De

motu perturbationibusque planetarum

secundum legem electrodynamicam Weberianam solem ambientium.

Dissertatio inauguralis

quam

ad summos in philosophia honores

in

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rite obtinendos

scripsit

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§. 1.

Argumentum disquisitionis consequentis est „inquirere in motum et perturbationes planetarum secundum legem electrodynamicam Weberianam solem ambientium". Cujus problematis solutio variis de causis haud inutilis videtur esse.

Quodsi hanc legem loco Neutonianae pro lege fundamentali attractionis, id quod valde verisimile, accipimus, facile demonstrari potest, etiam in hoc casu valere principium conservationis virium vivarum. Generaliter illud valet, si momentum virtuale vis attractivae in variationem totalem atque quotientem differentialem secundum tempus sumptum dispertiri possumus. Denotantibus m, m_1 massas corporum se invicem attrahentium, λ constantem gravitationis, c alteram constantem (velocitatem, qua propagatur vis attractiva in spatio) r distantiam corporum, fit in nostro casu

$$f\delta r = -\frac{\lambda m m_1}{rr}\left(1-\frac{1}{cc}\frac{dr^2}{dt^2}+\frac{2}{cc}r\frac{ddr}{dt^2}\right)\delta r = \delta\left(\frac{1}{r}+\frac{1}{cc}\cdot\frac{1}{r}\frac{dr^2}{dt^2}\right)-\frac{d}{dt}\left(\frac{2}{cc}\cdot\frac{1}{r}\frac{dr}{dt}\delta r\right) = \delta W-\frac{dU}{dt}$$

Aequationem fundamentalem dynamices

$$\Sigma\left(X - m\frac{ddx}{dt^2}\right)\delta x + \left(Y - m\frac{ddy}{dt^2}\right)\delta y + \left(Z - m\frac{ddz}{dt^2}\right)\delta z = 0$$

posito

$$T = \Sigma\tfrac{1}{2}m\left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2}\right)$$

$$\delta, S = \Sigma m\left(\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z\right)$$

$$\delta, V = \Sigma(X\delta x + Y\delta y + Z\delta z)$$

sic repraesentari licet

$$\delta, V + \delta T - \frac{d(\delta, S)}{dt} = 0$$

sive quum sit $\delta, V = \delta W - \frac{dU}{dt}$

$$\delta W + \delta T - \frac{d(\delta, S)}{dt} - \frac{dU}{dt} = 0$$

Quodsi hic statuimus $\delta x = dx$, $\delta y = dy$, $\delta z = dz$ obtinemus integrando

$$W - T - U = \text{const.}$$

quod est principium virium vivarum. Principia conservationis centri gravitatis atque arearum locum habere, per se clarum.

Problema primum nobis resolvendum in eo consistit, ut motum relativum duorum corporum secundum legem electrodynamicam mutuo se attrahentium definiamus. Quod problema iterum ad hoc simplicius reducere licet, ubi alterum corpus fixum concipimus, alterius massam statuimus $= 1$. Resoluto enim hoc problemate, habemus solutionem alterius, si pro λ statuimus λ multiplicatam in massam amborum corporum; id quod inde a Neutonis temporibus notum.

Jam procedimus ad eruendas tres constantes integrationis ita comparatas, ut ex iis reliquae tres per solas quadraturas inveniri possint. Docuit enim primus ill. Jacobi, inventa dimidia constantium problematis mechanici certis conditionibus satisfacientium, non modo reliquas per solas quadraturas haberi, sed etiam earum ope formulas elementorum perturbatorum sub forma simplicissima exhiberi posse. Sint $q_1, q_2 \dots q_n, t$, variabiles independentes problematis mechanici, ponimus

$$\frac{\vartheta(W+T)}{\vartheta q'_1} = p_1, \quad \frac{\vartheta(W+T)}{\vartheta q'_2} = p_2, \quad \dots \frac{\vartheta(W+T)}{\vartheta q'_n} = p_n,$$

ubi $q'_1 = \frac{dq_1}{dt} \dots q'_n = \frac{dq_n}{dt}$ atque differentiatio ϑ ita efficitur, ut sint variabiles independentes $q_1 \dots q_n, q'_1 \dots q'_n, t$. Tum posito

$$H = \Sigma p_i q'_i - W - T$$

aequationes differentiales motus sub forma Hamiltoniana exhibitae hae sunt

$$-\frac{dp_i}{dt} = -p'_i = \frac{dH}{dq_i}, \quad \frac{dq_i}{dt} = q'_i = \frac{dH}{dp_i}, \quad i = 1, 2, \dots n$$

Signum d hic et in sequentibus semper adhibebitur, ubi differentiatio ad variabiles independentes $t, p_1 \dots p_n, q_1 \dots q_n$ refertur. Inventae sint constantes integrationis $a_1, a_2 \dots a_n$ ita comparatae, ut satisfiat $\frac{n(n-1)}{2}$ aequationibus

$$0 = [a_h, a_k] = \sum_{i=n}^{i=1} \left(\frac{da_h}{dq_i} \frac{da_k}{dp_i} - \frac{da_h}{dp_i} \frac{da_k}{dq_i} \right)$$

$$h = 1, 2 \dots n, \quad k = 1, 2 \dots n$$

Ex hoc autem aequationum systemate sequitur

1. $$\frac{\delta p_h}{\delta q_k} = \frac{\delta p_k}{\delta q_h}$$

signo δ ad $t, q_1 \dots q_n, a_1 \dots a_n$ relato; quod ita demonstrari potest. Habemus

$$0 = \frac{\delta a_h}{\delta q_i} = \frac{da_h}{dq_i} + \sum_\iota \frac{da_h}{dp_\iota} \frac{\delta p_\iota}{\delta q_i}$$

$$0 = \frac{\delta a_k}{\delta q_i} = \frac{da_k}{dq_i} + \sum_\iota \frac{da_k}{dp_\iota} \frac{\delta p_\iota}{\delta q_i}$$

unde $$\frac{da_h}{dq_i} \frac{da_k}{dp_i} - \frac{da_h}{dp_i} \frac{da_k}{dq_i} = -\sum_\iota \left(\frac{da_h}{dp_\iota} \frac{da_k}{dp_i} - \frac{da_h}{dp_i} \frac{da_k}{dp_\iota} \right) \frac{\delta p_\iota}{\delta q_i}$$

α) $$0 = -[a_h, a_k] = \sum_i \sum_\iota \left(\frac{da_h}{dp_\iota} \frac{da_k}{dp_i} - \frac{da_h}{dp_i} \frac{da_k}{dp_\iota} \right) \left(\frac{\delta p_\iota}{\delta q_i} - \frac{\delta p_i}{\delta q_\iota} \right)$$

ubi duplex summatio tantum per omnes combinationes numerorum i, l extendenda est. Porro designantibus φ et ψ functiones ipsorum p et q, fit

$$\frac{d\varphi}{dp_\iota} = \sum_k \frac{\delta \varphi}{\delta q_k} \frac{dq_k}{dp_\iota} + \sum_h \frac{\delta \varphi}{\delta a_h} \frac{da_h}{dp_\iota} = \sum_h \frac{\delta \varphi}{\delta a_h} \frac{da_h}{dp_\iota}$$

$$\frac{d\psi}{dp_i} = \sum_h \frac{\delta \psi}{\delta q_h} \frac{dq_h}{dp_i} + \sum_k \frac{\delta \psi}{\delta a_k} \frac{da_k}{dp_i} = \sum_k \frac{\delta \psi}{\delta a_k} \frac{da_k}{dp_i}$$

quum sit $\frac{dq}{dp} = 0$. Inde sequitur

$$\frac{d\varphi}{dp_\iota} \frac{d\psi}{dp_i} - \frac{d\varphi}{dp_i} \frac{d\psi}{dp_\iota} = \sum_{h,k} \frac{\delta \varphi}{\delta a_h} \frac{\delta \psi}{\delta a_k} \left(\frac{da_h}{dp_\iota} \frac{da_k}{dp_i} - \frac{da_h}{dp_i} \frac{da_k}{dp_\iota} \right) = \sum_{(h,k)} \left(\frac{\delta \varphi}{\delta a_h} \frac{\delta \psi}{\delta a_k} - \frac{\delta \varphi}{\delta a_k} \frac{\delta \psi}{\delta a_h} \right) \left(\frac{da_h}{dp_\iota} \frac{da_k}{dpi} - \frac{da_h}{dp_i} \frac{da_k}{dp_\iota} \right)$$

Numeri h, k, uncis inclusi indicant, summationem per omnes combinationes ipsorum h, k extendendam esse. Posito $\varphi = p_i$, $\psi = p_\iota$, haec aequatio abit in

β) $$-1 = \sum_{(h,k)} \left(\frac{\delta p_i}{\delta a_h} \frac{\delta p_\iota}{\delta a_k} - \frac{\delta p_i}{\delta a_k} \frac{\delta p_\iota}{\delta a_h} \right) \left(\frac{da_h}{dp_\iota} \frac{da_k}{dpi} - \frac{da_h}{dp_i} \frac{da_k}{dp_\iota} \right)$$

Tribuendo φ et ψ valores p_λ, p_μ, ubi indices λ, μ non simul eidem atque i et l, expressiones in sinistra parte identice evanescere patet, ita ut sit

γ) $$0 = \sum_{(h,k)} \left(\frac{\delta p_\lambda}{\delta a_h} \frac{\delta p_\mu}{\delta a_k} - \frac{\delta p_\lambda}{\delta a_k} \frac{\delta p_\mu}{\delta a_h} \right) \left(\frac{da_h}{dp_\iota} \frac{da_k}{dp_i} - \frac{da_h}{dp_i} \frac{da_k}{dp_\iota} \right)$$

Quodsi igitur aequationes α) quamque per factorem aptum $\frac{\delta p_\lambda}{\delta a_h} \frac{\delta p_\mu}{\delta a_k} - \frac{\delta p_\lambda}{\delta a_k} \frac{\delta p_\mu}{\delta a_h}$ multiplicamus omniaque producta in unam summam conflamus, secundum γ) factores quantitatum $\frac{\delta p_\iota}{\delta q_i} - \frac{\delta p_i}{\delta q_\iota}$ evanescunt, excepto factore ipsius $\frac{\delta p_\mu}{\delta q_\lambda} - \frac{\delta p_\lambda}{\delta q_\mu}$ qui est $= -1$ secundum β). Fit igitur generaliter $\frac{\delta p_\mu}{\delta q_\lambda} - \frac{\delta p_\lambda}{\delta q_\mu} = 0$, q. e. d. Determinantem systematis α) non evanescere, non sine ambagibus demonstrari potuisset. Porro quum sit

$$\frac{dp_\iota}{dt} = \frac{\delta p_\iota}{\delta t} + \sum_i \frac{\delta p_\iota}{\delta qi} q'_i = \frac{\delta p_\iota}{\delta t} + \sum_i \frac{\delta p_\iota}{\delta q_i} \frac{dH}{dp_i} = \frac{\delta p_\iota}{\delta t} + \sum_i \frac{\delta p_i}{\delta q_\iota} \frac{dH}{\delta p_i} = \frac{\delta p_\iota}{\delta t} + \frac{\delta H}{\delta q_\iota} - \frac{dH}{dq_\iota}$$

nec non $$\frac{dp_\iota}{dt} = -\frac{dH}{dq_\iota}$$

2. evenit $$\frac{\delta p_\iota}{\delta t} = -\frac{\delta H}{\delta q_\iota}$$

Ex aequationibus 1, et 2, videmus, expressionem

$$p_1 dq_1 + p_2 dq_2 + \dots p_n dq_n - Hdt$$

totale esse differentiale, cujus integrale designamus per V. Erit igitur

3) $$\frac{\delta V}{\delta t} = -H, \quad \frac{\delta V}{\delta q_1} = p_1 \ldots \frac{\delta V}{\delta q_n} = p_n$$

Habemus etiam

4) $\frac{\delta V}{\delta a_1} = -b_1 \ldots \frac{\delta V}{\delta a_n} = -b_n$, designantibus $b_1 \ldots b_n$ n novas constantes. Hoc ita probatur. Est

$$\frac{d(-b_\iota)}{dt} = \frac{\delta\delta V}{\delta t \delta a_\iota} + \sum_i \frac{\delta\delta V}{\delta q_i \delta a_\iota} q'_i = -\frac{\delta H}{\delta a_\iota} + \sum_i \frac{\delta p_i}{\delta a_\iota} \frac{dH}{dp_i} = -\frac{\delta H}{\delta a_\iota} + \frac{\delta H}{\delta a_\iota} = 0. \quad \text{q. e. d.}$$

3) et 4) completum systema aequationum integralium constituunt. Inventis igitur $a_1, a_2, a_3 \ldots a_n$ problema ad quadraturas revocatum est. Nam ex aequationibus $a_1 = \text{const.} \ldots a_n = \text{const.}$ quantitatibus $p_1, p_2 \ldots p_n$ deductis atque in expressione dV substitutis, reliquas problematis mechanici constantes $b_1 \ldots b_n$ per solas differentiationes inveniri posse, e 4) intelligimus.

§. 2.

Sint F_1, F_2, F_3 integralia conservationis arearum, H integrale conservationis virium vivarum, demonstramus, $F_1 = a_2$, $\varphi = \sqrt{F_1 F_1 + F_2 F_2 + F_3 F_3} = a_3$ $H = a_1$ satisfacere aequationibus $[a_1\, a_2] = 0$, $[a_2\, a_3] = 0$, $[a_3\, a_1] = 0$. Designantibus $r \sin\eta \cos\xi$, $r \sin\eta \sin\xi$, $r \cos\eta$ coordinatas orthogonales corporis moti, cujus massa $= 1$ statuta, vis viva ita exprimitur $T = \frac{1}{2}\left(\frac{dr^2}{dt^2} + rr\frac{d\eta^2}{dt^2} + rr \sin\eta^2 \frac{d\xi^2}{dt^2}\right)$. Quodsi loco q_1, q_2, q_3 sumimus variabiles indepedentes r, η, ξ, fit

$$p_1 = q'_1 + \frac{\lambda}{cc}\frac{2}{q_1} q'_1, \quad p_2 = q_1 q_1 q'_2, \quad p_3 = q_1 q_1 q'_3 \sin^2 q_2$$

Integralia

$$H = \tfrac{1}{2}(q'_1 q'_1 + q_1 q_1 q'_2 q'_2 + q_1 q_1 q'_3 q'_3 \sin^2 q_2) - \frac{\lambda}{q_1} + \frac{\lambda}{cc}\frac{1}{q_1} q'_1 q'_1 = a_1$$

$$F_1 = q_1 q_1 q'_3 \sin^2 q_2 = a_2$$

$$\varphi = q_1 q_1 \sqrt{q'_2 q'_2 + q'_3 q'_3 \sin^2 q_2} = a_3$$

ope expressionum pro p_1, p_2, p_3 propositarum transeunt in haec

$$a_1 = \tfrac{1}{2}\frac{p_1 p_1}{1 + \frac{\lambda}{cc}\frac{2}{q_1}} + \tfrac{1}{2}\frac{1}{q_1 q_1}\left(p_2 p_2 + \frac{p_3 p_3}{\sin^2 q_2}\right) - \frac{\lambda}{q_1}$$

1) $$a_2 = p_3$$

$$a_3 = \sqrt{p_2 p_2 + \frac{p_3 p_3}{\sin^2 q_2}}$$

quae facillime probantur satisfacere aequationibus $[a_1, a_2] = 0$, $[a_2, a_3] = 0$, $[a_3, a_1] = 0$

Ex 1) sequitur

$$p_1 = \sqrt{\left\{2\left(a_1 + \frac{\lambda}{q_1}\right) - \frac{a_3 a_3}{q_1 q_1}\right\}\left\{1 + \frac{\lambda}{cc}\cdot\frac{2}{q_1}\right\}}$$

$$p_2 = \sqrt{a_3 a_3 - \frac{a_2 a_2}{\sin^2 q_2}}, \quad p_3 = a_2$$

Invenitur igitur

$$V = \int\sqrt{\left\{2\left(a_1 + \frac{\lambda}{q_1}\right) - \frac{a_3 a_3}{q_1 q_1}\right\}\left\{1 + \frac{\lambda}{cc}\frac{2}{q_1}\right\}}\, dq_1 + \int\sqrt{a_3 a_3 - \frac{a_2 a_2}{\sin^2 q_2}}\, dq_2 + a_2 q_3 - a_1 t$$

quae expressio, quum variabiles sint separatae, revera est integrabilis. Hinc sequitur

$$\frac{\delta V}{\delta a_1} = -b_1 = \int \frac{\sqrt{1 + \frac{\lambda}{cc}\cdot\frac{2}{q_1}}}{\sqrt{2\left(a_1 + \frac{\lambda}{q_1}\right) - \frac{a_3 a_3}{q_1 q_1}}}\, dq_1 - t$$

$$\frac{\delta V}{\delta a_2} = -b_2 = -\int \frac{a_2\, dq_2}{\sqrt{a_3 a_3 - \frac{a_2 a_2}{\sin^2 q_2}}\, .\, \sin^2 q_2} + q_3$$

$$\frac{\delta V}{\delta a_3} = -b_3 = -\int \frac{a_3\sqrt{1 + \frac{\lambda}{cc}\cdot\frac{2}{q_1}}}{\sqrt{2\left(a_1 + \frac{\lambda}{q_1}\right) - \frac{a_3 a_3}{q_1 q_1}}}\cdot\frac{dq_1}{q_1 q_1} + \int \frac{a_3\, dq_2}{\sqrt{a_3 a_3 - \frac{a_2 a_2}{\sin^2 q_2}}}$$

§. 3.

Aequatio $t - b_1 = \int \frac{\sqrt{1 + \frac{\lambda}{cc}\cdot\frac{2}{q_1}}}{\sqrt{2\left(a_1 + \frac{\lambda}{q_1}\right) - \frac{a_3 a_3}{q_1 q_1}}}\, dq_1$, quae relationem praebet inter tempus et radium vectorem, nobis demonstrat, q_1 contineri inter valores $-\frac{\lambda}{2a_1} + \frac{\sqrt{\lambda\lambda + 2a_1 a_3 a_3}}{2a_1}$ et $-\frac{\lambda}{2a_1} - \frac{\sqrt{\lambda\lambda + 2a_1 a_3 a_3}}{2a_1}$ atque igitur $-\frac{\lambda}{2a_1}$ esse distantiam mediam a sole.

Jam propius consideremus integrale ellipticum

$$\int \frac{\sqrt{1 + \frac{\lambda}{cc}\cdot\frac{2}{q_1}}}{\sqrt{2\left(a_1 + \frac{\lambda}{q_1}\right) - \frac{a_3 a_3}{q_1 q_1}}}\, dq_1 = \sqrt{-2a_1}\int \frac{\sqrt{q_1\left(q_1 + \frac{2\lambda}{cc}\right)}}{\sqrt{\lambda\lambda + 2a_1 a_3 a_3 - (\lambda + 2a_1 q_1)^2}}\, dq_1 = I$$

Posito $\lambda + 2a_1 q_1 = -\sqrt{\lambda\lambda + 2a_1 a_3 a_3}\,.\,y$ transit in

$$-\frac{\lambda\lambda + 2a_1 a_3 a_3}{(-2a_1)^{\frac{3}{2}}}\int\frac{\sqrt{y + \dfrac{\lambda}{\sqrt{\lambda\lambda + 2a_1 a_3 a_3}}\cdot y - \dfrac{4\lambda a_1 - cc\lambda}{cc\sqrt{\lambda\lambda + 2a_1 a_3 a_3}}}}{i\sqrt{yy - 1}}\,dy$$

Cum sit
$$\frac{\lambda}{2a_1} + \frac{\sqrt{\lambda\lambda + 2a_1 a_3 a_3}}{2a_1} < q_1 < -\frac{\lambda}{2a_1} - \frac{\sqrt{\lambda\lambda + 2a_1 a_3 a_3}}{2a_1}$$

y continetur inter limites -1 et $+1$; qua de re si brevitatis gratia ponimus

$$\frac{\lambda}{\sqrt{\lambda\lambda + 2a_1 a_3 . a_3}} = \gamma,\ -\frac{4\lambda a_1 - cc\lambda}{cc\sqrt{\lambda\lambda + 2a_1 a_3 a_3}} = \delta,\ -\frac{\lambda\lambda + 2a_1 a_3 a_3}{(-2a_1)^{\frac{3}{2}}} = B,$$

cum habeatur
$$1 > -1 > -\gamma > -\delta$$

ad revocandum integrale $I = B\int\frac{y + \gamma . y + \delta}{i\sqrt{yy - 1 . y + \gamma . y + \delta}}\,dy$ ad formam canonicam integralium ellipticorum formulam B. II, pag. 15 Fund. nov. adhibere possumus.

Secundum significationem illic adhibitam si statuimus

$$m = \sqrt[4]{\gamma + 1 . \delta - 1 .\ \delta + 1 . \gamma - 1}$$

$$n = \frac{\sqrt{\gamma + 1 . \delta - 1} + \sqrt{\delta + 1 . \gamma - 1}}{2} = \frac{m}{k'}$$

$$k = \sqrt{1 - k'k'} = \frac{\sqrt{\gamma + 1 . \delta - 1} - \sqrt{\delta + 1 . \gamma - 1}}{\sqrt{\gamma + 1 . \delta - 1} + \sqrt{\gamma - 1 . \delta + 1}}$$

$$M = \sqrt[4]{\frac{\gamma + 1 . \delta + 1}{\gamma - 1 . \delta - 1}},\quad tang\,\tfrac{1}{2}\left(\frac{\pi}{2} + amv\right) = M\sqrt{\frac{1 + y}{1 - y}}$$

est
$$\frac{dy}{i\sqrt{yy - 1 . y + \gamma . y + \delta}} = \frac{d\left(\dfrac{\pi}{2} + amv\right)}{\sqrt{mm\ sin^2 amv + nn\ cos^2 amv}}$$
$$= \frac{k'}{m}\cdot\frac{damv}{\sqrt{1 - kk\ sin^2 amv}} = \frac{k'}{m}\,dv$$

Porro reductionibus rite perfectis, habemus

$$y + \gamma . y + \delta = \sqrt{\gamma\gamma - 1 . \delta\delta - 1}.\left\{\frac{\sqrt{\gamma + 1 . \delta - 1} + \sqrt{\gamma - 1 . \delta + 1}}{\sqrt{\gamma - 1 . \delta - 1} + \sqrt{\gamma + 1 . \delta + 1}}\right\}^2 \frac{\varDelta^2 amv}{\left\{1 + \dfrac{\sqrt{\gamma - 1 . \delta - 1} - \sqrt{\gamma + 1 . \delta + 1}}{\sqrt{\gamma - 1 . \delta - 1} + \sqrt{\gamma + 1 . \delta + 1}}\,sin\,amv\right\}^2}$$

ideoque

$$I = 2B\frac{\sqrt{\gamma\gamma - 1 . \delta\delta - 1}.(\sqrt{\gamma + 1 . \delta - 1} + \sqrt{\gamma' - 1 . \delta + 1})}{(\sqrt{\gamma - 1 . \delta - 1} + \sqrt{\gamma + 1 . \delta + 1})^2}\int\frac{\varDelta^2 amv\ dv}{\left\{1 + \dfrac{\sqrt{\gamma - 1 . \delta - 1} - \sqrt{\gamma + 1 . \delta + 1}}{\sqrt{\gamma - 1 . \delta - 1} + \sqrt{\gamma + 1 . \delta + 1}}\,sin\,amv\right\}^2}$$

Quod integrale ut ad formam usitatam integralium ellipticorum tertii ordinis reducamus, novum parametrum h per aequationem $\frac{\sqrt{\gamma - 1 . \delta - 1} - \sqrt{\gamma + 1 . \delta + 1}}{\sqrt{\gamma - 1 . \delta - 1} + \sqrt{\gamma + 1 . \delta + 1}} = k\,sin\,amh$ introducimus, unde $sin\,amh = \frac{\sqrt{\gamma\gamma - 1} + \sqrt{\delta\delta - 1}}{\sqrt{\gamma\gamma - 1} - \sqrt{\delta\delta - 1}}$

$$\varDelta\,amh = \frac{2\sqrt[4]{\gamma\gamma - 1 . \delta\delta - 1}}{\sqrt{\gamma - 1 . \delta - 1} + \sqrt{\gamma + 1 . \delta + 1}}$$

Ex expressionibus pro $k\,sin\,amh$ et $sin\,amh$ obtinemus

$$\gamma - 1 = \frac{1 - k}{k}\cdot\frac{1 + k\,sin\,amh}{1 - sin\,amh},\quad \gamma + 1 = \frac{1 + k}{k}\cdot\frac{1 - k\,sin\,amh}{1 - sin\,amh}$$

$$\delta - 1 = -\frac{1 + k}{k}\cdot\frac{1 + k\,sin\,amh}{1 + sin\,amh},\quad \delta + 1 = -\frac{1 - k}{k}\cdot\frac{1 - k\,sin\,amh}{1 + sin\,amh}$$

unde
$$\sqrt{\gamma + 1 . \delta - 1} + \sqrt{\gamma - 1 . \delta + 1} = -\frac{2i}{k}\cdot\frac{\varDelta\,amh}{cos\,amh}$$

Sic constanti factore integralis transformato, I hanc formam tractabiliorem induit

$$I = -B\int\frac{i\varDelta^3 amh}{k\,cos\,amh}\left\{\frac{\varDelta\,amv}{1 + sin\,amh\,sin\,amv}\right\}^2 dv$$

Quod integrale ut ad functiones Θ revocemus, hanc facimus considerationem. Posito

$$Q = \int\frac{\varDelta^2 amv}{sin\,am\alpha + sin\,amv}\,dv$$

fit
$$\frac{dQ}{d\alpha} = -\int\frac{cos\,am\alpha\ \varDelta\,am\alpha\ \varDelta^2 amv}{(sin\,am\alpha + sin\,amv)^2}\,dv$$

sive faciendo $\alpha = h + iK'$

$$K' = \int_0^{\frac{\pi}{2}}\frac{dv}{\sqrt{1 - k'k'\ sin^2 v}}$$

habemus
$$\frac{dQ}{dh} = \int\frac{k\,\varDelta\,amh\ cos\,amh\ \varDelta^2 amv}{(1 + k\,sin\,amh\,sin\,amv)^2}\,dv$$

$$I = -\frac{Bi}{kk}\,\frac{\varDelta^2 amh}{cos^2 amh}\,\frac{dQ}{dh}$$

Problema igitur ad investigandum integrale Q reductum est. Est vero

$$Q = \int \frac{\Delta^2 am v}{\sin am\alpha + \sin am v}\, dv = \int \frac{\sin am\alpha\, \Delta^2 am v\, dv}{\sin^2 am\alpha - \sin^2 am v} - \int \frac{\sin am v\, \Delta^2 am v\, dv}{\sin^2 am\alpha - \sin^2 am v}$$

E theoremate fundamentali ill. Jacobi de integralibus ellipticis tertiae speciei

$$\int_0^v \frac{kk \sin am\alpha \cos am\alpha\, \Delta am\alpha \sin^2 am v}{1 - kk \sin^2 am\alpha \sin^2 am v}\, dv = v\, \frac{d \log \Theta(\alpha)}{d\alpha} + \tfrac{1}{2} \log \frac{\Theta(v-\alpha)}{\Theta(v+\alpha)}$$

ubi $\qquad \Theta \dfrac{2Kv}{\pi} = 1 - 2q^2 \cos 2v + 2q^4 \cos 4v - 2q^9 \cos 6v + \ldots$

$\log q = -\dfrac{\pi K'}{K}$, $am K = \frac{1}{2}\pi$, $am(K', k') = \frac{1}{2}\pi$, statim sequitur, posito loco v et α resp. $v + K + K'i$ et $\alpha + K$

$$\int \frac{\sin am\alpha\, \Delta^2 am v\, dv}{\sin^2 am\alpha - \sin^2 am v} = \frac{\Delta am\alpha}{\cos am\alpha} \left\{ - v\, \frac{d \log \Theta(\alpha + K)}{d\alpha} + \tfrac{1}{2} \log \frac{H(\alpha + v)}{H(\alpha - v)} \right\}$$

$$H\left(\frac{2Kv}{\pi}\right) = 2\sqrt[4]{q}\, \sin v - 2\sqrt[4]{q^9}\, \sin 3v + 2\sqrt[4]{q^{25}}\, \sin 5v - \ldots$$

Alterum integrale

$$\int \frac{\sin^2 am v\, \Delta^2 am v}{\sin^2 am\alpha - \sin^2 am v}\, dv$$

quod t pro $\sin^2 am v$ substituto, tantum radicem e polynomio secundi gradus continet, invenitur post reductiones obvias

$$= -\tfrac{1}{2} \frac{\Delta am\alpha}{\cos am\alpha} \log \left\{ \frac{\sin coam v - \sin coam\alpha}{\sin coam v + \sin coam\alpha} \right\} + \frac{k}{2} \log \left\{ \frac{k \sin coam v - 1}{k \sin coam v + 1} \right\}$$

ita ut jam

$$Q = \int \frac{\Delta^2 am v\, dv}{\sin am\alpha + \sin am v} = \frac{\Delta am\alpha}{\cos am\alpha} \left\{ - v\, \frac{\Theta'(\alpha + K)}{\Theta(\alpha + K)} + \tfrac{1}{2} \log \frac{H(\alpha + v)}{H(\alpha - v)} + \tfrac{1}{2} \log \frac{\sin coam v - \sin coam\alpha}{\sin coam v + \sin coam\alpha} \right\}$$
$$- \frac{k}{2} \log \left\{ \frac{k \sin coam v - 1}{k \sin coam v + 1} \right\}$$

Qua in expressione secundum α differentiata pro α scribendum $h + iK'$. Sic oritur respectu aequationum

$$\Theta(v + iK') = H(v + K)\, e^{-\frac{i\pi}{4K}(2v + iK')}$$

$$H(v + iK') = i\Theta(v)\, e^{-\frac{i\pi}{4K}(2v + iK')}$$

$$\frac{dQ}{dh} = -\frac{kk'k' \sin am h}{\Delta^2 am h} \left\{ - v\, \frac{H'(h + K)}{H(h + K)} + \tfrac{1}{2} \log \frac{\Theta(h + v)}{\Theta(h - v)} \cdot \frac{k \sin coam h \sin coam v - 1}{k \sin coam h \sin coam v + 1} \right\}$$
$$+ \frac{k \cos am h}{\Delta am h} \left\{ - v\, \frac{dd \log H(h + K)}{dh^2} + \tfrac{1}{2} \frac{\Theta'(h + v)}{\Theta(h + v)} + \tfrac{1}{2} \frac{\Theta'(h - v)}{\Theta(h - v)} - \frac{kk'k' \sin am h \sin coam v}{(kk \sin^2 coam h \sin^2 coam v - 1)\, \Delta^2 am h} \right\}$$

Jam cum sit $I = -\dfrac{Bi}{kk} \dfrac{\Delta^2 am h}{\cos^2 am h} \dfrac{dQ}{dh}$, tandem evenit

$$I = B\, \frac{ik'k'}{k} \frac{\sin am h}{\cos^2 am h} \left\{ - v\, \frac{H'(h + K)}{H(h + K)} + \tfrac{1}{2} \log \frac{\Theta(h + v)}{\Theta(h - v)} \cdot \frac{k \sin coam h \sin coam v - 1}{k \sin coam h \sin coam v + 1} \right\}$$
$$- B\, \frac{i}{k} \frac{\Delta am h}{\cos am h} \left\{ - v\, \frac{dd \log H(h + K)}{dh^2} + \tfrac{1}{2} \frac{\Theta'(h + v)}{\Theta(h + v)} + \tfrac{1}{2} \frac{\Theta'(h - v)}{\Theta(h - v)} - \frac{kk'k'}{\Delta^2 am h} \cdot \frac{\sin am h \sin coam v + 1}{kk \sin^2 coam h \sin^2 coam v - 1} \right\}$$
$$= t - b_1$$

§. 4.

Porro habemus

$$1) \qquad q_3 + b_2 = \int \frac{a_2\, dq_2}{\sqrt{a_3 a_3 - \frac{a_2 a_2}{\sin^2 q_2} \cdot \sin^2 q_2}} = -\int \frac{a_2\, d \cot g\, q_2}{\sqrt{a_3 a_3 - a_2 a_2 - a_2 a_2 \cot^2 q_2}}$$
$$= \arccos \left(\sqrt{\frac{a_2 a_2}{a_3 a_3 - a_2 a_2}}\, \cot g\, q_2 \right)$$

Restat, ut tertiam constantem evolvamus hac aequatione definitam

$$b_3 = \int \frac{a_3 \sqrt{1 + \frac{\lambda}{cc} \cdot \frac{2}{q_1}}}{\sqrt{2\left(a_1 + \frac{\lambda}{q_1}\right) - \frac{a_3 a_3}{q_1 q_1}}} \cdot \frac{dq_1}{q_1 q_1} - a_3 \int \frac{dq_2}{\sqrt{a_3 a_3 - \frac{a_2 a_2}{\sin^2 q_2}}}$$

Primum est

$$\int \frac{a_3\, dq_2}{\sqrt{a_3 a_3 - \frac{a_2 a_2}{\sin^2 q_2}}} = -\int \frac{a_3\, d \cos q_2}{\sqrt{a_3 a_3 - a_2 a_2 - a_3 a_3 \cos^2 q_2}} = \arccos \left(\sqrt{\frac{a_3 a_3}{a_3 a_3 - a_2 a_2}}\, \cos q_2 \right)$$

Deinde integrale

$$a_3 \int \frac{\sqrt{1 + \frac{\lambda}{cc} \cdot \frac{2}{q_1}}}{\sqrt{2\left(a_1 + \frac{\lambda}{q_1}\right) - \frac{a_3 a_3}{q_1 q_1}}} \cdot \frac{dq_1}{q_1 q_1} = a_3 a_3 \int \frac{\sqrt{1 + \frac{\lambda}{cc} \cdot \frac{2}{q_1}}}{\sqrt{\lambda\lambda + 2 a_1 a_3 a_3 - \left(\lambda - \frac{a_3 a_3}{q_1}\right)^2}} \cdot \frac{dq_1}{q_1 q_1}$$

per substitutionem $\lambda - \frac{a_3 a_3}{q_1} = \sqrt{\lambda\lambda + 2a_1 a_3 a_3} \, . x$ et brevitatis causa posito $\frac{cca_3 a_3 + 2\lambda\lambda}{2\lambda\sqrt{\lambda\lambda + 2a_1 a_3 a_3}} = \varrho$,

$A = \sqrt{\frac{2\lambda\sqrt{\lambda\lambda + 2a_1 a_3 a_3}}{ca_3}}$, abit in

$$A \int \frac{x - \varrho}{\sqrt{x - \varrho \, . \, xx - 1}} \, dx = I_1$$

Cum x contineatur inter limites $+1$ et -1, atque $\varrho > 1$, formulis A I pag. 16 Fund. nov. uti possumus. Ponimus igitur $m = \sqrt[4]{\varrho\varrho - 1}$

$$n = \frac{\sqrt{\varrho - 1} + \sqrt{\varrho + 1}}{2} = \frac{m}{\mu'}, \quad \mu\mu = 1 - \mu'\mu'$$

$$M = \sqrt[4]{\frac{\varrho - 1}{\varrho + 1}}, \quad tg \, \tfrac{1}{2}\left(\frac{\pi}{2} + amu\right) = M \sqrt{\frac{1 + x}{1 - x}}$$

fit

$$\frac{dx}{\sqrt{x - \varrho \, . \, xx - 1}} = \frac{\mu'}{m} \, du$$

et reductionibus rite perfectis

$$x - \varrho = -\sqrt{\varrho\varrho - 1} \left\{ \frac{1 - \mu \sin amu}{1 + \mu \sin amu} \right\}$$

ita ut

$$I_1 = -A \frac{\mu'\mu'}{\sqrt{2\mu}} \int \frac{1 - \mu \sin amu}{1 + \mu \sin amu} \, du$$

Integrale

$\int \frac{1 - \mu \sin amu}{1 + \mu \sin amu} \, du$ pendet a valoribus $\int \frac{\sin amu \, du}{1 - \mu\mu \sin^2 amu}$ et $\int \frac{du}{1 - \mu\mu \sin^2 amu}$

Primum integrale facile invenitur esse aequale

$$-\frac{1}{\mu'\mu'} \frac{\cos amu}{\Delta amu} = -\frac{1}{\mu'\mu'} \sin coam \, u$$

Primum est altioris indaginis. Expressio

$$\frac{dd \log \Theta(u + M)}{du^2} = \frac{\Theta(u + M)\,\Theta''(u + M) - \Theta'^2(u + M)}{\Theta^2(u + M)}$$

ubi $M = \int_0^{\frac{\pi}{2}} \frac{du}{\sqrt{1 - \mu\mu \sin^2 u}}$, functio est monodroma dupliciter periodica secundi ordinis cum periodis $2M$ et $2M'i$; $M' = \int_0^{\frac{\pi}{2}} \frac{du}{\sqrt{1 - \mu'\mu' \sin^2 u}}$. Fit infinita, ubi $\Theta = 0$. Functio $\frac{1}{\Delta^2 amu}$

easdem habet periodos atque evanescit pro iisdem valoribus, pro quibus $\frac{dd \log \Theta(u + M)}{du^2}$. Secundum theoriam functionum complexarum igitur altera per alteram exprimi potest. Qua de re ponimus

$$\frac{dd \log \Theta(u + M)}{du^2} = \frac{A}{\Delta^2 amu} + B$$

ubi constantes A et B adhuc determinandae sunt. Posito $u = 0$, $u = M$ obtinemus

$$\frac{\Theta''(M)}{\Theta(M)} = \sqrt{\frac{\pi}{2M}}\, \Theta''(M) = A + B, \quad \frac{\Theta''(0)}{\Theta(0)} = \sqrt{\frac{\pi}{2\mu' M}}\, \Theta''(0) = \frac{A}{\mu'\mu'} + B$$

unde

$$A \frac{\mu\mu}{\mu'\mu'} = \sqrt{\frac{\pi}{2M}} \left\{ \frac{\Theta''(0)}{\sqrt{\mu'}} - \Theta''(M) \right\}$$

$$-B\mu\mu = \sqrt{\frac{\pi}{2M}} \left\{ \Theta''(0)\, \mu'^{\frac{3}{2}} - \Theta''(M) \right\}$$

quas expressiones ut simplificemus, bis differentiamus aequationem

$$\sqrt{\mu'}\, \Theta(u + M) = \Theta(u)\, \Delta amu$$

ponimusque $u = 0$. Ita respectu aequationis $\Theta(0) = \sqrt{\frac{2\mu' M}{\pi}}$ evenit

$$\Theta''(M) = \frac{1}{\sqrt{\mu'}} \left\{ \Theta''(0) - \mu\mu \sqrt{\frac{2\mu' M}{\pi}} \right\}$$

quo valore in expressione pro A illato, fit simpliciter $A = \mu'\mu'$. Porro bis differentiamus aequationem $H(u) = \sqrt{\mu}\, \Theta(u) \sin amu$ differentiationeque peracta statuimus $u = M$. Sic invenitur

$$H''(M) = -\mu'\mu' \sqrt{\frac{2\mu M}{\pi}} + \sqrt{\mu}\, \Theta''(M).$$

Harum aequationum ope nanciscimur

$$\Theta''(M) - \sqrt{\mu'^3}\, \Theta''(0) = \sqrt{\mu^3}\, H''(M)$$

$$B = \sqrt{\frac{\pi}{2\mu M}}\, H''(M)$$

ita ut tandem

$$\frac{dd \log \Theta(u + M)}{du^2} = \frac{\mu'\mu'}{\Delta^2 amu} + \sqrt{\frac{\pi}{2\mu M}}\, H''(M)$$

ideoque

$$\int \frac{du}{\Delta^2 amu} = -\frac{u}{\mu'\mu'} \sqrt{\frac{\pi}{2\mu M}}\, H''(M) + \frac{1}{\mu'\mu'} \frac{d \log \Theta(u + M)}{du}$$

Singulis jam erutis integralibus, habemus

$$\int \frac{1-\mu \sin amu}{1+\mu \sin amu} du = -\frac{2u}{\mu'\mu'}\sqrt{\frac{\pi}{2\mu M}}\, H''(M) - u + \frac{2\mu}{\mu'\mu'}\frac{\cos amu}{\Delta amu} + \frac{2}{\mu'\mu'}\frac{d \log \Theta(u+M)}{du}$$

$$I_1 = A\left\{\frac{u}{\mu}\sqrt{\frac{\pi}{M}}\, H''(M) + \frac{\mu'\mu'}{\sqrt{2\mu}} u - \frac{\sqrt{2\mu}\cos amu}{\Delta amu} - \sqrt{\frac{2}{\mu}}\frac{d \log \Theta(u+M)}{du}\right\}$$

denique

$$b_3 = I_1 - arc \cos\left(\sqrt{\frac{a_3 a_3}{a_3 a_3 - a_2 a_2}}\cos q_2\right)$$

Ut omnia absoluta habeamus, adhuc functionis V expressio finita nobis quaerenda est

$$V = \int\sqrt{\left\{2\left(a_1 + \frac{\lambda}{q_1}\right) - \frac{a_3 a_3}{q_1 q_1}\right\}\left\{1 + \frac{\lambda}{cc}\cdot\frac{2}{q_1}\right\}}\, dq_1 + \int\sqrt{a_3 a_3 - \frac{a_2 a_2}{\sin^2 q_2}}\, dq_2 + a_2 q_3 - a_1 t$$

Prius integrale iisdem quibus initio §. 3 adhibitis transformationibus, transit in

$\sqrt{-2a_1}\int\frac{1-yy \,.\, y+\delta}{y+\gamma}\cdot\frac{dy}{i\sqrt{yy-1 \,.\, y+\gamma \,.\, y+\delta}}$, ubi γ, δ significationem, illic iis tributam, retinent. Quodsi porro ponimus

$$k' = \frac{2\sqrt[4]{\gamma\gamma-1 \,.\, \delta\delta-1}}{\sqrt{\gamma+1 \,.\, \delta-1} + \sqrt{\delta+1 \,.\, \gamma-1}}, \quad k = \sqrt{1-k'k'} = \frac{\sqrt{\gamma+1 \,.\, \delta-1} - \sqrt{\delta+1 \,.\, \gamma-1}}{\sqrt{\gamma+1 \,.\, \delta-1} + \sqrt{\delta+1 \,.\, \gamma-1}}$$

$$M = \sqrt[4]{\frac{\gamma+1 \,.\, \delta+1}{\gamma-1 \,.\, \delta-1}}, \quad tg\left(\frac{\pi}{4} + \tfrac{1}{2} amv\right) = M\sqrt{\frac{1+y}{1-y}}$$

habemus

$$\frac{dy}{i\sqrt{yy-1 \,.\, y+\gamma \,.\, y+\delta}} = \frac{k'}{m} dv$$

$$1 - yy = \frac{4M^2}{(1+M^2)^2}\frac{\cos^2 amv}{\left\{1 + \left\{\frac{\sqrt{\gamma-1 \,.\, \delta-1} - \sqrt{\gamma+1 \,.\, \delta+1}}{\sqrt{\gamma-1 \,.\, \delta-1} + \sqrt{\gamma+1 \,.\, \delta+1}}\right\}\sin amv\right\}^2}$$

$$\frac{y+\delta}{y+\gamma} = \frac{\sqrt{\delta\delta-1}}{\sqrt{\gamma\gamma-1}}\,\frac{1-k \sin amv}{1+k \sin amv}$$

Introducendo parametrum h aequatione

$$k \sin amh = \frac{\sqrt{\gamma-1 \,.\, \delta-1} - \sqrt{\delta+1 \,.\, \gamma+1}}{\sqrt{\gamma-1 \,.\, \delta-1} + \sqrt{\gamma+1 \,.\, \delta+1}}$$

fit

$$\frac{k'}{m}\cdot\frac{4M^2}{(1+M^2)^2}\cdot\frac{\sqrt{\delta\delta-1}}{\sqrt{\gamma\gamma-1}} = ik \cos amh\, \Delta amh\, \frac{\sin amh - 1}{\sin amh + 1} = G$$

atque integrale ad formam canonicam functionum ellipticarum revocandum

$$= \sqrt{-2a_1}\, G\int\frac{\cos^2 amv}{(1 + k \sin amh \sin amv)^2}\cdot\frac{1-k \sin amv}{1+k \sin amv} dv$$

$$= \sqrt{-2a_1}\, G \,.\, \Sigma$$

Est autem $\dfrac{\cos^2 amv \,.\, (1 - k \sin amv)}{(1 + k \sin amh \sin amv)^2\,(1 + k \sin amv)}$

$$= \frac{A}{(1 + k \sin amh \sin amv)^2} + \frac{B}{1 + k \sin amh \sin amv} + \frac{C}{1 + k \sin amv} + D$$

ubi

$$A = -\frac{\Delta^2 amh}{kk \sin^2 amh}\cdot\frac{\sin amh + 1}{\sin amh - 1}$$

$$C = -\frac{k'k' \,.\, (k+1)}{k^3}\cdot\frac{1}{(\sin amh - 1)^2}$$

$$1 = A + B + C + D$$

$$0 = 1 - \frac{\sin amh}{1 + \sin amh} A + \sin amh\, D + \frac{\sin amh - 1}{2}\cdot C$$

Ita definitis A, B, C, D, habemus

$$\Sigma = A\int\frac{dv}{(1 + k \sin amh \sin amv)^2} + B\int\frac{dv}{1 + k \sin amh \sin amv} + C\int\frac{dv}{1 + k \sin amv} + Dv$$

$$= A\Sigma' + B\Sigma'' + C\Sigma''' + Dv$$

Jam quodque integrale seorsim tractamus atque initium facimus a Σ'. Quodsi ponimus

$$P = \int\frac{k \sin amh\, dv}{1 + k \sin amh \sin amv}$$

fit

$$\frac{dP}{dh} = \int\frac{k \cos amh\, \Delta amh\, dv}{(1 + k \sin amh \sin amv)^2} = k \cos amh\, \Delta amh\, \Sigma'$$

$$\Sigma' = \frac{1}{k \cos amh\, \Delta amh}\,\frac{dP}{dh}$$

$$P \text{ autem } = k \sin amh\int\frac{dv}{1 - kk \sin^2 amh \sin^2 amv} - kk \sin^2 amh\int\frac{\sin amv\, dv}{1 - kk \sin^2 amh \sin^2 amv}$$

Valor ipsius $\int\frac{dv}{1 - kk \sin^2 amh \sin^2 amv}$ sequitur e theormate fundamentali cl. Jacobi jam supra in usum vocato, ponendo loco v et h resp. $v + iK'$, $h + iK'$.

Ita evenit

$$\int \frac{dv}{1 - k^2 \sin^2 amh \sin^2 amv} = \frac{1}{\Delta amh \, cotg \, amh} \left\{ v \frac{H'(h)}{H(h)} + \tfrac{1}{2} \log \frac{\Theta(v-h)}{\Theta(v+h)} \right\}.$$

Porro est

$$\int \frac{\sin amv \, dv}{1 - kk \sin^2 amh \sin^2 amv} = \frac{1}{2\Delta amh \cos amh} \int [\sin am(v+h) + \sin am(v-h)] \, dv$$

$$= \frac{1}{2k \Delta amh \cos amh} \log \frac{\Delta am(v+h) - k \cos am(v+h)}{\Delta am(v-h) + k \cos am(v-h)}$$

quum sit

$$\int \sin amv \, dv = \frac{1}{k} \log [\Delta amv - k \cos amv]$$

His substitutionibus factis, obtinemus

$$P = \frac{k \sin^2 amh}{\Delta amh \cos amh} \left\{ v \frac{H'(h)}{H(h)} + \tfrac{1}{2} \log \frac{\Theta(v-h)}{\Theta(v+h)} \cdot \frac{\Delta am(v-h) - k \cos am(v-h)}{\Delta am(v+h) + k \cos am(v+h)} \right\}$$

unde tandem

$$\Sigma' = \frac{1}{k \cos amh \, \Delta amh} \cdot \frac{dP}{dh}$$

qua differentiatione supersedere possumus.

Valor ipsius Σ'' in antecedentibus contentus est. Nempe

$$\Sigma'' = \frac{1}{k \sin amh \cdot P}$$

Denique $$\Sigma''' = \int \frac{dv}{1 + k \sin amv} = \int \frac{dv}{\Delta^2 amv} - \int \frac{k \sin amv \, dv}{\Delta^2 amv}$$

Ut supra demonstratum, fit

$$\int \frac{dv}{\Delta^2 amv} = -\frac{v}{k'k'} \sqrt{\frac{\pi}{2kK}} H''(K) + \frac{1}{k'k'} \frac{d \log \Theta(v+K)}{dv}, \text{ atque } \int \frac{k \sin amv \, dv}{\Delta^2 amv} = -\frac{k}{k'k'} \sin coamv$$

qua de re

$$\Sigma''' = -\frac{v}{k'k'} \sqrt{\frac{\pi}{2kK}} H''(K) + \frac{1}{k'k'} \frac{d \log \Theta(v+K)}{dv} + \frac{k}{k'k'} \sin coamv$$

Singulis jam erutis integralibus Σ', Σ'', Σ''', habetur

$$\Sigma = A\Sigma' + B\Sigma'' + C\Sigma''' + Dv$$

Alterum quod in V exstat integrale

$$S = \int \sqrt{a_3 a_3 - \frac{a_2 a_2}{\sin^2 q_2}} \, dq_2$$

haud difficile invenitur

$$= a_3 \arccos \left(\sqrt{\frac{a_3 a_3}{a_3 a_3 - a_2 a_2}} \cos q_2 \right) - a_2 \arccos \left(\sqrt{\frac{a_2 a_2}{a_3 a_3 - a_2 a_2}} \, cotg \, q_2 \right)$$

Tandem

$$V = \sqrt{-2a_1} \cdot G \, \Sigma + S + a_2 q_2 - a_1 t$$ Q. E. F.

§. 5.

Significatio astronomica illarum constantium integrationis quoad tempus atque situm orbitae a corpore descriptae spectant haud difficile intelligitur. Ex aequatione 1) §. 4

$$\cos(q_3 + b_2) = \sqrt{\frac{a_3 a_3}{a_3 a_3 - a_2 a_2}} \, cotg \, q_2$$

sequitur, posito $\frac{a_2}{a_3} = \cos i$

$$\cos(q_3 + b_2) = cotg \, i \; cotg \, q_2$$

unde videmus orbitam corporis moti esse simpliciter curvatam, denotare i inclinationem orbitae atque $-b_2$ longitudinem nodi ascendentis. Per $\frac{\lambda}{2a_1}$ exprimi distantiam mediam a sole, quam tamquam elementum orbitae introducere convenit, jam supra diximus. Aequationes inter tempus radiumque vectorem atque anomaliam veram radiumque vectorem, quibus definitur motus corporis, in expressionibus pro constantibus inventis contentae sunt. Qualis est aequatio $t - b_1 = I$, ubi b_1 pro tempore perihelii assumi posse perspicuum. Porro habetur, denotante φ anomaliam veram, $d\varphi = \frac{a_3}{q_1 q_1} dt$ sive

$$\varphi = \int \frac{a_3 \sqrt{1 + \frac{\lambda}{cc} \cdot \frac{2}{q_1}}}{\sqrt{2\left(a_1 + \frac{\lambda}{q_1}\right) - \frac{a_3 a_3}{q_1 q_1}}} \cdot \frac{dq_1}{q_1 q_1}$$

$$= \frac{\sqrt{2\lambda} \sqrt{\lambda\lambda + 2a_1 a_3 a_3}}{c a_3} \left\{ \frac{u}{\mu} \sqrt{\frac{\pi}{M}} H''(M) + \frac{\mu' \mu'}{\sqrt{2\mu}} u + \frac{\sqrt{2\mu} \cos amu}{\Delta amu} + \sqrt{\frac{2}{\mu}} \frac{d \log \Theta(u+M)}{du} \right\}$$

quae est aequatio orbitae a corpore descriptae.

Integrale I_1 §i anteced. igitur nihil aliud est nisi anomalia vera φ. Qua de re habemus

$$\cos(\varphi - b_3) = \frac{\cos q_2}{\sin i} = \sin\left(\varphi + \frac{\pi}{2} - b_3\right)$$

unde $\frac{\pi}{2} - b_3$ esse distantiam perihelii a nodo ascendente elucet, siquidem angulum φ a perihelio numeramus. Quodsi denique a_3 sive a_2, quae relatione $a_2 = a_3 \cos i$ ab invicem dependent, tamquam parametrum motum corporis definientem spectamus, omnes sex constantes integrationis geometrice interpretati sumus. Motus ille hac potissimum re a motu elliptico differt, quod perihelium corporis attracti non eundem locum in spatio obtinet, sed in circulo circa solem movetur. Posito enim in aequatione

$$\varphi + \varphi_0 = \frac{\sqrt{2\lambda\sqrt{\lambda\lambda + 2a_1 a_3 a_3}}}{ca_3}\left\{\frac{u}{\mu}\sqrt{\frac{\pi}{M}}\, H''(M) + \frac{\mu'\mu'}{\sqrt{2\mu}}\, u - \sqrt{2\mu}\,\frac{\cos amu}{\Delta amu} - \sqrt{\frac{2}{\mu}}\,\frac{d \log \Theta(u+M)}{du}\right\}$$

$\varphi = 0$ et $q_1 = \dfrac{-\lambda}{2a_1} + \dfrac{\sqrt{\lambda\lambda + 2a_1 a_3 a_3}}{2a_1} = Q_2$, qui est minimus valor ipsius q_1, habemus, cum sit $amu = -\dfrac{\pi}{2}$, $u = -M$,

$$\varphi_0 = -\frac{\sqrt{2\lambda\sqrt{\lambda\lambda + 2a_1 a_3 a_3}}}{ca_3}\left\{\frac{\sqrt{\pi M}}{\mu}\, H''(M) + \frac{\mu'\mu'}{\sqrt{2\mu}}\, M\right\}$$

Tum aequato q_1 maximo ejus valori $\dfrac{-\lambda - \sqrt{\lambda\lambda + 2a_1 a_3 a_3}}{2a_1} = Q_1$, fiat $\varphi = \varphi_1$; cum sit porro $amu = \dfrac{\pi}{2}$, $u = M$ erit

$$\varphi_1 + \varphi_0 = \frac{\sqrt{2\lambda\sqrt{\lambda\lambda + 2a_1 a_3 a_3}}}{ca_3}\left\{\frac{\sqrt{\pi M}}{\mu}\, H''(M) + \frac{\mu'\mu'}{\sqrt{2\mu}}\, M\right\}$$

atque igitur

$$\varphi_1 = \frac{2\sqrt{2\lambda\sqrt{\lambda\lambda + 2a_1 a_3 a_3}}}{ca_3}\left\{\frac{\sqrt{\pi M}}{\mu}\, H''(M) + \frac{\mu'\mu'}{\sqrt{2\mu}}\, M\right\}$$

qui valor a π diversus. Ut vero discamus, quantum a π differat, expressio in seriem evolvenda est. Quem in finem ab aequatione primitiva integrali proficimur

$$\varphi + \varphi_0 = \int \frac{a_3\sqrt{1 + \dfrac{\lambda}{cc}\cdot\dfrac{2}{q_1}}}{\sqrt{2\left(a_1 + \dfrac{\lambda}{q_1}\right) - \dfrac{a_3 a_3}{q_1 q_1}}}\,\frac{dq_1}{q_1 q_1}$$

ponimusque $\dfrac{a_3 a_3}{q_1} - \lambda = \sqrt{\lambda\lambda + 2a_1 a_3 a_3}\cos\theta$, unde evenit

$$\varphi + \varphi_0 = \sqrt{1 + \frac{2\lambda\lambda}{cca_3 a_3}}\int\sqrt{\left\{1 + \frac{2\lambda\sqrt{\lambda\lambda + 2a_1 a_3 a_3}}{cca_3 a_3 + 2\lambda\lambda}\cos\theta\right\}}\, d\theta$$

$$= \sqrt{1 + \frac{2\lambda\lambda}{cca_3 a_3}}\int\left\{1 + \frac{1}{2\varrho}\cos\theta - \frac{1}{8\varrho\varrho}\cos^2\theta + \frac{1}{16\varrho^3}\cos^3\theta - \frac{5}{128\varrho^4}\cos^4\theta + \ldots\right\} d\theta$$

$$\varrho = \frac{cca_3 a_3 + 2\lambda\lambda}{2\lambda\sqrt{\lambda\lambda + 2a_1 a_3 a_3}}.$$

Valoribus ipsius q_1, Q_2 et Q_1 quum respondeat resp. $\theta = 0$, $\theta = \pi$, obtinemus valorem ipsius φ_1, qui distantiam angularem perihelii ab aphelio denotat, si integrale inter limites 0 et π sumimus. Ita respectu aequationum

$$\int_0^{\pi}\cos^{2n+1} x\,dx = 0, \quad \int_0^{\frac{\pi}{2}}\cos^{2n} x\,dx = \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\cdot\frac{\pi}{2}$$

fit

$$\varphi_1 = \pi\sqrt{1 + \frac{2\lambda\lambda}{cca_3 a_3}}\left\{1 - \sum_{n=1}^{n=\infty}\frac{1.3.5\ldots.4n-3}{(2_n.2.4.6\ldots 2n)}\cdot\frac{1}{\varrho^{2n}}\right\}$$

Quodsi potestates $\dfrac{1}{c^4}$ altioresque propter magnitudinem quantitatis c negligimus, habemus

$$\varphi_1 = \pi\left(1 + \frac{\lambda\lambda}{cca_3 a_3}\right)$$

unde distantia duorum periheliorum se consequentium

$$= 2\pi\,\frac{\lambda\lambda}{cca_3 a_3}$$

Hac expressione φ_1 comparata cum supra inventa, cum sit $\varrho = \dfrac{1 + \mu\mu}{2\mu}$, habemus adhuc relationem elegantem

$$1 - \sum_{n=1}^{n=\infty}\frac{1.3.5..4n-3}{(2^n.2.4.6\ldots 2n)^2}\left(\frac{2\mu}{1+\mu\mu}\right)^{2n} = \frac{2}{\pi}\sqrt{\frac{2\mu}{1+\mu\mu}}\left\{H''(M)\,\frac{\sqrt{\pi M}}{\mu} + \frac{\mu'\mu'}{\sqrt{2\mu}}\, M\right\}$$

§. 6.

Problemate simplici de motu relativo duorum corporum in antecedentibus resoluto, ad considerationem perturbationum a gravitatione aliarum massarum resultantium transeamus. Sit Ω functio perturbatrix, a_1, a_2, a_3, b_1, b_2, b_3 systema elementarum canonicorum conditionibus in §° 1) explicatis satisfacientium, secundum theorema egregium ill. Jacobi formulae perturbatoriae prorsus ejusdem sunt formae atque aequationes differentiales motus; etenim

$$\frac{da_1}{dt} = -\frac{d\Omega}{db_1}, \quad \frac{da_2}{dt} = -\frac{d\Omega}{db_2}, \quad \frac{da_3}{dt} = -\frac{d\Omega}{db_3},$$

$$\frac{db_1}{dt} = \frac{d\Omega}{da_1}, \quad \frac{db_2}{dt} = \frac{d\Omega}{da_2}, \quad \frac{db_3}{dt} = \frac{d\Omega}{da_3},$$

ubi Ω tamquam functio elementorum a, b considerata. Designante m corpus, cujus motum relativum circa M perscrutati sumus, denotantibus porro m', m'', m''' etc. massas perturbatrices,

$r, r_{,}, r_{,,}$ etc. distantias ipsarum $m, m', m'' \dots$ ab M; r'_0, r''_0, r'''_0 distantias ipsarum $m', m'', m''' \dots$ ab m; r''_1, r'''_1 etc. distantias ipsarum m'', m''' etc. ab $m_{,}$, et sic porro, ita ut sit $m^{(i)}_{(k)}$ distantia corporis $m^{(i)}$ ab $m^{(k)}$, functio perturbatrix Ω hanc induit formam

$$\lambda m' \left(\frac{rr + r'r' - r'_0 r'_0}{2r'^3}\right)\left(1 - \frac{1}{cc}\frac{dr'^2}{dt^2} + \frac{2}{cc} r' \frac{ddr'}{dt^2}\right)$$

$$+ \lambda m'' \left(\frac{rr + r''r'' - r''_0 r''_0}{2r''^3}\right)\left(1 - \frac{1}{cc}\frac{dr''^2}{dt^2} + \frac{2}{cc} r'' \frac{ddr''}{dt^2}\right) + \text{etc.}$$

$$- \frac{\Pi}{m}$$

ubi $$\Pi = \frac{mm'\lambda}{r'_0}\left(1 - \frac{1}{cc}\frac{dr'_0}{dt}\right) + \frac{mm''}{r''_0}\left(1 - \frac{1}{cc}\frac{dr''_0}{dt}\right) + \frac{m'm''}{r''_1}\left(1 - \frac{1}{cc}\cdot\frac{dr''_1}{dt}\right) + \cdots$$

Hic tantummodo perturbationes saeculares, quae e terminis non periodicis functionis Ω oriuntur, consideramus quum propter gravitatem earum, tum quia earum ope constantem c determinari posse verisimile est.

Jam $r' \dots r'_0 \dots$ ut functiones temporis constantiumque $a_1 \dots b_3$ exprimendae atque in Ω substituendae sunt. Quem in finem expressionibus supra evolutis propter inextricabilem calculi complicationem uti nequimus; evolutiones in series nobis adhibendae sunt, tertiis altioribusque potestatibus ipsius c neglectis.

Denotante φ longitudinem corporis in orbita, erat

$$\varphi + \varphi_0 = \int \frac{a_3\sqrt{1 + \frac{\lambda}{cc}\cdot\frac{2}{q_1}}}{\sqrt{2\left(a_1 + \frac{\lambda}{q_1}\right) - \frac{a_3 a_3}{q_1 q_1}}} \cdot dq_1 = a_3 \int \frac{1 + \frac{\lambda}{cc}\cdot\frac{1}{q_1} - \frac{1}{2}\frac{\lambda\lambda}{c^4}\cdot\frac{1}{q_1 q_1}}{\sqrt{2\left(a_1 + \frac{\lambda}{q_1}\right) - \frac{a_3 a_3}{q_1 q_1}}}\, dq_1$$

Quodsi angulum u introducimus aequatione

$$\frac{a_3 a_3}{r} - \lambda = \sqrt{\lambda\lambda + 2a_1 a_3 a_3}\, \cos u$$

ubi pro q_1 scripsimus r, hoc integrale abit in

$$\varphi + \varphi_0 = u\left(1 + \frac{\lambda\lambda}{cc\, a_3 a_3} - \frac{1}{2}\frac{\lambda^4}{c^4}\cdot\frac{1}{a_3^4} - \frac{1}{4}\frac{\lambda\lambda(\lambda\lambda + 2a_1 a_3 a_3)}{c^4 a_3^4}\right)$$

$$+ \left(\frac{\lambda\sqrt{\lambda\lambda + 2a_1 a_3 a_3}}{cc\, a_3 a_3} - \frac{\lambda^3\sqrt{\lambda\lambda + 2a_1 a_3 a_3}}{c^4 a_3^4}\right) \sin u$$

$$- \frac{1}{8}\frac{\lambda\lambda(\lambda\lambda + 2a_1 a_3 a_3)}{c^4 a_3^4} \sin 2u$$

Angulum φ a perihelio computare convenit. Tum fit $\varphi_0 = 0$. Porro erat

$$t - b_1 = \int \frac{\sqrt{1 + \frac{\lambda}{cc}\cdot\frac{2}{r}}}{\sqrt{2\left(a_1 + \frac{\lambda}{r}\right) - \frac{a_3 a_3}{rr}}}\, dr = \int \frac{\left(1 + \frac{\lambda}{cc}\cdot\frac{1}{r} - \frac{1}{2}\frac{\lambda\lambda}{c^4}\cdot\frac{1}{rr}\right)}{\sqrt{2\left(a_1 + \frac{\lambda}{r}\right) - \frac{a_3 a_3}{rr}}}\, dr$$

Ponimus $\lambda + 2a_1 r = \sqrt{\lambda\lambda + 2a_1 a_3 a_3}\, \cos E$, fit

$$t - b_1 = \left(\frac{\lambda}{(-2a_1)^{\frac{3}{2}}} + \frac{1}{cc}\cdot\frac{\lambda}{(-2a_1)^{\frac{1}{2}}}\right) E - \frac{\sqrt{\lambda\lambda + 2a_1 a_3 a_3}}{(-2a_1)^{\frac{3}{2}}} \sin E$$

$$- \frac{1}{2}\frac{\lambda\lambda}{c^4}\cdot\frac{1}{a_3} \arccos \frac{\lambda \cos E - \sqrt{\lambda\lambda + 2a_1 a_3 a_3}}{\lambda - \sqrt{\lambda\lambda + 2a_1 a_3 a_3}\, \cos E}$$

sive ratione habita aequationis

$$\cos u = \frac{\lambda \cos E - \sqrt{\lambda\lambda + 2a_1 a_3 a_3}}{\lambda - \sqrt{\lambda\lambda + 2a_1 a_3 a_3}\, \cos E}$$

qua anguli u et E inter se conjuncti sunt

$$t - b_1 = \left(\frac{\lambda}{(-2a_1)^{\frac{3}{2}}} + \frac{1}{cc}\cdot\frac{\lambda}{(-2a_1)^{\frac{1}{2}}}\right) E - \frac{\sqrt{\lambda\lambda + 2a_1 a_3 a_3}}{(-2a_1)^{\frac{3}{2}}} \sin E - \frac{1}{2}\frac{\lambda\lambda}{c^4}\cdot\frac{1}{a^3}\, u$$

Posito $c = \infty$, formulae pro t et φ erutae in formulas notas motus elliptici transeunt, si pro $\frac{\sqrt{\lambda\lambda + 2a_1 a_3 a_3}}{\lambda}$ scribimus e, i. e. excentricitatem ellipseos, pro $(t - b_1)\frac{(-2a_1)^{\frac{3}{2}}}{\lambda}$ anomaliam mediam ϑ. A quantitate $\frac{\sqrt{\lambda\lambda + 2a_1 a_3 a_3}}{\lambda}$ etiam in nostro problemate pendere ellipticitatem curvae elucet, qua de re pro ea significationem e retinebimus. Utique in hoc calculo approximativo ad interpretendas res pro a_1, a_2, a_3 elementa elliptica retinere convenit. Designat autem

$\frac{\lambda}{-2a_1}$ .... semiaxem majorem A, sive potius in nostro casu distantiam mediam a sole

$\frac{a_2}{\sqrt{\lambda}}$ ... radicem quadraticam semiparametri p multiplicatam per cosinum inclinationis i

$\frac{a_3}{\sqrt{\lambda}}$ .... radicem quadraticam semiparametri p

$\frac{\sqrt{\lambda\lambda + 2a_1 a_3 a_3}}{\lambda}$ ... excentricitatem e

Ita habemus hoc formularum systema, si pro $1 + \frac{\lambda}{cc}\cdot\frac{1}{p} - \frac{1}{2}\frac{\lambda\lambda}{c^4}\cdot\frac{1}{pp}$ scribimus $\sqrt{1 + \frac{\lambda}{cc}\cdot\frac{2}{p}}$, quod ab eo tantum quantitatibus tertii ordinis differt

$$\varphi = u\left(\sqrt{1+\frac{\lambda}{cc}\cdot\frac{2}{p}} - \tfrac{1}{4}\cdot\frac{\lambda\lambda}{c^4}\cdot\frac{ee}{pp}\right) + \left(\frac{\lambda}{cc}\cdot\frac{e}{p} - \frac{\lambda\lambda}{c^4}\cdot\frac{e}{pp}\right)\sin u - \tfrac{1}{8}\,\frac{\lambda\lambda}{c^4}\cdot\frac{ee}{pp}\sin 2u$$

$$\vartheta = \left(1+\frac{\lambda}{cc}\cdot\frac{1}{A}\right)E + e\sin E - \tfrac{1}{2}\cdot\frac{\lambda\lambda}{c^4}\cdot\frac{1}{\sqrt{p}\,.\,A^{\frac{3}{2}}}\,u$$

$$r = \frac{p}{1+e\cos u} = A(1-e\cos E)$$

$$\cos u = \frac{\cos E - e}{1-e\cos E},\quad tg\,\frac{u}{2} = \sqrt{\frac{1+e}{1-e}}\,tg\,\frac{E}{2}$$

Motus igitur e lege Weberiana oriundus e motu elliptico derivatus concipi potest, siquidem correctiones aliquot illis formulis expressas adhibemus. Atque etiam in formulis perturbatoriis quantitates A et e sive $p = A(1-ee)$, quippe quarum a variationibus maxime pendeat stabilitas systematis mundi, retinere convenit. Si igitur per ω longitudinem nodi ascendentis, per π distantiam perihelii ab eo, per t_0 tempus perihelii designamus, habetur

$$\lambda\frac{dA}{dt} = 2A^2\frac{d\Omega}{dt_0},\qquad \lambda\frac{dt_0}{dt} = -2A^2\frac{d\Omega}{dA}$$

$$\sqrt{\lambda}\,\frac{d\pi}{dt} = \frac{d\Omega}{d\sqrt{p}},\qquad \sqrt{\lambda}\,\frac{d\sqrt{p}}{dt} = -\frac{d\Omega}{d\pi}$$

$$\sqrt{\lambda}\,\frac{d\omega}{dt} = \frac{d\Omega}{d\sqrt{p}\cos i},\qquad \sqrt{\lambda}\,\frac{d\sqrt{p}\cos i}{dt} = -\frac{d\Omega}{d\omega}$$

Quibus in formulis pro r et φ expressiones earum in t aliisque constantibus sunt substituendae. Hoc in calculo e et i tamquam quantitates perparvas primi ordinis concipimus, id quod in systemate nostro planetarum bono jure licet, negligimusque tertias altioresque dimensiones harum quantitatum. Neglecto $\left(\frac{1}{cc}\right)^3$, $\frac{1}{cc}$ ut perparva primi ordinis supponenda est, ita ut producta hujus quantitatis in e et i plus duarum dimensionum omnino sint rejicienda. Notum est, u exprimi per E hac serie

$$\alpha)\qquad u = E + 2\left(\mu\sin E + \frac{\mu\mu}{2}\sin 2E + \frac{\mu^3}{3}\sin 3E + \ldots.\right)$$

ubi $$\mu^p = \frac{e^p}{2^p} + \frac{p}{2^{p+2}}e^{p+2} + \frac{p(p+3)}{2\,.\,2^{p+4}}e^{p+4} + \frac{p(p+4)(p+5)}{2\,.\,3\,.\,2^{p+6}}e^{p+6} + \ldots$$

unde videmus, in ϑ pro u simpliciter scribendum esse E. Patet igitur, si pro

$$1+\frac{\lambda}{cc}\cdot\frac{1}{A} - \tfrac{1}{2}\,\frac{\lambda\lambda}{c^4}\cdot\frac{1}{A^{\frac{3}{2}}\sqrt{p}}\quad\text{statuimus}\quad\sqrt{1+\frac{\lambda}{cc}\cdot\frac{2}{A}}$$

esse $$\vartheta = \sqrt{1+\frac{\lambda}{cc}\cdot\frac{2}{A}}\cdot E + e\sin E.$$ Hinc fit, posito

$$\theta = \frac{\vartheta}{\sqrt{1+\frac{\lambda}{cc}\cdot\frac{2}{A}}},\quad \frac{e}{\sqrt{1+\frac{\lambda}{cc}\cdot\frac{2}{A}}} = \varepsilon,$$

$$E = \theta + \varepsilon\sin\theta + \frac{\varepsilon\varepsilon}{2}\sin 2\theta + \frac{\varepsilon^3}{2^3}(3\sin 3\theta - \sin\theta) + \ldots$$

$$\sin E = \sin\theta + \frac{\varepsilon}{2}\sin 2\theta + \frac{\varepsilon\varepsilon}{2^3}(3\sin 3\theta - \sin\theta) + \ldots$$

$$\sin 2E = \sin 2\theta + \varepsilon(\sin 3\theta - \sin\theta) + \varepsilon\varepsilon(\sin 4\theta - \sin 2\theta) + \ldots$$

etc.

$$\cos E = \cos\theta + \frac{\varepsilon}{2}(\cos 2\theta - 1) + \frac{\varepsilon\varepsilon}{8}(3\cos 3\theta - 3\cos\theta) + \ldots$$

Quibus expressionibus substitutis in aequatione α), evenit neglectis negligendis

$$u = \theta + (e+\varepsilon)\sin\theta + \tfrac{5}{4}ee\sin 2\theta$$
$$= \theta + (e+\varepsilon)\sin\theta + \tfrac{5}{4}ee\sin 2\vartheta$$

Hoc pacto e $\varphi = u\sqrt{1+\frac{\lambda}{cc}\cdot\frac{2}{p}} + \frac{\lambda}{cc}\cdot\frac{e}{p}\sin u$ eruimus, quum pro $\sin u$ scribi possit $\sin\vartheta$, nec non ϑ pro $\theta\sqrt{1+\frac{\lambda}{cc}\cdot\frac{2}{p}} = \vartheta\,\frac{\sqrt{1+\frac{\lambda}{cc}\cdot\frac{2}{p}}}{\sqrt{1+\frac{\lambda}{cc}\cdot\frac{2}{A}}}$,

$$\varphi = \vartheta + 2e\left(1+\frac{\lambda}{cc}\cdot\frac{1}{p}\right)\sin\vartheta - \frac{2\lambda}{cc}\cdot\frac{e}{A}\cos\vartheta + \tfrac{5}{4}ee\sin 2\vartheta$$

Tandem $r = A(1-e\cos E)$ invenitur esse

$$= A\left[1 - e\left(\cos\vartheta + \frac{\lambda}{cc}\;\frac{1}{A}\sin\vartheta\right) - \frac{ee}{2}(\cos 2\vartheta - 1)\right]$$

§. 7.

Quibus omnibus rite praeparatis, ad evolvendam functionem Ω procedamus; et quidem primo partem ejus

$$\Omega' = \lambda m'\,\frac{rr + r'r' - r'_0r'_0}{2r'^3}\left(1 - \frac{1}{cc}\cdot\frac{dr'^2}{dt^2} + \frac{2}{cc}\cdot r'\frac{ddr'}{dt^2}\right) - \frac{\lambda m'}{r'_0}\left(1 - \frac{1}{cc}\cdot\frac{dr'_0}{dt}\right)$$

consideremus. Sint x, y, z, x_1, y_1, z_1 coordinatae orthogonales corporum m et m', fit

$$rr = xx + yy + zz,\quad r'r' = x'x' + y'y' + z'z',\quad r'_0r'_0 = (x-x')^2 + (y-y')^2 + (z-z')^2$$

Secundum formulas in opere celeberrimo Mécanique analytique evolutas habemus

$$x = r\,(\alpha\ \cos\varphi + \beta\ \sin\varphi)$$
$$y = r\,(\alpha_{,}\cos\varphi + \beta_{,}\ \sin\varphi)$$
$$z = r\,(\alpha_{,,}\cos\varphi + \beta_{,,}\ \sin\varphi)$$

ubi

$$\begin{aligned} \alpha &= \cos\pi\cos\omega - \sin\pi\sin\omega\cos i \\ \beta &= -\sin\pi\cos\omega - \cos\pi\sin\omega\cos i \\ \alpha_{,} &= \cos\pi\sin\omega + \sin\pi\cos\omega\cos i \\ \beta_{,} &= -\sin\pi\sin\omega + \cos\pi\cos\omega\cos i \\ \alpha_{,,} &= \sin\pi\sin i \\ \beta_{,,} &= \cos\pi\sin i \end{aligned}$$

Quantitates ad corpus m' pertinentes ductibus $'$ denotabuntur. Ita habetur

$$r_0{}'r_c{}' = rr + r'r' - (A + B)\,rr'\cos(\varphi - \varphi') - (A - B_{,})\,rr'\cos(\varphi + \varphi') - (A_{,} - B)\,rr'\sin(\varphi - \varphi')$$
$$- (A_{,} + B)\,rr'\sin(\varphi + \varphi')$$

ubi

$$\begin{aligned} A &= \alpha\alpha' + \alpha_{,}'\alpha_{,} + \alpha_{,,}\alpha_{,,}', \\ B &= \alpha\beta' + \alpha_{,}\beta_{,}' + \alpha_{,,}\beta_{,,}', \\ A_{,} &= \alpha'\beta + \alpha_{,}'\beta_{,} + \alpha_{,,}'\beta_{,,}, \\ B_{,} &= \beta\beta' + \beta_{,}\beta_{,}' + \beta_{,,}\beta_{,,}' \end{aligned}$$

Quas quantitates Lagrange demonstravit ita exprimi posse

$$\begin{aligned} A &= \cos\pi_0{}'\cos\omega_0{}' - \sin\pi_0{}'\sin\omega_0{}'\cos i_0{}' \\ B &= -\sin\pi_0{}'\cos\omega_0{}' - \cos\pi_0{}'\sin\omega_0{}'\cos i_0{}' \\ A_{,} &= \cos\pi_0{}'\sin\omega_0{}' + \sin\pi_0{}'\cos\omega_0{}'\cos i_0{}' \\ B_{,} &= -\sin\pi_0{}'\sin\omega_0{}' + \cos\pi_0{}'\cos\omega_0{}'\cos i_0{}'. \end{aligned}$$

ubi $\pi_0{}'$, $\omega_0{}'$, $i_0{}'$ significant resp. long. per., long. nodi asc., inclinationem orbitae ab m' descriptae planitie orbitae corporis m pro basi assumpta. Harum expressionum ope eruitur

$$r_0{}'r_0{}' = rr + r'r' - 2rr'\cos(\varphi - \varphi' - \omega_0{}' - \pi_0{}')\cos^2\tfrac{1}{2}i_0{}' - 2rr'\cos(\varphi + \varphi' - \omega_0{}' + \pi_0{}')\sin^2\tfrac{1}{2}i_0{}'$$

unde

$$\frac{rr + r'r' - r_0{}'r_0{}'}{2r'^3} = \frac{r\cos(\varphi - \varphi' - \omega_0{}' - \pi_0{}')}{r'r'} + \frac{r[\cos(\varphi - \varphi' - \omega_0{}' - \pi_0{}') - \cos(\varphi + \varphi' - \omega_0{}' + \pi_0{}')]}{r'r'}\sin^2\tfrac{1}{2}i_0{}'$$

Porro neglectis tertiis potestatibus ipsius $\sin\frac{1}{2}i_0{}'$ habemus

$$\frac{1}{r_0{}'} = \frac{1}{\sqrt{[rr + r'r' - 2rr'\cos(\varphi - \varphi' - \omega_0{}' - \pi_0{}')]}} + \frac{rr'[\cos(\varphi + \varphi' - \omega_0{}' + \pi_0{}') - \cos(\varphi - \varphi' - \omega_0{}' - \pi_0{}')]}{[rr + r'r' - 2rr'\cos(\varphi - \varphi' - \omega_0{}' - \pi_0{}')]^{\frac{3}{2}}}\sin^2\tfrac{1}{2}i_0{}'$$

Radicalia in denominatore hujus expressionis existentia in series secundum cosinus multiplorum anguli $\varphi - \varphi' - \omega_0{}' - \pi_0{}'$ progredientes evolvi possunt, ita ut sit retentis significationibus ill. Lagrange, atque statuto $\Phi = \varphi - \varphi' - \omega_0{}' - \pi_0{}'$

$$(rr + r'r' - 2rr'\cos\Phi)^{-\frac{1}{2}} = (r,r') + (r,r')_1\cos\Phi + (r,r')_2\cos 2\Phi + (r,r')_3\cos 3\Phi + \text{etc.}$$
$$(rr + r'r' - 2rr'\cos\Phi)^{-\frac{3}{2}} = [r,r'] + [r,r']_1\cos\Phi + [r,r']_2\cos 2\Phi + [r,r']_3\cos 3\Phi + \text{etc.}$$

Habemus igitur

$$\frac{1}{r_0{}'} = (r,r') + (r,r')_1\cos\Phi + (r,r')_2\cos 2\Phi + \text{etc.}$$
$$+ rr'\left\{[r,r'] + [r,r']_1\cos\Phi + [r,r']_2\cos 2\Phi + \ldots\right\}\left\{\cos\Psi - \cos\Phi\right\}\sin^2\tfrac{1}{2}i_0{}'$$

ubi $\Psi = \varphi + \varphi' - \omega_0{}' + \pi_0{}'$. Qua in expressione valores ipsarum r, r', φ, φ' in §° 7 eruti sunt substituendi. Erat

$$r = A\left[1 - e\cos\vartheta - \frac{\lambda}{cc}\cdot\frac{e}{A}\sin\vartheta - \frac{ee}{2}(\cos 2\vartheta - 1)\right]$$

$$r' = A'\left[1 - e'\cos\vartheta' - \frac{\lambda}{cc}\cdot\frac{e'}{A'}\sin\vartheta' - \frac{e'e'}{2}(\cos 2\vartheta' - 1)\right]$$

$$\varphi = \vartheta + 2e\sin\vartheta + 2\frac{\lambda}{cc}\cdot\frac{e}{p}\sin\vartheta - 2\frac{\lambda}{cc}\cdot\frac{e}{A}\cos\vartheta + \tfrac{5}{4}ee\sin 2\vartheta$$

$$\varphi' = \vartheta' + 2e'\sin\vartheta' + 2\frac{\lambda}{cc}\cdot\frac{e'}{p'}\sin\vartheta' - 2\frac{\lambda}{cc}\cdot\frac{e'}{A'}\cos\vartheta' + \tfrac{5}{4}e'e'\sin 2\vartheta'$$

In functione Ω', $\frac{1}{r_0{}'}$ exsistit multiplicatum in $1 - \frac{1}{cc}\cdot\frac{dr_0{}'}{dt}$. In expressione $\frac{1}{cc}\cdot\frac{dr_0{}'}{dt}$ terminus $\sin^2\frac{1}{2}i_0{}'$ continens, rejiciendus est, qua de re nanciscimur

$$\frac{1}{cc}\cdot\frac{dr_0{}'}{dt} = \frac{1}{cc}\cdot\frac{1}{r_0{}'}\left(r\frac{dr}{dt} + r'\frac{dr'}{dt} - r'\cos\Phi\frac{dr}{dt} - r\cos\Phi\frac{dr'}{dt} - rr'\sin\Phi\frac{d\Phi}{dt}\right)$$

In parenthesi tantum quantitates primi ordinis retinendas esse perspicuum. Quodsi igitur introducimus pro

$$r\frac{dr}{dt} + r'\frac{dr'}{dt} \ldots\ldots e\sqrt{A\lambda}\sin\vartheta + e'\sqrt{A'\lambda\sin\vartheta'}$$

$$r'\frac{dr}{dt} + r\frac{dr'}{dt} \ldots\ldots \frac{A\sqrt{\lambda}}{\sqrt{A'}}e'\sin\vartheta' + \frac{A'\sqrt{\lambda}}{\sqrt{A}}e\sin\vartheta$$

$$rr'\frac{d\Phi}{dt} \ldots\ldots \sqrt{\lambda}\left\{\left(\frac{A'}{\sqrt{A}} - \frac{A}{\sqrt{A'}}\right)(1 - e\cos\vartheta - e'\cos\vartheta') + 2\frac{A'e}{\sqrt{A}}\cos\vartheta - 2\frac{Ae'}{\sqrt{A'}}\cos\vartheta'\right\}$$

nec non pro $\cos\Phi$, cujus valorem exactius exhibemus,

$$\cos\Phi = \cos(\vartheta - \vartheta' - \omega_0{}' - \pi_0{}') -$$

$$-\sin(\vartheta-\vartheta'-\omega_0'-\pi_0')\left\{\begin{matrix}2e\left(1+\frac{\lambda}{cc}\cdot\frac{1}{p}\right)\sin\vartheta-\frac{2\lambda}{cc}\cdot\frac{e}{A}\cos\vartheta+\frac{5}{4}ee\sin 2\vartheta\\-2e'\left(1+\frac{\lambda}{cc}\cdot\frac{1}{p'}\right)\sin\vartheta'-\frac{2\lambda}{cc}\cdot\frac{e'}{A'}\cos\vartheta'+\frac{5}{4}e'e'\sin 2\vartheta'\end{matrix}\right\}$$

$$-\cos(\vartheta-\vartheta'-\omega_0'-\pi_0')\left\{ee-ee\cos 2\vartheta+e'e'-e'e'\cos 2\vartheta'-2ee'\left[\cos(\vartheta-\vartheta')-\cos(\vartheta+\vartheta')\right]\right\}$$

invenimus

$$\frac{1}{cc}\cdot\frac{dr_0'}{dt}=\frac{\sqrt{\lambda}}{cc}\cdot\frac{1}{r_0'}\left\{e\sqrt{A}\sin\vartheta+e'\sqrt{A'}\sin\vartheta'-\left(\frac{A}{\sqrt{A'}}e'\sin\vartheta'+\frac{A'}{\sqrt{A}}e\sin\vartheta\right)\cos(\vartheta-\vartheta'-\omega_0'-\pi_0')\right.$$

$$-\left[\left(\frac{A'}{\sqrt{A}}-\frac{A}{\sqrt{A'}}\right)(1-e\cos\vartheta-e'\cos\vartheta')+\frac{2A'e}{\sqrt{A}}\cos\vartheta-\frac{2Ae'}{\sqrt{A'}}\cos\vartheta'\right]\sin(\vartheta-\vartheta'-\omega_0'-\pi_0')$$

$$\left.-2\left(\frac{A'}{\sqrt{A}}-\frac{A}{\sqrt{A'}}\right)(e\sin\vartheta-e'\sin\vartheta')\cos(\vartheta-\vartheta'-\omega_0'-\pi_0')\right\}$$

In functione Ω' $\frac{1}{cc}\cdot\frac{dr_0'}{dt}$ multiplicata invenitur per $\frac{1}{r_0'}$; factor expressionis $\frac{1}{cc}\cdot\frac{1}{r_0'}\cdot\frac{dr_0'}{dt}$ igitur est

$$\frac{1}{r_0'r_0'}=\frac{1}{rr+r'r'-2rr'\cos\Phi}+\frac{2rr'(\cos\Psi-\cos\Phi)}{(rr+r'r'-2rr'\cos\Phi)^2}\sin^2\tfrac{1}{2}i_0'$$

cujus primam partem tantum retinendam esse patet. Illa autem hoc modo in seriem secundum multipla ipsius Φ progredientem evolvi potest.

Habemus

$$\frac{1}{rr+r'r'-2rr'\cos\Phi}=\frac{1}{r-r'e^{\Phi i}}\cdot\frac{1}{r-r'e^{-\Phi i}}$$

denotante e numerum, cujus log. hyp. = 1. Porro

$$\frac{1}{r-r'e^{\Phi i}}=\frac{1}{r}\left\{1+\frac{r'}{r}e^{\Phi i}+\frac{r'r'}{rr}e^{2\Phi i}+\frac{r'^3}{r^3}e^{3\Phi i}+\dots\right\}$$

$$\frac{1}{r-r'e^{-\Phi i}}=\frac{1}{r}\left\{1+\frac{r'}{r}e^{-\Phi i}+\frac{r'r'}{rr}e^{-2\Phi i}+\frac{r'^3}{r^3}e^{-3\Phi i}+\dots\right\}$$

Quae series modo tum convergunt, si $r>r'$. Quum vero r et r' expressionem similiter ingrediantur, modo r cum r' permutare licet, si contrarium locum habet. Quibus seriebus in se ductis, revocatisque quantitatibus exponentialibus ad cosinus multiplorum ipsius Φ, obtinemus

$$\frac{1}{rr+r'r'-2rr'\cos\Phi}=\{r,r'\}+\{r,r'\}_1\cos\Phi+\{r,r'\}_2\cos 2\Phi+\dots$$

ubi

$$\{r,r'\}=\frac{1}{rr}\left(1+\frac{r'r'}{rr}+\frac{r'^4}{r^4}+\dots\right)=\frac{1}{rr-r'r'}$$

$$\{r,r'\}_1=\frac{2}{rr}\left(\frac{r'}{r}+\frac{r'^3}{r^3}+\frac{r'^5}{r^5}+\dots\right)=2\frac{r'}{r}\cdot\frac{1}{rr-r'r'}$$

$$\{r,r'\}_2=\frac{2}{rr}\left(\frac{r'r'}{rr}+\frac{r'^4}{r^4}+\frac{r'^6}{r^6}+\dots\right)=2\frac{r'r'}{rr}\cdot\frac{1}{rr-r'r'}$$

etc.

Quae cum ita sint, tandem eruitur, neglectis negligendis

$$\frac{1}{cc}\cdot\frac{1}{r_0'}\cdot\frac{dr_0'}{dt}=$$

$$\frac{\sqrt{\lambda}}{cc}\left\{e\sqrt{A}\sin\vartheta+e'\sqrt{A'}\sin\vartheta'-\left[\frac{A}{\sqrt{A'}}e'\sin\vartheta'+\frac{A'}{\sqrt{A}}e\sin\vartheta+2\left(\frac{A'}{\sqrt{A}}-\frac{A}{\sqrt{A'}}\right)(e\sin\vartheta-e'\sin\vartheta')\right]\cos(\vartheta-\vartheta'-\omega_0'-\pi_0')\right.$$

$$\left.+\left[\left(\frac{A'}{\sqrt{A}}-\frac{A}{\sqrt{A'}}\right)(e\cos\vartheta-e'\cos\vartheta')-\frac{2A'e}{\sqrt{A}}\cos\vartheta-\frac{2A}{\sqrt{A'}}e'\cos\vartheta'\right]\sin(\vartheta-\vartheta'-\omega_0'-\pi_0'\right\}\times$$

$$\left\{\{A,A'\}+\{A,A'\}_1\cos(\vartheta-\vartheta'-\omega_0'-\pi_0')+\{A,A'\}_2\cos 2(\vartheta-\vartheta'-\omega_0'-\pi_0')+\dots\right\}$$

$$-\frac{\sqrt{\lambda}}{cc}\left(\frac{A'}{\sqrt{A}}-\frac{A}{\sqrt{A'}}\right)\sin(\vartheta-\vartheta'-\omega_0'-\pi_0')\left\{\{r,r'\}+\{r,r'\}_1\cos\Phi+\{r,r'\}_2\cos 2\Phi+\dots\right\}$$

ubi adhuc ponendum

$$\{r,r'\}=\{A,A'\}+\frac{d\{A,A'\}}{dA}\left(-Ae\cos\vartheta-\frac{\lambda}{cc}e\sin\vartheta+\frac{Aee}{2}-\frac{Aee}{2}\cos 2\vartheta\right)$$

$$+\frac{dd\{A,A'\}}{2dA^2}\times\frac{AAee}{2}(1+2\cos 2\vartheta)$$

$$+\frac{d\{A,A'\}}{dA'}\left(-A'e'\cos\vartheta'-\frac{\lambda}{cc}\cdot e'\sin\vartheta'+\frac{A'e'e'}{2}-\frac{A'e'e'}{2}\cos 2\vartheta'\right)$$

$$+\frac{dd\{A,A'\}}{2dA'^2}\times\frac{A'A'e'e'}{2}(1+2\cos 2\vartheta')$$

$$+\frac{dd\{A,A'\}}{dA\,dA'}\times\frac{AA'ee'}{2}\left[\cos(\vartheta-\vartheta')+\cos(\vartheta+\vartheta')\right]$$

similiterque pro $\{r,r'\}_1$ etc. Valor ipsius $\cos\Phi$ jam supra propositus. Attamen tantummodo quantitatum primi ordinis ratio habenda est propter factorem $\frac{\sqrt{\lambda}}{cc}$; functionem $\{r,r'\}$ ideo exactius exhibui, ut ad ejus normam parenthesibus tantum commutatis formentur functiones (r,r'), $[r,r']$ mox in usum vocandae. Expressionem pro $\frac{1}{cc}\frac{1}{r_0'}\frac{dr_0'}{dt}$ propositam si accuratius examinaverimus, nullum eam continere terminum non-periodicum inveniemus, ita ut ad variationes saeculares planetarum inde nihil redundet.

Inquiramus in functionem $\frac{1}{r_0{}'}$. Erat

$$\frac{1}{r_0{}'} = (r,r') + (r,r')_1 \cos \Phi + (r,r')_2 \cos 2\Phi + \text{etc.}$$

$$+ rr' \{[r,r'] + [r,r']_1 \cos \Phi + [r,r']_2 \cos 2\Phi + ..\} \{\cos \Psi - \cos \Phi\} \sin^2 \tfrac{1}{2} i_0$$

ubi
$$\Phi = \varphi - \varphi' - \omega_0{}' - \pi_0{}', \quad \Psi = \varphi + \varphi' - \omega_0{}' + \pi_0{}'$$

In termino, qui continet $\sin^2 \frac{1}{2} i_0{}'$, jam quantitates primi ordinis excludendae sunt; pro r, r' igitur scribendum A, A', pro Φ et Ψ resp. $\vartheta - \vartheta' - \omega_0{}' - \pi_0{}'$, $\vartheta + \vartheta' - \Omega_0{}' + \pi_0{}'$. Quum porro tantum terminos a ϑ et ϑ' independentes retineamus, inde habetur nihil nisi

$$- \tfrac{1}{2} A A' [A, A']_1 \sin^2 \tfrac{1}{2} i_0{}'$$

E (r, r') nanciscimur terminum non periodicum

$$(A,A') + \left\{\frac{d(A,A')}{dA} \frac{A}{2} + \frac{dd(A,A')}{dA^2} \frac{AA}{4}\right\} ee + \left\{\frac{d(A,A')}{dA'} \frac{A'}{2} + \frac{dd(A,A')}{dA'^2} \frac{A'A'}{4}\right\} e'e'$$

E $(r,r')_1 \cos \Phi$, multiplicatione recte instituta, restat

$$\left\{(A,A')_1 + \frac{d(A,A')_1}{dA} \frac{A}{2} + \frac{d(A,A')_1}{dA'} \frac{A'}{2} + \frac{dd(A,A')}{dA\,dA'} \frac{AA'}{4}\right\} ee' \cos(\omega_0' - \pi_0')$$

Expressiones sequentes $(r,r')_2 \cos 2\Phi$ etc. nihil contribuunt ad perturbationes saeculares. Termini inventi prorsus congruunt cum iis quas ill. Lagrange poposuit. —

Jam procedamus ad alteram partem functionis Ω'

$$\lambda m_, \frac{rr + r'r' - r_0{}'r_0{}'}{2r'^3} \left(1 - \frac{1}{cc} \cdot \frac{dr'^2}{dt^2} + \frac{2}{cc} r' \frac{ddr'}{dt^2}\right)$$

E quantitate uncis inclusa facile invenitur restare

$$1 + \frac{2\lambda}{cc} \cdot \frac{e'}{A'} \cos \vartheta'$$

Porro erat

$$\frac{rr + r'r' - r_0{}'r_0{}'}{2r_,^3} = \frac{r \cos(\varphi - \varphi' - \omega_0{}' - \pi_0{}')}{r'r'} + \frac{r[\cos(\varphi - \varphi' - \omega_0{}' - \pi_0{}') - \cos(\varphi + \varphi' - \omega_0{}' + \pi_0{}')]}{r'r'} \sin^2 \tfrac{1}{2} i_0{}'$$

Inde sequitur

$$\frac{rr + r'r' - r_0{}'r_0{}'}{2r'^3} \left(1 - \frac{1}{cc} \cdot \frac{dr'^2}{dt^2} + \frac{2}{cc} r' \frac{ddr'}{dt^2}\right) =$$

$$\frac{r \cos(\varphi - \varphi' - \omega_0{}' - \pi_0{}')}{r'r'} + \frac{r[\cos(\varphi - \varphi' - \omega_0{}' - \pi_0{}') - \cos(\varphi + \varphi' - \omega_0{}' + \pi_0{}')]}{r'r'} \sin^2 \tfrac{1}{2} i_0{}'$$

$$+ \frac{2\lambda}{cc} \cdot \frac{e'}{A'} \cdot \frac{r \cos(\varphi - \varphi' - \omega_0{}' - \pi_0{}')}{r'r'} \cos \vartheta'$$

Duo termini posteriores neglectis quantitatibus tertii ordinis, abeunt in hos

$$\frac{A[\cos(\vartheta - \vartheta' - \omega_0{}' - \pi_0{}') - \cos(\vartheta + \vartheta' - \omega_0{}' + \pi_0{}')]}{A'A'} \sin^2 \tfrac{1}{2} i_0{}'$$

$$+ \frac{2\lambda}{cc} \cdot \frac{e'}{A'^3} A \cos(\vartheta - \vartheta' - \omega_0{}' - \pi_0{}') \cos \vartheta'$$

patetque nullos inde fluere terminos constantes. Substitutionibus factis pro r et r', fit

$$\frac{r}{r'r'} = \frac{A}{A'A'} \Big\{1 - e\left(\cos \vartheta + \frac{\lambda}{cc} \cdot \frac{1}{A} \sin \vartheta\right) + 2e'\left(\cos \vartheta' + \frac{\lambda}{cc} \cdot \frac{1}{A'} \sin \vartheta'\right)$$

$$+ \frac{ee}{2}(1 - \cos 2\vartheta) + \frac{e'e'}{2}(1 + 5 \cos 2\vartheta) - ee'[\cos(\vartheta - \vartheta') + \cos(\vartheta + \vartheta')]\Big\}$$

Quodsi hunc valorem ducimus in valorem ipsius cos Φ supra exhibitum, videmus terminos constantes tertium ordinem non egredientes se invicem destruere. Fit enim

$$\frac{A}{A'}\left(-\frac{ee'}{2} - ee' + \frac{ee'}{2} + ee'\right) \cos(\omega_0{}' - \pi_0{}') = 0$$

Si igitur per $[\Omega']$ designamus partem constantem ipsis Ω', habemus

$$[\Omega'] = \lambda m' \Big\{- \tfrac{1}{2} AA' [A,A']_1 \sin^2 \tfrac{1}{2} i_0{}' + (A,A') + \left(\frac{d(A,A')}{dA} \cdot \frac{A}{2} + \frac{dd(A,A')}{dA^2} \cdot \frac{AA}{4}\right) ee$$

$$+ \left(\frac{d(A,A')}{dA'} \cdot \frac{A'}{2} + \frac{dd(A,A')}{dA'^2} \cdot \frac{A'A'}{4}\right) e'e'$$

$$+ \left((A,A')_1 + \frac{d(A,A')_1}{dA} \cdot \frac{A}{2} + \frac{d(A,A')_1}{dA'} \cdot \frac{A'}{2} + \frac{dd(A,A)_1}{dA\,dA'} \cdot \frac{AA'}{4}\right) ee' \cos(\omega_0{}' - \pi_0{}')\Big\}$$

qua cum expressione ill. Lagrange comparata, eas omnino congruere videmus, id quod valde memorabile. Functio $\Omega = \Omega' + \Omega'' + \Omega''' + \ldots$, ubi Ω' definitur vis perturbatrix massa m' erercita etc.; igitur $[\Omega] = [\Omega'] + [\Omega''] + [\Omega'''] + \cdots$

Omnia, quae Lagrange de perturbationibus saecularibus protulit, etiam hic valent.

Ex aequatione

$$\lambda \frac{dA}{dt} = 2A^2 \frac{d[\Omega]}{dt_0}$$

sequitur $dA = 0$, quum t_0 functionem $[\Omega]$ omnino non ingrediatur. Mediae distantiae planetarum a sole, etiamsi valeat lex electrodynamica, loco Neutonianae, variationibus saecularibus non afficiuntur. Secundum Lagrange est simplicius

$$[\Omega] = \tfrac{1}{8} \lambda m' \begin{Bmatrix} 8(A,A') + AA'[A,A']_1 (ee + e'e') \\ - 2AA'[A,A']_2\, ee' \cos(\omega_0{}' - \pi_0{}') \end{Bmatrix} - \frac{\lambda m'}{4} AA'[A,A'] \sin^2 \tfrac{1}{2} i_0{}'$$

$$+ \tfrac{1}{8} \lambda m'' \begin{Bmatrix} 8(A,A'') + AA''[A,A'']_1 (ee + e''e'') \\ - 2AA''[A,A'']_2\, ee'' \cos(\omega_0{}'' - \pi_0{}'') \end{Bmatrix} - \frac{\lambda m''}{4} AA''[A,A''] \sin^2 \tfrac{1}{2} i_0{}''$$

$+$ etc.

$$\cos i_0' = \cos i \cos i' + \cos(\omega - \omega') \sin i \sin i'$$

$$\cos i_0'' = \cos i \cos i'' + \cos(\omega - \omega'') \sin i \sin i''$$

Quodsi loco π_0' introducitur angulus χ, quo definitur rotatio perihelii in plano orbitae, notum est esse

$$\omega_0' - \pi_0' = \chi' - \chi$$

$$\omega_0'' - \pi_0'' = \chi'' - \chi$$

$$d\chi = d\pi + \cos i d\omega$$

formulaeque perturbatoriae angulum π continentes abeunt in has

$$\frac{dp}{dt} = -\frac{2\sqrt{p}}{\sqrt{\lambda}} \cdot \frac{d[\Omega]}{d\chi}, \quad \frac{d\chi}{dt} = \frac{2\sqrt{p}}{\sqrt{\lambda}} \cdot \frac{d[\Omega]}{dp}$$

Habetur etiam

$$\frac{de}{dt} = \frac{\sqrt{1 - ee}}{e\sqrt{\lambda A}} \cdot \frac{d[\Omega]}{d\chi}$$

Formulas adhuc faciles demonstratu, quibus variationes excentricitatum, sive potius in nostro problemate quantitatum ellipticitatem curvae definientium, nec non velocitatum angularium periheliorum, variationes inclinationum intra fixos limites coërcentur, appono, quae inveniuntur apud Laplace et Lagrange, quaeque etiam hic valent

$$m\sqrt{A}\,.\,ee + m'\sqrt{A'}\,.\,e'e' + m''\sqrt{A''}\,e''e'' + \ldots = \text{const.}$$

$$m\sqrt{A}\,.\,\frac{ee\,d\chi}{dt} + m'\sqrt{A'}\frac{e'e'\,d\chi'}{dt} + m''\sqrt{A''}\frac{e''e''\,d\chi''}{dt} + \ldots = \text{valori finito}$$

$$m\sqrt{A}\sin^2\frac{i}{2} + m'\sqrt{A'}\sin^2\frac{i'}{2} + m''\sqrt{A''}\sin^2\frac{i''}{2} + \ldots = \text{const.}$$

Sed haec addigitasse sufficit.
