Music and Measurement:

On the Eidetic Principles of Harmony and Motion

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Beatae Mariae Semper Virgini,
Mediatrici, Coredemptrici, et Advocatae

ταὐτὸν γὰρ ποιοῦσι τοῖς ἐν τῇ ἀστρονομίᾳ· τοὺς γὰρ ἐν ταύταις ταῖς συμφωνίαις ταῖς ακουομέναις ἀριθμοὺς ζητοῦσιν, ἀλλ᾽ οὐκ εἰς προβλήματα ἀνίασιν ἐπισκοπεῖν, τίνες ξύμφωνοι ἀριθμοὶ καὶ τίνες οὔ, καὶ διὰ τί ἑκάτεροι.[1]

For they do the same thing as those in astronomy: they seek the numbers in these harmonies being heard, but they do not ascend to contemplate the problems of which numbers harmonize and which do not, and the reason for each.

— Plato, *Republic*

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1. Musica Abscondita

1.1 The Definition of Music

It is widely accepted that one ought to define a subject's matter and form before proceeding to its investigation, but this supposes in the reader a framework in which to locate and interpret the author's definition. A mathematical theory of music presents difficulties in this regard. Indeed, few musicians today would take seriously the assertion of Leibniz that "music is the hidden arithmetical exercise of the mind unaware that it is counting." (*Musica est exercitium arithmeticae occultum nescientis se numerare animi.*[2]) Even if this definition does not adequately capture the true mathematical character of music, it certainly specifies music either as an application of mathematics to physical phenomena or as a mathematical discipline in its own right. Concert music, however, has run far beyond mathematical theory, and is commonly classed among the fine arts, rather than among the liberal arts of the *quadrivium*. Eduard Hanslick's opinion better captures the ethos of today's musician:

> Perhaps more out of caution than from need, we may add in conclusion that the musically beautiful has nothing to do with mathematics. This notion, which laymen (sensitive authors among them) cherish concerning the role of mathematics in music, is a remarkably vague one. Not content that the vibrations of tones, the spacing of intervals, and consonances and dissonances can be traced back to mathematical proportions, they are also convinced that the beauty of a musical work is based on number. The study of harmony and counterpoint is considered a kind of cabala which teaches compositional calculus.
>
> Even though mathematics provides an indispensable key for the investigation of the physical aspects of musical art, its importance with regard to completed musical works ought not to be overrated. In a musical composition, be it the most beautiful or the ugliest, nothing at all is mathematically worked out. The creations of the imagination are not sums. All monochord experiments, acoustic figures, proportions of intervals, and the like, are irrelevant. The domain of aesthetics begins where these elementary relationships, however important, have left off. Mathematics merely puts in order the rudimentary material for artistic treatment and operates secretly in the simplest relations.[3]

Excessive mathematical claims regarding music—claims that take the promise for the deed—produce an impatience with mathematical theory that is rather excusable, if regrettable. For the assertion of Leibniz may be taken to denigrate musical practice, as if mathematicians had easily achieved in the realm of discursive thought what the whole tradition of concert music has accomplished only through genius and untiring effort. Such an interpretation would be unjust to Leibniz no less than to these musicians.

The lack of *conscious* mathematical application, duly recognized by Leibniz, in no way excludes a thoroughgoing mathematical structure to the art. It simply means that, in view of the complexity of composition, mathematics can only describe, rather than prescribe, musical practice. Consider sculpture. The sculptor, for all his technique, cannot escape the mathematical demands of solid geometry, and all his efforts—conscious or unconscious—tend toward a definite geometric form, however complex, however expressive of his culture. This sensible geometric form persists in the medium, and is the only true artistic subject, its shape and proportions being the indispensable components of any aesthetic response. In like manner, musical composition is the molding and direction of sound in time, a motion wholly reducible to mathematical relations. Whatever the impulse or intent of the artist, no matter how much meaning accrues to music from personal, cultural, political, or spiritual concerns, the potential forms of music, though infinite in number, can be fully specified through mathematics. If, as Hanslick claims, there is a properly musical aesthetic, it is surely mathematical. The music theorist must consequently strive to understand how mathematical form satisfies aesthetic criteria, and how it acts as a substrate to which human mores and motives may be subsequently related. Sheet music, a symbolic shorthand that directs the making of music, *but not its hearing*, is a valuable tool, yet by no means an ultimate description.

The fault lies with mathematics, which has long been unmoored from the metaphysical distinctions that lend themselves to aesthetic analysis. A reconciliation cannot be achieved, therefore, by a critique of musical practice, but only by an instauration of modern mathematics, a sifting and reconstruction on more solid principles. How are calculations involving measurable quantities, rather than numbers, to be justified scientifically? This problem of *theoretical logistic*, first enunciated by Plato[4] and lately resurrected by Jacob Klein[5], can be solved by recourse to infinite ensembles of natural numbers belonging to a *common arithmetical species* (εἴδη). The conditions of musical harmony then follow naturally from this new *eidetic* theory. Music may consequently be defined as *the art of measurement and the science of the measurable*. By way of comparison, arithmetic may be defined as *the art of numeration and the science of the numerable*, and geometry as *the art of extension and the science of the extended*. Arithmetic, music, and geometry—each a liberal art in its own right—form a hierarchy. Arithmetic supplies principles to music, which, together with arithmetic, supplies principles to geometry. Music thus finds its purest physical expression in sound, a temporal phenomenon that is both quantitative and subject to counting, but without bodily extension. Music, in turn, is divisible into two branches. In meter, *completed* motions are related through *different numbers of equal measures*, whereas, in harmonics, *states* of motion are related through number-species *without recourse to the terms of motion*.

Eidetic theory justifies much that is beyond the scope of prior theories. It explains why every rise or fall of an octave yields tones that are *identical* in pitch class, rather than similar. It not only accurately predicts harmonious intervals, but explains by the very same principles how these intervals may be combined to form complex chords, thus revealing the foundations of tonality. The diatonic scale is but the superposition of two such chords. Harmonic and metrical descriptions clarify the foundations of modern physics by revealing the intrinsic difference in mathematical kind between motion and the terms of motion, thus undermining the philosophically dubious methods of the differential calculus. Yet music is of greater consequence than the production of beautiful sound—or even the understanding of the cosmos. In the classical tradition, music has a psychological and ethical importance. In this regard, eidetic theory sustains a traditional notion of the soul and of the mean belonging to moral virtue.

1.2 Mathematical, Physical, and Intermediate Sciences of Music

To justify the place of music among the sciences, it is helpful to consider the musical claims of modern mathematical physics. Although there are variations, these theories share a common foundation. In conformity with mathematical practice dating from Fourier, a periodic wave, that is, a waveform that repeats itself indefinitely, can be uniquely composed out of pure sinusoidal waves, called *harmonics* or *partials*, the frequencies of which are exact multiples of the frequency of the composite waveform. Conversely, the composition of partial waveforms yields a single waveform that repeats itself with this fundamental frequency, allowing the composed sinusoids to be grasped in their unity.

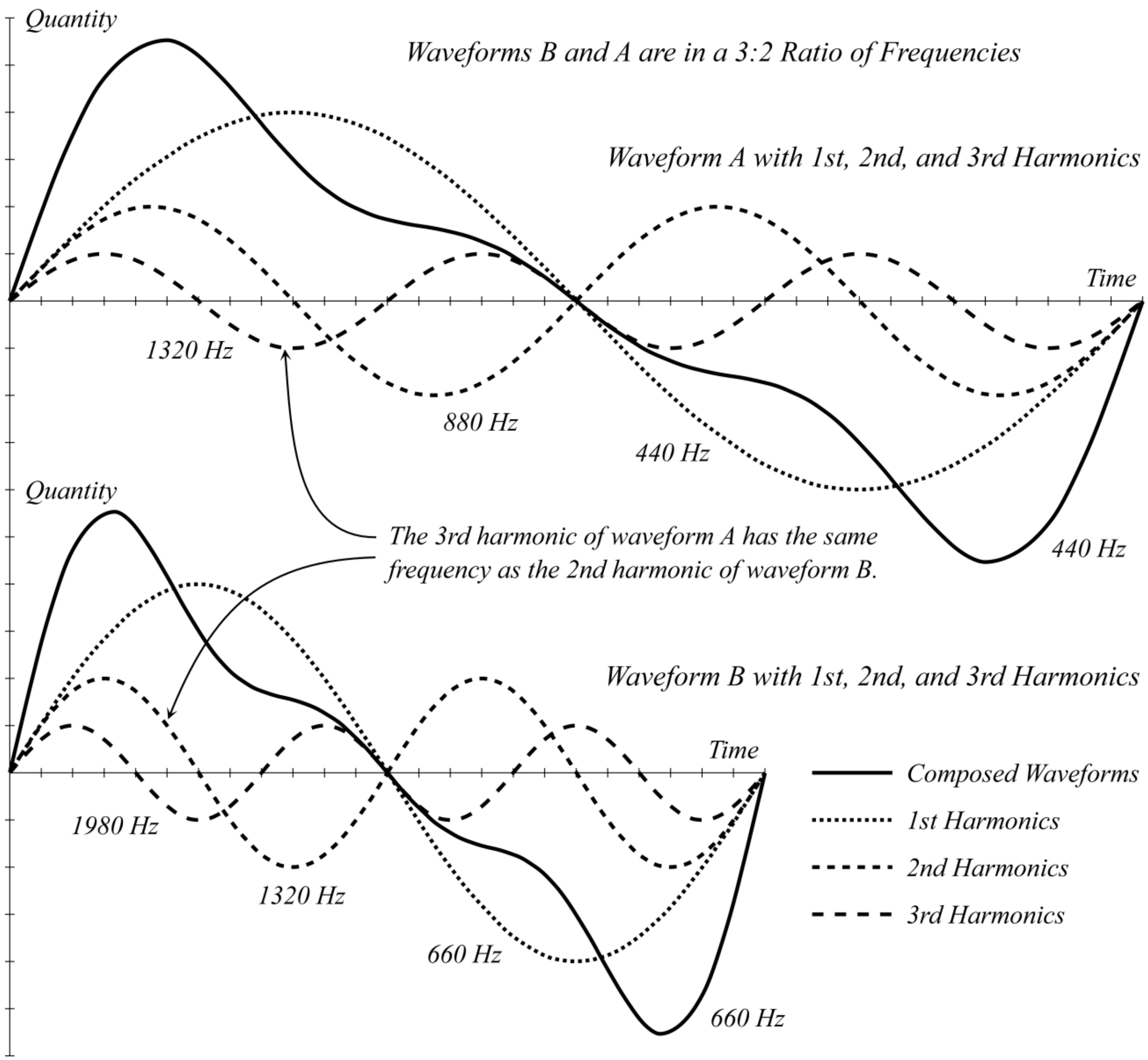


Figure 1: The Decomposition of a Periodic Waveform into Harmonics

Figure 1 illustrates these principles for the case of two composite waveforms, waveform A with a frequency of 440 Hz, and waveform B with a frequency of 660 Hz. Each of these waves can be decomposed into three elemental sinusoidal waves, called harmonics, each having a frequency that is a multiple of the frequency of the composite waveform, that is, of the fundamental frequency. These harmonics can be composed once again by adding the magnitudes of their displacements at every instant in time. The principle of harmony consists then in this: tones harmonize with each other insofar as they share common multiples of their fundamental frequencies or, what is mathematically the same, insofar as their fundamental frequencies are themselves multiples of a common frequency. The tones represented by these two composite waveforms will harmonize because they share a common frequency, 1320 Hz, among their partials. The third harmonic of tone A oscillates with the same frequency as the second harmonic of tone B. Alternatively, one may observe that two full cycles or periods of waveform A will take the same amount of time as three periods of waveform B. In other words, the frequency of waveform A is twice 220 Hz, whereas that of waveform B is three times 220 Hz.

The condition for harmony endorsed by this school is entirely mathematical in its formal description. This has indeed been the aim of the sciences since Galileo. St. Thomas Aquinas employs a distinction useful in this regard:

> And thence it is that three orders of sciences are found concerning natural and mathematical things. For some, such as physics and agriculture and the like, which consider the properties of natural things as such, are purely natural. But some are purely mathematical: those which set limits to quantities absolutely, such as geometry to magnitude and arithmetic to number. But some, which apply mathematical principles to natural things, as do music, astronomy, and the like, are intermediate. These sciences are nevertheless more closely related to mathematics because, in their consideration, that which is physical is as if material, but that which is mathematical is as if formal—just as music considers sounds, not inasmuch as they are sounds, but inasmuch as they make proportions according to numbers, and it is similar in other things. And on account of this they demonstrate their conclusions about natural things, but through mathematical means. And therefore nothing prevents if, inasmuch as they share in the natural, they have regard for sensible matter, whereas, inasmuch as they share in mathematics, they are abstract.[6]

Regardless of whether one acknowledges the possibility of purely physical sciences as understood by St. Thomas, the whole of modern mathematical physics clearly belongs to the class of intermediate sciences (*scientiae mediae*), for mathematical description and demonstration alone are countenanced, yet that which is described and demonstrated pertains to physical objects apprehended through the senses. In the modern theory of harmony there are no formal principles transcending mathematics. Since it is the formal principle that permits scientific demonstration, the modern theory, though often put forth as physical, is in fact intermediate.

The distinction between mathematical and physical sciences is linked to their respective degrees of abstraction from matter. The senses abstract from particular matter, that is, from the matter of *individual* substance, yielding the sensible matter treated by the physicist. The intellect, in turn, abstracts from this sensible matter, yielding the intelligible matter of the mathematician. St. Thomas argues:

> ... mathematical objects are not wholly abstracted from matter, but only from sensible matter. Moreover, the parts of quantity from which the demonstration seems in some manner to be derived from a material cause are not sensible matter, but pertain to intelligible matter, which is found even in mathematical objects...[7]

Proclus, in his *Commentary on the First Book of Euclid's Elements*, explains this process of discovery and demonstration in greater detail:

> ... the discovery of theorems does not occur without recourse to matter, that is, intelligible matter. In going forth into this matter and shaping it, our ideas are plausibly said to resemble acts of production; for the movement of our thought in projecting its own ideas is a production, we have said, of the figures in our imagination and of their properties. But it is in imagination that the constructions, sectionings, superpositions, comparisons, additions, and subtractions take place, whereas the contents of our understanding all stand fixed, without any generation or change.[8]

Because the mathematical sciences demand intelligible matter for a cause, a purely mathematical theory of music is not only possible, but requisite to an intermediate science of music. The intermediate science merely applies these mathematical forms to hearing, implying a reunion of intelligible with sensible matter. Mathematical deduction in music is consequently applicable to audible concords. It follows that one cannot reject mathematical theories of harmony as such by offering putatively *physical* objections that already depend upon a mathematical form of explanation, or that beg the question by adverting to sensory organs conceived and explained mathematically. The burden rests upon those who espouse a physical theory to propose a principle of harmony that does not already contain virtually a mathematical one.

To further clarify the distinction of Aquinas, it is helpful to consider a more genuinely physical theory of music. Henri Bergson's theory is but an application of his general philosophy of becoming and duration. He maintains that audible tones are fundamentally qualitative in nature. Bergson further adds that tones are quantified only by the projection of these interpenetrating qualities into the world of body and space:

> Are the differences in pitch, such as our ear perceives, quantitative differences? I grant that a sharper sound calls up the picture of a higher position in space. But does it follow from this that the notes of the scale, as auditory sensations, differ otherwise than in quality? Forget what you have learnt from physics, examine carefully your idea of a higher or lower note, and see whether you do not think simply of the greater or less effort which the tensor muscle of your vocal chords has to make in order to produce the note? As the effort by which your voice passes from one note to another is discontinuous, you picture to yourself these successive notes as points in space, to be reached by a series of sudden jumps, in each of which you cross an empty separating interval: this is why you establish intervals between the notes of the scale.[9]

Bergson claims that the notions of higher and lower pitch derive from the extent of the body from which the vibrations originate: low pitches from as deep as the abdomen, higher pitches from the throat.[10] His philosophy is a reaction to the imperial claims of Newtonian space and time, an attempt to delimit the competence of quantity, thereby preserving a domain in which the purely qualitative may abide. Bergson is to philosophy what Monet is to painting and DeBussy is to music, a thoroughgoing impressionist.

Though Bergson's aim is in some respects noble, his method is deeply flawed. It suffices here to show the weakness of his theory of music. Let it be granted that there are qualitative differences between one tone and another. Bergson's explanation is nevertheless deficient in two very serious respects. The first is that the human voice is incapable of producing by physical effort the full range of sound audible to the human ear. Yet all these tones are experienced as higher or lower, harmonious or inharmonious. The second is that the human voice is capable of producing tones by a continuous or near-continuous increase of tension in the vocal chords. There is no reason to leap from one tone to another, rather than to progress through the intermediate tones. To explain the phenomenon of music scientifically, one must explain why intervallic change elicits a distinctive aesthetic response, and why only certain intervals can produce this response. Bergson can do no more than give names to harmonious experiences. He posits no knowledge of the causes themselves.

Then the question to be investigated in this essay is not whether there is an intermediate science of music, but what prior mathematical science informs it. Aristotle takes music to be an immediate application of *arithmetic* to sound:

> But "the reason why" and "the fact that" differ in another way, namely, that each is contemplated by a different science. This is so whenever it holds that one science is subordinated to another, as are optics to geometry, mechanics to stereometry, harmonics to arithmetic, and appearances to astronomy. Some of these sciences have almost the same name, such as mathematical astronomy and nautical astronomy, mathematical harmonics and audible harmonics. Here it belongs to the observers to know "the fact that" and to the mathematicians to know "the reason why." For the latter possess the demonstration of the causes, but often do not know the facts, just as those who contemplate the universal often do not know each instance, for a lack of observation.[11]

The eidetic theory of music is a mathematical science distinct from and subordinate to arithmetic. It will be shown that this science of quantity and measurement, rather than counting, provides the formal basis for the intermediate science of music.

1.3 Against the Beat Theory of Harmony

Before attempting a serious critique of current theories of number and quantity, it is necessary to clear away some common prejudices. Helmholtz's beat theory of harmony[12] continues to be accepted uncritically, despite both its serious explanatory flaws and the adequate empirical evidence against it. In modern mathematical analysis, when two sinusoidal waves with unequal but very close frequencies are superimposed, the resulting waveform can be described as a sinusoid with a frequency equal to the average of the two frequencies, but bounded within another sinusoidal envelope that both distorts somewhat the bounded sinusoid and varies the amplitude of its oscillation. *Figure 2* illustrates this behavior.

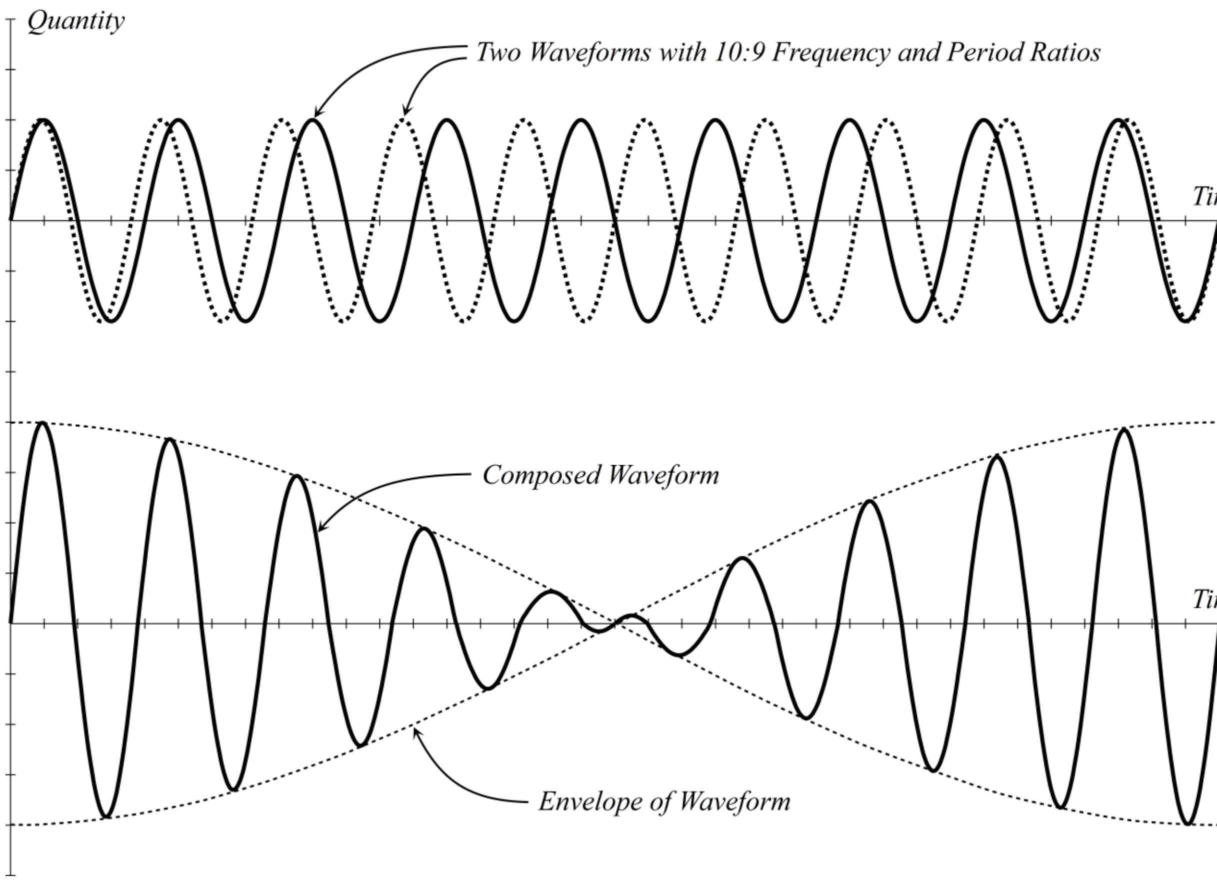


Figure 2: The Formation of Beats

On the first horizontal axis are two waves with frequencies of oscillation in a ratio of 10:9. Initially, both waves rise together, but the difference in their frequencies causes them to gradually lose their coherence, until one is rising just as the other is falling. This continues until they once again regain their coherence. The waves are initially in phase, then out of phase, and then back in phase again. When waves are in phase, they reinforce each other, yielding maximal amplitude, a process known as *constructive interference*. When they are out of phase, however, there is *destructive interference*, as the fall of one cancels the rise of the other, yielding minimal amplitudes in the combined waveform. This alternating constructive and destructive interference produces an oscillation in the amplitude of the wave, a phenomenon called *beats*. In the case of sound, these beats correspond to a modulation in perceived loudness. Helmholtz explains harmony as a lack of beats when tones are played together. Disharmony, on the other hand, is the presence of beats in music. He argues mathematically that beats are minimized when tonal frequencies are in the ratio of small whole numbers. Tones in these ratios thus harmonize, whereas tones differing perceptibly from these ratios do not.[13]

Both musical experience and common sense militate against this theory. Tones separated by a very large interval will lack harmonics close enough to produce beats, and should therefore always harmonize. Although the physical production of a single pitch typically involves the simultaneous production of many harmonics, the higher of these are quickly damped out, leaving only the lower harmonics. Yet there is no harmonizing a B and C, even when separated by three or more octaves. Such is the difficulty in a theory that defines harmony as a privation of disharmony. Moreover, dynamics, or modulations in intensity, are an indispensable part of musical art. The *pianoforte* was designed precisely to accommodate this need on a keyboard instrument, and violinists are sometimes directed to play their instruments *tremolo*. Since the perception of dynamics appears to be largely distinct from the perception of harmony, how can one legitimately substitute aesthetic criteria governing loudness for criteria governing pitch? While the presence of beats can certainly undermine the beauty of music, one should not confuse this aesthetic experience with disharmony.

There is also adequate *experimental* evidence to disprove Helmholtz's theory. "Demonstration 31: Tones and Tuning with Stretched Partials"[14] of Houtsma, Rossing, and Wagenaars shows that harmony cannot be explained in terms of beats. Tones are produced when standing waves are generated within a physical instrument. The production of a given pitch yields not just a sinusoid of that frequency, but many superimposed harmonics, all having multiples of the fundamental frequency, as in *Figure 3*:

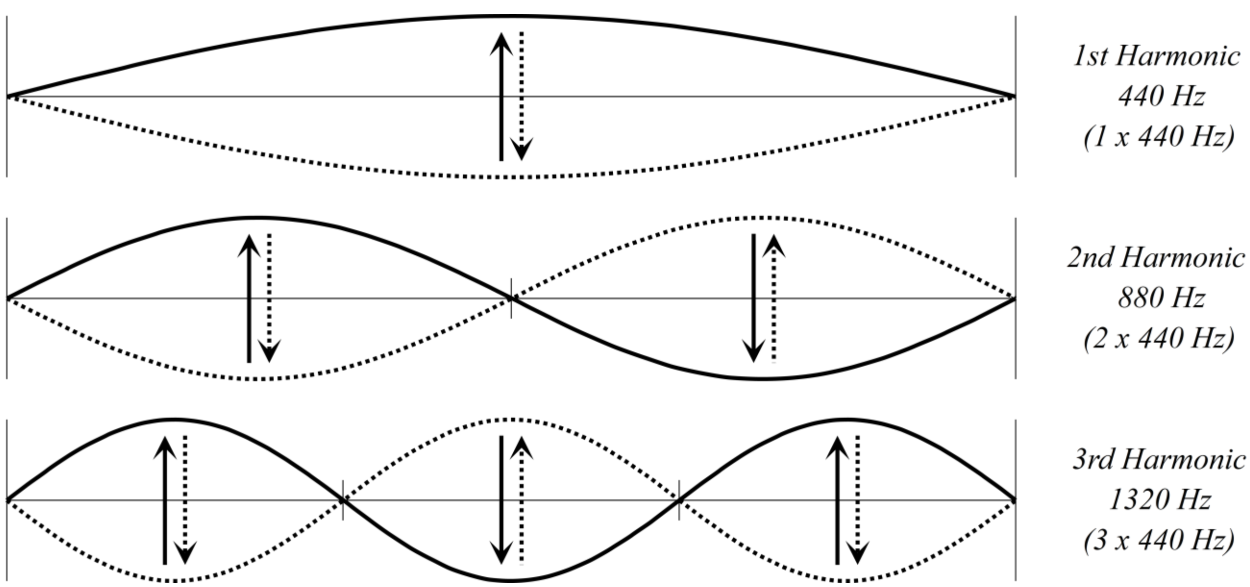


Figure 3: Standing Waves Formed on a Vibrating String

The whole ratios of the harmonics produced by physical instruments make them inadequate for testing the beat theory of harmony. A controlled experiment requires variation in the intervals separating the higher harmonics from the first harmonic, or fundamental. The experimenters achieved this by synthesizing tones. As a test vehicle, they chose *Als der gutige Gott* of J. S. Bach (BWV 264), using harmonics that varied inversely in amplitude with the number of the harmonic.

Four trials were made. The first was a musical synthesis using equal temperament, with the higher harmonics integral multiples of the first. The composition is tuneful and harmonious, without the appearance of beats. In the second trial, both the melodic scale and the partials belonging to individual tones were stretched uniformly. Each octave was stretched by a factor of 1.05. Each of the twelve semitones in the equally tempered scale was thus dilated by a factor of $(1.05)^{1/12}$. The partials were stretched proportionally as well. The n^{th} harmonic, instead of being a whole multiple, n, of the fundamental frequency, was made a multiple, n', of the fundamental frequency, where, in accordance with the modern conception,

$$n' = n^{\log_2 2.1} = n^{1.070389}$$

Thus, the second harmonic has a ratio of 2.1:1, the third has a ratio of 3.24:1, and the fourth has a ratio of 4.41:1, *etc.*, each with respect to the fundamental frequency. This uniform stretching of both melodic intervals and harmonics guarantees that no new interference between harmonics is introduced, while preserving any existing interference. This uniform dilation would, if anything, tend to reduce beats, because the intervals separating every pair of harmonics would increase. Nevertheless, Bach's composition loses its natural harmony, proving that harmony, in the precise sense of the term, is not a matter of beats.

Moreover, the individual voices in the second trial are blurred and difficult to distinguish. The results of the third and fourth trials are consistent with the first and second. In the third, the melodic scale was stretched, while the harmonics were not. The composition sounds out of tune, but the voices remain distinct. In the fourth, the melodic scale was not stretched, but the harmonics were. The voices are once again indistinct. This loss of voice is not difficult to understand. The rule of whole multiples in harmonics is an artifact of how a given tone is *physically produced*. When the superimposed harmonics of several tones are grouped cognitively by their ability to form whole multiples in frequency, each bundle of harmonics is separated from others that cannot enter the same relationship, allowing one to distinguish tones from different sources even when they mingle. A significant dilation of the harmonics destroys this apprehension, and voice is lost, unless the sources of sound can be localized by other means. The relative intensities of these bundled harmonics determine the *timbre* of an instrument, the quality of voice that distinguishes it from other instruments playing the very same note. Without voice, harmony remains unnoticed, for, to experience a fusion of two tones, one must first distinguish the tones. *Thus, a single voice consisting of many harmonics is not heard as a harmony, but as a distinct timbre.*

1.4 The Classical Theory of Number and Magnitude

There exist today serious mathematical impediments to the correct understanding of music, and these concern the very nature of number. To the classical or medieval mind, the mathematical arts of the *quadrivium*, no less than the verbal arts of the *trivium*, were grounded in familiar human operations. The Greek concept of number (ἀριθμός) was inseparably bound to such an operation. As Jacob Klein repeatedly points out:

> The fundamental phenomenon which we should never lose sight of in determining the meaning of arithmos (ἀριθμός) is counting, or more exactly, the *counting-off*, of some number of things.[15]

Arithmetic treats of a multitude capable of indefinite increase, whereas geometry treats of magnitude capable of indefinite division. The distinction is fundamental to classical and medieval music theory, as is clear from the treatise of Boethius:

> For multitude, beginning from finite quantity and increasing indefinitely, proceeds so that there is no end of increasing. It is limited with respect to the smallest, but unlimited with respect to the greater, and its principle is unity, than which there is nothing smaller. It increases through numbers and is stretched out indefinitely, nor is there any number that makes a limit to its increase. Magnitude, on the other hand, takes a finite quantity as its measure, but decreases indefinitely. For if there is a line of length one foot, or any other length whatsoever, it can be divided into two equals, and its half can be cut in half, and its half again into another half, so that never is any limit made to the cutting of magnitude.[16]

One observation suffices to show the distance between modern and ancient concepts of number. According to the classical mathematician, when magnitude is divided, number is multiplied: To divide a magnitude in half is to produce two lesser magnitudes. The modern mathematician would see in this same division a reduction in number, for his concept of number has been fully assimilated to measurable quantity: To divide a magnitude in half is to divide its number in half. Whereas the classical mathematician retained an ontological distinction between arithmetic and geometry, implying thereby a difference in manner of intellection, the modern mathematician circumvents this distinction by redefining number itself.

Although one can count that which is divisible—extended units for instance—no less than that which is indivisible, it does not follow that arithmetical units must be at least potentially divisible. This would be to define the science of arithmetic materially, rather than formally. Arithmetic is formally derived from the operation of counting, and the objects that are counted, whether divisible or not, are treated by that science as numerable units only. Consequently, any divisibility in the object under consideration must be treated by a different science, the proper and original operations of which imply that divisibility. Because the unit is integral, the only truly *arithmetical* ratios are those between a number and its multiples, that is, ratios that can be reduced to a whole number ratio with the unit. For instance, 6:3 reduces to 2:1 because 6 is twice 3. Proportionality in arithmetic can likewise only mean that two arithmetical ratios are reducible to the same ratio with unity. For example, both 6:3 and 4:2 reduce to 2:1, and therefore form the proportion, 6:3::4:2. But 6:4 cannot be reduced arithmetically to 3:2, because neither is an arithmetical ratio to begin with. In other words, 6:4::3:2 is arithmetically unintelligible, and *alternando*, the transposition of the two means of a proportion, is usually illegitimate within the domain of pure arithmetic. A proportion such as 6:4::3:2 can only be applied metaphorically to arithmetic. For this reason, music, which employs such ratios and proportions, cannot be an immediate application of arithmetic to hearing. The intermediate science of music presupposes a mathematical science capable of treating quantity and division, a science subalternate to arithmetic, but informed by additional principles. The definition of mathematical disciplines posited by Nicomachus of Gerasa is also therefore untenable:

> Again, to start fresh, since of quantity one kind is viewed by itself, having no relation to anything else, as "even," "odd," "perfect," and the like, and the other is relative to something else and is conceived of together with its relationship to another thing, like "double," "greater," "smaller," "half," "one and one-half times," and so forth, it is clear that two scientific methods will lay hold of and deal with the whole investigation of quantity; arithmetic, absolute quantity, and music, relative quantity.[17]

Since arithmetic itself distinguishes between number taken "absolutely" (*e.g.* 2) and "relatively" (*e.g.* 2:1), these terms do not adequately distinguish arithmetic and music. These relations, understood as existing between numbers, being nothing more than their ability to form countable products, belong to the science of arithmetic, but, taken metrically, belong to music. "Relative quantity," as described by Nicomachus, equivocates between the countable and the measurable.

1.5 Theoretical Logistic

The exegetical work of Jacob Klein is crucial to understanding both classical and modern mathematics. Klein took as his starting point the difficulty of justifying calculation scientifically, of finding the *theoretical logistic* to which Plato had once alluded:

> Plato's special demand for a theoretical logistic corresponds to the understanding that within the *unified* framework of the purely noetic [intellectual] sciences there should also be a science addressed to the pure relations of numbers as such, which would correspond to the common art of calculation and provide its foundation. This science, which is "in the service of the search for the noble and the good" (πρὸς τὴν τοῦ καλοῦ τε καὶ ἀγαθοῦ ζήτησιν), inquires into the *presuppositions* of common calculation and also of harmony, and ignores the manner in which these sciences might be pursued in other contexts.[18]

Among these presuppositions of common calculation, one difficulty stands out, specifically, that the unit of calculation must often be divided:

> But the crucial obstacle to theoretical logistic—keeping in mind its connection with calculation—arises from fractions, or more exactly, from the fractionalization of the unit of calculation.[19]

The challenge, far from trivial, is to justify intellectually the common practice of adding, subtracting, multiplying, and dividing fractions. If division of the unit is impossible at the noetic level, how can the unit be fractioned? As Klein explains, the Neoplatonic commentators usually located this divisibility in the senses:

> In every calculation or counting process we work from the outset with many units, and if in the course of the calculation we are forced to partition one of these units, then what we do is precisely to substitute something else for the indivisible unit which is subject to partitioning, while the unit itself *is not partitioned but only further multiplied*... Hence arises the necessity of making a strict distinction between the one object of sense which is subject to counting and calculation and the one as such, *i.e.*, of keeping each "one" thing strictly separate from all "ones." Each single thing can be infinitely partitioned because of its bodily nature as an object of sense. The unit which can only be grasped in thought is, on the other hand, indivisible simply, precisely in virtue of its purely noetic [intellectual] character... This property of the noetic [intellectual] unit clearly precludes calculation with it.[20]

Although this resolves the tension between the indivisibility of the unit and the practice of division, it does so at the expense of making a truly theoretical account of calculation impossible. For, according to Neoplatonic principles, the senses are instruments not of science (ἐπιστήμη), but of opinion (δόξα). The instability of sensation makes it unfit for demonstration. Consequently, unlike arithmetic, logistic is for the Neoplatonic philosopher an essentially practical and servile art. Klein sees Aristotle, rather than Plato or his followers, as the true originator of theoretical logistic, for Aristotle takes the unit of arithmetic to be a unit of measurement, understood broadly, not a separate and subsisting reality to be apprehended intellectually:

> From Aristotle's ontological conception insofar as it affects the problem of *theoretical logistic*, we can, however, draw another consequence, far more central in our context. We saw ... that the crucial difficulty of theoretical logistic as the theory of those mutual relations of numbers that provide the basis of all calculations lay in the concept of the monad [unit], insofar as it is understood as an independent and, as such, simply indivisible object. *Aristotle's criticism obviates this difficulty* by showing that this "indivisibility" does not accrue to the *monas* [unit] as a self-subsisting *hen* [one], but by virtue of the measuring character of any such unit, be it of an aisthetic [sensible] or noetic [intellectual] nature.[21]

A theoretical science of logistic is possible because unity does not belong essentially to the monad, but only insofar as it is employed as a measure. Klein explains the import of this change:

> Nothing now stands in the way of changing the unit of measurement in the course of the calculation and of transforming all the fractional parts of the original unit into "whole" numbers consisting of the new units of measurement. Thus even fractions can now be treated "scientifically." If we disregard for a moment the fact that Plato's demand for a theoretical logistic is in fact realized within a different context, namely in the general theory of proportions ... there can be no doubt that only Aristotle's conception of *mathematika* makes possible that "theoretical logistic" which suffered from the dilemma of being at the

> same time postulated by the *chorismos* [separation] thesis and yet precluded from realization by that very thesis.[22]

The unit can be simultaneously sensible and intelligible because, according to Aristotle, the intellect abstracts an intelligible species from the sensible image. The unit thus conceived possesses an intellectual character, making it suitable for demonstration, and a sensible character, making it fit for practical measurement.

Yet the essential difficulty remains. Although Aristotle's account of the monad's being does allow one to change the unit during calculation, it in no way guarantees that the relations that one commonly understands to exist in calculation will be preserved by the substitution. This is precisely why Plato's demand for a theoretical logistic arises within the *theory of proportions*, for it is the intelligibility and stability of these ratios that are at stake. Partition, which is sufficient to produce a multitude, does not of itself imply *equality* of parts, but Aristotle assumes that the production of a new unit suitable for counting also implies the production of a new common measure. His theory depends upon the identification of unit and measure, an equivocation preserved and expanded in the unit-measure of modern analysis. To the contrary, fractional measurement requires, together with the indivisible *unit*, a divisible *measure* and a relation of equality that guarantees that the fractional parts are adequately produced. It is clearly the measure that is divided in calculation, not the unit, whereas the number of units is multiplied. Number (ἀριθμός) can only be a *marker* of quantitative relation under the assumption of equal measures, not the relation itself. Aristotle begs the question, for, presumably, the equality of the fractioned parts will itself be established through measurement, which requires the discovery of still another common measure beyond the last. This process must continue indefinitely. Contrary to Klein's contention, therefore, the Platonic separation of an indivisible monad from the sensible particular has little bearing upon the question, and Aristotle fails to provide a sufficient ground for measurement and calculation.

1.6 Common Axioms

Jacob Klein did, nevertheless, identify a supposition fundamental to modern mathematics, namely, that there must to the common axioms of mathematics be a corresponding mathematical object.[23] In other words, because there are methods common to arithmetic and geometry, there must be a "quantity in general" to serve as the subject of a superior science. Or, to put it more concretely, because numbers can be "equal" and magnitudes can be "equal," because numbers can be "added" and magnitudes can be "added," numbers and magnitudes are embraced under a common genus, "quantity in general," from which they derive their common properties. Proclus, commenting on Euclid's *Elements*, affirms the existence of common axioms:

> Returning to axioms, we note that some of them are peculiar to arithmetic, some peculiar to geometry, and some common to both. That every number is measured by the number one is an arithmetic axiom; to geometry belong the principles that two equal straight lines will coincide with each other and that every magnitude can be divided indefinitely; but that

> two things equal to the same thing are equal to each other, and similar axioms, are common to both sciences. But each of them makes use of them only so far as its subject matter requires, geometry for magnitudes, arithmetic for numbers. In the same way some postulates are peculiar to certain sciences, others are common. That a number can be divided into least parts we should say is a postulate peculiar to arithmetic, that every finite straight line can be produced in a straight line peculiar to geometry, and that quantity is capable of indefinite increase common to both. For both number and magnitude are capable of such increase.[24]

In other words, "addition" and "increase," "equality" and "inequality," are predicated of both arithmetical and geometrical objects. The question is whether these terms refer to a common genus embracing two species of "quantity" or to an analogical community subordinating the two sciences. In the first case, the common axioms would belong to a mathematical science superior to both, that is, to universal mathematics. In the second, the common axioms would not belong to a superior mathematical science, but to each science analogously, with metaphysics assigning to each the being and order proper to it. The very character of first philosophy is at stake. Proclus maintains that the common axioms form a distinct science:

> Just as we have noted these common principles and seen that they pervade all classes of mathematical objects, so let us enumerate the simple theorems that are common to them all, that is, the theorems generated by the single science that embraces alike all forms of mathematical knowledge; and let us see how they fit into all these sciences and can be observed alike in numbers, magnitudes, and motions. Such are the theorems governing proportion, namely, the rules of compounding, dividing, converting, and alternating; likewise the theorems concerning ratios of all kinds, multiple, superparticular, superpartient, and their counterparts; and the theorems about equality and inequality in their most general and universal aspects, not equality or inequality of figures, numbers, or motions, but each of the two by itself as having a nature common to all its forms and capable of more simple apprehension.[25]

Just as the geometer can prove general theorems about triangles without considering different species of triangle (*e.g.* scalene or isosceles), so also the mathematician can prove general theorems about quantity without considering different species of quantity. These common theorems are demonstrated without reference to arithmetic ("number"), music ("motion"), or geometry ("magnitudes") and therefore supply principles to each:

> Consequently, we must not regard these common theorems as subsisting primarily in these many separate forms of being, nor as later born and deriving their origin from them, but as prior to their instances and superior in simplicity and exactness. For this reason, knowledge of them takes precedence over the particular sciences and furnishes to them their principles; that is, these several sciences are based upon this prior science and refer back to it.[26]

Moreover, Proclus goes beyond a generalized method dealing with the separate mathematical sciences. He also assigns a distinct and more perfect mathematical object to this common science:

> Then whose function is it to know the principle of alternation alike in magnitudes or numbers and the principles governing the division of compound magnitudes or numbers and the compounding of separate ones? It cannot be that we have sciences of particular areas of being and knowledge of them but have no single science of the immaterial objects that stand much closer to intellectual inspection. Knowledge of those objects is by far the prior science, and from it the several sciences get their common propositions, our knowledge ascending from the more partial to the more general until at last we reach the science of being.[27]

Proclus clearly objectifies "quantity in general," as indeed must any genuine Platonist who accepts the common axioms. In this respect, the early modern innovators understood Proclus well enough, and were certainly justified in seeing in him a philosophical precedent. He simply had not *substituted* this higher object for the objects of arithmetic and geometry, which is the essence of the *symbolic* intentionality uncovered by Jacob Klein.

In modern hands, symbolic intentionality has become an instrument for the reduction of all reality to a homogeneous mathematical order. Through it, the mathematician slips unawares between diverse modes of intellection, ignoring the intrinsic barriers between sciences. The history of modern mathematics is a gradual oblivion of these ontological barriers. Universal mathematics (*mathesis universalis*) has usurped the place of metaphysics, in the process spawning philosophies inimical to right reason. Any general and genuine restoration of metaphysics therefore demands an *ontology of mathematical types*. The notion of common axioms, accepted by Aristotelians and Platonists alike, is erroneous. It is but the misconception and misplacement of a science *ontologically intermediate* to arithmetic and geometry, namely music, which therefore relates these disciplines, *but not as a science superior to both*.

1.7 Symbolic Mathematics

Klein argues forcefully for a historical rupture in the inherited concept of number, a rupture that was formalized with the work of François Viète, and then expanded and developed under subsequent thinkers, such as Stevin, Wallace, and Descartes. The substance of his analysis is that modern mathematical ideation derives from a new process of concept formation, which he calls "symbolic." In this new symbolic intentionality, first intentions (concepts that intend the object) are in some manner identified with or replaced by second intentions (concepts that intend the intelligibility of the object). Though Klein maintains that this process is characteristic of modernity as such, it is with quantity that he primarily concerns himself. In the analytic art of symbolic quantities, more generally known as algebra, the concept of this or that quantity is replaced with "quantity in general," and calculations are performed upon this "quantity in general" as though it were itself some definite quantity. The relative ease of calculating with "quantity in general," instead of definite quantities, liberated mathematicians to build the impressive edifice of modern mathematical physics.

A concrete example of this kind of concept formation will make the process clearer. It is well known that the addition of an odd number and an even number will yield an odd number, whereas the addition of an even to an even or an odd to an odd will produce an even number. One can be convinced of this by induction, but it can also be deduced from the definition of odd and even numbers. Still, what is always meant is the addition of two definite numbers, even if they are unknown. The concepts of oddness and evenness (second intentions) can always be led back to that of definite collections of units (first intentions). Such is the mathematics of the Greeks and their ancient and medieval successors. Consider, on the other hand, if the abstract properties of oddness and evenness were *substituted* for concrete odd or even numbers, respectively, and subsequently used *as if they were numbers*. A symbolic calculus of the odd and even could be created in this way, and the abstracted properties rendered in symbolic axioms:

Axioms of Addition:	Axioms of Multiplication
o + o → e	o × o → o
e + e → e	e × e → e
o + e → o	o × e → e
e + o → o	e × o → e

In these expressions "o" and "e" are simply markers for the *properties* of oddness or evenness, now treated as if they were themselves being added. The fact that "e" can simultaneously stand in a single expression for two different numbers (*e.g.* 4 and 6) proves that these symbols no longer signify a definite, if yet unknown, even number, but rather *the power to specify to human intellect* the concept of evenness. By this calculus, one can rapidly evaluate the oddness or evenness of complex expressions—without ever adding or multiplying the numbers—by successively applying the symbolic axioms:

Symbolic Expression	Symbolic Axiom Applied	Parallel Numerical Expression
1. o × (o × (e + o) + o)	1. Given.	1. 5 × (7 × (2 + 3) + 1)
2. o × (o × o + o)	2. e + o → o	2. 5 × (7 × 5 + 1)
3. o × (o + o)	3. o × o → o	3. 5 × (35 + 1)
4. o × e	4. o + o → e	4. 5 × 36
5. e	5. o × e → e	5. 180

It is therefore concluded that o × (o × (e + o) + o) → e. This calculus has been completely detached from the nature of oddness and evenness as verified in actual assemblages of units. While there is great economy in this procedure, this purely formalistic treatment divorces mathematical deduction from construction. In symbolic mathematics, the mathematician can impose axioms arbitrarily, rather than abide by the limits of the imagination. The mere consistency and completeness of axiomatic systems, rather than their mathematical *truth*, thus become the overriding concern of foundational mathematics.

1.8 The Stepwise Symbolic Origin of Algebra

Algebra originates in this kind of symbol-generating abstraction. As in the calculus of odd and even, an intelligible property is substituted for the object that originally bears that property. Consider the development of "negative numbers." It is universally recognized that one may subtract a smaller number from a larger, yielding their difference. However, the intelligibility of subtraction, unlike that of oddness and evenness, does not belong to all numbers indiscriminately. So, while the difference of 5 and 3 is 2, there cannot, in accordance with the natural concept of number, be a difference of 3 and 5. The nature of number poses limitations to the operations that can be performed upon it. Nevertheless, the property of being able to form a difference is common to an arbitrarily large collection of number-pairs, and can at least be conceived generally. When this intelligibility is substituted for numbers themselves, that is, when being a number means being subject to subtraction, the intellect is led perforce to accept the reality of operations such as $3 - 5$, resulting in "negative numbers," in this case, -2. The possibility of operation no longer derives from the nature of number. Rather, the nature of number is reconceived for the sake of operation. The new "numbers," called "integers," putatively absorb the original natural numbers as special cases of the more general concept of "being subject to the operation of subtraction."

Mathematicians have subsequently justified "integers" as ordered pairs of natural numbers that obey newly defined operations. Two ordered pairs of counting numbers, (a_1, a_2) and (b_1, b_2), are said to be equivalent if $a_1 + b_2 = a_2 + b_1$. "Integers" can be "added," "subtracted," and "multiplied":

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$
$$(a_1, a_2) - (b_1, b_2) = (a_1 + b_2, a_2 + b_1)$$
$$(a_1, a_2) \cdot (b_1, b_2) = (a_1b_1 + a_2b_2, a_1b_2 + a_2b_1)$$

These operations upon pairs of natural numbers, together with the previous equivalence relation—none of which invoke the subtraction of natural numbers—express the once fully *imposed* but now *contrived* operability of "integers," the coherence of which depends upon the complete replacement of the original natural numbers with number-pairs. A "positive integer" and a natural number are very different in their intelligibility. In fact, a given "positive integer" is not even a single number-pair, but rather an infinite class of number-pairs satisfying the equivalence relation. The natural numbers are a subset of the integers only in a metaphorical sense, for an ontological limitation of natural numbers has been subverted.

The same sort of substitution is necessary to overcome the limitation of divisibility. Although 6 can be divided evenly by 3, yielding 2, 8 cannot be so divided, for it is not a multiple of 3. The property of divisibility is imposed as an additional condition of "being a number," yielding a new number-concept, the "rational numbers." The ratio, 8:3, which was to both ancient and medieval mathematicians an intelligible *relationship* between two numbers, is itself henceforth to be understood as a "number." As with integers, this was later justified by an artificial construction contrived precisely to preserve the *operations* that centuries before had been imposed universally. Once again, it is the desired operation that determines the new concept of "number," not the nature of number that determines possible operations. Algebraic abstraction involves at every turn the inversion of the medieval principle, *operatio sequitur esse*. A "rational number" is defined in terms of ordered pair of *integers*, (n, d), a numerator and denominator, with d not equal to the "integer," 0. Two such "rational numbers," (n_1, d_1) and (n_2, d_2), are said to be equivalent if $n_1d_2 = n_2d_1$. Just as an "integer" signifies an entire class of ordered pairs of natural numbers, so also a "rational number" signifies a whole class of ordered pairs of "integers." Operations upon these "rational numbers" are defined as follows:

$$(n_1, d_1) + (n_2, d_2) = (n_1d_2 + n_2d_1, d_1d_2)$$
$$(n_1, d_1) - (n_2, d_2) = (n_1d_2 - n_2d_1, d_1d_2)$$
$$(n_1, d_1) \cdot (n_2, d_2) = (n_1n_2, d_1d_2)$$
$$(n_1, d_1) \div (n_2, d_2) = (n_1d_2, n_2d_1)$$

This new number-concept is then substituted for that of the integers. The natural number, 3, the "integer," 3, and the "rational number," 3, are by no means the same. Each has a different and, strictly speaking, irreconcilable form deriving from different levels of symbolic intentionality. The natural numbers are a subset of the integers and the integers, in turn, a subset of the rational numbers *only* under the condition that the more primitive classes have been discarded.

Richard Dedekind brought this revolution to its fulfillment in his definition of the "real numbers," which are simply ways of partitioning the so-called "rational numbers" into two classes, much as a point is supposed to divide a line into two rays.[28] Partition no longer yields an increase in number. Rather, partition is itself the number. Since Dedekind's cut (*schnitt*) cannot itself be cut, one has the impression, despite the ensuing history of paradox and conflict, that the science of calculation has been ultimately and adequately grounded. Number is therefore nothing more than the property of partibility belonging to extension. The incoherence of Dedekind's definition can only be perceived by studying the concatenation of sophisms that prepared the way for it.

The process of substitution is thus continued, so that roots of "positive rational numbers" are accommodated by the introduction of "irrational numbers," and then roots of "negative rational numbers" by "imaginary numbers." At each step, a new number-concept is formed precisely to permit the more universal application of an operation, and at each step the old number-concept gives way to the new, with *anti-ontological consequences* for the substitution. "Negative number" places privation on an equal footing with actuality. "Rational number" confuses being in relation with being taken absolutely. "Irrational number" collapses the distinction between act and potency. In this way, Klein's analysis can be fruitfully expanded to include a whole hierarchy of symbol-generating abstractions. "Number" at each new level is conceived as *matter capable of being worked*, and belongs to a higher symbolic order than the previous concept. The imposition of operability goes hand in hand with the desire, voiced by Descartes, to become "like masters and possessors of nature,"[29] but it does so by closing every mathematical door leading to genuine first philosophy. The demand for operational closure in mathematics is a revolutionary act against the metaphysical order. It is a rejection of nature as given in favor of its specification in accord with human technique and desire. Reversing this intellectual subversion therefore entails the counter-revolutionary process of putting mathematical objects back into a coherent ontological whole.

As Klein has argued, calculation with species (*logistice speciosa*), that is, with the property of "being a number," is grounded in an original calculation with numbers (*logistice numerosa*). The former derives its warrant from the latter, but allows the mathematician to deal with entire classes of mathematical objects. The progression from $3^2 + 4^2 = 5^2$ to $a^2 + 4^2 = 5^2$ involves a substitution of a, the property of being a number, for 3, the number itself. One could not otherwise justify the manipulation of an *unknown* number. The substitution, however, is tentative; it demands to be resolved into a definite number. This is an indeterminacy with respect to the intellect alone, and when asked, "What is the value of a?" a proper response is, "I do not know." When, on the other hand, one writes $a^2 + b^2 = c^2$, the literals, a, b, and c, reflect only a *variable* relationship between three subjects, each being nothing more than the property of "being a number." If asked the "value" of the variable, a, one should not respond, "I do not know," but rather, "Whatever you wish." The modern variable is primarily indeterminate with respect to the will, and only in consequence with respect to the intellect. This is exactly what is needed by the scientist and engineer to manipulate and control nature in accordance with human desires. However, because the symbolic intentionality of a variable abstracts from numbers themselves, there must always be an independent means of generating the definite number, which presupposes an independent means of knowing mathematical possibilities. Symbolic mathematics is always parasitic upon the mathematical imagination.

1.9 Symbolic Abstraction and the Problem of Measurement

The facility and scope of symbolic mathematics have made it a potent instrument in the hands of its modern practitioners, but at the price of obscuring the act of measurement. There cannot be a *theoretical* problem of measurement when the intelligibility of quantity is taken for quantity itself. When the identity of quantitative *concepts* and *constructs* is presupposed, there is no need for an *act* of measurement to mediate them. (The modern fate of incommensurables is a salient example.) Lagrange announces this final break with traditional mathematics in the preface to the 1788 edition of his *Méchanique Analitique*: "One will find no figures in this work. The methods that I set forth here require neither constructions, nor geometric or mechanical reasoning, but solely algebraic operations, subject to a regular and uniform procedure."[30] There can henceforth be only the *practical* problem of measurement, specifically, of presenting an already *assumed* intelligibility to the senses, isolating it physically, and determining in a controlled experiment its relationship to other intelligibilities. A distinction between theoretical and experimental physics thus develops, together with a suitable division of labor. This distinction informs the modern epistemological debates between rationalists and empiricists. Rationalism is essentially the philosophical assertion of the identity of the objects of human knowledge with their intelligibility. If all human concepts include the complete intelligibility of the objects they intend, then, in principle, all things are knowable to man. Empiricism is the recognition that, with the theoretical problem of measurement discarded, only the senses are capable of distinguishing between true and false physical theories. The argument between rationalists and empiricists therefore hides a deeper agreement about the nature of both mathematical and physical objects, an understanding pre-modern thinkers would have rejected for serious philosophical reasons.

When properties are only with difficulty distinguished from the objects bearing the properties, symbolic substitution can be insidious. The results of the last century suggest that further progress in physics now demands a mathematical understanding of measurement. This act of measurement, banned *a priori* by the modern mathematical apparatus, has already reasserted itself in distorted forms, first in relativity theory, and then in quantum mechanics. One cannot expect to understand the counterintuitive aspects of modern theory without going beyond—or, better yet, *getting behind*—the inherited symbolic calculus. It must truly be wondered to what extent the so called "laws of physics" are merely the residue of unacknowledged mathematical limitations.

1.10 The Act of Measurement

In view of these grave confusions, it is important to clarify by suitable terms the distinctions already made. *Parity* will henceforth signify sameness in number; two numbers will be described as either *parate* ($\doteq$) or *disparate* ($\dot{\neq}$). When two numbers are disparate, one will be said to be *fewer* ($\dot{<}$) or *more* ($\dot{>}$) than the other. *Equality* will signify sameness in quantity; two quantities will be described as *equal* ($=$) or *unequal* ($\neq$). When two quantities are unequal, one will be described as *greater* ($>$) or *less* ($<$) than the other. *Congruence* will signify sameness in linear extension; two extensions will be described as *congruent* ($\cong$) or *incongruent* ($\tilde{\neq}$). When two extensions are incongruent, one will be described as *larger* ($\tilde{>}$) or *smaller* ($\tilde{<}$) than the other. The relations that exist between diverse orders of mathematical being can thus be expressed without confusion.

The act of measurement adds three conditions to that of enumeration. First, the enumerated units must include quantity. Second, the quantities in these units must be equal, making of them a *common measure*. Third, the quantities must be juxtaposed. With these conditions met, the equality of two quantities is verified when the number of such measures is the same. Failure to establish equality can therefore occur in several ways:

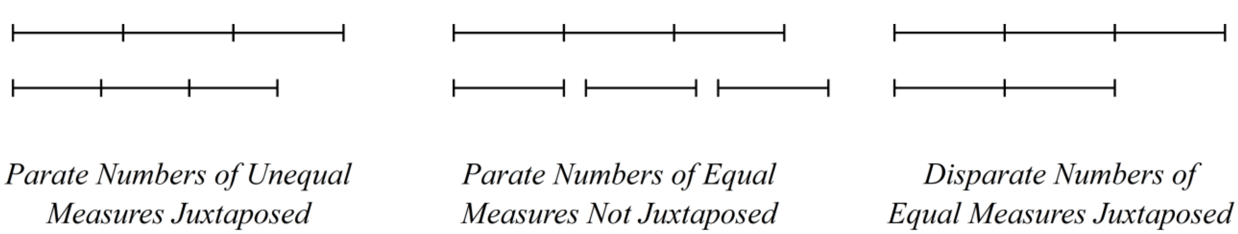


Figure 4: Failure in Measurement

The quantity measured by m juxtaposed units of measure [μ] will be represented as m[μ]. If $m \doteq n$, then $m[\mu] = n[\mu]$. If $m \dot{>} n$, then $m[\mu] > n[\mu]$. If $m \dot{<} n$, then $m[\mu] < n[\mu]$. Because the operation of counting is superimposed upon quantity, it is easy to abstract from the three conditions of measurement. The mathematician routinely does so by signifying measurable quantity through numbers, as when he represents fractioned *quantities* as a numerator and denominator. This abstraction can be legitimate, so long as he keeps in mind the aspects of measuring from which he prescinds: quantity, equality, and juxtaposition. When, however, the mathematician speaks of "rational" or "irrational" numbers, he has surely lost sight of the distinctions between number, ratio, and quantity. This confusion of number and quantity, which forms the basis of algebra, is only further exacerbated by the Cartesian assimilation of geometric relations to algebra.

1.11 Fraction and Meter

In meter one uses numbers, together with the relation of equality, to measure quantities. It must be emphasized that the fractional quantities of which it treats are not *fractional numbers,* but rather, *numbers of fractioned measures*. These fractional measures can themselves be understood metrically. For example, a *third-measure*, [μ/3], is defined in relation to [μ] as that measure for which [μ] = 3[μ/3]. Meter considers *eight third-measures*, designated 8[μ/3], but never *eight-thirds of a measure*, designated (8/3)[μ]. Every fraction of the original measure permits a subsequent recounting of units having the new measure, but the unit as such, which is the basis of counting, is not thereby divided. Fractional quantities are treated formally by a double relation, and therefore do not possess the unity and status of number. To take the previous example, *eight third-measures* posits two metrical relations:

$$[\xi] = 8[\mu/3], \text{ with } [\mu] = 3[\mu/3]$$

There is no metrical relation between [ξ] and [μ], but each has a metrical relation to a common measure, [μ/3]. It is the formal treatment that is at issue. The quantities, materially speaking, are such that there remains the possibility of developing a relation between [ξ] and [μ] unmediated by a common measure. One would be wholly justified in calling such a relation "eight-thirds," but it simply cannot be justified by the principles of meter.

David Rapport Lachterman observes that classical mathematicians generally kept these limitations in sight. Nevertheless, their methods sometimes outstripped their rationality:

> A ratio for Euclid ... is a *relation* between magnitudes; it is not a magnitude or a quantity in its own right. (Hence, it is most emphatically *not* a "rational number"...) Therefore, operations to which magnitudes are "naturally" subject (such as addition of line-segments, multiplication of numbers and its geometrical counterpart...) would appear to be alien intruders once transplanted to the domain of ratios (or, indeed, the domain of proportions, as will happen in algebra when equations are added to, or multiplied by, one another, and so on.) Nonetheless, on the most plausible reading of "compound ratio" in Euclid, we are being asked to allow some such alien operation to be applied to a ratio or, more precisely, to a pair of ratios.[31]

The compounding of ratios used in Euclid's *Elements* presupposes that the ratios expressing fractional quantities can themselves be multiplied. But ratio is neither magnitude nor number:

> Once more, however, we need to keep in mind that magnitudes are *in a relation* (ratio) *with respect to* "size"; their relation does not itself *have* a "size," nor would the product of two relations, if we could make sense of that, *be* a "size."[32]

Aristotle succumbs to a similar temptation. He treats the numerical ratios that are the basis of classical Greek harmony as quantities that can be compared, and even measured:

> In music it [the measure] is the diesis, since it is the smallest, and in articulate sound it is the phoneme... But the measure is not always one in number, rather sometimes several, such as two dieses—not according to hearing, but in the ratios—and the several articulate sounds used for measuring [speech]...[33]

The diesis is a ratio, not a quantity—and consequently not a measure. To be sure, Aristotle uses "measure" in a general and fluid sense, but this is precisely why attributes belonging to quantity alone are too easily transferred to quantitative relations. Lachterman explains how this false union of ratio and quantity ultimately permits the transformation of classical proportion into the modern equation:

> The early modern "heir" to the seat of power occupied by proportions in Greek and medieval mathematics is the equation, and an equation, as the early moderns were always aware, is nothing other than a series of ratios which can be combined and, for practical purposes, set equal to zero. Viète speaks of an *analogism* as the transformation of an equation into the original form of a proportion [*analogia*]; similarly, Leibniz talks of the *conversio aequationis in analogiam vel contra* ("the conversion of an equation into a proportion or the reverse"). The entire format of algebraic analysis and geometrical construction which is based on the use of equations presupposes a single meaning for the Euclidean criterion of "being in the same ratio," namely, that the two (or more) ratios are *quantitatively* equivalent.[34]

Modern mathematical analysis therefore rests on an illegitimate identification. Despite the practical successes of modern technique, five hundred years of mathematical philosophy have been vitiated from the start.

1.12 A Critique of Metrical Theories of Harmony

It is now possible to critique metrical theories of harmony in light of the foregoing principles. Although the most basic intervals of music can certainly be expressed as a pair of natural numbers, one numerating, the other denominating, this metrical treatment cannot be said to have attained the essential principles informing harmonic science. Recall that meter does not express an immediate relation between two quantities, but a pair of arithmetical relations to a third that measures each. Harmony must be described by the *formal* part of the two metrical relations that compose it, which are the numerator and denominator, respectively. These forms are counting numbers, and it is difficult to see what arithmetical property these two numbers could have in common that would in any way represent a harmony.

Metrical theories of harmony suffer from another closely connected flaw. Real tones are never exactly related by the theoretical ratio. Their quantities, materially speaking, can be brought extremely close to these ratios in sensible matter, and arbitrarily close in intelligible matter. The difficulty is that as one brings the material relation of quantities closer and closer to the ideal, the numbers through which the relations are formally mediated do not converge upon, but, in fact, diverge from the ideal numbers—a result well known by number theorists. Consider an example: The ideal interval called the *perfect fifth* is designated metrically by the ratio, 3:2. In practice, the ratio of two tones signified by it will diverge from it. Let the actual ratio of audible frequencies be 299:199, a ratio that will be perceived by the ear as a perfect fifth. These two ratios are materially quite close, but, formally speaking, the latter departs numerically from the former. Let the two sounds now be brought into a closer material relation, designated metrically by 2999:1999. As the relation converges materially upon that designated by the ratio, 3:2, the formal part of the metrical relations, the numerators and denominators, becoming ever larger, diverge from the ideal. Since the relation between sounds can only be apprehended through form, harmony cannot be grounded in metrical principles. Yet the material relation between the quantities designated by this formal 3:2 ratio is somehow correct. There must be another form of quantitative relation.

The metrical theory of harmony also fails to account for octave equivalency, whereby tones separated by an octave, that is, tones in a 2:1 ratio of frequencies, are in some respect heard identically. Using octave equivalency, one can distinguish between pitch height, which signifies the absolute frequency of a tone, and pitch class, or relative position within the octave. (This equivalence is also preserved in compounding the octave ratio, so that tones in a frequency ratio of 2:1, 4:1, or 8:1 all have an identical quality.) Understood metrically, the ratio, 2:1, should not yield qualitative identity in the two tones, but only similarity, because their fundamental frequencies *differ*, and because their harmonics (frequency multiples) are *not fully shared*. This applies even more to the ratios, 4:1 and 8:1. One may object that tones separated by an octave are indeed merely similar, not in any sense identical, but a little reflection will dispose of this claim. A proportional mistuning of each tone, by one part in ten for example, will change pitch heights, but keep them within the same pitch class, because they will retain their octave ratio, and with it their equivalency. One part of hearing is changed, the other unchanged. Pitch height is absolute in character, whereas pitch class exists only in relation. This critique of metrical theories of harmony concerns only relations between pitches, not pitches taken absolutely. An unwillingness to distinguish absolute and relative properties is therefore no counter-argument.

Lastly, one may add empirical difficulties to the metrical theory of harmony. The notion that harmony derives from ratios involving small natural numbers can be traced to antiquity. The prejudice is almost ineradicable, despite the interminable debates about the true mathematical ratios of intervals beyond the octave (2:1), the perfect fifth (3:2), and the perfect fourth (4:3). Yet the perfect fifth (3:2) is no more harmonious than the perfect fourth (4:3), despite the simpler ratio, for the related tones are in the same pitch classes respectively. This suggests that what has been taken as a fundamental principle is more likely a mathematical coincidence.

1.13 The Meaning of Exponents

The early modern revolution in mathematical ideation can be fruitfully approached through a commonplace of modern mathematical expression: the exponent. Greek mathematicians understood well that lengths, areas, and volumes can all be compared in ratios. Yet they were very careful to maintain homogeneity between the terms of the ratio. Lengths can only be compared to lengths, areas to areas, and volumes to volumes. To speak of a ratio between area and volume is utterly meaningless, for there is no measure common to the two. This distinction of kind serves as the basis for a distinction in numbers: length-numbers, area-numbers, and volume-numbers. These are no less heterogeneous than the lengths, areas, and volumes they count. Nevertheless, an unresolved tension exists between the kinds. For example, the relationship between the diagonal and side of a square cannot be expressed metrically, for the magnitudes are incommensurable. And yet the area of a square formed on the diagonal can be represented metrically as twice the area of the original square. This certainly implies some hidden relationship not only between lengths and areas, but between length-numbers and area-numbers. Ancient mathematicians refused, however, to treat magnitudes as numbers on account of the philosophical incoherence it entails.

Algebra uses an operational technique that bypasses these impediments. The homogeneity of the "quantity in general" posited in modern algebra necessarily abolishes dimensional difference. Whereas classical mathematicians expressed the area of one square as a multiple of another, modern mathematicians typically express it as an iterated operation. Areas and lengths are heterogeneous in classical geometry, but in algebra the quantity resulting from an iterated operation is homogenous with the original. The concrete example in *Figure 5* illustrates this.

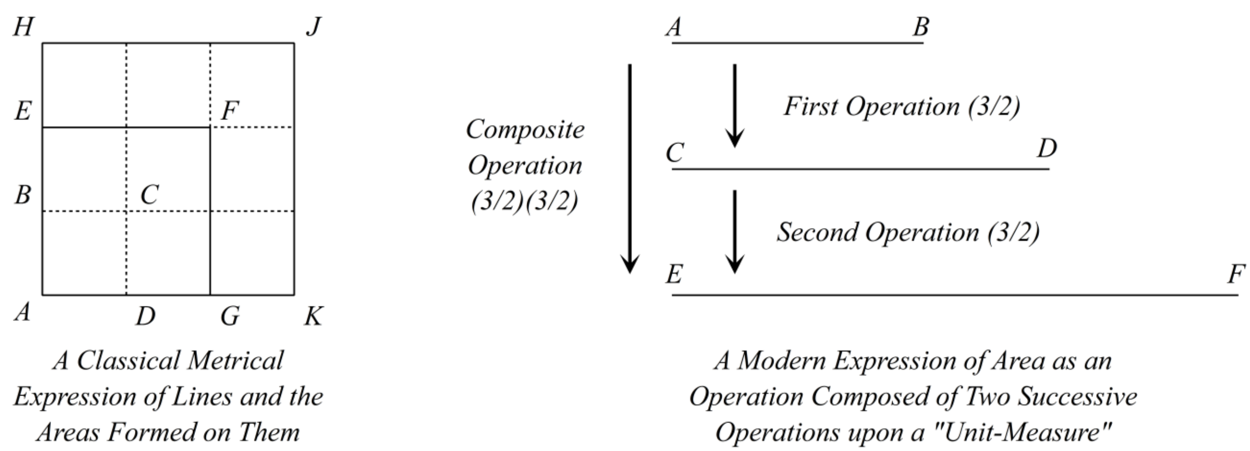


Figure 5: The Classical and Modern Expressions of Area

The classical mathematician conceived the relation of length and area somewhat as follows: Length *AD* measures *AG* and *AK*, yielding 2 and 3 units, respectively. Likewise, *AB* measures *AE* and *AH*, also yielding 2 and 3 units, respectively. The area formed by *AD* and *AB* is *ABCD*. Likewise, *AEFG* is the area formed by *AG* and *AE*, and *AHJK* is the area formed by *AH* and *AK*. *ABCD* is a measure of the area of both *AEFG* and *AHJK*, yielding 4 and 9 units, respectively. The ratio of area *AHJK* to area *AEFG* is therefore 9:4, whereas the ratios of lengths *AK* to *AG* and *AH* to *AE* are 3:2. The composition of these length ratios is $(3\cdot3):(2\cdot2)$ or 9:4, which means that magnitudes act analogously to numbers, even if they cannot really form products in the manner of numbers.

The metrical reduction of one length to another thus extends to the power to produce area, in such a way that the formation of an area from two lines corresponds to the product of the two numbers of the linear measures. Let $[\mu_1\mu_2]$ signify the area formed on lengths $[\mu_1]$ and $[\mu_2]$, and $[\xi_1\xi_2]$ the area formed on lengths $[\xi_1]$ and $[\xi_2]$. If lengths $[\xi_1]$ and $[\xi_2]$ are measured by $[\mu_1]$ and $[\mu_2]$, such that $[\xi_1] = m[\mu_1]$ and $[\xi_2] = n[\mu_2]$, then $[\xi_1\xi_2] = (m\cdot n)[\mu_1\mu_2]$. To speak of the composition of ratios, then, is to say: If $[\xi_1] = m[\mu_1]$ and $[\xi_2] = n[\mu_2]$, and likewise $[\sigma_1] = p[\mu_1]$ and $[\sigma_2] = q[\mu_2]$, it follows that $[\xi_1\xi_2] = (m\cdot n)[\mu_1\mu_2]$ and $[\sigma_1\sigma_2] = (p\cdot q)[\mu_1\mu_2]$. This twofold relation can be abbreviated in fractional terms: If $[\xi_1] = m[\sigma_1/p]$ and $[\xi_2] = n[\sigma_2/q]$, then $[\xi_1\xi_2] = (m\cdot n)[\sigma_1\sigma_2/(p\cdot q)]$. That is, if two lengths are in the ratio m:p and two others are in the ratio n:q, then the areas formed from the length pairs are in the ratio $(m\cdot n)$:$(p\cdot q)$.

The predominant modern conception is wholly different. The mathematician posits operations capable of transforming one quantity into another homogeneous with the first. Without philosophical justification, he presupposes an immediate relation between quantities. "Area" is but the composition of two of these operations, which together yield a quantity homogeneous with the original. In the concrete example considered above, the "rational number" (3/2) operates upon a quantity designated by *AB*, yielding *CD*. This "rational number" then operates upon *CD*, yielding *EF*. The "rational number" relating *EF* to *AB*, formed by compounding these "rational numbers," is consequently $(3/2)^2$ or (9/4). The exponent, 2, signifies primarily the number of iterations of the operation, and only secondarily the number of dimensions. Modern ideation can be expressed thus: If $[\mu_2] = (p/m)[\mu_1]$ and $[\mu_3] = (q/n)[\mu_2]$, then $[\mu_3] = (q/n)(p/m)[\mu_1]$, or $[\mu_3] = ((q\cdot p)/(n\cdot m))[\mu_1]$. On account of the homogeneity of quantity, there is no reason to stop the compounding of ratios at the third. Mathematicians and natural scientists are emboldened today to speak of dimensions beyond the three extended dimensions precisely because their ideation is operational, and consequently severed from the limitations of extension. Fractal geometers go so far as to posit *non-integral (Haussdorff) dimensions*. These developments, however, are a mere artifact of the reduction of length, area, and volume to homogeneous quantity.

This dimensional revolution can be observed in Descartes' seminal work in analytic geometry. Area and volume have ceased to exist in their traditional and intuitive meanings. What has replaced them is a system of linear geometric ratios doubling as algebraic quantities. He announces this reduction at the commencement of *La Géometrie*:

> Any problem in geometry can easily be reduced to such terms that a knowledge of the lengths of certain straight lines is sufficient for its construction.[35]

Exponents have become entirely operational, signifying a number of operations on homogeneous linear quantity, yet still somehow referring to the areas and volumes that they have replaced:

> Here it must be observed that by a^2, b^3, and similar expressions, I ordinarily mean only simple lines, which, however, I name squares, cubes, *etc.*, so that I may make use of the terms employed in algebra.[36]

This new operational mathematic is so tenuously linked to traditional areas and volumes that the mathematician must be reminded to keep operational consistency with regard to exponents:

> It should also be noted that all parts of a single line should always be expressed by the same number of dimensions, provided unity is not determined by the conditions of the problem. Thus, a^3 contains as many dimensions as ab^2 or b^3 ... It is not, however, the same thing when unity is determined, because unity can always be understood, even where there are too many or too few dimensions; thus, if it be required to extract the cube root of $a^2b^2 - b$, we must consider the quantity a^2b^2 divided once by unity, and the quantity b multiplied twice by unity.[37]

This dimensional requirement does not follow from the nature of operational mathematics itself, but from the desire to translate traditional geometric problems into this new mathematical idiom.

1.14 The Algebraic Division of Operation

Much that would have appeared absurd to the classical mathematician is meaningful to the modern only because the latter conceives "number" in an operational fashion that preserves the homogeneity of quantity. Take, for instance, a "polynomial function," such as $f(x) = x^1 + x^2 + x^3$. To the Greek mathematician, the first term could only have signified a length-number, the second term the product of two length-numbers, that is, an area-number, and the third term the product of three length-numbers, that is, a volume-number. Since these quantities, thus interpreted, are heterogeneous, they cannot be added, and the entire expression is meaningless. However, conceived from the modern operational standpoint, the first term signifies a single operation upon a "unit-measure," the second an iteration of that operation upon the result, and the third a further iteration of the operation upon that result. The quantities understood in each term are homogeneous with the unit-measure and therefore with each other, permitting them to be "added." This homogeneity of all quantity is what allows the natural scientist to write such quantitative relations between "physical variables" as $y = y_0 + v_0t + at^2/2$. Since these "variables" do not express their kinds in the imagination, it becomes necessary to append to them physical "units"—for example, meters (m), seconds (s), meters per second (m/s), and meters per second per second (m/s^2)—designating precisely the kind of quantity that belongs to each. These modern "units" merely guarantee that homogeneous quantity, itself undifferentiated, is coherently applied to a differentiated world. Ratios formed from two or more different "units," m/s for example, mark quantities that depend upon the homogeneity of operational mathematics for their very intelligibility, yet refer ultimately to a real heterogeneity. This indicates that the relationships posited by modern mathematical physics have a derived character, and that the underlying reality is not truly apprehended by them. Modern scientific "units" are part of an interpretation that refers theories that are ever more remote from human experience back to that experience. As the mathematical forms have become increasingly abstruse, so also has that link become more fragile.

Yet algebra goes far beyond a merely operational viewpoint in mathematics, which is coherent, if not yet properly developed or legitimately applied. Its acceptance presupposes the indefinite division of operation, as though the form of operation were itself matter. In a previous example, the exponent signified a certain number of iterated operations upon some quantity, but algebra also uses fractional exponents. This can sometimes be justified. If there exists an operation, (8/3), one may write (64/9)[μ] as (8/3)(8/3)[μ], or $(64/9)[\mu] = (8/3)^2[\mu]$, that is, as an iteration of (8/3). What is meant, then, by $(64/9)^{1/2}$ is the decomposition of (64/9) into two identical operations, namely, (8/3). This decomposition is possible because the operation is indeed composite. One may therefore write $(64/9)^{1/2}[\mu] = (8/3)[\mu]$. On the other hand, the rational operation, (2/1)[μ], cannot be decomposed into two identical rational relations. Unless there exists an immediate quantitative relation that cannot be metrically expressed, $(2/1)^{1/2}[\mu]$ is wholly meaningless. To perform one of two operations is intelligible, but to perform half an operation is nonsensical. The iteration of operation belongs to the domain of the numerable, not the measurable. There can be no fraction of an operation as there can be a fraction of a quantity. *Operation can only be divided to the extent that the number of iterations is composite and can itself be divided*.

There is a twofold conceptual displacement in algebra, in both cases illegitimate. The first is to transfer the formal character of the operation to its product, so that the quantity produced by the operation attains the status of "number." The second is to transfer the original quantity, taken as a "unit-measure," to the form of the operation. The form thus appropriates the indefinite divisibility that is proper to quantity. This arbitrary imposition of divisibility at the level of operation leads to the logarithm, by which a "product" (or "quotient") of "rational quantities" is treated as a "sum" (or "difference") of "operational quantities." This infinite divisibility of operation also yields a uniquely modern form of materialism. It is not the static materialism of the ancient atomists, who sought to explain all reality in terms of geometric solids, thus resolving all substance into the disposition of bodies. It is rather the dynamic materialism of modern physics, in which the form and power of operation, including that of the soul, is assimilated to extended matter.

1.15 Algebra as Confused Music

The ancient theory of Aristoxenus, which is grounded in the scientific method of Aristotle, approaches harmony from an empirical standpoint, with results that coincide in many ways with modern mathematical practice. Aristoxenus posits a different relationship between the senses and reason than do the Pythagoreans or Platonists:

> Through hearing we assess the magnitudes of intervals and through reason we apprehend their functions [δύναμεις]. We must therefore become practiced in assessing particulars accurately. While it is usual in dealing with geometrical figures to say, "Let this be a straight line," we must not be satisfied with similar remarks in relation to intervals. The geometer makes no use of the faculty of perception: he does not train his eyesight to assess the straight or the circular or anything else of that kind either well or badly; it is rather the carpenter, the wood-turner, and some of the other crafts that concern themselves with this. But for the student of music accuracy of perception stands just about first in order of importance, since if he perceives badly it is impossible for him to give a good account of the things which he does not perceive at all.[38]

According to Aristoxenus, the senses lead and intelligence is concerned primarily with establishing an order among the intervals apprehended by the senses. His opposition of music to geometry implies that music does not possess an intelligible matter of its own upon which judgments may be exercised apart from the senses.

Nevertheless, Aristoxenus maintains that pitch forms a continuum analogous to that of extension. Two notes terminate an interval, just as two points terminate a line segment:

> Now that this is understood we must say what a note [φθόγγος] is. To put it briefly, a note is the incidence of the voice on one pitch: for it is when the voice appears to rest at one pitch that there seems to be a note capable of being put into a position in a harmonically attuned melody [μέλος ἡμόσμενον]. That, then, is the sort of thing a note is. An interval [διάστημα] is that which is bounded by two notes which do not have the same pitch, since an interval appears, roughly speaking, to be a difference between pitches, and a place capable of receiving notes higher than the lower of the pitches which bound it, and lower than the higher of them.[39]

For Aristoxenus, intervals are not ratios of pitches, but differences between pitches, which presupposes that intervals are themselves quantities as well as relations between quantities. It is on the basis of this conception that Aristoxenus rejects the assertion of Epigonus that a note has breadth.[40] For the latter's assertion, judged according to the former's analogy, is absurd: Aristotle had already demonstrated that the terminus of extension, namely the point, cannot in turn be extended. Yet the probable rationale of Epigonus is not so difficult to grasp: One perceives the same harmony even when the real interval is inexact, implying that there is something flexible in the apprehension of the interval itself. The intervals in music admit some limited degree of practical variation.

By its identification of ratio with quantity, algebra is the mathematical fulfillment of Aristoxenian harmonic theory. The algebraic understanding of music is best observed in the theory of Simon Stevin, particularly in his Second Postulate: "All whole tones to be equal and likewise all semitones."[41] Such a postulate is unintelligible unless one conceives pitch as an Aristoxenian continuum. Classical theorists of harmony from Pythagoras onwards understood the impossibility of preserving even the perfect fifth (3:2) if one posits identical subintervals of the octave (2:1). Such an imposition destroys the very concepts of number and measure by introducing incommensurable magnitudes into music. Stevin, reinforced by the practical formalism of algebra, posits the division of the octave into 12 “equal” semi-tones, with the “product” of two such “ratios” being “equal” to a whole tone. The “ratio” corresponding to a semitone is, tellingly, incapable of rational expression, but can be approximated as 1.0595. The “ratio” of the whole tone is the “product” of the “ratios” of two semitones, approximately 1.1225. The counting of semitones corresponds to the number of iterations of the operation corresponding to the semitone. To further aid in comparison of intervals, the semitones were each later subdivided into 100 cents, so that the entire interval of the octave is divided into 1200 such cents. To speak to the current musical and mathematical culture it is necessary at times to employ these logarithmically based descriptions, despite their philosophical incoherence.

Number of Semitones	***Stevin’s “Ratio”***	***Decimal Approximation***	***Interval in Cents***
0	$2^{0/12} : 1$	1.0000	0
1	$2^{1/12} : 1$	1.0595	100
2	$2^{2/12} : 1$	1.1225	200
3	$2^{3/12} : 1$	1.1892	300
4	$2^{4/12} : 1$	1.2599	400
5	$2^{5/12} : 1$	1.3348	500
6	$2^{6/12} : 1$	1.4142	600
7	$2^{7/12} : 1$	1.4983	700
8	$2^{8/12} : 1$	1.5874	800
9	$2^{9/12} : 1$	1.6818	900
10	$2^{10/12} : 1$	1.7818	1000
11	$2^{11/12} : 1$	1.8877	1100
12	$2^{12/12} : 1$	2.0000	1200

Table 1: The Harmonic Intervals of Simon Stevin

Because the “ratios” produced by this division of the octave into 12 “equal” semitones approximate the whole-number ratios that had come to be accepted as the basis of harmony, the system of Stevin is in use even today, and goes by the name of “equal temperament.” By “temperament” is understood an intentional modification of a true interval for the sake of musical practice. Stevin, too easily impressed by algebraic technique, falsely took these “equal” semitones and whole tones to be the true foundation of harmony.

2. Musica Instrumentalis

2.1 The Logistical Basis of Equality

The modern mathematician does not seek out the ground of calculation as traditionally understood, but imposes a ground, implicitly accepting the mathematical consequences of such a *fiat*—even if this means *abolishing* the natural concept of number. But if the modern treatment is unacceptable, how are quantities to be grounded? As already noted, to ground measure metrically is merely to beg the question. For every equality established by meter presupposes a division into measurable units, the equality of which must again be established, a recursive procedure that must continue indefinitely. There is no smallest measure that can serve as the foundation of the science of meter. There remains, nevertheless, the possibility that equality may be apprehended through the *company of all possible measures*, the collective properties of which are irreducible to an absolute measure. On this basis, equality would be established by a means distinct from the metrical treatment, and the infinite recursion would be broken. An impossible infinity of *actual partitions* would then be replaced by an actual infinity of *possible divisions*. This foundation would not alter the ancient and intuitive understanding of number and measure.

To establish a collective property of all possible measures of a quantity, one must turn to what is coincidentally called a "harmonic series." Such a series is typically represented by:

$$1/1 + 1/2 + 1/3 + 1/4 + \cdots + 1/n$$

where n indicates the number of the latest term in the series. However, to unequivocally communicate the true meaning of the series, it must be presented as an expression of quantities that can serve as measures of some reference quantity, designated by [μ]:

$$[\mu/1] + [\mu/2] + [\mu/3] + [\mu/4] + \cdots + [\mu/n]$$

It is well known that, as n increases without bound, the series does not converge to any definite quantity, but likewise increases without bound. Nor does it bear a constant relation with the number of terms. Consider, in turn, the common measure of all the terms in that finite series. This will be the quantity denominated by the least common multiple of each of the denominators that appear in the series. Let the least common multiple (LCM) of the first n counting numbers be designated by $\Lambda f(n)$, where here $f(n) \doteq n$. That is,

$\Lambda f(1) \doteq \mathrm{LCM}(1) \doteq 1$
$\Lambda f(2) \doteq \mathrm{LCM}(1, 2) \doteq 2$
$\Lambda f(3) \doteq \mathrm{LCM}(1, 2, 3) \doteq 6$
$\Lambda f(4) \doteq \mathrm{LCM}(1, 2, 3, 4) \doteq 12$
...
$\Lambda f(n) \doteq \mathrm{LCM}(f(1), f(2), f(3), \ldots, f(n))$, with $f(n) \doteq n$

or simply,

$\Lambda_{\{n\}}(n) \doteq \mathrm{LCM}(1, 2, 3, 4, \ldots, n)$

The common measure of all the terms in the previous series will therefore be $[\mu/\Lambda_{\{n\}}(n)]$. Now consider instead the harmonic series once removed from the original:

$$[\mu/1] + [\mu/2] + [\mu/3] + [\mu/4] + \cdots + [\mu/\Lambda_{\{n\}}(n)]$$

This series also grows without bound, but for very large n, the series approximates $n[\mu]$. To obtain an exact correspondence, one must consider the quantity produced by the series as itself divided by n:

$$\left[\frac{[\mu/1] + [\mu/2] + [\mu/3] + [\mu/4] + \cdots + [\mu/\Lambda_{\{n\}}(n)]}{n}\right] \approx [\mu]$$

The mathematician may write it thus:

$$\lim_{n\to\infty}\left[\frac{[\mu/1] + [\mu/2] + [\mu/3] + [\mu/4] + \cdots + [\mu/\Lambda_{\{n\}}(n)]}{n}\right] = [\mu]$$

and by this is simply meant that the quantity expressed on the left side can be made arbitrarily close to the quantity on the right by choosing a sufficiently large number, n. Thus a relation of equality with the original measure is obtained.

2.2 How Division and Partition Differ

At this point the thoughtful reader will naturally object that a ground for the science of meter can hardly be overcome by a relation that invokes this very science for its understanding. To put it succinctly, does not this approach beg the question as much as the attempt to give a metrical ground for equality? To anticipate the conclusion, it is on account of the *arithmetical* elements of meter that the recursive metrical procedure for establishing equality must be rejected. Quantity, by its very nature, admits arbitrary denomination, and so this harmonic basis of equality, which does not invoke the counting of absolutes, is both coherent and adequate.

When the equality of two quantities is to be determined metrically, both quantities must be expressed as the same number of units of equal measure. That is, if $[\mu] = n[\varphi]$ and $[\xi] = n[\varphi]$, then $[\mu] = [\xi]$. Equality can be established metrically only by counting. The operation of counting implies a unit in addition to a measure, and this unit must have certain attributes. First, it must possess integrity *ad intra* in order that it may be a countable item. Second, it must have a bearing of alterity or otherness *ad extra*. Alterity is necessary so that the unit may stand apart from all other units being counted. Without either of these closely related attributes, the notion of number as a multitude of units becomes incoherent.

The terms in the harmonic series do not represent partitions into units to be counted, but pure denominations of quantity—not actual, but potential divisions. To verify that the measures are not separate parts, one may observe that, for a suitable choice of n, they would collectively exceed the original quantity by any multiple desired. These measures therefore lack mutual alterity and are not the countable units required by the metrical science. Nor can one find in the harmonic series the true indicator of counting, which is a *numerator*, rather than a denominator. The relation of equality derived from the series is not superimposed arithmetically, but is intrinsic to quantity itself. *Partition implies division, but division does not imply partition*.

The critic will likely answer that insofar as this ground of equality invokes denominated quantities, for example, [μ/1], [μ/2], and [μ/3], it implies a counting process. Indeed, it holds true for these fractioned measures that [μ] = 1[μ/1], [μ] = 2[μ/2], and [μ] = 3[μ/3]. However, these relations do not enter formally into the ground of equality; only the measures, [μ/1], [μ/2], and [μ/3] do. The intelligibility of the measure is prior to that of quantity measured and is independent of it. There are no operations, "divide by 1", "divide by 2", or "divide by 3", that will, beginning with [μ], yield these quantities. There is no inverse operation for the act of measurement. The science of meter cannot produce these fractional measures, but can only test their suitability. One must say, therefore, that the science of meter enters harmonics materially, but not formally, just as in providing the ground for equality of measure the science of harmonics enters materially, but not formally, into meter. These sciences concern the same subject, quantity, yet have *complementary formal principles*, making them distinct, but mutually dependent. The formal principle in meter is *number*. The formal principle in harmonics, as will now be shown, is the *arithmetical species*. Each is thus subordinate to arithmetic, but in a different manner.

2.3 The Derivation of Eidetic Ratios from Arithmetical Species

The relation of equality is fundamental to the possibility of measurement, and the exposition has thus far emphasized this role. However, the immediate relations that can be investigated eidetically are unlimited in number. A proper development of music demands that these immediate relations be elucidated, for they form the intervals of musical harmony. In the derivation of the relation of equality, a harmonic series was made to converge upon a single definite quantitative relation. This was accomplished by using a number once removed from n, the number of terms in the original series. To be more exact, that number was the least common denominator of all the terms in the original series, $\Lambda_{\{n\}}(n) \doteq \mathrm{LCM}(1, 2, 3, 4, ..., n)$. Different species of numbers bring the series into differing relations with the original quantity. Rather than the first n counting numbers, $\{1, 2, 3, 4, ..., n\}$, one may choose the first n odd numbers $\{1, 3, 5, 7, ..., 2n-1\}$. In this case, a different logistical function is defined in terms of n:

$\Lambda f(1) \doteq \mathrm{LCM}(1) \doteq 1$
$\Lambda f(2) \doteq \mathrm{LCM}(1, 3) \doteq 3$
$\Lambda f(3) \doteq \mathrm{LCM}(1, 3, 5) \doteq 15$
$\Lambda f(4) \doteq \mathrm{LCM}(1, 3, 5, 7) \doteq 105$
$\Lambda f(5) \doteq \mathrm{LCM}(1, 3, 5, 7, 9) \doteq 315$
...
$\Lambda f(n) \doteq \mathrm{LCM}(1, 3, 5, 7, ..., f(n))$, with $f(n) \doteq 2n-1$

or simply,

$$\Lambda_{\{2n-1\}}(n) \doteq \mathrm{LCM}(1, 3, 5, 7, ..., 2n-1)$$

Once again, rather than considering the series,

$$[\mu/1] + [\mu/2] + [\mu/3] + [\mu/4] + [\mu/5] + ... + [\mu/n]$$

one must consider the series

$$[\mu/1] + [\mu/2] + [\mu/3] + [\mu/4] + [\mu/5] + ... + [\mu/\Lambda f(n)]$$

With $f(n) = 2n-1$, a different quantitative relation ensues. Let the quantity to which the series converges be designated by the eidetic ratio, $\varepsilon_{\{2n-1\}}$:

$$\varepsilon_{\{2n-1\}}[\mu] = \lim_{n\to\infty}\left[\frac{[\mu/1] + [\mu/2] + [\mu/3] + [\mu/4] + \cdots + \left[\mu/\Lambda_{\{2n-1\}}(n)(n)\right]}{n}\right]$$

A metrical relation holds between the two quantities, such that $1[\varepsilon_{\{2n-1\}}[\mu]] = 2[\mu]$, but the harmonic relation is *unmediated*, so it may be written $\varepsilon_{\{2n-1\}}[\mu] = (2/1)[\mu]$. However, it must always be remembered that *the reality of the relation inheres in the species, not in the numerator and denominator of the metrical relationship*, in this case (2/1). These numbers are merely appropriated to the relation from the corresponding metrical relationship; they do not define it. To give a further example, let $f(n) \doteq 3n-2$. Then

$$\begin{aligned}
&\Lambda f(1) \doteq \mathrm{LCM}(1) \doteq 1\\
&\Lambda f(2) \doteq \mathrm{LCM}(1, 4) \doteq 4\\
&\Lambda f(3) \doteq \mathrm{LCM}(1, 4, 7) \doteq 28\\
&\Lambda f(4) \doteq \mathrm{LCM}(1, 4, 7, 10) \doteq 140\\
&\Lambda f(5) \doteq \mathrm{LCM}(1, 4, 7, 10, 13) \doteq 1820\\
&...\\
&\Lambda f(n) \doteq \mathrm{LCM}(1, 4, 7, 10, ..., f(n)), \text{ with } f(n) \doteq 3n-2
\end{aligned}$$

or simply,

$$\Lambda_{\{3n-2\}}(\mathrm{n}) \doteq \mathrm{LCM}(1, 4, 7, 10, ..., 3n-2)$$

Then

$$\varepsilon_{\{3n-2\}}[\mu] = \lim_{n\to\infty}\left[\frac{[\mu/1] + [\mu/2] + [\mu/3] + [\mu/4] + \cdots + \left[\mu/\Lambda_{\{3n-2\}}(n)\right]}{n}\right]$$

A metrical relation likewise holds between these two quantities, such that $4[\varepsilon_{\{3n-2\}}[\mu]] = 9[\mu]$, and the unmediated harmonic relation may therefore be written $\varepsilon_{\{3n-2\}}[\mu] = (9/4)[\mu]$. To restate, what has been found is not an ultimate and absolute measure of quantity, something which does not exist, but a mathematical *relation* between quantities specified through an infinite ensemble of a definite numerical species. This harmonic science is therefore rightly called *eidetic* from these numerical species (εἴδη) that inform the quantitative relations.

2.4 Bateman's Rule for Eidetic Ratios

One may dispense with the brute-force determination of eidetic ratios for a certain class of number-species (εἴδη), namely, those defined by polynomials of the first degree. Dr. Paul Bateman[42] has shown that the eidetic ratios of species of the form, $\{qn-p\}$, with p and q relatively prime (*e.g.* $\{8n-7\}$) can be obtained by the following rule:

$$\varepsilon_{\{qn-p\}}[\mu] = (q/\varphi(q)) \cdot (1/\beta_1 + 1/\beta_1 + \ldots + 1/\beta_{\varphi(q)})[\mu]$$

where $\varphi(q)$ is Euler's function, the number of natural numbers between 1 and q inclusive that do not share a common factor (other than 1) with q, and where β_1 through $\beta_{\varphi(q)}$ are these very numbers. An example will make this much clearer. In the case of $\varepsilon_{\{8n-7\}}$ there are four numbers between 1 and 8 inclusive that do not share a common factor with 8, namely, 1, 3, 5, and 7. So $\varphi(8) \doteq 4$. By appropriating the numerators and denominators from the metrical expressions, Bateman's rule yields:

$$\varepsilon_{\{8n-7\}}[\mu] = (8/\varphi(8)) \cdot (1/1 + 1/3 + 1/5 + 1/7)[\mu]$$

$$\varepsilon_{\{8n-7\}}[\mu] = (8/4) \cdot (105/105 + 35/105 + 21/105 + 15/105)[\mu]$$

$$\varepsilon_{\{8n-7\}}[\mu] = (2/1) \cdot (176/105)[\mu]$$

$$\varepsilon_{\{8n-7\}}[\mu] = (352/105)[\mu]$$

Bateman's formula suffices for all the species considered in this essay. For a non-linear species, the eidetic ratio remains uncertain in view of the approximations involved in any computational approach involving infinite terms. It is a subject ripe for investigation.

Traditionally, harmonious intervals, in themselves and their combinations, have been judged by the satisfaction afforded to the musically trained listener. The first advantage of eidetic theory is that it displaces the demands of investigation from the proficiency of the ear, which is generally wanting, into the domain of pure mathematical calculation, which, even if a rule is lacking, is capable of indefinite refinement by taking an ever-larger ensemble of counting numbers. The second advantage, as will become clear later, is that the principles of harmony are not determined formally by these ratios, but by the number-species (εἴδη) that govern them. As the size of the ensemble increases, the ratio becomes more exact, but the governing form of the relation, namely, the species, remains the same. One will therefore hear a perfect fifth whether the metrical ratio is approximately 299:199, 2999:1999, or exactly 3:2. *Table 2* lists the eidetic ratios for irreducible first-degree number-species with first coefficients less than or equal to 16. The leading coefficient, q, of the linear term will henceforth be referred to as the *order* of the species and, at least for first degree polynomials, it is this order that determines the eidetic ratio. The order will also be written in shorthand as Ω_q.

Order	*Symbol*	*Species*	*Eidetic Ratio*
1st	Ω_1	$\{n\}$	(1/1)
2nd	Ω_2	$\{2n-1\}$	(2/1)
3rd	Ω_3	$\{3n-1\}$, $\{3n-2\}$	(9/4)
4th	Ω_4	$\{4n-1\}$, $\{4n-3\}$	(8/3)
5th	Ω_5	$\{5n-1\}$, $\{5n-2\}$ $\{5n-3\}$, $\{5n-4\}$	(125/48)
6th	Ω_6	$\{6n-1\}$, $\{6n-5\}$	(18/5)
7th	Ω_7	$\{7n-1\}$, $\{7n-2\}$ $\{7n-3\}$, $\{7n-4\}$ $\{7n-5\}$, $\{7n-6\}$	(343/120)
8th	Ω_8	$\{8n-1\}$, $\{8n-3\}$ $\{8n-5\}$, $\{8n-7\}$	(352/105)
9th	Ω_9	$\{9n-1\}$, $\{9n-2\}$ $\{9n-4\}$, $\{9n-5\}$ $\{9n-7\}$, $\{9n-8\}$	(1,863/560)
10th	Ω_{10}	$\{10n-1\}$, $\{10n-3\}$ $\{10n-7\}$, $\{10n-9\}$	(250/63)
11th	Ω_{11}	$\{11n-1\}$, $\{11n-2\}$ $\{11n-3\}$, $\{11n-4\}$ $\{11n-5\}$, $\{11n-6\}$ $\{11n-7\}$, $\{11n-8\}$ $\{11n-9\}$, $\{11n-10\}$	(81,191/25,200)
12th	Ω_{12}	$\{12n-1\}$, $\{12n-5\}$ $\{12n-7\}$, $\{12n-11\}$	(1,656/385)
13th	Ω_{13}	$\{13n-1\}$, $\{13n-2\}$ $\{13n-3\}$, $\{13n-4\}$ $\{13n-5\}$, $\{13n-6\}$ $\{13n-7\}$, $\{13n-8\}$ $\{13n-9\}$, $\{13n-10\}$ $\{13n-11\}$, $\{13n-12\}$	(1,118,273/332,640)
14th	Ω_{14}	$\{14n-1\}$, $\{14n-3\}$ $\{14n-5\}$, $\{14n-9\}$ $\{14n-11\}$, $\{14n-13\}$	(81,634/19,305)
15th	Ω_{15}	$\{15n-1\}$, $\{15n-2\}$ $\{15n-4\}$, $\{15n-7\}$ $\{15n-8\}$, $\{15n-11\}$ $\{15n-13\}$, $\{15n-14\}$	(271,125/64,064)
16th	Ω_{16}	$\{16n-1\}$, $\{16n-3\}$ $\{16n-5\}$, $\{16n-7\}$ $\{16n-9\}$, $\{16n-11\}$ $\{16n-13\}$, $\{16n-15\}$	(182,144/45,045)

Table 2: The First Sixteen Eidetic Ratios

2.5 The Octave (2/1)

Musical harmony has both quantitative and qualitative features. Audible concords exhibit a fusion of tones, as if there were some sensible community between them. Boethius explains:

> For as often as two strings, one of them heavier, are stretched and at the same time plucked, thus rendering a mingled and sweet sound, and the two voices coalesce into one as if joined together, then is made what is called "consonance." But when, struck at the same time, each voice desires to go on its own, and they do not mix for the ear into one sweet sound composed out of the two, then this is what is called "dissonance."[43]

To repeat, there is a common prejudice among mathematical theorists of music that the simplicity of the numerators and denominators in the metrical ratio is decisive in apprehending harmony, as per Boethius:

> That consonance must be posited as first and sweetest, the quality of which the sense comprehends more clearly. For just as each and every thing is of such a kind unto itself, so is it apprehended even as such by the sense. If therefore that consonance which consists of the duple ratio [2:1] is better known than all others, then there is no doubt that the diapason [octave] is the first consonance of all and exceeds the others in merit, since it precedes them in understanding[44]

Although the simplest ratios, 1:1, 2:1, 3:2, and 4:3, certainly do describe the most consonant intervals, the ratios themselves shed no light upon the qualitative experience of these consonances. The intervals, 1:1 and 2:1, the unison and octave, are experienced as perfectly fused together, even if, in the case of the octave, one also hears the second tone at a higher pitch than the first. One cannot say the same about the other harmonious intervals, for, in addition to a difference in pitch, they exhibit an imperfect blending. Even further removed are those tones that do not sound well together, but out of which beautiful melodies may yet be constructed. This division is summed up by Ptolemy:

> Let us define as homophones those which, when played together, create for the ear the impression of a single note, as do octaves and those that are composed of octaves; as concordant those closest to the homophones, like fifths and fourths and those composed of these and the homophones; and as melodic those closest to the concords, like tones [*toniaioi*] and the others of that sort. Thus in a way the homophones go together with the concords, and the concords with the melodic.[45]

The true principle of harmony must account for both the qualitative experience of the fusion of tones and the commonly perceived division outlined by Ptolemy. The metrical description of harmonic ratios is so remote from musical experience that one cannot look for a solution there. One must look instead to the species (εἴδη) of numbers that give rise to these ratios.

Harmony is ultimately an inclusive ordering of two species, or to be more precise, of the infinite ensembles of natural numbers determined by those species. Consider three species:

$\{n\}$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	$\cdots$
$\{2n-1\}$	1		3		5		7		9		11		$\cdots$
$\{2n\}$		2		4		6		8		10		12	$\cdots$

$\{n\}$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	$\cdots$
$\{2n-1\}$	1		3		5		7		9		11		$\cdots$
$\{2n\}$		2		4		6		8		10		12	$\cdots$

The first ensemble, $\{n\}$, is the most comprehensive possible, and to it corresponds the unison, which is perfect equality, the unmediated ratio (1/1). The species of the odd, to which the ratio (2/1) corresponds, is consonant with the first species because it is fully comprehended by it. This explains the fusion experienced when tones separated by an octave are heard simultaneously. The species, $\{n\}$ and $\{2n-1\}$, have, moreover, a property unique to themselves. If one separates the odd numbers, $\{2n-1\}$, from the natural numbers, $\{n\}$, leaving the even numbers, $\{2n\}$, this residue retains the eidetic fullness of the natural numbers, $\{n\}$, for each number in $\{2n\}$ is simply twice a corresponding number in $\{n\}$. The remaining sequence of even numbers,

2	4	6	8	10	12	14	16	18	20	22	$\cdots$

can be written as

2·1	2·2	2·3	2·4	2·5	2·6	2·7	2·8	2·9	2·10	2·11	$\cdots$

The factor, 2, in $\{2n\}$ is common to all the terms, and will be represented only once in any least common multiple, so $\Lambda_{\{2n\}}(n) \doteq 2{\cdot}\Lambda_{\{n\}}(n)$. Since the single factor of 2 in the expression becomes negligible for large n, $\varepsilon_{\{2n\}}[\mu] = \varepsilon_{\{n\}}[\mu] = (1/1)[\mu]$. (Any whole multiple of a species has the same eidetic ratio as the species itself.) So far as its logistical character is concerned, the ensemble of even numbers can therefore be reduced to that of the ensemble of natural numbers:

1	2	3	4	5	6	7	8	9	10	11	$\cdots$

Separating the odd numbers, that is, an octave, yields a return without end to the natural numbers through the even numbers. Thus, the ratio of equality, $\varepsilon_{\{n\}}[\mu] = (1/1)[\mu]$, without loss to itself, begets the ratio, $\varepsilon_{\{2n-1\}}[\mu] = (2/1)[\mu]$, and the entire system of octaves. This is the only case in which such a reduction can be effected. Every other separation reduces the logistical perfection of the comprehending species.

Because the logistical properties of the residual even numbers are essentially the same as those of the natural numbers, the octave relates tones that sound qualitatively the same, even though they differ in pitch height. It is this qualitative unity that gives rise to the distinction of Ptolemy. While other harmonious intervals are called *consonances*, the octave alone is called *equisonance* or *homophone*. One may therefore form a pitch class of tones separated by octaves and assign them a common name. For instance, if one tone is designated as "C," all tones separated from it by one or more octaves will likewise carry this designation. Homophony permits one to normalize intervals by compounding octaves, for an interval in harmony with one octave is in harmony with all. Thus, all harmonious intervals can be represented as a ratio "between" (1/1) and (2/1), to speak metaphorically.

2.6 The Perfect Fourth (4/3) and Perfect Fifth (3/2)

Because the unison and octave are so intimately linked to each other, the octave system is a feature of all advanced music. Considering, therefore, only those intervals which harmonize with the octave, one necessarily passes over two species, namely $\{3n-1\}$ and $\{3n-2\}$, for these species, though compassed by the natural numbers, are not compassed by the species that gives rise to the octave, namely $\{2n-1\}$. Similar considerations bar the introduction of other species with first coefficients that are odd. One may proceed to the next two species, namely $\{4n-1\}$ and $\{4n-3\}$. (Note that $\{4n-2\}$ is reducible to $\{2n-1\}$, just as $\{4n\}$ is to $\{2n\}$ and $\{n\}$.) Each of these two species harmonizes with the octave interval, $\{2n-1\}$, because each ensemble is contained within the ensemble of the odd:

$\{2n-1\}$	**1**	**3**	**5**	**7**	**9**	**11**	**13**	**15**	**17**	**19**	**21**	**23**	$\cdots$
$\{4n-3\}$	1		5		9		13		17		21		$\cdots$
$\{4n-1\}$		3		7		11		15		19		23	$\cdots$

In this case, however, the logistical perfection of the species of the odd is diminished after the separation of one of the comprehended species. The ensemble remaining after the separation of $\{4n-1\}$ from $\{2n-1\}$, namely, $\{4n-3\}$, cannot be reduced to $\{2n-1\}$. Nor can $\{4n-1\}$, the ensemble remaining after the separation of $\{4n-3\}$ from $\{2n-1\}$, be likewise reduced. In other words, there is an imperfect fusion of tones. The harmony between these species is consequently less perfect than that between octave and unison. The intervals produced by these and other included species correspond to the concords mentioned by Ptolemy.

Whereas the odd species, $\{2n-1\}$, has the ratio, (2/1), both $\{4n-3\}$ and $\{4n-1\}$ have the ratio, (8/3). Each ratio expresses a potential quantity in relation to an actual reference quantity. One may therefore calculate the ratio of the two potential quantities from their specific relations to the reference quantity. If precedence is given to the comprehended species over the comprehending species, the potential quantities are in a ratio of (8/3)/(2/1), or (4/3), a *perfect fourth*. *Table 3* indicates this interval in several common musical tunings. The eidetic theory is in perfect agreement with each system, except equal temperament. Perfect fourths in an equally tempered system are acknowledged to differ slightly from a truly harmonious interval, and are therefore unworthy of scientific consideration. The equally tempered system has been included merely for the sake of practical reference.

Complementing the perfect fourth is the perfect fifth, which is formed by giving precedence to the comprehending species over the comprehended. When the interval, (2/1)/(8/3), or (3/4), is normalized with respect to the octave (*i.e.* doubled), it yields the simple ratio, (3/2). This ratio is also in agreement with previous theory, as can be seen in *Table 3*:

Perfect Fourth	***Interval***	***Decimal***
Pythagorean Interval	4 : 3	1.3333
Natural Interval	4 : 3	1.3333
Eidetic Interval	4 : 3	1.3333
Equal Temperament	$2^{5/12}$: 1	1.3348

Perfect Fifth	***Interval***	***Decimal***
Equal Temperament	$2^{7/12}$: 1	1.4983
Eidetic Interval	3 : 2	1.5000
Natural Interval	3 : 2	1.5000
Pythagorean Interval	3 : 2	1.5000

Table 3: A Comparison of Perfect Fourths and Perfect Fifths

Whereas the relation of octave to unison is univocal with respect to both species, this is not the case with imperfect harmonies. In perfect fourths and fifths, the comprehending species, $\{2n-1\}$, is designated univocally by the ratio, (2/1), but the two comprehended species, namely, $\{4n-1\}$ and $\{4n-3\}$, are designated equivocally by the ratio, (8/3). In other words, $\{4n-1\}$ and $\{4n-3\}$, despite their specific difference, share a common eidetic ratio.

2.7 The Major Third (44/35) and Minor Sixth (35/22)

The species, $\{6n-5\}$ and $\{6n-1\}$, both harmonize with $\{2n-1\}$, but not with $\{4n-3\}$ and $\{4n-1\}$, already introduced, so discussion of these will be deferred. On the other hand, $\{8n-7\}$, $\{8n-5\}$, $\{8n-3\}$, and $\{8n-1\}$ do harmonize with one or the other of the species, $\{4n-3\}$ and $\{4n-1\}$, and therefore with the octave, $\{2n-1\}$, as well. Both the comprehended and comprehending species are designated equivocally by their eidetic ratios, (352/105) and (8/3) respectively. Two species, $\{8n-7\}$ and $\{8n-3\}$, are comprehended by $\{4n-3\}$ in a manner reminiscent of the treatment of the perfect fourth and fifth:

$\{4n-3\}$	**1**	**5**	**9**	**13**	**17**	**21**	**25**	**29**	**33**	**37**	**41**	**45**	$\cdots$
$\{8n-7\}$	1		9		17		25		33		41		$\cdots$
$\{8n-3\}$		5		13		21		29		37		45	$\cdots$

Two other species, $\{8n-5\}$ and $\{8n-1\}$, are comprehended by $\{4n-1\}$:

$\{4n-1\}$	**3**	**7**	**11**	**15**	**19**	**23**	**27**	**31**	**35**	**39**	**43**	**47**	$\cdots$
$\{8n-5\}$	3		11		19		25		35		43		$\cdots$
$\{8n-1\}$		7		15		23		31		39		47	$\cdots$

There are thus four distinct manners of forming this interval through these number species. When the comprehended species is given precedence over the comprehending species, one has (352/105)/(8/3) or (44/35). This is the *major third*. Eidetic theory yields a value very close to equal temperament. It falls about midway between the natural and Pythagorean tuning systems. When precedence is given to the comprehending species, one obtains (8/3)/(352/105), or (35/44), which can be normalized to (35/22), the *minor sixth*. These eidetic intervals are compared in with the Pythagorean, Natural, and Equally Tempered intervals in *Table 4*:

Major Third	***Interval***	***Decimal***
Natural Interval	5 : 4	1.2500
Eidetic Interval	44 : 35	1.2571
Equal Temperament	$2^{1/3}$: 1	1.2599
Pythagorean Interval	81 : 64	1.2656

Minor Sixth	***Interval***	***Decimal***
Pythagorean Interval	128 : 81	1.5802
Equal Temperament	$2^{2/3}$: 1	1.5874
Eidetic Interval	35 : 22	1.5909
Natural Interval	8 : 5	1.6000

Table 4: A Comparison of Major Thirds and Minor Sixths

2.8 The Major Sixth (176/105) and Minor Third (105/88)

If a first ensemble comprehends a second, and the second comprehends a third, then the first also comprehends the third. This is the case for the species already considered, and so one may derive another harmonious interval without introducing a new species. Consider the ensembles, $\{8n-7\}$, $\{8n-5\}$, $\{8n-3\}$, and $\{8n-1\}$, inasmuch as they are comprehended by the ensemble of odd numbers, $\{2n-1\}$:

$\{2n-1\}$	**1**	**3**	**5**	**7**	**9**	**11**	**13**	**15**	**17**	**19**	**21**	**23**	$\cdots$
$\{8n-7\}$	1				9				17				$\cdots$
$\{8n-5\}$		3				11				19			$\cdots$
$\{8n-3\}$			5				13				21		$\cdots$
$\{8n-1\}$				7				15				23	$\cdots$

The interval considered here is univocal with respect to the comprehending species, but equivocal with respect to the comprehended species, $\{8n-7\}$, $\{8n-5\}$, $\{8n-3\}$, and $\{8n-1\}$. When the comprehended species are given precedence over the comprehending species, this composition of ratios yields (352/105)/(2/1), or (176/105). This is the major sixth. A comparison of this derived interval with those posited by other tuning systems can be found in *Table 5*. Eidetic principles conform admirably with musical practice.

Major Sixth	***Interval***	***Decimal***
Natural Interval	5 : 3	1.6667
Eidetic Interval	176 : 105	1.6762
Equal Temperament	$2^{3/4}$: 1	1.6818
Pythagorean Interval	27 : 16	1.6875

Minor Third	***Interval***	***Decimal***
Pythagorean Interval	32 : 27	1.1852
Equal Temperament	$2^{1/4}$: 1	1.1892
Eidetic Interval	105 : 88	1.1932
Natural Interval	6 : 5	1.2000

Table 5: A Comparison of Major Sixths and Minor Thirds

The minor third is formed by giving precedence to the comprehending species over the comprehended species and normalizing with respect to the octave. That is to say that (2/1)/(352/105) yields (105/176), which, when doubled, is (105/88).

2.9 The Tritone

The next ensembles that harmonize with all that went before are those of the sixteenth-order. However, for reasons that will become clearer, discussion of these species has been postponed. Proceeding to the thirty-second-order species, one can derive the tritone. The tritone is equivocal with respect to both the comprehending and comprehended species. Eighth-order and thirty-second-order species harmonize in the following ways:

$\{8n-7\}$	**1**	**9**	**17**	**25**	**33**	**41**	**49**	**57**	**65**	**73**	**81**	**89**	⋯
$\{32n-31\}$	1				33				65				⋯
$\{32n-23\}$		9				41				73			⋯
$\{32n-15\}$			17				49				81		⋯
$\{32n-7\}$				25				57				89	⋯

$\{8n-5\}$	**3**	**11**	**19**	**27**	**35**	**43**	**51**	**59**	**67**	**75**	**83**	**91**	⋯
$\{32n-31\}$	3				35				67				⋯
$\{32n-23\}$		11				43				75			⋯
$\{32n-15\}$			19				51				83		⋯
$\{32n-7\}$				27				59				91	⋯

$\{8n-3\}$	**5**	**13**	**21**	**29**	**37**	**45**	**53**	**61**	**69**	**77**	**85**	**93**	⋯
$\{32n-31\}$	5				37				69				⋯
$\{32n-23\}$		13				45				77			⋯
$\{32n-15\}$			21				53				85		⋯
$\{32n-7\}$				29				61				93	⋯

$\{8n-1\}$	**7**	**15**	**26**	**31**	**39**	**47**	**55**	**63**	**71**	**79**	**87**	**93**	⋯
$\{32n-31\}$	7				39				71				⋯
$\{32n-23\}$		15				47				79			⋯
$\{32n-15\}$			26				55				87		⋯
$\{32n-7\}$				31				63				93	⋯

There are two tritone intervals, both of which involve numerators and denominators of impractical size. If the thirty-second-order species takes precedence over the eighth, then the interval is approximated by the decimal fraction, 1.4128, the lesser tritone. If reversed and normalized with respect to the octave, it is approximated as 1.4156, the greater tritone. A true octave can only be produced by compounding one of each, rather than two of the same. *Table 6* compares the tritones of various tuning systems:

Tritone	***Interval***	***Decimal***
Natural Interval	45 : 32	1.4063
Eidetic Interval		1.4128
Equal Temperament	$2^{1/2}$: 1	1.4142
Eidetic Interval		1.4156
Pythagorean Interval	729 : 512	1.4238

Table 6: A Comparison of Tritones

The tritone in an equally tempered system lies exactly between the two intervals predicted by eidetic theory, and, in practice, the three cannot be distinguished by the ear.

2.10 The Minor Seventh and Major Second

One may also form concords between thirty-second-order and fourth-order ensembles. These harmonies are again equivocal with respect to both comprehending and comprehended species:

$\{4n-3\}$	**1**	**5**	**9**	**13**	**17**	**21**	**25**	**29**	**33**	**37**	**41**	**45**	**49**	**53**	**57**	**61**	$\cdots$
$\{32n-31\}$	1								33								$\cdots$
$\{32n-27\}$		5								37							$\cdots$
$\{32n-23\}$			9								41						$\cdots$
$\{32n-19\}$				13								45					$\cdots$
$\{32n-15\}$					17								49				$\cdots$
$\{32n-11\}$						21								53			$\cdots$
$\{32n-7\}$							25								57		$\cdots$
$\{32n-3\}$								29								61	$\cdots$

$\{4n-1\}$	**3**	**7**	**11**	**15**	**19**	**23**	**27**	**31**	**35**	**39**	**43**	**47**	**51**	**55**	**59**	**63**	$\cdots$
$\{32n-31\}$	3								35								$\cdots$
$\{32n-27\}$		7								39							$\cdots$
$\{32n-23\}$			11								43						$\cdots$
$\{32n-19\}$				15								47					$\cdots$
$\{32n-15\}$					19								51				$\cdots$
$\{32n-11\}$						23								55			$\cdots$
$\{32n-7\}$							27								59		$\cdots$
$\{32n-3\}$								31								63	$\cdots$

Once again, the metrical expressions for the ratios involve numerators and denominators of impractical size. If the thirty-second-order species takes precedence over the fourth, then the interval is approximated by the decimal fraction, 1.7761, which corresponds to the minor seventh. If the priority is reversed and normalized with respect to the octave, it is approximated as 1.1261, a major second. These intervals are compared in *Table 7* to those of traditional tuning systems:

Minor Seventh	*Interval*	*Decimal*
Eidetic Interval		1.7761
Natural Interval	16 : 9	1.7778
Pythagorean Interval	16 : 9	1.7778
Equal Temperament	$2^{5/6}$: 1	1.7818

Major Second	*Interval*	*Decimal*
Equal Temperament	$2^{1/6}$: 1	1.1225
Pythagorean Interval	9 : 8	1.1250
Natural Interval	9 : 8	1.1250
Eidetic Interval		1.1261

Table 7: A Comparison of Minor Sevenths and Major Seconds

Judged by the eidetic theory of harmony, traditional tunings of minor sixths make the ratio a bit high, whereas the corresponding tunings of the major second make the ratio a bit low.

2.11 The Weak Minor Third and Weak Major Sixth

The various thirty-second-order species also harmonize with the second-order species, that is, with the octave species. The resulting harmonies are univocal with respect to the comprehending species and equivocal with respect to the comprehended species:

$\{2n-1\}$	**1**	**3**	**5**	**7**	**9**	**11**	**13**	**15**	**17**	**19**	**21**	**23**	**25**	**27**	**29**	**31**	**⋯**
$\{32n-31\}$	1																⋯
$\{32n-29\}$		3															⋯
$\{32n-27\}$			5														⋯
$\{32n-25\}$				7													⋯
$\{32n-23\}$					9												⋯
$\{32n-21\}$						11											⋯
$\{32n-19\}$							13										⋯
$\{32n-17\}$								15									⋯
$\{32n-15\}$									17								⋯
$\{32n-13\}$										19							⋯
$\{32n-11\}$											21						⋯
$\{32n-9\}$												23					⋯
$\{32n-7\}$													25				⋯
$\{32n-5\}$														27			⋯
$\{32n-3\}$															29		⋯
$\{32n-1\}$																31	⋯

By giving precedence to the thirty-second-order species, one obtains a ratio of approximately 1.1841, whereas, by giving precedence to the second-order species, one obtains an approximate ratio of 1.6891. These correspond, respectively, to another minor third and major sixth. There are, then, more than one of these traditional intervals. However, because the thirty-second-order species is so far removed from the octave species, these secondary minor thirds and major sixths are weaker. Moreover, in the stronger interval of the minor third, the comprehending species is given precedence to the comprehended, whereas, in this weaker interval, the comprehended species is given precedence to the comprehending. The reverse is true for major sixths. Compare these weaker harmonies to the traditional tuning systems:

Major Sixths	*Interval*	*Decimal*
Natural Interval	5 : 3	1.6667
Equal Temperament	$2^{3/4}$: 1	1.6818
Pythagorean Interval	27 : 16	1.6875
Eidetic Interval		1.6891

Minor Thirds	*Interval*	*Decimal*
Eidetic Interval		1.1841
Pythagorean Interval	32 : 27	1.1852
Equal Temperament	$2^{1/4}$: 1	1.1892
Natural Interval	6 : 5	1.2000

Table 8: A Comparison of Weak Major Sixths and Minor Thirds

The eidetic interval corresponds closely to that of the Pythagorean tuning. The inference to be drawn is that traditional tuning systems, though discarded, cannot be so easily dismissed, for they may pick up harmonic species that modern systems fail to capture. There is an indefinitely large multitude of harmonies, even if most are far too weak to be perceived by the human ear.

2.12 Sixteenth-Order Species

Among harmonic intervals, the sixteenth-order ensembles, $\{16n-1\}$, $\{16n-3\}$, $\{16n-5\}$, $\{16n-7\}$, $\{16n-9\}$, $\{16n-11\}$, $\{16n-13\}$, and $\{16n-15\}$, were intentionally passed over. The exact eidetic ratio of these species is 182,144/45,045, which provides little intuition as to the material relations of quantities. It can be approximated decimally by 4.0436. Normalized with the octave, this yields 2.0218, and then 1.0109. For comparison to existing tuning systems, ratios will be approximated by the nearest decimal fraction, and even expressed in logarithmic "cents." Due to the proximity of this eidetic ratio to two octaves, intervals formed from sixteenth-order species will shadow other theoretical intervals. These shadow intervals, which are in fact harmonious intervals, but far weaker, will all differ from the common intervals by a ratio materially approximated by 1.0109, which corresponds to a "difference" of 18.8 cents. This difference may account in some situations for the phenomenon of "octave stretching," by which the interval of an octave is increased up to 20 cents from the correct value. The major third, minor sixth, major second, minor seventh, and tritone lack shadow intervals, because they are not formed from the octave species, $\{2n-1\}$. Although the shadow octave (approximately 2.0218/1) forms a quantity materially close to that formed by the octave, (2/1), it does not possess the property of octave equivalency. Consequently, it has no bearing on pitch class, and cannot be used to normalize eidetic ratios to within the octave. In other words, "octave stretching" cannot be compounded. The weakness of these higher-order harmonies will make them far less distinct than their traditional counterparts. One may, nevertheless, form a consistent tuning system by replacing traditional intervals with shadow intervals. *Table 9* details the shadow intervals and contrasts them with the basic intervals:

	Basic Intervals				Shadow Intervals			
Interval Name	***Order of 1st Species***	***Order of 2nd Species***	***Approx. Ratio***	***Cents***	***Order of 1st Species***	***Order of 2nd Species***	***Approx. Ratio***	***Cents***
Unison	Ω_1	Ω_1	1.0000	0.0	Ω_1	Ω_{16}	1.0109	18.8
Minor Second	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
Major Second	Ω_{32}	Ω_4	1.1261	205.5	N/A	N/A	N/A	N/A
Minor Third	Ω_2	Ω_{32}	1.1841	292.5	Ω_{16}	Ω_{32}	1.1713	273.7
	Ω_8	Ω_2	1.1932	305.8	Ω_8	Ω_{16}	1.2062	324.6
Major Third	Ω_4	Ω_8	1.2571	396.2	N/A	N/A	N/A	N/A
Perfect Fourth	Ω_2	Ω_4	1.3333	498.0	Ω_{16}	Ω_4	1.3478	516.8
Tritone	Ω_8	Ω_{32}	1.4128	598.3	N/A	N/A	N/A	N/A
	Ω_{32}	Ω_8	1.4156	601.7	N/A	N/A	N/A	N/A
Perfect Fifth	Ω_4	Ω_2	1.5000	702.0	Ω_4	Ω_6	1.4838	683.2
Minor Sixth	Ω_8	Ω_4	1.5909	903.8	N/A	N/A	N/A	N/A
Major Sixth	Ω_2	Ω_8	1.6762	894.2	Ω_{16}	Ω_8	1.6581	875.4
	Ω_{32}	Ω_2	1.6891	907.5	Ω_{32}	Ω_{16}	1.7075	926.3
Minor Seventh	Ω_4	Ω_{32}	1.7761	994.5	N/A	N/A	N/A	N/A
Major Seventh	N/A	N/A	N/A	N/A	N/A	N/A	N/A	N/A
Octave	Ω_1	Ω_2	2.0000	1200.0	Ω_{16}	Ω_1	1.9784	1181.2

Table 9: Basic Intervals and their Shadow Intervals

2.13 The Variety of Eidetic Hierarchies

This inquiry considers primarily those intervals formed from the second, fourth, eighth, and thirty-second-order species, henceforth called the Ω_2-Ω_4-Ω_8-Ω_{32} hierarchy, which explains the harmonious intervals typically found in Western music. *Table 10* records these intervals and compares them with equal temperament, Pythagorean tuning, and Just Intonation. The three-species hierarchy, Ω_2-Ω_4-Ω_8, consists of the simplest harmonious species, and these happen to correspond exactly to the major triad—or minor triad, as will be explained. There are, nevertheless, other hierarchies, the intervals of which also tend to cluster about traditional Western intervals. The Ω_2-Ω_6-Ω_{12}-Ω_{36} hierarchy produces intervals very much like those of Ω_2-Ω_4-Ω_8-Ω_{32}, so much so that one may surmise that the former can often be heard as well. The Ω_2-Ω_6-Ω_{12}-Ω_{36} hierarchy lacks a tritone, but possesses a minor second and major seventh. The chord structures also differ greatly. *Table 11* records the intervals formed by this hierarchy, again comparing them to the three other tuning systems. Finally, *Figure 6* presents the intervals of both hierarchies graphically in terms of logarithmic cents, permitting an easier comparison both to equal temperament and to each other.

2.14 The Major Tetrad

The metrical formulation of harmony offers no mathematical guarantee that the compounding of two intervals, themselves harmonious, will produce an interval that is also harmonious. For instance, compounding a perfect fifth and a major third, two highly concordant intervals, forms a major seventh, which exhibits very little harmony. On the other hand, a minor third, which is less concordant than a perfect fifth, when compounded with the same major third, produces a perfect fifth, which is, along with the perfect fourth, the most concordant interval after the octave. Musicians have found rules for compounding intervals in harmonious ways, but there has been no genuine science behind them.

The eidetic theory of harmony has been thus far limited to pair-wise relationships between ensembles, but the very same principle of inclusion can be applied to three or even four ensembles to form triadic and tetradic chords. If a first ensemble includes a second ensemble, and the second includes a third, then the first ensemble will likewise include the third. This principle was used to derive new intervals without introducing new species, but it is also the eidetic foundation of chord formation, because each species will harmonize with all the others. Take G as a reference tone. To this one may assign the first-order species (Ω_1) which yields the unison (G). Included within the first-order species (Ω_1) is the second-order species (Ω_2), which yields the octave (G-G), the fourth-order species (Ω_4), which, normalized by the octave, yields the perfect fourth (G-C), the eighth-order species (Ω_8) which, normalized by the octave, yields the major sixth (G-E), and the thirty-second-order species (Ω_{32}) which, normalized by the octave, yields the weak minor third (G-B♭). Together, they form a chord (C-E-G-B♭), called C^7, which is a "dominant seventh." Of this chord a prominent theorist has said, "The dominant seventh is, in fact, the central propulsive force in our music; it is unambiguous and unequivocal."[46]

Table 10: Intervals of the Ω_2-Ω_4-Ω_8-Ω_{32} Eidetic Hierarchy

Twelve-Tone Intervals		*Eidetic Transitions*		*Equal Temperament*		*Pythagorean Tuning*		*Just Intonation*	
Ascending Interval	***Half Tones***	***Decimal Ratio***	***Major Tonality***	***Decimal Ratio***	***Difference in Cents***	***Decimal Ratio***	***Difference in Cents***	***Decimal Ratio***	***Difference in Cents***
Unison	0	1.0000	$\Omega_1 \rightarrow \Omega_1$	1.0000	+0.0	1.0000	+0.0	1.0000	+0.0
Minor Second	1	—	—	1.0595	—	1.0535	—	1.0667	—
Major Second	2	1.1261	$\Omega_{32} \rightarrow \Omega_4$	1.1225	−5.5	1.1250	−1.7	1.1250	−1.7
Minor Third	3	1.1841	$\Omega_2 \rightarrow \Omega_{32}$	1.1892	+7.5	1.1852	+1.6	1.2000	+23.1
		1.1932	$\Omega_8 \rightarrow \Omega_2$		−5.8		−11.6		+9.8
Major Third	4	1.2571	$\Omega_4 \rightarrow \Omega_8$	1.2599	+3.8	1.2656	+11.7	1.2500	−9.8
Perfect Fourth	5	1.3333	$\Omega_2 \rightarrow \Omega_4$	1.3348	+1.3	1.3333	+0.0	1.3333	+0.0
Tritone	6	1.4128	$\Omega_8 \rightarrow \Omega_{32}$	1.4142	+1.7	1.4238	+13.4	1.4063	−8.0
		1.4156	$\Omega_{32} \rightarrow \Omega_8$		−1.7		+10.0		−11.4
Perfect Fifth	7	1.5000	$\Omega_4 \rightarrow \Omega_2$	1.4983	−1.3	1.5000	+0.0	1.5000	+0.0
Minor Sixth	8	1.5909	$\Omega_8 \rightarrow \Omega_4$	1.5874	−3.8	1.5802	−11.7	1.6000	+9.8
Major Sixth	9	1.6762	$\Omega_2 \rightarrow \Omega_8$	1.6818	+5.8	1.6875	+11.6	1.6667	−9.8
		1.6891	$\Omega_{32} \rightarrow \Omega_2$		−7.5		−1.6		−23.1
Minor Seventh	10	1.7761	$\Omega_4 \rightarrow \Omega_{32}$	1.7818	+5.5	1.7778	+1.7	1.7778	+1.7
Major Seventh	11	—	—	1.8877	—	1.8984	—	1.8750	—
Octave	12	2.0000	$\Omega_1 \rightarrow \Omega_2$	2.0000	+0.0	2.0000	+0.0	2.0000	+0.0

Table 11: Intervals of the Ω_2-Ω_6-Ω_{12}-Ω_{36} Eidetic Hierarchy

Twelve-Tone Intervals		*Eidetic Transitions*		*Equal Temperament*		*Pythagorean Tuning*		*Just Intonation*	
Ascending Interval	***Half Tones***	***Decimal Ratio***	***Minor Tonality***	***Decimal Ratio***	***Difference in Cents***	***Decimal Ratio***	***Difference in Cents***	***Decimal Ratio***	***Difference in Cents***
Unison	0	1.0000	$\Omega_1 \to \Omega_1$	1.0000	+0.0	1.0000	+0.0	1.0000	+0.0
Minor Second	1	1.0753	$\Omega_2 \to \Omega_{12}$	1.0595	−17.8	1.0535	−24.6	1.0667	−9.6
Major Second	2	1.1111	$\Omega_6 \to \Omega_2$	1.1225	+12.2	1.1250	+14.9	1.1250	+14.9
Minor Third	3	1.1948	$\Omega_6 \to \Omega_{12}$	1.1892	−5.6	1.1852	−9.7	1.2000	+5.2
Major Third	4	1.2561	$\Omega_{12} \to \Omega_{36}$	1.2599	+3.6	1.2656	+9.0	1.2500	−5.8
Perfect Fourth	5	1.3326	$\Omega_{36} \to \Omega_6$	1.3348	+2.0	1.3333	+0.6	1.3333	+0.6
		1.3507	$\Omega_2 \to \Omega_{36}$		−14.2		−15.5		−15.5
Tritone	6	—	—	1.4142	—	1.4238	—	1.4063	—
Perfect Fifth	7	1.4807	$\Omega_{36} \to \Omega_2$	1.4983	+14.2	1.5000	+15.5	1.5000	+15.5
		1.5008	$\Omega_6 \to \Omega_{36}$		−2.0		−0.6		−0.6
Minor Sixth	8	1.5922	$\Omega_{36} \to \Omega_{12}$	1.5874	−3.6	1.5802	−9.0	1.6000	+5.8
Major Sixth	9	1.6739	$\Omega_{12} \to \Omega_6$	1.6818	+5.6	1.6875	+9.7	1.6667	−5.2
Minor Seventh	10	1.8000	$\Omega_2 \to \Omega_6$	1.7818	−12.2	1.7778	−14.9	1.7778	−14.9
Major Seventh	11	1.8599	$\Omega_{12} \to \Omega_2$	1.8877	+17.8	1.8984	+24.6	1.8750	+9.6
Octave	12	2.0000	$\Omega_1 \to \Omega_2$	2.0000	+0.0	2.0000	+0.0	2.0000	+0.0

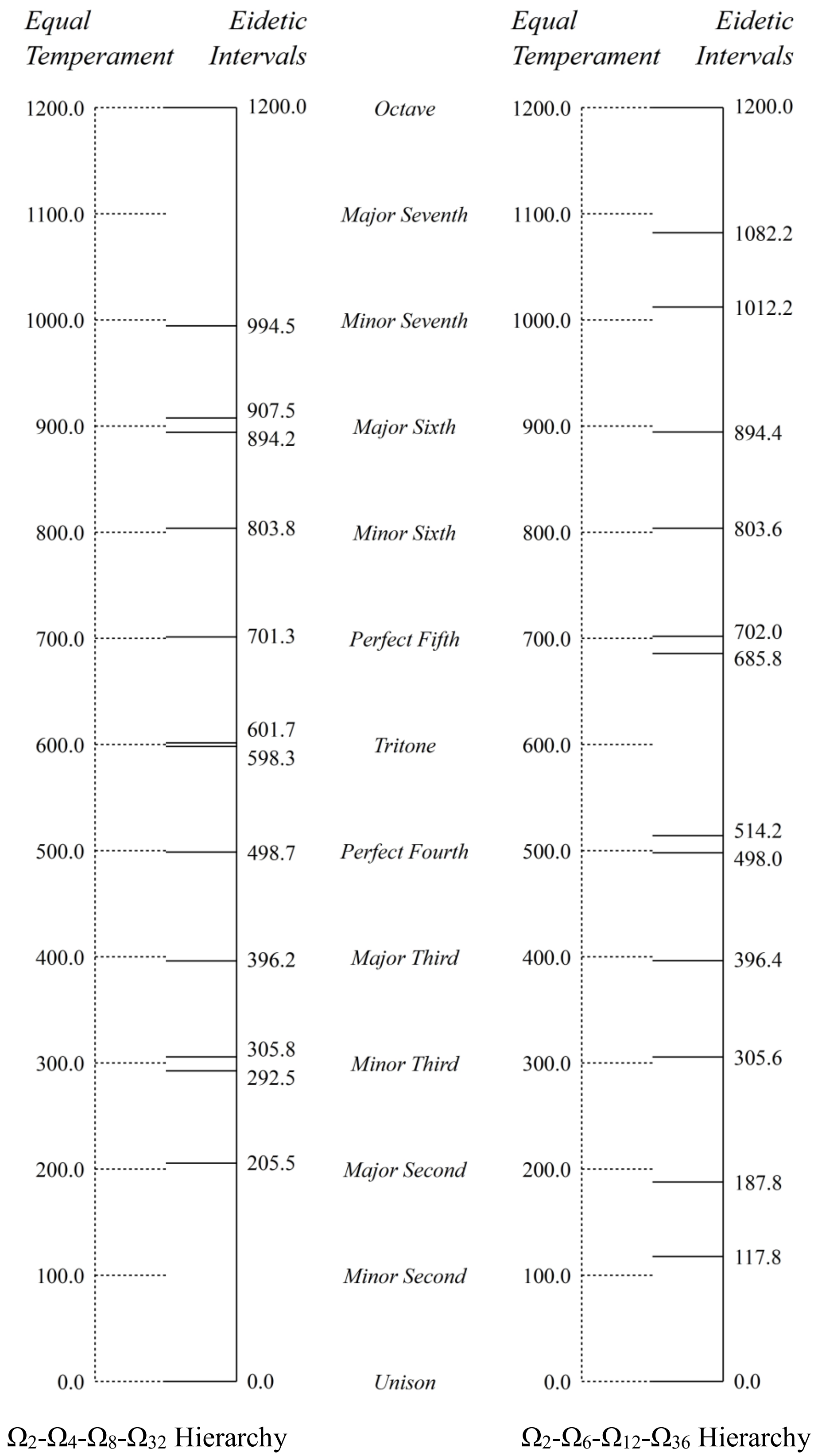


Figure 6: Intervals Formed from Two Simple Eidetic Hierarchies

Every interval within this tetrad is also harmonious, though not all to the same degree. The interval, C-G, is a major fifth; E-G is a strong minor third; B♭-G is a weak major sixth; C-E is a major third; E-C is a minor sixth; C-B♭ is a minor seventh; B♭-C is the major second; E-B♭ is the lesser tritone; B♭-E is the greater tritone. Three more eidetic tetrads can be produced by inversion, a process in which the highest tone is displaced an octave lower, or the lowest tone displaced an octave higher. From the major eidetic tetrad, including inversion, twelve triadic chords can be formed. Yet these eidetic chords by no means exhaust all the chords used by musicians. They are simply *complete*, in the sense that these triadic chords necessarily share with the tetrad the property that all intervals within it are eidetically harmonious. Musicians are always free to produce chords that include another form of harmony, or even various discords. These chords may be classified as *incomplete* to distinguish them from the complete chords considered here.

2.15 The Minor Tetrad

Minor chords present a mathematical difficulty arising out of the compounding of intervals. The difficulty is most easily perceived by considering the tones of triads in a rising or falling sequence, a mathematical *arpeggio*, so to speak. Consider the major triad in C. It can be described by the transition, $\Omega_4 \rightarrow \Omega_8 \rightarrow \Omega_2$. Beginning at C ($\Omega_4$), it ascends by a major third to E (Ω_8), and then by a minor third to G (Ω_2). The minor triad in C (C-E♭-G) ascends first by a minor third, and then by a major third, but this reversal in the order of intervals cannot be effected by reversing the order of the species. The transition, $\Omega_4 \rightarrow \Omega_8 \rightarrow \Omega_2$, when reversed, yields $\Omega_2 \rightarrow \Omega_8 \rightarrow \Omega_4$. This corresponds not to the rise of a minor third followed by the rise of a major third, but to a fall of a minor third, followed by a fall of a major third. What is required is $\Omega_8 \rightarrow \Omega_2$ followed by $\Omega_4 \rightarrow \Omega_8$, which is incoherent, because the middle species is not common. Nor can this be resolved by employing the weak minor third in place of the strong. This would correspond to $\Omega_2 \rightarrow \Omega_{32}$ and then $\Omega_4 \rightarrow \Omega_8$, an impossibility. Nevertheless, it seems unlikely that the minor triad is incomplete and that harmony can only be experienced pair-wise between tones. This runs counter to musical experience. A proper understanding of the minor triad therefore demands a deeper investigation of intervallic change.

A logical way to address this difficulty is to consider it from the point of view of plenitude and privation. Although privation is not itself a thing, it possesses an intelligibility that can yet be apprehended. Consider a hole in the ground. When a man digs another hole one meter to the right, filling up the first with the excavated contents of the second, one may say that the soil has been moved to the left. Nevertheless, one can equally describe it as the motion of the hole one meter to the right. If this is performed again and again by digging holes always one meter to the right, it is sometimes easier to describe the one consecutive motion of the hole to the right rather than the several motions of distinct clods of earth to the left. Mathematical physicists use such techniques all the time to describe "anti-particles." To any change from one privation to another, there must always correspond an underlying change of plenitude, but the latter description is sometimes more complex and less easily apprehended.

In the major tetrad, intervallic motion is from the plenitude of one species to the plenitude of another. In the minor tetrad, intervallic motion is from the privation of one species to the privation of another. This privation may even be the reason that minor chords produce an emotional response of sadness. Privation will be expressed by underlining the order of the species. The privation of the octave species will, for example, be designated $\underline{\Omega}_2$. Whereas the transition, $\Omega_1 \rightarrow \Omega_2$, represents the rise of an octave, the transition $\underline{\Omega}_1 \rightarrow \underline{\Omega}_2$, represents the fall of an octave. However, reversing the order of the privative species, the transition, $\underline{\Omega}_2 \rightarrow \underline{\Omega}_1$, also yields a rise of an octave. There is a minor tetrad entirely parallel to that of major. As the transitions, $\Omega_4 \rightarrow \Omega_8$ and $\Omega_8 \rightarrow \Omega_2$, yield a rise of a major third followed by the rise of a minor third, so also do $\underline{\Omega}_8 \rightarrow \underline{\Omega}_4$ and $\underline{\Omega}_2 \rightarrow \underline{\Omega}_8$. The latter can be reordered and compounded to form $\underline{\Omega}_2 \rightarrow \underline{\Omega}_8 \rightarrow \underline{\Omega}_4$. This is a rise of a minor third followed by a rise of a major third, thus forming a minor triad. Whereas the major eidetic tetrad of tones is described by the transitions, $\Omega_4 \rightarrow \Omega_8 \rightarrow \Omega_2 \rightarrow \Omega_{32}$, the minor eidetic tetrad consists of tones described by the transitions, $\underline{\Omega}_2 \rightarrow \underline{\Omega}_8 \rightarrow \underline{\Omega}_4 \rightarrow \underline{\Omega}_{32}$. The major and minor eidetic tetrads are shown in *Figure 7*:

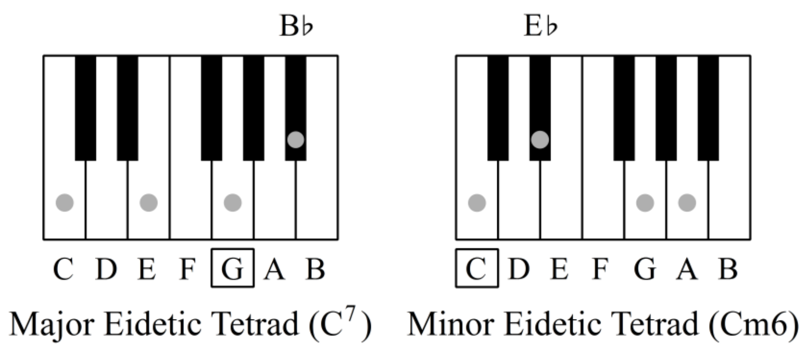


Figure 7: Major and Minor Eidetic Tetrads

The reference tone for the major eidetic tetrad in C (C^7) is G, whereas that for the minor eidetic tetrad in C (Cm6) is C. Thus, in the major eidetic tetrad, the reference tone is on the fifth, whereas on the minor eidetic tetrad, the reference tone is on the root of the chord.

2.16 The Diatonic Scale

A musician's choice of musical system is not unlike a painter's choice of palette. As a palette of colors enables a painter to paint a variety of subjects in an aesthetically satisfying manner, so also a system of music enables a composer to compose with exactness and clarity what he wishes to express. There have been a multitude of musical systems—in antiquity and modernity, in the East and West—not one of which may be properly called "incorrect," for each is instrumental to some end. The most that one can hope to explain is why some system or another has been successful, and why it exhibits certain characteristics. The diatonic scale has for a long time been a dominant force in western music, a music which has been accepted and, in many cases, adopted globally. To examine the underlying structure of this scale, together with major and minor tonalities, is a worthy inquiry. This corresponds to the investigation of Ptolemy's third class of tones, the melodic.

Nichomachus of Gerasa records that the diatonic scale was first created by Pythagoras when he combined two tetrachords,[47] but the musical properties of the diatonic scale are better understood as the superposition of major and minor eidetic tetrads. The diatonic scale in C major is formed by the superposition of the dominant seventh chord in C (C^7) and the minor sixth chord in D (Dm6). The reference tones are G for the major tetrad and D for the minor tetrad, each again designated by a box. This is represented in *Figure 8*, as on a piano keyboard. The natural minor scale, also shown, is obtained by shifting both the major and minor eidetic tetrads upward by thee "half tones," so to speak. This is equivalent to superimposing $E\flat^7$ and Fm6, the eidetic reference tones of which are B♭ and F.

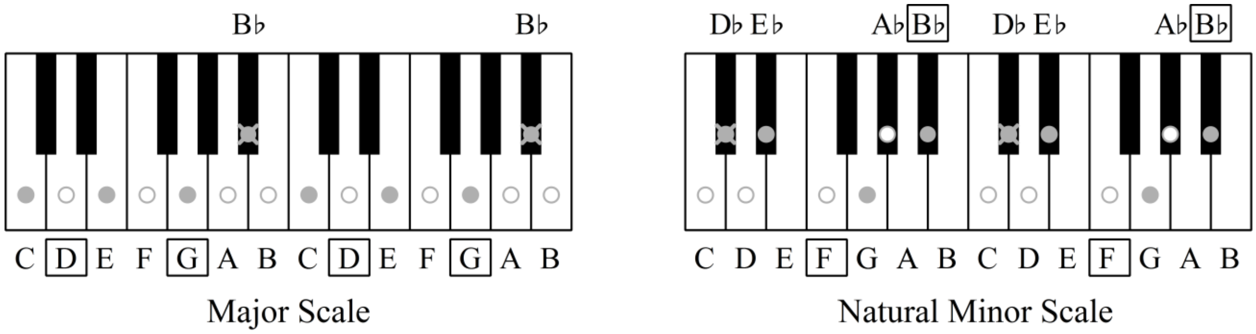


Figure 8: Major and Natural Minor Scales

Every tone in the diatonic scale is included in these superimposed tetrads. However, because there are only seven pitch classes in the diatonic scale, there is an additional tone, B♭ in the C major scale. This might seem to undermine the claim that the scale is formed from eidetic tetrads, but the addition only serves to prove the utility of eidetic theory. In Gregorian chant, the starting point for the modern tradition of music in the West, there are indeed eight permissible tones. Gregorian chant allows only one accidental, precisely the flattening of B (*ti*) to B♭ (*te*), usually in modes IV, V, and VI. This suggests that medieval chant was already being heard differently than ancient music had been, and in such a way as to permit the subsequent development of polyphony. This hypothesis concerning scales is testable. If the scales are indeed heard as two tetrads, the ear should be, *melodically* speaking, less sensitive to a collective mistuning of one or the other of the tetrads, so long as the intervals within each tetrad are preserved.

A striking characteristic of the rising diatonic scale is the alternation between the two eidetic tetrads. Consecutive tones in the scale harmonize only weakly, if at all, but tones separated by another tone harmonize well. This alternation gives temporal impetus to the music, driving it forward melodically, but fails when one arrives at B (*ti*), creating an instability that can be tentatively resolved by flattening the note to B♭ (*te*). Yet this flattening only means a subsequent loss of alternation when one arrives at C (*do*). The instability cannot be eliminated in a heptatonic scale, only deferred. It should not be surprising, then, to find that different Gregorian modes use either one or the other note. This alternation between tetrads also explains why the seventh tone in the major scale, which in C major is B, acts as a *leading tone*. Ascending the diatonic scale from C, and arriving at B, the pattern of alternation is broken, and the listener's expectation is unfulfilled. This creates melodic tension, which is released by attaining C, in which the expectation of a return to the major eidetic tetrad is finally realized. Therefore, B leads naturally to C in the C major scale. Further confirmation of this structure is provided by the *leading chord*, which in C major is formed from B, D, and F. All the tones of this chord belong to the minor eidetic tetrad, which likewise defers the resolution into the major tetrad, producing tension.

2.17 A Short Account of Key Coloration

Different musical keys are commonly held to evoke different emotions in the listener, a phenomenon called *key coloration*. The cause of this experience is the modification in intervals produced by transposing each note by the same number of "half-tones." Because the intervals of the original "half-tones" are not the same, the transposition is uneven, causing the mathematical ratios between consecutive tones to vary. (The use of equal temperament, on the other hand, eliminates key coloration by making all "half-tones" the same.) Eidetic theory provides a deeper insight into why these key changes produce different musical experiences. As has been observed, there are often several eidetic intervals bunched about what is traditionally held to be a single interval. Indeed, there are many that are not explored in this essay. Slight changes in tuning can therefore cause meaningful changes in hearing, as the ear apprehends different species in varying degrees. Musical audition is complex, escaping the limitations of what can be written on staff lines.

2.18 On the Ptolemaic Critique of Aristoxenus

The fundaments of musical harmony having been set forth, it is fitting to once again take up the opinions of the great classical theorists. Unlike the Pythagoreans, Ptolemy does not identify sensible sounds with number, but makes a distinction between sense and reason: The perception of harmony by the senses is imperfect, admitting variability, whereas the apprehension by reason is exact, allowing demonstration. Thus, harmony truly consists in the ratios of simple whole numbers, and the flexibility observed in practical music is attributable to the defects of hearing:

> Harmonic knowledge is the power that grasps the distinctions related to high and low pitch in sounds [ψόφοι]: sound is a modification [πάθος] of air that has been struck (this is the first and most fundamental of things heard): and the criteria of *harmonia* are hearing and reason, not however in the same way. Rather, hearing is concerned with the matter and the modification, reason with the form and the cause, since it is in general characteristic of the senses to discover what is approximate and to adopt from elsewhere what is accurate, and of reason to adopt from elsewhere what is approximate, and to discover what is accurate. For since matter is determined and bounded only by form, and modifications only by the causes of movements, and since of these the former [i.e. matter and modifications] belong to sense perception, the latter to reason, it follows naturally that the apprehensions of the senses are determined and bounded by those of reason, first submitting to them the distinctions that they have grasped in rough outline—at least in the case of things that can be detected through sensation—and being guided by them towards distinctions that are accurate and accepted.[48]

The senses, then, prefigure and suggest rational relations. They merely grope toward what they cannot attain. The relations can only be apprehended by the intellect, which confirms the hearing in its perceptions. The Aristoxenian emphasis on the senses is consequently misplaced. More to the point, Ptolemy shows the implausibility of the Aristoxenian theory of music, which treats the relations existing between magnitudes as a continuum, instead of ratios:

> This shows, then, that we should not find fault with the Pythagoreans in the matter of the discovery of the ratios of the concords, for here they are right, but in that of the investigation of their causes, which has led them astray from the objective; but we should find fault with the Aristoxenians, since they neither accepted these ratios as clearly established, nor, if they really lacked confidence in them, did they seek more satisfactory ones—assuming that they were genuinely committed to the theoretical study of music. For they must necessarily agree that such experiences come to the hearing from a relation that the notes have to one another, and further that where the impressions are the same, the differences are determinate and the same. Yet in what relation, for each species [of concord], the two notes that make it stand, they neither say nor enquire, but as if the notes themselves were bodiless and what lie between them were bodies, they compare only the intervals [διάστασεις] belonging to the species, so as to appear to be doing something with number and reason.[49]

Ptolemy repeats this criticism in other contexts. The citation of these similar texts is well worth any potential tedium, for the point is crucial for modern mathematics:

> And indeed we could now call sounds of this sort "notes" [φθόγγοι], since a note is a sound that retains the same tone [τόνος]. Hence each taken alone has no ratio, for it is one and undifferentiated in relation to itself, whereas ratio is a relation and occurs first in two terms.[50]

> In general, it would seem an absurdity to think that the differences possess a ratio that is not exhibited through the magnitudes that make the differences, and to suppose that the magnitudes have none—the magnitudes from which it is possible immediately to derive the ratio of the differences. And if they were to deny that their comparisons are of the differences between the notes, they would be unable to say of what other things they are. For the concordant or the melodic is not just some empty distance or mere length, nor is it bodily, and predicated of one single thing, the magnitude: rather, it is predicated of two things at least, these being unequal—that is, the sounds that make them—so that it is not possible to say that the comparisons in respect of quantity are of anything but the notes and the differences between them, neither of which do they [the Aristoxenians] make known or provide with a common definition, a definition, that is, that is one and the same, and through which it is shown how the sounds are related both to one another and to the difference [between them].[51]

> From these facts too, therefore, it seems that Aristoxenus gave no thought to ratio [λόγος] but defined the genera only by what lies between the notes, and not by their differences considered in relation to one another, passing over the causes of the differences as being no cause, as nothings, as mere limits, while attaching the distinctions to things that are bodiless and empty.[52]

Musical interval, as Ptolemy correctly points out, designates not a place bounded by points, but a ratio of quantities. Yet there is a hidden truth in the doctrine of Aristoxenus, deriving perhaps from his musical experience. For his theory concerns not the tone, which he incorrectly defines as the limit of an interval, but the interval itself. Consequently, his rule of harmony must determine which *intervals* can be harmoniously combined. This is also the case in the eidetic theory of harmony.

Metrical theories of harmony, whether of Pythagoras, Archytas, or Ptolemy—or even one of the moderns—seriously distort the phenomenon. Since matter is only intelligible through form, harmony must exist between two forms. Metrical theories give the illusion of satisfying this requirement, because a metrical ratio is expressed as a numerator and a denominator, each of which is formal in its mathematical character. However, these numbers have no commonality in the arithmetical sense. Eidetic theory shows that harmony between tones is grounded in the specific forms of two *immediate* ratios. That is, two tones harmonize only because each frequency is in a definite rational relation to a reference quantity, and the eidetic forms of these relations harmonize. Even the octave is the harmonious blending of two species, $\{2n-1\}$ and $\{n\}$. Harmony is primarily and formally between intervals, and only secondarily and materially between the terms of intervals. The form of the relation is not constituted by the quantitative terms; the quantitative terms are constituted by the form of the relation. Indeed, only because the relations share a common reference quantity can the other two quantities be compared. To summarize, Aristoxenus was wrong to maintain that intervals are places bounded by tones, but Ptolemy was wrong to accord primacy to the tones themselves, rather than to the intervals that relate them.

2.19 The Aesthetic Criteria of Harmony

Traditional aesthetic theory specifies three attributes essential to beauty: clarity (*claritas*), integrity (*integritas*), and proportion (*proportio*). Clarity is the degree to which a work of art corresponds to the concept it instantiates. Integrity is the degree to which every part of the work befits the whole. Proportion is the degree to which the work achieves the ideal relationships between its parts. Because beauty is, in certain respects, a transcendental attribute, these criteria are applied analogously to each form of art. Since the application of these principles thus varies with the medium, a mathematical theory of harmony must include an explanation of how it concretely satisfies these aesthetic criteria.

Harmonic clarity is the degree to which an ensemble of natural numbers corresponds to the arithmetical species it is supposed to express. Lack of clarity can result from either an insufficiently large ensemble, or from deviations from the species resulting from either gaps or inclusions. As the ensemble grows without bound, the clarity of a given interval becomes more and more perfect. *Harmonic integrity is the degree to which ensembles are comprehended one within another, and ultimately within the ensemble of the unison*. Insofar as the comprehended ensemble contains elements that cannot be found in the comprehending, there will be a lack of harmonic integrity. Integrity therefore presupposes clarity. *Harmonic proportion is conformity to the ideal quantitative ratios determined by the arithmetical species*. Hence, proportion presupposes integrity, which in turn presupposes clarity. Although “ratio” and “proportion” are often used synonymously, ratio exists between only two terms, whereas proportion exists between four. Eidetic ratios between frequencies arise out of a pair of relations, each relating a quantity to a common reference quantity. So, proportion, rather than ratio, is truly the foundation of harmony. The nature of this proportion, however, differs from arithmetical proportions. To establish a harmonic proportion, it is not necessary that each pair of frequencies be related to a reference quantity through a common form. It suffices that the form of one relation be *included* within the other. To summarize then, eidetic theory is fully capable of expressing mathematically the classical criteria for beauty.

3. Musica Mundana et Humana

3.1 Uniform States of Motion

The eidetic ground of logistic is a refinement and concrete application of the Aristotelian definition of motion. Aristotle declares that "Motion is the entelechy of a being in potency as such."[53] "Entelechy" (ἐντελέχεια) literally means a state of being in its end (τέλος). Aristotle's specification has the merit of defining motion without recourse to motion. Any definition of motion as change, actualization, or realization merely begs the question. Change, actualization, and realization are themselves motions, with the result that motion will itself be defined as a type of motion. Nevertheless, although Aristotle's definition is metaphysically elegant, it is also problematic. The difficulty lies in conceiving this "entelechy" as anything other than a partial actualization of the end toward which a being is moving, in which case this act of motion will be nothing but a subsequent actuality of the moving subject.

Philoponus makes clear in his commentary that actuality and potentiality must somehow coexist for motion to be something different than either:

> The word, '*entelechy*' in Aristotle signifies actuality and completion, for it is a compound of the words *hen* ('one'), *teleion* ('complete') and *ekhein* ('have a certain state). When any particular thing possesses its own completion, it is said to exist in *entelechy*. The added words 'in so far as it is such' stand for 'in so far as it is potentially that thing in particular', in order that the *entelechy* may come into being while the potentiality, whose *entelechy* it was, persists. For the entelechy in the case of any thing is of two kinds, one when the thing is already in its completed state, and has rid itself of all potentiality, one of the thing when, having changed from its potential state, it undergoes mutation according to its potentiality, and is being turned into its form.[54]

Entelechy is thus of two sorts: that which coexists with potentiality, and that which is the completion of potentiality. Motion, according to Philoponus, is a coexistence of the actual and potential in the same subject. He explains that Themistius has a slightly different interpretation, but one that again associates entelechy with the actuality of one of the terms of motion:

> But Themistius explains change with a slight alteration, viz. that it is the initial *entelechy* of that which exists potentially, in so far as it is such. For the final *entelechy*, he says, is the change to the form in which it then remains at rest, while the first one is the approach to this, which is change. And the previous *entelechy* is an incomplete actuality (for it is *en route* to form and the absolute and simple *entelechy*), while the latter one is complete. For when the subject of change attains this state it is at rest and is completely free from potentiality.[55]

According to these commentators, to understand correctly Aristotle's definition it is necessary to distinguish "entelechy" in two senses: as either coexisting with potency, or as excluding potency. The former is motion, the latter a term of motion. St. Thomas restates Aristotle in similar terms: "motion is an entelechy, that is, the act of a thing existing in potency insofar as it is such."[56] Following Aristotle, he identifies the act of motion with the action of the mover in the moved:

> He [Aristotle] shows that the act of the mover and the moved are the same. For something is called a mover inasmuch as it acts, but moved inasmuch as it is passive. But that which the mover causes by acting and that which the moved receives by being passive are the same. And this is what he says, that the mover is active upon the mobile thing, that is, it causes the act of the mobile thing. For which reason, it is necessary that there be one act of both, that is, of the mover and the moved: For that which is from the mover as an agent cause and that which is in the moved as patient and recipient are the same.[57]

In this sense motion is simultaneously act and potency, for the mover is active in moving the moved, and the moved passive in its being moved. Motion is therefore legitimately in the causal junction of mover and moved.

Although Aquinas thus distinguishes metaphysically between motion and the terms of motion, concrete physical application proves elusive. In local motion the final term of motion appears arbitrary, and with it the application of the definition. Aristotle's analogy of the road between Athens and Thebes[58], or alternatively between Thebes and Athens, aptly expresses how motion can be simultaneously active and passive, but this same road can illustrate the difficulty of applying Aristotle's definition to continuous motion. Suppose that the road between Athens and Thebes passes through Eurythrae, and that the man walking this road is departing from Athens. Then one may define his motion as either the act of the man in Athens insofar as he is potentially in Thebes, or as the act of the man in Athens insofar as he is potentially in Eurythrae. The potentiality of a man being in Thebes differs from that of a man being in Eurythrae to the degree that these actualities also differ, and so, by the definition, there are two distinct, but simultaneous, motions.

The only way to make sense of this is to say that the latter motion exists virtually within the former, but this is merely to say that one stretch of road is a part of the whole. Instead of being reduced to the terms of motion, local motion is thereby reduced to the path traversed. One can make the same inference by another means. Consider that there are many ways to travel from Athens to Thebes, and that there will therefore be differing motions corresponding to the different paths taken. The only way that an individual motion can be specified under Aristotle's definition is to identify the act of motion with the path of travel. If a man is traveling from Athens to Thebes via Eurythrae, then Eurythrae must be included in the actuality of that motion. Aristotle appears to conclude that the points traversed along the way are only potential, but not actual, terms of motion, just as a line is only potentially, but not actually, divisible at every point. Granting the metaphysical value of the traditional interpretation when applied to a hierarchy of movers, it has nonetheless failed to explain the *empirical* fact of motion, the experience of something not yet become, but becoming. Given the difficulty of the question, it is understandable, though certainly not laudable, that some have mocked Aristotle's definition as simply beside the point—the point being, in their minds, the treatment of quantity without regard for metaphysical scruples. However, the current state of mathematical physics no longer permits this luxury.

If the act of motion is conceived as proportional to its current potency, then it cannot be clearly distinguished from that potency's perfect fulfillment in act, namely, the end or term of motion. Or if the act of motion is taken to be the path, then motion itself will be extended. The "act of a thing existing in potency insofar as it is such" must, therefore, be an act of a different sort. The act of continuous motion is not a partial actualization of the final term of the motion, or even the extension between terms, but rather that actuality *through which* terms are related at that instant. It is an act that *constitutes* a subject's potency for further act in terms of its present actuality, and this according to a definite ratio or relation. In other words, the act of motion specifies what something is able to be in terms of what it is. The act of motion is thus ontologically posterior to the *terminus a quo* of motion, but prior to the potency of the *terminus ad quem*, all of which exist in the present instant and are subject to continuous change. The act of motion thus relates the before and after that are the foundation of time. The difficulty with the traditional interpretation of Aristotle's definition is that the potency in motion is understood as ontologically prior to the act of motion itself, causing the act to be identified with the fulfillment of that potency, that is, the term of motion. In truth, the moving subject is not passive and static with respect to its own motion, but active and dynamic. Borrowing a medieval distinction, one may say that, although the mobile subject is in passive potency (*potentia passiva*) with respect to its mover, it is in active potency (*potentia activa*) with respect to its subsequent states.

How then is local motion to be defined? Infinite ensembles of natural numbers establish an intelligible ratio between actual and potential quantity. To be more exact, *an infinite ensemble is the act of a quantity in potency (to every division) precisely insofar as it is in potency (to every division)*. It thus expresses a state of *quantitative motion*. The ground of equality, and therefore of measurement, is consequently the apprehension of the *act* of a uniform generative motion. *Becoming is thus subordinated to being*. Moreover, because these ensembles cannot be reduced to the term of motion, but specify that term, motion itself can be something distinct from its terms, a state that persists—or varies—as the mobile being progresses in its actuality.

3.2 Mathematics and Motion

Eidetic theory commits its adherents to a very definite position regarding the division of the sciences. Aristotelians maintain that mathematics does not treat of motion, which is rather the domain of physical science. At a superficial level the question is hardly worthy of discussion, for it does not concern realities so much as human conventions: Both Aristotle and St. Thomas grant that the natural scientist employs mathematics in the study of motion, so the question seems not to be whether mathematics can be used to describe motion, but how one chooses to categorize that use. At a deeper level, however, there is a serious question at stake: Is the intelligible matter belonging to mathematics itself capable of motion? Is there a genuinely *theoretical* mechanics, or is mechanics a merely practical art posterior to the study of immobile mathematical objects?

If the latter, then the path of motion could still be studied in the senses by means of these immobile canons drawn from mathematics. Motion itself, however, would have to be apprehended by extra-mathematical means or conceived by intellectual reflection on the inadequacy of mathematics. Both alternatives present serious difficulties. The understanding of local motion is inseparable from a mathematical account. It is meaningless to describe local motion without adverting to quantity, for the removal of quantity removes the motion as well. The definition of motion can only be applied to various kinds by analogy. On the other hand, reflection on the impotence of mathematics to describe motion does not ascribe to it any positive content. Motion would forever lie beyond the understanding; it would be a mere limit of physical knowledge.

The answer that better accords with experience is that intelligible matter is indeed mobile, but that the mathematician can fix it, or at least consider it as fixed, in his study. If anyone doubts the fluidity of intelligible matter, let him withdraw from the sensible image and try to imagine to himself a true equilateral triangle, retaining unchanged for a while not only the concept, but the intelligible image itself. To reason geometrically without the diagrams that steady the imagination is a matter of great difficulty. One can develop this skill to a high degree, but that which is acquired by practice does not belong by nature, and so intelligible matter as such appears to be no less mobile than sensible matter. Intelligible matter differs from sensible by the intellect's capacity to determine images produced out of its own power. The decisive difference between sensible and intelligible matter, then, leaving aside their respective origins, is that intelligible matter is subject to human will, whereas sensible matter is not. Mathematical abstraction from the senses does not therefore imply abstraction from motion, but only from an inferior mode of imagination. If both physics and mathematics treat of motion, then physics cannot, properly speaking, be the study of mobile being (*ens mobile*). Rather, *physics is the science of substance as sensible*.

3.3 Aristoxenus on Intervallic Motion

Aristoxenus begins his treatment of harmony by dividing movement into two classes, the continuous and the intervallic. This distinction between continuous and discrete change in pitch is the basis of an elegant distinction between speech and music:

> First of all, then, we must discuss the different kinds of movement with respect to place [τόπος], and try to understand what they are. While every vocal sound can move in the manner mentioned, there are two forms of this movement, the continuous and the intervallic. In the continuous form the voice seems to perception to traverse a place [τόπος] in such a way as never to stand still even at the extremities themselves, at least so far as its representation in perception is concerned, moving continuously to the point of silence, whereas in the other, which we call intervallic, it seems to move in the opposite way. During its course it brings itself to rest at one pitch and then another; it does this continuously (I mean continuously in respect of time), passing over the places bounded by the pitches, but coming to rest on the pitches themselves and sounding them alone, and is described as singing, and as moving in intervallic motion.[59]

While the distinction between speech and singing given by Aristoxenus is eminently useful, his theory depends upon a deceptive comparison: He understands the sounding *motions* treated by music to occupy *places*. Aristoxenus is not unaware of the criticism to which he opens himself:

> Let us not be disturbed by the opinions of those who reduce notes to movements, and who say quite generally that sound is movement, as though we should be obliged to say that it sometimes happens that movement does not move, but is stationary and at rest. It makes no difference to us if pitch is called evenness or sameness of movement, or if some yet more learned name than these is invented. We shall say, none the less, that the voice stands still when perception exhibits it to us not setting off towards the high or the low; and all we are doing is attaching this name to that sort of qualification of the voice. The voice appears to do this when it sings; for it moves while it makes an interval, and stands still on a note. If when it moves in what we call motion it is acquiring a difference in speed in the motion of which these people speak, and if when being at rest in what we call rest its speed remains constant and retains one and the same pace, that need make no difference to us. For it is clear enough what we call movement and rest of the voice, and what they call movement.[60]

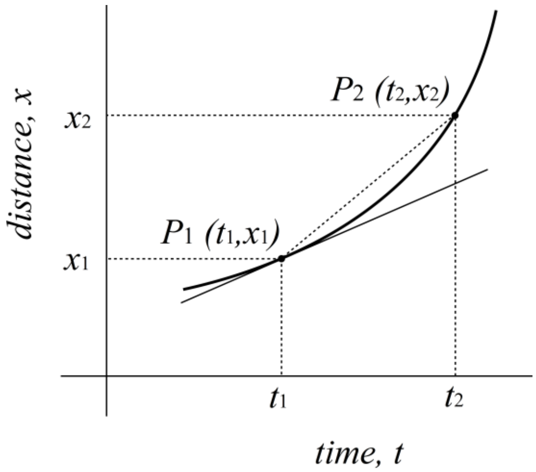
Aristoxenus dismisses this criticism as a mere matter of words, but it is far from clear that he apprehends the philosophical incoherence of describing a change in the state of local motion as itself a local motion—and therefore capable of analysis on the very same basis. Yet in conceiving notes as the mere limits of intervals in a tonal continuum, he can hardly escape such a conclusion. He prefigures the modern physicist, who, to define the *acceleration* (or rate of change in velocity) of an object, uses the same spatial technique as he does to define its *velocity* (or rate of change in displacement). This incoherence belongs equally to the analysis of average and instantaneous rates.

3.4 Bergson's Critique of Modern Kinematics

Though Henri Bergson errs in elevating becoming over being by reducing being to the become, his critiques are often worthy of attention, for they point out defects in modern physics that are congenital. Differential calculus is the natural language of the modern mechanics of motion, and it is by no means accidental that both ripened together in the mind of a single man. Newton understood his "method of fluxions," as the name itself suggests, to be a mathematics of motion:

> Now those Quantities which I consider as gradually and indefinitely increasing, I shall hereafter call *Fluents*, or *Flowing Quantities*, and shall represent them by the final letters of the Alphabet v, x, y, and z; that I may distinguish them from other Quantities, which in equations are to be consider'd as known and determinate, and which therefore are represented by the initial Letters a, b, c, &c. And the Velocities by which every Fluent is increased by its generating Motion, (which I may call *Fluxions*, or simply Velocities or Celerities,) I shall represent by the same Letters pointed thus $\dot{v}$, $\dot{x}$, $\dot{y}$, and $\dot{z}$.[61]

To be sure, his mathematical treatment is unable to apprehend the state of becoming that he expresses by these terms, so he reverts to infinitesimal increments, which, though indefinitely small, retain the *terminus a quo* and *terminus ad quem* of motion. These methods have become quite sophisticated over time, but the central assumption remains that an instantaneous state of motion or becoming can be attained through an ever more accurate calculation from two terms.



$$rate\ of\ change = \frac{dx}{dt} = \lim_{t_2 \to t_1} \frac{x_2 - x_1}{t_2 - t_1}$$

Figure 9:
The Calculation of Tangential Slope

To be exact, the *average rate of chang*e of a variable quantity, displacement for example, is calculated by taking the ratio of the change in that variable to the change in time. This necessarily involves the incoherent but now familiar notion of a ratio of heterogeneous quantities. The differential calculus merely adds to this the notion that an *instantaneous rate of change* can be obtained by reducing the time interval indefinitely. This can be understood from *Figure 9*. By fixing t_1 and allowing t_2 to become arbitrarily close to t_1, the point P_2 will likewise become arbitrarily close to P_1. The slope of the secant joining P_1 and P_2, that is, $(x_2 - x_1)/(t_2 - t_1)$, will simultaneously become arbitrarily close to the slope of the tangent line at P_1. The ratio to which the average rate of change, $(x_2 - x_1)/(t_2 - t_1)$, converges is taken to be an instantaneous rate of change. Instantaneous velocities may therefore be derived by considering the variable slope of tangents to the graph relating displacement to time. However, the velocity is in turn understood to be a quantity varying in time, and is subsequently graphed against time. This new graph likewise has at every "point" in time a tangent of definite slope. Instantaneous acceleration is consequently derived as the instantaneous rate of change of velocity with respect to time. This well-known method is fundamental to the entire project of modern mathematical physics. But does it truly capture a state of motion?

Bergson points out rather convincingly that it does not. He argues that what is apprehended in such a process is always the term of motion, even when the increment of motion becomes arbitrarily small. Consider his argument, shorn of his faulty metaphysical and epistemological interpretations:

> When I follow with my eyes on the dial of a clock the movement of the hand which corresponds to the oscillations of the pendulum, I do not measure duration, as seems to be thought; I merely count simultaneities, which is very different. Outside of me, in space, there is never more than a single position of the hand and the pendulum, for nothing is left of the past positions.[62]

> To measure the velocity of a movement, as we shall see, is simply to ascertain a simultaneity; to introduce this velocity into calculations is simply to use a convenient means of anticipating a simultaneity. Thus mathematics confines itself to its own province as long as it is occupied with determining the simultaneous positions of Achilles and the tortoise at a given moment, or when it admits *a priori* that the two moving bodies meet at a point *X*—a meeting which is itself a simultaneity. But it goes beyond its province when it claims to reconstruct what takes place in the interval between two simultaneities; or rather it is inevitably led, even then, to consider simultaneities once more, fresh simultaneities, the indefinitely increasing number of which ought to be a warning that we cannot make movement out of immobilities, nor time out of space.[63]

The differential calculus, together with the physical sciences that employ it, fail to make intelligible what joins the terms of motion, except in multiplying those terms of motion by further dividing the interval. The apprehension of motion itself is *systematically* deferred by the differential calculus. Bergson locates these difficulties in the limitations of algebra:

> Now, in this analysis of variable motion, it is a question only of spaces once traversed and of simultaneous positions once reached. We were thus justified in saying that, while all that mechanics retains of time is simultaneity, all that it retains of motion itself—restricted, as it is, to a *measurement* of motion—is immobility.
>
> The result might have been foreseen by noticing that mechanics necessarily deals with equations, and that an algebraic equation always expresses something already done. Now, it is of the very essence of duration and motion, as they appear to our consciousness, to be something that is unceasingly being done; thus algebra can represent the results gained at a certain moment of duration and the positions occupied by a certain moving body in space, but not duration and motion themselves. Mathematics may, indeed, increase the number of simultaneities and positions which it takes into consideration by making the intervals very small: it may even, by using the differential instead of the difference, show that it is possible to increase without limit the number of these intervals of duration. Nevertheless, however small the interval is supposed to be, it is the extremity of the interval at which mathematics always places itself. As for the interval itself, as for the duration and the motion, they are necessarily left out of the equation.[64]

The modern calculus presupposes that the whole trajectory of a particle can be given in advance, so that one can, by surveying the whole variable relationship between the particle's displacement and time, arrive at these instantaneous velocities. Such a presupposition, which is, in fact, quite dubious, was first employed analogically by Nicole Oresme[65] in the fourteenth century when he represented time as a magnitude in a graph. Time, however, is by no means extended like the magnitudes of geometry; it cannot be given all at once to our understanding, nor can the terms of motion be simultaneously compared.

3.5 The Attribution of Spatial Properties to States of Motion

It is often said that the theories of relativity and quantum mechanics, verified empirically over the past century, run counter to the common sense founded upon daily experience. However, it would be more accurate to say that these theories violate the common sense developed in the schools, an intellectual habit based not upon the careful observation of nature, but upon the cultivation of algebraic patterns of thinking. In the special theory of relativity, velocities, which purport to represent states of motion, do not add in the manner of displacements. This contradicts the doctrine that is taught implicitly, if not explicitly, from grammar school onward: If an observer moving at velocity, v, away from an observer at rest observes in turn an object moving away at velocity, v_m, the observer at rest will observe the object moving away at a velocity, $v_r = v_m + v$. This follows naturally from the assumption that displacements can be added, that time is common, and that states of motion are nothing more than the evanescent ratios of displacements to time.

Consider carefully the mutual dependence of classical physics and modern mathematical technique. Let there be an object moving in a single dimension and observed by one person at rest and another who is also moving in the same direction. Let d_{ri} and d_{rf} signify the initial and final displacement of the object in relation to the observer at rest, and d_{mi} and d_{mf} signify the initial and final displacement of the object in relation to the observer in motion. Also, let d_i and d_f signify the initial and final displacements of the moving observer in relation to the observer at rest, with the result that $d_{ri} = d_{mi} + d_i$ and $d_{rf} = d_{mf} + d_f$. Finally, let t represent the quantity of time that passes during the motion. Then the average velocity of the object with respect to the moving observer will be $(d_{mf} - d_{mi})/t$, while the average velocity of the object with respect to the observer at rest will be $(d_{rf} - d_{ri})/t$. Under the assumption that displacements are additive, one may substitute the expressions, $d_{ri} = d_{mi} + d_i$ and $d_{rf} = d_{mf} + d_f$, into the second expression to derive the relationship:

$$(d_{rf} - d_{ri})/t = [(d_{mf} + d_f) - (d_{mi} + d_i)]/t$$
$$(d_{rf} - d_{ri})/t = (d_{mf} - d_{mi})/t + (d_f - d_i)/t$$
$$v_r = v_m + v$$

This relationship also holds as these quantities are made arbitrarily small, and is therefore no less valid in the case of the "instantaneous velocities" of differential calculus. This result, however, turns out to be a mere approximation, a relationship that fails to hold as velocities approach c, the "speed of light," the upper bound of all velocities. More generally, $v_r = (v_m + v)/(1 + v_m v/c^2)$.

Is special relativity truly contrary to common sense? For one who is unaccustomed to the modern mathematical treatment of motion, it is difficult to even conceive how two *states of motion* can in any way be "added." It is only the algebraic formalism, which treats motion as a ratio between heterogeneous quantities—and itself a quantity—that convinces. This treatment has colored the fundamental intuitions of the physicist. As Bergson notes:

> We generally say that a movement takes place *in* space, and when we assert that motion is homogenous and divisible, it is of the space traversed that we are thinking, as if it were interchangeable with the motion itself.[66]

States of motion therefore cannot be divided and cannot be expressed in metrical terms. However, as already shown, they can be treated mathematically through infinite ensembles of counting numbers. This applies not only to the experience of motion, but also to the experience of temporal flow, which is grounded in motion. The Aristotelian physics so clearly expressed by St. Thomas must in this matter be corrected:

> When, therefore, we sense one "now" and do not discern in motion an earlier and later, or when we discern in motion an earlier and later but take the same "now" as the end of the earlier and the beginning of the later, then it does not seem that time passes, for neither is there motion. But when we take an earlier and later and number them, then we say that time passes. And this is because time is nothing else than "the number of motion in respect to the earlier and later." For we perceive time, as has been said, when we number the earlier and later in motion. It is therefore obvious that time is not motion, but follows upon motion insofar as it is numbered.[67]

Aristotle conceives time metrically, ignoring the equally valid experience of the flow of time. Eidetic theory supplies the mathematical ground for this universal experience.

3.6 A Metrical Paradox in Relativity Theory

Modern mathematical physicists aim to express notions that contradict Newtonian natural philosophy, while still employing the same mathematical principles Newton used to buttress them, as if mathematics were a philosophically neutral medium of expression. Relativistic physics is advanced by a legal fiction analogous to those used in Roman or common law. A legal fiction is a contra-factual assumption used to apply law to a situation in which it otherwise could not. In the case of relativity, this legal fiction consists in positing a second non-existent observer in the thought experiment. It is formally stated in the principle: *The laws of physics are the same in every inertial (unaccelerated) frame of reference.* Application of this principle permits one to draw conclusions that cannot be drawn from the perspective of a single observer. While this legal fiction has certainly enabled more accurate mathematical predictions, it has done so without reforming fundamental intuitions. The latter can only be accomplished by revising the answers given to the most basic questions about number, quantity, motion, extension, and space.

Special Relativity is derived from several assumptions, including the homogeneity and isotropy of space, and the conservation of momentum and energy. However, the assumption that truly sets it apart from Newtonian mechanics is the constancy of the speed of light in all inertial frames of reference. There is but one standard upon which all observers will agree. This standard is not a length, but absolute and perfect motion, or better yet, perfect mobility. In relativistic mechanics, the measured length of an object is understood to contract because of its motion. The object will appear shorter if viewed from an inertial frame of reference in which it is perceived to be moving than in an inertial frame of reference in which it is seen to be at rest. This contraction of length is governed by:

$$L = L_o \sqrt{1 - \left(\frac{v}{c}\right)^2}$$

where v is the velocity of the object in relation to the observer, c is the speed of light, L is the length measured by the observer, and L_o is the length that would be measured by one observing the object at rest, that is, by one whose frame of reference is moving with the object.

The difficulty for metrical interpretation is real. Consider the measurement performed by an observer who perceives a rigid rod to be moving uniformly in the direction of its length. In the traditional science of measurement, the length of the rod must be expressed as a certain number of measures. For instance, taking a meter-stick as a standard, the observer will say that the rod is a certain number of meter-sticks long. In which frame of reference, then, if any, is the meter stick itself at rest? If the meter-stick is taken to be at rest in the moving frame, and therefore moving with the rod from the perspective of the observer's frame, the meter-stick will contract by the same factor as the rod being measured. The *number* of meter-sticks commensurate to its length will consequently be the same whether the rod moves or not. One may draw from this a startling inference: A *part cannot serve as the measure of the whole*. For if the whole of the length is moving, then so also must the parts of that length, with the result that the *number* of measures will be the same in all frames of reference. If relativistic mechanics is to be coherent, the measure cannot be moving with the object measured. On the other hand, how can a meter-stick taken to be at rest in the "rest" frame serve as the measure of a moving object that is displaced from it by some distance? Superposition is certainly not possible in this case, if ever. A complete reform of measurement is demanded. To apprehend the meaning of relativistic mechanics, one must therefore grasp the role of motion—*real* motion, *states* of motion.

3.7 The Role of Ensembles in the Science of Meter

Motion plays a fundamental role in the act of measurement, that is, in the science of meter. Although the mathematical ratios established by infinite ensembles of natural numbers express states of uniform motion, these uniform motions can nevertheless yield measures. In harmony, two potential quantities are related to a common actual quantity through their respective eidetic relations. Suppose that the potential quantity of both relations is joined instead, allowing the actual quantities to differ. The ratio of two actual quantities will be the same as the ratio of the two potential quantities previously considered, but inverted. For example, the octave relation yields a potential quantity that is twice that of the unison. Inverted, the octave relation yields an actual quantity that is one-half of that produced by the unison. The pairs of inverted ratios act much as temporal frequency and period (or spatial frequency and wavelength) do in modern physics:

$$\nu \cdot \tau = 1 \qquad \xi_x \cdot \lambda_x = 1 \qquad \xi_x \cdot \lambda_y = 1 \qquad \xi_y \cdot \lambda_z = 1$$

Wavelengths and actual quantities are analogues. Both determine within their respective theories the accuracy with which one can measure lengths. However, in eidetic theory both frequency and reference quantity are quantitative *relations*. Reference quantity is not an absolute, for it can be applied to any definite extension.

Just as in the apprehension of tones there is an inexhaustible community of species between the octave and the unison, despite their differing states of motion, so also in the generation of measures. As in the generation of octaves, the removal of odd numbers from the counting numbers leaves a residue that has the same eidetic character as the original. Thus, the residue can itself be treated as a unison, to which there is a corresponding octave. This process can be carried out indefinitely. Because the unison reference is exactly twice that of the octave reference, a whole chain of equal juxtaposed measures can be generated. The process is expressed below:

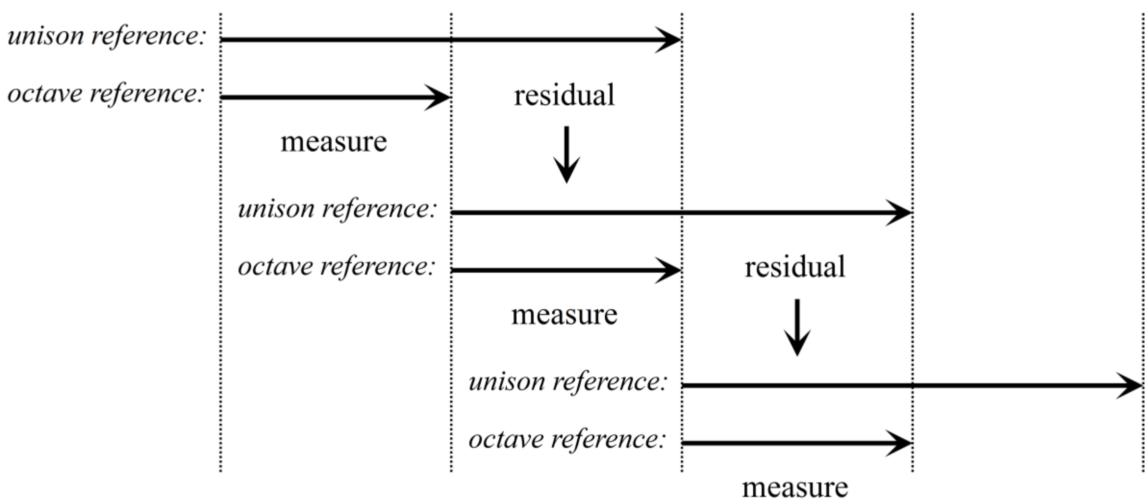


Figure 10: The Generation of Equal and Juxtaposed Measures

The equality and juxtaposition of measures thus generated suffices as a ground for the act of measurement without assuming the subdivision of quantity *ad infinitum*. The relationship between octave and unison is an analogue to Plato's ἀόριστος δυάς, or indefinite dyad, which he understood to operate on the ἕν, or one, to produce multitude. Such a notion applies better to the musical domain than to the arithmetical.

3.8 Physical and Statistical Time

In classical modern physics, it is possible to trace the exact trajectory of a particle as it travels through space, for it has but a single position and velocity at every instant. Subsequent positions and velocities are fully determined by the present through reciprocal forces acting between particles. Measurement itself does not enter the description of the evolving system of particles. In quantum mechanics, however, the observer has no privileged access to the position and velocity of an object, but must perform a measurement. The result of this measurement is not itself the bearer of the system's subsequent states. Rather, quantum mechanics specifies a *wave function* bearing only the *intrinsic probabilities* of measurement. In the non-relativistic case, that is, when the speed of the particle is far less than the speed of light, this wave function is governed by the Schrödinger wave equation:

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)+V(x,y,z,t)\psi=-\mathrm{i}\hbar\frac{\partial\psi}{\partial t}$$

Since the wave function, ψ, from which the probabilities of the various results of a measurement are derived, is a function of both position and time, the probabilities themselves are functions of time. And since the equation includes a derivative with respect to time, these probabilities likewise have a rate of change.

However, the idea of an instantaneous rate of change in probability is incoherent, for such a thing can never be empirically validated. The definition of the time derivative demands that the increment of time go to zero ($\Delta t \rightarrow 0$). This time increment belongs to *physical time* because it purports to describe how the wave function and its attendant probabilities will evolve. However, the wave function represents an infinitely large distribution of quantities, and can therefore only be constructed statistically by taking an arbitrarily large number of measurements. As each measurement demands time, so also this ever-increasing collection of measurements requires an ever-increasing increment of time to perform them. Thus, to accurately specify the wave function, *statistical time* must increase without limit ($\Delta t \rightarrow \infty$). Nor is it legitimate to substitute single measurements on an indefinitely large number of "equivalent" systems for the indefinitely large number of measurements on the single evolving system, for this does not constitute a temporal evolution of the sort described by the Schrödinger equation. This latter procedure, which is casually accepted today, represents an error in logical types. In the limit of small measures, just as in that of high speeds, the mathematical ground of the modern project fails, and the difficulty of measurement is felt acutely.

To resolve the logical contradiction implied by the divergence of these two species of time, it is necessary to reformulate the calculus of motion on a new basis. Rather than relying on the *differential calculus* of the infinitesimally small, one must resort to a *statistical calculus* based on infinitely large ensembles. In such a calculus, "irrational numbers" are mathematically unnecessary, because uncertainty can be represented by distributions of commensurable quantities. Consider the collection of all possible metrically described quantities represented in *Table 12*:

		Numerator						
		1	**2**	**3**	**4**	**5**	**6**	⋯
Denominator	**1**	1[μ/1]	2[μ/1]	3[μ/1]	4[μ/1]	5[μ/1]	6[μ/1]	⋯
	2	1[μ/2]	2[μ/2]	3[μ/2]	4[μ/2]	5[μ/2]	6[μ/2]	⋯
	3	1[μ/3]	2[μ/3]	3[μ/3]	4[μ/3]	5[μ/3]	6[μ/3]	⋯
	4	1[μ/4]	2[μ/4]	3[μ/4]	4[μ/4]	5[μ/4]	6[μ/4]	⋯
	5	1[μ/5]	2[μ/4]	3[μ/5]	4[μ/5]	5[μ/5]	6[μ/5]	⋯
	6	1[μ/6]	2[μ/6]	3[μ/6]	4[μ/6]	5[μ/6]	6[μ/6]	⋯
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Table 12: An Infinite Metrical Ensemble

There is little difficulty apprehending indefinitely large sums of quantities expressed with the same measure. The sum of the first n terms in row d of the table yields $((n^2 + n)/2)[\mu/d]$. It is a purely arithmetical problem with a simple solution, because the measure is equally denominated in each term. The real difficulty lies in finding the expectation values of an indefinitely large number of quantities *differently denominated*, which corresponds to finding the sum down a column. For purposes of comparison, consider the first d rows of column n. There is a common numerator, but every possible denomination of quantity is represented by the denominators. The sum would be:

$$n[\ [\mu/1] + [\mu/2] + [\mu/3] + [\mu/4] + [\mu/5] + \ldots + [\mu/d]\ ]$$

Such a series must be made to converge. The eidetic principles by which this is accomplished therefore represent a necessary first step in developing a complete statistical calculus.

3.9 Quantum Physics, Statistical Mechanics, and Eidetic Theory

Quantum mechanics treats of intervallic as well as continuous change. To borrow the distinction of Aristoxenus, it has ceased to be mere speech, and has become musical. Eidetic theory has a profound affinity to quantum mechanics. To begin, it posits a mathematical distinction between motion and the terms of motion. Quantum mechanics admits four *kinematic* variables: time (t) and the three spatial variables (x, y, and, z). Corresponding to the spatiotemporal variables are four *dynamic* variables: energy (E), and a component of momentum in each spatial direction (p_x, p_y, and p_z). These dynamic variables are related to frequency by the relations,

$$E = h \cdot \nu \qquad p_x = h \cdot \xi_x \qquad p_y = h \cdot \xi_x \qquad p_y = h \cdot \xi_y$$

where h is Planck's constant, ν is the temporal frequency of the wave, and ξ_x, ξ_y, and ξ_z are the spatial frequencies of the wave in the x, y, and z dimensions. The four pairs of kinematic and dynamic variables are easily assigned a metrical or harmonic character, for the former signify a term of motion, the latter a state of motion:

	Metrical Treatment	*Harmonic Treatment*
Temporal Variables	t (time)	E (energy)
Spatial Variables	x, y, and z (position)	p_x, p_y, and p_z (momentum)

As there are but two complementary treatments of quantity, it suffices to consider motion in its terms (position and time) or in its process (momentum and energy). Higher-order derivatives in modern mechanics, such as acceleration, are *a priori* incapable of referring to states, and can only signify changes in states.

There is, moreover, a profound connection between the mathematical science of harmony and statistical mechanics. Recall that eidetic ratios are the results of a limiting process:

$$\lim_{n\to\infty}\left[\frac{[\mu/1]+[\mu/2]+[\mu/3]+[\mu/4]+\cdots+[\mu/\Lambda f(n)]}{n}\right]$$

The limit can be materially approximated in modern terms:

$$\lim_{n\to\infty}\left[\frac{\ln(\Lambda f(n))}{n}\right]=\nu$$

where ν is a temporal frequency ratio. When the ensemble is very large, that is, when n is very large, this becomes:

$$\ln(\Lambda f(n))/n \approx \nu$$

And this yields a "common measure" of:

$$1/\Lambda f(n) \approx e^{-n\nu}$$

This is comparable in form to the exponentials appearing in the canonical partition functions of statistical mechanics:

$$e^{\left(-\frac{E_i}{kT}\right)} \quad \text{or} \quad e^{\left(-\left(\frac{h}{kT}\right)\nu_i\right)}$$

Where E_i is the energy of the i^{th} microstate, k is Boltzmann's constant, and T is the absolute temperature. Statistical mechanics must make assumptions about energy levels (*e.g.* all microstates of a system having the same energy are equally probable.) It is quantum mechanics that supplies the relation of energy to frequency, $E = h \cdot \nu$. Eidetic theory itself generates the frequency relation, without demanding another theory. It therefore has the potential to link quantum mechanics and statistical mechanics more explicitly, by showing that each represents a limiting case of a more general theory of measurement.

3.10 Mathematical Constants in Physics

From the point of view of eidetic theory, *Euler's number*, *e*, also called *Napier's constant*, which is found not only in the canonical partition functions of statistical mechanics, but also in solutions to the Schrödinger wave equation, represents an approximation to reality. It is represented numerically as an unending decimal, 2.7182818284…, and appears in the base of the natural logarithm, $\ln(n)$ or $\log_e(n)$. This natural logarithm can be defined by:

$$\ln(n) = \int_1^n \frac{1}{x} dx$$

This function is represented graphically below as the lightly shaded area under the curve $y = 1/x$ between $x = 1$ and $x = n$:

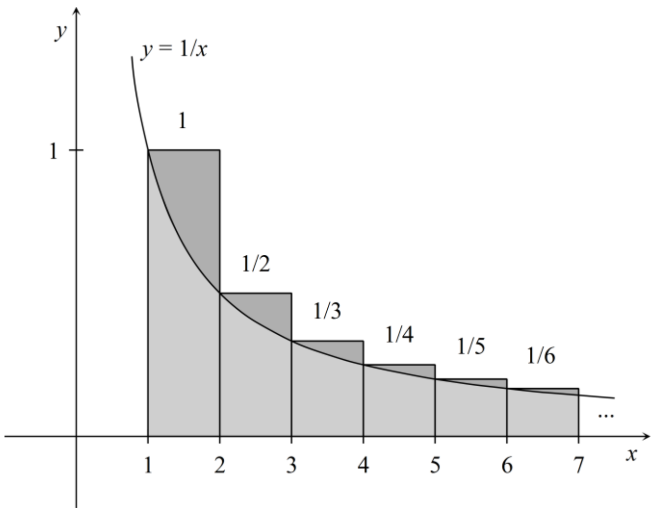


Figure 11: A Comparison of the Natural Logarithm and the Harmonic Series

This relationship approximates the harmonic series, previously discussed and also included in the graphical representation. The more darkly shaded area above the curve, $y = 1/x$, represents the deficiency of the approximation of $\ln(n)$ to the harmonic series. This deficiency converges materially to what is called the *Euler-Mascheroni constant*, γ:

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \int_1^n \frac{1}{x}dx\right) \approx 0.57721566\ldots$$

If the Napier constant arises in approximating the harmonic series, then the Euler-Mascheroni constant represents a correction to this approximation. The latter constant appears in some procedures of "renormalization," by which divergent series in modern physics are made to converge to "accepted" values. These procedures have little to no philosophical justification. Is it not possible that these series diverge precisely because the role of the harmonic series in measurement has not been properly understood? There is a historical precedent for this in the "ultraviolet catastrophe," in which the total black-body radiation of an emitter over all frequencies did not converge—until frequency and energy were quantized by Max Planck.

3.11 Quantitative Operation and Quantity as Such

One must distinguish carefully between the phenomenal and operational forms of music. Only the former have been thus far considered. As a phenomenon apprehended and experienced in audition or imagination, harmony relates states of uniform motion by means of arithmetical ensembles. The formal part of the ratio between actual and potential quantity therefore lacks the unity of a single form. Since a real ensemble can be incomplete or vary somewhat from the ideal, there can be some play in heard intervals. Octave equivalency and the formation of pitch classes also depend upon this dispersion of form. Considered rightly, this dispersion is a metaphysical imperfection, for unity is one of the transcendental attributes of being. Moreover, the inexactitude of any finite ensemble, no matter how large, makes operating with these forms impossible. They pertain rather to harmonic experience and the apprehension of motion in the phantasm.

Yet these phenomena cannot stand on their own unless grounded in the real transformation of one quantity into another. In consequence, there must correspond to each phenomenal form of music, that is, to each ensemble, an operational form possessing the requisite unity. A distinction of actuality and potency in the quantitative terms of the relation remains, but the relation takes quantity and transforms it in accordance with the same eidetic ratios that hold for the phenomenon of music. This new conceptual relation can be derived from the specific character of the ensemble itself by a transcendent reflection on the part of the observer. The operation contains the species as *implicate* form, whereas the phenomenon exhibits the species in *explicate* form. Or, to speak in terms of logic, the operational form includes the species as logical intension, the phenomenal form as logical extension.

Yet reason can attain a still more fundamental concept by means of such reflection. The operational forms relating actual and potential quantity still presuppose a concept of quantity as such to serve as the ground of all things musical. It is only through such a concept that one can even define the subject matter of music. From what species is such a concept formed? Quantity, taken simply, must be always and everywhere equal to itself. Therefore, any inequality between the actual and potential terms in a phenomenal relation of music makes the ensemble unsuitable to the formation of the concept. It follows that the concept of quantity as such must be derived from the most perfect of all ensembles, namely, the ensemble of all numbers. No quantitative distinction holds between the terms of the equality, only the metaphysical distinction of act and potency. Nevertheless, the concept of equality, whether phenomenal or operational, remains a *relative* one. There must be an absolute in which this relation itself is grounded. And so, the intellect, reflecting upon the ensemble of all numbers as the phenomenal form of the unison, forms an *absolute* concept having but a single term, that is, the concept of *quantity as such*. To be clear, quantity as such is not conceived symbolically as common to arithmetic and geometry, but as the subject of a distinct and definite mathematical science, the science of music. The single term of this concept transcends the distinction of act and potency in quantity. It thus serves in a double role. First, it is the ground of both terms, actual and potential, in the operational and phenomenal relations. Second, because of its superiority to every form of quantitative operation, including that of equality, it is the metaphysical envelope in which such transformations occur. Distinctions of act and potency at a higher grade than quantitative naturally remain untouched by this superiority.

3.12 The Analogy of Physical Natures

The observer of nature is presented everywhere with sensible states of motion. Indeed, the Latin, *natura*, and the Greek, φύσις, indicate the essence of a thing insofar as it is a principle of motion. It is the task of the physical scientist not only to describe these motions, but to investigate their causes. For this reason a correct assignation of ontological grades to mathematical objects is crucial to the physical scientist's understanding of reality. If ontological grades are confused, or discarded altogether, one's apprehension of reality will be proportionally degraded. Because reality proceeds from cause to effect, whereas the investigation of that reality proceeds from effects to their causes, the order of concept formation is generally the reverse of the order of nature. Practically speaking, then, one may often arrive at the order of nature by reversing the order of concept formation.

By applying this epistemological principle to the study of music, one can see clearly that the *real phenomena* of quantitative motion, as exact analogues of the mathematical relations, lie at a lower ontological level than the real intention of quantity as such—or even the quantitative operations that follow upon it. This empirical motion, studied mathematically, is nothing other than the intermediate science of harmonics. Lying behind these real phenomena are the real operations that continually transform the material subject from one quantitative disposition to another. These real operations are analogues of mathematical operations. To the highest concept of music, quantity as such, corresponds the highest musical analogue in nature. The character of this reality has been so obscured in modern times that it is necessary to reconsider the Thomistic doctrine of *substance*.

3.13 A Summary of the Thomistic Doctrine of Corporeal Substance

For St. Thomas, as for Aristotle, substance is the primary predicament. The other nine predicaments, the accidents, are ultimately grounded in a real relation to substance, because substance is the reality in which they inhere. There is a not a single material substance for the whole material creation, but many, each of which subsists in its own matter and possesses its own *substantial form*. To this form, which is the first act of the substance, there corresponds a first potency, to which the substantial form is joined. This first potency is called *primary matter*. Within the composite of substantial form and prime matter there exist many secondary forms and matters which together form accidents. Because secondary matter is not pure potency in the order of material substance, it necessarily has some prior form, not unlike the way that the bronze in which a bust is cast already has a different secondary form and matter than the gold, silver, iron, or any other material out of which it might otherwise have been cast. Primary matter, on the other hand, is a potency prior to all such accidents, because the substance, which depends upon prime matter as a principle, is the very subject in which these accidents can exist.

The substantial form as such communicates existence and its proper essence to prime matter and the body. The forms of the operations proper to a given substance exist virtually within the substantial form and work in the body in accordance with the secondary matter in each part. The dispositions of the body are necessary to these operations, but the root of operational power lies in the substantial form. Defects in operation are consequently due to inadequate disposition of matter:

> the species of operation follows upon the species of form, which is the principle of operation; but the inefficacy of operation follows form inasmuch as it inheres in a subject.[68]

The soul is simply the substantial or essential form of an animate body. It possesses operations more perfect than those of inert substance, and transforms the body in accord with them. Plants, animals, and men have souls, each with a perfection corresponding to its nature. The operations of the soul follow upon that nature, but work only in parts of the body that are properly disposed:

> Although the soul is a simple form according to essence, it is nevertheless manifold in power inasmuch as it is the principle of diverse operations. And because a form perfects matter, not only for the sake of to-be [*esse*], but for the sake of operating, it is therefore necessary that, although the soul is one form, the parts of the body are perfected by it in diverse ways, and each according to that which befits its operation. And following upon this it is also necessary that there be an order in the parts according to the order of operations, as has been said. But that order is according to the operation of the body with respect to the soul, which is a mover.[69]

> The soul, although it is one and simple in essence, still has power with respect to diverse operations. And inasmuch as the soul is the form of the body according to essence, it naturally gives to-be [*esse*] and species to what is perfectible by it. Those things, moreover, which exist naturally are on account of an end; it is necessary, then, that the soul establish in the body a diversity of parts, as befits diverse operations.[70]

> an organic body is perfectible by the soul primarily and through itself, but the individual organs and parts of organs as ordered to the whole...[71]

Man alone in the material creation has rational operations, which transcend matter, even if the senses, memory, and imagination are themselves bound up in material causes:

> Although the to-be [*esse*] of the soul is in some way that of the body, nevertheless the body does not achieve participation of the soul's to-be [*esse*] according to the soul's whole nobility and power; and there is consequently some operation of the soul in which the body does not share.[72]

Possessing incorporeal operations, the human soul must be able to subsist without a body.

3.14 Musical Harmony and the Immortal Soul

The doctrine that the soul is a harmony is a truly ancient one, originating, it seems, with the Pythagoreans. If harmony is understood metrically as a disposition of elements or parts of the body, then the objection of Simmias to the immortality of the soul, as presented in Plato's *Phaedo*, must be conceded:

> If, then, the soul happens to be some sort of harmony, it is clear that whenever the body is slackened or stretched disproportionately by diseases and other evils, the soul, though most divine, must straightaway be destroyed, just like the other harmonies, those in sounds and those in all the other works of craftsmen; and that the remains of each body stay behind for a long time until they are burned or decay.[73]

Plato, through the character of Socrates, argues that harmony is more fitting to virtue in the soul than to the existence of the soul itself. A more exact and penetrating rebuttal of the Pythagorean doctrine is given by St. Thomas Aquinas. To the argument that the departure of the soul coincides with the departure of harmony in the body, and that the soul is therefore nothing more than the form of this harmony, the Common Doctor gives a succinct reply:

> But this does not follow; because a proportion of this sort is not [substantial] form, as they used to believe, but is the disposition of matter to form. And if the harmony of a composition is properly taken for a disposition, it follows well that, with the disposition of matter to form remaining, the form remains, and with the disposition destroyed, the form is removed. Nevertheless, [it does] not [follow] that harmony is form, but the disposition of matter to form.[74]

> The body having been corrupted, there does not perish from the soul the nature according to which it befits the soul to be a form, even though it does not by its act perfect matter so as to be a form.[75]

St. Thomas denies that harmony pertains to the essential or substantial form of a living being. It is rather a disposition of matter enabling it to receive substantial form. It follows, then, that the destruction of this harmony will result in the death of the body, but not necessarily the destruction of the soul. The musical objection to the immortality of the human soul is therefore invalid.

This analysis is based on a metrical understanding of harmony, which does not rise above the dimensions of the body. The eidetic theory transcends the Angelic Doctor's observation about the character of harmony. To contrast, it pertains neither to the body nor to the soul, as such, but to the mode of their union. The human soul, though of itself subsistent, is distinguished from the subsisting forms of angels by its real intention toward body. Granting that the dispositive form of the body, which includes all its metrical relations, is requisite to the retention of substantial form, the substantial form must itself include operative forms that coincide with the dispositive forms. The implicate arithmetical species (εἶδος) within an operational form of harmony cannot plausibly belong to extension, for it is conceptual. It expresses a "quiddity" (*quidditas*) or "whatness" transcending the mere disposition of matter. The form of such an operation must exist virtually within the substantial form.

The operational forms of harmony, being directed not to individual magnitudes, but to all magnitudes that can be put into the quantitative relation, satisfy the demand that the soul be *operationally* present to every part of the body that is organically disposed to be worked upon. But this cannot explain the singular communication of existence to the whole body. The soul has but one termination whereby it communicates some part of its existence to the body:

> Although to-be [*esse*] is the most formal among all things, it is nonetheless also the most communicable, although it is not shared in the same mode by lower and higher things. In this way, therefore, the body participates in the to-be [*esse*] of the soul, but not so nobly as the soul itself.[76]

Only one musical form, the concept of quantity as such, can serve as an analogue for this intention of the soul. The soul is present to the body in its whole and parts, per St. Thomas Aquinas, just as the intention of quantity is present to all mathematical extension. The real intention of quantity contracts the body's participation in the soul's existence (*esse*), but without contracting essence.

3.15 The Algebraic Foundation of the Mechanical Philosophy

The philosophical methods and doctrines of Descartes are coordinated with his mathematical practice. Regarding the composition of substance, he considers the scholastic notion of prime matter to be denuded of every positive feature, and therefore devoid of any meaning. He proposes in its stead a matter identical with body extended in three dimensions:

> On the other hand, let us not also think that this matter is the 'prime matter' of the philosophers, which they have stripped so thoroughly of all its forms and qualities that nothing remains in it which can be clearly understood. Let us rather conceive it as a real, perfectly solid body which uniformly fills the entire length, breadth and depth of this huge space in the midst of which we have brought our mind to rest.[77]

In thus conceiving matter, he takes his cue from solid geometry. His matter is the subject of pure extension, definite in its disposition, a kind of secondary matter:

> In reality it is impossible to take even the smallest fraction from quantity or extension without also removing just as much from the substance; and conversely, it is impossible to remove the smallest amount of substance without taking away just as much from the quantity or extension.[78]

For Descartes, the very essence of matter is extension. He thus decisively sets aside the prime matter posited by scholastic philosophers as the necessary correlate of substantial form:

> Nevertheless, the philosophers are so subtle that they can find difficulties in things which seem extremely clear to other men, and the memory of their 'prime matter', which they

> know to be rather hard to conceive may divert them from knowledge of the matter of which I am speaking. Thus I must tell them at this point that, unless I am mistaken, the whole difficulty they face with their matter arises simply from their wanting to distinguish it from its own quantity and from its external extension—that is, from the property it has of occupying space. But they should also not find it strange if I suppose that the quantity of the matter I have described does not differ from its substance any more than number differs from the things numbered. Nor should they find it strange if I conceive its extension, or the property it has of occupying space, not as an accident, but as its true form and essence.[79]

Descartes is certainly correct that scholastic philosophers were motivated by a desire to distinguish extension and primary matter, for definite shape and size must be posterior to the composition of primary matter and substantial form, that is, to substance itself. Aristotle's category of disposition, or *situs*, is predicated accidentally, not substantially. The sitting Socrates is the same substance as the standing Socrates. Since material substance is a persistent subject for varying accidents, the matter corresponding to substantial form must be prior to any definite extension.

As body already extended in three dimensions, Cartesian matter dispenses with substantial form. The indefinite divisibility of mathematical operation in algebra, leading to its subsequent identification with extension, is the key to understanding the mechanical philosophy of nature. Since operations are considered only materially, the operations once reserved to substantial form are now taken to be intrinsic to matter itself. Matter is not only transformed, but self-transforming. An animal has no soul, the substantial form of an animate body, for it is just a machine obeying universal mathematical "laws." It does not rise above the surrounding inert matter by its more perfect operation, but is of a piece with it, differing not in nature, but in complexity.

Descartes preserves the human soul because it has immaterial operations, but severs it from any unconscious influence over the body. Since the formal role of the soul is to give life to the body, his musings on the process of dying are filled with consequence:

> Furthermore, although all these movements cease in a corpse, once the soul has quit the body, we must not infer that it is the soul which produces them; the only inference we may make is that it is one and the same cause which both make the body unfitted to produce these movements and makes the soul leave the body.
>
> It is true that we may find it hard to believe that the mere disposition of the bodily organs is sufficient to produce in us all the movements which are in no way determined by our thought. So I will now try to prove the point, and to give such a full account of the entire bodily machine that we will have no more reason to think that it is our soul which produces in it the movements which we know by experience are not controlled by our will than we have reason to think that there is a soul in a clock which makes it tell the time.[80]

In the mind of Descartes, the human body is nothing but a machine, the operations of which are wholly specified by the disposition of its parts. Although he allows that human will has some influence over the body, this influence must remain completely mysterious so long as the operational union of body and soul has been broken.

Since the soul is no longer present to each part or organ with respect to its respective operation, the soul can have a relation only to the whole body, and only insofar as it is a *working* mechanical whole:

> But in order to understand all these things more perfectly, we need to recognize that the soul is really joined to the whole body, and that we cannot properly say that it exists in any one part of the body to the exclusion of the others. For the body is a unity which is in a sense indivisible because of the arrangement of its organs, these being so related to one another that the removal of any one of them renders the whole body defective. And the soul is of such a nature that it has no relation to extension, or to the dimensions or other properties of the matter of which the body is composed: it is related solely to the whole assemblage of the body's organs. This is obvious from our inability to conceive of a half or a third of a soul, or of the extension which a soul occupies. Nor does the soul become any smaller if we cut off some part of the body, but it becomes completely separate from the body when we break up the assemblage of the body's organs.[81]

The integrity of a Cartesian body does not derive from its soul, but from the way its parts, or organs, are disposed correctly to act upon each other. It is the integrity of a mechanical design, something never adequately defined by Descartes. He imposes operability upon matter in a manner analogous to that in which algebraists impose operability upon number. Matter is reconceived at each stage in the development of modern physics to include the operability demanded by new experimental results. This, then, is the history of modern scientific progress. Mechanical integrity of the sort proposed by Descartes does not precede operability, but is derived from it: *Esse sequitur operatio.* This mechanical integrity is not, then, imposed upon the body by the soul, but merely read from the body as the necessary condition of the soul's union with it. How the soul can read these dispositions when it "has no relation to extension" is beyond all explaining, as is the manner in which bodies so perfectly fitted to the soul are somehow to arise independently of it.

According to Descartes, when one or more organs are destroyed, rendering the machine inoperable, this mechanical integrity is dissolved and the body dies. The body does not die because the soul, no longer able to reside in the body, departs and no longer animates it. Rather, the soul departs because the body has died:

> So as to avoid this error, let us note that death never occurs through the absence of the soul, but only because one of the principal parts of the body decays. And let us recognize that the difference between the body of a living man and that of a dead man is just like the difference between, on the one hand, a watch or other automaton (that is, a self-moving machine) when it is wound up and contains in itself the corporeal principle of the movements for which it is designed, together with everything else required for its operation; and, on the other hand, the same watch or machine when it is broken and the principle of its movement ceases to be active.[82]

Plato had considered self-motion to be the essential feature of soul itself, but the operations of a "self-moving machine" would inhere in matter apart from the activity of the soul.

3.16 Prime Matter and the Principle of Individuation

If, as has been argued, the to-be (*esse*) that is communicated to a body through its substantial form is contracted by quantity as such, it yet remains to ascertain the meaning that can be assigned to Thomistic prime matter. This prime matter can be neither quantity as such, as this signifies only an intention toward body, nor extension, which is already differentiated into parts. Rather, it must signify the real intention of quantity as such, but only insofar as it is united to real extension. From that simple intention, there is but one primary matter in substance corresponding to the one substantial form. Primary matter is, however, distinguishable from the real intention of quantity, as well as from the to-be (*esse*) delimited by that intention.

On this basis, it becomes possible to investigate the medieval problem of individuation through mathematical analogues. The principle of individuation (*principium individuationis*) in a material substance is what makes it integrally one, while setting it apart from other substances. The explanation espoused by the Thomistic school of philosophy comes nearest to fulfilling these demands. St. Thomas maintains that material substance is individuated by matter designated by quantity (*materia quantitate signata*). He adds that this quantity can only mean *indefinite quantity*, for, as already noted, the individual persists through changes in bodily disposition. Lacking a mathematical science intermediate to geometry and arithmetic, St. Thomas struggles to express how dimensions can serve to individuate a substance without themselves becoming accidents:

> Those dimensions, however, can be considered in a two-fold manner: In one manner according to their termination; and I say that they are terminated according to a determinate measure and figure, and in this manner they are placed together in the genus of quantity as complete entities. And in this way, they cannot be the principle of individuation, because, since such a termination of dimensions would vary frequently about the individual, it would follow that the individual would not remain always the same in number. In another manner, although dimensions can never be without some determination—just as the nature of color cannot be without the determination of white or black—they can be considered without that determination in the nature of dimension, and they are thus placed together in the genus of quantity as imperfect. And matter is made out of these indeterminate dimensions to be this designated matter. In this way it individuates form, and thus from matter is caused a diversity in number within each species.[83]

> Dimensions cannot be understood in matter except that matter be understood as constituted in its substantial and corporeal to-be [*esse*] by substantial form, which in a man is not done through any other form than the soul, as has been said. Hence dimensions of this sort are not understood in matter before the soul, absolutely speaking, but insofar as the final grades of perfection are concerned...[84]

From the point of view of geometry alone, a distinction between imperfect and perfect quantity makes little sense. There is no geometrical figure without definite shape and dimension. From the perspective of eidetic music theory, however, this imperfect quantity makes perfect sense. It can be identified as quantity as such. In a legitimate sense, quantity as such is unterminated.

Before determining the principle of individuation, it is important to distinguish unity and integrity. *Unity*, absolute indivisibility, and *identity*, its relational counterpart, are formally in the substantial form. Metaphysically, these pertain to *subsistence*. *Integrity*, which is the foundation of number and numerical difference, is, so to speak, the unity of making. It is formally in the maker, and only materially in that which is made. Its relational counterpart is *alterity*, which sets off one subject from another. Metaphysically, these derive from *existence*. Because matter of itself has no formal principle, integrity is formally in the substantial form through which matter receives its existence and only materially in the matter receiving it. Taken relatively, matter stands in opposition to all that does not partake of that integrity.

The principle of individuation must therefore belong to the order of existence, rather than subsistence. Moreover, it cannot be identified with the integrity and alterity belonging to matter as simply extended, for each part is distinguished as being other than the residue of the whole that does not belong to the part. Then every part would be individuated of itself, destroying the integrity of the whole. To put it in concrete terms, the hand of Socrates, having a certain integrity of its own and a bearing of alterity toward the remainder of the body, would thereby be individuated at the expense of the whole Socrates. The individuation of material substance demands that the whole matter of the substance possess integrity and a bearing of alterity regarding all else. The matter of individuation must therefore be primary matter.

According to St. Thomas, this matter of individuation is designated by indefinite quantity, which the musical analogue identifies as quantity taken simply, that is, quantity as such. The quantity by which matter is designated is therefore neither an accidental quantity posterior to substance nor a quantity prior to substance, but a quantity concomitant with substance. However, still greater refinement is demanded. In musical science, quantity as such serves two roles. First, it establishes the intelligibility of the terms between which quantitative operations work. That there should be a mathematical ratio between two line segments presupposes that they both be understood as quantities. Analogously, the real intention toward body must first unite quantitatively to each extended part of matter before that part can be worked quantitatively. Second, quantity as such is the existential envelope of extension. Extension is more narrowly contracted than quantity intended simply. Whereas substantial form is operationally in prime matter, prime matter is existentially in substantial form. It is by this second analogy that prime matter individuates, for it receives an integral existence and consequently has a single bearing of alterity toward other bodies. To summarize: *The principle of individuation for corporeal substance is prime matter designated by quantity as such, and taken existentially, rather than operationally.*

After death, the human soul continues to exist. This means that the existential intention toward body is retained, even if it is no longer terminated in a physical body:

> Separated souls do not differ in species, but in number, from the fact that they can be united to such or such body.[85]

Existential integrity and alterity are retained in the real intention of the human soul toward body, even after the loss of the body. Eidetic theory thus provides a mathematical analogue for this persisting individuation that geometry certainly cannot.

3.17 Music and Ethics

The moral influence of music has been investigated at least since the Pythagoreans. It cannot be doubted that the motions of audible music influence the operations of the soul. As St. Thomas observes, the musical impetus received by the passions can be helpful in moving men to act rightly:

> It is, moreover, obvious that the souls of men are affected in different ways in accord with different melodies... And, therefore, it was decided that hymns be incorporated into the divine praises in order that the minds of the weak might be roused to devotion.[86]

Audible music disposes the soul of the hearer to operate in accordance with the occasion. A march rouses the soldier to battle. Lyrical music inclines the hearer to affection or compassion. Because the source of this power and the ethical responsibilities of the musician are difficult matters, deference to the received wisdom is always a prudent course.

Eidetic theory does, however, illuminate one aspect of ethical theory, namely, the moral virtues, the chief being justice, prudence, temperance, and fortitude. These virtues direct the soul toward a mean (μέτρον) between extremes. For instance, fortitude leads the soul to seek the mean between cowardice and recklessness. Natural justice concerns either the proportional distribution of goods or the equality of value in their exchange. Each of these virtues involves quantity either literally or by analogy. Going beyond or falling short of the measure is a moral failure, just as tightening a violin string too tightly or loosely will produce a pitch that is too sharp or flat to harmonize with the other strings. It should not surprise then that harmony has long been a model of moral virtue, particularly for the Pythagoreans and Platonists, who were influenced greatly by mathematics.

To this traditional theory of the mean, eidetic theory adds the principle of hierarchy. Just as two quantitative motions harmonize if the species of one ensemble is comprehended under the species of the other, and, just as quantitative operations harmonize if the implicate form of one is comprehended virtually by the other, so the operations of soul harmonize only if subordinated one to another in their formal perfection. As phenomenal and operational harmonies in music yield the proper relational measure, namely, those intervals that harmonize, so the proper hierarchical ordering of operations in the soul yields the proper measure (μέτρον) of moral action. A soul is most truly musical when the passions are ordered by the natural virtues, in turn elevated by grace and perfected by the theological virtues of faith, hope, and charity. The supernatural clarity, integrity, and proportion of man’s actions will in this way surpass all the harmonies of nature.

End Notes

(Where original texts are given, translations are those of the author.)
(All texts of St. Thomas Aquinas are from www.corpusthomisticum.org.)

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1 Plato, *Republic*, Greek text of Shorey, in *Loeb Classical Library*, vol. 276 (Cambridge: Harvard University Press 1994) 531c

2 Leibniz, Gottfried Wilhelm, *Letter to Christian Goldbach*, dated April 17, 1712

3 Hanslick, Eduard, *On the Musically Beautiful*, translated by Geoffrey Payzant from *Von Musikalisch-Schönen* (1891) (Indianapolis: Hackett, 1986) p. 41

4 Plato, *Gorgias* 451a-c, *Charmides* 165e-166b

5 Klein, Jacob, *Greek Mathematical Thought and the Origin of Algebra*, trans. Eva Brann (New York: Dover, 1992)

6 Aquinas, Thomas, *Super Boetium De Trinitate*, pars 3 q. 5 a. 3 ad 6 (Et inde est quod de rebus naturalibus et mathematicis tres ordines scientiarum inveniuntur. Quaedam enim sunt pure naturales, quae considerant proprietates rerum naturalium, in quantum huiusmodi, sicut physica et agricultura et huiusmodi. Quaedam vero sunt pure mathematicae, quae determinant de quantitatibus absolute, sicut geometria de magnitudine et arithmetica de numero. Quaedam vero sunt mediae, quae principia mathematica ad res naturales applicant, ut musica, astrologia et huiusmodi. Quae tamen magis sunt affines mathematicis, quia in earum consideratione id quod est physicum est quasi materiale, quod autem est mathematicum est quasi formale; sicut musica considerat sonos, non in quantum sunt soni, sed in quantum sunt secundum numeros proportionabiles, et similiter est in aliis. Et propter hoc demonstrant conclusiones suas circa res naturales, sed per media mathematica; et ideo nihil prohibet, si in quantum cum naturali communicant, materiam sensibilem respiciunt. In quantum enim cum mathematica communicant, abstractae sunt.)

7 Aquinas, Thomas, *Super Boetium De Trinitate*, pars 3 q. 5 a. 3 ad 4 (...mathematica non abstrahuntur a qualibet materia, sed solum a materia sensibili. Partes autem quantitatis, a quibus demonstratio sumpta quodammodo a causa materiali videtur sumi, non sunt materia sensibilis, sed pertinent ad materiam intelligibilem, quae etiam in mathematicis invenitur, ut patet in VII metaphysicae.)

8 Proclus, *A Commentary on the First Book of Euclid's Elements*, translated by Glenn R. Morrow (Princeton: Princeton University Press, 1970) 78-79, p. 64

9 Bergson, Henri *Time and Free Will*, trans. F. L. Pogson from *Essai sur les données immédiates de la conscience* (1889) (New York: Humanities Press, 1971), Ch. I, p. 45

10 *Ibid*. 9, pp. 44-46

11 Aristotle, *Posterior Analytics*, Greek text ed. Hugh Tredennick, in *Loeb Classical Library*, vol. 391, (Cambridge: Harvard University Press, 1989) I.XIII.78b38-79a6, pp. 88, 90 (ἄλλον δὲ τρόπον διαφέρει τὸ διότι τοῦ ὅτι τῷ δι' ἄλλης ἐπιστήμης ἑκάτερον θεωρεῖν. τοιαύτα δ' ἐστὶν ὅσα οὕτως ἔχει πρὸς ἄλληλα ὥστ' εἶναι θάτερον ὑπὸ θάτερον, οἷον τὰ ὀπτικὰ πρὸς γεωμετρίαν καὶ τὰ μηχανικὰ πρὸς στερεομετρίαν καὶ τὰ ἁρμονικὰ πρὸς ἀριθμητικὴν καὶ τὰ φαινόμενα πρὸς ἀστρολογικήν. σχεδὸν δὲ συνώνυμοί εἰσιν ἔνιαι τούτων τῶν ἐπιστημῶν, οἷον ἀστρολογία ἥ τε μαθηματικὴ καὶ ἡ ναυτική, καὶ ἁρμονικὴ ἥ τε μαθηματικὴ καὶ ἡ κατὰ τὴν ἀκοήν. ἐνταῦθα γὰρ τὸ μὲν ὅτι τῶν αἰσθητικῶν εἰδέναι, τὸ δὲ διότι τῶν μαθηματικῶν· οὗτοι γὰρ ἔχουσι τῶν αἰτίων τὰς ἀποδείξεις, καὶ πολλάκις οὐκ ἴσασι τὸ ὅτι, καθάπερ οἱ τὸ καθόλου θεωροῦντες πολλάκις ἔνια τῶν καθ' ἕκαστον οὐκ ἴσασι δι' ἀνεπισκεψίαν.)

12 Helmholtz, Hermann *On the Sensations of Tone*, trans. Alexander J. Ellis (New York: Dover, 1954)

13 *Ibid.* 12, Ch. X., pp.179-197

14 A. J. M. Houtsma, T. D. Rossing, W. M. Wagenaars, et. al., *Auditory Demonstrations*, "Demonstration 31: Tones and Tuning with Stretched Partials", Prepared at the Institute for Perception Research (IPO) Eindhoven, The Netherlands, for the Acoustical Society of America. (The audio contents of the CD can also be found at https://ccrma.stanford.edu/~malcolm/correlograms/index.html.)

15 *Ibid.* 5, p. 46

16 Boethius, Anicius Manlius Severinus, *De Institutione Musica*, II.iii (Multitudo enim a finita inchoans quantitate crescens in infinita progreditur, ut nullus crescendi finis occurrat; estque ad minimum terminata, interminabilis ad maius, eiusque principium unitas est, qua minus nihil est. Crescit vero per numeros atque in infinita protenditur nec

ullus numerus, quominus crescat, terminum facit. Sed magnitudo finitam rursus suae mensurae recipit quantitatem, sed in infinita decrescit. Nam si sit pedalis linea vel cuiuslibet alterius modi, potest in duo aequa dividi, eiusque medietas in medietatem secari eiusque rursus medietas in aliam medietatem, ut nunquam ullus secandi magnitudinem terminus fiat.)

[17] Nicomachus of Gerasa, *Introduction to Arithmetic*, trans. Martin L. D'Ooge, in *Great Books of the Western World*, vol. 11 (Chicago: Encyclopedia Britannica, 1989) III.1, p. 812

[18] *Ibid.* 5, p. 38, [*nota bene*: Bracketed terms inserted by author.]

[19] *Ibid.* 5, p. 39

[20] *Ibid.* 5, pp. 40-41 [*nota bene*: Bracketed terms inserted by author.]

[21] *Ibid.* 5, p. 112 [*nota bene*: Bracketed terms inserted by author.]

[22] *Ibid.* 5, pp. 112-113 [*nota bene*: Bracketed terms inserted by author.]

[23] *Ibid.* 5, pp. 158-159

[24] *Ibid.* 8, 184, pp. 144-145

[25] *Ibid.* 8, 7-8, p. 6

[26] *Ibid.* 8, 8-9, p. 7

[27] *Ibid.* 8, 9, p. 8

[28] Dedekind, Richard, *Essays on the Theory of Numbers*, trans. W. W. Beman (New York: Dover, 1963) I.IV, pp. 12-13

[29] Descartes, René, *Discourse on Method*, trans. Elizabeth S. Haldane and G. R. T. Ross (Mineola: Dover, 2003) Book VI, p. 41

[30] Lagrange, Joseph-Louis, *Méchanique Analitique*, Avertissement (Paris: 1788) ("On ne trouvera point de Figures dans cet Ouvrage. Les méthodes que j'y expose ne demandent ni constructions ni raisonnemens géométriques ou méchaniques, mais seulement les opérations algébraiques, assujetties à une marche réguliere et uniforme.")

[31] Lachtermann, David Rapport, *The Ethics of Geometry* (New York: Routledge, 1989) p. 38

[32] *Ibid.* 31, p. 39

[33] Aristotle, *Metaphysics*, Greek text of Tredennick, *Loeb Classical Library*, vol. 287 (Harvard University Press, Cambridge, MA, 1990) X.I.1053a, p. 8 (καὶ ἐν μουσικῇ δίεσις, ὅτι ἐλάχιστον, καὶ ἐν φωνῇ στοιχεῖον... οὐκ ἀεὶ δὲ τῷ ἀριθμῷ ἓν τὸ μέτρον, ἀλλ' ἐνίοτε πλείω, οἷον αἱ διέσεις δύο, αἱ μὴ κατὰ τὴν ἀκοήν ἀλλ' ἐν τοῖς λόγοις, καὶ αἱ φωναὶ πλείους αἷς μετροῦμεν...)

[34] *Ibid.* 31, p. 45

[35] Descartes, René, *Géometrie*, trans. David Eugene Smith and Marcia L Latham (New York: Dover, 1954) Book I, p. 2

[36] *Ibid*. 35, Book I, p. 5

[37] *Ibid*. 35, Book I, p. 6

[38] Aristoxenus, *Elementa Harmonica*, in *Greek Musical Writings: II. Harmonic and Acoustic Theory*, ed. by Andrew Barker (New York: University of Cambridge, 1989), Book II., pp. 150-151

[39] *Ibid.* 38, Book I., p. 136 [*nota bene:* "space" has been emended by the author with "place" to better translate the Aristotelian term.]

[40] *Ibid.* 38, Book I., pp. 127-128

[41] Stevin, Simon, *On the Theory of the Art of Singing*, in *The Principal Works of Simon Stevin*, ed. A.D. Fokker (Amsterdam: C.V. Swets & Zeitlinger, 1966) p. 427

[42] Personal correspondence (1997)

[43] Boethius, Anicius Manlius Severinus, *De Institutione Musica*, I.28, (Quotiens enim duo nervi uno graviore intenduntur simulque pulsi reddunt permixtum quodammodo et suavem sonum, duaeque voces in unum quasi coniunctae coalescunt; tunc fit ea, quae dicitur consonantia. Cum vero simul pulsis sibi quisque ire cupit nec permiscent ad aurem suavem atque unum ex duobus compositum sonum, tunc est, quae dicitur dissonantia.)

[44] Boethius, Anicius Manlius Severinus, *De Institutione Musica*, II.18 (Haec enim ponenda est maxime esse prima suavisque consonantia cuius proprietatem sensus apertior conprehendit. Quale est enim unumquodque per semet ipsum, tale etiam deprehenditur sensu. Si igitur cunctis notior est ea consonantia, quae in duplicitate consistit, non est dubium, primam esse omnium diapason consonantiam meritoque excellere, quoniam cognitione praecedat.)

[45] Ptolemy, *The Harmonics*, in *Greek Musical Writings: II. Harmonic and Acoustic Theory*, ed. by Andrew Barker (New York: University of Cambridge, 1989) I.7, p. 289

[46] Goldman, Richard Franco, *Harmony in Western Music* (Norton, 1965) pp. 34–35

[47] Nichomachus of Gerasa, *The Manual of Harmonics*, trans. Flora R. Levin (Grand Rapids: Phanes Press, 1994) p. 73

[48] *Ibid.* 45, I.1, p. 276

[49] *Ibid.* 45, I.9, p. 293

[50] *Ibid.* 45, I.4, p. 284

[51] *Ibid.* 45, I.9, p. 295

[52] *Ibid.* 45, I.13, p. 303-304

[53] Aristotle, *Physics*, Greek text, *Loeb Classical Library*, vol. 228 (Cambridge: Harvard University Press, 1990) III.I.201a.10-11, p. 194 (ἡ τοῦ δυνάμει ὄντος ἐντελέχεια, ᾗ τοιοῦτον, κίνησίς ἐστιν)

[54] Philoponus, *On Aristotle's Physics 3*, trans. M. J. Edwards (Ithaca: Cornell University Press, 1994), 342.12-20, p. 14

[55] *Ibid.* 54, 351.9-15, p. 22

[56] Aquinas, Thomas, *In libros Physicorum*, lib. 3, l. 2, n. 3 (motus est *entelechia*, idest actus existentis in potentia secundum quod huiusmodi)

[57] Aquinas, Thomas, *In libros Physicorum*, lib. 3, l. 4, n. 10 (ostendit quod idem sit actus moventis et moti. Movens enim dicitur inquantum aliquid agit, motum autem inquantum patitur; sed idem est quod movens agendo causat, et quod motum patiendo recipit. Et hoc est quod dicit, quod movens est activum mobilis, idest actum mobilis causat. Quare oportet unum actum esse utriusque, scilicet moventis et moti: idem enim est quod est a movente ut a causa agente, et quod est in moto ut in patiente et recipiente.)

[58] Aristotle, *Physics*, Greek text, *Loeb Classical Library*, vol. 228 (Cambridge: Harvard University Press, 1990) III.III.202b, p. 212-213

[59] *Ibid.* 38, Book I., p. 132 [*nota bene:* "space" has been emended by the author with "place" to better translate the Aristotelian term.]

[60] *Ibid.* 38, Book I., pp. 134-135

[61] Newton, Isaac, *The Method of Fluxions and Infinite Series*, trans. John Colson, (London: Henry Woodfall, 1736) Introduction.60

[62] *Ibid.* 9, Ch. II, p. 107-108

[63] *Ibid.* 9, Ch. II, p. 114-115

[64] *Ibid.* 9, Ch. II, p. 119

[65] Oresme, Nicole, *De configurationibus qualitatum,* trans. Marshall Clagett, in *The Science of Mechanics in the Middle Ages* (Madison: University of Wisconsin Press, 1961) pp. 347-381

[66] *Ibid.* 9, Ch. II, p. 110

[67] Aquinas, Thomas, *In libros Physicorum*, lib. 4, l. 17, n. 10 (Quando igitur sentimus unum nunc, et non discernimus in motu prius et posterius; vel quando discernimus in motu prius et posterius, sed accipimus idem nunc ut finem prioris et principium posterioris; non videtur fieri tempus, quia neque est motus. Sed cum accipimus prius et posterius et numeramus ea, tunc dicimus fieri tempus. Et hoc ideo, quia tempus nihil aliud est quam "numerus motus secundum prius et posterius": tempus enim percipimus, ut dictum est, cum numeramus prius et posterius in motu. Manifestum est ergo quod tempus non est motus, sed sequitur motum secundum quod numeratur.)

[68] Aquinas, Thomas, *Questiones disputatae de Anima*, a. 2, ad 7 (species operationis consequitur speciem formae, quae est operationis principium; licet inefficacia operationis sequatur formam secundum quod inhaeret subiecto.)

[69] Aquinas, Thomas, *Questiones disputatae de Anima*, a. 9, ad 14 (licet anima sit forma simplex secundum essentiam, est tamen multiplex virtute secundum quod est principium diversarum operationum. Et quia forma perficit materiam, non solum quantum ad esse sed etiam ad operandum, ideo oportet quod licet anima sit una forma, partes corporis diversimode perficiantur ab ipsa, et unaquaeque secundum quod competit eius operationi. Et secundum hoc etiam oportet esse ordinem in partibus secundum ordinem operationum, ut dictum est; sed iste ordo est secundum operationem corporis ad animam, ut est motor.)

[70] Aquinas, Thomas, *Questiones disputatae de Anima*, a. 10, ad 17 (anima, quamvis sit una et simplex in essentia, habet tamen virtutem ad diversas operationes. Et quia naturaliter dat esse et speciem suo perfectibili in quantum est

forma corporis secundum essentiam; ea autem quae sunt naturaliter, sunt propter finem; oportet quod anima constituat in corpore diversitatem partium, prout congruit diversis operationibus.)

[71] Aquinas, Thomas, *De Spiritualibus Creaturis*, a. 4, ad 13 (corpus organicum est perfectibile ab anima primo et per se; singula autem organa et organorum partes, ut in ordine ad totum...)

[72] Aquinas, Thomas, *Questiones disputatae de Anima*, a. 1, ad 18 (quamvis esse animae sit quodammodo corporis, non tamen corpus attingit ad esse animae participandum secundum totam suam nobilitatem et virtutem; et ideo est aliqua operatio animae in qua non communicat corpus.)

[73] Plato, *Phaedo*, Greek text ed. by Fowler, *Loeb Classical Library*, vol. 36 (Cambridge: Harvard University Press 1995) 86c-d, pp. 298, 300, (εἰ οὖν τυγχάνει ἡ ψυχὴ οὖσα ἁρμονία τις, δῆλον ὅτι, ὅταν χαλασθῇ τὸ σῶμα ἡμῶν ἀμέτρως ἢ ἐπιταθῇ ὑπὸ νόσων καὶ ἄλλων κακῶν, τὴν μὲν ψυχὴν ἀνάγκη εὐθὺς ὑπάρχει ἀπολωλέναι, καίπερ οὖσαν θειοτάτην, ὥσπερ καὶ αἱ ἄλλαι ἁρμονίαι αἵ τ’ ἐν τοῖς φθόγγοις καὶ αἱ ἐν τοῖς τῶν δημιουργῶν ἔργοις πᾶσι, τὰ δὲ λείψανα τοῦ σώματος ἑκάστου πολὺν χρόνον παραμένειν, ἕως ἂν ἢ κατακαυθῇ ἢ κατασαπῇ.)

[74] Aquinas, Thomas, *Sentencia libri De anima*, lib. 1 l. 9 n. 13 (Sed hoc non sequitur; quia proportio huiusmodi non est forma, sicut ipsi credebant, sed est dispositio materiae ad formam. Et si accipiatur proprie harmonia compositionis pro dispositione, bene sequitur, quod manente dispositione materiae ad formam manet forma, et destructa dispositione, removetur forma. Non tamen quod harmonia sit forma, sed dispositio materiae ad formam.)

[75] Aquinas, Thomas, *Questiones disputatae de Anima*, a. 1, ad 10 (Corrupto corpore non perit ab anima natura secundum quam competit ei ut sit forma; licet non perficiat materiam actu, ut sit forma.)

[76] Aquinas, Thomas, *Questiones disputatae de Anima*, a. 1, ad 17 (Licet esse sit formalissimum inter omnia, tamen est etiam maxime communicabile, licet non eodem modo inferioribus et superioribus communicetur. Sic ergo corpus esse animae participat, sed non ita nobiliter sicut anima.)

[77] Descartes, *The World*, trans. Robert Stroothoff, in *The Philosophical Writings of Descartes, Vol I.,* (New York: Cambridge University Press, 1993) 6 [33], p. 91

[78] Descartes, *Principles of Philosophy*, trans. John Cottingham, in *The Philosophical Writings of Descartes, Vol I.* (New York: Cambridge University Press, 1993) II.8 [45], p. 226

[79] *Ibid.* 77, 6 [35-36], p. 92

[80] Descartes, *Description of the Human Body*, trans. John Cottingham, in *The Philosophical Writings of Descartes, Vol I.* (New York: Cambridge University Press, 1993) 1 [225-226], p. 315

[81] Descartes, *The Passions of the Soul*, trans. Robert Stroothoff, in *The Philosophical Writings of Descartes, Vol I.* (New York: Cambridge University Press, 1993) I.30 [351], p. 339

[82] *Ibid.* 81, I.6 [330-331], p. 329-330

[83] Aquinas, Thomas, *Super De Trinitate*, pars 2 q. 4 a. 2 co. 7 (Dimensiones autem istae possunt dupliciter considerari. Uno modo secundum earum terminationem; et dico eas terminari secundum determinatam mensuram et figuram, et sic ut entia perfecta collocantur in genere quantitatis. Et sic non possunt esse principium individuationis; quia cum talis terminatio dimensionum varietur frequenter circa individuum, sequeretur quod individuum non remaneret semper idem numero. Alio modo possunt considerari sine ista determinatione in natura dimensionis tantum, quamvis numquam sine aliqua determinatione esse possint, sicut nec natura coloris sine determinatione albi et nigri; et sic collocantur in genere quantitatis ut imperfectum. Et ex his dimensionibus indeterminatis materia efficitur haec materia signata, et sic individuat formam, et sic ex materia causatur diversitas secundum numerum in eadem specie.)

[84] Aquinas, Thomas, *Questiones disputatae de Anima*, a. 9, ad 17 (dimensiones non possunt intelligi in materia nisi secundum quod materia intelligitur constituta per formam substantialem in esse substantiali corporeo: quod quidem non fit per aliam formam in homine quam per animam, ut dictum est. Unde huiusmodi dimensiones non praeintelliguntur ante animam in materia totaliter, sed quantum ad ultimos gradus perfectionis...)

[85] Aquinas, Thomas, *Questiones disputatae de Anima*, a. 3, ad 15 (animae separatae non differunt specie, sed numero, ex eo quod sunt tali vel tali corpori unibiles.)

[86] Aquinas, Thomas, *Summa Theologica*, II[a]-IIae q. 91 a. 2 co. (Manifestum est autem quod secundum diversas melodias sonorum animi hominum diversimode disponuntur... Et ideo salubriter fuit institutum ut in divinas laudes cantus assumerentur, ut animi infirmorum magis provocarentur ad devotionem.)