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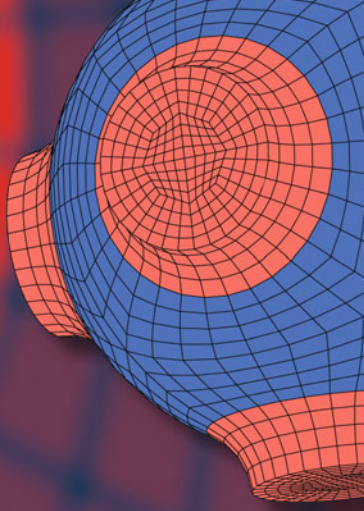

Gérard A. Maugin

Non-Classical Continuum Mechanics

A Dictionary

Springer

Advanced Structured Materials

Volume 51

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A Dictionary

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"Young men should prove theorems. Old men should write books"
(attributed to Godfrey Harold HARDY, British mathematician, 1877–1947)

Foreword

The new book of Gérard A. Maugin "Non-classical Continuum Mechanics: A dictionary" is one of the most interesting contributions to Continuum Mechanics of the recent years. The reason is very simple—the author is a witness of the renaissance in the late fifties and sixties of the last century of continuum mechanics theories based on the fundamental and pioneering ideas of Eugene & Francois Cosserat (Eugene 1866–1931, François 1852–1914), Pierre Duhem (1861–1916), Lord Kelvin (William Thomson 1824−1907) among others. In addition, he was influenced by two great schools in continuum Mechanics: the French school "organized" by Paul Germain and the American school (mostly related to the Rational Continuum Mechanics) connected with the outstanding scientists like Clifford A. Truesdell III (1919−2000), Walter Noll (born 1925), A. Cemal Eringen (1921−2009), Raymond D. Mindlin (1906−1987) among them. From this time, we have two directions of research

- The application of the Cosserat theory (and later other generalized theories) in practical cases and
- The creation and establishment of new theories partly being generalizations of the existing ones, partly being special cases (as usual connected with constraints which range of validity is usually unknown

At the moment we have hundreds of suggestions for new theories or improvements of existing ones and thousands of papers focusing our attention on various items of the theories. But up to now no systematization in this research field was presented and the present book is the first step in this direction.

The book consists of two parts:

- Part one: Prerequisites,
- Part two: The dictionary in alphabetic order

In the first part a brief introduction is given w.r.t. "What is classical continuum mechanics?" and "What is generalized continuum mechanics (GCM)?". The author tried two show the difference between the two directions by some equations and

statements since the terms “classical” and “non-classical” is not enough—it seems that the distinction by the year of formulation the relevant theories is not sufficient and one can see some ideas of, for example, Leonard Euler (1707−1783) are valid for both directions. So the term “Generalized Continuum Mechanics” is introduced since as usual one has more degrees of freedom in each material point of a generalized continuum in comparison with the classical one.

The dictionary contains some more than one hundred headings with explanations of various length. This part cannot be complete since the book is only a first step. The author of this foreword wishes all the best to Gérard A. Maugin. May he can collect more information—even from the responses of the readers to this book—for a new extended version of the book in a couple of years.

August 2016

Holm Altenbach
Magdeburg, Germany

Preface

This short preface has a main purpose to explain how I came to the idea of the present opus.

From my viewpoint, a short encyclopaedia or dictionary should give the basic definitions, main historic developments, a short technical description, directions of research, and a short selected but efficient bibliography for each item.[1] It should also point out unavoidable relationships between various entries, so that the redaction of such a work requires some technical experience from its author and also some benevolence and open mindedness. Because of my life story and experience I believe to have acquired the required conspectus, but also inquisitiveness, for this redaction in a selected subject matter, theories of continua that are decidedly *not classical*.

My experience in writing concise technical reports on various facets of science and technology goes back to a job I had to fulfil during my short stay in the French Air Force (since the French Ministry of Defence—note it's always "Defence"—had paid some of my studies). To say the truth I had to write reports on ongoing research in various countries from documents that were not always publicly accessible. This proved to be a good training. Now in my professional scientific career, which really started in 1968, I wrote an innumerable quantity of reports on already published papers (for *Mathematical Reviews*, *Applied Mechanics Reviews*, and *Zentralblatt für Mathematik*), more than seventy reviews (often as short essays) of published books, and also an incredible number of assessments of papers proposed for publication in many scientific journals relevant to continuum mechanics, applied mathematics and mathematical physics. I had the weakness practically

[1]We are here faithful to d'Alembert's preliminary discourse to the celebrated eighteenth-century grand encyclopaedia of Diderot and d'Alembert when this author wrote (cf. p. 4 in the English translation of "Preliminary discourse…" by R.N. Schwab, Bobbs-Merrill, Indianapolis, 1963): "As an Encyclopédie, it is set to forth as well as possible the order and connection of the parts of human knowledge. As reasoned Dictionary of the Sciences, Arts, and Trades, it is to contain the general principles that form the basis of each science and each art, liberal or mechanical, and the most essential facts that make up the body and substance of each".

to never refuse this duty for two reasons. One is that this duty provides an easy way to keep aware of recent developments without having to investigate too much by oneself in the ocean of publications, the other being that someone must do the job, menial as it is, and it better be someone well informed, smart (here no false modesty) and benevolent than someone inexperienced and grumpy. Of course, this is not really creative work, but it is a way to remain a perpetual student. This is not also high style literature, but not everyone is a born Marcel Proust. Anyway, as they say: "Proust is too long and life is too short". Smart editors-in-chief—from the UK, the USA, Germany, and France (I don't give names)—succeeded in exploiting my somewhat naïve vanity by using arguments like: "Only you can look at this paper, only you can make some sense out of this mess, etc" that reminds me of a song of my youth "Only you—*and you alone*—can make the darkness bright" by the famous vocal group of singers called the *Fabulous Platters* in the 1950s and 1960s.

More specifically, concerning the very subject matter of this book, it happened that most of my creative scientific career—roughly the period 1970–2010—took place in a time that witnessed the burgeoning of new ideas and new models to describe the continuum mechanics of materials at different scales while my direct masters had contributed to the emergence of a new *generalized continuum mechanics* (*GCM*) in the 1960s and 1970s, and my own research took me to little explored (at the time) fields such as a true nonlinear continuum mechanics of electromagnetic solids, coupled linear and nonlinear waves in such fields, and so-called configurational mechanics with the accompanying paraphernalia of non-Riemannian geometry. I had the chance to witness some of these developments in *GCM* at Princeton and in Summer schools held at the time. There were busy years such as 1964 that saw the simultaneous publication of at least four different expansions in *GCM*—by Toupin, Mindlin, Eringen, and Green and Naghdi—with harsh confrontation between the different tenants in the late 1960s and early 1970s, and new approaches to continuously dislocated bodies. The 1970s were also rich with the development of nonlocal theories of various types. Other complex theories, such as those of porous bodies, superfluids, liquid crystals, extended thermodynamics, generalized thermo-elasticity, were also born. It is all these aspects in their diversity and also in what they share in common that is the true subject matter of this short book, with a will to help those confused readers and scientists new to the field to apprehend it in the best, albeit concise, conditions. To some of them it will open new horizons, to others it may correct some misinterpretations and favour a revisited fruitful interest. In all it should satisfy the natural scientific curiosity of many readers, who I expect to be perpetual students just like myself.

The work is presented in two parts. Part I includes prerequisites in classical continuum mechanics, and elements of the mechanics of generalized continua. This provides a necessary background and a general view of non-classical continuum mechanics, especially in the form of generalized continuum mechanics. Part II constitutes the dictionary *per se* in *alphabetic order* of the entries—so that there is no real need for a subject index. This includes around a hundred entries with numerous reference citations and cross references. These entries are of various sizes and in-depth description extending from a few lines to several pages. For the most

largely expanded ones, historical background is given (all original sources were consulted but transcribed in a modern unifying notation) as well as basic formulation, further progress, contemporary references, and cross references. I am sure that this is not exhaustive and any gross error and absence of relevant items are due only to my own focused idiosyncrasy and my negligence. I expect the reader to forgive me as the field is open and infinite by its very definition.

Paris, France
June 2016

Gérard A. Maugin

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Part I
Prerequisites

Chapter 1
What Is Classical Continuum Mechanics?

Preliminary note. Like in the rest of this book, although a direct intrinsic notation (bold types) is used from time to time in equations where there is no ambiguity in notation, an unsophisticated Cartesian (indicial) tensor notation is mostly used because of the frequent occurrence of non-symmetric tensors and tensors of order higher than two. Divergence is always taken on the first index of such objects.

Introduction

A clear-cut definition of non-classical continuum mechanics can be given only by a negation, so that we need recall what is understood (by us) by "classical continuum mechanics".

We understand by "classical continuum mechanics" the kind of paradigm that was born with the combination of ideas from Leonard Euler (1707–1783), Joseph L. Lagrange (1736–1813), and Augustin L. Cauchy (1789–1857), and the invention of the divergence theorem by George Green (1793–1841), and that practically remained unaltered until rather recently. This is still the backbone of what is taught to engineers all around the world even at the beginning of this twenty-first century. These ideas are essentially the following ones: (i) the notion of contiguity introduced by Euler together with the global statement of the balance of linear and angular momenta; (ii) the generalization of Euler's notion of pressure in the notion of stress "tensor" by Cauchy, and (iii) the obvious necessity to apply Green's divergence theorem to transform the global balance laws of equilibrium or motion. "Contiguity" by Euler means a local action by contact. "Cauchy" refers to his 1822 astute introduction of the so-called Cauchy lemma that relates the unit normal and the applied traction at a cut, yielding thus the birth of a "linear vector function" (Gibbs) or "tensor" (Voigt) in modern terms[1]. A secondary working

[1]Historical developments are reported in Maugin (2013, 2014).

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hypothesis is the supposed absence of a density of volume couple, resulting in the symmetry of the introduced stress tensor (a fact rigorously proved by L. Boltzmann). An externally applied body force is allowed and the inertial term (acceleration) has the form prescribed by Newton, Lagrange and Cauchy, i.e., mass density multiplied by acceleration. The only vector degree of freedom considered is translation, giving rise to the notion of displacement. Everything occurs in three-dimensional Euclidean physical space, with a time parametrization by an absolute Newtonian time. Singular lines or points on the boundary of the body should not exist. Any deviation from this paradigm in any form contributes to the science of "generalized continuum mechanics" (Maugin 2011).

Balance Equations

More precisely, in the language of equations, using either a direct intrinsic formalism or a Cartesian tensor index notation, we have the following local equations at each regular point in a body B at time t (See Truesdell and Toupin 1960; Eringen 1962, etc.; $i, j = 1, 2, 3$):

Local balance of linear momentum:

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f} + div\, \mathbf{t} \quad or \quad \rho \frac{dv_i}{dt} = \rho f_i + \frac{\partial}{\partial x_j} t_{ji}; \tag{1.1}$$

Local balance of momentum of momentum (***T = transpose***):

$$\mathbf{t} = \mathbf{t}^T \quad or \quad t_{ji} = t_{ij}, \tag{1.2}$$

where $\mathbf{v} = \{v_i\}$ is the velocity, $\mathbf{f} = \{f_i\}$ is the body force per unit mass, and $\mathbf{t} = \{t_{ji}\}$ is Cauchy's stress tensor. The natural boundary condition associated with (1.1) at the regular boundary ∂B of B reads

$$\mathbf{T}^d = \mathbf{n}.\mathbf{t} \quad or \quad T_i^d = n_j t_{ji}, \tag{1.3}$$

where $\mathbf{n} = \{n_i\}$ denotes the unit outward pointing normal to ∂B, and $\mathbf{T}^d = \{T_i^d\}$ stands for the applied traction. Equation (1.3) is a statement of the Cauchy lemma for stresses at a regular boundary. By multiplying scalarly (1.1) by $\mathbf{v}$, one obtains

$$\rho \frac{d}{dt}\left(\frac{1}{2}\mathbf{v}^2\right) = \rho \mathbf{f}.\mathbf{v} + \left(t_{ij} v_i\right)_{,j} - t_{ji} v_{i,j}. \tag{1.4}$$

Or by integration over the body B while accounting for the continuity equation (conservation of mass),

$$\frac{d}{dt}\int_B \frac{1}{2}\,\rho \mathbf{v}^2 dv = \int_B \left(\rho\,\mathbf{f}.\mathbf{v} - t_{ji}v_{i,j}\right) dv + \int_{\partial B} \mathbf{T}^d\,.\,\mathbf{v}\,da, \tag{1.5}$$

while the global form of (1.1) reads

$$\frac{d}{dt}\int_B \rho\,\mathbf{v}\,dv = \int_B \rho\,\mathbf{f}\,dv + \int_{\partial B} \mathbf{T}^d da, \tag{1.6}$$

and the global expression of the balance of moment of momentum is

$$\frac{d}{dt}\int_B (\rho\mathbf{v}\times\mathbf{x})\,dv = \int_B (\rho\mathbf{f}\times\mathbf{x})\,dv + \int_{\partial B} \left(\mathbf{T}^d\times\mathbf{x}\right) da. \tag{1.7}$$

Equation (1.5) may be called the *theorem of the kinetic energy*.

What is remarkable here is that the given equations are *independent* of the precise behaviour of the material (within the given paradigm). This was the great achievement of Cauchy (1828) as indicated by the very title of Cauchy's contribution; but see the caveat below.

Now we go one step further by introducing the two laws of thermodynamics in global form. Of course they are of more recent extraction since they required the formulation of such laws by the combined action of Kelvin, Joule, Helmholtz, Mayer, Clausius, and Duhem in the nineteenth century.

First law of thermodynamics in global form:

$$\frac{d}{dt}\int_B \rho\left(\frac{1}{2}\mathbf{v}^2 + e\right) dv = \int_B \rho(\mathbf{f}.\mathbf{v} + h)\,dv + \int_{\partial B} \left(\mathbf{T}^d\,.\,\mathbf{v} - \mathbf{q}.\mathbf{n}\right) da; \tag{1.8}$$

Second law of thermodynamics in global form:

$$\frac{d}{dt}\int_B \rho\,\eta\,dv \geq \int_B \rho\tilde{\eta}\,dv - \int_{\partial B} \underline{\eta}.\mathbf{n}\,da, \tag{1.9}$$

where e is the internal energy per unit mass, h is the body source of heat per unit mass, $\mathbf{q}$ is the (in) flux of heat, η is the entropy density per unit mass, $\tilde{\eta}$ is the body source of entropy, and $\underline{\eta}$ is the (in) flux of entropy. In agreement with the results of thermo-statics, it is agreed upon that

$$\tilde{\eta} = \frac{h}{\theta}, \quad \underline{\eta} = \frac{\mathbf{q}}{\theta}, \tag{1.10}$$

where θ is the thermodynamic temperature such that $\theta > 0, \inf\,\theta = 0$. The inequality (1.9) is Clausius-Planck form improved by Duhem and Truesdell.

On combining (1.5) and (1.9) we obtain the equation that may be a called the *theorem of the internal energy*:

$$\frac{d}{dt}\int_B \rho e \, dv = \int_B \left(\rho h + t_{ji} v_{i,j}\right) dv - \int_{\partial B} \mathbf{q}.\mathbf{n} \, da, \tag{1.11}$$

the local form of which reads:

$$\rho \frac{de}{dt} = t_{ji} v_{i,j} - \nabla.\mathbf{q} + \rho h. \tag{1.12}$$

This eventually provides the equation of heat propagation.

By the same token the localization of (1.8) on account of (1.9) and (1.11) yields a form of the second law of thermodynamics often called the *Clausius-Duhem inequality*:

$$-\rho\left(\frac{d\psi}{dt} + \eta \frac{d\theta}{dt}\right) + t_{ji} v_{i,j} - \underline{\eta}.\nabla\theta \geq 0, \tag{1.13}$$

where we defined the Helmholtz free energy density ψ by

$$\psi = e - \eta\theta; \tag{1.14}$$

Now we can say that we have completed the paradigm of pure classical continuum mechanics in a paradigm of *classical thermo-mechanics of continua* (as presented in many contemporary textbooks). Equation (1.13) is conceived as a constraint imposed on constitutive equations, i.e., the set of relations needed to close the system of equations to be solved.

Caveat: There is a kind of very strong constitutive assumption hidden in the formulation (1.11) through (1.13). We note that

$$p_{int} = -t_{ji} v_{i,j} = -t_{ji} \quad D_{ij}, \quad D_{ij} = v_{(i,j)} \tag{1.15}$$

is none other than the elementary power expanded by the internal "force" of symmetric components t_{ji}. This essentially means in its inherent duality that this internal force is determined primarily by the *first* gradient of displacement (in solid mechanics) or the *first* gradient of the velocity (in media with viscosity). This is referred to as a *first-order gradient theory of continua* (cf. Maugin 1980). In other words, this corresponds to a theory of so-called "*simple*" materials according to W. Noll (cf. Truesdell and Noll 1965).

The second expression in the first of (1.15) follows from the symmetry of the stress. But another vision of this matter is as follows. The *principle of virtual power* for the present paradigm of continuum mechanics can be enunciated thus:

$$P^*_{inertia}(B) = P^*_{ext}(B, \partial B) + P^*_{int}(B), \tag{1.16}$$

where the vector field **v*** with dimension of a velocity field is a so-called *virtual* field and we have defined the virtual power of inertial forces, externally applied forces, and "internal forces" (here stresses) by

$$P^*_{inertia}(B) = \int_B \rho\, \dot{\mathbf{v}}.\mathbf{v}^*\, dv, \tag{1.17}$$

$$P^*_{ext}(B, \partial B) = \int_B \rho\, \mathbf{f}.\mathbf{v}^*\, dv + \int_{\partial B} \mathbf{T}^d.\mathbf{v}^*\, da, \tag{1.18}$$

$$P^*_{int}(B) = -\int_B t_{ji} D^*_{ij}\, dv, \tag{1.19}$$

with

$$D^*_{ij} \equiv \frac{1}{2}\left(v^*_{i\ ,j} + v^*_{j,i}\right) \tag{1.20}$$

Each of these expressions is a linear continuous functional. But expression (1.19) which concerns a constitutive quantity that must be *objective* (i.e., invariant by rotations in the actual configuration) must be linear in an objective velocity field, thus in the rate of strain only since this is the only part of the velocity gradient that is objective. Also, there is no term directly linear in the velocity field (a non-objective quantity) itself in (1.19) for the same reason. Hence the a priori introduced co-factor (the stress) is necessarily symmetric; Of course, the expression (1.19) vanishes identically for a virtual rigidifying velocity field for which $\mathbf{D}* = \mathbf{0}$ by virtue of Killing's theorem.

For a real velocity field (no asterisk), (1.16) reduces to the theorem (1.5) of the kinetic energy. Accordingly, this shows that instead of the global statements of the balance of linear momentum and moment of momentum, (1.6) and (1.7), we may as well consider (1.16) as an a priori statement in which the virtual power of internal forces (stresses) accounts for the objectivity of the stress tensor. In contrast to the Newton-Euler postulate of standard balance equations, this manner of constructing the equations of continuum mechanics is that recommended all along the nineteenth century in works by Piola, Kirchhoff, Helmholtz, Duhem, Poincaré, Hilbert, etc. In the modern framework, following Hellinger (1914) and Germain (1973), it is the best way not only for the formulation of classical continuum mechanics but also for all types of generalized continuum mechanics because it emphasizes the structure of the basic kinematics of the continuum, and thus, by duality, that of the field of generalized internal forces (cf. Maugin 1980). Thus thermo-mechanics will be based on the statements (1.16), (1.8) and (1.9).

Finally, it should be noted that in writing (1.8) and (1.9), it was assumed that both heat and entropy exchanges with the surroundings are of the *contact type* (i.e., flux) in the same way as the mechanical interaction of the Euler-Cauchy type (stress). This follows the basic idea of Fourier who was influenced by Cauchy. For the sake of further comparison it is salient to remind the most elementary classical behaviours

Reminder: The Most Classical Behaviours of Classical Continuum Thermo-Mechanics

Finite-Strain Thermoelasticity

In the general framework of nonlinear continuum mechanics (cf. Eringen 1967; Maugin 1988; Truesdell and Noll 1965; Truesdell and Toupin 1960), one usually distinguishes between the actual reference K_t at Newtonian time t and placement $\mathbf{x}$ of coordinates $x_i, i = 1, 2, 3$ in Euclidean physical space E^3 and a reference configuration K_R where material "particles" $\mathbf{X}$ are labelled by means of material coordinates $X^K, K = 1, 2, 3$ on a manifold M^3. Only Cartesian systems of coordinates are used here for the sake of simplicity. The deformation or motion is described by a sufficiently regular time-parametrized mapping

$$\mathbf{x} = \bar{\mathbf{x}}(\mathbf{X}, t) \quad or \quad x_i = \bar{x}_i\big(X^K, t\big). \tag{1.21}$$

Finite deformations are described by means of the object noted $\mathbf{F}$ such that

$$\mathbf{F} = \frac{\partial \bar{\mathbf{x}}}{\partial \mathbf{X}} = \nabla_R \bar{\mathbf{x}} = \big\{F_{iK} = \partial \bar{x}_i / \partial X_K = x_{i,K}\big\}, \quad J = J_{F.} = \det \mathbf{F} > 0. \tag{P.1.22}$$

so that the inverse motion $\mathbf{X} = \bar{\mathbf{X}}(\mathbf{x}, t)$ is well defined. Important measures of deformation $\mathbf{C}$ and $\mathbf{E}$ are given by

$$\mathbf{C} = \mathbf{F}^T \mathbf{F} = \big\{C_{KL} = x_{i,K} x_{i,L}\big\}, \quad \mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{1}_R) = \left\{E_{KL} = \frac{1}{2}(C_{KL} - \delta_{KL})\right\}. \tag{1.23}$$

It is easily proved that

$$\dot{\mathbf{E}} = \mathbf{F}^T \mathbf{D}\, \mathbf{F}, \tag{1.24}$$

where $\mathbf{D}$ is none other than the strain rate of components D_{ij} (Cf. Eq. 1.15).

Then *thermo-elasticity in finite strains* is defined by a free energy ψ and material heat-flux vector $\mathbf{Q} = J\mathbf{F}^{-1}\mathbf{q}$ such that

$$W = \rho_0 \psi = \overline{W}(\mathbf{E},\, \theta), \quad \mathbf{Q} = \bar{\mathbf{Q}}(\nabla_R \theta; \theta, \mathbf{E}), \quad \bar{\mathbf{Q}}(\mathbf{0}; \theta, \mathbf{E}) = \mathbf{0}, \tag{1.25}$$

where the mass conservation between K_R and K_t reads

$$\rho_0 = \rho J. \tag{1.26}$$

A standard exploitation of the Clausius-Duhem inequality (1.13)—in the manner of Coleman and Noll—yields the constitutive equations

$$t_{ji} = J^{-1} \frac{\partial \overline{W}}{\partial E_{KL}} x_{i,K} x_{j,L} = t_{ij}, \quad \eta = -\rho_0^{-1} \frac{\partial \overline{W}}{\partial \theta} \tag{1.27}$$

and the dissipation restriction

$$Q_K \, \theta_{,K} \leq 0. \tag{1.28}$$

It remains to specify the symmetry group of the considered material (here taken as materially homogeneous). Various approximations can be deduced from (1.27) and (1.28).

Linear Homogeneous Isotropic Elasticity

Let $\mathbf{u} = \{u_i\}$ the elastic displacement and $\mathbf{e} = \left\{e_{ij} = u_{(i,j)} = \left(u_{i,j} + u_{j,i}\right)/2\right\}$ the infinitesimal strain. Then the internal energy W per unit volume, the Cauchy stress t_{ji}, and the linearized balance of linear momentum are given by

$$W = \rho_0 e = \frac{1}{2}\left[\lambda \, (e_{kk})^2 + 2\mu \, e_{ij} \, e_{ij}\right], \tag{1.29}$$

$$t_{ji} = \lambda \, e_{kk} \delta_{ij} + 2\mu \, e_{ij}, \tag{1.30}$$

$$t_{lk,l} + \rho_0 \, (f_k - \ddot{u}_k) \tag{1.31}$$

or

$$(\lambda + 2\mu) \nabla \, \nabla . \mathbf{u} - \mu \, \nabla \times \nabla \times \mathbf{u} + \rho_0 (\mathbf{f} - \ddot{\mathbf{u}}) = 0, \tag{1.32}$$

where the Lamé coefficients λ and μ satisfy the inequalities

$$3\lambda + 2\mu \geq 0, \quad \mu \geq 0, \tag{1.33}$$

ρ_0 is the uniform constant matter density, and a superimposed dot denotes spatial time differentiation.

It must be noticed that the recovery of the displacement vector from a known deformation requires the satisfaction of the so-called Navier-Saint-Venant compatibility conditions:

$$\varepsilon_{ijk}\varepsilon_{pqr}e_{jp.kr} = 0 \quad or \quad \nabla \times \nabla \times \mathbf{e} = 0. \tag{1.34}$$

The last equation is written in the intrinsic formalism used by Kröner.

Linear Elastic Crystals

Then (1.21) is replaced by the general relation

$$t_{ji} = C_{jikl}\, e_{kl}, \tag{1.35}$$

with symmetry relations

$$C_{jikl} = C_{(ji)(kl)} = C_{klji} \tag{1.36}$$

for the tensor of elasticities that involves at most twenty one distinct coefficients.

Eulerian Fluids

In this case

$$\psi = \psi\left(\rho^{-1}, \theta\right), \quad t_{ji} = -\pi\, \delta_{ji}, \quad \pi = -\frac{\partial \bar{\psi}}{\partial(\rho^{-1})}, \quad \eta = -\frac{\partial\, \bar{\psi}}{\partial \theta}, \tag{1.37}$$

where π is the thermodynamic pressure and η is the entropy density. At rest, the fluid response is fully isotropic. This characterizes the *Eulerian behaviour*.

Newtonian-Stokesian Fluids

This is the standard viscous behaviour of isotropic fluids for which the Cauchy stress is given by the linear representation

$$\mathbf{t} = -\pi\, \mathbf{1} + \lambda_v D_{kk}\mathbf{1} + 2\mu_v \mathbf{D}, \quad \mathbf{D} = \left\{ D_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right) \right\}, \tag{1.38}$$

yielding the equation of linear momentum in the form

$$-\nabla\pi + (\lambda_v + 2\mu_v)\,\nabla\nabla\,.\,\mathbf{v} - \mu_v\,\nabla\times\nabla\times\mathbf{v} + \rho\,(\mathbf{f} - \dot{\mathbf{v}}) = \mathbf{0} \tag{1.39}$$

together with the continuity equation

$$\dot{\rho} + \rho\,\nabla.\mathbf{v} = 0, \tag{1.40}$$

and the inequalities

$$3\lambda_v + 2\mu_v \geq 0, \quad \mu_v \geq 0 \tag{1.41}$$

for the viscosity coefficients λ_v and μ_v.

A true *Stokesian fluid* is subjected to the constraint $3\lambda_v + 2\mu_v = 0$. For an *incompressible* fluid, the thermodynamic pressure π is replaced by the mechanical pressure p which is none other than a Lagrange multiplier introduced to account for the kinematic constraint

$$D_{kk} = trace\,\mathbf{D} = \nabla.\mathbf{v} = 0. \tag{1.42}$$

Fourier Heat Conduction and Linear Thermoelasticity (Duhamel-Neumann)

In agreement with the Clausius-Duhem inequality a simple form of the heat conduction equation for isotropic thermal behaviour is given by Fourier's law

$$\mathbf{q} = -\chi\,\nabla\theta, \quad \chi \geq 0. \tag{1.43}$$

Linear thermoelasticity is then described by this equation a nd a stress constitutive relation that directly generalizes (1.30) in the form

$$t_{ji} = [\lambda\,e_{kk} + m(\theta - \theta_0)]\,\delta_{ji} + 2\mu\,e_{ij}, \tag{1.44}$$

where θ_0 is a constant reference temperature and m is the stress-temperature coefficient. The latter is related to the more usual dilatation coefficient α by the relation $m = -\alpha\,(3\lambda + 2\mu)$. In addition to Eq. (1.31) the temperature field θ has to satisfy the linearized (internal energy) equation:

$$\rho_0 C\dot{\theta} - m\,\theta_0\dot{e}_{kk} = \chi\nabla^2\theta + \rho_0 h, \tag{1.45}$$

where

$$C = -\theta_0 \frac{\partial^2 \bar{\psi}}{\partial \theta^2}\bigg|_{\mathbf{e}=\mathbf{0},\theta=\theta_0} \geq 0. \tag{1.46}$$

This is the specific heat at constant strain whose positivity requires the free energy to be concave in the temperature variable. In many problems the second term in (1.45) is discarded and the pure temperature solution then is carried in the equation of motion. This scheme can be deduced as an approximation from that for finite-strain thermo-elasticity (see above).

Unfortunately, (1.43) and (1.46) yield a parabolic system and a resulting propagation of heat at infinite speed, so that some more satisfactory description may be required.

Main References for this chapter are: Maugin (1988, Chap. 1) and Nowacki (1975).

Linear Piezoelectricity

Like linear thermo-elasticity, this is another example of coupled-field theory that does not alter the essential hypotheses at the basis of classical continuum mechanics. Suffice it to notice that within a quasi-electrostatic approximation and assuming that the considered crystal admits no centre of symmetry—and thus allows for the existence of piezoelectricity—the elastic constitutive relation (1.35) is enriched by an electric-field **E** contribution to read

$$t_{ji} = C_{jikl}\, e_{kl} - e_{kji} E_k, \tag{1.47}$$

with symmetry relations

$$C_{jikl} = C_{(ji)(kl)} = C_{klji}, \quad e_{kji} = e_{kij}, \tag{1.48}$$

while the electric constitutive equation for the electric displacement **D** reads

$$D_i = \varepsilon_{ij} E_j + e_{ikl}\, e_{kl}, \quad \varepsilon_{ij} = \varepsilon_{ji}. \tag{1.49}$$

The electric equations in a dielectric and in quasi-electrostatics reduce to

$$D_{i,i} = 0, \nabla \times \mathbf{E} = 0 \Rightarrow \mathbf{E} = -\nabla \phi. \tag{1.50}$$

Here tensors of components ε_{ij} and e_{kij} stand for the dielectric coefficients and the piezoelectricity coupling constants. Note that in this simple linearized theory there are no other electromagnetic interactions between matter and fields than through Eqs. (1.47) and (1.49). In particular, there is neither "ponderomotive" force

present in (1.47) nor "ponderomotive couple" in the local balance of angular momentum, so that the Cauchy stress remains symmetric. Furthermore, even the mathematical character of the resulting system is unchanged, so that we can qualify piezoelectricity of a "weak" generalization compared to linear elasticity. The generalization just rests in the replacement of three dependent variables (the components of displacement) by four due to the addition of the electrostatic scalar potential ϕ [See Maugin (1988), Chap. 4].

Note: Here we do not speak about linear viscoelasticity and elasto-plasticity in small strains or non-Newtonian fluids as these mechanical behaviours of great interest do not alter the basic field equations of continuum mechanics, but they do come into the thermo-mechanical picture through their typical irreversible properties [Cf. Maugin (1992)].

For nonlinear elastic and fluid behaviours we refer the reader to the many existing treatises, in particular those of Truesdell and Toupin (1960) and Truesdell and Noll (1965).

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Chapter 2
What Is Generalized Continuum Mechanics (GCM)?

Introduction

We classify under the title "generalized continuum mechanics" all what is not covered in the restricted framework of the Cauchy model exposed in the prerequisite Chap. 1 under the title of "classical continuum mechanics". In a structured overview this generalization can be presented through the successive abandonment of the basic working hypotheses of standard continuum mechanics of Cauchy: that is, introduction of a density of bulk couple, of a rigidly rotating microstructure and *couple stresses* (Cosserat continua or *micropolar* bodies, nonsymmetric stresses), introduction of a truly deformable microstructure (*micromorphic* bodies), "weak" *nonlocalization* with *gradient theories* and the notion of *hyperstresses*, and the introduction of characteristic lengths, "strong" *nonlocalization* with space functional constitutive equations and the loss of the Cauchy notion of stress, and finally giving up the Euclidean and even Riemannian material background. We peruse these steps in this overview, referring the reader to specialized entries for technical details.

Asymmetric Stress

This asymmetry may be due to the existence of *body couples*; the only known physical example of these couples relates to the case of electromagnetic deformable continua where the volume magnetization is not aligned with the local magnetic field $\mathbf{M}$, or the dielectric polarization $\mathbf{P}$ is not aligned with the local electric field creating thus couples per unit volume in the form of vector products $\mathbf{M} \times \mathbf{H}$ or $\mathbf{P} \times \mathbf{E}$ in an obvious notation. Accounting for such terms in Eq. (1.7) will result in a deviation from the symmetry condition (1.2) with the existence of a nonzero skew part of the stress given by

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Advanced Structured Materials 51, DOI 10.1007/978-981-10-2434-4_2

$$t_{[ji]} = M_{[i}H_{j]} \quad or \quad P_{[i}E_{j]}. \tag{2.1}$$

In many materials this is strictly zero in reason of the proportionality of the field **M** in **H** or of **P** in **E**. Also, the situation described by Eq. (2.1) may be only transient as **M** may rapidly align with **H** or **P** with **E**. Of course interaction of electromagnetic fields with deformable matter may be much more complicated than that described by Eq. (2.1) involving *both* couple and force of electromagnetic origin, and an import of a specific energy. For a full development of this aspect in Galilean or relativistic dynamics we recommend the treatise of Eringen and Maugin (1990; reprint 2012).

Surface Couples

This concept may be harder to imagine physically. But there is no opposition of principle to introduce in strict parallel with an applied surface traction (in the Cauchy model), an applied surface couple $\mathbf{C}^d$ per unit surface. This is an axial vector. A reasoning *à la Cauchy* will yield the introduction of the notion of couple stress **m** such that

$$n_j m_{ji} = C_i^d. \tag{2.2}$$

The object of induced component m_{ji} still is "axial" in its second index i. Accordingly, we can introduce a geometrical object with three indices, m_{jik}, such that

$$m_{kji} = m_{k[ji]} = m_{kp}\varepsilon_{pji}, \tag{2.3}$$

where ε_{pji} is Levi-Civita alternating symbol. Inclusion of a surface contribution involving the expression in Eq. (1.7) will transform the local statement of the balance of moment of momentum (1.2) in the following more general form:

$$t_{[ji]} + m_{kji,k} = 0. \tag{2.4}$$

If in addition there exits a distribution of body couples per unit mass (rewritten as a skewsymmetric tensor $C_{ji} = -C_{ij}$), then Eq. (2.4) will be generalized to the following local equilibrium of couples:

$$t_{[ji]} + \rho\, C_{ij} + m_{kji,k} = 0. \tag{2.5}$$

Furthermore, if this additional effect is related to the existence of a true internal degree of freedom (of rotation) giving rise to some spin, then an inertial term will be added in the right-hand side of Eq. (2.5) that then becomes a true dynamic equation:

$$t_{[ji]} + \rho\, C_{ij} + m_{kji,k} = \rho\, \dot{S}_{ji}, \tag{2.6}$$

where $S_{ij} = -S_{ji}$ is a spin (i.e., an angular momentum).

Equation (2.5) was the equation to which the Cosserats (1909) were naturally led—together with the static form of (1.1)—by applying an invariance requirement (Euclidean action) that represents a pioneer's application of group theory in continuum mechanics, and requires from translation and possible rotational degrees of freedom of a material point to be on the same a priori footing. Of course, the expression of constitutive equations in order to close the system of field equations demands an elaboration of the associated generalized kinematics (see entry on Cosserat continua). Furthermore, Eringen (1966) formulated a law of conservation of micro-inertia that complements the usual conservation of mass in the dynamical case. Equation (2.6) corresponds to a rewriting of the global conservation law in the following generalized form:

$$\frac{d}{dt}\int_B \rho(\mathbf{v}\times\mathbf{x}+\mathbf{s})\,dv = \int_B \rho(\mathbf{f}\times\mathbf{x}+\mathbf{c})\,dv + \int_{\partial B}\left(\mathbf{T}^d\times\mathbf{x}+\mathbf{C}^d\right)da, \tag{2.7}$$

where **s** and **c** are the axial vectors dual of the skew tensors **S** and **C**, respectively.

Eringen-Mindlin Micromorphic Model of Microstructured Continua

In the previous section no mention of any microscopic definition of the newly introduced quantities was given. But we can well imagine in agreement with the original vision of Voigt, Duhem and the Cosserats that a material point that normally experiences a translation is now assimilated to a small rigid body that can also rotate, and is thus able to respond to local couples. A more refined vision would be to see this point itself as a small deformable body, hence exhibiting six degrees of freedom in addition to translation. This corresponds to the model of **micromorphic media** devised by Eringen and Suhubi in 1964; the model of **microstructured media** devised by Mindlin (1964) in the same year is equivalent (representing a homogeneous deformation within the small body) (see entry on Eringen-Mindlin model).

The local balance of equilibrium in a micromorphic body can be written as

$$\mu_{kij,k} + t_{ji} - s_{ji} + l_{ij} = 0, \quad t_{ji} = t_{(ji)} + t_{[ji]}, \quad s_{[ji]} = 0, \quad l_{ji} = C_{ji} + l_{(ji)} \tag{2.8}$$

where μ_{kji} is the called the hyperstress tensor, s_{ji} is the so-called symmetric micro-stress, and l_{ij} is the body-moment tensor of which the skew part represents a body couple $C_{ji} = -C_{ij}$:

$$\mu_{kij,k} + t_{ji} - s_{ji} + l_{ij} = 0, \quad t_{ji} = t_{(ji)} + t_{[ji]}, \quad s_{[ji]} = 0, \quad l_{ji} = C_{ji} + l_{(ji)}; \tag{2.9}$$

then the Cosserat or micropolar model is obtained by taking the skew part of the first of Eq. (2.9) and setting $m_{kji:} = \mu_{k[ji]}$.

Bodies with microstretch (Eringen 1969). This is a further reduction of the model Eq. (2.9) obtained by noting m_k the intrinsic dilatational stress or microstretch vector; l the body microstretch force such that $l_{(ij)} = (l/3)\delta_{ij}$, and t and s are intrinsic and micro scalar forces, so that we have

$$\mu_{klm} = \frac{1}{3} m_k \delta_{lm} - \frac{1}{2} \varepsilon_{lmr} m_{kr}, \tag{2.10}$$

hence

$$m_{kl,k} + \varepsilon_{lmn} t_{mn} + C_l = 0, \quad m_{k,k} + t - s + l = 0. \tag{2.11}$$

Note that an additional natural boundary condition involving the new higher-order stresses μ_{kij} and m_{ji} must complement the standard Cauchy condition of the Prequisite Chap. 1, e.g.,

$$n_k \mu_{kij} = C^d_{ij} \quad or \quad n_j m_{ji} = C^d_i, \tag{2.12}$$

where C^d_i is akin to a surface couple.

Finally, we note the further case of **dilatational elasticity** (Cowin and Nunziato 1983) [only the second of Eq. (2.11) is relevant]:

$$m_{k,k} + t - s + l = 0. \tag{2.13}$$

Here the additional natural boundary condition will be of the form

$$n_k m_k = M^d, \tag{2.14}$$

where M^d is akin to a *tension*.

All these equations are given here in Cartesian components in order to avoid any misunderstanding that can be created by a direct intrinsic notation: μ_{kij} is a new internal force having the nature of a third-order tensor. It has to start with no specific symmetry in Eq. (2.8) and it may be referred to as a ***hyperstress***. In the case of Eq. (2.10) this quantity is skewsymmetric in its last two indices and a second order tensor—called a *couple stress*—of components m_{ji} can be introduced having *axial* nature with respect to its second index. The fields s_{ji} and l_{ij} are, respectively, a symmetric second-order tensor and a general second-order tensor. The former is an *intrinsic interaction stress*, while the latter refers to an external source of *both* stress and couple according to the last of Eq. (2.9). Only the skew part of the later remains in the special case of micropolar materials. The skewsymmetric C_{ji} can be of

electromagnetic origin, and more rarely of pure mechanical origin. Equations (2.10) and (2.13) represent a kind of intermediate case between micromorphic and micropolar materials. The case of dilatational elasticity in Eq. (2.13) appears as a further reduction of that in Eq. (2.11). This will be useful in describing the mechanical behaviour of media exhibiting a distribution of holes or cavities in evolution.

Weakly Nonlocal Modelling

Cases examined in the preceding paragraphs do not question the notion of contiguity of Euler and Cauchy. They just add new fields of internal forces that still satisfy the same contiguity argument. Totally different is the viewpoint that envisages a more analytically precise definition of a classical quantity such as the elastic displacement. This is best emphasized by extending the obvious limited expansion of the power on internal forces considered in the first of Eq. (2.15) to higher order spatial gradients of the velocity field, for instance as

$$p_{int} = -(t_{ji}v_{i,j} + m_{kji}v_{i,jk} + \cdots), \tag{2.15}$$

where $m_{kji} = m_{(kj)i}$ may be called a stress of higher order or *hyperstress*. This a priori has at most eighteen independent components. In terms of the geometry of a bounding surface (so-called natural boundary condition) this new concept will require the consideration of the second-order geometrical description of the surface, hence the curvature. This destroys the standard Euler-Cauchy notion of contiguity. In pure elasticity, the effect of the contribution of the hyperstress will be of importance wherever the strain is not spatially uniform, and obviously where one observes a rapid variation of the elastic displacement, e.g. in boundary layers. This vision is quite different from the one considered in the preceding section, since now only one standard field, the displacement, or the velocity in the case of fluids, is involved. The Euler-Cauchy framework maybe referred to as a first—gradient theory—when referred to the expression of the power of internal forces. The theory described by Eq. (2.15) with an expansion limited to second-order is called a second-gradient theory. One can generalize this in principle to an nth-gradient theory (cf. Maugin 1980). The second-gradient is well exposed in Germain (1973). Such theories are often referred to as *weakly nonlocal theories*. The only complications are the statement of the relevant boundary conditions, and the obviously large number of material coefficients to be measured, save in carefully selected simple geometries. Analytically, the resulting problems will be *stiffer* than standard ones, but they may be approached by some approximations such as singular perturbations (as exemplified by boundary-value problems involving matched asymptotic expansions between inner and outer expansions).

Historically, the roots of the gradient theories may probably be found in the general presentation by G. Piola (in the 1840s–1860s), and more precisely in Barré de Saint-Venant (1869) and the original works of Le Roux (1911, 1913). In the 1960s we note the works of Mindlin and co-workers and Toupin who revived this approach in the modern framework (Mindlin and Eshel 1968; Mindlin and Tiersten 1962; Toupin 1962). Note that this modelling is sometimes mistaken for the Cosserat model even by the best authors. This may come from the fact that if one assumes in a Cosserat continuum that the rotational velocity of the microstructure is constrained to follow the usual rate of rotation of the Cauchy continuum, then we are led to a degenerate theory of the second-gradient type, which should be called the *constrained Cosserat continuum*. This appears to be badly conditioned for dynamical properties.

Strongly Nonlocal Modelling

Although the basic idea may be mentioned in Duhem (1893), a true development of this modelling took place in the 1960s with the works of Kröner and Datta (1966), Kunin (1966), and Rogula (1965). Later on Eringen and Edelen (1972) elaborated a more abstract formulation. Synthesis works on the subject are by Kunin (1982) and Eringen (2002). Technically, the Cauchy construct does *not* apply anymore since contiguity is lost altogether. In principle, only the case of infinite bodies should be considered as any cut would destroy the prevailing long-range ordering. Constitutive equations become integral expressions over space, perhaps with a more or less rapid attenuation with distance of the spatial kernel. This, of course, inherits from the action-at-a-distance dear to the Newtonians, while adapting the disguise of a continuous framework. This view is justified by the approximation of an infinite crystal lattice: the relevant kernels can be justified through this discrete approach. But this approach raises the matter of solving integro-differential equations—not always a pleasant task—instead of partial-differential equations. What about boundary conditions that are in essence foreign to this representation of matter-matter interaction? There remains a possibility of the existence of a "weak-nonlocal" limit by the approximation by gradient models. Typically one would consider in the linear elastic case a stress constitutive equation in the form

$$t_{ji}(\mathbf{x}) = \int_{\textit{all space}} C_{jikl}(|\mathbf{x} - \mathbf{x}'|)\; e_{kl}(\mathbf{x}')\; d^3\mathbf{x}', \tag{2.16}$$

where the constitutive functions C_{jikl} decreases markedly with the distance between material points $\mathbf{x}'$ and $\mathbf{x}$, that are equivalent with an obvious reciprocity. Note that standard local linear elasticity follows from Eq. (2.16) by considering the special case

$$C_{jikl}(|\mathbf{x}-\mathbf{x}'|) = C^0_{jikl}\,\delta(|\mathbf{x}-\mathbf{x}'|), \tag{2.17}$$

where δ is Dirac's delta generalized function, and the tensorial coefficient C^0_{jikl} depends at most on the point $\mathbf{x}$ alone (for inhomogeneous materials).

In one-dimensional space, a constitutive equation such as Eq. (2.16) will provide a balance of linear momentum in the following integro-differential form:

$$\rho_0 \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left[\int_{-\infty}^{+\infty} E\,\alpha(|x-x'|)\,\frac{\partial u(x',t)}{\partial x'}\,dx'\right] = 0, \tag{2.18}$$

or

$$\frac{\partial^2 u}{\partial t^2} - c_0^2 \frac{\partial}{\partial x}\left[\alpha * \frac{\partial u}{\partial x}\right] = 0, \tag{2.19}$$

where the symbol $*$ stands for the convolution product (in space) and we have set $c_0^2 = E/\rho_0$. One needs a sensible expression for the kernel (or influence function or weight function) α.

The historical moment in the recognition of the usefulness of strongly nonlocal theories was the EUROMECH colloquium on nonlocality organized by Dominik Rogula in Warsaw in 1977. Note in conclusion to this point that any field theory can be generalized to a nonlocal one while saving the notions of linearity and anisotropy, but losing the usual notion of flux.

The Loss of Euclidean Structure

In classical continuum mechanics the arena of regular deformations is the physical Euclidean space E^3 which is assimilated to R^3. That is fine for regular displacement fields. But in some materials such as metals there exists a huge quantity of dislocations, lines along which the displacement suffers a discontinuity measured by the so-called Burgers vector. There also exist other kinds of singularities such as disclinations (lines along which the rotation vector does not close up in a round circuit), and cavities (vacancies in the case of crystals) or micro inclusions (foreign atoms in the case of atoms). The existence of what may be called *defects* questions the generally accepted idea to represent a material manifold—the set of material points—as a simple Euclidean space. Something more sophisticated must be envisaged. This was achieved in the second half of the twentieth century with no unique answer. But the most frequent one seems to consider a more adapted geometric background that will be non-Euclidean or even non-Riemannian. This is exemplified by a manifold without curvature but with affine connection, or an Einstein-Cartan space with both torsion and curvature, etc. With this one enters a

true "geometrization" of continuum mechanics of which conceptual difficulties compare favourably with those met in modern theories of gravitation. Pioneers in the field in the years 1950–70 were Kondo (1955) in Japan, Kröner (1958) in Germany, Bilby (1955) and his group in the UK, Stojanovic (1969) in what was then Yugoslavia, and Noll (1967) and Wang (1967) in the USA. Main properties of this type of approach are: (i) the relationship to the multiple decomposition of finite strains (Bilby, Kröner, Lee) and (ii) the generalization of theories such as the theory of volumetric growth or the theory of phase transitions within a unified approach to local structural rearrangements (local evolution of reference).

Another complication may be the intrinsic difficulty to define analytically some fields, in particular gradients, when the material itself is viewed as a fractal set. This constitutes the last avatar of continuum mechanics with a possible relationship to fractional derivatives (see the dictionary entry "Fractal continua").

General references on generalized continuum mechanics are: Altenbach and Eremeyev (2013), Altenbach et al. (2011, 2013), Maugin (2010, 2011), Maugin and Metrikine (2010), and the historical proceedings (Kröner 1968).

Cross references in the dictionary part: Cosserat continua, Couple stress, Directors theory, Electromagnetic continua, Eringen-Mindlin medium, Fractal continua, Gradient elasticity, Higher-order gradient theories, Hyperstresses, Le Roux elasticity, Micromorphic continua, Microstructure, Non-locality (strong), Non-locality (weak).

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Part II
The Dictionary in Alphabetic Order

Chapter 3
A–B: From "Aifantis E.C" to "Biot's Poro-Elasticity"

Aifantis E.C

Elias C. Aifantis is a Greek born (1950) engineering scientist educated at the National Technical University in Athens (NTUA, Greece) and with a PhD obtained at the University of Minnesota. Most of his creative work has been devoted to mixtures, gradient fluids, dissipative structures, dislocation motion, and gradient theories of plasticity and elasticity. The expression "Aifantis elasticity model" is coined after him.

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Cross references: Gradient elasticity, gradient plasticity, gradient theories, nonlocal theory (weak)

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Anisotropic Fluids

According to the cannons of standard continuum mechanics formulated by Truedsell and Noll in the 1950–1970 period, all fluids of which the kinematics and mechanical behaviour are defined solely in terms of density and the velocity field at actual time are necessarily *isotropic*. This rather reductive result follows from the fact that fluids have no memory of any past configuration so that a strict application of *objectivity* (invariance under rigid-body motions in the actual configuration) yields an isotropic representation of the corresponding stress. A particular case is that of *Eulerian fluids* for which the stress involves only a pressure, e.g., $t_{ij} = -p\delta_{ij}$. Newtonian fluids of which the stress is linear in the rate of deformation tensor also obey this rule. This is also the case of non-Newtonian fluids which admit a non-linear behaviour that can be expressed in terms of higher-order time derivatives of the velocity gradient (e.g., by means of Rilvin-Ericksen tensors).

But there exist fluids that can flow although they clearly exhibit directional properties. This is exemplified by fluid polymeric solutions, some dilute suspensions of macromolecules, and some liquid crystals. This apparently antonymic association of the words "liquid" and "crystals" means that such materials partake of fluids (they can flow) and of crystals (they also possess some order akin to that of crystals). This is their great originality. It is Jerald L. Ericksen (1960) who pioneered in this direction by proposing a theory of anisotropic fluids exploiting the notion of *director*. This director here is noted $\mathbf{n}$ (not to be mistaken for a unit normal). This should be a special case of the theory of oriented media (with directors). We follow Ericksen (1960) and Stokes (1984, Chap. 4). Couple stresses and heat effects are assumed to be absent. Then the local balance laws are those for mass, linear momentum, angular momentum and energy. They read:

$$\dot{\rho} + \rho \nabla . \mathbf{v} = 0, \tag{3.1}$$

$$\rho \dot{v}_i = t_{ji,j} + \rho f_i, \tag{3.2}$$

$$\rho \dot{\sigma}_i = \varepsilon_{ijk} t_{jk} + \rho l_i, \tag{3.3}$$

$$\rho \dot{e} = t_{ji} D_{ij} + \rho \left(\omega_i \dot{\sigma}_i - \dot{k}_s \right) - q_{k,k} + \rho h. \tag{3.4}$$

Here ω_i stands for the rotational velocity associated with the spin (or intrinsic angular momentum), and k_s is the kinetic energy (per unit mass) associated with the microstructure represented by the director field. For a model of dumbell-shaped particles, using an elementary approach, we can write:

$$\sigma_i = (\mathbf{n} \times \dot{\mathbf{n}})_i, \quad k_s = \frac{1}{2} \dot{\mathbf{n}} . \dot{\mathbf{n}}. \tag{3.5}$$

An interesting vector quantity is the following objective time derivative (called the Jaumann derivative) of $\mathbf{n}$:

$$\hat{\mathbf{n}} = D_J\mathbf{n} := \dot{\mathbf{n}} - \Omega.\mathbf{n} = \left\{\dot{n}_i - \Omega_{ij}n_j\right\}, \tag{3.6}$$

with

$$\Omega_{ij} = v_{[i,j]} = \frac{1}{2}\left(v_{i,j} - v_{j,i}\right), \quad D_{ij} = v_{(i,j)} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right). \tag{3.7}$$

One obtains then

$$\rho(\omega_i\dot{\sigma}_i - k_s) = -\rho\hat{n}_i\ddot{n}_i.$$

On setting (cf. Ericksen 1960; a kind of balance equation without flux since there are no couple stresses)

$$\rho\ddot{n}_i = g_i, \tag{3.8}$$

we can rewrite Eq. (3.4) as

$$\rho\dot{e} = t_{ji}D_{ij} - g_i\hat{n}_i - q_{k,k} + \rho h, \tag{3.9}$$

while (3.3) provides the expression of the skewsymmetric part of the stress as

$$t_{[ij]} = n_{[i}g_{j]} - \frac{1}{2}\rho\varepsilon_{ijk}l_k. \tag{3.10}$$

It remains now to construct a set of constitutive equations for the fields t_{ij} and g_i.

One is tempted to try linear equations in the independent variables D_{ij} and $\hat{n}_i$. For a naturally transversely isotropic behaviour with privileged direction given by vector $\mathbf{n}$ itself, a theorem proved by Smith and Rivlin (1957) shows that the contributing terms are expressible as linear combinations of n_i and δ_{ij}. Assuming that directors $\mathbf{n}$ and $-\mathbf{n}$ are not distinguishable, this yields the following lengthy expressions:

$$\begin{aligned} t_{ij} = {} & (\alpha_0 + \alpha_1 D_{kk} + \alpha_2 D_{kl}n_k n_l + \alpha_3\hat{n}_k n_k)\delta_{ij} + (\alpha_4 + \alpha_5 D_{kk} + \alpha_6 D_{kl}n_k n_l + \alpha_7\hat{n}_k n_k)n_i n_j \\ & + \alpha_8 D_{ij} + \alpha_9 D_{ik}n_k n_j + \alpha_{10}D_{jk}n_k n_i + \alpha_{11}n_i\hat{n}_j + \alpha_{12}n_j\hat{n}_i \end{aligned} \tag{3.11}$$

and

$$g_i = (\gamma_0 + \gamma_1 D_{kk} + \gamma_2 D_{kl}n_k n_l + \gamma_3\hat{n}_k n_k)n_i + \gamma_4 D_{ik}n_k + \gamma_5\hat{n}_i, \tag{3.12}$$

where the coefficients $\alpha's$ and $\gamma's$ are still functions of density ρ and the length of $\mathbf{n}$ via $n^2 = n_i n_i$. Expressions (3.11) and (3.12) must be compatible with the second law of thermodynamics and also Eq. (3.10) that requires that

$$\gamma_4 = \alpha_{10} - \alpha_9, \quad \gamma_5 = \alpha_{11} - \alpha_{12}$$

in the absence of body couples.

An invariant (objective) form of the internal energy reads

$$e = e\left(\rho, n^2\right). \tag{3.13}$$

Then the elementary entropy production reads

$$\rho\theta\gamma = \left(t_{(ij)} + \rho^2 \frac{\partial\psi}{\partial\rho}\delta_{ij}\right)D_{ij} - \left(g_i + 2\rho\frac{\partial\psi}{\partial n^2}n_i\right)\hat{n}_i \geq 0, \tag{3.14}$$

where $\psi = e - \eta\theta$ is the free energy density, η is the entropy density, and θ is the thermodynamic temperature here kept constant so that we can indifferently use e or ψ as thermodynamic potential. The factors of D_{ij} and $\hat{n}_i$ in Eq. (3.14) are the dissipative parts of the constitutive equations which could be written linear in the corresponding rates. Ericksen has considered a much simplified theory of incompressible fluids in which the director's inertia is discarded so that $g_i = 0$. This allows one to extract an expression for the objective time rate $\hat{n}_i$ in the form

$$\hat{n}_i = (\mu_1 + \mu_2 D_{kl}n_k n_l)n_i + \mu_3 D_{ik}n_k, \tag{3.15}$$

where the coefficients are deduced from the $\gamma's$ and still depend on n^2.

Furthermore, in the absence of body couples and with $g_i = 0$, the stress t_{ij} becomes symmetric. This reduces the number of independent coefficients in the expression of the stress. Finally, if the director is of constant magnitude (taken equal to one without loss in generality), it is shown that the two constitutive equations finally read in a relatively simple form (Ericksen 1960; Stokes 1984, Chap. 4):

$$t_{ij} = -p\delta_{ij} + (\lambda_1 + \lambda_2 D_{kl}n_k n_l)n_i n_j + 2\lambda_3 D_{ij} + 2\lambda_4\left(D_{ik}n_k n_j + D_{jk}n_k n_i\right) \tag{3.16}$$

and

$$\hat{n}_i = \mu_3(D_{ik}n_k - D_{kl}n_k n_l n_i), \tag{3.17}$$

where the coefficients $\lambda's$ and $\mu's$ are pure constants. Equation (3.17) may be viewed as the evolution equation for the vector variable n_i considered as an *internal variable of thermodynamic state* (cf. Maugin and Drouot 1983, p. 718). The above deduced simplified equations were considered by Ericksen in a series of papers dealing with typical flow solutions (simple shear, orientationally induced flow, irrotational motion, Poiseuille flow, Couette flow) in the period 1960–1962.

Of course, the modelling would have become much more complex had we considered the possible existence of couple stresses, e.g., with an energy density such as

$$\psi = \tilde{\psi}(\rho, \mathbf{n}, \nabla \mathbf{n}). \tag{3.18}$$

Then one can define the following two fields:

$$b_i = -\rho \frac{\partial \tilde{\psi}}{\partial n_i}, \quad m_{ij} = \rho \frac{\partial \tilde{\psi}}{\partial n_{i,j}} \neq m_{ji}. \tag{3.19}$$

The rotational invariance of function $\tilde{\psi}$ [obtained by imposing on (3.18) an infinitesimal rotation represented by a skewsymmetric tensor as an infinitesimal generator of the rotation group] yields the mathematical constraint (Maugin and Drouot 1983)

$$n_{[i} b_{j]} + m_{k[i} n_{k,j]} + m_{[ik} n_{j],k} = 0. \tag{3.20}$$

One can define a *couple-stress tensor* $\mathbf{M}$ by

$$\mathbf{M} = \left\{ M_{jik} = m_{j[i} n_{k]} = -M_{jki} \right\}. \tag{3.21}$$

But since the medium is not supposed to globally respond to such couple stresses by virtue of the symmetry of the stress tensor, we have the constraint

$$div\mathbf{M} = 0 \; or \; \left(m_{j[i} n_{k]} \right)_{,j} = 0. \tag{3.22}$$

This allows one to show that the residual dissipation inequality reads

$$\Phi = \hat{t}_{ij} D_{ji} + \tilde{b}_i \hat{n}_i \geq 0, \tag{3.23}$$

wherein

$$\hat{t}_{jk} = \tilde{t}_{jk} + m_{i(j} n_{i,k)}, \tilde{b}_i = b_i + m_{ij,j}, \tilde{t}_{jk} = t_{jk} + p\delta_{jk}. \tag{3.24}$$

This approach, not pursued here, would yield a theory with a vectorial internal variable of state n_i which has diffusive nature since its gradient is introduced [this implies that the extra entropy flux is defined in a certain ad hoc way; see Maugin and Drouot (1983)]. It may be more realistic to introduce an internal variable that is a symmetric tensor. Then the theory is closer to the original proposal of Hand (1961). This tensor may be the "conformation" of macromolecules in the so described fluid solutions (See Entry: Solutions of macromolecules).

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Cross references: Oriented media (with directors), couple stresses, liquid crystals, solutions of macromolecules.

Asymmetric Elasticity

This was the first expression used to describe continua where the Cauchy stress tensor is not symmetric for some reason. Pioneering works in this direction are those of French crystallographers (Laval 1957a, b, c) and Le Corre (1956). More recent works that reached the scientific medium of continuum mechanics were those of Grioli (1960) and Palmov (1964). Nowacki (1986) continued for some time to refer to this material modelling by this expression. But in truth, it is nowadays admitted that this expression is most often synonymous with Cosserat continua or micropolar continua (in the classification of A.C. Eringen).

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Cross references: Cosserat continua, micropolar continua, polar continua, oriented media.

Auxetic Materials

The word "auxetic" derives from the Greek word "auxetikos" which means "has a tendency to increase", itself deriving from the noun "auxesis" for "increase". The English wording seems to be due to K. Evans in 1991, but the first example of a synthetic auxetic material was provided by Lakes (1987) from Iowa when he presented a structure with *negative Poisson's ratio*. This is indeed what characterizes such materials. We remember that most natural materials have a Poisson ratio ν with value between zero and one half.

[In isotropic linear elasticity, Poisson's ratio is defined by

$$\nu = \frac{\lambda}{2(\lambda + \mu)}, \tag{3.25}$$

in terms of Lamé's coefficients λ and μ while thermodynamics implies that

$$3\lambda + 2\mu \geq 0, \quad \mu \geq 0. \tag{3.26}$$

Thus most natural materials become thinner when they are stretched.

In contrast, auxetic materials become thicker perpendicular to an applied stretching force. This strange un-natural property of auxetics can occur only because they present a type of network with hinge-like structures. The latter flex when they are stretched. Evans (1991) speaks of auxetic polymers while Lakes (1987) considers foams. A similar phenomenon can be exhibited in composites with star-shaped inclusions (cf. Theocaris et al. 1997). The prevalent cellular or periodic-composite structure lends itself well to a homogenization procedure (cf. Theocaris and Stavroulakis 1998). These nonclassical continua can have interesting engineering applications (cf. Stavroulakis 2005). A comprehensive synthesis is given by Lim (2015).

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Cross references: Cellular materials, Homogenization, Metamaterials.

Biot's Theory of Poro-elasticity

Maurice A. Biot (1905–1985) was a Belgian-American physicist who always brought original solutions to problems of mechanics and/or physics. To him are due original works in the theory of finite strains in elastic bodies, an incremental theory of deformable solid mechanics, variational principles, irreversible thermodynamics, and, most relevant to the present contribution, a theory of *poro-elasticity* (then called "Biot's theory"). The latter was a real extension of constitutive equations to saturated poro-elastic solids (porous continua made of a connected elastic skeleton and fluid entirely filling the remaining available space). He paid special attention to the propagation of waves in such media, obtaining thus results that provided useful measurement techniques in geophysical applications (Biot 1956a, b). Biot generally is a follower of Terzaghi (Biot 1941). But, in 1955, he proposed an original theory with a set of two coupled linear momentum equations for the fluid (subscript F) and solid (subscript S) constituents with respective displacement fields $\mathbf{u}_F$ and $\mathbf{u}_S$ in the following form:

$$\rho^{11}\ddot{\mathbf{u}}_S + \rho^{12}\ddot{\mathbf{u}}_F + b(\dot{\mathbf{u}}_S - \dot{\mathbf{u}}_F) = div\underline{\sigma}^S + \rho^S\mathbf{b}, \tag{3.27}$$

$$\rho^{12}\ddot{\mathbf{u}}_S + \rho^{22}\ddot{\mathbf{u}}_F + b(\dot{\mathbf{u}}_F - \dot{\mathbf{u}}_S) = \nabla\sigma + \rho^F\mathbf{b}, \tag{3.28}$$

where the mass (or inertial) coupling quantity is such that $\rho^{12} < 0$; $\underline{\sigma}^S$ is the solid elastic stress, $\mathbf{b}$ is the external force per unit mass of each constituent, and $\sigma = -np$ is the real fluid pressure if n is the porosity. Finally, the interacting constant b is related to Darcy's law by an expression of the type

$$b = \frac{\mu n^2}{k}. \tag{3.29}$$

where k is the permeability and μ is the viscosity. The strange coupling inertia terms come from a kinetic energy that is jointly quadratic in the velocities $\dot{\mathbf{u}}_S$ and $\dot{\mathbf{u}}_F$. The coupling involving Darcy's concept is deduced from a dissipation potential quadratic in the relative velocity $(\dot{\mathbf{u}}_S - \dot{\mathbf{u}}_F)$—in the manner of Rayleigh and evoking a friction phenomenon between two phases. Indeed, the set (3.27)–(3.28) really follows from a Lagrangian-Hamiltonian variational principle with source terms in the resulting Euler-Lagrange equations derived from the dissipation potential. The inertial coupling term is very much like the mutual inductance term in Maxwell's dynamical theory of induction that he built in the Lagrangian way by analogy with a generalized quadratic kinetic energy. This, at the time (1956), was a formidable achievement and original development of Biot in the field of geophysical engineering; all the more that Biot could give a reasonable meaning to the inertial contributions in the left-hand side of Eqs. (3.27)–(3.28). Darcy's law was a decisive argument in the formulation.

Interesting dynamic properties were deduced by Biot (1956a, b) from his model (3.27)–(3.28) that is still considering small strains. But in one of his last papers Biot (1972) formulated a theory of finite deformations in porous solids (still in his own formalism and notation). Many subsequent works will reformulate Biot's theories in the modern nonlinear format of continuum thermo-mechanics, adding in fact little to their physics and main properties. In the case of poroelastic solids infused with compressible fluids, new results generalizing those of Biot were obtained by Quiligotti et al. (2004).

An interesting, but seldom cited, paper was proposed by Frenkel (1944) before most of the works by Biot. In modern notation Frenkel's equations of the fluid and solid phases of the porous medium are given by

$$\rho^F \frac{\partial \mathbf{v}_F}{\partial t} = -n\nabla p + \rho^F \mathbf{b} - \frac{\mu}{\kappa}(\mathbf{v}_F - \mathbf{v}_S), \tag{3.30}$$

$$\rho^S \frac{\partial \mathbf{v}_S}{\partial t} = div\underline{\sigma}^S - (1-n)\nabla p + \rho^S \mathbf{b} + \frac{\mu}{\kappa}(\mathbf{v}_F - \mathbf{v}_S), \tag{3.31}$$

where we identify the last terms in the right-hand side of these equations as due to Darcy's law.

To conclude this contribution we note Derski's (1978) model where the mass coupling of Biot between fluid and solid components is apparently neglected but it is assumed that the density of the fluid component can be divided into two parts, one part being the density of the free fluid ρ^{Ff}, moving with the velocity $\mathbf{v}_F$, and the other part being the density ρ^{Fs} of the "trapped" fluid that moves with the velocity of the solid skeleton $\mathbf{v}_S$. It was shown that Derski's equations of momenta are equivalent to those of Biot with the appropriate interpretation of densities in the kinetic energy (See de Boer 2000, p. 305).

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Cross references: Porous media (as seen in GCM), Mixtures (mechanics of).

Chapter 4
C: From "Capillarity" to "Couple Stress (in Medium with Constrained Rotation)"

Capillarity

The original theory of capillarity was developed by Thomas Young (1773–1829) and Pierre Simon de Laplace (1749–1827). It is Laplace who proposed the celebrated equation of capillarity. In both cases this theory was conceived as a Newtonian like theory involving an inverse-square law of interaction between "particles". A marked interest in this theory was still exhibited in the late nineteenth century and early twentieth century as shown by the lecture notes of Duhem (1903), Poincaré (1895) and Hilbert (1906–1907); See also the remarkable book by Bouasse (1924). It is interesting for researchers in continuum mechanics to note how Hilbert (above all, a mathematician) exposed in a few pages a continuum theory that relies on the existence of a *surface energy*. This in fact is not so brutal since this author first considered that the energy across a thin layer of thickness 2d varies like the normal derivative of the density across this layer, i.e., $e = a\,\partial\rho/\partial n$ where a is a scalar coefficient. By integration this yields

$$E = a \iint_F df \int_{-d}^{+d} \frac{\partial \rho}{\partial n} dl = a \iint_F (\rho_1 - \rho_2) df, \tag{4.1}$$

where F is a surface of element df between the two media in contact. A more mathematical viewpoint consists in parametrizing the surface F in a Gaussian way with squared line element $ds^2 = e\,du^2 + 2f\,du\,dv + g\,dv^2$. Then $df = \sqrt{eg - f^2}\,du\,dv = h\,du\,dv$. Therefore,

$$E = c \iint_{F_O} h\,du\,dv, \tag{4.2}$$

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assuming that $\rho_1 - \rho_2$ is a constant along the surface F. The final Laplace equation will be obtained by application of a Hamiltonian principle after appropriate definition of the variations of geometric and energy quantities:

$$p_2 - p_1 = \gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \tag{4.3}$$

where p denotes the pressure, γ is the coefficient of *surface tension*, and R_1 and R_2 are the principal curvature radii of the interface at the considered point.

To modern eyes this interest for the phenomenon of capillarity within continuum mechanics may seem a bit strange. But it is a subject matter of actuality at the time of Hilbert's lectures as proved by the interest manifested by other great scientists (e.g., Duhem, Poincaré) after a very interesting paper by Korteweg (1901) in the Netherlands.

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Cross references: Density-gradient fluids, Interstitial working, Korteweg fluids, Non-locality (weak), Surface tension.

Cellular Materials as Generalized Continua

Materials considered as *cellular* abound in nature but the most useful ones are those obtained as man-made constructs. In the last category we count the 2D honeycomb-like structures (with a 2D structure in the plane and uniformity along the third direction) that have good mechanical properties combined to low weight. Another man-made 3D example is provided by metallic foams with an obviously much less regular geometric arrangement. Natural examples are represented by the structure of trabecular bones, wood, coral and glass sponge. The shape and size of

the cells are of utmost importance for the resulting mechanical properties, the relative density that varies from 0.001 to 0.3 (in the case of porous solids) being a decisive factor. Cellular solids are the most studied ones (cf. Gibson and Ashby 1997), 3D metallic foams and 2D honeycomb-like structures being highlighted because of their industrial applications. Scale relations govern their mechanical properties (cf. Evans et al. 1998). A cellular solid is seen as an interconnected network of solid struts or plates and forming edges and faces of cells. Foams are more in the form of open more or less regular holes bounded by solid edges looking like solid membranes over faces of polyhedral cells. Overall, they can be visualized from regular periodic lattices (case of honeycomb-like structures) to random multiscale structures (case of foams); cf. Jeulin and Ostoja-Starzewski (2001). It was soon realized that such materials with a rather well defined grid framework could be macroscopically acknowledged as some generalized continua of the micropolar type (Bazant and Christensen 1972). With a marked periodic structure this working hypothesis is certainly justified as pondered by Kumar and McDowell (2004). This was a general problem thoroughly considered by Forest (2006). As recalled by this author, various *homogenization* techniques may be used to deduce manageable continuous models for such cellular materials. Most of these models belong in *non-classical continuum mechanics* including micropolar (cf. Kumar and McDowell 2004; Spadoni and Ruzzene 2012), micromorphic (cf. Forest 2006) and strain gradient (cf. Auffray et al. 2010) ones, while being not limited to an elastic behaviour.

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Cross references: Auxetic materials, Cosserat continua, Generalized continuum mechanics, Gradient elasticity, Homogenization, Metamaterials, Micromorphic continua, Micropolar continua.

Configurational Mechanics

"Configurational mechanics" is that part of continuum mechanics which deals with the energy type of forces that drive material inhomogeneities in the form of extended or abrupt defects (e.g., inclusions, dislocations, disclinations, fracture fronts, transformation fronts). It is thus an integral part of the theory of material inhomogeneities (cf. Maugin 1993, 2011). The interest for such a fruitful branch of continuum thermo-mechanics is enhanced by the fact that its fundamental equations (balance of so-called pseudo-momentum and of energy) apply simultaneously to all physical effects present in a system, and thus to both usual degrees of freedom—also in their higher-order gradient theory—and internal ones or other present fields—such as electromagnetic fields and internal variables of state. This can easily be acknowledged in the general theory of fields and the allied application of invariance theorems such as those proved of Emmy Noether (1918). The generality of the method is exemplified by the following abstract example. Let $\phi^{\alpha},\ \alpha = 1, 2\ldots$ a set of fields that depend on the classical parametrization of nonlinear continuum mechanics, Newtonian time t and material co-ordinates $X^K,\ K = 1, 2, 3$, i.e., $\phi^{\alpha} = \phi^{\alpha}(X^K, t)$. In the absence of dissipation we consider a variational formulation of the Lagangian-Hamiltonian type, with Lagrangian density per unit volume

$$L = L\left(\dot{\phi}^{\alpha}, \partial\phi^{\alpha}/\partial X^K\right) \tag{4.4}$$

for a so-called first-order *gradient theory*. The resulting Euler-Lagrange equations, also called field equations per se, are

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\phi}^{\alpha}}\right) + \frac{\partial}{\partial X^K}\left(\frac{\partial L}{\partial(\partial\phi^{\alpha}/\partial X^K)}\right) = 0,\ \alpha = 1, 2, \ldots$$

Appealing to the invariance of L under time translation and translation in X^K-space and applying Noether's theorem (see Chap. 4 in Maugin 2011), one obtains a *scalar* conservation law in the explicit form of an *energy equation*

$$\left.\frac{\partial H}{\partial t}\right|_X - \nabla_R.\mathbf{Q} = 0, \tag{4.5}$$

and a co-vectorial (material) balance law called the balance of *pseudo-momentum* as

$$\left.\frac{\partial \mathbf{P}}{\partial t}\right|_X - div_R\mathbf{b} = \mathbf{0}, \tag{4.6}$$

where we defined the following canonical quantities:

- *energy (Hamiltonian density)*:

$$H := \sum_{\alpha} \dot{\phi}^{\alpha} \left(\partial L / \partial \dot{\phi}^{\alpha} \right) - L, \ \dot{\phi}^{\alpha} \equiv \partial \phi^{\alpha} / \partial t; \tag{4.7}$$

- *energy flux vector*:

$$\mathbf{Q} = \left\{ Q^K := - \sum_{\alpha} \dot{\phi}^{\alpha} \frac{\partial L}{\partial (\partial_K \phi^{\alpha})} \right\}; \tag{4.8}$$

- *canonical (here material) momentum*:

$$\mathbf{P} = \left\{ P_K := - \sum_{\alpha} \frac{\partial \phi^{\alpha}}{\partial X^K} \frac{\partial L}{\partial (\partial \phi^{\alpha} / \partial t)} \right\}; \tag{4.9}$$

- *canonical stress tensor*:

$$\mathbf{b} = \left\{ b_L^K := - \left(L \delta_L^K - \sum_{\alpha} \frac{\partial \phi_{\alpha}}{\partial X^L} \frac{\partial L}{\partial (\partial \phi^{\alpha} / \partial X^K)} \right) \right\}. \tag{4.10}$$

What must be principally gathered from the above is the essentially different nature of the Euler-Lagrange equations for which one such equation is written for *each field*—or in a more mechanical jargon, *each degree of freedom*—and of the *canonical equations of energy and momentum*—e.g., (4.5) and (4.6)—which pertain to the ***whole*** physical system and by necessity consider all fields *simultaneously* [note the summation over α in the definitions (4.7) through (4.10)].

The above-given formulation is called *canonical* because it does not depend on the precise physical meaning of the fields ϕ^{α}. In classical small-strain elasticity the ϕ^{α}'s are the three Cartesian displacement components $u_i, \ i = 1, 2, 3$ while

$$L = \frac{1}{2} \rho_0 \dot{\mathbf{u}}^2 - W(\mathbf{e}), \quad \mathbf{e} = \left\{ e_{ij} = \frac{1}{2} \left(u_{i,j} + u_{j,i} \right) \right\}. \tag{4.11}$$

Then Equations the Euler-Lagrange equations and Eqs. (4.6) and (4.7) are given by (cf. Maugin 2011, Sect. 3.2.1)

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j} \sigma_{ji} = 0, \quad \sigma_{ji} = \frac{\partial W}{\partial e_{ij}}, \tag{4.12}$$

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho_0 \dot{\mathbf{u}}^2 + W \right) - \frac{\partial}{\partial x_j} \left(\sigma_{ji} \dot{u}_i \right) = 0, \tag{4.13}$$

and

$$\frac{\partial}{\partial t}p_i^w - \frac{\partial}{\partial x_j}b_{ji} = 0 \tag{4.14}$$

where the "wave momentum" p_i^w and the "Eshelby material tensor" b_{ji} are given by

$$p_i^w = -\rho_0 \dot{u}_j u_{j,i} \tag{4.15}$$

and

$$b_{ji} = -\left(L\delta_{ji} + \sigma_{jk}u_{k,i}\right). \tag{4.16}$$

Equation (4.14) is very useful in many applications, including in fracture theory where the celebrated *J*-integral follows from the volume integral of the second term. In the presence of material inhomogeneities, thermal effects and dissipative processes, source terms are present in the right hand side of Eqs. (4.13) and (4.14)—see Maugin (2011) for all these developments. But in the present context it is of greater interest to see how these equations are modified in some theories of *non-classical* continuum mechanics on account of the canonical nature of the definitions (4.7) through (4.10). First, these definitions are readily extended to the elasticity with higher-order gradients such as the *strain-gradient elasticity* [cf. Maugin and Trimarco (1992) in finite strains]. In the small-strain approximation of this theory, (4.15) is left unchanged (there are no additional dynamic degrees of freedom) while (4.16) is generalized to

$$b_{ji} = -\left(L\delta_{ji} + \bar{\sigma}_{jp}u_{p,i} + 2m_{jpq}u_{q,pi}\right) + \left(m_{jpq}u_{p,i}\right)_{,q}, \tag{4.17}$$

wherein

$$\sigma_{ji} = \bar{\sigma}_{ji} - \frac{\partial}{\partial x_k}m_{kji} \equiv \frac{\delta W}{\delta e_{ij}}, \tag{4.18}$$

with

$$\bar{\sigma}_{ji} = \frac{\partial W}{\partial e_{ij}}, \quad m_{kji} = \frac{\partial W}{\partial\left(e_{ji,k}\right)}, \quad W = W\left(e_{ij}, e_{ij,k}\right). \tag{4.19}$$

But the formalism (4.7) through (4.10) also admits the inclusion of additional dynamic degrees of freedom as they exist in the case of *Cosserat* or *micropolar continua*. The finite-strain, finite-internal rotation case of configurational mechanics was given by Maugin (1998). For the illustrative case of small strains and small angle excursions of the rigid internal microstructure, in the absence of externally applied body force and couple, we have first the basic field equations of linear and angular momenta in the form (cf. Eringen 1968)

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j}\sigma_{ji} = 0, \tag{4.20}$$

and

$$\rho_0 j_{ij} \frac{\partial^2 \phi_j}{\partial t^2} - \frac{\partial m_{ji}}{\partial x_j} - \varepsilon_{ipq}\sigma_{pq} = 0, \tag{4.21}$$

where ϕ_j are the vector components of a small rotation angle, m_{ji} are the components of the *couple-stress* tensor, and ε_{ipq} is the alternating symbol. Obviously, the Cauchy stress tensor of components σ_{ji} is no longer symmetric. For the sake of simplicity the micro-inertia $\boldsymbol{j}$ is taken isotropic: $j_{ji} = I\delta_{ij}$. The elastic constitutive equations are given by

$$\underline{\sigma} = \frac{\partial \widehat{W}}{\partial \mathbf{e}}, \quad \underline{m} = \frac{\partial \widehat{W}}{\partial \underline{\gamma}}, \quad W = \widehat{W}\left(\mathbf{e}, \underline{\gamma}\right) \tag{4.22}$$

with strain measures $\mathbf{e}$ and $\underline{\gamma}$ defined by

$$\mathbf{e} := (\nabla \mathbf{u})^T + dual\ \phi = \left\{e_{ji} = u_{i,j} - \varepsilon_{jik}\phi_k\right\}, \quad \underline{\gamma} := \nabla\phi = \left\{\gamma_{ji} = \phi_{i,j}\right\}. \tag{4.23}$$

On account of the above introduced approximations, the canonical balance equation of momentum takes on the following form for a homogeneous material

$$\frac{\partial}{\partial t} p_i^w - \frac{\partial}{\partial x_j} b_{ji} = 0, \tag{4.24}$$

wherein

$$p_i^w = -\rho_0\left(\dot{u}_j u_{j,i} + I\dot{\phi}_j \phi_{j,i}\right), \tag{4.25}$$

and

$$b_{ji} = -\left(L\delta_{ji} + \sigma_{jk}u_{k,i} + m_{jk}\phi_{k,i}\right), \tag{4.26}$$

with a Lagrangian density now given by

$$L = \frac{1}{2}\rho_0 \dot{\mathbf{u}}^2 + \frac{1}{2}\rho_0 I\dot{\phi}_k\dot{\phi}_k - \widehat{W}. \tag{4.27}$$

Note that although (4.24) basically is an equation of *linear momentum* [resulting from the invariance of the *whole* physical system under consideration by *translations* of the coordinate parametrization in space (i.e., material space in modern

jargon)], both the wave momentum (4.24) and the Eshelby stress (4.26) involve the internal degree of freedom of rotation on equal footing with the usual displacement. Equations (4.24) through (4.26) follow directly from the canonical expression of Noether's theorem applied to the basic fields (u_i, ϕ_i) with a visible summation over terms generated by these two fields. This opens the path to the study of fracture and the propagation of phase-transformation fronts in non-classical continuum mechanics (cf. Maugin 1998, 2011). Other books dealing with configurational forces are by Gurtin (2000) and Li and Wang (2008).

Caveat. Some authors have mistaken the equation of canonical (material) momentum with a simple material version of the equation governing solely a microstructure. This is erroneous as clearly emphasized above after Eq. (4.10) and illustrated in Eqs. (4.24)–(4.27)—see also the flow chart in Maugin (1998) or Fig. 9.1 in Maugin (2011, p. 260).

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Cross references: Cosserat continua, Couple stress, Gradient elasticity, Higher-order gradient theory, Material inhomogeneities, Micropolar continua, Micropolar elasticity.

Connection and Torsion

These two notions in a modern context of differential geometry are due to the French geometer Elie Cartan (1869–1951). This scientist was interested in more complex geometries than the Riemannian one that had just received a beautiful application with the gravitation theory of Einstein ("General relativity")—cf. Cartan

(1922, 1923, 1925, 1935). Unlike standard Riemannian spaces with their Christoffel symbols and Riemannian curvature, this non-Riemannian geometry considered a connection defined independently of the metric and a possible "torsion" in addition to, or independently of, any curvature. This is theoretically engaging but what about physical applications? Cartan immediately thought of the possible geometric representation of some rotational effects (but the physical spin of quantum mechanics was not yet discovered) and he—an expert in group theory—certainly perceived of some relationship with generalized continuum as he praised the original work of the Cosserat brothers of 1909, where these authors placed rotational and translational degrees of freedom on an equal footing in their variational formulation constrained by some "Euclidean invariance". Cartan could not imagine that his concepts would find applications in the continuum mechanics of defective materials such as those endowed with continuous distributions of dislocations and disclinations (i.e., plastically deformed bodies). Cartan's affine connection and torsion play a fundamental role in the theories of finite-strain elasto-plasticity and the theory of material growth as happens in soft biological tissues, and more generally in the geometric theory of evolving structural rearrangements. Authors like K. Kondo, E. Kröner, B. A. Bilby, W. Noll, C.C. Wang; M. Epstein and G.A. Maugin have been strongly influenced by this geometric vision. A few examples of their works are provided by Bilby et al. (1955), Ciarletta and Maugin (2011), Epstein and Maugin (1990, 2000), Noll (1967), Wang (1967), and Yavari and Goriely (2012). A historical perspective is given in Maugin (2014).

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Cross references: Continuously defective materials, Defects in GCM, Dislocations and disclinations, Material growth, Non-Euclidean geometry of defective materials.

Contiguity

The word "contiguity" (not to be confused with "continuity") refers to the fact that mechanical action on a point (the "element" of continuum mechanics) is due to its direct environment. The first manifestation of this may be traced back to John Bernoulli's (1739) consideration that the fluid on one side of an infinitesimal slice pressed normally upon that slice, so that Johann was close to the notion of internal pressure and thus a concrete view of *contiguity* of action in continuum mechanics, in a line that Leonhard Euler in the period 1749–1752 and Augustin L. Cauchy in the period 1823–1828 will expand, yielding finally the notion of *stress*. Indeed, pressure was seen as the action from all sides and from neighbouring elements of fluid on an isolated element of fluid (a "particle"). In modern terms, it is isotropic and, with Euler, will be viewed as *normal* force acting on an element of surface. The notion of contiguity is thus definitely reached while pressure becomes a true field that depends on both space and time in the general case of dynamics. Cauchy obtained his (our) notion of stress by considering the possibility of an obliquely applied force on an element of surface (cf. historical developments in Chaps. 2 and 3 in Maugin 2014). This is the very basis of classical continuum mechanics.

However, this notion of contiguity seems to be seriously questioned for the first time by Duhem (1893). With his usual inquisitiveness this author asks whether farther distant elements of the continuum, if not material points of the whole body, are causes of the mechanical response at a particular point. In modern terms, one would have to envisage a kind of *non-local interaction*, no longer by direct contact even though closer points may have a stronger influence than distant ones. This brings us to the modern vision of (strong*) non-locality* where space functionals have to replace point-like constitutive equations. *Peridynamics* is the last avatar of this notion. Some of these developments may find a vague ancestry in some posthumously published work of 1854 by Gabrio Piola (see Piola 2014). Between the contiguity (contact action) of Euler and Cauchy and the strongly non-local theory, one may find a weakly non-local theory where the mechanical response is still

point-like, but depending on further gradients of the displacement field or of density. Then one deals with *gradient* theories of which the merits can be compared to those of the strongly non-local one (cf. Maugin 1979). Both weakly and strongly non-local theories belong to non-classical continuum mechanics.

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Cross references: Gradient elasticity, Higher-order gradient theories, Hyperstresses (notion of), Lattice dynamics, Long-range interactions, Non-locality (strong), Non-locality (weak), Peridynamics.

Continua with Latent Microstructure

This denomination was coined by G. Capriz in a somewhat abstract framework and applies to a general class of microstructured continua in which each point has associated with it a set of "order parameters" (Note: This is *not* a good choice of vocabulary) which may be interpreted as coordinates of an element in an appropriate manifold M of dimension m. Examples of such continua are provided by continua with voids, liquids with non-diffusive bubbles, liquid crystals, Cosserat continua, biaxial nematics, continua with vector microstructure, micromorphic continua, bodies with continuous distribution of dislocations, superfluid helium. M can be straightforwardly defined for each of these cases (cf. Capriz 1989). Apart from the abstract and often useless mathematical formalism, this does not bring any really new features, save for the unified presentation. The notion of "latent microstructure" more precisely applies to the case where a constraint is applied to the microstructure. By accounting for this constraint, only an indirect trace of the microstructure remains—through contributions in the classical linear momentum equation. Only the apparent placement remains to be determined, so that the microstructure can be said to be "latent". An example of such a situation is provided by the case of a Cosserat continuum with constrained rotation (i.e., micro-rotation velocity always equal to the macro-rotation velocity).

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Cross references: Liquid crystals as continua, Micromorphic fluids, Microstructure, Oriented media (with directors).

Continuously Defective Materials

We are all familiar with the ideal picture of a perfect and regular atomic arrangement of "atoms" in a crystal. But this is rather infrequent in nature except perhaps with purest gems. Since industrial materials command a much lower price, most materials present defects, in fact plenty of them. These defects are identified under various names as dislocations, disclinations, point defects, vacancies, microcracks, twin boundaries, stacking faults, etc. They are singular points, lines and surfaces that are usually not allowed by the regularity assumptions at the basis of classical continuum mechanics (where most fields are assumed of continuity class C^2 in order to apply operations of vector and tensor analysis). On the other hand, structural defects are so numerous that in a continuum description one must envisage continuous distributions of them, with a well defined density (such is the case for dislocations that allow for the plasticity of many materials). We may then speak of *continuously defective materials*. Special mathematical tools may have to be introduced to cope with such situations. This includes generalized gradient operators and elements of Riemannian and non-Riemannian geometries.

Cross references: Configurational mechanics, Connection and torsion, Defects in GCM, Dislocations and disclinations, Gradient plasticity, Material inhomogeneities (theory of), Non-Euclidean geometry of defective materials.

Cosserat Continua

Following a pioneering work of E. & F. Cosserat (1909; see also Maugin 2014), one usually refers as "Cosserat continua" those continua which, in addition to the usual translational degree of freedom (displacement)—at each material point—of the Euler-Cauchy-Navier theory of continuum mechanics, exhibit also an

independent rotational degree of freedom that, by duality, generates couples both in volume and at surfaces. The corresponding internal force field at surfaces (a torque per unit area) is called *couple stress*. A consequence of the presence of such a field is the loss of symmetry of the Cauchy stress. The newly introduced internal degree of freedom is a field of a kind of rigid-body rotation. Cosserat continua are also called polar or *micropolar* continua (cf. Eringen 1968, 1999) or else *oriented continua* or still media with *asymmetric stress* (cf. Grioli 1960; Nowacki 1986). Early works on Cosserat media are by Günther (1958), Neuber (1964), and Schaeffer (1967). Other historical developments are reported in Maugin (2014). Technical details are to be found in other entries (see cross references).

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Cross references: Cosserat E. & F., Couple stress, Eringen A.C., Generalized continuum mechanics, Micropolar continua, Microstructured continuum theory (Eringen), Nowacki W., Oriented media (with directors).

Cosserat Continua (Experimental Confrontation)

Although one can easily identify media that deserve to be viewed as Cosserat continua, the presence of rather numerous new material coefficients in their theoretical formulation poses a problem from the experimental viewpoint, i.e., the estimate of these numerical values. One early attempt by means of an artificially conceived elastic material (matrix with embedded foreign particles) was offered by Gauthier and Jashman (1975) at the Colorado School of Mines. This was rather inconclusive. But the most extended investigation of the subject seems to be by Roderick S. Lakes in Wisconsin during an extended period (1980–1990s). Favoured materials of study of this author in this context are natural porous bone, metallic foams and cellular materials. These selected materials with microstructure are shown experimentally to obey the Cosserat theory (in its micropolar form rather) more closely than classical elasticity. The technical methods used in this study involve holography in particular. The most relevant typical effects include the size effect and the existence of length scales, enhanced toughness (reduction of stress concentration at small holes), enhanced effect in torsion, negative Poisson ratio, and frequency-dependence (i.e., dispersion) for some wave processes. The Cosserat elasticity constants obtained in different experimental modalities have been compared successfully for internal consistency.

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Cross references: Cosserat continua, Couple stress, Generalized continuum mechanics, Micropolar elasticity, Porous media as seen in GCM.

Cosserat Eugène and François

The two brothers Eugène COSSERAT (1866–1931) and François COSSERAT (1852–1914) were, respectively, a mathematician-astronomer (an alumnus of the *Ecole Normale Supérieure* in Paris and professor at the University of Toulouse in the south west of France) and a civil engineer (who belonged to the prestigious "Corps des Ponts et Chaussées" after graduating from the *Ecole Polytechnique* in Paris). Apart from their own professional occupations, they joined efforts to ponder fundamental points in the bases of continuum mechanics. Their main common works in the field are a pioneering study of finite deformations in elasticity (Cosserat E & F 1896) and a very original long memoir of 1909. In the last work with a rather general title, "Théorie des corps déformables" (Cosserats 1909), that was to become famous after its rediscovery by C.A. Truesdell, they offered a pioneers' vision of generalized continua (introduction of couple stresses). They more or less were forced to consider the possible existence of *internal couples* by imposing an invariance (so-called *Euclidean invariance*) in a Lagrangian-Hamiltonian formulation, which invariance treats on an equal footing translations and rotations. This gave rise to the possible existence of a new type of internal force, the *couple stress* along with that of stress, and the possibility to have *non-symmetric* stresses. This was a first application of an argument of elementary group theory in continuum mechanics. This work sparked a study of such generalized continua in the 1960s–1970s under different coinages: asymmetric elasticity, Cosserat continua, polar media, oriented continua, micropolar media.

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Cross references: Asymmetric elasticity, Cosserat continua, couple stress, generalized continuum mechanics, internal degrees of freedom, polar continua, oriented continua, micropolar continua, multipolar continua.

Cosserat Point

To understand this a priori strange concept one must recall the role played by «directors» in oriented media of various dimensions, in particular in the kinematic description of slender objects such as shells and plates or rods, objects thin in one or two of their dimensions, respectively. That is, in passing from a three-dimensional body to a thin one along it thickness—correctly, an asymptotic procedure—such as for a shell or a plate, one is tempted to introduce a kinematics that accounts for the material point on this essentially two-dimensional (Cosserat) "surface" equipped with a director $\mathbf{d}_3$ of varying length orthogonal to that surface and modelling the deformation through the small thickness [a basic idea of the Cosserat brothers (1909), but also Ericksen and Truesdell (1958), and Naghdi (1972)]. Similarly, in describing the deformation of an object such as a rod or a filament obtained by considering small deformation over the section of this essentially one dimensional (Cosserat) "curve", it seems attractive and natural to introduce two directors that describe the deformation across the section [also a basic idea due to the Cosserat brothers (1909), but taken over and entertained by many modern mechanicians such as Antman (1972), and Green et al. (1974)]. Pursuing along the same line of thought in reducing the essential physical dimension of a deformable object (passing from three dimensions to a "Cosserat surface" and then a "Cosserat curve") while increasing simultaneously the number of needed directors, one naturally arrives at the concept of a body that is a "*Cosserat point*"—a body "thin" in its three dimensions—but that requires three (or more) directors for the description of its small volume, more or less complicated, deformation. This type of approach, well cultivated by Rubin and co-workers (cf. Rubin 1985, 2000), is motivated and proved useful by its applicability to the numerical solution of continuum problems. Is thus introduced the notion of 3D brick Cosserat-point elements. For instance, a 3D eight-noded brick element involving seven directors allows the account of homogeneous and inhomogeneous deformations that include bending, torsion and higher-order "hour glassing" (nonphysical zero energy mode of deformation that produces zero strain and no stress). We refer the reader to Rubin and co-workers for developments of the allied numerical schemes and fruitful applications.

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Cross references: Oriented media (with directors), Cosserat continua.

Couple Stress

In the same way as the traditional stress is the internal force response to a force per unit area—according to Cauchy's celebrated argument—, *couple stress* is the internal force response to a *torque* per unit area, so that we can write in parallel the two formulas (cf. Prerequisites):

$$\mathbf{T}^d = \mathbf{n}.\underline{\sigma} \quad or \quad T_i^d = n_j \sigma_{ji} \tag{4.28}$$

and

$$\mathbf{M}^d = \mathbf{n}.\underline{\mu} \quad or \quad M_i^d = n_j \mu_{ji} \tag{4.29}$$

where the couple stress tensor of components μ_{ji} is axial on its second index *i*. Of course while the applied traction $\mathbf{T}^d$ is in duality with a displacement to produce a work, the applied torque $\mathbf{M}^d$ is in duality with a rotation. Whether this rotation is independent or related to the macro-rotation based on the displacement is a debated matter. The material realization of $\mathbf{M}^d$ may be a problem, and even difficult to conceive. But the strict parallelism between Eqs. (4.28) and (4.29) practically is a logical necessity according to the pioneering work of the Cosserat brothers (1909) where a physically accepted invariance (so-called *Euclidean invariance*) places translational and rotational degrees of freedom on equal footing. The notion of couple stress is a basic ingredient in many theories of nonclassical continua (cf. Cosserat continua, micropolar continua, oriented media, etc.).

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Cross references: Asymmetric elasticity, Cosserat E. & F., Cosserat continua, Cosserat continua (experimental confrontation), Couple stress (in medium with

constrained rotation), Dipolar materials, Generalized continuum mechanics, Micropolar continua, Micropolar elasticity, Micropolar fluids, Oriented media (with directors).

Couple Stress (in Medium with Constrained Rotation)

This is the original theory of nonsymmetric stresses proposed by Mindlin and Tiersten (1962) where, in addition to body forces ($\mathbf{f}$) and surface tractions ($\mathbf{T}^d$), there are applied body couples ($\mathbf{c}$) and surface couples ($\mathbf{M}^d$). The latter act in duality with the medium's angular velocity (defined from the velocity field: $\boldsymbol{\Omega} = \nabla \times \mathbf{v}/2$) so that no new kinematics need be introduced (i.e., no additional internal degrees of freedom). The integral form of the balance equations of mass, linear momentum, angular momentum and energy is given by the following set for a simply connected material body V of regular bounding surface $S = \partial V$ equipped with unit outward normal $\mathbf{n}$:

$$\frac{d}{dt}\int_V \rho \, dV = 0, \tag{4.30}$$

$$\frac{d}{dt}\int_V \rho \mathbf{v} \, dV = \int_S \mathbf{T}^d dS + \int_V \rho \mathbf{f} \, dV, \tag{4.31}$$

$$\frac{d}{dt}\int_V \mathbf{x} \times \rho \mathbf{v} \, dV = \int_S \left(\mathbf{x} \times \mathbf{T}^d + \mathbf{M}^d\right) dS + \int_V (\mathbf{x} \times \mathbf{f} + \mathbf{c}) \rho \, dV, \tag{4.32}$$

$$\frac{d}{dt}\int_V \rho \left(\frac{1}{2}\mathbf{v}^2 + e\right) dV = \int_S \left(\mathbf{T}^d.\mathbf{v} + \mathbf{M}^d.\boldsymbol{\Omega}\right) dS + \int_V \rho(\mathbf{f}.\mathbf{v} + \mathbf{c}.\boldsymbol{\Omega}) dV, \tag{4.33}$$

where e is the internal energy per unit mass. Application of Cauchy's tetrahedron argument to the second and third equations yields the introduction of the classical (force-) *stress tensor* $\underline{\sigma}$ and the *couple-stress tensor* $\underline{\mu}$ by the relations

$$\mathbf{T}^d = \mathbf{n}.\underline{\sigma}, \quad \mathbf{M}^d = \mathbf{n}.\underline{\mu}. \tag{4.34}$$

The local forms of (4.31) and (4.32) are shown to read

$$\sigma_{ji,j} + \rho f_i = \rho \frac{dv_i}{dt} \quad or \quad \nabla.\underline{\sigma} + \rho \mathbf{f} = \rho \frac{d\mathbf{v}}{dt} \tag{4.35}$$

and

$$\mu_{ji,j} + \rho c_i + \varepsilon_{ijk}\sigma_{jk} = 0, \tag{4.36}$$

where ε_{ijk} is Levi-Civita's alternating symbol. From the last equation we obtain the antisymmetric part of the stress as

$$\sigma^A_{kl} = \frac{1}{2}\varepsilon_{jlk}\mu_{ij,i} + \frac{1}{2}\varepsilon_{jlk}\rho c_i. \tag{4.37}$$

Hence (4.35) can be rewritten as

$$\nabla.\underline{\sigma^S} + \frac{1}{2}\nabla \times \nabla.\underline{\mu} + \rho\mathbf{f} + \frac{1}{2}\nabla \times \rho\mathbf{c} = \rho\frac{d\mathbf{v}}{dt}, \tag{4.38}$$

where $\underline{\sigma^S}$ is the symmetric part of the stress such that $\underline{\sigma} = \underline{\sigma^S} + \underline{\sigma^A}$.

On defining the deviator

$$\mu^D_{ij} = \mu_{ij} - \frac{1}{2}\mu_{kk}\delta_{ij} \tag{4.39}$$

and noting the identities

$$\varepsilon_{jkl}v_{l,kj} \equiv 0,\ \varepsilon_{lmj}\mu_{kk,jm} \equiv 0,$$

Equation (4.38) transforms to

$$\nabla.\underline{\sigma^S} + \frac{1}{2}\nabla \times \nabla.\underline{\mu^D} + \rho\mathbf{f} + \frac{1}{2}\nabla \times \rho\mathbf{c} = \rho\frac{d\mathbf{v}}{dt}, \tag{4.40}$$

while the localization of the energy balance (4.33) reduces to the following remarkable form:

$$\rho\frac{de}{dt} = \sigma^S_{ij}D_{ij} + \mu^D_{ij}\Omega_{j,i},\ D_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right),\ \Omega_j = \frac{1}{2}\varepsilon_{jkl}v_{l,k}. \tag{4.41}$$

This exhibits the thermodynamic duality between $\underline{\sigma^S}$ and the rate of strain $\boldsymbol{D}$ on the one hand and between the deviator $\underline{\mu^D}$ and the gradient of the angular rotation vector on the other.

In this state of generality it remains to specify the expression of the natural boundary conditions which are none other than (4.34) on the regular boundary $S = \partial V$. These are obtained by substituting from the detailed expressions of $\underline{\sigma}$ and $\underline{\mu}$. That is,

$$n_i\sigma^S_{ij} + \frac{1}{2}n_i\varepsilon_{kji}\mu^D_{lk,l} - \frac{1}{2}n_i\varepsilon_{kji}\left(n_l\mu^D_{lm}n_m\right)_{,ki} = T^d_i - \frac{1}{2}n_i\varepsilon_{kji}\rho c_k - \frac{1}{2}n_i\varepsilon_{kji}\left(n_lM^d_l\right)_{,k}, \tag{4.42}$$

and

$$\varepsilon_{lkj} n_k n_i \mu_{ij}^D = \varepsilon_{lkj} n_k M_j^d. \tag{4.43}$$

Along an edge (discontinuity line in the unit normal), the following jump condition applies

$$\left[\mathbf{n}.\underline{\mu^D}.\mathbf{n}\right] = [\mathbf{n}].\mathbf{M}^d. \tag{4.44}$$

Note that in usual conditions the scalar of the couple stress remains completely indeterminate, and this results in an indeterminacy of σ_{ki}^A

Remark

It is clear that the present theory is a very special case of a second-gradient theory as shown by the energy Eq. (4.41). The complexity of (4.42) casts doubts on the easy applicability of this theory. But, overall, it is the presence of the body couple in the final linear momentum equation that is perturbing (the situation is even worse when an internal spin exists). Recently, Hadjesfandiari and Dargush (2011) have proposed a consistent size-dependent couple-stress theory that claims to eliminate all inconsistencies in earlier proposed generalized continuum mechanics of the local type, where the couple stress now is necessarily symmetric (so as to avoid indeterminacy in the related boundary conditions).

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Cross references: Cosserat continua, Couple stress, Generalized continuum mechanics, Generalized internal forces, Gradient elasticity.

Chapter 5
D: From "Defects in GCM" to "Duhem Pierre"

Defects in GCM

The observed frequent occurrence of structural defects of various types (dislocations, disclinations, microcracks, etc) in elastic crystals coincides with the naturally introduced framework of generalized continuum mechanics. This is all the more supported by a historical perusal of the matter that the introduction of such generalized models of continua in the 1960s was often kindled by the will to come to grips with unphysical singularity fields. One main question at the time was whether these models could cope with—or eliminate—the physically inadmissible singularities exhibited in some prototypical problems of classical continuum mechanics. This concerns particularly models such as couple stresses, Cosserat continua, micromorphic and micropolar continua, multipolar continua, gradient models, and finally nonlocal ones. This is illustrated by a continuous quest for nonsingular solutions in many of these models, the stress concentration around a hole, and dislocation and crack-tip problems being emblematic. This quest started as early as the first published detailed papers on generalized continuum mechanics (cf. Koiter 1964; Kaloni and Ariman 1967; Sternberg 1968), but was accelerated in the 1970 and 1980s with the expansion of nonlocal continuum mechanics (cf. Eringen 1976, 1977a, b, 1979; Ari and Eringen 1983; Kim and Eringen 1973; Eringen et al. 1977) and the many works of Lazar and Maugin (2004a, b, 2005, 2006, 2007) and Lazar et al. (2005, 2006) which had for main purpose to establish and illustrate firm results and compare these results for various GCM theories. We can state that the crop was not so encouraging in spite of many technicalities, except perhaps in cases considering strong nonlocality (cf. Eringen 1979) or mixing the ideas of polar and gradient theories.

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Cross references: Asymmetric elasticity, Configurational mechanics, Continuously defective materials, Cosserat continua, Couple stress, Dislocations and disclinations, Gradient elasticity, Micropolar elasticity, Non-locality (strong), Oriented media (with directors).

Density-Gradient Fluids

[See Entries: Capillarity, Gradient elasticity, Higher-order gradient theories, Interstitial working, Korteweg fluids].

Differential Geometry in Nonclassical Continuum Mechanics

It took some time to mechanicians—who were mostly analysts—of the continuum in the nineteenth century to acknowledge the basic role played by differential geometry in their science. Some elements—infinitesimal distance, metric, curvature—of Riemannian geometry at most were involved. A more precise intervention of this geometry transpires in now well accepted treatises (e.g., Marsden and Hughes 1983; Frenkel 1997, Appendix A). With nonclassical continuum mechanics more recent advances such as non-Riemannian geometry, differential forms, connections and torsion, and fibre bundles had to be introduced in order to apprehend more inclusively the fundamental picture of the mechanics of defective bodies. A predominant role in this evolution was due to mechanicians (K. Kondo, E. Kröner, B.A. Bilby, W. Noll, C.C. Wang) who captured the significance of the innovative works of Elie Cartan (1869–1951) who himself was much influenced by those of Pfaff, Lie and Killing in the nineteenth century. This was permeated by an incursion of group theoretical concepts that can be traced back to the original works of the Cosserat brothers on generalized continua (so-called Cosserat continua). Some very meaningful works are represented by Bilby et al. (1955 and subsequent works), Kondo (1955), Kröner (1958), Noll (1967), Wang (1967). Recent trends are illustrated by the book of Epstein (2010) and the works of Zubov (1997) and Yavari and Goriely (2012). A historical perspective is given in Maugin (2014).

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Cross references: Connection and torsion, Continuously defective materials, Defects in GCM, Dislocations and disclinations, Generalized continuum mechanics, Kondo, Kröner.

Dilatational Elasticity

This is another name given to the theory of elasticity with voids (See Entry: Materials with voids).

Dipolar Continua

Dipolar continua are stricto sensu continua that obey a special kind of generalized continuum mechanics. They constitute a special class of *multipolar* continua, as originally introduced by Green and Rivlin (1964), and further expanded by Mindlin and Tiersten (1964). In such dipolar continua, each particle is composed of *sub-particles* that interact by means of so-called *dipolar* forces (first order approximation). The continuum representation of such microstructured continua yields a gradient-type of elasticity which, in dynamics, exhibits an additional inertia term involving the Laplacian of the acceleration. Typically, in the absence of prescribed body force and body couple, one obtains a local equation of linear momentum in the following form:

$$\left(\tau_{ij} - m_{ijk,k}\right)_{,j} = \rho\ddot{u}_i - \left(\eta\ddot{u}_{i,j}\right)_{,j}, \tag{5.1}$$

where

$$\eta = \rho h^2/3. \tag{5.2}$$

The last quantity is the inertia of the microstructure seen as a collection of sub-particles of cubic shape ($2h$ is the length of the cube edges). Here τ_{ij} and m_{kij} are standard symmetric stresses and hyperstresses, respectively. In the simplified framework proposed by Aifantis (1992), we will have macroscopically isotropic materials constitutive equations of the form

$$\tau_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}, \quad m_{ijk} = l^2\tau_{ij,k} \tag{5.3}$$

where l is a length scale.

The main interest of the model (5.1) is the intervening of the additional inertia in dynamic problems such as surface waves on an elastic microstructured substrate. This was dealt with by Georgiadis and co-workers (e.g., Georgiadis and Velgaki 2003; Georgiadis et al. 2004).

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Cross references: Gradient elasticity, Green A.E., Multipolar continua.

Directors' Theory

[See Entries: Generalized continuum mechanics, Liquid crystals as continua, Liquid crystals (Ericksen-Leslie theory), and, mainly, Oriented media (with directors)].

Dislocations and Disclinations

Dislocations and disclinations are structural defects, i.e., they are the manifestation of distortions in the perfect (periodic) ordering prevailing in a crystalline structure. But this matter must be examined in both discrete-crystal and continuous pictures. The mathematical notion was introduced by Volterra (1907)—without this naming—as manifestation of a discontinuity in elastic displacement and was illustrated by an astute thought experiment ("cut, displace, and glue back"). This creates a disturbance of the mechanical fields in the neighbourhood of this discontinuity. However, it is in the discrete approach that a more vivid picture can be obtained. For instance, for a so-called *edge* dislocation, it is the edge of the extra half-plane of atoms introduced in a regular crystal lattice which is called the *edge* dislocation. More generally, if we go along a circuit around the dislocation line of unit tangent $\boldsymbol{\tau}$, one observes a lack of closure that is a finite number of the basis vectors of the lattice. This is called the *Burgers vector* usually noted $\mathbf{b}$. This vector is parallel to $\boldsymbol{\tau}$ for a *screw* dislocation and perpendicular to $\boldsymbol{\tau}$ for an *edge* dislocation. In practice many dislocations are mixed (see e.g., Friedel 1964) when the line direction and the Burgers vector are at any angle. The relationship of the notion of *dislocation* (name given by Taylor) and the phenomenon of *plasticity* was established almost simultaneously by three authors: G.I. Taylor, M. Polanyi and E. Orowan (in particular Taylor 1934). They proposed that *shear* could be caused by the propagation of elementary linear defects they called *dislocations*. The insight of these scientists was all the more remarkable that the experimental proof of the existence of dislocations as individual objects had to await the 1950s with the invention of electronic microscopy.

Dislocations are observed to move under the influence of a change of the local state of stresses caused by the application of external loads. Dislocations can move by *slip* (in the plane formed by the Burgers vector and the line tangent) or by *climb* (motion outside the slip plane). Their motion propagates the plastic deformation, i.e., they ease the ductility of materials. They can interact between them, but also with the lattice and point defects. In the case of the lattice that is periodic the displacement of a dislocation needs to overcome energy barriers, a phenomenon akin to friction. The corresponding friction force is called the *Peierls-Nabarro force*. In the case of mobile point defects (foreign atoms, impurities), dislocations attract them to form so-called *Cottrell clouds*. This hinders the motion of dislocations. In turn this explains why pure metals are more ductile than alloys. When a dislocation is strongly pinned on immobile atoms, one observes a curving of the dislocation line ultimately forming a free circular dislocation and thereby a multiplication of dislocations. This is the *mechanism of Frank-Read*. The *Portevin-Lechatelier phenomenon*, observed as oscillations on a traction curve, is related to the successive pinning and unpinning of slowly moving dislocations on mobile atoms at different sites. Dislocations can also interact with precipitates. In mathematical terms dislocations can be viewed as topological defects akin to *solitons*.

Disclinations in principle are related to a discontinuity in a rotation angle instead of a displacement. They are particularly frequent in liquid crystals.

In the context of this book, the notion of *geometrically necessary dislocations* (for short GNDs) is of importance. Indeed, if one envisages the bending of a regular lattice, this can be realized only if dislocations are introduced to justify the angle difference between the original atomic planes. These are GNDs. This lattice curvature involves a gradient of strain, and will ultimately yields a necessary relationship between GNDs, plasticity and a strain-gradient theory. The basic relation here is that originally introduced by the crystallographer-glaciologist Nye (1953) between lattice curvature and the density tensor of dislocations. That is, if the curvature tensor κ_{ij} is defined as a small right-handed lattice rotation $\delta\theta$ about the i-axis for a unit change of position of magnitude δx in the j-direction, i.e.,

$$\delta\theta_i = \kappa_{ij}\delta x_j, \tag{5.4}$$

then Nye's tensor α_{ij} relates the GND density to the lattice curvature in the following manner (cf. Arsenlis and Parks 1999; this is minus Nye's original formula)

$$\kappa_{ij} = -\left(\alpha_{ji} - \frac{1}{2}\alpha_{kk}\delta_{ij}\right). \tag{5.5}$$

This can be derived by evaluating the gradient of the displacement gradient

$$u_{i,k} = \gamma_{ik} + \phi_{ik} + e^{el}_{ik}, \tag{5.6}$$

where the three contributions stand for the plastic slip tensor, the skew part of the rotation tensor and the symmetric elastic strain tensor, and then noting that $\phi_{ik} = \varepsilon_{ilk}\theta_l$—where θ_l is the lattice rotation vector, and finally forming the quantity

$$\varepsilon_{pjk}u_{i,kj} \equiv 0 \tag{5.7}$$

and evaluating the contribution $\varepsilon_{pjk}\gamma_{ik,j}$ in terms of the plastic deformation and the crystallographic shears (Schmidt's formula) and the dislocation densities (cf. Arsenlis and Parks 1999).

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Cross references: Continuously defective materials, Defects in GCM, Gradient plasticity, Material inhomogeneities, Solitons.

Double Force

While studying self-equilibrated systems of forces and the superposition of forces with non-common points of application, Ziegler (1995; pp. 71–73) specifies the notion of *double forces* giving rise to both forces and couples. This notion appears in the Green and Rivlin (1964b) theory of multipolar continua. Specializing to the theory of dipolar continua (Green and Rivlin 1964a where there is only one displacement vector field, but *dipole forces* are considered) as examined by Tiersten and Bleustein (1974), this can be illustrated by the exploitation of the equation of conservation of energy in global form

$$\frac{d}{dt}\int\limits_V \rho\left(\frac{1}{2}v_i v_i + e\right)dV = \int\limits_V \rho\left(f_i v_i + \psi_{ji} v_{i,j}\right)dV + \int\limits_S \left(T_i^d v_i + \Delta_{ji} v_{i,j}\right)dS. \tag{5.8}$$

Here f_i and T_i^d are standard body force (per unit mass) and standard surface-traction vector per unit area. The new quantities are so-called double force ψ_{ji} per unit mass and surface double force Δ_{ji} per unit area. The last two quantities are generally not symmetric and therefore accept a decomposition into symmetric and skewsymmetric contributions such as

$$\psi_{ji} = \psi_{(ji)} + \psi_{[ji]}, \Delta_{ji} = \Delta_{(ji)} + \Delta_{[ji]}. \tag{5.9}$$

If the symmetric parts are identically zero, then setting

$$\psi_{[ji]} = \psi_{ji} = \frac{1}{2}\varepsilon_{jik} c_k,\ \Delta_{[ji]} = \Delta_{ji} = \frac{1}{2}\varepsilon_{jik} M_k^d, \tag{5.10}$$

Equation (5.8) reduces to the couple-stress case where c_k and M_k^d are components of couples per unit mass and unit surface, respectively. Then Eq. (5.8) above reduces to Eq. (5.11) in the entry "Couple stress (in medium with constrained rotation)". Introduction of the classical (*force-*) *stress tensor* $\underline{\sigma}$ and the *couple-stress tensor* $\underline{\mu}$ will follow by an application of the tetrahedron argument yielding the Cauchy-like relations

$$\mathbf{T}^d = \mathbf{n}.\underline{\sigma}, \quad \mathbf{M}^d = \mathbf{n}.\underline{\mu}. \tag{5.11}$$

It is then shown that $\underline{\sigma}$ is not symmetric due to the existence of the couple $\mathbf{c}$.

All becomes more complicated with nonvanishing symmetric parts of ψ_{ji} and Δ_{ji}. It can well be conceived that the symmetric and skewsymmetric parts of ψ_{ji} are given in terms of another theory (e.g., in electromagnetic continua; see below). It remains difficult to discuss, but formally, the role played by the symmetric part of Δ_{ji}. This is thoroughly discussed by Tiersten and Bleustein (1974, pp. 79–84) to whom we refer.

[***Remark***: Electromagnetic continua. In such media considered in quasi-electromagnetostatics, (i.e., neglecting electromagnetic inertia) it is shown that the *ponderomotive force and couple* due to electromagnetic fields are given by the expressions

$$f_i^{em} = t_{ji,j}^{em}, \quad c_k^{em} = \varepsilon_{kji} t_{ji}^{em}, \tag{5.12}$$

where t_{ji}^{em} is a generally not symmetric stress tensor that is formally quadratic in the components of the electromagnetic fields. It is a true dyadic in the sense of Gibbs. Then the following identity is easily proved:

$$\int_V f_i^{em} v_i dV = \int_{\partial V} T_i^{em} v_i dS - \int_V t_{ji}^{em} v_{i,j} dV, \quad T_i^{em} = n_j t_{ji}^{em} \text{ at } \partial V. \tag{5.13}$$

Furthermore,

$$t_{ji}^{em} v_{i,j} = t_{(ji)}^{em} D_{ij} + t_{[ji]}^{em} v_{[i,j]} = t_{(ji)}^{em} D_{ij} + c_i^{em} \Omega_i, \tag{5.14}$$

where

$$D_{ij} = v_{(i,j)} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right), \quad \Omega_i = \varepsilon_{ijk} v_{k,j}. \tag{5.15}$$

Thus the electromagnetic interactions with deformable matter can be represented by a bulk double force (symmetric double force and a couple) and a surface force, as exploited in the formulation of the mechanics of electromagnetic continua by means of the principle of virtual power by the author (Maugin 1980, synthesizing works of the 1970s)].

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Cross references: Contiguity, Dipolar materials, Electromagnetic continua, Generalized internal forces, Gradient elasticity, Higher-order gradient theories, Multipolar continua Ponderomotive couple.

Duhem Pierre

Pierre Duhem (1861–1916), probably one of the most powerful intellects of his time, is a remarkable character. He combines in one person a brilliant and sharp mind, a prolific writer and contributor to phenomenological physics, the champion of energetics, a philosopher of science, and the true creator of the history of medieval science. He was a pioneer in many facets of continuum physics. He may be considered one of the founding fathers of generalized continuum mechanics, at least in two instances, thanks to one of his first deeply thought writings (Duhem 1893). First, according to Truesdell, he is the one who mentioned the possibility to introduce new internal degrees of freedom in the form of so-called "directors" (a triad of rigid vectors at each material point in order to describe orientational changes in some kind of internal rotation; cf. Ericksen and Truesdell 1958). However, credit should also be granted to Woldemar Voigt (1850–1919) who may have expressed a similar need in his study of crystals in 1887 that dealt with the elasticity of crystals involving polarized molecules. Second, according to Edelen (1976, p. 44), he may also be responsible for the idea of replacing the contiguity hypothesis of Euler-Cauchy with something more general, that is, some nonlocality, meaning by that expression the possible dependence of stress and body force at a point on the state of the whole body. The notion of directors was exploited in the kinematic description of anisotropic fluids and liquid crystals (Ericksen), as also for the deformation of slender structural elements such as rods, plates and shells (cf. Ericksen and Truesdell 1958), and jets in fluids (cf. works by Naghdi). Nonlocal continuum mechanics now is a chapter that pervades the whole of continuum mechanics, especially at a micro scale where scale effects are much relevant. The life and scientific achievements of Duhem were described in detail by Manville (1927) and Jaki (1984).

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Cross references: Anisotropic fluids, Cosserat continua, Liquid crystals, Non-locality (strong).

Chapter 6
E: From "Edelen D.G.B." to "Extra-Entropy Flux"

Edelen D.G.B

Dominic G.B. Edelen (1929–2010) was an American mathematician with an extremely wide spectrum of interests (general relativity, astrophysics, geometry, exterior calculus, mathematical theory of defects, gauge theory, thermodynamics). In the present context he is mostly known for his rational theory of nonlocal continuum mechanics, sometimes in association with A.C. Eringen (Eringen and Edelen 1972). His writings (Edelen 1962, 1969; Edelen and Lagoudas 1988; Edelen and Laws 1971; Kadic and Edelen 1983) are characterized by a high degree of sophistication.

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Cross references: Non-locality (strong), Dislocations and disclinations, Defects in GCM.

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EDGE FORCES

On the surface of bodies that present singular lines (edges) one may imagine in all generality the datum of *lineal forces*, also called *edge* forces. The standard continuum mechanics of Cauchy and others is not equipped to consider the existence of such data. To introduce such edge forces, one must revisit the celebrated Cauchy's argument that usually envisages bounding surfaces with continuous tangent plane, although this rarely happens in real situations. It is only recently that the matter was solved mathematically in two papers of great interest by Dell'isola and Seppecher (1995) and Noll and Virga (1990).

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Cross references: Gradient elasticity, Higher-order gradient theories.

Electric Quadrupoles

[See Entry: Ionic crystals (elasticity of)].

Electromagnetic Continua

For many engineers the continuum mechanics of electromagnetic stands like an over complex and intricate field that belongs more to theoretical and/or applied physics than to mechanical sciences. But modern applications have led to an unavoidable consideration of this field. Bases had to be pondered and revisited in the light of increasing physical knowledge and the whole had to be reformulated in order to provide a more or less easily exploitable field. This necessary evolution and resulting richness is illustrated by some treatises (e.g., Eringen and Maugin 1990; Maugin 1988).

The complexity of the field is reflected in the multiplicity of possible electro-magneto-mechanical interactions most of which in effect yielding a rather nonclassical continuum mechanics. These interactions belong in different classes. First, in a general nonlinear framework, there are direct interactions in the form of body forces and couples. These are called "ponderomotive" quantities although applied to volume elements of a body. They can be transformed into the notions of (Maxwell) electromagnetic stresses and of nonsymmetric Cauchy stress. The

second possibility is an energy kind of coupling that arises from a simultaneous dependency of internal energy on mechanical and electromagnetic entities, i.e., deformations and electromagnetic fields. Well known couplings in this class are piezoelectricity, electrostriction and magnetostriction that result from these couplings depending on the assumed material symmetry. Finally, the account of some microscopic properties (magnetic spin, permanent electric dipoles) will introduce the notion of *electromagnetic microstructure* in interaction with the deformation and stress fields. This is illustrated by the cases of ferromagnetism, micromagnetism, ferroelectric and ionic deformable crystals as documented in detail in other entries to this book and to which the reader is referred.

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Cross references: Electric quadrupoles, Ferroelectric crystals (elasticity of), Ferroic states, Ionic crystals, Micromagnetism in elastic solids, Polarization gradient, Ponderomotive couple.

Ericksen J.L

Jerald L. Ericksen (born 1924) is one of the most original and deep contributors to the renewal of continuum mechanics in the second half of the twentieth century. He made fundamental contributions to the mechanics of nonlinear solids and fluids (cf. Rivlin-Ericksen tensors, Rivlin-Ericksen fluids, etc; cf. Ericksen and Rivlin 1954). In the framework of nonclassical continuum mechanics, he was one of the most creative and influential contributor with his introduction of anisotropic fluids (Ericksen 1960), his modelling of liquid crystals (Ericksen 1961), and the application of the director theory while dealing with plates and shells (Ericksne and Truesdell 1958). A biography of Ericsken is given in Beatty and Hayes (Editors, 2005)—see also Ericksen (1979).

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Cross references: Anisotropic fluids, Couple stress (see Cosserat continua), Internal degrees of freedom (dynamics), Liquid crystals as continua, Liquid crystals (Ericksen-Leslie theory), Oriented media (with directors).

Eringen A. Cemal

A. Cemal Eringen (1921–2009) was an American Turkish-born engineer-scientist (cf. Maugin 2011). Educated as a mechanical engineer (PhD 1949 at the Brooklyn Polytechnic with N. Hoff), he became interested in the foundations of continuum thermo-mechanics and was one of the most articulated contributors to the theory of generalized continuum mechanics in its various forms: micromorphic and micropolar media (of which he coined the names) and nonlocal continuum mechanics, media in interaction with electromagnetic fields (cf. Eringen and Maugin 1990), theory of mixtures, liquid crystals, with applications in wave propagation and the theory of structural defects (cracks, dislocations). In addition to multiple original epoch making papers in the period 1962–1980, he has produced referential syntheses on these various developments (e.g., Eringen 1999, 2001, 2002).

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Cross references: Asymmetric elasticity, Cosserat continua, couple stress, Internal degrees of freedom, Liquid crystals (Eringen-Lee theory), Micromorphic continua, Micropolar continua, Microstretch, non-locality (strong).

Eringen-Mindlin Medium

This is sometimes the common name given to two closely related (but not identical in their basic assumptions) theories of microstructured media, that of Eringen and Suhubi (1964) and that of Mindlin (1964). The term "*micromorphic* media" following Eringen's classification is now commonly accepted (cf. Eringen 1999). Such media present a fully deformable microstructure at each material point, in addition to the usual translational degree of freedom.

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Cross references: Eringen A.C., Mindlin R.D., Micromorphic continua, Microstructured continuum theory (Eringen), Microstructured continuum theory (Mindlin).

Extended Thermodynamics

The classical theory of irreversible processes (cf. De Groot and Mazur 1962) as well as the rational mechanics of Coleman, Noll and Truesdell (cf. Truesdell 1984) admits the validity of the thermostatic definitions of entropy and thermodynamic temperature, and this in spite of the obvious more or less important deviation from thermodynamic equilibrium. This seems in contradistinction with results from the kinetic theory. An early proposal to remedy this deficiency was made by Machlup and Onsager (1953) and consists in involving the dissipative fluxes (e.g., viscous stresses, heat flux, electric conduction current) in the formulation of the entropy density. The definite step in that direction was taken by Müller and Ruggeri (1993)—also Jou et al. (1993).

Let $\underline{\sigma}^D$ denotes the dissipative stress. Then the free energy density ψ, entropy density η and entropy flux $\mathbf{s}$ will be given by general expressions

$$\psi = \psi\left(\theta, -; \underline{\sigma}^D, \mathbf{q}, \mathbf{J}\right), \tag{6.1}$$

$$\eta = \eta_s + N\left(\underline{\sigma}^D, \mathbf{q}, \mathbf{J}\right), \tag{6.2}$$

$$\mathbf{s} = \frac{\mathbf{q}}{\theta} + \mathbf{k}\left(\underline{\sigma}^D, \mathbf{q}, \mathbf{J}\right). \tag{6.3}$$

The vector **k** is referred to as the *extra-entropy flux* (see that Entry). It is expected that both η and θ will coincide with their thermostatic values when all dissipative fluxes vanish. Because of the tensorial orders involved, the deviations from the thermostatic definitions in Eqs. (6.1) and (6.2) are usually quadratic in the dissipative fluxes. For instance, in an isotropic body, ψ and η will certainly contain terms proportional to **q**.**q** and **J**.**J**, while **k** may in fact contain a term linear in **q** or **J**, and perhaps a term jointly quadratic in $\underline{\sigma}^D$ and **q**.

This methodology will in principle apply to all types of continua, whether classical or not classical (e.g., Cosserat fluids) with the appropriate list of dissipative fluxes. Moreover, these dissipative fluxes will themselves satisfy *evolution-diffusion equations* inspired by higher-order kinetic-theory developments. This thoughtful interaction between two different levels of description of physical reality is original but would be rejected by tenants of pure phenomenology. However, this new thermodynamic approach is certainly comforted by the fact that it allows a satisfaction of causality, resulting in the end in hyperbolic systems of equations with a bounded speed of propagation. This extended "rational" thermodynamics bears such a strong print from the kinetic theory of fluids that it is difficult to apply it to complex solid-like behaviours exhibiting hysteresis such as plasticity. Another avenue must be opened to cope with such cases. The thermodynamics with internal variables of state (see that entry and Maugin 1999) seems to be the looked for framework.

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Cross references: Extra-entropy flux, Internal variables of state.

Extra-Entropy Flux

The notions of thermodynamic temperature and entropy are well defined only in thermostatics (cf. Kestin 1966). However, in the "rational" thermodynamics advocated by B.D. Coleman, W. Noll and C.A. Truesdell (cf. Truesdell 1984), it is

a priori admitted that these notions also exist in the condition of thermodynamic non-equilibrium, even far from equilibrium. Furthermore, it is currently admitted that the imports of heat and entropy per unit mass and unit surface are related by the simple equations

$$s = \frac{h}{\theta}, \quad \mathbf{s} = \frac{\mathbf{q}}{\theta}, \tag{6.4}$$

where h and $\mathbf{q}$ are the body supply of heat per unit mass and the heat influx vector, and s and $\mathbf{s}$ are the corresponding supply of entropy per unit mass and the entropy influx vector while $\theta > 0, \quad \inf \theta = 0$, is the thermodynamic temperature (in Kelvin units). But it is more than natural to hypothesize that outside thermodynamic equilibrium the entropy influx will deviate from the expression recalled in the second of (6.4) by a vector field $\mathbf{k}$, called the *extra-entropy flux*, so that in lieu of $(6.4)_2$ there holds the more general relationship

$$\mathbf{s} = \frac{\mathbf{q}}{\theta} + \mathbf{k}, \tag{6.5}$$

where vector $\mathbf{k}$ has to be described by means of a constitutive relation typical of non-equilibrium for each material. Expression (6.5) was proposed by Müller (1973) who stated that the second of (6.4) is contradicted by kinetic theory. Vector $\mathbf{k}$ plays an essential role in extended thermodynamics (see that Entry, and Müller and Ruggeri 1993) and is of utmost importance in some theories of non-classical continuum mechanics (cf. Maugin 1990; Morro 2006) where it allows for an accommodation of non-classical contributions from the energy density.

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Cross references: Extended thermodynamics, Internal variables of state, Korteweg fluids, Solutions of macromolecules.

Chapter 7
F: From "Ferroelectric Crystals (Elasticity of)" to "Fractal Continua"

Ferroelectric Crystals (Elasticity of)

This entry concerns the continuum mechanics of ferroelectric crystals. This corresponds to the "electric" analogue of micromagnetism as an electric "ferroic" state, although the physical properties of the essential field (electric polarization) are quite different from those of the magnetization. In particular, it is a *polar* vector to be contrasted with the axial nature of the magnetization: its microscopic definition does not involve any time derivative, being based only on the notion of electric charge and geometric distance. Ferroelectric crystals can exhibit local nonzero electric polarization in the absence of applied electric field (cf. Lines and Glass 1972). They usually present strong interactions with the deformation field possibly through piezoelectricity or electrostriction. Their basic structure is in *domains*. Interaction between neighbouring electric dipoles and the resulting local ordering requires considering the influence of *polarization gradients*, so that a weakly nonlocal theory is to be envisaged from the electric viewpoint.

Modelling

We consider the case of the quasi-electrostatics of deformable dielectrics for the sake of simplicity. We can envisage a generalized motion described by the functions (compare to the micromagnetic case)

$$\mathbf{x} = \bar{\mathbf{x}}(\mathbf{X}, t), \pi = \bar{\pi}(\mathbf{X}, t), \tag{7.1}$$

where $\boldsymbol{\pi}$ is an electric polarization (polar vector) per unit mass in the deformed configuration. The second function defines a **polarization continuum**, *PC*. Accounting for a standard form of inertia for polarization (with coefficient d_E per

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unit mass), one is tempted to write down a balance equation for *PC for* the whole body B in a more or less standard form:

$$\frac{d}{dt}\int_B \rho d_E \dot{\pi}\, dv = \int_B \rho\left(\mathbf{E}+\mathbf{E}^L\right)dv + \int_{\partial B} \mathbf{A}\, da, \tag{7.2}$$

where $\mathbf{E}$ is the Maxwellian electrostatic field, and $\mathbf{E}^L$ is a quantity akin to an electric field and due to the possible interaction with the lattice continuum *LC*, whose deformation is described by the first of (7.1). Finally, $\mathbf{A}$, also akin to an electric field or a surface electric polarization, accounts in the form of a *contact* action for interactions between neighbouring electric dipoles that favour the ordering of electric dipoles. Applying to this the Cauchy principle, we can introduce a second order—nonsymmetric—tensor $\widehat{\mathbf{E}}$ such that

$$\mathbf{A} = \mathbf{n}.\widehat{\mathbf{E}} \; at \; \partial B. \tag{7.3}$$

Localization of (7.2) therefore yields the balance equation

$$d_E \ddot{\pi} = \mathbf{E} + \mathbf{E}^L + \rho^{-1}\, div\widehat{\mathbf{E}} \; in \; B. \tag{7.4}$$

It is difficult to grant a true physical meaning to the balance (7.2) which strongly resembles the balance law postulated, with the same degree of arbitrariness, in anisotropic fluids (nematic liquid crystals) by Ericksen (1960)—See Entries: Oriented media, Anisotropic fluids. One possible interpretation is that (7.4) is a standard equation of motion for a unit (hypothetical) electric charge (but the medium considered is a dielectric *free* of charges). In this interpretation *LC* and *PC* may be viewed as two interpenetrating continua. Such an a priori interpretation was advanced by Tiersten (1971). As to the surface condition (7.3) we can write it more explicitly as

$$\rho^{-1}\mathbf{n}.\widehat{\mathbf{E}} = \pi_S, \tag{7.5}$$

where π_S is a density of surface electric polarization (a polar vector).

The global balances of linear and angular momenta for the lattice continuum naturally read as

$$\frac{d}{dt}\int_B \rho\mathbf{v}\, dv = \int_B (\mathbf{f}+\mathbf{f}^{em})dv + \int_{\partial B} \mathbf{t}_{(n)} da; \tag{7.6}$$

and

$$\frac{d}{dt}\int_B (\mathbf{r}\times\rho\mathbf{v})dv = \int_B \left(\mathbf{r}\times(\mathbf{f}+\mathbf{f}^{em})+\mathbf{c}_{(PC/LC)}\right)dv + \int_{\partial B}\left(\mathbf{r}\times\mathbf{t}_{(n)}\right)da, \tag{7.7}$$

where $\mathbf{f}^{em}$ is what remains of the general *ponderomotive* force for dielectrics in quasi-electrostatics (cf. Maugin 1988) and $\mathbf{c}_{(PC/LC)}$ is a couple due to interactions between polarization and lattice continua. Then, artificial as this may look, the balance of angular momentum for the *PC* is given by:

$$\frac{d}{dt}\int_B \rho\pi\times\dot{\pi}\,dv = \int_B \left(\mathbf{c}^{em}+\mathbf{c}_{(LC/PC)}\right)dv + \int_{\partial B}\pi\times\mathbf{A}_{(n)}da; \tag{7.8}$$

This is complemented by the first law of thermodynamics for the combined continuum:

$$\frac{d}{dt}\int_B \rho\left(\frac{1}{2}\mathbf{v}^2+\frac{1}{2}d_E\dot{\pi}^2+e\right)dv = \int_B (\mathbf{f}.\mathbf{v}+w^{em}+\rho h)dv + \int_{\partial B}\left(\mathbf{t}_{(n)}.\mathbf{v}+\mathbf{A}_{(n)}.\dot{\pi}+q_{(n)}\right)da, \tag{7.9}$$

where e is the internal energy, and the second law of thermodynamics for the combined continuum (η is the entropy density and h is the bulk input of heat while θ is the thermodynamic temperature):

$$\frac{d}{dt}\int_B \rho\eta\,dv \geq \int_B \rho\theta^{-1}h\,dv - \int_{\partial B}\theta^{-1}\mathbf{q}.\mathbf{n}\,da. \tag{7.10}$$

In these equations,

$$\mathbf{c}^{em}=\rho\pi\times\mathbf{E},\ \mathbf{c}_{LC/PC}=\rho\pi\times\mathbf{E}^L,\ \mathbf{A}_{(n)}=\mathbf{n}.\widehat{\mathbf{E}} \tag{7.11}$$

and w^{em} is what remains of the electromagnetic energy contribution in the case of dielectrics in quasi-electrostatics (cf. Maugin 1988).

On account of (7.4) and the local form of (7.6) the local forms of (7.7) and (7.8) are easily established as

$$t_{[ji]}=\rho E^L_{[j}\pi_{i]}-\widehat{E}_{p[j}\pi_{i],p} \tag{7.12}$$

and

$$\rho \frac{d}{dt} d_E (\pi \times \dot{\pi})_i = c_i^{em} + c_{(LC/PC)i} + \varepsilon_{ijk} \left(\pi_j \widehat{E}_{pk} \right)_{,p}. \tag{7.13}$$

The rest of this approach consists in expressing (7.9) and (7.10). On introducing objective time rates such as [cf. the case of micromagnetism—Eqs. (7.35), (7.36)]

$$\hat{p}_i = (D_J \pi)_i \equiv \dot{\pi}_i - \Omega_{ij} \pi_j, \ \widehat{\Pi}_{ij} = (\dot{\pi}_i)_{,j} - \Omega_{ik} \pi_{k,j}, \tag{7.14}$$

one can show that (7.9) and (7.10) lead to the following local forms of the energy equation and of the Clausius-Duhem inequality:

$$\rho \dot{e} = tr\left(\mathbf{t}^S \mathbf{D} \right) - \rho \mathbf{E}^L . \hat{\mathbf{p}} + tr\left(\widehat{\mathbf{E}} \widehat{\Pi}^T \right) - \nabla . \mathbf{q} + \rho h, \tag{7.15}$$

$$-\rho \left(\dot{\psi} + \eta \dot{\theta} \right) + tr\left(\mathbf{t}^S \mathbf{D} \right) - \rho \mathbf{E}^L . \hat{\mathbf{p}} + tr\left(\widehat{\mathbf{E}} \widehat{\Pi}^T \right) - \theta^{-1} \mathbf{q} . \nabla \theta \geq 0, \tag{7.16}$$

with

$$t_{ji} = t_{ji}^S + t_{[ji]}, \ t_{[ji]} = \rho E_{[j}^L \pi_{i]} - \widehat{E}_{p[j} \pi_{i],p}. \tag{7.17}$$

Together with Maxwell's electrostatic equations for dielectrics,

$$\nabla \times \mathbf{E} = \mathbf{0}, \ \nabla . \mathbf{D} = 0, \ \mathbf{D} = \mathbf{E} + \rho \pi, \tag{7.18}$$

this concludes the formal construction of the theory before establishing constitutive equations constrained by the inequality (7.16).

Approach via the Principle of Virtual Power

It is now clear that an approach exploiting directly the principle of virtual power for the present theory will be very much like what is achieved for ferromagnets—see Entry: Micromagnetism—except for the essential difference regarding the inertial force of the polarization lattice *PC*. That is, with a general statement of the principle in the form (cf. Maugin 1980)

$$P_{inert}^*(B) = P_{int}^*(B) + P_{vol}^*(B) + P_{surf}^*(\partial B), \tag{7.19}$$

we shall a priori write

$$P_{inert}^*(B) = \int_B \left(\rho \dot{\mathbf{v}} . \mathbf{v}^* + \rho d_E \ddot{\pi}_i \dot{\pi}_i^* \right) dv, \tag{7.20}$$

where we clearly distinguish between real fields (actual solutions of a problem) and virtual ones (indicated with an upper right asterisk) at our disposal in this type of variational formulation. In particular, for real fields, this yields

$$P_{inert}(B) = \frac{d}{dt}\int_B \left(\frac{1}{2}\rho \mathbf{v}^2 + \frac{1}{2}\rho d_E \dot{\pi}^2\right) dv = \frac{d}{dt}K(B). \tag{7.21}$$

The other global virtual powers are directly written down as

$$P^*_{bulk}(B) = \int_B ((\mathbf{f} + \mathbf{f}^{em}).\mathbf{v}^* + \rho \mathbf{E}.(\dot{\pi})^*) dv, \tag{7.22}$$

$$P^*_{surf}(\partial B) = \int_{\partial B} \left(\left(\mathbf{t}_{(n)} + \mathbf{t}^{em}_{(n)}\right).\mathbf{v}^* + \rho \pi_S.(\dot{\pi})^*\right) da, \tag{7.23}$$

and

$$P^*_{int}(B) = -\int_B \left(t^S_{ji} v^*_{i,j} - \rho E^L_i \hat{p}^*_i + \widehat{E}_{ji}\widehat{\Pi}^*_{ij}\right) dv. \tag{7.24}$$

Here the last one is written as a linear continuous functional on a set of objective virtual velocity fields built from $v_i, v_{i,j}, \dot{\pi}_i$, and $\dot{\pi}_{i,j}$. From the standard application of the principle of virtual power for any volume and surface elements and for arbitrary members of the set $\{\mathbf{v}^*, (\dot{\pi})^*\}$, one deduces the local equations of linear momentum of the *LC* and the governing Eq. (7.4) of the *PC*, together with the accompanying natural boundary conditions. Then Eqs. (7.5) and (7.16) follow in the usual way, using the result (7.20). Equations (7.15) and (7.16) govern both recoverable and dissipative phenomena associated with the two fields (7.1). In the absence of dissipation, the field equations and associated boundary conditions can be deduced from a Hamiltonian-Lagrangian variational principle written in the reference configuration K_R with a Lagrangian density per unit volume in the form

$$L = \frac{1}{2}\rho_R \mathbf{v}^2 + \frac{1}{2}\rho_R d_E \dot{\pi}^2 - \Psi(\mathbf{F} = \nabla_R \bar{\mathbf{x}}, \pi, \nabla_R \bar{\pi}), \tag{7.25}$$

but one must add to this the electrostatic energy including both free-field and electric-dipole energies:

$$e^{elec} = \frac{1}{2}\mathbf{E}^2 + \rho_R \pi.\mathbf{E}. \tag{7.26}$$

The theory with main thermodynamic ingredients (7.15) and (7.16) was proposed by Maugin and Pouget (1981) for the study of coupled

polarization-deformation waves in a bulk or at surfaces (see Maugin 1988, Chap. 7). Pouget et al. (1986a, b) have given a lattice-dynamics justification of the above-proposed continuum equations. In particular, they gave an evaluation of the inertial factor d_E.

Analogy with Cosserat Continua

Applying the alternation symbol to Eq. (7.13) and using the identity

$$\varepsilon_{kli}\varepsilon_{ipq} = \delta_{kp}\delta_{lq} - \delta_{kq}\delta_{lp}, \tag{7.27}$$

or, equivalently, taking the tensor product of (7.4) with $\boldsymbol{\pi}$ and then the skew part of the result we obtain

$$\rho\frac{d}{dt}\left(d_E\dot{\pi}_{[i}\pi_{j]}\right) = E_{[i}P_{j]} + \left(\rho E^L_{[i}\pi_{j]} - \widehat{E}_{k[i}\pi_{j],k}\right) + \left(\widehat{E}_{k[i}\pi_{j]}\right)_{,k}, \tag{7.28}$$

or

$$\rho\dot{S}_{ij} = C^{em}_{ij} + t_{[ji]} + M_{kij,k}, \tag{7.29}$$

where we accounted for (7.17) and we set

$$S_{ij} = d_E\dot{\pi}_{[i}\pi_{j]},\ C^{em}_{ij} = E_{[i}P_{j]},\ M_{kij} = \widehat{E}_{k[i}\pi_{j]}. \tag{7.30}$$

Simultaneously, (7.5) yields the associated natural boundary condition at ∂B:

$$n_k M_{kij} = M_{(n)ij} \equiv \pi_{S[i}P_{j]}. \tag{7.31}$$

Equations (7.29) and (7.31) are in the canonical form of the local balance of angular momentum for a Cosserat or micropolar continuum in Eringen's classification, but all terms have an electric origin. These equations were obtained by the author (Maugin 1971, 1980, 2012). Nonlinear-wave applications are given in Maugin et al. (1992).

Reduction to a Model Without Microstructure

When pure ferroelectric features are ignored or neglecting polarization inertia and polarization-gradient effects Eq. (7.27) reduces to

$$t_{[ij]} = E_{[i}P_{j]}. \tag{7.32}$$

This corresponds to the classical theory of nonlinear dielectrics as originally built by Toupin (1956, 1963) and Eringen (1963)—see Chap. 7 in Eringen and Maugin (1990, Vol. 1). This nonlinear theory in finite strains applies in particular to *electroelastic polymers* with electrostriction as main electro-mechanical coupling (See works by Dorfmann, Ogden, Bustamante, and others in the 2000s). The skewsymmetric part of the stress vanishes when polarization and electric fields are aligned. This occurs in isotropic bodies. Still the ponderomotive force is present. However, if quadratic effects in the electric field are discarded altogether, corresponding to a fully linear theory, then both ponderomotive force and couple disappear leaving for only possible electromechanical couplings piezoelectricity, material symmetry permitting (no centre of symmetry).

Antiferroelectric Materials

It is easily imagined that a theory of deformable antiferroelectrics (e.g., lead zirconate or sodium niobate) in which an antiparallel arrangement of permanent electric dipoles can be devised by analogy with the theory of antiferromagnetics (see Entry: Micromagnetism), i.e., by considering the polarization density $\boldsymbol{\pi}$ as arising from the *vector* sum of two opposite polarization sub-lattices of equal magnitude. Such a model was constructed by Soumahoro and Pouget (1994) who also studied in detail its dynamical consequences.

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Cross references: Ferroïc states, Micromagnetism, Ponderomotive couple,

Ferroic States

The qualification of "ferroïc" applies to states of material substances that, at some point or in small regions, exhibit a nonvanishing field such as magnetization in a ferromagnet in the absence of applied field (magnetic field in the magnetic case). This is possible only in so-called *domains* of limited spatial extension (cf. Prirovoskii 1976), of which the assemblage forms, by statistic compensation in magnitude and orientation, the whole material specimen. Accordingly, a true "mono-domain" specimen cannot exist without the presence of closing domains at its boundary. An applied field of given direction can modify this arrangement by favouring the expansion of properly aligned domains at the expense of other domains. The primitive example of such material behaviour is provided by ferromagnetism—the paradigmatic case. Other examples are provided by adding the prefix "ferro" to other material responses, such as in "*ferro*-electric" materials and "*ferro*-elastic" bodies, exhibiting local nonzero spontaneous electric polarization (density of electric dipoles) in the former case, and nonzero spontaneous elastic deformation in the latter case. Other fields such as temperature and deformation in ferromagnetism, or magnetic field in a ferro-elastic can also alter the initial "ferro" arrangement. The existence of such states is directly related to the existence of a favourable material symmetry (cf. Kittel 1971; Aizu 1970). The spontaneous field of interest then is naturally considered as a primitive independent variable in a continuous field approach, the "effect" (spontaneous field) being possible in the absence of the "cause" (applied field).

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Cross references: Ferroelectric crystals (elasticity of), Micromagnetism in elastic solids

Fractal Continua

One must distinguish between continuity of a function and its differentiability. Many curves are continuous and present a well-defined slope everywhere except perhaps at some singular points where this slope can suffer a finite discontinuity. In classical continuum mechanics one is used to consider at the starting point space-time functions (e.g., the placement) that admit continuous first and second order derivatives almost everywhere, so that one can define unambiguously notions such as acceleration and second gradient of the displacement. The considered functions are said to be of class C^2. This is the case of the mass density and other material properties, save at the crossing of singular surfaces. Motion itself and its velocity and acceleration may be discontinuous at such surfaces (e.g., at material interfaces, in shock-wave theory or when studying acceleration waves). An altogether different situation occurs with the appearance of the notion of *fractal set*, largely due to Benoît Mendelbrot (1924–2010), a French-American engineer who became a creative mathematician with multiple scientific interests and an efficient populariser (see his wonderful book, Mandelbrot 1982). Mendelbrot was not the creator of the very notion of fractal (he coined the term with an acute flair for publicity). This was known as *Julia set* after the French mathematician Gaston Julia (1893–1978). But with the aid of computers he provided illustrations of what could be imagined but not visually represented before.

The notion of fractal relates to the fact that some functions, although themselves continuous, have *everywhere* discontinuous derivatives. The resulting strange object when the notion is applied to geometry (the field of Mendelbrot) exhibits the feature to present a similar picture at all scales, that is, independently of any magnification. Details keep the same appearance at all scales like a reduced-size copy of the whole. It was soon realized by Mandelbrot and others that many physical and biological objects present this invariance property, although a well defined quantity called the *Hausdorff dimension D* (after Felix Hausdorff 1868–

1942) correctly characterizes each class of fractal set. This dimension can be smaller than that of the space in which the fractal geometric set is embedded, and it can be a non integer.

Acknowledging the obvious existence of fractal geometric sets in materials, but still wanting to be able to solve mechanical problems with boundary and initial conditions, one needs an analytical way to account for the fractal feature in systems of equations that previously were well defined in terms of classical partial derivatives. At the time of writing this endeavour has been actively pursued but in several possible directions and exploiting various techniques. One is thus led to studying the notion of *fractal continua*. The main point will obviously be the appropriate definition of derivatives.

One possible avenue to fulfil the programme is an application of a homogenization method called *dimensional regularization*. This method was exploited by V.E. Tarasov—a most active scientist in the field—to transform fractional integrals over fractal sets into equivalent continuous integrals over Euclidean sets. This enabled Tarasov (2005) to map a mechanical problem of a fractal onto a problem in the Euclidean space in which the fractal is embedded. This approach yields the looked for continuum equations for the conservation of mass, linear momentum and energy. The method essentially uses a "product measure" that allows for the needed transformation. This is primarily related to the notion of mass. For instance, for the mass with power law $m(R) \approx R^D$, where R is a length scale of measurement (say, a resolution) and $D<3$ is the fractal dimension of mass, the fractional integral representing the mass in the region w embedded in E^3 is given by

$$m(w) = \int_w \rho(R) dV_D = \int_w \rho(R) c_3(D,R) dV_3, \tag{7.33}$$

where c_3 is defined in terms of R, D and Euler's gamma function. Other coefficients $c_2(D,R)$ and $c(D,d,R)$ can be defined, in terms of which one can express the generalized space and time derivatives such as

$$\nabla_k^D f = c_3^{-1}(D,R) \frac{\partial}{\partial x_k} [c_2(d,R) f], \ \frac{\partial}{\partial x_k} \equiv \nabla_k, \tag{7.34}$$

$$\left(\frac{d}{dt}\right)_D f = \frac{\partial f}{\partial t} + c(D,d,R) v_k \nabla_k f \tag{7.35}$$

Ostoja-Starzewski et al. have emphasized the appropriate validity of Tarasov's transformation for isotropic fractal media. In order to improve on the case of anisotropic fractal media, Li and Ostoja-Starzewski (2009) have proposed to introduce the notion of product measures (one of each in each direction in Euclidean space). With this they have at hand the whole machinery needed to proceed to the construction of all differential operators and both Green-Gauss and Reynolds theorems of continuum mechanics and the resulting balance equations as

also the wave equation in fractal elasticity (cf. Ostoja-Starzewski et al. 2013). We refer to these authors for the subtleties of the approach. In particular, for the sake of illustration, we note the following formula for small strains:

$$e_{ij} = \frac{1}{2}\left(\nabla_j^D u_i + \nabla_i^D u_j\right) = \frac{1}{2}\left[\frac{1}{c_1^{(j)}}\nabla_j u_i + \frac{1}{c_1^{(i)}}\nabla_i u_j\right], \tag{7.36}$$

where $c_1^{(k)}$ is the product measure in the direction of x_k. The authors pay a special attention to the case of micropolar materials because the Cauchy stress is generally not symmetric in fractal media, hence the possible importance of couple stresses.

In concluding, other approaches deserve to be mentioned. First of all, the tremendous amount of works by Balankin and co-workers (summarized in the synthetic contribution of Balankin 2013) would need more than a superficial review, but it goes along the same path as what was just exposed. Next, Epstein and Sniatycki (2006) have proposed an approach based on the notion of differential spaces of Sikorski. This allows for the introduction of generalized forces and stresses for fractal objects, exploiting an extended form of the principle of virtual work and finally yielding a study of structural self-similarity in elasticity. This seems to provide an efficient means to obtain effective properties in an inexpensive numerical scheme. Finally, Michelitsch et al. (2009, 2012, 2014) have dealt more deeply with the possible definition of differential operators—the basic problem in fractal theory—by having recourse to considering lattice models with self-similar harmonic interparticle interactions. This also leads to a self-similar Laplacian operator that takes the form of a combination of fractional integrals in a continuum limit. In the case of elasticity one obtains thus a Hooke law including a non-local convolution with an elasticity modulus function as a power law. Interesting properties of wave propagation follow thereof.

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Cross references: Couple stresses, Non-locality (strong).

Chapter 8
G: From "Generalized Continuum Mechanics" to "Green A.E."

Generalized Continuum Mechanics (GCM)

[See Chap. 2 in Part One].

Generalized Internal Forces

These are all types of co-factors that one introduces when writing formally the power expanded by all degrees of freedom and their spatial gradients of any order. They need to be expressed by thermodynamically admissible constitutive equations. In modern continuum thermodynamics, it is required that they should be objective quantities if they are really "internal" (i.e., invariant by time-dependent rotations of the actual frame). They include usual stresses, couple stresses, and hyperstresses of various orders as also multipolar stresses of any order. They are indeed representative of the complexity of nonclassical continuum mechanics.

Cross references: Cosserat continua, Couple stress, Double force, Gradient elasticity, Higher-order gradient theories, Hyperstresses (notion of), Micromorphic continua, Mutipolar continua, Oriented media.

Generalized Thermo-Elasticity

The classical theory of heat conduction equipped with Fourier's law provides a parabolic equation—the celebrated heat equation—that yields an instantaneous propagation of heat, i.e., an infinite speed of propagation of temperature. James C. Maxwell was probably the first to note that dynamic heat processes should exhibit a relaxation, hence a time scale. The well accepted theory of thermo-elasticity

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originally developed by Duhamel (1837) and Neumann (1885) in the nineteenth century consisted in coupling together (small strain) elasticity and Fourier's theory. This kept the defect of Fourier's theory in its dynamical framework since only the mechanical-wave properties could be reproduced albeit with a differentiation between isentropic and isothermal regimes. It is only in the 1940s–1950s that Maxwell's idea was revisited and appropriately generalized Fourier's laws of heat conduction were proposed by Cattaneo (1948; 1958) in Italy and Vernotte (1958) in France. These authors indeed introduced a heat relaxation time as follows. They complemented the local energy balance law in rigid bodies, i.e., with an obvious notation in 1D,

$$\rho_0 C \frac{\partial \theta}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{8.1}$$

by the improved Fourier's law

$$q = -k \frac{\partial \theta}{\partial x} + \sigma \frac{\partial^2 \theta}{\partial x \, \partial t}. \tag{8.2}$$

On substituting from (8.2) into (8.1), one obtains the equation

$$\tau \frac{\partial^2 \theta}{\partial t^2} + \frac{\partial \theta}{\partial t} = \left(\frac{k}{\rho_0 C} \right) \frac{\partial^2 \theta}{\partial x^2}, \tag{8.3}$$

where $\tau = \sigma / k$, with $k > 0$, is a relaxation time, and a fourth-order mixed space and time derivative has been discarded. The second-order time derivative in the left-hand side of (8.3) plays the role of an inertia term and transforms the original Fourier parabolic equation into a hyperbolic equation with attenuation. In the 3D isotropic case, the Cattaneo-Vernotte system reads:

$$\rho_0 C \frac{\partial \theta}{\partial t} + \frac{\partial q_i}{\partial x_i} = 0, \quad \tau \frac{\partial q_i}{\partial t} + q_i = -k \frac{\partial \theta}{\partial x_i}. \tag{8.4}$$

It is in the 1960s–1970s that specialists of continuum mechanics thought of accounting for such a possible heat-relaxation effect in the coupled theory of thermo-elasticity [originally as given in books such as Parkus (1968), Hetnarski and Eslami (2009) or Nowacki (1986)] thus giving rise to the notion of *generalized thermo-elasticity*. This was achieved via a multitude of modellings of which we can name a few: Lord and Shulman (1967), Fox (1969), Green and Lindsay (1972), Green and Naghdi (1993), Gurtin and Pipkin (1968), Hetnarski and Ignaczak (1999). Other modellings apply other concepts and generalize the thermo-mechanical case to more complex mechanical descriptions and electro-mechanical interactions. This is well described and documented in the syntheses of Chandrasekharaiah (1986, 1998) and that of Joseph and Preziosi

(1989), the first two chapters of Straughan's (2011) book, and in many entries in the *Encyclopaedia of Thermal Stresses* (Hetnarski 2014).

For the sake of example, we recall the linear anisotropic system of classical thermoelasticity:

$$\begin{aligned} \rho_0 \ddot{u}_i &= \left(c_{ijkh} u_{k,h}\right)_{,j} + \left(a_{ij}\theta\right)_{,j} + \rho_0 f_i, \\ C\dot{\theta} &= a_{ij}\dot{u}_{i,j} + \left(k_{ik}\theta_{,k}\right)_{,i} + \rho_0 r. \end{aligned} \tag{8.5}$$

with an obvious notation [c_{ijkh} = elasticity coefficients, a_{ij} = thermoelasticity coefficients, k_{ik} = thermal conductivity tensor]. In the case of the *linear Lord-Shulman theory* of generalized thermoelasticity, the following system replaces (8.5):

$$\begin{aligned} &\rho_0 \ddot{u}_i = (\lambda + \mu) u_{j,ij} + \mu \Delta u_i - (3\lambda + 2\mu)\alpha\theta_{,i}, \\ &\rho_0 C\left(\tau\ddot{\theta} + \dot{\theta}\right) + (3\lambda + 2\mu)\alpha\theta_0(\tau\ddot{e}_{kk} + \dot{e}_{kk}) = k\Delta\theta + \rho_0 r, \end{aligned} \tag{8.6}$$

where apart from τ all symbols are those of the classical theory.

In the case of the *linearized anisotropic case of Green and Lindsay*, the system (8.5) is replaced by the following one:

$$\begin{aligned} &\rho_0 \ddot{u}_i = \left(c_{ijkh} u_{k,h}\right)_{,j} + \left[a_{ij}\left(\theta + \alpha\dot{\theta}\right)\right]_{,j} + \rho_0 f_i, \\ &\rho_0\left(h\ddot{\theta} + d\dot{\theta} - a_{ij}\dot{u}_{i,j} - b_i\dot{\theta}_{,i}\right) = \rho_0\theta_0^{-1} r + \left(b_i\dot{\theta} + k_{ij}\theta_{,j}\right)_{,i}. \end{aligned} \tag{8.7}$$

Note that generalized thermoelasticity is active only in dynamics, as nothing is altered in statics where no time scale is involved. In true dynamics it gives rise to the notion of *second sound*, as indeed observed in some materials (solid Helium, sodium fluoride, bismuth, etc.).

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Cross references: Extended thermodynamics, Extra-entropy flux, Generalized continuum mechanics, Green A.E., Nowacki W.

Gradient Elasticity

It is quite natural in analysis to describe the behaviour of a function at a point with a sufficient degree of approximation by considering the value of successive derivatives as exemplified by the Cauchy expansion of a function; for instance, along a curve, the slope, the curvature, etc. In space, the successive gradients would have to be prescribed at a point. This in fact is part of the full field theory as illustrated in the book of Rzewuski (1964) with an expanded functional Euler-Lagrange derivative in the case of a variational formulation. Classical elasticity requires the consideration of the gradient of displacement—in order to define the strain, the thermodynamic dual of the classical (Cauchy) stress. This seems quite enough to most engineers and scientists in what we called a *first-gradient* theory (as the lowest case in a hierarchy; cf. Maugin 1980). But considering the influence of higher-order gradients of the displacement in a refined theory of elasticity is not such a strange idea. In fact, this was questioned by Barré de Saint-Venant (1869) and also by Piola (1856; See Piola 2014). The first true application—based on the remark that torsion introduces spatially non-uniform strains—is probably due to Le Roux (1911, 1913).

Further developments had to await the 1960s with works by Mindlin and his co-workers. No doubt that these works were inspired by the lattice-dynamics approach to materials as introduced by Born and von Kármán (1912) early in the twentieth century (See Born and Huang 1954). This approach provides a finite-difference description with interactions—usually represented by elastic springs (but this is only a powerful image)—between "particles" (mass points) that are more or less neighbours in a regular (lattice) arrangement. Note that the notion of lattice practically goes back to Cauchy himself. In the original Born-Kármán theory only nearest neighbours are considered providing at first finite differences of the second order only. Through a limit procedure known as the long-wavelength limit, this recurs to the standard continuum elasticity that involves only the first gradient of the elastic displacement in the potential or the second gradient of the displacement in the local expression of the momentum balance. By the same procedure, accounting for more distant neighbours—such as second ones—will naturally yield a theory of elasticity with higher-order gradients in the continuum limit. This we called the *Boussinesq paradigm* (cf. Christov et al. 2007) after the first example constructed by Boussinesq (1903) for elastic solids. Because of the spatial—but necessarily limited—extension of the lattice description with farther neighbours, this we also called "*weak nonlocality*" (cf. Maugin 1979), an approach which permits one to remain within the application domain of partial differential equations (as compared to "strong nonlocality"—see the corresponding entry—that will involve functionals over space). Only second-gradients of displacements or first-gradients of the strain are considered. One should note that the sign of the higher-order spatial interactions and those of the fourth-order derivatives play an important role in the discussion of stability of the resulting continuum model in the sense of Hadamard. In particular, some models may yield an *anomalous* dispersion of elastic waves in the crystal, although it is not reasonable to stretch the model to too small wavelengths. This is thoroughly debated by, e.g., Mülhaus and Oka (1996), Askes et al. (2008), and Askes and Aifantis (2011).

In a variational formulation, one would start with a Lagrangian volume density in the form

$$L = K - W, \tag{8.8}$$

$$K = \frac{1}{2}\rho \dot{u}_i \dot{u}_i,\ W = \overline{W}\left(u_{i,j}, u_{i,jk}\right)\ or\ W = \widetilde{W}\left(e_{ij}, e_{ij,k}\right) \tag{8.9}$$

with

$$e_{ij} = u_{(i,j)} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right). \tag{8.10}$$

The last form given in (8.9) accounts for the objectivity (material indifference) of the potential energy W. Note that the kinetic energy K assumes its classical expression. The derived local balance equation of linear momentum is obtained as

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j}\sigma_{ji} = 0, \tag{8.11}$$

where the Cauchy stress is given by

$$\sigma_{ji} = \bar{\sigma}_{ji} - \frac{\partial}{\partial x_k} m_{kji} \equiv \frac{\delta W}{\delta e_{ij}}, \tag{8.12}$$

with

$$\bar{\sigma}_{ji} = \frac{\partial W}{\partial e_{ij}} = \bar{\sigma}_{ij},\ m_{kji} = \frac{\partial W}{\partial (e_{ji,k})} = m_{kij}, \tag{8.13}$$

and $\delta/\delta e_{ij}$ denotes a functional (Euler-Lagrange) derivative. The symmetric tensor of components $\bar{\sigma}_{ji}$ may be called the *intrinsic stress* and is only part of the Cauchy stress. The tensor of (at most, eighteen independent) components m_{kji} is usually referred as the hyperstress tensor. Mindlin and co-workers (Mindlin 1964; Mindlin and Tiersten 1962; Mindlin and Eshel 1968) have proposed general quadratic expressions for the energy W. The number of independent elasticity coefficients for a body of general symmetry is much increased compared to standard elasticity and this poses a serious problem for their numerical and experimental evaluation. This is a drawback for this type of refined theory. That is why, for all practical purposes in establishing the main consequences of the theory, Aifantis (1992) proposed the following very convenient form of W in the linear case

$$W = \frac{1}{2}\bar{\sigma}_{ji} e_{ij} + \frac{1}{2}\delta^2 \frac{\partial}{\partial x_k}\left(\bar{\sigma}_{ji}\right)\frac{\partial}{\partial x_k}\left(e_{ji}\right), \tag{8.14}$$

where δ is a characteristic length. This means that once $\bar{\sigma}_{ji}$ is formally known, so is the case of m_{kji}, and then only δ is a new material parameter compared to the standard linear elasticity. The working hypothesis (8.14) is used so as to obtain the simplest generalization of standard elasticity. This is sufficient to study the characteristic singularities (dislocations, disclinations) and main wave properties in elasticity problems. In particular, the presence of the characteristic length δ results in a *dispersion* of linear waves.

The direct approach using a global statement of fundamental balance laws is not an obvious matter, in particular in formulating the required boundary conditions that may apply at a regular boundary but also at singular lines along which one observes a discontinuous tangent plane (not to speak of possible apices). An elegant way to deal with this matter is to exploit the *principle of virtual power* as a basic statement as in Germain (1973) because natural boundary conditions strictly unfold in parallel with the bulk equations. In particular, in formulating the power of internal forces in this formulation one can write for this quantity per unit volume

$$p_{int} = -\left(\bar{\sigma}_{ji} v_{i,j} + m_{kji} v_{i,jk}\right) = -\left(\bar{\sigma}_{ji} D_{ij} + m_{kji} v_{i,jk}\right), v_i = \dot{u}_i, \tag{8.15}$$

where objectivity of internal forces has been enforced in the last expression. In terms of the geometry of a bounding surface (so-called natural boundary condition) the new concept represented by m_{kji} requires the consideration of the second-order geometrical description of the surface, hence the curvature. This destroys the standard Euler-Cauchy notion of contiguity where only the unit normal is involved. In pure elasticity, the effect of the contribution of the hyperstress will be of importance wherever the strain is not spatially uniform, and obviously where one observes a rapid variation of the elastic displacement, e.g. in boundary layers. Let $D(.) := n_p \partial_p(.)$ the normal gradient operator at the boundary ∂B of unit outward normal of components n_p, $D_p(.) := \partial_p - n_p D$ the corresponding surface gradient operator, [..] the notation for a jump, and if s_r denotes the unit tangent vector to an oriented curve C on ∂B representing a singular line, the tractions boundary conditions are found in the following rather complex form:

$$P_q^{(n)} = n_p \sigma_{pq} - D_p\left(n_r m_{rpq}\right) + \left(D_j n_j\right) n_r n_p m_{rpq} \text{ at regular parts of } \partial B, \tag{8.16}$$

$$R_q^n = n_r n_p m_{rpq} \text{ at regular parts of } \partial B, \tag{8.17}$$

$$E_q = \left[n_r k_p m_{rpq}\right] \text{ along } C, \tag{8.18}$$

with $k_q = \varepsilon_{rpq} s_r n_p$. In these equations,

$$P_q^{(n)} := t_q^{(n)} + (D_r n_r) n_p T_{pq}^{(n)} - D_p T_{pq}^{(n)}, \tag{8.19}$$

$$R_q^{(n)} := n_p T_{pq}^{(n)}, \tag{8.20}$$

$$E_q := \left[k_p T_{pq}^{(n)}\right], \tag{8.21}$$

where $t_q^{(n)}$ is a true force surface traction, and $T_{pq}^{(n)}$ is a true double force surface traction (a notion issued from the mechanics of plates and shells). $P_q^{(n)}$ and $R_q^{(n)}$ are only auxiliary forces while E_q is a line load applied long C. The mean curvature of the surface is given by $\Omega = -D_j n_j/2$. The complexity of the above recalled boundary conditions is obvious.

An expression such as (8.14), rustic as it is, allows one to provide exemplary solutions to many problems such as those involving dislocations (cf. Lazar and Maugin 2005). Note that the Cauchy stress σ now reads

$$\sigma = \left(1 - c^2 \nabla^2\right) \sigma_c, \tag{8.22}$$

where σ_c is the usual stress tensor. We can say that this expression contains a *Helmholtz operator* in factor of σ_c.

Second-gradient models in finite strain have been devised by authors like Gurtin—for dislocation theory—, Cleja-Tigoiu—for elasto-plasticity—, Sciarra et al. (2007)—for porous media—and Ciarletta et al. (2012)—for the bio-mechanics of soft tissues. In this case, on must replace the second of (8.9) by a more general expression

$$W = \overline{W}(\mathbf{F}, \nabla_R \mathbf{F}), \tag{8.23}$$

per unit reference volume, where $\mathbf{F}$ denotes the gradient of the direct motion and ∇_R stands for the referential gradient.

Finally, form the standpoint of the basis principles of continuum mechanics, the notion of gradient theory requires revisiting Cauchy's approach by accounting for the possible existence of discontinuities in the tangent plane to the bounding surface and of possible edge forces [see the fundamental works of Noll and Virga (1990) and Dell'isola and Seppecher (1995)].

In conclusion, very interesting features of the gradient model of elasticity are:

- The inevitable introduction of characteristic lengths;
- The appearance of so-called capillarity effects (surface tension) due to the explicit intervening of curvature of surfaces;
- Correlative boundary layer effects;
- Dispersion of waves with a possible competition and balance between nonlinearity and dispersion, and the existence of solitonic structures,

and the eventual relationship with the Ginzburg-Landau theory of phase transitions.

But a rather unpleasant feature of this modelling is that the resulting mathematical problems become more *stiff* than before with its higher-order space derivatives, creating potential difficulties in dynamical computations unless one constructs appropriate finite-difference schemes (as done by C.I. Christov and G.A. Maugin in the 1990s). In analysis, it is wise to exploit the technique of matched asymptotic expansions in order to avoid solving the stiff problem unnecessarily in large parts of the body—outside boundary layers—, that are practically unaffected by the gradient description.

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Cross references: Capillarity, Contiguity, Double force, Le Roux elasticity, Non-locality (weak), Non-locality (strong).

Gradient Plasticity

In some situations of deformation like in torsion, Le Roux (1911) emphasized the role of nonuniform strains in elasticity, introducing thus a pioneer's gradient theory of elasticity. But in the framework of plasticity, it seems according to Nye (1953) and Kubin and Mortensen (2003) that Friedel (1964) contains the first mention of a hardening effect due to the plastic accommodation of elastic strain gradients by dislocations. In words, Friedel (1964, p. 254) wrote that "… the minimum dislocation density to produce the deformation [in the bending of a crystal to curvature c] is given by $\rho = c/b$ [where b is Burgers vector]… This density […] introduces short-range stresses on a scale comparable with the average distance l between dislocations […]. Ones expects therefore a parabolic law $\sigma \approx \sigma_0 + (\mu/2\pi)(bc)^{1/2}$[…]. Similar but more elaborate equations can be given in the same way for any type of macroscopic distorsion which is not a uniform shear". These are *geometrically necessary dislocations* (GNDs) in the sense of Nye (1953) and Ashby (1970)—See Entry: Dislocations and disclinations. From this remark there follows the conception of plasticity flow rules that involve not only the stress (like in classical plasticity; cf. Maugin 1992) but also the gradient of the stress with a characteristic length (Aifantis 1984, 1987)—of which the thermodynamic formulation can be disputed; cf. Maugin (1990)—or a real expansion of a phenomenological theory of strain gradient in plasticity (Fleck and Hutchinson 1993, 1997, 2001) comforted by experiments on the torsion of very thin copper wires (cf. Fleck et al. 1994), and made more "rational" and mathematical by Fleck and Willis (2009). Other much relevant works are Muhlaus and Aifantis (1991), Gutkin and

Aifantis (1999), and Gurtin and Anand (2009). Without dealing in detail with such developments, we note the following essential concepts under the working hypothesis of small strains.

The primary notion is that of Nye's (1953) tensor of dislocation density, usually noted α_{ij} in Cartesian components. In the discrete description, it is defined in terms of the number of dislocations piercing a unit area and of their Burgers vectors. In geometrical terms (Nye 1953) it is directly related to the curvature tensor κ_{ij} by the simple formula

$$\alpha_{ij} = \kappa_{ji} - \delta_{ij}\kappa_{kk}, \tag{8.24}$$

which by inversion yields

$$\kappa_{ij} = \alpha_{ji} - \frac{1}{2}\delta_{ij}\alpha_{kk}. \tag{8.25}$$

In his celebrated work Kröner's (1958) defines α_{ji} as minus the transposed of the above one. By Burgers' definition of Burgers vector and use of the Stokes theorem we can also write

$$B_i = \int_\Gamma \gamma_{ij} dx_j = \int_S \alpha_{ij} n_j dS, \tag{8.26}$$

where Γ is a circuit traversed in the sense of a right-handed screw motion along the unit normal $\mathbf{n}$ to the surface S encircled by Γ and γ_{ij} is the local distortion, and

$$\alpha_{ij} = \varepsilon_{jkp}\gamma_{ip,k}, \tag{8.27}$$

so that the local dislocation density tensor is proportional to a second gradient of the displacement. This agrees with (8.24) since κ_{ij} is a curvature. In Kröner's synthetic notation (8.27) can also be written as $\underline{\alpha} \approx \nabla \times \underline{\gamma}$. According to Fleck and Hutchinson, this can also be related to the constrained continuum theory of *couple stresses*. Indeed, the curvature is the spatial gradient of the material rotation of vector of components θ_i, so that

$$\kappa_{ij} = \theta_{i,j} = \frac{1}{2}\varepsilon_{ipk}u_{k,pj} = \varepsilon_{ipk}e_{kj,p}. \tag{8.28}$$

Equations (8.25) and (8.28) are identical if we account for a transposition. From (8.27) we note that $\partial\alpha_{ij}/\partial x_i = 0$, what is a conservation law. The complete relationship of the theory of geometrical necessary dislocations and the constrained couple stress theory of Toupin and others was established in Fleck and Hutchinson (1993; also 1997). They go one step further by incorporating their approach in a Mindlin type of gradient elasticity-plasticity by introducing the second gradient of

the displacement $\eta_{ijk} = u_{k,ij} = \eta_{jik}$ and the energy in terms of the usual strain and of this gradient by the energy variation

$$\delta W = \sigma_{ij}\delta e_{ij} + \tau_{ijk}\delta\eta_{ijk}, \tag{8.29}$$

from which there follow the constitutive equations

$$\sigma_{ij} = \frac{\partial W}{\partial e_{ij}},\ \tau_{ijk} = \frac{\partial W}{\partial \eta_{ijk}}, \tag{8.30}$$

where σ_{ij} is the second-order symmetric stress and τ_{ijk} is a third-order stress (or hyperstress). Note that the notion of applied couples and double forces will necessary appear at boundaries in addition to usual traction forces. For an *incompressible* displacement field (a current hypothesis in elasto-plasticity), only the "deviatoric" parts σ'_{ij} and τ'_{ijk} are retained. The associated space of relevant generalized stresses of vector $\Sigma = (\underline{\sigma'}, \underline{\tau'})$ is therefore a five + eighteen = 23-dimensional space, so is the space of dual strains $E = \left(\mathbf{e}', \underline{\eta'}\right)$. This allows for the formulation of a generalized flow rule in the Σ-space and a variational inequality formulation in the Drucker-Hill form (cf. Fleck and Hutchinson 1997), a normality law:

$$\left(\sigma_{ij} - \sigma^*_{ij}\right)\dot{e}^p_{ij} + \left(\tau_{ijk} - \tau^*_{ijk}\right)\dot{\eta}^p_{ijk} \geq 0, \tag{8.31}$$

where the stress state $(\underline{\sigma}, \underline{\tau})$ is associated with the plastic strain rate $\left(\dot{\mathbf{e}}^p, \underline{\dot{\eta}^p}\right)$ and $(\underline{\sigma}^*\underline{\tau}^*)$ is any other stress state on or inside the yield surface. Equation (8.31) is a direct generalization of the now classical variational inequation of elasto-plasticity theory sometimes known as the Hill–Mandel maximal-dissipation principle (cf. Maugin 1992, p. 55). Because of the presence of the third-order stress in the yield criterion, a length scale is involved and is characteristic of the strain-gradient theory.

A finite-strain theory has been expanded with a generalization of (8.27) in the self-explaining form $\underline{\alpha} \approx \nabla_R \times \mathbf{F}^p$, where $\mathbf{F}^p$ is the plastic part in the multiplication decomposition of the deformation gradient $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$, and ∇_R denotes a referential gradient. The corresponding theory rapidly acquires a high degree of complexity with which we shall not deal.

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Cross references: Couple stress, Couple stress (in medium with constrained rotation), Dislocations and disclinations, Double force, Gradient elasticity, Le Roux elasticity.

Granular Materials as Generalized Continua

Granular materials illustrated by coal, sand, rice, grinded coffee, sugar powder and other kinds of powders are viewed as conglomerations of discrete solid particles where the latter interact by friction. They can flow at more or less large speeds so that globally they present fluid properties with typical internal friction and are thus of fundamental rheological interest. The typical smaller size of grains is about one micrometer. Although they do not constitute a single phase of matter, they lend themselves to a continuum description with appropriate working hypotheses—the only description envisaged in this entry. They are ubiquitous in nature, agriculture, food processing and pharmaceutical industry. Historically, fruitful research on

granular materials can be traced back to the remarkable work of Coulomb (1781) on friction law that was indeed stated for such materials, and followed original works by Amontons (1699). Basing on his meticulous observation of natural processes (sand, dunes), Bagnold (1941) produced a remarkable work that pioneered the physics of granular matter. Powders are characterized by the smallness of the particle size and the corresponding property of being more cohesive. More on the basic physics of granular material is to be found in books such as Duran (1999) and Mester (2009) and journals devoted to the physics of grains and powders. Only the continuum viewpoint is approached here although we note that stress in a granular solid is not distributed uniformly; it is transmitted/conducted along a chain of forces. What we retain as requirements in a continuous approach is that the resulting continuum must be able to flow, to account for a time evolution of its internal structure, and to exhibit a Coulomb friction like behaviour in statics.

From the viewpoint of the present entry, the pioneering works are those of Goodman (1969), Goodman and Cowin (1972), Jenkins (1975), and Cowing and Goodman (1976). In these works a primary role is played by an independent kinematic variable called the *volume distribution function* noted v and such that at any current point $\mathbf{x}$ in the continuum

$$0 \leq v(\mathbf{x}, t) \leq 1. \tag{8.32}$$

The distribution mass density noted γ is such that the classical mass density function or bulk density ρ is given by

$$\rho = \gamma v. \tag{8.33}$$

Thus γ corresponds to the mass density of the granules themselves while v represents the granular volume distribution. This allows accounting for the concept of *dilatancy* as introduced by O. Reynolds in 1885. The idea of generalized continuum is directly related to the introduction of v considered as an additional independent kinematic variable. Accordingly, the following global balance relations can be set forth:

- Balance of energy

$$\frac{d}{dt}\int_V \rho\left(e + \frac{1}{2}v_i v_i + \frac{1}{2}k\dot{v}\dot{v}\right)dV = \int_{\partial V} n_j\left(t_{ji} + h_j\dot{v} - q_j\right)dS + \int_V \rho(f_i v_i + l\dot{v} + h)dV; \tag{8.34}$$

- Entropy inequality

$$\frac{d}{dt}\int_V \rho\eta\, dV \geq \int_V \rho\frac{h}{\theta}dV - \int_{\partial V} s_j n_j dS; \tag{8.35}$$

- Balance of equilibrated force

$$\frac{d}{dt}\int_V \rho k\dot{v}\, dV = \int_{\partial V} h_j n_j dS + \int_V \rho(l+g)dV; \tag{8.36}$$

- Balance of equilibrated inertia

$$\frac{d}{dt}\int_V \rho k\, dV = 0. \tag{8.37}$$

Here, $e, \eta, t_{ji}, q_j, f_i, h$ and θ have their usual meaning (cf. Chap. 1 in Part One). The newly introduced quantities are the equilibrated inertia k, the equilibrated stress vector h_i, the external equilibrated body force l, and the intrinsic equilibrated body force g. The power terms associated with v relate to the fact that this variable is a priori independent from the standard motion. They look very much like terms introduced in the Ericksen-Leslie theory of liquid crystals or in the theory of continua with voids, or the generalized continuum theories proposed by Mindlin, Green and Rivlin and others in the 1960s. Equation (8.37) is posited in this simple form in order to complete the theory. In addition, the influx of entropy is left some freedom by writing it as

$$s_j = \theta^{-1} q_j + k_j, \tag{8.38}$$

where k_j are the components of the so-called *extra entropy flux* (see Entry: Extra-entropy flux). By arguments of continuity and invariance (Green and Rivlin), the following local balance equations are deduced from Eq. (8.34):

$$\dot{\rho} + \rho \nabla . \mathbf{v} = 0, \tag{8.39}$$

$$\rho \dot{v}_i = t_{ji,j} + \rho f_i, \tag{8.40}$$

$$t_{ji} = t_{ij}, \tag{8.41}$$

in a very standard form for a one-phase continuum—and then (8.34)–(8.37) yield the following field expressions after some manipulation (cf. Goodman and Cowin 1972, p. 255; but with our notation)

$$\dot{k} = 0, \tag{8.42}$$

$$\rho k\ddot{v} = h_{i,i} + \rho(l+g), \tag{8.43}$$

$$\rho\dot{e} + p_{r(i)} = \rho h - q_{i,i}, \tag{8.44}$$

$$\rho\dot{\eta} \geq \rho h\theta^{-1} - s_{j,j}, \tag{8.45}$$

where we defined the power of *internal forces* per unit volume by

$$p_{(i)} = -\left(t_{ji}D_{ij} - \rho g\dot{v} + h_i(\dot{v})_i\right). \tag{8.46}$$

On introducing the free energy density $\psi = e - \eta\theta$ and accounting for (8.38) one is led to the following form of the Clausius-Duhem inequality for the present theory:

$$-\rho\left(\dot{\psi} + \eta\dot{\theta}\right) - p_{(i)} + (\theta k_i)_{,i} - s_j\theta_{,j} \geq 0. \tag{8.47}$$

This is practically in the classical form save for the more complex expression of $p_{(i)}$ and the presence of the divergence term. The inequality (8.47) is exploited as a constraint on the formulation of thermodynamically admissible constitutive equations. The critical point here is the possible dependence on the additional variables

$$v, \dot{v}, v_{,i}, \gamma, v_0, \tag{8.48}$$

where v_0 is the reference volume distribution. It is shown that

$$\psi = \psi\left(v_0, v, v_{,i}, \gamma, \theta\right) \tag{8.49}$$

only, with the constraint due to rotational invariance

$$\frac{\partial\psi}{\partial v_{,[i}} v_{,j]} = 0. \tag{8.50}$$

Two kinds of pressure can be defined thus:

$$p = \gamma^2 v\frac{\partial\psi}{\partial\gamma}, \ \hat{p} = \gamma v^2\frac{\partial\psi}{\partial v}. \tag{8.51}$$

Furthermore, vector k_i cannot depend on v_0, γ, D_{ij}. But it can also be taken nil in the isotropic case, so that vector h_i takes the simplified form

$$h_i = \rho\frac{\partial\psi}{\partial v_{,i}} = 2\alpha v_{,i}, \tag{8.52}$$

and is therefore known if ψ is known. In these conditions the residual dissipation inequality reads

$$\Phi = \left(t_{ji} + p\delta_{ji} + 2\alpha v_{,j} v_{,i}\right) D_{ij} - \left(\rho g - \frac{p - \hat{p}}{v}\right)\dot{v} - s_i \theta_{,i} \geq 0; \tag{8.53}$$

It governs viscosity, heat conduction, and the relaxation of v. We shall not elaborate further along this line. But it is shown by Goodman and Cowin (1972, p. 261) that the resulting equilibrium equations in the linear theory imbeds the Mohr-Coulomb theory of limiting equilibrium in agreement with Sokolovskii (1965). We refer to Goodman and Cowin (1972) for the case of granular materials with incompressible granules and to Cowin and Goodman (1976) for a variational formulation.

While we pointed out the similarity of the above sketched out theory with other theories of generalized continua, it was unavoidable that granular materials be approached by means of a Cosserat (or micropolar) continuum (cf. Kanatani 1979; Ahmadi 1982, Sadovskaya and Sadovskii 2005) or a second-gradient theory (cf. Yang and Misra 2012). A discrete scheme accounting for possible rotations of granules, interactions and couple-stresses and its continuum limit are also constructed by Pavlov et al. (2006) but in 2D.

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Cross-references: Cosserat continua, Dilatational elasticity, Extra-entropy flux, Internal degrees of freedom, Materials with voids, Micropolar continua, Porous media seen in GCM.

Green A.E

One of the world masters in continuum mechanics in the second half of the twentieth century, Albert E. Green (1912–1999), although educated in Cambridge, spent most of his active career at Oxford. He developed fruitful co-operations with Ronald S. Rivlin and Paul M. Naghdi. Apart from seminal contributions to linear and nonlinear elasticity, he expanded influential works in generalized continuum mechanics such as the *multipolar* theory, thermo-elasticity, and the theory of shells and rods. Some of his numerous works are listed below. Green provided some biographic information in Green (1974).

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Cross references: Dipolar stress field, Directors theory, Generalized continuum mechanics, Multipolar theory, Naghdi P.M., Oriented media (with directors), Rivlin R.S.

Chapter 9
H–I: From "Higher-Order Gradient Theories" to "Ionic Crystals (Elasticity of)"

Higher-Order Gradient Theories

The idea of a gradient theory of the n-th order seems to be a natural one in general field theory (cf. Rzewuski 1964). This consists in considering a priori a theory that involves a potential W that depends on a field $\phi(\mathbf{x}, t)$ through its successive gradients up to order n, i.e.,

$$W = \overline{W}\left(\phi, \phi_{,i}, \phi_{,ij}, \phi_{,ijk}, \ldots\right). \tag{9.1}$$

Most of the field theories (in mechanics, electromagnetism, etc) proposed since the early nineteenth century are of this type. In continuum mechanics, the basic field is that of (vector) displacement between a reference configuration and an actual configuration, whence (9.1) in principle translates to

$$W = \overline{W}\left(u_i, u_{i,j}, u_{i,jk}, \ldots\right). \tag{9.2}$$

Classical continuum mechanics is a theory of the *first-gradient* of the elastic displacement, so that (9.2) formally reduces to

$$W = \overline{W}\left(u_i, u_{i,j}\right) \tag{9.3}$$

only (but a direct dependency on displacement u_i is excluded by Galilean invariance).

Perhaps that a more illustrative viewpoint is given by the consideration of a power akin to the notion of principle of virtual power, where one writes the power expanded per unit volume of deformable matter (as introduced in Maugin 1980)

$$P(B) = \int_B \left(f_i v_i + \sigma_{ji} v_{i,j} + m_{kji} v_{i,jk} + \cdots\right) dv, \tag{9.4}$$

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where $v_i = \dot{u}_i$ is the velocity field, and f_i, σ_{ij}, and m_{ijk} are tensor coefficients of various orders. They may rightly be called a body *force*, a (nonsymmetric) *stress*, and a *hyperstress* (symmetric in its first two indices). Such a representation may probably be traced back to nineteenth-century authors [e.g., Piola (2014), Barré de Saint-Venant (1869) or Duhem (1893)] although not so explicitly. The first twentieth-century author to have toyed with the idea in elasticity seems to be Le Roux (1911). As a matter of fact, the introduction of forces and their higher-order tensor generalizations as *cofactors* in an expansion of the type (9.4)—usually limited to the first gradient—became a routine among theoreticians (e.g., Kirchhoff, Poincaré, Hilbert, Hellinger) of the late nineteenth and early twentieth century, at least on the European continent, yielding a direct introduction of the notion of stress without exploiting Cauchy's argument.

Application to a realistic and solution-wise useful formulation of a modelling such as (9.4) in contemporary *nonclassical* continuum mechanics, requires specific attention concerning three particular points. First, while mathematical physicists more than often consider a volume integral in (9.4) extending to infinity with all fields and their derivatives going to zero at infinite distances, mechanical engineers have to deal with bodies B that are of finite extension, and thus have to consider the application of data (forces) on the corresponding more or less regular boundaries. This means that (9.4) has to be postulated in conjunction with other integrals over manifolds of smaller dimensions. In turn, the corresponding evaluation of the global result will involve exploiting "divergence like" theorems (e.g., Green, Stokes, etc, theorems). This is formally exemplified in nD in a work by Dell'isola et al. (2012). But keeping with the physics of three-dimensional bodies is wise. In turn (second point), this means that considering at most a *second-gradient theory* with respect to the displacement is more than sufficient. Finally, in applying a formalism such as in (9.4) in modern continuum mechanics, one may distinguish between forces that are required to satisfy different invariances. This is particularly emphasized in Maugin (1980) where one must distinguish between de facto external forces (such as f_i—the dual of the non-objective field v_i) and true internal forces for which one must construct constitutive equations that must be objective (i.e., invariant by time-dependent rotation in the actual configuration). That is, for true internal forces, the expression (9.4) must reduce to a linear functional on a set of *objective* generalized velocities; for a second-gradient—and up—theory, we would have

$$P_{\text{int}}(B) = \int_B (\sigma_{(ji)} D_{ij} + m_{kji} v_{i,jk} + \cdots) dv, \tag{9.5}$$

as only the symmetric part $D_{ij} = v_{(i,j)}$, the rate of strain, is objective.

The best known second-gradient theory of continua is the *strain-gradient theory of elasticity* proposed by Mindlin and co-workers in the 1960s. But the notion of

gradient theory can be extended to many continuum theories, whether mechanical or not, where gradients of the original fields are introduced for a more accurate continuum description (e.g., in electromagnetism, micromagnetism, theory of polarization gradient, etc). Transition from the discrete to the continuum is examined by Triantafyllidis and Bardenhagen (1993) in 1D.

To conclude, the consideration of higher-order gradients is often referred to as considering a *weakly nonlocal* theory of continua as the chosen gradient order introduces length scales that characterize a rather limited spatial extension of interactions between material points. This is well illustrated in lattice dynamics and can be compared to truly nonlocal theories (as first discussed in Maugin 1979).

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Cross references: Capillarity, Density-gradient fluids, Generalized continuum mechanics, Generalized internal forces (see hyperstresses), Gradient elasticity, Gradient plasticity, Hyperstresses (notion of), Implicit gradient elasticity models, Lattice dynamics, Le Roux elasticity, Micromagnetism in elastic solids, Mindlin R. D., Non-locality (weak), Non-locality (strong), Polarization gradient.

Homogenization

In a general manner homogenization consists in replacing a heterogeneous material by a somewhat *equivalent* homogeneous material. To perform such an operation one must identify a representative volume element (RVE) that is most characteristic of the structure of the material at a small scale where some properties and fields vary rapidly. In most real cases the structure is not well ordered. But, remarkably enough, assuming a rather (in most cases unrealistic—unless man-made) ideal situation with a regular, e.g., periodic, pattern, one can conceive of an easily implemented scheme that is both mathematically sound and practically very efficient as proved by many cases—e.g., in porous media. The RVE then is a basic cell of this convenient periodic arrangement while in the averaging technique the RVE should contain enough statistical information about the heterogeneous medium in order to be truly representative. Anyway, we are facing two spatial length scales, those of the RVE (l) and of the macroscopic scale (L), typically in the ratio $\varepsilon = l/L$, where ε is an infinitesimally small quantity. In physical Euclidean space the macroscopic domain Ω of interest is considered periodic and the rescaled unit cell is $Y = (0,1)^3$ and the two coordinates ($\mathbf{x}$ macroscopically and $\mathbf{y}$—fast variable—in the cell) are given by

$$\mathbf{x} \in \Omega, \quad \mathbf{y} = \frac{\mathbf{x}}{\varepsilon} \in Y. \tag{9.6}$$

The result of this parametrization is that the problem now is embedded in a sequence of similar problems parametrized by a scaling ε with expressions of the type (asymptotic formal expansion, so-called Ansatz)

$$\mathbf{v}_\varepsilon(\mathbf{x}) = \mathbf{v}(\mathbf{x},\mathbf{y}) = \mathbf{v}_0(\mathbf{x},\mathbf{y}) + \varepsilon \mathbf{v}_1(\mathbf{x},\mathbf{y}) + \varepsilon^2 \mathbf{v}_2(\mathbf{x},\mathbf{y}) + O\left(\varepsilon^3\right) \tag{9.7}$$

for all fields. Then homogenization amounts to performing an asymptotic analysis when ε tends to zero. The limit is the solution of the homogenized problem. This mathematical technique is called *asymptotic periodic homogenization* (APH). It was essentially developed in France, Russia and Italy by authors such as Sanchez-Palencia (1980), Bensoussan et al. (1978), and Bakhvalov and Panasenko (1989), within the framework of appropriate functional spaces.

For a typical exemplary diffusion problem—akin to elasticity—with rapidly oscillating coefficients, we consider the partial differential equation:

$$\nabla . \left(\mathbf{A}\left(\frac{\mathbf{x}}{\varepsilon}\right) . \nabla \mathbf{v}_\varepsilon \right) = \mathbf{f}, \tag{9.8}$$

where vector-valued function $\mathbf{v}$ stands for the elastic displacement. This is to be replaced by the homogenized equation

$$\nabla.\left(\mathbf{A}^{\text{hom}}.\nabla\mathbf{v}\right) = \mathbf{f}, \tag{9.9}$$

where $\mathbf{A}^{\text{hom}}$ is the homogenized (constant in space) material tensor coefficient.

To arrive at this result and the expression of $\mathbf{A}^{\text{hom}}$, one has to consider the formal expansion (9.7) and substitute from it in Eq. (9.8) yielding thus a hierarchy of problems. The effective coefficients $\mathbf{A}^{\text{hom}}$ are determined by solving a set of boundary-value problems over the unit cell for the function $\mathbf{v}_1(\mathbf{x}, \mathbf{y})$. It is noted that only the first two terms in (9.7) are justified; to go further one would have to account for boundary-layer terms.

This homogenization technique is extremely valuable in the case of elastic composites. But it also applies to that of porous media (see Entry "Porous media"). In many cases the obtained homogenized material behaviour is of the same type as the components of the original periodic material. Much more fruitful from the viewpoint of this book is the case where this homogenization technique delivers a more complex behaviour that falls within the theory of non-classical continuum mechanics (e.g., Cosserat continua). This was achieved by Forest and Sab (1998) and is exposed in Forest (2006; in particular Figure 5.1) by polynomial expansion of the local displacement within the appropriately chosen RVE with specific data over these cells.

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Cross references: Porous media (as seen in GCM).

Hyperstress (Notion of)

"Hyperstress" is the name usually given to internal forces of a higher-order tensor order than the usual stresses. Thus they are the thermodynamic dual of a gradient of displacement of order greater than one (in the expression of a work), or of a gradient of order greater than one of the velocity (in the expression of a power)—see Entry: "Higher-order gradient theories". A hyperstress of order n is the internal response to a surface applied "force" of order $n - 1$. In the second-gradient theory (of the displacement field), the surface applied force is called a "*double force*" surface traction—a notion issued from the theory of plates and shells—represented

by the data of the components of a second-order tensor. The corresponding Cauchy relation for the hyperstress at a surface is a complicated affair that necessarily involves not only the unit normal but also its local variation and the curvature of the surface at the surface point and whether the surface is regular at that point (for an example, see Entry: "Gradient elasticity").

Cross references: Double force, Gradient elasticity, Higher-order gradient theories

Implicit Gradient Elasticity Models

These are formal models of gradient elasticity where—outside any energy considerations, there is posited an implicit constitutive relation of the form

$$f\left(\underline{\sigma}, \mathbf{e}, \nabla^2\mathbf{e}, \nabla^2\underline{\sigma}\right) = 0, \tag{9.10}$$

between the stress $\underline{\sigma}$ and the strain $\mathbf{e}$. The idea seems to go back to Morgan (1966), and more recently Rajagopal and Srivinisa (2009). The function f is taken as a general linear isotropic function of its arguments. An example of relation (9.10) is given by (Aifantis 2010)

$$\left(1 - l_1^2\nabla^2\right)\underline{\sigma} = \mathbf{C}\left(1 - l_2^2\nabla^2\right)\mathbf{e}, \tag{9.11}$$

where l_1 and l_2 are two characteristic lengths and $\mathbf{C}$ is the usual fourth-order tensor of elasticity stiffnesses. Model (9.11) is *not* hyperelastic.

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Cross reference: Gradient elasticity.

Internal Degrees of Freedom

Here we understand by internal degrees of freedom the new observable and controllable descriptors (independent fields) that are needed to complement the classical motion in order to obtain a sufficiently accurate description of the

deformation and kinematics of the medium. These new fields may be scalars, vectors, or tensors of appropriate orders. In contrast to internal variables of thermodynamic states (cf. that Entry), it is important to note that these quantities are not only measurable and observable, but also controllable by means of corresponding applied "forces" in volume and at surfaces. They exist whether the response of the system is thermodynamically reversible or not. Another essential property is to have associated with them inertia, so that they play an important role in dynamics. We illustrate these properties in the spirit of analytical mechanics by considering a formulation of the type of the principle of virtual power (PVP). This principle states for a body of volume B bounded by a regular surface ∂B that "the virtual power of inertia forces is balanced by the power expanded by internal, volume and surface forces". Let $q_\alpha,\ \alpha = 1, 2, \ldots, n$, denotes the set of these internal degrees of freedom and $\dot{q}_\alpha$ the corresponding rates (generalized velocities). Then the PVP can be stated in the form (cf. Maugin 1980):

$$P_{inertia}(B) = P_{internal}(B) + P_{vol}(B) + P_{surf}(\partial B), \tag{9.12}$$

where [compare to Eq. (1.16) through (1.20) in the chapter on prerequisites]

$$P_{inertia}(B) = \int_B \rho \left(\dot{\mathbf{v}}.\mathbf{v} + \sum_\alpha I_\alpha \ddot{q}_\alpha \dot{q}_\alpha \right) dV, \tag{9.13}$$

$$P_{internal}(B) = -\int_B \left(t_{ji} v_{i,j} + \sum_\alpha \left(g_\alpha \dot{q}_\alpha + G_{\alpha j} \dot{q}_{\alpha,j} \right) \right) dV, \tag{9.14}$$

$$P_{vol}(B) = \int_B \rho \left(\mathbf{f}.\mathbf{v} + \sum_\alpha f_\alpha \dot{q}_\alpha \right) dV, \tag{9.15}$$

and

$$P_{surf}(\partial B) = \int_{\partial B} \left(\mathbf{T}^d.\mathbf{v} + \sum_\alpha T^d_\alpha \dot{q}_\alpha \right) dS. \tag{9.16}$$

Expression (9.14) was written without further ado, but in practice it must be reduced to a linear continuous functional over a set of *objective* fields so that the introduced co-factors are indeed *internal forces* that are also objective and in need of constitutive equations. The way to construct this set from the set $\left\{ v_{i,j}, \dot{q}_\alpha, \dot{q}_{\alpha,j} \right\}$ is indicated in Maugin (1980). The result depends on the tensor character of the newly introduced degrees of freedom. The fields f_α and T^d_α are the bulk and surface applied "forces" that control these degrees of freedom in the same way as $\mathbf{f}$ and $\mathbf{T}^d$ represents external body force and applied surface traction.

The general framework (9.12)–(9.16) applies to a variety of modelling in non-classical continuum mechanics [e.g., anisotropic fluids, asymmetric elasticity, Cosserat continua, dipolar materials, ferroelectric crystals, micromagnetism, liquid crystals, micromorphic continua, materials with voids, micropolar continua, oriented media (with directors), etc]. It was limited here to a so-called first-order gradient theory both for the classical motion and the internal degrees of freedom. But it is readily generalized to higher-order gradient theory to the price, however, of a possible difficult interpretation with boundary conditions. It is not difficult to complement statement (9.12) with global statements of the first and second laws of thermodynamics to reach a complete thermo-mechanical theory that admits both reversible and irreversible phenomena and couples with the temperature field. An alternate formulation to (9.12) is the statement of global balance laws for the classical motion and the additional internal degrees of freedom. But the last equation may have a debated physical significance (e.g., the balance for the director field in Ericksen's theory of anisotropic fluids), a problem that is eschewed with a formulation (9.12) in the line of field theory.

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Maugin G.A. (1980). The principle of virtual power: Application to coupled fields, Acta Mechanica, 35: 1-70.

Cross references: Anisotropic fluids, Cosserat continua, Dipolar materials, Directors theory, Double force, Ferroelectric crystals, Generalized continuum mechanics, Higher-order gradient theories, Liquid crystals as continua, Materials with voids, Micromagnetism in elastic solids, Micromoprhic continua, Micropolar continua, Multipolar continua, Oriented media (wit directors).

Internal Variables of State

Many careless authors assimilate internal variables of thermodynamic state—for short *internal variables*, but sometimes also *called hidden variables*—and internal degrees of freedom (see above for the definition of the latter). This, from our viewpoint, is an unforgivable mistake. The idea of *internal variables* is probably traced back to Duhem (1911; according to Truesdell), but probably more to Bridgman (1943). A modern introduction is due to Coleman and Gurtin (1967). The best modern analyst and advocate of this thermodynamics with internal variables has been Joseph Kestin (see, e.g., Bataille and Kestin 1979). Maugin and Muschik (1994) have specified and analysed many of its facets—see also Maugin (1999). Its general features can be presented as follows.

Internal variables of state are introduced in addition to the usual observable variables of state (e.g., deformation, temperature). They are supposed to account for the complex internal microscopic processes that occur in the material and manifest themselves at a macroscopic scale in the form of *dissipation*. They are of pure

thermodynamic dissipative nature. Accordingly, being internal and not observable—although certainly identifiable by a gifted physicist—they do not appear in the usual statement of the first law of thermodynamics as they are not directly acted upon by bulk or surface actions. But they do evolve under the action of external loads as a result of complex processes that follow from a re-distribution of internal forces (internal rearrangements of matter, etc). For instance, the local density of dislocations (responsible for the macroscopic phenomenon of plasticity) evolves when a system of standard forces (tractions) is applied to a body. In compliance with this, the basic formulation of a theory of continua involving internal variables offers at first no "visual" difference with a classical theory. In particular, in the classical theory of the first gradient as sketched out in the Prerequisite Chap. 1, the basic statements are those of the principle of virtual power and the first and second laws of thermodynamics. That is, for a body B of regular boundary ∂B,

$$\int_B \rho \dot{\mathbf{v}}.\mathbf{v}\, dV = -\int_B t_{ji} v_{i,j}\, dV + \int_B \rho \mathbf{f}.\mathbf{v}\, dV + \int_{\partial B} \mathbf{T}^d.\mathbf{v}\, dS, \tag{9.17}$$

$$\frac{d}{dt}\int_B \rho\left(\frac{1}{2}\mathbf{v}^2 + e\right) dV = \int_B \rho(\mathbf{f}.\mathbf{v} + h) dV + \int_{\partial B} \left(\mathbf{T}^d.\mathbf{v} - \mathbf{q}.\mathbf{n}\right) dS, \tag{9.18}$$

$$\frac{d}{dt}\int_B \rho\eta\, dV \geq \int_B \rho s\, dV - \int_{\partial B} \mathbf{s}.\mathbf{n}\, dS, \tag{9.19}$$

where e is the internal energy per unit mass, η is the entropy per unit mass, h and s are the heat and entropy supplies per unit mass, while $\mathbf{q}$ and $\mathbf{s}$ are the influxes of heat and entropy through the surface of unit outward normal $\mathbf{n}$. A theory of internal variables that is essentially a thermodynamic theory is meaningless without the statement (9.19), as nothing distinguishes this theory from a classical continuum theory with valid statements (9.17) and (9.18). It is the inequality (9.19) that will govern the irreversible thermodynamics related to the presence of the internal variables. To progress further one needs to recall that the theory deviates the least from the classical thermo-mechanical theory by assuming that s and $\mathbf{s}$ are related to h and $\mathbf{q}$ by the (thermo-statics) relations

$$s = \frac{h}{\theta}, \quad \mathbf{s} = \frac{\mathbf{q}}{\theta}, \tag{9.20}$$

where θ is the thermodynamic temperature ($\theta > 0$, $\inf \theta = 0$). These relations in principle imply a small deviation from thermodynamic equilibrium, but this validity is stretched outside equilibrium without much worry (this follows the hypotheses of Coleman-Noll rational thermodynamics). Then one introduces the free (Helmholtz) energy density ψ by $\psi = e - \eta\theta$ and shows that the combination of the localized

form of Eq. (9.17) through (9.19) yields the now classical Clausius-Duhem inequality in the form

$$-\rho\left(\dot{\psi}+\eta\dot{\theta}\right)+t_{ji}D_{ij}-\mathbf{s}.\nabla\theta\geq 0. \tag{9.21}$$

The true original formulation of the theory comes with a specification of the dependence of functions such as e and ψ, e.g., $\psi(\chi,\alpha)$ where χ stands for the usual observable variables (deformation, temperature) and α for the ordered set of internal variables. We refer the reader to Rice (1971) and Maugin (1992) for the application to the theories of elasto-plasticity, viscoplasticity and damage, and Maugin and Drouot (1983) for the application to solutions of macromolecules, and in a more general physical framework to Maugin (1999). What are deduced from the thermodynamic reasoning applied to (9.21) are pure *evolution equations* (of a more or less regular form) for the internal variables (e.g., plastic strain, damage variable, conformation of macromolecules, etc). This is *not* non-classical continuum mechanics per se according to our definition. However, it is observed that some internal variables of state (e.g., concentration of some constituents) can be of a *diffusive* nature. This implies—in the line of Valanis (1996)—the intervening of the spatial gradient of the said variables so that we need to consider the following caveat. That is, we can consider a free energy of a more general form $\psi(\chi,\alpha,\nabla\alpha)$. The above scheme will be left unchanged but for a generalization of the second of (9.20) to the form

$$\mathbf{s}=\frac{\mathbf{q}}{\theta}+\mathbf{k}, \tag{9.22}$$

where $\mathbf{k}$ may be called the *extra entropy flux* (a notion dealt with by Müller 1973). Then (9.21) is shown to be replaced by the more general inequality

$$-\rho\left(\dot{\psi}+\eta\dot{\theta}\right)+t_{ji}D_{ij}+\nabla.(\theta\,\mathbf{k})-\mathbf{s}.\nabla\theta\geq 0. \tag{9.23}$$

Assuming that $\psi=\psi(\mathbf{F},\theta,\alpha,\nabla\alpha)$, where $\mathbf{F}$ is the deformation gradient, this equation yields the remaining *dissipation inequality*

$$t_{ji}^{dis}D_{ij}+A\,\dot{\alpha}-\mathbf{s}.\nabla\theta\geq 0, \tag{9.24}$$

where we have set

$$\eta=-\frac{\partial\psi}{\partial\theta},t_{ji}^{dis}=t_{ji}-t_{ji}^{elas},t_{ji}^{elas}=\rho\frac{\partial\psi}{\partial F_{(iK}}F_{j)K} \tag{9.25}$$

and ($\delta/\delta\alpha$ denotes a variational Euler-Lagrange derivative)

$$A = -\rho \frac{\delta \psi}{\delta \alpha} = -\rho \left(\frac{\partial \psi}{\partial \alpha} - \nabla . \frac{\partial \psi}{\partial (\nabla \alpha)} \right), \quad \mathbf{k} = -\frac{\rho}{\theta} \frac{\partial \psi}{\partial (\nabla \alpha)} \dot{\alpha}. \tag{9.26}$$

The usual thermodynamic reasoning on the new dissipation inequality (9.24) will yield an *evolution-diffusion equation* for the internal variable α (see Maugin 1990) if $\dot{\alpha}$ is linear in the conjugated thermodynamic force A. This is very much like a Landau-Ginzburg equation for the α field. Note that there is no change in the formulation (9.17) of the principle of virtual power contrary to what was advanced by some authors (e.g., Frémond and Nedjar 1993) and the thermodynamic dual of $\nabla\theta$ in the remaining dissipation inequality still is the entropy flux vector $\mathbf{s}$. Also, the last of Eq. (9.26) is not so original as such an extra entropy flux appears in the theory of fluid mixtures with α identified as a concentration (cf. De Groot and Mazur 1962).

An additional important difference with the theory of additional internal degrees of freedom is that where the latter can permit the exhibition of solitary wave and solitons because of the possible simultaneous presence and competition of nonlinearity and dispersion, the theory with gradients of the internal variables will yield *dissipative structures* because of the possible simultaneous presence and competition of nonlinearity and dissipation.

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Cross references: Extended thermodynamics, Internal degrees of freedom, Solutions of macromolecules.

Interstitial Working

"Interstitial working" is the name given by Dunn (1986)—also Dunn and Serrin (1985)—to an *additional energy flux* of which the existence must be assumed at a surface in order that the considered body enclosed in this surface be able to be a *non-simple* one (in the sense of Noll) or, in other words, be a continuum of higher-order gradient than one (e.g., second-order gradient model depending on the gradient of the deformation gradient $\mathbf{F}$, and higher). A priori, such a model will admit a density of free energy in the form (this is only an example)

$$\psi = \hat{\psi}\left(\mathbf{F}, \theta, \nabla_R \mathbf{F}, \nabla_R^2 \mathbf{F}, \nabla\theta, \dot{\mathbf{F}}\right), \tag{9.27}$$

where ∇ and ∇_R are the usual gradient and referential gradient, respectively, $\dot{\mathbf{F}}$ is the time derivative of $\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X}$, and θ is the thermodynamic temperature ($\theta > 0$; $\inf\theta = 0$). The global balance equations are similar to those of the classical ones for mass, linear momentum, moment of momentum, and the entropy inequality (See Chap. 1 of Prerequisites). The only change is the presence of an additional term $u = u(\mathbf{X}, t; \mathbf{n})$ in the statement of the first law of thermodynamics which therefore reads:

$$\frac{d}{dt}\int_B \rho\left(\frac{1}{2}\mathbf{v}^2 + e\right)dV = \int_B \rho(\mathbf{f}.\mathbf{v} + h)dV + \int_{\partial B}\left(\mathbf{T}^d.\mathbf{v} + u - \mathbf{q}.\mathbf{n}\right)dS, \tag{9.28}$$

assuming regularity of the bounding surface ∂B. Here $\mathbf{T}^d$ is the usual applied surface traction, and $\mathbf{q}$ is the usual influx vector. The main question is to find a plausible expression for u, when (9.27) is assumed, and the constraint of the second law (Clausius-Duhem inequality) is enforced. Introduction of u in (9.28) means that "we are allowing spatial interactions of longer range to engender a rate of supply u of mechanical energy across every material surface in the body" (cf. Dunn and Serrin 1985)—hence the name of "interstitial working". This new quantity is supposed to be objective. But while u appears together with the flux $\mathbf{q}.\mathbf{n}$ in (9.28), it does not a priori occur in the second law where the entropy flux $\mathbf{s}$ still has its classical definition in terms of $\mathbf{q}$ and θ, i.e., $\mathbf{s} = \mathbf{q}/\theta$. A standard application of a

Cauchy-type argument shows that u must be linear in the normal $\mathbf{n}$, so that we can write

$$u(\mathbf{X}, t; \mathbf{n}) = \mathbf{u}(\mathbf{X}, t).\mathbf{n}, \tag{9.29}$$

where $\mathbf{u}$ may be called the *interstitial work flux*; it has to be objective. An alternate formulation would keep only $\mathbf{q}$ in (9.28) but consider an entropy flux in the more general form

$$\mathbf{s} = \frac{\mathbf{q}}{\theta} + \mathbf{k} \tag{9.30}$$

where the *extra entropy flux* $\mathbf{k}$ stands for an influx of entropy due to longer range spatial interactions [compare to the case of gradient of internal variables in the Entry: Internal variables, seen as a weakly non-local thermodynamic property]. Returning to the first formulation with u, the local Clausius-Duhem inequality is obtained in the form:

$$-\rho\left(\dot{\psi} + \eta\dot{\theta}\right) + t_{ji}D_{ij} + \nabla.\mathbf{u} - \mathbf{s}.\nabla\theta \geq 0. \tag{9.31}$$

Exploitation of this yields in particular that

$$\frac{\partial\hat{\psi}}{\partial(\nabla_R^2\mathbf{F})} = \mathbf{0},\ \frac{\partial\hat{\psi}}{\partial\dot{\mathbf{F}}} = \mathbf{0},\ \frac{\partial\hat{\psi}}{\partial(\nabla\theta)} = \mathbf{0}, \tag{9.32}$$

so that

$$\psi = \hat{\psi}(\mathbf{F}, \theta, \nabla_R\mathbf{F}), \tag{9.33}$$

only. The medium is at most a *first-gradient model* of the deformation gradient, while $\mathbf{u}$ is represented by a general expression [cf. Equation (1.18) in Dunn and Serrin (1985)]

$$\mathbf{u} = \mathbf{V}(\mathbf{F}, \theta, \nabla_R\mathbf{F}).\dot{\mathbf{F}} + \underline{\Omega}\left(\mathbf{F}, \theta, \nabla_R\mathbf{F}, \nabla_R^2\mathbf{F}\right).\nabla\theta + \mathbf{w}^E\left(\mathbf{F}, \theta, \nabla_R\mathbf{F}, \nabla_R^2\mathbf{F}\right). \tag{9.34}$$

Here $\mathbf{V}$ is a third-order tensor, $\underline{\Omega}$ is a skew second-order tensor, and $\mathbf{w}^E$ is a vector that depends neither on $\dot{\mathbf{F}}$ nor on $\nabla\theta$. Equation (9.34) tells that three possible mechanisms can produce the interstitial-work flux, the first one being dynamic. The Cartesian components of the stress t_{ji} are easily deduced in terms of the derivatives of the energy (9.33) with respect to the components of $\mathbf{F}$ and its gradient $\nabla_R\mathbf{F}$.

For fluids, it is shown that the free energy ψ reduces to

$$\psi = \overline{\psi}(\rho, \theta, \nabla\rho). \tag{9.35}$$

This is of the *Korteweg* type (see the corresponding Entry). Simultaneously, **u** acquires the form

$$\mathbf{u} = \rho \frac{\partial \overline{\psi}}{\partial(\nabla \rho)} \dot{\rho} + \overline{\mathbf{w}}, \tag{9.36}$$

where the second contribution vanishes if the material possesses a centre of symmetry (e.g., isotropy). The corresponding stress components are given by

$$t_{ji} = \left[-\rho^2 \frac{\partial \overline{\psi}}{\partial \rho} + \rho \left(\rho \frac{\partial \overline{\psi}}{\partial \rho_{,k}} \right)_{,k} \right] \delta_{ji} - \rho \rho_{,j} \frac{\partial \overline{\psi}}{\partial \rho_{,i}}. \tag{9.37}$$

We need not elaborate further on this modelling.

But it is of interest to note that with the occurrence of discontinuity lines (edges) at surfaces, the problem is much more difficult. Dell'Isola and Seppecher (1995), generalizing some argument of Noll (1973) and Noll and Virga (1990) on Cauchy's postulate have then established the relationship between edge contact forces, double forces and interstitial working.

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Cross references: Dipolar materials, Double force, Edge forces, Extra entropy flux, Generalized internal forces, Gradient elasticity, Higher-order gradient theory, Korteweg fluids, Long-range interactions, Non-locality (weak), Surface tension.

Ionic Crystals (Elasticity of)

Electro-magneto-mechanical coupled effects of the *first order* are of various nature and complexity for they necessarily involve appropriate *symmetry properties* of the considered material. Thus, (inverse) *piezoelectricity*, discovered in 1881 by the Curie brothers after their discovery of the "direct" piezoelectric effect (1880;

appearance of electric charges at the boundary of a deformed body), provides a strain that is linear in the applied electric field but only for certain allowed material symmetries of crystals. This requires the absence of a centre of symmetry to allow for a direct linear coupling between a stress (essentially a second-order tensor variable) and an electric field (essentially a polar vector). But it happens that some substances of great interest possess potential electro-mechanical couplings although they exhibit a centre of symmetry. This is the case of so-called *ionic crystals* such as alkali halides (e.g., NaCl, NaI, KCl, LiCl, LiBr,…). Ionic nonpolar solid dielectrics in general contain more than one type of atoms, but no permanent dipoles. Their structure is characterized by a regular three-dimensional alternation of positive and negative ions, and hence the entire crystal has no permanent dipole moment. However, in the presence of an external electric field, the positive ion lattice will suffer a displacement relative too the negative ion lattice, resulting in ionic polarization. What about linear electromechanical couplings in the presence of a centre of symmetry? The idea was put forward by Mindlin (1968) that such couplings may exist in centrosymmetric crystals, even in materials of highest symmetry (e.g., centrosymmetric isotropy) on the condition that the *gradient of electric polarization* be included as a variable in the stored energy of deformation and polarization. This possibility is corroborated by the lattice dynamics of crystals of the ionic type in the so-called long-wave approximation (Mindlin 1972a; Askar 1986; Askar and Lee 1974; Askar et al. 1970). This makes that an electro-mechanical theory of elastic materials with polarization gradient will necessarily look *formally* very much like the theory of elastic ferroelectrics although the microscopic mechanisms at work in the two theories are quite different. Thus for infinitesimal strains and weak electric fields, we can directly state the following local governing equations for the *linear* theory of elastic dielectrics with polarization gradients:

- Balance of linear momentum:

$$t_{ji,j} + \rho_0 f_i = \rho_0 \ddot{u}_i, \tag{9.38}$$

- Balance of angular momentum:

$$t_{ji} = t_{ij}, \tag{9.39}$$

- Balance law for electric polarization:

$$E_i + E_i^L + \widehat{E}_{ji,j} = (d_E/\rho_0)\ddot{P}_i, \tag{9.40}$$

- Maxwell's equations for quasi-electrostatics:

$$\nabla^2 \phi = \nabla . \mathbf{P}, \tag{9.41}$$

at each regular point $\mathbf{x}$ in the body B, along with the following boundary conditions at the regular boundary ∂B of unit outward normal $\mathbf{n}$:

$$n_j t_{ji} = T_i^d, \tag{9.42}$$

$$n_j \widehat{E}_{ji} = 0, \tag{9.43}$$

$$(\partial\phi/\partial n) + n_i P_i = 0. \tag{9.44}$$

In these equations $\mathbf{P}$ is the electric polarization per unit volume, $\mathbf{E}$ is the Maxwellian electric field, ϕ is the electric scalar potential, t_{ji} is a symmetric Cauchy stress, $\mathbf{T}^d$ is an applied traction, $\mathbf{E}^L$ is a local electric field, and $\widehat{E}_{ji}$ are the Cartesian components of a tensor that accounts for the presence of the polarization gradient. Equations (9.41) and (9.44) follow from the reduced Maxwell equations

$$\nabla \times \mathbf{E} = \mathbf{0} \Rightarrow \mathbf{E} = -\nabla\phi; \quad \nabla.\mathbf{D} = 0, \mathbf{D} = \mathbf{E} + \mathbf{P}. \tag{9.45}$$

There is neither ponderomotive force nor ponderomotive couple present in these equations as a consequence of the absence of electric charges (the medium is a dielectric) and of linearization.

Typically, the nondissipative constitutive equations are deduced from an energy W per unit volume

$$W = \overline{W}\big(e_{ij}, P_i, P_{i,j}\big),\ e_{ij} = u_{(i,j)} = \frac{1}{2}\big(u_{i,j} + u_{j,i}\big), \tag{9.46}$$

in the form

$$t_{ji} = \frac{\partial \overline{W}}{\partial e_{ij}},\ E_i^L = -\frac{\partial \overline{W}}{\partial P_i},\ \widehat{E}_{ji} = \rho_0 \frac{\partial \overline{W}}{\partial P_{i,j}}. \tag{9.47}$$

For instance, in the case of an isotropic crystal a (at most) quadratic representation of W will contain a (non-forbidden) term linear in $\nabla.\mathbf{P}$ and coupling terms jointly quadratic in the components of the strain and the polarization gradient. The last terms will contribute to *a stress linear in the polarization gradient*, hence the looked for original electro-mechanical coupling. As to the term linear in $\nabla.\mathbf{P}$, it is of highest interest as it yields the notion of *surface bond energy*.

Indeed, as a matter of general philosophy, the introduction of polarization gradients—a manifestation of a weakly nonlocal theory—yields effects that are typical of all gradient-type theories. For instance, certain solutions in infinite media which presented point singularities in the absence of gradients are now smoothed out with the aid of gradients. Furthermore, where gradients are to be expected to be most important, thus essentially nearby boundaries and strong transition zones (e.g., at shock waves), there will occur boundary layer effects confined to a region of small thickness. With this type of phenomena in mind we need to ponder the existence of

surface bond energy. In effect, it must be realized that in order to separate an ionic crystalline body into two parts along a surface S, a certain bond energy must be overcome. Following Mindlin we can say that this is the energy that would be needed to break the atomic bonds across S if the strain and polarization were prevented from developing, for instance by some externally applied field. The release of the latter field would result in a deformation and a polarization localized in the vicinity of S, with a corresponding surface energy of polarization and deformation. The latter energy is always shown to be negative. In all, the energy required to separate an ionic-crystalline body into two parts along a surface S is the so-called bond energy from which must be subtracted the absolute value of the surface energy of deformation and polarization. The former is not accounted for in usual continuum formulations, but it naturally arises in a lattice-dynamics approach, and is evaluated in the present polarization-gradient theory as originally shown by Mindlin. Indeed, if $b_0\nabla.\mathbf{P}$ is the linear term present in the expansion of W for a centrosymmetric (cubic or isotropic) crystal, then the surface bond energy per unit surface is calculated as

$$w_S = \frac{1}{2} b_0 (\mathbf{n}.\mathbf{P})|_{\partial B},\ b_0 < 0. \tag{9.48}$$

Numerical values of b_0 for different alkali halides have been determined by Askar et al. (1970) This is also true of the other material coefficients that are involved along with the polarization gradient.

Other effects typically accounted for by the theory of polarization gradient are the explanation of the abnormal deviation of the capacitance of thin dielectric films and the acoustical activity of ionic crystals. These are described and documented in the review of Mindlin (1972b) and the book of Maugin (1988, Chap. 7). The influence of structural defects (dislocations, plastic deformation) is studied and reviewed in a book by Nowacki (2006) that synthesizes many of this author's works, including the coupling with thermal effects and the resulting thermo-elasticity. A Hamiltonian-Lagrangian variational formulation was proposed quite early by Suhubi (1969).

Remark on electric quadrupoles

The microscopic electric description considered in this entry views electric macroscopic polarization as a polar vector. Its thermodynamic dual is akin to an electric field $\left(\mathbf{E}^L\right)$. Its gradient has for thermodynamic dual $\widehat{\mathbf{E}}$ (dimensionally, an electric field multiplied by a length). However, another view consists in considering macroscopic electric polarization as made of an electric dipole density $\overline{\mathbf{P}}$, per se, and an electric quadrupole density, $\overline{\mathbf{Q}}$, so that $\mathbf{P} = \overline{\mathbf{P}} - div\,\overline{\mathbf{Q}}$ (and in fact further neglected electric multipoles)—a natural outcome of the Lorentz modelling as

thoroughly studied in Eringen and Maugin (1990, Vol. I)—and then considering $\overline{\mathbf{P}}$ and $\overline{\mathbf{Q}}$ as independent electric independent variables. The thermodynamic dual of $\overline{\mathbf{Q}}$ will then be a gradient of electric field. Such a description, envisaged by the author in the early 1970s, also yields an electric continuum endowed with a microstructure involving couple stresses. For instance, in quasi-electrostatics, we would have the following expressions of the ponderomotive force and couple, (Maxwell) electric stress, and electromagnetic energy source:

$$f_i^{em} = \overline{P}_j E_{i,j} + \overline{Q}_{jp} E_{u,jp},\ c_i^{em} = \varepsilon_{ijk}\left(\overline{P}_j E_k + \overline{Q}_{mj} E_{k,m}\right) \tag{9.49}$$

and

$$t_{ji}^{em} = D_j E_i + \overline{Q}_{jk} E_{i,k} - \frac{1}{2}\mathbf{E}^2 \delta_{ji},\ w^{em} = f_j^{em} v_j + \rho E_i \dot{\bar{\pi}}_i + \rho E_{i,j} d\left(\overline{Q}_{ji}/\rho\right)/dt. \tag{9.50}$$

More on these in the form of problems is to be found in pp. 87–89 in Eringen and Maugin (1990, Vol. I). A formulation by the principle of virtual power is given in Maugin (1980). This modelling has attracted little attention save in few independent works by H. Demiray and B. Collet. But because of the necessary presence of the gradient of the electric field (or the second gradient of the electrostatic potential) in the energy density of this theory, the latter is prone to providing size effects and exotic behaviours similar to those offered by the polarization-gradient model.

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Cross references: Ferroelectric crystal (elasticity of), Polarization gradient, Non-locality (weak).

Chapter 10
K–L: From "Kelvin Continuum" to "Long-Range Interactions"

Kelvin Continuum

Most engineers and scientists in continuum mechanics ignore that research in their field was kindled in the nineteenth century by the search for a model of elastic medium able to transmit light, the so-called "aether". This followed the discovery by Fresnel that light travels with vibrations orthogonal to the direction of propagation, in contrast with usual acoustic waves that are longitudinally polarized. Even the great Cauchy did not eschew this trend in the 1820s, but this led him to his foundational paper in true continuum mechanics. The story is well told by Whittaker (1951). The question was whether one can propose a model of "elasticity" that allows for the propagation of transversely polarized waves. Some works purporting to a solution were those of MacCullagh (1839), and William Thomson (Lord Kelvin) in 1867. The first author devised a model of elastic continuum that resists only to a rotation of the volume element. As to Kelvin, he in fact proposed a model of quasi-rigid aether built from *gyrostats* and this indeed provides a model medium that resists only to deformations caused by rotations. He became obsessed with the idea of an aether built of vortices. This is reported by the Cosserat brothers (Cosserat and Cosserat 1909), considered the initiators of generalized continuum mechanics, as a first example of a medium responding to a density of couples in its bulk. More on these vortices was pursued along this line by Larmor (1900) as a general model of atoms at the dawn of the twentieth century (see also Greco, undated).

The notion of Kelvin medium was revisited by mechanicians in St Petersburg in the 1990s and early 2000s [Gavrilov (1996), Grekova and Zhilin (2001)] under the supervision of P.A. Zhilin. They noticed that neither the rotation about the axis of gyrostats nor the gradient of this rotation should be involved in the energy density, what imposes constraints on the expression formula for the energy density. They also emphasized an analogy with the theory of continua endowed with electronic spin (seen as a minute gyroscope like in micromagnetism; see Maugin and Eringen

G.A. Maugin, *Non-Classical Continuum Mechanics*,
Advanced Structured Materials 51, DOI 10.1007/978-981-10-2434-4_10

1972). But contrary to the original Kelvin approach, the medium considered by these authors shows also some resistance to translational deformations, so that it is closer to a Cosserat/polar elastic medium (see also Grekova and Maugin 2005, for an application to magnetic crystals).

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Cross references: Cosserat continua, Couple stress, Micromagnetism in elastic solids.

Kondo K

Kazuo Kondo (1911–2001) was a Japanese aeronautical engineer who introduced in continuum mechanics notions of Riemannian geometry in order to deal with structural defects that are responsible for plasticity (1952). He was much influenced by his reading of Einstein's gravitation theory and the work of the geometer E. Cartan on non-Riemannian geometries. He extensively developed his original ideas in a series of memoirs (Kondo, 1951–1962) that provide hard reading due to Kondo's specific style and the introduction of many neologisms (See Croll 2006, for more on Kondo).

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Cross references: Continuously defective materials, Differential geometry in nonclassical continuum mechanics, Generalized continuum mechanics, Non-Euclidean geometry of defective materials.

Korteweg Fluids

Diederik Johannes Korteweg (1848–1941) was a Dutch mathematician whose fame nowadays is mostly attached to the Korteweg-de Vries (KdV) equation of nonlinear wave propagation in the form of *solitons*. As a disciple of Johannes Diederik van der Waals (1837–1923), he was much interested in the thermodynamics of phase transitions and criticality in fluid mixtures. In 1901, Korteweg (1901) proposed to replace the usual jump condition at a surface separating homogeneous fluids of different densities in the theory of capillarity (the celebrated Laplace equation) by smooth constitutive equations for stresses arising in response to *density gradients*. This introduced the *Korteweg stress* at an interface between two fluids. In modern format, the free energy density of Korteweg fluid admits the general expression

$$\psi = \overline{\psi}(\rho, \theta, \nabla\rho), \tag{10.1}$$

where θ is the thermodynamic temperature $(\theta > 0, \inf\theta = 0)$ and ρ is the density. The corresponding stress components of the Korteweg stress and entropy density are given by

$$t_{ji} = \left[-\rho^2\frac{\partial\overline{\psi}}{\partial\rho} + \rho\left(\rho\frac{\partial\overline{\psi}}{\partial\rho_{,k}}\right)_{,k}\right]\delta_{ji} - \rho\rho_{,j}\frac{\partial\overline{\psi}}{\partial\rho_{,i}}, \quad \eta = -\frac{\partial\overline{\psi}}{\partial\theta}. \tag{10.2}$$

If $\overline{\psi}$ is simply quadratic in $\nabla\rho$, then the first of these takes on the form

$$t_{ji} = -p\delta_{ji} - C\rho_{,j}\rho_{,i} \tag{10.3}$$

An expression of this type is obtained by Rocard (1967) with a constant factor C in his study of surface tension based on the kinetic theory of gases.

Equation (10.1) belongs in a nonclassical continuum theory of the *second gradient* (cf. Gouin 1988), $\nabla\rho$ for a fluid replacing the gradient $\nabla_R\mathbf{F}$ or $\nabla\nabla\mathbf{u}$ for an elastic solid in finite or small strains (remember that in general $\rho = \rho_0 J^{-1}$ with $J = \det\mathbf{F}$). The theory (10.1)–(10.2) can be constructed in a general framework—that admits the notion of interstitial working—as shown in Dunn and Serrin (1985). Following Korteweg (1901), Casal (1961, 1963, 1972) adopted he viewpoint expressed in (10.1) for his theory of internal capillarity. This line was pursued by Seppecher (1987) in his doctoral thesis—also Gatignol and Seppecher (1986), and Gouin (1988). As shown originally by Korteweg—see also Gouin (1988, p. 677)—the Laplace equation is recovered when the layer where a substantial variation in density takes place is very thin, and the difference in pressure is obtained as in Laplace equation as proportional to the mean curvature and the surface tension is given by an integral through the layer as

$$\gamma = \int_{Layer} C\left(\frac{\partial\rho}{\partial n}\right)^2 dx_3, \tag{10.4}$$

where $\mathbf{n}$ is the direction normal to the surface. Of course, this relates to a surface energy.

Truesdell and Noll (1865, p. 514) note that a most interesting aspect of Korteweg's theory is that it provides a correction to classical hydrostatics for equilibrium figures of compressible fluids. Other works related to density-gradient fluids in the Korteweg manner are by Blinowski (1975, 1979), Cahn and Hilliard (1958, 1959), Aifantis and Serrin (1983), and Casal and Gouin (1985)—see also Sonnet and Virga (2010) for a possible relationship with the theory of liquid crystals. Another possible approach to the modelling of the Korteweg type of media is to exploit the notion of extra-entropy flux as done by Morro (2006) following Maugin (1990).

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Cross references: Capillarity, Density-gradient fluids, Generalized continuum mechanics, Higher-order gradient theories, Interstitial working, Surface tension.

Kröner Ekkehart

The German mathematical physicist Ekkehart Kröner (1919–2000) is one of the most prominent figures in the field of crystal physics, the theory of structural defects, and generalized continuum mechanics in the second part of the twentieth century. After delayed studies due to the Second World War, he became a professor first in Clausthal and then in Stuttgart. He was a very deep and rigorous thinker and

a pioneer in the geometric theory of defective crystals introducing there notions such as the incompatibility tensor and the Einstein tensor (Kröner 1958, 1968, 1981). He paved the way for many future studies in the statistical theory of elastic crystals (Kröner 1972), the theory of homogenization, the nonlocal theory of continua (Kröner 1968; Kröner and Datta 1966), and the birth of a true configurational mechanics. His influence left a print on all developments in nonclassical continuum mechanics in the last fifty years.

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Cross references: Configurational mechanics, Connection and torsion, Continuously defective materials, Differential geometry in nonclassical continuum mechanics, Generalized continuum mechanics, Non-Euclidean geometry of defective materials, Non-locality (strong).

Kunin I.A

Isaak A. Kunin (born 1928) is a Russian/American mathematical scientist for a long time at Novosibirsk (1952–1979) and then at the University of Houston (1979–2003). He is a pioneer (Kunin 1966, 1970) in the study of nonlocal interactions in elasticity, in media with microstructure, and the consequences of these properties in wave propagation. He authored a remarkable monograph on the subject (cf. Kunin 1982/1983) of which the English translation was edited by his friend E. Kröner. More recently, he became interested in the gauge theory of materials with defects.

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Cross references: Continuously defective materials, Differential geometry in nonclassical continuum mechanics, Generalized continuum mechanics, Non-locality (strong).

Lattice Dynamics

The modelling by means of what is nowadays called *lattice dynamics* can be traced back to the notion of *finite difference* considered in the late eighteenth century and in one of Cauchy's original works on elasticity (Cauchy 1828). But the real breakthrough came with the work of Max Born and Theodor von Kármán (1912). The idea is to consider a regularly ordered lattice of mass points interacting between each other by means of forces. The latter can be of the linear spring type in the simplest case, and only nearest neighbours feel these mutual interactions. Generalizations include nonlinear forces of interaction and interactions between more distant neighbours (ideally, going to infinite distances of separation). Simple modelling considers one-dimensional (1D) periodic chains of such "atoms", although 2D and 3D lattices are not forbidden but then requires the consideration of some symmetry in the spatial arrangement of atoms since this is supposed to apply to crystals. The word "atom" is used for convenience to denote lattice mass points but without any modern atomic connotation.

In 1D, if u_n denotes the displacement of atom labelled n from its rest position, the second-order difference d_2 (or three-point difference) for similar elastic restoring between nearest neighbours to the left and the right is given by

$$d_2 = u_{n-1} - 2u_n + u_{n+1}. \tag{10.5}$$

The corresponding force per unit volume a^3—where a is the distance between neighbouring atoms at rest—reads

$$F = \left(\frac{k}{a}\right)\frac{d_2}{a^2} \tag{10.6}$$

with k a spring constant representative of elastic restoring and $\rho = m/a^3$ is the mass density of the corresponding medium with m the mass of individual identical atoms. In the so-called *long-wavelength* (λ) *limit* when each u remains small with respect to $\lambda \gg a$, the second factor in (10.6) becomes exactly the second-order space derivative $\partial^2 u/\partial x^2$ of a displacement u, considered as a continuous function of time *and* of the space coordinate x along the chain axis. One obtains thus a 1D dynamical representation of the linear momentum equation in elasticity with an elasticity coefficient $E = k/a$. This methodology—that does not aim at reproducing exactly a

true microscopic picture—is very convenient in constructing dynamical models of continua and is well documented in now classic books (e.g., Born and Huang 1954; Kosevich 1988; Maradudin et al. 1971).

If one accounts for the next-next neighbour interaction, then one is led to considering the following five-point differences

$$d_4 = u_{n-2} - 4u_{n-1} + 6u_n - 4u_{n+1} + u_{n+2}. \tag{10.7}$$

In passing to the long-wavelength limit this will result in the presence of a fourth-order space derivative $\partial^4 u/\partial x^4$ of the continuous displacement in the continuum model. This will correspond to a Boussinesq kind of elasticity also called *gradient elasticity*. One can pursue this more refined modelling as shown by Christov et al. (1996) to the price of having to solve analytically and numerically *stiff* partial differential equations with higher-order space derivatives.

This fruitful heuristic methodology can be generalized to include rotating "particles" in the chain (cf. Pavlov et al. 2006) with a view to describing the dynamics of some granular materials, a microstructure at each "point" (with dumbbell particles represented by trusses; cf. Askar 1986), lattices with electromechanical interactions in ionic crystals (cf. Askar et al. 1970), lattices offering a sensible schematization of ferroelectric crystals exhibiting permanent electric dipoles (cf. Pouget et al. 1986a, b), and diatomic lattices with chains containing two types of atoms (particles) in each cell (cf. Askar 1986; Flytzanis et al. 1986). At this point, it must be recalled that the lattice dynamics of nonlinear elastic chains has played a crucial role in the discovery of *solitons* in the celebrated numerical experiment of Pasta, Ulam and Fermi at Los Alamos in 1955. Since then many models exhibiting more or less exact soliton behaviour have been constructed on a lattice dynamical basis with different types of nonlinear interactions and various ranges of spatial interactions (see Maugin 1999). The Toda lattice (Toda 1970) with exponentially decreasing interaction potential is such a fruitful model. Recently, a lattice model has been proposed to study fractal elastic materials (cf. Michelitsch et al. 2009a, b) with interparticle interactions of all length scales.

In all, although not always appreciated by purists of continuum mechanics, lattice dynamics offers an efficient tool to justify or to construct many models of nonclassical continuum mechanics, in particular as it concerns dispersion and nonlinear properties or the combination and/or competition of both.

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Cross references: Fractal continua, Gradient elasticity, Granular materials as generalized continua, Higher-order gradient theories, Internal degrees of freedom (dynamics), Nonlinear waves in generalized continua, Solitons (in non-classical continua),

Le Roux Elasticity

This expression applies to a type of elasticity developed by J. (Jean-Marie) Le Roux (1863–1949) in the period 1910–1914. This is an original work for this period, in which we recognize the seeds of the theory of gradient elasticity. Le Roux received his doctoral degree in 1895 (Le Roux 1895) and obtained a professorship of applied mathematics at the University of Rennes in Western France (Bretagne = Brittany) only in 1902 and then stayed there until 1933. A specialist of linear partial

differential equations, he realized in 1910 that some problems of the continuum mechanics of deformable solids such as torsion (cf. Le Roux 1911), involve non-homogeneous strains and, therefore, the corresponding theory should include from the start not only strains in the classical sense but also their spatial gradient, or more generally the gradient of the displacement gradient. Accordingly, in a variational formulation, one should consider to start with a Hamiltonian-Lagrangian formulation of the type

$$\delta \int_B W\left(u_{i,j}, u_{i,jk}\right) dV = 0. \tag{10.8}$$

The equilibrium equation at each regular material point becomes the Euler-Lagrange equation

$$\frac{\partial}{\partial x_j}\left(\frac{\partial W}{\partial u_{i,j}} - \frac{\partial}{\partial x_k}\left(\frac{\partial W}{\partial u_{i,jk}}\right)\right) = 0. \tag{10.9}$$

There remains to settle the question of boundary conditions.

Some ideas akin to Le Roux's one may be found in the unpublished works of Gabrio Piola (see Piola 2014), and also in a note of Barré de Saint-Venant (1869). Strangely enough, although Le Roux was much appreciated by H. Poincaré as an analyst, and Le Roux applied some of the ideas of the Cosserat brothers (Cosserat and Cosserat 1896) in the case of finite deformations (Le Roux 1913),—where he also introduced a relationship with Christoffel's symbols—Le Roux's works did not find much response in France. But he seems to have been popular to some extent in Eastern European countries (Russia, Poland, East part of Germany, Czechoslovakia—in particular, see citations by Nowacki 1986). Now we can claim that Le Roux's works anticipated works of the 1960s and 1970s, in particular by R.D. Mindlin and his co-workers in the USA (Mindlin 1964; Mindlin and Eshel 1968), and may thus be considered one of the creators of gradient elasticity, an example of weakly nonlocal theory of continua.

Caveat: In his old age, Le Roux was unfortunate to misunderstand and harshly criticize Einstein's theory of relativity (Le Roux 1933).

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Cross references: Aifantis E.C., Gradient elasticity, Mindlin R.D., Non-locality (weak).

Liquid Crystals as Continua

Liquid crystals—almost an oxymoron—is a denomination that applies to substances that can flow, like liquids, but can also exhibit a kind of ordering, like crystals, i.e., the exhibition of some kind of long range orientational order. This dual response of course depends on specific physical conditions such as the range of temperature so that these substances can present several phases. They present so-called *mesophases*. Microscopically, liquid crystals exhibit molecules in the form of rods or disks. Discovered by Friedrick Reinitzer in 1885 with transition properties shown by Otto Lehman at the end of the nineteenth century, they were classified by Georges Friedel (1865–1933) in 1922 in a memoir of some two hundred pages! Two large classes are thus distinguished: *nematic* (from the Greek word meaning "thread") and *smectic*. In the first case the mass centres of the molecules are distributed randomly in three dimensions (no long-range order in positions of the centre of mass of molecules) but still the molecules have a tendency to align parallel. In smectics, this parallel alignment exists but molecules are arranged in equidistant planes. *Cholesteric* liquid crystals correspond to the case where the elongated molecules of the nematic phase are organized into adjacent planes with a slight rotation between planes, hence exhibiting a kind of stratification. This gives a kind of helical structure that may also be called *chiral nematic phase*.

Macroscopically, liquid crystals may be viewed as continua with specific properties due to the presence of elongated molecules, and thus locally preferred directional properties. This idea was exploited in the first continuum description of

such crystals by Oseen (1933) who introduced what is now called a unit *director* **n** as representative of a local average orientation of a bunch (assimilated to a material point) of elongated molecules. Frank (1958) took over Oseen's approach, so that the latter can be referred to as the Oseen-Frank (O-F) theory. The associated energy depends on the gradient of **n**, i.e., we can write

$$W = W(\rho, \theta, \nabla \mathbf{n}), \tag{10.10}$$

since $\mathbf{n}^2 = 1 = const$. The simplest dependency on $\nabla \mathbf{n}$ can be expressed in three terms:

$$\nabla.\mathbf{n}, \mathbf{n}.(\nabla \times \mathbf{n}), \mathbf{n} \times (\nabla \times \mathbf{n}), \tag{10.11}$$

and a quadratic "elastic" energy reads

$$2W = k_{11}(\nabla.\mathbf{n})^2 + k_{22}(\mathbf{n}.\nabla \times \mathbf{n})^2 + k_{33}[\mathbf{n} \times (\nabla \times \mathbf{n})]^2. \tag{10.12}$$

The three contributions correspond to different types of deformation called, respectively, pure *splay*, pure *twist*, and pure *bend*. The three coefficients are of the same order of magnitude so that an often considered simpler model reads

$$W = \frac{1}{2} k n_{i,j} n_{i,j}. \tag{10.13}$$

For chiral liquid crystals one can directly replace the variable $\mathbf{n}.(\nabla \times \mathbf{n})$ by $\mathbf{n}.(\nabla \times \mathbf{n}) + q_0$, where q_0 is related to the pitch P_0 of the cholesteric helix by $q_0 = 2\pi/P_0$. This "elastic" theory of liquid crystals is particularly powerful for modelling liquid-crystal devices and lipid layers.

Liquid crystals present physically and industrially interesting couplings with thermal effects and electromagnetic fields. Their flow behaviour is markedly different from that of standard Newtonian fluids, and of course liquid crystals differ at rest from Eulerian fluids by virtue of their directional properties. The best general reference to the physics of liquid crystals remains the book by de Gennes and Prost (1993)—see also Chandrasekhar (1992) and the very informative course by Andrienko (2006).

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Cross references: Anisotropic fluids, Liquid crystals (Ericksen-Leslie theory), Liquid crystals (Eringen-Lee theory), Oriented media (with directors).

Liquid Crystals (Ericksen-Leslie Theory)

J.L. Ericksen (born 1924) and his co-worker F.M. Leslie (1935–2000) have incorporated the theory of liquid crystals into the general theory of anisotropic or oriented media (with directors) and extended the Oseen-Frank theory so as to include the dissipative viscous behaviour. The departure point by Ericksen (1960, 1961) is the standard approach through global balance laws, one of which is a rather strange balance law for the dynamics of a single director **n** set as

$$\frac{d}{dt}\int_B \rho\dot{\mathbf{n}}dv = \int_{\partial B} \mathbf{k}.\mathbf{t}^m dS + \int_B \rho\mathbf{G}dv \tag{10.14}$$

where, *exceptionally*, **k** denotes the unit normal. This is perfectly analogous to the classical equation of linear momentum

$$\frac{d}{dt}\int_B \rho\dot{\mathbf{x}}\,dv = \int_{\partial B} \mathbf{k}.\mathbf{t}^M dS + \int_B \rho\mathbf{f}\,dv \tag{10.15}$$

Another way to ponder the same system is to write down the following global balances of linear and angular momenta and of energy (here also the unit normal is noted **k** so as to avoid any misunderstanding with the director field **n**):

$$\frac{d}{dt}\int_B \rho(\dot{\mathbf{x}}+\dot{\mathbf{n}})dv = \int_{\partial B} \mathbf{k}.\left(\mathbf{t}^M+\mathbf{t}^m\right)dS + \int_B \rho(\mathbf{f}+\mathbf{G})dv, \tag{10.16}$$

$$\frac{d}{dt}\int_B \rho(\mathbf{x}\times\dot{\mathbf{x}}+\mathbf{n}\times\dot{\mathbf{n}})dv = \int_{\partial B} \left(\mathbf{x}\times\left(\mathbf{k}.\mathbf{t}^M\right)+\mathbf{n}\times(\mathbf{k}.\mathbf{t}^m)\right)dS + \int_B \rho(\mathbf{x}\times\mathbf{f}+\mathbf{n}\times\mathbf{G}+\mathbf{C})dv, \tag{10.17}$$

and

$$\frac{d}{dt}\int_B \rho\left(e+\frac{1}{2}\left(\dot{\mathbf{x}}^2+\dot{\mathbf{n}}^2\right)\right)dv = -\int_{\partial B} \mathbf{k}.\mathbf{q}\,dS + \int_B \rho q\,dv \tag{10.18}$$

where $\mathbf{q}$ stands for the influx of *total* energy, i.e.,

$$\mathbf{q} = \dot{\mathbf{x}}.\mathbf{t}^M + \dot{\mathbf{n}}.\mathbf{t}^m - \mathbf{q}^h, \tag{10.19}$$

where $\mathbf{q}^h$ relates only to heat. Similarly for q,

$$q = \mathbf{f}.\dot{\mathbf{x}} + \mathbf{G}.\dot{\mathbf{n}} + h. \tag{10.20}$$

Here $\mathbf{G}$ is an externally applied "force" acting on the director field (e.g., via a magnetic field), and $\mathbf{C}$ is an externally applied couple density.

The artificiality of this construct must be underlined although it exploits a strict analogy between the classical continuum and the field of director.

From Eqs. (10.17) and (10.18) one deduces the following two equations:

$$t^M_{[kl]} = n_{[k,p}t^m_{l]p} + \rho C_{kl} \tag{10.21}$$

and

$$\rho\dot{e} = t^M_{kl}\dot{x}_{k,l} + t^m_{kl}\dot{n}_{k,l} - q_{k,k} + \rho h. \tag{10.22}$$

We prefer to exploit the most elegant and very efficient, albeit also formal, method of the principle of virtual power as generalized by Maugin (1980). For this we introduce the following objective time derivatives (Maugin 1980, Paragraph 7.3):

$$D_{ij} = v_{(i,j)} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right), \hat{n}_i = \left(D_J\mathbf{n}\right)_i := \dot{n}_i - \Omega_{ij}n_j, \hat{N}_{ij} := \dot{n}_{i,j} - \Omega_{ik}n_{k,j} \tag{10.23}$$

with

$$\Omega_{ij} = v_{[i,j]} = \frac{1}{2}\left(v_{i,j} - v_{j,i}\right). \tag{10.24}$$

Then the principle of virtual power for a continuum theory with degrees of freedom the actual position $\mathbf{x}$ and the director $\mathbf{n}$, and accounting for their first spatial gradients, reads:

$$P^*_{(a)} = P^*_{(int)} + P^*_{(vol)} + P^*_{(surf)}, \tag{10.25}$$

where a right asterisk indicates the application of virtual velocity fields, and the various powers of acceleration, internal forces, body forces and surface forces for a body B of regular boundary ∂B of unit outward normal $\mathbf{k}$ are given by

$$P^*_{(a)}(B) = \int_B \rho\left(\dot{v}_i v^*_i + I\ddot{n}_i\dot{n}^*_i\right)dv, \tag{10.26}$$

$$P^*_{(int)}(B) = -\int_B \left(\sigma_{ji} D^*_{ij} - g_i \hat{n}^*_i + \pi_{ji} \hat{N}^*_{ij}\right) dv, \tag{10.27}$$

$$P^*_{(surf)}(\partial B) = \int_{\partial B} \left(\mathbf{T}^d.\mathbf{v}^* + \mathbf{T}^n.\dot{\mathbf{n}}^*\right) ds, \tag{10.28}$$

$$P^*_{(vol)}(B) = \int_B \rho(\mathbf{f}.\mathbf{v}^* + \mathbf{G}.\dot{\mathbf{n}}^*) dv. \tag{10.29}$$

These are linear continuous functionals with the property that (10.27) is written on a set of *objective* velocities so that $P^*_{(int)}$ vanishes identically for a generalized rigid-body motion given by the vanishing of the three quantities defined in (10.23). Note that $\hat{n}^*_i$ and $\hat{N}^*_{ij}$ are defined in terms of $\dot{n}^*_i$, $\left(\dot{n}^*_i\right)_{,j}$ and Ω^*_{ij}. Application for sufficiently continuous fields, arbitrary virtual velocities and any elements of volume and surface yields the following local field equations:

$$t_{ji,j} + \rho f_i = \rho \dot{v}_i \quad in \quad B, \tag{10.30}$$

$$\pi_{ji,j} + g_i + \rho G_i = \rho I \ddot{n}_i \quad in \quad B, \tag{10.31}$$

$$k_j t_{ji} = T^d_i \quad on \quad \partial B, \tag{10.32}$$

$$k_j \pi_{ji} = T^n_i \quad on \quad \partial B, \tag{10.33}$$

with a Cauchy stress defined by

$$t_{ji} = \sigma_{ji} + g_{[i} n_{j]} - \pi_{k[i} n_{j],k}. \tag{10.34}$$

On taking the vector product of (10.31) with $\mathbf{n}$, we also obtain the equation

$$\rho I \ddot{n}_{[i} n_{j]} = \left(\rho G_{[i} + g_{[i} + \pi_{k[i,k}\right) n_{j]} \quad in \quad B, \tag{10.35}$$

while (10.33) yields at ∂B

$$k_k \pi_{k[i} n_{j]} = T^n_{[i} n_{j]}. \tag{10.36}$$

Equation (10.34) is a kind of canonical decomposition of the Cauchy stress in symmetric and skewsymmetric parts with

$$t_{[ji]} = g_{[i} n_{j]} - \pi_{k[i} n_{j],k} \tag{10.37}$$

In the original theory of anisotropic fluids of Ericksen (1960) where the gradient of **n** is not introduced, this reduces to (cf. Equation (10.23) in Entry "Anisotropic fluids")

$$t_{[ji]} = g_{[i}n_{j]}. \tag{10.38}$$

Returning to the general case of Eq. (10.25) the corresponding statement of the first law of thermodynamics (for real velocity fields) reads

$$\frac{d}{dt}\int_B \rho\left(e + \frac{1}{2}\left(\mathbf{v}^2 + I\dot{\mathbf{n}}^2\right)\right)dv = \int_B \rho(\mathbf{f}.\mathbf{v} + \mathbf{G}.\dot{\mathbf{n}} + \rho h)dv + \int_{\partial B}\left(\mathbf{T}^d.\mathbf{v} + \mathbf{T}^n.\dot{\mathbf{n}} - \mathbf{q}.\mathbf{k}\right)ds. \tag{10.39}$$

Combination of this with the principle (10.25) written for real fields provides after localization the local equation for the internal energy e in the form

$$\rho\dot{e} = \left(\sigma_{ji}D_{ij} - g_i\hat{n}_i + \pi_{ji}\hat{N}_{ij}\right) - q_{k,k} + \rho h. \tag{10.40}$$

Further combination of this with the local form of the second law of thermodynamics and introduction of the free energy density $\psi = e - \eta\theta$ lead to the so-called Clausius-Duhem inequality:

$$-\rho\left(\dot{\psi} + \eta\dot{\theta}\right) + \left(\sigma_{ji}D_{ij} - g_i\hat{n}_i + \pi_{ji}\hat{N}_{ij}\right) - (q_k/\theta)\theta_{,k} \geq 0. \tag{10.41}$$

This constrains the constitutive equations for the set

$$\psi, \eta, \sigma_{ji}, g_i, \pi_{ji}, q_k. \tag{10.42}$$

The thermodynamically recoverable parts of these are deduced from the free energy

$$\psi = \overline{\psi}\left(\rho, \theta, n_i, n_{i,j}\right). \tag{10.43}$$

On obtains thus

$$\sigma_{ji} = -\pi\delta_{ij}, g_i = -\rho\frac{\partial\overline{\psi}}{\partial n_i}, \pi_{ji} = \rho\frac{\partial\overline{\psi}}{\partial n_{i,j}}, \pi = -\frac{\partial\overline{\psi}}{\partial\rho^{-1}}, \tag{10.44}$$

and so

$$t^R_{ji} = -\pi\delta_{ij} - \rho\frac{\partial\overline{\psi}}{\partial n_{[i}}n_{j]} - \rho\frac{\partial\overline{\psi}}{\partial n_{k,[i}}n_{j],k} \tag{10.45}$$

The dissipative parts of the constitutive equations satisfy the remaining dissipation inequality:

$$\Phi = \left(\sigma_{ji}^D D_{ij} - g_i^D \hat{n}_i + \pi_{ji}^D \hat{N}_{ij}\right) - (q_k/\theta)\theta_{,k} \geq 0. \tag{10.46}$$

But the rotational invariance of $\overline{\psi}$ materializes in the following constraint

$$\frac{\partial \overline{\psi}}{\partial n_{[i}} n_{j]} + \frac{\partial \overline{\psi}}{\partial n_{k,[i}} n_{k,j]} + \frac{\partial \overline{\psi}}{\partial n_{[i,k}} n_{j],k} = 0. \tag{10.47}$$

On account of this result and of the expanded form of $\hat{N}_{ij}$, it is shown that (10.44) can be rewritten in the form

$$t_{ji}^R = -\pi\delta_{ij} - \rho \frac{\partial \overline{\psi}}{\partial n_{k,i}} n_{k,j}. \tag{10.48}$$

This is sometimes referred to as *Ericksen's tensor* for liquid crystals at rest (i.e., hydrostatics). This of course shows that liquid crystals do not satisfy Euler's hydrostatics.

In the *hydrodynamics* of liquid crystals that accounts for viscosity and director relaxation in orientation (neglecting the dissipative part theoretically associated with tensor π_{ij}—that accounts for the long-range orientation ordering), we have to formulate meaningful equations for σ_{ji}^D and g_i^D in accord with (10.45). This was achieved by Leslie (1968) for nematics for which one refers to "nematodynamics". We refer to this author and Chap. 5 in de Gennes and Prost (1995). This is also strongly correlated with, if not identical to, Eqs. (10.24) and (10.25) in the Entry "Anisotropic fluids". Experiments allowing the measure of the Leslie coefficients are described in the book of de Gennes and Prost, Chap. 5.

If the director $\mathbf{n}$ is of constant unit length one must account for the constraints $\mathbf{n}^2 = 1$ and $(\nabla \mathbf{n}).\mathbf{n} = 0$ by introducing Lagrange multipliers, a scalar α and a vector $\boldsymbol{\beta}$.

Interaction With Electromagnetic Fields

Liquid crystals are prone to interacting with electromagnetic fields in the following manners. First the externally applied force $\mathbf{G}$ introduced in Eqs. (10.29) and (10.31) may be due to a magnetic field, thus in fact causing a couple as shown in (10.35) when it is not aligned with $\mathbf{n}$. Second, a magnetic field orthogonal to $\mathbf{n}$ and exceeding a critical value makes the optical properties of the liquid crystal to change abruptly. This is known as *Frederiks transition* in nematics. Also, the dielectric susceptibility of a nematic is anisotropic resulting in the existence of

ordinary and extraordinary refractive indexes so that the medium is birefringent. This can lead to multicoloured images in the examination of liquid crystals under polarized white light. Finally, in cholesteric liquid crystals light is reflected with a wavelength proportional to the pitch. But the latter depends on the temperature so that the reflected colour also depends on the temperature (cf. liquid-crystal thermometers). This is one among many applications of liquid crystals.

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Cross references: Directors' theory, Ericksen J.L., Internal degrees of freedom, Liquid crystals as continua, Mesoscopic theory of complex media, Microstructure, Oriented media (with directors), Solutions of macromolecules.

Liquid Crystals (Eringen-Lee theory)

The theory of micropolar continua much contributed by A.C. Eringen and his co-workers lends itself quite naturally to a phenomenological approach to liquid crystals; this is particularly obvious if we remember the representation of the internal degree of rotation of micropolar theory by orthogonal transformations by means of the Gibbs picture that exploits a rotation vector $\underline{\phi}$ and a unit direction **n** (See the entry "Micropolar continua"). This approach to liquid crystals was expanded with some relative success by Eringen and some co-workers (e.g., James D. Lee) in the late 1960s and early 1970s (See, e.g., Lee and Eringen 1971a, b, c). This is reviewed in detail in Eringen (2001, Chap. 12). Here this is not expanded. What must be noted, however, is the essential role played by the notion of *wryness* (some kind of "contortion") in the elasticity behaviour and that of the generalized time rates in the case of hydrodynamics. It can be shown that this Eringen-Lee (E-L) theory reduces to the Oseen-Frank-Ericksen theory with appropriate approximations (cf. Eringen 2001, pp. 169–181, also Eringen 1993, 1997; Rymarz 1990).

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Cross references: Cosserat continua, Eringen A.C., Internal degrees of freedom, Liquid crystals as continua, Micropolar continua, Micropolar fluids, Microstructure, Oriented media (with directors), Solutions of macromolecules.

Liquid Crystals (Landau-De Gennes theory)

The Oseen-Frank-Ericksen-Leslie theory that involves a director **n** only is satisfactory in the solution of problems of statics and in nematodynamics. But to describe realistically the phase transition between an isotropic phase and an anisotropic phase the introduction of an *order parameter* that characterizes this transition is necessary (cf. De Gennes and Prost 1993). This order parameter cannot be a vector because liquid crystals possess a centre of symmetry. Suppose that the molecules composing the nematic crystal are rigid and rod-like. Let $\mathbf{n}^{(\alpha)}$ a unit vector along the axis of molecule labelled (α). A natural order parameter then is a second-order tensor

$$S_{ij}(\mathbf{x}) = \frac{1}{N}\sum_{\alpha}\left(n_i^{(\alpha)}n_j^{(\alpha)} - \frac{1}{3}\delta_{ij}\right). \tag{10.49}$$

Here the summation is over all N molecules in a small but macroscopic volume at point **x**. This tensor is symmetric and traceless. It vanishes in the isotropic phase as shown by a probabilistic argument applied to the orientation of the elongated molecules. Otherwise, it is very sensitive to the direction of the average orientation of the molecules. For a uniaxial nematic phase (axial symmetry of the orientational molecular distribution function), the tensor (10.49) reduces to the simple form

$$S_{ij} = S\left(n_i n_j - \frac{1}{3}\delta_{ij}\right). \tag{10.50}$$

Here the scalar S measures the degree of alignment of the elongated molecules.

In the spirit of the Landau theory of phase transitions (after Lev D. Landau, 1908–1968), the free energy function Ψ is considered as an analytic function of the order parameter. A possible expression is given by the following expansion limited to the fourth order in S_{ij}:

$$\Psi = \Psi_{iso} + \frac{1}{2}AS_{ij}S_{ij} - \frac{1}{3}BS_{ij}S_{jk}S_{ki} + \frac{1}{4}CS_{ij}S_{ij}S_{kl}S_{kl}, \tag{10.51}$$

where the traceless property has been taken into account in writing the last contribution. Here Ψ_{iso} is the free energy of the isotropic phase, and coefficients A, B and C may still depend on pressure and temperature. The phase transition takes place near the temperature at which A vanishes. Thus one writes

$$A = A'(\theta - \theta^*). \tag{10.52}$$

Coefficients B and C may be assumed constant without much loss.

For nematic liquid crystals where S_{ij} varies slowly in space, terms including the spatial gradient of S_{ij} should be added to expression (10.51), e.g., an *elastic* (or weakly non-local) contribution with elasticity coefficients L_1 and L_2:

$$\Psi_{elas} = \frac{1}{2}L_1 S_{ij,k}S_{ij,k} + \frac{1}{2}L_2 S_{ij,j}S_{ik,k}. \tag{10.53}$$

The coefficients are easily related to those in the Oseen-Frank energy by $k_{11} = k_{33} = 9S_b^2(L_1 + L_2/2)/2$ and $k_{22} = 9S_b^2 L_1/2$, where S_b is the bulk nematic order parameter.

For a uniaxial liquid crystal, using (10.50) for the order parameter, from (10.51) we obtain the expression

$$\Psi = \Psi_{iso} + \frac{1}{3}AS^2 - \frac{2}{27}BS^3 + \frac{1}{9}CS^4. \tag{10.53}$$

This can be used to discuss the phase transition in the Landau tradition. Other phase-transition theories exploiting more microscopic notions and statistical theory have been proposed by. Maier and Saupe (1958) in the period 1958–1960 (a particularly successful theory) and Onsager (1949). This is outside the scope of the present contribution.

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Cross references: Directors' theory, Internal degrees of freedom, Internal variables of state, Liquid crystals as continua, Mesoscopic theory of complex media, Microstructure, Solutions of macromolecules.

Long-Range Interactions

Classical continuum mechanics is based on the notion of *contiguity*, i.e., transmission of internal forces by *contact action*. This resulted from Euler's and Cauchy's innovative works. This is in contrast with *action-at-a-distance* of which the most popular example is provided by gravitation. This is carried into classical continuum in the form of the *body force*. But even here there is some ambiguity brought by Laplace with the notion of potential ϕ and the fact that the body force may be written as

$$f_i = -\phi_{,i} = -\Phi_{ji,j}, \Phi_{ji} = \phi\delta_{ji}, \tag{10.54}$$

and the consideration of gravitation as a field theory (this will in fact culminate in Einstein's theory of gravitation which is a continuum theory per se; see Chap. 3 in Maugin 2014). The situation is also ambiguous in electromagnetic continua (see that Entry) where the interaction of electromagnetic fields with deformable matter can alternately be viewed as a body force (the action-at-a-distance view) and a contact action by means of the Maxwell stress or its generalization (cf. Maugin 1988). By some strange face-about, the notion of action-at-a-distance was re-introduced in continuum mechanics in basic discussions by Duhem (1893)—as cited by Edelen (1976)—and then fully formalized in works by Kunin et al. (1968), Edelen and Eringen in the period 1965–1980 (cf. Eringen 2002) in the *non-local theory* of continua [See Entry: Non-locality (strong)]. In this theory the transmission of internal forces occurs between not only neighbouring points but also between *distant* material points in the body. This exemplifies the notion of *long-range interactions* and the necessarily associated notion of characteristic length (e.g., radius of sphere of interaction, coherence length, etc.). A new field theory was thus opened where integro-differential equations replace "nice" partial differential equations. Whether this is a clear advantage as a more physically justified vision or a drawback and unnecessary complication is a discussed matter (cf. Maugin 1979). Anyway, this new trend has been very attractive to many mechanicians with some mathematical dexterity. This notion of long-range interaction has also permeated the lattice dynamics (see that Entry) where interactions between *far* "neighbours" have been considered (e.g., between second-nearest neighbours) often with a view to justifying the non-local continuum theory (cf. Kunin 1982–3). Peridynamics also is a modern manifestation of this trend.

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Cross references: Edelen D.G.B., Electromagnetic continua, Eringen A.C., Generalized continuum mechanics, Kröner E., Kunin I.A., Non-locality (strong), Peridynamics, Rogula D.

Chapter 11
M: From "Material Growth (Theory of)" to "Micromagnetism in Elastic Solids"

Material Growth (theory of)

Material growth can occur in the bulk—so-called *volumetric growth*—or by accretion at a surface. Here we consider only the first kind. That is, we are concerned with a single continuum with mass balance law (in a reference configuration) written as

$$\frac{\partial}{\partial t}\rho_0(\mathbf{X},t)\Big|_{\mathbf{X}} \neq 0. \tag{11.1}$$

The nonvanishing right-hand side in this equation suggests that some external source is present such as occurs in biological growth where nutrients provide this source. Thus, in agreement with the formulation of standard processes in continuum mechanics, the following equation may hold for mass in the reference configuration K_{R} of a nonlinear deformable body:

$$\frac{\partial}{\partial t}\rho_0(\mathbf{X},\,t)\Big|_{\mathbf{X}} = \Pi_0 + \nabla_R \,.\, \mathbf{M}, \tag{11.2}$$

where Π_0 is a scalar source, $\mathbf{M}$ is a material vector such as a flux, and ∇_R is a referential gradient. The modelling problem consists in formulating possible expressions for Π_0 and $\mathbf{M}$ on a consistent thermodynamic basis. In this approach growth is seen as a special type of *local structural rearrangement* which consists in pushing in more "material particles" of the same type at a given material point. These are not foreign ones. They just look the same as those previously there. The result of this is the change of matter density in the reference configuration as stated in (11.2). But, as shown by Epstein and Maugin (2000), this equation is too general to be accepted in a classical theory for continua that considers only the *first* gradient of deformation; we must assume that $\mathbf{M} \equiv \mathbf{0}$, so that (11.2) reduces to

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$$\left.\frac{\partial}{\partial t}\rho_0(\mathbf{X},\, t)\right|_{\mathbf{X}} = \Pi_0. \tag{11.3}$$

Following some ideas of elasto-plasticity (Lee 1969), the deformation gradient $\mathbf{F}$ is considered as the multiplicative composition of an elasticity "gradient" $\mathbf{F}_e$ and a growth "gradient" $\mathbf{F}_g$—but none of these two is a true gradient:

$$\mathbf{F} = \mathbf{F}_e\,\mathbf{F}_g. \tag{11.4}$$

This follows the line of works by Takamizawa and Matsuda (1990), Rodriguez et al. (1994), Taber (1995), and more recent authors. This theory fits well in the configurational mechanics of materials (cf. Gurtin 1999; Maugin 2011, Chap. 10) in which $\mathbf{K} = \mathbf{F}_g^{-1}$ may be called a transplant and from which can be defined a *connection* in a geometric interpretation. We denote the inhomogeneity velocity gradient in the reference configuration K_R by

$$\mathbf{L}_K := \dot{\mathbf{K}}\,.\,\mathbf{K}^{-1} = \frac{\partial}{\partial t}\left(\mathbf{F}_g^{-1}\right)\Big|_{\mathbf{X}}^{\cdot}.\mathbf{F}_g. \tag{11.5}$$

Because the density in the reference crystal $\rho_c = \rho_0 J_K, J_K = \det\mathbf{K}$ is fixed, computing the time derivative of the reference density ρ_0, one obtains

$$\left.\frac{\partial}{\partial t}\rho_0\right|_{\mathbf{X}} = \rho_c\left(\frac{\partial}{\partial t}\,J_K^{-1}\right) = -\rho_0\; tr\,\mathbf{L}_K, \tag{11.6}$$

Thus, in this simplified theory,

$$\Pi_0 = -\rho_0\; tr\,\mathbf{L}_K. \tag{11.7}$$

This means that no separate time evolution has to be specified for the volumetric source Π_0 if we know the time evolution of the transplant $\mathbf{K}$ or of $\mathbf{F}_g$. Furthermore, the time evolution of the determinant of $\mathbf{K}$, $J_K = det\,\mathbf{K}$, tells us whether there is actually growth (negative time derivative) or resorption (positive time derivative). Accordingly, growth occurs at time t for negative $tr\,\mathbf{L}_K$, while J_K itself is a measure of accumulated growth from the time origin. If $\mathbf{K}$ is nonspherical, then there is a rotation and/or a distortion in addition to growth or resorption.

Finally, the phenomenon of growth is associated with *dissipation*. This is shown by uncovering the driving force behind growth and exploiting the Clausius-Duhem inequality in the usual manner. While the thermo-elastic behaviour is derived from the energy density, the residual dissipation inequality reduces to

$$\Phi = tr(\mathbf{M}\,.\,\mathbf{L}_K) - \theta^{-1}\mathbf{Q}\,.\,\nabla_R\theta \geq 0, \tag{11.8}$$

where

$$\mathbf{M} = \mathbf{T}.\mathbf{F} \tag{11.9}$$

is the so-called *Mandel stress tensor* (a part of the Eshelby stress of configurational mechanics) and $\mathbf{Q}$ is the material heat (in)flux. Thus we can say that the Mandel stress is the driving force behind growth as it is formally the driving force behind many bulk structural rearrangements. Leaving aside a conduction law of the Fourier type this yields a relationship between $\mathbf{M}$ and $\mathbf{L}_K$. If the latter involves one or several characteristic times, then the material finally behaves as a generally *non-linear*, heat conducting, *viscoelastic* material, in agreement with Cowin's (1996) proposal. The full theory, applications and numerics are given in Maugin (2011, Chap. 10).

Now back to Eq. (11.2) where $\mathbf{M}$ is a diffusion flux (not to be mistaken for the Mandel stress). Epstein and Maugin (2000) have shown that a standard theory of continua [dealing only with the deformation gradient (so-called "simple" materials)] cannot accommodate the presence of the last term in the evolution-diffusion Eq. (11.2). The technical reason for that is that this would yield a gradient of the energy, and thus a term linear in the second gradient of the deformation in the Clausius-Duhem inequality while the body is not equipped to respond to this kind of effect. Accordingly a solution consists in considering a *second-gradient* theory in finite strains in the line of *higher-order gradient theories*. Such an approach, obviously somewhat complicated, was proposed by Ciarletta and Maugin (2011) also Ciarletta et al. (2012), and in principle allows for a good, albeit complex, phenomenological theory of material growth and remodelling, as happens in biomechanics, on account of mass transport and morphogenetic species. It necessarily involves first-order and second-order transplants (local structural rearrangements) and two material connections on the material manifold from the geometric viewpoint. It is shown that the evolution of these structural changes or "material inhomogeneities" is governed by Eshelby-like stress *and* hyperstress tensors. A thermodynamically admissible set of constitutive equations was proposed being illustrated by a set of convincing applications. In such a theory for which we refer the reader to original works, the free energy density is given by an expression

$$\psi(\mathbf{F}, \nabla_R \mathbf{F}, c_\alpha, \theta; \mathbf{X}) = \left(\det \mathbf{F}_g\right)\, \psi_0(\mathbf{F}_e, \mathbf{Q}_e, c_\alpha, \theta), \tag{11.10}$$

where the introduced material isomorphism can be seen as a symmetry group of the strain energy so that $\mathbf{F}_e$ and $\mathbf{Q}_e$ are defined by

$$\mathbf{F}_e = \mathbf{F} \,.\, \mathbf{F}_g^{-1}, \mathbf{Q}_e = \nabla_R \mathbf{F} : \left[\mathbf{F}_g^{-1}, \mathbf{F}_g^{-1}\right] - \mathbf{F}_e \,.\, \mathbf{Q}_g : \left[\mathbf{F}_g^{-1}, \mathbf{F}_g^{-1}\right], \tag{11.11}$$

where $\mathbf{Q}_g = \nabla_R \mathbf{F}_g$, and the c_α's are the concentrations $c_\alpha(\mathbf{x}, t)$—per unit volume—of passive scalars that drive both growth (e.g., nutrients, growth factor) and mass transport phenomena (e.g., migration signals, morphogens) with $\alpha = 1, 2, \ldots$ the generic species. For these an evolution-diffusion equation in the form

$$\dot{c}_\alpha - \nabla_R . \mathbf{J}_{c\alpha} = \xi_\alpha(\mathbf{F}, \nabla_R \mathbf{F}), \tag{11.12}$$

holds where ξ_α is the absorption rate of the α-th species.

Similarly, $\mathbf{M}$ will be of the form

$$\mathbf{M}(\mathbf{F}, \nabla_R \mathbf{F}, c_\alpha, \theta; \mathbf{X}) = \left(\det \mathbf{F}_g\right) \mathbf{F}_g^{-1} . \mathbf{M}_0(\mathbf{F}_e, \mathbf{Q}_e, c_\alpha, \theta). \tag{11.13}$$

The evolution equations for both transplants of first and second order are constrained by the residual dissipation inequality (cf. Ciarletta and Maugin 2011).

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Cross references: Configurational mechanics, Connection and torsion, Higher-order gradient theories, Material inhomogeneities (theory of).

Material Inhomogeneities (Theory of)

The expression "material inhomogeneity" refers to the fact that material properties (density, elasticity coefficient, other material coefficients) of a body may vary more or less continuously depending on the material point and this, even in the absence of external loading and of motion. Accordingly, the property of material homogeneity or inhomogeneity of a body is reflected by the translational invariance or non-invariance of the physical system under consideration on the material manifold (i.e., the set of material "points"). Mathematically, this is materialized in the equation known as the balance of pseudomomentum or material momentum which is generated by an infinitesimal translation on the material manifold (cf. Maugin 1993). This is the arena of so-called *configurational forces* (cf. Maugin 2011). This applies to all field theories and thus in both classical and non-classical continuum mechanics.

References

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Cross references: Configurational mechanics, Material growth (theory of).

Materials with Voids

This expression is more particularly reserved to a class of materials introduced in a thermomechanical background by Cowin and Nunziato (1983). This class may be considered a very particular case of *micromorphic* continua, in which an internal degree of freedom of the dilatation type (hence described by a pure scalar variable ϕ) is attached to each material point. This may also be referred to as *dilatational elasticity* or else, it can be viewed as a special case of *microstretch elasticity* (cf. Eringen 1999) in which the internal-rotation component is discarded. The new equation that governs this internal dilatational deformation reads (case of quasi-statics)

$$m_{k,k} + \sigma - s + l = 0. \tag{11.14}$$

with an associated natural boundary condition in the form

$$n_k m_k = M^d, \tag{11.15}$$

where M^d is akin to a tension. The vector field of components m_k may be called the dilatational stress or microstretch vector and the scalar fields σ and s can be identified as the spherical parts of second order symmetric tensors and may be

referred to as intrinsic and micro-scalar forces, and l as the body microstretch force in the spirit of Eringen's micromorphic continua (See Entry: Micromorphic continua). This will be useful in describing the mechanical behaviour of media exhibiting a distribution of holes or cavities in evolution, e.g., a body containing non-interconnected spherical holes (thus not exactly a porous medium). This theory is related to Capriz's (1989) theory of microstructure. In the dynamical case an inertial term describing the time change of the radius r of the holes must be added in the right-hand side of (11.15). This looks much like the evolution of spherical bubbles in some bubbly fluids.

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Cross references: Generalized continuum mechanics, Micromorphic continua, Microstretch continua.

Mesoscopic Theory of Complex Continua

The so-called mesoscopic theory intends to go beyond classical continuum mechanics by extending the usual space-time $(\mathbf{x}, t)$ description of dependent variables to a mesoscopic space (hence of dimension four plus the number of mesoscopic variables). Let $\mathbf{m} \in M$ the set of those variables that is an element of a suitable manifold M on which integration can be defined. A statistical notion, the mesoscopic distribution function $f(\mathbf{m}, \mathbf{x}, t)$, is introduced such that

$$f(\mathbf{m}, \mathbf{x}, t) \equiv f(.), (.) \equiv (\mathbf{m}, \mathbf{x}, t) \in M \times R^3 \times R^1. \tag{11.16}$$

This function f describes the distribution of $\mathbf{m}$ in a volume element around point $\mathbf{x}$ at time t. It is normalized, i.e.

$$\int f(\mathbf{m}, \mathbf{x}, t) dM = 1. \tag{11.17}$$

Generalizing the usual local balance law for a quantity $\mathbf{A}$, we now have to consider balance laws in the following a priori form:

$$\frac{\partial}{\partial t}\mathbf{A}(.) + \nabla_x.[\mathbf{v}(.)\mathbf{A}(.) - \mathbf{S}(.)] + \nabla_m.[\mathbf{w}(.)\mathbf{A}(.) - \mathbf{R}(.)] = \Sigma(.). \tag{11.18}$$

Here ∇_x and ∇_m are the nabla gradient operators in physical ($\mathbf{x}$) space and in M, respectively. The fields $\mathbf{v}$ and $\mathbf{w}$ are the classical velocity field and the *mesoscopic change velocity* (noted $\mathbf{u}$ by Muschik et al.). We can state the following self-explaining definition:

$$(\mathbf{m}, \mathbf{x}, t) \rightarrow (\mathbf{m} + \mathbf{w}(.)\Delta t, \mathbf{x} + \mathbf{v}(.)\Delta t, t + \Delta t)$$

The fields $\mathbf{S}$, $\mathbf{R}$ and Σ must be identified. As examples of Eq. (11.18) we have the following local mesoscopic balance equations for mass and linear momentum:

$$\frac{\partial}{\partial t}\rho(.) + \nabla_x.[\mathbf{v}(.)\,\rho(.)] + \nabla_m.[\mathbf{w}(.)\rho(.)] = 0, \tag{11.19}$$

and

$$\frac{\partial}{\partial t}[\rho(.)\mathbf{v}(.)] + \nabla_x.[\mathbf{v}(.)\rho(.)\mathbf{v}(.) - \mathbf{t}(.)] + \nabla_m.[\mathbf{w}(.)\rho(.)\mathbf{v}(.) - \underline{\tau}] = \mathbf{k}(.) \tag{11.20}$$

Here $\mathbf{t}$ is the Cauchy stress (but function of $(\mathbf{m}, \mathbf{x}, t)$) and $\underline{\tau}$ is the analogous stress but on M; finally, $\mathbf{k}$ is the applied force density **[**also function of $(\mathbf{m}, \mathbf{x}, t)$]. An equation of angular momentum can be written down in the same formalism, involving possibly a couple stress [function of $(\mathbf{m}, \mathbf{x}, t)$], and an analogue acting on the mesoscopic variable $\mathbf{m}$. Usual macroscopic quantities will be obtained by integration over the mesoscopic variables. For instance, for mass and usual velocity field,

$$\rho(\mathbf{x}, t) = \int \rho(.)\, dM, \quad \mathbf{v}(\mathbf{x}, t) = \int f(.)\, \mathbf{v}(.) dM, \tag{11.21}$$

where $\rho(.) = \rho(\mathbf{x}, t)f(.)$. The latter describes the mass density of all molecules contained in a volume element for which the value of the mesoscopic variable is $\mathbf{m}$. Then the definition given in the first of (11.21) is analogous to a formula of mixture theory which states that the mass density of the mixture is the sum of the mass densities of their constituents. In this vision it may be said that the mesoscopic theory is akin to a mixture theory having a continuous index to describe the various species (cf. Muschik et al. 2004 for this illuminating analogy).

The mass balance is obtained in the usual form

$$\frac{\partial}{\partial t}\,\rho(\mathbf{x}, t.) + \nabla_x.[\mathbf{v}(\mathbf{x}, t)\,\rho(\mathbf{x}, t)] = 0, \tag{11.22}$$

because

$$\int \nabla_m.[\ldots] dM = 0; \tag{11.23}$$

In turn, the macroscopic mass balance induces a differential equation for the mesoscopic distribution function (Muschik et al. 1999):

$$\frac{\partial}{\partial t}f(.)+\nabla_x.[\mathbf{v}(.)f(.)]+\nabla_m.[\mathbf{w}(.)f(.)]+f(.)\left[\frac{\partial}{\partial t}+\mathbf{v}(.)f(.)\nabla_x\right]\ln\,\rho(\mathbf{x},t)=0. \tag{11.24}$$

This is an integro-differential equation for f since $\rho(\mathbf{x},t)$ is defined by the first of (11.21).

The mesoscopic approach can be applied to Cosserat continua. But the most straightforward and rich application (which may in fact have triggered this development) is to nematic liquid crystal (Blenk et al. 1992; Ehrentraut et al. 1997; Muschik et al. 1999, 2004). In such media the variable $\mathbf{m}$ is the mean orientation given by the director $\mathbf{n}$ of unit length for a crystal consisting of uniaxial molecules. Accordingly, M is the two-dimensional unit sphere S^2. The mesoscopic distribution function f now becomes an orientation distribution function $f(\mathbf{n},\mathbf{x},t)$. The $\mathbf{n}$ can be represented in terms of an angle and a unit vector of direction $\mathbf{n}_0$ (note that $\mathbf{n}_0$ and $-\mathbf{n}_0$ belong to the same rotation) in quaternion formalism. This results in a symmetry for f. The *mesoscopic change velocity* $\mathbf{w}$ is an element of the tangent space to S^3 (in this formalism), so that it satisfies the orthogonality condition

$$\mathbf{n}_0.\mathbf{w}=0. \tag{11.25}$$

With this one can proceed to the construction of the mesoscopic balance equations for nematic liquid crystals (cf. Blenk et al. 1992; Ehrentraut et al. 1997; Muschik et al. 1999, 2004). The mesoscopic equations of balance of angular momentum and spin play a fundamental role in this theory. The mesoscopic balances of energy and entropy are also needed for a fully expanded thermo-mechanical theory. This rather technical but straightforward development is clearly exposed in Muschik et al. (2004).

A final remark concerns the construction of representative order parameters that are needed in the discussion of phase transitions. This systematic construction indeed requires the exploitation of the mesoscopic background in which the mesoscopic distribution function necessarily intervenes. A hierarchy of such parameters can be defined as moments of this distribution function, e.g.,

$$\mathbf{a}(\mathbf{x},t)=\int f(.)\,\mathbf{m}\,dM, \tag{11.26}$$

$$\mathbf{a}_2(\mathbf{x},t)=\int f(.)\underline{\underline{\mathbf{m}\,\mathbf{m}}}\,dM, \tag{11.27}$$

$$\mathbf{a}_4(\mathbf{x},t)=\int f(.)\underline{\underline{\mathbf{m}\,\mathbf{m}\,\mathbf{m}\,\mathbf{m}}}\,dM,\,etc. \tag{11.28}$$

where the double underlining means the fully symmetric (traceless for $\mathbf{a}_2$) part of the underlined tensor product. Balance laws for the order parameters can be constructed. This is achieved by Muschik et al., in particular for the Landau type of second-order alignment tensor for liquid crystals with unaxial molecules. But a closing hypothesis is needed as the fourth-order non-traceless moment $\mathbf{a}_4$ is involved in the equation determining $\mathbf{a}_2$. A simple ansatz used to avoid this difficulty reads (cf. Muschik et al. 2004):

$$\int_{S^2} f(.)\,\mathbf{nnnn}\,d^2n = \int_{S^2} f(.)\,\mathbf{nn}\,d^2n \int_{S^2} f(.)\,\mathbf{nn}\,d^2n. \tag{11.29}$$

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Cross references: Liquid crystals, Cosserat continua

Metamaterials

In principle, "metamaterials" are materials that go beyond the usual notion of materials, i.e., those produced by Nature. They are in fact artificial structures with properties defined by their structure rather than their composition; they are designed so as to exhibit properties of which values cannot be found in Nature. This trend was first illustrated by optical metamaterials that exhibit a negative refraction index (as shown by Victor Veselago in 1967). Insofar as mechanical materials are concerned, this new scheme practically touches the domain of acoustic materials since they are the true acoustic properties that are most often altered, providing for instance negative effective bulk modulus (Lee et al. 2009a), negative effective mass density (Lee et al. 2009b), negative longitudinal and volume compressibility (cf. Nicolaou and Motter 2012), and obviously auxetic materials (with negative Poisson's ratio: see the Entry "Auxetic materials"), etc. There are even

metamaterials of the a priori solid type that behave like fluids (Page 2011), as they have a finite bulk modulus but a vanishing shear modulus. These are called *meta-fluids* and are illustrated by so-called "pentamode structures" [cf. Milton and Cherkaev (1995)]. General introductory references are Engheta and Ziolkowski (2006) and Zoudhi et al. (2008). Mechanical metamaterials achieve the desired effects by incorporating structural elements of sub-wavelength sizes, i.e., that are of smaller length than the wavelength of the waves they affect. Some of the modelling of three-dimensional composites of metal/non-metallic inclusions that are periodically or randomly embedded in a low permittivity matrix can be modelled by analytical methods akin to homogenization techniques, such as mixing formulas or scattering-matrix methods. In the case of solids, we refer to the works of Theocaris and Stavroulakis in the Entry "Auxetic materials".

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Cross reference: Auxetic materials.

Micromagnetism in Elastic Solids

Micromagnetics or micromagnetism is, in spite of its name, a macroscopic theory of magnetic materials but one that accounts for some features of these materials at a microscopic (quantum) scale, namely, magnetic spin and Heisenberg's exchanges forces. Without this peculiarity, some macroscopic effects (ferromagnetism itself) would not exist. This theory was fashionable in the 1960s–1970s when theorists from various horizons, electrical engineering (W.F. Brown Jr, F.R. Morgnenthaler), continuum mechanics (H.F. Tiersten, G.A. Maugin) and solid state physics (C. Kittel, A.I Akhiezer) combined efforts to examine the applicability of such an approach to wave conversion between mechanical and magnetic effects. Indeed, one would like to account for possible significant couplings between the dynamics

of magnetic spin and crystal deformation, essentially in ferromagnetic bodies. We recall elements of this theory by following Tiersten's (1964, 1965) works and the formalism and rational developments by Maugin (1971, 1988, Chap. 6; Eringen and Maugin 1990, Chap. 9). Because of its ferroïc nature (see the entry on "ferroics"), the theory must consider magnetization as a primitive independent variable (its local value may not be zero in the absence of applied magnetic field). Furthermore, *quasi-magnetostatics* of non-electrically polarized *insulators* only are considered for the sake of simplicity but also as most realistic cases.

Continuum Modelling

Using the standard notation of nonlinear continuum mechanics we may consider to start with the following *generalized motion* for deformable magnetized bodies of the ferroïc type:

$$\mathbf{x} = \bar{\mathbf{x}}(\mathbf{X}, t); \quad \mu = \bar{\mu}(\mathbf{X}, t), \tag{11.30}$$

where the first of these denotes the classical finite deformation at Newtonian time t between the reference configuration K_R and the actual configuration K_t. Here $\mathbf{x}$ is the placement of Euclidean coordinates $x_i, i = 1, 2, 3$, and $\mathbf{X}$ denotes the material point of coordinates $X_K, K = 1, 2, 3$ in material space. Borrowing the denomination introduced by Tiersten (1964), we may say that the first of (11.30) describes the time evolution of the **lattice continuum** or *LC* (standard matter in the macroscopic description), while the second of (11.30) provides the time evolution of the magnetization density or μ (magnetic or electronic) **spin continuum** or *SC*, with the *gyromagnetic* relation (see Entry: Ponderomotive couple)

$$\mathbf{s} = \gamma^{-1}\mu.$$

The two continua should be treated on an equal footing in the vision of *generalized continuum mechanics* (**GCM**). But they do not respond exactly to the same kind of loads while we must also envisage interactions between these two "continua". In particular, the spin continuum cannot translate with respect to the lattice continuum. It, therefore, "expands" and "contracts" with the lattice continuum and, accordingly, its volumetric behaviour is governed by the usual continuity equation. As usual, the lattice continuum is assumed to be able to respond to volume and surface forces—hence exhibits stresses—and to volume couples, so that stress is not expected to be symmetric. We assume that it is not equipped with any mechanism to respond to surface couples, so that it does not exhibit couple stresses of mechanical origin. The balance of linear (physical) momentum simply says that whatever force of magnetic origin—e.g., the reduced *ponderomotive force* of electromagnetism in deformable bodies—is applied to a point in the spin continuum, it is directly transferred to the lattice continuum at the same point. The spin

continuum, by its very nature, can respond only to **couples**, which may be either of the volume or of the surface type. Accordingly, we consider that the *ponderomotive couple* (See that entry, Eq. (11.35); here $\widetilde{\mathbf{M}} = \rho\,\mu$)

$$\mathbf{c}^{em} = \widetilde{\mathbf{M}} \times \mathbf{B} = \rho\,\mu \times \mathbf{B} \tag{11.31}$$

is directly applied to the spin continuum.

In so far as the interactions between lattice and spin continua are concerned, they must necessarily be of the couple type since the spin continuum is sensitive only to that type of interaction. Following Tiersten (1964), we naturally assume that this couple is due to a *local magnetic induction* $\mathbf{B}^L$—to be given a constitutive equation—so that we can apply the "recipe" [compare (11.31)]

$$\mathbf{c}_{(LC/SC)} = \mathbf{M} \times \mathbf{B}^L = \rho\,\mu \times \mathbf{B}^L. \tag{11.32}$$

Angular momentum being conserved (exchanged) between the two continua, an equal and opposite couple

$$\mathbf{c}_{(SC/LC)} = -\mathbf{c}_{(LC/SC)} = \mathbf{B}^L \times \mathbf{M} = \rho\,\mathbf{B}^L \times \mu \tag{11.33}$$

is exerted on the unit volume of the lattice continuum.

Finally, in order to account for ferromagnetic (Heisenberg) exchange forces of quantum origin (interactions between neighbouring spins) that cause magnetic ordering but fall off rapidly with distance, we can represent these "forces" in a continuum description by a *contact action* in much the same manner as the stress vector for a Cauchy deformable continuum, except that this "surface exchange contact force" must also obey the "recipe" of a magnetic couple (the density ρ is included in **A**):

$$\mathbf{c}_{(LC\text{-}surface)} = \mu \times \mathbf{A}, \tag{11.34}$$

where **A** is an axial vector that depends on the local unit normal **n** to the surface and can be written in the same way as the classical stress vector (Cauchy principle), i.e.,

$$\mathbf{A} = \mathbf{A}_{(n)} = \mathbf{A}(\mathbf{x}, t; \mathbf{n}) = \mathbf{n}.\hat{\mathbf{B}}\,(\mathbf{x}, t), \tag{11.35}$$

so that (11.34) yields the following surface couple density acting on the spin continuum:

$$\mathbf{c}_{(SC\text{-}surface)} = \mu \times \mathbf{A} = \mu \times \left(\mathbf{n}.\hat{\mathbf{B}}\right). \tag{11.36}$$

Because of this very expression we can surmise that only the portion of **A** orthogonal to **M** is effectively defined. Thus, without loss in generality we can set forth the following condition:

$$\mathbf{A}.\mathbf{M} = \mathbf{n}.\hat{\mathbf{B}}.\mathbf{M} = 0 \tag{11.37}$$

at any point at the surface of the body. A similar orthogonality condition can be imposed on $\mathbf{B}^L$ but at any point inside the body.

Global Balance Laws

Collecting now the various proposed expressions, we can write down the global balance laws at time t in K_t for a magnetic body of volume B and regular bounding surface ∂B [for the sake of simplicity we ignore any discontinuity surface within the body; for the equations at discontinuity surfaces, see Maugin (1988)]:

Balance of mass for the combined continuum:

$$\frac{d}{dt}\int_B \rho\, dv = 0; \tag{11.38}$$

Balance of linear momentum for the LC:

$$\frac{d}{dt}\int_B \rho\, \mathbf{v}\, dv = \int_B (\mathbf{f} + \mathbf{f}^{em}) dv + \int_{\partial B} \mathbf{t}_{(n)} da; \tag{11.39}$$

Balance of angular momentum for the LC:

$$\frac{d}{dt}\int_B (\mathbf{r} \times \rho\, \mathbf{v}) dv = \int_B \left(\mathbf{r} \times (\mathbf{f} + \mathbf{f}^{em}) + \mathbf{c}_{(SC/LC)}\right) dv + \int_{\partial B} \left(\mathbf{r} \times \mathbf{t}_{(n)}\right) da\,; \tag{11.40}$$

Balance of angular momentum for the SC:

$$\frac{d}{dt}\int_B \rho\, \gamma^{-1}\, \mu\, dv = \int_B \left(\mathbf{c}^{em} + \mathbf{c}_{(LC/SC)}\right) dv + \int_{\partial B} \mu \times \mathbf{A}_{(n)} da; \tag{11.41}$$

First law of thermodynamics for the combined continuum:

$$\frac{d}{dt}\int_B \rho \left(\frac{1}{2}\mathbf{v}^2 + e\right) dv = \int_B (\mathbf{f}.\mathbf{v} + w^{em} + \rho\, h) dv + \int_{\partial B} \left(\mathbf{t}_{(n)}.\mathbf{v} + \mathbf{A}_{(n)}.\dot{\mu} - q_{(n)}\right) da; \tag{11.42}$$

Second law of thermodynamics for the combined continuum:

$$\frac{d}{dt}\int_B \rho\,\eta\,dv \geq \int_B \rho\,\theta^{-1}h\,dv - \int_{\partial B} \theta^{-1}\,q_{(n)}\,da. \tag{11.43}$$

In these equations, $\mathbf{t}_{(n)}$ is the surface traction, $\mathbf{f}$ is a body mechanical force (e.g., gravity), e is the internal energy per unit mass, η is the entropy per unit mass, h is the body heat source, $q_{(n)}$ is the heat influx. Classically [compare (11.35)],

$$\mathbf{t}_{(n)} = \mathbf{n}.\mathbf{t}, \quad q_{(n)} = \mathbf{n}.\mathbf{q}, \tag{11.44}$$

where $\mathbf{t}$ is the Cauchy stress and $\mathbf{q}$ is the heat-(in)flux vector.

Local Balance Laws

Standard localization of these global equations on account of the assumed continuity of all fields yields the following *local equations* at any point in B:

$$\dot{\rho} + \rho\nabla.\mathbf{v} = 0, \tag{11.45}$$

$$\rho\,\dot{\mathbf{v}} = div\,\mathbf{t} + \mathbf{f} + \mathbf{f}^{em}, \tag{11.46}$$

$$\varepsilon_{ijk}\left(t_{jk} + \rho\,B^L_j\mu_k\right) = 0, \tag{11.47}$$

$$\gamma^{-1}\,\dot{\mu}_i = \left[\mu \times \left(\mathbf{B} + \mathbf{B}^L + \rho^{-1}div\,\hat{\mathbf{B}}\right)\right]_i + \rho^{-1}\varepsilon_{ijk}\,\hat{B}_{pk}\,\mu_{j,p}, \tag{11.48}$$

$$\rho\left(\dot{e} + \frac{d}{dt}\left(\frac{1}{2}\mathbf{v}^2\right)\right) = t_{ji}\,v_{i,j} + \left(t_{kj,k}\,v_j + \mathbf{f}.\mathbf{v}\right) + \hat{B}_{kj}\,\dot{\mu}_{j,k} + \hat{B}_{kj,k}\dot{\mu}_j + w^{em} + \rho h - \nabla.\mathbf{q}, \tag{11.49}$$

and

$$\rho\dot{\eta} \geq \theta^{-1}\rho h - \theta^{-1}\nabla.\mathbf{q} - \mathbf{q}.\nabla\left(\theta^{-1}\right), \tag{11.50}$$

where the divergence of nonsymmetric tensors is to be taken on the first index, and a superimposed dot denotes the classical material time derivative.

Equations (11.49), (11.47), (11.19) and (11.50) are transformed thus. In (11.49), we must account for the kinetic energy theorem obtained by taking the inner product of the motion Eq. (11.46) by $\mathbf{v}$:

$$\rho\,\frac{d}{dt}\left(\frac{1}{2}\mathbf{v}^2\right) = \left(t_{kj,k}v_j + \mathbf{f}.\mathbf{v}\right) + \mathbf{f}^{em}.\,\mathbf{v}, \tag{11.51}$$

while w^{em} is given in Eringen and Maugin (1990)—but reduced to the case of quasi-magnetostatics. Therefore, (11.49) reads

$$\rho\,\dot{\hat{e}} = t_{ji}\,v_{i,j} + \rho\mathbf{B}.\dot{\mu} + \hat{B}_{kj}\dot{\mu}_{jk} + \hat{B}_{kj,k}\dot{\mu}_j + \rho h - \nabla.\mathbf{q},\ \hat{e} = e - e^{mag} \equiv e + \mu.\mathbf{B}. \tag{11.52}$$

But in a magnetic domain where magnetization has reached saturation,$\mu.\,\mu = \mu^2 = \mu_S^2$, we must have $\mu_i\dot{\mu}_i = 0; \mu_i\mu_{i,K} = 0$, so that $\dot{\mu}$ must be of the purely *precessional* form

$$\dot{\mu} = \omega \times \mu. \tag{11.53}$$

As a consequence the last contribution in (11.48) must vanish:

$$\hat{B}_{k[j}\mu_{i],k} = 0. \tag{11.54}$$

On account of this one checks that

$$\left(div\,\hat{\mathbf{B}}\right).\dot{\mu} = -\rho\left(\mathbf{B} + \mathbf{B}^L\right).\dot{\mu}, \tag{11.55}$$

because

$$\omega = -\gamma\mathbf{B}^{\mathit{eff}}, \quad \mathbf{B}^{\mathit{eff}} = \mathbf{B} + \mathbf{B}^L + \rho^{-1}div\hat{\mathbf{B}}. \tag{11.56}$$

This may be viewed as a continuum generalization of the celebrated Larmor precession equation $\omega_{Larmor} = -\gamma\,\mathbf{B}$ for an isolated electron in a magnetic induction B (see Entry: Ponderomotive couple).

Finally, (11.52) transforms to the following form using an intrinsic notation (T = transpose):

$$\rho\dot{\hat{e}} = tr\left[\mathbf{t}(\nabla\mathbf{v})^T\right] - \rho\mathbf{B}^L.\dot{\mu} + tr\left[\hat{\mathbf{B}}(\nabla\dot{\mu})^T\right] - \nabla.\mathbf{q} + \rho h. \tag{11.57}$$

In the same conditions (11.50) provides the following *Clausius-Duhem inequality*:

$$-\rho\left(\dot{\hat{\psi}} + \eta\dot{\theta}\right) + tr\left[\mathbf{t}(\nabla\mathbf{v})^T\right] - \rho\mathbf{B}^L.\dot{\mu} + tr\left[\hat{\mathbf{B}}\,(\nabla\,\dot{\mu})^T\right] - \theta^{-1}\mathbf{q}.\nabla\theta \geq 0, \tag{11.58}$$

wherein the free energy density has been defined by

$$\hat{\psi} = \hat{e} - \eta\theta. \tag{11.59}$$

Equation (11.58) nowadays plays an essential role in the construction of constitutive equations that we need for the set of quantities

$$\left\{\hat{\psi}, \eta, \mathbf{t}, \mathbf{B}^L, \hat{\mathbf{B}}, \mathbf{q}\right\}. \tag{11.60}$$

Equation (11.58) represents a constraint imposed by thermodynamic irreversibility (in particular in the so-called Coleman-Noll exploitation that is adopted here). In this formulation we usually look for the expression of so-called objective (or materially indifferent) entities. To that purpose we should rewrite (11.58) in terms of such quantities. This is achieved as follows. On the one hand we note from (11.47) that the skewsymmetric part of $\mathbf{t}$ is given by

$$t_{[ji]} = \rho\, \mu_{[j} B^L_{i]} \tag{11.61}$$

and we can write

$$\mathbf{t} = \mathbf{t}^S + \mathbf{t}^A, \text{i.e.}, t_{ji} = t_{(ji)} + t_{[ji]}. \tag{11.62}$$

We introduce the following objective time rates (cf. Maugin 1980):

$$D_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right) \tag{11.63}$$

and

$$\hat{m}_i = (D_J \mu)_i \equiv \mu_i - \Omega_{ij}\mu_j, \hat{M}_{ij} = \left(\dot{\mu}_i\right)_{,j} - \Omega_{ik}\mu_{k,j}, \tag{11.64}$$

with

$$\Omega_{ij} = \frac{1}{2}\left(v_{i,j} - v_{j,i}\right). \tag{11.65}$$

The quantities defined in (11.63) and (11.64) are indeed *objective*. The first of (11.64) is none other than a so-called *Jaumann* derivative. The second of (11.64) is not exactly the Jaumann derivative of the gradient of $\boldsymbol{\mu}$, but it is closely related to it modulo a term involving the rate of strain (11.64). On account of these we show that

$$tr\left[\mathbf{t}(\nabla \mathbf{v})^T\right] - \rho\, \mathbf{B}^L.\ \dot{\mu} + tr\left[\hat{\mathbf{B}}\,(\nabla\, \dot{\mu})^T\right] \equiv tr\left(\mathbf{t}^S \mathbf{D}\right) - \rho\, \mathbf{B}^L.\ \hat{\mathbf{m}} + tr\left(\hat{\mathbf{B}}\, \hat{\mathbf{M}}^T\right), \tag{11.66}$$

whence the looked for reduced useful expression for (11.57) and (11.58).

In summary, for the present modelling the local field equations at any regular material point in the body B are provided by Eqs. (11.45), (11.46), (11.53), (11.57) and the reduced form of Maxwell's equations (magnetostatics, no conduction)

$$\nabla \times \mathbf{H} = \mathbf{0}, \quad \nabla.\mathbf{B} = 0, \ \mathbf{H} = \mathbf{B} - \rho\,\mu, \tag{11.67}$$

in which the Cauchy stress $\mathbf{t}$ is given by Eqs. (11.61) and (11.62), and the effective magnetic induction $\mathbf{B}^{eff}$ is given by (11.56). Equation (11.57) will eventually provide the heat-propagation equation while the inequality (11.58) constrains the constitutive behaviour. Interestingly enough, we note that the energy Eq. (11.42) does not contain any contribution due to the spin lattice inertia by virtue of the d'Alembertian nature of this quantity [cf. (11.53)]—more on this point in the following two paragraphs.

Approach via the Principle of Virtual Power

In modern continuum mechanics, an elegant and powerful means of constructing field equations and associated natural boundary conditions is provided by an algebraically structured formulation of the (d'Alembert) principle of virtual power as exposed at length in Maugin (1980). In this somewhat abstract formulation this principle is enunciated in the following form for global powers over the body B and its boundary ∂B: *The virtual power of inertial forces is, at each instant of time, balanced by the total virtual power of "internal forces" and that of externally applied forces both in the bulk and at the surface*, the word "force" being understood in a generalized manner. Inertial forces have an expression provided by physics, internal forces need to be given constitutive equations, and external forces are prescribed in form and perhaps in value. In mathematical terms (See the chapter on the classical theory in Part One):

$$P^*_{inert}(B) = P^*_{int}(B) + P^*_{extern}(B, \partial B), \tag{11.68}$$

where an asterisk will denote the value of an expression is a so-called virtual velocity field (itself noted with an asterisk). In the present case, the generalized kinematical description of the model (11.30) provides the basic virtual velocity field by

$$v^* = \left\{ v^*_i, (\dot{\mu}_i)^* = (\omega^* \times \mu)_i \right\}, \tag{11.69}$$

where ω^* is a virtual precessional velocity of the *SC*. Thus

$$P^*_{inert}(B) = \int_B \left(\rho\, \dot{\mathbf{v}}.\mathbf{v}^* + \gamma^{-1}\, \dot{\mu}.\omega^* \right) dv, \tag{11.70}$$

where we clearly distinguish between real fields (no asterisks; actual solutions of a problem) and virtual ones (noted with an asterisk; at our disposal in this type of

variational formulation). In particular, for real fields, because of (11.53), (11.69) yields

$$P_{inert}(B) = \frac{d}{dt}\int_B \left(\frac{1}{2}\rho\,\mathbf{v}^2\right)dv = \frac{d}{dt}\,K(B), \tag{11.71}$$

where $K(B)$ is the total kinetic energy of the traditional motion.

The total power of external forces is obviously given by the following expression:

$$P^*_{extern}(B,\,\partial B) = P^*(B) + P^*(\partial B) \tag{11.72}$$

wherein

$$P^*(B) = \int_B ((\mathbf{f} + \mathbf{f}^{em}).\mathbf{v}^* + \rho\,\mathbf{B}.(\dot{\mu})^*)dv, \tag{11.73}$$

and

$$P^*(\partial B) = \int_{\partial B} \left(\left(\mathbf{t}_{(n)} + \mathbf{t}^{em}_{(n)}\right).\mathbf{v}^* + \mathbf{A}.(\dot{\mu})^*\right) da, \tag{11.74}$$

where $\mathbf{t}^{em}_{(n)}$ is an eventual magnetic surface traction related to the possible existence of a magnetic field outside B (see Maugin 1988, Chap. 6).

Finally, the global virtual power of internal forces is constructed as follows. First a "gradient order" is selected for the kinematics associated with internal forces. Generalizing classical continuum mechanics (which is a first order gradient theory of displacement) we consider a first-order gradient theory based on (11.69). That is,

$$V = \left\{v_i, v_{i,j}, \dot{\mu}_i, \dot{\mu}_{i,j}\right\}. \tag{11.75}$$

But internal forces must be *objective*, i.e., frame indifferent, or invariant under changes of observer in the actual configuration (superimposition of a rigid body motion of dimension 6). Accordingly, one must extract from the 24-dimensional space spanned by (11.75) a set of objective quantities, this set V_{obj}, a quotient space, being necessarily of dimension 24 − 6 = 18. We have shown elsewhere (Maugin 1980) how to systematically construct such quotient spaces. In the present case a good set is given by

$$V_{obj} = \left\{D_{ij}, \hat{m}_i, \hat{M}_{ij}\right\}, \tag{11.76}$$

where it happens that the quantities thus formally introduced have already been defined in (11.63)–(11.65). Then the power P^*_{inter} is written as a continuous linear

form on the set V^*_{obj}, introducing thus formally internal forces $\left\{\mathbf{t}^S, -\rho\,\mathbf{B}^L,\ \hat{\mathbf{B}}\right\}$ as co-factors of the elements of V^*_{obj}. That is (signs are chosen for convenience),

$$P^*_{int}(B) = -\int_B \left(t^S_{ji} v^*_{i,j} - \rho\, B^L_i\, \hat{m}^*_i + \hat{B}_{ji}\hat{M}^*_{ij}\right) dv. \tag{11.77}$$

Collecting the various contributions and assuming that the obtained global expression is valid for any element of volume and surface and any virtual velocity field (11.69) we obtain the local equations

$$\rho\,\dot{\mathbf{v}} = div\,\mathbf{t} + \mathbf{f} + \mathbf{f}^{em}\ in\,B, \tag{11.78}$$

and

$$\gamma^{-1}\dot{\mu} = -\left(\mathbf{B}^{eff} \times \mu\right)\ in\,B, \tag{11.79}$$

with the nonsymmetric stress $\mathbf{t}$ given by

$$t_{ji} = t^S_{ji} - \rho\, B^L_{[j}\mu_{i]} \tag{11.80}$$

on account of the constraint (11.54), and $\mathbf{B}^{eff}$ given by the second of (11.56). Simultaneously, we obtain the natural boundary conditions (not given here—see Maugin 1988, Chap. 6) for $\mathbf{t}$ and $\mathbf{A}$.

It is readily checked that Eqs. (11.78) and (11.79), together with (11.80) and (11.56) are identical to the equations deduced in the foregoing paragraph. Pursuing along the same line, and considering the principle (11.68) for real velocity fields, on account of (11.71) we obtain the *global equation of kinetic energy* in the form:

$$\frac{d}{dt}K(B) = P_{int}(B) + P_{extern}(B, \partial B). \tag{11.81}$$

This is to be combined with the global statement of the first law of thermodynamics (11.42) to deduce the global form of the *internal-energy theorem*. By localization this will yield (11.57) with the already transformed expression involving the objective internal forces. The exploitation of the inequality (11.58) is unchanged.

The present formulation—formal as it is—has certain advantages, one of which being the account of the d'Alembert-inertia couple in the expression (11.69). But more interestingly, it provides a direct modelling of more general ferroïc cases such as in ferrimagnets and antiferromagnets.

Hamiltonian Variational Formulation

The above-given formulation is valid for both deformable solid and fluid behaviours and also in the presence of dissipative processes such as *viscosity* (via **D**) and *spin-lattice relaxation* (via $\hat{\mathbf{m}}$). In the absence of dissipative processes and for an a priori known behaviour—e.g., elasticity—it is possible to approach the present theory via a Hamiltonian variational principle. For this type of approach to *elastic ferromagnets* we refer to Tiersten (1965), Brown (1966), and Maugin and Eringen (1972). Note that in order to account for the d'alembertian nature of the magnetic spin inertia one must introduce an *already varied term* for this effect.

Ferrimagnetic and Antiferromagnetic Materials

The magnetic description considered in the foregoing paragraphs often is insufficient and not realistic enough for many magnetic materials such as *ferrites*. Louis Néel (Nobel prize in physics 1970 for this matter) introduced in the early 1940s a model in which the most general description of the magnetization field in a magnetically ordered crystal below its magnetic-phase-transition temperature consists in the vector resultant of the sum of n magnetization fields $\mu_\alpha, \alpha = 1, 2, \ldots, n$ per unit mass—referred to as *magnetic sub-lattices*—arising at each point from n different ionic species having different spectroscopic splitting factors, thus various gyromagnetic ratios γ_α, so that the total magnetic spin per unit mass is not necessarily aligned with the total magnetization. This model proved to be efficient in accounting for the unusual magnetic properties (e.g., susceptibility) of ferrites—iron oxides—for which Néel coined the behaviour name *ferrimagnetism*. Simple *antiferromagnetism* is the special case for which only two magnetic sub-lattices subsist, of equal magnitude and opposite direction, allowing for the absence of global magnetization in the absence of applied magnetic field. But the magnetic response is quite different from that of classical ferromagnetism when a magnetic field is applied (see Eringen and Maugin 1990, Vol. I, pp. 110–111). The resulting dynamics is also much more involved yielding a multiplicity of magnon branches in the case of ferrimagnetism. Models of elastic ferrimagnetic and antiferromagnetic solids have been proposed by Maugin and Sioké-Rainaldy in the 1970s and 1980s, together with the accompanying coupled wave-like studies.

Analogy with Cosserat Continua

Returning to the ferromagnetic case, we note that the spin-precession Eq. (11.79) deals with axial vectors. Accordingly, we can introduce dual skewsymmetric

tensors by applying the alternation symbol ε_{kli} to its i-component. On account of the well known formula

$$\varepsilon_{kli}\varepsilon_{ipq} = \delta_{kp}\delta_{lq} - \delta_{kq}\delta_{lp}, \tag{11.82}$$

this operation results in the equation

$$\frac{1}{2}\rho\,\gamma^{-1}\varepsilon_{kli}\dot{\mu}_i = \rho\,\mu_{[k}B_{l]} + \rho\,\mu_{[k}B^L_{l]} + \left(\mu_{[k}\hat{B}_{ml]}\right)_{,m} - \mu_{[k,m}\hat{B}_{ml]}. \tag{11.83}$$

But we note that the last term in the right-hand of this equation vanishes identically because of the constraint (11.54), while the resulting penultimate term is none other that the skew part of the Cauchy stress according to (11.80), and the first term is none other than the ponderomotive couple written as a skew tensor (dual of the axial vector $\mathbf{c}^{em}$). Thus Eq. (11.83) reads

$$\rho\,\dot{S}_{kl} = M_{pkl,p} + t_{[kl]} + C_{kl}, \tag{11.84}$$

wherein

$$S_{kl} = \frac{1}{2}\gamma^{-1}\varepsilon_{kli}\mu_i,\ M_{pkl} = \mu_{[k}\hat{B}_{pl]},\ C_{kl} = C^{em}_{kl} = \varepsilon_{kli}\,\tilde{c}^{em}_i. \tag{11.85}$$

Equation (11.84) is in the canonical form of the balance equation of angular momentum in *Cosserat or micropolar continua* except that all contributions here have a magnetic origin, the gyromagnetic relation for the inertial term S_{kl}, Heisenberg exchange forces for the couple stress tensor M_{pkl}, the applied coupled C_{ij}, and the skew part of the nonsymmetric Cauchy stress. Equation (11.84) and the accompanying boundary condition were deduced by Maugin in his PhD thesis (Princeton 1971).

Reduction to a Model Without Microstructure (Paramagnetic and Soft-ferromagnetic Bodies)

When true ferromagnetic effects (gyromagnetic effect, Heisenberg exchange forces) are discarded, Eq. (11.84) reduces to

$$t_{[kl]} = C^{em}_{[kl]} = -M_{[k}B_{l]} = B_{[k}M_{l]}. \tag{11.86}$$

This applies to the simpler cases of nonlinear paramagnetic and soft-ferromagnetic bodies. The resulting theory applies, in particular, to

magnetoelastic polymers as recently developed (on this subject, one can refer to the Udine course of 2009; Ogden and Steigman, Editors, 2010; and also Chap. 8, Vol. I, Eringen and Maugin 1990). Whenever field and magnetization are aligned (case of magnetically isotropic bodies) or in a purely linear theory in which one discards the right-hand side of (11.86) as being second order in the fields, the skew part of the stress is zero. The only remaining magneto-mechanical coupling in the first case remains magnetostriction for any symmetry, while in the second case only piezo-magnetism may exist, under severe symmetry conditions however.

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Cross references: Cosserat continua, Ferromagnetism, Ferroic states, Micropolar continua Ponderomotive couple.

From "Micromorphic Continua" to "Multipolar Continua (Green-Rivlin)"

Micromorphic Continua

This is the name granted by Eringen (e.g., in Eringen 1999) to a large class of continua in his classification of models of generalized continuum mechanics. In a general way this qualification applies to continua in which each material "point" is endowed with a *deformable microstructure*, in addition to the usual degree of freedom of translation provided by the displacement field. This modelling includes in particular models devised by Eringen and Suhubi (1964) and Mindlin (1964), and also multipolar continua (Green and Rivlin 1964) and media with a set of deformable directors at each material point. The general idea may be traced back to Duhem (1893) and Voigt (1887). The continuum is called a Cosserat or micropolar continuum when the microstructure is *rigid*, and therefore accounts only for a possible internal *rotation*.

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Cross references: Cosserat continua, Directors theory, Eringen A.C., Generalized continuum mechanics, Micromorphic fluids, Microstructure, Microstructured continuum theory (Eringen), Microstructured continuum theory (Mindlin), Microstructured fluids, Mindlin R.D., Oriented media (with directors), Relaxed micromorphic continua, Solutions of macromolecules.

Micromorphic Fluids

These are micromorphic continua with a fluid-like behaviour. Contrary to micro-stretch fluids, the whole microdeformation of micromorphic continua (see that entry) is kept in the kinematic description. They were originally introduced by Eringen (1964) with additional notions on micro-inertia (its conservation) in Eringen (1966).

We remind the reader [see Entry: "Microstructured continuum theory (Eringen)"] that the following two local field equations (in all twelve components)

must be considered as the expression of the laws of linear momentum and moment of momentum:

$$t_{kl,k} + \rho f_l = \rho\, \dot{v}_l, \tag{11.87}$$

$$m_{klm,k} + t_{ml} - s_{ml} + \rho\, l_{lm} = \rho\, \dot{\sigma}_{lm}, \tag{11.88}$$

where f_l and l_{lm} are densities of prescribed body vector force and (symmetric and skewsymmetric) tensor forces, t_{kl} is the nonsymmetric Cauchy stress, m_{klm} is the *hyperstress tensor*, and s_{ml} stands for an *intrinsic symmetric stress*. Equations (11.87) and (11.88) are to be complemented by the conservation laws of mass and micro-inertia. These read

$$\dot{\rho} + \rho\, \nabla.\mathbf{v} = 0, \tag{11.89}$$

$$\frac{d\, i_{kl}}{dt} - (i_{kr} v_{lr} + i_{lr}\, v_{kr}) = 0. \tag{11.90}$$

Here $i_{kl} = i_{lk}$ is the micro-inertia tensor, v_{kr} is the microgyration tensor, and the moment of moment/spin tensor $\dot{\sigma}_{lm}$ is related to the gyration tensor through the equation

$$\dot{\sigma}_{lm} = i_{jm}\left(\frac{d\, v_{lj}}{dt} + v_{li}\, v_{ij}\right). \tag{11.91}$$

The rates of strains are given by expressions

$$a_{kl} = v_{l,k} - v_{lk}, \quad b_{klm} = v_{kl,m}, \quad c_{kl} = \frac{1}{2}(v_{kl} + v_{lk}) = c_{lk}. \tag{11.92}$$

Both a_{kl} and b_{klm} admit symmetric and skewsymmetric parts in their indices k and l. The same holds true for t_{kl} and m_{pkl}. The local internal-energy equation and the Clausius-Duhem inequality read

$$\rho\, \dot{e} = t_{kl} a_{kl} + s_{kl} c_{lk} + m_{km}\, b_{lmk} - q_{k,k} + \rho\, h, \tag{11.93}$$

$$-\rho\left(\dot{\psi} + \eta\dot{\theta}\right) + t_{kl} a_{kl} + s_{kl} c_{lk} + m_{km} b_{lmk} - (q_k/\theta)\, \theta_{,k} \geq 0. \tag{11.94}$$

With a free energy density of the form

$$\psi = \overline{\psi}\left(\theta, \rho^{-1}, i_{kl}\right), \tag{11.95}$$

the thermodynamically recoverable parts of the constitutive behaviour are obtained as

$$\eta = -\frac{\partial \overline{\psi}}{\partial \theta}, t_{kl}^R = -\pi\, \delta_{kl}, m_{klm}^R = 0, s_{kl}^R = -\pi\delta_{kl} - \pi_{kl}, \tag{11.96}$$

with the "pressures"

$$\pi = -\frac{\partial \overline{\psi}}{\partial \rho^{-1}}, \pi_{kl} = -\rho\left(\frac{\partial \overline{\psi}}{\partial i_{rk}} i_{rl} + \frac{\partial \overline{\psi}}{\partial i_{rl}} i_{rk}\right). \tag{11.97}$$

The dissipative contributions (indexed with a superscript D) and heat conduction will have to satisfy the following remaining dissipation inequality:

$$t_{kl}^D a_{kl} + s_{kl}^D c_{lk} + m_{km}^D b_{lmk} - (q_k/\theta)\, \theta_{,k} \geq 0. \tag{11.98}$$

There is no need here to expand the long expressions of these dissipative parts even for the relatively simple case of linear isotropic thermo-viscous fluids (see Eringen 2001, pp. 283–285). When substituted for in the field equations, the resulting long coupled expressions provide a system of twenty equations for twenty unknown fields, this high number demonstrating thus the intractability of the modelling in general. But with some reasonable approximations this kind of involved modelling seems to provide satisfactory descriptions of some physical situations and solutions to exemplary problems. This is the case of the flow of dilute suspensions, blood flow in narrow arteries, and perhaps some possible approach to turbulence. The reader is referred to Eringen (2001, Chap. 17), Kang and Eringen (1976), Kirwan and Newman (1969), and Eringen (1972) for these applications.

For the sake of completeness should be mentioned the theory of macromolecule solutions by Maugin and Drouot (1983) where the notion of tensor internal variable of state (this is purely "dissipative") replaces that of the internal degree of freedom of micromotion, in the absence of micro-inertia, and also the theory of microstructured bodies proposed by Capriz (1989) which seems well adapted to an approach of microstructured continua with voids and bubbles or dilatant granules. In the theory of Maugin and Drouot (1983), the microdeformation is replaced by the "conformation" of macromolecules that swim in the liquid carrier. Chapter 5 in Stokes (1984) is also a relevant reference with a readable introduction to the subject.

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Cross references: Eringen-Mindlin medium, Generalized continuum mechanics, Internal degrees of freedom, Micromorphic continua, Microstructured fluids, Solutions of macromolecules.

Micropolar Continua (Cf. Cosserat Continua)

«Micropolar continua» is the denomination given by A.C. Eringen to a subclass of "micromorphic" bodies, in which the additionally introduced internal deformation at each material point reduces to a pure rotation. This is another naming for "Cosserat continua" or "polar continua" or "oriented media" (with a triad of rigidly rotating "directors" of unit length). Such media respond to couple stresses. Accordingly, the primary kinematics of these continua is described by the classical (macro-) motion and a micro-motion $\underline{\chi}$ such that in components (with the standard notation of nonlinear continuum mechanics between a reference configuration and the actual configuration)

$$x_k = \bar{x}_k(X_K, t), \quad \chi_{kK} = \bar{\chi}_{kK}(X_K, t), \tag{11.99}$$

where $\underline{\chi}$ reduces to a proper orthogonal transformation such that

$$\underline{\chi}^T = \underline{\chi}^{-1}, \quad \det \underline{\chi} = +1. \tag{11.100}$$

The most efficient representation of the micromotion is given by Gibbs' equation (Gibbs 1901; also Eringen 1999, Sect. 1.3) revived by Kafadar and Eringen (1971):

$$\chi_{kl} = \chi_{kK}\, \delta_{Kl} = \cos\phi\, \delta_{kl} - \sin\phi\, \varepsilon_{klm} n_m + (1 - \cos\phi)\, n_k n_l, \tag{11.101}$$

where

$$\phi = (\phi_k \phi_k)^{1/2}, \quad n_k = \phi_k/\phi. \tag{11.102}$$

Here ϕ is the angle of rotation about the axis of rotation $\mathbf{n}$, and δ_{Kl} denotes the director cosines of spatial and material frames.

The gyration tensor v_{kl} is such that

$$\dot{\chi}_{kK} = v_{kl}\,\chi_{lK} \tag{11.103}$$

The associated microgyration vector v_m is such that

$$v_{lk} = -\varepsilon_{lkm} v_m,\, v_m = -\frac{1}{2}\varepsilon_{mlk} v_{lk}. \tag{11.104}$$

It is shown that this vector is related to the rate of rotation vector $\dot{\phi}$ and $\dot{\mathbf{n}}$ via the equation

$$v_m = \dot{\phi}\, n_m + \sin\,\phi\, \dot{n}_m + (1 - \cos\,\phi)(\mathbf{n} \times \dot{\mathbf{n}})_m = \Lambda_{mp}\,\dot{\phi}_p, \tag{11.105}$$

with

$$\Lambda_{mp} = \frac{\sin\,\phi}{\phi}\delta_{mp} - \frac{1 - \cos\phi}{\phi^2}\varepsilon_{mpq}\phi_q + \left(1 - \frac{\sin\,\phi}{\phi}\right)\frac{\phi_m\,\phi_p}{\phi^2}. \tag{11.106}$$

Deformation rates are given by

$$a_{kl} = v_{l,k} - v_{lk} = v_{l,k} - \varepsilon_{klm} v_m, b_{kl} = v_{k,l}. \tag{11.107}$$

Linear Strain Measures

With a rotation tensor ϕ_{kK} defined by

$$\phi_{kK} = \chi_{kK} - \delta_{kK}, \tag{11.108}$$

in the linear approximation we have

$$\chi_{kK} \cong (\delta_{kl} + \phi_{kl})\,\delta_{lK}, \tag{11.109}$$

and we obtain linear strain tensors in the form

$$e_{kl} = u_{l,k} - \varepsilon_{klm}\phi_m, \quad \gamma_{kl} = \phi_{k,l}. \tag{11.110}$$

This is useful in dealing with linear micropolar elasticity (see that entry). The fully nonlinear theory of micropolar elasticity is of rare use except in solving problems such as the (Euler's) *elastica* (cf. Kafadar 1972). The theory of micropolar fluids is much exploited in many flow problems (see the entry "micropolar fluids").

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Cross references: Cosserat continua, Couple stress, Micropolar elasticity, Micropolar fluids, Microstructured continuum theory (Eringen), Oriented media.

Micropolar Elasticity

This is the theory of *elastic solids* deduced from a mechanics of generalized continua which has for basis the theory of micromorphic continua when the micromotion is reduced to a pure rotation. This is the naming given by A.C.Eringen, one of its most active proponents in the period 1964–1980. It was the object of a formidable reception with a huge quantity of papers dealing with various problems in media thought to correspond to such a description. However, it is basically the same as the theory of (elastic) Cosserat continua, also called polar continua or theory of oriented media or still theory of asymmetric elasticity by other authors. Such media respond to *couple stresses*. Historical papers in the field are by Aero and Kuvshinskii (1960) and Palmov (1964). Some of the ideas can be traced back to Pierre Duhem and Woldemar Voigt in the 1890s. A crystal clear exposition of the bases of the theory was given by Eringen in 1968. Another influential book was the one by Nowacki (1986). A more recent exposition is offered in Eringen's (1999, Chap. 5) synthesis. A sound introduction to the fully nonlinear theory is to be found in the paper of Kafadar and Eringen (1971) by using Gibbs' representation of orthogonal transformations. But, obviously, most papers and applications have been devoted to the case of small deformations and small rotations of the internal structure of material points as can be expected in solid crystals to which the theory clearly applies.

The primary kinematics of these continua is described by the classical (macro-) motion and a micro-motion $\underline{\chi}$ such that in components (with the standard notation of nonlinear continuum mechanics between a reference configuration and the actual configuration)

$$x_k = \bar{x}_k(X_K, t), \quad \chi_{kK} = \bar{\chi}_{kK}(X_K, t), \tag{11.111}$$

where $\underline{\chi}$ reduces to an orthogonal transformation such that

$$\underline{\chi}^T = \underline{\chi}^{-1}, \quad \det \underline{\chi} = +1. \tag{11.112}$$

The most efficient representation of the micromotion is given by Gibbs' equation (Gibbs 1901; also Eringen 1999, Sect. 1.3) revived by Kafadar and Eringen (1971):

$$\chi_{kl} = \chi_{kK}\,\delta_{Kl} = \cos\phi\,\delta_{kl} - \sin\phi\,\varepsilon_{klm}n_m + (1-\cos\,\phi)\,n_k n_l, \tag{11.113}$$

where

$$\phi = (\phi_k\phi_k)^{1/2}, \quad n_k = \phi_k/\phi. \tag{11.114}$$

Here ϕ is the angle of rotation about the axis of rotation of unit direction $\mathbf{n}$, and δ_{Kl} denotes the director cosines of spatial and material frames.

The gyration tensor v_{kl} is such that

$$\dot{\chi}_{kK} = v_{kl}\,\chi_{lK} \tag{11.115}$$

The associated microgyration vector v_m is such that

$$v_{lk} = -\varepsilon_{lkm}v_m, \quad v_m = -\frac{1}{2}\varepsilon_{mlk}v_{lk}. \tag{11.116}$$

It is shown that this vector is related to the rate of rotation vector $\dot{\phi}$ and $\dot{\mathbf{n}}$ via the equation

$$v_m = \dot{\phi}\,n_m + \sin\,\phi\,\dot{n}_m + (1-\cos\phi)(\mathbf{n}\times\dot{\mathbf{n}})_m = \Lambda_{mp}\,\dot{\phi}_p. \tag{11.117}$$

with

$$\Lambda_{mp} = \frac{\sin\,\phi}{\phi}\,\delta_{mp} - \frac{1-\cos\phi}{\phi^2}\varepsilon_{mpq}\phi_q + \left(1-\frac{\sin\,\phi}{\phi}\right)\frac{\phi_m\,\phi_p}{\phi^2}. \tag{11.118}$$

Lagrangian measures of finite deformations are deduced from (11.111) as

$$C_{KL} = x_{k,K}x_{k,L}, \quad \overline{C}_{KL} = x_{k,K}\chi_{kL}, \quad \Gamma_{KLM} = \chi^{-1}_{Kk}\,\chi_{kL,M}. \tag{11.119}$$

The first of these is the standard Cauchy-Green strain, the second may be called the *Cosserat strain*, and the last is referred to as the *wryness tensor* (by the unusual term "wryness" one understands something like "distortion", "contortion" or "twist"). As a consequence, from (11.112) we note that

$$\mathbf{C} = \overline{\mathbf{C}}\,\overline{\mathbf{C}}^T, \tag{11.120}$$

so that an energy density function $W = W\big(C_{KL}, \overline{C}_{KL}, \Gamma_{KLM}\big)$ can be replaced by an expression

$$W = \overline{W}\big(\overline{C}_{KL}, \Gamma_{KLM}\big). \tag{11.121}$$

Because of the skewsymmetry built in the tensor object Γ_{KLM}, this can be replaced by the object

$$\Gamma_{QL} = -\frac{1}{2}\varepsilon_{QPK}\Gamma_{KPL} \tag{11.122}$$

simply directly noted $\underline{\Gamma}$. Thus (11.121) is also replaced by

$$W = \widetilde{W}\left(\overline{\mathbf{C}}, \underline{\Gamma}\right). \tag{11.123}$$

The local coupled balance laws of linear and angular momenta in the actual configuration are given by the two equations

$$t_{lk,l} + \rho(f_k - \dot{v}_k) = 0, \tag{11.124}$$

$$m_{lk,l} + \varepsilon_{kmn}t_{mn} + \rho(l_k - \dot{\sigma}_k) = 0, \tag{11.125}$$

where m_{lk}, l_k and $\dot{\sigma}_k$ stand for the couple-stress tensor, the applied couple density and the density of internal-spin rate while t_{lk} is the nonsymmetric stress tensor. In addition we have the local statement of conservation of mass ρ and micro-inertia j_{kl}(that we do not recall here) so that

$$\dot{\sigma}_k = \frac{d}{dt}(j_{kl}v_l), \tag{11.126}$$

where v_l is the gyration (rate) vector introduced in (11.116). The local energy balance in the absence of thermal effects can be written as

$$\rho\,\dot{e} = t_{lk}\left(v_{k,l} - \varepsilon_{lkm}\,v_m\right) + m_{lk}v_{k,l}. \tag{11.127}$$

Equations (11.124), (11.125) and (11.127) admit a re-writing in the material (Piola-Kirchhoff) form as

$$T_{Ki,K} + \rho_0(f_i - \dot{v}_i) = 0, \tag{11.128}$$

$$M_{Ki,K} + \varepsilon_{ipq}x_{p,L}T_{Lq} + \rho_0(l_i - \dot{\sigma}_i) = 0 \tag{11.129}$$

and ($W = \rho_0 e$)

$$\dot{W} = T_{Ki}a_{pi}x_{p,K} + M_{Ki}b_{ij}x_{j,K}, \tag{11.130}$$

wherein (Piola transformations)

$$T_{Ki} = JX_{K,j}\,t_{ji}, \quad M_{Ki} = JX_{K,j}m_{ji}, \quad J = \rho_0/\rho, \tag{11.131}$$

and (rates of deformations)

$$a_{pi} = v_{i,p} - \varepsilon_{pim} v_m, \quad b_{ij} = v_{i,j} \tag{11.132}$$

From (11.130) and a direct computation of the time rates of $\overline{\mathbf{C}}$ and $\underline{\Gamma}$,

$$\dot{\overline{C}}_{KL} = x_{j,K}\, a_{ij}\, \chi_{iL}, \quad \dot{\Gamma}_{KL} = x_{j,K} b_{ij}\, \chi_{iL}, \tag{11.133}$$

there will follow the two constitutive equations

$$T_{Ki} = \frac{\partial \widetilde{W}}{\partial \overline{C}_{KL}} \chi_{iL}, \quad M_{Ki} = \frac{\partial\, \widetilde{W}}{\partial \Gamma_{LK}}\, \chi_{iL}; \tag{11.134}$$

and thus

$$t_{ji} = \frac{\rho_0}{\rho} x_{j,Q} T_{Qi} = \frac{\rho_0}{\rho} x_{l,K} \frac{\partial\, \widetilde{W}}{\partial \overline{C}_{KL}} \chi_{iL}, \quad m_{ji} = \frac{\rho_0}{\rho} x_{j,K} \frac{\partial \widetilde{W}}{\partial \Gamma_{LK}}\, \chi_{iL}. \tag{11.135}$$

These are quite general and, of course, of limited use. One r are exploitation of these equations in 2D is by Kafadar (1972) concerning the problem of the elastica (large deformation and finite internal rotation).

Theory for Small Strains and Small Internal Rotation Angles

This is also called the "*linear theory*" of micropolar elasticity. To reach this useful approximation we first set $\widetilde{\mathbf{C}} = \overline{\mathbf{C}} - \mathbf{I}_R$, and note that for small angle ϕ, Eq. (11.113) yields

$$\chi_{kl} = \chi_{kK}\, \delta_{Kl} \cong \delta_{kl} - \varepsilon_{klm} \phi_m, \quad \phi_m = \phi n_m, \tag{11.136}$$

so that

$$\widetilde{C}_{KL} \delta_{Ki} \delta_{Lj} \cong e_{ij} := u_{j,i} - \varepsilon_{ijk}\, \phi_k, \tag{11.137}$$

where u_i are the components of the usual displacement. Similarly,

$$\Gamma_{LK} \delta_{Li} \delta_{Kj} \cong \phi_{i,j} = \gamma_{ij}. \tag{11.138}$$

Then Eqs. (11.135) provide the "small strain" constitutive equations

$$t_{ji} = \frac{\partial \hat{W}}{\partial e_{ji}}, \quad m_{ji} = \frac{\partial \hat{W}}{\partial \phi_{j,i}}, \tag{11.139}$$

with $W = \hat{W}(e_{ij}, \gamma_{ij})$ reduced to a quadratic function of its two arguments. For example, for *isotropy*,

$$W = \frac{1}{2}\left[\lambda\, e_{kk}e_{ll} + (\mu+\kappa)e_{kl}\, e_{kl} + \mu\, e_{kl}e_{lk} + \alpha\, \phi_{k,k}\phi_{l,l} + \beta\phi_{k,l}\phi_{l,k} + \gamma\, \phi_{k,l}\phi_{k,l}\right], \tag{11.140}$$

where λ, μ, κ and α, β, γ are the six remaining material coefficients in this modelling of elasticity. Necessary and sufficient conditions for the energy density W to be nonnegative are

$$\begin{aligned} &3\lambda + 2\mu + \kappa \geq 0, \quad 2\mu + \kappa \geq 0, \;\; \kappa \geq 0, \\ &3\alpha + \beta + \gamma \geq 0, \quad \gamma \geq \beta \geq -\gamma, \;\; \gamma \geq 0. \end{aligned} \tag{11.141}$$

Coefficients λ and μ would be the Lamé coefficients in the absence of rotating microstructure. In the same conditions the inertial terms of Eqs. (11.124) and (11.125) reduce to

$$\dot{v}_i = \ddot{u}_i, \quad \dot{\sigma}_i = j\ddot{\phi}_i, \tag{11.142}$$

where j is a constant inertia factor.

On substituting from (11.139) and (11.140) in Eqs. (11.124) and (11.125) we finally obtain the following two coupled equations of motion written in direct intrinsic notation

$$(\lambda + 2\mu + \kappa)\nabla\,\nabla.\mathbf{u} - (\mu+\kappa)\nabla \times \nabla \times \mathbf{u} + \kappa\,\nabla \times \underline{\phi} + \rho_0(\mathbf{f} - \ddot{\mathbf{u}}) = \mathbf{0}, \tag{11.143}$$

$$(\alpha + \beta + \gamma)\nabla\nabla.\underline{\phi} - \gamma\nabla \times \nabla \times \underline{\phi} + \kappa\nabla \times \mathbf{u} - 2\kappa\underline{\phi} + \rho_0\left(\mathbf{l} - j\ddot{\underline{\phi}}\right) = \mathbf{0}. \tag{11.144}$$

Accompanying boundary conditions at a regular boundary will involve data in the components of the functions $\mathbf{u}$ and $\underline{\phi}$ or the normal components of the stress and couple-stress tensors.

As a special case one can construct a theory of indeterminate couple-stresses where the micro-rotation is slaved to the macro-rotation (deduced from the curl of the elastic displacement). This, as we know, yields an unsatisfactory dynamical theory (cf. Eringen 1968, Sect. XXIII).

Several books offer static solutions of boundary-value problems for linear micropolar elasticity (cf. Eringen 1968, 1999, Chap. 5; Nowacki 1986). Dynamical solutions involving bulk and surface modes are also studied in many works

including in the original paper of Eringen and Suhubi of 1964. Variational principles, uniqueness and reciprocity theorems have been established (e.g., by D. Iesan in Romania with co-operations with R. Quintanilla and A. Scalia) as also fundamental solutions (by L. Dragos)—see also Dyszlewicz (2004). Conservation laws (in the manner of E. Noether) associated with the balance laws have been discussed by various authors (e.g., Pucci and Saccomandi 1990; Maugin 1998; Eringen 1999, Sect. 5.10); the related crack and fracture problems, propagation of transformation surfaces, dislocations and disclinations have been examined in papers and books (e.g., Maugin 2010, Chap. 9). Nonlinear waves (with small deformations but finite microrotations) were studied by Maugin and Miled (1986), Erbay and Suhubi (1989) and others. Micropolar plates and shells are the object of thorough studies (e.g., by V.A. Eremeyev and H. Altenbach—see Altenbach and Eremeyev, Eds, 2013). In all we can say that micropolar elasticity has received almost as much due attention as standard elasticity.

However, two questions are of utmost importance. One is whether the continuum models of of micropolar elasticity can be deduced in some way from some microscopic model. Such a derivation was presented by Askar and Cakmak (1968) where in fact the microscopic medium was modelled by man-made structure-grid frameworks (see also Aksar 1986). This is rather astute albeit phenomenological at the utmost. Another possibility is to start from a lattice where rotations of molecules are allowed in addition to the translation of material points to which they are attached. This was expanded by Askar (1972, 1986) and Pouget et al. (1986) for elastic crystals that exhibit molecular groups. The identification with continuum equations is obtained in a so-called long-wavelength limit. It delivers plausible values for constitutive constants except perhaps for the micro-inertia. A more recent 2D model is by Pavlov et al. (2006). A completely different type of approach exploits the technique of homogenization in periodic media and yields possible models of generalized continua although the initial microscopic vision involves only standard models (cf. Forest 2006).

The second naturally raised question concerns the possibility to reach significant values of the material coefficients of micropolar elasticity by means of statical or dynamical experiments. The early dynamical (wave) experiments by Gauthier and Jashman (1975) using a man-made model material (epoxy matrix with uniformly distributed "rigid" aluminium inclusions; also Jashman and Gauthier 1980) were rather inconclusive.

The numerous experiments by Rod S. Lakes and co-workers in Wisconsin seem to be more reliable (see Lakes 1995). They use naturally structured animal bones. They exhibit a size effect that helps determining characteristic lengths necessarily involved in the micropolar model. They produce an enhancement of toughness in the experiment examining the stress concentration at a circular hole. As it should, the micropolar solid exhibits a new kind of wave associated with the micro-rotation (cf. Lakes 1995; much more on the website of R.S. Lakes). Of course the difficulty of carrying these experiments should not be overlooked; but they are absolutely

necessary to give sound credentials to a theory that appears too farfetched to many engineers, but is also more and more currently accepted by materials scientists despite the scarcity of reliable experimental results.

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Cross references: Asymmetric elasticity, Cosserat continua, Couple stress, Eringen A.C., Generalized continuum mechanics, Generalized internal forces, Micropolar continua.

Micropolar Fluids

This is the theory of *fluids* deduced from a mechanics of generalized continua which has for basis the theory of micromorphic continua where the micromotion is reduced to a pure rotation. It was the object of a formidable reception with thousands of papers dealing with various flows including in liquid crystals with rigid molecules, magnetic fluids, clouds with dust, muddy fluids, and biological fluids.

The balance laws of general micropolar fluids at any regular point in the flow are as follows (cf. Eringen 1966, 2001, Chap. 9):

Conservation of mass

$$\dot{\rho} + \rho \nabla . \mathbf{v} = 0; \tag{11.145}$$

Conservation of micro-inertia

$$\frac{d j_{kl}}{dt} + \left(\varepsilon_{kpr} j_{lp} + \varepsilon_{lpr} j_{kp}\right) v_r = 0; \tag{11.146}$$

Balance of linear momentum

$$t_{kl,k} + \rho(f_l - \dot{v}_l) = 0; \tag{11.147}$$

Balance of moment of momentum

$$m_{kl,k} + \varepsilon_{lmn}\, t_{mn} + \rho(l_l - \dot{\sigma}_l) = 0; \tag{11.148}$$

Balance of energy

$$-\rho\, \dot{e} + t_{kl}\, a_{kl} + m_{kl}\, b_{lk} - q_{k,k} + \rho h = 0. \tag{11.149}$$

This is complemented by the local statement of the second aw of thermodynamics in the usual form:

$$\rho\dot{\eta} + (q_k/\theta)_{,k} - \rho\, h/\theta \geq 0. \tag{11.150}$$

In these equations t_{kl} is the nonsymmetric Cauchy stress, m_{kl} is the couple stress tensor, e is the internal energy density, l_l is the applied body couple (if any), q_k is the heat (in)flux, h is the energy source density, η is the entropy density, θ is the thermodynamic temperature $(\theta > 0, \inf\, \theta = 0)$, and σ_l is the internal spin density (or angular momentum) such that

$$\dot{\sigma}_l = \frac{d}{dt}\,(j_{lk}\, v_k), \tag{11.151}$$

where j_{lk} is the micro-inertia density that must satisfy the conservation law (11.146) in parallel with the conservation of mass (11.145). In addition,

$$a_{kl} = v_{l,k} - v_{lk} = v_{l,k} - \varepsilon_{klm} v_m, \quad b_{kl} = v_{k,l} \tag{11.152}$$

are the relevant deformation-rate tensors.

Inequality (11.150) is viewed as a constraint imposed on the constitutive equations needed to close the system of field equations. For isotropic micropolar fluids, one takes $j_{kl} = j\,\delta_{kl}$ with $dj/dt = 0$, and we can introduce the free energy density ψ and a dissipation potential Φ

$$\psi = e - \eta\theta = \overline{\psi}\left(\rho^{-1}, j, \theta\right), \tag{11.153}$$

and

$$\Phi = \overline{\Phi}\left(a_{kl}, b_{kl}, \theta_{,k}; \theta, \rho^{-1}, j\right). \tag{11.154}$$

Exploitation of the Clausius-Duhem inequality

$$-\rho\left(\dot{\psi} + \eta\dot{\theta}\right) + t_{kl}a_{kl} + m_{kl}b_{lk} - q_k\left(\theta_{,k}/\theta\right) \geq 0 \tag{11.155}$$

will yield the following constitutive equations:

$$\eta = -\frac{\partial \overline{\psi}}{\partial \theta}, t_{kl} = -p\,\delta_{kl} + t^D_{kl}, p = -\frac{\partial \overline{\psi}}{\partial \rho^{-1}}, \tag{11.156}$$

and

$$t^D_{kl} = \frac{\partial \overline{\Phi}}{\partial a_{kl}},\ m_{kl} = m^D_{kl} = \frac{\partial\, \overline{\Phi}}{\partial b_{lk}}, q_k = -\frac{\partial \overline{\Phi}}{\partial\left(\theta_{,k}/\theta\right)}. \tag{11.157}$$

Examples of linear dissipative constitutive equations for "thermo-microfluids" are given by the following expressions (Eringen 1972, 2011, p 14):

$$t^D_{kl} = \lambda_v\, a_{mm}\delta_{kl} + (\mu_v + \kappa_v)a_{kl} + \mu_v\, a_{lk}, \tag{11.158}$$

$$m^D_{kl} = (\alpha/\theta)\varepsilon_{klm}\,\theta_{,m} + \alpha_v\, b_{mm}\delta_{kl} + \beta_v\, b_{kl} + \gamma_v b_{lk}, \tag{11.159}$$

$$q_k = K\,\theta_{,k}/\theta + \alpha\,\varepsilon_{klm}\,v_{m,l}, \tag{11.160}$$

with various scalar coefficients that have to check a set of inequalities in order to satisfy the non-negativeness of the dissipation function. These inequalities read (see Eqs. 11.9.4.12 in Eringen 2011)

$$\begin{aligned} &3\lambda_v + 2\mu_v + \kappa_v \geq 0,\ 2\mu_v + \kappa_v \geq 0,\ \kappa_v \geq 0,\\ &3\alpha_v + \beta_v + \gamma_v \geq 0,\ \gamma_v + \beta_v \geq 0,\ K(\gamma_v - \beta_v) \geq 2\,\alpha^2,\\ &K \geq 0,\ \gamma_v - \beta_v \geq 0\, when\, \alpha = 0 \end{aligned} \tag{11.161}$$

The first two of these reduce to the classical inequalities for viscosities in the absence of microstructure.

Equations (11.158) and (11.159) exhibit a possible original coupling between heat conduction and the viscosity effects related to the microstructure.

In the absence of heat effects, the local balance equations for isotropic micropolar fluids are given by the following set at any regular point in the fluid:

$$\dot{\rho} + \rho\nabla.\mathbf{v} = 0, \tag{11.162}$$

$$\frac{d\,j}{dt} = 0, \tag{11.163}$$

$$-\nabla p + (\lambda_v + 2\mu_v + \kappa_v)\nabla\,\nabla.\mathbf{v} - (\mu_v + \kappa_v)\,\nabla \times \nabla \times \mathbf{v} + \kappa_v \nabla \times \underline{v} + \rho\,(\mathbf{f} - \dot{\mathbf{v}}) = \mathbf{0}, \tag{11.164}$$

$$(\alpha_v + \beta_v + \gamma_v)\nabla\,\nabla.\underline{v} - \gamma_v \nabla \times \nabla \times \underline{v} + \kappa_v\,\nabla \times \mathbf{v} - 2\kappa_v \underline{v} + \rho\left(\mathbf{l} - j\,\frac{d\underline{v}}{dt}\right) = 0. \tag{11.165}$$

Typically associated boundary conditions involve a combination of data in velocities $\mathbf{v}$ and $\underline{v}$ and normal components of the stress tensor and couple-stress tensor. The conditions $\mathbf{v} = \mathbf{0}$ and $\underline{v} = \mathbf{0}$ at a boundary mean *strict adherence*. More involved boundary conditions involve both the gyration velocity $\underline{v}$ and the gradient of the usual velocity field $\mathbf{v}$ (cf. Kirwan 1986). The above set equations have been the subject of many applications (including generalization of standard flow problems for Newtonan viscous fluids). This is well documented in Eringen (2011, Chap. 9); Stokes [1984, Chap. 6 (Caution: different notation)], and the reviews of Ariman, Turk and Sylvester (1973, 1974); and Cowin (1974). Historical papers on the matter were by Aero et al. (1965) and Cowin (1968). Mathematical properties of the system of equations for micropolar fluids are studied in Lukaszewicz (1998).

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Cross references: Cosserat continua, Couple stress, Eringen A.C., Generalized continuum mechanics, Generalized internal forces, Micropolar continua.

Microstretch Continua

This class of continua was introduced by Eringen (1969) in the context of fluids. In Eringen's vision of microcontinuum theory this class is intermediate between micromorphic continua and micropolar (Cosser at) ones, in that it retains not only an internal rotation but also a scalar internal degree of elongation (as if a single

director was deformable in length). That is, we consider the general local balance equations of micromorphic bodies (cf. the Eringen-Mindlin micromorphic model of microstructured continua)

$$t_{kl,k} + \rho\,(f_l - \dot{v}_l) = 0 \tag{11.166}$$

and

$$\mu_{kij,k} + t_{ji} - s_{ji} + \rho\, l_{ij} - \rho\,\dot{\sigma}_{ij} = 0, \tag{11.167}$$

with

$$t_{ji} = t_{(ji)} + t_{[ji]}, \quad s_{[ji]} = 0, l_{ji} = C_{ji} + l_{(ji)} \tag{11.168}$$

where μ_{kji} is called the *hyperstress* tensor, s_{ji} is the so-called *symmetric microstress*, and l_{ij} is a density of *body-moment tensor* of which the skew part represents a density of body couple $C_{ji} = -C_{ij}$. Then the Cosserat or micropolar model is obtained by taking the skew part of (11.167) and setting $m_{kji:} = \mu_{k[ji]}$ the couple-stress tensor. The microstretch continuum is a further reduction of the model (11.167) obtained by noting m_k the intrinsic dilatational stress or *microstretch vector*; l the body microstretch force such that $l_{(ij)} = (l/3)\delta_{ij}$, and t and s are intrinsic and *micro scalar* forces, so that we have

$$\mu_{klm} = \frac{1}{3}\, m_k \delta_{lm} - \frac{1}{2}\, \varepsilon_{lmr} m_{kr}. \tag{11.169}$$

Hence (11.167) provides the following two equations for the couple stress and the microstretch

$$m_{kl,k} + \varepsilon_{lmn} t_{mn} + \rho\, C_l = \rho\,\dot{\sigma}_l, \quad m_{k,k} + t - s + \rho\, l = \rho\dot{\sigma}, \tag{11.170}$$

where C_l and σ_l stand for the axial vectors associated with the skewsymmetric tensors C_{ji} and $\sigma_{[ji]}$. Note that an additional natural boundary condition involving the new higher-order stresses μ_{kij} and m_{ji} must complement the standard Cauchy condition of the Prequisite Chap. 1, e.g.,

$$n_k\, \mu_{kij} = C^d_{ij} \quad or \quad n_j\, m_{ji} = C^d_i, \tag{11.171}$$

where C^d_i is akin to a *surface couple*. Similarly, a new boundary condition should involve the normal component of the vector m_k, e.g.,

$$n_k m_k = M^d, \tag{11.172}$$

where M^d is akin to a *tension*. The fields ϕ (a scalar not to be mistaken for the angle introduced in other sections) and the vector of components $\gamma_k = 3\,\phi_{,k}$ will

complement the kinematic description needed in the second of (11.170). In small stretches $\dot{\sigma} = j_0 \ddot{\phi}$ where j_0 is a scalar.

Finally, we note the further case of **dilatational elasticity** (Cowin and Nunziato 1983) [only the second of (11.170) is relevant]:

$$m_{k,k} + t - s + \rho\, l = \rho\, j_0 \ddot{\phi}. \tag{11.173}$$

for which (11.172) is still valid.

Constitutive Equations

For this we need some energy considerations. In a general heat conducting thermo-microstretch continuum the equation of the internal energy will read (cf. Eringen 1999, p. 251):

$$\rho\, \dot{e} = t_{kl} a_{kl} + m_{kl} b_{kl} + m_k v_{,k} + (s - t) v - q_{k,k} + \rho h, \tag{11.174}$$

where we have defined the following time rates:

$$a_{kl} = v_{l,k} - \varepsilon_{klp} v_p, \quad b_{kl} = v_{l,k}, \quad c_{kl} = v\, \delta_{kl}, \quad v = \frac{1}{3j} \frac{dj}{dt}, \tag{11.175}$$

where j represents the microvolume change with microdeformations (i.e., $j = \det \underline{\chi}$ if $\underline{\chi}$ is the initial microdeformation).

The local Clausius-Duhem inequality will follow by combining (11.174) with the local entropy inequality

$$\rho \dot{\eta} + (q_k/\theta)_{,k} - \rho\, h/\theta \geq 0, \tag{11.176}$$

and introducing the free energy density wherein η, ψ, θ stand for the densities of entropy, free energy $(\psi = e - \eta\theta)$, and the thermodynamic temperature $(\inf \theta = 0,\ \theta > 0)$.

Microstretch Elasticity

We consider only the small-strain case (cf. Eringen 1999, p. 67, for finite strains). Then the relevant measures of deformation are given by

$$e_{kl} = u_{l,k} - \varepsilon_{klm}\,\phi_m, \quad \gamma_{kl} = \phi_{k,l}, \quad \gamma_k = 3\,\phi_{,k}, \quad \varepsilon = 3\,\phi. \tag{11.177}$$

There exist direct relationships between expressions (11.175) and the time-rate of change of these deformations. With $W = \rho_0 e$, in the absence of thermal effects, (11.174) yields the following constitutive equations:

$$t_{kl} = \frac{\partial W}{\partial e_{kl}},\; m_{kl} = \frac{\partial W}{\partial \gamma_{lk}},\; m_k = \frac{\partial W}{\partial \phi_{,k}},\; s - t = \frac{\partial\, W}{\partial \phi}, \tag{11.178}$$

while Eqs. (11.170) read

$$t_{kl,k} + \rho_0(f_l - \ddot{u}_l) = 0, \tag{11.179}$$

$$m_{kl,k} + \varepsilon_{lmn} t_{mn} + \rho_0\left(l_l - j_{lm}\ddot{\phi}_m\right) = 0, \tag{11.180}$$

$$m_{k,k} + t - s + \rho_0\left(l - \frac{1}{2} j_0 \ddot{\phi}\right) = 0. \tag{11.181}$$

For a quadratic energy and the special case of isotropy, it is shown that the constitutive Eqs. (11.178) acquire the following form (cf. Eringen 1999, Sect. 6.1):

$$t_{kl} = \left(\lambda_0\,\phi + \lambda\,u_{r,r}\right)\delta_{kl} + \mu\left(u_{k,l} + u_{l,k}\right) + \kappa\left(u_{l,k} - \varepsilon_{klr}\phi_r\right), \tag{11.182}$$

$$m_{kl} = \alpha\phi_{r,r}\delta_{kl} + \beta\,\phi_{k,l} + \gamma\phi_{l,k} + b_0\varepsilon_{lkm}\,\phi_{,m}, \tag{11.183}$$

$$m_k = a_0\phi_{,k} + b_0\varepsilon_{klm}\phi_{l,m}, \tag{11.184}$$

$$s - t = \lambda_1\phi + \lambda_0 u_{k,k}. \tag{11.185}$$

These can be compared to the constitutive equations of the micropolar case. They exhibit the additional coefficients λ_0, λ_1, a_0 and b_0. What is changed from the inequalities that guarantee the non-negativeness of the energy is given by $\leq$ the following three inequalities:

$$3\lambda + 2\mu + \kappa \geq 3\lambda_0^2/\lambda_1, a_0 \geq 0, \lambda_1 \geq 0, \tag{11.186}$$

where the first one is obtained by replacing λ by $\lambda - \lambda_0^2/\lambda_1$.

Finally, the local field Eqs. (11.179) through (11.181) lead to the following set:

$$\lambda_0\nabla\phi + (\lambda + 2\mu + \kappa)\nabla\,\nabla.\mathbf{u} - (\mu + \kappa)\nabla \times \nabla \times \mathbf{u} + \kappa\nabla \times \underline{\phi} + \rho_0\,(\mathbf{f} - \ddot{\mathbf{u}}) = \mathbf{0}, \tag{11.187}$$

$$(\alpha+\beta+\gamma)\,\nabla\nabla.\underline{\phi}-\gamma\,\nabla\times\nabla\times\underline{\phi}+\kappa\,\nabla\times\mathbf{u}-2\kappa\underline{\phi}+\rho_0\left(\mathbf{l}-j\underline{\ddot{\phi}}\right)=\mathbf{0}, \tag{11.188}$$

$$a_0\nabla^2\phi-\lambda_1\phi-\lambda_0\nabla.\mathbf{u}+\rho_0\left(l-\frac{1}{2}j_0\ddot{\phi}\right)=0. \tag{11.189}$$

These must be complemented by appropriate initial conditions and boundary conditions. Harmonic bulk and surface wave solutions have been given by Eringen (1999, Sects. 6.3 and 6.6). This author also gives a modelling by means of a lattice approach from which he deduces estimates for the values of the newly introduced continuum coefficients in terms of the diatomic lattice parameters.

Full *thermo*-elastic equations are reported by Eringen (1999, pp. 254–255).

The above results also apply to the theory of **dilatational elasticity** (Cowin and Nunziato 1983) where the internal-rotation component is discarded.

Microstretch Fluids

It seems that many fluid-like materials may accept a valid description by means of a microstretch fluid. Such a description must be based on an exploitation of thermodynamic equations, in particular the energy Eq. (11.174) and the rates of strain (11.175). Let $\psi=e-\eta\theta=\overline{\psi}(\rho^{-1},j_{kl},j_0,\theta)$. The constitutive equations for the recoverable behaviour are easily shown to be of the form

$$\eta=-\frac{\partial\overline{\psi}}{\partial\theta},\ t^R_{kl}=-\pi\,\delta_{kl},\ m^R_{kl}=0, m^R_k=0,\ (s-t)^R=-\pi_0, \tag{11.190}$$

where

$$\pi=-\frac{\partial\overline{\psi}}{\partial\rho^{-1}},\ \pi_0=-2\rho\left(\frac{\partial\overline{\psi}}{\partial j_{kl}}j_{kl}+\frac{\partial\overline{\psi}}{\partial j_0}j_0\right), \tag{11.191}$$

the last defined quantity being a thermodynamic *micropressure*.

The dissipative parts (noted with a right superscript D) and heat conduction will satisfy the following remaining dissipation inequality:

$$t^D_{kl}a_{kl}+m^D_{kl}b_{lk}+m^D_k v_{,k}+(s-t)\,{}^D v-q_k\theta_{,k}/\theta\geq 0. \tag{11.192}$$

There is no need here to expand the whole theory (cf. Eringen 1999, Chap. 3). Linear representations of the dissipative contributions for an isotropic fluid can be proposed in the following form:

$$t_{kl}^{D} = \left(\lambda_0 v + \lambda_v a_{pp}\right)\delta_{kl} + \left(\mu_v + \kappa_v\right) a_{kl} + \mu_v a_{lk}, \tag{11.193}$$

$$m_{kl}^{D} = b_0 \varepsilon_{mlk}\, v_{,m} + \alpha_v b_{pp}\delta_{kl} + \beta_v b_{kl} + \gamma_v b_{lk}, \tag{11.194}$$

$$m_{k}^{D} = a_0\, v_{,k} + b_0 \varepsilon_{klm} b_{lm}, \tag{11.195}$$

$$(s-t)^{D} = \lambda_1 v + \lambda_0 a_{kk}, \tag{11.196}$$

where coupling with temperature gradient is discarded. The introduced coefficients have to satisfy a set of inequalities in order to obtain a non-negative dissipation (see Eringen 2001, p. 243).

An interesting special case corresponds to isothermal situations for a macroscopically inviscid fluid, and neglect of microrotational effects and of the microstretch viscosity a_0. The surviving local field equations are

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla.(\rho \mathbf{v}) = 0, \tag{11.197}$$

$$\frac{\partial j}{\partial t} + j_{,k} v_k - 2jv = 0, \tag{11.198}$$

$$-\nabla \pi + \lambda_0\, \nabla v - \rho \frac{d\mathbf{v}}{dt} = 0, \tag{11.199}$$

$$\pi_0 - \lambda_1 v - \lambda_0 \nabla.\mathbf{v} - \frac{1}{2}\rho j \frac{dv}{dt} = 0. \tag{11.200}$$

To the price of many approximations, Eringen (1990) has shown that this system could yield the equation of linear momentum for a bubbly liquid in the form

$$\frac{\partial^2 \mathbf{v}}{\partial t^2} = c_e^2 \nabla\, \nabla.\mathbf{v} + d\, \nabla\, \nabla.\frac{\partial \mathbf{v}}{\partial t} + \varepsilon\, \nabla\, \nabla.\frac{\partial^2 \mathbf{v}}{\partial t^2}, \tag{11.201}$$

where d and ε are appropriately defined coefficients, and c_e is the sound speed (supposed to be constant for the sake of simplicity in the deduction) in a bubbly liquid. Equation (11.201) has the same look as the equation obtained by Wijngaarden (1972) by means of a hydrodynamic model.

Another possible application of microstretch fluids is the description of blood flow in small arteries (cf. Eringen 2001, pp. 250–252).

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Cross references: Dilatational elasticity, Eringen A.C., Generalized continuum mechanics.

Microstructure

Microstructure is the observed fact that many, if not all, materials examined at the naked eye or with the help of an optical microscope, or even at a lower scale by more powerful magnification means, are obviously made of grains, molecules, parts, small regions, domains, and other substructures, so that these materials are not materially uniform and are certainly more complex in their mechanical response than a continuous assembly of material "points". In spite of this early observation, the first developments of continuum mechanics equipped with the simplest analytical tools considered these materials as uniform and homogeneous as a gross but sufficient and useful simplification, allowing thus for relatively easy computations.

But with progress in observational means and also a will to go further than the continuum description using a single vector field—the displacement with smoothly varying properties—and finally a real need in civil and mechanical engineering one had to envisage the truly non-uniform structure or a better approximation of the generalized kinetic description. The simplest path on this road is the consideration of the powerful idea of *averaging*, to start with, a volumetric averaging. This has the advantage to yield an effective replacement continuum that belongs to classical continuum mechanics with appropriately defined effective material coefficients. Most of classical continuum mechanics, applied whether to solid or fluid types of materials, was formulated this way. The justification for this approach is to be found in the fact that length scales considered (in the application of external loads statically or dynamically (then the wave length)) are much larger than the identified micro-scale. Improvement was brought by the consideration of stochastic processes, like in the mechanics of polycrystals with random-like arrangement or of fluid solutions in the case of suspensions. Of course the structural complexity of some materials is easily recognized in materials like liquid crystals or solutions of macromolecules, or else in grain-like solid materials (e.g., concrete). A modern re-evaluation of this technique is that of *homogenization*, which can be given a strict mathematical justification, especially in so-called asymptotic periodic homogenization (APH). Of course the question is more pregnant when the external stimuli have a length scale that approaches or even matches—resonates with—the identified scale of the microstructure. Each "grain" of the microstructure may then be excited. If one wants to keep a continuum description because of its clear

advantages in analysis, then one needs to account for the microstructure through astutely introduced additional kinematic descriptors in a true field theory. In the end, this boils down to attributing additional degrees of freedom—beyond the classical translational ones—to the "points" of the material continuum; We are then facing a true oxymoron, that of "*structured point*". Nonetheless, this can be achieved without much second thoughts or guilt feeling in a variety of ways as described in many entries in this dictionary (see cross references). This is the best and potentially rich materialization of non-classical continuum mechanics. In addition, to make things more complex, one may have to account for a microstructure of another nature, such as electromagnetic properties.

Cross references: Cellular materials as generalized continua, Continua with latent microstructure, Cosserat continua, Generalized continuum mechanics, Electromagnetic continua, Ferroelectric crystals (elasticity of), Generalized continuum mechanics, Homogenization, Internal degrees of freedom (dynamics), Liquid crystals as continua, Micromagnetism in elastic solids, Micromorphic continua, Micromorphic fluids, Micropolar continua, Micropolar elasticity, Micropolar fluids, Microstretch continua, Microstructured continuum theory (Eringen), Microstructured continuum theory (Mindlin), Microstructured fluids, Multipolar continua, Oriented media (with directors), Solutions of macromolecules.

Microstructured Continuum Theory (Eringen)

(Note: Divergence of tensors is taken on the first index).

This is a theory of microstructured continuum, also known as the theory of micromorphic *continua*, proposed by Eringen and Suhubi in 1964—on the basis of a microscopic space-averaging procedure—and further developed in all detail (finite strains, approximations, special cases, applications) in the course of years (cf. Eringen 1968, 2000). Although a complete nonlinear theory was expanded, here we are satisfied with elements of the theory of small perturbations (sometimes called the "*linear*" theory). The displacement field of Cartesian components u_i' in a macrovolume V' is the sum of the usual (macro) displacement u_i and a linear function of the internal coordinates ξ_k, i.e.,

$$u_i'(\mathbf{x}, \xi, t) = u_i(\mathbf{x}, t) + \chi_{ik}(\mathbf{x}, t)\, \xi_k. \tag{11.202}$$

where the object χ_{ik} in general has *nine* components. The natural measures of strains are defined as (Eringen's notation)

$$\varepsilon_{kl} = u_{l,k} - \phi_{lk}, \quad e_{kl} = \frac{1}{2}(\phi_{kl} + \phi_{lk}), \quad \gamma_{klm} = \phi_{kl,m} \tag{11.203}$$

where ϕ_{lk} is the small deviation of χ_{lk} from the unit δ_{lk}. With an energy density per unit volume of the form

$$W = W\left(\varepsilon_{ij}, e_{ij}, \gamma_{ijk}\right), \tag{11.204}$$

the relevant constitutive equations for micro-elastic solids are given by

$$t_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} \neq t_{ji}, \quad s_{ji} = \frac{\partial W}{\partial e_{ij}} = s_{ij}, \quad m_{kij} = \frac{\partial W}{\partial \gamma_{ijk}}. \tag{11.205}$$

These are involved in the following two local field equations (in all, twelve components) that may be considered the expression of the laws of linear momentum and moment of momentum:

$$t_{kl,k} + \rho f_l = \rho \ddot{u}_l, \tag{11.206}$$

$$m_{klm,k} + t_{ml} - s_{ml} + \rho l_{lm} = \rho \dot{\sigma}_{lm}, \tag{11.207}$$

where f_l and l_{lm} are densities of prescribed body vector and (symmetric and skewsymmetric) tensor forces, and

$$\dot{\sigma}_{lm} = i_{mj} \ddot{\phi}_{lj}, \tag{11.208}$$

if i_{mj} is some kind of inertia (here supposed to be constant; but in general there exist conservation laws of mass and inertia, the latter having been formulated by Eringen in 1964).

The associated natural boundary conditions at the boundary of the material volume read

$$n_k t_{kl} = T_l^d, \quad n_k m_{klm} = M_{lm}^d, \tag{11.209}$$

where T_l^d is an applied (vector) traction, while M_{lm}^d is an applied (tensor without specific symmetry) traction.

The presented theory is very much like Mindlin's (1964) theory of a microstructured continuum [see the entry "Microstructured continuum theory (Mindlin)"] if we note that the above introduced χ_{ij} is like the "transposed" of Mindlin' s ψ_{ij}(i.e., $\chi_{ij} = \psi_{ji}$) and we have the following identifications (with the convention: "Eringen" $\Leftrightarrow$"Mindlin"):

$$\varepsilon_{ji} \Leftrightarrow \gamma_{ji}, \quad \gamma_{klm} \Leftrightarrow \kappa_{lkm}, \tag{11.210}$$

$$t_{ij} \Leftrightarrow \sigma_{ij} + \tau_{ij}, \quad t_{ml} - s_{ml} \Leftrightarrow \tau_{lm}, \quad m_{klm} \Leftrightarrow \mu_{kml}. \tag{11.211}$$

The *symmetric* tensor of components s_{ml} may be called the *micro-stress*.

For further use, we note the following decompositions in the Eringen-Suhubi theory:

$$\varepsilon_{ji} = \varepsilon_{(ji)} + \varepsilon_{[ji]}, \quad \varepsilon_{(ji)} = \varepsilon\delta_{ji} + \bar{\varepsilon}_{ji}, \quad \bar{\varepsilon}_{jj} \equiv 0, \tag{11.212}$$

$$e_{(ij)} = e\,\delta_{ij} + \bar{e}_{ij}, \bar{e}_{jj} \equiv 0, \tag{11.213}$$

$$\gamma_{kij} = \gamma_k\,\delta_{ij} + \gamma_{k[ij]} + \bar{\gamma}_{k(ij)}, \gamma_k := \gamma_{kjj}/3, \tag{11.214}$$

$$m_{kij} = m_{k[ij]} + \frac{1}{3} m_k \delta_{ij} + \bar{m}_{k(ij)}. \tag{11.215}$$

These emphasize the shear and dilatational parts in the various strain measures and the hyperstress m_{kij}.

The compatibility conditions for recovering the initially introduced displacements from the strain measures are given by

$$\begin{aligned} \varepsilon_{kpq}\left(\varepsilon_{pl,q} + \gamma_{lpq}\right) &= 0, \\ \varepsilon_{kpq}\,\gamma_{lmp,q} &= 0, \\ 2e_{kl,m} - \gamma_{klm} - \gamma_{lkm} &= 0. \end{aligned} \tag{11.216}$$

Special cases

A. **Micropolar continua**:

In this case the general micro-deformation χ_{ij} reduces to a pure rotation hence an orthogonal tensor such that

$$\chi^{-1} = \chi^T, \quad \det\,\chi = 1. \tag{11.217}$$

The small deviation ϕ_{lk} introduced in (11.203) reduces to a skew-symmetric tensor, with which we can associate an axial vector of components ϕ_i such that

$$\phi_i = \frac{1}{2}\,\varepsilon_{ilk}\phi_{kl}, \quad \phi_{kl} = -\varepsilon_{klm}\phi_m. \tag{11.218}$$

Then the first and third of (11.203) yield

$$\varepsilon_{kl} = u_{(l,k)} + u_{[l,k]} + \varepsilon_{lkm}\phi_m = u_{(l,k)} + \varepsilon_{klm}(\omega_m - \phi_m), \quad \gamma_{klm} = \varepsilon_{kln}\,\phi_{n,m} \tag{11.219}$$

with a macro-rotation defined by

$$\omega_m = \frac{1}{2}\varepsilon_{mlk} u_{k,l}. \tag{11.220}$$

In this theory the micro-stress s_{ij} vanishes identically. Considering the skew part of Eq. (11.207) and associating axial vectors with the skew parts of the remaining tensors, we obtain local equations of linear and angular momenta in the form:

$$t_{kl,k} + \rho f_l = \rho \ddot{u}_l, \tag{11.221}$$

$$m_{kn,k} + \varepsilon_{nml} t_{ml} + \rho\, l_n = \rho\, \dot{\sigma}_n = \rho i \ddot{\phi}_n, \tag{11.222}$$

with an isotropic inertia, and

$$m_{kn} = \varepsilon_{nml} m_{k[lm]}, \quad l_n = \varepsilon_{nml} l_{[lm]}, \quad \dot{\sigma}_n = \varepsilon_{nml}\, \dot{\sigma}_{[ln]}, \tag{11.223}$$

so that m_{kn} here is a true couple-stress, l_n is a true body couple, and $\dot{\sigma}_n$ is a true spin (angular momentum rate). The natural boundary conditions reduce accordingly with true applied surface traction and applied surface couple only.

If we impose the kinematic constraint that $\omega_n = \phi_n$ always, then ε_{kl} reduces to the usual symmetric part of the displacement gradient, while γ_{klm} will reduce to a special expression of the second gradient of the usual displacement. This is called the *constrained theory of Cosserat (or micropolar) continua.*

B. **Continua with microstretch**

This is a modelling introduced by Eringen (1969). It is obtained by setting to zero the shear parts of the micro-fields in Eqs. (11.212)–(11.215), leaving only governing equations for $m_{k[ij]}$ and m_k i.e., in quasi-statics equations of the type

$$m_{kl,k} + \varepsilon_{lmn} t_{mn} + \rho l_l = 0, \quad m_{k,k} + t - s + \rho l = 0, \tag{11.224}$$

with an obvious interpretation for l_l (body couple) and the scalar forces t, s and l.

C. **Dilatation elasticity**

This is also called *elasticity with voids*. It was proposed by Cowin and Nunziato (1983). It consists in keeping only the dilatation [emphasized in the decompositions (11.212)–(11.215)] of the microstructure in addition to the macro-displacement, hence only the second of (11.224) is relevant,

$$m_{k,k} + t - s + \rho l = 0, \tag{11.225}$$

complementing the usual local equilibrium equation.

Remark The full theory with finite strains and finite internal motion is to be found in Eringen and Suhubi (1964) and Eringen (1999).

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Cross references: Cosserat continua, Couple stress (as medium with constrained rotation), Microstructured continuum theory (Mindlin), Micromorphic continua, Micropolar continua (cf. Cosserat continua), Microstretch (media with).

Microsctructured Continuum Theory (Mindlin)

(Note: Divergence of tensors is taken on the first index).

In this theory expanded by R.D. Mindlin in 1964 the total displacement field u_i' of a *micro* point is expanded as a power series in the micro-coordinates and retaining only the first two terms yielding

$$u_i'(\mathbf{x}', \mathbf{x}, t) = u_i(\mathbf{x}, t) + x_k' \psi_{ki}(\mathbf{x}, t), \tag{11.226}$$

where the internal coordinate x_k' is measured from the mass centre of the micro-volume V' while the two contributions in (11.226) can be interpreted as the ordinary *macro*-displacement and the *micro*-displacement. Macro- and micro-strains are defined by (Mindlin's notation)

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \quad \psi_{(ij)} = \frac{1}{2}\left(\psi_{ij} + \psi_{ji}\right). \tag{11.227}$$

Furthermore, the relative deformation and the micro-deformation gradient are defined by

$$\gamma_{ij} = u_{j,i} - \psi_{ij}, \quad \kappa_{ijk} = \psi_{ji,k}. \tag{11.228}$$

The following compatibility conditions apply:

$$\varepsilon_{mik}\,\varepsilon_{nlj} e_{kl,ij} = 0, \tag{11.229}$$

$$\varepsilon_{mij}\;\kappa_{jkl,i} = 0, \tag{11.230}$$

$$\left(e_{jk} + \omega_{jk} - \gamma_{jk}\right)_{,i} = \kappa_{ijk}. \tag{11.231}$$

The kinetic energy per unit of macro-volume is defined by

$$K = \frac{1}{2V'} \int_{V'} \rho\prime \dot{u}'_i\, \dot{u}'_i\, dV'. \tag{11.232}$$

On using (11.226), this is evaluated as

$$K = \frac{1}{2}\rho\, \dot{u}_i \dot{u}_i + \frac{1}{2} I_{kl}\; \dot{\psi}_{kj}\; \dot{\psi}_{lj}, \tag{11.233}$$

on account of the definitions

$$\rho = \frac{1}{V'} \int_{V\prime} \rho'\, dV', \quad I_{kl} = \frac{1}{V'} \int_{V'} \rho'\, x'_k\, x'_l\, dV', \tag{11.234}$$

and the condition of mass centre

$$\int_{V'} \rho' x'_k dV\prime = 0. \tag{11.235}$$

Field Equations

They can be deduced by means of a Hamiltonian principle with prescribed surface data (Mindlin 1964) and an internal energy density

$$e = \bar{e}\left(e_{ij},\, \gamma_{ij},\, \kappa_{ijk}\right) \tag{11.236}$$

for small macro- and micro-strains. The twelve stress equations of motion are obtained as

$$\sigma_{ij,\,i} + \tau_{ij,i} + \rho f_j = \rho \ddot{u}_j, \tag{11.237}$$

$$\mu_{ijk,i} + \tau_{jk} + \rho\, \Phi_{jk} = I_{lj} \ddot{\psi}_{lk}, \tag{11.238}$$

with the constitutive equations

$$\sigma_{ij} = \rho \frac{\partial e}{\partial e_{ij}}, \quad \tau_{ij} = \rho \frac{\partial e}{\partial \gamma_{ij}}, \quad \mu_{ijk} = \rho \frac{\partial e}{\partial \kappa_{ijk}}, \tag{11.239}$$

while the associated twelve natural boundary conditions read

$$n_i\left(\sigma_{ij} + \tau_{ij}\right) = T_j^d, \quad n_i \mu_{ijk} = M_{jk}^d. \tag{11.240}$$

Here f_i and Φ_{jk} are externally applied densities of forces and "double" forces". T_j^d and M_{jk}^d are the corresponding applied surface forces. Note that only σ_{ij} is a symmetric stress. The new (nonsymmetric) tensor τ_{ij} measures the relative influence of macro- and micro- deformations and may be called the *micro-stress*. But this terminology may be misleading. A whole theory can be expanded once the expression of the energy e is prescribed for a given material symmetry. For a quadratic energy, Mindlin (1964) originally noted that only 903 independent material coefficients (out of 1764) at most are involved. This is a lot indeed! This greatly reduces to a more reasonable number of 18 in the case of centrosymmetric isotropy.

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Cross references: Microstructured continuum theory (Eringen), Micro-stress,

Microstructured Fluids

This is essentially the same as "micromorphic fluids" (see that entry).

Mindlin R.D

Raymond D. Mindlin (1906–1978) was a New Yorker who obtained all his diplomas and spent most of his professional career at Columbia University in New York. Although he achieved international renown with his initial works in solving difficult problems of elasticity (e.g., Mindlin 1936) and performing pioneering experiments in photo-mechanics, he became one of the best specialists of the theory of vibrations in plates, and expanded with success several of the branches of generalized continuum mechanics, including the mechanics of granular materials, the classical piezoelectricity of structures applied to signal processing (cf. Mindlin

2007), so-called *gradient elasticity* (cf. Mindlin and Eshel 1968), the elasticity of microstructured solids (cf. Mindlin 1964), the theory of elastic dielectric crystals with polarization gradient (cf. Mindlin 1968) and crystal-lattice dynamics (cf. Mindlin 1972). He is acknowledged as one of the most innovative engineering scientist in the field of generalized continuum mechanics (see a description of most of his influential works in Herrmann Editor 1974), and his collected works in Mindlin (1989).

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Cross references: Gradient elasticity, Microstructured continuum theory (Mindlin), Polarization gradient, Non-locality (weak).

Mixtures (Mechanics of)

Rational bases for the thermo-mechanical *continuum theory of mixtures* were formulated by Truesdell (1957). There followed in the 1960s–1970s a multitude of works in the same line, among these, Kelly (1964), Green and Naghdi (1965, 1971), Eringen and Ingram (1965, 1967), Bowen (1967, 1976), Müller (1968), and Atkin and Craine (1976).

The theory of mixtures uses the concept of volume fraction n of a species (or constituent), defined by

$$n = \frac{volume\ occupied\ by\ one\ species}{total\ volume}. \tag{11.241}$$

De Boer (2000, p. 31) attributes the introduction of this important concept (Woltman 1794). to the German hydraulician Reinhard Woltman (1757–1837). The

general thermodynamic theory of mixtures is a complicated affair that still poses fundamental questions, for example: (i) should a temperature be introduced for each constituent (same question for entropy); (ii) should global thermodynamic laws, in particular the second one, be defined for each constituent or for the global medium; (iii) how the stress condition at a boundary should be formulated since only one global applied traction can be specified while the mixture involves partial stresses? The only constraint on which all authors agree is that by summation over the constituents one obtains the thermo-mechanical equations that govern a unique standard continuum. The notions proper to the theory of mixtures are those of volume fractions, partial pressures and stresses, and interaction forces between species. The original role of the latter must be emphasized. The theory is particularly complicated when species exchange not only momentum but also mass as when chemical reactions occur. The theory is not expanded here so that we refer the readers to the already mentioned contributions. More recent references are Grinfeld (1991) and Rajagopal and Tao (1995). For reactive situations and combustion problems in fluids, see Prud'homme (1988, 2010). In this dictionary we consider only the application to porous media (see the corresponding entry) in the absence of chemical reactions and mass transfer.

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Cross reference: Porous media (as seen in GCM).

Multipolar Continua (Green-Rivlin)

In the fever of research for generalized continuum mechanics that occurred in the first half of the 1960s, A.E. Green and R.S. Rivlin (1964) proposed a kind of original approach that used the notion of multipoles. This notion is well known in gravitation theory.

Multipolar body forces can be defined through the power they expand; e.g. for a whole body of volume V,

$$P(V) = \int_V \rho f_{ij_1 \ldots j_\beta} v_{ij_1 \ldots j_\beta} dV \tag{11.242}$$

defines the body force 2^β-*pole* of the $(\beta + 1)$th kind per unit mass where $v_{ij_1 \ldots j_\beta}$ are velocities associated with the multipolar displacement fields (or simple 2^β-pole displacement fields) noted $x_{ij_1 \ldots j_\beta}$. The latter form a set of kinematic variables which may be changed independently of the classical motion x_i. Then, if $v_{ij_1 \ldots j_\beta, i_1 \ldots i_\alpha}$ denotes the αth space gradient of $v_{ij_1 \ldots j_\beta}$, the body force $2^{\alpha+\beta}$-pole of the $(\beta + 1)^{th}$ kind per unit mass is introduced in the body power

$$P(V) = \int_V \rho f_{ij_1 \ldots j_\beta : i_1 \ldots i_\alpha} v_{ij_1 \ldots j_\beta, i_1 \ldots i_\alpha} dV. \tag{11.243}$$

Of course, $f_{ij_1 \ldots j_\beta : i_1 \ldots i_\alpha}$ may be considered completely symmetric in the indices $i_1, \ldots, i_\alpha$.

For $\beta = 0$, (11.242) provides the power expanded by a classical body force vector of components f_i. Still for $\beta = 0$, but $\alpha = 1$, (11.243) provides the expression

$$P(V) = \int_V \rho f_{ij} v_{i,j} \, dV, \tag{11.244}$$

so that f_{ij} are the components of a (non-necessarily symmetric) stress per unit mass.

For $\alpha = 2$, (11.243) yields the expression

$$P(V) = \int_V \rho f_{ijk} v_{i,jk}\, dV, \tag{11.245}$$

and the object $f_{ijk} = f_{ikj}$ may be called a *hyperstress* per unit mass, and so on. Summing over the various gradient orders for $\beta = 0$ we will obtain an expression in the form of an expansion

$$P(V) = \int_V \left(\rho f_i v_i + \sigma_{ji} v_{i,j} + m_{jik} v_{i,jk} + \cdots\right) dV \tag{11.246}$$

where we have set $\sigma_{ji} = \rho f_{ij}$ and $m_{kji} = \rho f_{ijk} = m_{jki}$. We recognize in (11.246) an expression of the type considered by Mindlin (1964) in his gradient theory of elasticity—See also Eq. (11.245) in Entry: Higher-order gradient theories.

But Green and Rivlin (1964) go further by considering the notion of *multipolar surface forces and stresses*. This is where things become a little messy. In parallel with (11.242), but for a regular surface S, Green and Rivlin envisage powers of the form

$$P(S) = \int_S t_{ij_1\ldots j_\beta : i_1\ldots i_\alpha} v_{ij_1\ldots j_\beta, i_1\ldots i_\alpha} dS. \tag{11.247}$$

defining thus a surface force $t_{ij_1\ldots j_\beta : i_1\ldots i_\alpha}$ called the surface force $2^{\alpha+\beta}$-pole of the $(\beta+1)$th kind per unit area. This last quantity is associated with a surface whose unit normal at the point is n_k.When this n_k is a unit normal to the x_k-plane through the point we denote the corresponding tensor by $\sigma_{kij_1\ldots j_\beta : i_1\ldots i_\alpha}$. These are the components of a *surface stress tensor* $2^{\alpha+\beta}$-pole of the $(\beta+1)$th kind on an element of area at the point normal to the x_k-axis. For $\alpha = \beta = 0$, we recover the classical stress tensor σ_{ki} and traction $T_i^d = n_k \sigma_{ki}$ since (11.247) then reads

$$P(S) = \int_S n_k \sigma_{ki} v_i dS. \tag{11.248}$$

For $\beta = 0$, $\alpha = 1$, we shall obtain an expression

$$P(S) = \int_S n_k \sigma_{kij} v_{i,j} dS. \tag{11.249}$$

where the (non-necessarily) symmetric quantity $n_k \sigma_{kij}$ is called a *surface double force* (See Entry: Double force).

It is clear that the general model elaborated upon by Green and Rivlin rapidly becomes extremely complicated, and probably much too much so, as we haven't seen any useful application of this general model during the fifty years that followed

its inception. It is even surprising that Rivlin, usually much more pragmatic than this, went so far in this formalism. As a matter of fact, Green and Rivlin went as far as stating the material form (Piola- Kirchhoff), exploiting irreversible thermodynamics (the Clausius-Duhem inquality), and associating a kinetic energy with the multipolar displacement $x_{ij_1 \ldots j_\beta}$. Their paper remains only as a historical landmark. What we retain from it, however, is the case of a single displacement field and the notion of surface double force for the so-called *dipolar model* (see that entry), in which case elasticity theory is the one expanded by Mindlin (1964).

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Cross references: Dipolar materials, Double force, Generalized internal forces, Green A.E., Higher-order gradient theories, Hyperstresses (notion of).

Chapter 12
N: From "Naghdi P.M." to "Nowacki W."

Naghdi P.M

One of the world masters in continuum mechanics from the USA in the second half of the twentieth century, Paul M. Naghdi (1924–1994)—originally from Iran—spent most of his active career at Berkeley. He has touched upon all aspects of generalized continuum mechanics, often in co-operation with Albert E. Green from Oxford. In particular, he developed original views on continuum thermo-dynamics (Green and Naghi 1977, 1993, 1995), the theory of mixtures (Green and Naghdi 1965), and the theory of polar materials using the director approach (e.g., in Cosserat surfaces; cf. Green and Naghdi 1967; Naghdi 1972). A selected list of references is given below. Biographical elements on Naghdi are given by his former students Casey and Crochet (1995).

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G.A. Maugin, *Non-Classical Continuum Mechanics*,
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Cross references: Cosserat continua, Dipolar materials, Directors theory, Green A.E., Oriented media (with directors).

Non-euclidean Geometry of Defective Materials

[See entries: Connection and torsion, Defects in GCM, Dislocations and disclinations, Non-holonomic continua].

Non-holonomic Continua

We agree to call *non-holonomic continua* those continua in which the classical compatibility conditions for strains are not satisfied. These are the conditions initially proposed by Navier and Barré de Saint-Venant in small strains that a strain must satisfy to be unequivocally integrated into a displacement. An everywhere dislocated material—that is, a material with a very high density of dislocations (discontinuities of displacement), a practically necessary thermodynamic requirement—is an example of such continua. Remember that in classical continuum mechanics it is assumed that all material bodies are embedded in a Euclidean space. Following Maugin (1993, Chap. 3), this space may be characterized by the existence of a global Cartesian covering over the whole space. In this global Cartesian system the components of a vector remain unaltered in a parallel transport (e.g., around a closed contour). The criterion of holonomy on the material manifold is that the system of material coordinates $X^K, K = 1, 2, 3$ provides a *global* or *holonomic* (i.e., integrable) system of coordinates. The position vector $\mathbf{R}$ of a material point is a continuous and single-valued function of the $X^{K'}s$. In other words, with $\mathbf{G}_K, K = 1, 2, 3$, a set of basis vectors, $d\mathbf{R} = \mathbf{G}_K dX^K$ is an *exact* differential, or equivalently, the integral of this along a closed contour C is path-independent, i.e.,

$$\oint_C d\mathbf{R} = \oint_C \mathbf{G}_K dX^K = \mathbf{0}. \tag{12.1}$$

On using Stokes' theorem, this yields locally

$$\left(\Gamma^L_{JK} - \Gamma^L_{Kj}\right)\mathbf{G}_L = \mathbf{0}, \tag{12.2}$$

where

$$\Gamma^L_{JK}(\mathbf{G}) := \frac{\partial \mathbf{G}_K}{\partial X^J}.\mathbf{G}^L = \frac{\partial^2 \mathbf{R}}{\partial X^J \partial X^K}.\mathbf{G}^L \tag{12.3}$$

are *connection* coefficients (here Christoffel symbols of the second kind since material space is Euclidean) and $\mathbf{G}_L, L = 1, 2, 3$ denotes a reciprocal basis. Equation (12.2) is valid for any $\mathbf{G}_L$ so that we must have

$$\Gamma^L_{JK}(\mathbf{G}) = \Gamma^L_{KJ}(\mathbf{G}). \tag{12.4}$$

and the system $X^K, K = 1, 2, 3$ is said to be *holonomic*. A second criterion is that of *vanishing curvature*. This is established by requiring that the change in the components of a vector, say $\mathbf{A}$, should be zero after parallel transport around a closed circuit C, i.e.,

$$\oint_C \Gamma^L_{JK} A^K dX^J = 0. \tag{12.5}$$

Since

$$\frac{\partial A^L}{\partial X_J} = -\Gamma^L_{JK} A^K, \tag{12.6}$$

after application of Stokes' theorem and localization condition (12.5) for any vector $\mathbf{A}$ implies the following geometric condition:

$$R^{\cdots L}_{JMK} = 0, \tag{12.7}$$

where the components to this *Riemann-Christoffel curvature tensor* are defined by

$$R^{\cdots L}_{JMK} = \frac{\partial \Gamma^L_{MK}}{\partial X^J} - \frac{\partial \Gamma^L_{JK}}{\partial X^M} + \Gamma^L_{JN}\Gamma^N_{MK} - \Gamma^L_{MN}\Gamma^N_{JK}. \tag{12.8}$$

In 3D, due to the skewsymmetry of this tensor in the indices J and M and considering the totally covariant curvature tensor

$$R_{JKMQ} = G_{OL} R^{\cdots L}_{JKM}, \quad G_{QL} = \mathbf{G}_Q.\mathbf{G}_L, \tag{12.9}$$

which is also skewsymmetric in indices M and Q, we can introduce a second-order tensor (called Einstein tensor) such that

$$S^{AB} = \frac{1}{4} \varepsilon^{AJK} \varepsilon^{BMQ} R_{JKMQ}, \tag{12.10}$$

where ε^{AJK} is the alternating symbol. In small trains, (12.10) reduces to

$$S_{ab} = -\varepsilon_{ajk}\varepsilon_{bli}\frac{\partial^2 e_{ki}}{\partial x_j \partial x_l}. \tag{12.11}$$

The requirement that this vanishes is none other than the Navier-Saint-Venant condition of compatibility [Equations (P1.34) in Chap. 1 in Part One].

Non-holonomy is defined by the lack of satisfaction of the above enunciated conditions, hence its prevailing role in the theory of continuous distributions of dislocations (cf. Kröner's theory and the introduction of the incompatibility tensor; Kröner 1958). Then an *affine connection* (as defined by the geometer E. Cartan) and a *torsion tensor* can still be defined. Other much relevant works in this context are the series of papers started by Bilby et al. (1955) and works by Noll (1967) and Wang (1967).

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Cross references: Connection and torsion, Continuously defective materials, Defects in GCM, Differential geometry and nonclassical continuum mechanics, Dislocations and disclinations, Material inhomogeneities, Non-Euclidean geometry of defective materials.

Nonlinear Waves in Generalized Continua

Classical continuum mechanics in the sense of Chap. 1 in Part One (Prerequisites) admits the propagation of nonlinear waves (simple waves, shock waves, propagating phase-transition fronts) when some nonlinearity is active. This is the case in nonlinear elastic bodies and also in compressible fluids. The deformation or motion then is strongly coupled with thermal properties. We refer to Maugin (1999) and Maugin et al. (1992; with electro-magneto-elastic interactions) for general features of such propagation and illustrative examples—also Ani and Maugin (1988).

Generalized continua of various types also admit such wave-propagation phenomena with the appropriate dose of nonlinearity. But these continua (Cosserat continua, gradient elasticity, strongly nonlocal continua, and some electro-magneto-deformable bodies) all involve one or several characteristic lengths. This means that they are prone to the phenomenon of *dispersion*. Nonlinearity and dispersion are two ingredients that may compete and balance one another to favour the existence of other dynamical phenomena of high interest, namely, solitary waves and solitons. This is true in *Cosserat continua* and *oriented elastic solids* (e.g., Erbay and Suhubi 1989; Erbay et al. 1991; Erofeev and Potapov 1993; Kunin 1982; Maugin and Miled 1986a; Potapov and Pavlov 1995; Potapov et al. 1998; Pouget and Maugin 1989), *gradient continua* (e.g., Christov and Maugin 1995; Christov et al. 1996; Maugin and Cadet 1991; Pouget 1990), *nonlinear elastic crystals with magnetic or electric microstructure* (e.g., Kivshar and Malomed 1990; Maugin 1986; Maugin and Miled 1986b; Pouget and Maugin 1984, 1985), and *nonlocal continua* (e.g., Kunin 1982, Vol.2). Historically, the work of Frenkel and Kontorova (1938) was the first one to relate the mechanics of deformation (with dislocations) to a nonlinear dispersive equation recognized later on as the celebrated *sine-Gordon equation* of soliton theory—more on the historical development in time in Maugin (2011).

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Cross references: Cosserat continua, Couple stress, Dislocations and disclinations, Ferroelectric crystals (elasticity of), Gradient elasticity, Higher-order gradient theories, Internal degrees of freedom (dynamics), Micromagnetism in elastic solids, Micropolar elasticity, Non-locality (strong), Oriented media (with directors), Solitons (in non-classical continua).

Nonlocal Damage

Although of recent introduction in continuum mechanics, *damage* is now classically understood as the irreversible loss of elasticity due to the decrease of areas that transmit internal forces, through the appearance and subsequent growth of micro-cracks and micro-cavities. This occurs in the condition of *creep* (increase of strain in time while the load remains constant). Damage is a particularly important phenomenon in materials of civil engineering (e.g., concrete). Initial ideas for its modelling are due to Kachanov (1958) and Rabotnov (1963). A thermomechanical theory exploiting the notion of internal variable of state was presented by Lemaître (1985) and Lemaître and Chaboche (1985) and met a great success. It considers a

local theory in which the damage variable is a scalar D (for the sake of simplicity) such that $0 \leq D \leq 1$, where $D = 0$ corresponds to a virgin element and $D = 1$ corresponds to a fully damaged element (i.e., fracture). In small strain isotropic elasticity and one-dimensional setting, the energy density per unit volume can be proposed as

$$W(\mathbf{e}, D) = \frac{1}{2}(1 - D)E\left(e^{elas}\right)^2. \tag{12.12}$$

Within the framework of the theory with internal variables of state, the associated thermodynamic force for damage Y is such that

$$Y = -\frac{\partial W}{\partial D} \equiv \frac{1}{2}E\left(e^{elas}\right)^2, \Phi = Y \quad \dot{D} \geq 0\,, \tag{12.13}$$

where Φ is the dissipation that an evolution law of damage should respect, and the first of (12.13) shows that the driving force is none other than the elastic energy of the undamaged specimen. Anisotropic three-dimensional generalizations for the material, tensor generalization for the damage variable, and coupling with other irreversible effects (plasticity, viscoplasticity, porosity) have been proposed in a multitude of works (in particular, see Chaboche 1988; Krajcinovic 1989; Lemaître and Chaboche 1985; Lemaître 1996; Voyiadjis and Kattan 2005). Rather different approaches are presented by Grabacki (1989) and Rabier (1989). Though, all these are *purely local* from the view point of continuum mechanics.

An altogether different viewpoint and more deeply analyzed survey of the phenomenon were expanded by Bazant and his co-workers (in particular, Pijaudier-Cabot and Bazant 1987). While (12.12) reflects the notion of *contiguity* as accepted in classical continuum mechanics, the multiple reasons why damage should be viewed as a *nonlocal* phenomenon were clearly exposed by Bazant (1991). First, it should be realized that the formation and growth of a micro-crack depends on the strain energy stored in a nonzero volume of the material surrounding the micro-crack. The release of this energy drives the growth of this micro-crack. A micromechanical analysis then reveals that damage is a function of the spatially averaged fracturing strain of the macroscopic smoothing continuum. Thus damage is *nonlocal*. One has thus shown that

$$D \approx F(\langle e \rangle), \tag{12.14}$$

where $\langle e \rangle$ is an average (nonlocal) strain. The latter can be defined as a statistical average of the form

$$\langle \mathbf{e}(\mathbf{x}) \rangle = \frac{1}{V_e} \int_{V_e} \alpha(\mathbf{x} - \mathbf{s})\mathbf{e}(\mathbf{s}) dV(\mathbf{s}), \tag{12.15}$$

where $V_e \approx l^3$ is the representative volume element (of size l) and $\alpha(\mathbf{x} - \mathbf{s})$ is a weight function that ensues from statistical averaging over all values of l. A second argument relates to the interactions among micro-cracks examined as a superposition of a number of problems of elasticity. A result of this analysis is that the weight function for spatial integration (or summation in a discrete approximation) is not a fixed material property because the micro-crack system evolves with the progress of loading and is affected by the presence of a boundary and the shape of this boundary. In partial conclusion, interactions also imply nonlocality. Finally, a role is played by microstructural inhomogeneities such as in concrete specimens which can be represented by a randomly generated configuration of hard aggregate pieces—interacting via central forces—with a prescribed distribution embedded in a relatively soft matrix (here, mortar). What is most interesting in this deeply thought analysis is that only the damage variable (i.e., the fracturing strain) need be considered as nonlocal while all other mechanical fields can be accepted in their usual local description. This was indeed implemented in the celebrated paper of Pijaudier-Cabot and Bazant (1987) where only those variables that control strain softening are subjected to a nonlocal treatment while the elastic part of the strain is treated as local. This "simplification" eschews the occurrence of formerly met spurious mesh sensitivity and incorrect convergence features in computations. Here the usual local damage energy release rate is replaced with its spatial average over the representative volume element whose size is a characteristic property of the material. This looks a bit far from the more formal developments of Eringen's theory of (strong) nonlocality—see that entry—but some results of Eringen and co-workers (e.g., Eringen and Ari 1983) on the crack problem in nonlocal elasticity are accounted for in some part of the reasoning.

To be complete we cite the stress-based nonlocal damage model of Giry et al. (2011), the work of Xia et al. (1987), and the variational formulation of Challamel (2010).

Weak Nonlocality

In the spirit of weak nonlocality (see that entry), damage can also be approached with the notion of *gradient of damage*. This was achieved by Markov (1995) and Frémond and Nedjar (1996)—but see the remarks on the related thermodynamics of internal variables of state and their gradients in Maugin (1990); See also variational formulations by Lorentz and Andrieux (1999) and Pham and Marigo (2010).

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Cross references: Gradient elasticity, Higher-order gradient theories, Internal variables of state, Nonlocality (as opposed to contiguity), Nonlocality (strong), Nonlocality (weak).

Nonlocality (as Opposed to Contiguity)

Nonlocality or action at a distance is defined in contrast with contiguity or action by contact. The latter in fact is strongly correlated with a continuum vision since it underlines the fact that across a surface drawn in a body there is direct action of the matter on one side with the matter on the other side. In classical physics this is illustrated by the two cases of mechanical action via the notion of stresses (pressure in the case of non-viscous fluids) and heat flux. In contrast, *nonlocality* or *action at a distance* was first concerned with the interaction between distant point masses as illustrated by Newton's law of gravitation. In practice this is not limited to a region of space; the whole space is involved, but the very form of Newton's law (inverse square law for the force, inverse of distance for the potential) indicates a stronger influence of closer points. R. Boscovich (1711–1787) proposed that if attraction is certainly active at large distances between two particles, the interaction becomes of the *repulsive* type at short distances, thus avoiding a problem of infinities. Nonetheless, the Newtonian model became some kind of dogma or paradigm with an application to electrostatics (with Coulomb) between electric charges, electromagnetism (with Ampère) between elementary currents, and even capillarity (with Laplace), so that this permeated through the whole field of physics of the period with the powerful notion of potential, a rather evasive mathematical notion but that suits well our modern vision of mathematical physics.

The discussion between contact and at-a-distance actions may have continued uninterrupted during the whole nineteenth century especially insofar as passing at the limit from a discrete description of matter to a continuous one. But in continuum mechanics the notion of contiguity seems to be seriously questioned for the first time by Pierre Duhem (1893)—as emphasized by Edelen (1976, p. 44). Duhem asked whether farther distant elements of the continuum, if not material points of the whole body, are causes of the mechanical response at a particular point. In modern terms, one would have to envisage a kind of *non-local interaction*, no longer by direct contact even though closer points may have a stronger influence than distant ones. Some of these developments may find a vague ancestry in some posthumously published work of 1854 by Gabrio Piola (see Piola 2014). Nonlocality of this type was introduced in different fields of physics in the 1940s to the 1960s (crystal optics, radiation, superconductivity) as exemplified by the book of Agranovich and Ginzburg (1984). These works emphasize the necessary occurrence of characteristic lengths (such as a coherence length) and thus the importance of *dispersion* from the point of view of wave propagation.

A blossoming of nonlocal mechanical continuum theories occurred in the 1970s, especially with the works of E. Kröner, B.K. Datta, B.K.D. Gairola, I.A. Kunin, D. Rogula, D.G.B. Edelen and A.C. Eringen (synthesized in Eringen 2002). Like Newton's theory of gravitation, these nonlocal theories in principle apply to the whole space and the notion of stress is not primary if ever likely to exist since any cut by a surface would destroy the nonlocality. For practical aspects, however, it is desirable to limit this action at-a-distance to some reasonable spatial range (a kind

of sphere of influence) while noting the obvious rather rapid decrease of the action with distance. This is also true in lattice dynamics which is often called as a support for nonlocality in crystals (cf. Kunin 1982; Eringen 2002) where interaction between non-immediate neighbours has to be limited to a certain range (rarely more than second nearest neighbours) except in fractal approaches where there is no privileged spatial scale (see works by T. Michelitsch et al.).

This brings us to the modern vision of (strong*) non-locality* where space functionals have to replace point-wise constitutive equations. But between the contiguity of Euler and Cauchy and the strongly non-local theory, one may find a *weakly non-local theory* where the mechanical response is still point-wise, but depending on further gradients of the displacement field or of density at the same point. Then one deals with *gradient* theories (or non-simple materials) of which the merits can be compared to those of the strongly non-local one (cf. Maugin 1979). Both weakly and strongly non-local theories belong to non-classical continuum mechanics.

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Cross references: Contiguity, Edelen D.G.B., Eringen A.C., Gradient elasticity, Kröner E., Lattice dynamics, Nonlocal damage, Non-locality (strong), Non-locality (weak), Peridynamics, Rogula D.

Nonlocality (Strong)

One distinguishes between "strong" nonlocality and "weak" nonlocality. Here we consider the former. Original works in this field are due to Kröner and Datta (1966) in elastostatics, Gairola (also in Kröner's environment), Kunin (1966), Rogula (1965), Edelen and Laws (1971), and Edelen and Eringen (1972) and many other works by these two authors. Detailed reviews and books are due to Kunin (1982)

and Eringen (2002). The essence of nonlocal linear elasticity was reviewed in Chap. 2 of Part One in the present book [in particular, see Equations (P2.18) and (P2.19) in the 1D case]. This approach requires the solution of integro-differential equations and the a priori knowledge of a kernel that provides an idea of the influence of more or less distant points ($\mathbf{x}'$) on the mechanical response at a selected material point ($\mathbf{x}$) with the natural understanding that distant points influence much less than close points (fast decrease with distance), and the possible existence of a cut-off distance that defines a limited zone of influence. Examples of such kernels are:

- a triangular kernel with finite support (Eringen 1972a):

$$\alpha(|\mathbf{x}' - \mathbf{x}|) = \frac{1}{a}\left(1 - \frac{|x' - x|}{a}\right) \textit{ for } \frac{|x' - x|}{a} \leq 1 \textit{ and } 0 \textit{ otherwise}, \tag{12.16}$$

  where a is a typical lattice spacing;
- Exponentially decreasing kernel:

$$\alpha(|\mathbf{x}' - \mathbf{x}|) = \alpha_0 \exp\left(-\frac{k^2}{l^2}|\mathbf{x}' - \mathbf{x}|^2\right), \tag{12.17}$$

  where α_0 and k are constants and l is a constant internal characteristic length.
- Oscillatory decreasing function:

$$\alpha(x; v) = \frac{1}{\pi}\frac{\sin(vx)}{x}, 0 < v < \infty. \tag{12.18}$$

- Bessel function of imaginary argument:

$$\alpha(|\mathbf{x}|) = \left(2\pi l^2\right)^{-1} K_0(|\mathbf{x}|/l),\ 0 < l < \infty. \tag{12.19}$$

Note that kernels such as (12.16) are suggested by wave studies in crystal lattices and the presence of characteristic lengths generally favours *dispersion* in wave propagation.

The introduction of such nonlocal theories was mildly appreciated at the beginning of these studies—looking down at these as useless exercises, but in time it gained a tremendous interest, so that it seemed that everything had to become "nonlocal" (including, plasticity, Cosserat theory, electromagnetic bodies, damage, friction, etc.) with the unlimited enthusiasm of new converts for the involved scientists. The unexpected liking for such developments may have been kindled by the fact that nonlocality avoids the singular behaviour of solutions in the classical problem of the crack tip (cf. Eringen and Kim 1974a, b; Eringen et al. 1977; Eringen and Ari 1983) and of the infinities at the core of dislocations (cf. Eringen 1977a, b). All these problems are solved in an infinite space or with a kernel with limited support.

It is Edelen and Eringen who proposed a rather formal ("rational") approach to strong nonlocality in a series of papers (Edelen 1969; Edelen and Laws 1971; Edelen and Eringen 1972). This is rather technical and unfortunately afflicted by a heavy formalism. The only notion that may be of general interest is that of *nonlocal residuals*. It is rightly emphasized by these authors that localization of global balance laws poses a problem in the nonlocal theory if we admit that only global balance laws are posited to be true. In classical continuum mechanics the global balance laws are posited to be valid—for sufficiently regular continuous fields—for any volume and surface elements. This is the so-called *localization* argument. This is hindered by strong nonlocality for which, for example, an attempt at localization for linear momentum will yield the "local" equations (cf. Eringen 2002, pp. 20–21):

$$t_{kl,k} + \rho(f_l - \dot{v}_l) = \rho\hat{f}_l \text{ in } V - \sigma, \tag{12.20}$$

$$[t_{kl} - \rho v_l(v_k - u_k)]n_k = \hat{F}_l \text{ on } \sigma, \tag{12.21}$$

where σ is a discontinuity surface moving at velocity of components u_k and equipped with unit normal n_k. Here t_{kl} has the usual meaning of a locally defined stress. The *residuals* in the right-hand side of (12.20) and (12.21) are subjected to the constraint

$$\int_{V-\sigma} \rho\hat{f}_l \, dv + \int_{\sigma} \hat{F}_l \, da = 0. \tag{12.22}$$

If we keep the form (12.20), we will need constitutive equations for both volume and surface residuals. This is tantamount to including surface physics in the domain of continuum mechanics.

An equation of local "conservation" of energy with energy residual can then be deduced.

But Equations such as (12.20) can also be given a classical outlook by introducing

$$\rho\hat{f}_l = -\hat{t}_{kl,k}, T_{kl} = t_{kl} + \hat{t}_{kl}, \tag{12.23}$$

so that (12.20) now reads

$$T_{kl,k} + \rho(f_l - \dot{v}_l) = 0, \tag{12.24}$$

but then T_{kl} is a *nonlocal field* of which the constitutive equation must reflect the nonlocal character, hence a space-integral form as presented in Chap. 2 of Part One. A similar manipulation can be effected on the energy equation; this will make appear a *surface energy*.

To conclude we first note that nonlocal fluid mechanics was also expanded (see Eringen 2002). *Peridynamics* (see that entry) is the last avatar of the strong nonlocal

theory. Nonlocal damage was introduced in 1987 (cf. Bazant and co-workers at the entry: Nonlocal damage) and nonlocal friction by Duvaut (1982). What is more exciting is the relationship established between nonlocality and *fractional calculus* (cf. Carpinteri et al. 2009; Atanakovic and Stankovic, 2009; and recent works by Michelitsch et al. 2013—initially based on the notion of fractal lattices).

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Cross references: Contiguity, Edelen D.G.B., Eringen A.C., Fractal continua, Kröner E., Kunin I.A., Lattice dynamics, Long-range interactions, Non-locality (as opposed to contiguity), Non-locality (weak), Nonlocal damage, Peridynamics.

Nonlocality (Weak)

It is reasonable to assume that the zone of marked influence introduced in the case of strong nonlocality is fairly small, perhaps of the order of a few lattice spaces in a solid crystal or a short coherence distance in fluids. Accordingly, the idea to replace this spatial functional formulation of strong nonlocality by the influence of the very first spatial gradients of the independent fields (e.g., the deformation) at a point seems to be salient. This naturally yields the notion of *gradient* theories illustrated by the notions of hyperstress, Le Roux theory of elasticity, Mindlin's gradient elasticity, capillarity, and surface tension [see the corresponding entries] while allowing a return to a treatment of mathematical problems by means of partial differential equations, with possible necessary special attention paid to boundary conditions. The relative interest between, and comparative practical convenience of, strong and weak nonlocalities can be discussed (cf. Maugin 1979).

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Cross references: Capillarity, Density-gradient fluids, Double force, Edge forces, Gradient elasticity, Gradient plasticity, Higher-order gradient theories, Hyperstress (notion of), Lattice dynamics, Le Roux elasticity, Nonlocal damage, Polarization gradient, Surface tension.

Nowacki W

Witold Nowacki (1911–1986) was a Polish applied mathematician-civil engineer who became one of the most articulate and active contributors to several facets of generalized continuum mechanics with a specific interest in dynamical aspects, including: thermo-elasticity, electro-magneto-mechanical interactions, and oriented (micropolar) continua. As one of the twentieth-century masters of Polish mechanics, he was extremely influential in his native country and abroad with books having received a large diffusion (e.g., Nowacki 1975, 1983, 1986a, b). Witold Nowacki has published an autobiography (Nowacki 1986c).

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Cross references: Asymmetric elasticity, Cosserat continua, Electromagnetic continua, Micropolar continua, Oriented media.

Chapter 13
O–P: From "Oriented Media (with Directors)" to "Porous Media as Seen in GCM"

Oriented Media (with Directors)

These are continuous media described by a classical motion and a set of unit "directors" attached to each material point that evolves in time forming a deformable or rigid triad. In the rigid case, the kinematics of these directors will describe a kind of internal rotation, or *orientation* (an additional degree of freedom). They have no real precise material meaning except perhaps in crystals and plasticity and dislocated bodies where they would be related to crystallographic directions. According to C.A. Truesdell, it seems that the basic idea goes back Pierre Duhem (1893). Such objects were considered (without this specific name) by the Cosserat brothers (1909) in the description of the deformation of material plates and filaments, adapting geometrical ideas of Darboux (the mobile triad). The idea was taken over by Ericksen and Truesdell (1958) in their deformation theory of rods and shells, and by other authors such as Toupin (1964) and Naghdi (1972) or Hayart (1966). Exploiting the concept of a unique director, Ericksen singlehandedly expanded a theory of anisotropic fluids that was to develop in a universally accepted theory of nematic liquid crystals (Ericksen 1960). Later on this was to be generalized by Frank M. Leslie (from Scotland) and him to include dissipative effects. In the framework of solid mechanics, we note the little known works of Stojanovic (1969) and his co-workers in Yugoslavia. A more recent work exploiting the notion of directors in nonlinear dynamics is by Pouget and Maugin (1989). Maugin and Eringen (1972) had proposed an attempt at a four-dimensional covariant formulation in order to describe relativistic continua with spin (exploiting thus the notion of *tetrad* in space-time).

Let $\mathbf{d}_\lambda, \lambda = 1, 2, 3$ denotes the set of three directors that depend on the material point $\boldsymbol{X}$ and time. In the case of a rigid triad of unit directors (case of a Cosserat continuum), we have the following constrains:

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$$\mathbf{d}_\lambda.\mathbf{d}_\mu = \delta_{\lambda\mu}, \mathbf{d}_\lambda.\mathbf{d}_\lambda(\lambda fixed) = 1. \tag{13.1}$$

But this is not necessarily the case. In general, the equation satisfied by the directors mimics that of the usual motion, so that we have the following field equations at a current point $\boldsymbol{x}$ in a regular body:

- Balance of linear momentum:

$$\rho\ddot{x}_i = t_{ji,j} + \rho f_i, \tag{13.2}$$

- Field equation for the director field:

$$\rho i_{\lambda\mu}\ddot{d}_{\mu i} = h_{\lambda ji;j} + \rho k_{\lambda i}, \tag{13.3}$$

- Balance of momentum of momentum (cf. Stojanovic 1969; summation over λ applies here)

$$t_{[ji]} = m_{kij,k} + d_{\lambda[i,k}h_{\lambda,j]k} + \rho L_{ij}, \tag{13.4}$$

where m_{kij} is a couple-stress tensor, and f_i and L_{ij} are densities of body force and couple, respectively. The factor $i_{\lambda\mu}$ is a kind of symmetric inertia tensor.

A more classical form of the equation of balance of angular momentum is obtained by introducing a rate of angular momentum $\dot{s}_{ij}$, a total couple stress μ_{kij} and an extrinsic couple l_{ij} by

$$\dot{s}_{ij} = i_{\lambda\mu}\ddot{d}_{\lambda[i}d_{\mu j]}, \tag{13.5}$$

$$\mu_{kij} = m_{kij} + h_{\lambda k[i}d_{\lambda j]}, \tag{13.6}$$

$$l_{ij} = L_{ij} + d_{\lambda[i}k_{\lambda j]}, \tag{13.7}$$

and taking the tensor product of (13.3) with $\mathbf{d}_\lambda$ and then the skew part of the result. One obtains thus the following equation:

$$\rho\dot{s}_{ij} = \mu_{kij,k} - t_{[ij]} + \rho l_{ij}. \tag{13.8}$$

This is the standard dynamic form of the balance of angular momentum in many theories of generalized continua (e.g. Toupin 1964). In an elastic body m_{kij} and $h_{\lambda ki}$ are primarily determined by the dependence of the energy density on the second gradient of the classical motion and the gradient of the director field, respectively.

It must be acknowledged that this theory had a rather limited legacy save perhaps in relation with an approach to dislocated bodies (e.g., in Stojanovic 1969;

Eringen and Clauss 1970), the dynamics of granular media, and the theory of Cosserat surfaces (Green et al. 1965).

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Cross references: Anisotropic fluids, Cosserat continua, Cosserat surfaces, Couple stress, Directors, liquid crystals.

Peridynamics

Introduction

The word "peridynamics" is a neologism created from the Greek word "peri" meaning "the surrounding" and "dynamics" that needs no explanation. It in fact is a special version of the *nonlocal theory of continua* (see Entry: Non-locality (strong)). As such, it will replace the usual partial differential equations by integro-differential equations. In truth, it essentially provides a bridge between standard local continuum models and nonlocal atomistic models. It is an extension of classical continuum mechanics suitable for modelling discontinuous phenomena such as discontinuous displacement (e.g., in fracture). The main idea is to avoid the difficulty presented by the existence of field singularities such as happens in the study of fracture for which the partial differential equations of classical mechanics are not well equipped. It directly provides an efficient numerical method to deal with these cases. The early developments of this approach are principally due to a small group of authors around Stewart A. Silling (among them, Richard B. Lehoucq, F. Bobaru, W Hu, Y.D. Ha, and E. Askari) starting in year 2000 [cf. Silling (2000), Silling and Askari (2005), Silling et al. (2007), Silling and Lehoucq (2010)].

The Main Idea

We consider an equation of motion of the continuum in the vector form

$$\rho(x)\ddot{u}(x,t) = F_{int} + \rho b(x,t) \tag{13.9}$$

where the "internal (body) force" is given by a space integral expression

$$F_{int}(x,t) = \int_B f(u(x',t) - u(x,t), x' - x, x) dV_{x'}. \tag{13.10}$$

Here x and x' are two points in the body B, and u and u' are the corresponding displacements. The vector-valued function f depends on both relative displacement and relative position. The initiators of this approach have introduced a special wording such as "pairwise force function" for f and "bond" for the interaction between points x and x'. It is reasonable to introduce a critical distance of interactions. This is called the "horizon" (but this is not a very good denomination; the radius of the sphere of influence on point x would be more appropriate). Since x and x' play symmetric roles, we have (equality of action and reaction)

$$f(u-u',x-x',x') = -f(u'-u,x'-x,x), \tag{13.11}$$

while the balance of angular momentum requires that

$$((x'+u')-(x+u)) \times f(u'-u,x'-x,x) = 0. \tag{13.12}$$

If we call *bond elongation* the scalar

$$e := |(x'+u')-(x+u)| - |x'-x|, \tag{13.13}$$

it might be physically justified to introduce a maximum value e_{max} at which there occurs a breakage of the bond. This phenomenon of bond breakage and the resulting load redistribution in the body is the process by which a crack expands according to this modelling. This appears to be a very efficient method in the numerical simulation of such a process. Various sophistications have been introduced to improve the method. For instance, the notion of "*peridynamic states*" has been introduced to eschew the oversimplification that bonds respond independently of all the others in the body. This consists in considering that the force density f in each bond in fact depends on the stretches in all the bonds connected to its endpoints, and this in addition to its own stretch. The long synthesis by Silling and Lehoucq (2010) is the most instructive one, having been written by the main original contributors to peridynamics. An introduction to practical peridynamics with examples of computations is given in a book by Gerstle (2016) that can serve as a textbook.

Fields of Applications: Brittle fracture, Modelling of membranes and fibres, kinetics of phase transformations, fracture of nano-fibre networks, molecular dynamics, multi-scale modelling, study of crack nucleation, of crack branching, transient heat conduction, phonon dispersion, etc.

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Cross references: Non-locality (as opposed to continuity), Non-locality (weak).

Polarization Gradient

[See Entry: Ionic crystals (elasticity of)].

Ponderomotive Couple

A magnetic dipole **m** placed in a magnetic field **H** is subjected to a torque

$$\mathbf{c} = \mathbf{m} \times \mathbf{H}. \tag{13.14}$$

This is obviously illustrated by the alignment of a compass needle with the local earth magnetic field. The alignment takes place so as to minimize the energy $E = -\mathbf{m}.\mathbf{H}$. In a continuous body (13.14) translates to a density of volume couple

$$\mathbf{C}_M = \mathbf{M} \times \mathbf{H}, \tag{13.15}$$

where **M** is a volume density of magnetization. It was soon realized that the existence of such a couple would cause an asymmetry of the stress tensor in a magnetizable elastic body. As noted by Hayart (1966, p. 3), this was remarked by Bouasse (1931) in his course on the strength of materials.

In an electrically polarized continuum with polarization **P** per unit volume, a density of volume couple

$$\mathbf{C}_P = \mathbf{P} \times \mathbf{E} \tag{13.16}$$

may exist when the averaged electric dipole is not exactly aligned with the local electric field **E**. Thus, globally, in an electromagnetic continuum we may have a density of body couples of the form

$$\mathbf{C}^{em} = \mathbf{M} \times \mathbf{H} + \mathbf{P} \times \mathbf{E}. \tag{13.17}$$

Because there hold the definitions (here written in so-called Lorentz-Heaviside units)

$$\mathbf{B} = \mathbf{H} + \mathbf{M}, \mathbf{D} = \mathbf{E} + \mathbf{P}, \tag{13.18}$$

where **B** is the magnetic induction and **D** is the electric displacement, Eq. (13.17) can also be written as

$$\mathbf{C}^{em} = \mathbf{M} \times \mathbf{B} + \mathbf{P} \times \mathbf{D}. \tag{13.19}$$

Couple (13.7) or (13.19) is usually called the *ponderomotive couple* in parallel with an electromagnetic body force $\mathbf{f}^{em}$ called the ponderomotive force. The

adjective "ponderomotive" may not be spot on as it recalls the notion of point, but it is traditional. Both expressions of $\mathbf{C}^{em}$ and $\mathbf{f}^{em}$ can be established by computing volume or statistical averages in a modelling that starts from a set of electric charges such as in a famous evaluation made by H.A. Lorentz (1853–1928) in his celebrated "theory of electrons" (*not* the electrons of modern physics)—cf. Lorentz (1909); Eringen and Maugin (1990).

In simple phenomenological (macroscopic) physics, **M** and **P** are given constitutive equations of the form

$$\mathbf{M} = \mathbf{M}(\mathbf{H}, .), \mathbf{P} = \mathbf{P}(\mathbf{E}, .), \tag{13.20}$$

where the missing independent variable may be temperature and/or strain. In an isotropic body where **M** and **P** are strictly proportional to **H** and **E**, respectively, the couple (13.19) vanishes identically. This may be the case in fluids (e.g., in magneto-hydrodynamics and electro-hydrodynamics). But this is not true in general, so that the local balance of angular momentum with stress tensor **t** will read in components (in the absence of internal spin and couple stresses)

$$\varepsilon_{klm} t_{lm} = -C_k^{em} \text{ or } t_{[ji]} = -\frac{1}{2}\varepsilon_{jik} C_k^{em}, \tag{13.21}$$

or

$$t_{[ji]} = M_{[i} B_{j]} + P_{[i} E_{j]}, \tag{13.22}$$

where $t_{[ji]}$ denotes the skewsymmetric (or antisymmetric) part of the stress tensor defined by

$$t_{[ji]} = \frac{1}{2}\left(t_{ji} - t_{ij}\right), \tag{13.23}$$

and ε_{ijk} is Levi-Civitta's alternating symbol.

Equation (13.21) or (13.22) provides the simplest example of a nonclassical stress expression in so-called generalized continuum mechanics.

Remark

At a microscopic scale magnetic moment **m** and magnetic spin **s** are related by the gyromagnetic ratio γ_e and **s** satisfies a gyroscopic like equation

$$\dot{\mathbf{s}} = \Omega \times \mathbf{s} \tag{13.24}$$

where $\Omega = -\gamma_e \mathbf{B}$ is Larmor's precessional velocity. In the macroscopic phenomenological theory called *micromagnetism*, Eq. (13.24) generalizes to a continuum equation of the same form, i.e.

$$\dot{\mathbf{M}} = \Omega \times \mathbf{M}, \Omega = -\gamma \mathbf{B}^{eff}, \tag{13.25}$$

where γ is an effective gyromagnetic ratio and $\mathbf{B}^{eff}$ is an effective magnetic induction which, in particular, involves the magnetic induction **B**, so that the couple (13.15) is involved in Eq. (13.25). In turn, this will necessarily be coupled with the skew part of the stress tensor in a deformable body (see the entry "micromagnetism").

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Cross references: Couple stress, Electromagnetic continua, Ferroic states, Generalized continuum mechanics, Micromagnetism.

Porous Media (as Seen in GCM)

A porous medium can be seen as a heterogeneous body made of two constituents, the matrix and the interconnected voids filled of a liquid or really empty. When the voids are completely filled in the medium is said to be *fluid-saturated*. One may think of representing this global medium by assuming that the constituents are interpenetrating and in fact co-existent in certain relative proportions at each point but without real mass exchange between them. Then an averaging can be envisaged that will provide macroscopic laws of, say, a *poro-elastic* medium. Such an approach was used early in the mechanics of composites and obviously in mixtures of fluids. In more recent times, a rather mathematical way to deduce basic laws of porous media is the technique of *homogenization* (See the entry "Homogenization").

In most cases the structure is not well ordered as a porous medium obviously presents a random distribution of pores, themselves of various sizes (for this aspect, see Adler 1992). But, remarkably enough, assuming a rather (in most cases unrealistic—unless man-made) ideal situation with a regular, e.g., periodic, pattern of pores, one can conceive of an easily implemented scheme that is both mathematically sound and practically very efficient as proved by the case of porous media. The *representative volume element* (RVE) then is a basic cell of this convenient periodic arrangement while in the averaging technique the RVE should contain

enough statistical information about the heterogeneous medium in order to be truly representative. Anyway, we are facing two spatial length scales, those of the RVE (l) and of the macroscopic scale (L), typically in the ratio $\varepsilon = l/L$, where ε is an infinitesimally small quantity. In physical Euclidean space the macroscopic domain Ω of interest is considered periodic, the rescaled unit cell is $Y = (0,1)^3$, and the two coordinates ($\mathbf{x}$ macroscopically and $\mathbf{y}$—fast variable—in the cell) are given by

$$\mathbf{x} \in \Omega, \quad \mathbf{y} = \frac{\mathbf{x}}{\varepsilon} \in Y. \tag{13.26}$$

The result of this parametrization is that the problem now is embedded in a sequence of similar problems parametrized by a scaling ε with expressions of the type (asymptotic formal expansion, so-called Ansatz)

$$\mathbf{v}_\varepsilon(\mathbf{x}) = \mathbf{v}(\mathbf{x},\mathbf{y}) = \mathbf{v}_0(\mathbf{x},\mathbf{y}) + \varepsilon\mathbf{v}_1(\mathbf{x},\mathbf{y}) + \varepsilon^2\mathbf{v}_2(\mathbf{x},\mathbf{y}) + O\left(\varepsilon^3\right) \tag{13.27}$$

for all fields, e.g., the velocity field $\mathbf{v}$. Then homogenization amounts to performing an asymptotic analysis when ε tends to zero. The limit is the solution of the homogenized problem. This mathematical technique is called *asymptotic periodic homogenization* (APH). Application of the APH to porous media was first given by Lévy and Sanchez-Palencia (1975) and Ene and Sanchez-Palencia (1975) by applying the APH scheme to the *steady Stokes equation for an incompressible viscous fluid* with vanishing velocity $\mathbf{v}$ on the boundary S of Ω, i.e.,

$$\nabla p_\varepsilon - \varepsilon^2\mu\nabla^2\mathbf{v}_\varepsilon = \mathbf{f}, \nabla.\mathbf{v}_\varepsilon = 0, \mathbf{v}_\varepsilon|_S = \mathbf{0}, \tag{13.28}$$

a set of equations which admits a unique solution in the appropriate functional spaces for $\mathbf{v}_\varepsilon$ and p_ε. The authors deduced the *Darcy-Hagen law*

$$\mathbf{q} = -\frac{k}{\mu}(\nabla p - \rho\mathbf{g}) \tag{13.29}$$

where $\mathbf{q}$ is the flux or discharge per unit area, p is the pressure, ∇p is the pressure gradient, $\mathbf{g}$ is the gravity vector, k is the permeability, and μ is the dynamic viscosity. The so-called Darcy velocity is defined as

$$v_{Darcy} = \frac{\mathbf{q}}{A} = v_{pore}n, \tag{13.30}$$

where $\mathbf{q}$ is the filtration flux (discharge per unit area), A is the filtration surface, and n is called *porosity* (sometimes noted ϕ) given by

$$n = \frac{volume\ of\ void\ (pores)}{total\ volume}. \tag{13.31}$$

Lévy (1983) went further by studying the Stokes flow through a swarm of fixed (non-connected) particles and justified thus *Brinkman's law* in the form

$$6\pi\mu ac\,\mathbf{v} = -\nabla p_0 + \mu\nabla^2\mathbf{v} + \mathbf{f} \tag{13.32}$$

by a technique of matched asymptotic expansions involving a boundary layer around the particles. Notice that this model is a strange one for a porous medium because of the lack of connectivity between particles (the same remark applies to Brinkman's work).

[The original Brinkman (1947) law reads

$$\frac{\mu}{k}\mathbf{v} = -\nabla p + \lambda\nabla^2\mathbf{v}, \tag{13.33}$$

where λ is called *effective viscosity*].

Another generalization is due to Forchheimer (1901) when the fluid velocity is large enough, yielding a *nonlinear* generalization of Darcy's law in the form

$$\frac{\mu}{k}\mathbf{v} + a|\mathbf{v}|\mathbf{v} = -\nabla p. \tag{13.34}$$

This equation can be said to be *nonlinear* and to account for *inertial* effects.

In a convective flow where the viscosity varies considerably with temperature, it may be necessary to combine Eqs. (13.33) and (13.34). Higher order nonlinear terms can also be added in the left-hand side of (13.35). All these equations are valid for incompressible fluids for which $\nabla.\mathbf{v} = 0$. There also exist anisotropic generalizations that involve a second-order tensor of permeability $\mathbf{k}$ instead of the scalar k. Finally, in the same line of thought as what was proposed for heat conduction by Cattaneo and Vernotte in the 1940s–1950s to give a finite speed to the heat propagation phenomenon (See the entry "Generalized thermo-elasticity"), for very short times Eq. (13.29)—without gravity—can be improved to read

$$\tau\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v} = -\frac{k}{\mu}\nabla p, \tag{13.35}$$

where τ is a very small time constant. Such a form will result in a *hyperbolic* groundwater flow equation, perhaps not of much practical use (cf. Cattaneo's modelling applied to porous media in Straughan 2011, pp.238–240).

The question of the validity of the various approximations obtained in the homogenization procedure (Lévy, Sanchez-Palencia, Auriault et al.), calls for a thorough examination of the respective domains of validity in terms of non-dimensional numbers (scales). For this we refer the reader to Auriault (2009) who underlines the case of so-called "poor separation of scales", i.e., when ε, if we may say so, is not so small (Auriault et al. 2005). In this analysis, Brinkman's law is shown to be invalid for connected porous matrices so that this casts some doubt on the real physical meaning of Brinkman's modelling for porous media.

Porous Media and the Theory of Mixtures

It is easily conceived that a porous medium may be considered a mixture, but a mixture reduced to two constituents or two phases, a solid one (the matrix) and a fluid one likely to flow through the interstices left by the solid with interconnected pores. This is the view adopted by, e.g., Rajagopal and Tao (1995). This is the *continuum theory of mixtures* for which Clifford Truesdell (1957) provided the first thermodynamically based rational theory. There followed in the 1960s–1970s a multitude of works in the same line.

The theory of mixtures uses the concept of volume fraction, an example of which is given by Eq. (13.31) in the case of porous media. The general *thermodynamic* theory of mixtures is a complicated affair that still poses fundamental questions, for example: (i) should a temperature be introduced for each constituent (same question for entropy); (ii) should global thermodynamic laws, in particular the second one, be defined for each constituent or for the global medium; (iii) how the stress condition at a boundary should be formulated since only one global applied traction can be specified while the mixture involves partial stresses? The only constraint on which all authors agree is that by summation over the constituents one obtains the thermo-mechanical equations that govern a standard *unique* continuum. We shall not delve into this generality as temperature effects are altogether ignored in the rest of this section. We prefer to examine the advances made by a few remarkable contributors during the period 1910–1960. In this period that preceded the full implementation of nonlinear continuum thermomechanics, we can identify the breakthroughs made by four specialists, civil engineers or physicists, who brought new constructive ideas in the mechanics of porous media, and more generally to soil mechanics, with many potential applications to the construction of dams, the exploration and exploitation of oil resources, etc. These remarkable scientists are: Paul Fillunger (1883–1937); Karl von Terzaghi (1883–1963), Maurice A. Biot (1905–1985), and Gerhard Heinrich (1902–1983), among whom three (that is, save Biot) were affiliated with the Technical University of Vienna (Wien, Austria). For a full exposition of their works we refer to De Boer (2000). Both Fillunger and Terzaghi were outstanding professors but with very different personalities who entered into a dramatic conflict. But while Terzaghi (1923), often considered the father of soil mechanics as a science, and much more internationally known than Fillunger, proposed a rather intuitive law for the vertical consolidation of porous media in a simple (parabolic) form reminiscent of Fourier's and Fick's laws, Fillunger (1936) published an approach that would perfectly fit in a modern presentation in the way of the theory of mixtures. Considering a one-dimensional setting for the ease in presentation and the case of incompressible fluid and solid components, Fillunger wrote the local equations of conservation of linear momentum and mass (no exchange of mass between the two constituents with a rigid solid constituent) as the following system in his notation:

$$\begin{aligned}
&\frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial z} = \frac{1}{\rho_1}\left(-Z - \frac{\partial p_1}{\partial z}\right),\\
&\frac{\partial v_2}{\partial t} + v_2 \frac{\partial v_2}{\partial z} = \frac{1}{\rho_2}\left(Z - \frac{\partial p_2}{\partial z}\right),\\
&\frac{\partial \rho_1}{\partial t} + \frac{\partial}{\partial z}(\rho_1 v_1) = 0, \frac{\partial \rho_2}{\partial t} + \frac{\partial}{\partial z}(\rho_2 v_2) = 0,
\end{aligned} \tag{13.36}$$

where subscripts 1 and 2 refer to the fluid and solid constituents, respectively, and Z is an *interaction force*. If n is the porosity, $p = p_1 + p_2$ the total pressure in terms of the partial pressures, and γ_1 and γ_2 are the specific weights, we have

$$p_1 = np, p_2 = (1-n)p, \tag{13.37}$$

and

$$\frac{1}{\rho_1} = \frac{g}{n\gamma_1}, \frac{1}{\rho_2} = \frac{g}{(1-n)\gamma_2}. \tag{13.38}$$

This allows us to transform the last two of Eqs. (13.36) to the form

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial z}(nv_1) = 0, \quad -\frac{\partial n}{\partial t} + \frac{\partial}{\partial z}(1-n)v_2 = 0. \tag{13.39}$$

Finally, the system is closed on account of Darcy's law here simply written as

$$Z = \frac{nv_1}{k}, \tag{13.40}$$

where k is the permeability. Of course, it is Darcy's law that is instrumental in the final argument. The above given system can be extended to the case of a deformable solid matrix and to three dimensions of space. It contains the quintessence of the theory of mixtures applied to porous media, and thus is extremely modern. Heinrich, also an Austrian engineer and professor of mechanics at TH Wien, was a direct disciple of Fillunger and formulated with Desoyer (1961) the proper three-dimensional formulation.

The work of Biot on porous media is quite original and is examined in its own entry as it provides a true nonclassical modelling.

The modern *thermo-mechanics* of porous media is expanded in books such as De Boer (2000), Ehlers (2010), Wilmanski (1998, 2003), Rajagopal and Tao (1995) and Coussy (1995, 2010). These works apply the now admitted methodology, in particular with the implementation of the Clausius-Duhem inequality as a constraint imposed on the constitutive equations. The deduced constitutive equations thus obtained can be rather cumbersome and sometimes of an unnecessary complexity and generality. Noteworthy are the original works of Wilmanski (1996, 2005a) where this author considers an equation governing the porosity and also the notion

of *tortuosity* in the continuum framework. As to the work of Shriram and Rajagopal (2014), it exploits the rational thermodynamics of mixtures and the criterion of maximal rate of entropy production, and reproduces the Darcy, Brinkman and Forchheimer laws with appropriate working hypotheses.

The present contribution would not be complete without acknowledging the role played by porosity in the plastic behaviour of some materials, in particular among geomaterials and even in metals with growing cavities. The resulting effect of porosity is an alteration in the yield criterion, whether of the Mises or Coulomb type. In the case of metals, porosity intervenes in the ductile fracture of these materials. The breakthrough work of Gurson (1977) can be mentioned in this context, as also improvements on his proposal by other authors, e.g., Cologanu et al. (1997). Finally, we note a growing interest in so-called *microporomechanics* (cf. Dormieux et al. 2006).

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Cross references: Biot’s theory of poro-elasticity, Homogenization, Mixtures (mechanics of),

Chapter 14
Q–R: From "Quasi-crystals (Elasticity of)" to "Rogula R.D."

Quasi-crystals (Elasticity of)

Introduction

The elasticity of quasi-crystals belongs in a class by itself although it may be approached from two different sides that we shall rapidly survey. Without entering the technicalities of condensed matter physics, we note that *quasi-crystals* experimentally discovered in certain alloys in 1982 by D. Shechtman (Nobel Prize 2011) and with an initial symmetry theory proposed in 1984 (see Lubensky 1988; Fan 2011, Trebin 2003; for the history and main structural and physical properties) present a challenge for their modelling as deformable continua. It is acknowledged that they present some *aperiodicity* (or quasi-periodicity along certain directions or in planes) which is their main characteristic symmetry property. This allows for the acceptance of symmetries (e.g., five-fold orientational symmetry) that were heretofore forbidden but are now revealed in diffraction experiments for certain alloys. Furthermore, their strange symmetry behaviour is macroscopically represented by the co-existence of two elementary excitations (in the sense of Lev D. Landau), a rather classical one, known as *phonons* (the typical vibrations of regular—periodic—crystal lattices) in physical—so-called *parallel*—space and a new internal one—in so-called *orthogonal* space—referred to as *phason* that corresponds to a relative motion of the constituent density waves of condensed matter physics (in other words, the internal rearrangements of an environment—cf. Gähler et al. 2003). Phenomenologically, one displacement is associated with each of these but whereas one interpretation favours a vision somewhat parallel for the two displacements with particular structural disorder or structure fluctuations associated with the phasons (here referring to Bak's vision), another interpretation sees the phasons as diffusive with large diffusive time (Lubensky et al.). These two possibilities can be examined from the point of view of the phenomenological

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(continuum) thermomechanics of continua, keeping also in mind the importance of the role played by defects and some non-linearities in such crystals.

General Field Equations

Only a Cartesian tensor notation (with its usual conventions) that is accessible to all, engineers and physicists, is used. We are interested in the *dynamics* of elastic quasi-crystals so that inertia and evolution in time are taken into account, depending on the selected model. Two such models are here referred to as the "Bak" (inspired) model and the "Lubensky" (inspired) model according to the main authors of these two possibilities (Bak 1985a, b), in the first case; and Lubensky and co-workers (1985, 1988) in the second case.

A. **The "Bak" dynamic model**

In this case the equations of linear momentum associated with both phonons and phasons have a standard form with inertial terms, i.e., in Cartesian tensor notation

$$\frac{\partial}{\partial x_j}\sigma_{ji} + f_i = \rho \frac{\partial^2 u_i}{\partial t^2} \ (phonons), \tag{14.1}$$

$$\frac{\partial}{\partial x_j}H_{ji} + g_i = \rho \frac{\partial^2 w_i}{\partial t^2} \ (phasons), \tag{14.2}$$

where σ_{ji} is the *symmetric* stress tensor for "phonons", H_{ji} is the generally non-symmetric stress tensor for "phasons", f_i and g_i are the corresponding (if any) external forces per unit volume, u_i is the standard elastic displacement vector, w_i is the "phason" displacement vector, and ρ is the matter density. In the absence of dissipative effects, the two stresses are derived from a volume energy density $W\left(e_{ij}, w_{ij}\right)$ by

$$\sigma_{ji} = \frac{\partial W}{\partial e_{ij}}, H_{ji} = \frac{\partial W}{\partial w_{ij}}, \tag{14.3}$$

where

$$e_{ij} = e_{(i,j)} := \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \ w_{ij} := w_{i,j}. \tag{14.4}$$

Here w_{ij} is not reduced to its symmetric part since the "phason" field gradient is not subjected to rotational invariance, having in fact two indices that refer to directions in two different spaces (the internal or orthogonal space and the physical

space). In this framework the global medium may be viewed as the interaction—via the strain energy function—between two continua that are essentially elastic. Only the peculiarity of the second definition in (14.4)—cf. the already noted absence of symmetrization—and the application of material symmetry conditions will clearly distinguish the behaviour of one component from the other. The theory can be deduced from a standard Lagrangian-Hamiltonian variational principle fundamentally based on the expression (T is a time interval, and B stands for a regular three-dimensional body)

$$\delta \int_T \int_B L d^3x\, dt = 0,\ L = K - W, \tag{14.5}$$

in the absence of external body forces, and

$$K = \frac{1}{2}\rho \dot{u}_i \dot{u}_i + \frac{1}{2}\rho \dot{w}_i \dot{w}_i, \tag{14.6}$$

where a superimposed dot denotes the partial time derivative.

B. **The "Lubensky" dynamic model**

In this case Eqs. (14.1) and (14.2) are replaced by the equations

$$\frac{\partial}{\partial x_j}\sigma_{ji} + f_i = \rho \frac{\partial^2 u_i}{\partial t^2}\ (phonons), \tag{14.7}$$

$$\frac{\partial}{\partial x_j} H_{ji} + g_i = \kappa \frac{\partial w_i}{\partial t}\ (phasons), \tag{14.8}$$

where the stress tensors are still given by constitutive equations. (14.3) with definitions (14.4), and κ is the reciprocal of the kinetic coefficient for the phason field. The system (14.7), (14.8) couples a wave equation with an evolution-diffusion equation so that we can expect a dynamic response quite different from that derivable from the set (14.1), (14.2). Moreover, system (14.7), (14.8) is no longer derivable from a Lagrangian-Hamiltonian variational formulation. Another type of approach involving irreversible thermodynamics is necessary to justify this system on a continuum basis. This can be formally accommodated within the thermodynamics of continua which involves so-called *internal variables of state* (see the corresponding entry), the phason displacement being here a vectorial internal variable of state (*not* an internal degree of freedom). The related "internal" variables are not directly controllable by external forces (thus g_i should vanish) and it simply contributes to the expression of the free energy of the system while its evolution is governed by the second law of thermodynamics. To formulate this approach one needs to return to the general equations of a continuum with special caution concerning the entropy flux. A priori no internal variable of state shows up in the basic

equations. We consider only small deformations and an obvious notation. We have the following local equations:

- the local balance of linear and angular momenta:

$$\frac{\partial}{\partial x_j}\sigma_{ji} + f_i = \rho \frac{\partial^2 u_i}{\partial t^2}, \quad \sigma_{ij} = \sigma_{ij}; \tag{14.9}$$

- the first law of thermodynamics:

$$\frac{\partial}{\partial t}(K+E) - \frac{\partial}{\partial x_j}\left(\sigma_{ji}\dot{u}_i - q_j\right) = \rho h; \tag{14.10}$$

- the second law of thermodynamics:

$$\frac{\partial}{\partial t}S \geq \frac{\partial s_i}{\partial x_i} - \rho\frac{h}{\theta}; \tag{14.11}$$

where E is the internal energy per unit volume, q_j and s_i stand for the heat flux vector and entropy flux vector, respectively, and S is the entropy per unit volume. Finally, θ is the thermodynamic temperature ($\theta > 0,\ \inf\theta = 0$). Here,

$$K = \frac{1}{2}\rho\dot{u}_i\dot{u}_i \tag{14.12}$$

only, and

$$s_i = k_i + \frac{q_i}{\theta}. \tag{14.13}$$

Vector k_i is called the *extra* entropy flux; it vanishes in most continuum theories save in the presence of diffusive processes (cf. Maugin 1999), which will be the case in this paragraph.

Introducing the free energy per unit volume W by $W = E - \theta S$, and accounting for (14.9), (14.12) and (14.13), combination of (14.10) and (14.11) yields the following *Clausius-Duhem inequality*:

$$-\left(\dot{W} + S\dot{\theta}\right) + \sigma_{ji}\dot{e}_{ij} + (\theta k_i)_{,i} - s_i\theta_{,i} \geq 0. \tag{14.14}$$

Now we introduce the dependence of the free energy on *both* observable and internal variables of state by considering the function

$$W = \overline{W}\left(e_{ij}, w_i, w_{i,j}, \theta\right). \tag{14.15}$$

Substituting from this into (14.14) we obtain for the latter *sufficient* conditions for its satisfaction as

$$\sigma_{ji} = \frac{\partial \overline{W}}{\partial e_{ij}},\ S = -\frac{\partial \overline{W}}{\partial \theta}, \tag{14.16}$$

and

$$A_i \dot{w}_i - s_i \theta_{,i} \geq 0, \tag{14.17}$$

with

$$A_i \equiv -\frac{\delta \overline{W}}{\delta w_i} := -\left(\frac{\partial \overline{W}}{\partial w_i} - \frac{\partial}{\partial x_j}\frac{\partial \overline{W}}{\partial w_{i,j}}\right),\ k_i = -\frac{1}{\theta}\frac{\partial \overline{W}}{\partial w_{i,j}}\dot{w}_j. \tag{14.18}$$

A frequent working hypothesis in *T.I.V* is to split the remaining dissipation inequality (14.17) in two parts related to heat conduction and entropy production due to the internal variable, yielding thus

$$s_i \theta_{,i} \leq 0,\ A_i \dot{w}_i \geq 0. \tag{14.19}$$

The first of these will eventually yield Fourier's law of heat conduction. As to the second, a simple evolution-diffusion model is obtained by taking W independent of w_i but quadratic in the gradient of this variable and assuming that A_i is directly proportional to $\dot{w}_i$—and thus guaranteeing the satisfaction of the second of (14.19). One obtain thus

$$\frac{\partial H_{ji}}{\partial x_j} = \kappa \frac{\partial w_i}{\partial t},\ H_{ji} \equiv \frac{\partial \overline{W}}{\partial w_{i,j}},\ \kappa \geq 0. \tag{14.20}$$

We have thus formally—but within thermodynamical admissibility—obtained the basic equations of the thermoelasticity of quasicrystals in the "Lubensky" format of the so-called *elasto-hydrodynamics* of such materials. This can obviously be discussed since this is purely phenomenological and the true "quasi-crystal" nature of the studied material will be made evident only after specifying W and imposing appropriate symmetry conditions.

Nonlinearity and Plasticity of Quasicrystals

Nonlinearity usually calls for a formulation in finite strains and the introduction of geometrical nonlinearities together with physical nonlinearities. This is not the case here where we are satisfied with physical nonlinearities only. This primarily means

a potential energy that is not quadratic in the relevant (small) deformation measures. Note that Eq. (14.3) are still formally valid in this case in the absence of irreversibilities. Should we know the mechanical response curves, we could in fact define the associated free energy by the formula proposed by Fan (2011, p. 295)

$$W\left(e_{ij}, w_{i,j}\right) = \int_0^{e_{ij}} \sigma_{ji} de_{ij} + \int_0^{w_{i,j}} H_{ji} dw_{ij}. \tag{14.21}$$

This does not tell much about the exact constitutive equations, save for the fact that the existence of a potential requires the absence of dissipation and the reversibility of the path in generalized stress-strain space.

However, for many engineers (see Chap. 1 in Maugin 1992) nonlinearity is synonymous with *plasticity*, i.e., an inherently dissipative behaviour marked by unloading along response curves that differ from those of the loading phases, and thus exhibiting a kind of hysteresis (of course, in the absence of unloading, nobody can say that the body is not "simply" nonlinear elastic, even though of a special type!). The accompanying thermo-mechanical description of true plasticity is necessarily much more involved and, in spite of the presence and importance of structural defects in quasicrystals (cf. Levine et al. 1985), we do not know yet if there is real need for a mathematical theory of plasticity of quasicrystals expanded along the lines given in Maugin (1992) for usual crystals. Such proposal is sketched out by Fan (2011, Sect. 14.2). This formulation proposing in parallel "plastic" evolution equations for both generalized strains e_{ij} and $w_{ij} \equiv w_{i,j}$, and including the effect of the phason stress H_{ij} on the generalized effective stress used to define the yield surface, may be surmised by analogy in the "Bak" inspired formulation, but it is doubtful in the "Lubensky" formulation where the phason displacement w_i itself already satisfies an evolution-diffusion equation.

Conclusion

In the foregoing paragraphs we have tried to elucidate some of the critical points concerning the continuum thermodynamics of quasicrystals at the little cost of the implementation of now currently accepted formalisms. However, the main point remains that of the explicit writing of the coupled generalized "Hooke" constitutive equations when the relevant energy is simply jointly quadratic in the deformation measures (14.4), i.e., when Eq. (14.3) result in the linear relations

$$\sigma_{ji} = C_{jilk} e_{lk} + R_{jilk} w_{k,l}, \; H_{ji} = K_{jilk} w_{k,l} + R_{lkji} e_{kl}, \tag{14.22}$$

wherein

$$C_{jilk} = C_{(ji)(lk)} = C_{lkji},\ K_{jilk} = K_{lkji},\ R_{jilk} = R_{(ji)lk}. \tag{14.23}$$

Specific expressions of the tensor coefficients for various material symmetries admitting the existence of quasicrystal symmetries are documented in Fan (2011; Chaps. 5 and 6) and references therein. Fortunately, only a few nonzero coefficients are necessary in many of the applications for so-called "one-dimensional" quasi-crystals (that are nonetheless three-dimensional structures, standard symmetry still applying in a plane orthogonal to this peculiar direction).

What remains unclear is the physical significance of a quantity such as the prescribed force of component g_i—when it exists—and of the "natural" boundary condition (involving the normal vector component $n_j H_{ji}$) for the phason field. The work of Cimmelli (2002) does not help to answer this query. It is claimed by some authors (e.g., Ding et al. 1995) that the g_i force is related to the first gradient of the phason eigen-distortion and phonon eigen-distortion, but this is outside continuum mechanics.

It can also be noted that there exists strong disagreement—accompanied by harsh comments—between tenants of various physical interpretations of dynamical properties of quasi-crystals [e.g., opposite schools represented by Coddens (2006) and Francoual et al. (2003) and further discussions by these authors]. Furthermore, a third model viewing the phason basic equation as a true wave equation but with a dissipative term (and thus finally yielding a kind of telegraphy equation) was proposed by Agiasofitou and Lazar (2014) while phonons remain giving rise to undamped elastic waves.

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Cross references: Internal degrees of freedom, Internal variables of state.

Relaxed Micromorphic Continua

The expression "relaxed micromorphic continua" has been recently introduced by a group of authors, mainly P. Neff and A. Madeo, in the 2010s (cf. the synthetic view in Neff et al. 2014). The authors rightly note that a theory of anisotropic continua of the micromorphic type such as in Eringen-Mindlin media with intrinsically non-symmetric force stresses, would involve, even in linear form, an incredible number of material coefficients, rendering the theory practically un-exploitable. It was proposed to *relax* the general theory to one with symmetric Cauchy force stresses and curvature response only due to dislocation energy. The resulting framework is very close to that of the Claus-Eringen model (cf. Claus and Eringen 1969; Eringen and Claus 1970) but with symmetric force stresses and in the absence of mixed coupling terms.

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Cross references: Dislocations and disclinations, Eringen-Mindlin medium, Micromorphic continua,

Rivlin R.S

Ronald S. Rivlin (1905–2005), a British-American applied mathematician, was one of the most creative contributors to the revival of continuum mechanics in the 1940–1970 period. His contributions span all of solid and fluid mechanics with particular attention paid to finite deformations, nonlinear elasticity (e.g., the Mooney-Rivlin energy) and non-Newtonian fluid mechanics (Reiner-Rivlin fluids, Rivlin-Ericksen fluids). He contributed to non-classical continuum mechanics, together with A.E. Green, with a rather complex theory of multipolar continua. Rivlin (1996) has produced an interesting autobiography.

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Cross references: Dipolar materials, Double force, Generalized continuum mechanics, Multipolar continua (Green-Rivlin).

Rogula D

Dominik Rogula (born 1940) is a Polish mechanician who pioneered the theory of *strong nonlocality* in continuum mechanics, i.e., accounting for the possibility of long-rang interaction in contrast to the contiguity hypothesis introduced by Euler and Cauchy (cf. Rogula 1965). He was instrumental in the popularization of the concept of this strong nonlocality that necessarily yields dispersion of elastic waves as it introduces the notion of spatial scale (cf. Rogula 1965, 1976). This he did in correlation with the concept of structural defect (e.g., dislocations).

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Cross reference: Non-locality (strong).

Chapter 15
S–T: From "Solitons (in on-Classical Continua)" to "Truesdell C.A."

Solitons (in Non-classical Continua)

Many non-classical continua offer an ideal frame for the exhibition of non-linear waves akin to so-called *solitons*. The reason for this is that they often present *dispersion* (existence of length-scales). Combination or balance of this property with *non-linearity* favours the existence of this dynamic phenomenon: that is, the propagation of strongly localized nonlinear signals that propagate over long distances without alteration in their shape and a possible "particle-like" interaction with their fellows. This is the case in gradient elasticity with an appropriate dose of non-linearity, and more generally, in what we called the *Boussinesq paradigm*—wave systems endowed with non-linearity and dispersion of various orders (cf. Christov et al. 2007). Note that non-linearity is not enough to warrant the existence of such waves. That is, a non-linear classical continuous medium may favour the existence of wave dynamic phenomena such as shock waves, but not of solitons for they are missing in their internal structure the dispersive element. Examples of soliton solutions in non-classical continua of the solid type are provided in works by Maugin and Miled (1986), Erbay (1996), Erbay et al. (1992) for micropolar media, Erofeev and Potapov (1993) in media with couple stresses, Pouget and Maugin (1989) for oriented media with directors, and Maugin and Cadet (1991) in martensitic alloys. An overview is given in a book (Maugin 1999) and a historical perspective is drawn in Maugin (2011).

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Cross references: Cosserat continua, Couple stress, Gradient elasticity, Lattice dynamics, Micropolar continua, Micropolar elasticity, Nonlinear waves in generalized continua, Oriented media (with directors).

Solutions of Macromolecules

Introduction

Here we are concerned with a phenomenological and thermodynamical approach to dilute solutions of polymers with the ultimate view to comprehend such phenomena as the dynamics of the deformation of macromolecules in the fluid carrier, the damping of turbulence by polymers, hence the reduction of friction in turbulent regimes, and the stress-induced diffusion of macromolecules. Known approaches to the mechanical properties of macromolecule solutions divide in two large classes, the first one involving "kinetic-theory" arguments, hence statistical features, and the second one being purely phenomenological and having, in principle, no recourse to microscopic features but perhaps for some intuitive notions (see below). The first of these is illustrated by the magisterial books of Bird et al. (1977a, b) while the second one could be illustrated by the works of Hand (1962), Lhuillier and Ouibrahim (1980), Maugin and Drouot (1983, 1991) and others—also Maugin (1999, Chap. 6). Obviously, all authors—whatever their approach—are aware of the presence of some *microstructure* in the solution. Macromolecules are, in general, to be modelled by deformable "particles" of which the instantaneous shape and orientation constitute a microstructure in the sense granted, say, by Eringen in so-called *micromorphic continua* (see that Entry). Many chemical physicists are interested in the flow-induced changes in the microstructure, that is, the influence of a prescribed flow of the surrounding carrier fluid (more than often a usual Newtonian fluid) on the shape and orientation of this microstructure. In this vision, the macromolecule is considered as a *test* particle since it is influenced by, but does not influence itself, the macroscopic flow. For very dilute solutions, where the mean

distance between deformable particles is much larger than the spatial scale of variation in the fluid flow, it is easily conceived that this one-way influence can be studied for one isolated particle of a certain type (spheroid, ellipsoid, rod-like particle). This provides a splendid problem to hydrodynamicists. For not so dilute solutions the situation becomes much more complex, neighbouring particles interacting possibly with one another and yielding eventually some kind of ordering akin to a nematic phase in the case of elongated particles (cf. liquid-crystal—Ericksen-Leslie theory). Anyhow, it is now the equation which governs the evolution in the deformation of the particle which then is of interest and the carrier's flow behaviour or the behaviour of the overall mixture must intervene in this equation. On the other hand, one may be interested in the overall flow changes due to the presence of macromolecules (their deformability in particular). The two effects may be mixed. Then the main question in a phenomenological theory will be what kind of *descriptor* should one introduce to describe the evolution of the microstructure?

Microstructure and Conformation

Being entangled in an intricate manner, long polymeric chains in their equilibrium configuration K_0 may be assumed to roughly have the shape of a spherical ball (so-called spherically coiled *conformation*, of radius r_0 of the order of 20 Å) under the influence of Brownian agitation. These "spheres" will slightly deform but will keep an essentially three-dimensional sphere (spheroid, ellipsoid) in the presence of a weakly elongational flow of the surrounding fluid in a configuration K_t, while they will more or less take a one-dimensional (rod-like) structure (with a length of the order of 2500 Å) in strongly elongational flows. Let $\mathbf{r}$ be, for instance, the directed distance between extreme monomers in a molecule chain in K_t. In the first case, a second-order moment (lowest harmonic departure from sphericity), hence a symmetric second-order tensor, must be introduced to describe the three-dimensional deformability of the molecules. Such a tensor, an inertia per unit mass, may be defined as

$$\mathbf{R} := \langle \mathbf{r} \otimes \mathbf{r} \rangle = \mathbf{R}^T. \tag{15.1}$$

But at K_0, $\langle \mathbf{r}_0 \rangle = \mathbf{0}$, and $\mathbf{R}_0 = r_0^2 \mathbf{1}/3$, so that a good relative measure of the deformation of molecules is given by

$$\mathbf{C} = \mathbf{R} - \mathbf{R}_0 \; or \; \mathbf{K} = \left(3 \frac{\mathbf{R}}{r_0^2} - \mathbf{1} \right). \tag{15.2}$$

This is called the *conformation*. In the continuum theory that we approach here this will be considered as an *internal variable of state*, so that it does not appear a priori in the basic conservation laws of the global continuum and it will simply

contribute to the free energy and be governed by the second law of thermodynamics (See Entry: Internal variable of state). Let c the ratio of the mass density ρ_P of polymer molecules to the solvent density ρ_S, so that the concentration in macromolecules is given by

$$c = \rho_P/\rho, \quad \rho_P = \rho - \rho_S. \tag{15.3}$$

The local field and thermodynamic equations which govern the macroscopic fluid considered as a mixture with prevalent solvent mass at the current point $\mathbf{x}$ at time t in K_t read:

$$\dot{\rho} + \rho \nabla.\mathbf{v} = 0, \tag{15.4}$$

$$\rho \dot{c} + \nabla.\mathbf{J} = 0, \tag{15.5}$$

$$\rho \dot{v}_i = t_{ji,j} + \rho f_i, \tag{15.6}$$

$$t_{ji} = t_{ij}, \tag{15.7}$$

$$\rho \dot{e} = t_{ji} d_{ij} - \nabla.\mathbf{q} + \rho h, \tag{15.8}$$

together with the Clausius-Duhem inequality

$$-\rho\left(\dot{\psi} + \eta\dot{\theta}\right) + t_{ji} D_{ij} + \nabla.(\theta\mathbf{k}) - \mathbf{S}.\nabla\theta \geq 0, \tag{15.9}$$

where $\mathbf{J}$ is a diffusion flux, $\mathbf{t}$ is Cauchy's symmetric stress, and $\mathbf{k}$ is an extra-entropy flux (see that entry) such that

$$\mathbf{k} = \mathbf{S} - \theta^{-1}\mathbf{q} \tag{15.10}$$

if $\mathbf{S}$ is the entropy (in) flux. The volumetric behaviour of the macroscopic fluid is assumed to be the same as that of the solvent, and is therefore considered incompressible, so that

$$D_{ii} = \nabla.\mathbf{v} = 0. \tag{15.11}$$

Constitutive Relations

The two main equations of the theory will be the *constitutive equation for the stress* that will slightly deviate from that of a Newtonian incompressible fluid, i.e., formally

$$\mathbf{t} = 2\eta_v(c)\mathbf{D} + \mathbf{t}_P(.,.,\mathbf{C}), \tag{15.12}$$

and the *evolution equation* for the internal variable a priori written as

$$D\mathbf{C} = \mathbf{A}(\theta, \mathbf{D}, .; \mathbf{C}), \tag{15.13}$$

where D is a kind of time derivative to be specified. In particular, we impose that both Eqs. (15.12) and (15.13) be objective, i.e., form invariant under rotation of the actual frame. It is shown that the contributions $\mathbf{t}_P$ and $\mathbf{A}$ are not independent. An appropriate choice of the D derivative that is objective is provided by a Jaumann (co-rotational) derivative such that

$$(D_J\mathbf{C})_{ij} = \dot{C}_{ij} - \Omega_{ik}C_{kj} - \Omega_{jk}C_{ki}, \quad \Omega_{ij} = \frac{1}{2}\left(v_{i,j} - v_{j,i}\right). \tag{15.14}$$

so that (15.13) will be replaced by

$$\hat{\mathbf{C}} := D_J\mathbf{C} = \tilde{\mathbf{A}}(\theta, \mathbf{D}, .; \mathbf{C}). \tag{15.15}$$

The free energy has the following functional dependence:

$$\psi = \tilde{\psi}(\theta, c, \mathbf{C}), \tag{15.16}$$

so that one can define

$$\tilde{\eta} = -\frac{\partial\tilde{\psi}}{\partial\theta}, \mu = \frac{\partial\tilde{\psi}}{\partial c}, \underline{\alpha} = -\rho\frac{\partial\tilde{\psi}}{\partial\mathbf{C}} = \underline{\alpha}^T. \tag{15.17}$$

On assuming that

$$\eta = \tilde{\eta}, \tilde{\mathbf{t}} = \mathbf{t} + p\mathbf{1}, \mathbf{k} = -(\mu/\theta)\mathbf{J}, \tag{15.17}$$

where $\tilde{\mathbf{t}}$ is the stress tensor deprived of its mechanical pressure contribution, the Clausius-Duhem inequality (15.9) reduces to the following *residual dissipation inequality*:

$$\Phi = trace\left(\tilde{\mathbf{t}}\mathbf{D} + \underline{\alpha}\tilde{\mathbf{A}}\right) - (\mathbf{J}.\nabla\mu + \mathbf{S}.\nabla\theta) \geq 0. \tag{15.18}$$

The latter governs the phenomena of viscosity, relaxation of $\mathbf{C}$, diffusion of polymer macromolecules, and heat conduction. The last of (15.17) has been chosen so as to eliminate a divergence term in the inequality.

Then the theory is completed in the following way. First a more specific expression must be given to $\tilde{\psi}$. Because of rotational invariance the latter must satisfy the constraint

$$\alpha_{i[j}C_{k]i} = 0. \tag{15.19}$$

This requires that $\tilde{\psi}$ depends on $\mathbf{C}$ only through its three elementary invariants C_α, i.e.,

$$\psi = \bar{\psi}(\theta, c, C_\alpha), \quad \alpha = I, II, III. \tag{15.20}$$

Next, in the inequality (15.18) the fluxes are given by the set $\{\tilde{\mathbf{t}}, \tilde{\mathbf{A}}, \mathbf{J}, \mathbf{S}\}$ while the conjugated forces are given by the set $\{\mathbf{D}, \underline{\alpha}, -\nabla\mu, -\nabla\theta\}$. On noting that $\underline{\alpha}$ is even while $\mathbf{D}$ is odd under time reversal, the a priori admissible linear isotropic *dissipative* contributions are given by

$$\mathbf{t}^D = 2\eta_v\mathbf{D}, \quad \mathbf{A}^D = \zeta(tr\underline{\alpha})\mathbf{1} + \kappa\underline{\alpha}, \tag{15.21}$$

$$\mathbf{J} = -(D\nabla\mu + A\nabla\theta), \quad \mathbf{S} = -(K\nabla\theta + A\nabla\mu), \tag{15.22}$$

where the various scalar coefficients have to satisfy a set of inequalities (cf. Maugin and Drouot 1983, p. 712). But there is more than that in the expression (15.18) as astutely remarked by Lhuillier and Ouibrahim (1980). There is the possibility of existence of so-called *gyroscopic* contributions—our coinage—$\mathbf{t}^G$ and $\mathbf{A}^G$ such that we have the following *orthogonality* condition (in the appropriate twelve-dimensional space) between generalized "forces" and "velocities"—hence the qualification of "gyroscopic":

$$\Phi^G := trace\left(\mathbf{t}^G\mathbf{D} + \underline{\alpha}\mathbf{A}^G\right) = 0. \tag{15.23}$$

with

$$\mathbf{t}^G = \bar{\mathbf{t}}(\mathbf{D}, \underline{\alpha}; \mathbf{C}, \theta, c), \quad \mathbf{A}^G = \bar{\mathbf{A}}(\mathbf{D}, \underline{\alpha}; \mathbf{C}, \theta, c), \tag{15.24}$$

where the last three arguments—which in fact define the laws of state (15.16)–(15.17)—here are parameters. Simple examples of relations (15.24) are given by

$$\mathbf{t}^G = \lambda\underline{\alpha} - \beta(\mathbf{C}\underline{\alpha} + \underline{\alpha}\mathbf{C}), \quad \mathbf{A}^G = -\lambda\mathbf{D} + \beta(\mathbf{C}\mathbf{D} + \mathbf{D}\mathbf{C}), \tag{15.25}$$

where λ and β are scalars which depend at most on θ, c, and the nonvanishing elementary invariants of $\mathbf{C}$. Their sign is not constrained by any thermodynamic inequality, but they are the *same* coefficients in both contributions as otherwise they would not be consistent with the orthogonality condition (15.23). On account of (15.21) and assuming some kind of Hookean elasticity H for the macromolecules in terms of $\mathbf{C}$, we are led to Eqs. (15.12)–(15.13) in the following form:

$$\mathbf{t} = -p\mathbf{1} + 2\eta_v + 2\beta H\mathbf{C}, \tag{15.26}$$

$$D\mathbf{C} := D_J\mathbf{C} - \beta(\mathbf{CD} + \mathbf{DC}) = -\frac{1}{\tau_C}\mathbf{C} - \lambda\mathbf{D}, \tag{15.27}$$

where τ_C is a relaxation time and $D\mathbf{C}$ is an admissible objective derivative of $\mathbf{C}$. It is a kind of Gordon-Showalter (1972) derivative where β plays the role of a slip coefficient. As to coefficient λ it can ultimately be related to the osmotic pressure of the solution.

This case illustrates here the power of the theory of internal variables of state in constructing a useful physically justified model of complex continuum akin to a micromorphic one. Note that in the case of strong elongational flows, the full tensor $\mathbf{C}$ need not be considered. One can be satisfied with considering only the direction $\underline{\lambda}$ associated with the flow direction and to replace the dependency of the theory on $\mathbf{R}$ defined in (15.1) by that on the quantity $R_\lambda\underline{\lambda}$ with

$$R_\lambda = \underline{\lambda}.(\mathbf{R}.\underline{\lambda}) = \left\langle (\mathbf{r}.\underline{\lambda})^2 \right\rangle. \tag{15.28}$$

The resulting theory will be one with an internal variable of state represented by a vector field as described in Maugin and Drouot (1983, Sect. 5), a theory very close to one of anisotropic fluids.

Finally, we note that Drouot and Berrajaa (1993, 1996) have introduced some *weak nonlocality* in the above sketched out theory by transforming the evolution equation for the conformation into an *evolution-diffusion equation* accounting for the spatial gradient of the conformation. This allows one to study boundary-layer effects at walls.

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Cross references: Anisotropic fluids, Internal variables of state, Liquid crystals (Landau-de Gennes theory), Micromorphic continua, Mixtures (mechanics of), Non-locality (weak).

Superfluids

Superfluidity is a surprising state of matter that materializes in the non-viscous flow of liquid Helium (Helium II or ^{4}He) with zero entropy and no dissipation through extremely narrow channels, at very low temperature, while viscosity is exhibited above a certain phase-transition temperature (the so-called lambda point at 2.2 K). The phenomenon was discovered simultaneously by Pyotr Kapitsa (1894–1984, NP 1978) and John F. Allen in 1937. Soon afterwards theoretical explanations were proposed by several authors, among them Fritz London and Laszlo Tisza in 1938 and 1940. Although a quantum mechanical support is to be found in the theory of Bose-Einstein condensation—clearly, outside the scope of the present work—the idea emerged that a sufficiently good description would be based on a *hydrodynamical* model considering a *mixture of two fluids*, a fluid called normal component, and a fluid called superfluid component. This was a masterpiece achievement by the famous Russian physicist Lev D. Landau (in 1941) who received the Nobel Prize in 1962 for this exploit (but he contributed many other works to theoretical physics). In truth, Landau's theory is a phenomenological and semi-microscopic theory. Its great success is that it forecasts the propagation of two types of waves. One of these corresponds to usual sound waves that are associated with fluctuations in the density, and another one, called *second sound*, that is associated with fluctuations in entropy or temperature. The existence of the latter was confirmed in experiments by Vasilii Peskhov in 1944–47. The story is well told in the expert review by Donnelly (2009). Following this author, indexing with s and n quantities that refer to the superfluid and normal fluid components, one can deduce from Tisza's two-fluid model, linearized equations for the two-fluid motion in the form:

$$\rho_n \frac{\partial \mathbf{v}_n}{\partial t} = -\frac{\rho_n}{\rho}\nabla p + \eta \nabla^2 \mathbf{v}_n + \rho_s S \nabla \theta, \tag{15.29}$$

$$\rho_s \frac{\partial \mathbf{v}_s}{\partial t} = -\frac{\rho_s}{\rho}\nabla p - \rho_s S \nabla \theta, \tag{15.30}$$

where $\rho = \rho_n + \rho_s$, p is the pressure, η is the viscosity, S is entropy and θ is temperature. Neglecting the viscous term in (15.29), some manipulations lead to two wave equations in the following form:

$$\frac{\partial^2 \rho}{\partial t^2} = \nabla^2 p, \quad \frac{\partial^2 S}{\partial t^2} = \frac{\rho_s}{\rho_n} S^2 \nabla^2 \theta, \tag{15.31}$$

from which there follows the two velocities

$$c_1 = \sqrt{(\partial p/\partial \rho)_S}, \quad c_2 = \sqrt{\frac{\rho_s}{\rho_n C_p} \theta S^2}, \tag{15.32}$$

where C_p is the specific heat. The second of these—that strongly depends on temperature with c_2 going to zero as θ goes to zero—is the *second-sound* velocity given by Landau. Such heat pulses travel at a velocity of the order of 20 m/s in Helium II at about 1.8 K. Main physical properties of, and basic models for, superfluid helium are given in books by pioneers in the field (Kalatnikov 1965; Wilks 1967; Putterman 1974), and in more recent syntheses (Annett 2005; Guénault 2003; Tilley and Tilley 1990).

Two-Fluid Model and Internal Momentum

It is clear that neither an Eulerian fluid model nor a Navier-Stokes viscous one is appropriate for the description of the strange behaviour of superfluids at low temperature, whence the idea of considering such fluids as if they were *mixtures* of two fluids (London, Tisza, Landau). The *normal* one has finite, although small, viscosity and carries all the entropy, while the *superfluid* one has no viscosity or entropy and can flow without dissipation. The corresponding velocity $\mathbf{v}_s$ satisfies the irrotationality condition $\nabla \times \mathbf{v}_s = \mathbf{0}$. The superfluid can be seen as a background fluid that is at absolute zero. There are two kinds of quasiparticles, phonons and "rotons" (according to Landau). The "rotons" are connected to the vortex density—hence their name—but Landau's equations are strictly valid only in helium flow without vortices (Landau's equations were improved in 1961 by Bekarevich and Kalatnikov to account for vortex motion as can occur in superfluid in rotation according to R.P. Feynman in 1951). Landau's equation for $\mathbf{v}_s$ reads

$$\frac{\partial \mathbf{v}_s}{\partial t} + \nabla\left(\mu + \frac{1}{2}\mathbf{v}_s^2\right) = \mathbf{0}, \tag{15.33}$$

and obviously guarantees that the condition $\nabla \times \mathbf{v}_s = \mathbf{0}$ is kept in time in ideal situations if true initially. As a matter of fact, the non-dissipative Landau-Kalatnikov equations in the case of irrotationality of the superfluid components are given by Eq. (15.33) and the following equations:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s) = 0, \tag{15.34}$$

$$\frac{\partial}{\partial t}(\rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s) + \nabla.(p\mathbf{1} + \rho_n \mathbf{v}_n \otimes \mathbf{v}_n + \rho_s \mathbf{v}_s \otimes \mathbf{v}_s) = \mathbf{0}, \tag{15.35}$$

$$\frac{\partial S}{\partial t} + \nabla.(S\mathbf{v}_n) = 0. \tag{15.36}$$

It is possible to transform this set of Eqs. (15.33)–(15.36) into a new set where (15.35) no longer appears explicitly, and which looks much more symmetric; that is (Putterman 1974),

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{v}_s + S\mathbf{A}) = 0 \tag{15.36}$$

$$\frac{\partial \mathbf{v}_s}{\partial t} + \nabla\left(\mu + \frac{1}{2}\mathbf{v}_s^2\right) = \mathbf{0}, \tag{15.37}$$

$$\frac{\partial \mathbf{A}}{\partial t} + \nabla(\theta + \mathbf{v}_n.\mathbf{A}) = \mathbf{v}_n \times (\nabla \times \mathbf{A}), \tag{15.38}$$

$$\frac{\partial S}{\partial t} + \nabla.(S\mathbf{v}_n) = 0, \tag{15.39}$$

where vector $\mathbf{A}$ is defined by

$$\mathbf{A} = \frac{\rho_n}{S}(\mathbf{v}_n - \mathbf{v}_s). \tag{15.40}$$

The latter is related to an *internal momentum* $\mathbf{P}$ by

$$\mathbf{A} = \mathbf{P}/S, \quad \mathbf{P} \equiv \rho_n(\mathbf{v}_n - \mathbf{v}_s). \tag{15.41}$$

Lhuillier et al. (1975b) have proposed a variational formulation for the above recalled set taking account of vortex dynamics and a superfluid component that is no longer irrotational. In a very thoughtful paper, the same authors (Lhuillier et al. 1975a) have indeed proposed a model of superfluid helium with an internal momentum $\mathbf{P}$ in presence of vortices and dissipative processes governed by the classical theory of irreversible processes. Their main arguments in an astute construct are to start from elementary excitations of Landau and to make an efficient use of Galilean transformations. In their model the fluid considered is characterized by a unique velocity field (say, $\mathbf{v}_s$) so that the total linear momentum reads

$$\mathbf{J} = \rho \mathbf{v}_s + \mathbf{P}. \tag{15.42}$$

An evolution equation (in fact a balance equation) for $\mathbf{P}$ is needed to close the system of equations. In this scheme entropy is transported at velocity $\mathbf{v}_s + \mathbf{c}$. The Landau model is recovered with the identification mentioned in the second part of (15.41) and

$$\mathbf{c} = \mathbf{v}_n - \mathbf{v}_s. \tag{15.44}$$

Accordingly, Landau's model can be considered as a model with internal momentum but with a law of irrotationality for $\mathbf{v}_s$ replacing the conservation law for the internal momentum. The linearized (about ρ_0 and S_0) acoustic approximation of the Lhuillier et al. equations reads (cf. Lhuillier et al. 1975a, p. 783)

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho_0 \mathbf{v}_s + \mathbf{P}) = 0, \tag{15.45}$$

$$\frac{\partial \mathbf{v}_s}{\partial t} + \nabla \mu = \mathbf{0}, \tag{15.46}$$

$$\frac{\partial \mathbf{P}}{\partial t} + S_0 \nabla \theta = \mathbf{0}, \tag{15.47}$$

$$\frac{\partial S}{\partial t} + S_0 \nabla.(\mathbf{v}_s + \mathbf{c}) = 0, \tag{15.48}$$

to be compared to the system (15.29)–(15.30). This indeed yields an acoustic mode with vibrations of pressure and a second mode (second sound) with vibrations of temperature and internal momentum. One advantage of the Lhuillier et al. model is that it explains the presence of *friction forces* that are experimentally observed.

An approach to superfluidity via extended thermodynamics was proposed by Greco and Müller (1984) but, in our opinion, is not convincing. As a matter of fact, their approach is one, among others, that allows for the existence of thermal or heat waves (cf. Straughan 2011) at finite speed.

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Cross references: Extended thermodynamics, Generalized thermoelasticity, Internal degrees of freedom, Mixtures (mechanics of).

Surface Tension

The first clear appearance of the notion of *surface tension* undoubtedly is in the original theory of capillarity as developed by Thomas Young (1773–1829) and Pierre Simon de Laplace (1749–1827). It is Laplace who proposed the celebrated equation of capillarity in the form.

$$p_2 - p_1 = \gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \tag{15.49}$$

where p denotes the pressure, γ is the coefficient of *surface tension*, and R_1 and R_2 are the principal curvature radii of the interface at the considered point. Since the pressure, like a stress, is energy per unit volume, surface tension is *energy per unit area*. The original derivation of Eq. (15.49) was in the spirit of Newton's attraction theory and involved an inverse-square law of interaction between "particles". It is only in the twentieth century that *continuum* theories were proposed by introducing the idea of material surfaces endowed with an elastic surface energy density (cf. Gurtin and Murdoch 1975). This formed the basis for a modern theory of *capillarity*. By invoking smoothly but rapidly varying densities, hence a gradient of density, this theory is very close to the theory of Korteweg fluids, density-gradient fluids, and gradient elasticity (see these Entries). The notion of surface energy can also be considered in an approach to structures made of a bulk material covered with a strictly adhering very thin elastic layer, called a "lid" in several works. This, in turn, provides interesting properties to propagating surface acoustic waves (e.g., existence of a dispersive monomode of the shear-horizontal type)—see, e.g., Murdoch (1976), Maugin and Hadouaj (1991)—and alterations to Rayleigh surface waves [cf. Vlasie-Beloncle and Rousseau (2006)]. The continuum model originally

expanded by Casal (1963) finds its best applications in the study of fluid interfaces and contact lines [cf. Gatignol and Seppecher 1986) and Seppecher (1987)].

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Cross references: Capillarity, Density-gradient fluids, Gradient elasticity, Interstitial working, Korteweg fluids.

Toupin R.A

Richard A. Toupin (born 1926) is an American physicist-applied mathematician—mechanician who spent most of his career at IBM. Although the author of seminal works dealing with fundamental problems of continuum mechanics, he is mostly worldwide known for his epoch-making contributions to various aspects of generalized continuum mechanics including nonlinear elastic electrically polarized materials (Toupin 1956), gradient theory and couple-stress theory (Toupin 1962, 1964), and obviously his monumental contribution—together with C.A. Truesdell—to the Encyclopaedia of Physics (Handbuch der Physik 1960).

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Cross references: Cosserat continua, Couple stress, Couple stress (as medium with constrained rotation), Electromagnetic continua, Generalized continuum mechanics, Gradient elasticity, Higher-order gradient theories, Hyperstress (notion of).

Truesdell C.A

Clifford A. Truesdell (1919–2000) was an American applied mathematician who, through his scientific and historic studies (e.g., Truesdell 1952), is considered to be the most articulated contributor to the formulation of modern continuum mechanics. He introduced a rigorous and well documented presentation, the spirit of which is fully expanded in his phenomenal contributions to the *Handbuch der Physik* together with Toupin (1960) and Noll (1965), and his unconditional support to rational thermo-dynamics (due mostly to B.D. Coleman and W. Noll; cf. Truesdell 1969). In the framework of nonclassical continuum mechanics, he is thought to be responsible for unearthing ideas advanced by P. Duhem and the Cosserat brothers in the period 1890–1910. He thus re-introduced the fruitful idea of "directors" in the study of slender structural elements together with Ericsken (1958). A biography of Truesdell was produced by Ignatieff and Willig (1999).

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Conclusion

The present book is an original one, in both subject matter and form. The subject matter is the large variety of non-classical theories of continua that burgeoned in the second half of the twentieth century and led to some understandable confusion and difficulty in discriminating among the wealth of proposed theories. Thoughtful as it is, the book ambitions to bring some order, comparison and clarification among these numerous theories and a quantity of sometimes fuzzy concepts. Its practical aim is to provide a handy compendium and help those confused readers and also scientists new to the field to apprehend it in the best, albeit concise, conditions. For convenience and to favour easy consultation, a format of "dictionary" (alphabetic ordering) has been adopted with about a hundred entries after two more classical chapters of reminding prerequisites. These entries are of various sizes and in depth description extending from a few lines to several pages. For the most largely expanded ones, historical background is given as well as basic formulation, further progress, contemporary references, and enlightening cross references. Typical entries cover anisotropic fluids, Cosserat media, micromorphic elasticity, couple stress, gradient theories, directors' theories, electromagnetic continua of various types, internal degrees of freedom, internal variables of state, interstitial working, Korteweg fluids, liquid crystals, materials with voids, metamaterials, superfluids, quasi-crystals, and general notions such as contiguity, non-locality, fractality, homogenization, and a wealth of other relevant subjects. In producing this unique tool for the large scientific community of students and professionals in mechanics, applied mathematics, and materials science, the author has exploited his long-time experience as an often called referee, reviewer of papers and books, editor in specialized journals and proceedings, and as a direct witness of original developments, and himself an enthusiastic and never tired contributor to the field.

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