reprint from its first English version included in the above mentioned edition of Tarski's papers (some bibliographical references, however, have been expanded to become comprehensible to the reader deprived of the larger context of Tarski's book).

**) To avoid misunderstandings it should be stated that the present article does not contain results discovered by both the authors jointly, but is a compilation of theorems and concepts belonging to five different persons. Each theorem and concept the notion of a (*meaningful*) *sentence* and that of a *consequence of a set of sentences* with respect to the sentential calculus.

DEFINITION 1. *The set S of all sentences is the intersection of all those sets which contain all sentential variables (elementary sentences) and are closed under the operations of forming implications and negations.*[1])

The concepts of *sentential variable*, of *implication* and of *negation* cannot be explained further; they must rather be regarded as primitive concepts of the metatheory of sentential calculus (i.e. of that part of metamathematics in which sentential calculus is investigated). The fundamental properties of these concepts, which suffice for the construction of the part of metamathematics with which we are here concerned, can be expressed in a series of simple sentences (axioms) which need not now be stated. Usually the letters "p", "q", "r", etc., are used as sentential variables. In order to express in symbols the sentences "p implies q" (or "if p, then q") and "it is not the case that p", Łukasiewicz employs the formulas "Cpq" and "Np" respectively.[2]) With this notation the use of such technical signs as parentheses, dots, etc., is rendered unnecessary. We shall encounter several examples of sentences written in this symbolism in subsequent sections. In addition to the formation of implications and negations other similar operations are commonly used in the sentential calculus. But as these are all definable by means of the two mentioned above they will not be considered here.

is ascribed to its respective originator. Theorem 3, for instance, is not a theorem of Łukasiewicz and Tarski, but a theorem of Lindenbaum. Nevertheless, some scholars mistakenly referred to both authors, Łukasiewicz and Tarski, the many-valued systems of logic ascribed in the article to Łukasiewicz alone. In spite of a correction which appeared in 1933 in the *Journal of Philosophy*, vol. 30, p. 364, this mistake persists till today. It clearly follows from § 3 and notes of this article that the idea of a logic different from the ordinary system called by Łukasiewicz the *two-valued logic*, and the construction of many-valued systems of logic described here, are entirely due to Łukasiewicz alone and should not be referred to Łukasiewicz and Tarski.

[1]) A set—according to the usual terminology of abstract set theory—is said to be *closed under given operations* if as the result of carrying out these operations on elements of the set in question one always obtains elements of this same set.

[2]) Cf. J. Łukasiewicz, "O znaczeniu i potrzebach logiki matematycznej" (On the significance and needs of mathematical logic), *Nauka Polska* 10(1929), pp. 604–620, p. 610 note, and Łukasiewicz *Elementy logiki matematycznej*, p. 40, 1st edition, Warsaw, 1929. [An English translation—*Elements of Mathematical Logic*—was published in 1963 and reprinted in 1966.]

The symbol "Cpq", which in the sentential calculus *expresses* the implication between "p" and "q", is to be clearly distinguished from the metamathematical symbol "$c(x, y)$", which *denotes* the implication with the antecedent x and the consequent y. The expression "Cpq" is a *sentence* (in the sentential calculus) whilst the expression "$c(x,y)$" is the *name of a sentence* (in the metasentential calculus). An analogous remark applies to the symbolic expressions "Np" and "$n(x)$".

The consequences of a set of sentences are formed with the help of two operations, that of *substitution* and that of *detachment* (the modus ponens scheme of inference). The intuitive meaning of the first operation is clear; we shall not, therefore, discuss its character more closely. The second operation consists in obtaining the sentence y from the sentences x and $z = c(x, y)$.

We are now in a position to explain the concept of consequence:

DEFINITION 2. *The set of consequences $Cn(X)$ of the set X of sentences is the intersection of all those sets which include the set $X \subseteq S$ and are closed under the operations of substitution and detachment.*

From this we obtain:

THEOREM 1. *The concepts S and $Cn(X)$ satisfy the axioms* 1–5 *given in article* III.[3])

We are especially interested in those parts X of the set S which form *deductive systems*, i.e. which satisfy the formula $Cn(X) = X$. Two methods of constructing such systems are available to us. In the first, the so-called *axiomatic method*, an arbitrary, usually finite, set X of sentences—an *axiom system*—is given, and the set $Cn(X)$, i.e. the smallest deductive system over X, is formed. The second method, which can best be called the *matrix method*, depends upon the following definitions of Tarski:[4])

[3]) See article III, p. 31, in A. Tarski, *op. cit.*

[4]) The origin of this method is to be sought in the well-known verification procedure for the usual two-valued sentential calculus (see below Def. 5), which was used by Peirce ("On the Algebra of Logic", *Am. Journ. of Math.* 7(1885), p. 191) and Schröder. This was thoroughly treated in J. Łukasiewicz, ["Two-Valued Logic", pp. 89–109 of this book]. Łukasiewicz was also the first to define by means of a matrix a system of the sentential calculus different from the usual one, namely his three-valued system (see below, p. 126, note **). This he did in the year 1920. Many-valued systems, defined by matrices, were also known to Post (see E. L. Post, "Introduction

DEFINITION 3. *A (logical) matrix is an ordered quadruple* $\mathfrak{M} = [A, B, f, g]$ *which consists of two disjoint sets (with elements of any kind whatever) A and B, a function f of two variables and a function g of one variable, where the two functions are defined for all elements of the set* $A+B$ *and take as values elements of* $A+B$ *exclusively.*

The matrix $\mathfrak{M} = [A, B, f, g]$ *is called normal if the formulas* $x \in B$ *and* $y \in A$ *always imply* $f(x, y) \in A$.

DEFINITION 4. *The function h is called a value function of the matrix* $\mathfrak{M} = [A, B, f, g]$ *if it satisfies the following conditions: (1) the function h is defined for every* $x \in S$; *(2) if x is a sentential variable, then* $h(x) \in A+B$; *(3) if* $x \in S$ *and* $y \in S$, *then*

$$h(c(x, y)) = f(h(x), h(y));$$

(4) if $x \in S$ *then* $h(n(x)) = g(h(x))$.

The sentence x is satisfied (or verified) by the matrix $\mathfrak{M} = [A, B, f, g]$, *in symbols* $x \in \mathfrak{E}(\mathfrak{M})$, *if the formula* $h(x) \in B$ *holds for every value function h of this matrix.*

The elements of the set B are, following Bernays,[5] called *designated* elements.

In order to construct a system of the sentential calculus with the help of the matrix method a matrix $\mathfrak{M}$ (usually normal) is set up and the set $\mathfrak{E}(\mathfrak{M})$ of all those sentences which are satisfied by this matrix is considered. This procedure rests upon the following easily provable theorem:

THEOREM 2. *If* $\mathfrak{M}$ *is a normal matrix, then* $\mathfrak{E}(\mathfrak{M}) \in \mathfrak{S}$.

If the set $\mathfrak{E}(\mathfrak{M})$ forms a system (as it always will, according to Th. 2, if the matrix $\mathfrak{M}$ is normal), it is called the *system generated by the matrix* $\mathfrak{M}$.

to a General Theory of Elementary Propositions", *Am. Journ. of Math.* 43(1921), pp. 180 ff.). The method used by P. Bernays ("Axiomatische Untersuchung des Aussagenkalküls der *Principia Mathematica*", *Math. Z.* 25 (1926), pp. 305–320) for the proof of his theorems on independence also rests on matrix formation. The view of matrix formation as a general method of constructing systems is due to Tarski.

[5]) See P. Bernays, "Axiomatische Untersuchung des Aussagenkalküls der *Principia Mathematica*", *Math. Z.* 25(1926), p. 316.

The following converse of Th. 2, which was proved by Lindenbaum, makes evident the generality of the matrix method described here:

THEOREM 3. *For every system* $X \in \mathfrak{S}$ *there exists a normal matrix* $\mathfrak{M} = [A, B, f, g]$, *with an at most denumerable set* $A+B$, *which satisfies the formula* $X = \mathfrak{E}(\mathfrak{M})$. *)

Each of the two methods has its advantages and disadvantages. Systems constructed by means of the axiomatic method are easier to investigate regarding their axiomatizability, but systems generated by matrices are easier to test for completeness and consistency. In particular the following evident theorem holds:

THEOREM 4. *If* $\mathfrak{M} = [A, B, f, g]$ *is a normal matrix and* $A \neq 0$, *then* $\mathfrak{E}(\mathfrak{M}) \in \mathfrak{W}$.

2. The ordinary (two-valued) system of the sentential calculus

In the first place we consider the most important of the systems of the sentential calculus, namely the well-known *ordinary system* (also called by Łukasiewicz[6]) the two-valued system), which is here denoted by "L".

Using the matrix method, the system L may be defined in the following way:

DEFINITION 5. *The ordinary system L of the sentential calculus is the set of all sentences which are satisfied by the matrix* $\mathfrak{M} = [A, B, f, g]$ *where* $A = \{0\}$, $B = \{1\}$[7]) *and the functions f and g are defined by the formulas:* $f(0, 0) = f(0, 1) = f(1, 1) = 1$, $f(1, 0) = 0$, $g(0) = 1$, $g(1) = 0$.

From this definition it follows easily that the system L is consistent and complete:

THEOREM 5. $L \in \mathfrak{S}.\mathfrak{W}.\mathfrak{V}$.

[6]) See note 4, p. 133.

[7]) The set having a as its only element is denoted by $\{a\}$.

*) A proof of this theorem has recently been published in J. Łoś, "O matrycach logicznych" ("On logical matrices", in Polish), *Travaux de la Société des Sciences et des Lettres de Wrocław*, Ser. B, No. 19, Wrocław (1949), 42 pp. See also H. Hermes, "Zur Theorie der aussagenlogischen Matrizen", *Math. Z.* 53(1951), pp. 414–418.

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The system L can also be defined by means of the axiomatic method. The first axiom system of the sentential calculus was given by C. Frege.[8]) Other axiom systems have been given by Whitehead and Russell[9]) and by Hilbert.[10]) Of the systems at present known the simplest is that of Łukasiewicz; he also has proved in an elementary manner the equivalence of the two definitions of L.[11]) His result may be stated thus:

THEOREM 6. *Let X be the set consisting of the three sentences:*

"CCpqCCqrCpr", "CCNppp", "CpCpNpq";

then $X \in \mathfrak{Ax}(L)$. Consequently L is axiomatizable, $L \in \mathfrak{A}$.

According to a method for investigating the independence of a set X of sentences developed by Bernays and Łukasiewicz,[12]) a normal matrix $\mathfrak{M}_y$ is constructed for every sentence $y \in X$ which verifies all sentences of the set X with the exception of y.

[8]) *Begriffsschrift*, Halle a/S (1879), pp. 25–30. Frege's system is based upon the following six axioms: "*CpCqp*", "*CCpCqrCCpqCpr*", "*CCpCqrCqCpr*", "*CCpqCNqNp*" "*CNNpp*", "*CpNNp*". Łukasiewicz has shown that in this system the third axiom is superfluous since it can be derived from the preceding two axioms, and that the last three axioms can be replaced by the single sentence "*CCNpNqCpq*".

[9]) *Principia Mathematica*, 1 (1925), p. 91.

[10]) See D. Hilbert, "Die logischen Grundlagen der Mathematik", *Math. Ann*, 88 (1923), p. 153.

[11]) Cf. J. Łukasiewicz, *Elements of Mathematical Logic*, pp. 45 and 121 ff. The proof of the equivalence of the two definitions of L amounts to the same thing as proving the completeness of the system L when defined by means of the axiomatic method. The first proof of completeness of this kind is found in Post, *op. cit.*.

[12]) Bernays has published, in his article (note 5), which dates from the year 1926 (but according to the author's statement contains results from his unpublished *Habilitationsschrift* presented in the year 1918), a method based upon matrix formations, which enables us to investigate the independence of given sets of sentences. The method given by Bernays was known before its publication to Łukasiewicz who, independently of Bernays, and following a suggestion of Tarski (cf. "On the Primitive Term of Logistic", published as paper I in A. Tarski, *op. cit.*, pp. 8–14), first applied his many-valued systems, defined by means of matrices, to the proofs of independence, and subsequently discovered the general method. On the basis of this method Łukasiewicz had already in 1924 investigated the independence of the axiom systems given by Whitehead and Russell and by Hilbert, and had shown that neither of them is independent. These results (without proof) are contained in the following note by Łukasiewicz: "Démonstration de la compatibilité des axiomes de la théorie de la déduction", *Ann. Soc. Pol. Math.* 3 (1925), p. 149.

With the help of this method Łukasiewicz proved that in contrast to the previously mentioned axiom system the following theorem holds:

THEOREM 7. *The set X of sentences given in Th. 6 is independent; consequently X is a basis of L, $X \in \mathfrak{B}(L)$.*

Tarski developed another *structural* method for the study of independence. Although less general than the method of matrix formation, this can be successfully in some cases.

The following general theorem is due to Tarski:*)

THEOREM 8. *The system L, as well as every axiomatizable system of the sentential calculus which contains the sentences: "CpCqp" and "CpCqCCpCqrr" (or "CpCqCCpCqrCsr"), possesses a basis consisting of a single sentence.*[13])

The proof of this theorem enables us in particular to give effectively a basis of the system L which contains a single element.[14])**) Łukasiewicz has simplified Tarski's proof and, with the help of previous work of B. Sobociński, has established the following:

THEOREM 9. *The set which consists of the single sentence z:*

"CCCpCqpCCCNrCsNtCCrCsuCCtsCtuvCwv"

is a basis of the system L, i.e. $\{z\} \in \mathfrak{B}(L)$.

[13]) An analogous, but quite trivial, theorem applies to all axiomatizable systems of those deductive disciplines which already presuppose the sentential calculus and satisfy not only Axs. 1–5, but also Axs. 6*–10* of "On Some Fundamental Concepts of Mathematics" in A. Tarski, *op. cit.*

[14]) This result was obtained by Tarski in the year 1925; cf. S. Leśniewski, "Grundzüge eines neuen Systems der Grundlagen der Mathematik", *Fund. Math.* 14 (1929), p. 58. An axiom system of the ordinary sentential calculus consisting of a single axiom was set up by Nicod in the year 1917 (see J. Nicod, "A Reduction in the Number of the Primitive Propositions of Logic", *Proc. Cambridge Phil. Soc.* 19 (1917), pp. 32–41). The axiom of Nicod is constructed with the Sheffer disjunction "$p|q$" as the only primitive term, and the rule of detachment formulated by Nicod in connexion with this term is stronger than the rule of detachment for implication. This facilitated the solution of the problem.

*) Compare in this connexion a recent paper of K. Schröter "Deduktive abgeschlossene Mengen ohne Basis", *Mathematische Nachrichten* 7 (1952), in particular pp. 294 ff.

**) The axiom originally found by Tarski is explicitly formulated in the article by B. Sobociński, "Z badań nad teorią dedukcji" (Some investigations upon the theory of deduction), *Przegląd Filozoficzny* 35 (1932), pp. 172–193, in particular p. 189. It consists of 53 letters.

This sentence z which has 33 letters, is the shortest sentence known at the present time which suffices as the only axiom for developing the system L. The sentence z is not organic with respect to the system L. For a sentence $y \in X$ is said to be *organic with respect to a system* X if no (meaningful) part of y is an element of X (the term "organic" comes from S. Leśniewski, and we owe the definition of organic sentence to M. Wajsberg). The sentence z is not organic with respect to L because it contains parts, e.g. "*CpCqp*", which are elements of L. Sobociński has given an organic axiom for the system L which contains 47 letters.*)

The following theorem is a generalization of Th. 8:

THEOREM 10. *The system L, as well as every axiomatizable system of the sentential calculus which contains the sentences "CpCqp" and "CpCqCCpCqrr", possesses for every natural number m a basis containing exactly m elements.*

For the system L Sobociński has effectively proved this theorem; the generalization to other systems is due to Tarski.**)

In contrast to this property of the system L, Tarski has effectively shown that:

THEOREM 11. *For every natural number m, systems of the sentential calculus exist every basis of which contains exactly m elements.*

The following considerations of Tarski concern the special case of this theorem when $m = 1$ (Def. 6 and Ths. 12–14).

DEFINITION 6. *The sentence x is called indecomposable if $x \in S$ and if every basis of the system $Cn(\{x\})$ consists of only one sentence (i.e. if no independent set of sentences containing more than one element is equivalent to the set $\{x\}$).*

If this condition is not satisfied then the sentence x is said to be decomposable.

It is found that almost all known sentences of the system L are indecomposable; in particular:

THEOREM 12. *The sentences*

"*Cpp*", "*CpCCpqq*", "*CCCpqpp*", "*CCCpqqCCqpp*",
"*CCpqCCqrCpr*", "*CCqrCCpqCpr*"

are indecomposable.

THEOREM 13. *If $x \in S$, $y \in S$, and $z \in S$ then the sentences $n(x)$, $c(n(x), y)$, $c(c(n(x), y), z)$, $c(x, c(n(y), z))$ are indecomposable; in particular, this holds for the sentences:*

"*CNNpp*", "*CpNNp*", "*CNpCpq*", "*CpCNpq*",
"*CCNppp*", "*CCpNpNp*".

From Ths. 12 and 13 it results that the set of sentences given in Th. 6 consists exclusively of indecomposable sentences.

On the other hand the following theorem has been proved:

THEOREM 14. *The sentences*

"*CpCqp*", "*CCCpqrCqr*", *and* "*CCpCqrCqCpr*"

*are decomposable.**)

*) The results discussed in the last paragraph of the text were improved after the original publication of this article. In fact Łukasiewicz found in 1932 a single non-organic axiom consisting of 29 letters: see Sobociński "Z badań nad teorią dedukcji" (Some researches on the theory of deduction), *Przegląd Filozoficzny* 35 (1932), pp. 171–193, especially pp. 181 ff. In 1936 on the ground of a result of Sobociński he published without proof a single organic axiom of 23 letters in J. Łukasiewicz "Logistic and Philosophy", pp. 218–235 of this book, p. 224, note 10. The shortest hitherto known single organic axiom consisting of 21 letters was found in 1952 by C. A. Meredith; see his article, "Single axioms for the systems (C, N), (C, O), and (A, N) of the two-valued propositional calculus", *The Journal of Computing Systems* 1 (1953), pp. 155–164.

**) See B. Sobociński, "Z badań nad teorią dedukcji" ("Some researches on the theory of deduction"), *Przegląd Filozoficzny* 35 (1932), pp. 178 ff.

*) The following remarks may help the reader to reconstruct the proofs of Ths. 12–14:

(i) Let $x =$ "*Cpp*". It can easily be shown that the system $Cn(\{x\})$ consists of all those and only those sentences which can be obtained from x by substitution; more generally, if Y is any subset of $Cn(\{x\})$, then $Cn(Y)$ consists of those and only those sentences which are obtainable from sentences of Y by substitution. Hence we conclude without difficulty that every independent set of sentences which is equivalent to $\{x\}$ consists of just one sentence, in fact, of a sentence $c(v, v)$, where v is an arbitrary variable.

By means of a similar argument many of the sentences mentioned in Ths. 12 and 13 can be proved to be indecomposable.

(ii) Let $x =$ "*CpCqp*", $y =$ "*CCpCqpCpCqp*" and $z = c(y, x)$. Clearly the set $\{x\}$ is equivalent to the set $\{y, z\}$. Also it can easily be shown that the set $\{y, z\}$ is independent. Hence the sentence x is decomposable.

A noteworthy theorem on axiom systems of L has been proved by Wajsberg:

THEOREM 15. *In every basis (and in general in every axiom system) of the system L, as well as of every sub-system of L which contains the sentence "CpCqCrp", at least three distinct sentential variables occur.*[15]) *In other words, if X is the set of all those sentences of the system L in which at most two distinct variables occur, then $L - Cn(X) \neq 0$; in particular, the sentence "CpCqCrp" belongs to L but not to $Cn(X)$.**)

3. Many-valued systems of the sentential calculus

In addition to the ordinary system of the sentential calculus there are many other systems of this calculus which are worthy of investigation. This was first pointed out by Łukasiewicz who has also singled out a specially important class of such systems.[16]) The systems founded by Łukasiewicz are here called *n-valued systems* of the sentential calculus and denoted by the symbol "L_n" (where either n is a natural number or $n = \aleph_0$). These systems can be defined by means of the matrix method in the following way:

DEFINITION 7. *The n-valued system L_n of the sentential calculus (where n is a natural number or $n = \aleph_0$) is the set of all sentences which are satisfied by the matrix $\mathfrak{M} = [A, B, f, g]$ where, in the case $n = 1$ the set A is null, in the case $1 < n < \aleph_0$ A consists of all fractions of the form $k/(n-1)$ for $0 \leqslant k < n-1$, and in the case $n = \aleph_0$ it consists of all fractions k/l for $0 \leqslant k < l$; further the set B is equal to $\{1\}$ and the functions f and g are defined by the formulas: $f(x, y) = \min(1, 1-x+y)$, $g(x) = 1-x$.*

As Lindenbaum has shown, the system $L_{\aleph_0}$ is not changed if, in the definition of this system, the set A of all proper fractions is replaced by another infinite sub-set of the interval $\langle 0, 1 \rangle$:

THEOREM 16. *Let $\mathfrak{M} = [A, B, f, g]$ be a matrix where $B = \{1\}$, the functions f and g satisfy the formulas*

$$f(x, y) = \min(1, 1-x+y), \quad g(x) = 1-x,$$

and A be an arbitrary infinite set of numbers which satisfies the condition: $0 \leqslant x < 1$ for every $x \in A$, and is closed under the two operations f and g; then $\mathfrak{E}(\mathfrak{M}) = L_{\aleph_0}$.[17])

From Def. 7 the following facts established by Łukasiewicz are easily obtained:

THEOREM 17. (a) $L_1 = S$, $L_2 = L$;

(b) *if $2 \leqslant m < \aleph_0$, $2 \leqslant n \leqslant \aleph_0$ and $n-1$ is a divisor of $m-1$, then $L_m \subseteq L_n$;*

(c) $L_{\aleph_0} = \prod_{1 \leqslant n < \aleph_0} L_n$.

THEOREM 18. *All systems L_n for $3 \leqslant n \leqslant \aleph_0$ are consistent but not complete: $L_n \in \mathfrak{S}.\mathfrak{W} - \mathfrak{B}$.*

The converse of Th. 17(b) was proved by Lindenbaum:

[15]) It is not necessary to explain any further the meaning of the expression "in the sentence x two or three distinct variables occur" since it is intuitively clear. "Distinct" here means the same as "not equiform" (cf. III, p. 31, note 3 in Tarski, *op. cit.*).

[16]) What is called the *three-valued* system of the sentential calculus was constructed by Łukasiewicz in the year 1920 and described in a lecture given to the Polish Philosophical Society in Lwów. A report by the author, giving the content of that lecture fairly throughly was published in the journal *Ruch Filozoficzny* 5 (1920), p. 170 (in Polish). A short account of the n-valued systems, the discovery of which belongs to the year 1922, is given in J. Łukasiewicz, *Elementy logiki matematycznej*, pp. 115 ff. The philosophical implications of n-valued systems of sentential calculus are discussed in the article of Łukasiewicz, "Philosophical Remarks on Many-Valued Systems of Propositional Logic", pp. 153–178 of this book.

*) Wajsberg's proof of Th. 15 is given at the end of his paper "Aksjomatyzacja trójwartościowego rachunku zdań" (Axiomatization of the three-valued sentential calculus), *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie* 24 (1931), cl. iii. Another proof of the result discussed can be obtained by the use of the method developed in the note of A. H. Diamond and J. C. C. McKinsey, "Algebras and their subalgebras", *Bulletin of the American Mathematical Society* 53 (1947), pp. 959–962. For another proof of that part of Th. 15 which concerns the whole system L, see also S. Jaśkowski, "Trois contributions au calcul des propositions bivalent", *Studia Societatis Scientiarum Torunensis*, section A, 1 (1948), pp. 3–15, in particular pp. 9 ff.

[17]) Lindenbaum gave a lecture at the first congress of the Polish mathematicians (Lwów, 1927) on mathematical methods of investigating the sentential calculus in which, among other things, he formulated the above-mentioned theorem. Cf. his note "Méthodes mathématiques dans les recherches sur le système de la théorie de déduction", *Księga Pamiątkowa Pierwszego Polskiego Zjazdu Matematycznego*, Kraków, 1929, p. 36.

THEOREM 19. *For* $2 \leqslant m < \aleph_0$ *and* $2 \leqslant n < \aleph_0$ *we have* $L_m \subseteq L_n$ *if and only if* $n-1$ *is a divisor of* $m-1$.

Th. 17(c) was improved by Tarski by means of Th. 16:

THEOREM 20. $L_{\aleph_0} = \prod_{1 \leqslant i < \aleph_0} L_{n_i}$ *for every increasing sequence* n_i *of natural numbers.*

Concerning the problem of the degree of completeness for systems L_n the following partial result has been obtained.

THEOREM 21. *If* $n-1$ *is a prime number* (*in particular if* $n = 3$), *then there are only two systems, namely* S *and* L, *which contain* L_n *as a proper part; in other words, every sentence* $x \in S - L_n$ *satisfies one of the formulas*: $Cn(L_n + \{x\}) = S$ *or* $Cn(L_n + \{x\}) = L$; $\gamma(L_n) = 3$.

This theorem was proved for $n = 3$ by Lindenbaum; the generalization to all prime numbers given in the theorem is due to Tarski.*)

Regarding the axiomatizability of the system L_n we have the following theorem which was first proved by Wajsberg for $n = 3$ and for all n for which $n-1$ is a prime number, and was later extended to all natural numbers by Lindenbaum:

THEOREM 22. *For every* n, $1 \leqslant n < \aleph_0$, *we have* $L_n \in \mathfrak{A}$.

The effective proof of Th. 22 enables us to give a basis for every system L_n where $1 \leqslant n < \aleph_0$. In particular Wajsberg has established:

THEOREM 23. *The set* X *consisting of the sentences*

"*CpCqp*", "*CCpqCCqrCpr*", "*CCNpNqCqp*", "*CCCpNppp*"

forms a basis of L_3, *i.e.* $X \in \mathfrak{B}(L_3)$.

The following theorem of Wajsberg is one of the generalizations of Th. 22 at present known:

*) In May 1930 while the original printing of this article was in progress, Th. 21 was improved and the problem of the degree of completeness was solved for systems L_n with an arbitrary natural n; this was a joint result of members of a proseminar conducted by Łukasiewicz and Tarski in the University of Warsaw. A proof of Th. 21 and its generalizations appeared in print recently; see A. Rose, "The degree of completeness of m-valued Łukasiewicz propositional calculus", *The Journal of the London Mathematical Society* 27 (1952), pp. 92–102. The solution of the same problem for $L_{\aleph_0}$ has been given in A. Rose, "The degree of completeness of the $\aleph_0$-valued Łukasiewicz propositional calculus", *The Journal of the London Mathematical Society* 28 (1953), pp. 176–184.

THEOREM 24. *Let* $\mathfrak{M} = [A, B, f, g]$ *be a normal matrix in which the set* $A + B$ *is finite. If the sentences*

"*CCpqCCqrCpr*", "*CCqrCCpqCpr*", "*CCqrCpp*",
"*CCpqCNqNp*", "*CNqCCpqNp*"

are satisfied by this matrix, then $\mathfrak{E}(\mathfrak{M}) \in \mathfrak{A}$.*)

The Ths. 8, 10, and 15 of § 2 can be applied to the systems L_n. Accordingly we have:

THEOREM 25. *Every system* L_n, *where* $2 \leqslant n < \aleph_0$, *possesses, for every natural number* m (*and in particular for* $m = 1$), *a basis which has exactly* m *elements.*

THEOREM 26. *In every basis* (*and in general in every axiom system*) *of the system* L_n *at least three distinct sentential variables occur.*

As regards the problem of extending Th. 22 to the case $n = \aleph_0$ Łukasiewicz has formulated the hypothesis that *the system* $L_{\aleph_0}$ *is*

*) There is a comprehensive literature related to Ths. 22–24 and, more generally, concerning the axiomatizability of various systems of sentential calculus. We list a few papers on this subject in which further bibliographical references can also be found: M. Wajsberg, "Aksjomatyzacja trójwartościowego rachunku zdań" ("Axiomatization of the three-valued sentential calculus", in Polish), *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie* 24 (1931), cl. iii, pp. 126–148. A further relevant paper by the same author: "Beiträge zum Metaaussagenkalkül I", *Monatshefte für Mathematik und Physik* 42 (1935), pp. 221–242. B. Sobociński, "Aksjomatyzacja pewnych wielowartościowych systemów teorii dedukcji" ("Axiomatization of certain many-valued systems of the theory of deduction", in Polish), *Roczniki prac naukowych Zrzeszenia Asystentów Uniwersytetu Józefa Piłsudskiego w Warszawie*, 1 (1936), Wydział Matematyczno Przyrodniczy Nr. 1, pp. 399–419. J. Słupecki, "Dowód aksjomatyzowalności pełnych systemów wielowartościowych rachunku zdań" ("A proof of the axiomatizability of functionally complete systems of many-valued sentential calculus", in Polish), *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie* 32 (1939), cl. iii, pp. 110–128; and by the same author: "Pełny trójwartościowy rachunek zdań" ("The full three-valued sentential calculus", in Polish), *Annales Universitatis Mariae Curie-Skłodowska* 1 (1946), pp. 193–209. J. B. Rosser and A. R. Turquette, "Axiom schemes for m-valued propositional calculi", *Journal of Symbolic Logic* 10 (1945), pp. 61–82, and "A note on the deductive completeness of m-valued propositional calculi", ibid., 14 (1949), pp. 219–125.

In the first paper of Wajsberg listed above we find a proof of Th. 23, in the second a proof of Th. 24.

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axiomatizable and that *the set consisting of the following five sentences*

$$\text{“}CpCqp\text{”},\ \text{“}CCpqCCqrCpr\text{”},\ \text{“}CCCpqqCCqpp\text{”},$$
$$\text{“}CCCpqCqpCqp\text{”},\ \text{“}CCNpNqCqp\text{”}$$

forms an axiom system for $L_{\aleph_0}$.*)

It must be emphasized that, as defined here, the systems L_n for $n > 2$ have a fragmentary character, since they are incomplete and are only sub-systems of the ordinary system L. The problem of supplementing these systems to form complete and consistent systems which are at the same time distinct from L can be positively solved, but only in one way, namely by widening the concept of meaningful sentence of the sentential calculus, and by introducing, beside the operations of forming implications and negations, other analogous operations which cannot be reduced to these two (cf. also § 5).

Finally we may add that the number of all possible systems of the sentential calculus was determined by Lindenbaum.

THEOREM 27. $\overline{\overline{\mathfrak{S}}} = 2^{\aleph_0}$, *but* $\overline{\overline{\mathfrak{S}.\mathfrak{A}}} = \aleph_0$.**)

This result was improved by Tarski as follows:

THEOREM 28. $\overline{\overline{\mathfrak{S}.\mathfrak{W}.\mathfrak{V}}} = 2^{\aleph_0}$, *but* $\overline{\overline{\mathfrak{S}.\mathfrak{W}.\mathfrak{V}.\mathfrak{A}}} = \aleph_0$.***)

*) This hypothesis has proved to be correct; see M. Wajsberg, "Beiträge zum Metaaussagenkalkül I", *Monatshefte für Mathematik und Physik* 42 (1935), pp. 221–242, in particular p. 240. As far as we know, however, Wajsberg's proof has not appeared in print.

The axiom-system above is not independent: C. A. Meredith has shown that "$CCCpqCqpCqp$" is deducible from the remaining axioms.

**) A proof of the first part of Th. 27 (and in fact of a somewhat stronger result) can be found in the paper of K. Schröter, *op. cit.*, pp. 301 ff. The proof of the second part of Th. 27 is almost obvious.

***) The proof of Th. 28 can be outlined as follows: For any given natural number $n = 1, 2, 3, \ldots$ let x_n be the sentence which is formed by n symbols "C" followed by $n+1$ variables "p". Given any set N of natural numbers, let X_N be the set consisting of all sentences x_{3n} where n belongs to N and of all sentences x_{3n+1} where n does not belong to N. It can easily be shown that the set $Cn(X_N)$ coincides with the set of all those sentences which can be obtained from sentences of X_N by substitution. Hence the set X_N is consistent and can therefore be extended to form a complete and consistent system X_N^*. On the other hand, if M and N are two different sets of natural numbers, then the sum of X_M and X_N is clearly inconsistent, and hence the systems X_M^* and X_N^* cannot be identical. The remaining part of the proof is obvious.

4. The restricted sentential calculus

In investigations into the sentential calculus attention is sometimes restricted to those sentences in which no negation sign occurs. This part of the sentential calculus can be treated as an independent deductive discipline, one which is still simpler than the ordinary sentential calculus and will be called here the *restricted sentential calculus*.

For this purpose we must first of all modify the concept of meaningful sentence by omitting the operation of forming negations from Def. 1. In a corresponding way the concept of substitution is also simplified, and this brings with it a change in the concept of consequence. *After these modifications Th.* 1 *remains valid.*

For the construction of closed systems of the restricted sentential calculus both of the methods described in § 1 are used: the axiomatic and the matrix method. But a logical matrix is now defined as an ordered triple $[A, B, f]$ and not as an ordered quadruple (Def. 3); consequently condition (4) in Def. 4 of a value function disappears. *Ths.* 2–4 *remain valid.*

The definition of the ordinary system L^+ of the restricted sentential calculus is completely analogous to Def. 5, with one obvious difference which is called for by the modification in the concept of matrix. This system has been investigated by Tarski. From the definition of the system its consistency and completeness are easily derivable; hence *Th.* 5 *holds also in the restricted sentential calculus.* The axiomatizability of the system is established in the following theorem:

THEOREM 29. *The set* X *consisting of the three sentences* "$CpCqp$", "$CCpqCCqrCpr$", "$CCCpqpp$" *forms a basis of the system* L^+; *consequently* $L^+ \in \mathfrak{A}$.

This theorem originates with Tarski; it contains, however, a simplification communicated to the authors by P. Bernays. In fact the original axiom system of Tarski included, instead of the sentence "$CCCpqpp$", a more complicated sentence, "$CCCpqrCCprr$".*) The independence of both axiom systems was established by Łukasiewicz.

*) The original proof of Th. 29 has not been published. But a proof of this result can easily be obtained by means of a method developed in M. Wajsberg, "Metalogische Beiträge", *Wiadomości Matematyczne* 18 (1936), pp. 131–168, in particular

Ths. 8, 10, 11, 12, 14, *and* 15 *from* § 2 *have been extended to the restricted sentential calculus* by their originators. Tarski, in particular, has succeeded in setting up a basis of the system L^+ consisting of only a single sentence. Two simple examples of such sentences, each containing 25 letters, are given in the next theorem. The first is an organic sentence and was found by Wajsberg, the second is not organic and is due to Łukasiewicz:

THEOREM 30. *The set of sentences consisting either of the single sentence*

"*CCCpqCCrstCCuCCrstCCpuCst*"

or of the single sentence

"*CCCpCqpCCCCCrstuCCsuCruvv*"

*forms a basis of the system L^+.**)

Def. 7 of the n-valued system L_n can be applied at once to the restricted sentential calculus provided only that the concept of matrix is suitably modified. *Ths.* 16–22 *as well as* 24–26, which describe the mutual relations among the various systems L_n^+, determine the degree of completeness of the systems and establish their axiomatizability, *have been extended to the restricted sentential calculus* by their originators. (In the case of Th. 21 this was done by Tarski; for Th. 22 by Wajsberg. In Th. 24 the sentences with negation signs are to be omitted.) The problem of the axiomatizability of the system $L_{\aleph_0}^+$ is left open.

Finally, *the number of all possible systems of the sentential calculus*, which was determined by Lindenbaum and Tarski in Ths. 27 and 28, *also remains unchanged in the restricted sentential calculus.***)

pp. 154–157; the derivations which are needed for applying Wajsberg's method can be found, for example, in: W. V. Quine, *System of Logistic*, Cambridge, Mass. 1934, pp. 60 ff.

*) More recently Łukasiewicz has shown that the sentence "*CCCpqrCCrpCsp*" can also serve a single axiom for system L^+ and that there is no shorter sentence with this property. See J. Łukasiewicz, "The Shortest Axiom of the Implicational Calculus of Propositions", p. 295–305 of this book.

**) The footnote concerning Th. 27 on p. 144 applies to the restricted sentential calculus as well.

5. The extended sentential calculus

By the *extended sentential calculus* we understand a deductive discipline in the sentences of which there occur what are called *universal quantifiers* in addition to sentential variables and the implication sign.[18]) For the universal quantifier Łukasiewicz uses the sign "$\prod$" which was introduced by Peirce.[19]) With this notation the formula "$\prod pq$" is the symbolic expression of the sentence "for all p, q (holds)". The operation which consists in putting the universal quantifier "$\prod$" with a sentential variable x in front of a given sentence y is called *universal quantification of the sentence y with respect to the sentential variable x*, and is denoted

[18]) In his article "Grundzüge eines neuen Systems der Grundlagen der Mathematik" Leśniewski has described the outlines of a deductive system, called by him *Protothetic*, which, compared with the extended sentential calculus, goes still further beyond the ordinary sentential calculus in the respect that, in addition to quantifiers, variable functors are introduced. (In the sentence "*Cpq*" the expression "*C*" is called a *functor*, and "*p*" and "*q*" are called the *arguments*. The word functor we owe to Kotarbiński. In both the ordinary and the extended sentential calculus only constant functors are used.) In addition to this principal distinction, there are yet other differences between the extended sentential calculus and the protothetic as it is described by Leśniewski. In contrast to the extended sentential calculus, in the protothetic only those expressions are regarded as meaningful sentences in which no free, but only bound (apparent) variables occur. Some new operations (rules of inference or directives) are also introduced by means of which consequences are derived from given sentences, such, for example, as the operation of distributing quantifiers, which is superfluous in the extended sentential calculus. Finally it must be emphasized that Leśniewski has formulated with the utmost precision the conditions which a sentence must satisfy if it is to be admitted as a definition in the system of the protothetic, whereas in the present work the problem of definitions has been left untouched. Article I belongs to protothetic. A sketch of the extended sentential calculus is given in J. Łukasiewicz, *Elements of Mathematical Logic*, pp. 154–169; this sketch rests in great part on results of Tarski (cf. A. Tarski, *op. cit.* Preface, p. vii). The two-valued logic of Łukasiewicz ("The Shortest Axiom....", pp. 295–305 of this book) has many points of contact with the extended sentential calculus. Finally, there are many analogies between the extended sentential calculus and the functional calculus of Hilbert and Ackermann (see Hilbert, D., and Ackermann, W., *Grundzüge der theoretischen Logik*, Berlin, 1928, especially pp. 84–85).

[19]) The expression "quantifier" occurs in the work of Peirce (note 4), p. 197, although with a somewhat different meaning.

by "$\pi_x(y)$" in metamathematical discussions. This concept it to be regarded as a primitive concept of the metasentential calculus.[20]

Definition 8. *The set $S^{\times}$ of all meaningful sentences (of the extended sentential calculus) is the intersection of all those sets which contain all sentential variables and are closed under the two operations of forming implications and of universal quantification (with respect to an arbitrary sentential variable).*[21]

The operations of forming negations and of existential quantification (which consists in prefixing to a given sentence y the existential quantifier "$\sum$" with a sentential variable x), are not considered here because, in the system of the extended sentential calculus in which we are interested, they can be defined with the help of the two operations previously mentioned. For example, we can use the formula "$Cp\prod qq$" as definiens for "Np".

In deriving consequences from an arbitrary set of sentences Łukasiewicz and Tarski make use of the operations of *insertion* and *deletion of quantifiers*, in addition to those of substitution[22] and detachment. The first of these operations consists in obtaining a sentence $y = c(z, \pi_t(u))$ from a sentence of the form $x = c(z, u)$, where $z \in S^{\times}$ and $u \in S^{\times}$ under the assumption that t is a sentential variable which is not free in z.[23] The second operation is the inverse of the first and consists in deriving the sentence $x = c(z, u)$ from the sentence $y = c(z, \pi_t(u))$ (in this case without any restriction concerning the variable t).[24]

20) Cf. the remarks following Def. 1 in § 1.

21) In contrast to the above mentioned book of Hilbert and Ackermann, p. 52, as well as to the standpoint taken in Łukasiewicz, *Elements of Mathematical Logic*, p. 155, the expression $\pi_x(y)$ is also regarded as meaningful when x either occurs as a bound variable in y or does not occur in y at all.

22) The operation of substitution undergoes certain restrictions in the extended sentential calculus (cf. J. Łukasiewicz, *Elements* ... p. 160, and Hilbert and Ackermann, *op. cit.* p. 54.)

23) We do not need to discuss the meaning of the expression "x occurs in the sentence y as a free (or bound) variable" since it is sufficiently clear (cf. J. Łukasiewicz, *Elements* ..., p. 156, and Hilbert and Ackermann, *op. cit.*, p. 54).

24) In the restricted functional calculus only the first operation is used. Instead of the second an axiom is set up. (cf. Hilbert and Ackermann, *op. cit.*, pp. 53–54). An analogous procedure would not be possible in our calculus; for if we drop the second operation the system $L^{\times}$ to be discussed below would not have a finite basis.

Definition 9. *The set $Cn^{\times}(X)$ of consequences of the set X of sentences (in the sense of the extended sentential calculus) is the intersection of all those sets which include the given set $X \subseteq S^{\times}$ and are closed under the operations of substitution and detachment, as well as insertion and deletion of quantifiers.*

With this interpretation of the concepts $S^{\times}$ and $Cn^{\times}(X)$ *Th. 1 from § 1 remains valid.*

As before, two methods are available for the construction of deductive systems: the axiomatic and the matrix methods. The second method has not yet received a sufficiently clear general formulation, and in fact the problem of a simple and useful definition of the concept of matrix still presents many difficulties. Nevertheless this method has been successfully applied by Łukasiewicz in special cases, namely for the construction of the n-valued systems $L^{\times}_n$ (for $n < \aleph_0$) and in particular for the construction of the ordinary system $L^{\times}$ of the extended sentential calculus. The construction of the systems $L^{\times}_n$ is precisely described in the following

Definition 10. *First let us introduce the following auxiliary notation: $b =$ "p", $g = \pi_b(b)$ (falsehood), $n(x) = c(x, g)$ for every $x \in S^{\times}$ (the negation of the sentence x), $a(x, y) = c(c(x, y), y)$ and $k(x, y) = n(a(n(x), n(y)))$ for every $x \in S^{\times}$ and every $y \in S^{\times}$ (the disjunction or rather alternation, and the conjunction of the sentences x and y);*[25] *furthermore $k^m_{i=1}(x_i) = x^1$ for $m = 1$ and $k^m_{i=1}(x_i) = k(k^{m-1}_{i=1}(x_i), x_m)$ for every arbitrary natural number $m > 1$, where $x_i \in S^{\times}$ for $1 \leqslant i \leqslant m$ (the conjunction of the sentences $x_1, x_2, \ldots, x_m$). Further we put $b_m = b$ for $m = 1$, $b_m = c(n(b), b_{m-1})$ for every natural number $m > 1$, and finally $a_m = \pi_b(c(b_m, b))$ for every natural number m.*[26]

25) The logical expressions "Apq" ("p or q") and "Kpq" ("p and q") correspond, in the symbolism introduced by Łukasiewicz, to the metalogical expressions "$a(x, y)$" and "$k(x, y)$" respectively. Of the two possible definitions of the alternation, which in the two-valued, but not in the n-valued, system are equivalent: $a(x, y) = c(c(x, y), y)$ and $a(x, y) = c(n(x), y)$, the first was chosen by Łukasiewicz for various, partly intuitive, reasons (cf. J. Łukasiewicz, "Two-Valued Logic", p. 89–109 of this book).

26) For example,

$$b_1 = \text{"}p\text{"},\ b_2 = \text{"}Cp\Pi ppp\text{"},\ b_3 = \text{"}Cp\Pi ppCCp\Pi ppp\text{"},$$

and

$$a_1 = \text{"}\Pi pCpp\text{"},\ a_2 = \text{"}\Pi pCCCp\Pi pppp\text{"},$$

$$a_3 = \text{"}\Pi pCCCp\Pi ppCCp\Pi pppp\text{"}.$$

Now let n be a definite natural number > 1. We choose n sentences called *basic sentences*, and denote them by the symbols "g_1", "g_2",..., "g_n"; in fact we put $g_1 = g$, $g_2 = a_{n-1}$, and $g_m = c(n(g_2), g_{m-1})$ for every m, $2 < m \leqslant n$.[27] Let G be the smallest set of sentences which contains all sentential variables and basic sentences and is closed with respect to the operation of forming implications.

A function h is called a *value function* (*of the n-th degree*) if it satisfies the following conditions: (1) the function h is defined for every sentence $x \in G$; (2) if x is a sentential variable, then $h(x)$ is a fraction of the form $(m-1)/(n-1)$, where m is a natural number and $1 \leqslant m \leqslant n$; (3) for every natural number m, $1 \leqslant m \leqslant n$, we have $h(g_m) = (m-1)/(n-1)$; (4) if $x \in G$ and $y \in G$, then $h(c(x, y)) = \min(1, 1-h(x)+h(y))$.[28]

With every sentence $x \in S^{\times}$ a sentence $f(x) \in G$ is correlated by recursion in the following way: (1) if x is a sentential variable or a basic sentence, then $f(x) = x$; (2) if $x \in S^{\times}$, $y \in S^{\times}$, and $c(x, y)$ is not a basic sentence, we put $f(c(x, y)) = c(f(x), f(y))$; (3) if x is a sentential variable which is not free in the sentence $y \in S^{\times}$, then $f(\pi_x(y)) = f(y)$; (4) but if the sentential variable x is free in the sentence $y \in S^{\times}$ and $\pi_x(y)$ is not a basic sentence, then we put $f(\pi_x(y)) = k_{i=1}^{n}(f(y_i))$ where the sentence y_i for every i, $1 \leqslant i \leqslant n$, arises from y by the substitution of the basic sentence g_i for the free variable x.

The *n-valued system* $L_n^{\times}$ *of the extended sentential calculus*, where $2 \leqslant n < \aleph_0$, is now defined as the set of all those sentences $x \in S^{\times}f$ which satisfy the formula $h(f(x)) = 1$ for every value function h (of the nth degree); in addition $L^{\times}$ is set equal to $S^{\times}$. The system $L_2^{\times} = L^{\times}$ is also called the *ordinary system of the extended sentential calculus.*[29]

[27]) For example, for $n = 3$:

$$g_1 = \text{"}\Pi pp\text{"},\ g_2 = \text{"}\Pi pCCCp\Pi pppp\text{"},$$
$$g_3 = \text{"}CC\Pi pCCCp\Pi pppp\Pi pp\Pi pCCCp\Pi pppp\text{"}.$$

[28]) Cf. Defs. 4 and 7 above.

[29]) In the definition adopted by Łukasiewicz, instead of the basic sentences g_1, g_2, ..., g_n, there occur what are called sentential constants, $c_1, c_2, \ldots, c_n$, i.e. special signs distinct from sentential variables. The concept of meaningful sentence is thereby temporarily extended. The rest of the definition runs quite analogously to the definition in the text. In the final definition of the systems $L_n^{\times}$ all expressions which contain sentential constants are eliminated, and the concept of meaningful sentences is reduced

From this definition of the systems $L_n^{\times}$ the following facts easily result (they are partly in opposition to Ths. 18 and 19 of § 3):

THEOREM 31. *$L_n^{\times} \in \mathfrak{S}.\mathfrak{W}.\mathfrak{B}$ for every natural n, $2 \leqslant n < \aleph_0$.*

THEOREM 32. *For $2 \leqslant m < \aleph_0$ and $2 \leqslant n < \aleph_0$ we have $L_m^{\times} \subseteq L_n^{\times}$ if and only if $m = n$ (no system of the sequence $L_n^{\times}$, where $2 \leqslant n < \aleph_0$, is included in another system of this sequence).*

THEOREM 33. *The set of all sentences of the system $L_n^{\times}$ (where $1 \leqslant n < \aleph_0$) in which no bound variables occur is identical with the corresponding system $L_n^{\times}$ of the restricted sentential calculus.*

Regarding the axiomatizability of these systems Tarski has shown that *Ths.* 8, 10, 22, 29, *and* 30 *also hold in the extended sentential calculus.* In this connexion Tarski has also proved the following:

THEOREM 34. *Every axiom system of the system $L_2^{\times} = L^{+}$ in the restricted sentential calculus is at the same time an axiom system of the system $L_2^{\times} = L^{\times}$ in the extended sentential calculus.*[30]

On the other hand, not every basis of the system L^{+} in the restricted sentential calculus is at the same time a basis in the extended calculus (and not every set of sentences which is independent in the restricted sentential calculus remains independent in the extended calculus).

THEOREM 35. *For $3 \leqslant n < \aleph_0$ universal quantifiers and bound variables occur in at least one sentence of every basis (and in general of every axiom system) of the system $L_n^{\times}$.*

It is worthy of note that the proof given by Tarski of Th. 22 in the extended sentential calculus makes it possible to construct effectively

to the original expressions. By means of the modification introduced in the text, which is due to Tarski, the definition of the systems $L_n^{\times}$ certainly takes on a simpler form from the metalogical standpoint, but at the same time it becomes less perspicuous. In order to establish the equivalence of the two definitions it suffices to point out that the expressions chosen as basic sentences satisfy the following condition: *for every value function h* (in the sense of the original definition of Łukasiewicz),

$$h(f(g_m)) = h(c_m) = \frac{m-1}{n-1}, \text{ where } 1 \leqslant m \leqslant n.$$

[30]) The completeness and axiomatizability of the system $L_2^{\times}$ was proved by Tarski in the year 1927. His proof was subsequently simplified by S. Jaśkowski.

an axiom system for every system $L_n^{\times}$ ($3 \leqslant n < \aleph_0$). Relatively simple axiom systems of this kind were constructed by Wajsberg; in the case $n = 3$ his result is as follows:

THEOREM 36. *Let X be the set consisting of the following sentences:*

$$\text{“}CCCpqCrqCCqpCrp\text{”}, \text{“}CpCqp\text{”}, \text{“}CCCpCpqpp\text{”},$$
$$\text{“}C\prod pCCCp\prod ppppC\prod pCCCp\prod pppp\prod pp\text{”},$$
$$\text{“}CC\prod pCCCp\prod pppp\prod pp\prod pCCCp\prod pppp\text{”};$$

then $X \in \mathfrak{A}\mathfrak{x}(L_3^{\times})$.

An exact definition of the denumerably-valued system $L_{\aleph_0}^{\times}$ of the extended sentential calculus presents much greater difficulties than that of the finite-valued systems. This system has not yet been investigated.

Ths. 27 *and* 28, which determine the number of all possible systems, *remain correct in the extended sentential calculus.*

In conclusion we should like to add that, as the simplest deductive discipline, the sentential calculus is particularly suitable for metamathematical investigations. It is to be regarded as a laboratory in which metamathematical methods can be discovered and metamathematical concepts constructed which can then be carried over to more complicated mathematical systems.

PHILOSOPHICAL REMARKS ON MANY-VALUED SYSTEMS OF PROPOSITIONAL LOGIC *)

In the communication “Untersuchungen über den Aussagenkalkül” (Investigations into the Sentential Calculus) which appeared in this issue under Tarski's and my name, Section 3 is devoted to the “many-valued” systems of propositional logic established by myself. Referring the reader to this communication as far as logical questions are concerned, I here propose to clarify the origin and significance of those systems from a philosophical point of view.

1. Modal propositions

The three-valued system of propositional logic owes its origin to certain inquiries I made into so-called “modal propositions” and the notions of possibility and necessity closely connected with them.[1])

[1]) I read a paper on these inquiries at the meeting on 5 June 1920 of the Polish Philosophical Society at Lwów. The essential parts of this paper were published in the Polish periodical *Ruch Filozoficzny* 5 (1920), pp. 170–171. [The first English translation of that text was published as paper 1 in the McCall edition and translated by H. Hiż from the version published in *Ruch Filozoficzny*. In this book (pp. 87–88) the translation was made by O. Wojtasiewicz from the text of the lecture itself, read by Łukasiewicz on 5 June, 1920.]

*) [Editorial note from the McCall edition: This paper appeared originally under the title “Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenkalküls” in *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie* 23 (1930), cl. iii, pp. 51–77. Translated by H. Weber], reprinted in the 1961 edition *Z zagadnień logiki i filozofii*.

By *modal propositions* I mean propositions that have been formed after the pattern of one of the following four expressions:

(1) It is possible that p	in symbols Mp.
(2) It is not possible that p	in symbols NMp.
(3) It is possible that not-p	in symbols MNp.
(4) It is not possible that not-p	in symbols $NMNp$.

The letter "p" designates here any proposition; "N" is the symbol of *negation* ("Np" = "not-p"); "M" corresponds to the words "it is possible that". Instead of saying "it is not possible that not-p", one can also use the phrase "it is necessary that p".

The expressions listed here are not identical with Kant's "problematical" and "apodictic" judgements. Rather they correspond to the modal propositions of medieval logic originating in Aristotle and formed from the four "modes": *possibile* (e.g. *Socratem currere est possibile*), *impossibile*, *contingens*, and *necessarium*. Besides these four modes, two more modes were cited by the logicians of the Middle Ages; namely, *verum* and *falsum*. However, these modes were given no further consideration, as the modal propositions corresponding to them, "it is true that p" and "it is false that p", were regarded as being equivalent to the propositions "p" and "Np".[2])

The expression "it is possible that" is not defined here; its sense is made clear by the theorems which hold for modal propositions.

2. Theorems concerning modal propositions

In the history of logic we meet with three groups of theorems concerning modal propositions.

Among the *first* group I count those well-known theorems which have been handed down to us from classical logic and have been regarded by it as truths evident without demonstration:

(a) *Ab oportere ad esse valet consequentia.*

(b) *Ab esse ad posse valet consequentia.*

By contraposition we get from (b) a third proposition:

(c) *Ab non posse ad non esse valet consequentia.*

[2]) Cf. Prantl, *Geschichte der Logik im Abendlande*, vol. iii, p. 14, note 42; p. 117, note 542.

The latter proposition means: "The inference from not-being-possible to not-being is valid". For instance: It is not possible to divide a prime number by four; therefore no prime number is divisible by four. This example is plausible, and just as plausible is the following general theorem which we shall keep in mind as representative of the first group:

I. *If it is not possible that p, then not-p.*

Less well known, but no less intuitive, seems to be the following theorem of the *second* group quoted by Leibniz in the *Théodicée*:[3])

(d) *Unumquodque, quando est, oportet esse.*

"Whatever is, *when* it is, is necessary." This theorem dates back to Aristotle, who, to be sure, holds that not everything which is is necessary and not everything which is not is impossible, but when something which is is, then it is also necessary; and when something which is not is not, then it is also impossible.[4])

The theorems just quoted are not easily interpreted. First I shall give some examples.

It is not necessary that I should be at home this evening. But *when* I am at home this evening, then on this assumption it is *necessary* that I should be at home this evening. A second example: It rarely happens that I have no money in my pocket, but *if* I have now (at a certain moment t) no money in my pocket, it is *not possible*, on this assumption, that I have money (at just the same moment t) in my pocket.

Note has to be taken of two things about these examples. First, the propositions: "I am at home this evening" and "I have (at the moment t) no money in my pocket" are supposed to be *true*, and on this supposition the necessity or impossibility respectively is inferred. Secondly, the word *quando* in (d), and the corresponding ὅταν of Aristotle, is not a conditional, but a temporal particle. Yet the temporal merges into the conditional, if the determination of time in the temporally connected propositions is included in the content of the propositions.

The examples given are, moreover, evident enough to establish the following general theorem, which we shall keep in mind as representative of the second group:

[3]) *Philos. Schriften* (ed. Gerhardt), vol. 6, p. 131.

[4]) *De interpr.* 9. 19a23: Τὸ μὲν οὖν εἶναι τὸ ὂν ὅταν ᾖ, καὶ τὸ μὴ ὂν μὴ εἶναι, ὅταν μὴ ᾖ, ἀνάγκη· οὐ μὴν οὔτε τὸ ὂν ἅπαν ἀνάγκη εἶναι οὔτε τὸ μὴ ὂν μὴ εἶναι.

II. *If it is supposed that not-p, then it is (on this supposition) not possible that p.*

The *third* group consists of only *one* theorem based on the Aristotelian concept of "two-sided" possibility. According to Aristotle there are some things which are possible in both directions, i.e. which *can* be, but *need* not be. It is possible, for instance, that this cloak should be cut; but it is also possible that it should not be cut. [5])

Again, it is possible that the patient will die, but it is also possible that he will recover, and therefore not die. This concept of two-sided possibility is deeply rooted in everyday thinking and speech. The following theorem, to which we will return, seems therefore to be just as evident as the two preceding ones:

III. *For some p: it is possible that p and it is possible that not-p.*

3. Consequences of the first two theorems concerning modal propositions

We shall now draw some inferences from theorems I and II cited above. For this purpose we shall first represent those theorems in the symbolism of propositional logic.

Let "*Cpq*" symbolize the implication: "if *p*, then *q*", "*p*" and "*q*" denoting any proposition. It is evident that theorem I can be expressed in the form of an implication, which I call "thesis" 1: [6])

1 *CNMpNp*.

Meaning: "If it is not possible that *p*, then not-*p*."

It is not equally evident, but can be proved, that theorem II can be represented as an implication which is the converse of 1. For if a proposition "β" is valid on the assumption "α", this means no more than that "β" is true if "α" is true. The implication "if α, then β" therefore holds, if "α" is true. Since this implication must also hold if "α" is false, it holds in both cases. We thus arrive at the thesis:

2 *CNpNMp*.

[5]) *De interpr.* 9. 19a9: ὅλως ἔστιν ἐν τοῖς μὴ ἀεὶ ἐνεργοῦσι τὸ δυνατὸν εἶναι καὶ μὴ ὁμοίως· ἐν οἷς ἄμφω ἐνδέχεται, καὶ τὸ εἶναι καὶ τὸ μὴ εἶναι...οἷον ὅτι τουτὶ τὸ ἱμάτιον δυνατόν ἐστι διατμηθῆναι...ὁμοίως δὲ καὶ τὸ μὴ διατμηθῆναι δυνατόν.

[6]) Following Leśniewski, I understand by "theses" axioms as well as theorems of a deductive system.

This means: "If not-*p*, then it is not possible that *p*." Theorem II cannot be expressed in any other way in the two-valued propositional calculus.

From these theses and using the usual propositional calculus, we shall prove several consequences. All the following demonstrations are strictly formalized and carried out by means of two rules of inference: *substitution* and *detachment*. These well-known rules of inference will not be discussed here. I will only explain how formalized proofs are recorded in the symbolism which I introduced.

Before each thesis to be proved (to which consecutive numbers are assigned for purposes of identification) is an unnumbered line, which I call the "derivational line". Each derivational line consists of two parts separated by the sign "×". The symbols before and after the separation sign denote the same expression, but in different ways. Before the separation sign, a substitution is indicated, which is to be carried out on a thesis already proved. In the first derivational line, for example, the expression "3 *q*/*Mp*" means that "*Mp*" should be substituted for "*q*" in 3. The resultant thesis, which is omitted in the proof for the sake of brevity, would be:

3′ *CCNMpNpCpMp*.

The expression "*C*1–7" after the separation sign refers to this thesis 3′ and indicates that the rule of detachment can be applied to 3′. Thesis 3′ is asserted as a substitution instance of thesis 3; but since it is an implication whose antecedent is thesis 1, its consequent may be detached and asserted as thesis 7. In the second derivational line the number "8" denotes the thesis obtained from 7 by the substitution "*p*/*Np*". In the derivational line of thesis 10, the rule of detachment is used twice. After these explanations, I believe the reader will have no difficulty in understanding the demonstration below.

In addition to theses 1 and 2, which appear as axioms, four well-known auxiliary theses from the ordinary propositional calculus appear in the demonstration: three laws of transposition, numbered 3–5; and the principle of the hypothetical syllogism, thesis 6. All of these theses I place at the head of the demonstration as premisses.

1 *CNMpNp*.
2 *CNpNMp*.
3 *CCNqNpCpq*.

4 $CCNpqCNqp$.

5 $CCpNqCqNp$.

6 $CCpqCCqrCpr$.

*

$3\ q/Mp \times C1 - 7$.

7 $CpMp$.

$7\ p/Np \times 8$.

8 $CNpMNp$.

$4\ q/MNp \times C8 - 9$.

9 $CNMNpp$.

$6\ p/NMNp,\ q/p,\ r/Mp \times C9 - C7 - 10$.

10 $CNMNpMp$.

$4\ p/MNp,\ q/Mp \times C10 - 11$.

11 $CNMpMNp$.

*

$3\ q/p,\ p/Mp \times C2 - 12$.

12 $CMpp$.

$12\ p/Np \times 13$.

13 $CMNpNp$.

$5\ p/MNp,\ q/p \times C13 - 14$.

14 $CpNMNp$.

$6\ p/Mp,\ q/p,\ r/NMNp \times C12 - C14 - 15$.

15 $CMpNMNp$.

$5\ p/Mp,\ q/MNp \times C15 - 16$.

16 $CMNpNMp$.

Theses 7–11 are consequences of 1; 12–16 result from 2. Thesis 7 says: "if *p*, then it is possible that *p*". Thesis 9 says: "if it is not possible that not-*p*, then *p*". The latter thesis corresponds to theorem (a) in classical logic, cited above, the first to theorem (b). Both are evident. In fact, all theses of the first group, 7–11, are evident.

Not so evident are the theses of the second group, 12–16. Thesis 12 reads: "if it is possible that *p*, then *p*". On the basis of this thesis we can infer: It is possible that the patient will die; hence he will die. This inference will be admitted only by those making no distinction between possibility and being. Theses of the second group, 12–16, are the converses of theses of the first group, 7–11. Whoever admits both groups of theses must assume the following propositions to be equivalent: "*p*", "it is possible that *p*", and "it is not possible that not-*p*", or "it is necessary that *p*". Also the propositions "not-*p*", "it is possible that not-*p*", and "it is not possible that *p*". But then the concepts of possibility and necessity become dispensable. This unpleasant consequence results from the acceptance of our symbolic formulation of theorem II, which is evident in ordinary language and can be recognized as being true without reservation. Nevertheless it seems to me impossible to express proposition II in the symbolic language of the two-valued propositional calculus in any other way than by a simple implication which is the converse of thesis 1.

4. Consequences of the third theorem on modal propositions

The symbolic formulation of the third theorem leads to another unwelcome result.

Theorem III can be expressed only by means of the symbolism of the extended propositional calculus. Let "$\sum$" be the *existential quantifier*, and let "$\sum p$" denote the expression "for some *p*". Let "*Kpq*" be the symbol of *conjunction*, "*p* and *q*", where "*p*" and "*q*" denote any propositions. Theorem III can then be expressed symbolically as follows:

17 $\sum pKMpMNp$.

This means verbally: "For some *p*: it is possible that *p*, and it is possible that not-*p*."

The existential quantifier "$\sum$" can be expressed by means of the *universal* quantifier "$\prod$". If "$\prod p$" says: "for every *p*", and if "$\alpha(p)$" represents any expression containing "*p*", the following definition is evident:

D1 $\sum p\alpha(p) = N\prod pN\alpha(p)$.

D1 states that the expressions: "for some *p*, $\alpha(p)$ (holds)" and "it is not true that for each *p* not-$\alpha(p)$ (holds)" mean the same thing. Thesis 17 then becomes the following thesis:

18 $N\prod pNKMpMNp$.

There is, however, besides the extended propositional calculus, a still more general logical system created by Leśniewski, which he has termed "protothetic". [7]) The main difference between protothetic and the extended propositional calculus is the occurrence in the latter of variable "functors" [8]) as well as constants.

Denoting a variable functor to which *one* proposition only is attached as argument by "ϕ", we can prove the following proposition in protothetic:

$$CK\phi p\phi Np\phi q.$$

In words: "If ϕ of p and ϕ of not-p, then ϕ of q."

Since this proposition is valid for all functors with one argument, it is also valid for the functor "M". We thus obtain:

19 $\quad CKMpMNpMq.$

Theses 18 and 19, as well as two auxiliary theses from the ordinary propositional calculus, viz. the principle of transposition 4 mentioned above, and another rule of transposition, thesis 20, are premisses of the formalized proof given below. Besides substitution and detachment, the rule for the *introduction of a quantifier* is used in the proof. This rule runs thus: If in the consequent of an implication which is a thesis there occurs a free propositional variable "p" which does not occur in the antecedent of that implication, the symbol "$\prod p$" may be put before the consequent. This rule of inference is denoted below by "$+\prod$." Beginning with the premisses, our demonstration then reads thus:

18 $\quad N\prod pNKMpMNp.$

19 $\quad CKMpMNpMq.$

20 $\quad CCpqCNqNp.$

$\quad 20\overset{*}{p}/KMpMNp,\ q/Mq\times C19–21.$

21 $\quad CNMqNKMpMNp.$

$\quad 21+\prod\times 22.$

22 $\quad CNMq\prod pNKMpMNp.$

[7]) S. Leśniewski, "Grundzüge eines neuen Systems der Grundlagen der Mathematik", introduction and §§ 1–11, *Fund. Math.* 14(1929).

[8]) In the function "Cpq", "C" is the "functor", and "p" and "q" the "arguments". The term "functor" was introduced by Kotarbiński.

$\quad 4p/Mp,\ q/\prod pNKMpMNp\times C22q/p-C18-23.$

23 $\quad Mp.$

The result obtained, thesis 23, has to be admitted as being true. This thesis, which in words reads "it is possible that p", holds for any p. We therefore have to admit as true the proposition "it is possible that 2 is a prime number", as well as the proposition "it is possible that 2 is not a prime number". Freely speaking, we have been led to admit everything as possible by reason of theorem III. Yet if everything is possible, then nothing is impossible and nothing necessary. For if the proposition "Mp" is admitted, we obtain from it by substitution the proposition "MNp", and the expressions "NMp" and "$NMNp$" have to be rejected as negations of those preceding.

These are consequences running contrary to all of our intuitions. Yet I see no possibility of expressing theorem III, in the symbolism of the extended propositional calculus, in any other form than that of thesis 17 or 18.

5. Incompatibility of the theorems on modal propositions in the two-valued propositional calculus

The unpleasant consequences to which we were led by theorems II and III considered separately become wholly unacceptable when we consider both theorems together.

Indeed, when we combine thesis 12, resulting from the symbolic formulation of theorem II, with thesis 23:

12 $\quad CMpp$

23 $\quad Mp$

we immediately obtain:

$\quad 12\times C23-24,$

24 $\quad p.$

If therefore theses 12 and 23 are valid, any proposition p is valid too. Hence we arrive at the inconsistent system of all propositions. Theorems II and III are incompatible when symbolically represented as theses 2 and 18.

We can obtain the same result without employing thesis 19, which presupposes a proposition from protothetic. In the following demonstration we use the theses 12, 13, and 20 alone, as well as certain auxiliary theses of the ordinary propositional calculus:

25 *CpCqp.*

26 *NKpNp.*

27 *CCpqCCrsCKprKqs.*

*

$27\,p/Mp,\ q/p,\ r/MNp,\ s/Np \times C12 - C13 - 28.$

28 *CKMpMNpKpNp.*

$20\,p/KMpMNp,\ q/KpNp \times C28 - C26 - 29.$

29 *NKMpMNp.*

$25\,p/NKMpMNp \times C29 - 30.$

30 *CqNKMpMNp.*

$30 + \prod \times 31.$

31 $Cq\prod pNKMpMNp.$

$31\,q/CpCqp \times C25 - 32.$

32 $\prod pNKMpMNp.$

Theses 18 and 32 contradict each other. Therefore propositions II and III are incompatible.

The demonstration given above could be made intuitively plausible in the following manner: If according to proposition III the expressions "$M\alpha$" and "$MN\alpha$" were jointly true for a certain proposition "α", then the propositions "α" and "$N\alpha$" would also have to be true according to theses 12 and 13. Yet this is impossible, because "α" and "$N\alpha$" contradict each other.

In view of this fact the problem of modal propositions could be solved in two ways, taking the two-valued propositional calculus as a basis. Theorem I and those theses of the first group connected with it (viz. theses 1 and 7–11) have to be accepted unconditionally; they were actually never called in question. Of theorems II and III only one can be selected. If we decide in favour of theorem II and those theses of the second group connected with it (viz. theses 2 and 12–16), then all modal propositions become equivalent to non-modal ones. The consequence of this is that it is not worth while to introduce modal propositions into logic. Also, the extremely intuitive concept of two-sided possibility

must then be rejected as being inconsistent. If, on the other hand, we decide in favour of proposition III, we are compelled to admit the paradoxical consequence that everything is possible. On this condition again it is senseless to introduce modal propositions into logic; moreover, we would then have to do without the intuitively evident theorem II in order to avoid contradiction. None of these solutions can claim to be satisfactory.

A different result was not to be expected. This becomes especially clear when the system of the two-valued propositional calculus is defined by the so-called matrix method. On the basis of this method it is assumed that all propositional variables can take only two constant values, namely "0" or "the false" and "1" or "the true". It is further laid down that:

$C00 = C01 = C11 = 1, \quad C10 = 0, \quad N0 = 1, \quad \text{and} \quad N1 = 0.$

These equations are recorded in the following table, which is the "matrix" of the two-valued propositional calculus based on "*C*" and "*N*".

C	0	1	*N*
0	1	1	1
1	0	1	0

In a two-valued system only four different functions of *one* argument can be formed. If "ϕ" denotes a functor of one argument, then the following cases are possible: (1) $\phi 0 = 0$ and $\phi 1 = 0$; this function we denote by "*Fp*" ("*falsum* of *p*"). (2) $\phi 0 = 0$ and $\phi 1 = 1$; ϕp is equivalent to *p*. (3) $\phi 0 = 1$ and $\phi 1 = 0$; this is the negation of *p*, "*Np*". (4) $\phi 0 = 1$ and $\phi 1 = 1$; this function we denote by "*Vp*" ("*verum* of *p*").

"*Mp*" must be identical with one of these four cases. But each of theses 1, 2, and 18 excludes certain cases. By direct verification with "0" and "1" it can be ascertained that:

$$(A)\begin{cases} 1 & CNMpNp \quad \text{holds only for } Mp = p \text{ or } Mp = Vp. \\ 2 & CNpNMp \quad \text{holds only for } Mp = p \text{ or } Mp = Fp. \\ 18 & N\prod pNKMpMNp \quad \text{holds only for } Mp = Vp. \end{cases}$$

Thesis 18 is verified by the statement: $\prod p\alpha(p) = K\alpha(0)\alpha(1)$. One then obtains:

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$$N\prod pNKMpMNp = NKNKM0MN0NKM1MN1$$
$$= NKNKM0M1NKM1M0$$
$$= NKNKM0M1NKM0M1$$
$$= NNKM0M1 = KM0M1.$$

The last conjunction obtains only on the condition that

$$M0 = M1 = 1.$$

The conditions (A) make it evident that theses 1 and 2 can be valid jointly only for $Mp = p$; just as theses 1 and 18 can be valid only for $Mp = Vp$. Theses 2 and 18 are incompatible, as there is no function for "Mp" which would simultaneously verify both theses.

6. Modal propositions and the three-valued propositional calculus

When I recognized the incompatibility of the traditional theorems on modal propositions in 1920,[9]) I was occupied with establishing the system of the ordinary "two-valued" propositional calculus by means of the matrix method.[10]) I satisfied myself at that time that all theses of the ordinary propositional calculus could be proved on the assumption that their propositional variables could assume only two values, "0" or "the false", and "1" or "the true".

To this assumption corresponds the basic theorem that *every proposition is either true or false*. For short I will term this the *law of bivalence*. Although this is occasionally called the law of the excluded middle, I prefer to reserve this name for the familiar principle of classical logic that two contradictory propositions cannot be false simultaneously.

The law of bivalence is the basis of our entire logic, yet it was already much disputed by the ancients. Known to Aristotle, although contested for propositions referring to future contingencies; peremptorily rejected by the Epicureans, the law of bivalence makes its first full appearance with Chrysippus and the Stoics as a principle of their dialectic, which represents the ancient propositional calculus.[11]) The quarrel about the law of bivalence has a metaphysical background, the advocates of the law being decided determinists, while its opponents tend towards an indeterministic *Weltanschauung*.[12]) Thus we have re-entered the area of the concepts of possibility and necessity.

The most fundamental law of logic seems after all to be not quite evident. Relying on venerable examples, which go back to Aristotle, I tried to refute the law of bivalence by pursuing the following line of thought.*)

I can assume without contradiction that my presence in Warsaw at a certain moment of next year, e.g. at noon on 21 December, is at the present time determined neither positively nor negatively. Hence it is *possible*, but not *necessary*, that I shall be present in Warsaw at the given time. On this assumption the proposition "I shall be in Warsaw at noon on 21 December of next year", can at the present time be neither true nor false. For if it were true now, my future presence in Warsaw would have to be necessary, which is contradictory to the assumption. If it were false now, on the other hand, my future presence in Warsaw would have to be impossible, which is also contradictory to the assumption. Therefore the proposition considered is at the moment *neither true nor false* and must possess a third value, different from "0" or falsity and "1" or truth. This value we can designate by

[9]) In the report cited in note 1 (p. 153) I had defined the concept of two-sided possibility more strictly by assuming that the propositions "it is possible that p" and "it is possible that not-p" must always hold simultaneously, which in conjunction with propositions of the two first groups leads to numerous contradictions. I had in mind here the Aristotelian concept of "pure" possibility. It seems that Aristotle distinguished between two essentially different kinds of possibility: possibility in the proper sense or pure possibility, by which something is only possible if it is not necessary; and possibility in the improper sense, which is connected with necessity and results from it according to our thesis 10. Cf. H. Maier, *Die Syllogistik des Aristoteles*, part i (Tübingen, 1896), pp. 180, 181.

[10]) The results of these inquiries have been published in my article "Logika dwuwartościowa" (Two-valued Logic), which appeared in the Polish philosophical review *Przegląd Filozoficzny* (Studies in honour of Professor Twardowski) 23 (1921), pp. 189–205 [pp. 89–109 of this book].

[11]) Cf. the appendix: "On the history of the law of bivalence", pp. 176 ff.

[12]) In the inaugural address which I delivered as Chancellor of the University of Warsaw in 1922, I tried to solve the problem of an indeterministic philosophy by three-valued logic. A revised version of this lecture will be published shortly in Polish. [In fact, this text ("On Determinism") was published 16 years later by J. Słupecki in the 1961 edition *Z zagadnień logiki i filozofii*, and next, in an English translation in the McCall edition as paper 2, reprinted in this book on pp. 110–128.]

*) In the paper "On Determinism" mentioned above Łukasiewicz gives an example of the reasoning of the same kind.

"$\frac{1}{2}$". It represents "the possible", and joins "the true" and "the false" as a third value.

The three-valued system of propositional logic owes its origin to this line of thought. Next the matrix had to be given by which this new system of logic could be defined. Immediately it was clear to me that if the proposition concerning my future presence in Warsaw took the value $\frac{1}{2}$, its negation must take the same value $\frac{1}{2}$. Thus I obtained the equation $N\frac{1}{2} = \frac{1}{2}$. For implication I still had to determine the five equations containing the value $\frac{1}{2}$, namely $C0\frac{1}{2}$, $C\frac{1}{2}0$, $C\frac{1}{2}\frac{1}{2}$, $C\frac{1}{2}1$, and $C1\frac{1}{2}$. Equations not containing the value $\frac{1}{2}$, I took over from the two-valued system of propositional logic, as well as the values for "$N0$" and "$N1$". The desired equations I obtained on the basis of detailed considerations, which were more or less plausible to me. In this way I finally arrived at the formulation of a three-valued propositional calculus, defined by the matrix below. The system originated in 1920.[13])

C	0	$\frac{1}{2}$	1	N
0	1	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1	0

7. Definition of the concept of possibility

On the basis of this system I then tried to construct a definition of the concept of possibility which would allow me to establish all the intuitive traditional theorems for modal propositions without contradiction. I did this with regard to the concept of "pure" possibility, and soon found a satisfactory definition.[14]) Later on, however, I became

[13]) I reported on this system to the Polish Philosophical Society at Lwów on 19 June, 1920. The essential contents of this report have been published in *Ruch Filozoficzny* 5 (1920), p. 170 [pp. 87–88 of this book].

[14]) The definition found was rather complicated and read thus:

D*1 $Mp = AEpNp\Pi qNCpKqNq.$

That is: The expression "it is possible that p" means "either p and not-p are equivalent to one another, or there is no pair of contradictory propositions implied by p". "A" is the sign of *alternation*; "E" the sign of *equivalence*. In three-valued logic the following definitions hold:

D*2 $Apq = CCpqq.$

D*3 $Kpq = NANpNq.$

convinced that the wider concept of possibility *in general* was to be preferred to the more narrow concept of *pure* possibility. In what follows, therefore, I discuss a definition of the latter concept, which satisfies all the requirements of theorems I–III.

The definition in question was discovered by Tarski in 1921 when he attended my seminars as a student at the University of Warsaw. Tarski's definition is as follows:

D2 $Mp = CNpp.$

Expressed verbally this says: "it is possible that p" means "if not-p then p".

One must grasp the intuitive meaning of this definition. The expression "$CNpp$" is according to the three-valued matrix false if and only if "p" is false. Otherwise "$CNpp$" is true. Thus we obtain the equations:

$$M0 = 0, \quad M\tfrac{1}{2} = 1, \quad M1 = 1.$$

D*4 $Epq = KCpqCqp.$

The definition of "impossibility" is more evident:

D*5 $NMp = KNEpNp\Sigma qCpKqNq.$

That is, the expression "it is not possible that p" means "p and not-p are not equivalent to one another, and there is a pair of contradictory propositions implied by p".

From D*1 the following equations are obtained for "M": $M0 = 0$, $M\frac{1}{2} = 1$, $M1 = \frac{1}{2}$. By means of these equations and the matrix of the three-valued propositional calculus the following theses can be easily verified:

(1) $CpCpNMNp.$

(2) $CNpCNpNMp.$

(3) $CMpCMpMNp.$

(4) $CMNpCMNpMp.$

(5) $CNMpCNMpNp.$

(6) $CNMNpCNMNpp.$

Thesis (5) allows us to obtain by two detachments, in accordance with theorem I and on the basis of the admitted proposition "it is not possible that α" ("$NM\alpha$"), the proposition "not-α" ("$N\alpha$"). Conversely we get by two detachments the proposition "it is not possible that α" ("$NM\alpha$") from thesis (2), in accordance with theorem II, on the basis of the admitted proposition "not-α" ("$N\alpha$"). Furthermore, if one of the propositions "it is possible that α" ("$M\alpha$"), and "it is possible that not-α" ("$MN\alpha$") is admitted, the other of these propositions has to be admitted too, by theses (3) and (4). From the admitted propositions "α" and "it is necessary that α" no inference can be made to the proposition "it is possible that α", since we are dealing here with "pure" possibility, which is incompatible with necessity. Cf. note 9, page 164.

Hence, if any proposition "α" is false, the proposition "it is possible that α" is false too. And if "α" is true, or if it takes the third value, that of "possibility", then the proposition "it is possible that α" is true. This agrees very well with our intuitions.

In two-valued logic the expression "*CNpp*" is equivalent to the expression "*p*"; but not in three-valued logic. The thesis "*CCNppp*", valid in the two-valued calculus, and appearing as an axiom in my system of the ordinary propositional calculus,[15] is not valid for $p = \frac{1}{2}$ in the three-valued system. Vailati has written an interesting monograph on the thesis "*CCNppp*",[16] in which it is shown that Euclid made use of this thesis in demonstrating one of his theorems, without formulating it expressly.[17] It was Clavius, a commentator on Euclid from the second half of the sixteenth century, a Jesuit and the constructor of the Gregorian calendar, who first paid attention to this thesis.[18] Since that time it appears to have acquired a certain popularity among Jesuit scholars under the name *consequentia mirabilis.*[19] The notable Jesuit Gerolamo Saccheri in particular was so taken by the

[15]) Cf. *Elementy logiki matematycznej* (Elements of mathematical logic), a lithographed edition of lectures given by me at the University of Warsaw in the autumn of 1928–1929, revised by M. Presburger (Warsaw, 1929), p. 45. [An English translation made by O. Wojtasiewicz and edited by J. Słupecki (*Elements of Mathematical Logic*) was published as co-edition by PWN and Pergamon Press in 1963 and reprinted in 1966.]

[16]) *Scritti di G. Vailati*, Leipzig–Firenze, 1911. *CXV. A proposito d'un passo del Teereto e di una dimostrazione di Euclide*, pp. 516–527.

[17]) Cf. Vailati, op. cit., pp. 518 ff. It seems to have escaped Vailati that the above-mentioned thesis was already known to the Stoics, although not in its pure form. We read in Sextus Empiricus, *Adv. math.* viii. 292; εἰ τὸ πρῶτον, τὸ πρῶτον· εἰ οὐ τὸ πρῶτον, τὸ πρῶτον· ἤτοι τὸ πρῶτον ἢ οὐ τὸ πρῶτον· τὸ πρῶτον ἄρα. If in this schema the self-evident premisses εἰ τὸ πρῶτον, τὸ πρῶτον and ἤτοι τὸ πρῶτον ἢ οὐ τὸ πρῶτον are omitted, we obtain the consequence εἰ οὐ τὸ πρῶτον, τὸ πρῶτον· τὸ πρῶτον ἄρα, which corresponds to the thesis "*CCNppp*".

[18]) Cf. Vailati, op. cit., p. 521.

[19]) I find the name *consequentia mirabilis* for this thesis in the writings of Polish Jesuits. Adam Krasnodębski, in his *Philosophia Aristotelis explicata* (Warsaw, 1676), *Dialecticae Prolegomenon* 21, writes, for instance, the following: *Artificium argumentandi per consequentiam mirabilem in hoc positum est* (*uti de re speculativa optime in Polonia meritus. R. P. Tho. Młodzianowski Tr. I de Poenit. disp.* 1. *quae.* 1. *difficul.* 1 No. 20 *refert*), *ut ex propositione quam tuetur respondens, ab argumentante eliciatur contradictoria.*

thesis "*CCNppp*" that he attempted to demonstrate Euclid's parallel postulate on the basis of it. The attempt failed, but Saccheri gained the title of being a precursor of non-Euclidean geometry.[20])

The thesis "*CCNppp*" states that if for a certain proposition, say "α", the implication "$CN\alpha\alpha$" holds, then "α" holds too. The implication "if not-α, then α" does not, to be sure, mean the same as the expression "α can be inferred from not-α", yet the more general concept of implication covers the more special case of inference. If therefore from a proposition "not-α" the proposition "α" can be inferred, then "α" is true. It would, however, not be correct to assume with Saccheri that the fact "from not-α is inferred α" stamps the proposition "α" as a *prima veritas.*[21]) On the contrary, the thesis "*CCNppp*" strikes us as outrightly paradoxical, as is also indicated by its name, *consequentia mirabilis. This* alone is certain: if any proposition *can* be inferred from its contradictory opposite, it is certainly not false, hence not impossible either. It is *possible*, as Tarski's definition states. This definition will perhaps be even more obvious, if it is applied to the concept of necessity. For we obtain in accordance with D2:

D3 $NMNp = NCpNp,$

which says that "it is necessary that *p*" means "it is not true that if *p*, then not-*p*". Freely speaking, we can then assert that a certain proposition "α" is necessary, if and only if it does not contain its own negation.

Without stressing the intuitive character of the above definition, we have to admit in any case that this definition meets all of the requirements of theorems I–III. Indeed, as Tarski has shown, it is the *only* positive definition in the three-valued system which meets these requirements. We will now proceed to demonstrate these last assertions.

[20]) Cf. Vailati, op. cit. *CIX. Di un'opera dimenticata del P. Gerolamo Saccheri* ('*Logica demonstrativa*' 1697), pp. 477–484.

[21]) Cf. Vailati, op. cit., p. 526, where the following words of Saccheri are quoted: "Nam hic maxime videtur esse cuiusque primae veritatis veluti character ut non nisi exquisita aliqua redargutione ex suo ipso contradictorio assumpto ut, vero illa ipsi sibi tandem restitui possit" (*Euclides ab omni naevo vindicatus*, p. 99).

8. Consequences of the definition of the concept of possibility

From definition D2 it follows that all theses of the first group are verified, i.e. thesis 1, corresponding to theorem I, and theses 7–11. For in three-valued propositional logic the thesis

T1 $CpCqp$

holds good. We thus obtain:

$T1q/Np \times T2.$

T2 $CpCNpp.$

$T2.D2 \times T3.$

T3 $CpMp.$

In the derivational line belonging to thesis T3 a rule of inference has been used which permits us to replace the right side of a definition by its left. Since all laws of transposition as well as the principle of the syllogism hold true in the three-valued calculus, we obtain all of the remaining theses of the first group from T3. All these theses are perfectly evident.

The theses of the second group are not valid. However, not all these theses are evident in any case. Two of them, of which one corresponds to theorem II, are in a certain sense valid, though not as simple implications. To be exact, by definition D2 the following propositions hold true in the three-valued calculus:

$$CpCpNMNp \quad \text{and} \quad CNpCNpNMp,$$

although the expressions

$$CpNMNp \quad \text{and} \quad CNpNMp$$

are not valid. This is caused by the fact that in the three-valued calculus the thesis "$CCpCpqCpq$" does not hold, and because of this the expressions "$C\alpha C\alpha\beta$" and "$C\alpha\beta$" are not equivalent to each other as they are in the ordinary two-valued calculus. The above-mentioned propositions can be demonstrated by means of the following auxiliary theses, which also hold true in three-valued propositional logic:

T4 $CpCCpqq.$

T5 $CpCCNNpqq.$

T6 $CCpCqrCpCNrNq.$

T7 $CCpCqNrCpCrNq.$

*

$T6p/Np, q/CNpp, r/p \times CT4p/Np, q/p\text{–}T8.$

T8 $CNpCNpNCNpp.$

$T8.D2 \times T9.$

T9 $CNpCNpNMp.$

$T7q/CNNpNp, r/p \times CT5q/Np\text{–}T10.$

T10 $CpCpNCNNpNp.$

$T10.D2p/Np \times T11.$

T11 $CpCpNMNp.$

If the proposition "not-α" is admitted, then by double detachment applied to thesis T9, the proposition "it is not possible that α" is obtained. If the proposition "α" is admitted, then, by T11 and double detachment, one arrives at the proposition: "it is not possible that not-α" which means the same as "it is necessary that α". It can therefore be correctly inferred: "I have no money in my pocket; hence it is not possible that I have money in my pocket." Or again, "I am at home in the evening; hence it is necessary that I am at home in the evening". The intuitively evident theorem II has been shown to hold good, moreover, in such a way that the Aristotelian maxime is maintained, according to which not everything which is is necessary and not everything which is not is impossible. For the expressions "α" and "$NMN\alpha$" as well as "$N\alpha$" and "$NM\alpha$" are not equivalent to each other. Nor can being be inferred from possibility, as long as "Mp" means the same as "$CNpp$", since neither "$CMpp$" nor "$CMpCMpp$" holds true in the three-valued propositional calculus.

Finally, theorem III is verified in the form of the theses:

T12 $\sum pKMpMNp$

or

T13 $N\prod pNKMpMNp,$

in which the following definitions are assumed:

D4 $Apq = CCpqq.$

D5 $Kpq = NANpNq.$

Theses T12 and T13 are easily verified with the help of the matrix of the three-valued calculus and the equations given for "M" in the preceding section. For $p = \frac{1}{2}$ we obtain:

$$KM\tfrac{1}{2}MN\tfrac{1}{2} = K1M\tfrac{1}{2} = K11 = 1.$$

There is, therefore, a value for p for which the expression "$KMpMNp$" is correct.

As a résumé of the above findings we are now able to establish the following theorem:

All the traditional theorems for modal propositions have been established free of contradiction in the three-valued propositional calculus, on the basis of the definition "$Mp = CNpp$".

This result seems to me highly significant. For it appears that those of our intuitions which are connected with the concepts of possibility and necessity point to a system of logic which is fundamentally different from ordinary logic based on the law of bivalence.

It remains to prove that the definition given by Tarski is the only one in the three-valued calculus which meets the requirements of theorems I–III. This can be shown in the following manner. Since according to theorem I the proposition "$N\alpha$" follows from the proposition "$NM\alpha$", by the law of transposition "$M\alpha$" must follow from "α". Hence, if $\alpha = 1$, then $M\alpha = M1 = 1$. We thus obtain the equation $M1 = 1$. On the other hand, according to theorem II the proposition "$NM\alpha$" follows from the proposition "$N\alpha$". Hence if $\alpha = 0$, or $N\alpha = 1$, then $NM\alpha = NM0 = 1$. But $NM0$ can equal 1 only under the condition that $M0 = 0$. We thus obtain the second equation: $M0 = 0$. Finally also theorem III, "$\sum pKMpMNp$", must be true. But it is not true for $p = 0$ or $p = 1$, for in both cases one term of the conjunction is false; hence the conjunction itself must be false too. We therefore have to assume that $M\frac{1}{2} = 1$, since only then does the conjunction "$KMpMNp$" equal 1 for $p = \frac{1}{2}$. In this way the function "Mp" is fully determined for the three-valued propositional calculus, and can be defined only by "$CNpp$" or by some other expression equivalent to it.

9. Philosophical significance of many-valued systems of propositional logic

Besides the three-valued system of propositional logic, I discovered an entire class of closely related systems in 1922, which I defined by means of the matrix method in the following manner:

When "p" and "q" denote certain numbers of the interval (0, 1), then:

$$\begin{aligned} Cpq &= 1 && \text{for } p \leqslant q, \\ Cpq &= 1-p+q && \text{for } p > q, \\ Np &= 1-p. \end{aligned}$$

If only the limiting values 0 and 1 are chosen from the interval (0, 1), the above definition represents the matrix of the ordinary two-valued propositional calculus. If, in addition, the value $\frac{1}{2}$ is included, we obtain the matrix of the three-valued system. In a similar manner 4, 5, ..., n-valued systems can be formed.

It was clear to me from the outset that among all the many-valued systems only two can claim any philosophical significance: the three-valued and the infinite-valued ones.*) For if values other than "0" and "1" are interpreted as "the possible", only two cases can reasonably be distinguished: either one assumes that there are no variations in degree of the possible and consequently arrives at the three-valued system; or one assumes the opposite, in which case it would be most natural to suppose (as in the theory of probabilities) that there are infinitely many degrees of possibility, which leads to the infinite-valued propositional calculus. I believe that the latter system is preferable to all others. Unfortunately this system has not yet been investigated sufficiently; in particular the relation of the infinite-valued system to the calculus of probabilities awaits further inquiry.[22])

If the definition of possibility established by Tarski is assumed for the infinite-valued system, there result, as in the three-valued system, all theses mentioned in the preceding section. The intuitively evident theorems I–III are therefore also verified in the infinite-valued propositional calculus.

The three-valued system is a proper part of the two-valued, just as the infinite-valued system is a proper part of the three-valued one. This means that all theses of the three- and infinite-valued systems (without quantifiers) hold true for the two-valued system. There are, however, theses which are valid in the two-valued calculus but not

[22]) My little book *Die logischen Grundlagen der Wahrscheinlichkeitsrechnung*, Cracow, 1913, Akad. d. Wiss., tries to base the notion of probability on quite a different idea.

*) In his "A System of Modal Logic" (pp. 352–390 of this book) Łukasiewicz holds a clearly different opinion on this issue.

in the infinite-valued system. But when it is a question of the best known propositional theses—for instance those listed in *Principia Mathematica*[23])—the difference between the three-valued and the infinite-valued propositional calculus is minimal. To be sure, I cannot find a single thesis in this work that would be valid in the three-valued system without being also true in the infinite-valued one.

The most important theses of the two-valued calculus which do not hold true for the three- and infinite-valued systems concern certain apagogic inference schemata that have been suspect from time immemorial. For example, the following theses do not hold true in many-valued systems: "*CCNppp*", "*CCpNpNp*", "*CCpqCCpNqNp*", "*CCpKqNqNp*", "*CCpEqNqNp*". The first of these theses has been discussed above; the second differs from the first only by the introduction of the negation of *p* for *p*. The two other theses justify us in assuming a proposition "$N\alpha$" to be true, when from its opposite "α" two mutually contradictory propositions can be derived. The last thesis asserts that a proposition from which the equivalence of two contradictory propositions follows is incorrect. There are modes of inference in mathematics, among others the so-called "diagonal method" in set theory, which are founded on such theses not accepted in the three- and infinite-valued systems of propositional logic. It would be interesting to inquire whether mathematical theorems based on the diagonal method could be demonstrated without propositional theses such as these.

Although many-valued systems of propositional logic are merely fragments of the ordinary propositional calculus, the situation changes entirely when these systems are extended by the addition of the universal quantifier. There are theses of the extended many-valued systems which are not valid in the two-valued system. T13 serves as an example of such a thesis. If the expression "*Mp*" in T13 is replaced in accordance with D2 by "*CNpp*", and "*MNp*" by "*CNNpNp*", we obtain the thesis:

T14 $\quad N\prod pNKCNppCNNpNp,$

which is false in the two-valued calculus. The three-valued system of propositional logic with quantifiers, which owing to the research of Tarski and Wajsberg can be represented axiomatically, is the simplest

[23]) Cf. A. N. Whitehead and B. Russell, *Principia Mathematica* (Cambridge, 1910), vol. i, pp. 94–131.

example of a consistent logical system which is as different from the ordinary two-valued system as any non-Euclidean geometry is from the Euclidean.

I think it may be said that the system mentioned is the *first* intuitively grounded system differing from the ordinary propositional calculus. It was the main purpose of this communication to prove that this intuitive basis lay in the theorems I–III, which are intuitively evident for modal propositions, but which are not jointly tenable in ordinary logic. It is true that Post has investigated many-valued systems of propositional logic from a purely formal point of view, yet he has not been able to interpret them logically.[24]) The well-known attempts of Brouwer,*) who rejects the universal validity of the law of the excluded middle and also repudiates several theses of the ordinary propositional calculus, have so far not led to an intuitively based *system*. They are merely fragments of a system whose construction and significance are still entirely obscure.[25])

It would perhaps not be right to call the many-valued systems of propositional logic established by me "non-Aristotelian" logic, as Aristotle was the first to have thought that the law of bivalence could not be true for certain propositions. Our new-found logic might be rather termed "non-Chrysippean", since Chrysippus appears to have been the first logician to consciously set up and stubbornly defend the theorem that every proposition is either true or false. This Chrysippean theorem has to the present day formed the most basic foundation of our entire logic.

It is not easy to foresee what influence the discovery of non-Chry-

[24]) See E. L. Post, "Introduction to a general theory of elementary propositions", *Am. Journ. of Math.* 43 (1921), p. 182: "... the highest dimensioned intuitional proposition space is two."

[25]) Cf., e.g., L. E. J. Brouwer, "Intuitionistische Zerlegung mathematischer Grundbegriffe", *Jahresber. d. Deutsch. Math.-Vereinigung* 33 (1925), pp. 251 ff.; "Zur Begründung der intuitionistischen Mathematik. I", *Math. Ann.* 93 (1925), pp. 244 ff.

*) In 1930, when this article appeared, the results obtained by A. Heyting and expressing Brouwer's intuitions in the form of a formalized logical system were not yet published. In his paper "On the Intuitionistic Theory of Deduction" (pp. 325–340 of this book) Łukasiewicz says of that system: "It seems to me that among the hitherto known many-valued systems of logic the intuitionistic theory is the most intuitive and elegant".

sippean systems of logic will exercise on philosophical speculation. However, it seems to me that the philosophical significance of the systems of logic treated here might be at least as great as the significance of non-Euclidean systems of geometry.

APPENDIX

On the history of the law of bivalence

The law of bivalence, i.e. the law according to which every proposition is either true or false, was familiar to Aristotle, who explicitly characterized a proposition, ἀπόφανσις, as discourse which is either true or false. We read in *De interpr.* 4. $17^{a}2$: ἀποφαντικὸς δὲ (scil. λόγος; λόγος ἀποφαντικός = ἀπόφανσις) οὐ πᾶς, ἀλλ' ἐν ᾧ τὸ ἀληθεύειν ἢ ψεύδεσθαι ὑπάρχει. Aristotle, however, does not accept the validity of this law for propositions dealing with contingent future events. The famous chapter 9 of *De interpretatione* is devoted to this matter. Aristotle believes that determinism would be the inevitable consequence of the law of bivalence, a consequence he is unable to accept. Hence he is forced to restrict the law. He does not, however, do this decisively enough, and for this reason his way of putting the matter is not quite clear. The most important passage reads as follows (*De interpr.* 9.$19^{a}36$): τούτων γὰρ (scil. τῶν μὴ ἀεὶ ὄντων ἢ μὴ ἀεὶ μὴ ὄντων) ἀνάγκη μὲν θάτερον μόριον τῆς ἀντιφάσεως ἀληθὲς εἶναι ἢ ψεῦδος, οὐ μέντοι τόδε ἢ τόδε ἀλλ' ὁπότερ ἔτυχε, καὶ μᾶλλον μὲν ἀληθῆ τὴν ἑτέραν, οὐ μέντοι ἤδη ἀληθῆ ἢ ψευδῆ. Another passage of *De interpretatione*, viz. $18^{b}8$: τὸ γὰρ ὁπότερ' ἔτυχεν οὐδὲν μᾶλλον οὕτως ἢ μὴ οὕτως ἔχει ἢ ἕξει, allowed the Stoics to maintain that Aristotle denied the law of bivalence. Thus we find in Boethius, *Ad Arist. de interpr.*, ed. secunda, rec. Meiser, p. 208 (ed. Bas., p. 364), the passage: "putaverunt autem quidam, quorum Stoici quoque sunt, Aristotelem dicere in futuro contingentes nec veras esse nec falsas". The Peripatetics attempted to defend Aristotle against this objection by puzzling out a "distinction" between the *definite verum* and the *indefinite verum*, non-existent in the Stagirite's works. Thus Boethius says (*Ad Arist. de interpr.*, ed. prima, rec. Meiser, p. 125): "manifestum esse non necesse esse omnes adfirmationes et negationes *definite* veras esse (sed deest 'definite' atque ideo subaudiendum est)". The sentence in parentheses has been taken almost literally from Greek commentators. Cf. Ammonius, *In librum Arist. de interpr.*, ed. Busse, p. 141, 20: προσυπακουομένου δηλονότι τοῦ "ἀφωρισμένως".

There can be no doubt that the Epicureans, who embraced an indeterministic *Weltanschauung*, made Aristotle's idea their own. One of the most important passages bearing witness to this has been transmitted to us by Cicero, *De fato* 37: "Necesse est enim in rebus contrariis duabus (contraria autem hoc loco ea dico, quorum alterum ait quid, alterum negat) ex his igitur necesse est, invito Epicuro, alterum verum esse, alterum falsum: at 'sauciabitur Philocteta', omnibus ante seculis verum fuit, 'non sauciabitur', falsum. Nisi forte volumus Epicureorum opinionem sequi, qui tales enuntiationes *nec veras nec falsas esse dicunt*: aut, cum id pudet, illud tamen dicunt, quod est impudentius, veras esse ex contrariis disiunctiones; sed, quae in his enuntiata essent, eorum neutrum esse verum." Cicero opposes this opinion and then continues: "Tenebitur ergo id quod a Chrysippo defenditur: omnem enuntiationem aut veram aut falsam esse". That not only the Epicureans shared the opinion of Aristotle, follows from a passage of Simplicius, *In Arist. cat.*, ed. Kalbfleisch, p. 406 (f. 103A ed. Bas.): "ὁ δὲ Νικόστρατος αἰτιᾶται κἀνταῦθα λέγων μὴ ἴδιον εἶναι τῶν κατὰ ἀντίφασιν ἀντικειμένων τὸ διαιρεῖν τὸ ἀληθὲς καὶ τὸ ψεῦδος. ... αἱ γὰρ εἰς τὸν μέλλοντα χρόνον ἐγκεκλιμέναι προτάσεις οὔτε ἀληθεῖς εἰσιν οὔτε ψευδεῖς διὰ τὴν τοῦ ἐνδεχομένου φύσιν· οὔτε γὰρ τὸ 'ἔσται ναυμαχία' ἀληθὲς οὔτε τὸ 'οὐκ ἔσται', ἀλλ ὁπότερον ἔτυχεν". The latter example is borrowed from Aristotle's *De interpr.* 9. $19^{a}30$. For Nikostratos see Prantl, vol. i, pp. 618–620.

In conscious opposition to this, the Stoics, as outspoken determinists, and especially Chrysippus, established the law of bivalence as the fundamental principle of their dialectic. As evidence the following quotations, taken from J. v. Arnim's *Stoicorum veterum fragmenta*, vol. ii, may be cited: (1) Page 62, fr. 193: *Diocles Magnes apud Diog. Laert.* vii. 65: ἀξίωμα δέ ἐστιν ὅ ἐστιν ἀληθὲς ἢ ψεῦδος. (2) Page 63, fr. 196: Cicero, *Acad. Pr.* ii. 95: "Fundamentum dialecticae est, quidquid enuntietur (id autem appellant ἀξίωμα—) aut verum esse aut falsum." (3) Page 275, fr. 952: Cicero, *De fato* 20: "Concludit enim Chrysippus hoc modo: 'Si est motus sine causa, non omnis enuntiatio, quod ἀξίωμα dialectici appellant, aut vera aut falsa erit; causas enim efficientis quod non habebit, id nec verum nec falsum erit. Omnis autem enuntiatio aut vera aut falsa est. Motus ergo sine causa nullus est. 21. Quod si ita est, omnia, quae fiunt, causis fiunt antegressis. Id si ita est, fato omnia fiunt Efficitur igitur fato fieri, quaecunque fiant.' ... Itaque contendit omnis nervos Chrysippus ut persuadeat omne ἀξίωμα aut verum esse aut falsum."

I have compiled thus many quotations on purpose, for, although they illuminate one of the most important problems of logic, it nevertheless appears that many of them were either unknown to the historians of logic, or at least not sufficiently appreciated. The reason for this is in my opinion that the history of logic has thus far been treated by philosophers with insufficient training in logic. The older authors cannot be blamed for this, as a scientific logic has existed only for a few decades. *The history of logic must be written anew*, and by an historian who has a thorough command of modern mathematical logic. Valuable as Prantl's work is as a compilation of sources and materials, from a logical point of view it is practically worthless. To give only one illustration of this, Prantl, as well as all the later authors who have written about the logic of the Stoa, such as Zeller and Brochard, have entirely misunderstood this logic. For anybody familiar with mathematical logic it is self-evident *that the Stoic dialectic is the ancient form of modern propositional logic.*[26])

Propositional logic, which contains only propositional variables, is as distinct from the Aristotelian syllogistic, which operates only with name variables, as arithmetic is from geometry. The Stoic dialectic is not a development or supplementation of Aristotelian logic, but an achievement of equal rank with that of Aristotle. In view of this it seems only fair to demand of an historian of logic that he know something about logic. Nowadays it does not suffice to be merely a philosopher in order to voice one's opinion on logic.

26) I have already expressed this idea, in 1923, in a paper read to the first congress of Polish philosophers in Lwów. A short summary of it appeared in *Przegląd Filozoficzny* 30 (1927), p. 278. [Łukasiewicz develops his historical analysis of Stoic logic in his article "On the History of the Logic of Propositions" (pp. 197–217 of this book).]

COMMENTS ON NICOD'S AXIOM AND ON "GENERALIZING DEDUCTION" *)

In the present paper I use the following bibliographical abbreviations:

"Ajdukiewicz" for "*Główne zasady metodologii nauk i logiki formalnej* (Fundamental principles of the methodology of science and of formal logic). Lectures delivered by Professor K. Ajdukiewicz at the University of Warsaw in the academic year 1927/1928. Authorized lecture notes edited by M. Presburger. Publications of the Association of Students of Mathematics and Physics of the University of Warsaw. Vol. XVI, 1928."

"Kotarbiński" for "Tadeusz Kotarbiński, *Elementy teorii poznania, logiki formalnej i metodologii nauk* (Elements of epistemology, formal logic, and the methodology of science). The Ossolineum Publishers, Lwów 1929."**)

"Leśniewski" for "Dr. Phil. Stanisław Leśniewski, a.o. Professor der Philosophie der Mathematik an der Universität Warszawa, *Grundzüge eines neuen Systems der Grundlagen der Mathematik*, Einleitung und §§ 1–11. Sonderabdruck (mit unveränderter Pagination) aus dem XIV. Bande der *Fundamenta Mathematicae*. Warsaw, 1929."

"Łukasiewicz (1)" for "Jan Łukasiewicz, *O znaczeniu i potrzebach logiki matematycznej* (On the significance and requirements of mathematical logic). *Nauka Polska*, Vol. X, Warsaw 1929."

"Łukasiewicz (2)" for "Dr. Jan Łukasiewicz, Professor of the University of Warsaw, *Elementy logiki matematycznej* (Elements of mathematical logic). Authorized lecture notes prepared by M. Presburger.

*) First published as "Uwagi o aksjomacie Nicoda i 'dedukcji uogólniającej'" in *Księga pamiątkowa Polskiego Towarzystwa Filozoficznego*, Lwów, 1931. Reprinted in the 1961 edition *Z zagadnień logiki i filozofii*.

**) Available in English under the title *Gnosiology* (published jointly in 1966 by Zakład Narodowy im. Ossolińskich and Pergamon Press).

FILOSOFIA Y LETRAS

Publications of the Association of Students of Mathematics and Physics of the University of Warsaw. Vol. XVIII, 1929."*)

The items "Ajdukiewicz" and "Łukasiewicz (2)" are lithographed.

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At the beginning of his German-language treatise on the foundations of mathematics Dr. Leśniewski parenthetically mentions (with my approval) a certain "simplification" of Nicod's axiom, made by me in 1925 and consisting in the reduction of the different variables that occur in this axiom from five to four.[1]) Since I have not so far published this result of my research, I shall do it in this paper so that Dr. Leśniewski's reference, based only on a manuscript source, may have a foundation in a printed publication.

I do this the more willingly as I can at the same time settle another issue. In transforming Nicod's axiom I encountered for the first time a case of deductive inference in which the conclusion is more general than the premiss. The second part of the present paper is concerned with that "generalizing deduction", which may prove to be of interest not only to logicians, but to philosophers as well.

I

1. Nicod's axiom can, with the use of parentheses, be written in the following way:[2])

(K) $\{p/(q/r)\}/[\{t/(t/t)\}/\{(s/q)/((p/s)/(p/s))\}]$.

In the parenthesis-free symbolism this becomes:[3])

(N) $DDpDqrDDtDttDDsqDDpsDps$.

The symbol "D", which corresponds to the symbol "/", is the only constant occurring in this axiom; all other symbols, that is lower-case

[1]) Cf. Leśniewski, p. 10.

[2]) Cf. Kotarbiński, p. 247 (quoted after the English translation).

[3]) I came upon the idea of a parenthesis-free notation in 1924. I used that notation for the first time in my article Łukasiewicz (1), p. 610, footnote. See also Łukasiewicz (2) pp. 7 and 38, and Kotarbiński, p. 244.

*) An English translation entitled *Elements of Mathematical Logic* is available now (published jointly in 1963 by the Polish Scientific Publishers and Pergamon Press).

letters, are propositional variables. A function of the type "$D\alpha\beta$" means the same as "if α, then it is not true that β", or "it is not true that (α and β)". Thus "D" is a proposition-forming functor of two propositional arguments; this means that in functions of the type "$D\alpha\beta$" both "α" and "β" are propositions, and "$D\alpha\beta$" is a proposition too.[4]) Dr. Sheffer has demonstrated that a function of this type can be used to define all other functions of the theory of deduction.*) Nicod's axiom, together with definitions, suffices to lay the foundations for the entire theory of deduction.[5])

If we bear in mind that the functor "D" always precedes its arguments, and that its two arguments are propositions, we can easily analyse the structure of axiom (N). We only have to realize which propositions belong to the various occurrences of "D" as their arguments. For instance, the third "D" has the proposition "q" as its first argument, and the proposition "r" as its second argument. The second "D" has the proposition "p" as its first argument, and the proposition "Dqr" as its second argument. Further analysis is made easier by the comparison of expressions (K) and (N).

Nicod's axiom is not self-evident. I shall not try to explain its content. That it is a true proposition one can verify by the zero-one verification method, assuming the following equations:[6])

$$D00 = 1, \qquad D01 = 1,$$
$$D10 = 1, \qquad D11 = 0.$$

"0" here stands for a false proposition, while "1" stands for a true proposition. By substituting in (N) 0's and 1's for the variables in any combinations we always obtain 1 after reductions performed in accord-

[4]) The term "functor" comes from Kotarbiński. Cf. Ajdukiewicz, p. 147. The term "proposition-forming" was, as far as I know, first used by Ajdukiewicz. Cf. Ajdukiewicz, p. 16.

[5]) Historical and bibliographical information concerning the works of Sheffer and Nicod can be found in Leśniewski, pp. 9–10. Definitions of some functions, best known in propositional calculus, by means of the symbol "/" or "D" are given in Kotarbiński, p. 172, and Łukasiewicz (2), pp. 56–57.

[6]) For the zero-one verification method see Kotarbiński, pp. 159–163

*) Today, instead of "theory of deduction" we prefer the term "propositional calculus".

ance with the equations quoted above. For instance, if we put $p/0$, $q/1$, $r/0$, $s/1$, $t/0$, we obtain:

$$DD0D10DD0D00DD11DD01D01 = DD01DD01D0D11 = D1D1D00 = D1D11 = D10 = 1.$$

In deducing consequences from his axiom Nicod uses the rule of substitution and the rule of detachment.

The rule of substitution, which he does not formulate,[7]) permits us to join to the system those theses which are obtained from theses already belonging to the system by the substitution for variables of significant expressions of the system. In the system in question, every lower-case letter is a significant expression, as is the expression "$D\alpha\beta$" if both "α" and "β" are significant expressions. All significant expressions are propositions.

The rule of detachment is adopted by Nicod in the form which is equivalent to the following formulation: if a thesis of the type "$D\alpha D\beta\gamma$" belongs to the system, as does a thesis of the form of "α", then a thesis of the form of "γ" may be joined to the system. This rule becomes self-evident if we note that the expression "$D\alpha D\beta\gamma$" means the same as "if α, then it is not true that $D\beta\gamma$", and the expression "it is not true that $D\beta\gamma$" means the same as "it is not true that [it is not true that (β and γ)]", that is, "β and γ". Hence the expression "$D\alpha D\beta\gamma$" means the same as "if α, then β and γ". Hence if the whole of such an expression is asserted, and if "α" is also asserted, we may assert both "β" and "γ". But Nicod's rule of detachment disregards the expression "β" so that we may not assert that expression on the strength of that rule, nor is it necessary for us to know that it has been asserted, if we want to apply that rule.

2. The transformation which I made in Nicod's axiom consists in this, that I replaced the variable "t" by "s", thus obtaining the following thesis:

(Ł) $\qquad DDpDqrDDsDssDDsqDDpsDps.$

The thesis (Ł) includes four different variables, "p", "q", "r", and "s", whereas Nicod's axiom (N) includes five, that is the four enumerated above and also the variable "t". Nevertheless, the theses (N) and (Ł)

7) Cf. Leśniewski, p. 10.

are equivalent, for it can be shown, by means of the rules of substitution and detachment accepted in the system, that (Ł) is a consequence of (N) and conversely (N) is a consequence of (Ł).

The proof of the first theorem, stating that (Ł) is a consequence of (N), is very easy, for it suffices to substitute "s" for "t" in (N) to obtain (Ł). The proof of the other theorem, stating that (N) is a consequence of (Ł), is not so simple and requires repeated application of the rules of substitution and detachment. That proof is recorded below by a method which must first be explained.

The starting point of the proof is the thesis (Ł), which in the proof is marked by the ordinal number "1". The terminal point of the proof is Nicod's axiom (N), which is marked by the ordinal number "11". All theses marked with ordinal numbers, except for the first, are those steps of the proof which are obtained on the strength of the rule of detachment. In the proof I note down only the theses obtained by detachment; I do not note down the theses obtained by substitution, but only mark the substitutions to be performed in order to obtain these theses.

Each thesis, except for the first, is preceded by a non-numbered line which is the "proof line" of the thesis that follows. Each proof line consists of two parts, separated from one another by a cross.*) In the first part, which precedes the cross, I mark the substitutions to be performed in some earlier thesis, already recorded in the proof. In the second part, which follows the cross, I note down the structure of the thesis obtained by means of the substitution marked before the cross, and I do it in such a way as to make it clear that the rule of detachment may be applied to that thesis. For instance, in the first part of the proof line of Thesis 2: "$1p/DpDqr$, $q/DsDss$, $r/DDsqDDpsDps$, s/t" I mark that a thesis is to be formed by substituting in 1 the expression "$DpDqr$" for "p", the expression "$DsDss$" for "q", the expression "$DDsqDDpsDps$" for "r", and the expression "t" for "s". On performing these substitutions we obtain Thesis (A), which is a step in the proof, but is not recorded in the proof in order to make the proof shorter:

*) In this paper, and in some others of his works, Łukasiewicz used an asterisk instead of a cross. For the sake of uniformity, in this volume the asterisk has been replaced everywhere by the cross.

(A) *DDDpDqrDDsDssDDsqDDpsDpsDDtDttDDtDsDssDDDpDqrtDDp Dqrt.*

In the second part of this proof line: "*D*1*D*6-2", I mark what is the structure of Thesis (A) just formed. It begins with the letter "*D*", followed by an expression of the form of Thesis 1, next followed by another "*D*" and an expression of the form of Thesis 6, and ends with an expression of the form of Thesis 2. This shows that the rule of detachment may be applied to Thesis (A), for it is a thesis of the type "$D\alpha D\beta\gamma$", belongs to the system as a substitution of Thesis 1, and the expression which occurs in place of "α" also belongs to the system as it is of the form of Thesis 1. Thus the expression which occurs in place of "γ" may be "detached" from Thesis (A) and joined to the system as Thesis 2. An expression of the form of Thesis 6, to be obtained later on, occurs in the place of "β", but we know already that the expression "β" does not intervene in the application of the rule of detachment.

Now that the reader understands the method of writing down the proof, he can easily check all the proof lines. The best way is to take two sheets of paper, perform on one of them all the substitutions marked in the first part of a given proof line, and write out on the other the thesis occurring in the second part of that proof line. In this way the reader should obtain on both sheets identical expressions. Note that the sequence of symbols "*qr*/*DDpDqrt*" indicates that the expression "*DDpDqrt*" is to be substituted for both "*q*" and "*r*".

Here is the proof of the theorem stating that (N) is a consequence of (Ł):

1 *DDpDqrDDsDssDDsqDDpsDps.* (Ł)

1 *p*/*DDpDqr*, *q*/*DsDss*, *r*/*DDsqDDpsDps*, *s*/*t* × *D*1*D*6 – 2.

2 *DDtDsDssDDDpDqrtDDpDqrt.*

1 *p*/*DtDsDss*, *qr*/*DDpDqrt*, *s*/*w* × *D*2*D*6*t*/*w* – 3.

3 *DDwDDpDqrtDDDtDsDsswDDtDsDssw.*

3 *w*/*DpDqr*, *pqr*/*s*, *t*/*DDsqDDpsDps*, *s*/*t* × *D*1*D*4 – 4.

4 *DDDDsqDDpsDpsDtDttDpDqr.*

2 *t*/*DDDstDDtsDtsDtDtt*, *s*/*t* × *D*4*qpr*/*tD*5 – 5.

5 *DDpDqrDDDstDDtsDtsDtDtt.*

5 *p*/*DtDsDss*, *qr*/*DDpDqrt* × *D*2*D*7 – 6.

6 *DtDtt.*

1 *pqr*/*t* × *D*6*D*6*t*/*s* – 7.

7 *DDstDDtsDts.*

1 *p*/*Dst*, *qr*/*Dts*, *s*/*r* × *D*7*D*6*t*/*r* – 8.

8 *DDrDtsDDDstrDDstr.*

7 *s*/*DDDsqDDpsDpsDtDtt*, *t*/*DpDqr* × *D*4*D*9 – 9.

9 *DDpDqrDDDsqDDpsDpsDtDtt.*

8 *r*/*DpDqr*, *t*/*DDsqDDpsDps*, *s*/*DtDtt* × *D*9*D*10 – 10.

10 *DDDtDttDDsqDDpsDpsDpDqr.*

7 *s*/*DDtDttDDsqDDpsDps*, *t*/*DpDqr* × *D*10*D*11 – 11.

11 *DDpDqrDDtDttDDsqDDpsDps.* (N)

In this proof the rule of detachment is used 10 times, and the rule of substitution 11 times, for we have to count not only those substitutions which are marked on the left side of each of the 10 proof lines, but also the substitution "4*qpr*/*t*", marked in the second part of the proof line of Thesis 5. On the other hand, I do not count the substitutions "6*t*/*w*", "6*t*/*s*", "6*t*/*r*", marked in the proof lines of Theses 3, 7, and 8, since they pertain to those expressions which are disregarded in the detachment. Thus, in order to pass from Nicod's axiom (N) to my axiom (Ł) it is necessary to perform 21 steps of proof. I do not know how to reduce that number.[8]

The proof is complete, although it is recorded in an abbreviated manner. Moreover, the proof is formalized, which means that any one who knows the rules of inference used in the proof can verify the correctness of the proof by referring exclusively to the form of the theses and disregarding their meanings.

My axiom may be considered as a simplification of Nicod's axiom if both are noted down not by means of real, i.e., free, variables, that is if both axioms are preceded by universal quantifiers which bind the variables occurring in the axioms. On introducing an expression of the type "$\prod\alpha$", which means "for every α" and using the parenthesis-free notation of expressions with quantifiers,[9] we obtain the following

[8]) Leśniewski, p. 10, mentions 24 steps of the proof. In fact, the proof in my manuscript of 1925, which was the basis of Dr. Leśniewski's reference, had that many steps. Now, then preparing that proof for publication I have succeeded in simplifying it by reducing the number of steps by three.

[9]) Cf. Łukasiewicz (2), pp. 78 ff.

theses:

(No) $\prod p \prod q \prod r \prod s \prod t DDpDqrDDtDttDDsqDDpsDps.$

(Ło) $\prod p \prod q \prod r \prod s DDpDqrDDsDssDDsqDDpsDps.$

In this form my axiom is shorter, and hence simpler, than Nicod's axiom.[10]

II

3. The above considerations would, perhaps, have little significance had they not revealed a certain logical fact which at first seemed paradoxical to me. Let us compare once more the axioms (N) and (Ł):

(N) $DDpDqrDDtDttDDsqDDpsDps.$

(Ł) $DDpDqrDDsDssDDsqDDpsDps.$

Both axioms are valid for any values of the variables occurring in them. But whereas in axiom (N) we may substitute for the variables "*s*" and "*t*" any propositions, either the same, i.e., of identical form, or different, in the corresponding places of axiom (Ł) we may substitute only the same propositions. This is so because only one variable "*s*" in axiom (Ł) corresponds to the different variables "*s*" and "*t*" in axiom (N). (Ł) can be obtained from (N) by the "identification" of the variables "*s*" and "*t*", that is, by the substitution of the variable "*s*" for the variable "*t*", but (N) can in no way be obtained from (Ł) by substitution alone. Axiom (N) is more general than axiom (Ł), and axiom (Ł) is a special case of axiom (N). And yet there is a deductive proof which demonstrates that the more general thesis (N) follows as a conclusion from the less general thesis (Ł) as its only premiss. I have thus encountered a previously unknown and unexpected case of generalizing deduction.

[10]) Axiom (N) and (Ł) are not organic. We call "organic" a thesis of a system, no part of which is a thesis of that system. The term "organic" was in that sense first used by Dr. Leśniewski, while the definition of an "organic" thesis comes from Mr. Wajsberg. Axioms (N) and (Ł) are not organic, since some of their parts, namely "*DtDtt*" or "*DsDss*", respectively, are theses of the system. In 1927, when he knew the result of my research presented in this paper, Mr. Wajsberg demonstrated that Nicod's axiom can be equivalently replaced by the following organic thesis:

(W) $DDpDqrDDDsrDDpsDpsDpDpq.$

This result forms part of Mr. Wajsberg's M. A. thesis, not published.

When I realized the significance of this fact I started a search for similar examples in the ordinary system of the theory of deduction. I soon found such examples among the theses which include implication only. I shall discuss below the simplest of those examples.

In the implicational system the sole primitive expression is a function of the type "$C\alpha\beta$".[11]) By the expression "$C\alpha\beta$" I mean the conditional proposition, i.e., the implication, "if α, then β". In this expression "*C*" is a proposition-forming functor of two propositional arguments. Implicational theses are noted down without parentheses in a way similar to that used above with the theses with the functor "*D*".

The rule of substitution in this system is the same as in Nicod's system. Any expression that is significant in the system may be substituted for a variable. Any lower-case letter and any expression of the type "$C\alpha\beta$", if "α" and "β" are significant expressions, is a significant expression. The rule of detachment is formulated as follows: if a thesis of the type "$C\alpha\beta$" belongs to the system, and if the thesis of the form of "α" also belongs to the system, then the thesis of the form of "β" may be joined to the system.

By means of these rules we may demonstrate the equivalence of the following two theses:[12])

1 $CqCqCrCsr,$

5 $CpCqCrCsr.$

Thesis 5 includes four different variables, while Thesis 1 includes only three such variables. Thesis 1 can be deduced from Thesis 5 by substituting in 5 the variable "*q*" for the variable "*p*". Thesis 5 can be inferred from Thesis 1 by substitution and detachment.

Here is the complete proof, noted down in an abbreviated form in a manner analogous to the proof of thesis (N) on the strength of thesis (Ł):

1 $CqCqCrCsr.$

$1\ q/CqCqCrCsr \times C1-2.$

2 $CCqCqCrCsrCrCsr.$

[11]) On the meaning of this function cf. Łukasiewicz (2), pp. 28–31. On the axioms of the implicational system see Łukasiewicz (2), p. 47 [see also the end of the present article and footnote*), p. 196 of this article].

[12]) Cf. Łukasiewicz (2), pp. 44–45, where this example is given for the first time, together with a mention about generalizing deduction.

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$2 \times C1-3.$

3 $CrCsr.$

$3r/CrCsr, s/q \times C3-4.$

4 $CqCrCsr.$

$3r/CqCrCsr, s/p \times C4-5.$

5 $CpCqCrCsr.$

Let me add for explanation's sake that in this proof, on the left side of the first proof line, there is an indication of the substitution "$1\, q/CqCqCrCsr$", the performance of which yields the following thesis, not recorded in the proof:

(T) $CCqCqCrCsrCCqCqCrCsrCrCsr.$

The structure of Thesis (T) is recorded on the right side of this proof line in the symbols "$C1$–2". This thesis begins with the letter "C", followed first by an expression of the form of Thesis 1, and next by an expression of the form of Thesis 2. This shows that the rule of detachment is applicable to Thesis (T). This is so because it is a thesis of the type "$C\alpha\beta$", it belongs to the system as a substitution of Thesis 1, and the expression which occurs in it in place of "α" also belongs to the system, since it is of the form of Thesis 1. Hence we may detach from Thesis (T) the expression which occurs in it in place of "β" by joining to the system, as Thesis 2, an expression of the form of "β". Further proof lines can easily be checked by the reader himself. The whole proof consists of seven steps, three substitutions and four detachments. I shall now analyse the proof in detail.

4. My intention is to explain first the meanings of Theses 1 and 5 and to convince the reader that they are true and in confirmity with intuition.

The proof given above shows that Thesis 3 is a consequence of Thesis 1, and Thesis 5 is a consequence of Thesis 3. Since in turn Thesis 1 is, by substitution, a consequence of Thesis 5, it follows that all three theses, 1, 3, and 5, are equivalent with one another. Let us now examine the meaning of the shortest of them, i.e., Thesis 3.

This thesis reads: "if r, then if s, then r". The terms "r" and "s" stand for any propositions. This thesis will not appear self-evident to everyone. And yet it can be deduced from the most self-evident theses. No one will deny that whatever propositions "r" and "s" we consider it is true that:

I. If r and s, then r.

Nor will anyone deny that the two formulations: "if r and s, then t" and "if r, then if s, then t", are equivalent. For instance, the following formulations are equivalent: "if a number x is even and divisible by 3, then it is divisible by 6" and "if a number x is even, then if it is divisible by 3, it is divisible by 6". Hence it follows that if we consider any propositions "r", "s", and "t", then it is true that:

II. If (if r and s, then t), then [if r, then (if s, then t)].

These two theorems may be written in symbols as follows:

I $CKrsr.$

II $CCKrstCrCst.$

The formula "Krs" stands for a conjunction of the propositions "r" and "s".[13]) By substituting in II the variable "r" for "t" and by applying the rule of detachment we obtain our Thesis 3:

$\mathrm{II}\; t/r \times C1-3.$

3 $CrCsr.$

Thus Thesis 3 is a consequence of self-evident theses. Its meaning might by approximately formulated thus: if one asserts a proposition "r" unconditionally, then he is also authorized to assert it on a condition "s", so that he has the right to state: "if s, then r".

Now Thesis 3 is asserted unconditionally; hence we have the right to assert it on a condition "q", that is, we are authorized to state: "if q, then if r, then if s, then r". This is Thesis 4 formulated verbally.

Thesis 4 also is asserted unconditionally; hence we have the right to assert it on any condition, be it the old condition "q" or the new condition "p". In this way we obtain verbal formulations of Theses 1 and 5:

1. If q, then if q, then if r, then if s, then r.
5. If p, then if q, then if r, then if s, then r.

Thus the meanings of these theses are established. The theses are true and in agreement with intuition.

13) Cf. Łukasiewicz (2), p. 36.

My point now is to convince the reader beyond all doubt that the following theorems are true:

(a) The proof demonstrating that Thesis 5 is a consequence of Thesis 1 is based on Thesis 1 as its only premiss.

(b) The rules of inference used in that proof have long been known and accepted as rules of deductive inference.

(c) Conclusion 5 is more general than premiss 1.

As to (a): The completeness of the proof shows that Thesis 1 is the only premiss used in the proof. The proof has no gaps; every step of the proof is recorded or marked and is based on rules of inference specified in advance.

As to (b): The rules of inference used in the proof correspond to the rules of deductive inference already known in antiquity. All the theses considered are true for any propositions "p", "q", "r", and "s", which occur in them. Hence they are also true for certain propositions, namely conditional propositions, which we substitute in the theses. For whatever is valid for any objects of a kind, is also valid for certain objects of that kind. In applying the rule of substitution we base ourselves on the principle *dictum de omni*, which was not explicitly formulated by Aristotle, but which has always been considered the foundation of this theory of the syllogism. And the theory of the Aristotelian syllogism to this day is believed to form the nucleus of deductive logic.

In applying the rule of detachment we base our argument on the Stoic syllogism called *modus ponens*:

$$\begin{array}{l} \text{If } \alpha\text{, then } \beta, \\ \text{Now } \alpha, \\ \hline \text{Hence } \beta. \end{array}$$

No one has ever denied that this is a mode of deductive inference.

As to (c): Thesis 5 is more general than Thesis 1, since it covers all cases covered by Thesis 1 and also cases which Thesis 1 does not cover. This will become clear when we enumerate the types of these cases:

The truth of both theses in question, like all theses in the theory of deduction, depends not on the contents of the sentences "p", "q", "r", and "s", but only on their truth or falsehood. The zero-one verification method is based precisely on that fact. If we represent a false proposition by "0", and a true proposition by "1", we obtain all the types of cases covered by Thesis 5, when in that thesis we substitute for the variables 0's and 1's in all possible combinations. The number of such combinations is 16:

$$\begin{array}{ll} C0C0C0C00 & C1C1C0C00 \\ C0C0C0C10 & C1C1C0C10 \\ C0C0C1C01 & C1C1C1C01 \\ C0C0C1C11 & C1C1C1C11 \\ C0C1C0C00 & C1C0C0C00 \\ C0C1C0C10 & C1C0C0C10 \\ C0C1C1C01 & C1C0C1C01 \\ C0C1C1C11 & C1C0C1C11. \end{array}$$

All these combinations are covered by Thesis 5; on the other hand, Thesis 1 covers only the first 8 combinations written out in the upper half, that is only those in which the term that follows the first "C" is equiform with the term that follows the second "C". Hence it is evident that Thesis 1 is a special case of Thesis 5. And yet Thesis 5, more general than Thesis 1, is a consequence of the latter on the strength of deductive inference.

I realize that this is a very particular case of generalization, since it refers to only one class of objects, namely to propositions. We infer that something is true for any propositions "p" and "q", either the same or different, on the strength of the fact that something is true for the proposition "q". Nevertheless this case shows that at least in the sphere of these objects, generalizing deduction is possible.

5. In textbooks on logic we often encounter the view that deduction is an inference from the general to the particular. This opinion is erroneous even in the field of traditional logic[14]) for that inference by which from the sentence "no even number is an odd number" we obtain the sentence "no odd number is an even number" is certainly deductive, since it is based on the law of conversion of general negative propositions, accepted in Aristotelian logic. Yet it may not be asserted that in that inference the relation between the premiss and the conclusion is the same as between the general and the particular. Now that we have demonstrated that in certain cases we can pass, in a deductive

[14]) Cf. Kotarbiński, p. 233

manner, from the particular to the general, the incorrectness of the above characterization of deductive inference becomes even more striking.

Together with that erroneous characterization of deduction the view that deduction does not widen our knowledge is also definitively refuted. It seems that these two opinions had their source in the conviction that the principle *dictum de omni* is the foundation of Aristotelian logic, and that Aristotelian logic exhausts deductive logic. But both these convictions are erroneous. Neither is Aristotle's theory of the syllogism based exclusively on the idea contained, although not precisely formulated, in the principle *dictum de omni*, nor does that theory cover the whole of deductive logic.[15]) Along with Aristotelian logic, which is a "logic of terms", there has for ages been Stoic logic,*) which is a "logic of propositions" and which corresponds to the present-day theory of deduction.[16])

These two logical systems are essentially different, since they are concerned with different semantic categories. No Stoic syllogism, including the law of inference called *modus ponens*, is deducible from Aristotelian logic.

As long as the principle *dictum de omni* was supposed to be the foundation of all deductive logic it was possible to think that deduction is inference from the general to the particular and that it does not widen our knowledge. But when the modern "theory of deduction" was formed, and when both the Aristotelian principle *dictum de omni* in the form of the rule of substitution, and the Stoic syllogism *modus ponens* as the rule of detachment, were applied to it, it became clear that deductive inference may be as "creative" as inductive inference, without thereby losing anything of its certainty.

I disregard here further philosophical consequences connected with these results of research in order to conclude by reverting to those problems which can be handled on the basis of mathematical logic.

[15]) Concerning the axioms on which Aristotle's theory of syllogism is based see Łukasiewicz (2), p. 87 ff. See also his *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic*, Oxford, 1951.

[16]) Cf. Łukasiewicz (2), pp. 19 ff.

*) Stoic logic was discussed by Łukasiewicz in detail in his paper "On the History of the Logic of Propositions", (see pp. 197–219 of the present volume).

For brevity's sake, let us call "generalizing theses" those theses from which more general theses can be deduced by the rules of substitution and detachment. I am concerned here above all with the following problem, which so far I have been unable to solve: what, if any, characteristics are shared by all generalizing theses? For the sake of those who might wish to investigate this problem I quote here a number of facts which I have established.

I have verified that the following theses are equivalent to one another:

(A1)	*CsCCCpqrCqr*.
(A2)	*CCstCCCpqrCqr*.
(A3)	*CCptCCCpqrCqr*.
(A4)	*CCrtCCCpqrCqr*.

Of these, Thesis (A1) is the most general. Thesis (A2) is obtained from it by the substitution "*s*/*Cst*", and Theses (A3) and (A4) are obtained from (A2) by the respective substitutions "*s*/*p*" and "*s*/*r*". Conversely, Theses (A1) and (A2) are obtained from both (A3) and (A4) by substitution and detachment, and moreover (A1) is obtained from (A2). Thus the generalizing theses here are Theses (A2), (A3), and (A4). Note that all four of these theses are equivalent to the thesis "*CCCpqrCqr*".

Further, I have verified that the following theses are equivalent to one another:

(B1)	*CtCCCpqrCCCsprr*.
(B2)	*CCutCCCpqrCCCsprr*.
(B3)	*CCrtCCCpqrCCCsprr*.

Here, too, Thesis (B1) is the most general. Thesis (B2) is a consequence of (B1) on the strength of the substitution "*t*/*Cut*", and (B3) is a consequence of (B2) on the strength of the substitution "*u*/*r*". Conversely, both (B1) and (B2) are consequences of (B3) on the strength of substitution and detachment; in the same way (B1) is a consequence of (B2). Thus, the generalizing theses here are (B2) and (B3). All three are equivalent to the thesis "*CCCpqrCCCsprr*".

The above examples of generalizing theses have the property in common that their consequences include a thesis of the form "*CrCsr*". This property is also shared by Thesis 1, given in Section 3 as an example of a generalizing thesis But a conclusion stating that all generalizing theses share that property, would be erroneous. Here is an example to the contrary. The following theses are equivalent:

(F1) *CpCqCrCsCtr*.

(F2) *CqCqCrCsCtr*.

The generalizing thesis here is (F2). But that thesis does not have among its consequences any thesis of the form "*CrCsr*"; it has as a consequence only a thesis of the form "*CrCsCtr*". These two theses: "*CrCsr*" and "*CrCsCtr*", are independent of one another,[17]) but nevertheless they have a property in common: they make it possible to form, from any asserted thesis "α", a thesis of the type "$Cs\alpha$", where "s" is a variable that does not occur in "α". It is to be investigated whether this property is common to the generalizing theses.

All the examples of generalizing theses adduced so far, not excluding Axiom (Ł), which is a transformation of Nicod's axiom, are non-organic theses.[18]) But it would be erroneous to conclude that all generalizing theses are non-organic. In 1926, Wajsberg demonstrated that every implicational thesis that does not include negation can be deduced by substitution and detachment from the following organic thesis: *)

(W1) *CCCpqCCrstCCuCCrstCCpuCst*,

which can thus serve as the sole axiom of the implicational system.[19]) I have ascertained that (W1) has as a consequence the following more general thesis:

(W2) *CCCpqCCrstCCuCCwstCCpuCst*.

(W1) is obtained from (W2) by the substitution "*w/r*". Thesis (W1) is thus an example of an organic generalizing thesis. The consequences of this thesis include all implicational theses.

[17]) For the method of proving the independence of theses of the propositional calculus, see Łukasiewicz (2), pp. 109 ff.

[18]) Cf. footnote 10 above.

[19]) The result obtained by Mr. Wajsberg, as given in the present paper, was part of his M. A. thesis.

*) Cf. footnote *), p. 196

I also wish to point out that all the "sole" axioms of the implicational and the implicational-negational system known to me, whether organic or non-organic, share the property that they are either generalizing theses, like Wajsberg's axiom (W1), or are "generalized" theses, like Thesis (W2), which means that they are equivalent to some of their special cases. But it would be premature to conclude that all the sole axioms of the implicational or the implicational-negational system are either generalizing or generalized theses. I have the impression that Wajsberg's organic axiom with the primitive term "*D*" has neither of these two properties, though I have not been able to prove this fact beyond all doubt. Should it be confirmed, then it could be expected that the implicational system also includes sole axioms that have neither of these two properties.

It would be interesting to solve the problems raised above, since their solution might shed some light on generalizing deduction and thus explain on what those strange facts depend.

Added while text was in proof:

Since two years have elapsed from the completion of the present paper I wish to add here some comments and some results which I have obtained in the meantime.

As to Part I

a) Dr. Leśniewski noticed many years ago that Nicod's deduction of the thesis "*DtDtt*" from Axiom (N) contains an error. As far as I am aware, that error has not been corrected. I would not mention this fact even now were it not that 1931 saw the appearance of a comprehensive three-volume treatise by Jörgen Jörgensen, Professor of the University of Copenhagen, *A Treatise in Formal Logic* (Copenhagen–London), which, following Nicod in that respect, repeats his mistakes (cf. vol. I, p. 258, and vol. II, p. 151); in particular, Theorem (17), from which the thesis "*DtDtt*" is directly deduced, is erroneous. In drawing attention to that error I also wish to state that the deduction of Thesis 6, i.e., "*DtDtt*", from Axiom (Ł), and hence, indirectly, from Axiom (N), as given in the present paper, seems to be the first correct proof of that thesis in Nicod's system.

b) In connection with the concluding remark in the Addendum to Part I, I wish to add that in 1931 I found an organic thesis which is equivalent to Nicod's axiom and which differs from Wajsberg's thesis. The thesis in question is as follows:

(M1) *DDpDqrDDpDrpDDsqDDpsDps.*

As to Part II

c) The supposition is untenable that the generalizing theses of the implicational system all have the property that their consequences include a thesis that makes it possible to form, from any thesis "α", a thesis of the type "$Cs\alpha$", where "s" is a variable that does not occur in "α". Here is an example to the contrary:

(G1) *CsCCpCpqCrCpq.*

(G2) *CCsCstCCpCpqCrCpq.*

(G3) *CCpCpqCCpCpqCrCpq.*

All these theses are equivalent to one another, from which it follows that (G2) and (G3) are generalizing theses. Yet these theses do not have the property referred to above.

d) In connection with Wajsberg's sole axiom (W1) of the implicational system, I wish to add that in 1930 I found the following axiom of the implicational system, which is the shortest of all those known to me so far:

(Ł2) *CCCpqCrsCtCCspCrp.* *)

This axiom is equivalent to the following thesis that is a special case of it:

(Ł3) *CCCpqCrsCCtuCCspCrp.*

Wajsberg's axiom (W1) of the implicational system was published in the article: J. Łukasiewicz und A. Tarski, "Untersuchungen über den Aussagenkalkül, *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie* 23 (1930), cl. iii. **)

*) In 1936, Łukasiewicz found a 13-letter axiom of the implicational propositional calculus. In this connection see his paper "In Defence of Logistic" in the present volume, pp. 236–249. He also discussed that axiom in a separate paper, "The Shortest Axiom of the Implicational Calculus of Propositions" (see pp. 295–305 of the present volume) where he proved that there is no sole axiom of the implicational propositional calculus consisting of less than 13 letters.

**) See "Investigations into the Sentential Calculus", pp. 131–152 of the present volume.

ON THE HISTORY OF THE LOGIC OF PROPOSITIONS *)

Modern mathematical logic has taught us to distinguish within formal logic two basic disciplines, no less different from one another than arithmetic and geometry. These are, the *logic of propositions* and the *logic of terms*. The difference between the two consists in the fact that in the logic of propositions there appear, besides logical constants, only propositional variables, while in the logic of terms term variables occur.

The simplest way of making this difference clear is to examine the Stoic and the Peripatetic versions of the law of identity. To avoid misunderstanding let me at once say that, so far as our sources indicate, the two laws of identity were only incidentally formulated by the ancients, and in no way belong to the basic principles of either logic. The *Stoic* law of identity reads "if the first, then the first", and is to be found as a premiss in one of the inference-schemata cited by Sextus Empiricus.[1]) The *Peripatetic* law of identity is "*a* belongs to all *a*", and is not mentioned by Aristotle, but can be inferred from a passage in Alexander's commentary on the *Prior Analytics.*[2]) Using variable letters we can write the Stoic law of identity in the form "if *p* then *p*"; the Peripatetic law can be recast in the form "all *a* is *a*". In the first law the expression "if ... then" is a logical constant, and "*p*" a propositional

[1]) Sextus, *Adv. Math.* viii. 292 (missing in Arnim): εἰ τὸ πρῶτον, τὸ πρῶτον. Good as H. von Arnim's collection is (*Stoicorum veterum fragmenta*, vol. ii, Leipzig 1903), it does not begin to serve as source material for Stoic dialectic.

[2]) Alexander, *In anal. pr. comm.*, ed. Wallies, p. 34, 1. 19: γίνεται...τὸ Α τινὶ τῷ Α μὴ ὑπάρχον, ὅπερ ἄτοπον.

*) [Editorial note from the McCall edition: This paper originally appeared under the title "Z historii logiki zdań" in *Przegląd Filozoficzny* 37 (1934), pp. 417–437. It is reprinted in a collection of Łukasiewicz's papers entitled *Z zagadnień logiki i filozofii*, edited by J. Słupecki, Warsaw, 1961. A German translation by the author appeared as "Zur Geschichte der Aussagenlogik" in *Erkenntnis* 5 (1935), pp. 111–131. Translated from the German version by S. McCall.]

variable; only *propositions* such as "it is day" can be meaningfully substituted for "p". This substitution yields a special case of the Stoic law of identity: "if it is day, it is day". In the second law the expression "all ... is" is a logical constant, and "a" a term variable; "a" can be meaningfully replaced only by a term, and, in accordance with a tacit assumption of Aristotelian logic, only by a general term at that, such as "man". Upon substitution we get a special case of the Peripatetic law of identity: "all man is man". The Stoic law of identity is a thesis of the *logic of propositions*, whereas the Peripatetic law is a thesis of the *logic of terms*.

This fundamental difference between the logic of propositions and the logic of terms was unknown to any of the older historians of logic. It explains why there has been, up to the present day, no history of the logic of propositions, and, consequently, no correct picture of the history of formal logic as a whole. Indispensable as Prantl's [3]) work is, even today, as a collection of sources and material, it has scarcely any value as an historical presentation of logical problems and theories. The history of logic must be written anew, and by an historian who has fully mastered mathematical logic. I shall in this short paper touch upon only three main points in the history of propositional logic. Firstly I wish to show that the Stoic dialectic, in contrast to the Aristotelian syllogistic, is the *ancient* form of propositional logic; and, accordingly, that the hitherto wholly misunderstood and wrongly judged accomplishments of the Stoics should be restored their due honour. Secondly I shall try to show, by means of several examples, that the Stoic propositional logic lived on and was further developed in *medieval* times, particularly in the theory of "consequences". Thirdly I think it important to establish something that does not seem to be commonly known even in Germany, namely that the founder of *modern* propositional logic is Gottlob Frege.

The Stoic law of identity mentioned above, which belongs to propositional logic, bears witness that the Stoic dialectic is a logic of propositions. However, an isolated theorem proves nothing. We shall accordingly take into consideration the well-known inference-schema

[3]) K. Prantl, *Geschichte der Logik im Abendlande*, vols. i–iv, Leipzig, 1855–1870; vol. ii, 2nd edition, Leipzig, 1885.

which the Stoics placed at the head of their dialectic as the first "indemonstrable" syllogism:

> If the first, then the second;
> but the first;
> therefore the second. [4])

In this formula the words "the first" and "the second" are *variables*, for the Stoics denote variables not with letters, but with ordinal numbers. [5]) It is clear that here too only *propositions* may be meaningfully substituted for these variables; e.g. "it is day", and "it is light". When this substitution is made we get the inference which occurs again and again as a school example in Stoic texts: "If it is day, then it is light; but it is day; therefore it is light." That indeed *propositions* and not *terms* are to be substituted for the variables in the above formula is not only evident from its sense, but is clearly implied by the following example: "If Plato lives, then Plato breathes; but the first; therefore the second." Here "the first" plainly refers to the *proposition* "Plato lives", and "the second" to the *proposition* "Plato breathes". [6])

The fundamental difference between Stoic and Aristotelian logic does not lie in the fact that hypothetical and disjunctive propositions occur in Stoic dialectic, while in Aristotelian syllogistic only categorical propositions appear. Strictly speaking, hypothetical propositions can be found in Aristotle's syllogistic also, for each proper Aristotelian syllogism is an *implication*, and hence a hypothetical proposition. For example, "If a belongs to all b and c belongs to all a, then c belongs to all b". [7]) The main difference between the two ancient systems of logic lies rather in the fact that in the Stoic syllogisms the variables are *propositional variables*, while in Aristotle's they are *term variables*. This crucial difference is completely obliterated, however, if we translate the above-mentioned Stoic syllogism as Prantl does (i, p. 473):

[4]) Sextus, *Adv. math.* viii. 227 (Arnim, ii. 242, p. 81, 1. 22): εἰ τὸ πρῶτον, τὸ δεύτερον· τὸ δέ γε πρῶτον· τὸ ἄρα δεύτερον.

[5]) Apuleius, *De interpr.* 279 (Arnim, ii, p. 81 note): "Stoici porro pro litteris numeros usurpant, ut 'si primum, secundum; atqui primum; secundum igitur'".

[6]) Diogenes Laert. vii. 76 (quoted in Prantl, i, p. 471, note 177; missing in Arnim): εἰ ζῇ Πλάτων, ἀναπνεῖ Πλάτων· ἀλλὰ μὴν τὸ πρῶτον· τὸ ἄρα δεύτερον.

[7]) Aristotle, *An. pr.* ii. 11. 61b34: εἰ γὰρ τὸ Α παντὶ τῷ Β καὶ τὸ Γ παντὶ τῷ Α, τὸ Γ παντὶ τῷ Β [sc. ὑπάρχει].

If the first is, the second is
But the first is
―――――――――――――
Therefore the second is.

By adding to each variable the little word "is", which occurs *nowhere* in the ancient texts, Prantl, without knowing or wishing it, falsely converts Stoic propositional logic into a logic of terms. For in Prantl's schema only terms, not propositions, can be meaningfully substituted for "the first" and "the second". As far as we can judge from the fragmentary state of the Stoic dialectic that has come down to us, *all* Stoic inference-schemata contain, besides logical constants, only propositional variables. Stoic logic is therefore a logic of propositions.[8]

There is yet a second important difference between the Aristotelian and the Stoic syllogisms. Aristotelian syllogisms are logical theses, and a *logical thesis* is a proposition which contains, besides logical constants, only propositional or term variables, and which is true for all values of its variables. Stoic syllogisms are inference-schemata, in the sense of rules of inference, and a *rule of inference* is a prescription empowering the reasoner to derive new propositions from ones already admitted. We should examine this difference somewhat more closely.

The Aristotelian syllogism quoted above, which can also be written "if all b is a and all a is c, then all b is c", is an implication of the form "if α and β, then γ", whose antecedent is a conjunction of the premisses α and β, and whose consequent is the conclusion γ. As an implication, this syllogism is a *proposition* which Aristotle recognizes as true; one that does indeed hold for all values of its variables "a", "b", and "c". If constant values are substituted for these variables, we get true *propositions*. Inasmuch as the syllogism in question contains, besides variables, only the logical constants "if ... then", "and", and "all ... is", it is, like all other Aristotelian syllogisms, a logical thesis.

It is otherwise in Stoic logic. The Stoic syllogism given above, which

[8] I have defended this interpretation of the Stoic dialectic since 1923; see J. Łukasiewicz, "Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenkalküls", *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie* 23 (1930), cl. iii, pp. 51–77. ["Philosophical Remarks on Many-Valued Systems of Propositional Logic", pp. 153–178 of this volume.] I rejoice in having found in H. Scholz, *Geschichte der Logik* (Berlin, 1931), p. 31, a supporter of this point of view.

with the help of letters can be written "if p, then q; but p; therefore q", consists, as does the Aristotelian syllogism, of two premisses and a conclusion. But here the premisses are *not* bound up together with the conclusion in a single unified proposition. This is plain from the word "therefore" which introduces the conclusion. The syllogism in question is consequently *not* a proposition. Since it is not a proposition, it can be neither true nor false; for it is acknowledged that truth and falsehood belong to propositions alone. Hence the Stoic syllogism is *not* a logical thesis: if constant values are substituted for its variables the result is not a proposition, but an *inference*. The syllogism is accordingly an inference-schema, having the force of a *rule of inference* which can be more accurately expressed in the following way: whoever accepts as true both the implication "if p, then q" and its antecedent "p", also has the right to accept as true the consequent "q" of this implication—i.e. to detach "q" from "p". This rule of inference, under the name of the "rule of detachment", has become almost a classic in modern logic.

All Stoic syllogisms are formulated as *rules of inference*. In this way Stoic dialectic differs not only from Aristotelian syllogistic, but also from modern propositional logic, which is a system of *logical theses*.

However, the Stoics were acquainted with a clear and simple method of converting all their rules of inference into theses. This involves a distinction between binding and non-binding inferences. An inference with premisses α and β and conclusion γ they call *binding* [*bündig*], if the *implication*, whose antecedent is the conjunction of the two premisses α and β and whose consequent is the conclusion γ, is valid. For example, the following inference is binding: "if it is day, then it is light; but it is day; therefore it is light", for the corresponding implication is correct: "if it is day and if it is day then it is light, then it is light".[9]

[9] Sextus, *Hyp. pyrrh.* ii. 137 (missing in Arnim, who nevertheless in ii. 239, p. 78, l. 15, quotes the parallel passage from *Adv. math.* viii. 415 (416)): ἐν τούτῳ τῷ ⟨λόγῳ⟩ "εἰ ἡμέρα ἔστι, φῶς ἔστιν· ἀλλὰ μὴν ἡμέρα ἔστιν· φῶς ἄρα ἔστιν" τὸ μὲν "φῶς ἄρα ἔστιν" συμπέρασμά ἐστι, τὰ δὲ λοιπὰ λήμματα. τῶν δὲ λόγων οἱ μέν εἰσι συνακτικοὶ οἱ δὲ ἀσύνακτοι, συνακτικοὶ μέν, ὅταν τὸ συνημμένον τὸ ἀρχόμενον μὲν ἀπὸ τοῦ διὰ τῶν τοῦ λόγου λημμάτων συμπεπλεγμένου, λῆγον δὲ εἰς τὴν ἐπιφορὰν αὐτοῦ, ὑγιὲς ᾖ, οἷον ὁ προειρημένος λόγος συνακτικός ἐστιν, ἐπεὶ τῇ διὰ τῶν λημμάτων αὐτοῦ συμπλοκῇ ταύτῃ "ἡμέρα

This just observation makes possible the conversion of inferences into propositions. When it is applied to the rule of inference "if p then q; but p; therefore q" we obtain the implication "if p and if p then q, then q", which is a thesis of propositional logic, since besides propositional variables only the logical constants "if ... then" and "and" occur in it.

It is not possible for me to go into all the details of Stoic logic here. I wish only to comment upon the most important points. The Stoic logic of propositions is a *two-valued* logic. In it the basic principle holds, that every proposition is *either true or false*, or, as we say today, can take one of only *two* possible "truth-values", "the true" or "the false".[10] This principle is laid down in conscious opposition to the view that there are propositions which are *neither true nor false*, namely those which treat of future contingent events. This view, which was particularly widespread among the Epicureans, was also ascribed by the Stoics to Aristotle.[11]

In Stoic propositional logic the following functions occur: negation, implication, conjunction, and disjunction. The first three functions are defined, as is normally said nowadays, as "truth-functions". By a *truth-function* is meant a function whose arguments are propositions, and whose truth-value depends only on the truth-value of its arguments.

According to the Stoics one obtains the *negation* or the contradictory of a proposition when the sign of negation is placed *in front of* the proposition.[12] This theoretically correct and practically valuable rule continues to be operative in the Middle Ages.[13] It is universally recognized in modern logic.

ἔστι καὶ εἰ ἡμέρα ἔστι, φῶς ἔστιν" ἀκολουθεῖ τὸ "φῶς ἔστιν" ἐν τούτῳ τῷ συνημμένῳ· "εἰ ἡμέρα ἔστι καὶ εἰ ἡμέρα ἔστι, φῶς ἔστι, φῶς ἔστιν".

[10]) Cicero, *Acad. pr.* ii. 95 (Arnim, ii. 196, p. 63): "Fundamentum dialecticae est, quidquid enuntietur, id autem appellant ἀξίωμα ..., aut verum esse aut falsum."

[11]) Boethius, *Ad Arist. de interpr.* ed. secunda, Meiser, p. 208 (missing in Arnim): "Putaverunt autem quidam, quorum Stoici quoque sunt, Aristotelem dicere in futuro contingentes nec veras esse nec falsas." See on this matter my earlier paper cited above, pp. 75 ff. [This volume, pp. 176 ff.]

[12]) Apuleius, *De interpr.* 266 (Arnim, ii. 204a, p. 66): "Solum autem abdicativum vocant, cui negativa particula praeponitur." The word "οὐχί" serves as the sign of propositional negation.

[13]) See note 3, p. 198.

There are many disputes in ancient times over the meaning of the implication "if p then q".[14] The argument seems to have been started by Philo the Megarian, who was the first to define implication as a truth-function in much the same way as is done today. According to Philo an implication is true if and only if it does not begin with truth and end with falsehood. An implication is accordingly true in three cases~~: firstly, if its antecedent and its~~ conse~~quent are both true; secondly~~ if its antecedent and its consequent are both false; and thirdly, if the antecedent is false and the consequent true. Only in *one* case is the implication false, namely when the antecedent is true and the consequent false.[15] Another Megarian, Diodorus Cronus, maintained on the other hand that an implication is true if and only if it neither *was* nor *is possible* for it to begin with truth and end with falsehood.[16] This ancient dispute concerning the concept of implication, immortalized by Callimachus in an epigram ("Even the ravens on the roof tops are croaking about which conditionals are true"),[17] is reminiscent of the polemic waged by one of the modern followers of Diodorus, C. I. Lewis, against the other advocates of mathematical logic.[18]

[14]) Cicero, *Acad. pr.* ii. 143 (Arnim, ii. 285, p. 93): "In hoc ipso, quod in elementis dialectici docent, quo modo iudicare oporteat, verum falsumne sit, siquid ita conexum est, ut hoc: 'si dies est, lucet', quanta contentio est. Aliter Diodoro, aliter Philoni, Chrysippo aliter placet."

[15]) Sextus, *Adv. math.* viii. 113: ὁ μὲν φίλων ἔλεγεν ἀληθὲς γίνεσθαι τὸ συνημμένον, (= implication), ὅταν μὴ ἄρχηται ἀπ' ἀληθοῦς καὶ λήγῃ ἐπὶ ψεῦδος, ὥστε τριχῶς μὲν γίνεσθαι κατ' αὐτὸν ἀληθὲς συνημμένον, καθ' ἕνα δὲ τρόπον ψεῦδος. There follows the enumeration of all four cases with examples.

[16]) Sextus, *Adv. math.* viii. 115: Διόδωρος δὲ ἀληθὲς εἶναι φησι συνημμένον ὅπερ μήτε ἐνεδέχετο μήτε ἐνδέχεται ἀρχόμενον ἀπ' ἀληθοῦς λήγειν ἐπὶ ψεῦδος.

[17]) Sextus, *Adv. math.* i. 309: τὸ ὑπὸ τοῦ Καλλιμάχου εἰς Διόδωρον τὸν Κρόνον συγγραφέν [sc. ἐπιγραμμάτιον]· ἠνὶ δέ κου κόρακες τεγέων ἔπι κοῖα συνῆπται κρώζουσι. . . .

[18]) Being of the opinion that the concept of "material implication", which comes from Philo, leads to paradoxes, such as "a false proposition implies any proposition", and "a true proposition is implied by any proposition" (compare the passage from Duns Scotus in note 44 of p. 214 below), Lewis wishes to replace "material implication" by "strict implication", the latter being defined in the following way. "p implies q" or "p strictly implies q" is to mean "it is false that it is possible that p should be true and q false". See C. I. Lewis and C. H. Langford, *Symbolic Logic*, New York and London, 1932, pp. 122 and 124.

In the Stoic school, Philo's definition was accepted. At least, Sextus ascribes this concept directly to the Stoics.[19])

The *conjunction* "p and q" is defined by the Stoics as a truth-function. It is true if and only if both its members are true; otherwise it is false.[20])

An analogous definition of the *disjunction* "p or q" does not occur in the fragments of Stoic logic which have come down to us. We gather, from the rules of inference for disjunction laid down by Chrysippus, that he considered disjunction as an exclusive "either-or" connective. Thus according to Chrysippus the two members of a true disjunction cannot both be true at the same time. This seems to have been changed later. The conviction arises that the expression "p or q" is synonymous with the implication "if not p then q".[21]) In this case we would no longer be dealing with exclusive disjunction, but with non-exclusive *alternation*. In the Middle Ages, as we shall see later, the non-exclusive character of disjunction comes clearly to light.

All the above-mentioned logical functions are to be found in the inference-schemata of Stoic dialectic. Of these inference-schemata, some are considered to be "indemonstrable", that is to say accepted *axiomatically* as correct, while the others are reduced to the indemonstrable ones. It is Chrysippus who is supposed to have laid down the indemonstrable inference-schemata or syllogisms. These consist of the following five (in which I denote the variables not by ordinal numerals, but by letters):

I. If p then q; but p; therefore q.

II. If p then q; but not-q; therefore not-p.

III. Not both p and q; but p; therefore not-q.

[19]) *Hyp. pyrrh.* ii. 104, and *Adv. math*, viii. 245 (Arnim, ii. 221, p. 72, l. 32). Cf. also Diogenes Laert. vii. 81 (Arnim, ii. 243, p. 81).

[20]) Sextus, *Adv. math.* viii. 125 (Arnim, ii. 211, p. 69): [λέγουσιν] ὑγιὲς εἶναι συμπεπλεγμένον (= conjunction) τὸ πάντ' ἔχον ἐν αὐτῷ ἀληθῆ, οἷον τὸ "ἡμέρα ἔστι καὶ φῶς ἔστιν", ψεῦδος δὲ τὸ ἔχον ψεῦδος.

[21]) Galen, *Institutio Logica*, ed. Kalbfleisch, p. 9, l. 13: τὸ τοιοῦτον εἶδος τῆς λέξεως "εἰ μὴ νύξ ἐστιν, ἡμέρα ἐστίν" διεζευγμένον ἐστὶν ἀξίωμα τῇ φύσει τῶν πραγμάτων αὐτῇ, συνημμένου δὲ ἰδέαν ἔχει τῇ λέξει. Exclusive disjunction is called "διεζευγμένον". For non-exclusive alternation Galen uses the expression "παραδιεζευγμένον" (p. 35. l. 6).

IV. Either p or q; but p; therefore not-q.

V. Either p or q; but not-q; therefore p.[22])

It is apparent from the fourth syllogism that disjunction is conceived of as an exclusive "either-or" connective. For non-exclusive alternation this syllogism is not valid.[23])

The reduction of the derived inference-schemata to the indemonstrables is a masterpiece of logical acumen. The source of our information in this matter is Sextus, who thoroughly understands the dialectical technique of the Stoics and must be considered among the best sources of Stoic logic. With a clarity that leaves nothing to be desired he informs us, for example, how the Stoics reduced the inference-schema "if p and q, then r; not-r, but p; therefore not-q" to the second and third indemonstrable syllogisms. From the premisses "if p and q, then r" and "not-r" we get, using the second syllogism, the conclusion "not both p and q". This conclusion and the remaining premiss "p" yield, by the third syllogism, "not-q".[24])

[22]) Galen, *Inst. log.*, ed. Kalbfleisch, p. 15 (Arnim, ii. 245, p. 82): ὃν ὁ Χρύσιππος ὀνομάζει πρῶτον ἀναπόδεικτον, ὁ τοιοῦτος τρόπος ἐστίν. "εἰ τὸ α', τὸ β'· τὸ δὲ α'· τὸ ἄρα β'·" ὃν ὁ Χρύσιππος δεύτερον ἀναπόδεικτον ὀνομάζει, τοιοῦτός ἐστιν· "εἰ τὸ α', τὸ β'· οὐχὶ δὲ τὸ β'· οὐκ ἄρα τὸ α'".—Κἀπὶ τοῦ τρίτου κατὰ τοῦτον… τοιοῦτος ὁ τρόπος ἐστίν· "οὐχὶ τό τε α' καὶ τὸ β'· ⟨τὸ δὲ α'· οὐκ ἄρα τὸ β'⟩".—κἀπὶ τοῦ τετάρτου κατὰ τὸν αὐτόν… τοιοῦτός τις ὁ τρόπος ἐστίν. "ἤτοι τὸ α' ἢ τὸ β'· τὸ δὲ α'· οὐκ ἄρα τὸ β'".—κἀπὶ τοῦ πεμπτου… τοιοῦτός ἐστιν ὁ τρόπος. "ἤτοι τὸ α' ἢ τὸ β'· ⟨οὐχὶ δὲ τὸ β'· τὸ ἄρα α'⟩". Cf. also Arnim, ii. 241 and 242, pp. 79–81.

[23]) Prantl, who actually hates Stoic logic, writes on this matter as follows (i, p. 474): "Here the enormous stupidity of the distinction between the moods IV and V does not have to be especially remarked upon." It is disgraceful to encounter such an assertion in a learned work, particularly as it rests upon ignorance of logic. Prantl further supposes that Chrysippus took the five syllogisms from Theophrastus, and "anyone who copies completely unfamiliar material thereby runs the risk of only displaying his own ignorance". Herein lies another historical error. It cannot be shown from our sources that Theophrastus constructed or even knew of the above-mentioned syllogisms.

[24]) Sextus, *Adv. math.* viii. 235, 236 (missing in Arnim). The inference-schema reads "εἰ τὸ πρῶτον καὶ τὸ δεύτερον, τὸ τρίτον· οὐχὶ δέ γε τὸ τρίτον, ἀλλὰ καὶ τὸ πρῶτον· οὐκ ἄρα τὸ δεύτερον." To the end of the reduction it runs: ὥστε δύο εἶναι ἀναποδείκτους, ἕνα μὲν τοιοῦτον· "εἰ τὸ πρῶτον καὶ τὸ δεύτερον, τὸ τρίτον· οὐχί δέ γε τὸ τρίτον· οὐκ ἄρα τὸ πρῶτον καὶ τὸ δεύτερον", ὅς ἐστι δεύτερος ἀναπόδεικτος, ἕτερον δὲ τρίτον τὸν οὕτως ἔχοντα· "οὐχὶ τὸ πρῶτον καὶ τὸ δεύτερον· ἀλλὰ μὴν τὸ πρῶτον· οὐκ ἄρα τὸ δεύτερον".

Another example given by Sextus, in which the first syllogism is used twice, remained unintelligible to Prantl. The example, in general form, reads: "if p, then if p then q; but p; therefore q". The reduction proceeds as follows. From the premisses "if p, then if p then q" and "p" we get, by the first syllogism, the conclusion "if p then q". From this conclusion and the premiss "p" we obtain, again by the first syllogism, "q".[25]) The inference-schema dealt with here is most interesting: it corresponds to a thesis of the propositional calculus that was recently raised to the rank of an axiom by Hilbert and Bernays.[26])

The number of derived inference-schemata is supposed to have been very great.[27]) Of those that have come down to us the following syllogism, quoted by Origen, merits our attention: "If p then q; if p then not-q; therefore not-p." The example of it, given in addition, is also very interesting: "If you know that you are dead, then you are dead (for nothing false can be known); if you know that you are dead, then you are not dead (for the dead know nothing); therefore you do not know that you are dead."[28]) The above passage from Origen is also important in that it gives us information about the meaning of a hitherto erroneously interpreted expression of Stoic dialectic.[29])

[25]) Sextus, *Adv. math.* viii. 230–3 (missing in Arnim). The text is corrupt, although unambiguously clear. It was corrected first by E. Kochalsky in his dissertation *De Sexti Empirici adversus logicos quaestiones criticae*, Marburg, 1911, pp. 83–85. Nevertheless he finishes his corrections with an appeal to Zeller and Prantl in the following way: "Nimirum huiusmodi argumentum non simplex indemonstrabile per se est absurdissimum, sed Stoicos in syllogismis inveniendis incredibilia paene gessisse inter omnes constat." One sees from this how pernicious Prantl's influence was.

[26]) Hilbert and Bernays, *Grundlagen der Mathematik*, vol. i, Berlin, 1934, p. 66. The thesis in question is, in words, "if [if p, then (if p then q)] then (if p then q)".

[27]) Cicero, *Topica* 14, 57 (quoted by Zeller, *Die Philosophie der Griechen*, iii. 1, 5th edition 1923, p. 114, note I; missing in Arnim): "ex iis modis conclusiones innumerabiles nascuntur."

[28]) Origen, *Contra Celsum*, vii. 15 (*Works*, vol. ii, ed. Koetschau, 1899, p. 166, missing in Arnim): ὅταν δὲ δύο συνημμένα λήγῃ εἰς τὰ ἀλλήλοις ἀντικείμενα τῷ καλουμένῳ "διὰ δύο τροπικῶν" θεωρήματι ἀναιρεῖται τὸ ἐν ἀμφοτέροις τοῖς συνημμένοις ἡγούμενον ... καὶ ὑπάγεταί γε ὁ λόγος τρόπῳ τοιούτῳ· "εἰ τὸ πρῶτον, καὶ τὸ δεύτερον· εἰ τὸ πρῶτον, οὐ τὸ δεύτερον· οὐκ ἄρα τὸ πρῶτον". φέρουσι δὲ καὶ ἐπι ὕλης τὸν τρόπον τοῦτον οἱ ἀπὸ τῆς Στοᾶς λέγοντες τό· "εἰ ἐπίστασαι ὅτι τέθνηκας, ⟨τέθνηκας· εἰ ἐπίστασαι ὅτι τέθνηκας⟩, οὐ τέθνηκας". ἀκολουθεῖ τό· "οὐκ ἄρα ἐπίστασαι, ὅτι τέθνηκας".

[29]) Neither Prantl nor Zeller knows the passage, although Fabricius had already

In connexion with the Stoic logic of propositions I would like to touch upon one or two questions of a general nature. The Stoics are constantly criticized for the fact that in their logic the most trivial empiricism as well as the most empty formalism appears. Thus Prantl (i, p. 457) says, in citing the examples given by the Stoics for implication, that these are "examples, from which it is sufficiently obvious both that ~~the crudest empirical criterion~~ is ~~displayed, and that there is total~~ lack of any understanding of the causal nexus between essences and inherences". Prantl's unfavourable judgement is not justified. If empirical examples are given for logical formulae, the criterion of truth for these examples must also be in some way empirical. However, the examples do not belong to logic, and in Stoic logic itself we do not find the slightest trace of empiricism. When it is asserted that the Stoics lacked an understanding of the causal nexus, we may conclude only that Prantl fails to grasp the Philonian concept of implication accepted by the Stoic. In two-valued logic there *can* be no other concept of implication than the Philonian. This has nothing to do with either empiricism or the causal nexus, for the expression "if p then q" does not mean the same as "q follows from p".

The accusation of formalism, which was often made even in ancient times,[30]) is quite justified, only in our eyes it is not an accusation at all. Formalism, or better *formalization*, means the ideal of exactitude that each deductive system strives to attain. We say that a deductive, axiomatically constructed system is *formalized* when the correctness of the deductions in the system can be verified without having to refer back to the *meaning* of the expressions and symbols used in the deductions. They may be verified, that is, by anyone who understands the rules of inference of the system. In this sense the Stoics prepared the way for formalism, and they cannot be credited highly enough for that. They held strictly to *words* and not to their *meanings*, which is

referred to it (*Sexti Empirici Opera*, 2nd edition 1840, vol. i, p. 112). The expression in question is "διὰ δύο τροπικῶν" ("τροπικόν" is a non-simple premiss, e.g. an implication). It is wrongly interpreted by Prantl (i, p. 480) and Zeller (iii. 1, pp. 114–115 note 5); it means a syllogism in which two τροπικά, in this case two implications, occur as premisses.

[30]) Galen, *Inst. log.*, ed. Kalbfleisch, p. 11, l. 6 (Arnim, ii. 208, p. 69, l. 4): ἀλλ' οἱ περὶ Χρύσιππον κἀνταῦθα τῇ λέξει μᾶλλον ἢ τοῖς πράγμασι προσέχοντες τὸν νοῦν. ...

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the principal requirement of formalization, and they even did so in conscious opposition to the Peripatetics. Alexander occasionally expresses the opinion that the essence of the syllogism lies not in words but in what the words mean.[31]) The Stoics would undoubtedly maintain the opposite. For in spite of the fact that, for example, they took the expressions "if p then q", and "q follows from p" to be synonymous (which is incorrect), they did *not* describe the inference-schema "q follows from p; but p; therefore q" as a syllogism, although the following schema, which in their opinion is synonymous with it, *is* a syllogism: "if p then q; but p; therefore q".[32])

In connexion with this controversy between the Stoic and the Peripatetic schools, we are ultimately confronted with the question, whether the Stoics understood anything about the meaning in principle of their propositional logic, and, in particular, whether they were aware of having created a system of logic different from Aristotle's. Scholz believes that we must answer the first part of this question in the negative.[33]) For the second part of the question we have at our disposal two hitherto little-noticed accounts.

In his commentary on Aristotle's *Topics* Alexander enumerates, under the heading "syncritical problems", certain controversial questions discussed in ancient times—as, for example, whether the moon is bigger than the earth, or whether surgical treatment is to be preferred to medical. In so doing he also mentions the following comparable problems from logic: "whether induction is more convincing than the syllogism; and which syllogism is the first, the categorical or the hypothetical; and which syllogistic figure is the first or the better".[34])

[31]) Alexander, *In anal. pr. comm.*, ed. Wallies, p. 372, l. 29: οὐκ ἐν ταῖς λέξεσιν ὁ συλλογισμὸς τὸ εἶναι ἔχει ἀλλ᾽ ἐν τοῖς σημαινομένοις.

[32]) Alexander, op. cit., p. 373, l. 29 (Arnim, ii. 253, p. 84): οἱ δὲ νεώτεροι ταῖς λέξεσιν ἐπακολουθοῦντες οὐκέτι δὲ σημαινομένοις, οὐ ταὐτόν φασι γίνεσθαι ἐν ταῖς εἰς τὰς ἰσοδυναμούσας λέξεις μεταλήψεσι τῶν ὅρων· ταὐτὸν γὰρ σημαίνοντος τοῦ "εἰ τὸ A, τὸ B" τῷ "ἀκολουθεῖ τῷ A τὸ B" συλλογιστικὸν μὲν λόγον φασὶν εἶναι τοιαύτης ληφθείσης τῆς λέξεως· "εἰ τὸ A, τὸ B· τὸ δὲ A· τὸ ἄρα B", οὐκέτι δὲ συλλογιστικὸν ἀλλὰ περαντικὸν τὸ "ἀκολουθεῖ τῷ A τὸ B· τὸ δὲ A· τὸ ἄρα B".

[33]) *Geschichte der Logik*, p. 32.

[34]) Alexander, *In top. comm.*, ed. Wallies, p. 218: ἔστιν ἔτι καὶ ἐν τῇ λογικῇ συγκριτικῶς τινα ζητούμενα, ὡς τὸ πότερον πειστικώτερον, ἐπαγωγὴ ἢ συλλογισμός, καὶ ποῖος πρῶτος συλλογισμός, ὁ κατηγορικὸς ἢ ὁ ὑποθετικός, καὶ ποῖον σχῆμα πρῶτον ἢ βέλτιον.

It is the *second* question that interests us here: which syllogism is the first, the categorical or the hypothetical. Now the categorical syllogism is the Aristotelian, the hypothetical, the Stoic. Our controversy accordingly concerns the relation of Aristotelian to Stoic logic, and aims at establishing which of these systems is the *first*, i.e., as I understand it, the *logically prior*.

An answer to this question is to be found in the highly interesting introduction to logic written by Galen. Galen reports that Boethius, who according to Ammonius was the eleventh head of the Peripatetic school after Aristotle, and who was reckoned one of the most acute logicians of his time, himself considered, although he was a Peripatetic, the hypothetical and not the categorical syllogisms to be the first. Against this Galen raises the objection that the categorical premisses, as simple propositions, are logically prior to the hypothetical ones constructed out of them. However, he does not appear to attach any great importance either to this argument or to the whole controversy, for he thinks that there is not much to be gained or lost in the dispute. One should become as familiar with the one kind of syllogism as with the other, but in what order this should take place, or which of them should be referred to as primary, may be left to one's own discretion.[35])

From these two fragments we may, in my opinion, conclude not only that the Stoics were aware of the difference between their own logical system and the Aristotelian, but also that they correctly judged the relation between the two systems. We know today that propositional logic is logically prior to the logic of terms. If we analyse the proofs that Aristotle uses in the *Analytics* to reduce syllogisms of the second and third figures to syllogisms of the first figure, we see clearly that theses of propositional logic must be employed throughout. The syllogism

[35]) Galen, *Inst. log.*, ed. Kalbfleisch, p. 17; καὶ μέντοι καὶ τῶν ἐκ τοῦ Περιπάτου τινὲς ὥσπερ καὶ Βοηθὸς οὐ μόνον ἀναποδείκτους ὀνομάζουσιν τοὺς ἐκ τῶν ἡγεμονικῶν λημμάτων συλλογισμούς, ἀλλὰ καὶ πρώτους· ὅσοι δὲ ἐκ κατηγορικῶν προτάσεων εἰσὶν ἀναπόδεικτοι συλλογισμοί, τούτους οὐκ ἔτι πρώτους ὀνομάζειν συγχωροῦσι· καίτοι καθ᾽ ἕτερόν γε τρόπον οἱ τοιοῦτοι πρότεροι τῶν ὑποθετικῶν εἰσιν, εἴπερ γε καὶ αἱ προτάσεις αὐτῶν ἐξ ὧν σύγκεινται πρότεραι βεβαίως εἰσίν· οὐδεὶς γὰρ ἀμφισβητήσει τὸ μὴ οὐ πρότερον εἶναι τὸ ἁπλοῦν τοῦ συνθέτου. ἀλλὰ περὶ μὲν τῶν τοιούτων ἀμφισβητήσεων οὔτε εὑρεῖν οὔτε ἀγνοῆσαι μέγα· χρὴ γὰρ ἀμφότερα τὰ μέρη γιγνώσκειν τῶν συλλογισμῶν, καὶ τοῦτ᾽ ἔστι τὸ χρήσιμον, ὀνομάζειν δὲ τοὺς ἑτέρους ἢ διδάσκειν προτέρους ὡς ἑκάστῳ φίλον.

which later received the name "Baroco" cannot be formally reduced to "Barbara" without the propositional thesis "if (if p and q, then r), then (if p and not-r, then not-q)." Now this thesis corresponds to an inference-schema which, as we saw above, was well known to the Stoics. It is highly probable that the application of this inference-schema to Aristotle's syllogisms did not escape the Stoics. We know also that the logic of propositions is of far greater importance than the meagre fragment of the logic of terms that is incorporated in Aristotle's syllogistic. The logic of propositions is the basis of all logical and mathematical systems. We must be thankful to the Stoics for having laid the foundations of this admirable theory.

A great deal about how Stoic influences continue to be at work in the Middle Ages may be found in Prantl. That, however, the *propositional logic* created by them undergoes a further development in that period seems to have been realized by no one up to now. Once again it is not possible for me to go into details here, especially since the sources for medieval logic are not easily accessible. I shall in what follows merely give a short account of what is to be found of propositional logic in the *Summulae logicales* of Petrus Hispanus, that classic manual of medieval logic, together with the commentary on it by Versorius; as well as what can be found in the writings of the subtle Duns Scotus. The Philonian criterion of a true implication, already disputed in ancient times, seems not to have been known to Petrus Hispanus. To make up for this there appears in his work, under the name of *disjunction* and replacing Chrysippus' "either-or" connective, non-exclusive *alternation* as a truth-function.[36]) We learn that a disjunction, i.e. the joining together of two propositions by means of the connective '*vel*', is false if and only if its two members are false. Otherwise it is true, even when *both* its members are true—though this was admitted with a certain reluctance.[37])

[36]) Prantl (iii, p. 43) has nothing to report on this, for he is not aware of the difference between disjunction and alternation.

[37]) *Summulae*, tract. i, *De disiunctiva* (quoted only in abridged form in Prantl iii, p. 43, note 158; I quote from a comparatively later edition, *Petri Hispani Summulae Logicales cum Versorii Parisiensis clarissima expositione, Venetiis* 1597 *apud Matthaeum Valentinum*, which differs from the text quoted by Prantl in various places): "Disiunctiva est illa, in qua coniunguntur duae propositiones categoricae per hanc coniunctionem 'vel' aut aliam sibi aequivalentem, ut 'Socrates currit vel Plato disputat'. Ad veritatem disiunctivae sufficit, alteram partem esse veram, ut 'homo est animal vel equus est asinus', tamen permittitur, quod utraque pars eius sit vera, sed non ita proprie, ut 'homo est animal vel equus est hinnibilis'. Ad falsitatem eius oportet, utramque partem eius esse falsam, ut 'homo est asinus vel equus est lapis'."

In the commentary the two following rules of inference are laid down for disjunction. Firstly, from a disjunction and the negation of one member the other member may be inferred; e.g. "Man is an animal or a horse is a stone; but a horse is not a stone; therefore man is an animal." This is precisely the fifth indemonstrable syllogism of the Stoics; the fourth is, of course, missing, since it is valid only for exclusive disjunction. Secondly, from the truth of one member the truth of the disjunction may be inferred; e.g. "Man runs, therefore man is an ass or man runs".[38]) The examples are grotesque, but none the less clear enough. The second rule is new, not occurring in the Stoic texts. Moreover, it is correct only on condition that disjunction is taken as non-exclusive alternation.

Conjunction, which here bears the name of *copulative* assertion, is defined by Petrus Hispanus as a truth-function, just as it was by the Stoics. The only rule of inference which seems to be new is one which is added in the commentary: from a conjunction each of its members may be inferred; e.g. "Man is an animal and God exists, therefore man is an animal."[39])

In this connexion we find in the commentary on Petrus Hispanus the following beautiful remark: a conjunction and a disjunction with mutually contradictory members contradict one another.[40]) That is to say, the following propositions stand in contradiction to one another:

[38]) *Summulae*, loc. cit.: "dupliciter arguitur a disiunctivis. Uno modo, a tota disiunctiva cum destructione unius partis ad positionem alterius, ut 'homo est animal vel equus est lapis; sed equus non est lapis, igitur homo est animal'. Secundo modo, arguendo a veritate unius partis ad veritatem totius, et est bona consequentia, unde bene sequitur, haec est vera 'homo currit', igitur haec est vera 'homo est asinus vel homo currit'."

[39]) *Summulae*. tract. i, *De copulativa*: "arguendo a tota copulativa ad veritatem cuiuslibet partis eius seorsum, est bona consequentia. Ut bene sequitur 'homo est animal et Deus est, ergo homo est animal'."

[40]) *Summulae*, loc. cit.: "copulativa et disiunctiva de partibus contradicentibus contradicunt." The same thought, which is here stated somewhat too concisely, is expressed by Occam much more clearly: "Opposita contradictoria disiunctivae est una copulativa composita ex contradictoriis partium ipsius disiunctivae." (See Prantl, iii, p. 396, note 958.)

"*p* and *q*" and "not-*p* or not-*q*", as well as "*p* or *q*" and "not-*p* and not-*q*". In other words, "*p* and *q*" is equivalent to the negation of "not-*p* or not-*q*", and "*p* or *q*" to the negation of "non-*p* and not-*q*". From which it follows that the so-called De Morgan's laws were known long before De Morgan.

Finally we read, at the same place in the commentary, that the contradictory opposite of a proposition cannot be "more truly" formed than by *prefixing* the negation sign to the proposition.[41]) Here the Stoic influence, which we mentioned above, emerges particularly clearly. All the above rules of inference, together with the last remark, are also found in Duns Scotus. It seems therefore that they were generally recognized in the Middle Ages.

The survival of Stoic propositional logic in the Middle Ages is particularly evident in the theory of "consequences". By a *consequence* the medieval logicians understand not only an implication, but also an inference-schema of the type "*p*, therefore *q*", in which "*p*" and "*q*" are propositions. As a rule, however, consequences are represented as *inference-schemata.*[42]) Consequences are divided into material and formal. A consequence is *formal* if it holds for all terms in the same arrangement and form; otherwise it is *material.* Formal consequences are, as laws of logic, always correct. A material consequence is correct or "good" (*bona*) only if it can be reduced to a formal consequence through the assumption of a true proposition as premiss. If the assumed proposition is necessarily true, the consequence is called *bona simpliciter*; if it is only contingently true, the consequence is called *bona ut nunc.* The latter distinction seems to me to be of no great significance.[43])

[41]) *Summulae,* loc. cit.: "non est verius dare contradictionem, quam toti propositioni *praeponere* negationem."

[42]) Duns Scotus, *Quaestiones super anal. pr.* i, 10 (Prantl, iii, p. 139 note 614): "Consequentia est propositio hypothetica composita ex antecedente et consequente mediante coniunctione conditionali vel rationali." As a *coniunctio conditionalis* the word *si* is used; as a *coniunctio rationalis* either *igitur* or *ergo*.

[43]) Loc. cit. (Prantl, iii, p. 139 note 615, p. 140 notes 617, 619): "Consequentia sic dividitur: quaedam est materialis, quaedam formalis. Consequentia formalis est illa, quae tenet in omnibus terminis stante consimili dispositione et forma terminorum." (There follows a precise setting-forth of what belongs to the form of a consequence.) "—Consequentia materialis est illa, quae non tenet in omnibus terminis retenta consimili dispositione et forma. Et talis est duplex, quia quaedam est vera simpliciter, et alia est vera ut nunc. Consequentia vera simpliciter est illa, quae potest reduci ad formalem per assumptionem unius propositionis necessariae.—Consequentia materialis bona ut nunc est illa, quae potest reduci ad formalem per assumptionem alicuius propositionis contingentis verae."

Later medieval handbooks of logic, in chapters entitled *De consequentiis,* introduced among various other formal consequences some which belong to propositional logic. Several of these consequences we have already become acquainted with above. It would be worth while for someone to take the trouble to assemble *all* of them, for then we would have a *complete* picture of the medieval logic of propositions.

The theory of consequences deserves the closest attention for another reason, however. From the concept of material consequence described above, the Philonian concept of implication, forgotten in the Middle Ages, can be derived in a logical but quite unexpected way. It is worth going into this derivation more closely.

The implication "if *p* then *q*" corresponds to the inference-schema "*p*, therefore *q*"; both forms are even characterized in the same way as consequences. A true implication corresponds to a good consequence, and vice versa. A material consequence is good if it can be transformed into a formal consequence by the assumption of a true premiss. It follows from this, first, that every implication whose consequent is true must itself be true. Thus if "*q*" is true, the material consequence "*p* therefore *q*" is good for all *p*; for if the proposition "*q*", true by assumption, be added as a premiss, we obtain the inference-schema "*p* and *q*, therefore *q*", and this inference-schema is, as we have seen above, a *formal* consequence. Secondly it follows that every implication whose antecedent is false must also be true. Thus if "*p*" is false, the material consequence "*p*, therefore *q*" is good for all *q*; for if the true proposition "not-*p*" (i.e. the contradictory of the proposition *p*, false by assumption) be added as a premiss, we get the rule of inference "*p* and not-*p*, therefore *q*", and this rule of inference is a *formal* consequence, as we shall see below. In three cases therefore ("true-true", "false-true", and "false-false") is an implication true; in the fourth case ("true-false") it is of course false. Implication is therefore strictly defined as a truth-function, according to the Philonian model.

This conclusion seems to have escaped Duns Scotus. Still, he was clearly aware of all the assumptions that led up to it. He knows, that is, that from any false proposition any other proposition follows in a good material consequence, and that any true proposition results from any other proposition in a good material consequence. And finally he proves that, from a proposition which contains a formal contradiction, any proposition at all can be obtained in a *formal* consequence.[44]) The proof is given by means of an example and goes as follows. The consequence "Socrates runs and Socrates does not run, therefore you are in Rome" is formally correct. From the conjunction "Socrates runs and Socrates does not run" the proposition "Socrates runs", as well as the proposition "Socrates does not run", follows in formal consequence. From the proposition "Socrates runs" there follows further, in formal consequence, the disjunction "Socrates runs or you are in Rome". Finally, from this disjunction and the negation of its first member we obtain, in formal consequence, the proposition "you are in Rome".[45]) With the collapse of medieval scholasticism all these fine investigations fell into total oblivion.

The "philosophical" logic of modern times is infected through and through with psychology and epistemology. It has no understanding of nor interest in questions of formal logic. Aristotelian syllogistic is at best taken account of in its traditional distortion, and we find scarcely a trace of propositional logic. In vain does one seek for problems that are new, precisely formulated, and methodically solved. Everything dissolves in vague philosophical speculations.

Modern logic is reborn out of the spirit of mathematics. With "mathematical" logic or *logistic* a new logic arises and comes into full bloom

[44]) Loc. cit. (Prantl, iii, p. 141 note 621): "Ad quamlibet propositionem falsam sequitur quaelibet alia propositio in consequentia bona materiali ut nunc.— Omnis propositio vera sequitur ad quamcunque aliam propositionem in bona consequentia materiali ut nunc.—Ad quamlibet propositionem implicantem contradictionem de forma sequitur quaelibet alia propositio in consequentia *formali*."

[45]) Duns Scotus, *Quaestiones super anal. pr.* ii, 3 (not quoted by Prantl): "'Socrates currit et Socrates non currit; igitur tu es Romae.' Probatur, quia ad dictam copulativam sequitur quaelibet eius pars gratia formae. Tunc reservata ista parte 'Socrates non currit', arguatur ex alia sic: "Socrates currit, igitur Socrates currit vel tu es Romae', quia quaelibet propositio infert seipsam formaliter cum qualibet alia in una disiunctiva. Et ultra sequitur: 'Socrates currit vel tu es Romae; sed Socrates non currit (ut reservatum fuit); igitur tu es Romae', quod fuit probatum per illam regulam: ex disiunctiva cum contradictoria unius partis ad reliquam partem est bona consequentia."

in the space of a few decades. With it the logic of propositions again comes into its own. And here we encounter all at once a phenomenon unique in the history of logic: suddenly, without any possible historical explanation, modern propositional logic springs with almost perfect completeness into the gifted mind of Gottlob Frege, the greatest logician of our time. In 1879 Frege published a small but weighty treatise entitled *Begriffschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens* ("Begriffschrift", a formalized language of pure thought modelled upon arithmetic). In this treatise the whole logic of propositions is for the first time laid down as a deductive system in strict axiomatic form.[46]) The Fregean system of propositional logic is built upon two fundamental concepts, negation and implication. Implication is defined as a truth-function in just the same way as was done by Philo more than 2000 years before. Other functions are not introduced, although the expression "if not-p then q" can be read also as "p or q", and the expression "not-(if p then not-q)" as "p and q". With the help of the fundamental concepts six *Kernsatze* or axioms are laid down, from which all other theorems of propositional logic can be derived by means of rules of inference—the *rule of detachment*, which is explicitly formulated as a rule, and the *rule of substitution*, which is used without being formulated. Serving as the rule of detachment (the name does not originate with Frege) is the first indemonstrable syllogism of the Stoics: if the implication "if α then β" together with the antecedent "α" of this implication are admitted as theses of the system, then the consequent "β" may also be admitted and detached from the implication as a new thesis. As for the rule of substitution, it allows meaningful expressions only to be substituted for the variables. *Meaningful* expressions (this concept does not appear in the *Begriffschrift*) include firstly variables, then negations of the type "not-α", where α is a meaningful expression, and finally implications of the type "if α then β", where α and β are meaningful expressions. The theses of the system, i.e. the axioms and theorems, are expressed in a symbolism consisting of vertical and horizontal lines that take up an excessive amount of space. This symbol-

[46]) See on this matter Łukasiewicz and Tarski, "Untersuchungen über den Aussagenkalkül", *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie* 23 (1930), cl. iii, p. 35, note 9. ["Investigations into the Sentential Calculus", pp. 131–152 of this volume.]

ism of Frege's does, however, have the advantage of avoiding all punctuation marks, such as brackets, dots, and so on. I have succeeded in devising a simpler bracket-free symbolism, requiring the least possible space. Brackets are eliminated by placing the functions "if" and "not" before their arguments. The expression "if *p* then *q*" is represented in my symbolism by "*Cpq*", and "not-*p*" by "*Np*". Each "*C*" has as arguments the two meaningful expressions immediately following it, and each "*N*" has one such expression. Frege's axioms assume, in this symbolism, the following form:

I.	*CpCqp*.	IV.	*CCpqCNqNp*.
II.	*CCpCqrCCpqCpr*.	V.	*CNNpp*.
III.	*CCpCqrCqCpr*.	VI.	*CpNNp*.

This axiom system is complete: that is, all correct theses of propositional logic can be derived from it by means of the two rules of inference. It is deficient only in "elegance": the system is not independent, for the third axiom can be deduced from the first two. The deduction, which is performed below, gives one an idea of how a modern formalized system of propositional logic appears. To explain the deductive technique used I add the following notes.[47] Before every thesis to be proved (each of which is provided with a consecutive number and can thereby be recognized as a thesis), there is an unnumbered line, which I shall call the "derivational line". Each derivational line consists of two parts, which are separated by the sign "×". What stands before and after this separation designates the same formula, but in a different way. Before the separation sign is given the substitution which is to be performed on an already asserted thesis. For example, in the derivational line that belongs to thesis 1, the expression "I *p*/*CCpCqrCCpqCpr*, *q*/*Cqr*" means that in I "*CCpCqrCCpqCpr*" is to be substituted for "*p*" and "*Cqr*" for "*q*". The thesis resulting from this substitution is omitted from the proof for the sake of brevity—it looks like this:

1′ *CCCpCqrCCpqCprCCqrCCpCqrCCpqCpr*.

The expression "*C*II – 1" after the separation sign indicates the construc-

[47]) Cf. Łukasiewicz, "Ein Vollständigkeitsbeweis des zweiwertigen Aussagenkalküls", *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie* 24 (1931), cl. iii, p. 157.

tion of the same thesis 1′, and in such a way as to make it clear that the rule of detachment can be applied to 1′. We see, that is, that the thesis 1′ is of the type "$C\alpha\beta$", where "α" denotes axiom II. Hence "β", or 1, can be detached from it as a new thesis. Up to thesis 3 the deduction exactly follows Frege's train of thought.

I	*CpCqp*.	
II	*CCpCqrCCpqCpr*.	
	*	
	I *p*/*CCpCqrCCpqCpr*, *q*/*Cqr* × *C* – 1.	
1	*CCqrCCpCqrCCpqCpr*.	
	II *p*/*Cqr*, *q*/*CpCqr*, *r*/*CCpqCpr* × *C*1 – 2.	
2	*CCCqrCpCqrCCqrCCpqCpr*.	
	2 × *C*I*p*/*Cqr*, *q*/*p* – 3.	
3	*CCqrCCpqCpr*.	
	II *p*/*Cqr*, *q*/*Cpq*, *r*/*Cpr* × *C*3 – 4.	
4	*CCCqrCpqCCqrCpr*.	
	I*p*/*CpCqp*, *q*/*r* × *C*I – 5.	
5	*CrCpCqp*.	
	4 *q*/*Cpq*, *p*/*q* × *C*5*r*/*CCpqr*, *p*/*q*, *q*/*p* – 6.	
6	*CCCpqrCqr*.	
	3 *q*/*CCpqr*, *r*/*Cqr*, *p*/*s* × *C*6 – 7.	
7	*CCsCCpqrCsCqr*.	
	7 *s*/*CpCqr*, *r*/*Cpr* × *C*II – 8.	
8	*CCpCqrCqCpr*.	(III)

The two-valued logic of propositions, founded by the Stoics, carried on by the Scholastics, and axiomatized by Frege, stands now as a completed system before us. Scholarly research, however, knows no limits. With "many-valued" systems of propositional logic a new domain of investigation has, in recent years, come into being; a domain which opens up surprising and unsuspected vistas. History, however, need only report about this new logic in the future.

LOGISTIC AND PHILOSOPHY *)

The direct impulse to write the present artictle is due to Father Augustyn Jakubisiak's book *Od zakresu do treści* (From extension to intension).[1]) His book, which is a collection of philosophical papers, is preceded by his introduction in which he attacks those philosophical currents which, in his opinion, are connected with logistic. I regret that Father Jakubisiak, who lives in Paris, did not take the trouble to become aquainted with the milieus and opinions that he criticizes. In that way he might have avoided certain formulations which discredit his attack. Here are some examples.

Claiming that the philosophical currents connected with logistic have declared a merciless war on the philosophical doctrines of the past, Father Jakubisiak says on page 11: "Such an attitude towards the philosophy of the past is found in Russell, in Whitehead, and in Kreis, Witgenstein, Schlick, Carnap, and many others, among whom a prominent place is held by Polish logicians of the notorious 'Warsaw school'." I have never heard of any philosopher whose name is Kreis and who might be mentioned in this connection, but I do know that Schlick and Carnap belong to a group of philosophers which in philosophical circles is known as "Wiener Kreis", i.e., "Vienna Circle". Has Father Jakubisiak mistaken the name of a group for the surname of an individual?[2])

Father Jakubisiak further quotes extracts from my address at the Second Conference of Polish Philosophers in 1927, summarized in my

[1]) Augustyn Jakubisiak, *Od zakresu do treści* (From extension to intension), Warsaw, 1936, p. 301.

[2]) For accuracy's sake let it be noted that "Wittgenstein" is spelled with a double "t" in the first syllable. I also take the liberty to remark that the Warsaw school of logistic has already won some renown both in Poland and abroad, but the first time it has been called "notorious" is by Father Jakubisiak.

*) First published as "Logistyka a filozofia" in *Przegląd Filozoficzny* 39 (1936), pp. 115–131. Reprinted in 1961 edition *Z zagadnień logiki i filozofii.*

paper *O metodę w filozofii* (Towards a method in philosophy), published in the Conference Book.[3]) On page 12 Father Jakubisiak repeats my words that "the logic created by mathematicians, which sets a new standard of scientific precision, much higher than all the previous standards of precision, has opened [4]) our eyes to the uselessness of philosophical speculation. Hence, as at the time of Kant, the need arises for a reform of philosophy. Yet a reform not in the name of some vague criticism and in the spirit of a non-scientific theory of cognition, but a reform in the name of science and in the spirit of mathematical logic." Less than two pages further, on page 14, Father Jakubisiak writes: "While the human mind, contrary to Kant's prohibitions, gains ever deeper insight into the surrounding reality, the defenders of the logistic reform of philosophy want to forbid to that same human mind all contact with reality and to make it concentrate on a sterile study of *a priori* forms without content, on idle talk." The reader who remembers the fragment of my address quoted above and also knows that "mathematical logic" means the same as "logistic" is fully entitled to suppose that exactly I, as a defender of a logistic reform of philosophy, and even, as I find on page 11 of the book, as one of the "promoters" of that new philosophy, want to forbid to the human mind all contact with reality and yet in my paper quoted above I wrote quite explicitly: "We must incessantly strive for contact with reality, so that we do not produce mythical entities like Platonic ideas and Kantian things-in-themselves, but understand the essence and structure of that real world in which we live and act and which we somehow want to improve". Has, then, Father Jakubisiak failed to read to the end my paper which has only two pages?

In one of his papers Professor Zawirski, of Poznań, is concerned with an argument of Heisenberg which might be summarized as follows.[5]) In the principle of causality, which states: "if we know the present exact-

[3]) In *Przegląd Filozoficzny* 21 (1928), pp. 3–5. Its Polish title is misquoted by Father Jakubisiak on p. 12 as "O metodzie w filozofii" (On a method in philosophy).

[4]) The author misquotes as "will open".

[5]) Zygmunt Zawirski, "W sprawie indeterminizmu fizyki kwantowej" (Concerning the indeterminism of quantum physics), in *Księga Pamiątkowa Towarzystwa Filozoficznego we Lwowie* (Commemorative book of the Lwów Philosophical Society), Lwów, 1931, pp. 456–483. See in particular pp. 478–479. This paper, too, is misquoted by Father Jakubisiak as "W sprawie indeterminizmu".

ly, we can predict the future", the antecedent is false, for we are unable to know the present exactly. Hence the principle of causality is not valid. Professor Zawirski raises against this reasoning the following objection which is quoted verbatim by Father Jakubisiak in his footnote on page 17: "We may not speak of the falsehood of the principle of causality, even if we consider it in Heisenberg's formulation. It has the form of an implication; in that implication the antecedent is false, hence the principle is wrong, says Heisenberg. Now, Professor Zawirski writes, one may not reason so. It is precisely the property of an implication that it remains true even if its antecedent is false". If Heisenberg's idea is rendered correctly, which is not questioned by Father Jakubisiak, then Professor Zawirski's objection is correct, for we know from the logic of propositions that an implication having a false antecedent is true. Nor can I say that Professor Zawirski overestimates the weight of his argument. He would agree, as Father Jakubisiak mentions, with Born's opinion that should it be impossible to ascertain the antecedent, the principle of causality would be "ein leeres Gerede" and would not be applicable. Why then does our author pile denunciations upon Professor Zawirski's objection? He writes: "Mr. Zawirski accuses Heisenberg of ignorance of the rules of logic (...) poor Heisenberg does not even guess the simple and profound critical operations with which Mr. Zawirski undermines his main thesis!; ... it is only to be regretted that Heisenberg does not, and probably never will, know what formidable opponents he has in Poznań University". Can Father Jakubisiak not know the rule of logic to which Professor Zawirski refers?

In view of such formulations found in the book under discussion, Father Jakubisiak's attack on logistic and logistic philosophy could be passed over in silence. If I have decided otherwise, I have done so in order to avail myself of the opportunity provided by that attack to clarify some misunderstandings, which are not lacking when it comes to the relationship between logistic and philosophy, and to formulate precisely my own opinion on the matter.

I

Father Jakubisiak begins his attack with the statement (p. 11): "The defence of the essential postulates of criticism is also undertaken, though in a different way, by the latest philosophical currents, called either logical empiricism, or mathematical logic, or just logistic". This sentence

contains two misunderstandings. The first is implied by the insinuation that the said philosophical "currents" undertake the defence of the essential principles of criticism, i.e., Kantian philosophy, which is entirely at variance with the truth. I shall refer to this point later on. The second misunderstanding consists in identifying logical empiricism with mathematical logic, or logistic. The misunderstanding consists in this, that by "empiricism", be it logical or any other empiricism, we mean a philosophical current or trend, and "logistic" is not the name of a trend in philosophy, or even a trend in logic, but is the name of a discipline, like "arithmetic" or "psychology". In this connection let me repeat what I said at the Eighth International Congress of Philosophers in Prague, 1934:[6] "Logistic, also called 'mathematical logic', still seems to some philosophers to be only a certain trend which exists within logic along with other equally legitimate trends, while to some mathematicians it seems to have only the value of an auxiliary discipline, initiated for the purpose of laying the foundations of mathematics. In view of this I wish to emphasize that I treat logistic as an autonomous discipline which embodies modern formal scientific logic, and that it would be impossible for me to accept the existence, outside logistic, of any 'trend' in logic that might pass for scientific logic. Historically and this point I would like to stress in particular, modern logic is a higher stage of development of ancient formal logic, which can develop fully only now owing to the fact that with the co-operation of mathematicians it has succeeded in liberating itself from obscure philosophical speculations which for so long hindered its progress". Logistic, as I see it, and I do not doubt that all scientists who pursue this branch of research see it in the same way, is thus nothing else than the contemporary form of formal logic and would be fully entitled to call itself just logic, since formal logic forms the nucleus of logic.

Now, no one doubts that logic is neither a trend nor a current in philosophy but at the most is a branch of philosophy. Contemporary formal logic, or logistic, has, however, expanded so much and has grown so independent of philosophy that it is, like psychology, to be treated as a separate discipline. In view of its method and the precision of its results, and also in view of the problems with which it is con-

[6] Jan Łukasiewicz, "Znaczenie analizy logicznej dla poznania" (The significance of logical analysis for cognition), in *Przegląd Filozoficzny* 37 (1934), p. 369.

cerned, that discipline now comes closer to mathematics than to philosophy.

Further, I wish to point out that logistic not only is not a philosophical trend, but is not associated with any trend in philosophy. Let those philosophers who are not acquainted with logic and so cannot ascertain this for themselves just consider that logistic is something related to Aristotle's theory of the syllogism. Like Aristotelian syllogistic, it investigates forms of reasoning and lays down the methods of correct inference and proof. Now, it is certainly obvious that a person may do research work on syllogistic, and analogously on proof theory, while professing in philosophy indifferently empiricism or rationalism, realism or idealism, or taking on these issues no standpoint at all. In logistic, as in arithmetic, no definite philosophical point of view is either explicitly assumed or clandestinely accepted. Logistic is not philosophy nor does it pretend to replace philosophy.

It does not follow, of course, that in logistic there are no issues that have philosophical importance. Every discipline has such issues, and Father Jakubisiak knows this best, since in his collection of philosophical essays he refers incessantly to mathematics or physics or biology or even history. Disregarding here the issue of many-valued logics, which in my opinion are of the greatest importance to philosophy, I wish to mention briefly a certain other problem of logistic, which is most closely associated with philosophy.

Contemporary logic has a nominalistic guise. It refers not to concepts and judgements, but to terms and propositions, and treats those terms and propositions not as *flatus vocis*, but—having a visual approach—as inscriptions having certain forms. In accordance with that assumption, logistic strives to formalize all logical deductions, that is, to present them so that their agreement with the rules of inference, i.e., the rules of transforming inscriptions, can be checked without any reference to the meanings of the inscriptions. This striving, which in antiquity was initiated by the Stoics, who, in that respect, opposed the Peripatetics, is intended to reduce all logical self-evidence to visual self-evidence with a disregard for all elusive elements of a conceptual nature.[7])

[7]) Examples of formalized logical proofs can be found in my paper quoted in footnote 6 above (on p. 375), and also in the following two items: Jan Łukasiewicz, "O znaczeniu i potrzebach logiki matematycznej" (On the significance and requirements of mathematical logic), *Nauka Polska* 10 (1929), p. 610, footnote; Jan Łukasiewicz, "Z historii logiki zdań" (On the history of the logic of propositions), *Przegląd Filozoficzny* 37 (1934), p. 437. The last-named paper also includes (p. 428) quotations from Alexander which clarify the stand-point of the Stoics and the Peripatetics on this matter. [Of these three items only the last-quoted is included in the present volume, pp. 197–217. Pages 437 and 428, referred to above, correspond to pp. 217 and 208, respectively, of the present book.]

While in practice they adopted the nominalistic standpoint, the logicians, as far as I see, have not yet discussed nominalism thoroughly enough as a philosophical doctrine. But I consider a future discussion of this matter most desirable for the following reason.

If we treat propositions as inscriptions and inscriptions as products of human activity, then we must assume that the set of propositions is finite. No one doubts that we are able to produce only a finite number of inscriptions. On the other hand, in any logical system we assume rules of inference which lead to an infinite set of theses, that is, propositions which are asserted in that system. For instance, in the propositional calculus from any thesis we can obtain a new, longer, thesis by substituting for a variable a formula that is a negation or an implication. Hence there is no longest logical thesis, in the same way as there is no greatest natural number. Hence it follows that the set of logical theses is infinite. That infinity manifests itself at every step even in such an elementary logical system as the two-valued propositional calculus. For we can very easily establish a one-to-one correspondence between the set of all theses of two-valued logic and a set of theses that is only a proper part of the former set, thus revealing, in the case of the logical theses, a property which according to Dedekind is typical of infinite sets.[8])

How can we reconcile these facts with nominalism? We might simply disregard them and maintain that only those theses exist which have been written by someone. Then the set of theses would always be finite, and there would always exist a longest thesis. Such a point of view

[8]) For that purpose it suffices, in the implicational-negational system, to associate with all implicational theses their equiform implicational theses, and with those theses which include negation to associate formulae that differ from those theses only by having a formula "$C\alpha N\alpha$" in place of the negation "$N\alpha$". The latter set will also be a set of theses and will be equinumerous with, but only a proper part of the former, i.e., the set of all theses, since it undoubtedly will not include, for instance, the thesis "$CpCNpq$".

would be consistent, yet it seems that on such a basis it would be difficult to engage in logistic, and in particular metalogistic, research, in the same way as it would be difficult to build arithmetic on the basis of the assumption that the set of natural numbers is finite. In doing so we would make logic depend on certain empirical facts, that is, on the existence of certain inscriptions, which would be hardly acceptable. Further, following Dr. Tarski, we might consider as inscriptions not only products of human activity, but all physical bodies of definite size and shape, and assume that there are infinitely many such bodies.[9]) But then we would have to make logic depend on a hardly probable physical hypothesis, which is not desirable in any case. How, then, are we to avoid all these difficulties without abandoning nominalism?

We have so far been little worried by these difficulties, and this is the strangest point. It was so probably because, while we use nominalistic terminology, we are not true nominalists but incline toward some unanalysed conceptualism or even idealism. For instance, we believe that in the two-valued implicational-negational propositional calculus there exists a "sole" shortest axiom, although so far no one knows what that axiom looks like, and hence no one can write it down.[10]) It is as if that axiom existed as some ideal entity which we may discover some day. It would be worth while to analyse in detail all such beliefs, bearing in mind the principle, expounded by the *Venerabilis Inceptor* of nominalism, that *entia non sunt multiplicanda praeter necessitatem.*[11])

II

"The denial of metaphysics, so strongly emphasized by logicians, is a bequest of the philosopher from Königsberg", Father Jakubisiak continues on page 12 of his book. This proposition includes two new mis-statements, one historical, one factual in nature. The latter consists in this: that the students of logistic are supposed to deny metaphysics. I have already said that logistic is not philosophy, hence it is not concerned with metaphysics. Logistic neither denies not affirms metaphysics because it is not concerned with it. What is true is only that some philosophers, who along with philosophy also engage in logistic, deny metaphysics. They include, above all, the representatives of the Vienna Circle. I shall refer to this point later on.

For the time being I wish to discuss the former, historical, mis-statement. I am, of course, not authorized to speak on behalf of the Vienna Circle, but I am sure that its representatives would protest most vigorously against the supposition that the denial of metaphysics, which they propound, is a bequest of the philosopher from Königsberg. I am convinced that Kant's transcendental philosophy, which assumes, on the one hand, the existence of things in themselves, unrecognizable to us, and on the other supposes the existence of the mind endowed with some *a priori* forms of cognition, must in the eyes of the members of the Vienna Circle pass for metaphysics of the worst kind. The denial of metaphysics by the Vienna Circle is much more radical than Father Jakubisiak imagines and is a bequest not of Kant, but of Hume. It is

[9]) Alfred Tarski, "Pojęcie prawdy w językach nauk dedukcyjnych" in *Prace Towarzystwa Naukowego Warszawskiego* (The Works of the Warsaw Scientific Society), Section III, Warsaw, 1933. An English version is now available as "The Concept of Truth in Formalized Languages" in Alfred Tarski, *Logic, Semantics, Metamathematics*, Oxford, 1956, pp. 152–278. For the problem raised here cf. footnote 2 on p. 174 of the text in English.

[10]) Information on the sole axioms of the implicational-negational system can be found in Bolesław Sobociński, "Z badań nad teorią dedukcji" (Some research on the theory of deduction), *Przegląd Filozoficzny* 35 (1932), pp. 172–176, and footnote 5 on pp. 187–190. The details given there should be augmented by the fact that on February 2, 1933, Mr Sobociński found the following organic axiom consisting of 27 letters:

CCCpqCCCNpNrsCrtCuCCtpCvCrp,

which I next reduced to 25 letters:

CCCpqCCCNpNrsCrtCuCCtpCrp.

This is one of the two shortest known axioms of the implicational-negational system. The other, found by me, has the form:

CCCpqCCNrsCNtCrtCCtpCuCrp.

It may be supposed with considerable probability that neither of these two axioms is the desired shortest one. Such research is, however, so laborious that it cannot be said when, if ever, it will be completed. At the moment of sending this paper to the printers I have found out that there is an axiom of the implicational-negational system which consists of 23 letters. Its form is as follows:

CCCpqCCCNrNstrCuCCrpCsp.

[Cf. footnote *) on p. 138]

[11]) The need for a discussion of nominalism was pointed out by Father Jan Salamucha in his paper "Logika zdań u Wilhelma Ockhama" (Logic of propositions in the works of William Ockham), *Przegląd Filozoficzny* 38 (1933), p. 210. Father Salamucha's mention induced me to include these remarks on this matter.

to Hume that Professor Carnap, a leading representative of the Vienna Circle, refers when quoting his well known words:

"It seems to me, that the only objects of abstract science or of demonstration are quantity and number (...) All other enquiries of men regard only matter of fact and existence; and these are evidently incapable of demonstration (...) When we run over libraries, persuaded of these principles, what havoc must we make? If we take in our hand any volume; of divinity or school metaphysics, for instance; let us ask. Does it contain any abstract reasoning concerning quantity or number? No. Does it contain any experimental reasoning concerning matter of fact or existence? No. Commit in them to the flames; for it can contain nothing but sophistry and illusion."[12] Carnap considers these words—though I doubt that he is right—to be the classical formulation of the view that only mathematical propositions and propositions about facts are meaningful (sinnvoll), while metaphysical propositions are meaningless (sinnlos). This is the gist of Carnap's denial of metaphysics; let it be added that, according to Carnap, mathematical propositions include logical propositions and propositions of the logical syntax of language which, in his opinion, is nothing else than the mathematics of language.

I should like here to formulate my own opinion on this matter and to dissociate myself from the opinions of the Vienna Circle and from Carnap's opinion in particular. I have shifted my interests from philosophy to logistic, and the latter, not because of its content but because of its method, has greatly affected my opinion of philosophy. All this had happened even before the Vienna Circle was formed. I gave it a forceful expression in a now forgotten article, written in 1924 to mark the two-hundredth anniversary of Kant's birth:[13] "I realize", I wrote then, "that my critical opinion about the scientific value of Kant's philosophy and modern philosophy in general may be too subjective; but that opinion forces itself upon me the more strongly the further away I go from philosophy and look back at it from the

12) Rudolf Carnap, "Die Aufgabe der Wissenschaftslogik", Einheitswissenschaft, No. 3, 1934, pp. 7 and 21. Hume's words, quoted by Carnap, are in the 12th Chapter of his work *An Enquiry Concerning Human Understanding*.

13) Jan Łukasiewicz, "Kant i filozofia nowożytna" (Kant and modern philosophy), *Wiadomości Literackie* vol. I, No. 19, of May 11, 1924.

distance that separates philosophical speculation from scientific method." And my opinion on Kant's philosophy, formulated in that article, was as follows: "That philosophy calls itself critical. But how far away it stays from true, scientific criticism! Even the very differentiation between analytic and synthetic judgements is not scientifically formulated by Kant. We are not entitled to assert that the space around us must comply with certain geometrical truths, for we do not know whether that space is Euclidean or perhaps of some other kind. It is impossible to understand what are those allegedly pure ideas of space and time that are said to be inherent in us. The world of things in themselves is a metaphysical fiction that can vie with Leibniz's monadology. When we apply to it the requirements of scientific criticism, Kantian philosophy collapses like a house of cards. At every step we find vague concepts, incomprehensible statements, unjustified assertions, contradictions, and logical errors. Nothing is left except a few perhaps inspired ideas, a raw material that awaits scientific elaboration. That is why that philosophy has not performed its task, although its influence has been great. After Kant, people have not started to philosophize more critically, more reasonably, more cautiously. Kant gave rise to German idealistic philosophy, whose flights of fancy and non-scientific character has surpassed all pre-Kantian systems. Metaphysical problems have been left unsolved, though, I think, they are not unsolvable. But they must be approached with a scientific method, the same well-tested method which is used by a mathematician or a physicist. And above all people have to learn to think clearly, logically, and precisely. All modern philosophy has been incapacitated by the inability to think clearly, precisely, and in a scientific manner."

Whoever reads carefully these words which now, twelve years later, I can ratify with equal conviction will probably understand both the origin and the intention of my coming out against philosophical speculation. Such comprehension may be improved by the following comments. My critical appraisal of philosophy as it has existed so far is the reaction of a man who, having studied philosophy and read various philosophical books to the full, finally came into contact with scientific method not only in theory, but also in the direct practice of his own creative work. This is the reaction of a man who experienced personally that specific joy which is a result of a correct solution of a uniquely

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formulated scientific problem, a solution which at any moment can be checked by a strictly defined method and about which one simply knows that it must be that and no other and that it will remain in science once and for all as a permanent result of methodical research. This is, it seems to me, the normal reaction of every scientist to philosophical speculation. Only a mathematician or a physicist who is not versed in philosophy and comes into casual contact with it usually lacks the courage to express aloud his opinion of philosophy. But he who has been a philosopher and has become a logician and has come to know the most precise methods of reasoning which we have at our disposal today, has no such scruples. He knows what is the value of philosophical speculation as it has existed so far. And he knows what can be the value of reasoning carried out, as it usually happens, in inexact, ambiguous words of everyday language and based neither on empirical data nor on the precise framework of a symbolic language. Such work can have no scientific value and is a waste of time and mental energy.

But someone may say: "It seems to follow from these remarks that you only consider scientific those reasonings which are based on empirical data or on a precise symbolic language, which is the language of mathematics. Is that not exactly the standpoint of Hume? And does it not include a denial of metaphysics?" Not at all, I reply. My standpoint is quite different. Hume thought that the mathematical or "demonstrative" method can be applied only to magnitudes and numbers. Logistic has demonstrated that it has much wider application. It must be applied to metaphysical problems as well. In my article referred to above I wrote: "Metaphysical problems have been left unsolved, though, I think, they are not unsolvable. But they must be approached with a scientific method, the same well-tested method which is used by a mathematician or a physicist." I tried to outline such a method in my already mentioned paper "O metodę w filozofii." I also wrote that "a future scientific philosophy must start its own construction from the very beginning, from the foundations. And to start from the foundations means to make first a review of the philosophical problems and to select from among them only those problems that can be formulated in a comprehensible manner and to reject all the others." When referring to the problems that would have to be rejected, I meant first of all the problems concerned with the essence of the world or things in themselves, for I did not, and do not now, know how to formulate these problems in a comprehensible manner. "Next," I continued, "the task would be to try to solve those philosophical problems that can be formulated in a comprehensible manner. The most appropriate method for this purpose again seems to be the method of mathematical logic, the deductive, axiomatic, method. We would have to base our arguments on propositions which are as clear and certain as possible from the intuitive point of view and to adopt such statements as axioms. As the primitive or undefined terms we would have to select concepts whose meanings can be explained from all sides by examples. We would have to strive for a reduction of the number of axioms and primitive concepts to a minimum and to count them all carefully. All other concepts would have to be defined unconditionally by means of primitive terms, and all other theorems would have to be proved unconditionally by means of axioms and the rules of proof as adopted in logic. The results obtained in this way would have to be checked incessantly against intuitive and empirical data and with the results obtained in other disciplines, in particular in the natural sciences. In case of disagreement the system would have to be improved by the formulation of new axioms and the choice of new primitive terms. I thought then, and today I do not think otherwise, that that method could be applied to the problems of the finiteness or infinity of the world, to the problems of space, time, causality, teleology, and determinism. In particular, I have always been most interested in the issue of determinism and indeterminism; I have associated it with the problem of many-valued logics and thought that the method outlined above might serve as an approach to the solution of that issue."

In the light of these considerations, the difference between my standpoint on metaphysics and that of the Vienna Circle, and Carnap in particular, becomes clear. Carnap rejects metaphysical issues as meaningless because, following Kant, he counts as metaphysical propositions only those which claim to represent knowledge about something which remains completely outside all experience, e.g., the essence of things, things-in-themselves, the absolute, etc.[14]) With such an interpretation

[14]) Rudolf Carnap, "Philosophy and Logical Syntax", *Psyche Miniatures*, General Series No. 70, London, 1935, p. 15: "I will call *metaphysical* all those propositions which claim to represent knowledge about something which is over or beyond all

of metaphysics I can agree with Carnap's opinion. But in fact we are not concerned with such a concept of metaphysics, which, as is commonly known, emerged from an erroneous interpretation of a title of Aristotelian works. There are problems, for instance those of the structure of the universe, which have always been included in philosophy, and in particular in metaphysics, regardless of whether or not one is inclined to call them metaphysical. For Carnap, all these questions are only problems of language or, more strictly, problems of the syntax of language. Now I fully approve of Carnap's precise studies in the syntax of language; research in that field originated in Warsaw, where the first impulse was given by Professor Leśniewski and systematic foundations were later laid by Dr Tarski, whose works were not without effect on Carnap's later research.[15]) But I can in no way agree with such a formulation by Carnap as: "Thus all questions about the structure of space and time are syntactical questions about the structure of the language, and especially the structure of the formation and transformation rules concerning space- and time-coordinates."[16]) In the same place Carnap has a similar formulation about the problems of causality and determinism. A detailed refutation of such opinions would require a separate paper. Here I can only outline my view point on the issue.

I reason quite simply, perhaps naively, but no one has convinced me so far that I reason incorrectly. I would include among problems resolvable on the basis of language only such questions as whether all bodies are extensive, on the assumption that by a "body" I mean something extensive and define the term in that way. These are analytic propositions, and in my opinion only such propositions can be decided on the basis of language. On the other hand, I do not understand how we could decide on the basis of language whether the universe is spatially finite or infinite. For, by "the universe" I do not mean anything finite

experience, e.g., about the real Essence of things, about Things in themselves, the Absolute and such like".

[15]) Alfred Tarski, "Über einige fundamentale Begriffe der Metamathematik", *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie* 23 (1930), cl. iii, pp.22–29. [An English translation is now available in Alfred Tarski, *Logic, Semantics, Metamathematics*, Oxford, 1956, pp. 30–37]. In this paper, Tarski introduced the concepts of "sentence" and "consequence", fundamental for the syntax of language, on which Carnap later also based his ideas.

[16]) Cf. the English text of Carnap's work quoted above, p. 86.

or infinite, and hence I have to do with a synthetic, and not an analytic, proposition. Further, I know that to be finite and to be infinite are two different things and that they are incompatible with one another, but which one is in fact true does not in the least depend on us and our linguistic rules. The same applies to the issues of determinism and causality. In the world, causal necessity either is or is not the omnipotent ruler and everything either is or is not determined in advance, but this again cannot depend on any of our rules of the syntax of language. These problems are for me factual, real, and objective, and not purely formal, linguistic problems. I raise far-reaching objections against the way in which Carnap tries to reduce objective problems to linguistic problems. In addition to objective propositions which reproduce facts, e.g., "this rose is red", he distinguishes pseudo-objective propositions which are formed when we speak, as he puts it, "content-wise". Each such content-wise mode of speech has its counterpart in the formal mode of speech, and according to Carnap the latter is the only proper one. For instance, the proposition "The fact that the body *a* now expands is a naturally necessary consequence of the fact that the body *a* is being heated" is such a pseudo-objective proposition, formulated content-wise. It has its counterpart in the following proposition formulated in the formal mode of speech: "The proposition '*a* expands' is a consequence of the proposition '*a* is being heated' and of physical laws (at present accepted by science)." Carnap adds that content-wise formulated propositions result in the illusion that there exist some factual relationships—he uses here the rather obscure term "Objektbezogenheit" — which in reality do not exist, so that these propositions easily lead to misunderstandings and even contradictions. That is why, at least in the decisive places, we ought to avoid the content-wise mode of speech and replace it by the formal mode of speech [17]). I could agree

[17]) Cf. Carnap's work in German quoted in footnote 12 above. On p. 14 he writes: "Inhaltliche Redeweise: 2a. Der Umstand, dass der Körper *a* sich jetzt ausdehnt, ist eine naturnotwendige Folge des Umstandes, dass *a* erwärmt wird.—2b. Formale Redeweise: Der Satz '*a* dehnt sich aus' ist eine Folge aus dem Satz '*a* wird erwärmt' und den (gegenwärting wissenschaftlich anerkannten) physikalischen Gesetzen.— Die Sätze der inhaltlichen Redeweise täuschen Objektbezogenheit vor, wo keine vorhanden ist. Sie führen dadurch leicht zu Unklarheiten und Scheinproblemen, ja sogar zu Widersprüchen. Daher ist es ratsam, die inhaltliche Redeweise an den entscheidenden Stellen nach Möglichkeit zu vermeiden und statt dessen die formale anzuwenden".

that in the example quoted above the formal mode of speech corresponds to the content-wise. But how does Carnap know, as it seems to result from his formulation that he knows, that there is no factual relationship between the expansion of a body and its being heated? Why does he think that in this case the content-wise mode of speech may mislead us? These are dogmatic assertions which lack all justification. In the second example, to be found in Carnap's book in English, I do not even see the correspondence which is said to hold between the content-wise, or "material", as Carnaps calls it there, and the formal mode of speech. Carnap asserts that the pseudo-objective proposition, which in the material mode of speech is "the evening-star and the morning-star are identical", has its counterpart in the "syntactical" proposition formulated in the formal mode of speech: "the words 'evening-star' and 'morning-star' are synonymous." Here too reference is made to the deceptive character of the material mode of speech [18]). It seems to me that many empirical observations were needed to realize that the star which appears in the western section of the sky soon after sunset is the same planet that we see in the eastern section of the sky shortly before sunrise. The comprehension of *this* fact is something entirely different from the statement of the fact that two terms are synonymous. I can readily agree that the terms "bay horse" and "reddish-brown horse" are synonyms, since by a "bay horse" I mean exactly a reddish-brown horse. But that "evening star" and "morning star" denote the same object cannot be decided on the basis of language.

I think that in Carnap the attempt to reduce certain objective problems to linguistic ones results from his erroneous interpretation of the *a priori* sciences and their role in the study of reality. That erroneous opinion was taken over by Carnap from Wittgenstein, who considers all *a priori* propositions, that is, those belonging to logic and mathematics, to be tautologies. Carnap calls all such propositions analytic.

[18]) Cf. Carnap's work in English quoted in footnote 14 above. On p. 61 he writes: "*Pseudo-object-sentences. Material mode of speech.* 4b. The evening-star and the morning-star are identical.—*Syntactical sentences. Formal mode of speech.* 4c. The words 'evening-star' and 'morning-star' are synonymous." With reference to the same example he writes on p. 67: "Here we find again that deceptive character of the material mode as-to-the-subject-matter of its sentences. Most of the sentences of philosophy *deceive* us in this way, because, as we shall see, most of these are formulated in the material mode of speech."

I have always opposed that terminology, since the associations it evokes may make it misleading. Moreover, Carnap believes, together with Wittgenstein, that *a priori* propositions do not convey anything about reality. For them the *a priori* disciplines are only instruments which facilitate the cognition of reality, but a scientific interpretation of the world could, if necessary, do without those *a priori* elements. Now my opinion on the *a priori* disciplines and their role in the study of reality is entirely different. We know today that not only do different systems of geometry exist, but different systems of logic as well, and they have, moreover, the property that one cannot be translated into another. I am convinced that one and only one of these logical systems is valid in the real world, that is, is real, in the same way as one and only one system of geometry is real. Today, it is true, we do not yet know which system that is, but I do not doubt that empirical research will sometime demonstrate whether the space of the universe is Euclidean or non-Euclidean, and whether relationships between facts correspond to two-valued logic or to one of the many-valued logics. All *a priori* systems, as soon as they are applied to reality, become natural-science hypotheses which have to be verified by facts in a similar way as is done with physical hypotheses. My approach to the problems of metaphysics is connected with this opinion. *)

Carnap's analyses in this field I consider to be a risky philosophical speculation which will die away as all similar speculations have died away. I think that my standpoint is more cautious and more rational than the radical standpoint of Carnap and the Vienna Circle. Professor Ajdukiewicz was right when he wrote about the logistic anti-irrationalism in Poland that he did not know any Polish philosopher who would accept the material theses of the Vienna Circle as his own [19]). We are, it seems, too sober to do so.

[19]) Kazimir Ajdukiewicz, "Der logistische Antiirrationalismus in Polen", *Erkenntnis* 5 (1935), pp. 151–161; "Direkte Anhänger des Wiener Kreises haben wir in Polen nicht, d.h. ich kenne keinen polnischen Philosophen, der die sachlichen Thesen des Wiener Kreises sich zu eigen gemacht hätte." (First published in Polish in *Przegląd Filozoficzny* 37 (1934).)

*) On the relationship between logic and reality cf. "In Defence of Logistic" in the present volume, pp. 236–249, where Łukasiewicz's standpoint is somewhat different. In "On the Intuitionistic Theory of Deduction", also in the present volume, pp. 325–340 he refers to this matter once more (p. 333), but this time his point of view is markedly different.

BIBLIOTECA DE FILOSOFIA Y LETRAS

III

"This, then, is the ultimate goal of scientific philosophy. It begins with a denial of metaphysics, and ends in the denial of God." So wrote Father Jakubisiak in his book, on page 23.

I am truly grateful to Father Jakubisiak for not having written "logistic" in the place of "scientific philosophy", for I need not defend logistic against the blame of godlessness. But since Father Jakubisiak does not always distinguish between logistic on the one hand and logical empiricism and scientific philosophy on the other, it will do no harm if I add a few words on this matter.

Logistic is an exact, mathematical, discipline and has nothing to say on the issues of religion and the existence of God. The logisticians, according to their personal convictions, include both believers and non-believers. Father Jakubisiak mentions in his book the name of a professor at the University of Warsaw who is not a logistician but knows and values logistic and has a lively interest in it, and who is supposed, to use Father Jakubisiak's words, "to combat religion on behalf of science" (p. 22). Even if this were so, should logistic be charged with godlessness? I could mention the name of another Warsaw philosopher, who also knows and values logistic and has a lively interest for it, and who would be willing to apply that discipline to theological theories as well. [20]) And do we not now have priests who acknowledge the value of logistic?

I have the feeling now that I am beating on an open door. It suffices to say that neither does logistic include, explicitly or implicitly, any definite philosophical doctrine, nor does it clandestinely patronize any antireligious tendency.

The same applies to scientific philosophy, as I understand it here. Scientific philosophy does not want to combat anyone, for it has a great positive task to carry out: it has to construct a new view of the world and of life, based on exact, methodical thinking. "The work that faces future scientific philosophers", I wrote in my paper "O metodę w filozofii", "is immense as it is; it will be performed by minds much more powerful than those which have ever existed on our globe". I believe that a man who believes in the existence of a good and wise Force that rules this world, a man who believes in the existence of God, can view with confidence the future results of this work.

[20]) Jan Franciszek Drewnowski, "Zarys programu filozoficznego" (An outline of a philosophical programme), *Przegląd Filozoficzny* 37 (1934). See in particular §§ 169–174.

Logistic and scientific philosophy are, above all, products of the intellect. I ascribe to reason and to precise logical thinking much more importance than is usually done. History has shown us that methodical research, based on empirical data and strict reasoning, has a great and lasting value not only in science, but in practical life as well. When discussing that problem I often used to refer to the example provided by the world war. All those human activities from the period of the world war which were based on disciplines grounded on method have proved effective. Technical installations, airplanes, telephones, radio apparatus worked effectively, for a good or a bad purpose, since they were based on mathematical and physical laws. Medicines to combat disease and prevent epidemics worked effectively — these for a good purpose only — since they were based on biological research. Only those human activities failed which had no support in disciplines grounded in method, since the humanities are not usually so grounded. People failed to control effectively and to put into rational and purposeful order the economic and social phenomena, whether during the war or after the war. I believe that when the knowledge of logistic, and hence the ability to think in a precise manner, becomes common among all research workers, we shall be able to overcome the methodological defects of those most difficult disciplines which are concerned with Man and with human society.

Although I am an intellectual — indeed, precisely because of this fact—I realize, perhaps better than other people do, the great truth that intellect is not everything. I know that reason has two limits, the upper and the lower. The upper limit is formed by the axioms on which our scientific systems are based. We cannot go beyond that limit, and in the choice of axioms we must be guided not by reason but by what we usually call intuition. The lower limit is formed by individual unrepeatable facts which cannot be interpreted by any consequences deduced from general laws and from axioms. A direct observation of such facts and some kind of intuitive comprehension of them must replace reason for us. In those fields which lie outside the limits of reason there is room enough as well for religious sentiments and convictions, which also ought to permeate the whole of our rational activity.

IN DEFENCE OF LOGISTIC *)

In a discussion at the Roman Catholic Scientific Institute I delivered two speeches in defence of logistic. Following a suggestion by the Editor of the present periodical I have expanded them into the article that follows.

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Logistic, created by mathematicians in the 19th century, did not appear to have deeper connections with traditional logic as pursued by philosophers. Boole's algebra of logic, being a theory of classes, referred, it is true, to Aristotelian logic, but the propositional calculus, originated by Frege in 1879 and placed at the forefront of logistic by Russell and Whitehead, the authors of *Principia Mathematica*, seemed to have nothing in common with philosophers' logic. It is not to be wondered then that logistic has not enjoyed, and still does not enjoy, approval in philosophical circles. It is alien to them since it has not developed from the logical tradition which they know, and its strangeness is intensified by its mathematical attire.

As I had been long concerned with logistic, and with the propositional calculus in particular, I became interested in the problem as to whether that fundamental section of mathematical logic had been known before the existence of logistic. For the purpose of informing myself on this matter I consulted textbooks on the history of logic and monographs dealing with that discipline. But I soon realized that I would not learn much from those books, for they were written by philosophers who either underestimated formal logic and its problems or did not have a proper knowledge and understanding of the subject and either disregarded or misrepresented it. It was necessary to go to the sources. This I did and I discovered in Stoic logic, so much disparaged by Prantl

* First published as "W obronie logistyki" in *Studia Gnesnensia* 15 (1937).

and Zeller, the ancient prototype of modern propositional logic.[1]) Stoic logic had been known almost continuously from its creation, but none realized that that logic, as propositional logic, differs essentially from Aristotle's syllogistic as the logic of terms. It was only logistic which, by making us more sensitive to logical problems, made it possible for us to notice that difference. Today we know that the two main branches of modern logistic, logic of propositions and logic of terms, Stoic logic and Aristotelian logic, existed even in antiquity. Aristotle's logic always held a superior position since it was backed by the immense authority of the greatest ancient philosopher, with whom no representative of the Stoic school, Chrysippus included, could vie in importance. But along with Aristotelian logic there existed throughout the centuries the weaker current of Stoic logic, well known in the Middle Ages to scholastic logicians who pursued it in their commentaries to Aristotle and treatises *De consequentiis* and contributed to it many a new truth.

In this way I have been able to reunite, in an important place, the broken thread of tradition between ancient logic and logistic. We could find more such threads connecting old formal logic with modern logistic. I shall mention only the axiomatic method, so characteristic of logistic, which was already used by Aristotle when he constructed his theory of syllogism. But that fact, like many other facts and opinions in the field of logic, fell into complete oblivion in the period of modern philosophy which, as a reaction to mediaeval scholastic philosophy, totally neglected formal logic, replacing it by what was called the theory of cognition. Formal logic, dominated by philosophers, suffered a decline, from which it was rescued by mathematicians, who imparted to it the form of logistic.

Thus today's logistic is nothing more nor less than a continuation and expansion of ancient formal logic. It is not a trend in logic, along with which some other trends might exist, but it is precisely contemporary scientific formal logic which bears a similar relation to ancient logic as, for instance, contemporary mathematics bears to Euclid's *Elements*. Now, it is obvious that whoever wants to learn mathematics

[1]) Jan Łukasiewicz, "Z historii logiki zdań" (On the History of the Logic of Propositions), *Przegląd Filozoficzny* 37 (1934), pp. 97–117. [See the present volume, pp. 197–217.]

cannot today confine himself to Euclid; it is equally clear that whoever wants to become acquainted with formal logic cannot confine himself to Aristotle. Moreover, in order to comprehend properly Aristotle's syllogistic and to appreciate both its rigour and its beauty, one has first to study the contemporary propositional calculus, for in the proofs of syllogistic moods Aristotle intuitively used theses of that calculus.

In the light of such an interpretation of logistic, its relation to philosophy becomes clear. Although I have formulated my opinion on that issue elsewhere,[2] in order to avoid recurring misunderstandings I shall try here to describe my standpoint with precision. Now I have to state first of all that although logic used to pass for a branch of philosophy, contemporary formal logic, or logistic, has expanded so much and has grown so independent of philosophy that it is to be treated as a separate discipline. In view of its method and the precision of its results, and also in view of the content of its problems, that discipline today comes closer to mathematics than to philosophy. I have to state further that not only is logistic not philosophy or any branch of philosophy, but it is also not associated with any trend in philosophy. The principal task of logistic is to establish methods of correct inference and proof. This is the same task which Aristotle set himself when he originated his theory of the syllogism. Now, it is obvious that a person can pursue syllogistic and investigate proof theory as well, whether he accepts in philosophy empiricism or rationalism, realism or idealism, spiritualism or materialism, or does not adopt any viewpoint on those issues. In logistic, I emphasize once more, no definite philosophical doctrine is contained explicitly or implicitly. Logistic does not claim to replace philosophy; its only task is to provide philosophy, like any other discipline, with the best instruments to make research more efficient.

These statements summarize the whole of my view on the relation between logistic and philosophy. And although these statements have been made in all sincerity and although their justification seems to be clear, I am not at all astonished that they fail to convince everyone. To all these assurances an opponent of logistic might always say: "And yet I assert, for I feel it intuitively, that logistic grew out of quite

[2]) Jan Łukasiewicz, "Logistyka a filozofia" (Logistic and Philosophy), *Przegląd Filozoficzny* 39 (1936), pp. 115–131. [See the present volume, pp. 218–235.]

definite philosophical substratum—which fact may not even be realized by the founders of logistic—and that therefore it favours certain trends in philosophy and is hostile to others." I have in fact encountered such objections, raised from various quarters, that logistic professes or favours not one but a whole bunch of philosophical trends, with which not all agree, such as nominalism, formalism, positivism, conventionalism, pragmatism, and relativism. I shall deal with these objections one by one.

I must admit frankly that had I been asked not very long ago whether as a logistician I profess nominalism, I would without hesitation have replied in the affirmative. I did not reflect more deeply on the nominalist doctrine itself, and I only paid attention to the actual practice of the logisticians. Now, logisticians strive for the greatest possible rigour, and that can be achieved by the construction of as precise a language as possible. Our own thought, when not formulated in words, is hard even for us to grasp, and another person's thought, when not clad in any sensory form, can be grasped only by clairvoyants. Any thought, if it is to become a scientific truth that every man can learn and verify, must assume some perceivable form, must be given some linguistic formulation. All these are, I think, indisputable statements. It follows from them that precision of thought can be guaranteed only by precision of language. This was already known to the Stoics, who in that respect opposed the Peripatetics. That is also why logistic attaches most attention to the signs and inscriptions which it handles. Let me give at least one example, which will demonstrate better than all general formulations in what the supposed nominalism, and also formalism, of logistic consists. There is in logistic a rule of inference, called the rule of detachment, which states that whoever asserts the conditional proposition of the form "if α, then β", and also asserts the antecedent of that proposition, "α", may assert the consequent of the proposition, "β". In order to be able to apply this rule we must know that the proposition "α", which we assert separately, expresses "the same" thought which in the conditional proposition is expressed by the antecedent, for it is only then that we may draw the inference. And that can be ascertained only if both propositions represented by "α" have the same outward appearance, i.e. are equiform. We cannot grasp the thoughts expressed by these propositions directly, and the equiformity of propositions expressing certain thoughts is a necessary condition, though not a sufficient one for the identity of the thoughts. Should a person who asserts the proposition "if every man is fallible, then every logician is fallible" at the same time assert the proposition "any man is fallible", we could not arrive at the conclusion "hence every logician is fallible", because there would be no guarantee that the proposition "any man is fallible" expresses the same thought as the proposition "every man is fallible", which is not equiform with the former. It would be necessary to state by definition that the word "any" means the same as "every", replace in the proposition "any man is fallible" the word "any" by the word "every" on the strength of the rule of replacement by definition, and only then, having asserted the proposition "every man is fallible", equiform with the antecedent of the asserted conditional proposition, may we arrive at the conclusion. In this way we try to formalize all logical deductions, that is, to interpret them as inscriptions constructed so that we can check the correctness of the reasoning without referring to the meanings of those inscriptions. We do so because we are unable to grasp the meanings, whereas the signs are visible and clear, and in comparing them we can rely entirely on visual obviousness.

Is this concern for the precision of the language and the formalization of the proofs in themselves tantamount to nominalism? It would seem not. Logistic would adopt the nominalist standpoint if it treated terms and propositions exclusively as inscriptions of certains forms, without being concerned about whether they mean anything and what they mean. Logistic would then become a science of ornaments or figures, which we draw and combine in accordance with certain rules, toying with them as if in a game of chess. Today, I could not accept such a standpoint, and that not only for the reason formulated by me not so long ago,*) that the set of inscriptions is always finite, while the set of logical theses, in logic of propositions alone, is infinite: all my intuitions object to the ultimate consequences of nominalism. By difficult mental work, going on for years and surmounting enormous difficulties, we are step by step acquiring new logical truths. And with what are these truths to be concerned? With empty inscriptions and spatial ornaments? I am not a graphic artist or a calligrapher, and I am

*) In the article "Logistic and Philosophy", included in the present volume. The remarks to which Łukasiewicz refers here are on p. 223.

not interested in ornaments and inscriptions. The whole difference between logistic and a game of chess consists precisely in this, that chessmen do not mean anything, while logical symbols have meaning. We are concerned with that meaning, with the thoughts and ideas expressed by signs, even if we do not know what these meanings are, and not with the signs as such. Through the intermediary of these signs we want to grasp some laws of thought that would be applicable to mathematics and philosophy and to all disciplines that make use of reasoning. That goal is worthy of the greatest effort. We formalize logical deductions and we are right in doing so; but formalization is only a means of acquiring knowledge and certainty about something, and what is important for us is not the means but that of which we obtain cognition through those means.

Today I can no longer adopt a nominalist standpoint in logistic. But I say that as a philosopher, and not as a logician. Logistic cannot settle the question, because it is not philosophy. *A fortiori* it cannot be blamed for nominalism.

Other objections are being raised in cennection with formalism, not against logistic itself, but against the attempts to apply it to philosophy. It is said that logistic would like to axiomatize and formalize everything, but that is impossible to achieve, because reality is richer than its rationalized, logistic formalization. It can be grasped not only by discursive thinking, but also by thinking in terms of images, by concrete, emotional and intuitive thinking. I should like to reply briefly to this objection as well.

I do not know what intuitive thinking is and I do not feel competent to explain it. But I am convinced that besides discursive thinking there may be some other way of arriving at the truth, because such facts are known to logisticians from their own experience. It does happen sometimes that either as a result of the subconscious work of the mind, or owing to a fortunate association of ideas, or thanks to an instinctive sense of truth, a creative and fertile idea, which removes our difficulties and shows new paths of research, appears in our consciousness quite unexpectedly, as it were by inspiration.[3]) This happens in particular

[3]) In this connection cf. Jan Łukasiewicz, "O nauce" (On science), *Biblioteczka Filozoficzna*, 5, Polskie Towarzystwo Filozoficzne, Lwów, 1934. [Included in the present volume as "Creative Elements in Science", pp. 1–15]

on the front lines of human thought, where we face territories not yet conquered by science, not illuminated by thought, dark and unknown. There intuition often replaces discursive thinking, which in such cases is usually helpless, and makes the first pioneer conquests in the new territories. But once the territory has been conquered, then it should be occupied by discursive thinking with the full apparatus of logistic, so that the spoils of intuition, which can easily prove fallible, may be checked, ordered, and rationalized. For in my opinion only such mental territory may be considered definitively won for science which has been ordered by methods approved by logic. This is how I imagine co-operation between intuitive and discursive thinking.

To the objection of positivism I have replied comprehensively in my paper "Logistic and Philosophy", mentioned above. There I have discussed, in particular, my attitude towards the views of the Vienna Circle. In this place I should only like to make a brief remark in connection with that objection.

The concept of positivism is somewhat elastic. A man who is guided by reason, without succumbing to his emotions, and sticks to reality, without yielding to fantasy, is often considered a positivist. I have to admit that in this respect I am also a positivist. I firmly believe in reason, though I know its limitations, and I take reality into account, while trying to restrain my emotions and fantasy. Logistic could but intensify these inclinations. This explains my dislike of philosophical speculations. I do not reject metaphysics, I do not condemn philosophy, I am not biased in advance against any philosophical trend, but I disapprove of sloppy mental work. And it is probably neither my fault, nor that of logistic, that it sharpens criticism and discloses many defects in philosophical speculation. I predict that any person who receives a good logistical training will view these problems in the same way I do.

Further, contemporary logistic is blamed for being based on conventionalism. That this is so is supposed to be proved by the fact that present-day systems of logistic are not constrained in the structure of their axiomatic systems by any absolute rules or ideas, but are built in an arbitrary way. I should like to examine this objection in greater detail.

Let us consider first what is called the two-valued propositional

calculus. It is known that this calculus can be presented axiomatically in various ways, which depend above all on the primitive terms selected and the rules of inference adopted. But even with the same primitive terms, e.g., implication and negation, and with the same rules of inference, e.g., substitution and detachment, we can select the axioms of the propositional calculus in many ways. Does it follow that the propositional calculus is constructed in an arbitrary manner? Not in the least. We may not impart to just any theses the status of axioms, because in our calculus the axiomatic system must satisfy very rigorous conditions: it must be consistent, independent, and complete, which means that it must potentially contain all the true theses of the system. Only such a system of axioms is good, but at the same time any such system of axioms is good, since all of them are equivalent to one another and all of them generate the same system of the propositional calculus. In choosing this or that system of axioms out of all possible ones, we need not be constrained by any absolute principles, for we know in advance that such principles, e.g., the principle of consistency, are satisfied by all systems of axioms, and we are guided only by practical or didactic considerations. I do not see in all this even a trace of conventionalism, which I have never favoured and do not favour now. To put it simply, the two-valued propositional calculus has the property that it can be constructed axiomatically in different ways, and that property is a logical fact which does not depend on our will and which we have to accept whether we like it or not.

That property, by the way, is shared by the two-valued propositional calculus with other axiomatic systems, including Aristotle's theory of the syllogism. The Stagirite tried to axiomatize his theory of the syllogism, but his system of axioms was insufficient. I have solved this problem in former papers by adopting as the primitive formulae of this syllogistic the propositions "all A is B" and "some A is B", and as axioms the theses "all A is A", "some A is A" and the syllogistic moods *Barbara* and *Datisi*.[4]) To these I joined the rules of substitution, detachment, and definitional replacement, and the propositional calculus as an auxiliary system. I could, of course, have chosen other primi-

[4]) Jan Łukasiewicz, *Elementy logiki matematycznej*, Warsaw, 1939, pp. 86–96. [Cf. the English-language version, *Elements of Mathematical Logic*, Warsaw–Oxford 1963, pp. 103–117.]

tive formulae, e.g., the propositions "all *A* is *B*" and "no *A* is *B*". I would then have had to adopt a different system of axioms. But even for the primitive formulae which I have chosen I could have selected different axioms, for instance, instead of the thesis "some *A* is *A*" I could have used the law of conversion of universal affirmative propositions, and instead of the mood *Datisi* I could have adopted the mood *Dimatis* of the fourth figure. Thus Aristotelian syllogistic, too, can be constructed axiomatically in many ways. There is no conventionalism behind this fact, since all these systems of axioms are equivalent to one another and generate the whole of Aristotelian logic with the same moods of syllogisms.

The underlying bias against these allegedly arbitrary systems of axioms seems to consist subconsciously in a requirement of the theory of cognition which might be formulated thus: "In every deductive system there is only one directly self-evident principle on which all the theses of that system are to be based." The stress is laid both on "only one" and on "directly self-evident". It already pleased Kant to be able to deduce something, as he put it, according to his wish, *nach Wunsch*, from a single principle, *aus einem einzigen Prinzip*. How beautiful it would be if such a principle were the one and only in this sense, too, that the system could not be based on any other, and if it also were directly self-evident, and hence somehow necessary and absolute! But that would be too beautiful to be true. It is a fact that the two-valued implicational-negational propositional calculus which makes use of the rules of substitution and detachment can be based on a single axiom, but that, too, can be done in many ways. Hence in that calculus there are many "sole" axioms. Moreover, none of the axioms which we have come to know so far is directly self-evident, bacause all of them are too long to have their truth grasped intuitively. In what concerns the last-named point, the situation usually is such that self-evident theses are deductively weak, and those theses which are deductively strong—and only such can serve as axioms—are not self-evident. In the implicational propositional calculus, which includes only implications without negation, probably the most self-evident thesis is the law of identity "if *p*, then *p*", i.e., in symbolic notation, *Cpp*. But with the rules of substitution and detachment that law enables only a deduction of its own substitutions and hence is very weak deductively and of course cannot serve as the sole axiom of that calculus. On the other hand, the only axioms of the implicational calculus are not self-evident. Last year I succeeded in finding the shortest axiom of that calculus. In the parenthesis-free notation which I have conceived it has only 13 letters and the following form: *CCCpqrCCrpCsp*. *) But this axiom, too, is not quite self-evident, and in any case it is less self-evident than the law of hypothetical syllogism *CCpqCCqrCpr*, or even than Frege's law (which is not shorter than my axiom) *CCpCqrCCpqCpr*, neither of which serves as a sole axiom of the system.

I now pass to the final objections listed above, namely those which blame logistic for pragmatism and relativism. I am most concerned with these objections since they have been raised in connection with the many-valued systems of propositional logic. That is why I should like to reply to them in greater detail.

First of all, as the founder of many-valued systems of propositional logic I state that historically those systems have not developed on the basis of conventionalism or relativism, but have emerged from logical researches concerned with modal propositions and the related concepts of possibility and necessity.[5]) In the construction of such systems I made use of the matrix method, invented by Peirce as early as 1885. My students, Messers Słupecki, Sobociński, and Wajsberg continued my research and applied the axiomatic method to many-valued systems.[6])

[5]) Jan Łukasiewicz, "O pojęciu możliwości" (On the concept of possibility), (report on a lecture), *Ruch Filozoficzny* 5 (1920), pp. 169a–170a; Jan Łukasiewicz, "O logice trójwartościowej" (On three-valued logic), (report on a lecture), ibid., pp. 170a–171a; Jan Łukasiewicz, "Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenkalküls", *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie*, 23 (1930), cl. iii, pp. 51–77. [The first report is not included in the present publication, for the second see pp. 87–88 of this volume. For the third item, see "Philosophical Remarks on Many-Valued Systems of Propositional Logic", pp. 153–178 of this volume.]

[6]) M. Wajsberg, "Aksjomatyzacja trójwartościowego rachunku zdań" (Axiomatization of the three-valued propositional calculus), *Sprawozdania z posiedzeń Towarzystwa Naukowego Warszawskiego* 24 (1931), Wydział III; J. Słupecki, "Der volle dreiwertige Aussagenkalkül", *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie*, 29 (1936), cl. iii; B. Sobociński, "Aksjomatyzacja pewnych wielowartościowych systemów teorii dedukcji" (Axiomatization of some many-valued systems of the theory of deduction), *Roczniki prac naukowych Zrzeszenia Asystentów Uniwersytetu Józefa Piłsudskiego*, vol. I, Warsaw, 1936.

*) Cf. footnote on p. 196.

In particular, owing to the work done by Słupecki we know today how to base what is called the full three-valued propositional calculus with one selected value on a system of axioms which is consistent, independent, and complete in the same sense as the axiomatic systems of the two-valued calculus. I specify these facts in order to state on their basis that the existence of systems of many-valued logic is today to be taken into account in the same way as, e.g., the existence of systems of non-Euclidean geometry is to be taken into account. Those systems do not depend on any philosophical doctrine, for they would fall with the collapse of the doctrine, but are as much an objective result of research as any established mathematical theory. Thus one cannot state: "I reject contemporary logistic, because it has resulted in many-valued logic, and I revert to traditional logic," just as he may not say: "I reject contemporary geometry, because it has resulted in non-Euclidean geometry, and I revert to Euclidean geometry." Such a standpoint would not only cancel the achievements of contemporary science, but would be, I dare say, an ostrich policy consisting in the belief that what is ignored does not exist. We cannot disregard the systems of many-valued logic once they have been constructed; we can only argue whether they can be interpreted intuitively as well as two-valued logic can, and whether they will find any application. I want to enlarge a little on this issues.

The deepest foundation of all logic known so far, whether logic of propositions or logic of terms, whether Stoic or Aristotelian logic, is the principle of bivalence which states that every proposition is either true of false, that is, has one, and only one, of these two logical values.*) Logic changes from its very foundations if we assume that in addition to truth and falsehood there is also some third logical value or several such values. I made my assumption referring to the authority of Aristotle himself, for no one other than the Stagirite seemed to believe that propositions concerning future fortuitous events are today neither true nor false. This is how some formulations made by Aristotle in the ninth chapter of his *Hermeneutics* are to be interpreted and how they were interpreted by the Stoics, as testified by Boetius. In stating

*) For the details of the principle of bivalence see "Philosophical Remarks on Many-Valued Systems of Propositional Logic" in the present volume, pp. 153–178.

this the Stagirite tried to avoid determinism, which to him seemed to be unavoidably connected with the principle of bivalence.

If that standpoint of Aristotle is correct, and if among propositions about events taking place in the universe there are propositions which at the present moment are still neither true nor false, then those propositions must have some third logical value. But then the world of facts around us is ruled not by a two-valued logic, but by a three-valued or some other many-valued logic, if the number of those new values is greater. Then many-valued systems of logic would acquire both an intuitive justification and a vast field of application.

I have often considered the problem of how to determine whether there do exist propositions about facts that have that third logical value. Here a logical problem becomes an ontological issue concerned with the structure of the universe. Has everything that happens in the universe been determined for centuries, or are certain future facts not yet determined today? Does there exist in the universe a sphere of contingency, or is everything inevitably ruled by necessity? And is that sphere of contingency, if its exists, to be sought only in the future, or can it also be found in the past? These are questions which it is very difficult to answer. I have always believed that answers to these questions can be provided only by empirical data, in the same way that only empirical data can tell us whether the space in which we move about is Euclidean or non-Euclidean. Here is the origin of the imputations of pragmatism to logistic, imputations that are unjustified as far as logistic is concerned, since these imputations might be addressed only to me personally. Nor can I accept such imputations. I do not accept pragmatism as a theory of truth, and I think that no reasonable person would accept that doctrine. Nor have I ever thought of verifying pragmatically the truth of logical systems. Those systems do not need such a verification. I well know that all logical systems which we construct are necessarily true under the assumptions made in their construction. The only point would be to verify the ontological assumptions that underlie logic, and I think that I act in accordance with the methods universally adopted in natural science if I strive to verify the consequences of those assumptions in the light of facts.*) On this issue my opinion is contrary to that of the Vienna Circle positivists,

*) Cf. footnote *) on p. 233.

for they deny that such questions are subject to empirical verification and claim that they belong exclusively to the syntax of language. That opinion of the members of the Vienna Circle, which I do not share, I would call conventionalism.

I do not consider the problem of the interpretation of many-valued systems to be settled definitively. Our knowledge of those systems, which have been developed so recently, is still inadequate. They will have to be thoroughly examined, both from the formal and from the intuitive point of view. But even today I can state one thing: relativism is not a consequence of the existence of those systems. It cannot be inferred from the possibility of different systems of logic, and hence of different concepts of truth, depending on what logical system is adopted, that there are no absolute truths. I adduce this argument here for there is a scientist who has drawn such conclusions from the existence of different systems of logic. Two years ago E.T. Bell, an American professor of mathematics, published a popular book entitled *The Search for Truth.*[7] As the motto of his book he took the following words from *St. John's Gospel* (XVIII, 38): "Pilate saith unto him, What is truth?" That question, Professor Bell claims, ceased to have sense when systems of many-valued logic were made known in 1930.

In view of this I state: that question has never ceased and will never cease to have sense. Absolute truths of thought did not collapse in 1930. Whatever discredit anyone may try to cast upon many-valued logics, he cannot deny that their existence has not invalidated the principle of exclusive contradiction. This is an absolute truth which holds in all logical systems under the penalty that should this principle be violated then all logic and all scientific research would lose their purpose. Also valid remain the rules of inference, namely the rule of substitution, which corresponds to the Aristotelian *dictum de omni*, and the rule of detachment, analogous to the Stoic syllogism called *modus ponens*. Owing precisely to these rules we are building today not one but many logical systems, each of which is consistent and free of contradiction. It may be that other absolute principles, with which all logical systems must comply, also exist. I think that it is one of the main tasks of future logistic and philosophy to bring out all those principles.

[7] Eric Temple Bell, *The Search for Truth*, Allen and Unwin, London, 1934, in particular pp. 245–247.

In concluding these remarks I should like to outline an image which is connected with the most profound intuitions which I always experience in the face of logistic. That image will perhaps shed more light on the true background of that discipline, at least in my case, than all discursive description could. Now, whenever I work even on the least significant logistic problem, for instance, when I search for the shortest axiom of the implicational propositional calculus I always have the impression that I am facing a powerful, most coherent and most resistant structure. I sense that structure as if it were a concrete, tangible object, made of the hardest metal, a hundred times stronger than steel and concrete. I cannot change anything in it; I do not create anything of my own will, but by strenuous work I discover in it ever new details and arrive at unshakable and eternal truths. Where is and what is that ideal structure? A believer would say that it is in God and is His thought.

THE EQUIVALENTIAL CALCULUS *)

1. Equivalence and the equivalential system

By an *equivalence* I mean an expression of the type "α if and only if β", in symbols "$E\alpha\beta$", where α and β are propositions or propositional functions. The equivalence is true if α and β have the same truth-

*) [Editorial note from the McCall edition: This paper was intended to appear, under the title "Der Äquivalenzenkalkül", in vol. 1 of the Polish periodical *Collectanea Logica*, Warsaw, 1939, pp. 145–169. The following short history of this periodical is taken from the introduction to B. Sobociński's "An investigation of protothetic", published as No. 5 of the *Cahiers de l'Institut d'Études polonaises en Belgique*, Brussels, 1949.

"In 1937, at the suggestion of Mr. Jan Łukasiewicz, we founded in Poland a periodical devoted to Logic, its history and its applications, under the title *Collectanea Logica*. It was to be issued as one large volume each year, and would be international in character, containing different papers in Polish, English, French, German, Italian, and Latin. The editor of *Collectanea Logica* was Łukasiewicz, and its managing editor myself. ... On the first of September 1939 the first part of the volume, which would have had 500 pages, was printed, the second part already collected and in proof. Moreover, the first five papers from the prepared part were already published as offprints. At the siege of Warsaw in September 1939 the printing-house of the periodical was completely burned, with all the prepared type, blocks, and offprints. The final proofs of the first volume, most of the prepared offprints, and the archives of the publication escaped in my flat, but all this was destroyed in August 1944 during the Warsaw Insurrection." Sobociński follows this by giving a brief description of the contents of the first volume.

Only one copy of "Der Äquivalenzenkalkül", sent as a review copy to Scholz in Münster, is known to have survived the war, and is now in Poland. [Translated by P. Woodruff.] Polish translation is included in the 1961 edition *Z zagadnień logiki i filozofii*.

value, i.e. either both are true or both are false; otherwise it is false. If we denote the True by "1" and the False by "2", the following equations hold:

$$E11 = 1, \quad E12 = 2,$$
$$E21 = 2, \quad E22 = 1.$$

These equations are indicated in the following matrix

$$\begin{array}{c|cc} E & 1 & 2 \\ \hline 1 & 1 & 2 \\ 2 & 2 & 1 \end{array}$$

which I shall call the *normal* matrix for equivalence. The first argument is written to the left of the vertical stroke, the second above the horizontal line.

An equivalence $E\alpha\beta$, in which besides propositional variables only functors of the propositional calculus appear, is called an equivalence of *propositional logic*. For example *EKpqKqp*, in words "p and q if and only if q and p", is an equivalence of propositional logic. If in an equivalence of propositional logic no functor of the propositional calculus other than E appears, I shall call it a *pure* equivalence of propositional logic.

By the ordinary or two-valued equivalential system I mean the set of all pure equivalences of propositional logic which satisfy the normal matrix for equivalence. The matrix is said to be satisfied by a given equivalence, if all replacements of the propositional variables of the equivalence with the values 1 or 2 yield expressions which after reduction according to the matrix assume the value 1. For example, the equivalence *Epp* satisfies the matrix, for we get:

$$\text{for } p/1 \quad E11 = 1,$$
$$\text{for } p/2 \quad E22 = 1.$$

Likewise *EEpqEqp* satisfies the matrix, since the following equations hold:

$$\text{for } p/1, q/1 \quad EE11E11 = E11 = 1,$$
$$\text{for } p/1, q/2 \quad EE12E21 = E22 = 1,$$
$$\text{for } p/2, q/1 \quad EE21E12 = E22 = 1,$$
$$\text{for } p/2, q/2 \quad EE22E22 = E11 = 1.$$

On the other hand, the matrix is not satisfied by *EpEqp*, for we have:

$$\text{for} \quad p/2,\, q/2 \qquad E2E22 = E21 = 2.$$

The two-valued equivalential system is one of the simplest sub-systems of the propositional calculus. Numerous methodological questions can be formulated with particular clarity and simplicity in this system, and can be easily solved. For this reason it is worth while to subject the system to a detailed examination, for we obtain thereby an easily grasped introduction to the problems and methodology of the propositional calculus.

2. On the history of the equivalential calculus

Two pure equivalences are to be found in *Principia Mathematica*, [1]) namely

$$p \equiv p \quad \text{and} \quad p \equiv q \,.\equiv.\, q \equiv p,$$

to which correspond the following theses in my bracket-free symbolism:

$$Epp \quad \text{and} \quad EEpqEqp.$$

The first thesis says that equivalence is *reflexive*, the second, that it is *commutative*. Now, I long ago noted that equivalence is also *associative*, and accordingly I established the following thesis in the symbolism of *Principia*:

$$p \equiv .\, q \equiv r :\equiv: p \equiv q \,.\equiv r.$$

This thesis, which in my symbolism can be expressed by

$$EEpEqrEEpqr,$$

is cited by Tarski [2]) in his doctoral thesis of 1923.

Leśniewski [3]) was in 1929 the first to recognize that the two-valued equivalential system can be axiomatized. In particular, this can be done with the help of two rules of inference; the rule of substitution and the rule of detachment. The rule of detachment for equivalence is analogous to that for implication. Leśniewski characterizes this rule approximately thus: If an equivalence A belongs to the system whose right side is equiform with S, and if a theorem belongs to the system which is equiform with the left side of the equivalence A, then a theorem equiform with S may be added to the system. In other words, if the expressions "$E\alpha\beta$" and "α" belong to the system, then "β" may also be added to it. Leśniewski shows that with the help of this rule of detachment and the rule of substitution all pure equivalences provable in the ordinary propositional calculus can be deduced from the following two axioms:

A1. $p \equiv r \,.\equiv.\, q \equiv p :\equiv. r \equiv q,$

A2. $p \equiv .\, q \equiv r :\equiv: p \equiv q \,.\equiv r.$

The second axiom is the law of associativity for equivalence, discovered by myself. From these axioms, which in my symbolism read

$$EEEprEqpErq \quad \text{and} \quad EEpEqrEEpqr,$$

Leśniewski first derives seventy-nine theses in symbolic form, and then proves with the help of reasoning conducted in ordinary language that the above axiomatic system is complete. Proofs of consistency and independence are not found in Leśniewski.

After Leśniewski, Wajsberg [4]) in 1932 published simpler axiom-systems for the equivalential system, at first without completeness-proofs. Two of these systems consist of two axioms each, two others of one axiom apiece. The four systems of Wajsberg, given by the writer in my symbolism, are

(a) *EEEpqrEpEqr* and *EEpqEqp*,

(b) *EEpEqrErEqp* and *EEEpppp*,

(c) *EEEpEqrEErssEpq*,

(d) *EEEEpqrsEsEpEqr*.

Wajsberg gives completeness-proofs for these four axiom-systems in

[1]) A. N. Whitehead and B. Russell, *Principia Mathematica*, vol. i, Cambridge, 1910, p. 121, theorems *4.2 and *4.21.

[2]) A. Tajtelbaum-Tarski, *O wyrazie pierwotnym logistyki* (On the primitive term of logistic), Doctoral thesis, *Przegląd Filozoficzny* 26 (1923), p. 72 n. See also A. Tajtelbaum, "Sur le terme primitif de la logistique", *Fundamenta Mathematicae* 4 (1923), p. 199 n.

[3]) S. Leśniewski, "Grundzüge eines neuen Systems der Grundlagen der Mathematik", *Fundamenta Mathematicae* 14 (1929), § 3, pp. 15–30.

[4]) M. Wajsberg, "Ein neues Axiom des Aussagenkalküls in der Symbolik von Sheffer", *Monatshefte für Mathematik und Physik* 39 (1932), p. 262.

a later work[5]) by reducing them in part directly and in part indirectly to that of Leśniewski.

To Wajsberg belongs the credit of having first shown that the equivalential calculus can be based on a *single* axiom. The two single axioms of Wajsberg consist of fifteen letters each. It soon turned out that there are other equivalences, each of fifteen letters, which can be postulated as single axioms of the system. In a paper of 1932 Sobociński[6]) gives the following six axioms, which were discovered by various authors:

(e) *EEpEqrEEqEsrEsp* discovered by Bryman.
(f) *EEpEqrEEqErsEsp* discovered by Łukasiewicz.
(g) *EEsEpEqrEEpqErs* discovered by Łukasiewicz.
(h) *EEpEqrEEpErsEsq* discovered by Sobociński.
(i) *EEpEqrEEpEsrEsq* discovered by Sobociński.
(j) *EEpEqrEEpErsEqs* discovered by Sobociński.

Axiom (g) is obtained as a first detachment from Wajsberg's axiom (d). Completeness-proofs are not given by Sobociński.

In 1937 the Rumanian mathematical logician E. Gh. Mihailescu published a paper devoted specifically to the equivalential calculus.[7]) Mihailescu bases his work on the above-mentioned work of Leśniewski, and uses the bracket-free notation which I introduced. In metalogical investigations he makes use of Tarski's terminology. Wajsberg's work is apparently unknown to him. In his essay, the equivalential calculus is based on the two axioms *EEpqEqp* and *EEEpqrEpEqr*, discovered by Wajsberg. For this axiom-system he gives a new completeness-proof by reducing all meaningful expression to certain normal forms. For this purpose ninety-three theses are deduced from the axioms by substitution

[5]) M. Wajsberg, "Metalogische Beiträge", *Wiadomości Matematyczne* 43 (1936), pp. 132–133 and 163–166. Instead of axiom-system (a) the writer considers here the following axiom-system:

(a′) *EEpEqrEEpqr* and *EEpqEqp*,

which is obviously deductively equivalent to (a).

[6]) B. Sobociński, "Z badań nad teorią dedukcji" (Investigations into the theory of deduction), *Przegląd Filozoficzny* 35 (1932), pp. 186–187 and 192–193, nn. 35–37.

[7]) E. Gh. Mihailescu, "Recherches sur un sous-système du calcul des propositions" *Annales scientifiques de l'Université de Jassy* 23 (1937), pp. 106–124.

and detachment. The consistency and independence of the axiom-system is shown by the matrix method.

The present work contains my results from the year 1933. Among the most important results I wish to present are the discovery of the shortest axiom of the equivalential system as well as a new completeness-proof, which seems to me to be simpler than those of Leśniewski and Mihailescu.

3. Meaningful expressions and the rule of substitution

All expressions of our system are formed by juxtaposition of capital *E*'s, the sign of equivalence, and small Latin letters, the propositional variables. Not every expression thus formed is, however, *meaningful*; i.e. not every expression represents a proposition, or, more precisely, a propositional function. For example, "*pq*", "*E*", "*EE*", "*pEq*", "*Epqr*" are not meaningful expressions, for they do not represent propositional functions. On the other hand, propositional variables such as "*p*", "*q*", "*r*", etc., as well as equivalences both of whose members are meaningful expressions, such as "*Epq*", "*EEpqr*", "*EpEqr*", etc., are evidently meaningful. In the following I give a purely *structural* definition of "meaningful expression" for the equivalential system, by slightly modifying a definition found for the implication-negation system by Jaśkowski:[8])

An expression made up of the letter "*E*" and small Latin letters is *meaningful* if, and only if, it fulfils the following two conditions:

1. The number of "*E*'s" occurring in the expression must be *one less* than the number of small letters.

2. In every segment, which begins at an arbitrary point in the expression and reaches to the end of the expression, the number of "*E*'s" must be *less* then the number of small letters.

The two conditions are independent, as examples readily show. Thus, "*EpEqr*" fulfils both conditions and is therefore meaningful. The expressions "*EpqEr*" fulfils the first condition but not the second,

[8]) See J. Łukasiewicz, "Ein Vollständigkeitsbeweis des zweiwertigen Aussagenkalküls", *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie* 24 (1931), cl. iii, p. 156, n. 5.

since in the segment beginning with the second "*E*" the number of "*E*'s" is not less than the number of small letters. The expression "*pEEqrs*" satisfies the second condition but not the first, since the number of "*E*'s" in the expression is not one but two less than the number of small letters. Finally, it is clear that the expression "*pqEErs*" fulfils neither the first nor the second condition. The last three expressions are meaningless.

These conditions yield, among others, the following consequences:

(a) Propositional variables are meaningful expressions, for they satisfy both conditions.

(b) All composite meaningful expressions must begin with an "*E*". For if the second condition is to be fulfilled, then in the segment beginning with the second letter the number of "*E*'s" must be at least one less than the number of small letters. If a small letter is now added at the beginning, then the number of "*E*'s" in the whole expression must be less than that of the small letters by at least two letters, which contradicts the first condition.

In connexion with this definition there is a simple practical rule which enables us to decide at once if a given expression, composed of the letter "*E*" and small letters, is meaningful or not.[9] One first assigns each "*E*" the number -1 and each small letter the number $+1$. Then one adds these numbers sequentially, starting with the number assigned to the last letter on the right of the expression and proceeding by steps to the left, to the beginning of the expression. The following example illustrates this process:

EEEpqErsEtu
1 23 43 232121

The "*u*" is assigned $+1$, also the "*t*"; 1 plus 1 is 2, "*E*" is -1, $2-1=1$, etc. If the expression is meaningful, then the first condition says that the *sum*, which corresponds to the whole expression and stands at the very beginning, must be equal to 1; the second condition says that all *partial sums*, which correspond to single segments, must be positive, i.e. greater than 0. A glance at the number-series which belongs to the expression in the above example suffices to determine that this expression is mean-

[9] The idea behind this rule is not mine, but rather—as far as I know—that of a student of L. Chwistek.

ingful. If such a number-series does not begin with 1, if a 0 or even a negative number appears, then the expression is meaningless; e.g.

pEEqrs *EpqEr*
2 12321 12101

In the first example the sum which corresponds to the whole expression equals 2, which is incompatible with the first condition. In the second example the partial sum which corresponds to the segment "*Er*" equals 0, contrary to the second condition.

Not all meaningful expressions belong to the system. Those which do I call theses. In our equivalential system the theses are distinguished by the fact that they satisfy the normal matrix for equivalence.

Since we now have the concept of a meaningful expression at our disposal, we can formulate the *rule of substitution* precisely. On the basis of this rule one obtains a new thesis from a given thesis by replacing one or more of the propositional variables of the given thesis by *meaningful* expressions, where all equiform variables must be replaced by equiform expressions. For example, if in the thesis with which we are already familiar,

EEpqEqp,

the meaningful expression "*Eqr*" is substituted for "*q*" (which transformation I denote by "*q*/*Eqr*") we obtain a new thesis:

EEpEqrEEqrp.

The other rule which is used to derive theses is the previously characterized *rule of detachment*: If $E\alpha\beta$ and α are theses, β is also a thesis and hence can be detached from $E\alpha\beta$. For example, let the following two theses be given:

1 *EEpqEqp*.
2 *EEpEqrEEpqr*.

If in 1 the substitution "*p*/*EpEqr*, *q*/*EEpqr*" is made, one obtains

1′ *E* $\underbrace{EEpEqrEEpqr}_{2}$ $\underbrace{EEEpqrEpEqr}_{3}$.

Thesis 1′ begins with an "*E*" followed by thesis 2 as its first member; consequently its second member, in accordance with the rule of detachment, can be detached as a new thesis:

3 $EEEpqrEpEqr.$

This derivation I indicate briefly as follows:

$1\ p/EpEqr,\ q/EEpqr \times E2-3.$
3 $EEEpqrEpEqr.$

In the *derivational line* which precedes thesis 3, the series of expressions both before and after the *separation sign* "×" designates the thesis 1′, which is omitted for the sake of brevity.

4. The shortest axiom

The shortest axiom for the equivalential system, which I discovered, consists of eleven letters and reads: $EEpqEErqEpr$. From this axiom I will first derive Leśniewski's two axioms, as well as those theses which are required for the completeness-proof to be given later. All these theses are marked by an asterisk. I will then prove that no shorter axiom possesses the property of being a single axiom of the system. The following deductions, which are constructed with the sole use of the rules of substitution and detachment mentioned above, should be clear enough after what I have said above.[10])

*1 $EEpqEErqEpr.$
$1\ p/Epq,\ q/EErqEpr,\ r/s \times E1-2.$
2 $EEsEErqEprEEpqs.$
$2\ s/Epq \times E1-3.$
3 $EEpqEpq.$
$1\ p/Epq,\ q/Epq \times E3-4.$
4 $EErEpqEEpqr.$
$4\ r/Epq,\ p/Erq,\ q/Epr \times E1-5.$
5 $EEErqEprEpq.$
$5\ r/p,\ q/p \times E3\ q/p-6.$
*6 $Epp.$

[10]) For the derivational technique see p. 157 of my article cited in note 8 above, as well as my essay "Zur Geschichte der Aussagenlogik", *Erkenntnis* 5 (1935), p. 126. [p. 216 of this volume.]

$1\ p/q,\ r/p \times E6\ p/q-7.$
*7 $EEpqEqp.$
$1\ p/Epq,\ q/Eqp \times E7-8.$
8 $EErEqpEEpqr.$
$7\ p/ErEqp,\ q/EEpqr \times E8-9.$
9 $EEEpqrErEqp.$
$2\ s/EEprEpq,\ r/p,\ p/r \times E9\ q/r,\ r/Epq-10.$
10 $EErqEEprEpq.$
$5\ r/Epq,\ p/EpEpq \times E10\ r/Epq-11.$
*11 $EEpEpqq.$
$7\ p/EpEpq \times E11-12.$
*12 $EqEpEpq.$
$1\ p/EpEpq \times E11-13.$
13 $EErqEEpEpqr.$
$2\ s/EEpqr,\ r/q,\ q/Eqr \times E13\ r/Epq,\ q/r,\ p/q-14.$
*14 $EEpEqrEEpqr.$
$7\ p/EpEqr,\ q/EEpqr \times E14-15.$
*15 $EEEpqrEpEqr.$
$9\ p/Erq,\ q/p,\ r/EpEqr \times E9\ p/r,\ r/p-16.$
*16 $EEpEqrEpErq.$
$16\ p/EEprEqp \times E5\ r/p,\ q/r,\ p/q-17.$
*17 $EEEprEqpErq.$
$16\ p/EEpqr,\ q/r,\ r/Eqp \times E9-18.$
18 $EEEpqrEEqpr.$
$10\ r/EEqrs,\ q/EErqs \times E18\ p/q,\ q/r,\ r/s-19.$
*19 $EEpEEqrsEpEErqs.$
$10\ r/EEqrs,\ q/EqErs \times E15\ p/q,\ q/r,\ r/s-20.$
*20 $EEpEEqrsEpEqErs.$
$7\ p/EpEEqrs,\ q/EpEqErs \times E20-21.$
*21 $EEpEqErsEpEEqrs.$

Of the derived theses, *14 and *17 are Leśniewski's axioms. Herewith the proof is given, indirectly, that our axiom comprehends all theses of the system. It is, moreover, not the only "shortest" axiom of the equivalential system; I have found two other theses of eleven letters which can be likewise postulated as single axioms of the system. These are *EEpqEEprErq*, and *EEpqEErpEqr*. From each of these theses *1 can be derived in the following manner:

A

1 *EEpqEEprErq*.
1 p/Epq, q/EEprErq, r/s×E1–2.
2 *EEEpqsEsEEprErq*.
2 p/Epq, q/s, s/EsEEprErq×E2–3.
3 *EEsEEprErqEEEpqrErs*.
3 s/Epq×E1–4.
4 *EEEpqrErEpq*.
4 r/EEprErq×E1–5.
5 *EEEprErqEpq*.
5 r/Epr, q/ErEpr×E1 q/Epr–6.
6 *EpErEpr*.
1 q/ErEpr, r/q×E6–7.
7 *EEpqEqErEpr*.
2 s/EqErEpr×E7–8.
8 *EEqErEprEEprErq*.
8 q/p×E6–9.
9 *EEprErp*.
2 s/Eqp×E9 r/q–10.
10 *EEqpEEprErq*.
3 s/Eqp×E10–11.
11 *EEEpqrErEqp*.
11 p/Epr, q/Erq, r/Epq×E5–12.
12 *EEpqEErqEpr*.

B

1 *EEpqEErpEqr*.
1 p/Epq, q/EErpEqr, r/s×E1–2.
2 *EEsEpqEEErpEqrs*.
2 s/Epq, p/Erp, q/Eqr, r/s×E1–3.
3 *EEEsErpEEqrsEpq*.
3 r/Eqr, p/q, q/ErEqr×E2 p/Eqr–4.
4 *EqErEqr*.
2 s/q, p/r, q/Eqr, r/s×E4–5.
5 *EEEsrEEqrsq*.
5 r/Eqq, q/Eqq×E2 p/q, r/q–6.
6 *Eqq*.
1 p/q×E6–7.
7 *EErqEqr*.
2 s/Epq, p/q, q/p×E7 r/p–8.
8 *EEErqEprEpq*.
7 r/EErqEpr, q/Epq×E8–9.
9 *EEpqEErqEpr*.

5. The completeness-proof

The new completeness-proof, which I intend to give here, rests on a concept of completeness which is essentially due to the American mathematical logician Post.[11] I intend to prove the following:

Every meaningful expression in the equivalential system has either the property that it can be derived by the rules of inference from axiom *1 or the property that, when it is added to axiom *1, every meaningful expression is derivable.

The first property I call ξ_1, the second ξ_2.

The "either-or" in this case is non-exclusive. However, it will later turn out that the equivalential system constructed on the basis of our axiom is *consistent*, i.e. does not include all meaningful expressions of the system. Thus the properties ξ_1 and ξ_2 do in fact exclude each other.

The proof is based essentially on the previously stressed fact that every meaningful expression either is a propositional variable or begins

[11]) See in this connexion p. 161, n. 10 of my article cited in note 8 above, as well as the essay of H. Hermes and H. Scholz, "Ein neuer Vollständigkeitsbeweis für das reduzierte Fregesche Axiomensystem des Aussagenkalküls", *Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften*, New Series, vol. 1 (1937), p. 6, n. 5 .

with an "E", thus is of the type "$E\alpha\beta$", where "α" and "β" are understood to be meaningful expressions. In general, I will designate arbitrary meaningful expressions by the first few letters of the Greek alphabet, while designating propositional variables by "π".

The proof divides into eight sections, (a) to (h). These exhaust all possible cases which can occur when any meaningful expression is given.

(a) The given expression is a propositional variable. Then it has the property ξ_2, for from a variable all meaningful expressions can be derived through substitution.

(b) The given expression begins with more than one "E". Then on the basis of the theses

*15 $EEEpqrEpEqr$,

*14 $EEpEqrEEpqr$,

which are derivable from axiom *1, it can be transformed into a deductively equivalent and not longer expression which begins with only one "E".

Proof. Two expressions are called *deductively equivalent*[12] with respect to axiom *1, if on the basis of this axiom either expression can be derived from the other by means of the established rules of inference. Expressions which begin with more than one "E", i.e. have the form "$EE\alpha\beta\gamma$", are deductively equivalent to expressions of the form $E\alpha E\beta\gamma$, for in view of *15 and *14 we have:

I. $EE\alpha\beta\gamma$,	II. $E\alpha E\beta\gamma$,
*15 p/α, q/β, $r/\gamma \times E\mathrm{I}-\mathrm{II}$,	*14 p/α, q/β, $r/\gamma \times E\mathrm{II}-\mathrm{I}$,
II. $E\alpha E\beta\gamma$,	I. $EE\alpha\beta\gamma$.

"$E\alpha E\beta\gamma$" is no longer than "$EE\alpha\beta\gamma$" and has one "E" less at the beginning. If "α" again begins with an "E", the same transformation can be made and repeated until one obtains an expression which begins with one "E", and is hence of the form "$E\pi\delta$".

(c) The given expression begins with one "E" followed by a propositional variable, i.e. is of the type "$E\pi\delta$", where in "δ" no variable equiform with "π" occurs. Then the expression has the property ξ_2,

[12]) The term "deductively equivalent" I owe to the above-mentioned paper of Hermes and Scholz.

i.e. if it is taken together with the axiom, all meaningful expressions are derivable from it.

Proof. If "$E\pi\delta$" is conjoined to the axiom, one obtains on the basis of thesis

*7 $EEpqEqp$

the expression "$E\delta\pi$". On the other hand, we can derive the expression "δ" from "$E\pi\delta$" by the substitution $\pi/E\pi\delta$, since "δ" contains no variable of the same shape as "π" and hence is not changed by the substitution. From "$E\delta\pi$" and "δ" we get the variable "π" by detachment, and from "π" by substitution all meaningful expressions. The formal derivation has the form:

I	$E\pi\delta$.
	*7 p/π, $q/\delta \times E-\mathrm{III}$.
II	$E\delta\pi$.
	I $\pi/E\pi\delta \times E\mathrm{I}-\mathrm{III}$.
III	δ.
	$\mathrm{II} \times E\mathrm{III}-\mathrm{IV}$
IV	π.

In the following it is assumed that in expressions of the type "$E\pi\delta$" the expression "δ" always contains a variable equiform with "π". Furthermore, "δ" is either a variable or an expression of the form "$E\alpha\beta$". We examine first the latter case.

(d) The given expression has the form "$E\pi E\alpha\beta$", where the equivalence beginning with the second "E", i.e. "$E\alpha\beta$", contains a variable equiform with "π". If this variable is in the second but not in the first member of "$E\alpha\beta$", then the expression "$E\pi E\alpha\beta$" can be transformed on the basis of the thesis

*16 $EEpEqrEpErq$

into a deductively equivalent and not longer expression, namely "$E\pi\ E\beta\alpha$", in which the variable equiform with "π" appears in the first member of the equivalence beginning with the second "E".

Proof. "$E\pi E\alpha\beta$" and "$E\pi E\beta\alpha$" are by *16 deductively equivalent, for we have:

I. $E\pi E\alpha\beta$,	II. $E\pi E\beta\alpha$,
*16 p/π, q/α, $r/\beta \times E\mathrm{I}-\mathrm{II}$,	*16 p/π, q/β, $r/\alpha \times E\mathrm{II}-\mathrm{I}$,
II. $E\pi E\beta\alpha$,	I. $E\pi E\alpha\beta$.

If "π" does not occur in "α", it must be contained in "β", since *ex hypothesi* it occurs in "$E\alpha\beta$".

On the basis of this section we may assume subsequently that in expressions of the form "$E\pi E\alpha\beta$" the variable equiform with "π" occurs in the first member of the equivalence "$E\alpha\beta$", i.e. in "α". Now "α" is either a variable or an expression of the form "$E\alpha\beta$". We consider first the latter case.

(e) The given expression has the form "$E\pi EE\alpha\beta\gamma$" where the equivalence beginning with the third "E", i.e. "$E\alpha\beta$", contains a variable equiform with "π". If this variable occurs not in the first but in the second member of the equivalence beginning with the third "E", then the expression "$E\pi EE\alpha\beta\gamma$" can be transformed, in virtue of thesis

*19 *EEpEEqrsEpEErqs*,

into a deductively equivalent not longer expression, namely "$E\pi EE\beta\alpha\gamma$", in which the variable equiform with "π" appears in the first member of the equivalence beginning with the third "E".

Proof. "$E\pi EE\alpha\beta\gamma$" and "$E\pi EE\beta\alpha\gamma$" are deductively equivalent by *19, for we have:

I. $E\pi EE\alpha\beta\gamma$,	II. $E\pi EE\beta\alpha\gamma$,
*19 p/π, q/α, r/β, $s/\gamma \times E\mathrm{I}-\mathrm{II}$,	*19 p/π, q/β, r/α, $s/\gamma \times E\mathrm{II}-\mathrm{I}$,
II. $E\pi EE\beta\alpha\gamma$,	I. $E\pi EE\alpha\beta\gamma$.

If "π" does not appear in "α", it must in "β", since by assumption the equivalence "$E\alpha\beta$" contains a variable equiform with "π".

By reason of this section we may assume in what follows that, in expressions of the form "$E\pi EE\alpha\beta\gamma$", the variable equiform with "π" is contained in "α".

(f) The given expression has the form "$E\pi EE\alpha\beta\gamma$", where "α" contains a variable equiform with "π". Then by the theses

*20 *EEpEEqrsEpEqErs*,

*21 *EEpEqErsEpEEqrs*

the expression "$E\pi EE\alpha\beta\gamma$" can be transformed into the deductively equivalent not longer expression "$E\pi E\alpha E\beta\gamma$".

Proof. "$E\pi EE\alpha\beta\gamma$" and "$E\pi E\alpha E\beta\gamma$" are deductively equivalent, since by *20 and *21 we obtain:

I. $E\pi EE\alpha\beta\gamma$,	II. $E\pi E\alpha E\beta\gamma$,
*20 p/π, q/α, r/β, $s/\gamma \times E\mathrm{I}-\mathrm{II}$,	*21 p/π, q/α, r/β, $s/\gamma \times E\mathrm{II}-\mathrm{I}$,
II. $E\pi E\alpha E\beta\gamma$,	I. $E\pi EE\alpha\beta\gamma$.

In the expression "$E\pi E\alpha E\beta\gamma$", "$\alpha$" contains a variable of the same form as "π". If "α" is an equivalence, the transformations described under (e) and (f) can again be carried out, and repeated until a propositional variable is obtained in place of "α". Herewith we come back to the unresolved case mentioned at the end of section (d): the given expression has the form "$E\pi E\alpha\beta$" where "α" is a variable and also contains a variable equiform with "π". "α" must then, of course, be equiform with "π", and we have the case:

(g) The given expression has the form $E\pi E\pi\alpha$. Then on the basis of the theses

*11 *EEpEpqq*,

*12 *EqEpEpq*

it can be transformed into the deductively equivalent and *shorter* expression "α".

Proof. "$E\pi E\pi\alpha$" and "α" are deductively equivalent by reason of *11 and *12:

I. $E\pi E\pi\alpha$,	II. α,
*11 p/π, $q/\alpha \times E\mathrm{I}-\mathrm{II}$,	*12 q/α, $p/\pi \times E\mathrm{II}-\mathrm{I}$,
II. α,	I. $E\pi E\pi\alpha$.

To this shorter expression we can again apply the transformation rules mentioned in (a) to (g). If sections (a) or (c) are applicable, then the investigation is finished, for it is clear that the given expression has the property ξ_2. If this does not happen, one obtains progressively shorter expressions, until one reaches the shortest expression which possesses property ξ_1. This is, of course, the expression "$E\pi\pi$". And therewith the final outstanding case is resolved which was mentioned at the end of section (c): the given expression has the form "$E\pi\delta$", where "δ" is a variable and contains a variable equiform with "π". "δ" must then be equiform with "π", and we obtain the case:

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(h) The given expression has the form "$E\pi\pi$". Then it has property ξ_1 as a substitution-instance of the thesis

*6 *Epp*.

With this the completeness-proof is finished. The proof rests on ten theses, all of which are deducible from our axiom *1: *6, *7, *11, *12, *14, *15, *16, *19, *20, *21. Thus the proof is *effective*; i.e. with the aid of the enumerated theses it can always be decided whether a given expression has property ξ_1 or property ξ_2, and in each case the procedure which one must follow to exhibit one or the other property is exactly specified. This will be clarified subsequently by two examples, which are furthermore intended to make the completeness-proof here presented more intelligible.

6. Examples for the completeness-proof

As examples I choose two expressions, of which one exhibits property ξ_1, the other property ξ_2. The examples are so chosen that all the transformation rules enumerated in sections (a) to (h) are used in one or the other.

The first expression reads "*EEEpEqpEqrr*", and thus begins with more than one "*E*". Hence it falls under section (b). On the grounds of the schema:

$$EE\alpha\beta\gamma \sim E\alpha E\beta\gamma,$$

where "α" is "*EpEqp*", "β" is "*Eqr*", "γ" is "*r*", and the sign "$\sim$" denotes deductive equivalence, we have:

EEEpEqpEqrr $\sim$ *EEpEqpEEqrr* [section (b), theses *15 and *14].

The expression on the right obtained by the transformation still begins with more than one "*E*". So we apply rule (b) a second time:

EEpEqpEEqrr $\sim$ *EpEEqpEEqrr* [section (b), theses *15 and *14].

Now we have obtained an expression of the form "$E\pi EE\alpha\beta\gamma$" where the variable equiform with "π" is contained in "β". This comes under section (e), therefore under the schema:

$$E\pi EE\alpha\beta\gamma \sim E\pi EE\beta\alpha\gamma.$$

This schema yields the deductive equivalence:

EpEEqpEEqrr $\sim$ *EpEEpqEEqrr* [section (e), thesis *19].

The new expression is of the form "$E\pi EE\alpha\beta\gamma$", the variable equiform with "π" being contained in "α". Hence we must now apply section (f), i.e. the schema:

$$E\pi EE\alpha\beta\gamma \sim E\pi E\alpha E\beta\gamma,$$

from which we get:

EpEEpqEEqrr $\sim$ *EpEpEqEEqrr* [section (f), theses *20 and *21].

Now it is the turn of rule (g), since the new expression is of the type "$E\pi E\pi\alpha$" and according to the schema

$$E\pi E\pi\alpha \sim \alpha$$

can be transformed into the shorter expression "α". Thus we have:

EpEpEqEEqrr $\sim$ *EqEEqrr* [section (g), theses *11 and*12].

This shorter expression yields after two transformations:

EqEEqrr $\sim$ *EqEqErr* [section (f), theses *20 and *21],
EqEqErr $\sim$ *Err* [section (g), theses *11 and *12],

the shortest expression with the property ξ_1, namely

Err [section (h), thesis *6].

The analysis is ended. Now comes the synthesis, namely the derivation of the given expression, which has property ξ_1, from the theses adduced in the analysis, and hence indirectly from our axiom *1. We begin with the last expression to which the analysis led and climb back up step by step. In this process we shall not, however, use all the theses mentioned, but in every case when two theses are given in connexion with a deductive equivalence, we shall use only the second. Thus the derivation is based on theses *6, *12, *14, *19, and *21, which appear in this order: *12, *6, *21, *12, *21, *19, *14, and *14.

*12 $q/Err,\ p/q \times E$*6p/r – I.
I *EqEqErr*.
*21 $p/q,\ s/r \times E$I – II.
II *EqEEqrr*.

	*12 $q/EqEEqrr \times E$II – III.
III	*EpEpEqEEqrr.*
	*21 $q/p,\ r/q,\ s/EEqrr \times E$III – IV.
IV	*EpEEpqEEqrr.*
	*19 $q/p,\ r/q,\ s/EEqrr \times E$IV – V.
V	*EpEEqpEEqrr.*
	*14 $q/Eqp,\ r/EEqrr \times E$V – VI.
VI	*EEpEqpEEqrr.*
	*14 $p/EpEqp,\ q/Eqr \times E$VI – VII.
VII	*EEEpEqpEqrr.*

This completes the proof that the given expression "*EEEpEqpEqrr*" possesses the property ξ_1.

As a second example I choose the expression "*EEpEqrEps*" which, as it will turn out, has the property ξ_2. That is, added to the axioms, it entails the derivability of all meaningful expressions. After what I have said above, the following should be clear without further ado:

EEpEqrEps ~ *EpEEqrEps*	[section (b), theses *15 and *14],
EpEEqrEps ~ *EpEEpsEqr*	[section (d), thesis *16],
EpEEpsEEqr ~ *EpEpEsEqr*	[section (f), theses *20 and *21],
EpEpEsEqr ~ *EsEqr*	[section (g), theses *11 and *12],
EsEqr ~ *EEqrs*	[section (c), thesis *7].

"*EEqrs*" yields in conjunction with "*Eqr*", which follows from "*EsEqr*", the variable "*s*", and hence by (a) all meaningful expressions.

The synthetic construction begins with the given expression "*EEpEqrEps*" and descends to the variable "*s*". If, in the process, two theses are mentioned in a deductive equivalence, we use only the first. The deduction is thus based on theses *15, *16, *20, *11, and *7:

I	*EEpEqrEps.*
	*15 $q/Eqr,\ r/Eps \times E$I – II.
II	*EpEEqrEps.*
	*16 $q/Eqr,\ r/Eps \times E$II – III.
III	*EpEEpsEqr.*
	*20 $q/p,\ r/s,\ s/Eqr \times E$III – IV.
IV	*EpEpEsEqr.*

	*11 $q/EsEqr \times E$IV – V.
V	*EsEqr.*
	*7 $p/s,\ q/Eqr \times E$V – VI.
VI	*EEqrs.*
	V $s/EsEqr \times E$V – VII.
VII	*Eqr.*
	VI $\times E$VII – VIII.
VIII	*s.*

This constitutes the proof that the given expression "*EEpEqrEps*" possesses the property ξ_2.

7. Consistency of the equivalential system

As noted above, the equivalential system is *consistent*, i.e. not all meaningful expressions belong to the system, nor are they all, accordingly, derivable from our axiom. A simple proof of consistency is provided by the normal matrix for equivalence. However, I will here give in addition a purely *structural* proof of the consistency of the system, using an idea of Leśniewski's.

Leśniewski was the first to note that in all theses of the equivalential system the number of equiform variables of each shape, e.g. the number of "*p*" s, the number of "*q*"s, etc., is *even*. [13]) Let us designate this property by "G". It can now easily be shown, as Leśniewski and Tarski long since realized, that the property G is *hereditary* with respect to the rules of substitution and detachment. This means that all expressions which are derived from given G-expressions by means of these rules of inference also have the property G. This is evident in the case of the rule of substitution. For if an arbitrary variable "π" appears an even number of times in an expression, and for "π" any meaningful expression "α" is substituted, the number of equiform variables of any shape is changed by an even number. In the case of the rule of detachment, the assertion can be proved as follows. If the number of variables of each shape in "$E\alpha\beta$" and "α" is even, two cases can be distinguished. First, "β" contains a variable "π" such that no variable in "α" is equi-

[13]) Cf. op. cit., p. 26, point 6 and p. 29, point 11.

form with "π". Then "π" must appear an even number of times in "β", since the total number of equiform variables in "$E\alpha\beta$" is even. Second, "β" contains a variable "π" equiform with some variables in "α". Since in "$E\alpha\beta$" as well as "α" all equiform variables appear an even number of times, the number of variables "π" in "β" must also be even; for evens subtracted from evens yield evens. Hence if the expressions "$E\alpha\beta$" and "α" have property G, then "β", which follows from them by detachment, also has property G.

We now determine that in our axiom *1 all equiform variables of any shape, i.e. all "*p*"s, all "*q*"s, and all "*r*"s appear exactly twice, hence an even number of times. Therefore axiom *1 has the property G. Since this property is hereditary with respect to the rules of inference assumed in the system, it must belong to all consequences of the axioms, i.e. all theses of the equivalential system. From this it follows that not all meaningful expressions of the system are derivable from our axiom. For expressions such as "*p*", "*Epq*", "*EpEqp*" and, in general, expressions in which at least one variable occurs an odd number of times, cannot be derived from the axiom. With this the consistency of our system is proved.

8. Proof that axiom *1 is the shortest

The proof that *EEpqEErqEpr* is the shortest axiom of the equivalential system is divided into two parts. First I set down all theses which are shorter than axiom *1, i.e. number less than eleven letters; then I show that none of these theses can be the axiom.

To obtain all theses which number less than eleven letters, we must first remind ourselves of the following two points. Firstly we ascertained in 3 that in all meaningful expressions of our system, hence in all theses thereof, the number of "*E*"s is one less than the number of small letters. This yields the conclusion that every thesis of our system consists of an *odd* number of letters, hence must number 1, 3, 5, 7, or 9 letters if it is to be shorter than our axiom. Secondly we know from the previous paragraph that in all theses of the equivalential system the number of variables is *even*. It follows that no thesis of our system may consist of 1, 5, or 9 letters, for in all such expressions the number of variables is odd, being respectively 1, 3, or 5 letters. Thus we see

that theses which number less than 11 letters must consist of either 3 or 7 letters.

There is only one thesis of three letters, namely

Epp.

Theses of seven letters divide according to the order of the functors into the following five groups:

I. *EEExxxx*.
II. *EExExxx*.
III. *EExxExx*.
IV. *ExEExxx*.
V. *ExExExx*.

In each group the variables "*p*" and "*q*" (more than two different variables cannot occur) can be ordered in three ways: *ppqq*, *pqpq*, and *pqqp*. The remaining three orderings, *qppq*, *qpqp*, and *qqpp*, result from the first three by a change of variables. We thus obtain the following fifteen theses:

I_1. *EEEppqq*. II_1. *EEpEpqq*. III_1. *EEppEqq*.
I_2. *EEEpqpq*. II_2. *EEpEqpq*. III_2. *EEpqEpq*.
I_3. *EEEpqqp*. II_3. *EEpEqqp*. III_3. *EEpqEqp*.
IV_1. *EpEEpqq*. V_1. *EpEpEqq*.
IV_2. *EpEEqpq*. V_2. *EpEqEpq*.
IV_3. *EpEEqqp*. V_3. *EpEqEqp*.

These are *all* the theses which come under consideration, for theses which result from the above by identification of variables are weaker and hence may be disregarded.

Now we must show that none of these theses can be the axiom of our system. We do this in the following manner. For each thesis we give a matrix, preserving the property of deducibility, which is so constituted as to be satisfied by the given thesis but *not* by our axiom. This suffices to prove that from such a thesis our axiom cannot be derived. But if even *one* thesis of our system is not deducible from a given thesis, the latter can certainly not be the axiom of the system.

In all the matrices below, the first argument is written at the left, the second at the top. All contain only one designated value, denoted by "1". A matrix *M* is *satisfied* by a given thesis, if this thesis, for all assignments of values to its variables, yields an expression which, after reduction according to matrix M, yields the value 1. The designated value 1 appears only once in the one-line of the matrix, and always in the first position, so that only $E11 = 1$ while, for all β other than 1, $E1\beta \neq 1$. This suffices to ensure that satisfaction of the matrix is preserved by deductions using the rule of detachment. For if the expressions "$E\alpha\beta$" and "α" equal 1, so must "β". Satisfaction of all matrices is preserved in deductions using the rule of substitution. Thus if such a matrix is satisfied by a given thesis, it must be satisfied by all consequences of this thesis. Our axiom cannot be among these consequences if it does not satisfy the matrix.

To begin with, it is clear that our axiom

$$EEpqEErqEpr$$

does not satisfy the two-valued matrix M_1 below—the normal matrix for implication.

E	1	2
1	1	2
2	1	1

M_1

For $p/1$, $q/1$, $r/2$ we get:

$$EE11EE21E12 = E1E12 = E12 = 2.$$

On the other hand this matrix is satisfied by Epp as well as the theses: I_1. $EEppqq$, III_1. $EEppEqq$, III_2. $EEpqEpq$, IV_1. $EpEEpqq$, IV_3. $EpEEqqp$, V_1. $EpEpEqq$, V_2. $EpEqEpq$, V_3. $EpEqEqp$. For one sees at once that all these theses retain their validity when E is interpreted as the sign of implication. From this it follows that none of these theses can be the axiom.

E	1	2	3	4
1	1	2	3	4
2	4	1	1	3
3	2	4	1	1
4	3	1	2	1

M_2

For $p/1$, $q/3$, $r/2$ we get:

$$EE13EE23E12 = E3E12 = E32 = 4.$$

On the other hand it is satisfied by theses I_2. $EEEpqpq$ and II_2. $EEpEqpq$, as may be seen from the following tables:

p	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4
q	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
Epq	1	2	3	4	4	1	1	3	2	4	1	1	3	1	2	1
$EEpqp$	1	4	2	3	1	2	2	4	1	2	3	3	1	4	3	4
$EEEpqpq$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

p	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4
q	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
Eqp	1	4	2	3	2	1	4	1	3	1	1	2	4	3	1	1
$EpEqp$	1	4	2	3	1	4	3	4	1	2	2	4	1	2	3	3
$EEpEqpq$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

The three-valued matrix M_3

E	1	2	3
1	1	2	3
2	2	1	2
3	3	3	1

M_3

is not satisfied by the axiom for $p/1$, $q/3$, $r/2$; for we have:

$$EE13EE23E12 = E3E22 = E31 = 3,$$

but it is satisfied by thesis I_3. $EEEpqqp$.

p	1	1	1	2	2	2	3	3	3
q	1	2	3	1	2	3	1	2	3
Epq	1	2	3	2	1	2	3	3	1
$EEpqq$	1	1	1	2	2	2	3	3	3
$EEEpqqp$	1	1	1	1	1	1	1	1	1

Likewise the axiom does not satisfy the three-valued matrix M_4,

E	1	2	3
1	1	2	3
2	2	1	3
3	3	2	1

M_4

since we have for $p/1$, $q/3$, $r/2$:

$$EE13EE23E12 = E3E32 = E32 = 2,$$

while the matrix is satisfied by theses II_1. *EEpEpqq* and II_3. *EEpEqqp*.

p	1	1	1	2	2	2	3	3	3
q	1	2	3	1	2	3	1	2	3
Epq	1	2	3	2	1	3	3	2	1
$EpEpq$	1	2	3	1	2	3	1	2	3
$EEpEpqq$	1	1	1	1	1	1	1	1	1

p	1	2	3
Eqq	1	1	1
$EpEqq$	1	2	3
$EEpEqqp$	1	1	1

Furthermore, the three-valued matrix M_5 is not satisfied by the axiom;

E	1	2	3
1	1	2	3
2	2	1	2
3	3	2	1

M_5

for $p/1$, $q/3$, $r/2$ yields:

$$EE13EE23E12 = E3E22 = E31 = 3.$$

However, it is satisfied by thesis III_3. *EEpqEqp*.

p	1	1	1	2	2	2	3	3	3
q	1	2	3	1	2	3	1	2	3
Epq	1	2	3	2	1	2	3	2	1
Eqp	1	2	3	2	1	2	3	2	1
$EEpqEqp$	1	1	1	1	1	1	1	1	1

Finally, the four-valued matrix M_6 is not satisfied by the axiom for $p/1$, $q/3$, $r/2$;

E	1	2	3	4
1	1	4	2	4
2	3	1	4	1
3	3	1	1	2
4	4	3	3	1

M_6

for we have:

$$EE13EE23E12 = E2E44 = E21 = 3,$$

but is satisfied by thesis IV_2. *EpEEqpq*.

p	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4
q	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
Eqp	1	3	3	4	4	1	1	3	2	4	1	3	4	1	2	1
$EEqpq$	1	1	1	1	4	4	2	2	3	3	2	2	4	4	4	4
$EpEEqpq$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

From all this it follows that none of the sixteen shorter theses can be the axiom. Hence our axiom *1 is the shortest axiom for the equivalential system.

9. "Creative" definitions

In closing I would like to touch on an important methodological question which stands out with particular clarity in the context of the equivalential system.

In mathematical logic definitions are normally introduced by means of a special sign of definition. So, for example, one could introduce into the equivalential calculus the expression "*Vp*", read "verum of *p*", in accordance with the usage of *Principia Mathematica*[14]) as follows:

I $\qquad Vp = Epp$ Df.

Here the identity-sign together with the following letters "Df." indicate that the *definiendum* "*Vp*" means the same as the "*Epp*". Thus one may always replace "*Epp*" by "*Vp*" and vice versa, and every substitution-instance of the one expression may be replaced by a corresponding substitution-instance of the other.

There are, however, mathematical logicians who, in order to avoid a special definition-sign, introduce definitions as equivalences. This can happen in systems in which equivalence occurs as a primitive concept. In such systems the above definition of "*Vp*" can be written in the following manner:

II $\qquad$ *EVpEpp*.

Now, this definition is methodologically different from the first. For cases may occur in which the second definition yields more than the first, in that it can have—I can find no better term for it—"creative"

[14]) Op. cit., p. 11.

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effects. The following example makes clear what is to be understood by this.

The expression

III $EEsEppEEsEppEEpqEErqEpr$

is easily verified to be a thesis of the equivalential system. This thesis has the peculiarity that its consequences can be obtained only by substitution, not by detachment. It is "undetachable", as can be shown by a method deriving from Tarski. If the above thesis is to yield a new one by detachment, it must be possible to obtain two substitution-instances of thesis III, of which one is of the type "$E\alpha\beta$" and the other of the type "α". These conditions may be expressed as follows:

(a) $E\alpha\beta \cong EE\gamma E\delta\delta EE\gamma E\delta\delta EE\delta\epsilon EE\zeta\epsilon E\delta\zeta$,

(b) $\alpha \cong EE\varrho E\sigma\sigma EE\varrho E\sigma\sigma EE\sigma\tau EE\nu\tau E\sigma\nu$.

The sign of congruence "$\cong$" means here that the left expression is equiform with that on the right. Now in the first congruence the expression "$E\gamma E\delta\delta$" corresponds to the letter "α". Hence the following congruences must also hold:

(c) $\alpha \cong E\gamma E\delta\delta \cong EE\varrho E\sigma\sigma EE\varrho E\sigma\sigma EE\sigma\tau EE\nu\tau E\sigma\nu$.

This yields the further result, that the following expressions must be equiform:

(d) $\gamma \cong E\varrho E\sigma\sigma$, (e) $\delta \cong E\varrho E\sigma\sigma$, (f) $\delta \cong EE\sigma\tau EE\nu\tau E\sigma\nu$.

From (e) and (f) finally we get the following congruences:

(g) $\varrho \cong E\sigma\tau$, (h) $\sigma \cong E\nu\tau$, (i) $\sigma \cong E\sigma\nu$.

The last congruence yields an absurdity; for it is impossible for σ to be equiform with an expression which contains σ as a proper part. From this we conclude that it is not impossible to find two substitution-instances of thesis III of the forms "$E\alpha\beta$" and "α". Hence thesis III is undetachable. From this it follows immediately that no *shorter* thesis can be derived from thesis III, in particular not our axiom $EEpqEErqEpr$.

If, however, definitions of the second sort are now introduced in the equivalential system, e.g. definition II, then it is easily shown that thesis III can be postulated as the axiom of the system. For we have:

III $EEsEppEEsEppEEpqEErqEpr$.

II $EVpEpp$.

III $s/Vp \times E\text{II} - E\text{II} - \text{IV}$.

IV $EEpqEErqEpr$.

Thesis IV is the axiom of our system. From III by itself it cannot be derived; it can be inferred only with the help of definition II. The new term "V", however, which was introduced by the definition, does not appear in this thesis. Hence definition II leads to theses which cannot be derived from the thesis assumed as an axiom, although in these theses only primitive concepts of the system occur. Such definitions I call "creative". It is clear that one cannot get thesis IV from III if one introduces "Vp" by a definition of the first sort.

In deductive systems the role of definitions would seem to consist mainly in allowing us to replace longer and more complicated expressions by shorter and simpler ones. Moreover, some definitions can bring with them new, intuitively valuable insights. Under no circumstances, however, do definitions seem to be intended to give new properties to the undefined primitive concepts of the system. Primitive concepts should be characterized solely by *axioms*. If one takes this position, one should avoid the use of creative definitions whenever possible.*)

*) Łukasiewicz discussed once more the problem of definitions in the propositional calculus in his article "On Variable Functors of Propositional Arguments" (pp. 311–324 of this volume).

LOGIC AND THE PROBLEM OF THE FOUNDATIONS OF MATHEMATICS *)

1. The propositional calculus is the fundamental logical discipline. Other logical disciplines, in particular the functional calculus, are built on the propositional calculus, and the whole of mathematics is in turn based on logic. Thus the propositional calculus forms the deepest foundation of all deductive sciences. The present lecture is concerned with that fundamental calculus and its importance for mathematics.

2. Propositional logic has always been neglected. It was not known to Aristotle and was originated only by the Stoics. Yet Stoic propositional logic was, in antiquity as well as in the Middle Ages and modern times, always suppressed by Aristotelian syllogistic. The work of Frege, the brilliant German logician who in 1879 created the propositional calculus in an almost complete form, received at first almost no attention. It was only after 1910, when Russell and Whitehead in their fundamental work *Principia Mathematica* placed the propositional calculus at the forefront of mathematical logic, that it was realized what essential importance that discipline has within the science of mathematics.

3. Yet even to this day most mathematicians seem to know little about the propositional calculus. As Aristotle did, they make use quite intuitively of some of the simplest rules of inference of that logic, without even suspecting how rich it is in theorems and what a wealth of problems it offers. To acquaint the readers with these problems I shall refer to two rules of inference from those which are most commonly used by mathematicians.

4. If the following two premisses are given: "if p, then q" and "if q, then r", we may draw from them the conclusion: "if p, then r". In symbols (where C stands for "if ..., then ..."):

*) First published as "Die Logik und das Grundlagenproblem", *Les Entretiens de Zürich sur les fondements et la méthode des sciences mathématiques* 6–9, 12 (1938), Zürich, 1941, pp. 82–100.

$$\begin{array}{ll} Cpq & P_1 \\ \underline{Cqr} & P_2 \\ Cpr & S \end{array}$$

If the first premiss is represented by P_1, the second by P_2, and the conclusion by S, then obviously the following formula is valid:

1) $$CP_1CP_2S.$$

This means: "if P_1, then if P_2, then S". If in this formula the letters P_1, P_2 and S are replaced by the expressions which they represent, we obtain the following thesis of the propositional calculus:

2) $$CCpqCCqrCpr.$$

In words: "If (if p, then q), then (if (if q, then r), then (if p, then r))." This is the law of the hypothetical syllogism, the best known form of direct inference.

5. Another very common rule of inference is indirect inference. A proposition p is proved indirectly by first taking its negation Np (where N stands for "not") as the starting point of the proof, and deducing from Np a proposition q, which is known to be false. Hence it is deduced that Np must be false and therefore p must be true. Indirect inference thus has the form:

$$\begin{array}{ll} CNpq & P_1 \\ \underline{Nq} & P_2 \\ p & S \end{array}$$

If formula 1) is applied to this form of inference, another thesis of the propositional calculus is obtained:

3) $$CCNpqCNqp.$$

In words: "if (if not-p, then q), then (if not-q, then p)"; this is a form of the law of transposition.

6. The latter form of inference is contested by Professor Brouwer, the eminent mathematician, because it can be used to prove the existence of numbers which cannot be built effectively, that is by construction. For instance, the existence of even prime numbers is proved effectively on the strength of the following law of the functional calculus:

4) $$CFa\sum xFx.$$

In words: "if F of a, then there is an x such that F of x". If, now, Fx stands for "x is an even prime number", and if in 4) the number 2 is substituted for a, the following deduction process is obtained:

(1) $CF2\sum xFx$.
(2) $F2$.
$(1) = C(2)(3)$.
(3) $\sum xFx$.

In a free verbal rendering:

1° If 2 is an even prime number, then there are even prime numbers.
2° 2 is an even prime number.

Hence by detachment:

3° There are even prime numbers.

7. The existential proposition $\sum xFx$ can also be proved indirectly. The negation of that proposition, i.e. $N\sum xFx$, is taken as the starting point of the proof, a false proposition a is deduced from that negation, and on the strength of Thesis 3) the existential proposition is arrived at. The deduction process is as follows:

(1) $CCNpqCNqp$.
(2) $CN\sum xFxa$.
(3) Na.
$(1)\ p/\sum xFx,\ q/a = (4)$.
(4) $CCN\sum xFxaCNa\sum xFx$.
$(4) = C(2)(5)$.
(5) $CNa\sum xFx$.
$(5) = C(3)(6)$.
(6) $\sum xFx$.

It is assumed here that (2) and (3) are true premisses; (4) is obtained by a substitution in (1), and (4) yields the existential proposition (6) by double detachment.

8. Followers of intuitionistic logic do not accept the validity of an existential proposition obtained in this way, that is, non-effectively. Accordingly, they are forced to reject the law of transposition *CCNpqCNqp*. There are also other theses in propositional logic which the intuitionists do not consider universally valid. Among the prescribed theses is another form of the law of transposition *CCNpNqCpq*, the law of double negation with the negations in the antecedent *CNNpp*, and in particular the law of the excluded middle *ApNp* ("*A*" is the symbol of alternation "or"; "*ApNp*" is read "*p* or not-*p*"). On the other hand, valid are the two remaining laws of transposition, *CCpqCNqNp* and *CCpNqCqNp*, the law of double negation with the negations in the consequent *CpNNp*, and the law of excluded contradiction *NKpNp* ("*K*" is the symbol of conjunction "and"; "*NKpNp*" is read "not both *p* and not-*p*"). The issue is serious: the controversy is within the simplest and the most fundamental logical discipline, it is a true controversy over the foundations.

9. Now, the propositional calculus is not a heap of stones, which remains even if a few stones are removed from it. It is rather a mechanism of the greatest precision, which breaks down after the removal of a single cog-wheel and must then be reconstructed. That is why we must be most grateful to Mr Heyting for undertaking, in 1930, to formalize the propositional calculus in the spirit of intuitionism. He succeeded in constructing a system of axioms for the intuitionistic propositional calculus. I shall not discuss these axioms here, but I shall present here a result I obtained in May of this year following a suggestion of my respected friend, Professor Scholz of Münster, which will make it easier to compare ordinary and intuitionistic propositional logic.

10. The following independent system of axioms, which consists of four groups of axioms, suffices to construct the ordinary propositional calculus:

I	1	*CpCqp*.
	2	*CCpCpqCpq*.
	3	*CCpqCCqrCpr*.
II	4	*CKpqp*.
	5	*CKpqq*.
	6	*CCpqCCprCpKqr*.
III	7	*CpApq*.
	8	*CqApq*.
	9	*CCprCCqrCApqr*.
IV	10	*CCpNqCqNp*.
	11	*CNpCpq*.
	12	*CCCpNpqCCpqq*.

The axioms of Group I contain only the implication symbol "C". They characterize what is called "positive logic" in the sense of Professor Bernays. The axioms of Group II also contain the conjunction symbol "K", and those of Group III, the alternation symbol "A". The axioms of the first three groups have been formulated by Professor Bernays. The axioms of Group IV pertain to the negation symbol "N". The system also includes two rules of inference: the rule of substitution, which permits us to substitute any significant expressions for the variables, and the rule of detachment, which states that from the expressions $C\alpha\beta$ and α we can always deduce β.

11. The above system of 12 axioms is valid, as has been said, for the ordinary, or classical, propositional calculus. If Axiom 12 is dropped, then we obtain an axiom system of intuitionistic logic, which is equivalent to the axiom system formulated by Heyting with all the rules of inference belonging to it. If Axioms 11 and 12 are dropped, we obtain what is called the minimal calculus of Johannsson. The relationship between classical propositional logic and intuitionistic propositional logic is now clear: intuitionistic propositional logic covers a proper part, strictly limited, of the theses of the classical propositional calculus and is consequently essentially weaker than the latter. It is up to mathematicians to find out what can be built on this weaker foundation of mathematics. Research already carried out and still to be done can be as fertile and important for the problem of the foundations of logic as research on Zermelo's axiom of choice and its role in set theory and analysis, initiated by my respected colleague from Warsaw, Professor Sierpiński.

12. I shall not here go into the problem of whether it is justified to reject certain forms of inference of classical propositional logic. One thing is clear to me: this controversy cannot be settled now, either in the sphere of logic or in that of mathematics. Philosophical arguments, which are proposed from different quarters, are in my opinion not conclusive. The problem must first be studied more profoundly. I shall do so, although I realize how difficult it is to explore the depths. In this connection I might mention four points that may serve as road-signs: a) there are matrices in propositional logic; b) an adequate matrix corresponds to each system of propositional logic; c) matrices of many-

valued propositional logics can also be interpreted intuitively; d) in the case of intuitively interpreted matrices, all the logical functions which are possible with respect to a given matrix must be taken into account.

13. The matrix method was devised in 1885 by the eminent American logician Charles Peirce. In propositional logic the truth of theses depends not on their content, but on their truth value. In the classical propositional calculus there are two truth values: truth and falsehood. If truth is represented by "1", and falsehood by "2", the following equations can be formulated:

Negation	Implication	Conjunction	Alternation
$N1 = 2$	$C11 = 1$	$K11 = 1$	$A11 = 1$
$N2 = 1$	$C12 = 2$	$K12 = 2$	$A12 = 1$
	$C21 = 1$	$K21 = 2$	$A21 = 1$
	$C22 = 1$	$K22 = 2$	$A22 = 2$

All these equations together can briefly be represented in the form of tables (in the case of functions of two arguments the first argument being written in the column of the left, and the second, in the row above):

$$\begin{array}{c|c} N & \\ \hline 1 & 2 \\ 2 & 1 \end{array} \qquad \begin{array}{c|cc} C & 1 & 2 \\ \hline 1 & 1 & 2 \\ 2 & 1 & 1 \end{array} \qquad \begin{array}{c|cc} K & 1 & 2 \\ \hline 1 & 1 & 2 \\ 2 & 2 & 2 \end{array} \qquad \begin{array}{c|cc} A & 1 & 2 \\ \hline 1 & 1 & 1 \\ 2 & 1 & 2 \end{array}.$$

They are called the matrix for N, C, K, and A. Each matrix has at least one selected value; here it is truth, hence 1.

14. A verification method is associated with each matrix. We say that an expression of the propositional calculus satisfies a matrix if, for all valuation of its variables by the values included in the given matrix, the expression takes on the selected value after reduction. For instance, the thesis $CCCpNpqCCpqq$ satisfies the matrix given above, since we obtain:

for $p/1$, $q/1$: $CCC1N11CC111 = CCC121C11 = CC211 = C11 = 1$,
for $p/1$, $q/2$: $CCC1N12CC122 = CCC122C22 = CC221 = C11 = 1$,
for $p/2$, $q/1$: $CCC2N21CC211 = CCC211C11 = CC111 = C11 = 1$,
for $p/2$, $q/2$: $CCC2N22CC222 = CCC212C12 = CC122 = C22 = 1$.

All matrices are hereditary with respect to the rule of substitution, which means that if an expression satisfies a given matrix, then that matrix is also satisfied by all substitutions of that expression. For a matrix to be hereditary with respect to the rule of detachment, it is sufficient, though not necessary, that the function of two arguments $F\alpha\beta$ to which the rule of detachment is applied (usually it is implication) has the selected value for the selected α if β is also selected. Thus $C1\beta$ equals 1 only if β also equals 1. Such a matrix is called normal by my Warsaw colleague, Tarski. All normal matrices are hereditary with respect to the rule of detachment; hence, if a normal matrix is satisfied by $F\alpha\beta$ and α, then it must also be satisfied by β.

15. The matrix method was first used for the verification of theses of the classical propositional calculus. But it soon turned out that this method must be credited with an incomparably greater importance. It makes it possible to carry out proofs of independence in the sphere of propositional logic that were unknown to Frege and Russell. The merit of demonstrating how the matrix method can be used in proofs of independence goes to Professor Bernays. The same method was also known to me even before it was published by Professor Bernays. The idea of these proofs of independence can best be explained by an example. To prove, for the axiom system given above, that Axiom 12 is independent of the remaining axioms, it would obviously suffice to find a property which is hereditary with respect to the rules of inference and is characteristic of all the axioms with the exception of Axiom 12. If an axiom satisfies a normal matrix, there is accordingly a property of that axiom which is hereditary with respect to the rules of substitution and detachment. We construct the following three-valued matrix for N, C, K, and A:

N	
1	3
2	3
3	1

C	1	2	3
1	1	2	3
2	1	1	3
3	1	1	1

K	1	2	3
1	1	2	3
2	2	2	3
3	3	3	3

A	1	2	3
1	1	1	1
2	1	2	2
3	1	2	3

Again, 1 is the designated value. This matrix is normal, since $C1\beta$ equals 1 only if β also equals 1. It can easily be seen that the matrix is satisfied by the first eleven axioms; in fact for all value functions of the variables acquiring the values 1, 2, or 3, these axioms, according to the matrix, yield 1, after reduction. Only Axiom 12 does not satisfy the matrix, since for $p/2$ and $q/2$ we obtain:

$$CCC2N22CC222 = CCC232C12 = CC322 = C12 = 2.$$

16. This settles the first of the four problems raised above. There are matrices in propositional logic, and these play an important role in the propositional calculus. I now come to the second point: Tarski's metalogical researches make it possible for us to give a strict definition of the concept of a system of propositional logic. By a system of propositional logic we mean a set of significant expressions of propositional logic which is closed under the given rules of inference. As rules of inference we consider primarily the rules of substitution and detachment. It follows that any significant expressions, e.g.

$$CCppp \quad \text{and} \quad CCCpqqCCqpp,$$

together with all their consequences derived by use of the specified rules of inference, constitute a system of propositional logic. It is now clear that each normal matrix defines a system of propositional logic. Strikingly enough, the converse statement is also valid: Lindenbaum, one of my Warsaw colleagues, has proved that for every system of propositional logic there is an adequate normal matrix with at most a denumerable set of values. A matrix is called adequate with respect to a system if it is satisfied by all the expressions of that system and by them only. This important theorem was published without proof in 1930 in the paper "Untersuchungen über den Aussagenkalkül", written by Tarski and me.*)

17. I should now like to draw some consequences from that theorem. First of all, it is obvious that all axiomatic systems of the propositional calculus are systems of propositional logic in the sense of Tarski's definition. In accordance with Lindenbaum's theorem, every such system must have an adequate normal matrix. For the axiomatic classical propositional calculus the two-valued normal matrix given above is adequate, as it has often been proved that that matrix is satisfied by all theses of the classical propositional calculus and by them only. This is why the classical propositional calculus is termed two-valued. For every weaker system, that is every system in which certain theses of the two-valued calculus are not valid, the adequate matrix is no longer two-valued, but many-valued. Heyting's axiomatic intuitionistic

*) See pp. 131–152 of this volume.

propositional calculus is such a system. This calculus must accordingly have a many-valued adequate matrix. In fact, Gödel has proved that for Heyting's system there is no adequate normal matrix with a finite number of values. The same result has been obtained, independently of Gödel, by Jaśkowski, one of my former disciples in Warsaw, who has actually constructed for intuitionistic propositional logic the matrix with an infinite number of values.

18. I cannot engage here in a more detailed discussion of the very complicated issue of the adequate matrix of the intuitionistic propositional calculus. For my purpose it suffices to choose a simpler example. I have given above a three-valued normal matrix which is satisfied by the first eleven axioms of the system of axioms I have constructed. These first eleven axioms represent intuitionistic propositional logic. But the said matrix, which, by the way, is due to Heyting, is not adequate for the intuitionistic calculus, because it is satisfied not only by all the theses of that calculus, but also by other theses that do not belong to the intuitionistic calculus. Thus that matrix defines a stronger system. I have succeeded in axiomatizing that stronger system.

19. If in the system of axioms quoted under 10 above, Axiom 12 is replaced by the following Axiom 12a:

12a $\qquad CCNpqCCCqpqq,$

then Axioms 1–11 and 12a form an independent system for which the three-valued normal matrix constructed by Heyting and quoted in 15 is adequate. On the one hand, Axiom 12a is not deducible from the axioms of the intuitionistic calculus, which can be proved by means of a four-valued matrix, and on the other hand, that axiom satisfies the three-valued matrix given by Heyting. Thus, if taken together with the remaining axioms, it does not suffice to form the foundations of the two-valued calculus. We thus have a simple example of a system of propositional logic represented by axioms and weaker than the classical propositional calculus. Like that calculus, it has an adequate normal matrix, though it is not two-valued, but three-valued. For all systems that are weaker than the two-valued propositional calculus there are adequate, normal, many-valued matrices. This is not incidental; this is a law. And this law imparts an essential importance to

the matrix method. We have now exhausted the discussion of the second of the four issues raised.

20. The matrix for the two-valued propositional calculus was developed on an intuitive basis. The values of that matrix were interpreted as truth values: 1 as truth and 2 as falsehood. But later, when the matrix method came to be used in proofs of independence and numerous many-valued matrices were invented for that purpose, the intuitive interpretation of matrix values became lost. It was simply not necessary to interpret these values in any intuitive way. The matrices served the purpose of finding properties for given theses that are hereditary with respect to the rules of inference. The matrix values were reduced to the status of meaningless constants, and the formation of matrices became a purely formal procedure. Yet, in many-valued matrices, too, it is possible to interpret the values intuitively. As the first example I shall quote the matrix introduced by Heyting and discussed above.

21. In his fundamental work on the formal rules of intuitionistic logic Heyting states: "Group XII (that is the matrix now in question, only with renamed values) can be interpreted as follows: let 2 stand for any correct proposition, which cannot be false, but whose correctness cannot be proved. Then we obtain the tables given above." It follows from this explanation by the author that, on the basis of certain sequences of ideas, he believed it was obviously certain that he could construct the said matrix. We shall try to examine these sequences of ideas more closely.

22. Of the 30 equations which that matrix contains, 14 are taken from the two-valued calculus:

$N1 = 3.$	$C11 = 1.$	$K11 = 1.$	$A11 = 1.$
$N3 = 1.$	$C13 = 3.$	$K13 = 3.$	$A13 = 1.$
	$C31 = 1.$	$K31 = 3.$	$A31 = 1.$
	$C33 = 1.$	$K33 = 3.$	$A33 = 3.$

1 stands for truth, and 3 for falsehood. Another 10 equations, for conjunction and for alternation, are obtained on the basis of the following considerations: the new value 2 is the attribute of those propositions which cannot be false, but are not proved. This value is obviously weaker than truth but stronger than falsehood. Now, obviously the value of conjunction follows the value of the weaker argument, and

the value of alternation, that of the stronger argument. For the same values of the arguments the value of either function equals the value of the arguments. We thus obviously obtain the equations:

$$\begin{array}{ll} K12 = 2. & A12 = 1. \\ K21 = 2. & A21 = 1. \\ K22 = 2. & A22 = 2. \\ K23 = 3. & A23 = 2. \\ K32 = 3. & A32 = 2. \end{array}$$

Two other equations, namely those for implication:

$$C21 = 1 \quad \text{and} \quad C32 = 1,$$

are obtained on the strength of the rule valid in the two-valued calculus which states that implication with a true consequent or with a false antecedent must be true regardless of the value of the other argument.

The third equation for implication:

$$C22 = 1,$$

results from the quite intuitive law of identity. There might be difficulty only with the determination of the value of the expressions $C12$, $C23$, and $N2$. $C12$ cannot be truth, since then 2 would have to be truth also. But it cannot be falsehood either, since it does not have a false consequent. Thus we arrive at:

$$C12 = 2.$$

On the other hand, $C23$ is evidently falsehood, since an antecedent that cannot be false cannot yield a false consequent. Thus follows:

$$C23 = 3.$$

As the last equation we have:

$$N2 = 3,$$

since it is clear that the negation of a proposition that cannot be false is false.

23. I grasped this sequence of ideas the more easily as years before I had been the first to construct an intuitive three-valued matrix, though I was then guided by different ideas. Following the famous example of Aristotle, I came to the conclusion that propositions about possible

future events are neither true nor false at present. That I will be in Warsaw at noon on December 8, 1939, is a statement which today cannot be properly said to be either true or false. Hence it must have a third truth value. That third truth value is in the same relation to possibility as truth is to being and falsehood to non-being. On the basis of this idea I constructed the following three-valued matrix as early as 1920:

$$\begin{array}{c|c} N & \\ \hline 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{array} \qquad \begin{array}{c|ccc} C & 1 & 2 & 3 \\ \hline 1 & 1 & 2 & 3 \\ 2 & 1 & 1 & 2 \\ 3 & 1 & 1 & 1 \end{array} \qquad \begin{array}{c|ccc} K & 1 & 2 & 3 \\ \hline 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 3 & 3 & 3 & 3 \end{array} \qquad \begin{array}{c|ccc} A & 1 & 2 & 3 \\ \hline 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{array}$$

Here 1 is the designated value, i.e., truth, 3 stands for falsehood, and 2 for the third value, "possibility". The matrix is normal.

24. The comparison of this matrix with that of Heyting is extremely instructive. Since here, too, the third value, that is possibility, is weaker than truth and stronger than falsehood, the same equations must hold for conjunction and alternation. On the other hand, there is a difference in the equations for implication and negation, though in two points only: in the expressions $C23$ and $N2$. $C23 = 2$, and not 3 as with Heyting, since possibility can turn into either truth or falsehood. In the first case $C23$ becomes $C13$, that is falsehood, in the second, it becomes $C33$, that is truth. Hence $C23$ is neither true nor false and thus must have the third value. $N2$ also must have the third value, since it is obvious that both the proposition "I will be in Warsaw at noon on December 8, 1939" is today neither true nor false, but merely possible, and its negation, "I will not be in Warsaw at noon on December 8, 1939" can be neither true nor false, but merely possible. Thus two examples settle the third issue of the four specified above: many-valued matrices can also be interpreted intuitively.

25. There are partial systems of the two-valued propositional calculus, that is systems in which not all the functions of that calculus can be defined. Thus the implicational system, which is based on implication as its only term and on the thesis *CCCpqrCCrpCsp* (the shortest thesis from which all true implicational theses follow) as its sole axiom, is a partial system. Partial systems are incomplete, and hence imperfect. Thus in the implicational system neither negation nor conjunction

is definable. If we want to have a complete logic that can be used on all occasions, we must strive to construct systems of propositional logic in which as many of the functions in the system as possible are definable. In the two-valued propositional calculus there are $2^2 = 4$ possible functions of one argument, and $2^{2^2} = 16$ possible functions of two arguments. Even if not all of these functions can be used in practical inference and can be expressed by words of everyday language, they are nevertheless definable in the system based on implication and negation. That system is consequently complete.

26. And what about this problem in many-valued systems? As the number of matrix values increases, the number of possible functions increases, too. From elementary combinatory analysis we know that for n-valued matrices there are n^n possible functions of one argument and n^{n^n} possible functions of two arguments. The numbers increase rapidly. In three-valued systems the number of the possible functions of one argument amounts to $3^3 = 27$, and that of the possible functions of two arguments to $3^{3^2} = 3^9 = 19{,}683$. In four-valued systems the analogous numbers $4^4 = 256$ and $4^{4^2} = 4^{16} = 4{,}294{,}967{,}296$, that is more than four thousand million. In matrices with denumerably many values the set of possible functions is not denumerable.

27. Let us now revert to our examples. In a three-valued propositional logic, especially one in which the truth values can be interpreted intuitively, we may demand, as in the case of the two-valued calculus, that all the functions be definable. But this is not the case in the three-valued systems described above. It can, for instance, easily be proved that the function "*Tp*" (to be read as "true third value of *p*"), which takes on the constant value 2 (so that $Tp = 2$), cannot be defined in either system. Thus both systems are incomplete. Intuitionistic calculus is incomplete, too; it is not even possible to see how the infinitely many values of its matrix can be interpreted intuitively. Let us try to make one of the two systems complete. Since work has not been finished on the system based on Heyting's matrix, the three-valued system I have constructed must serve as the basis of the following analysis.

28. I wish to state first that in my three-valued propositional calculus both conjunction and alternation are definable. These two definitions

are:

$$Apq = CCpqq \quad \text{and} \quad Kpq = NANpNq.$$

(I may add here that in Heyting's three-valued calculus alternation is definable, namely

$$Apq = KCCpqqCCqpp,$$

but conjunction is not; in the intuitionistic propositional calculus none of the four functions *N*, *C*, *K*, and *A* is definable by means of the remaining ones.) It must also be mentioned that in my calculus the set of theses which make use of the terms *C* and *N* and satisfy the matrix can be axiomatized. The proof is due to Wajsberg, one of my former disciples, who has constructed the following system of axioms for that calculus:

(1) *CpCqp*.
(2) *CCpqCCqrCpr*.
(3) *CCCpNppp*.
(4) *CCNpNqCqp*.

The three-valued matrix I have constructed is adequate for this system of axioms.

29. The three-valued propositional calculus, which I defined by the matrix method and which Wajsberg axiomatized, is, as has been said, not complete, since not all of the 27 functions of one argument and the 19,683 functions of two arguments are definable in it. Słupecki, another of my disciples, has succeeded in making that system complete by adding a new function, and in axiomatizing the system thus made complete. Słupecki has proved that with the addition of the said function *Tp*, all functions of the system can be defined, and he has formulated two new axioms for this new function. I shall once more submit to the readers the complete system of axioms and also its adequate matrix:

(1) *CpCqp*.
(2) *CCpqCCqrCpr*.
(3) *CCCpNppp*.
(4) *CCNpNqCqp*.
(5) *CTpNTp*.
(6) *CNTpTp*.

N	
1	3
2	2
3	1

T	
1	2
2	2
3	2

C	1	2	3
1	1	2	3
2	1	1	2
3	1	1	1

The rules of substitution and detachment are valid in this system. The system of axioms is independent, consistent, and complete in the sense that every significant expression in the system either is deducible from axioms or, if joined to the axioms, results in a contradiction, that is, yields all significant expressions. The system is also complete in the sense that all the functions possible in the system are definable. Thus this system has all the properties which are the attributes of the classical two-valued propositional calculus. In this way the fourth of the issues mentioned above has been settled.

30. We have made a difficult progress in depth. The progress has been difficult not only bacause the problems to be solved are technically complicated (I have spared you the technicalities and presented the results), but especially because it involves entirely new ideas and methods. I can quite properly state that many fine brains have taken great pains to arrive at these results. To conclude this lecture I shall refer briefly to the significance of these results and their connection with the problem of the foundations of logic.

31. At the beginning of this lecture I stated that intuitionistic propositional logic, which rejects various theses of the two-valued propositional calculus, is a weaker system than that calculus. This system can be made stronger in various ways; to do so we choose rejected theses of the two-valued calculus and join them one by one to the system, until we obtain the strongest system, namely two-valued propositional logic. The complete system of three-valued propositional logic, formulated by me and axiomatized by Słupecki, which we may call the *S*-system in brief, is formed in quite a different way. If we join to it a thesis which is not deducible from its axioms but which is valid in the two-valued calculus, we obtain not a stronger system but a contradiction. This important fact can be explained by an example:

(7) *CCNppp*.

(7) $p/Tp = C(6)(8)$.

(8) *Tp*.

(5) $= C(8)(9)$.

(9) *NTp*.

(2) $q/Cqp = C(1)(10)$.

(10) *CCCqprCpr*.

(10) $q/Np, p/Nq, r/Cqp = C(4)(11)$.

(11) *CNqCqp*.

(11) $q/Tp = C(9)C(8)(12)$.

(12) *p*.

32. The thesis *CCNppp* holds neither in the intuitionistic system nor in the *S*-system. If this thesis is joined to the intuitionistic system, we obtain the two-valued calculus, but if it is joined to the *S*-system, we obtain a contradiction. This contradiction can be demonstrated as follows:

CCNppp is joined to Słupecki's system of axioms as Thesis (7). From that thesis and from Axioms (6) and (5) we obtain, by substitution and detachment, two contradictory theses, *Tp* and *NTp*. This contradiction can be made even stronger, since on the basis of the thesis *CNqCqp*, which is deducible in the system, (8) and (9) yield as a conclusion the propositional variable *p*, from which any significant expression can be obtained by substitution.

33. This shows that the *S*-system is not weaker than the two-valued calculus, but is a different system. On the one hand, in the *S*-system there are theses, such as *CTpNTp* and *CNTpTp*, which cannot be interpreted in the two-valued calculus; on the other hand, we may indicate theses of the two-valued calculus, such as *CCNppp*, which in the *S*-system result in a contradiction. Quite a new logic has developed before our eyes—namely modal logic, which was the goal of Aristotle and the Scholastics. This is not the only possible form of three-valued propositional logic; there are various types of three-valued systems, not reducible to one another, and innumerable forms of higher many-valued systems. These various forms of many-valued propositional logic are more or less in the same relation to the classical two-valued propositional calculus as the various systems of non-Euclidean geometry are to the Euclidean. There is, however, one difference: while the non-Euclidean geometries can be interpreted in the Euclidean, the interpretation of many-valued systems in the two-valued system seems out

of the question. Conversely, it is possible to interpret two-valued propositional logic in the *S*-system in many ways, so that the three-valued calculus proves to be stronger and richer than the two-valued.

34. We have thus come to the most essential issue of the problem of foundations that has been formulated in mathematics. The propositional calculus is the fundamental logical discipline, on which the whole of logic is based, while mathematics is in turn based on logic. As there are different systems of propositional logic, not reducible to one another, so there must also be different systems of predicate logic, and on these logical systems different systems of set theory and arithmetic should depend. As yet no works exist in this field. So far we have succeeded only in constructing many-valued systems of propositional logic with the utmost formal precision. Should these systems be applicable to mathematics, they would have to be worked out from the intuitive point of view as well. That this is possible has been demonstrated by the example of intuitionistic propositional logic. This is why these all-important and fundamental researches should be taken up by all logicians and mathematicians.

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Discussion concentrated almost entirely on the problem as to whether Łukasiewicz's three-valued logic can be subject to an intuitive interpretation.

Łukasiewicz stated in an additional explanation that *Tp* is the most convenient term to use in the axiomatizing his logic, but another term, symbolized by *Mp*, could be used for the same purpose and would leave nothing to be desired from the point of view of its intuitive meaning, since it could be interpreted as "possible". The corresponding system of axioms would then be as follows:

1. *CNMpCNMpNp.*
2. *CNMNpCNMNpp.*
3. *CMpCMpMNp.*
4. *CCpqCCNpqCCMpqq.*

This system can be interpreted intuitively. Yet it is not possible to interpret intuitively all the functions definable in that calculus. Their number (3^9) is too large for everyday language to have expressions corresponding to each of those functions.

This is, however, also the case of two-valued logic. This lack of representation for a part (or even an overwhelming majority) of possible functions is thus not an argument against the intuitive character of the system.

THE SHORTEST AXIOM OF THE IMPLICATIONAL CALCULUS OF PROPOSITIONS *)

1. Introductory remarks

The Implicational Calculus of Propositions constitutes that part of the Complete Propositional Calculus in which implication occurs as the only functor. I denote this functor by the letter "*C*" and put it before its arguments, thus dispensing with brackets. So the expression "*Cpq*" means "if *p*, then *q*." Two propositional expressions belong to each "*C*" as its arguments and follow it immediately. By propositional expressions I understand propositional variables denoted by the small letters of the Latin alphabet or expressions of the form "$C\alpha\beta$" in which "α" and "β" are already propositional expressions. Propositional expressions which are either axioms or theorems derived from the axioms will be called theses. In derivations I will make use of the rule of substitution, according to which I can add to a set of theses a propositional expression derived from a thesis of the set by substituting any propositional expressions for the variables of the thesis, and the rule of detachment which enables me to add to a set of theses a propositional expression "β" provided expressions of the form "$C\alpha\beta$" and "α" are already members of the set.

In this article I intend to prove that all theses of the Implicational Calculus of Propositions can be derived from the following axiom

(1) *CCCpqrCCrpCsp*

by applying the rule of substitution and the rule of detachment.

*) The lecture read on 23 June, 1947, at a meeting of the Royal Irish Academy. First published in *Proceedings of the Royal Irish Academy*, vol. 52, Section A, No. 3 (April 1948), pp. 25–33.

Axiom (1) consists of 13 letters, and is the shortest on the basis of which one can construct the Implicational Calculus of Propositions. I have mentioned this axiom twice in my previous articles, but on both occasions without proof.[1] The proof I am giving below will show that the following three theses can be derived from axiom (1):

$$CpCqp,$$
$$CCCpqpp,$$
$$CCpqCCqrCpr.$$

These three theses are known as the "Tarski–Bernays" set of axioms, and —as A. Tarski has proved—form a sufficient basis for the Implicational Calculus of Propositions.[2] The first one is the so-called law of simplification; I have called the second one Peirce's law; the third thesis is the law of hypothetical syllogism. As the derivation of the thesis is the law of hypothetical syllogism. As the derivation of the law of syllogism is particularly difficult, it may prove useful to show how it can be done. The derivation of the law of syllogism will be followed by a certain theorem concerning this law. In the last paragraph I wish to outline a proof that there is no shorter thesis which could function as a sole axiom of the Implicational Calculus of Propositions.

2. History of the problem

The problem of how to construct the Complete Propositional Calculus as well as the Implicational Calculus of Propositions on the basis of a single axiom was raised and solved in 1925 by Tarski, who gave a method of combining several axioms by applying the rule of substitution and the rule of detachment.[3] The first axioms arrived at, in accordance with this method, were very long. I tried to shorten them by modifying Tarski's method, and finally discovered the following axiom consisting of 25 letters:

(2) $CCCpCqpCCCCCrstuCCsuCruvv.$[4]

This axiom is non-organic, as some constituents of it, namely:

$$CpCqp \quad \text{and} \quad CCCCrstuCCsuCru,$$

are theses of the Calculus making the whole expression a conglomeration of two theses. Later I abandoned the idea of constructing shorter axioms in the way just mentioned, as in 1926 M. Wajsberg has shown that one could base the Implicational Calculus of Propositions on the following organic axiom, that is, on an axiom no constituent of which was a thesis of the Calculus. Wajsberg's axiom consisted too of 25 letters:

(3) $CCCpqCCrstCCuCCrstCCpuCst.$[5]

This discovery made me hope that there might exist shorter organic axioms, while, at the same time, I realized that the shortest axiom must

[1]) For the first time in the article "W obronie logistyki", *Studia Gnesnensia* XV, Poznań, 1937, p. 11 of the reprint; for the second time in the lecture "Die Logik und das Grundlagenproblem", *Les Entrietiens de Zürich sur les fondements et la méthode des sciences mathématiques* (1938) Zürich, 1941, p. 95. [See pp. 245 and 289 of this volume.]

[2]) J. Łukasiewicz und A. Tarski, "Untersuchungen über den Aussagenkalkül", *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie* 23 (1930), cl. iii, Satz 29. [See p. 145 of this volume, Theorem 29.]

[3]) Łukasiewicz–Tarski, l.c. Satz 8 and 25. [See p. 137 of this volume, Theorem 8, and p. 143, Theorem 25.]

[4]) Łukasiewicz–Tarski, l.c. Satz 30. [See p. 146 of this volume, Theorem 30.] I am giving here the first derivational steps based on this axiom as they are not easy. To be brief I denote the thesis

$$CCCCrstuCCsuCru$$

contained in the axiom by the letter "α." This letter can denote any thesis in which the variables "p," "q" and "v" do not appear. It will be shown how thesis "α" can be derived from the axiom. As to the technique applied in derivational procedure see explanations given in § 3 of this article.

1 $CCCpCqpC\alpha vv.$

$1\,p/C\alpha Cq\alpha, q/\alpha, v/Cq\alpha \times C1\,p/\alpha, v/C\alpha Cq\alpha - 2.$

2 $Cq\alpha.$

$2\,q/1 \times C1 - 3.$

3 $\alpha.$

I think that further steps should not be hindered by any difficulty.

[5]) Łukasiewicz–Tarski, l.c. Satz 30. [See Theorem 30, p. 146 of this volume.] As to the terms "organic" and "non-organic" see l.c. Satz 9. [See Theorem 9, p. 137 of this volume.] See also J. Łukasiewicz, "Uwagi o aksjomacie Nicoda i o 'dedukcji uogólniającej,'" *Księga Pamiątkowa Polskiego Towarzystwa Filozoficznego we Lwowie*, Lwów 1931, p. 15 of the reprint [p. 194 of this volume.] The proof that thesis (3) can be a sole axiom of the Implicational Calculus of Propositions, was given by M. Wajsberg, in his article: "Ein Neues Axiom des Aussagenkalküls in der Symbolik von Sheffer, *Monatshefte f. Math. u. Phys.* 35 (1932).

be organic, as conglomerations of several axioms are naturally bound to be longer. In 1930 I found an organic axiom which was shorter than Wajsberg's thesis and consisted of 17 letters:

(4) *CCCpqCrsCtCCspCrp.*

In 1932 I found another such axiom:

(5) *CCCpqCrsCCspCtCrp.*[6])

Then in 1936 I discovered the shortest axiom (1), cited above, and thus terminated the examination of the problem.

3. Derivation of Tarski–Bernays set of axioms from axiom (1)

The proof which follows is fully formalized in accordance with the method adopted by me in my previous publications.[7]) Every thesis which is not the axiom—all theses have their numbers and thus are distinguished as theses—is preceded by a line without its number. I call this line a derivational line. Every derivational line consists of two parts separated from each other by the cross "×". The cross is preceded by the substitution which has to be performed on a previously given thesis, and followed by detachment, which has to be performed on the thesis arrived at by the substitution. An example will clarify the methods: In the derivational line belonging to thesis 2 the expression "1 *p*/*Cpq*, *q*/*r*, *r*/*CCrpCsp*, *s*/*r*" means that in thesis 1 "*Cpq*" has to be substituted for "*p*", "*r*" for "*q*", "*CCrpCsp*" for "*r*" and "*r*" for "*s*". The thesis generated by this substitution is omitted in the actual proof to save space. It would be of the following form:

1′ *CCCCpqrCCrpCspCCCCrpCspCpqCrCpq.*

The expression following the cross "×", i.e. "*C*1–2," shows how thesis 1′ is constructed, making obvious that the rule of detachment can be applied to thesis 1′. Thesis 1′ begins with "*C*", then follows

[6]) See e.g. J. Łukasiewicz, "Uwagi o aksjomacie Nicoda," p. 17 of the reprint, [p. 196 of this volume] and B. Sobociński, "Z badań nad teorią dedukcji," *Przegląd Filozoficzny* 35 (1932), pp. 7 and 8.

[7]) See e.g. J. Łukasiewicz, "Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenkalküls," *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie* 23 (1930), cl. iii, p. 56 [p. 157 of this volume].

axiom (1) as the antecedent and thesis 2 as the consequent. Thus thesis 1′ is an expression of the form "$C\alpha\beta$," and both "$C\alpha\beta$" and "α" are theses. One can therefore detach "β," i.e.: 2 as a new thesis. In some derivational lines detachment is performed twice; as, for instance, in the derivational line preceding thesis 4. In the same line, instead of substituting the whole first thesis I substitute only its number. The stroke "/" is the sign of substitution and the dash "–" is the sign of detachment. I think that after these explanatory remarks the reader will be able to understand and check the proof without any difficulty.

1 *CCCpqrCCrpCsp.*

1 *p*/*Cpq*, *q*/*r*, *r*/*CCrpCsp*, *s*/*r* × *C*1 – 2.

2 *CCCCrpCspCpqCrCpq.*

1 *p*/*CCrpCsp*, *q*/*Cpq*, *r*/*CrCpq*, *s*/*t* × *C*2 – 3.

3 *CCCrCpqCCrpCspCtCCrpCsp.*

3 *r*/*Cpq*, *t*/1 × *C*1 *r*/*Cpq* – *C*1 – 4.

4 *CCCpqpCsp.*

1 *p*/*Cpq*, *q*/*p*, *r*/*Csp*, *s*/*r* × *C*4 – 5.

5 *CCCspCpqCrCpq.*

1 *p*/*Csp*, *q*/*Cpq*, *r*/*CrCpq*, *s*/*t* × *C*5 – 6.

6 *CCCrCpqCspCtCsp.*

1 *p*/*CrCpq*, *q*/*Csp*, *r*/*CtCsp*, *s*/*u* × *C*6 – 7.

7 *CCCtCspCrCpqCuCrCpq.*

7 *t*/*Cpq*, *p*/*q*, *r*/*CCsqp*, *q*/*p*, *u*/1 × *C*1 *r*/*Csq*, *s*/*q* – *C*1 – 8.

8 *CCCsqpCqp.*

8 *s*/*Cpq*, *q*/*r*, *p*/*CCrpCsp* × *C*1 – 9.

9 *CrCCrpCsp.*

1 *p*/*r*, *r*/*CCCrqpCsp*, *s*/*t* × *C*9 *r*/*Crq* – 10.

10 *CCCCCrqpCsprCtr.*

1 *p*/*CCCrqpCsp*, *q*/*r*, *r*/*Ctr*, *s*/*u* × *C*10 – 11.

11 *CCCtrCCCrqpCspCuCCCrqpCsp.*

1 *p*/*Ctr*, *q*/*CCCrqpCsp*, *r*/*CuCCCrqpCsp*, *s*/*v* × *C*11 – 12.

12 *CCCuCCCrqpCspCtrCvCtr.*

1 *p*/*CuCCCrqpCsp*, *q*/*Ctr*, *r*/*CvCtr*, *s*/*w* × *C*12 – 13.

13 *CCCvCtrCuCCCrqpCspCwCuCCCrqpCsp.*

$13\ v/CCspq,\ u/CCtrCsp,\ w/1 \times C1p/Csp,\ r/Ctr,\ s/CCrqp - C1 - 14.$

14 $CCCtrCspCCCrqpCsp.$

$14\ t/Cpq,\ s/Crp,\ p/Csp \times C1 - 15.$

15 $CCCrqCspCCrpCsp.$

$15\ s/CCrqp,\ p/Csp \times C9\ r/Crq - 16.$

16 $CCrCspCCCrqpCsp.$

$16\ r/CCpqr,\ s/Crp,\ p/Csp,\ q/t \times C1 - 17.$

17 $CCCCCpqrtCspCCrpCsp.$

$1\ p/CCCpqrt,\ q/Csp,\ r/CCrCsp,\ s/u \times C17 - 18.$

18 $CCCCrpCspCCCpqrtCuCCCpqrt.$

$18\ r/Crp,\ p/Csp, s/CCpqr,\ t/CCCpqrCsp,\ u/18 \times C18t/Csp,\ u/CCCspqCrp - C18 - 19.$

19 $CCCCspqCrpCCCpqrCsp.$

$14\ t/CCspq,\ r/Crp,\ s/CCpqr,\ p/Csp,\ q/p \times C19 - 20.$

20 $CCCCrppCspCCCpqrCsp.$

$20\ r/q,\ p/Cpr,\ s/Cqr \times C15r/q,\ q/Cpr,\ s/p,\ p/r - 21.$

21 $CCCCprqqCCqrCpr.$

$5\ s/Cpq,\ q/p,\ r/4 \times C4s/p - C4 - 22.$

22 $Cpp.$

$20\ s/Crp \times C22p/CCrpp - 23.$

23 $CCCpqrCCrpp.$

$8\ s/Cpq,\ q/r,\ p/CCrpp \times C23 - 24.$

24 $CrCCrpp.$

$15\ r/p,\ q/r,\ s/CCprq,\ p/q \times C24\ r/Cpr,\ p/q - 25.$

25 $CCpqCCCprqq.$

$25\ p/Cpq,\ q/CCCprqq,\ r/CCqrCpr - C25 - 26.$

26 $CCCCpqCCqrCprCCCprqqCCCprqq.$

$8\ s/Csq,\ q/p,\ p/Cqp \times C8 - 27.$

27 $CpCqp.$

$25\ q/p,\ r/q \times C22 - 28.$

28 $CCCpqpp.$

$21\ p/Cpq,\ r/CCqrCpr,\ q/CCCprqq \times C26 - C21 - 29.$

29 $CCpqCCqrCpr.$

4. A certain theorem concerning the law of syllogism

A formalized proof can be checked mechanically but cannot be mechanically discovered. I do not know of any other method of finding proofs in the Propositional Calculus than the method of "trial and error." In the above proof the most difficult step was to find the hypothetical syllogism. The task was made easier, thanks to a certain theorem discovered by myself in 1933. This theorem is not without more general importance.

I have derived the law of syllogism from two theses:

25 $CCpqCCCprqq,$

and

21 $CCCCprqqCCqrCpr.$

From thesis 25 follows thesis 26, and from theses 21 and 26 follows thesis 29, i.e. the law of syllogism. It had been known to me before that if we have two expressions of the form:

$$CCpq\alpha \quad \text{and} \quad C\alpha CCqrCpr,$$

where "α" is so constructed that the two expressions are theses, we can always derive the law of syllogism by applying the rules of substitution and detachment to these theses.

I write this theorem in the following symbols:

(A) $CCpq\alpha,\ C\alpha CCqrCpr \rightarrow CCpqCCqrCpr.$

It is easy to see that "α" must include both "p" and "q". Because if, for instance, "p" does not appear in "α," then by substituting "q" for "p" we can derive "$CCqq\alpha$," and as "Cqq" is a thesis, "α" must be a thesis also, and so must "$CCqrCpr$," which is not possible. The same reasoning applies to the variable "q." The variable "r" and other variables, e.g.: "s," can be constituents of "α" or not, as it does not affect the proof. While performing substitution for "p" and "q" we change "α", but if the substitution is of the same kind "S", "α" changes always into the same expression, which I denote by "$\alpha[S]$." The proof of the theorem (A) is based on this observation.

1 $CCpq\alpha.$

2 $C\alpha CCqrCpr.$

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1 p/Cpq, q/α, $r/CCqrCpr \times C1-3$.
3 $\alpha[p/Cpq, q/\alpha, r/CCqrCpr]$.
2 p/Cpq, q/α, $r/CCqrCpr \times C3-C2-4$.
4 $CCpqCCqrCpr$.

Let the premisses 1 and 2 be theses. I perform the same substitution on both of them: "1 p/Cpq, q/α, $r/CCqrCpr$." So in both cases "α" changes into the following expression: "$\alpha[p/Cpq, q/\alpha, -2r/CCqrCpr]$," which is a thesis. In the proof given in Section 3 "α" is of the form "$CCCprqq$." Here are other expressions which can function as "α", the premisses 1 and 2 remaining theses: "$CpCpq$," "$CCCqspq$," "$CCCpqrr$," "$CCrsCpq$."

5. Outline of a proof that axiom (1) is the shortest possible

I proved that there exists no thesis, shorter than axiom (1), on which one could construct the Implicational Calculus of Propositions by examining all shorter theses and not finding one among them sufficient to be the sole axiom of the Calculus. I cannot give here the full proof as it would take too much space, but I wish to outline the way of reasoning which led me to arrive at the above result.[8])

All propositional expressions of the Implicational Calculus of Propositions, therefore all its theses, consist of an odd number of letters, as in every propositional expression the number of variables is greater by one than the number of functors. The shortest implicational thesis is the law of identity "Cpp," which consists of 3 letters. The theses shorter than axiom (1), consisting itself of 13 letters, are theses consisting of 3, 5, 7, 9 or 11 letters. For our purpose it is enough to examine the theses consisting of 11 letters as if a shorter thesis "α" were a sole axiom, then it is easy to prove that the thesis "$Cz\alpha$," longer by two letters than "α" and in which "z" is a variable not appearing in "α", would also have been a sole axiom. By substituting "$Cz\alpha$" for "z" we derive "$CCz\alpha\alpha$" and then "α" by detachment. As "α" was supposed to be a sole axiom, "$Cz\alpha$" must be a sole axiom also.

After careful scrutiny I have come to the conclusion that there are 92 theses consisting of 11 letters if one disregards theses derived from

[8]) J. Słupecki has constructed a shorter proof based on certain general theorems. The proof as far as I know has not yet been published.

shorter theses by applying the rule of substitution. For example, the thesis "$CCCpCqCrspp$" is derived from the thesis "$CCCpqpp$" by substituting the expression "$CqCrs$" for "q." One can also disregard theses derived from theses consisting of 11 letters by identifying some of their variables. The thesis "$CCCpqqCCqpp$" is derived, for instance, from the thesis "$CCCpqrCCrpp$" by identifying the variables "q" and "r." The set of 92 theses can be divided into three groups. The first group, which is most numerous, contains theses belonging to the so-called Positive Logic in the meaning introduced by Bernays. These theses are generated by the following three axioms:

$$\text{(B)} \qquad \begin{array}{l} CpCqp. \\ CCpCpqCpq. \\ CCpqCCqrCpr. \end{array}$$

There are 64 such theses. None of them can function as the sole axiom, the Positive Logic being a fragment of the Implicational Calculus of Propositions. Neither Peirce's law nor axiom (1) can be derived from it. The strict proof is based on matrix I shown below:

$$\text{I} \qquad \begin{array}{c|ccc} C & 1 & 2 & 3 \\ \hline {}^*1 & 1 & 2 & 3 \\ 2 & 1 & 1 & 3 \\ 3 & 1 & 1 & 1 \end{array}$$

In this matrix values of the implication "$C\alpha\beta$" are given with respect to "α" and "β" assuming values: 1, 2 and 3. The first argument is in the left column, the second one in the top line of the matrix. Thus in accordance with the matrix "$C23$" has the value "3". For every combination made by substituting the figures 1, 2 and 3 for the variables in the axioms of set (B), reduction having been done according to the matrix, we obtain "1," i.e. the selected value marked with the asterisk. For example, if on the third axiom we perform the following substitution: "$p/1$," "$q/2$," "$r/3$" we obtain:

$$CC12CC23C13 = C2C33 = C21 = 1.$$

A thesis is verified by a matrix if for every combination of substitutions of figures for variables it generates the selected value provided the reduction has been done according to the matrix. The axioms of set (B) are verified by matrix I, which is hereditary with regard to the rule of substitution and the rule of detachment, i.e. all consequences of theses verified by it are also verified. All of the 64 theses that consist of 11 letters and belong to the Positive Logic are verified by matrix I, whereas axiom (1) is not verified by this matrix; if we perform the following substitution: "$p/2$," "$q/3$," "$r/3$," "$s/1$," we obtain:

$$CCC233CC32C12 = CC33C12 = C12 = 2.$$

Thus axiom (1) cannot be a consequence of any of those 64 theses, neither can any of them be a sole axiom of the Implicational Calculus of Propositions, as it is unable to generate all implicational theses.

The same method is applied to the remaining 28 theses which depend in one way or other on Peirce's law. They can be divided into two groups: the first one contains 24 theses, which can be deduced from the following set of axioms:

(C)

CpCqp.
CpCCpqq.
CCpqCCCprqq.
CCpCqrCCCqrsCps.

All the theses of this set are verified by the four-valued matrix given below, "1" being again the selected value.

II

C	1	2	3	4
*1	1	2	3	3
2	1	1	3	3
3	1	2	1	1
4	1	1	1	1

Axiom (1) cannot be a consequence either of set (C) or of any of the 24 theses verified by matrix II, because if we perform the following substitution, "$p/2$," "$q/1$," "$r/4$," "$s/3$," we obtain:

$$CCC214CC42C32 = CC14C12 = C32 = 2.$$

The remaining 4 theses consisting of 11 letters, but not verified by either matrix I or II, are the following:

CCCpqpCCprr.
CCCpqrCCrpp.
CCpqCCCprpq.
CCpqCCCqrpq.

These theses are verified by matrix III, four-valued with "1" as the selected value.

III

C	1	2	3	4
*1	1	2	3	3
2	1	1	3	3
3	1	2	1	1
4	1	2	2	1

Axiom (1) is not verified by matrix III because, if we perform the following substitution, "$p/3$," "$q/2$," "$r/1$," "$s/4$," we obtain:

$$CCC321CC13C43 = CC21C32 = C12 = 2.$$

It follows from the above considerations that every thesis shorter than axiom (1) is verified by at least one of the matrices I, II, or III, whereas axiom (1) is not verified by any of them. Therefore, there does not exist a thesis shorter than one consisting of 13 letters, which could be used as the sole axiom of the Implicational Calculus of Propositions. Whether besides axiom (1) there are any other theses consisting of 13 letters which could function as sole axioms of the Calculus is not known.

ON THE SYSTEM OF AXIOMS OF THE IMPLICATIONAL PROPOSITIONAL CALCULUS *)

Dedicated to the memory of M. Wajsberg

Mordchaj Wajsberg has demonstrated that in the system of axioms of the implicational propositional calculus, due to Tarski and Bernays,

$$CpCqp, \quad CCCpqpp, \quad CCpqCCqrCpr,$$

the first axiom may equally well be replaced by any of the following theses:

$$CpCCpqq,\ CpCCqpp,\ CpCCqqp,\ CqCpp,\ CqCCCpppp,\ CqCCCpppCCppp,$$
$$CqCCppCpp,\ CqCCppCCppCpp,\ CpCCppp,\ CpCpp.$$[1]

He has also succeeded in partly generalizing these results by stating that the axiom $CpCqp$ may be replaced by any thesis of the form $CpC\alpha p$, if α is a consequence of that new system of axioms, or by any thesis of the form $Cq\alpha$, if α does not contain variables equiform with q. He proved these generalizations by induction.[2]

In the present note I shall prove the following theorem, which includes all the quoted results obtained by Wajsberg, both particular and general:

If to Peirce's law $CCCpqpp$ and the law of the syllogism $CCqrCCqrCpr$ we join any thesis of the form $CpC\alpha\beta$, that is a thesis whose antecedent is a variable and whose consequent is an implication, then we obtain the law of simplification $CpCqp$ and hence a complete system of axioms of the implicational propositional calculus.

[1]) M. Wajsberg, "Metalogische Beiträge," in *Wiadomości Matematyczne* 43 (1936), pp. 131–168. "Metalogische Beiträge II," ibid., 47 (1939), pp. 119–139. In Part II see Theorems 2b, 2c and 2i, which repeat the results published in Part I, and Theorems 11, 13, 15, 16, 17, 34, and 42.

[2]) "Metalogische Beiträge II", Theorems 37 and 38. In lemmata 35 and 39 to these theorems M. Wajsberg makes use of inductive considerations.

*) Published in VI Zjazd Matematyków Polskich, Warszawa 20–23. IX. 1948 (The Sixth Congress of Polish Mathematicians, Warsaw, September 20 to 23, 1948), issued as a Supplement to vol. 22 of *Rocznik Polskiego Towarzystwa Matematycznego* (1950).

Here is the proof of this theorem, carried out exclusively by means of the rules of substitution and detachment:

1 $CpC\alpha\beta$.

2 $CCCpqpp$.

3 $CCpqCCqrCpr$.

3 $p/Cpq, q/CCqrCpr, r/s \times C3 - 4$.

4 $CCCCqrCprsCCpqs$.

4 $q/Cqr, r/Csr, s/CCsqCpCsr \times C4p/s, s/CpCsr - 5$.

5 $CCpCqrCCsqCpCsr$.

5 $p/CCCCprtCCqrts, q/CCqrCpr, r/s, s/Cpq \times C4q/Cpr, r/t, p/Cqr - C3 - 6$.

6 $CCCCCprtCCqrtsCCpqs$.

6 $p/q, t/Cpr, q/s, s/CCpqCCsrCpr \times C4s/CCsrCpr - 7$.

7 $CCqsCCpqCCsrCpr$.

5 $p/Cqs, q/Cpq, r/CCsrCpr, s/t \times C7 - 8$.

8 $CCtCpqCCqsCtCCsrCpr$.

7 $q/CCsqs \times C2p/s - 9$.

9 $CCpCCsqsCCsrCpr$.

7 $q/CtCCsqs, s/CCsrCtr, r/u \times C9\, p/t - 10$.

10 $CCpCtCCsqsCCCCsrCtruCpu$.

3 $q/C\alpha\beta, r/CCCsq\alpha CC\beta sCCsqs \times C1 - C7\, q/\alpha, s/\beta, p/Csq, r/s - 11$.

11 $CpCCCsq\alpha sCCsqs$.

3 $q/CCCsq\alpha CC\beta sCCsqs, r/CCCC\beta sCCsqsCCCsq\alpha Csq \times C11 - C3p/CCsq\alpha, q/CC\beta sCCsqs, r/Csq - 12$.

12 $CpCCCC\beta sCCsqsCsqCCCsq\alpha Csq$.

10 $t/CCC\beta sCCCsqsCsq, s/Csq, q/\alpha, r/q, u/CCsCCC\beta sCCsqsCsqCCCsqqCsq \times C12 - C5p/CCsqq, q/CCC\beta sCCsqsCsq, r/q - 13$.

13 $CpCCsCCC\beta sCCsqsCsqCCCsqqCsq$.

10 $t/CsCCC\beta sCCsqsCsq, s/Csq, r/s, u/CCsrCCCsqsr \times C13 - Cqp/CCsqs, q/CCC\beta sCCsqsCsq - 14$.

14 $CpCCsrCCCsqsr$.

$8\ t/p, p/Crq, q/CCCrtrq, r/q \times C14 s/r, r/q, q/t - 15$.

15 $CCCCCrtrqsCpCCsqCCrqq$.

$6\ p/Crt, t/q, s/CpCCCCqrqqCCrqq \times C15\ s/CCqrq - 16$.

16 $CCCrtqCpCCCCqrqqCCrqq$.

$2\ p/Cp\ CCCCqpqqCCpqq \times C16\ r/p, t/CCCCqpqqCCpqq - 17$.

17 $CpCCCCqpqqCCpqq$.

$17\ p/CCCpppp \times C2\ q/p - C2\ p/q, q/CCCpppp - 18$.

18 $CCCCCppppqq$.

$10\ p/CCppp, t/q, s/CCppp, q/p, r/p, u/Cqp \times C16\ r/p, t/p, q/p, p/q - C18\ q/Cqp - 19$.

19 $CCCpppCqp$.

$9\ s/CCppp, q/p, r/Cqp \times C17\ q/p - C19 - 20$.

20 $CpCqp$.

If, in the foregoing proof, we replace the letters α and β, wherever they occur, that is both in the theses and in the proof lines, by significant formulae of the implicational propositional calculus, selected so that $CpC\alpha\beta$ is a thesis, then we obtain the proof of the law $CpCqp$, based on that thesis, Peirce's law and the law of syllogism. For instance, if we replace α by p and β by p, then $CpC\alpha\beta$ becomes $CpCpp$, which thesis, together with Peirce's law and the law of the syllogism, yields the proof of the law $CpCqp$; we thus obtain the last of Wajsberg's particular results. If we put $\alpha = Cpq$, $\beta = q$, we obtain the proof of the law $CpCqp$ from the axioms $CpCCpqq$, $CCCpqpp$ and $CCpqCCqrCpr$, that is, the first of Wajsberg's particular results. Or, finally, if $\alpha = CqCpr$, $\beta = Cqr$ then we have the proof of the law $CpCqp$ from the theses $CpCCqCprCqr$, $CCCpqpp$ and $CCpqCCqrCpr$, where, as is known, the first of these theses is equivalent to the law of commutation. The letters α and β may be treated as abbreviations either of the formulae $CqCpr$ and Cqr, or of any other formulae, provided that these are selected so that $CpC\alpha\beta$ is a thesis. If such abbreviations are introduced, the *coherence* of the proof is not impaired, since in those theses which include these Greek letters — and they are only the theses 1, 11, 12 and 13 — I do not perform any substitutions, but I use the theses only as premisses, that is as antecedents of implications, from which I detach consequents. Thus the operation of substitution does not pertain to the meaning of the Greek letters and cannot change that meaning, nor can the operation of detachment. Thus the foregoing proof becomes a *schema*, according to which we can prove the law of simplification from any thesis of the form $CpC\alpha\beta$ by making use of Peirce's law and the law of the syllogism.

Let us consider, however, what will happen if the letters α and β are given values such that $CpC\alpha\beta$ does *not* become a thesis. For instance, let $\alpha = p$ and $\beta = s$, so that as the first axiom we adopt the formula $CpCps$, which is not a thesis. In this case, too, we obtain the law $CpCqp$, and hence we have a complete system of axioms of the implicational propositional calculus. It is known that if to such a system we join any formula that is not a thesis, we arrive at a contradiction in the sense that we obtain all significant formulae. This must also be true in our case. In fact, the very formula $CpCps$ leads to a contradiction, for if we substitute for p the formula $CpCps$, then after two detachments we obtain the variable s, and hence every significant formula. It is also worth while examining what happens to the remaining parts of the proof which contain the Greek letters, i.e., the formulae 11, 12, and 13. In our example, formula 11 also results in a contradiction, for if we put $\alpha = p$, $\beta = s$, then from $CpCCCsq\alpha CC\beta sCCsqs$ we obtain the formula $CpCCCsqpCCssCCsqs$, which on the substitutions p/Cpp and q/s and after four detachments again yields the variable s, that is, a contradiction. On the contrary, formula 13, which does not contain α but only β, is a thesis regardless of the value given to β. This is explained by the fact that for any β we can select an α such that $CpC\alpha\beta$ becomes a thesis: for instance, if we put $\alpha = Cp\beta$. Since such a possibility exists, and in the case of such a possibility all parts of the proof are theses, hence formula 13 must, in this case, be a thesis, too, and that for any β. Formula 12 is a thesis, too, although it contains both Greek letters, α and β. But in that formula α occurs in the string of letters, $CCCsq\alpha Csq$, and by Peirce's law and the law of simplification that string is equivalent to the formula Csq, which no longer contains the letter α. Hence that formula, too, must be a thesis for the same reasons for which formula 13 is. Hence, if α and β are selected so that $CpC\alpha\beta$ leads to a contradiction, then of all the parts of the proof only formula 11 can still lead to a contradiction; it is to be stressed that it *can*, but *need not*, since, for $\alpha = p$ and $\beta = Csq$, $CpC\alpha\beta$ becomes $CpCpCsq$, which yields a contradiction, but *CpCCCsqαCCβsCCsqs* becomes *CpCCCsqpCCCsqsCCsqs*, and hence a thesis.

The formula *CpCαβ* cannot be replaced in the proof by the more general formula *Cpα*; in other words, it is not true that if to Peirce's law and the law of the syllogism we join any thesis whose antecedent is a variable and whose consequent is arbitrary, i.e., any thesis of the form *Cpα*, then we obtain the law *CpCqp*. For if α stands not for an implication but for a variable, then the thesis *Cpα* becomes *Cpp*, and *CpCqp* does not follow from the system of theses *Cpp*, *CCCpqpp* and *CCpqCCqrCpr*; this is proved by the matrix below.

C	1	2	3	4
1	1	2	4	4
2	1	1	4	4
3	2	2	1	2
4	1	2	1	1

This matrix, in which 1 is the selected value, verifies the law of identity *Cpp*, Peirce's law, and the law of the syllogism, but it verifies neither the law *CpCqp* nor any thesis of the form *CpCαβ*. This is so because the formula *Cqp*, and also *Cαβ*, being an implication, can, in accordance with the matrix, take on only the values 1, 2, or 4, from which it follows that *CpCqp*, and also *CpCαβ*, for $p = 3$ always becomes 2. It is also worth mentioning that if to the formula *C*33 we give the value 2, and not 1, then the matrix modified in this way still verifies Peirce's law and the law of the syllogism, but no longer verifies the law of identity or any implicational thesis in which the antecedent is a variable.

ON VARIABLE FUNCTORS OF PROPOSITIONAL ARGUMENTS *)

1. A statement of Aristotle disproved. —2. The meaning of the variable functor δ. —3. Variable functors as applied to definitions. —4. The principle of bivalence. —5. Peirce's method of verification by 0 and 1.

1. A statement of Aristotle disproved

In his theory of modal syllogisms Aristotle accepts the statement: "If it is possible that *A* should belong to *B*, it is possible also that it should not belong to *B*." [1]) Replacing in this statement the proposition "*A* belongs to *B*" by the propositional variable "*p*," and denoting the functor "it is possible that" by "*M*," we get the formula:

1 *CMpMNp*

which may be read: "If it is possible that *p*, it is possible that not-*p*." [2])

Formula 1 leads to paradoxical consequences. This can be proved by means of the following thesis containing a variable functor of one propositional argument:

2 *CδpCδNpδq*,

in words: "if δ of *p*, then if δ of *Np*, δ of *q*." That means: if something is true of a proposition *p*, and the same is true of the negation of this

[1]) *Analytica Priora* I 13, 32 a 37: εἰ ἐνδέχεται τὸ Α τῷ Β ὑπάρχειν, ἐνδέχεται καὶ μὴ ὑπάρχειν. (Oxford Translation).

[2]) I am using throughout this paper my own symbolic notation without brackets, and my own arrangement of proofs. A full explanation of both these points is given in my paper "The Shortest Axiom of the Implicational Calculus of Propositions," these *Proceedings*, vol. 52 A 3, Dublin, 1948. [See pp. 295–305 of this volume.]

*) The lecture read on 30 November, 1949, at a meeting of the Royal Irish Academy. First published in *Proceedings of the Royal Irish Academy*, vol. 54, Section A, No. 2 (January 1951), pp. 24–35. Polish translation is included in the 1961 edition *Z zagadnień logiki i filozofii*.

proposition *Np*, it is true of an arbitrary proposition *q*. The variable functor is δ.

We need as an auxiliary thesis the so-called Frege's law:

3 $CCpCqrCCpqCpr$.

From 1, 2 and 3 we deduce the following consequences:

$2\ \delta/M \times 4.$

4 $CMpCMNpMq$.

$3\ p/Mp,\ q/MNp,\ r/Mq \times 5.$

5 $CCMpCMNpMqCCMpMNpCMpMq$.

$5 \times C4 - C1 - 6.$

6 $CMpMq$.

Theses 4 and 5 are got by substitution ("/" is the sign of substitution), 6 is got from 5 by two applications of the rule of detachment, i.e. the modus ponens of the Stoics: "if α, then β; but α; therefore β" ("–" is the sign of detachment).

It is obvious that there can be only two cases: Either

something is possible, i.e. $\sum pMp$

in words: "for some *p*, it is possible that *p*," or

nothing is possible, i.e. $N\sum pMp$,

in words: "it is not true that for some *p*, it is possible that *p*."

We agree with Aristotle that the second case cannot occur. Therefore, we may add to our premisses 1–3 as a new premiss the thesis:

7 $\sum pMp$.

According to a rule of particular quantifiers we may put before the antecedent of the implication 6 the quantifier $\sum$ binding the variable *p*, as this variable does not occur in the consequent as a free variable. We get thus the formula:

8 $C\sum pMpMq$.

From 8 we deduce 9 by detachment and substitution:

$8\ q/p \times C7 - 9$

9 Mp,

and by putting the universal quantifier before 9 we get 10:

$9 \prod p \times 10$

10 $\prod pMp$.

That means: "For all *p*, it is possible that *p*," or all is possible. This consequence is not true, because we agree with Aristotle that something is not possible, e.g. that an even number should be equal to an odd number. Thesis 2 containing a variable functor has been useful to disprove a basic principle of the Aristotelian theory of modal syllogisms. [3])

2. The meaning of the variable functor δ

Neither Frege, the founder of the modern propositional calculus, nor Russell, its propagator, have introduced variable functors into this calculus. The Polish logician Leśniewski (1886–1939) has added to the "theory of deduction" of the *Principia Mathematica* variable functors as well as quantifiers, calling the thus extended system of the propositional calculus "protothetic." [4]) The above cited thesis 2 which I owe to Leśniewski's protothetic has first drawn my attention to the importance of theses with variable functors. My own ideas are as follows:

A variable is a single letter considered with respect to a range of values that may be substituted for it. To substitute means practically to write in

[3]) Thesis 2 in its conjunctional form, $CK\phi p\phi Np\phi q$, and with ϕ instead of δ, I have used for the first time in my paper: "Philosophische Bermerkungen zu mehrwertigen Systemen des Aussagenkalküls", *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie* 23 (1930), cl. iii, p. 59. The same thesis I have mentioned again in 1947, in connection with the Aristotelian theory of modal syllogisms, in a letter from Dublin to Prof. I. M. Bocheński in Fribourg. The contents of this letter has been published in Bocheński's work: "La Logique de Théophraste", *Collectanea Friburgensia, Nouvelle Série*, Fasc. XXXII, Fribourg en Suisse, 1947, p. 99. I deduce here the conclusion *Mp* starting with the stronger premiss *EMpMNp* (*E* means equivalence). The weaker premiss, however, *CMpMNp*, is not only sufficient for this purpose, but it is at the same time in strict accordance with the clearest statement of Aristotle in this matter.

[4]) S. Leśniewski, "Grundzüge eines neuen Systems der Grundlagen der Mathematik," *Fundamenta Mathematicae* 14 (1929), pp. 1–81. About Leśniewski, see the paper of Z. Jordan, "The Development of Mathematical Logic and of Logical Positivism in Poland between the two Wars," *Polish Science and Learning* 6 (1945), Oxford University Press, esp. pp. 24–26. [See also "Introductory Remarks to the Continuation of my Article: 'Grundzüge eineis neuen Systems der Grundlagen der Mathematik'" by S. Leśniewski, *Polish Logic 1920–1939*, ed. by Storrs McCall, Oxford 1967.]

a formula instead of a variable one of its values, the same for the same variable. In the propositional calculus the range of values of propositional variables, as p or q, consists of all propositional expressions senseful in the calculus, besides this there may be two constants, 0 and 1, i.e. a constant false and a constant true proposition. What is the range of values of the functorial variable δ?

It is obvious that for δ in $\delta\alpha$, where α is a propositional expression, we may substitute any value which gives together with α a senseful expression of the propositional calculus. Such values are not only constant functors of one propositional argument, as N, negation, for example, but also complex expressions working like a functor of one argument, as Cr or $CC00$. By the substitution δ/Cr we get from the thesis 2, $C\delta pC\delta Np\delta q$, the expression $CCrpCCrNpCrq$, and by $\delta/CC00$ the expression $CCC00pCCC00NpCC00q$. It is evident, however, that this kind of substitutions does not cover all possible cases. We can get in this way from 2 neither $CpCNpq$, as it is impossible to blot the functors out of existence by means of a substitution, nor $CCprCCNprCqr$, because by no substitution for δ in δp or δq can the final p or q be removed from its place. There is, of course, no doubt whatever that the last two expressions are as good consequences of $C\delta pC\delta Np\delta q$, as $CNpCNNpNq$ or $CCrpCCrNpCrq$.

There are two ways to meet this difficulty. One of them, which I shall explain on an example, was chosen by Leśniewski. In order to get $CCprCCNprCqr$ from $C\delta pC\delta Np\delta q$ we must introduce by means of a definition a new function, say Grp, that would mean the same as Cpr. By the substitution δ/Gr we can get the expression $CGrpCGrNpGrq$, and then we can transform this expression by help of the definition into the required thesis. This way, however, is artificial and awkward. I have found another way that leads straight to our aim and is intuitively more convincing, than the roundabout way by means of a definition. It is a new kind of substitution.

The symbol $\delta\alpha$, where α is a propositional expression, represents, as I understand it, all senseful expressions of the propositional calculus containing α. E.g., δp represents Crp as well as Cpr, Cpp as well as $C\delta pp$, shortly, it represents all propositional expressions containing p including p and δp itself; similarly $\delta C00$ represents all expressions containing $C00$, as $C00$, $\delta C00$, $\delta\delta C00$, $CC00CpC00$, $C\delta C00p$, and so on. A rule

of substitution for $\delta\alpha$ must be formulated in a manner that we might get by substitution any expression represented by $\delta\alpha$. This seems to be intuitively clear. This new rule of substitution I shall again explain on examples.

If I want to get by substitution from thesis 2 the expression $CCprCCNprCqr$, I denote the required substitution by $\delta/C'r$. That means: in formula 2 instead of δ should be written an expression beginning with C, ending with r, and the place marked by the apostrophe "'" should be everywhere filled up by the argument belonging to δ. Another example: when I want to get by substitution from thesis 2 the formula $CpCNpq$, I denote the substitution by $\delta/'$ (I owe this suggestion to Mr. C. A. Meredith). That means: instead of δ should be written everywhere only the argument belonging to δ, in other words, δ should be everywhere omitted. A third example: from the formula $C\delta C00C\delta 0\delta p$ I get by substitution δ/C'' the expression $CCC00C00CC00Cpp$, because from $\delta\alpha$ results by this substitution $C\alpha\alpha$. It is evident that in this way we may get any expression represented by a formula containing δ's.

The substitution with an apostrophe can be also applied to substitutions of the first kind. From thesis 2 we get by the substitution δ/Cr as well as by δ/Cr' the same formula: $CCrpCCrNpCrq$. We shall see shortly that the substitution with an apostrophe is one of the most powerful rules of inference known today in logic.

3. Variable functors as applied to definitions

There are two ways of introducing definitions into the propositional calculus. One adopted by the authors of the *Principia Mathematica*, consists of expressing definitions by means of a special symbol, another way, adopted by Leśniewski, considers definitions as equivalences. Each way has its merits and faults.

In the *Principia Mathematica*, where the theory of deduction is based on two primitive terms, viz., negation ("$\sim p$") and disjunction ("$p \vee q$"), the definition of the implication ("$p \supset p$") is stated in the form:

$$p \supset q . = . \sim p \vee q \ \text{Df.}^{5)}$$

[5]) A. N. Whitehead and B. Russell, *Principia Mathematica*, vol. I, Cambridge, 1910, p. 11.

In words: "if p, then q" means the same as "not-p or q." The sign "=" and the letters "Df" are to be regarded as together forming one special symbol. This special symbol is connected with a special rule of inference allowing the replacement of the definiens by the definiendum and vice versa. This is the merit of this kind of definitions: the result is given immediately. But it has the defect of increasing the number of primitive terms which should be as small as possible.

Leśniewski would write the same definition as an equivalence, thereby introducing into his system no new primitive term, because for this very purpose he chose equivalence as a primitive term of protothetic. This is the merit of his standpoint. But on the other hand he cannot replace immediately the definiens by the definiendum or vice versa, because equivalence has its proper rules and does not allow for a rule of replacement.

Now there exists among the theses of protothetic besides thesis 2 still another thesis of great importance, called sometimes the law of extensionality. It runs thus:

11 $\quad CEpqC\delta p\delta q.$

In words: "If p then and only then when q, then if δ of p, δ of q." It means, roughly speaking: if p and q are equivalent, then whatever may be said of p, may be said also of q. Let us denote by P and Q two propositional expressions, where one of them, it does not matter which, is in a definition the definiens, and the other the definiendum. It is supposed that neither of them contains δ. As rightly constructed definitions may be regarded as true propositions, we accept the sentence:

12 $\quad EPQ.$

From 11 and 12 we deduce by substitution and detachment 13:

$\quad 11\, p/P, p/Q \times C12 - 13.$

13 $\quad C\delta P\delta Q.$

This new proposition is equivalent to EPQ, because from 13 we get again 12 by the law of identity:

14 $\quad Epp,$

and by two substitutions and one detachment:

$\quad 14\, p/P \times 15.$

15 $\quad EPP.$

$\quad 13\, \delta/EP' \times C15 - 12.$

12 $\quad EPQ.$

It follows from this consideration that instead of writing a definition as an equivalence we may use for this purpose the implication 13 having a variable functor before the definitions and the definiendum. This form has the merits of the two other forms of definitions mentioned above, without having their faults. As implication is the most convenient primitive term of any propositional calculus, the introduction of definitions by the formula 13 would not increase the number of primitive terms. On the other side, the use of variable functors allows immediate transformation of the definiens into the definiendum and conversely. Let us take an example.

In the system of the propositional calculus, based on implication C and a constant false proposition 0 as primitive terms, we may define the negation of a proposition p, i.e. Np, in the following way:

$$C\delta Np\delta Cp0.$$

That means, roughly speaking, to say "it is not true that p" is to say "if p, then 0." Applying this definition to thesis 2:

2 $\quad C\delta pC\delta Np\delta q,$

we get by substitution for δ:

$\quad 16\, \delta/C\delta pC\delta'\delta q \times C2 - 17.$

17 $\quad C\delta pC\delta Cp0\delta q.$

The definiendum Np is here replaced by the definiens $Cp0$. If we want to replace conversely the definiens by the definiendum, we ought to have the converse implication:

18 $\quad C\delta Cp0\delta Np.$

This converse implication is given together with the first. That is to say, we get from 13, without a new thesis and using only substitution and detachment, the converse formula $C\delta Q\delta P$ in this way:

13 $\quad C\delta P\delta Q.$

$\quad 13\, \delta/C\delta'\delta P \times 19.$

19 $\quad CC\delta P\delta PC\delta Q\delta P.$

$13\ \delta/CC\delta P\delta' C\delta Q\delta P \times 20.$

20 $CCC\delta P\delta PC\delta Q\delta PCC\delta P\delta QC\delta Q\delta P.$

$20 \times C19 - C13 - 21.$

21 $C\delta Q\delta P.$

This last deduction reveals the power of substitutions by apostrophe.

4. The principle of bivalence

In my paper of 1930 on multivalent systems of logic I have mentioned a principle that lies, in my opinion, at the bottom of our whole logic. I have called it the "principle of bivalence." [6]) A system of logic is called "bivalent," when it is based on the principle that every proposition is either true or false, i.e. when it takes for granted that there are only two possible values in logic, truth and falsity. This principle is different from the law of the excluded middle, according to which of two contradictory propositions one must be true.

At the time I formulated this principle of bivalence, I did not know that a thesis which embodied this principle might be taken as a single axiom of the whole propositional calculus. I learnt this as late as 1947.[7])

Let us look at such a thesis.

From 17 we get by substitution:

$17\ p/0,\ q/p \times 22.$

22 $C\delta 0C\delta C00\delta p.$

In words: "If δ of 0, then if δ of $C00$, δ of p." Now 0 is the symbol of a constant false proposition, and $C00$ ("if 0, then 0") can be regarded as a symbol of a constant true proposition, for $C00$ does not contain variables and is a true proposition. Thesis 22 means therefore: if some-

[6]) See my paper cited in note 3, where I say, p. 63: "Der Zweiwertigkeitssatz (= the principle of bivalence) ist die tiefste, jedoch schon im Altertum heftig umstrittene Grundlage unserer gesamten Logik". [See p. 165 of this volume.] In an appendix to this paper, pp. 75–77, I give a short history of this principle in the antiquity. [See pp. 176–178 of this volume.]

[7]) It is Dr. B. Sobociński, before the war of 1939 assistant and collaborator of Prof. Leśniewski, who has made the supposition (during his visit to Dublin in 1947) that thesis 22 or 24 might be sufficient as an axiom of the propositional calculus. His supposition was based on some results of Tarski quoted by Leśniewski, op. cit. p. 50.

thing is valid of a constant false proposition, and the same is valid of a constant true proposition, it is valid of any proposition. It follows from this statement that, in accordance with the principle of bivalence, only two kinds of propositions are supposed to exist, false and true ones.

From 22 we get by help of the law of commutation:

23 $CCpCqrCqCpr$

another thesis of the same kind, viz.:

$23\ p/\delta 0,\ q/\delta C00,\ r/\delta p \times C22 - 24.$

24 $C\delta C00C\delta 0\delta p.$

Here the constant true proposition is in the first place, and the constant false proposition in the second. My present purpose is to show that all theses of the theory of deduction in C and 0, i.e. theses without δ's and quantifiers, may be deduced from thesis 24 as a single axiom by means of the rules of substitution and detachment. I give first some consequences of this axiom needed for the proof in the subsequent chapter.

24 $C\delta C00C\delta 0\delta p.$

$24\ p/q \times 25.$

25 $C\delta C00C\delta 0\delta q.$

$24\ \delta/C'' \times 26.$

26 $CCC00C00CC00Cpp.$

$24\ \delta/' \times 27.$

27 $CC00C0p.$

$26\ p/0 \times 28.$

28 $CCC00C00CC00C00.$

$27\ p/C00 \times 29.$

29 $CC00C0C00.$

$24\ \delta/CC''C'C00 \times C28 - C29 - 30.$

30 $CCppCpC00.$

$27\ p/0 \times 31.$

31 $CC00C00.$

$30\ p/CC00C00 \times C28 - C31 - 32.$

32 $C00.$

$26 \times C31 - C32 - 33.$

33 $Cpp.$

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$27 \times C32 - 34.$

34 $C0p.$

$34\, p/Cq0 \times 35.$

35 $C0Cq0.$

$26\, p/C00 \times C31 - 36.$

36 $CC00CC00C00.$

$25\ \delta/CC00C'C00 \times C36 - C29 - 37.$

37 $CC00CqC00.$

$24\delta/C'Cq' \times C37 - C35 - 38.$

38 $CpCqp.$

$38p/Cpp \times C33 - 39.$

39 $CqCpp.$

$33p/CC00CC000 \times 40.$

40 $CCC00CC000CC00CC000.$

$39q/C0CC000, p/C00 \times 41.$

41 $CC0CC000CC00C00.$

$24\delta/CC'CC000CC00C'0 \times C40 - C41 - 42.$

42 $CCpCC000CC00Cp0.$

$33p/CC000 \times 43.$

43 $CCC000CC000.$

$42p/CC000 \times C43 - C32 - 44.$

44 $CCC0000.$

$39q/CC000, p/0 \times 45.$

45 $CCC000C00.$

$24\delta/CCC000' \times C45 - C44 - 46.$

46 $CCC000p.$

$46p/C\delta C00\delta 0 \times 47.$

47 $CCC000C\delta C00\delta 0.$

$39q/C00, p/\delta 0 \times 48.$

48 $CC00C\delta 0\delta 0.$

$24\delta/CC'0C\delta'\delta 0 \times C47 - C48 - 49.$

49 $CCp0C\delta p\delta 0.$

$49p/CC000 \times C44 - 50.$

50 $C\delta CC000\delta 0.$

$34p/C\delta 0\delta C00 \times 51.$

51 $C0C\delta 0\delta C00.$

$39q/C00, p/\delta C00 \times 52.$

52 $CC00C\delta C00\delta C00.$

$24\delta/C'C\delta'\delta C00 \times C52 - C51 - 53.$

53 $CpC\delta p\delta C00.$

$53p/CC00C00 \times C31 - 54.$

54 $C\delta CC00C00\delta C00.$

$35q/0 \times 55.$

55 $C0C00.$

$53p/C0C00 \times C55 - 56.$

56 $C\delta C0C00\delta C00.$

It seems to me that all steps of this deduction are perfectly clear and do not need an explanation. Theses 50, 54 and 56 required by the next chapter could be derived also from thesis 22 which is equivalent to 24. This other way, however, would be longer and more difficult.

5. Peirce's method of verification by 0 and 1

It is not my intention to explain here this method, as it should be known to all students of symbolic logic, but to describe its theoretical foundations in connexion with axiom 24. For this purpose we must introduce the definition of a constant true proposition, which may be done by taking the implication "if 0, then 0" as the meaning of "1".

$\mathrm{Df}1 \times 57.$

57 $C\delta C00\delta 1.$ 57.1 $C\delta 1\delta C00.$

With the implication 57 is given together the converse implication 57.1 according to our argument exhibited in Chapter 3. The same argument may be applied not only to definitions, but to any true implication of the form $C\delta P\delta Q$ provided neither P nor Q contains δ. With the help of 57 we may transform the axiom 24 and the theses 50, 54 and 56 into formulae having everywhere 1 instead of $C00$.

$57\ \delta/C\delta C0'\delta' \times C56 - 58.$

58 $C\delta C01\delta 1.$ 58.1 $C\delta 1\delta C01.$

$57\ \delta/C\delta C'0\delta 0 \times C50 - 59.$

59 $C\delta C10\delta 0.$ 59.1 $C\delta 0\delta C10.$

$57\ \delta/C\delta C''\delta' \times C54 - 60.$

60 $C\delta C11\delta 1.$ 60.1 $C\delta 1\delta C11.$

$57\ \delta/C\delta' C\delta 0\delta p \times C24 - 61.$

61 $C\delta 1C\delta 0\delta p.$

$61\ p/q \times 62.$

62 $C\delta 1C\delta 0\delta q.$

From the law of identity Cpp we get not only $C00$, which was already proved as thesis 32, but also $C11$:

$33\ p/1 \times 63.$

63 $C11.$

Let us now see how these theses work as a theoretical foundation for Peirce's method of verification. If we want to verify by this method a senseful expression of the theory of deduction in C and 0, we have to substitute instead of the variables occurring in this expression the symbols 0 and 1 in all possible combinations, reducing the thus obtained formulae on the ground of equalities: $C00 = 1$, $C01 = 1$, $C10 = 0$, and $C11 = 1$. If after the reduction all formulae give 1 as the final result, the expression is true or a thesis, if even one of them gives 0 as the final result, the expression is false. Let us take as an example of the first kind the law of Peirce $CCCpqpp$. We get by substitution four formulae each of them giving 1 as the final result:

$$p/0,\ q/0:\quad CCC0000 = CC100 = C00 = 1,$$
$$p/0,\ q/1:\quad CCC0100 = CC100 = C00 = 1,$$
$$p/1,\ q/0:\quad CCC1011 = CC011 = C11 = 1,$$
$$p/1,\ q/1:\quad CCC1111 = CC111 = C11 = 1.$$

Hence Peirce's law is verified as a thesis.

This purely practical method can be replaced by our inferential method in the following way:

$59.1\ \delta/C'0 \times C32 - 64.$

64 $CC100.$

$57.1\ \delta/CC'00 \times C64 - 65.$

65 $CCC0000.$

$58.1\ \delta/CC'00 \times C64 - 66.$

66 $CCC0100.$

$62\ \delta/CCC0'00 \times C66 - C65 - 67.$

67 $CCC0q00.$

$58.1\ \delta/C'1 \times C63 - 68.$

68 $CC011.$

$59.1\ \delta/CC'11 \times C68 - 69.$

69 $CCC1011.$

$60.1\ \delta/C'1 \times C63 - 70.$

70 $CC111.$

$60.1\ \delta/CC'11 \times C70 - 71.$

71 $CCC1111.$

$62\ \delta/CCC1'11 \times C71 - C69 - 72.$

72 $CCC1q11.$

$61\ \delta/CCC'q'' \times C72 - C67 - 73.$

73 $CCCpqpp.$

This way is longer than the practical procedure of verification, but it establishes the practical method on a solid theoretical basis. It is evident that any other thesis, the law of syllogism $CCpqCCqrCpr$ for instance, could be proved in the same manner. Now it has been stated that the law of simplification $CpCqp$ (thesis 38), Peirce's law $CCCpqpp$ (thesis 73) and the law of syllogism $CCpqCCqrCpr$ give together with $C0p$ (thesis 34) a complete system of axioms for the theory of deduction in C and 0.

Let us now consider an example of the second kind, when an expression is not verified, as in case of $CqCqp$. We get:

$$p/0,\ q/1:\quad C1C10 = C10 = 0.$$

From this one case we argue that $CqCqp$ is false. In our axiomatical system we can always prove that false expressions are not only not deducible from the axiom, but yield any expression whatever when added to the system. In our example the proof proceeds as follows:

I $CqCqp.$

$\mathrm{I}\ p/0,\ q/1 \times \mathrm{II}.$

II $C1C10.$

	59 $\delta/C1' \times C\mathrm{II} - \mathrm{III}$.
III	$C10$.
	59 $\delta/' \times C\mathrm{III} - \mathrm{IV}$.
IV	0.
	34 $\times\ C\mathrm{IV} - \mathrm{V}$.
V	p.

From p we may get by substitution any senseful expression we like. It is evident that the same procedure may be applied to any other false expression senseful in the system.

* * *

The results I have described in the chapters above are only a beginning. Their continuation is due to Mr. C. A. Meredith who has attended my lectures on Mathematical Logic at the Royal Irish Academy since 1946. Meredith[8] has shown that from axiom 24 or 22 can be deduced not only all theses of the theory of deduction, but also all theses with variable functors and quantifiers of propositional as well as functorial variables. Moreover, Meredith has found the shortest axiom of the thus extended system of the propositional calculus. It is a thesis which was already known as a curiosity to Leśniewski, and was brought to Dublin by Sobociński in 1947. It runs $C\delta\delta0\delta p$ and contains only six letters. To deduce from this thesis the whole calculus of propositions by means of the rule of substitution, the rule of detachment, and the rules of quantifiers, must be regarded as a masterpiece of deductive power. However important these results may be, the most important effect of this development is, in my opinion, the fact that a new and vast field of logical problems has been opened which deserves the attention of all students of logic.

8) See his paper: "On an Extended System of the Propositional Calculus", these *Proceedings*, vol. 54.

ON THE INTUITIONISTIC THEORY OF DEDUCTION *)

Introduction. —1. Axioms and rules of the intuitionistic theory of deduction. —2. A partial *T–N*-system. —3. Definition of *Cpq* and axioms of the *C–N*-system. —4. The rule of detachment for *C*. —5. Intuitionistic and classical functors. —6. The controversy about the principle of excluded middle. —7. Conclusion. —Appendix.

Introduction

The purpose of this paper is to prove that the intuitionistic theory of deduction contains as its proper part the classical theory of deduction. K. Gödel observed in 1932 that all the theses of the classical theory, in which no other functions occur except conjunction and negation, are also provable in the intuitionistic theory, but he did not prove that they form an axiomatized system.[1] B. Sobociński axiomatized in 1939 the conjunctive-negative system of the classical theory, but he did not show that this system can be established on an intuitionistic basis.[2] The present paper is independent of the results of Gödel and Sobociński.

1) See K. Gödel, "Zur intuitionistischen Arithmetik und Zahlentheorie", *Ergebnisse eines mathematischen Kolloquiums* (ed. by Menger), Heft 4 (Wien 1931/2), pp. 34–38. I owe this reference to the courtesy of Mr Johan J. de Jongh in Amsterdam, as the paper of Gödel was not accessible to me in Dublin.

2) The paper of B. Sobociński, "Aksjomatyzacja konjunkcyjno-negacyjnej teorii dedukcji", was printed in *Collectanea Logica*, (Warszawa, 1939), pp. 179–195. The volume of which I was the editor was almost ready for publication, when all its copies have been destroyed by bombs in the printers' office during the siege of Warsaw in September 1939. Sobociński's paper can be known by a detailed review of Prof. H. Scholz in the *Jahrbuch für mathematische Forschung*, 65 I (1939), pp. 24–25. See also B. Sobociński, "An Investigation of Protothetic", *Cahiers de l'Institut d'Etudes Polonaises en Belgique*, 5, 7 foll. (Bruxelles, 1949).

*) First published in *Konikl. Nederl. Akademie van Wetenschappen, Proceedings, Series A* (1952), No. 3, pp. 202–212. Polish translation is included in the 1961 edition *Z zagadnień logiki i filozofii*.

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1. Axioms and rules of the intuitionistic theory of deduction

I am using throughout this paper my own symbolic notation without brackets, denoting the functor of the classical implication by C, of the intuitionistic implication by F, of the classical conjunction by K, of the intuitionistic conjunction by T, of the classical alternative by A, of the intuitionistic alternative by O. The functor of negation is denoted in both systems by N.

I assert as axioms of the intuitionistic theory of deduction the following ten formulae:

1	*FqFpq.*	6	*FpOpq.*
2	*FFpFqrFFpqFpr.*	7	*FqOpq.*
3	*FTpqp.*	8	*FFprFFqrFOpqr.*
4	*FTpqq.*	9	*FFpNqFqNp.*
5	*FpFqTpq.*	10	*FpFNpq.*

All the other asserted expressions I derive from these ten by means of two rules of inference:

(a) The rule of substitution: If α is asserted, and β is a substitution of α, then β must be asserted.

(b) The rule of detachment: If $F\alpha\beta$ is asserted, and α is asserted, then β must be asserted.

Any significant expression may be substituted for a propositional variable, the same for the same variable. Significant expressions of the intuitionistic system are propositional variables $p, q, r, s, \ldots$, function $N\alpha$ provided α is a significant expression, and functions $F\alpha\beta$, $T\alpha\beta$, $O\alpha\beta$ provided α and β are significant expressions. Derivative functions introduced on the basis of the primitive ones by abbreviative definitions also are significant expressions of the system. Asserted expressions I call "theses".

The axioms 1–10 and the rules (a) and (b) are deductively equivalent to the well-known set of axioms and rules given by A. Heyting for the intuitionistic theory of deduction. [3])

[3]) See A. Heyting. "Die formalen Regeln der intuitionistischen Logik", *Sitzungsberichte der Preussischen Akademie der Wissenschaften*, Phys.-Math. Kl. (Berlin, 1932), pp. 42–56.

2. A partial *T–N*-system

An axiomatized set of theses in which no other primitive functors occur except T and N I call a "T–N-system". As rules of inference for a T–N-system I accept the rule of substitution (a) restricted to significant expressions of the system, and the rule of detachment running thus:

(c) If $NT\alpha N\beta$ is asserted, and α is asserted, then β must be asserted.

From the axioms 1–5 and 9 of the intuitionistic system there can be deduced the following three T–N-theses: [4])

58 *NTNTNpNpNp.*

59 *NTpNNTNpNq.*

60 *NTNTpNqNNTNTqNrNNTpNr.*

These theses taken as axioms form together with the rules (a) and (c) a partial T–N-system. The proof that they are not sufficient to build up the whole T–N-system is given by the matrix M_1.

T	1	2	3	N
*1	1	1	3	3
2	2	2	3	1
3	3	3	3	1

M_1

The first argument of T and the argument of N is in the column on the left (under T), the second argument of T is in the line on the top; figure 1 marked by an asterisk is the selected value. Matrix M_1 fulfils the rule of detachment (c) and verifies the formulae 58–60, but does not verify the T–N-thesis:

61 *NTNpp,*

because we get for $p/2$: $NTN22 = NT12 = N1 = 3$.

This partial T–N-system contains as its proper part the classical theory of deduction.

3. Definition of *Cpq* and axioms of the *C–N*-system

The functor of the classical implication C can be introduced into the intuitionistic system by an abbreviative definition on the basis of the

[4]) All deductions are given in the Appendix.

functors T and N. The definition is formulated as an F-implication by help of a variable functor δ: [5])

63 $F\delta NTpNq\delta Cpq$,

and means: $NTpNq$ can be replaced everywhere by Cpq and conversely, and any substitution of $NTpNq$ can be replaced by the same substitution of Cpq and conversely. I do not introduce δ into the system, I use it only in definitions as a more convenient symbol, than the sign "= Df." employed by the authors of *Principia Mathematica*. A special rule of substitution for δ is explained in the Appendix.

Now I shall prove on the ground of the intuitionistic system that the functor introduced by definition 63 has all the properties of the classical C. We get by this definition from the theses 58–60 the following formulae:

65 $CCNppp$.

67 $CpCNpq$.

72 $CCpqCCqrCpr$.

Formula 65 represents the principle of Clavius, formula 67 the principle of Duns Scotus, and formula 72 the principle of the syllogism. These three principles, as I have stated many years ago, [6]) are sufficient to establish the whole classical theory of deduction, provided the rule of detachment can be applied to them. This rule would run thus:

(d) If $C\alpha\beta$ is asserted, and α is asserted, then β must be asserted.

4. The rule of detachment for *C*

Rule (d) is not valid, if we admit that α and β may be any significant expression of the intuitionistic theory of deduction. This can be shown by an example. The following formulae:

87 $CNNOpNpOpNp$

[5]) See J. Łukasiewicz, "On Variable Functors of Propositional Arguments", *Proceedings of the Royal Irish Academy*, vol. 54, Sect. A, No. 52, pp. 28–30, Dublin, 1951. [See pp. 311–324 of this volume.] The method of expressing definitions described in this paper is valid with any implicational functor.

[6]) See J. Łukasiewicz, "O znaczeniu i potrzebach logiki matematycznej", *Nauka Polska* 10 (1929), pp. 610–612.

and

86 $NNOpNp$

which are of the form $C\alpha\beta$ and α, are provable in the intuitionistic system, but β, i.e. $OpNp$, is not provable. Heyting's three-valued matrix M_2 which verifies the rule of detachment (b) and all the axioms 1–10 does

F	1	2	3	N	T	1	2	3	O	1	2	3
*1	1	2	3	3	1	1	2	3	1	1	1	1
2	1	1	3	3	2	2	2	3	2	1	2	2
3	1	1	1	1	3	3	3	3	3	1	2	3

M_2

not verify $OpNp$, because we get for $p/2$: $O2N2 = O23 = 2$.

For our purpose, however, it is not necessary to prove that rule (d) is generally valid; it suffices to show that it is valid for the partial system C–N. In other words, it is sufficient to prove that rule (d) is valid, when α and β are significant expressions in which no other primitive functors occur except C and N. This can be proved by means of the following three theses derived from the axioms 1–5 and 9 by help of the definition 63:

73 $FCpqFpNNq$.

75 $FCpNqFpNq$.

77 $FCpCqrFpCqr$.

Now, every significant expression of the C–N-system is either a variable, or a negation beginning with N, or an expression beginning with C. If $C\alpha\beta$ and α are asserted, and β is $N\gamma$, i.e. begins with N, we get from 75 by two detachments $N\gamma$; and if β is $C\gamma\varepsilon$, i.e. begins with C, we get from 77 by two detachments $C\gamma\varepsilon$. In both cases rule (d) is satisfied. If β is a variable, we can always assume without loss of generality that β is q. But $C\alpha q$ and α cannot be asserted both, because if they were, we would get from 73 the asserted consequence NNq, therefore by substitution $NNNq$, and from the thesis:

$$FNNqFNNNqq,$$

which follows from axiom 10 by the substitution p/NNq, there would result by two detachments the variable q. Rule (d) would be satisfied again, but it is clear that it could not be applied in this case. The validity

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of the rule of detachment for C is thus proved for all significant expressions of the system C–N, and this completes the proof that the intuitionistic theory of deduction contains as its proper part the classical theory of deduction.

It should be observed that the role of axiom 10 in the last argument is essential. If this axiom is dropped, as in the "Minimalcalculus" of I. Johannsen, the rule of detachment for C could not be proved.

5. Intuitionistic and classical functors

The classical theory of deduction is included in the intuitionistic theory as the C–N-system. We can enrich this latter system by introducing into it the usual definitions of conjunction and alternative, denoted respectively by K and A:

93 $F\delta NCpNq\delta Kpq$,

90 $F\delta CNpq\delta Apq$,

getting thus all the classical theses in K and A. Between the classical functors C, K and A, and the corresponding intuitionistic functors F, T and O there exists a simple logical relation: all those classical functors are weaker than the corresponding intuitionistic ones. C is weaker than F, because the implication:

78 $FFpqCpq$

holds in the intuitionistic system, but its converse $FCpqFpq$ is not provable in it. Similarly the conjunctive functor K is weaker than T, because the implication

94 $FTpqKpq$

is provable in the intuitionistic system, whereas its converse $FKpqTpq$ is not provable. For the same reason A is weaker than O, as we can prove the thesis

91 $FOpqApq$,

but not its converse $FApqOpq$.

All these converse expressions can be disproved, if we add to the matrix M_2 the matrix M_3 for C, K and A, constructed on the basis of the matrix M_2 according to the definitions 63, 93 and 90 respectively.

C	1	2	3
*1	1	1	3
2	1	1	3
3	1	1	1

K	1	2	3
1	1	1	3
2	1	1	3
3	3	3	3

A	1	2	3
1	1	1	1
2	1	1	1
3	1	1	3

M_3

It follows from M_2 and M_3 that $FCpqFpq$ is not verified, because we get for $p/1$, $q/2$: $FC12F12 = F12 = 2$; similarly neither $FKpqTpq$ nor $FApOqpq$ is verified, as we get for $p/2$, $q/2$ in the first case: $FK22T22 = F12 = 2$, in the second case: $FA22O22 = F12 = 2$.

All the theses in F, T or O remain true, if we replace these stronger functors by the corresponding weaker ones. On the contrary, it is not always the case that a thesis in C, K or A remains true, if we replace these weaker functors by the corresponding stronger ones. The "strong" principle of Clavius $FFNppp$, the "strong" principle of double negation with negations in the antecedent $FNNpp$, the "strong" principle of excluded middle $OpNp$, are not accepted by the intuitionists. Nevertheless the corresponding weaker theses:

65 $CCNppp$,

80 $CNNpp$,

92 $ApNp$,

are provable in the intuitionistic system, and must be consequently accepted by the intuitionists.

6. The controversy about the principle of excluded middle

The most famous thesis not accepted by the intuitionists is the principle of excluded middle. This principle is very evident, if it is applied to such examples as: "Either it rains here and now, or it does not rain here and now". Its general formula, however, i.e. the principle "either p or not-p", cannot be based on examples; it must be either accepted as an axiom or proved on the ground of some other principles. In both cases it cannot be taken in isolation, but must belong to a logical system. Let us describe a system in which the principle of excluded middle is true.

Two functions, the alternative and the negation, occur in this principle, and two evident statements are connected with them:

(r) An alternative "either p or q" is true, if at least one of its components is true.

This is essential for the alternative and is accepted by the intuitionists as well as by the followers of the classical logic.

(s) The negation of a false proposition is true.

This also is accepted by all logicians, as it is the very essence of the negation. If we add to (r) and (s) the principle of bivalence:

(t) Every proposition is either true or false,

we can establish the principle of excluded middle in its general form. For either p is true, and then the principle "either p or not-p" is true according to (r), or p is false, and then the principle is true according to (r), because not-p is true according to (s).

The principle of bivalence has been stated by myself in 1921 as the fundamental of all hitherto known systems of logic. [7] This fundamental, however, is not so evident as the statements (r) and (s): it has been denied for future contingent events by Aristotle and the Epicureans, [8] and rejected in modern times by the systems of the so called "many-valued logic". [9] Such a many-valued system is the intuitionistic theory of deduction the adequate matrix of which is infinite according to Gödel. [10] The principle of bivalence cannot be applied to this system.

Nevertheless the principle of excluded middle can be proved in the intuitionistic theory of deduction, because the whole classical theory of deduction is contained in it. No real controversy results from this fact. The meaning of this principle depends on the meaning of two functions, the alternative and the negation. The essential properties of these functions, expressed by the statements (r) and (s), are satisfied by the intuitionistic theory as well as by the classical, so that we are entitled to call them in both systems "alternative" and "negation". But not all properties of the alternative are the same in both systems. I was therefore anxious to denote the intuitionistic alternative by another symbol than the classical one. The controversy disappears, for there cannot be a controversy between two different formulae such as *OpNp* and *ApNp*. *OpNp* may be rejected; *ApNp* must be accepted.

[7]) See J. Łukasiewicz, "Logika dwuwartościowa", *Przegląd Filozoficzny*, 23 (Warszawa, 1921). [See pp. 89–109 of this volume.] The passage concerning the principle of bivalence has been literally translated into French by W. Sierpiński, "Algèbre des ensembles", *Monografie Matematyczne* 23, Warszawa–Wrocław, 1951, p. 2.

[8]) See J. Łukasiewicz, "Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenkalküls", *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie* 23 (1930), cl. iii, pp. 75–77: Zur Geschichte des Zweiwertigkeitssatzes. [See pp. 176–178 of this volume.]

[9]) The first system of this kind, a three-valued modal theory of deduction defined by a matrix, has been constructed by myself in 1920. See *Ruch Filozoficzny*, 5 (1920), p. 170. [See pp. 87–88 of this volume.]

[10]) See K. Gödel, "Zum intuitionistischen Aussagenkalkül", *Anzeiger der Akademie der Wissenschaften in Wien*, Math.-Nat. Kl. 69 (Wien, 1932), p. 85.

7. Conclusion

We have no means to decide which of the n-valued systems of logic, $n \geqslant 2$, is true. Logic is not a science of the laws of thought or of any other real object; it is, in my opinion, only an instrument which enables us to draw asserted conclusions from asserted premisses. The classical theory of deduction which is verified by a two-valued matrix is the oldest and simplest logical system, and therefore the best known and widely used. But for some purposes, for instance in modal logic, an n-valued system, $n > 2$, might be more suitable and useful. The more useful and richer a logical system is, the more valuable it is.

At the first "Entretiens de Zurich" in 1938 I set forth the opinion that the intuitionistic calculus of propositions is only a part of the classical calculus, and therefore essentially weaker than the latter. [11]) I see today that just the contrary is the fact. The intuitionistic theory is richer and consequently more powerful than the classical one. All the applications of the classical theory to mathematics are also valid in the intuitionistic theory, but besides many subtle mathematical problems can be dealt with in the intuitionistic theory which cannot be formulated in the classical system. It seems to me that among the hitherto known many-valued systems of logic the intuitionistic theory is the most intuitive and elegant.

Gödel observed that in the intuitionistic system an alternative can be asserted only when one of its components is asserted. In my recently published work on Aristotle's Syllogistic I gave reasons for introducing "rejection" into the classical theory of deduction as a complement of

[11]) See J. Łukasiewicz, "Die Logik und das Grundlagenproblem", *Les entretiens de Zürich* 1938, publiés par F. Gonseth, 1941, p. 86. [See pp. 278–294 of this volume.]

"assertion".[12]) I reject axiomatically the propositional variable *p*, and I state the following two rules of rejection:

(e) If α is rejected, and α is a substitution of β, then β must be rejected.

(f) If $C\alpha\beta$, in our case $F\alpha\beta$, is asserted, and β is rejected, than α must be rejected.

If we add to these general rules a special rule of rejection which is valid according to Gödel in the intuitionistic system:

(g) If α and β are rejected, then $O\alpha\beta$ must be rejected,

we get, as far as I see, a categorical system in which all the classical theses not accepted by the intuitionists can easily be disproved.

APPENDIX

Every derived thesis is preceded by a "proof-line" consisting of two parts separated from each other by a cross. Various kinds of proof are explained by examples.

I. *Proofs by substitution only*. Proof-line belonging to thesis 13: "1 *q*/*Fqr* × 13". Put in 1 *Fqr* for *q* ("/" is the symbol of substitution); the result is thesis 13.

II. *Proofs by substitution and detachment*. Proof-line 14: "12 *s*/*Fqr* × *F*13–14". Perform the substitution as in I; the thesis obtained by this substitution: *FFFqrFpFqrFFqrFFpqFpr* is omitted to save space. It begins with an *F* and has as its antecedent thesis 13, as its consequent thesis 14. By detachment ("–" is the symbol of detachment) we get 14. In some cases (see for instance proof-line 18) the rule of detachment is applied twice.

III. *Proofs by δ-substitution*. (a) Proof-line 73: "63 *δ*/*F'FpNNq* × *F*46 – 73". Drop in 63 the *δ*'s and write instead *F'FpNNq* filling up the gaps marked by the apostrophe by the arguments of *δ*. You get *FFNTpNqFpNNqFCpqFpNNq*, i.e. *F*46 – 73. (b) Proof-line 64: "63 *δ*/*NT'Np*, *p*/*Np*, *q*/*p* × *F*58 – 64". Perform first the substitutions for the propositional variables, *p*/*Np* and *q*/*p*, getting *FδNTNpNpδCNpp*, and then proceed as in III (a).

[12]) See J. Łukasiewicz, *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic*, Oxford, 1951, p. 109.

The numbers in brackets refer to theses to which a given thesis is applied. For instance, thesis 3 is applied to theses 32, 42, 56.

Axioms

1	*FqFpq* (11, 13, 17, 19, 44).
2	*FFpFqrFFpqFpr* (11, 12, 16, 18, 32).
3	*FTpqp* (32, 42, 56).
4	*FTpqq* (31, 55).
5	*FpFqTpq* (20, 38).
6	*FpOpq* (82).
7	*FqOpq* (83).
8	*FFprFFqrFOpqr* (88).
9	*FFpNqFqNp* (24, 26, 29, 33, 43, 81).
10	*FpFNpq*.

F-theses

	1 *q*/*FFpFqrFFpqFpr*, *p*/*s* × *F*2 – 11.
11	*FsFFpFqrFFpqFpr* (12).
	2 *p*/*s*, *q*/*FpFqr*, *r*/*FFpqFpr* × *F*11 – 12.
12	*FFsFpFqrFsFFpqFpr* (14, 21).
	1 *q*/*Fqr* × 13.
13	*FFqrFpFqr* (14).
	12 *s*/*Fqr* × *F*13 –14.
14	*FFqrFFpqFpr* (15, 20, 25, 26, 28, 31, 34, 35, 37, 52, 75).
	14 *q*/*Fqr*, *r*/*FFpqFpr*, *p*/*s* × *C*14 – 15.
15	*FFsFqrFsFFpqFpr* (16, 50).
	15 *s*/*FpFqr*, *q*/*Fpq*, *r*/*Fpr*, *p*/*q* × *F*2 – 16.
16	*FFpFqrFFqFpqFqFpr* (18).
	1 *q*/*FqFpq*, *p*/*FpFqr* × *F*1 – 17.
17	*FFpFqrFqFpq* (18).
	2 *p*/*FpFqr*, *q*/*FqFpq*, *r*/*FqFpr* × *F*16 – *F*17 – 18.
18	*FFpFqrFqFpr* (19, 29, 39, 51).
	18 *p*/*q*, *q*/*p*, *r*/*q* × *F*1 – 19.
19	*FpFqq* (22).

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F–T-theses

14 *q/p, r/FqTpq, p/r* × *F*5 – 20.
20 *FFrpFrFqTpq* (21).

12 *s/Frp, p/r, r/Tpq* × *F*20 – 21.
21 *FFrpFFrqFrTpq* (40, 56, 84).

F–N-theses

19 *p/FpFqq, q/Np* × *F*19 – 22.
22 *FNpNp* (23, 40, 61, 79).

22 *p/q* × 23.
23 *FNqNq* (24).

9 *p/Nq* × *F*23 – 24.
24 *FqNNq* (25).

14 *r/NNq* × *F*24 – 25.
25 *FFpqFpNNq* (27, 36, 55).

14 *q/FpNq, r/FqNp, p/r* × *F*9 – 26.
26 *FFrFpNqFrFqNp* (27, 30, 47).

26 *r/Fpq, q/Nq* × *F*25 – 27.
27 *FFpqFNqNp* (28, 41, 57, 82, 83, 85).

14 *q/Fpq, r/FNqNp, p/r* × *F*27 – 28.
28 *FFrFpqFrFNqNp* (38, 49).

18 *p/FpNq, r/Np* × *F*9 – 29.
29 *FqFFpNqNp* (30).

26 *r/q, p/FpNq, q/p* × *F*29 – 30.
30 *FqFpNFpNq* (31).

F–T–N-theses

14 *r/FpNFpNq, p/Tpq* × *F*30 – *F*4 – 31.
31 *FTpqFpNFpNq* (32).

2 *p/Tpq, q/p, r/NFpNq* × *F*31 – *F*3 – 32.
32 *FTpqNFpNq* (33).

9 *p/Tpq, q/FpNq* × *F*32 – 33.
33 *FFpNqNTpq* (34, 58, 61).

14 *q/FpNq, r/NTpq, p/r* × *F*33 – 34.
34 *FFrFpNqFrNTpq* (35, 36).

14 *q/FrFpNq, r/FrNTpq, p/s* × *F*34 – 35.
35 *FFsFrFpNqFsFrNTpq* (53).

34 *r/Fpq, q/Nq* × *F*25 – 36.
36 *FFpqNTpNq* (37, 59, 60, 78).

14 *q/Fpq, r/NTpNq, p/r* × *F*36 – 37.
37 *FFrFpqFrNTpNq* (44, 54).

28 *r/p, p/q, q/Tpq* × *F*5 – 38.
38 *FpFNTpqNq* (39).

18 *q/NTpq, r/Nq* × *F*38 – 39.
39 *FNTpqFpNq* (46, 75).

21 *r/Np, p/Np, q/Np* × *F*22 – *F*22 – 40.
40 *FNpTNpNp* (41).

27 *p/Np, q/TNpNp* × *F*40 – 41.
41 *FNTNpNpNNp* (58).

3 *p/Np, q/Nq* × 42.
42 *FTNpNqNp* (43).

9 *p/TNpNq, q/p* × *F*42 – 43.
43 *FpNTNpNq* (59, 88).

37 *r/q* × *F*1 – 44.
44 *FqNTpNq* (45).

44 *p/Np* × 45.
45 *FqNTNpNq* (88).

39 *q/Nq* × 46.
46 *FNTpNqFpNNq* (47, 52, 73).

26 *r/NTpNq, q/Nq* × *F*46 – 47.
47 *FNTpNqFNqNp* (48).

47 *p/q, q/r* × 48.
48 *FNTqNrFNrNq* (49).

28 *r/NTqNr, p/Nr, q/Nq* × *F*48 – 49.
49 *FNTqNrFNNqNNr* (50).

15 *s/NTqNr, q/NNq, r/NNr* × *F*49 – 50.
50 *FNTqNrFFpNNqFpNNr* (51).

18 *p/NTqNr, q/FpNNq, r/FpNNr* × *F*50 – 51.
51 *FFpNNqFNTqNrFpNNr* (52).

14 *q/FpNNq, r/FNTqNrFpNNr, p/NTpNq* × *F*51 – *F*46 – 52.

52 *FNTpNqFNTqNrFpNNr* (53).

35 *s/NTpNq, r/NTqNr, q/Nr* × *F*52 – 53.

53 *FNTpNqFNTqNrNTpNr* (54).

37 *r/NTpNq, p/NTqNr, q/NTpNr* × *F*53 – 54.

54 *FNTpNqNTNTqNrNNTpNr* (60).

25 *p/Tpq* × *F*4 – 55.

55 *FTpqNNq* (56).

21 *r/Tpq, q/NNq* × *F*3 – *F*55 – 56.

56 *FTpqTpNNq* (57).

27 *p/Tpq, q/TpNNq* × *F*56 – 57.

57 *FNTpNNqNTpq* (74).

T–N-theses

33 *p/NTNpNp, q/Np* × *F*41 – 58.

58 *NTNTNpNpNp* (64).

36 *q/NTNpNq* × *F*43 – 59.

59 *NTpNNTNpNq* (66).

36 *p/NTpNq, q/NTNTqNrNNTpNr* × *F*54 – 60.

60 *NTNTpNqNNTNTqNrNNTpNr* (68).

33 *p/Np, q/p* × *F*22 – 61.

61 *NTNpp* (62).

61 *p/Np* × 62.

62 *NTNNpNp* (80, 86).

Theses with C

Df *Cpq* × 63.

63 *FδNTpNqδCpq* (64 – 74, 77, 78, 80, 89).

63 *δ/NT'Np, p/Np, q/p* × *F*58 – 64.

64 *NTCNppNp* (65).

63 *δ/', p/CNpp, q/p* × *F*64 – 65.

65 *CCNppp*.

63 *δ/NTpN', p/Np* × *F*59 – 66.

66 *NTpNCNpq* (67).

63 *δ/', q/CNpq* × *F*66 – 67.

67 *CpCNpq*.

63 *δ/NT'NNTNTqNrNNTpNr* × *F*60 – 68.

68 *NTCpqNNTNTqNrNNTpNr* (69).

63 *δ/NTCpqNNT'NNTpNr, p/q, q/r* × *F*68 – 69.

69 *NTCpqNNTCqrNNTpNr* (70).

63 *δ/NTCpqNNTCqrN', q/r* × *F*69 – 70.

70 *NTCpqNNTCqrNCpr* (71).

63 *δ/NTCpqN', p/Cqr, q/Cpr* × *F*70 – 71.

71 *NTCpqNCCqrCpr* (72).

63 *δ/', p/Cpq, q/CCqrCpr* × *F*71 – 72.

72 *CCpqCCqrCpr*.

63 *δ/F'FpNNq* × *F*46 – 73.

73 *FCpqFpNNq*.

63 *δ/F'NTpq, q/Nq* × *F*57 – 74.

74 *FCpNqNTpq* (75, 81).

14 *q/NTpq, r/FpNq, p/CpNq* × *F*39 – *F*74 – 75.

75 *FCpNqFpNq* (76).

75 *p/TqNr* × 76.

76 *FCpNTqNrFpNTqNr* (77).

63 *δ/FCp'Fp', p/q, q/r* × *F*76 – 77.

77 *FCpCqrFpCqr*.

63 *δ/FFpq'* × *F*36 – 78.

78 *FFpqCpq* (79).

78 *p/Np, q/Np* × *F*22 – 79.

79 *CNpNp* (92).

63 *δ/', p/NNp, q/p* × *F*62 – 80.

80 *CNNpp* (87).

9 *p/CpNq, q/Tpq* × *F*74 – 81.

81 *FTpqNCpNq* (94).

Theses with O

27 *q/Opq* × *F*6 – 82.

82 *FNOpqNp* (84).

27 *p/q, q/Opq* × *F*7 – 83.
83 *FNOpqNq* (84).
21 *r/NOpq, p/Nq, q/Np* × *F*83 – *F*82 – 84.
84 *FNOpqTNqNp* (85).
27 *p/NOpq, q/TNqNp* × *F*84 – 85.
85 *FNTNqNpNNOpq* (86).
85 *q/Np* × *F*62 – 86.
86 *NNOpNp*.
80 *p/OpNp* × 87.
87 *CNNOpNpOpNp*.
8 *r/NTNpNq* × *F*43 – *F*45 – 88.
88 *FOpqNTNpNq* (89).
63 *δ/FOpq', p/Np* × *F*88 – 89.
89 *FOpqCNpq* (91).

Theses with A

Df *Apq* × 90.
90 *FδCNpqδApq* (91, 92).
90 *δ/FOpq'* × *F*89 – 91.
91 *FOpqApq*.
90 *δ/', q/Np* × *F*79 – 92.
92 *ApNp*.

Theses with K

Df *Kpq* × 93.
93 *FδNCpNqδKpq* (94).
93 *δ/FTpq'* × *F*81 – 94.
94 *FTpqKpq*.

FORMALIZATION OF MATHEMATICAL THEORIES *)

1. The starting point for my research on the formalization of mathematical theories is a tentative application to such theories of many-valued logics. I have chosen for that purpose a non-modal three-valued logical system and I have succeeded in basing on it a part of the theory of natural numbers. Since many formulae of the logical system I employed differ from two-valued logic, it was necessary to be very careful in handling them correctly, and formalization was the best way, if not the only one, of avoiding mistakes. In order to compare the results thus obtained with the ordinary theory it was also necessary to formalize the theory of natural numbers on the basis of two-valued logic. I soon realized that the latter theory is very interesting in itself and that it reveals not only many new proofs of known theorems, but also many new theorems.

I have chosen the theory of natural numbers as the object of my logical investigations because I noticed that this theory, elementary as it is, is also very much neglected. In any textbook on the theory of numbers we find a number of elementary theorems on natural numbers, which are not proved correctly. Since it is known that these theorems are true, no one bothers to prove them in a precise manner. Take, for instance, Euclid's theorem stating that there is no greatest prime number. To prove this theorem the product of all prime numbers up to p is formed and the formula: [1])

$$q = 2 \cdot 3 \cdot 5 \cdot \ldots \cdot p + 1$$

is discussed. Since q is not divisible by any prime number up to p, it

[1]) Cf. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford, 1938, p. 12.

*) First published in: *Colloques internationaux du Centre national de la recherche scientifique*, XXXVI, *Les méthodes formelles en axiomatique*, Paris 1953.

follows that it is either itself a prime number greater than p, or that it is divisible by a prime number greater than p. Everyone seems to fail to notice that this proof, which is intuitively convincing, is not a real proof, since the gap marked by the dots between 5 and p cannot be filled in, as there is no general formula for a product of prime numbers. Hence the expression "$2 \cdot 3 \cdot 5 \cdot \ldots \cdot p$" lacks any definite meaning and must not occur in an exact proof. In many other proofs of theorems concerned exclusively with natural numbers, algebraic, rational, irrational, and even complex numbers are introduced, in spite of the methodological requirement that everything concerned with natural numbers should be proved by means of an axiomatic system constructed specifically for natural numbers and without introduction of numbers of any other kind.

2. Before illustrating these remarks with examples I ought to explain the parenthesis-free symbolism I have adopted in the formalization of theses and proofs. In the text that follows the letter C stands for the functor "if ... then", K for the functor "and", N for the negation "not"; 0 is a constant false proposition; $p, q, r, \ldots$ are propositional variables; δ is a variable proposition-forming functor of one propositional argument. $\prod$ ("for all") is the universal quantifier, $\sum$ ("for at least one" or "there is ... such that ...") is the existential quantifier. ζ ("plus") is a functor of two numerical arguments forming a number; ε ("equal to") and α ("less than") are proposition-forming functors of two numerical arguments; $a, b, c, d, \ldots$ are numerical variables; φ is a variable proposition-forming functor of one numerical argument. In the proof lines the stroke / is the symbol of substitution, the dash – is the symbol of detachment, and the cross $\times$ is a punctuation mark. All the functors are written before their arguments in order to avoid the use of parentheses.

Two logical theories, the propositional calculus and the functional calculus, are necessary and sufficient to formalize axiomatic mathematical theories, in particular the theory of natural numbers. The propositional calculus can be based on the principle of bivalence as its sole axiom: [2])

[2]) Cf. J. Łukasiewicz, "On Variable Functors of Propositional Arguments", *Proceedings of the Royal Irish Academy*, vol. 54, Sect. A, 2, Dublin, 1951. [See pp. 311–324 of this volume.] C. A. Meredith has found an even shorter axiom, namely $C\delta\delta 0\delta p$. Cf. his article "On an Extended System of the Propositional Calculus." Ibid., 54, A, 3, Dublin, 1951.

(1) $C\delta 0 C\delta C00\delta p.$

This formula is read: "If 0 satisfies δ, then if $C00$ (a constant true proposition) satisfies δ, then any proposition p satisfies δ". All the theses of the theory of deduction, which is the most elementary part of the propositional calculus, follow from this axiom by two rules of inference: the rule of substitution and the rule of detachment.

Quantification theory does not require any axioms, since it can be based on four rules of inference (the arrow $\rightarrow$ indicates that the implication on the left, provided that it is true, implies the implication on the right):

(2) $C\varphi a\beta \rightarrow C\prod a\varphi a\beta$ (symbolized by $\prod 1a$),

(3) $C\beta\varphi a \rightarrow C\beta\sum a\varphi a$ (symbolized by $\sum 2a$),

(4) $C\varphi a\beta \rightarrow C\sum a\varphi a\beta$ (symbolized by $\sum 1a$),

(5) $C\beta\varphi a \rightarrow C\beta\prod a\varphi a$ (symbolized by $\prod 2a$).

Rules (2) and (3) are unconditionally valid, while rules (4) and (5) are valid only on the condition that β (any propositional expression) does not contain a as a free variable.

3. The formalized fragment of the theory of natural numbers which I intend to outline now is based on two primitive terms, two definitions, and three axioms. The primitive terms are ("plus") and 1 ("one", the least natural number). The relations ε ("equal to") and α ("less than") are defined. The definitions are noted down as implications by means of the variable functor δ: [3])

(6) $C\delta\prod\varphi C\varphi a\varphi b\delta\varepsilon ab,$

(7) $C\delta\sum d\varepsilon\zeta dab\delta\alpha ab.$

Definition (6) states that the expression $\prod\varphi C\varphi a\varphi b$ ("for all φ, if a satisfies φ, then b satisfies φ") may be replaced by the expression εab ("a is equal to b"), and conversely. Definition (7) states that the expression $\sum d\varepsilon\zeta dab$ ("there is a d such that d plus a equals b") may be replaced by the expression αab ("a is less than b"), and conversely.

The following three theses are adopted as axioms:

[3]) Definitions which make use of the symbol δ are explained in my article quoted above, pp. 28–30 [pp. 311–324 of this volume].

(8) $\varepsilon\zeta\zeta abc\zeta b\zeta ac$ principle of associativity,
(9) $CN\alpha 1a\varepsilon 1a$ principle of dichotomy,
(10) $C\prod aC\prod bC\alpha ba\varphi b\varphi a\varphi a$ principle of ascent.

Axiom (8) expresses, in my notation, the formula $(a+b)+c = b+(a+c)$. Axiom (9) states: "If it is not true that 1 is less than a, then 1 is equal to a". The last axiom is, so far as I know, a new one and requires a more detailed explanation. To understand it, it is necessary to begin with the expression $\prod bC\alpha ba\varphi b$, which states: "For all b, if b is less than a, then b satisfies φ", or, more briefly, "All numbers less than a satisfy φ". This expression is the antecedent of another implication, $C\prod bC\alpha ba\varphi b\varphi a$, which can be read: "If all numbers less than a satisfy φ, then a satisfies φ". If this second implication is true for all a, i.e., if the expression $\prod aC\prod bC\alpha ba\varphi b\varphi a$ is true, then φa is true: all numbers satisfy φ. The last a is a free variable and may be replaced by another letter, for instance c, and preceded by the universal quantifier. The idea underlying this principle of ascent may be formulated as follows: "If a natural number satisfies φ on the assumption that all the preceding numbers satisfy φ, then all the natural numbers satisfy φ." [4])

4. All the properties of the relations ε and α, including the following theorems, can be deduced from these definitions and axioms by means of logical theses and rules:

(11) $N\alpha aa$ principle of irreflexivity of α,
(12) $C\alpha abN\alpha ba$ principle of asymmetry of α,
(13) $C\alpha abC\alpha bc\alpha ac$ principle of transitivity of α,
(14) $CN\alpha abCN\alpha ba\varepsilon ab$ principle of trichotomy.

We may then derive the laws of addition, such as the commutative law:

(15) $\varepsilon\zeta ab\zeta ba$,

[4]) A principle of this kind was formulated by P. Bernays in his article "Sur les questions méthodologiques actuelles de la théorie hilbertienne de la démonstration", *Les Entretiens de Zürich*, 1941. Mr. Bernays writes (p. 149) that it is necessary to justify the following reasoning: "If a property $B(\alpha)$ pertaining to an ordinal number α holds for 0 (the least of all the α's) and if it holds for α provided that it also holds for the preceding ordinal numbers, then it holds for all α". Mr. Bernays adds that this principle is a generalization of mathematical induction. Mr. Bernays's formulation, while not incorrect, is not quite exact: the condition that B should hold for the least α may be dropped.

the associative law in its ordinary form:

(16) $\varepsilon\zeta\zeta abc\zeta a\zeta bc$,

and the monotonic law:

(17) $C\alpha ab\alpha\zeta ac\zeta bc$.

Of those theorems which are little known I quote the following:

(18) $CK\alpha a\zeta b1\alpha b\zeta a1\varepsilon ab$,

which states: "If a is less than b plus 1, and b is less than a plus 1, then a equals b".

The two principles of mathematical induction, the strong and the weak, are also among the consequences of the system. The strong principle has the form:

(19) $C\prod aC\varphi a\varphi\zeta a1C\varphi 1\varphi a$,

and states: "If, for all a, if a satisfies φ then a plus 1 satisfies φ, then, if 1 satisfies φ every number satisfies φ". The weak principle is a consequence of the strong and is written:

(20) $C\prod a\varphi\zeta a1C\varphi 1\varphi a$.

Formula (20) states: "If, for all a, a plus 1 satisfies φ, then, if 1 satisfies φ every number satisfies φ". This principle, little known by mathematicians, is very useful in many proofs. I shall quote just one very simple example:

As logical premisses we adopt the principle of simplification

(21) $CpCqp$,

and the principle of Duns Scotus

(22) $CpCNpq$.

and as mathematical premisses we adopt the following two theses:

(23) $\varepsilon 11$ (1 equals 1),

(24) $\alpha 1\zeta a1$ (1 is less than a plus 1).

From these four premisses we can derive the principle of dichotomy by means of the weak principle of mathematical induction:

(22) $p/\alpha 1\zeta a1,\ q/\varepsilon 1\zeta a1 \times C(24) - (25)$

(25) $CN\alpha 1\zeta a1\varepsilon 1\zeta a1$.

(21) $p/CN\alpha 1\zeta a1\varepsilon 1\zeta a1 \times C(25)-(26)$.

(26) $CqCN\alpha 1\zeta a1\varepsilon 1\zeta a1$.

(26) $\prod 2a \times (27)$.

(27) $Cq\prod aCN\alpha 1\zeta a1\varepsilon 1\zeta a1$.

(27) $q/CpCqp \times C(21)-(28)$.

(28) $\prod aCN\alpha 1\zeta a1\varepsilon 1\zeta a1$.

(21) $p/\varepsilon\ 11, q/N\alpha 11 \times C(23)-(29)$.

(29) $CN\alpha 11\varepsilon 11$.

(20) $\varphi/CN\alpha 1'\varepsilon' \times C(28)-C(29)-(30)$.

(30) $CN\alpha 1a\varepsilon 1a$.

I hope that this derivation does not present any difficulty to the reader. The last transformation is the most difficult: to carry out the substitution $\varphi/CN\alpha 1'\varepsilon 1'$ it is necessary to replace φ by the expression $CN\alpha 1'\varepsilon 1'$ and at the same time to replace the apostrophes by the argument of φ. This yields the formula:

(31) $C\prod aCN\alpha 1\zeta a1\varepsilon 1\zeta a1CCN\alpha 11\varepsilon 11CN\alpha 1a\varepsilon 1a$,

from which thesis (30) can be obtained by two detachments.

Yet the most interesting are the consequences of the principle of ascent. That principle can be transformed, by purely logical means, into two other principles that are already known: the principle of the least number and Fermat's principle of descent. The principle of the least number has the following form:

(32) $C\varphi a\sum aK\varphi a\prod bC\alpha baN\varphi b$,

and states: "If a satisfies φ then there is a number a that satisfies φ, and for all b, if b is less than a, then b does not satisfy φ". This states simply, that if there is a natural number that satisfies φ, then there is always a least number satisfying φ. Fermat's principle of descent has a positive and a negative form:

(33) $C\prod aCN\varphi a\sum bK\alpha baN\varphi b\varphi a$ positive form,

(34) $C\prod aC\varphi a\sum bK\alpha ba\varphi bN\varphi a$ negative form.

The positive form has the following meaning: "If for any number that does not satisfy a certain condition there is a smaller number which has the same property, then every number satisfies that condition."

Analogously, the negative form means: "If for any number that satisfies a certain condition there is a smaller number which has the same property, then no number satisfies that condition." Principles (33) and (34) were used by Fermat in the proofs of several theorems in number theory. [5])

5. To give a sample of a formalized mathematical theory, I shall now prove that the principle of the least number and the principle of descent are deductively equivalent to the principle of ascent. Then I shall derive from the last-named principle the law of irreflexivity of the relation α. All the proofs will be based on the two-valued propositional calculus and the functional calculus, and do not require any other mathematical thesis except for the principle of ascent.

Axioms

1 $C\prod aC\prod bC\alpha ba\varphi b\varphi a\varphi a$ principle of ascent.

Auxiliary theses of deduction theory:

2 Cpp.

3 $CpCqp$.

4 $CCqrCCpqCpr$.

5 $CNNpp$.

6 $CCpNqCqNp$.

7 $CCNpqCNqp$.

8 $CCpCqNqCpNq$.

9 $CCpqCCqNrCrNp$.

10 $CCNpqCCqrCNrp$.

11 $CCNpqCCqNrCrp$.

12 $CCpNqNKpq$.

13 $CNCpqKpNq$.

14 $CNCpNqKqp$.

[5]) W. W. Rouse Ball in *A Short Account of the History of Mathematics*, London, 1940, pp. 296–298, quotes a letter by Fermat, kept in the Leyden University Library, which proves beyond doubt that Fermat discovered both forms of the principle of descent mentioned above.

A. From the principle of ascent to the principle of the least number.

14 $p/\prod bC\alpha baN\varphi b,\ q/\varphi a \times 15$.

15 $CNC\prod bC\alpha baN\varphi bN\varphi aK\varphi a\prod bC\alpha baN\varphi b$.

15 $\sum 2a \times 16$.

16 $CNC\prod bC\alpha baN\varphi bN\varphi a\sum aK\varphi a\prod bC\alpha baN\varphi b$.

7 $p/C\prod bC\alpha baN\varphi bN\varphi a,\ q/\sum aK\varphi a\prod bC\alpha baN\varphi b \times C16-17$.

17 $CN\sum aK\varphi a\prod bC\alpha baN\varphi bC\prod bC\alpha baN\varphi bN\varphi a$.

17 $\prod 2a \times 18$.

18 $CN\sum aK\varphi a\prod bC\alpha baN\varphi b\prod aC\prod bC\alpha baN\varphi bN\varphi a$.

1 $\varphi/N\varphi \times 19$.

19 $C\prod aC\prod bC\alpha baN\varphi bN\varphi aN\varphi a$.

11 $p/\sum aK\varphi a\prod bC\alpha baN\varphi b,\ q/\prod aC\prod bC\alpha baN\varphi bN\varphi a,\ r/\varphi a \times C18-C19-C20$.

20 $C\varphi a\sum aK\varphi a\prod bC\alpha baN\varphi b$, the principle of the least number.

B. From the principle of the least number to the principle of descent.

12 $p/\alpha ba,\ q/\varphi b \times 21$.

21 $CC\alpha baN\varphi bNK\alpha ba\varphi b$.

21 $\prod 1b \times 22$.

22 $C\prod bC\alpha baN\varphi bNK\alpha ba\varphi b$.

6 $p/\prod bC\alpha baN\varphi b,\ q/K\alpha ba\varphi b \times C22-23$.

23 $CK\alpha ba\varphi bN\prod bC\alpha baN\varphi b$.

23 $\sum 1b \times 24$.

24 $C\sum bK\alpha ba\varphi bN\prod bC\alpha baN\varphi b$.

4 $q/\sum bK\alpha ba\varphi b,\ r/N\prod bC\alpha baN\varphi b,\ p/\varphi a \times C24-25$.

25 $CC\varphi a\sum bK\alpha ba\varphi bC\varphi aN\prod bC\alpha baN\varphi b$.

12 $p/\varphi a,\ q/\prod bC\alpha baN\varphi b \times 26$.

26 $CC\varphi aN\prod bC\alpha baN\varphi bNK\varphi a\prod bC\alpha baN\varphi b$.

4 $q/C\varphi aN\prod bC\alpha baN\varphi b,\ r/NK\varphi a\prod bC\alpha baN\varphi b,\ p/C\varphi a\sum bK\alpha ba\varphi b \times C26-C25-27$.

27 $CC\varphi a\sum bK\alpha ba\varphi bNK\varphi a\prod bC\alpha baN\varphi b$.

27 $\prod 1a \times 28$.

28 $C\prod aC\varphi a\sum bK\alpha ba\varphi bNK\varphi a\prod bC\alpha baN\varphi b$.

6 $p/\prod aC\varphi a\sum bK\alpha ba\varphi b,\ q/K\varphi a\prod bC\alpha baN\varphi b \times C28-29$.

29 $CK\varphi a\prod bC\alpha baN\varphi bN\prod aC\varphi a\sum bK\alpha ba\varphi b$.

29 $\sum 1a \times 30$.

30 $C\sum aK\varphi a\prod bC\alpha baN\varphi bN\prod aC\varphi a\sum bK\alpha ba\varphi b$.

9 $p/\varphi a,\ q/\sum aK\varphi a\prod bC\alpha baN\varphi b,\ r/\prod aC\varphi a\sum bK\alpha ba\varphi b \times C20-C30-31$.

31 $C\prod aC\varphi a\sum bK\alpha ba\varphi bN\varphi a$, the principle of descent, negative form.

31 $\varphi/N\varphi' \times 32$.

32 $C\prod aCN\varphi a\sum bK\alpha baN\varphi bNN\varphi a$.

5 $p/\varphi a \times 33$.

33 $CNN\varphi a\varphi a$.

4 $q/NN\varphi a,\ r/\varphi a,\ p/\prod aCN\varphi a\sum bK\alpha baN\varphi b \times C33-C32-34$.

34 $C\prod aCN\varphi a\sum bk\alpha baN\varphi b\varphi a$, the principle of descent, positive form.

C. From the principle of descent to the principle of ascent.

13 $p/\alpha ba,\ q/\varphi b \times 35$.

35 $CNC\alpha ba\varphi bK\alpha baN\varphi b$.

35 $\sum 2b \times 36$.

36 $CNC\alpha ba\varphi b\sum bK\alpha baN\varphi b$.

7 $p/C\alpha ba\varphi b,\ q/\sum bK\alpha baN\varphi b \times C36-37$.

37 $CN\sum bK\alpha baN\varphi bC\alpha ba\varphi b$.

$37\ \prod 2b \times 38.$

38 $CN\sum bK\alpha baN\varphi b\prod bC\alpha ba\varphi b.$

$10\ p/\sum bK\alpha baN\varphi b,\ q/\prod bC\alpha ba\varphi b,\ r/\varphi a \times C38 - 39.$

39 $CC\prod bC\alpha ba\varphi b\varphi aCN\varphi\sum bK\alpha baN\varphi b.$

$39\ \prod 1a,\ \prod 2a \times 40.$

40 $C\prod aC\, bC\alpha ba\varphi b\varphi a\prod aCN\varphi a\sum bK\alpha baN\varphi b.$

$4\ q/\prod aCN\varphi a\sum bK\alpha baN\varphi b,\ r/\varphi a,\ p/\prod aC\prod bC\alpha ba\varphi b\varphi a \times C34 - C40\text{–}1.$

1 $C\prod aC\prod bC\alpha ba\varphi b\varphi a\varphi a$, the principle of ascent.

Since the principle of the least number results from the principle of ascent, the principle of descent results from the principle of the least number, and the principle of ascent results from the principle of descent, these three principles are deductively equivalent to one another.

D. From the principle of ascent to the irreflexivity of α.

$2\ p/C\alpha baN\alpha bb \times 41.$

41 $CC\alpha baN\alpha bbC\alpha baN\alpha bb.$

$41\ \prod 1b \times 42.$

42 $C\prod bC\alpha baN\alpha bbC\alpha baN\alpha bb.$

$42\ b/a \times 43.$

43 $C\prod bCab\alpha N\alpha bbC\alpha aaN\alpha aa.$

$8\ p/\prod bC\alpha baN\alpha bb,\ q/\alpha aa \times C43 - 44.$

44 $C\prod bC\alpha baN\alpha bb\alpha aa.$

$3\ p/C\prod bC\alpha baN\alpha bbN\alpha aa \times C44 - 45.$

45 $CqC\prod bC\alpha baN\alpha bbN\alpha aa.$

$45\ \prod 2a \times 46.$

46 $Cq\prod aC\prod bC\alpha baN\alpha bbN\alpha aa.$

$46\ q/CpCqp \times C3 - 47.$

47 $\prod aC\prod bC\alpha baN\alpha bbN\alpha aa.$

$1\ \varphi/N\alpha \times 48.$

48 $C\prod aC\prod bC\alpha baN\alpha bbN\alpha aaN\alpha aa.$

$48 \times C47 - 49.$

49 $N\alpha aa$, the principle of irreflexivity of α.[6])

I do not think that it would be possible to derive the theorems proved in this contribution without the powerful instrument of symbolic logic and without the formalization of proofs.

[6]) C. A. Meredith has proved that the principle of ascent can be derived from the principle of asymmetry of α, that is, a principle stronger than that of irreflexivity. His proof has not been published so far.

A SYSTEM OF MODAL LOGIC *)

The present essay consists of two parts: the first contains general remarks on systems of modal logic, the second is an exposition of a new modal system.

I

1. *What is modal logic*. A logical system is usually called "modal logic", if there occur in it modal expressions such as "possible" or "necessary". Instead of this rather vague characterization I shall try to give a precise definition of modal logic according to the tradition initiated by Aristotle.

First I shall explain what I understand by "basic modal logic". I am calling thus a system containing the expressions:

"It is possible that p" denoted by "Δp",

and

"It is necessary that p" denoted by "Γp",

if and only if they satisfy the following eight conditions:

I. The implication "If p, then it is possible that p" is asserted, in symbols:

1.1 $\vdash Cp\Delta p$.

"C" means "if – then", "p" is a propositional variable, and "$\vdash$" is the sign of assertion.[1])

II. The implication "If it is possible that p, then p" is rejected, in symbols:

[1]) The idea of assertion and its sign "$\vdash$" were introduced into logic by Frege in 1879, and afterwards accepted by the authors of the *Principia Mathematica*. In my previous papers I always omitted this sign, but here I am bringing it in because, besides assertion, I introduce rejection.

*) First published in *The Journal of Computing Systems*, 1 (1953), pp. 111–149. Polish translation is included in the 1961 edition *Z zagadnień logiki i filozofii*.

1.2 $\dashv C\Delta pp$.

"$\dashv$" is the sign of rejection.[2])

III. The proposition "It is possible that p" is rejected, in symbols:

1.3 $\dashv \Delta p$.

IV. The implication "If it is necessary that p, then p" is asserted, in symbols:

1.4 $\vdash C\Gamma pp$.

V. The implication "If p, then it is necessary that p" is rejected, in symbols:

1.5 $\dashv Cp\Gamma p$.

VI. The proposition "It is not necessary that p" is rejected, in symbols:

1.6 $\dashv N\Gamma p$.

"N" means "not".

VII. The equivalence "It is possible that p — if and only if — it is not necessary that not p" is asserted, in symbols:

1.7 $\vdash E\Delta pN\Gamma Np$.

"E" means "if and only if". In my symbolic notation the functors are always put before their arguments.

VIII. The equivalence "It is necessary that p — if and only if — it is not possible that not p" is asserted, in symbols:

1.8 $\vdash E\Gamma pN\Delta Np$.

The first condition corresponds to the principle: *Ab esse ad posse valet consequentia.*

The second condition corresponds to the saying: *A posse ad esse non valet consequentia.*

The third condition states that not all formulae beginning with Δ are asserted, because otherwise Δp would be equivalent to the function "*verum* of p" which is not a modal function.

The fourth condition corresponds to the principle: *Ab oportere ad esse valet consequentia.*

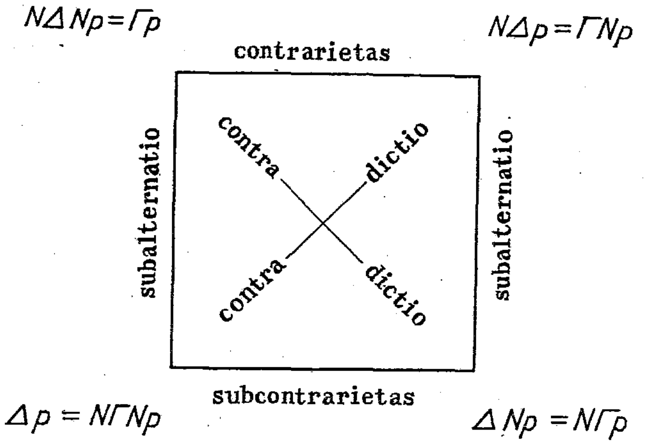
[2]) The idea of rejection was introduced into logic by myself in 1951. See J. Łukasiewicz, *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic*, Oxford, 1951, p. 109. I denote rejection by an inverted sign of assertion following a suggestion of Ivo Thomas.

The fifth condition corresponds to the saying: *Ab esse ad oportere non valet consequentia.*

The sixth condition states that not all formulae beginning with $N\Gamma$ are asserted, because otherwise Γp would be equivalent to the function "*falsum* of p" which is not a modal function.

The last two conditions are evident relations between possibility and necessity.

The above conditions, except perhaps the third and the sixth, which seem to be unknown to the traditional logicians, are embodied in the following "square of modalities".



I call a system "modal logic" if and only if it includes the basic modal logic as its part.

I accept throughout the paper that both Δ and Γ are proposition-forming functors of one propositional argument, and that both Δp and Γp are truth-functions, i.e., their truth-values depend only on the truth-values of their arguments. As there exists in the two-valued logic no functor of one argument which would satisfy the formulae 1.1, 1.2, and 1.3, or 1.4, 1.5, and 1.6, it is plain that the basic modal logic, and, consequently, every system of modal logic is a many-valued system.

2. *Axiomatization of the basic modal logic.* The next step to throw some light upon the modal logic is to axiomatize the basic modal logic

on the ground of the classical calculus of propositions. It can be easily seen that of the two modal functors, Δp and Γp, one may be taken as the primitive term, and the other can be defined. Let us take Δ as the primitive term. It would seem that, accepting the first three Δ-formulae as axioms, we could deduce from them the remaining four Γ-formulae. This is, however, not the case: formula 1.7, which contains besides Δ the defined functor Γ, cannot be got in this way, and must be accepted axiomatically. It is not elegant to use defined terms in axioms; we take, therefore, as the fourth axiom, instead of 1.7, the formula $E\Delta p\Delta NNp$, which is equivalent to $E\Delta pN\Gamma Np$. We get thus the following set of axioms:

2.1 $\vdash C p\Delta p$ (= 1.1),

2.2 $\dashv C\Delta pp$ (= 1.2),

2.3 $\dashv \Delta p$ (= 1.3),

2.4 $\vdash E\Delta p\Delta NNp$.

Γp is defined by the equivalence:

Df $\Gamma p \times$ 2.5.

2.5 $\vdash E\Gamma pN\Delta Np$. (= 1.8).

I accept the usual rules of substitution and detachment for the asserted formulae. The analogous rules for the rejected expressions run thus:

(a) Rule of substitution: If α is rejected, and α is a substitution of β, then β must be rejected.

(b) Rule of detachment: If $C\alpha\beta$ is asserted, and β is rejected, then α must be rejected.

Both rules are evident. Rule (a) is applied below to prove 2.11 and 2.14, rule (b) to prove 2.10 and 2.13.

The deduction.[3])

Auxiliary formulae of the propositional calculus:

T1 $\vdash CEpNqEErNpErq$.

T2 $\vdash CEpqCqp$.

T3 $\vdash CEpNqCCNrqCpr$.

[3]) For an explanation of the symbolism used in the deduction see my book on Aristotle's Syllogistic, pp. 81 and 96. [See also p. 342 of this volume.]

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Derived formulae of the basic modal logic:

T1 $p/\Gamma Np, q/\Delta NNp, r/\Delta p \times C2.5p/Np - 2.6$.

2.6 $\vdash EE\Delta pN\Gamma NpE\Delta p\Delta NNp$.

T2 $p/E\Delta pN\Gamma Np, q/E\Delta p\Delta NNp \times C2.6 - C2.4 - 2.7$.

2.7 $\vdash E\Delta pN\Gamma Np$ (= 1.7).

T3 $p/\Gamma p, q/\Delta Np, r/p \times C2.5 - C2.1p/Np - 2.8$.

2.8 $\vdash C\Gamma pp$ (= 1.4).

T3 $p/\Delta p, q/\Gamma Np, r/p \times C2.7 - 2.9$.

2.9 $\vdash CCNp\Gamma NpC\Delta pp$.

$2.9 \times C2.10 - 2.2$.

2.10 $\dashv CNp\Gamma Np$.

$2.10 \times 2.11\, p/Np$.

2.11 $\dashv Cp\Gamma p$ (= 1.5).

T2 $p/\Delta p, q/N\Gamma Np \times C2.7 - 2.12$.

2.12 $\vdash CN\Gamma Np\Delta p$.

$2.12 \times C2.13 - 2.3$.

2.13 $\dashv N\Gamma Np$.

$2.13 \times 2.14p/Np$.

2.14 $\dashv N\Gamma p$ (= 1.6).

Any of the four Δ-axioms is independent of the remaining three. This is easy to prove for the first three axioms. We take for C and N the normal two-valued matrix M_1, and show the independence of 2.1,

C	1	2	N
*1	1	2	2
2	1	1	1

M_1

2.2 and 2.3 by interpreting Δp as Np, p and Vp respectively. I shall explain the last proof. Vp, i.e., "*verum* of p", has for all truth-values of p the asserted truth-value of 1. If $\Delta p = Vp$, then $\Delta p = 1$, since $Vp = 1$. Δp, therefore, is asserted, i.e., axiom 2.3 is not verified. Axiom 2.1 is verified, because $Cp\Delta p = Cp1 = 1$; similarly, 2.4 is verified, because $E\Delta p\Delta NNp = E11 = 1$. From $C\Delta pp$ we get for $p/2$: $C\Delta 22 = C12 = 2$, and as 2 is rejected, $C\Delta pp$ is rejected too. All our axioms are verified by the matrix M_1 and the interpretation $\Delta p = Vp$

except 2.3 which, therefore, is independent of the remaining axioms. The other proofs are of a similar kind.

The proof of independence of the fourth axiom requires a three-valued matrix, M_2.[4]) M_2 verifies the C–N-axioms of the classical cal-

C	1	2	3	N	Δ	Γ
*1	1	2	3	3	1	1
2	1	1	1	1	1	3
3	1	1	1	1	3	3

M_2

culus of propositions, Δ is defined independently, and Γ is got from N and Δ according to the definition 2.5. 1 is the asserted truth-value, the other truth-values are rejected. It can be easily seen that $Cp\Delta p$ is asserted. $C\Delta pp$ is rejected, because we have for $p/2$: $C\Delta 22 = C12 = 2$. Similarly, Δp is rejected, because for $p/3$ we have $\Delta 3 = 3$. The axioms 2.1, 2.2 and 2.3 are thus verified, but axiom 2.4 is not verified, because its consequence $C\Delta p\Delta NNp$ is rejected for $p/2$: $C\Delta 2\Delta NN2 = C1\Delta N1 = C1\Delta 3 = C13 = 3$. Since 2.4, i.e., $E\Delta p\Delta NNp$, is equivalent to $E\Delta pN\Gamma Np$, this last formula is not verified too. Formula $Cp\Gamma p$ is also not verified, since it must be asserted according to the matrix, whereas it should be rejected. It is clear, therefore, that axiom 2.4 is indispensable for the axiomatization of the basic modal logic.

We get a corresponding set of axioms of the basic modal logic, if we take Γ as the primitive term and accept as axioms the following four Γ-formulae:

2.15 $\vdash C\Gamma pp$ (= 1.4),

2.16 $\dashv Cp\Gamma p$ (= 1.5),

2.17 $\dashv N\Gamma p$ (= 1.6),

2.18 $\vdash E\Gamma p\Gamma NNp$.

Δ is introduced by the definition:

Df $\Delta p \times 2.19$.

2.19 $\vdash E\Delta pN\Gamma Np$ (= 1.7).

Formula $E\Gamma p\Gamma NNp$ is equivalent to $E\Gamma pN\Delta Np$. This can be proved by T1 in the same way as the equivalence of $E\Delta p\Delta NNp$ and $E\Delta pN\Gamma Np$,

[4]) I owe this matrix to C. A. Meredith.

by interchanging in 2.6 the Δ's and Γ's. 2.18 is independent of the remaining axioms. The proof is given by the matrix M_3 with two asserted truth-values, which verifies the C–N-axioms of the classical calculus

C	1	2	3	N	Γ
*1	1	2	3	3	1
*2	1	1	3	3	3
3	1	2	1	1	3

M_3

of propositions and the axioms 2.15–2.17, but does not verify 2.18, as the consequence of 2.18, $C\Gamma NNp\Gamma p$, is rejected for $p/2$: $C\Gamma NN2\Gamma 2 = C\Gamma N33 = C\Gamma 13 = C13 = 3$.

3. *Aristotle's theorems of the propositional modal logic.* It is a pity that the formulae of the modal square never were correctly axiomatized on the basis of the classical calculus of propositions, and that even the problem of such an axiomatization never was clearly seen.[5]) Nobody, therefore, could observe that two odd formulae are hidden in the square, viz.:

$$E\Delta p\Delta NNp \quad \text{and} \quad E\Gamma p\Gamma NNp$$

which are indispensable for a correct axiomatization. These formulae throw a light on the modal logic just because of their similar shape: they suggest the idea that there must be a general principle independent of the modal square from which they may be deduced. There are still other reasons to suppose that the basic modal logic is not complete and requires the addition of some new principles. So, for instance, we believe that if a conjunction is possible, each of its factors should be possible, in symbols:

3.1 $\vdash C\Delta Kpq\Delta p$,

3.2 $\vdash C\Delta Kpq\Delta q$;

and if a conjunction is necessary, each of its factors should be necessary, in symbols:

[5]) The only logician, so far as I know, who saw this problem and tried to solve it, was I. M. Bocheński. His solution, however, is not correct, since the equivalence $E\Gamma pN\Delta Np$ is not deducible from his axioms. See I. M. Bocheński, "La logique de Théophraste", *Collectanea Friburgensia*, Fasc. 32, Fribourg en Suisse 1947, p. 92 Sect. 31.

3.3 $\vdash C\Gamma Kpq\Gamma p$,

3.4 $\vdash C\Gamma Kpq\Gamma q$.

None of these formulae can be deduced from the modal square, i.e., from the basic modal system.

It is strange enough that the only two theorems of the modal logic which Aristotle expressly states with propositional variables can be interpreted so as to give us the general principle we are looking for. Referring to his syllogisms, Aristotle writes in the *Prior Analytics*: "If one should denote the premisses by α, and the conclusion by β, it would not only result that if α is necessary, then β is necessary, but also if α is possible, then β is possible."[6]) There are two different ways of interpreting these theorems as formulae of modal logic, although it is highly improbable that Aristotle was aware of their difference. Let us explain these two interpretations.

All the Aristotelian syllogisms are implications of the form $C\alpha\beta$ where α is the conjunction of the two premises and β the conclusion. E.g., "If all a is b and all b is c, then all a is c", in symbols:

$$C\underbrace{KAabAbc}_{\alpha}\underbrace{Aac}_{\beta}.^{7)}$$

According to the above quotation, we get two modal theorems taking $C\alpha\beta$ as the antecedent, and $C\Delta\alpha\Delta\beta$ or $C\Gamma\alpha\Gamma\beta$ as the consequent, in symbols:

3.5 $\vdash CC\alpha\beta C\Delta\alpha\Delta\beta$

and

3.6 $\vdash CC\alpha\beta C\Gamma\alpha\Gamma\beta$.

The letters α and β stand here for the premisses and the conclusion of an Aristotelian syllogism. We may treat these theorems as special examples of general principles which we get by replacing the Greek letters by propositional variables:

3.7 $\vdash CCpqC\Delta p\Delta q$

and

[6]) *An. pr.*, A 15, 34a22: εἴ τις θείη τὸ μὲν Α τὰς προτάσεις, τὸ δὲ Β τὸ συμπέρασμα, συμβαίνοι ἂν οὐ μόνον ἀναγκαίου τοῦ Α ὄντος ἅμα καὶ τὸ Β εἶναι ἀναγκαῖον, ἀλλὰ καὶ δυνατοῦ δυνατόν.

[7]) See my book on Aristotle's Syllogistic, pp. 20 ff.

3.8 $\vdash CCpqC\Gamma p\Gamma q.$

This is the first interpretation. The principle for Δ seems to be confirmed by Aristotle himself in a second passage which reads quite generally: "It has been proved that if (if α is, β is), then (if α is possible, β is possible)."[8] Formula 3.7 is accepted as Aristotelian by A. Becker and I.M. Bocheński.[9]

We get a second interpretation if we draw attention to the fact that according to Aristotle the connection between the premisses α of a syllogism and its conclusion β is necessary. This gives us the special theorems:

3.9 $\vdash C\Gamma C\alpha\beta C\Delta\alpha\Delta\beta$

and

3.10 $\vdash C\Gamma C\alpha\beta C\Gamma\alpha\Gamma\beta,$

which we may extend into the principles:

3.11 $\vdash C\Gamma CpqC\Delta p\Delta q$

and

3.12 $\vdash C\Gamma CpqC\Gamma p\Gamma q.$

The principle 3.11 for Δ seems to be corroborated by Aristotle himself, as we read at the beginning of the same chapter where the other modal theorems occur: "First it has to be said that if (if α is, β must be), then (if α is possible, β must be possible too)."[10] The second "must" evidently refers to the necessary connection between the antecedent and the consequent, but the first "must" seems to state a necessary connection between α and β in the antecedent. There is no reference to a syllogism.

The formulae got by the first interpretation are stronger than those got by the second, as it is shown by the following deduction:

[8]) Ibid. 34a29: δέδεικται ὅτι εἰ τοῦ A ὄντος τὸ B ἔστι, καὶ δυνατοῦ ὄντος τοῦ A ἔσται τὸ B δυνατόν.

[9]) See A. Becker, *Die Aristotelische Theorie der Möglichkeitsschlüsse*, Berlin, 1933, p. 42 note, and I. M. Bocheński, "Ancient Formal Logic," *Studies in Logic and the Foundations of Mathematics*, Amsterdam, 1951, p. 71. Both authors refer to the passage 34a5 quoted in note 10, which rather supports the second interpretation.

[10]) Ibid. 34a5: πρῶτον δὲ λεκτέον ὅτι εἰ τοῦ A ὄντος ἀνάγκη τὸ B εἶναι, καὶ δυνατοῦ ὄντος τοῦ A δυνατὸν ἔσται καὶ τὸ B ἐξ ἀνάγκης.

T4 $\vdash CCpqCCqrCpr.$

1.4 $\vdash C\Gamma pp.$

$\mathrm{T4}\, p/\Gamma Cpq,\ q/Cpq,\ r/C\Delta p\Delta q \times C1.4 p/Cpq - C3.7 - 3.11.$

3.11 $\vdash C\Gamma CpqC\Delta p\Delta q.$

$\mathrm{T4}\, p/\Gamma Cpq,\ q/Cpq,\ r/C\Gamma p\Gamma q \times C1.4 p/Cpq - C3.8 - 3.12.$

3.12 $\vdash C\Gamma CpqC\Gamma p\Gamma q.$

We see that 3.11 follows from 3.7 and 3.12 from 3.8 by means of 1.4 and the principle of the hypothetical syllogism T4. The converse deduction is not valid. This can be proved by the matrix M_4 which results for C and N from the multiplication of M_1 by itself, verifies the C–N-axioms, the basic modal system, and the formulae 3.11 and 3.12, but does not verify 3.7 and 3.8, as we have for $p/4$, $q/2$: $CC42C\Delta 4\Delta 2 = C1C32 = C12 = 2$, and for $p/3$, $q/1$: $CC31C\Gamma 3\Gamma 1 = C1C32 = C12 = 2$.

C	1	2	3	4	N	Γ	Δ
*1	1	2	3	4	4	2	1
2	1	1	3	3	3	2	2
3	1	2	1	2	2	3	3
4	1	1	1	1	1	4	3

M_4

4. *Possible extensions of the basic modal logic.* All the four newly introduced principles, the stronger 3.7 and 3.8 as well as the weaker 3.11 and 3.12, are independent of the basic modal system on the ground of the classical calculus of propositions. It suffices to prove this for the weaker principles, because if these are shown to be independent, the stronger must be independent too. The proof is given by the eight-valued matrix M_5 which results for C and N from the multiplication of the matrix M_1 by the matrix M_4. M_5 verifies the C–N-axioms and the basic modal logic, but does not verify 3.11 and 3.12, as we get for $p/5$, $q/6$:

$$C\Gamma C56C\Delta 5\Delta 6 = C\Gamma 2C16 = C26 = 5$$

and for $p/3$, $q/4$:

$$C\Gamma C34C\Gamma 3\Gamma 4 = C\Gamma 2C38 = C26 = 5$$

C	1	2	3	4	5	6	7	8	N	Γ	Δ
*1	1	2	3	4	5	6	7	8	8	1	1
2	1	1	3	3	5	5	7	7	7	2	1
3	1	2	1	2	5	6	5	6	6	3	1
4	1	1	1	1	5	5	5	5	5	8	1
5	1	2	3	4	1	2	3	4	4	8	1
6	1	1	3	3	1	1	3	3	3	8	6
7	1	2	1	2	1	2	1	2	2	8	7
8	1	1	1	1	1	1	1	1	1	8	8

M_5

The new principles are not only independent of, but also consistent with the basic modal logic on the ground of the C–N-system. The proof of consistency is given by the matrix M_6 which is identical with M_4

C	1	2	3	4	N	Γ	Δ
*1	1	2	3	4	4	2	1
2	1	1	3	3	3	2	1
3	1	2	1	2	2	4	3
4	1	1	1	1	1	4	3

M_6

for C and N, but different for Γ and Δ. M_6 verifies the C–N-axioms, the basic modal logic, and all the four principles got by interpretation of Aristotle's modal theorems. Any such principle, when added to the basic modal logic, will expand this system into a fuller one.

The formula $CEpqC\phi p\phi q$ is called in logic "the principle of extensionality for ϕ". In a wider sense we may also thus call the formulae $CCpqC\phi p\phi q$ and $CCpqC\phi q\phi p$, because we get from them by $CEpqCpq$ or $CEpqCqp$ and the hypothetical syllogism the principle $CEpqC\phi p\phi q$. For instance, the principle of transposition $CCpqCNqNp$ is in a wider sense a principle of extensionality for N, because we get from it the formula $CEpqCNpNq$. The principles $CCpqC\Delta p\Delta q$ and $CCpqC\Gamma p\Gamma q$ are in a wider sense principles of extensionality for Δ and Γ.

These two principles are equivalent to each other on the ground of the C–N-system and the basic modal logic. Starting from 3.7 ($CCpqC\Delta p\Delta q$) we get 3.8 by means of the formulae:

T5 $\vdash CCNqNpCrsCCpqCNsNr$.

T6 $\vdash CEsqCEtrCCpCqrCpCst$.

2.5 $\vdash E\Gamma pN\Delta Np$.

T5 $r/\Delta Nq, s/\Delta Np \times C3.7p/Nq, q/Np - 4.1$.

4.1 $\vdash CCpqCN\Delta NpN\Delta Nq$.

T6 $s/\Gamma p, q/N\Delta Np, t/\Gamma q, r/N\Delta Nq, p/Cpq \times C2.5-$
$C2.5p/q-C4.1 - 3.8$.

3.8 $\vdash CCpqC\Gamma p\Gamma q$.

The converse deduction from 3.8 to 3.7 can be performed in the same way by interchanging Δ and Γ.

If we add 3.7 to the Δ-axioms of the basic modal logic we get by the laws of the C–N-system the formula $E\Delta p\Delta NNp$:

T7 $\vdash CpNNp$.

T8 $\vdash CNNpp$.

T9 $\vdash CCpqCCqpEpq$.

3.7 $q/NNp \times CT7 - 4.2$.

4.2 $\vdash C\Delta p\Delta NNp$.

3.7 $p/NNp, q/p \times CT8 - 4.3$.

4.3 $\vdash C\Delta NNp\Delta p$.

T9 $p/\Delta p, q/\Delta NNp \times C4.2 - C4.3 - 2.4$.

2.4 $\vdash E\Delta p\Delta NNp$.

In the same way we can prove $E\Gamma p\Gamma NNp$ starting from 3.8.

Owing to the stronger interpretation of the Aristotelian theorems we have found in the principle of extensionality for modal functors the general law from which the formulae $E\Delta p\Delta NNp$ and $E\Gamma p\Gamma NNp$ of the modal square can be deduced.

The extended modal system which arises by the addition of $CCpqC\Delta p\Delta q$ to the basic modal logic and is expounded in the second part of this article is the simplest complete modal logic with an adequate four-valued matrix. It is, in my opinion, both logically and philosophically of the highest importance. Nevertheless, it is wholly unknown. All the existing systems of modal logic, as far as I see, extend the basic modal logic by weaker principles, assuming either such formulae as $C\Gamma CpqC\Gamma p\Gamma q$ or $C\Gamma CpqC\Delta p\Delta q$ which correspond to the weaker interpretation of the Aristotelian theorems or rules of extensionality instead of principles. The principles of extensionality for modal functors are not accepted. In Von Wright's system, for instance, formula $CCpqCN\varDelta qN\varDelta p$, which is equivalent to $CCpqC\varDelta p\varDelta q$, is expressly disproved.[11]) All these modal systems are possible extensions of the basic modal logic and may have their own merits; perhaps we shall be able to decide some day which of them is the best.

II

5. *Axioms of the Ł-modal system.* The modal system expounded in this part presupposes the classical calculus of propositions, accepts $\varDelta$ as the only primitive modal term, and is built up on the basic modal logic with the addition of only one new modal principle, viz., the strict principle of extensionality for $\varDelta$.

The general principle of extensionality, taken *sensu stricto*, has the form:

5.1 $\vdash CEpqC\delta p\delta q$

where δ is a variable functor.[12]) This principle I extend to the modal functor $\varDelta$ getting thus the formula:

5.1 $\delta/\varDelta' \times$ 5.2,

5.2 $\vdash CEpqC\varDelta p\varDelta q.$

Formula 5.2 seems to be intuitively evident. We say that if p and q are equivalent to each other, then "If p is true, q is true", and "If p is false, q is false"; so we may also say that under the same condition "If p is possible, q is possible". Von Wright accepts in his system the rule of extensionality:

5.3 $\vdash E\alpha\beta \rightarrow \vdash E\varDelta\alpha\varDelta\beta,$

in words: "α if and only if β; therefore, α is possible, if and only if β is possible."[13]) The arrow is the sign of "therefore". Rule 5.3 follows from the formula 5.2.

[11]) See G. H. Von Wright, "An Essay in Modal Logic," *Studies in Logic and the Foundations of Mathematics*, Amsterdam, 1951, p. 22/23.

[12]) A short explanation of δ-substitution and δ-definition is given in the Appendix. For a detailed explanation see J. Łukasiewicz, "On Variable Functors of Propositional Arguments," *Proceedings of the Royal Irish Academy*, vol. 54 A 2, Dublin, 1951. [See pp. 311–324 of this volume.]

[13]) See Von Wright, l. c., p. 85. The symbolism and wording is mine.

The general principle of extensionality 5.1 must be accepted in the Ł-modal system, as it is valid for all functors of one argument of the classical calculus of propositions, and is admitted for the modal functor $\varDelta$. This leads to a simplification of the axiom-set of the system. From 5.1 by means of the formulae of the classical C–N-system:

T10 $\vdash CCpqCpCrq,$

T11 $\vdash CCpCqrCpCqCsr,$

T12 $\vdash CCprCCqrCApqr,$

T13 $\vdash AEpqENpq,$

we get the following consequences:

T10 $p/ENpq,\ q/C\delta Np\delta q,\ r/\delta p \times C5.1p/Np - 5.4.$

5.4 $\vdash CENpqC\delta pC\delta Np\delta q,$

T11 $p/Epq,\ q/\delta p,\ r/\delta q,\ s/\delta Np \times C5.1 - 5.5.$

5.5 $\vdash CEpqC\delta pC\delta Np\delta q.$

T12 $p/Epq,\ r/C\delta pC\delta Np\delta q,\ q/ENpq \times C5.5 - C5.4 - CT13 - 5.6.$

5.6 $\vdash C\delta pC\delta Np\delta q.$

It was shown by C. A. Meredith—in an unpublished paper—that formula 5.6 may be taken as the sole axiom of the classical C–N–δ–p-calculus, i.e., the classical C–N-calculus of propositions extended by the addition of variable functors. I accept, therefore, as the first axiom of the Ł-modal system the formula:

1 $\vdash C\delta pC\delta Np\delta q.$

From this axiom I derive by substitution and detachment the three axioms of the C–N-system:[14])

22 $\vdash CCpqCCqrCpr,$

20 $\vdash CCNppp,$

10 $\vdash CpCNpq,$

the principle of extensionality:

73 $\vdash CEpqC\delta p\delta q,$

[14]) Formulae marked with numbers without the decimal point are given in the Appendix.

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and the fourth axiom of the basic modal logic, 2.4:

89 $\vdash E\Delta p\Delta NNp$.

The other three axioms of the basic modal logic are independent of axiom 1, and must be taken axiomatically:

2 $\vdash Cp\Delta p$,

3 $\dashv C\Delta pp$,

4 $\dashv \Delta p$,

so that our modal system is based on four axioms: 1, 2, 3 and 4. As rules of inference I accept the rules of substitution and detachment for asserted and rejected formulae.

The proofs of independence are the same as in the basic modal logic. The independence of axiom 1 is proved by the matrix M_2, because we have for δ/Δ' and $p/2$, $q/3$: $C\Delta 2C\Delta N2\Delta 3 = C1C\Delta 13 = C1C13 = C13 = 3$.

The Aristotelian principle:

3.7 $\vdash CCpqC\Delta p\Delta q$

which is a principle of extensionality in a wider sense, is stronger than the strict principle of extensionality 5.2. Nevertheless, it is deducible in our system by means of the law

30 $\vdash C\delta CpqC\delta p\delta q$,

a consequence of the axiom 1. We get from this law by substitution δ/Δ' the formula:

77 $\vdash C\Delta CpqC\Delta p\Delta q$,

and as $CCpq\Delta Cpq$ is true according to our axiom 2, we get

78 $\vdash CCpqC\Delta p\Delta q$

by the help of the syllogism. We may say, therefore, that our system arises from the basic modal logic by the addition of an Aristotelian principle.

6. *Matrix of the Ł-modal system.* We get an adequate matrix of the Ł-modal system by "multiplying" the matrices M_7 and M_8, both identical with the adequate matrix M_1 of the two-valued calculus, but with different figures as elements in order to avoid misunderstandings. The

figures 5 and 7 marked by an asterisk are the selected elements, i.e., the asserted values, 6 and 8 are rejected.

C	5	6	N
*5	5	6	6
6	5	5	5

M_7

C	7	8	N
*7	7	8	8
8	7	7	7

M_8

The process of multiplication can be described as follows:

First, we form ordered pairs of elements of both matrices by combining an element of M_7 with an element of M_8; we get thus four combinations:

*(5, 7), (5, 8), (6, 7), (6, 8).

These combinations are the elements of the new matrix. The selected element is (5, 7), as 5 and 7 are the selected elements of the original matrices.

Secondly, we determine the truth-values of the functions C, N and Δ by means of the following equalities (a, b, d represent the elements of M_7, x, y, z the elements of M_8):

6.1 $C(a, x)(b, y) = (Cab, Cxy)$,

6.2 $N(a, x) = (Na, Nx)$,

6.3 $\Delta(a, x) = (a, Cxx)$.

Substituting for a and b the values 5 and 6, for x and y the values 7 and 8, and evaluating the functions on the right according to the matrices M_7 and M_8, e.g.: $C(6, 7)(6, 8) = (C66, C78) = (5, 8)$, we get from these equalities the following matrix M_9:

C	(5, 7)	(5, 8)	(6, 7)	(6, 8)	N	Δ
*(5, 7)	(5, 7)	(5, 8)	(6, 7)	(6, 8)	(6, 8)	(5, 7)
(5, 8)	(5, 7)	(5, 7)	(6, 7)	(6, 7)	(6, 7)	(5, 7)
(6, 7)	(5, 7)	(5, 8)	(5, 7)	(5, 8)	(5, 8)	(6, 7)
(6, 8)	(5, 7)	(5, 7)	(5, 7)	(5, 7)	(5, 7)	(6, 7)

M_9

M_9 is an adequate matrix of the system, i.e., it verifies all its formulae and no other formulae besides. This can be seen by the following consideration.

First, M_9 verifies the axioms of the C–N-calculus, $CCpqCCqrCpr$, $CCNppp$, and $CpCNpq$. Putting in these axioms for the variables arbitrary elements of the matrix, (a, x), (b, y), and (d, z), we get by means of 6.1 and 6.2 the following equalities:

6.4 $CC(a, x)(b, y)CC(b, y)(d, z)C(a, x)(d, z)$
$= C(Cab, Cxy)C(Cbd, Cyz)(Cad, Cxz)$
$= C(Cab, Cxy) = (CCbdCad, CCyzCxz)$
$= (CCabCCbdCad, CCxyCCyzCxz) = (5, 7)$.

6.5 $CCN(a, x)(a, x)(a, x) = CC(Na, Nx)(a, x)(a, x)$
$= C(CNaa, CNxx)(a, x)(CCNaaa, CCNxxx) = (5, 7)$.

6.6 $C(a, x)CN(a, x)(b, y) = C(a, x)C(Na, Nx)(b, y)$
$= C(a, x)(CNab, CNxy)(CaCNab, CxCNxy) = (5, 7)$.

The final result in all cases is the selected element $(5, 7)$ of M_9, as $CCabCCbdCad$, for instance, always gives 5 according to M_7, and $CCxyCCyzCxz$ always gives 7 according to M_8.

Secondly, M_9 verifies the axioms 2, 3, and 4. We have by 6.1 and 6.3:

6.7 $C(a, x)\Delta(a, x) = C(a, x)(a, Cxx) = (Caa, CxCxx) = (5, 7)$.

6.8 $C\Delta(a, x)(a, x) = C(a, Cxx)(a, x) = (Caa, CCxxx)$,

which gives $(5, 8)$ for $x/8$, a rejected element of M_9.

6.9 $\Delta(a, Cxx)$,

which gives $(6, 7)$ for $a/6$, again a rejected element of M_9.

Thirdly, in order to prove that $C\delta pC\delta Np\delta q$ is verified by M_9, it suffices to show that the principle of extensionality $CEpqC\delta p\delta q$ or $CCpqCCqpC\delta p\delta q$ is verified by M_9 for all functors of one argument definable by M_9. There are 16 such functors, as we can combine in 16 ways the four functions of M_7, V (verum), S (assertion), N (negation), and F (falsum) with the analogous four functions of M_8, e.g., (Va, Nx), (Sa, Fx), and so on. All these functions, however, are reducible to C–N-formulae, because $Va = Caa$, $Sa = a$, $Fa = NCaa$, and likewise $Vx = Cxx$, $Sx = x$ and $Fx = NCxx$. By substituting, therefore, the new functors for δ we get C–N-formulae, and all such formulae are verified by M_9. Take, for example, the principle of extensionality for Δ:

6.10 $CC(a, x)(b, y)CC(b, y)(a, x)C\Delta(a, x)\Delta(b, y)$
$= C(Cab, Cxy)C(Cba, Cyx)C(a, Cxx)(b, Cyy)$
$= C(Cab, Cxy)C(Cba, Cyx)(Cab, CCxxCyy)$
$= C(Cab, Cxy)(CCbaCab, CCyxCCxxCyy)$
$= (CCabCCbaCab, CCxyCCyxCCxxCyy) = (5, 7)$.

It follows from this consideration that all the formulae of our modal logic based on the axioms 1–4 are verified by the matrix M_9. It also follows that no other formulae besides can be verified by M_9; this results from the fact that the classical C–N-propositional calculus, to which all the formulae of our modal logic are matrically reducible, is "saturated", i.e., any formula must be either asserted on the ground of its asserted axioms, or rejected on the ground of the axiom of rejection $\dashv p$, which easily follows by substitution from our axiom 3 or 4. M_9, therefore, is an adequate matrix of the Ł-modal logic.

Let us now write, for the sake of abbreviation, 1 for $(5, 7)$, 2 for $(5, 8)$ 3 for $(6, 7)$, and 4 for $(6, 8)$; we get from M_9 the matrix M_6 which is the adequate matrix of our modal system in its simplest form.

C	1	2	3	4	N	Δ
*1	1	2	3	4	4	1
2	1	1	3	3	3	1
3	1	2	1	2	2	3
4	1	1	1	1	1	3

M_6

7. *The twin possibilities.* A curious logical fact is connected with the definition of Δ, which, as far as I know, has not yet been observed. The formulae with Δ are obviously a product of formulae verified by S (assertion) and V (verum). $Cp\Delta p$ is asserted because it is asserted for $\Delta = S$ and $\Delta = V$. $C\Delta pp$ and Δp are rejected because the first formula is rejected for $\Delta = V$, and the second for $\Delta = S$. Now we can obtain a product of S and V by multiplying S by V, getting thus the function $\Delta(a, x) = (Sa, Vx) = (a, Cxx)$, or by multiplying V by S getting $(Va, Sx) = (Caa, x)$. Let us denote this latter function by an inverted Δ:

7.1 $\nabla(a, x) = (Caa, x)$.

From 6.1, 6.2, and 7.1 there results the following matrix:

C	(5, 7)	(5, 8)	(6, 7)	(6, 8)	N	∇
*(5, 7)	(5, 7)	(5, 8)	(6, 7)	(6, 8)	(6, 8)	(5, 7)
(5, 8)	(5, 7)	(5, 7)	(6, 7)	(6, 7)	(6, 7)	(5, 8)
(6, 7)	(5, 7)	(5, 8)	(5, 7)	(5, 8)	(5, 8)	(5, 7)
(6, 8)	(5, 7)	(5, 7)	(5, 7)	(5, 7)	(5, 7)	(5, 8)

M_{10}

I shall now abbreviate this matrix by replacing the pairs of elements by single figures. As it does not matter which figures we choose, let us write 1 for (5, 7), 2 for (6, 7), 3 for (5, 8), and 4 for (6, 8). We get the matrix M_{6a} which is identical with M_6, as we can easily see by inter-

C	1	3	2	4	N	∇
*1	1	3	2	4	4	1
3	1	1	2	2	2	3
2	1	3	1	3	3	1
4	1	1	1	1	1	3

M_{6a}

changing the middle lines and columns. Consequently, M_{10} is identical with M_9, and the functor ∇ defined in this way is identical with the functor Δ.

We encounter here a logical paradox: although Δ and ∇ can be defined by the same matrix, they are not identical. Let us apply to ∇ in M_{10} the abbreviation of M_9: 1 for (5, 7), 2 for (5, 8), 3 for (6, 7), and 4 for (6, 8): we get for C, N, and Δ the matrix M_6, and for ∇ the matrix M_{11},

	∇
1	1
2	2
3	1
4	2

M_{11}

which is different from Δ. Δ and ∇ are undistinguishable when they occur separately, but their difference appears at once when they occur in the same formula. They are like twins who cannot be distinguished when met separately, but are instantly recognized as two when seen together. Take, for instance, the formulae $\Delta\Delta p$, $\nabla\nabla p$, $\Delta\nabla p$ and $\nabla\Delta p$. $\Delta\Delta p$ is equivalent to Δp which is rejected, and likewise $\nabla\nabla p$ is equivalent to ∇p which is rejected too. But $\nabla\Delta p$ and $\Delta\nabla p$ must be asserted according to M_6 and M_{11}. We cannot, therefore, replace in the two last formulae Δ by ∇, or vice versa, although both functors can be defined by the same matrix.

In the two-valued logic the asserted value, denoted by 1, is called "truth", the rejected value, denoted usually by 0, "falsity". When I

had discovered in 1920 a three-valued system of logic, I called the third value, which I denoted by $^1/_2$, "possibility".[15] Later on, after having found my n-valued modal systems, I thought that only two of them may be of philosophical importance, viz., the 3-valued and the $\aleph_0$-valued system. For we can assume, I argued, that either possibility has no degrees at all, getting thus the 3-valued system, or that it has infinitely many degrees, as in the theory of probabilities, and then we have the $\aleph_0$-valued system.[15] This opinion, as I see it today, was wrong. The Ł-modal logic is a 4-valued system with two values, 2 and 3, denoting possibility, but nevertheless, both values represent one and the same possibility in two different shapes. The values 2 and 3 are playing in the system exactly the same role which can be seen by the following table of the 16 functions of one argument:

(A)	(B)	(D)	(G)		(H)
(a, x)	(5, 7) (5, 8) (6, 7) (6, 8)	1234	p	p	1234
(a, Nx)	(5, 8) (5, 7) (6, 8) (6, 7)	2143	$C\Delta NpNC\Delta pp$	$C\nabla pNC\nabla NpNp$	3412
$(a, \nabla x)$	(5, 7) (5, 7) (6, 7) (6, 7)	1133	Δp	$C\nabla pp$	1212
(a, Fx)	(5, 8) (5, 8) (6, 8) (6, 8)	2244	$N\Delta Np$	$NC\nabla NpNp$	3434
(Na, x)	(6, 7) (6, 8) (5, 7) (5, 8)	3412	$C\Delta pNC\Delta NpNp$	$C\nabla NpNC\nabla pp$	2143
(Na, Nx)	(6, 8) (6, 7) (5, 8) (5, 7)	4321	Np	Np	4321
$(Na, \nabla x)$	(6, 7) (6, 7) (5, 7) (5, 7)	3311	ΔNp	$C\nabla NpNp$	2121
(Na, Fx)	(6, 8) (6, 8) (5, 8) (5, 8)	4422	$N\Delta p$	$NC\nabla pp$	4343
$(\nabla a, x)$	(5, 7) (5, 8) (5, 7) (5, 8)	1212	$C\Delta pp$	∇p	1133
$(\nabla a, Nx)$	(5, 8) (5, 7) (5, 8) (5, 7)	2121	$C\Delta NpNp$	∇Np	3311
$(\nabla a, \nabla x)$	(5, 7) (5, 7) (5, 7) (5, 7)	1111	Cpp	Cpp	1111
$(\nabla a, Fx)$	(5, 8) (5, 8) (5, 8) (5, 8)	2222	$N\Delta NCpp$	$\nabla NCpp$	3333
(Fa, x)	(6, 7) (6, 8) (6, 7) (6, 8)	3434	$NC\Delta NpNp$	$N\nabla Np$	2244
(Fa, Nx)	(6, 8) (6, 7) (6, 8) (6, 7)	4343	$NC\Delta pp$	$N\nabla p$	4422
$(Fa, \nabla x)$	(6, 7) (6, 7) (6, 7) (6, 7)	3333	$\Delta NCpp$	$N\nabla NCpp$	2222
(Fa, Fx)	(6, 8) (6, 8) (6, 8) (6, 8)	4444	$NCpp$	$NCpp$	4444

The first column (A) represents the 16 functions, the second (B) contains their matrices for $a/5$, $x/7$; $a/5, x/8$; $a/6, x/7$; $a/6, x/8$, the

[15] See J. Łukasiewicz, "Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenkalküls," *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie* 23 (1930), cl. iii, pp. 65 ff. and 72 [pp. 165 ff. and 171 of this volume].

third (D) is the translation of these matrices according to the equalities: $(5, 7) = 1$, $(5, 8) = 2$, $(6, 7) = 3$, $(6, 8) = 4$, the fourth column (G) gives the formulae corresponding to (D) for $p = 1, 2, 3, 4$ according to the matrices M_6 and M_{11} (e.g., $C\Delta NpNC\Delta pp$ in the second line has the value 2 for $p/1$, 1 for $p/2$, 4 for $p/3$ and 3 for $p/4$), and the last column (H) is the translation of the matrices (B) according to the equalities: $(5, 7) = 1$, $(5, 8) = 3$, $(6, 7) = 2$ and $(6, 8) = 4$, ordered for $p = 1, 2, 3, 4$. We get the figures of the last column from the column (D) by writing 3 for 2 and 2 for 3, and then by interchanging the middle figures, e.g., from 2143 we get first 3142, and then 3412.

It can be easily seen that the figures of the last column are matrices of the corresponding formulae with ∇ and, if we define Δp as $C\nabla pp$, also of the formulae with Δ. Assuming 1133 as the matrix of ∇p we get, e.g., for $C\nabla pp$ the matrix 1212, because $C\nabla 11 = C11 = 1$, $C\nabla 22 = C12 = 2$, $C\nabla 33 = C33 = 1$, $C\nabla 44 = C34 = 2$, and defining Δp as $C\nabla pp$ we have for $C\Delta pp$ the matrix 1133, because $C\Delta 11 = C11 = 1$, $C\Delta 22 = C22 = 1$, $C\Delta 33 = C13 = 3$, and $C\Delta 44 = C24 = 3$. Now, the formulae with ∇ and their corresponding matrices of column (H) are identical with the formulae with Δ and their corresponding matrices of column (D), as ∇ and Δ are identical functors. We see, therefore, that we get the same formulae by interchanging 2 and 3, and that these twin values of possibility play in the system the same role.

It also follows from the table that although all those 16 functions are reducible to the C–N-system by the matrices (B), the functions corresponding to the abbreviated matrices (D) and (H) cannot be defined in this way. The modal functor Δ or its twin ∇ is necessary and sufficient to represent them together with C and N.

8. *Some formulae of the Ł-modal logic.* The classical system of the propositional calculus extended by the addition of variable functors is not yet universally known. This system, inspired by Leśniewski's "Protothetic", was modified by myself by introduction of the rule of δ-substitution. Owing to this rule, we get easy and elegant proofs. By means of them I deduce in the Appendix from axiom 1 first the three axioms of the C–N-calculus, viz.:

10 $\vdash CpCNpq$ the principle of Duns Scotus,
20 $\vdash CCNppp$ the principle of Clavius,
22 $\vdash CCpqCCqrCpr$ the principle of the syllogism,

and then some other formulae without and with δ needed for the modal logic. Among the latter formulae the most important are the following ones: The principle of extensionality,

27 $\vdash CCpqCCqpC\delta p\delta q$

or

73 $\vdash CEpqC\delta p\delta q$,

the principles of δ-distribution with respect to C and A,

30 $\vdash C\delta CpqC\delta p\delta q$

and

71 $\vdash C\delta ApqA\delta p\delta q$,

and a principle of conjunction,

60 $\vdash C\delta pC\delta q\delta Kpq$.

From these auxiliary theses a considerable number of Δ- and Γ-formulae are derived in the Appendix. Here is an account of the most important of them.

(a) The basic modal logic is a part of our system. Three formulae of this system:

2 $\vdash Cp\Delta p$,

3 $\dashv C\Delta pp$

and

4 $\dashv \Delta p$,

are taken as axioms, and the remaining five:

129 $\vdash C\Gamma pp$,

156 $\dashv Cp\Gamma p$,

157 $\dashv N\Gamma p$,

124 $\vdash E\Gamma pN\Delta Np$

and

128 $\vdash E\Delta pN\Gamma Np$,

are proved as consequences.

(b) There are only four modal functors of one argument in the system, viz., $\Delta p\ (= N\Gamma Np)$, $N\Delta p\ (= \Gamma Np)$, $\Delta Np\ (= N\Gamma p)$, and $N\Delta Np\ (= \Gamma p)$.

This easily results from two principles of reduction for Δ:

94 $\quad \vdash E\Delta\Delta p\Delta p$

and

98 $\quad \vdash E\Delta N\Delta p\Delta Np$.

The corresponding principles for Γ run:

136 $\quad \vdash E\Gamma\Gamma p\Gamma p$

and

141 $\quad \vdash E\Gamma N\Gamma p\Gamma Np$.

It should be stressed that according to these principles a problematic proposition is equivalent to a problematic one, and an apodeictic proposition to an apodeictic one.

(c) There are three principles of Δ-distribution for C, K, and A (a fourth one, for E, is easily deducible from the first):

84 $\quad \vdash E\Delta CpqC\Delta p\Delta q$,
109 $\quad \vdash E\Delta KpqK\Delta p\Delta q$,
114 $\quad \vdash E\Delta ApqA\Delta p\Delta q$;

and two principles of Γ-distribution for K and A:

149 $\quad \vdash E\Gamma KpqK\Gamma p\Gamma q$

and

154 $\quad \vdash E\Gamma ApqA\Gamma p\Gamma q$.

The principle of Γ-distribution for C is not valid, because the formula:

162 $\quad \dashv CC\Gamma p\Gamma q\Gamma Cpq$

is rejected.

(d) No apodeictic proposition, i.e., no proposition beginning with Γ or with $N\Delta$, can be asserted in the system. This follows from the formulae:

134 $\quad \vdash C\Gamma qCp\Gamma p$

and

101 $\quad \vdash CN\Delta qC\Delta pp$.

Both formulae are asserted, but their consequents $Cp\Gamma p$ and $C\Delta pp$ are rejected; their antecedents, therefore, Γq and $N\Delta q$, must be rejected too. Now, from the rejected formulae Γq or $N\Delta q$ nothing can be got by substitution, not even the formula:

160 $\quad \dashv \Gamma Cpp$;

on the contrary, from 160 there results by substitution $\dashv \Gamma q$. On the other side, it is obvious that in the asserted formulae 134 and 101 any expression whatever may be put for q, and all formulae got in this way from Γq and $N\Delta q$ must be rejected. In order to express the fact that any proposition beginning with Γ or $N\Delta$ should be rejected, I employ Greek variables, calling them "interpretation-variables" in opposition to the "substitution-variables" denoted by Latin letters. We have, therefore:

$$\dashv \Gamma\alpha \quad \text{and} \quad \dashv N\Delta\alpha,$$

where α may be any formula, i.e., any significant expression of the system.

(e) On the contrary, our system contains many asserted problematic propositions. It follows from axiom 2 that if α is an asserted proposition then the proposition "It is possible that α" must be asserted too. We have, for instance:

102 $\quad \vdash \Delta Cpp$.

There are besides Δ-formulae whose argument is rejected, e.g.:

92 $\quad \vdash \Delta C\Delta pp$.

This problematic proposition is asserted, although its argument $C\Delta pp$ is rejected. Another interesting example is given by the

163 $\quad \vdash \Delta\nabla p$.

It is most difficult to express this formula in the ordinary language. Both Δ and ∇ may be rendered by the phrase "it is possible that", as both have exactly the same meaning. Nevertheless, they are different, and we cannot say "It is possible that it is possible that p", because this may have the meaning $\Delta\Delta p$, and $\Delta\Delta p$ cannot be asserted being equivalent to Δp.

The list of modal formulae given in the Appendix should be completed by modal formulae with δ. So, for instance, it can be proved that the following formulae are asserted:

$$\Delta C\delta\Delta p\delta p, \ \Delta C\delta p\delta\Delta p, \ \Delta C\delta p\delta\Gamma p, \ \Delta C\delta\Gamma p\delta p, \ \Delta E\delta p\delta\Delta p, \ \Delta E\delta p\delta\Gamma p.$$

All such formulae are put off to a further investigation.

9. *Some controversial problems.* Hitherto, the best known systems of modal logic are originated by C. I. Lewis. It is difficult to compare

my own modal logic with them, as they are based on the so-called "strict implication" which is stronger than the "material implication" employed by myself. I shall compare, therefore, my system with the systems of G. H. Von Wright, which are also based on the material implication and are equivalent, according to its author, to some systems of Lewis.

There are three modal systems presented by Von Wright in axiomatic form, and called by him M, M', and M''.[16]) All are based on the classical calculus of propositions and on two additional rules of transformation:

9.1 The "Rule of Extensionality": "If $f_1 \leftrightarrow f_2$ is provable, then $Mf_1 \leftrightarrow Mf_2$ is also provable." That means in my symbolism:

$$\vdash E\alpha\beta \rightarrow \vdash E\Delta\alpha\Delta\beta \quad (\text{“}M\text{”} = \text{“}\Delta\text{”}).$$

9.2 The "Rule of Tautology". "If f is provable, then Nf is provable". That means:

$$\vdash \alpha \rightarrow \vdash \Gamma\alpha \quad (\text{“}N\text{”} = \text{“}\Gamma\text{”}).$$

System M is established on two modal axioms:

9.3 $a \rightarrow Ma$ the "Axiom of Possibility",

which corresponds to our asserted formula 2 $Cp\Delta p$, and

9.4 $M(a \vee b) \leftrightarrow Ma \vee Mb$ the "Axiom of Distribution",

which corresponds to our asserted formula 114 $E\Delta Apq A\Delta p\Delta q$.

System M' arises from M by addition of the "First Axiom of Reduction":

9.5 $MMa \rightarrow Ma$,

which corresponds to our asserted formula 93 $C\Delta\Delta p\Delta p$, and M'' is got by addition of the "Second Axiom of Reduction":

9.6 $M \sim M\alpha \rightarrow \sim M\alpha$,

which corresponds to our rejected formula 121 $C\Delta N\Delta pN\Delta p$.

This last axiom gives, together with its converse formula $CN\Delta p\Delta N\Delta p$ (which results from $Cp\Delta p$ by the substitution $p/N\Delta p$), the equivalence $E\Delta N\Delta pN\Delta p$. Here a problematic proposition $\Delta N\Delta p$ appears to be equivalent to an apodeictic proposition $N\Delta p$, which is against our logical intuitions. The author himself seems to be doubtful about this

[16]) See l. c. p. 84, 85.

axiom. I think that it should be rejected, and the system M'' is not acceptable.

The systems M and M' are clearly incomplete, as 9.6, not being inconsistent with them, does not follow from them. The consistency of 9.6 with the rest of the system results from the one fact, among others, that for the interpretation $Ma = a$ all the axioms and rules remain valid. It may be added that in this case the system ceases to be a modal logic. As Von Wright does not accept rejection, I do not know how he can disprove the fromula $Ma \rightarrow a$ ($= C\Delta pp$) on the ground of his axiomatic system.

All his other axioms and rules, i.e., 9.1, 9.3, 9.4, and 9.5 are valid in my Ł-modal logic, except rule 9.2. This controversial rule, first stated by Aristotle, but not exactly enough, was the cause of many philosophical and theological discussions.[17]) After a long, but—in my opinion—unconvincing argumentation Von Wright says: "... the proposition that a tautology is necessary and a contradiction impossible are truths of logic. This certainly agrees with our logical intuitions."[18]) I am not certain that it does agree. I think, roughly speaking, that true propositions are simply true without being necessary, and false propositions are simply false without being impossible. This certainly does not hurt our logical intuitions, and may settle many controversies.

It may be asked, however: Why should we introduce necessity and impossibility into logic if true apodeictic propositions do not exist? I reply to this objection that we are primarily interested in problematic propositions of the form $\Delta\alpha$ and $\Delta N\alpha$, which may be true and useful, although their arguments are rejected, and introducing problematic propositions we cannot omit their negations, i.e., apodeictic propositions, as both are inextricably connected with each other.

The second controversial problem concerns the formula 108 $\vdash CK\Delta p\Delta q\Delta Kpq$. In some of his systems Lewis accepts the formula 106 $\vdash C\Delta Kpq K\Delta p\Delta q$, but rejects its converse 108 by the following argument: "If it is possible that p and q be both true, then p is possible and q

[17]) In an essay on Aristotle's Modal Logic, which will be published elsewhere, I am expounding at length the Aristotelian opinions on this subject. [See J. Łukasiewicz, "On Controversial Problem of Aristotle's Modal Syllogistic", *Dominican Studies* 7 (1954), pp. 114–128.]

[18]) See l. c., pp. 14, 15.

is possible. This implication is not reversible. For example: it is possible that the reader will see this at once. It is also possible that he will not see it at once. But it is not possible that he will both see it at once and not see it at once."[19]) As this argument is stated in words, and not in symbols, it is equally applicable to the strict as to the material implication. But its evidence is illusive. What is meant by "the reader"? If an individual reader, say R, is meant, then R either will see this at once, or R will not see this at once. In the first case, the proposition, "It is possible that R will see this at once," is true; but how can it be proved that R will possibly not see this at once? In the second case, the proposition, "It is possible that R will not see this at once", is true; but how can it be proved that R will possibly see this at once? The two premisses of the formula 108 are not both provable, and the formula cannot be refuted in this way.

Take another example. Let n be a positive integer. I contend that the following implication is true for all values of n: "If it is possible that n is even, and it is possible that n is not even, then it is possible that n is even and n is not even." If $n = 4$, it is true that n is possibly even, but it is not true that n is possibly not even; if n is 5, it is true that n is possibly not even, but it is not true that n is possibly even. The both premisses are never true together, and the formula cannot be refuted.

If again by "the reader" some reader is meant, then the propositions, "It is possible that some reader will see this at once", and "It is possible that some reader will not see this at once", may be both true, but in this case the consequent, "It is possible that some reader will see this at once and some reader will not see this at once", is obviously also true. It is, of course, not the same reader who will possible see this and possibly not see this at once. I cannot find an example which would refute formula 108; on the contrary, all seem to support its correctness.

I am fully aware that other systems of modal logic are possible based on different concepts of necessity and possibility. I firmly believe that we shall never be able to decide which of them is true. Systems of logic are instruments of thought, and the more useful a logical system is, the more valuable it is. I hope that the Ł-modal system expounded

[19]) See C. I. Lewis and C. H. Langford, *Symbolic Logic*, New York and London, 1932, p. 167.

above will be a useful instrument, and deserves a further investigation and development.

APPENDIX

Examples of δ-substitution and δ-definitions.

(a) Proof-line of formula 16:

$$1\,\delta/CpC\text{'}p \times C15\,q/p - C10\,q/p - 16.$$

Write $CpC\text{'}p$ instead of the δ's filling up the gaps marked by the apostrophe with the arguments of δ. You get thus from

$$C\delta pC\delta Np\delta q$$

the formula

$$CCpCppCCpCNppCpCqp,$$

from which there follows by two detachments $CpCqp$.

(b) Proof-line of formula 10:

$$1\,\delta/\text{'} \times 10.$$

Cancel simply the δ's in 1.

(c) Proof-line of formula 13:

$$1\,\delta/C\text{''}, p/CpCNpNq,\ q/p \times C11 - C12 - 13.$$

Perform first the substitutions for the propositional variables

$$C\delta CpCNpNqC\delta NCpCNpNq\delta p,$$

and write instead of the δ's their arguments α in form of $C\alpha\alpha$:

$$C\underbrace{CCpCNpNqCpCNpNq}C\underbrace{CNCpCNpNqNCpCNpNq}\underbrace{Cpp}$$

which is $C11 - C12 - 13$.

(d) All δ-definitions have the form $C\delta P\delta Q$, where P and Q are the *definiens* and the *definiendum*. P may be replaced everywhere by Q. Take as an example the proof-line of formula 55:

$$5\,\delta/C\text{'}p \times C52 - 55.$$

By replacing $NCpNq$ by Kpq according to example (a) we get from 52 formula 55.

The numbers in brackets after a formula F refer to formulae to which F is applied. For instance, 3 is applied to 118.

Axioms

1 $\vdash C\delta pC\delta Np\delta q$ (10, 11, 13, 15, 16, 18, 19, 22, 23, 25, 27, 29, 30, 38, 44, 48, 50, 60, 70).

2 $\vdash Cp\Delta p$ (74, 75, 78, 90, 94, 102).

3 $\dashv C\Delta pp$ (118).

4 $\dashv \Delta p$ (115).

Definitions

Df $Kpq \times 5$.
5 $\vdash C\delta NCpNq\delta Kpq$ (55, 56, 57).

Df $Apq \times 6$.
6 $\vdash C\delta CNpq\delta Apq$ (66, 67, 68, 69, 71).

Df $Epq \times 7$.
7 $\vdash C\delta KCpqCqp\delta Epq$ (72, 73).

Df $\Gamma p \times 8$.
8 $\vdash C\delta N\Delta Np\delta\Gamma p$ (123, 124, 126, 129, 131, 132).

Df $\nabla p \times 9$.
9 $\vdash C\delta C\Delta pp\delta\nabla p$ (163).

Consequences of Axiom 1

$1\ \delta/{}' \times 10$.
10 $\vdash CpCNpq$ (11, 12, 14, 16, 17, 28, 35, 66, 139).

$1\ \delta/CpCNp',\ p/q,\ q/Nq \times C10 - 11$.
11 $\vdash CCpCNpNqCpCNpNq$ (13).

$10\ p/CpCNpNq,\ q/NCpCNpNq \times C10\ q/Nq - 12$.
12 $\vdash CNCpCNpNqNCpCNpNq$ (13).

$1\ \delta/C'',\ p/CpCNpNq,\ q/p \times C11 - C12 - 13$.
13 $\vdash Cpp$ (14, 15, 18, 19, 20, 23, 32, 34, 45, 102, 123, 135, 155).

$10\ p/Cpp \times C13 - 14$.
14 $\vdash CNCppq$ (15).

$1\ \delta/C'Cpp,\ p/Cpp \times C13\ p/Cpp - C14\ q/Cpp - 15$.
15 $\vdash CqCpp$ (16, 23, 26, 30, 38, 47).

$1\ \delta/CpC'p \times C15\ q/p - C10\ q/p - 16$.
16 $\vdash CpCqp$ (17, 21, 22, 24, 26, 49, 53, 59, 67, 79, 91, 133).

$16\ p/CqCNqr,\ q/CNqCqr \times C10\ p/q,\ q/r - 17$.
17 $\vdash CCNqCqrCqCNqr$ (18).

$1\ \delta/CC'CqrCqC'r,\ p/q,\ q/p \times C13\ p/CqCqr - C17 - 18$.
18 $\vdash CCpCqrCqCpr$ (20, 32, 33, 34, 35, 51, 63, 100, 134).

$1\ \delta/C'p \times C13 - 19$.
19 $\vdash CCNppCqp$ (20, 21).

$18\ p/CNpp,\ q/Cpp,\ r/p \times C19\ q/Cpp - C13 - 20$.
20 $\vdash CCNppp$ (24, 28, 36).

$16\ p/CCNrrCpr,\ q/CpNr \times C19\ p/r,\ q/p - 21$.
21 $\vdash CCpNrCCNrrCpr$ (22).

$1\ \delta/CCp'CC'rCpr,\ p/r \times C16\ p/Cpr,\ q/Crr - C21 - 22$.
22 $\vdash CCpqCCqrCpr$ (33, 36, 50, 62, 64, 75, 78, 81, 85, 86, 90, 99, 101, 133, 139, 142).

$1\ \delta/CCpNpC'Np \times C13\ p/CpNp - C15\ q/CpNp,\ p/Np - 23$.
23 $\vdash CCpNpCqNp$ (25, 34).

$16\ p/CCNppp \times C20 - 24$.
24 $\vdash CqCCNppp$ (25).

$1\ \delta/CCpNpCCNpp' \times C24\ q/CpNp - C23\ q/CNpp - 25$.
25 $\vdash CCpNpCCNppq$ (27).

$16\ p/CCppC\delta p\delta p,\ q/Cpp \times C15\ q/Cpp,\ p/\delta p - 26$.
26 $\vdash CCppCCppC\delta p\delta p$ (27).

$1\ \delta/CCp'CC'pC\delta p\delta' \times C26 - C25\ q/C\delta p\delta Np - 27$.
27 $\vdash CCpqCCqpC\delta p\delta q$ (28, 40, 58, 65).

$27\ q/CNpp \times C10\ q/p - C20 - 28$.
28 $\vdash C\delta p\delta CNpp$ (29, 45, 47, 49).

$28\ \delta/C\delta' C\delta Nq\delta q,\ p/q \times C1\ p/q - 29$.
29 $\vdash C\delta CNqqC\delta Nq\delta q$ (30).

$1\ \delta/C\delta C'qC\delta'\delta q,\quad p/q,\quad q/p \times C15\ q/\delta Cqq,\quad p/\delta q - C29 - 30$.
30 $\vdash C\delta CpqC\delta p\delta q$ (31, 77, 130).

30 δ/Cp', p/q, q/r × 31.
31 ⊢ CCpCqrCCpqCpr (63, 135).

18 p/Cpq, q/p, r/q × C13 p/Cpq − 32.
32 ⊢ CpCCpqq (103, 155).

18 p/Cpq, q/Cqr, r/Cpr × C22 − 33.
33 ⊢ CCqrCCpqCpr (37, 143).

18 p/CpNp, q/Cpp, r/Np × C23 q/Cpp − C13 − 34.
34 ⊢ CCpNpNp (54).

18 q/Np, r/q × C10 − 35.
35 ⊢ CNpCpq (36, 52, 80, 90, 99).

22 p/NNp, q/CNpp, r/p × C35 p/Np, q/p − C20 − 36.
36 ⊢ CNNpp (37, 39, 40, 88, 117).

33 q/NNp, r/p, p/q × C36 − 37.
37 ⊢ CCqNNpCqp (38).

1 δ/CCNpN'C'p × C15 q/CNpNp − C37 q/Np − 38.
38 ⊢ CCNpNqCqp (39, 41, 42, 50, 101, 158).

38 p/NNp, q/p × C36 p/Np − 39.
39 ⊢ CpNNp (40, 87).

27 p/NNp, q/p × C36 − C39 − 40.
40 ⊢ CδNNpδp (41, 42, 43, 76, 83, 128).

40 δ/CCNp'CNqp, p/q × C38 q/Nq − 41.
41 ⊢ CCNpqCNqp (43, 52, 53, 126).

40 δ/CC'NqCqNp × C38 p/Np − 42.
42 ⊢ CCpNqCqNp (54, 121, 125).

40 δ/CC'qCNqNp × C41 p/Np − 43.
43 ⊢ CCpqCNqNp (74, 85, 86, 96, 137).

1 δ/C'q, q/r × 44.
44 ⊢ CCpqCCNpqCrq (46, 48).

28 δ/Cδ'δp × C13 p/δp − 45.
45 ⊢ CδCNppδp (46).

45 δ/CCpqCCNpq', p/q × C44 r/Nq − 46.
46 ⊢ CCpqCCNpqq (92, 140).

28 δ/CCprCCprC'r × C15 q/Cpr, p/Cpr − 47.
47 ⊢ CCprCCprCCNppr (48).

1 δ/CCprCC'rCCNp'r × C47 − C44 q/r, r/CNpNp − 48.
48 ⊢ CCprCCqrCCNpqr (68, 82).

28 δ/Cδ'CNδpδp × C16 p/δp, q/Nδp − 49.
49 ⊢ CδCNppCNδpδp (69).

22 p/Nδp, q/CNδNpNδq, r/CδqδNp × C1 δ/Nδ' − C38 p/δNp, q/δq − 50.
50 ⊢ CNδpCδqδNp (51, 140).

18 p/Nδp, q/δq, r/δNp × C50 − 51.
51 ⊢ CδqCNδpδNp (70).

41 q/CpNq × C35 q/Nq − 52.
52 ⊢ CNCpNqp (55).

41 p/q, q/CpNq × C16 p/Nq, q/p − 53.
53 ⊢ CNCpNqq (56).

42 p/CpNp, q/p × C34 − 54.
54 ⊢ CpNCpNp (57).

5 δ/C'p × C52 − 55.
55 ⊢ CKpqp (58, 62, 104, 144).

5 δ/C'q × C53 − 56.
56 ⊢ CKpqq (63, 105, 145).

5 δ/Cp', q/p × C54 − 57.
57 ⊢ CpKpp (58).

27 q/Kpp × C57 − C55 q/p − 58.
58 ⊢ CδpδKpp (59).

16 p/CδpδKpp, q/δp × C58 − 59.
59 ⊢ CδpCδpδKpp (60).

1 δ/CδpCδ'δKp' × C59 − C1 q/KpNp − 60.
60 ⊢ CδpCδqδKpq (61, 72, 107, 147).

60 δ/Cp', p/q, q/r × 61.
61 ⊢ CCpqCCprCpKqr (106, 146).

22 $p/Kpq, q/p \times C55 - 62$.
62 $\vdash CCprCKpqr$ (64).

18 $p/CKpqCqr, q/CKpqq, r/CKpqr \times C31p/Kp - qC56 - 63$.
63 $\vdash CCKpqCqrCKpqr$ (64).

22 $p/CpCqr, q/CKpqCqr, r/CKpqr \times C62\, r/Cqr - C63 - 64$.
64 $\vdash CCpCqrCKpqr$ (65, 108, 148).

64 $p/Cpq, q/Cqp, r/C\delta p\delta q \times C27 - 65$.
65 $\vdash CKCpqCqpC\delta p\delta q$ (73).

6 $\delta/Cp' \times C10 - 66$.
66 $\vdash CpApq$ (110, 150).

6 $\delta/Cq' \times C16\, p/q, q/Np - 67$.
67 $\vdash CqApq$ (111, 151).

6 $\delta/CCprCCqrC'r \times C48 - 68$.
68 $\vdash CCprCCqrCApqr$ (112, 152).

6 $\delta/C\delta' CN\delta p\delta p, q/p \times C49 - 69$.
69 $\vdash C\delta AppCN\delta p\delta p$ (70).

1 $\delta/C\delta Ap' CN\delta p\delta' \times C69 - C51\, q/ApNp - 70$.
70 $\vdash C\delta ApqCN\delta p\delta q$ (71).

6 $\delta/C\delta Apq', p/\delta p, q/\delta q \times C70 - 71$.
71 $\vdash C\delta ApqA\delta p\delta q$ (113, 153).

7 $\delta/CCpqCCqp' \times C60\delta/', p/Cpq, q/Cqp - 72$.
72 $\vdash CCpqCCqpEpq$ (84, 89, 94, 98, 109, 114, 124, 127, 136, 141, 149, 151).

7 $\delta/C'C\delta p\delta q \times C65 - 73$.
73 $\vdash CEpqC\delta p\delta q$.

Δ-*Formulae*

43 $q/\Delta p \times C2 - 74$.
74 $\vdash CN\Delta pNp$ (75, 76, 95).

22 $p/N\Delta p, q/Np, r/\Delta Np \times C74 - C2p/Np - 75$.
75 $\vdash CN\Delta p\Delta Np$ (81).

40 $\delta/CN\Delta Np' \times C74p/Np - 76$.
76 $\vdash CN\Delta Npp$ (129).

30 $\delta/\Delta' \times 77$.
77 $\vdash C\Delta CpqC\Delta p\Delta q$ (78, 84, 93, 97).

22 $p/Cpq, q/\Delta Cpq, r/C\Delta p\Delta q \times C2p/Cpq - C77 - 78$.
78 $\vdash CCpqC\Delta p\Delta q$ (79, 80, 85, 87, 88, 91, 95, 96, 104, 105, 110, 111).

78 $p/q, q/Cpq \times C16p/q, q/p - 79$.
79 $\vdash C\Delta q\Delta Cpq$ (82).

78 $p/Np, q/Cpq \times C35 - 80$.
80 $\vdash C\Delta Np\Delta Cpq$ (81).

22 $p/N\Delta p, q/\Delta Np, r/\Delta Cpq \times C75 - C80 - 81$.
81 $\vdash CN\Delta p\Delta Cpq$ (82).

48 $p/N\Delta p, r/\Delta Cpq, q/\Delta q \times C81 - C79 - 82$.
82 $\vdash CCNN\Delta p\Delta q\Delta Cpq$ (83).

40 $\delta/CC'\Delta q\Delta Cpq, p/\Delta p \times C82 - 83$.
83 $\vdash CC\Delta p\Delta q\Delta Cpq$ (84).

72 $p/\Delta Cpq, q/C\Delta p\Delta q \times C77 - C83 - 84$.
84 $\vdash E\Delta CCpqC\Delta p\Delta q$.

22 $p/Cpq, q/C\Delta p\Delta q, r/CN\Delta qN\Delta p \times C78 - C43p/\Delta p, q/\Delta q - 85$.
85 $\vdash CCpqCN\Delta qN\Delta p$ (86, 99).

22 $p/Cpq, q/CNqNp, r/CN\Delta NpN\Delta Nq \times C43 - C85p/Nq, q/Np - 86$.
86 $\vdash CCpqCN\Delta NpN\Delta Nq$ (131).

78 $q/NNp \times C39 - 87$.
87 $\vdash C\Delta p\Delta NNp$ (89).

78 $p/NNp, q/p \times C36 - 88$.
88 $\vdash C\Delta NNp\Delta p$ (89, 115).

72 $p/\Delta p, q/\Delta NNp \times C87 - C88 - 89$.
89 $\vdash E\Delta p\Delta NNp$.

22 $p/N\Delta p, q/C\Delta pp, r/\Delta C\Delta pp \times C35p/\Delta p, q/p - C2p/C\Delta pp - 90$.
90 $\vdash CN\Delta p\Delta C\Delta pp$ (92).

78 q/CΔpp × C16q/Δp − 91.
91 ⊢ CΔpΔCΔpp (92).
46 p/Δp, q/ΔCΔpp × C91 − C90 − 92.
92 ⊢ ΔCΔpp (93, 96, 163).
77 p/Δp, q/p × C92 − 93.
93 ⊢ CΔΔpΔp (94).
72 p/ΔΔp, q/Δp × C93 − C2p/Δp − 94.
94 ⊢ EΔΔpΔp.
78 p/NΔp, q/Np × C74 − 95.
95 ⊢ CΔNΔpΔNp (98).
78 p/CΔpp, q/CNpNΔp × C43p/Δp, q/p − C92 − 96.
96 ⊢ ΔCNpNΔp (97).
77 p/Np, q/NΔp × C96 − 97.
97 ⊢ CΔNpΔNΔp (98).
72 p/ΔNΔp, q/ΔNp × C95 − C97 − 98.
98 ⊢ EΔNΔpΔNp.
22 p/Np, q/Cpq, r/CNΔqNΔp × C35 − C85 − 99.
99 ⊢ CNpCNΔqNΔp (100).
18 p/Np, q/NΔq, r/NΔp × C99 − 100.
100 ⊢ CNΔqCNpNΔp (101).
22 p/NΔq, q/CNpNΔp, r/CΔpp × C100 − C38q/Δp − 101.
101 ⊢ CNΔqCΔpp (118).
2 p/Cpp × C13 − 102.
102 ⊢ ΔCpp (103).
32 p/ΔCpp × C102 − 103.
103 ⊢ CCΔCppqq (119).
78 p/Kpq, q/p × C55 − 104.
104 ⊢ CΔKpqΔp (106).
78 p/Kpq × C56 − 105.
105 ⊢ CΔKpqΔq (106).
61 p/Kpq, q/Δp, r/Δq × C104 − C105 − 106.
106 ⊢ CΔKpqKΔpΔq (109).

60 δ/Δ' × 107.
107 ⊢ CΔpCΔqΔKpq (108).
64 p/Δp, q/Δq, r/ΔKpq × C107 − 108.
108 ⊢ CKΔpΔqΔKpq (109).
72 p/ΔKpq, q/KΔpΔq × C106 − C108 − 109.
109 ⊢ EΔKpqKΔpΔq.
78 q/Apq × C66 − 110.
110 ⊢ CΔpΔApq (112).
78 p/q, q/Apq × C67 − 111.
111 ⊢ CΔqΔApq (112).
68 p/Δp, r/ΔApq, q/Δq × C110 − C111 − 112.
112 ⊢ CAΔpΔqΔApq (114).
71 δ/Δ' × 113.
113 ⊢ CΔApqAΔpΔq (114).
72 p/ΔApq, q/AΔpΔq × C113 − C112 − 114.
114 ⊢ EΔApqAΔpΔq.

* * *

88 × C115 − 4.
115 ⊣ ΔNNp (116).
115 × 116p/Np.
116 ⊣ ΔNp (117).
36 p/ΔNp × C117 − 116.
117 ⊣ NNΔNp (157).
101 q/NΔCpp × C118 − 3.
118 ⊣ NΔNΔCpp (119).
103 q/NΔNΔCpp × C119 − 118.
119 ⊣ CΔCppNΔNΔCpp (120).
119 × 120p/Cpp.
120 ⊣ CΔpNΔNΔp (121, 122).
42 p/ΔNΔp, q/Δp × C121 − 120.
121 ⊣ CΔNΔpNΔp.
120 × 122p/Δp.
122 ⊣ CpNΔNp (156).

Γ-Formulae

8 $\delta/C\delta'\delta N\Delta Np\times C13p/\delta N\Delta Np-123$.
123 $\vdash C\delta\Gamma p\delta N\Delta Np$ (124, 125, 156, 157).

72 $p/\Gamma p,\ q/N\Delta Np\times C123\,\delta/'-C8\,\delta/'-124$.
124 $\vdash E\Gamma pN\Delta Np$.

42 $p/\Gamma Np,\ q/\Delta NNp\times C123\,\delta/',\ p/Np-125$.
125 $\vdash C\Delta NNpN\Gamma Np$ (127).

41 $p/\Delta NNp,\ q/\Gamma Np\times C8\,\delta/',\ p/Np-126$.
126 $\vdash CN\Gamma Np\Delta NNp$ (127).

72 $p/\Delta NNp,\ q/N\Gamma Np\times C125-C126-127$.
127 $\vdash E\Delta NNpN\Gamma Np$ (128).

40 $\delta/E\Delta' N\Gamma Np\times C127-128$.
128 $\vdash E\Delta pN\Gamma Np$.

8 $\delta/C'p\times C76-129$.
129 $\vdash C\Gamma pp$ (136, 137, 142).

30 $\delta/\Gamma'\times 130$.
130 $\vdash C\Gamma CpqC\Gamma p\Gamma q$.

8 $\delta/CCpqC'N\Delta Nq\times C86-131$.
131 $\vdash CCpqC\Gamma pN\Delta Nq$ (132).

8 $\delta/CCpqC\Gamma p',\ p/q\times C131-132$.
132 $\vdash CCpqC\Gamma p\Gamma q$ (133, 138, 139, 144, 145, 150, 151).

22 $p/q,\ q/Cpq,\ r/C\Gamma p\Gamma q\times C16p/q,\ q/p-C132-133$.
133 $\vdash CqC\Gamma p\Gamma q$ (134, 135).

18 $q/\Gamma q,\ r/\Gamma p\times C133q/p,\ p/q-134$.
134 $\vdash C\Gamma qCp\Gamma p$ (160).

31 $p/\Gamma p,\ q/\Gamma p,\ r/\Gamma\Gamma p\times C133q/\Gamma p-C13p/\Gamma p-135$.
135 $\vdash C\Gamma p\Gamma\Gamma p$ (136).

72 $p/\Gamma\Gamma p,\ q/\Gamma p\times C129p/\Gamma p-C135-136$.
136 $\vdash E\Gamma\Gamma p\Gamma p$.

43 $p/\Gamma p,\ q/p\times C129-137$.
137 $\vdash CNpN\Gamma p$ (138).

132 $p/Np,\ q/N\Gamma p\times C137-138$.
138 $\vdash C\Gamma Np\Gamma N\Gamma p$ (141).

22 $p/\Gamma p,\ q/CN\Gamma pNp,\ r/C\Gamma N\Gamma p\Gamma Np\times C10p/\Gamma p,$
$q/Np-C132p/N\Gamma p,\ q/Np-139$.
139 $\vdash C\Gamma pC\Gamma N\Gamma p\Gamma Np$ (140).

46 $p/\Gamma p,\ q/C\Gamma N\Gamma p\Gamma Np\times C139-C50\,\delta/\Gamma',\ q/N\Gamma p-140$.
140 $\vdash C\Gamma N\Gamma p\Gamma Np$ (141, 142).

72 $p/\Gamma N\Gamma p,\ q/\Gamma Np\times C140-C138-141$.
141 $\vdash E\Gamma N\Gamma p\Gamma Np$.

22 $p/\Gamma N\Gamma p,\ q/\Gamma Np,\ r/Np\times C140-C129p/Np-142$.
142 $\vdash C\Gamma N\Gamma pNp$ (143).

33 $q/\Gamma N\Gamma p,\ r/Np,\ p/N\Gamma p\times C142-143$.
143 $\vdash CCN\Gamma p\Gamma N\Gamma pCN\Gamma pNp$ (159).

132 $p/Kpq,\ q/p\times C55-144$.
144 $\vdash C\Gamma Kpq\Gamma p$ (146).

132 $p/Kpq\times C56-145$.
145 $\vdash C\Gamma Kpq\Gamma q$ (146).

61 $p/\Gamma Kpq,\ q/\Gamma p,\ r/\Gamma q\times C144-C145-146$.
146 $\vdash C\Gamma KpqK\Gamma p\Gamma q$ (149).

60 $\delta/\Gamma'\times 147$.
147 $\vdash C\Gamma pC\Gamma q\Gamma Kpq$ (148).

64 $p/\Gamma p,\ q/\Gamma q,\ r/\Gamma Kpq\times C147-148$.
148 $\vdash CK\Gamma p\Gamma q\Gamma Kpq$ (149).

72 $p/\Gamma Kpq,\ q/K\Gamma p\Gamma q\times C146-C148-149$.
149 $\vdash E\Gamma KpqK\Gamma p\Gamma q$.

132 $q/Apq\times C66-150$.
150 $\vdash C\Gamma p\Gamma Apq$ (152).

132 $p/q,\ q/Apq\times C67-151$.
151 $\vdash C\Gamma q\Gamma Apq$ (152).

68 $p/\Gamma p,\ r/\Gamma Apq,\ q/\Gamma q\times C150-C151-152$.
152 $\vdash CA\Gamma p\Gamma q\Gamma Apq$ (154).

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$71\ \delta/\Gamma' \times 153.$

153 $\vdash C\Gamma ApqA\Gamma p\Gamma q$ (154).

$72\ p/\Gamma Apq,\ q/A\Gamma p\Gamma q \times C153 - C152 - 154.$

154 $\vdash E\Gamma ApqA\Gamma p\Gamma q.$

$32\ p/C\Gamma p\Gamma p,\ q/\Gamma Cpp \times C13p/\Gamma p - 155.$

155 $\vdash CCC\Gamma p\Gamma p\Gamma Cpp\Gamma Cpp$ (161).

* * *

$123\ \delta/Cp' \times C156 - 122.$

156 $\dashv Cp\Gamma p$ (158, 160).

$123\ \delta/N' \times C157 - 117.$

157 $\dashv N\Gamma p.$

$38\ p/\Gamma p,\ q/p \times C158 - 156.$

158 $\dashv CN\Gamma pNp$ (159).

$143 \times C159 - 158.$

159 $\dashv CN\Gamma p\Gamma N\Gamma p.$

$134\ q/Cpp \times C160 - 156.$

160 $\dashv \Gamma Cpp$ (161).

$155 \times C161 - 160.$

161 $\dashv CC\Gamma p\Gamma p\Gamma Cpp$ (162).

$161 \times 162\ q/p.$

162 $\dashv CC\Gamma p\Gamma q\Gamma Cpq.$

* * *

$9\ \delta/\Delta' \times C92 - 163.$

163 $\vdash \Delta Vp.$

ARITHMETIC AND MODAL LOGIC *)

1. It is well known that the theory of equality can be based on the following two axioms:

1.1 $\vdash \Theta aa$, i.e. "a is equal to a",

1.2 $\vdash C\Theta abC\phi a\phi b$, i.e. "If a is equal to b, then if ϕa, ϕb."

ϕ is a variable proposition-forming functor of one numerical argument, and we are allowed to substitute for ϕ any expression which gives together with the argument of ϕ a significant arithmetical formula.[1])

If we introduce into arithmetic the modal functor Γ ("it is necessary that") or Δ ("it is possible that"), then expressions as: $\Gamma\Theta ab$, i.e. "It is necessary that a should be equal to b", or $\Delta N\Theta ab$, i.e. "It is possible that a is not equal to b", will be significant arithmetical formulae. We have therefore:

$1.2\ \phi/\Gamma\Theta a' \times 1.3,$

1.3 $\vdash C\Theta abC\Gamma\Theta aa\Gamma\Theta ab,$

in words: "If a is equal to b, then if it is necessary that a should be equal to a, it is necessary that a should be equal to b". From 1.3 we get by the law of commutation

1.4 $\vdash CCpCqrCqCpr$

the formula 1.5:

$1.4\ p/\Theta ab,\ q/\Gamma\Theta aa,\ r/\Gamma\Theta ab \times C1.3 - 1.5,$

1.5 $\vdash C\Gamma\Theta aaC\Theta ab\Gamma\Theta ab.$

Most logicians would assert the antecedent of 1.5:[2])

[1]) For the explanation of my symbolic notation in arithmetic see: J. Łukasiewicz, "Sur la formalisation des théories mathematiques", *Colloques internationaux du Centre national de la recherche scientifique*, XXXVI *Les méthodes formelles en axiomatique*, Paris, 1953, pp. 11–19. [See p. 342 of this volume.]

[2]) W. V. Quine, "Three Grades of Modal Involvement", *Proceedings of the XIth International Congress of Philosophy*, vol. XIV, Brussels, 1953, p. 80: "... surely 'nec $(x = x)$' is true for all x." —'nec' means in our notation "Γ".

*) First published in *The Journal of Computing Systems* 1 (1954), pp. 213–219.

(a) $\Gamma\Theta aa$,

in words: "It is necessary that a should be equal to a." Hence they must accept the consequent:

(b) $C\Theta ab\Gamma\Theta ab$.

That means: Equality holds necessarily if it holds at all.

From (b) there follows by means of the modal theorem

1.6 $\vdash C\Gamma pN\Delta Np$,

and the principle of the syllogism

1.7 $\vdash CCpqCCqrCpr$

the consequence (c):

$1.7p/\Theta ab,\ q/\Gamma\Theta ab,\ r/N\Delta N\Theta ab \times C(\text{b}) - C1.6p/\Theta ab - (\text{c})$,

(c) $C\Theta abN\Delta N\Theta ab$,

and from (c) we get by the law of transposition

1.8 $\vdash CCpNqCqNp$

the formula (d):

$1.8\ p/\Theta ab,\ q/\Delta N\Theta ab \times C(\text{c}) - (\text{d})$,

(d) $C\Delta N\Theta abN\Theta ab$.

That means: "If it is (only) possible that a is not equal to b, then a is (factually) not equal to b."

2. I have not prefixed the sign of assertion "$\vdash$" to the formulae (a) (b), (c), and (d), as these formulae are in my opinion wrong and should be rejected. In particular formulae (b) and (d) are obviously false. Quine gives an example for the falsity of (b): Let a denote "the number of planets", and b the number "9". It is a factual truth that the number of (major) planets is equal to 9, but it is by no means necessary that it should be equal to 9. Quine tries to meet this difficulty by raising objections to the substitution of such singular terms for the variables. "Such instantiation—he writes—is allowable, certainly, in extensional logic; but it is a question of good behaviour of constant singular terms, and ... such behaviour is not to be counted on when there is a 'nec' in the wood-pile."[2]) Quine does not explain, however, how this "naughty" behaviour of singular terms should be corrected. His remark is a desideratum rather than a solution of the problem.

For the falsity of formula (d), not mentioned by Quine, take the following example: Let us suppose that the number a has been thrown with a die. It is possible that the number b next thrown with the die will be different from a. But if it is only possible that a will be different from b, i.e. not equal to b, then according to (d) a will factually be different from b. This consequence is obviously wrong, as it is possible to throw the same number twice.

3. The Ł-modal system expounded by myself in this *Journal*[3]) gives a satisfactory solution of the above difficulties. This system is based on two asserted axioms:

3.1 $\vdash C\delta pC\delta Np\delta q$,

3.2 $\vdash Cp\Delta p$.

The first axiom which means: "If p satisfies the condition δ, then if Np satisfies δ, any proposition q satisfies δ", yields the whole two-valued classical calculus of propositions and all the asserted δ-formulae. The second is a well-known modal theorem which gives together with the axiom 3.1 a complete system of modal logic. Both axioms are perfectly evident. Two axioms of rejection ("$\dashv$" is the sign of rejection):

3.3 $\dashv C\Delta pp$,

3.4 $\dashv \Delta p$,

are needed to characterize the system as a modal logic.

The rules of inference are the rule of substitution and detachment for the asserted and rejected formulae.

The system has the following adequate four-valued matrix M_1:

C	1	2	3	4	N	Δ	∇	Γ
*1	1	2	3	4	4	1	1	2
2	1	1	3	3	3	1	2	2
3	1	2	1	2	2	3	1	4
4	1	1	1	1	1	3	2	4

M_1

Γp is defined by $N\Delta Np$, ∇ by $C\Delta pp$.

[3]) J. Łukasiewicz, "A System of Modal Logic", *The Journal of Computing Systems*, 1 (1953), pp. 111–159. [See pp. 352–390 of this volume.]

A consequence of the system is the asserted formula:

3.5 $\vdash C\Gamma qCp\Gamma p$.

The consequent of this formula is rejected:

3.6 $\dashv Cp\Gamma p$.

We have therefore for any proposition a:

$3.5\ q/a \times C3.7 - 3.6,$

3.7 $\dashv \Gamma a$.

That means that no apodeictic proposition can be asserted in the Ł-modal system. Asserted propositions are merely true without being necessary. From 3.7 we get by interpreting a as Θaa:

3.8 $\dashv \Gamma\Theta aa$,

i.e. formula (a) is rejected. From

3.9 $\vdash CpCCpqq$

and

3.10 $\vdash \Theta aa$

there follows the consequence 3.11:

$3.9\ p/\Theta aa,\ q/\Gamma\Theta aa \times C3.10 - 3.11,$

3.11 $\vdash CC\Theta aa\Gamma\Theta aa\Gamma\Theta aa$.

Applying to 3.11 the rule of detachment for rejected formulae we get 3.12

$3.11 \times C3.12 - 3.8,$

3.12 $\dashv C\Theta aa\Gamma\Theta aa$,

and by the rule of substitution for rejected formulae we have:

$3.12 \times 3.13\ b/a,$

3.13 $\dashv C\Theta ab\Gamma\Theta ab$.

That means that formula (b) is rejected. In a similar way we can prove that formulae (c) and (d) must be rejected too. Here are the respective deductions:

3.14 $\vdash CN\Delta Np\Gamma p$ (follows from the definition of Γp).

3.15 $\vdash CCqrCCpqCpr$.

$3.15\ q/N\Delta N\Theta ab,\ r/\Gamma\Theta ab,\ p/\Theta ab \times C3.14p/\Theta ab - 3.16.$

3.16 $\vdash CC\Theta abN\Delta N\Theta abC\Theta ab\Gamma\Theta ab$.

$3.16 \times C3.17 - 3.13,$

3.17 $\dashv C\Theta abN\Delta N\Theta ab$, i.e., formula (c) is rejected.

3.18 $\vdash CCpNqCqNp$.

$3.18\ p/\Delta N\Theta ab,\ q/\Theta ab \times 3.19.$

3.19 $\vdash CC\Delta N\Theta abN\Theta abC\Theta abN\Delta N\Theta ab$.

$3.19 \times C3.20 - 3.17.$

3.20 $\dashv C\Delta N\Theta abN\Theta ab$, i.e., formula (d) is rejected.

On the other side, the asserted formula $C\Theta abC\Gamma\Theta aa\Gamma\Theta ab$, which is correctly deduced from the axiom $C\Theta abC\phi a\phi b$, is easily verified by the matrix M_1. As Θaa is asserted and has the value 1, $\Gamma\Theta aa = \Gamma 1 = 2$. We have therefore to verify the formula

$$C\Theta abC2\Gamma\Theta ab$$

for all possible value of Θab. We get:

For $\Theta ab = 1$: $C1C2\Gamma 1 = C1C22 = C11 = 1$;
For $\Theta ab = 2$: $C2C2\Gamma 2 = C2C22 = C21 = 1$;
For $\Theta ab = 3$: $C3C2\Gamma 3 = C3C24 = C33 = 1$;
For $\Theta ab = 4$: $C4C2\Gamma 4 = C4C24 = C43 = 1$.

4. Propositions, as "a is equal to a", are called "analytic". The well-known doctrine that all analytic propositions are necessary goes back to Aristotle who distinguishes between essential and accidental properties and asserts that essential properties belong to the things with necessity. [4])

Essential properties are based on definitions, i.e. on the meaning of words. So for instance, "Man is necessarily an animal" [5]), because "man" is defined as an "animal". In view of the formulae $\vdash C\Gamma pp$ and $\dashv Cp\Gamma p$ it is commonly held that apodeictic propositions have a higher dignity and are more reliable than corresponding assertoric ones. This consequence is for me by no means evident. I cannot understand why the true proposition based on the meaning of words "I am an animal" should be more reliable than the factual truth based on experience "I have brown eyes". Another Aristotle's argument connected with the subject and sometimes called "the Aristotelian paradox" is still less evident. Aristotle asserts: "If it is true to say that something is white or not

[4]) *An. post.* A 6, 74 b 6: τὰ δὲ καθ' αὑτα ὑπάρχοντα ἀναγκαῖα τοῖς πράγμασιν.

[5]) *An. pr.* A 9, 30 a 30: ζῷον μὲν γὰρ ὁ ἄνθρωπος ἐξ ἀνάγκης ἐστί.

white, it is necessary that it should be white or not white."[6] As it is impossible to translate this statement by the false formula $C p \Gamma p$, some logicians accept as a rule that it is allowed to infer from an asserted (analytic) proposition α the asserted apodeictic proposition $\Gamma\alpha$. This again leads to asserted apodeictic propositions, and if we accept such propositions at all, we are bound to assert the necessity of the principle "a is equal to a". In view of the difficulties which would result from this fact, I am inclined to think that all systems of modal logic which accept asserted apodeictic propositions are wrong.

A system of this kind is my three-valued modal logic constructed by the matrix M_2 in 1920, and developed in 1930.[7]

C	1	2	3	N
*1	1	2	3	3
2	1	1	2	2
3	1	1	1	1

M_2

C is here a kind of implication different from the ordinary one, and N a kind of negation. The system for C and N was axiomatized by M. Wajsberg,[8] and extended to a complete system by the addition of a new function by J. Słupecki.[9] In my paper of 1930 I accepted the definition of possibility suggested by A. Tarski:

(t) $\Delta p = CNpp$

which is equivalent to the definition of necessity:

(u) $\Gamma p = NCpNp.$

[6] *De int.* 9,18 a 39: εἰ γὰρ ἀληθὲς εἰπεῖν ὅτι λευκὸν ἢ ὅτι οὐλευκόν ἐστιν, ἀνάγκη εἶναι λευκὸν ἢ οὐ λευκόν.

[7] J. Łukasiewicz, "O logice trójwartościowej", *Ruch Filozoficzny* 5 (1920). [See pp. 87–88 of this volume.]—J. Łukasiewicz, "Philosophische Bermerkungen zu mehrwertigen Systemen des Aussagenkalküs", *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie* 23 (1930), cl. iii [See pp. 153–178 of this volume.]

[8] M. Wajsberg, "Aksjomatyzacja trójwartościowego rachunku zdań", *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie* 24 (1931), cl. iii.

[9] J. Słupecki, "Pełny trójwartościowy rachunek zdań", *Annales Universitatis Mariae Curie-Skłodowska*, vol. I, Nr 3, Sectio F, Lublin 1946.

It is easy to see on the ground of the matrix M_2 that in this system the formula

(v) $CpCp\Gamma p = CpCpNCpNp$

is verified, and yields not only ΓCpp for p/Cpp, but also the Aristotelian paradoxical rule. I think therefore that my three-valued modal logic cannot be regarded as an adequate system of modal logic.

The same remark should be made about the Lewis's systems of "strict implication". The function "p strictly implies q" is defined by Lewis by the expression "It is not possible that p and not q".[10] From this definition we can easily deduce that the formula "p strictly implies q" is an asserted apodeictic proposition. The systems of Lewis are certainly very interesting and may have their own merits; I think, however, that they cannot be regarded as adequate systems of modal logic.

5. Modal logic is important as a theory of possibility. There exist true problematic propositions which would not be true as assertoric propositions. There are other true propositions which cannot be proved without introducing possibility. Both kinds of proposition extend our knowledge beyond the stock of truths which can be got by the non-modal logic.

I shall explain here an application of my Ł-modal system to arithmetic which throws a singular light on the meaning of the so called "existential quantifier". From the principle of identity

5.1 $\vdash Cpp$

we get by substitution

$5.1\, p/\phi a \times 5.2.$

5.2 $\vdash C\phi a\phi a,$

and, by the rule of quantifiers denoted by $\sum 2$, formula 5.3:

$5.2 \sum 2a \times 5.3.$

5.3 $\vdash C\phi a \sum a\phi a.$

In words: "if ϕ of a, then for some a ϕ of a", or "if ϕ of a, then there exists such an a that ϕ of a". "a" denotes any positive integer, i.e. any number of the sequence 1, 2, 3, ..., in inf.

[10] C. I. Lewis and C. H. Langford, *Symbolic Logic*, New York and London, 1932, p. 124.

By means of the functors Δ and ∇ (see above section 3) it is possible to construct true expressions of the form $\sum a\phi a$, though there exists no positive integer a that would verify ϕa. The simplest expression of this kind is the following one:

5.4 $\vdash \sum aK\Delta\Theta 1a\nabla L1a$,

where "$L1a$" means "1 is less than a". For $a = 1$, $\Theta 1a$ is true and $L1a$ is false, or $\Theta 1a = 1$ and $L1a = 4$; for $a > 1$, $\Theta 1a$ is false and $L1a$ is true, or $\Theta 1a = 4$ and $L1a = 1$. We have therefore:

if $a = 1$, then $K\Delta\Theta 1a\nabla L1a = K\Delta 1\nabla 4 = K12$,
if $a > 1$, then $K\Delta\Theta 1a\nabla L1a = K\Delta 4\nabla 1 = K31$.

Now according to the matrix for $Kpq = NCpNq$:

K	1	2	3	4
1	1	2	3	4
2	2	2	4	4
3	3	4	3	4
4	4	4	4	4

the conjunction $K12 = 2$, and $K31 = 3$, i.e. none of them is equal to 1. Hence it appears that no positive integer verifies the conjunction $K\Delta\Theta 1a\nabla L1a$. Nevertheless the quantified expression $\sum aK\Delta\Theta 1a\nabla L1a$ is true. I give here a full proof of this theorem based on asserted formulae of the Ł-modal system and on three elementary arithmetical theses.

The premises

1 $\vdash Cpp$ (14).
2 $\vdash CCpqCCqrCpr$ (17, 22, 23).
3 $\vdash CNpCpq$ (20).
4 $\vdash CpCqKpq$ (16).
5 $\vdash CpCqKqp$ (19).
6 $\vdash CCprCCqrCApqr$ (24).
7 $\vdash Cp\Delta p$ (15).
8 $\vdash Cp\nabla p$ (18).
9 $\vdash CCpqC\Delta p\Delta q$ (21).
10 $\vdash A\Delta p\nabla p$ (24).

11 $\vdash \Theta aa$ (15) $(a = a)$.
12 $\vdash NLaa$ (20) $(a < a)$.
13 $\vdash La\Xi a1$ (18) $(a < a+1)$.

The deduction

$1\,p/K\Delta\Theta 1a\nabla L1a, \sum 2a \times 14$.
14 $\vdash CK\Delta\Theta 1a\nabla L1a \sum aK\Delta\Theta 1a\nabla L1a$ (17, 23).

$7\,p/\Theta 11 \times C11a/1 - 15$.
15 $\vdash \Delta\Theta 11$ (16).

$4\,p/\Delta\Theta 11, q/\nabla L11 \times C15 - 16$.
16 $\vdash C\nabla L11K\Delta\Theta 11\nabla L11$ (17).

$2\,p/\nabla L11, q/K\Delta\Theta 11\nabla L11, r/\sum aK\Delta\Theta 1a\nabla L1a \times C16 -$
$- C14a/1 - 17$.

17 $\vdash C\nabla L11 \sum aK\Delta\Theta 1a\nabla L1a$ (24).

$8\,p/L1\Xi 11 \times C13a/1 - 18$.
18 $\vdash \nabla L1\Xi 11$ (19).

$5\,p/\nabla L1\Xi 11, q/\Delta\Theta 1\Xi 11 \times C18 - 19$.
19 $\vdash C\Delta\Theta 1\Xi 11K\Delta\Theta 1\Xi 11\nabla L1\Xi 11$ (22).

$3\,p/L11, q/\Theta 1\Xi 11 \times C12a/1 - 20$.
20 $\vdash CL11\Theta 1\Xi 11$ (21).

$9\,p/L11, q/\Theta 1\Xi 11 \times C20 - 21$.
21 $\vdash C\Delta L11\Delta\Theta 1\Xi 11$ (22).

$2\,p/\Delta L11, q/\Delta\Theta 1\Xi 11, r/K\Delta\Theta 1\Xi 11\nabla L1\Xi 11 \times C21 -$
$- C19 - 22$.

22 $\vdash C\Delta L11K\Delta\Theta 1\Xi 11\nabla L1\Xi 11$ (23).

$2\,p/\Delta L11, q/K\Delta\Theta 1\Xi 11\nabla L1\Xi 11, r/\sum aK\Delta\Theta 1a\nabla L1a \times C22 -$
$- C14a/\Xi 11 - 23$.

23 $\vdash C\Delta L11 \sum aK\Delta\Theta 1a\nabla L1a$ (24).

$6\,p/\Delta L11, q/\nabla L11, r \sum aK\Delta\Theta 1a\nabla L1a \times C23 - C17 -$
$- C10p/L11 - 24$.

24 $\vdash \sum aK\Delta\Theta 1a\nabla L1a$.

If follows from this consideration that it would be wrong to translate the expression $\sum a$ by the phrase "for some a" or "there exists such an a that". In order to express in words a formula of the shape $\sum a\phi a$, we must first transform it into the equivalent $N \prod aN\phi a$, and then say accordingly: "It is not the case that for all a not ϕ of a." It seems to me that the philosophical implications of this logical fact may be of some importance.

LIST OF PUBLICATIONS BY JAN ŁUKASIEWICZ

(The abbreviations *P.F.* and *R.F.* stand for the titles of two Polish philosophical journals, namely *Przegląd Filozoficzny* and *Ruch Filozoficzny*, respectively)

1. Summary: "Vierteljahrschrift für wissenschaftliche Philosophie 1899", No. 3–4, *P.F.* 5 (1902), pp. 232–236.
2. O indukcji jako inwersji dedukcji" (On induction as the inversion of deduction), *P.F.* 6 (1903), pp. 9–24, 138–152.
3. Review: T. Mianowski, *O tzw. pojęciach wrodzonych u Locke'a i Leibniza* (On innate concepts in Locke and Leibniz), *P.F.* 7 (1904), pp. 94–95.
4. "O stosunkach logicznych" (On logical relations), *P.F.* 7 (1904), p. 245.
5. "Teza Husserla o stosunku logiki do psychologii" (Husserl's thesis on the relationship between logic and psychology), *P.F.* 7 (1904), pp. 476–477.
6. "Z psychologii porównywania" (On the psychology of comparison), *P.F.* 8 (1905), pp. 290–291.
7. "O dwóch rodzajach wniosków indukcyjnych" (On two kinds of inductive conclusions), *P.F.* 9 (1906), pp. 83–84.
8. Analiza i konstrukcja pojęcia przyczyny" (An analysis and construction of the concept of cause), *P.F.* 9 (1906), pp. 105–179.
9. "Tezy Höflera w sprawie przedstawień i sądów geometrycznych" (Höfler's thesis concerning geometrical ideas and judgements), *P.F.* 9 (1906), pp. 451–452.
10. "Co począć z pojęciem nieskończoności?" (What to do with the concept of infinity?), *P.F.* 10 (1907), pp. 135–137.
11. Review: H. Struve, *Die polnische Philosophie der letzten zehn Jahre (1894–1904)*.
12. "O wnioskowaniu indukcyjnym" (On inductive reasoning), *P.F.* 10 (1907), pp. 474–475.
13. "Logika a psychologia" (Logic and psychology), *P.F.* 10 (1907), pp. 489–491.
14. "Pragmatyzm, nowa nazwa pewnych starych kierunków myślenia" (Pragmatism, a new name of certain old trends in thinking), *P.F.* 11 (1908), pp. 341–342.
15. "Sprawozdanie z dwóch prac Stumpfa" (A review of two papers by Stumpf), *P.F.* 11 (1908), pp. 342–343.
16. An introduction to a talk on M. Borowski's paper: *Krytyka pojęcia związku przyczynowego* (A criticism of the concept of causal nexus), *P.F.* 11 (1908), p. 343.
17. "Zagadnienia i znaczenie ogólnej teorii stosunków" (The tasks and the meaning of a general theory of relations), *P.F.* 11 (1908), pp. 344–347.
18. "O prawdopodobieństwie wniosków indukcyjnych" (On the probability of inductive conclusions), *P.F.* 12 (1909), pp. 209–210.

19. "O poglądach filozoficznych Meinonga" (On Meinong's philosophical views), *P.F.* 12 (1909), pp. 559.
20. "O zasadzie wyłączonego środka" (On the principle of excluded middle), *P.F.* 13 (1910), pp. 372–373.
21. "Über den Satz von Widerspruch bei Aristoteles". *Bulletin international de Académie des Sciences de Cracovie, Classe de Philosophie* (1910), pp. 15–38.
22. "O zasadzie sprzeczności u Arystotelesa. Studium krytyczne" (On the principle of contradiction in Aristotle. A critical study), Kraków, 1910.
23. Review: Wł. Tatarkiewicz, *Die Disposition der aristotelischen Prinzipien*, *R.F.* 1 (1911), pp. 20–21.
24. "O wartościach logicznych" (On the logical values), *R.F.* 1 (1911), p. 52.
25. "O rodzajach rozumowania. Wstęp do teorii stosunków" (On the types of reasoning. An introduction to the theory of relations), *R.F.* 1 (1911), p. 78.
26. Review: P. Natorp, *Die logischen Grundlagen der exakten Wissenschaften*, *R.F.* 1 (1911), pp. 101–102.
27. Review: H. Struve, *Historia logiki jako teorii poznania w Polsce* (A history of logic as cognition theory in Poland), 2nd edition, *R.F.* 1 (1911), pp. 115–117.
28. "O potrzebie założenia instytutu metodologicznego" (On the need of founding an institute of methodology), *R.F.* 2 (1912), pp. 17–19.
29. Review: Wł. Biegański, *Czym jest logika*? (What is logic?), *R.F.* 2 (1912), p. 145.
30. "O twórczości w nauce" (Creative elements in science), *Księga pamiątkowa ku uczczeniu 250 rocznicy założenia Uniwersytetu Lwowskiego*, Lwów, 1912, pp. 1–15.
31. "Nowa teoria prawdopodobieństwa" (New probability theory), *R.F.* 3 (1913), p. 22.
32. Review: J. Kleiner, *Zygmunt Krasiński. Dzieje myśli* (Zygmunt Krasiński. A history of ideas), *R.F.* 3 (1913), pp. 109–111.
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37. "O nauce" (On science), *A guide for self-educated persons*, New edition, vol. I, 1915, pp. XV–XXXIX. Reprinted: Lwów, 1934, 1936, p. 40.
38. "O nauce i filozofii" (On science and philosophy), *P.F.* 28 (1915), pp. 190–196.
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40. Farewell lecture delivered in the Warsaw University Lecture Hall on March 7, 1918. Warszawa, 1918.
41. "O pojęciu możliwości" (On the concept of possibility), *R.F.* 6 (1919/20), pp. 169–170.

42. "O logice trójwartościowej" (On three-valued logic), *R.F.* 5 (1920), pp. 170–171.
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