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Ampère’s Force Law

A Modern Introduction

May 29, 2018

Springer

To my wife, who made this translation possible, to the Blessed Virgin Mother, and to the Holy Trinity, Who makes all things possible

Foreword

Note on the Translation

Anything in a

gray box,

which indicates a comment or a modernized version of an equation, and additions in [brackets] belong to the translator.

Alan Aversa

May 29, 2018

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Part I
Mathematical Introduction to Electrodynamics[1]

[1] [Duhem (1892, 1-46)]

Chapter 1
On Curvilinear Integrals[1]

1.1 Parameters that define the relative placement of two linear elements

In studying Electrodynamics and Electromagnetism, one constantly appeals to a certain number of propositions from Analytic Geometry rarely employed outside the domain of these sciences. We will collect here the most important of these propositions.

Let x, y, z $[\mathbf{M}]$ be the rectangular coordinates[2] of a point M of a curve on which a sense of direction is chosen. Let MM′ be an element of this curve, issuing from the M, and having length ds. The point M has coordinates

$$x' = x + \frac{dx}{ds}ds,$$
$$y' = y + \frac{dy}{ds}ds,$$
$$z' = z + \frac{dz}{ds}ds.$$

$$\mathbf{M'} = \mathbf{M} + \frac{d\mathbf{M}}{ds}ds$$

Let MT be the tangent in M to the curve under consideration, directed in the direction of travel chosen on the curve. The ray[3] MT makes, with the coordinate

[1] *See*, on the subject of curvilinear integrals and surface integrals, Tome I of the *Traité d'Analyse* by É. Picard. In this beautiful work, the theory of these integrals is treated with some great developments and by methods often different from those that are expressed here.

[2] In all that follows, except where the contrary is indicated, non-rectangular coordinates will never be used.

[3] [See Hadamard (2008, 3) for the definition of a *demi-droite*.]

axes Ox, Oy, Oz, angles α, β, γ, and it is known that

$$\cos\alpha = \frac{dx}{ds}, \qquad \cos\beta = \frac{dy}{ds}, \qquad \cos\gamma = \frac{dz}{ds}. \tag{1.1}$$

One often has to consider the system formed in space by two linear elements

$$\mathrm{MM}_1 = ds, \qquad \mathrm{M}'\mathrm{M}'_1 = ds'.$$

A similar system (Figure 1.1) is evidently defined by the following parameters:

1. The lengths ds, ds' of the two elements;

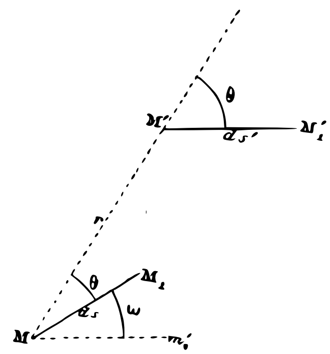

Fig. 1.1 [Relative positions of two line elements]

2. The distance r from the origin M of the first to the origin M′ of the second;
3. The three angles θ, θ', ω, which themselves are defined in the following manner:

- θ is the smallest angle that the direction MM_1 of the element ds makes with the direction MM′ of the line that joins the origin of the element ds with the origin of the element ds';
- ω' is the smallest of angle that the direction $\mathrm{M}'\mathrm{M}'_1$ of the element ds' makes with the direction MM′ itself;
- ω is the smallest of the two angles that the directions MM_1, $\mathrm{M}'\mathrm{M}'_1$ make with each other.

The knowledge of the parameters r, ds, ds', θ, θ', ω do not *unambiguously* define the system of two elements MM_1, $\mathrm{M}'\mathrm{M}'_1$; the element MM_1 being arbitrarily placed in space, the knowledge of these parameters defines, by the element $\mathrm{M}'\mathrm{M}'_1$, two possible positions, symmetric with respect to the plane $\mathrm{M}_1\mathrm{MM}'$. But, in a great number of cases, the function of the system of two elements which we will have to consider will have the same value for these two distinct systems. In these cases, one will be able to regard the system of two elements as completely defined by the knowledge of the parameters

$$ds, \qquad ds', \qquad r, \qquad \theta, \qquad \theta', \qquad \omega.$$

The three angles θ, θ', ω being, by definition, taken between 0 and π, are defined by their cosines. One can thus say, in the case of which we have just spoken, that a function of the system of the two elements is defined when one knows the parameters

$$ds, \qquad ds', \qquad r, \qquad \cos\theta, \qquad \cos\theta', \qquad \cos\omega.$$

These parameters, whose consideration returns at every moment in the following Chapters, are susceptible to many expressions which are indispensable to know.

Let x, y, z be the coordinates of the point M, and x', y', z' the coordinates of the point M′. We will have, in the first place,

$$r^2 = (x'-x)^2 + (y'-y)^2 + (z'-z)^2 . \tag{1.2}$$

Let α, β, γ be the angles of the direction MM$_1$ with the axes Ox, Oy, Oz and α', β', γ' be the angles of the direction M′M$_1'$ with the same axes. We will have, according to the equations (1.1),

$$\begin{aligned} \cos\alpha &= \frac{dx}{ds}, & \cos\beta &= \frac{dy}{ds}, & \cos\gamma &= \frac{dz}{ds}, \\ \cos\alpha' &= \frac{dx'}{ds'}, & \cos\beta' &= \frac{dy'}{ds'}, & \cos\gamma' &= \frac{dz'}{ds'}. \end{aligned}$$

Now, one knows that

$$\cos\omega = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'.$$

One thus has

$$\cos\omega = \frac{dx}{ds}\frac{dx'}{ds'} + \frac{dy}{ds}\frac{dy'}{ds'} + \frac{dz}{ds}\frac{dz'}{ds'}. \tag{1.3}$$

$$\cos\omega = \frac{d\mathbf{M}}{ds} \cdot \frac{d\mathbf{M'}}{ds'}$$

The line MM′ makes with Ox, Oy, Oz the angles λ, μ, ν, and one has

$$\cos\lambda = \frac{x'-x}{r}, \qquad \cos\mu = \frac{y'-y}{r}, \qquad \cos\nu = \frac{z'-z}{r}.$$

Now

$$\begin{aligned} \cos\theta &= \cos\lambda\cos\alpha + \cos\mu\cos\beta + \cos\nu\cos\gamma, \\ \cos\theta' &= \cos\lambda\cos\alpha' + \cos\mu\cos\beta' + \cos\nu\cos\gamma'. \end{aligned}$$

One thus has

$$\left.\begin{aligned} \cos\theta &= \frac{x'-x}{r}\frac{dx}{ds} + \frac{y'-y}{r}\frac{dy}{ds} + \frac{z'-z}{r}\frac{dz}{ds}, \\ \cos\theta' &= \frac{x'-x}{r}\frac{dx'}{ds} + \frac{y'-y}{r}\frac{dy'}{ds} + \frac{z'-z}{r}\frac{dz'}{ds}. \end{aligned}\right\} \tag{1.4}$$

$$\left.\begin{aligned} \cos\theta &= \frac{\mathbf{M'}-\mathbf{M}}{r}\cdot\frac{d\mathbf{M}}{ds} \\ \cos\theta' &= \frac{\mathbf{M'}-\mathbf{M}}{r}\cdot\frac{d\mathbf{M'}}{ds} \end{aligned}\right\}$$

Equation (1.2) gives

$$\begin{aligned} \frac{\partial r}{\partial x'} &= -\frac{\partial r}{\partial x} = \frac{x'-x}{r}, \\ \frac{\partial r}{\partial y'} &= -\frac{\partial r}{\partial y} = \frac{y'-y}{r}, \\ \frac{\partial r}{\partial z'} &= -\frac{\partial r}{\partial z} = \frac{z'-z}{r}, \end{aligned}$$

$$\nabla' r = -\nabla r = \frac{\mathbf{M'}-\mathbf{M}}{r}$$

relations by means of which the equations (1.4) become

$$\begin{aligned} \cos\theta &= -\left(\frac{\partial r}{\partial x}\frac{dx}{ds} + \frac{\partial r}{\partial y}\frac{dy}{ds} + \frac{\partial r}{\partial z}\frac{dz}{ds}\right), \\ \cos\theta' &= \frac{\partial r}{\partial x'}\frac{dx'}{ds} + \frac{\partial r}{\partial y'}\frac{dy'}{ds} + \frac{\partial r}{\partial z'}\frac{dz'}{ds} \end{aligned}$$

$$\begin{aligned} \cos\theta &= \nabla r\cdot\frac{d\mathbf{M}}{ds}, \\ \cos\theta' &= \nabla' r\cdot\frac{d\mathbf{M'}}{ds} \end{aligned}$$

or

$$\cos\theta = \frac{\partial r}{\partial s}, \qquad \cos\theta' = \frac{\partial r}{\partial s'}. \tag{1.5}$$

The collection of equations (1.4) and (1.5) gives

$$\frac{\partial r}{\partial s'} = \frac{x'-x}{r}\frac{dx'}{ds'} + \frac{y'-y}{r}\frac{dy'}{ds'} + \frac{z'-z}{r}\frac{dz'}{ds'}.$$

$$\frac{\partial r}{\partial s'} = \frac{\mathbf{M'}-\mathbf{M}}{r}\cdot\frac{d\mathbf{M'}}{ds'}$$

From which one easily deduces

$$\begin{aligned}\frac{\partial^2 r}{\partial s\,\partial s'} = &-\frac{1}{r}\left(\frac{dx}{ds}\frac{dx'}{ds'} + \frac{dy}{ds}\frac{dy'}{ds'} + \frac{dz}{ds}\frac{dz'}{ds'}\right)\\ &+\frac{1}{r}\left(\frac{x'-x}{r}\frac{dx}{ds} + \frac{y'-y}{r}\frac{dy}{ds} + \frac{z'-z}{r}\frac{dz}{ds}\right)\\ &\times\left(\frac{x'-x}{r}\frac{dx'}{ds'} + \frac{y'-y}{r}\frac{dy'}{ds'} + \frac{z'-z}{r}\frac{dz'}{ds'}\right).\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 r}{\partial s\,\partial s'} = &-\frac{1}{r}\left(\frac{d\mathbf{M}}{ds}\cdot\frac{d\mathbf{M'}}{ds'}\right)\\ &+\frac{1}{r}\left(\frac{\mathbf{M'}-\mathbf{M}}{r}\cdot\frac{d\mathbf{M}}{ds}\right)\\ &\times\left(\frac{\mathbf{M'}-\mathbf{M}}{r}\cdot\frac{d\mathbf{M'}}{ds'}\right)\end{aligned}$$

If on takes equations (1.3) and (1.4) into account, this equation becomes

$$\frac{\cos\theta\cos\theta'}{r} - \frac{\cos\omega}{r} = \frac{\partial^2 r}{\partial s\,\partial s'} \tag{1.6}$$

or, taking equations (1.5) into account,

$$\cos\omega = -\left(\frac{\partial r}{\partial s}\frac{\partial r}{\partial s'} + r\frac{\partial^2 r}{\partial s\,\partial s'}\right). \tag{1.7}$$

The line[4] MM′ and the ray MM_1 determine the first half-plane[5]. The line MM′ and the ray $\mathrm{M'M'_1}$ determine a second half-plane.

Let ε be the smallest dihedral angle[6] formed by these two half-planes. This angle being, by definition, between 0 and π, is determined by its cosine.

Through M we place $\mathrm{M}m'_1$ parallel to $\mathrm{M'M'_1}$ (Figure 1.2). In the trihedron[7]

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[4] [Duhem has "*droite indéfinie*" (a line not terminated on either end). See Hadamard (2008, 3).]

[5] [See Hadamard (1901, 6) for the definition of a "*demi-plan*".]

[6] [See Hadamard (1901, 24) for a definition of an "*angle diédre*".]

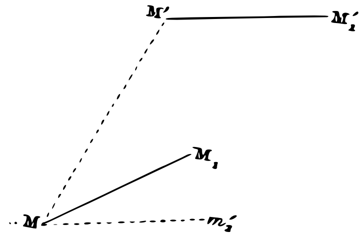

Fig. 1.2 [Lines determining two planes with a dihedral angle]

$\mathrm{MM}_1 m_1' \mathrm{M}'$, the angle ε is the dihedron opposite the angle $\mathrm{M}_1\mathrm{M}M_1'$ or ω; it is included between the faces $\mathrm{M}'\mathrm{MM}_1$, or θ and $\mathrm{M}'\mathrm{M}m_1'$, or $\theta' z$. One thus has

$$\cos\omega = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\varepsilon. \tag{1.8}$$

This equation shows us that, if a function dependent on the relative position of two elements ds and ds' depends, in a uniform manner, on the parameters

$$\theta, \qquad \theta', \qquad \omega,$$

then it depends in a uniform manner on the parameters

$$\theta, \qquad \theta', \qquad \varepsilon,$$

and *vice versa*; moreover, the angles θ, θ', ω, ε are all between 0 and π and, thus, defined in a uniform manner by their cosines.

The comparison of equations (1.6) and (1.7) gives

$$\sin\theta\sin\theta'\cos\varepsilon = -r\frac{\partial^2 r}{\partial s\,\partial s'}. \tag{1.9}$$

The various equations that we have just written are constantly used in the study of Electrodynamics.

We saw that the knowledge of the angles θ, θ', ω—or, what amounts to the same, of the angles θ, θ', ε—do not unambiguously define the relative direction of the two elements MM_1, $\mathrm{M}'\mathrm{M}_1'$.

Imagine a half-plane, limited by the line MM', and turning from left to right around this axis. Make this half-plane coincide at first with the half-plane $\mathrm{M}'\mathrm{MM}_1$. To come to coincide with the plane $\mathrm{MM}'\mathrm{M}_1'$, it will need to turn and angle e, between 0 and 2π. The knowledge of the angles θ, θ', e define *unambiguously* the relative direction of the two elements MM_1, $\mathrm{M}'\mathrm{M}_1'$.

If the angle e kis between 0 and π, one has

$$\varepsilon = e.$$

[7] [cf. Hadamard (1901, 41) for a definition of "trihedral angles" ("*angles trièdres*")]

If, on the contrary, the angle e is between π and 2π, on has

$$\varepsilon = 2\pi - e.$$

1.2 On the curvilinear integral. Definition. Fundamental theorem.

Let $U, V, W \left[\equiv \mathbf{U}\left(\mathbf{M}, \dot{\mathbf{M}}, \ddot{\mathbf{M}}, \ldots, \mathbf{M}^{(n)}\right)\right]$ be three uniform and continuous functions of the following variables:

$$\begin{array}{rrr} x, & y, & z, \\ \frac{dx}{ds}, & \frac{dy}{ds}, & \frac{dz}{ds}, \\ \frac{d^2x}{ds^2},, & \ldots, & \ldots, \\ \ldots, & \ldots, & \frac{d^n z}{ds^n}. \end{array}$$

We imagine that x, y, z are the coordinates of a variable point M of a curve AMB (Figure 1.3). Let s be the arc AM. One can always imagine that the curve is

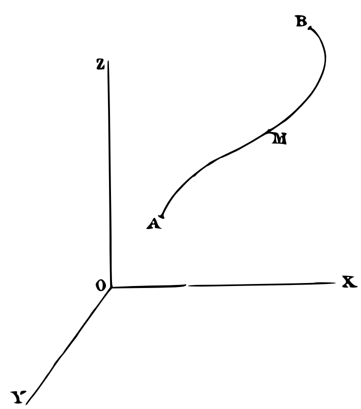

Fig. 1.3 [Variable point M on a curve AMB]

represented by the equations

$$\begin{aligned} x &= f(s), \\ y &= g(s), \\ z &= h(s), \end{aligned}$$

$$\mathbf{M} = \mathbf{M}(s)$$

f, g, h [or $\mathbf{M}$] being finite, uniform, and continuous functions of s, whose derivatives with respect to s are uniform up to order n exist and are finite and continuous functions of s, except at a limited number of points of the curve.

By means of these equations, the quantities

$$\begin{array}{ccc} \dfrac{dx}{ds}, & \dfrac{dy}{ds}, & \dfrac{dz}{ds}, \\ \dfrac{d^2x}{ds^2},, & \ldots, & \ldots, \\ \ldots, & \ldots, & \dfrac{d^nz}{ds^n}. \end{array}$$

$$\begin{array}{c} \dfrac{d\mathbf{M}}{ds}, \\ \dfrac{d^2\mathbf{M}}{ds^2}, \\ \dfrac{d^n\mathbf{M}}{ds^n} \end{array}$$

will become uniform functions of s; these functions can be infinite or discontinuous at certain points or in certain regions of the curve AMB. It will be the same for the functions $u(s), v(s), w(s)$ $\left[\equiv \tilde{\mathbf{M}}(s)\right]$, obtained by replacing the variables that figure in the functions U, V, W [$\equiv \mathbf{U}$] with their expressions as a function of s.

Let

$$\frac{dx}{ds} = \varphi(s), \qquad \frac{dy}{ds} = \psi(s), \qquad \frac{dz}{ds} = \theta(s).$$

Let, moreover, S be the length of the arc AMB. If the definite integral

$$\int_0^S [u(s)\varphi(s) + v(s)\psi(s) + w(s)\theta(s)]\, ds$$

$$\int_0^S \left[\tilde{\mathbf{M}} \cdot \frac{d\mathbf{M}}{ds}\right] ds$$

exists, we will represent it by the symbol

$$\int_{\text{AMB}} (U\,dx + V\,dy + W\,dz)\,,$$

$$\int_{\text{AMB}} \mathbf{U} \cdot d\mathbf{M}$$

and we will say that this symbol represents a *curvilinear integral* performed along the curve AMB.

It is necessary to remark that this symbol do not in general have any meaning if one does not suppose that the arc AMB is completely known; it is only when one supposes that this arc is known that it takes on a meaning, that of a definite integral, and for each different arc joining the point A to the point B corresponds a different meaning of this symbol, this meaning being translated by a different definite integral.

To define this integral, we have assumed the coordinates of a point of the curve AMB expressed by means of the arc s of this curve; but we may also just as well be able to assume them expressed by means of a parameter t that varies continuously along the curve AMB.

Almost all the properties of curvilinear integrals are deduced from a fundamental proposition that we are going to demonstrate.

We suppose that the three functions U, V, W [$\equiv \mathbf{U}$] depend only on x, y, z [$\equiv \mathbf{M}$] and, in addition, that we have

$$U = \frac{\partial F\,(x, y, z)}{\partial x},$$
$$V = \frac{\partial F\,(x, y, z)}{\partial y},$$
$$W = \frac{\partial F\,(x, y, z)}{\partial z},$$

$$\mathbf{U} = \nabla F$$

F being, in all space, a uniform, finite, and continuous function of x, y, z.

Let us consider any curve AMB given by the equations

$$x = f(s),$$
$$y = g(s),$$
$$z = h(s).$$

If in $F(x, y, z)$ one replaces x, y, z with these uniform, finite, and continuous functions of s, then $F(x, y, z)$ will be transformed into a uniform, finite, continuous function of s

$$F\,[f(s), g(s), h(s)] = \Phi(s).$$

The curvilinear integral

$$\int_{\mathrm{AMB}} (U\,dx + V\,dy + W\,dz)\,,$$

$$\int_{\mathrm{AMB}} \mathbf{U}\cdot d\mathbf{M}$$

will be equal, by definition, to

$$\int_0^S \left[\frac{\partial F}{\partial f(s)}\frac{\partial f(s)}{\partial s} + \frac{\partial F}{\partial g(s)}\frac{\partial g(s)}{\partial s} + \frac{\partial F}{\partial h(s)}\frac{\partial h(s)}{\partial s}\right]$$

or to

$$\int_0^S \frac{\partial \Phi(s)}{\partial s} ds.$$

$\Phi(s)$ being a uniform, finite, and continuous function of s, this latter quantity has the value

$$\Phi(S) - \Phi(0).$$

Let x_0, y_0, z_0 be the coordinates of the point A and x_1, y_1, z_1 the coordinates of the point B. We will have

$$\Phi(0) = F(x_0, y_0, z_0),$$
$$\Phi(S) = F(x_1, y_1, z_1)$$

and, consequently,

$$\int_{\mathrm{AMB}} (U\,dx + V\,dy + W\,dz) = F(x_1, y_1, z_1) - F(x_0, y_0, z_0).$$

$$\int_{\mathrm{AMB}} \mathbf{U}\cdot d\mathbf{M} = F(x_1, y_1, z_1) - F(x_0, y_0, z_0)$$

So the curvilinear integral considered depends exclusively on the origin and the extremity of the curve along which it is taken and not on the form of these curve.

In this particular case, one sees that one can attribute a meaning to the symbol

$$\int_{\mathrm{AMB}} (U\,dx + V\,dy + W\,dz)\,,$$

$$\int_{\text{AMB}} \mathbf{U} \cdot d\mathbf{M}$$

provided that one only knows the two points A and B, without it being necessary to know the curve AMB. This meaning is that of the difference

$$F(x_1, y_1, z_1) - F(x_0, y_0, z_0).$$

Suppose that the curve AMB is a closed curve; the point B coinciding with the point A, the coordinates x_1, y_1, z_1 are identical to the coordinates x_0, y_0, z_0, respectively. As, moreover, the function $F(x, y, z)$ is a uniform, finite, and continuous function of x, y, z, one will certainly have

$$F(x_1, y_1, z_1) - F(x_0, y_0, z_0) = 0.$$

Also, when U, V, W $[\equiv \mathbf{U}]$ are three partial derivatives of the same uniform, finite, and continuous function of x, y, z, the curvilinear integral

$$\int (U\,dx + V\,dy + W\,dz)\,,$$

$$\int \mathbf{U} \cdot d\mathbf{M}$$

evaluated over any closed curve, is equal to 0.

Before demonstrating the converse of this proposition, one remark is necessary.

If, for any open curve AMB, whose origin A has coordinates x_0, y_0, z_0 and whose extremity B has coordinates x_1, y_1, z_1, a certain curvilinear integral verifies the relation

$$\int_{\text{AMB}} (U\,dx + V\,dy + W\,dz) = F(x_1, y_1, z_1) - F(x_0, y_0, z_0).$$

$$\int_{\text{AMB}} \mathbf{U} \cdot d\mathbf{M} = F(x_1, y_1, z_1) - F(x_0, y_0, z_0)$$

$F(x, y, z)$ being a uniform, finite, and continuous function of x, y, z, one will have, for any closed curve,

$$\int (U\,dx + V\,dy + W\,dz) = 0.$$

$$\int \mathbf{U} \cdot d\mathbf{M} = 0$$

Conversely, we will consider a curvilinear integral such that, for any closed curve, one has

$$\int (U\,dx + V\,dy + W\,dz) = 0.$$

$$\int \mathbf{U} \cdot d\mathbf{M} = 0$$

and we look for the value of the integral

$$\int_{\text{AMB}} (U\,dx + V\,dy + W\,dz) = [\text{AMB}].$$

$$\int \mathbf{U} \cdot d\mathbf{M} = [\text{AMB}].$$

To obtain this value, we will remark in the first place that the integral AMB changes sign, without changing value, when one keeps the curve AMB and reverses its direction of travel: a relation that can be written symbolically

$$[\text{AMB}] + [\text{BMA}] = 0.$$

Indeed, the sum that we have written is none other than the value of the curvilinear integral considered along the particular closed curve AMBMA, and we know that this value is 0.

In the second place, let AMB, AM′B be two arcs of different curves joining the point A to the point B. The curve AMBM′A being a closed curve, one has

$$[\text{AMBM}'\text{A}] = 0,$$

which can also be written

$$[\text{AMB}] + [\text{BM}'\text{A}] = 0.$$

But, according to the previous remark,

$$[\text{BM}'\text{A}] + [\text{AM}'\text{B}] = 0.$$

One thus has, as we had said,

$$[\mathrm{AMB}] + [\mathrm{AM'B}].$$

These two remarks stated, we arbitrarily choose (Figure 1.4) a point Π, with coordinates α, β, γ. Let $P(x, y, z)$ be another point of the plane. The integral

$$\int_{\Pi\mathrm{MP}} (U\,dx + V\,dy + W\,dz)\,,$$

$$\int_{\Pi\mathrm{MP}} \mathbf{U}\cdot d\mathbf{M}$$

taken along any curve ΠMP joining the point Π to the point P, will have a value independent of the form of this curve and depending only on the position of the points Π and P. In addition, the position of the point Π being taken arbitrarily once and for

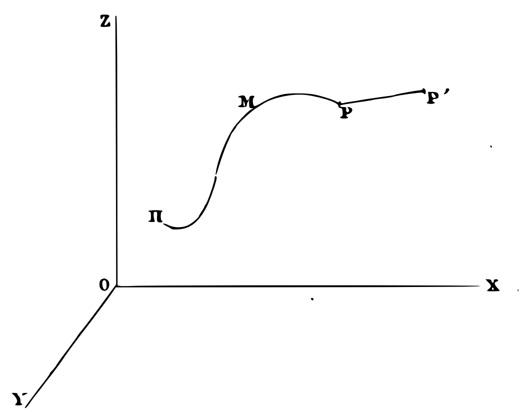

Fig. 1.4 [The curve ΠMPP′]

all, one sees that the value in question defines a uniform function of coordinates x, y, z of the point P. We denote this value by $F(x, y, z)$.

If the functions U, V, W $[\equiv \mathbf{U}]$ are of finite quantities, it is easy to see that this quantity is finite. It is also easy to see that it is continuous. Let, indeed, $P'(x', y', z')$ be a point near point P. The function $F(x', y', z')$ is the value of the curvilinear integral takes along any curve joining point Π to point P′. Now, as one such curvature, one can take the curve ΠMP following the line PP′. One then easily sees that

$$F(x', y', z') = F(x, y, z) + \int_{\mathrm{PP'}} (U\,dx + V\,dy + W\,dz)\,,$$

$$F(x', y', z') = F(x, y, z) + \int_{\mathrm{PP}'} \mathbf{U} \cdot d\mathbf{M}$$

and the integral on the right hand side is evidently infinitely small with PP′, which demonstrates the said theorem.

Having thus defined the uniform, finite, and continuous function of x, y, z that we have denoted $F(x, y, z)$, we arrive at the evaluation of [AMB].

If we note that ΠAMB (Figure 1.5) is a line that leads from point Π to point B, we will find

$$[\Pi\mathrm{AMB}] = F(x_1, y_1, z_1).$$

Moreover,

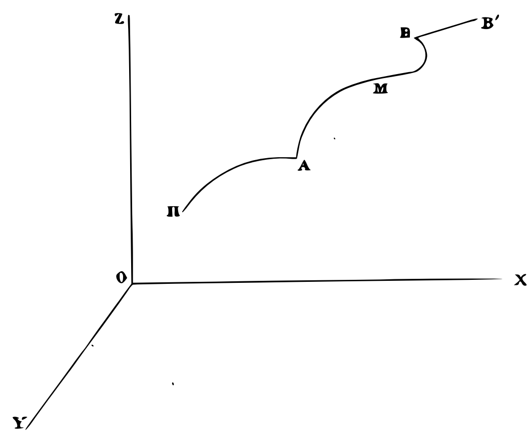

Fig. 1.5 [The curve ΠAMBB′]

$$[\Pi\mathrm{AMB}] = [\Pi\mathrm{A}] + [\mathrm{AMB}]$$

and

$$[\Pi\mathrm{A}] = F(x_0, y_0, z_0).$$

We thus find that

$$[\mathrm{AMB}] = \int_{\mathrm{AMB}} (U\,dx + V\,dy + W\,dz) = F(x_1, y_1, z_1) - F(x_0, y_0, z_0).$$

$$[\mathrm{AMB}] = \int_{\mathrm{AMB}} \mathbf{U} \cdot d\mathbf{M} = F(x_1, y_1, z_1) - F(x_0, y_0, z_0).$$

Thus: *to say that the curvilinear integral*

$$\int (U\,dx + V\,dy + W\,dz)\,,$$

$$\int \mathbf{U}\cdot d\mathbf{M}$$

evaluated along any closed contour is equal to 0*—or to say that the same integral evaluated over any curve is the difference that a uniform, finite, and continuous function of coordinates takes at the two extremities of the curve—is to state two equivalent propositions*

Now let us find what form the quantities U, V, W $[\equiv \mathbf{U}]$ should have so that one can state these two propositions.

We have

$$\int_{\text{AMB}} (U\,dx + V\,dy + W\,dz) = F(x_1, y_1, z_1) - F(x_0, y_0, z_0).$$

$$\int_{\text{AMB}} \mathbf{U}\cdot d\mathbf{M} = F(x_1, y_1, z_1) - F(x_0, y_0, z_0).$$

Let B′ be a point situated at an infinitely small distance ds from point B. Let α, β, γ be the cosines of the angles that the line BB′ make with Ox, Oy, Oz. We will have, for coordinates of point B,

$$x_1 + \alpha ds, \qquad y_1 + \beta ds, \qquad z_1 + \gamma ds.$$

We will thus have

$$\begin{aligned}\int_{\text{AMBB}'} (U\,dx + V\,dy + W\,dz) = F(x_1 + \alpha ds, y_1 + \beta ds, z_1 + \gamma ds)\\ - F(x_0, y_0, z_0).\end{aligned}$$

$$\begin{aligned}\int_{\text{AMBB}'} \mathbf{U}\cdot d\mathbf{M} = F(x_1 + \alpha ds, y_1 + \beta ds, z_1 + \gamma ds)\\ - F(x_0, y_0, z_0)\end{aligned}$$

But the first member can be written

$$\int_{\text{AMB}} (U\,dx + V\,dy + W\,dz) + (U_1\alpha + V_1\beta + W_1\gamma)ds,$$

$$\int_{\text{AMB}} \mathbf{U} \cdot d\mathbf{M} + (U_1\alpha + V_1\beta + W_1\gamma)ds$$

U_1, V_1, W_1 [$\equiv \mathbf{U_1}$] being the values of U, V, W [$\equiv \mathbf{U}$] at a certain point of the line BB′. One thus has

$$U_1 dx + V_1 dy + W_1 dz = F(x_1 + dx, y_1 + dy, z_1 + dz) - F(x_1, y_1, z_1),$$

$$\mathbf{U_1} \cdot d\mathbf{M} = F(x_1 + dx, y_1 + dy, z_1 + dz) - F(x_1, y_1, z_1)$$

i.e.,

$$U = \frac{\partial F}{\partial x}, \qquad V = \frac{\partial F}{\partial y}, \qquad W = \frac{\partial F}{\partial z}.$$

$$\mathbf{U}_i = \frac{\partial F}{\partial \mathbf{M}_i}$$

If one compares this result with the one whe obtained at the beginning of this paragraph, one sees that:

The necessary and sufficient condition for the curvilinear integral

$$\int (U\,dx + V\,dy + W\,dz),$$

$$\int (\mathbf{U} \cdot d\mathbf{M})$$

evaluated over a closed any closed curve to be equal to 0 *is that the three quantities* U, V, W [$\equiv \mathbf{U}$] *be the partial derivatives with respect to* x, y, z [$\equiv \mathbf{M}$] *of the same uniform, finite, and continuous function of* x, y, z [$\equiv \mathbf{M}$].

This is the fundamental theorem upon which the theory of curvilinear integrals rests.

The quantity

$$\frac{\partial r}{\partial s'} = \frac{x' - x}{r}\frac{dx'}{ds'} + \frac{y' - y}{r}\frac{dy'}{ds'} + \frac{z' - z}{r}\frac{dz'}{ds'}.$$

$$\frac{\partial r}{\partial s'} = \frac{\mathbf{M'} - \mathbf{M}}{r} \cdot \frac{d\mathbf{M'}}{ds'}$$

is a uniform, finite, and continuous function of coordinates x, y, z [$\equiv \mathbf{M}$] of a point of the curve s. Thus the integral

$$\int \frac{\partial^2 r}{\partial s\, \partial s'},$$

evaluated over any closed curve, is equal to 0.

Now equation (1.6) gives us

$$\frac{\cos\theta \cos\theta'}{r} - \frac{\cos\omega}{r} = \frac{\partial^2 r}{\partial s\, \partial s'},$$

the two integrals extending over the same closed curve.

A fortiori, if s and s' are any two closed curves, we will have

$$\iint \frac{\cos\theta \cos\theta'}{r} ds ds' = \iint \frac{\cos\omega}{r} ds ds'. \tag{1.10}$$

This equation plays, in Electrodynamics, an important role; it was demonstrated, in 1847, by F.-E. Neumann[8], in his task of comparing the results of his theory with the theory given by W. Weber.

1.3 Bertrand's Theorem

The fundamental theorem that we have just demonstrated will supply us with a proposition that we will frequently use. This prosition was given by J. Bertrand[9] in the course of his beautiful researches on Ampère's law.

This proposition is stated thus:

If the curvilinear integral

$$\int G\left(x, y, z, \frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right) ds,$$

---

[8] F.-E. Neumann, *Ueber ein allgemeines Princip der mathematischen Theorie inducirter elektrischer Ströme*. Read at the Academy of Sciences of Berlin, 9 August 1847.

[9] J. Bertrand, *Sur la démonstration de la formule qui reprȩsente l'action élémentaire de deux courants* (*Comptes rendus*, t. LXXV, p. 733; 1872.)

$$\int G\left(\mathbf{M}, \frac{d\mathbf{M}}{ds}\right)$$

evaluated over a closed contour, is infinitely small in the second order all the time that

$$\int ds$$

is infinitely small in the first order, the function G is linear and homogeneous in $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$ $\left[\equiv \frac{d\mathbf{M}}{ds}\right]$.

Let us consider, in fact, an infinitely small closed contour (Figure 1.6). Let $\mu(\xi, \eta, \zeta)$ be a fixed point, taken arbitrarily on this contour. Let $M(x, y, z)$ be a vari-

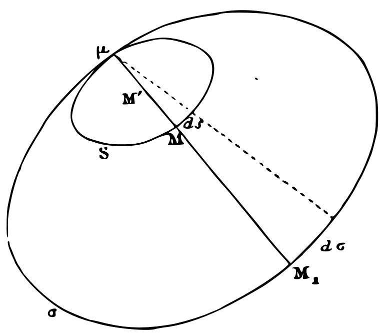

Fig. 1.6 [Bertrand's theorem]

able point of this contour. Let $M'(x', y', z')$ be a certain point conveniently chosen between the two preceding ones on the line that joins them. We will have

$$\begin{aligned}
&\int G\left(x, y, z, \frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right) ds \\
&= \int G\left(\xi, \eta, \zeta, \frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right) ds \\
&+ \int \left[(x' - \xi)\frac{\partial}{\partial x'} G\left(x', y', z', \frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right)\right. \\
&+ \int (y' - \eta)\frac{\partial}{\partial y'} G\left(x', y', z', \frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right) \\
&+ \int \left.(z' - \zeta)\frac{\partial}{\partial z'} G\left(x', y', z', \frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right)\right]
\end{aligned}$$

$$\begin{aligned}
&\int G\left(\mathbf{M}, \frac{d\mathbf{M}}{ds}\right) ds \\
&= \int G\left(\boldsymbol{\xi}, \frac{d\mathbf{M}}{ds}\right) ds \\
&+ \int \left[(\mathbf{M'} - \boldsymbol{\xi}) \cdot \nabla G\left(\mathbf{M'}, \frac{d\mathbf{M'}}{ds}\right)\right] ds.
\end{aligned}$$

The integral in the first member is, by hypothesis, infinitely small compared to $\int ds$. The quantities $(x' - \xi)$, $(y' - \eta)$, $(z' - \zeta)$ $[\equiv \mathbf{M'} - \boldsymbol{\xi}]$ being infinitely small, the last integral of the second member is also infinitely small compared to $\int ds$. Consequently, the quantity

$$\int G\left(\xi, \eta, \zeta, \frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right) ds$$

$$\int G\left(\boldsymbol{\xi}, \frac{d\mathbf{M}}{ds}\right) ds$$

must be at least infinitely small in the second order when

$$\int ds$$

is infinitely small in the first order.

Let us imagine any closed contour σ and, on this contour, any fixed point $M(\xi, \eta, \zeta)$. Let $M_1(x_1, y_1, z_1)$ be a variable point of this contour. I say that the integral

$$\int G\left(\xi, \eta, \zeta, \frac{dx_1}{d\sigma}, \frac{dy_1}{d\sigma}, \frac{dz_1}{d\sigma}\right) d\sigma,$$

$$\int G\left(\boldsymbol{\xi}, \frac{d\mathbf{M_1}}{d\sigma}\right) d\sigma$$

evaluated along this contour, is necessarily equal to 0.

Let us, indeed, imagine that one forms a contour s homothetic[10] to the preceding one, the center of homothecy[11] being at μ and the ratio of homothecy[12] having the value $\frac{1}{\lambda}$, the quantity λ being able to grow without limit.

The contour s is infinitely small.

If we note that, to the homologous points[13] of the two homothetic curves, the tangents to these two curves are parallel; if we denote by $M(x, y, z)$ the point of the contour s homologous to the point $M_1(x_1, y_1, z_1)$ of the contour σ; if ds and $d\sigma$ are the homologous elements of these two contours, we will have

$$\begin{aligned} \frac{dx_1}{d\sigma} &= \frac{dx}{ds}, \\ \frac{dy_1}{d\sigma} &= \frac{dy}{ds}, \\ \frac{dz_1}{d\sigma} &= \frac{dz}{ds}, \\ d\sigma &= \lambda ds. \end{aligned}$$

$$\begin{aligned} \frac{d\mathbf{M_1}}{d\sigma} &= \frac{d\mathbf{M}}{ds}, \\ d\sigma &= \lambda ds \end{aligned}$$

Putting

$$\int G\left(\xi, \eta, \zeta, \frac{dx_1}{d\sigma}, \frac{dy_1}{d\sigma}, \frac{dz_1}{d\sigma}\right) d\sigma = A,$$

$$\int G\left(\boldsymbol{\xi}, \frac{d\mathbf{M_1}}{d\sigma}\right) d\sigma = A$$

we will have the two equations

$$\begin{aligned} \int G\left(\xi, \eta, \zeta, \frac{dx_1}{d\sigma}, \frac{dy_1}{d\sigma}, \frac{dz_1}{d\sigma}\right) d\sigma &= \lambda \int G\left(\xi, \eta, \zeta, \frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right) ds, \\ \int d\sigma &= \lambda \int ds; \end{aligned}$$

10 [See Hadamard (2008, 145) for the definition of homothecy (*homothétie*) and related terms.]

11 [*centre d'homothétie*]

12 [*rapport d'homothétie*]

13 ["*Points homologues*" "is the name given to pairs of corresponding points in the two figures." (Hadamard, 2008, 50).]

$$\int G\left(\boldsymbol{\xi}, \frac{d\mathbf{M_1}}{d\sigma}\right) d\sigma = \int G\left(\boldsymbol{\xi}, \frac{d\mathbf{M}}{ds}\right) ds$$
$$\int d\sigma = \lambda \int ds$$

from which one deduces, by replacing

$$\int G\left(\xi, \eta, \zeta, \frac{dx_1}{d\sigma}, \frac{dy_1}{d\sigma}, \frac{dz_1}{d\sigma}\right)$$

$$\int G\left(\boldsymbol{\xi}, \frac{d\mathbf{M_1}}{d\sigma}\right) d\sigma$$

by A,

$$\int G\left(\xi, \eta, \zeta, \frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right) ds = \frac{A}{\int d\sigma} \int ds.$$

$$\int G\left(\boldsymbol{\xi}, \frac{d\mathbf{M_1}}{d\sigma}\right) d\sigma = \frac{A}{\int d\sigma} \int ds$$

According to this equation, the integral of the first member would be, contrary to what it should be, on the order of $\int ds$.

We are thus obliged to admit that the integral

$$\int G\left(\xi, \eta, \zeta, \frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right) ds,$$

$$\int G\left(\boldsymbol{\xi}, \frac{d\mathbf{M}}{ds}\right) ds$$

in which (ξ, η, ζ) is a fixed point of *any* closed contour over which the integral extends and (x, y, z) a variable point of the same contour, is equal to 0.

According to the fundamental proposition demonstrated in the previous paragraph, it is necessary and sufficient that a uniform, finite, and continuous functions of x, y, z exists, such that one have

$$G\left(\xi,\eta,\zeta,\frac{dx}{ds},\frac{dy}{ds},\frac{dz}{ds}\right)=\frac{\partial F}{\partial x}\frac{dx}{ds}+\frac{\partial F}{\partial y}\frac{dy}{ds}+\frac{\partial F}{\partial z}\frac{dz}{ds}.$$

$$G\left(\boldsymbol{\xi},\frac{d\mathbf{M}}{ds}\right)=\nabla F\cdot\frac{d\mathbf{M}}{ds}$$

The first member not depending on x, y, z, it must be the same for the second. Thus, the quantities $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial z}$ $[\equiv \nabla F]$ must be any simple functions of ξ, η, ζ $[\equiv \boldsymbol{\xi}]$. We should thus have

$$\begin{aligned}G\left(\xi,\eta,\zeta,\frac{dx}{ds},\frac{dy}{ds},\frac{dz}{ds}\right)=&P(\xi,\eta,\zeta)\frac{dx}{ds}\\&Q(\xi,\eta,\zeta)\frac{dy}{ds}\\&R(\xi,\eta,\zeta)\frac{dz}{ds};\end{aligned}$$

$$G\left(\boldsymbol{\xi},\frac{d\mathbf{M}}{ds}\right)=\mathbf{P}(\boldsymbol{\xi})\cdot\frac{d\mathbf{M}}{ds}$$

and, consequently, ξ, η, ζ $[\equiv \boldsymbol{\xi}]$ being anything,

$$\begin{aligned}G\left(x,y,z,\frac{dx}{ds},\frac{dy}{ds},\frac{dz}{ds}\right)=&P(x,y,z)\frac{dx}{ds}\\&Q(x,y,z)\frac{dy}{ds}\\&R(x,y,z)\frac{dz}{ds}.\end{aligned}$$

$$G\left(\mathbf{M},\frac{d\mathbf{M}}{ds}\right)=\mathbf{P}(\mathbf{M})\cdot\frac{d\mathbf{M}}{ds}$$

Bertrand's proposition is thus demonstrated.

Chapter 2
Stokes's and Ampère's Theorems[1]

2.1 Some definitions and lemmas of Geometry

We are going to examine, in the present Chapter, a new general property of curvilinear integrals; but this study will be preceded by a statement of some definitions and a presentation of some lemmas of general Geometry.

Let AB, CD (Figure 2.1) be two rays that do not intersect and form right angles with each other.

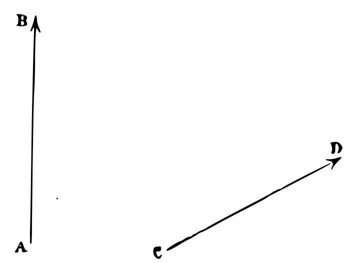

Fig. 2.1 [Two non-intersecting, perpendicular rays AB, CD]

Suppose that an observer, placed according to AB and viewing the point C, sees the ray CD directed toward his left; an observer, placed according to CD and viewing the point A, would then see the ray AB also directed toward his left. In these conditions, the system of the two directions AB, CD forms a system whose *sense of rotation is positive*. In the inverse conditions, the sense of rotation is negative.

This definition extends to two rays that are not perpendicular. The sense of rotation of the system of two rays AB, CD (Figure 2.2) will be, by definition, the sense of rotation of the system formed by the ray AB, and by the ray C*d*, the projection of CD on a plane perpendicular to AB.

[1] Several parts of this Chapter are extracted, almost verbatim, from the remarkable Work of Carl Neumann: *Die elektrischen Kräfte*. Leipzig, 1873 [Neumann (1898)]

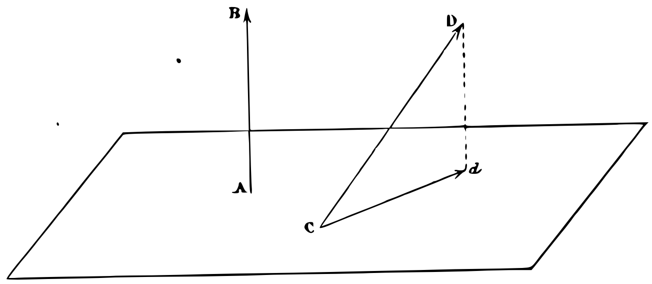

Fig. 2.2 [Two non-perpendicular rays AB, CD with projected ray C*d*]

Consider a circle (Figure 2.3) and a ray AB, normal to the plane of this circle and originating from its center. The side of the plane of this circle where the ray

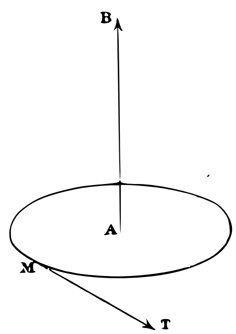

Fig. 2.3 [Defining the top side of a plane]

AB is found is called the *top side* of this plane. The circumference of this circle will be traversed *in a positive direction* if the tangent MT, directed in the direction of traversal, forms with AB a system with positive rotation. Ones sees that, if an observer standing on the top side of the plane goes on the circumference and traverses it in the positive direction, the area of the circle would be on his left.

A direction of traversal being chosen on the circumference of a circle, one will always be able to make this direction of traversal positive, by conveniently choosing the top side of the plane. The side of the plane that it is necessary to choose for the top side is called the *positive side*. One sees that an observer that lies along the tangent MT to the circumference of the circle, in the chosen direction of traversal, and who would see the center of the circle, would have the positive side on his left.

This definition can be extended to a very general class of closed curves.

Let us consider a closed curve that verifies the following requirements:

1. Traversing the curve in the chosen direction, one does not pass by the same point twice, and one cannot return to the point of departure without having traversed all the intermediate points.

2. Through the curve C, one can make a surface S pass such that the curve C forms, on this surface, the *contour of a closed and linearly connected area A*.
   These latter words require some explanation.
   The closed area A, having C for its contour, is called *linearly connected* when any two points M, M′, belonging to the area A, can be joined by a line situated entirely in the area A and not intersecting the curve C.
3. In each point M, the area A admits one and only one tangent plan whose orientation varies in a continuous manner when the point M moves on the area A.
4. The area A is a *two-sided* surface. This latter word requires some definitions.

Let M be a point of the area A; let MN be a ray normal to this plan, and invariably linked to this plane.

Let us displace the point M on the surface of area A. It drags with it the tangent plane and the normal MN, which is displaced with a continuous movement.

If, according to a certain displacement on the area A, the point M returns to its original position, the tangent plane will also reassume its original position. But, for the normal MN, two cases can occur:

Either the ray MN regains its original position, whatever the displacement of the point M be. One then says that *the area* A *is two-sided*.

Or, for certain conveniently chosen displacements of the point M, the ray MN will come to coincide, not with its original direction, but with the opposite direction. One then says that *the area* A *is single-sided*.

One easily makes a similar surface by taking a rectangular band ABCD (Figure 2.4) of paper and gluing the ends such that the point A comes to the point D and

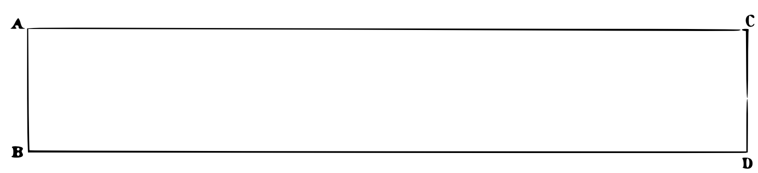

Fig. 2.4 [Strip of paper to be made into a Möbius strip]

the point B to the point C. One thus obtains the following surface (Figure 2.5).

It is easy to see that, if one makes the point M follow the path MPQRSM, the ray MN will come back following MN′.

Certain minimal surfaces even supply some remarkable examples of single-sided areas.

We will suppose that the area A is a two-sided area.

On the curve C (Figure 2.6) we choose a direction of traversal and propose, with respect to this direction of traversal, to define the *positive face* of the area A.

We take, on the curve C, a point M, and at this point draw the tangent MT to this curve in the chosen direction of traversal. We take on the area A a point M′, infinitely close to the point M, and, at M′, draw the normal M′N to the area A in a

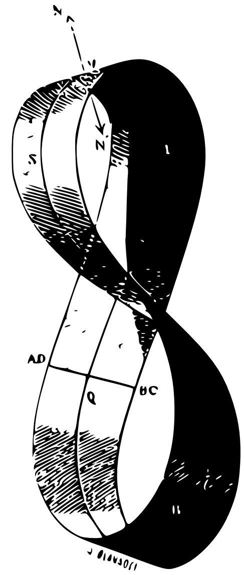

Fig. 2.5 [Möbius strip]

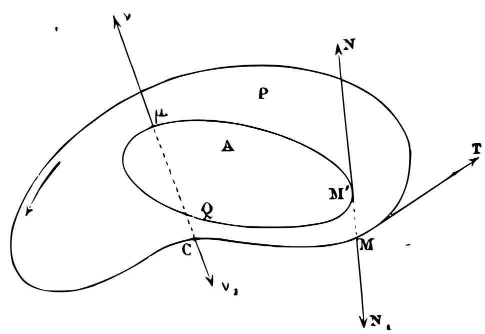

Fig. 2.6 [curve C]

direction such that the system of the two lines MT, M′N forms a system whose sense of rotation is positive.

This done, if we move the point M′ on the area A, we will be able to bring it successively to coincide with each of the points μ of this area, because this area is linearly connected by hypothesis.

If we bring the point M′ to the point μ by a determinate path M′Pμ, the ray M′N will vary continuously, so as to occupy a perfectly determinate position μν.

Firstly, one can show that the line M′N will again situate itself according to μν, if the point M′ comes to the point M by another path M′Qμ.

Indeed, the plane tangent to the point μ in the area A being unique, the line M′N can only assume the orientation μν or the directly opposite orientation $\mu\nu_1$. We suppose that, when the point M′ comes to μ, following the path M′Qμ, the line M′N will position itself according to μν. Conversely, the point μ coming to M′ along the path μQM′, the line $\mu\nu_1$ will position itself according to M′N, and the line μν according to the direction $M'N_1$ directly opposite to M′N.

That posed, we imagine that we will follow to the point M′ the closed path M′PμQM′. One sees that the line M′N will come, according to this traversal, to situate itself according to $M'N_1$, which is impossible, because the area is, by hypothesis, a two-sided area.

Secondly, one can prove that the direction μν, thus determined on the normal at μ, remains the same, whatever the position of the point M is on the curve C.

We suppose, indeed (Figure 2.7), that instead of initially choosing the system with a positive sense of rotation, formed by the tangent MT and the normal M′N, one had chosen the system with a positive sense of rotation formed by the tangent *mt* and the normal *m*′*n*.

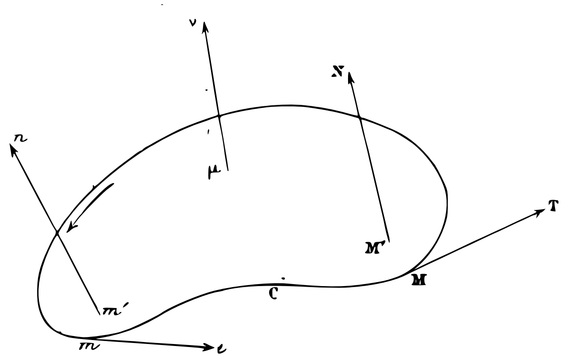

Fig. 2.7 [system with positive rotation formed by tangent *mt* and normal *m*′*n*]

In whatever way that the point M′ is brought to the point μ of the area A, the ray M′N will assume a determinate direction μν.

Now one can suppose that the point M′ is moved to the point μ by the following route:

1. The point M goes to the point *m* following the curve C, which is always possible, because any two points of the curve C are assumed to be always linked by this line. The point M′ comes at the same time to the point *m*′, resting infinitely close to M.
   The tangent MT will coincide with the tangent *mt*. The line M′N remains perpendicular to MT, and the system formed by these two lines unceasingly keeps a positive sense of rotation. Thus M′N will coincide with *m*′*n*.
2. We bring the point M′ from *m*′ to μ. M′N comes to μν; *m*′*n*, which coincides with M′N, also necessarily comes to μν. We so obtain, by the ray normal to the point μ, the same direction μν, as we took the point *m* or the point M as the point of departure.

We have so defined, unambiguously, a certain side of the area A limited by the curve C. This side is called the *positive face of the area A*.

According to what we have said, this positive face is always recognizable by the following characteristics:

1. An observer, lying along the tangent MT to the curve C in the direction of traversal of this curve and viewing the part close to the area A, has the positive face of the area A *on his left*;
2. An observer, standing on the positive face of the area A, in the vicinity of the curve C, and viewing the nearby parts of the curve C, marks, *by his left hand*, the direction of traversal of this curve.

We consider three rays, OA, OB, OC (Figure 2.8), originating from the same point O, and forming a perfectly defined trihedron. They pierce at A, B, C a spherical surface having O as its center. Let OMN be a ray, inside the trihedron, piercing the surface of the sphere at M. Let ABC be a circle traced on the surface of the sphere, and passing through the points A, B, C, this circle divides the sphere into two calottes[2], one of which, MABC, contains the point M. Suppose the circle ABC is traversed in the direction indicated by the letters. If MN marks the positive face of the calotte MABC, one says that *the trihedron* OABC *has a positive sense of rotation*. If, on the contrary, as occurs in (Figure 2.8), MN marks the negative face of the same calotte, one says that the trihedron OABC has a negative sense of rotation.

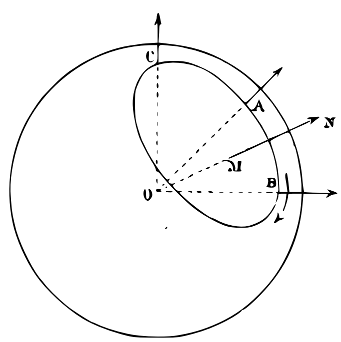

Fig. 2.8 [Sphere divided into two calottes]

When the trihedron OABC has a positive sense of rotation, one easily sees that, if an observer is placed along OA and views OB, the ray OC will be on his left.

We will assume, conforming to usage, that the trihedron Ox, Oy, Oz, formed by the positive directions of the coordinate axes, *have a negative sense of rotation*.

We will look for some analytic characteristics that permit us to recognize the sign of the sense of rotation of a trihedron or of a pair of lines.

Let us first consider a trihedron.

[2] ["*Calottes sphériques*" are the two regions that a circle dividing a sphere produces (Hadamard, 1901, 151).]

If we suppose that one continuously varies the orientation of the three rays that form the trihedron, without at any moment these three rays situating themselves in the same plane, it is easy to see that the sign of the trihedron will not change.

By such a displacement, we will be able to bring the trihedron OABC to be trirectangular; then the two lines OA, OB to coincide respectively with Ox, Oy. OC will then be situated along Oz if the trihedron OABC is negative, and along Oz' if this trihedron is positive.

That posed, let us adopt the following notations for the angles of the rays OA, OB, OC with the axes:

	Ox	Oy	Oz
OA	α_1	β_1	γ_1
OB	α_2	β_2	γ_2
OC	α_3	β_3	γ_3

and consider the determinant

$$\Delta = \begin{vmatrix} \cos\alpha_1 & \cos\beta_1 & \cos\gamma_1 \\ \cos\alpha_2 & \cos\beta_2 & \cos\gamma_2 \\ \cos\alpha_3 & \cos\beta_3 & \cos\gamma_3 \end{vmatrix}.$$

This determinant varies continuously with the orientation of the rays OA, OB, OC; it only becomes equal to 0 if the three rays are placed in the same plane.

Suppose the trihedron OABC is positive; we can, without at any moment the three rays that comprise it being in the same plane, bring it to coincide with the trihedron Oxyz'. The determinant Δ, without even changing sign, will then coincide with the determinant

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix},$$

which is negative; it was thus originally negative.

Suppose, on the contrary, that the trihedron OABC is negative; we will be able, without at any moment the three rays that comprise it being in the same plane, bring it to coincide with the trihedron Oxyz. The determinant Δ, without ever changing sign, will then coincide with the determinant

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix},$$

which is positive; it was thus originally.

Thus the trihedron OABC *has a sense of rotation whose sign is the opposite of that of the determinant*

$$\begin{vmatrix} \cos\alpha_1 & \cos\beta_1 & \cos\gamma_1 \\ \cos\alpha_2 & \cos\beta_2 & \cos\gamma_2 \\ \cos\alpha_3 & \cos\beta_3 & \cos\gamma_3 \end{vmatrix}.$$

Now we consider a pair of two lines PQ, P′Q′ (Figure 2.9). It is easy to see,

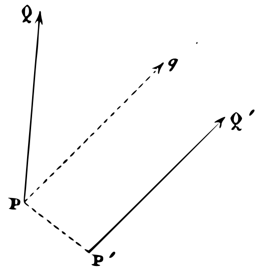

Fig. 2.9 [Determining the sense of rotation of a pair of lines]

according to the given definitions, that the sense of rotation of this pair is identical to the sense fo rotation of the trihedron PQP′*q*, P*q* being a parallel to the direction P′Q′ brought to the point P.

Let
x_0, y_0, z_0 [$\equiv \mathbf{P}$] be the coordinates of point P,
x_0', y_0', z_0' [$\equiv \mathbf{P'}$] be the coordinates of point P′,
α, β, γ [$\equiv \boldsymbol{\alpha}$] be the angles of the line PQ with the axes,
α', β', γ' [$\equiv \boldsymbol{\alpha'}$] be the angles of the line P′Q′ with the axes;
r the distance PP′.

The sign of the trihedron PQP′*q* is, according to what preceded, opposite to that of the determinant

$$\begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{x_0'-x_0}{r} & \frac{y_0'-y_0}{r} & \frac{z_0'-z_0}{r} \\ \cos\alpha' & \cos\beta' & \cos\gamma' \end{vmatrix}.$$

One thus sees that *the sign of the sense of rotation of the system of two lines* PQ, P′Q′ *is identical to the sign of the determinant*

$$\begin{vmatrix} x_0'-x_0 & x_0'-x_0 & x_0'-x_0 \\ \cos\alpha & \cos\beta & \cos\gamma \\ \cos\alpha' & \cos\beta' & \cos\gamma' \end{vmatrix}.$$

Now we imagine a linearly connected plane area A limited by a convex curve C (Figure 2.10). Let $M(x,y,z)$ and $M(x+dx,y+dy,z+dz)$ be two points near the curve C, following the direction of traversal. Let $\mu(\xi,\eta,\zeta)$ be a point inside area A.

At μ, we erect a normal μν on the positive side of the area A. It is easy to see that the normal μν forms a system with a positive sense of rotation with the tangent MT to the curve C at M.

Indeed, we enclose μM. On this line we take a point M′, infinitely close to the point M. It will be inside the curve A, because the area is assumed to be convex.

At M′ we draw M′N′ parallel to μν. The line M′N′, being normal to the positive face of A, will form with MT a system whose sens of rotation will be positive.

The same is obviously true of the system μν, MT, the line μν, and the line M′N being parallel, of the same direction, and situated on the same side of MT.

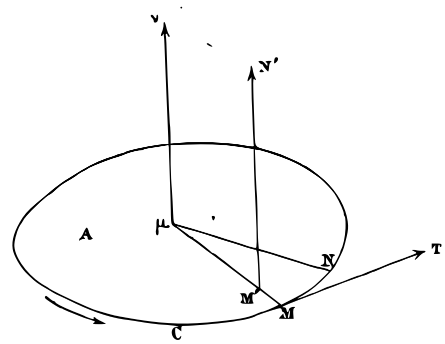

Fig. 2.10 [A linearly connected plane area limited by a convex curve]

Let $a, b, c\ [\equiv \mathbf{a}]$ be the direction cosines of the normal μν. We will have, according to what preceded,

$$\begin{vmatrix} x-\xi & y-\eta & z-\zeta \\ a & b & c \\ \frac{dx}{ds} & \frac{dy}{ds} & \frac{dz}{ds} \end{vmatrix},$$

or, evaluating the determinant,

$$\begin{cases} \quad a\left[(y-\eta)\dfrac{dz}{ds}-(z-\zeta)\dfrac{dy}{ds}\right] \\ +b\left[(z-\zeta)\dfrac{dx}{ds}-(x-\xi)\dfrac{dz}{ds}\right] \\ +c\left[(x-\xi)\dfrac{dy}{ds}-(y-\eta)\dfrac{dx}{ds}\right]<0. \end{cases} \tag{α}$$

$$\mathbf{a}\cdot\frac{d\mathbf{M}}{ds}\times(\mathbf{M}-\boldsymbol{\xi})<0$$

But, on the other hand, one has

$$\begin{cases} a=k\left[(y-\eta)\dfrac{dz}{ds}-(z-\zeta)\dfrac{dy}{ds}\right], \\ b=k\left[(z-\zeta)\dfrac{dx}{ds}-(x-\xi)\dfrac{dz}{ds}\right], \\ c=k\left[(x-\xi)\dfrac{dy}{ds}-(y-\eta)\dfrac{dx}{ds}\right], \end{cases} \tag{β}$$

$$\mathbf{a} = k\frac{d\mathbf{M}}{ds} \times (\mathbf{M} - \boldsymbol{\xi})$$

with the condition

$$a^2 + b^2 + c^2 = 1.$$

$$|\mathbf{a}| = 1$$

According to the equations (β) themselves, this becomes

$$k^2 \left\{ \left[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\right] \left[\left(\frac{dx}{ds}\right)^2 + \left(\frac{dx}{ds}\right)^2 + \left(\frac{dx}{ds}\right)^2 \right] - \left[(x-\xi)\frac{dx}{ds} + (y-\eta)\frac{dy}{ds} + (z-\zeta)\frac{dz}{ds} \right]^2 \right\} = 1,$$

$$k^2 \left\{ |\mathbf{x} - \boldsymbol{\xi}|^2 \left|\frac{d\mathbf{M}}{ds}\right|^2 - \left[(\mathbf{x} - \boldsymbol{\xi}) \cdot \frac{d\mathbf{M}}{ds} \right]^2 \right\} = 1$$

or, according to a known relation,

$$k^2 \frac{4\delta^2}{ds^2} = 1,$$

δ being the surface of the triangle MμN.

Thus if we denote by ε a quantity equal to $+1$ or -1, we will be able to write the equations (β):

$$a = \varepsilon \frac{(y-\eta)dz - (z-\zeta)dy}{2\delta},$$
$$b = \varepsilon \frac{(z-\zeta)dx - (x-\xi)dz}{2\delta},$$
$$c = \varepsilon \frac{(x-\xi)dy - (y-\eta)dx}{2\delta}.$$

Plugging these values into equation (α), we see that ε necessarily has the value -1, and we find at last the relations

$$\begin{cases} 2a\delta = -[(y-\eta)dz - (z-\zeta)dy], \\ 2b\delta = -[(z-\zeta)dx - (x-\xi)dz], \\ 2c\delta = -[(x-\xi)dy - (y-\eta)dx]. \end{cases} \tag{γ}$$

We formulate, for all the elements $\mathrm{MM}' = ds$ of the curve C, the equations analogous to the first of the equations (γ), and add them member by member. We will have

$$2a\sum\delta = \int(z\,dy - y\,dz) + \eta\int dz - \zeta\int dy.$$

Now the quantities $\int dy$ and $\int dz$, which represent the projections of the closed curve C on O*y* and on O*z*, are equal to 0, and one also finds the equation

$$2a\sum\delta = \int(z\,dy - y\,dz),$$

which one can also transform by noting that

$$\sum\delta = \Omega,$$

is the area enclosed by the curve C.

To demonstrate this equations, we supposed the curve C is convex. But it is easy to extend this demonstration to the case where the curve C is not convex.

Take, for example, the non-convex planar area A surrounded by the curve ABCDA (Figure 2.11). It is the excess of the convex area A, surrounded by the

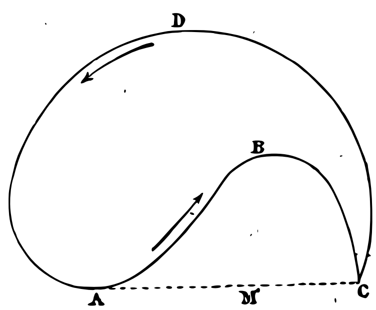

Fig. 2.11 [Non-convex area A surrounded by ABCDA]

curve AMCDA on the convex area A_2 surrounded by ABCMA. If Ω, Ω_1, Ω_2 are the values of the areas A, A_1, A_2, we will have

$$\Omega = \Omega_1 - \Omega_2.$$

The area A_1 has the same positive face as the area A; a thus has the same value for these two areas, and will be able to write

$$2a\Omega_1 = \int_{\text{AMC}} (z\,dy - y\,dz) + \int_{\text{CDA}} (z\,dy - y\,dz).$$

The positive face of the area A_2 coincides with the negative face of the area A. The normal to the positive face of the area A_2 thus has $-a$, $-b$, $-c$ for direction cosines, and one has

$$-2a\Omega_2 = \int_{\text{ABC}} (z\,dy - y\,dz) + \int_{\text{CMA}} (z\,dy - y\,dz) = 0,$$

and we will have

$$2a\Omega = \int_{\text{ABCDA}} (z\,dy - y\,dz),$$

which is the formula already obtained for a convex curve.

Let x, y, z be the coordinates of a point that traverses a closed planar curve C, *in a given direction; let Ω be the area enclosed by this curve; finally, let* (N, x), (N, y), (N, z) *be the angles that the normal to the positive face of these area makes with the axes. We have*

$$\begin{cases} 2\Omega\cos(\text{N}, x) = \int_{\text{C}} (z\,dy - y\,dz), \\ 2\Omega\cos(\text{N}, y) = \int_{\text{C}} (x\,dz - z\,dx), \\ 2\Omega\cos(\text{N}, z) = \int_{\text{C}} (y\,dx - x\,dy). \end{cases} \tag{2.1}$$

These equations are going to serve us in the demonstration of the important theorem which is the object of the following section.

2.2 Stokes's Theorem

Consider a closed, planar, infinitely small curve C endowed with a direction of traversal.

Let $U(x, y, z)$, $V(x, y, z)$, $W(x, y, z)$ $[\equiv \mathbf{U}]$ be three functions of x, y, z that are uniform, finite, and continuous, along with their first-order partial derivatives in a domain inside of which the curve C is situated. We are going to transform the integral

$$\int_{\text{C}} (U\,dx + V\,dy + W\,dz).$$

$$\int_{\text{C}} \mathbf{U} \cdot d\mathbf{M}$$

Let $\mu(\xi, \eta, \zeta)$ be a point inside the area bounded by the curve C. We will have

$$\begin{aligned}U(x,y,z) = U(\xi,\eta,\zeta) &+ (x-\xi)\frac{\partial}{\partial \xi}U(\xi,\eta,\zeta)\\ &+ (y-\eta)\frac{\partial}{\partial \eta}U(\xi,\eta,\zeta)\\ &+ (z-\zeta)\frac{\partial}{\partial \zeta}U(\xi,\eta,\zeta)\end{aligned}$$

$$U(x,y,z) = U(\xi,\eta,\zeta) + (\mathbf{M}-\boldsymbol{\xi})\cdot\nabla_{\boldsymbol{\xi}}U$$

and, consequently,

$$\begin{aligned}\int_{\mathrm{C}} U(x,y,z)dx = U(\xi,\eta,\zeta)\int_{\mathrm{C}} dx &\\ + \frac{\partial}{\partial \xi}U(\xi,\eta,\zeta)\int_{\mathrm{C}}(x-\xi)dx&\\ + \frac{\partial}{\partial \eta}U(\xi,\eta,\zeta)\int_{\mathrm{C}}(y-\eta)dx&\\ + \frac{\partial}{\partial \zeta}U(\xi,\eta,\zeta)\int_{\mathrm{C}}(z-\zeta)dx.&\end{aligned}$$

$$\int_{\mathrm{C}} U(x,y,z)dx = U(\xi,\eta,\zeta)\int_{\mathrm{C}} dx + \nabla_{\boldsymbol{\xi}}U\cdot\int_{\mathrm{C}}(\mathbf{M}-\boldsymbol{\xi})\,dx$$

We have, according to the fundamental theorem of curvilinear integrals (section 1.2),

$$\int_{\mathrm{C}} dx = 0,$$

$$\int_{\mathrm{C}} x\,dx = \int_{\mathrm{C}} d\left(\frac{x^2}{2}\right) = 0.$$

One then sees that the preceding equality can be written

$$\left\{\begin{aligned}\int_{\mathrm{C}} U(x,y,z)dx = \frac{\partial}{\partial \eta}U(\xi,\eta,\zeta)\int_{\mathrm{C}} y\,dx&\\ + \frac{\partial}{\partial \zeta}U(\xi,\eta,\zeta)\int_{\mathrm{C}} z\,dx.&\end{aligned}\right. \tag{α}$$

But one has, according to the fundamental property of curvilinear integrals,

$$\int_{\mathrm{C}}(y\,dx + x\,dy) = \int_{\mathrm{C}} d(xy) = 0,$$

and, according to the last equation (2.1),

$$\int_{\mathrm{C}} (y\,dx - x\,dy) = 2\Omega\cos(\mathrm{N}, z).$$

Thence, one easily concludes

$$\int_{\mathrm{C}} z\,dx = -\Omega\cos(\mathrm{N}, y), \qquad \int_{\mathrm{C}} x\,dz = \Omega\cos(\mathrm{N}, y).$$

Equation (α) thus becomes

$$\int_{\mathrm{C}} U(x, y, z)dx = \Omega\left[\cos(\mathrm{N}, z)\frac{\partial}{\partial\eta}U(\xi, \eta, \zeta) - \cos(\mathrm{N}, y)\frac{\partial}{\partial\zeta}U(\xi, \eta, \zeta)\right].$$

Adding member by member this equation and two other analogous ones obtained in the same way, one arrives at the identity

$$\left\{\begin{aligned} &\int_{\mathrm{C}} [U(x, y, z)dx + V(x, y, z)dy + W(x, y, z)dz] \\ &\quad = \Omega\Bigg\{\left[\cos(\mathrm{N}, z)\frac{\partial}{\partial\eta}U(\xi, \eta, \zeta) - \cos(\mathrm{N}, y)\frac{\partial}{\partial\zeta}U(\xi, \eta, \zeta)\right] \\ &\qquad + \left[\cos(\mathrm{N}, x)\frac{\partial}{\partial\zeta}V(\xi, \eta, \zeta) - \cos(\mathrm{N}, z)\frac{\partial}{\partial\xi}V(\xi, \eta, \zeta)\right] \\ &\qquad + \left[\cos(\mathrm{N}, y)\frac{\partial}{\partial\xi}W(\xi, \eta, \zeta) - \cos(\mathrm{N}, x)\frac{\partial}{\partial\eta}W(\xi, \eta, \zeta)\right]\Bigg\}. \end{aligned}\right. \tag{2.2}$$

This identity can be extended to any curve, if one can pass by this curve an area verifying all the necessary conditions so that one define the positive face. This extension rests upon a lemma that we will establish.

Consider a two-sided area *a* (Figure 2.12). Let ABCDA be the contour that limits

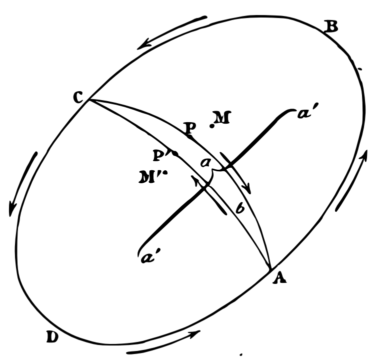

Fig. 2.12 [Two-sided area *a*]

it, with its direction of traversal. Join the point A to the point C by two infinitely close paths APC, AP′C, which have no common points besides A and C, and include between them an infinitely narrow area *b* contained in the considered area *a*.

If, from the considered area *a* one removes this infinitely narrow area *b*, an area *a*′ remains, whose contour is either ABCPAP′CDA or ABCP′APCDA, according to how the letters P and P′ are placed. Suppose these letters are placed such that the contour in question is traversed in the direction indicated by the first series of letters.

I say that this contour in question limits not only a single linearly connected area, but two distinct linearly connected area, such that it is impossible to pass a point of the one and a point of the author by a path situated entirely on the total considered area *a*′ and not encounter the contour.

To demonstrate it, first I note that *any area traced inside a two-sided area is a two-sided area*.

Indeed, let A be a two-sided area (Figure 2.13); A′ an area traced inside of it; let P and P′ be two points of the area A′; suppose that one departs from point P with a given orientation of the normal to the area A′, and that one arrives at the point P′ with an orientation normal to the area A′ that depends on the path traced on the area A′ which one has followed; it is to admit that, departing from the point P of the area A with a different normal orientation to the area A, along the path, traced on the area A, which one would have followed; it is to admit, in other words, that, contrary to the hypothesis, the area A would not be a two-sided area.

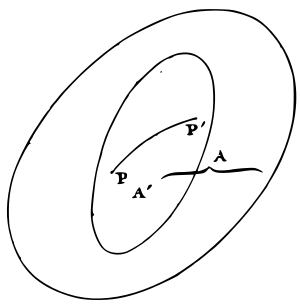

Fig. 2.13 [Two-sided area A]

According to the proposition we have just established, the area *a*′ (Figure 2.12), if it forms a single linearly connected area, should be, like the area *a*, a two-sided area; furthermore, if one observes that these two areas have a part of their contour and the direction of traversal on this part of the contour in common, one easily sees that their positive sides coincide at all points.

Now, we take two points P, P′ infinitely close on the paths CPA, AP′C. Let M be a point of the area *a*’, which can be brought to the point P by an infinitely small path situated on the area *a*′; let M′ be a point of the area *a*′ which can be brought to the point P′ by an infinitely small path situated on the area *a*′.

The normal to the positive face of the area a' at M forms a system of positive rotation with the tangent at P to the path CPA; the normal to the positive face of the area a' at M′ forms a system of positive rotation with the tangent at P′ to the path AP′C. Now the tangents at P and P′ to the paths CPA, AP′C are noticeably in opposite directions.

On the other hand, from the point M to the point M′, one can pass, according to the hypotheses made, along an infinitely small path MPP′M′ on the area a'. Thus, according to the hypotheses made on this latter, the normals to the positive face of A at M and at M′ are noticeably in the same direction, a result that contradicts the preceding one.

One cannot suppose that the ling ABCPAP′CDA forms the contour of a linearly connected area a'. Moreover, it cannot be decomposed into more than two closed curves and thus cannot bound more than two linearly connected areas.

One must necessarily suppose that the following theorem is exact:

Given a two-sided linearly connected area a *bounded by the curve ABCDA, take two points* A, C*, on this curve; join them by a path APC, traced on the given area, and do not twice pass by the same point; the two contours* ABCPA, CPADC *will each bound a linearly connected two-sided area, whose positive face will coincide with the area* a.

Consider the integral

$$\int (U\,dx + V\,dy + W\,dz),$$

$$\int \mathbf{U} \cdot d\mathbf{M}$$

evaluated along the contour ABCD. Denote it by

$$[\mathrm{ABCD}].$$

We will have

$$[\mathrm{ABCD}] = [\mathrm{ABC}] + [\mathrm{CDA}].$$

We note that obviously

$$[\mathrm{CPA}] + [\mathrm{APC}] = 0,$$

and we will have

$$[\mathrm{ABCD}] = [\mathrm{ABC}] + [\mathrm{CPA}] + [\mathrm{APC}] + [\mathrm{CDA}].$$

But one has

$$[\mathrm{ABC}] + [\mathrm{CPA}] = [\mathrm{ABCPA}],$$
$$[\mathrm{APC}] + [\mathrm{CDA}] = [\mathrm{APCDA}].$$

On thus has

$$[\text{ABCDA}] = [\text{ABCPA}] + [\text{APCDA}].$$

Thus, one can add the following proposition to the preceding theorem:
The integral

$$\int (U\,dx + V\,dy + W\,dz),$$

$$\int \mathbf{U} \cdot d\mathbf{M}$$

taken along the contour ABCDA, *is equal to the sum of the analogous integrals taken along the contours* ABCPA, ABCDA.

Demonstrating the accuracy of these theorems for two-sided areas mattered because they are not correct for single-sided areas; if one cuts the surface represented by (Figure 2.5) along AB, one does not separate it into two areas; one forms a single area, applicable to the rectangle ABCD (Figure 2.4) that served to form the surface.

One can, on each of the two contours ABCPA, APCDA, take up some demonstrations analogous to the preceding ones, then reason similarly along the areas in which one will have divided those that contain these two contours, and so on indefinitely.

One will thus arrive at justifying the following statement:

With two systems of conveniently drawn lines, divide the area A (Figure 2.14) into surface elements. Suppose the contour γ of each of these elements is traversed

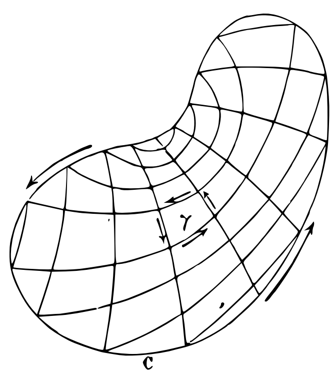

Fig. 2.14 [Area A divided into elements]

in a direction such that this element has the same positive face as the area A. We will have

$$\int_{\mathrm{C}} (U\,dx + V\,dy + W\,dz) = \sum \int_{\gamma} (U\,dx + V\,dy + W\,dz).$$

$$\int_{\mathrm{C}} \mathbf{U} \cdot d\mathbf{M} = \sum \int_{\gamma} \mathbf{U} \cdot d\mathbf{M}.$$

This posed, we note that each of the surface elements that we have just considered can be regarded as a plane element situated in the plane tangent to the surface A at a point of this element; apply to it identity (2.2); add member by member all these identities, and we will have demonstrated the following theorem:

Let x, y, z be the coordinates of a point that describes a closed curve C *in a determinate sense, and let $U(x, y, z)$, $V(x, y, z)$, $W(x, y, z)$ $[\equiv \mathbf{U}]$ be three finite, continuous, and uniform functions of x, y, z, along with their first-order partial derivatives, in the space where the curve* C *is.*

Put a two-sided area A *through the curve* C*; let $d\Omega$ be an element of the area* A*; ξ, η, ζ the coordinates of a point of this element;* N *the direction of the normal of the positive face of the area* A *at the point (ξ, η, ζ) $[\equiv \boldsymbol{\xi}]$.*

One has the identity

$$\left\{\begin{aligned} &\int_{\mathrm{C}} [U(x,y,z)dx + V(x,y,z)dy + W(x,y,z)dz] \\ &\quad = \int_{\mathrm{A}} \left\{ \left[\cos(\mathrm{N},z)\frac{\partial}{\partial\eta}U(\xi,\eta,\zeta) - \cos(\mathrm{N},y)\frac{\partial}{\partial\zeta}U(\xi,\eta,\zeta)\right] \right. \\ &\qquad\quad + \left[\cos(\mathrm{N},x)\frac{\partial}{\partial\zeta}V(\xi,\eta,\zeta) - \cos(\mathrm{N},z)\frac{\partial}{\partial\xi}V(\xi,\eta,\zeta)\right] \\ &\qquad\quad \left. + \left[\cos(\mathrm{N},y)\frac{\partial}{\partial\xi}W(\xi,\eta,\zeta) - \cos(\mathrm{N},x)\frac{\partial}{\partial\eta}W(\xi,\eta,\zeta)\right]\right\} d\Omega. \end{aligned}\right. \tag{1.3}$$

Regrouping the terms on the right-hand side of this identity differently, one can even write

$$\left\{\begin{aligned} &\int_{\mathrm{C}} [U(x,y,z)dx + V(x,y,z)dy + W(x,y,z)dz] \\ &\quad = \int_{\mathrm{A}} \left\{ \left[\frac{\partial}{\partial\zeta}V(\xi,\eta,\zeta) - \frac{\partial}{\partial\eta}W(\xi,\eta,\zeta)\right]\cos(\mathrm{N},x) \right. \\ &\qquad\quad + \left[\frac{\partial}{\partial\xi}W(\xi,\eta,\zeta) - \frac{\partial}{\partial\zeta}U(\xi,\eta,\zeta)\right]\cos(\mathrm{N},y) \\ &\qquad\quad \left. + \left[\frac{\partial}{\partial\eta}U(\xi,\eta,\zeta) - \frac{\partial}{\partial\xi}V(\xi,\eta,\zeta)\right]\cos(\mathrm{N},z)\right\} d\Omega \end{aligned}\right. \tag{1.4}$$

$$\int_{\mathrm{C}} \mathbf{U} \cdot d\mathbf{M} = \int_{\mathrm{A}} \boldsymbol{\nabla} \times \mathbf{U} \cdot d\mathbf{a},$$

where $\mathbf{a} \equiv (\cos(\mathrm{N}, x), \cos(\mathrm{N}, y), \cos(\mathrm{N}, z))$

This identity is due to Stokes.[3] It allows one to transform a simple curvilinear integral, evaluated over a closed curve, into a double integral, evaluated over a closed area limited by this closed curve. It plays a role very analogous to Green's identity, which permits one to transform a double integral, evaluated over a closed surface, into a triple integral, evaluated over the space that contains this surface.

2.3 Ampère's Theorem

A long time before Stokes gave this theorem in its general form, Ampère[4], in his Electrodynamics researches, employed particular propositions related to it.

Let C and C′ be two closed curves (Figure 2.15), through which one passes the two-sided areas A and A′. Let D and D′ be two domains containing within them-

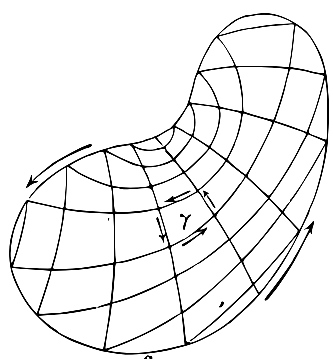

Fig. 2.15 [Two closed curves A and A′]

selves the areas A and A′. Let M(x, y, z) be a point of the domain D and M′(x', y', z') be a point of the domain D′. Finally, let r be the distance of the two points M and M′.

The distance r is susceptible to vary within certain limits. Let $f(r)$ be a function of r that, for all the values of r included between these two limits, is uniform, finite, and continuous, as well as its first-order derivative, its seconder-order derivative being finite.

We propose to transform the curvilinear double integral

$$\int_{\mathrm{C}}\int_{\mathrm{C}'} f(r)\left(\frac{dx}{ds}\frac{dx'}{ds'}+\frac{dy}{ds}\frac{dy'}{ds'}+\frac{dz}{ds}\frac{dz'}{ds'}\right)ds'ds,$$

[3] [See Katz (1979) for a history of Stokes's theorem.]

[4] Ampère, *Mémoire sur la théorie mathématique des phénomènes électrodynamiques, uniquement déduite de l'expérience* (*Mémoires de l'Académie des Sciences*, t. VI, p. 175 [Ampère (2015, 342)]; 1827). *See* also Gauss, *Werke*, Bd. V, p. 606 and 625.

$$\int_{\mathrm{C}} \int_{\mathrm{C}'} f(r) \frac{d\mathbf{M}}{ds} \cdot \frac{d\mathbf{M'}}{ds} \, ds' ds$$

Let
$d\Omega$ be an element of the area A;
N the normal to the positive face of the element $d\Omega$;
$d\Omega'$ an element of the area A$'$;
N$'$ the normal to the positive face of the element $d\Omega'$.

According to equation (1.3), we will have

$$\begin{aligned}
&\int_{\mathrm{C}} \left[f(r) \frac{dx}{ds} \frac{dx'}{ds'} + f(r) \frac{dy}{ds} \frac{dy'}{ds'} + f(r) \frac{dz}{ds} \frac{dz'}{ds'} \right] \\
&\quad = \int_{\mathrm{A}} \left\{ \left[\cos\,(\mathrm{N}, z) \frac{\partial f(r)}{\partial y} - \cos\,(\mathrm{N}, y) \frac{\partial f(r)}{\partial z} \right] \frac{x'}{s'} \right. \\
&\qquad + \left[\cos\,(\mathrm{N}, x) \frac{\partial f(r)}{\partial z} - \cos\,(\mathrm{N}, z) \frac{\partial f(r)}{\partial x} \right] \frac{y'}{s'} \\
&\qquad + \left. \left[\cos\,(\mathrm{N}, y) \frac{\partial f(r)}{\partial x} - \cos\,(\mathrm{N}, x) \frac{\partial f(r)}{\partial y} \right] \frac{z'}{s'} \right\} d\Omega.
\end{aligned}$$

Thus if one puts

$$\begin{aligned}
U' &= \cos(\mathrm{N}, z) \frac{\partial f(r)}{\partial y} - \cos(\mathrm{N}, y) \frac{\partial f(r)}{\partial z}, \\
V' &= \cos(\mathrm{N}, x) \frac{\partial f(r)}{\partial z} - \cos(\mathrm{N}, z) \frac{\partial f(r)}{\partial x}, \\
W' &= \cos(\mathrm{N}, y) \frac{\partial f(r)}{\partial x} - \cos(\mathrm{N}, x) \frac{\partial f(r)}{\partial y},
\end{aligned}$$

our double integral will be able to be written

$$\int_{\mathrm{A}} d\Omega \int_{\mathrm{C}} (U'\, dx' + V'\, dy' + W'\, dz').$$

$$\int_{\mathrm{A}} d\Omega \int_{\mathrm{C}} \mathbf{U'} \cdot d\mathbf{M'}$$

Reapplying equation (1.3) will give it the form

$$\int_{\mathrm{A}}\int_{\mathrm{A}'}\left\{\left[\cos(\mathrm{N}',z)\frac{\partial U'}{\partial y'}-\cos(\mathrm{N}',y)\frac{\partial U'}{\partial z'}\right]\right.$$
$$+\left[\cos(\mathrm{N}',x)\frac{\partial V'}{\partial z'}-\cos(\mathrm{N}',z)\frac{\partial V'}{\partial x'}\right]$$
$$\left.+\left[\cos(\mathrm{N}',y)\frac{\partial W'}{\partial x'}-\cos(\mathrm{N}',x)\frac{\partial W'}{\partial y'}\right]\right\}d\Omega\, d\Omega'.$$

Now, if one refers to the meaning of the functions U', V', W', one finds

$$\begin{aligned}
&\cos(\mathrm{N}',y)\frac{\partial W'}{\partial x'}-\cos(\mathrm{N}',z)\frac{\partial V'}{\partial x'}\\
&=[\cos(\mathrm{N},x)\cos(\mathrm{N}',x)+\cos(\mathrm{N},y)\cos(\mathrm{N}',y)+\cos(\mathrm{N},z)\cos(\mathrm{N}',z)]\frac{\partial^2 f}{\partial x\,\partial x'}\\
&\quad-\cos(\mathrm{N},x)\left[\cos(\mathrm{N}',x)\frac{\partial^2 f}{\partial x\,\partial x'}+\cos(\mathrm{N}',y)\frac{\partial^2 f}{\partial y\,\partial x'}+\cos(\mathrm{N}',z)\frac{\partial^2 f}{\partial z\,\partial x'}\right].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{\partial^2 f}{\partial x\,\partial x'}&=-\frac{1}{r}\frac{df}{dr}+\left(\frac{x'-x}{r}\right)^2\left(\frac{1}{r}\frac{df}{dr}-\frac{d^2 f}{dr^2}\right),\\
\frac{\partial^2 f}{\partial y\,\partial x'}&=\frac{(x'-x)(y'-y)}{r^2}\left(\frac{1}{r}\frac{df}{dr}-\frac{d^2 f}{dr^2}\right),\\
\frac{\partial^2 f}{\partial z\,\partial x'}&=\frac{(x'-x)(z'-z)}{r^2}\left(\frac{1}{r}\frac{df}{dr}-\frac{d^2 f}{dr^2}\right).
\end{aligned}$$

We thus have

$$\begin{aligned}
&\cos(\mathrm{N}',y)\frac{\partial W'}{\partial x'}-\cos(\mathrm{N}',z)\frac{\partial V'}{\partial x'}\\
&=[\cos(\mathrm{N},x)\cos(\mathrm{N}',x)+\cos(\mathrm{N},y)\cos(\mathrm{N}',y)+\cos(\mathrm{N},z)\cos(\mathrm{N}',z)]\\
&\times\left[\left(\frac{x'-x}{r}\right)^2\left(\frac{1}{r}\frac{df}{dr}-\frac{d^2 f}{dr^2}\right)-\frac{1}{r}\frac{df}{dr}\right]\\
&+\cos(\mathrm{N},x)\cos(\mathrm{N}',x)\frac{1}{r}\frac{df}{dr}\\
&-\cos(\mathrm{N},x)\frac{x'-x}{r}\left[\cos(\mathrm{N}',x)\frac{x'-x}{r}+\cos(\mathrm{N}',y)\frac{y'-y}{r}+\cos(\mathrm{N}',z)\frac{z'-z}{r}\right]\\
&\times\left(\frac{1}{r}\frac{df}{dr}-\frac{d^2 f}{dr^2}\right)
\end{aligned}$$

and consequently

$$
\begin{aligned}
&\cos(\mathrm{N}',y)\frac{\partial W'}{\partial x'} - \cos(\mathrm{N}',x)\frac{\partial W'}{\partial y'}\\
&+\cos(\mathrm{N}',z)\frac{\partial U'}{\partial y'} - \cos(\mathrm{N}',y)\frac{\partial U'}{\partial z'}\\
&+\cos(\mathrm{N}',x)\frac{\partial V'}{\partial z'} - \cos(\mathrm{N}',z)\frac{\partial V'}{\partial x'}\\
=& -[\cos(\mathrm{N},x)\cos(\mathrm{N}',x)+\cos(\mathrm{N},y)\cos(\mathrm{N}',y)+\cos(\mathrm{N},z)\cos(\mathrm{N}',z)]\\
&\times\left(\frac{1}{r}\frac{df}{dr}+\frac{d^2f}{dr^2}\right)\\
&-\left[\cos(\mathrm{N},x)\frac{x'-x}{r}+\cos(\mathrm{N},y)\frac{y'-y}{r}+\cos(\mathrm{N},z)\frac{z'-z}{r}\right]\\
&\times\left[\cos(\mathrm{N}',x)\frac{x'-x}{r}+\cos(\mathrm{N}',y)\frac{y'-y}{r}+\cos(\mathrm{N}',z)\frac{z'-z}{r}\right]\\
&\times\left(\frac{1}{r}\frac{df}{dr}-\frac{d^2f}{dr^2}\right).
\end{aligned}
$$

If one designates by r the direction that points from the point (x,y,z) to the point (x',y',z') and notes that one has

$$
\begin{aligned}
\cos(\mathrm{N},\mathrm{N}') &= \cos(\mathrm{N},x)\cos(\mathrm{N}',x)+\cos(\mathrm{N},y)\cos(\mathrm{N}',y)+\cos(\mathrm{N},z)\cos(\mathrm{N}',z),\\
\cos(\mathrm{N},r) &= \cos(\mathrm{N},x)\frac{x'-x}{r}+\cos(\mathrm{N},y)\frac{y'-y}{r}+\cos(\mathrm{N},z)\frac{z'-z}{r},\\
\cos(\mathrm{N}',r) &= \cos(\mathrm{N}',x)\frac{x'-x}{r}+\cos(\mathrm{N}',y)\frac{y'-y}{r}+\cos(\mathrm{N}',z)\frac{z'-z}{r},\\
\cos\omega &= \frac{dx}{ds}\frac{dx'}{ds'}+\frac{dy}{ds}\frac{dy'}{ds'}+\frac{dz}{ds}\frac{dz'}{ds'},
\end{aligned}
$$

$$
\begin{aligned}
\cos(\mathrm{N},\mathrm{N}') &= \mathbf{a}\cdot\mathbf{a'}\\
\cos(\mathrm{N},r) &= \mathbf{a}\cdot\frac{\mathbf{M'}-\mathbf{M}}{r}\\
\cos(\mathrm{N}',r) &= \mathbf{a'}\cdot\frac{\mathbf{M'}-\mathbf{M}}{r}\\
\cos\omega &= \frac{d\mathbf{M}}{ds}\cdot\frac{d\mathbf{M'}}{ds'}
\end{aligned}
$$

one will have

$$\left\{\begin{aligned} &\int_{\mathrm{C}}\int_{\mathrm{C}'} f(r)\cos\omega\, ds'\, ds \\ &\quad = -\int_{\mathrm{A}}\int_{\mathrm{A}'} \left[\cos(\mathrm{N},\mathrm{N}')\left(\frac{1}{r}\frac{df}{dr}+\frac{d^2 f}{dr^2}\right)\right. \\ &\qquad\qquad\qquad \left. + \cos(\mathrm{N},r)\cos(\mathrm{N}',r)\left(\frac{1}{r}\frac{df}{dr}-\frac{d^2 f}{dr^2}\right)\right] d\Omega'\, d\Omega. \end{aligned}\right. \tag{2.3}$$

Let us apply this important identity to the case where

$$f(r) = \frac{1}{r}.$$

This functions will satisfy the imposed conditions if the two curves C and C′ and the two-sided areas A and A′ passing through these two curves can be respectively enclosed inside the two domains D and D′ entirely outside of each other; because then the distance r from one point of the domain D to a point of the domain D′ will not become equal to 0. This condition can be stated simply by saying that the two areas A and A′ do not share any point.

We will have

$$\begin{aligned} \frac{df(r)}{dr} &= -\frac{1}{r^2}, \\ \frac{d^2 f(r)}{dr^2} &= \frac{2}{r^3} \end{aligned}$$

and consequently

$$\begin{aligned} \frac{1}{r}\frac{df}{dr}+\frac{d^2 f}{dr^2} &= \frac{1}{r^3}, \\ \frac{1}{r}\frac{df}{dr}-\frac{d^2 f}{dr^2} &= \frac{3}{r^3}. \end{aligned}$$

We will thus have

$$\begin{aligned} &\int_{\mathrm{C}}\int_{\mathrm{C}} \frac{\cos\omega}{r} ds'\, ds \\ &\quad = -\int_{\mathrm{A}}\int_{\mathrm{A}'} \left[\cos(\mathrm{N},r)\cos(\mathrm{N}',r)\frac{d^2\frac{1}{r}}{dr^2}\right. \\ &\qquad\qquad\qquad \left. - \frac{\cos(\mathrm{N},r)\cos(\mathrm{N}',r)-\cos(\mathrm{N},\mathrm{N}')}{r}\frac{d\frac{1}{r}}{dr}\right] d\Omega'\, d\Omega. \end{aligned}$$

But equations (1.5) and (1.6) of (chapter 1) give

$$\cos(\mathrm{N}, r) = -\frac{\partial r}{\partial \mathrm{N}}, \qquad \cos(\mathrm{N}', r) = -\frac{\partial r}{\partial \mathrm{N}'},$$
$$\frac{\cos(\mathrm{N}, r)\cos(\mathrm{N}', r) - \cos(\mathrm{N}, \mathrm{N}')}{r} = \frac{\partial^2 r}{\partial \mathrm{N}\, \partial \mathrm{N}'}.$$

Moreover,

$$\frac{d^2 \frac{1}{r}}{dr^2} \frac{\partial r}{\partial \mathrm{N}} \frac{\partial r}{\partial \mathrm{N}'} + \frac{d \frac{1}{r}}{dr} \frac{\partial^2 r}{\partial \mathrm{N}\, \partial \mathrm{N}'} = \frac{\partial^2 \frac{1}{r}}{\partial \mathrm{N}\, \partial \mathrm{N}'}.$$

Thus we arrive at the following identity:

If ds and ds' are the elements of two closed curves C *and* C′*; if $d\Omega$ and $d\Omega'$ are the elements of areas* A, A′*, passing through these two curves; if ω is the angle of the two elements ds and ds'; if, finally,* N *and* N′ *are the normals to the positive faces of the two elements $d\Omega$ and $d\Omega'$, one has*

$$\int_{\mathrm{C}} \int_{\mathrm{C}} \frac{\cos\omega}{r} ds'\, ds = -\int_{\mathrm{A}} \int_{\mathrm{A}'} \frac{\partial^2 \frac{1}{r}}{\partial \mathrm{N}\, \partial \mathrm{N}'} d\Omega\, d\Omega'. \tag{2.4}$$

We recall that it supposes that the two areas A, A′ do not share any point.

The transformation, made possible by Stokes's theorem, of a curvilinear integral evaluated over a curve and a double integral evaluated over the area that this curve limits, constitutes in Physics a method entirely analogous to the very fertile method, generalized by Green's identity, which consists in transforming an integral evaluated over a closed surface into an integral evaluated over the volume that this surface encloses. Similarly, the transformation that Ampère performed, of a curvilinear double integral into a quadruple integral evaluated over the two areas, constitutes a process analogous to the transformation of a sextuple integral evaluated over two volumes into a quadruple integral evaluated over the surfaces that limit these volumes. The role that this transformation plays in Gauss's theory of capillarity is known.

Part II
On Ampère's Law[5]

[5] [Duhem (1892, 309-332)]

Chapter 3
Ampère's law and demonstration

The various works by which Ampère arrived at formulating the law of action that a closed and uniform current exerts on a uniform current element are summarized in the great Memoir that he published in 1826[1]. Ampère's demonstration rests on six hypotheses and on three experimental laws.

First hypothesis. — *Let* C *be a uniform current that acts on an uniform current element* ds'. *We decompose in our mind the current* C *into elements* ds_1, ds_2, … *The action of the current* C *on the element* ds' *is the resultant of elementary actions exerted by the elements* ds_1, ds_2, … *on the element* ds'.

Second hypothesis. — *The action that the element* ds *exerts on the element* ds' *is a force, applied at a point of the element* ds' *and directed along the line that joins a point of the element* ds *to a point of the element* ds'. *The action of the element* ds' *on the element* ds *is equal and directly opposite of the former.*

Third hypothesis. — *The action of the element* ds *on the element* ds' *depends uniquely on the intensities* J *and* J' *of the currents that traverse the elements* ds *and* ds', *on the length, and on relative position of these two elements.*

From this hypothesis one easily deduces that the force exerted by the element ds on the element ds' is proportional to the product JJ'.

Consider a first element ds_1, traversed by a current of intensity J_1. It exercises on the element ds' a repulsive force that we will represent by $f(J_1, J')$.

Right next to the element ds_1 we place an element ds_2 of the same length, traversed by a current of intensity J_2. It will exert on the element ds' a repulsive action whose expression will only differ from the former by an exchange of the quantities J_1, J_2. This action will have the value $f(J_2, J')$.

The set of the two elements ds_1, ds_2 thus exerts on the element ds' a repulsive force whose value is

$$f(J_1, J') + f(J_2, J').$$

But this set can be regarded as a unique element, of the same length as each of the two

[1] Ampère, *Mémoire sur la théorie mathématique des phénomènes électrodynamiques, uniquement déduite de l'expérience* (*Mémoires de l'Académie des Sciences*, 1826 [Ampère (2015, 342-476)]).

But this set can be regarded as a unique element, of the same length as each of the two former ones, placed like each of the two former ones, and traversed by a current of intensity $(J_1 + J_2)$. The action of this element on the element ds' should thus have the value

$$f(J_1 + J_2, J').$$

Consequently, one has the identity

$$f(J_1, J') + f(J_2, J') = f(J_1 + J_2, J'),$$

an identity that demonstrates that the action of the element ds on the element ds' is proportional to J. One would similarly demonstrate that it is proportional to J' and, consequently, to the product JJ'.

The preceding hypothesis equally proves that the action exerted by the element ds on the element ds' is proportional to the product $ds\, ds'$.

Indeed, imagine that a first element ds_1, traversed by a current of intensity J, exerts on the element ds', traversed by a current of intensity J', a repulsion that we will represent by $f(ds_1, ds')$.

Prolong the element ds_1 with an infinitely small length ds_2. Suppose the element ds_2 is also traversed by a current of intensity J. It will exert on the element ds' a repulsive action whose direction will obviously be the same as the preceding one and whose value will obviously be $f(ds_2, ds')$.

The set of the two elements ds_1, ds_2 thus exerts on the element ds' a repulsive force whose value is

$$f(ds_1, ds_1) + f(ds_2, ds_1).$$

But, on the other hand, the set of these two elements can be considered as a unique element, of length $(ds_1 + ds_2)$ and with the same intensity and position as each of the elements ds_1, ds_2. Its repulsive action on the element ds' can be written

$$f(ds_1 + ds_2, ds_1).$$

Consequently, one has the identity

$$f(ds_1, ds') + f(ds_2, ds_1) = f(ds_1 + ds_1, ds'),$$

an identity which shows that the action of the element ds on the element ds' is proportional to ds; on would similarly show that this action is proportional to ds', and, consequently, to the product $ds\, ds'$.

The propositions that we have just demonstrated lead to the following conclusion: *The action that the element ds, traversed by a current of uniform intensity J, exerts on an element ds', traversed by a uniform current of intensity J', an action considered positive when it is repulsive, has for value*

$$F = JJ'\Phi\, ds\, ds', \tag{3.1}$$

Φ *depending only on the relative position of the two elements* ds, ds', *and not on their length.*

Fourth hypothesis[2]. — The two elements $\mathrm{MM}_1 = ds_1$, $\mathrm{MM}_2 = ds_2$, originating from a same point M, having the same length, and traversed by currents of the same intensity, exert the same action on the element $\mathrm{M'M'_1} = ds'$, if they are symmetric to one another with respect to the plane $\mathrm{MM'M'_1}$.

If one then refers to the considerations expressed above (section 1.1), one sees that this hypothesis entails the following consequence:

The function Φ *is a uniform function of four variables*

$$r, \qquad \cos\theta, \qquad \cos\theta', \qquad \cos\omega,$$
$$\Phi = \varphi(r, \cos\theta, \cos\theta', \cos\omega). \tag{3.2}$$

First experimental law (principle of sinuous currents). — *When a closed and uniform current traverses the contour of a two-sided area, all of whose dimensions are infinitely small, the action of this current on any current element is infinitely small as the product of the length of the element that suffers the action by the area that encompasses the acting circuit.*

It is unnecessary to recall here the classic experiment by which Ampère demonstrated this proposition.

This proposition granted, we will consider two elements AB$= ds$ and $\mathrm{A'B'} = ds'$ (Figure 3.1).

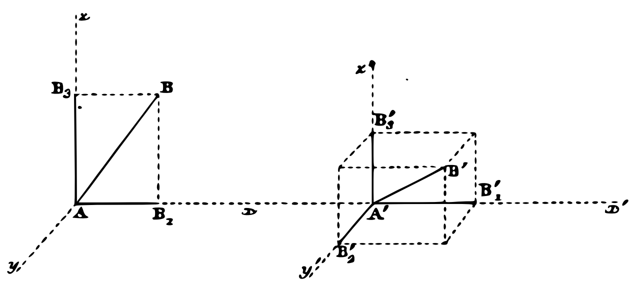

Fig. 3.1 [Elements ds and ds']

We take the line AA′ as the direction of the axes A*x*, A*x*′. In the half-plane BAA′, we take the normal to the line AA′ for the direction of the axes A*z*, A′*z*'. We take the normal to the plane BAA′, from the side of this plane where the element A′B′ is found, for the direction of the axes A*y*, A′*y*′.

[2] This hypothesis, or at least a particular case of this hypothesis, sufficient for demonstration, is specified by Ampère (*Théorie mathématique…*, A. Hermann reprint, p. 20 [Ampère (2015, 357)]) as a *theorem*; but the demonstration of this theorem implies another hypothesis on the mutual action of two perpendicular currents. The hypothetical character of this proposition is quite obvious if one observes that one would err in stating the same proposition after having replaced the element of *current* $\mathrm{M'M'_1}$ with a *magnetic element* $\mathrm{M'M'_1}$.

Let AB_1, AB_2 be the projections of AB on A*x* and on A*z*.

The area AB_1B being infinitely small with respect to ds, the action of a uniform current of intensity J, traversing the circuit AB_1BA, on the element ds', traversed by a current of intensity J', is infinitely small with respect to $JJ'\,ds\,ds'$. The action of the two elements AB_1 and B_1B, reduced to quantities on the order of $JJ'\,ds\,ds'$, amounts to the action of the element AB on the element ds'. The element B_1B itself can be replaced by the element AB_3.

One would prove, by an analogous reasoning, that instead of determining the action of any element on the element $A'B'$, one can determine the actions of the same element on the elements $A'B'_1$, $A'B'_2$, $A'B'_3$ and compose among them these latter actions.

We thus return to evaluate the action of each of the two elements

$$\begin{aligned} AB_1 &= ds\cos\theta, \\ AB_3 &= ds\sin\theta, \end{aligned}$$

on each of the three elements

$$\begin{aligned} A'B'_1 &= ds'\cos\theta', \\ A'B'_2 &= ds'\cos\theta'\sin\varepsilon, \\ A'B'_3 &= ds'\cos\theta'\cos\varepsilon, \end{aligned}$$

By reasoning as in Book 13, Chapter 2,[3] we will prove that one can neglect the action of

$$\begin{aligned} &AB_1 \text{ on } A'B'_2, \\ &AB_1 \text{ on } A'B'_3, \\ &AB_3 \text{ on } A'B'_1, \\ &AB_3 \text{ on } A'B'_2. \end{aligned}$$

If we then designate the repulsive action of AB_1 on $A'B'_1$ by

$$JJ'f(r)\overline{AB_1}.\overline{A'B'_1},$$

we will have

$$F = JJ'\,ds\,ds'[f(r)\cos\theta\cos\theta' + g(r)\sin\theta\sin\theta'\cos\varepsilon], \tag{3.3}$$

or even, noting that one has (equation 1.8),

$$\sin\theta\sin\theta'\cos\varepsilon = \cos\omega - \cos\theta\cos\theta',$$

and upon putting

[3] [Duhem (1892, 94-105)]

$$h(r) = f(r) - g(r),$$

$$F = JJ'\,ds\,ds'[h(r)\cos\theta\cos\theta' + g(r)\cos\omega]. \tag{3.3b}$$

Second experimental law. — *The action that any closed and uniform current exerts on a current element is normal to this element.*

Let ds be an element of the acting circuit. Let ds' be the element on which the action is exerted.

The element ds exerts on the element ds' an action whose component along ds' has the value

$$F\cos\theta'.$$

The entire acting circuit will thus exert on the element ds' an action whose component along the element ds' will have the value

$$\sum F\cos\theta',$$

or, according to equation (3.3),

$$JJ'\,ds'\int[f(r)\cos\theta\cos\theta' + g(r)\sin\theta\sin\theta'\cos\varepsilon]\cos\theta' ds$$

So that the preceding proposition is correct, it is necessary and sufficient that this quantity equal 0.

But we have (equations 1.5 and 1.9)

$$\cos\theta = \frac{\partial r}{\partial s}, \qquad \cos\theta' = \frac{\partial r}{\partial s'},$$
$$\sin\theta\sin\theta'\cos\varepsilon = -r\frac{\partial^2 r}{\partial s\,\partial s'}.$$

Consequently, so that the preceding proposition is correct, it is necessary and sufficient that the integral

$$\int\left[f(r)\frac{\partial r}{\partial s}\frac{\partial r}{\partial ds'} + rg(r)\frac{\partial^2 r}{\partial s\,\partial s'}\right]\frac{\partial r}{\partial s'}ds,$$

evaluated over any closed curve, equal 0.

This equation can also be written in the following way:

$$\int_S\left[f(r)\left(\frac{\partial r}{\partial s'}\right)^2 dr + \frac{1}{2}rg(r)d\left(\frac{\partial r}{\partial s'}\right)^2\right] = 0.$$

If one observes that when one traverses the circuit s, the two quantities r and $\left(\frac{\partial r}{\partial s}\right)^2$ vary in a continous manner, one arrives at the following conclusion:

So that the preceding equality occurs, it is necessary and sufficient that the quantity

$$f(r)\left(\frac{\partial r}{\partial s'}\right)^2 dr + \frac{1}{2} r g(r) d\left(\frac{\partial r}{\partial s'}\right)^2$$

be the total differential of a uniform and continuous function of r and $\frac{\partial r}{\partial s'}$.

This condition is translated by the equality

$$\frac{\partial}{\partial\left(\frac{\partial r}{\partial s'}\right)^2}\left[f(r)\left(\frac{\partial r}{\partial s'}\right)^2\right] = \frac{\partial}{\partial r}\left[\frac{1}{2} r g(r)\right]$$

where

$$2f(r) = \frac{d}{dr}[rg(r)]. \tag{3.4}$$

By virtue of the equation (3.4), equality (3.3) becomes

$$F = JJ'\,ds\,ds'\left\{g(r)\sin\theta\sin\theta'\cos\varepsilon + \frac{1}{2}\frac{d}{dr}[rg(r)]\cos\theta\cos\theta'\right\}. \tag{3.5}$$

Fifth hypothesis. — *The function g(r) is of the form*

$$g(r) = \frac{A}{r^n},$$

A being a constant and n a positive integer.

Equation (3.5) then takes the form

$$F = \frac{AJJ'\,ds\,ds'}{r^n}\left(\sin\theta\sin\theta'\cos\varepsilon - \frac{n-1}{2}\cos\theta\cos\theta'\right). \tag{3.6}$$

Third experimental law. — *In two similar electrodynamic systems, the actions that are exerted on two homologous elements are the same, if the intensities of the currents that traverse the various conductors are the same.*

Let there be two similar electrodynamic systems S and S_1, the similarity ratio[4] of the second to the first being k.

In the first S, the element ds' bears, on the part of the element ds, a repulsive action given by formula (3.6).

In the second, S_1, we will consider the two elements ds_1, ds'_1, homologues of ds, ds'; the element ds'_1 suffers, on the part of the element ds_1, a repulsive force F_1 given by the formula

$$F_1 = \frac{AJJ'\,ds_1\,ds'_1}{r_1^n}\left(\sin\theta_1\sin\theta'_1\cos\varepsilon_1 - \frac{n-1}{2}\cos\theta_1\cos\theta'_1\right). \tag{3.7}$$

But one has

[4] [*rapport de similitude* = "ratio of homothecy" (Hadamard, 2008, 145)]

$$\theta_1 = \theta, \qquad \theta'_1 = \theta', \qquad \varepsilon_1 = \varepsilon,$$
$$ds_1 = k\,ds, \qquad ds'_1 = k\,ds', \qquad r_1 = kr.$$

Formula (3.7), compared to formula (3.6), thus gives

$$F_1 = k^{2-n} F.$$

The elementary actions suffered by an element ds'_1 of the system S_1 thus form a system similar to that of the elementary actions that act on the homologous element ds' of the system S, the similarity ratio being k^{2-n}.

Now these two systems should have resultants equal to one another. It is thus necessary that

$$k^{2-n} = 1$$

or

$$n = 2.$$

This relation, plugged into formula (3.6), gives

$$F = \frac{AJJ'\,ds\,ds'}{r^2}\left(\sin\theta\sin\theta'\cos\varepsilon - \frac{1}{2}\cos\theta\cos\theta'\right). \tag{3.8}$$

Sixth hypothesis. — *Two parallel current elements, in the same direction, perpendicular to the line that joins them, attract.*

In this case, one has

$$\cos\theta = 0, \qquad \cos\theta' = 0,$$
$$\sin\theta = 1, \qquad \sin\theta' = 1, \qquad \cos\epsilon = 1.$$

Formula (3.8) should give a negative value for F. The constant A should have a negative value.

If we put

$$-A = \mathfrak{A}^2,$$

formula (3.8) becomes

$$F = \frac{\mathfrak{A}^2}{2}\frac{JJ'\,ds\,ds'}{r^2}(\cos\theta\cos\theta' - 2\sin\theta\sin\theta'\cos\varepsilon). \tag{3.9}$$

This is, as we saw in Book 14, Chapter 10, equation (7),[5] one of the forms of Ampère's law.

[5] [Duhem (1892, 274)]

Chapter 4
Ampère's law, J. Bertrand's demonstration

Ampère's demonstration relies on using three experimental laws. J. Bertrand[1] showed that the second experimental law that Ampère invoked implies the first, the *principle of sinuous currents*, such that the first experimental law cannot be kept as a principle.

The demonstration that J. Bertrand gave is the following:

Ampère's first four hypotheses entail equations (3.1) and (3.2), i.e., the following proposition:

The repulsive action of the element ds on the element ds' as given by the formula

$$F = JJ' \, ds \, ds' \varphi(r, \cos\theta, \cos\theta', \cos\omega). \tag{4.1}$$

The equations (1.5) and (1.7),

$$\begin{aligned} \cos\theta &= \frac{\partial r}{\partial s}, \\ \cos\theta' &= \frac{\partial r}{\partial s'}, \\ \cos\omega &= -\left(\frac{\partial r}{\partial s}\frac{\partial r}{\partial s'} + r\frac{\partial^2 r}{\partial s \, \partial s'}\right), \end{aligned}$$

allow putting that equation in the form

$$F = JJ' \, ds \, ds' \psi\left(r, \frac{\partial r}{\partial s}, \frac{\partial r}{\partial s'}, \frac{\partial^2 r}{\partial s \, \partial s'}\right). \tag{4.2}$$

We now invoke the second of the experimental laws that Ampère as first principles. We saw that this law is expressed by the following condition: The sum

[1] J. Bertrand, *Sur la lémonstration de la formule qui représente l'action élémentaire de deux courants* (*Comptes rendus*, vol. 75, p. 733; 1872). — *Démonstration des théorèmes relatifs aux actions électrodynamiques* (*Journal de Physique*, 1st series, vol. 3, p. 297; 1874). — *Leçons sur la théorie mathématique de l'Électricité*, professed at the College of France, p. 166. Paris, 1890.

$\sum F \cos\theta'$, extended to all the elements ds of a closed and uniform current, is equal to 0.

By virtue of equation (3.9), this condition can also be stated thus:

The integral

$$\int \psi\left(r, \frac{\partial r}{\partial s}, \frac{\partial r}{\partial s'}, \frac{\partial^2 r}{\partial s\,\partial s'}\right)\frac{\partial r}{\partial s'}ds,$$

evaluated over any closed curve, is equal to 0.

The quantities r and $\frac{\partial r}{\partial s'}$ surely vary continuously when one traverses a curve s, whereas the quantities $\frac{\partial r}{\partial s'}$ and $\frac{\partial^2 r}{\partial s\,\partial s'}$ can vary in any discontinuous manner in the case this curve has angular points. So that the preceding statement be correct, it is necessary and sufficient that one has

$$\begin{cases} \psi\left(r, \frac{\partial r}{\partial s}, \frac{\partial r}{\partial s'}, \frac{\partial^2 r}{\partial s\,\partial s'}\right)\frac{\partial r}{\partial s'}ds \\ \quad = \frac{\partial \Phi\left(r, \frac{\partial r}{\partial s'}\right)}{\partial r}\frac{\partial r}{\partial s}ds + \frac{\partial \Phi\left(r, \frac{\partial r}{\partial s'}\right)}{\partial\left(\frac{\partial r}{\partial s'}\right)}\frac{\partial^2 r}{\partial s'\,\partial s}ds, \end{cases} \tag{4.3}$$

$\Psi\left(r, \frac{\partial r}{\partial s'}\right)$ being a uniform and continuous function of the variables r, $\frac{\partial r}{\partial s'}$ and not depending on the variables $\frac{\partial r}{\partial s}$, $\frac{\partial^2 r}{\partial s\,\partial s'}$.

The second member of identity (4.3) is linear and homogeneous in $\frac{\partial r}{\partial s}$ and $\frac{\partial^2 r}{\partial s\,\partial s'}$. It must be the same for the first member. Thus the function

$$\psi\left(r, \frac{\partial r}{\partial s}, \frac{\partial r}{\partial s}, \frac{\partial^2 r}{\partial s\,\partial s'}\right)$$

is linear and homogeneous in $\frac{\partial r}{\partial s}$, $\frac{\partial^2 r}{\partial s\,\partial s'}$.

The law of the equality of action and reaction, which constitutes Ampère's second hypothesis, immediately leads to this consequence: the function ψ should not change sign when one permutes the letters s and s'. The function ψ is thus also linear and homogeneous in $\frac{\partial r}{\partial s'}$, $\frac{\partial^2 r}{\partial s\,\partial s'}$.

So one should have

$$\psi = A\frac{\partial r}{\partial s}\frac{\partial r}{\partial s'} + B\frac{\partial^2 r}{\partial s\,\partial s'},$$

the two quantities A and B being independent of the variables

$$\frac{\partial r}{\partial s}, \qquad \frac{\partial r}{\partial s'}, \qquad \frac{\partial^2 r}{\partial s\,\partial s'}$$

and, consequently, depending only on the fourth variable of which ψ can depend, the variable r. One thus has

$$\psi = A(r)\frac{\partial r}{\partial s}\frac{\partial r}{\partial s'} + B(r)\frac{\partial^2 r}{\partial s\,\partial s'}$$

or, according to equation (4.2),

$$F = JJ'\,ds\,ds'\left[A(r)\frac{\partial r}{\partial s}\frac{\partial r}{\partial s'} + B(r)\frac{\partial^2 r}{\partial s\,\partial s'}\right]. \tag{4.4}$$

As we have

$$\frac{\partial r}{\partial s} = \cos\theta, \qquad \frac{\partial r}{\partial s'} = \cos\theta', \qquad \frac{\partial^2 r}{\partial s\,\partial s'} = -\frac{\sin\theta\sin\theta'\cos\varepsilon}{r},$$

if we put

$$A(r) = -f(r),$$
$$B(r) = -\frac{1}{r}g(r),$$

equation (4.3) will reproduce the equation

$$F = JJ'\,ds\,ds'[f(r)\cos\theta\cos\theta' + g(r)\sin\theta\sin\theta'\cos\varepsilon]. \tag{3.3}$$

Now it is easy to see that equation (3.3) is precisely equivalent to the principle of sinuous currents.

Indeed, we have already seen that the principle of sinuous currents joined to Ampère's first three hypotheses leads to equation (3.3). We now prove that one can, from equation (3.3), deduce the principle of sinuous currents.

Choose any rectangular coordinate system. Let (x, y, z) $[\equiv \mathbf{x}]$ be a point of the element ds and (x', y', z') $[\equiv \mathbf{x'}]$ a point of the element ds'. A closed circuit s will exert on the element ds' a force whose three components will be

$$X\,ds' = JJ'\,ds'\int[f(r)\cos\theta\cos\theta' + g(r)\sin\theta\sin\theta'\cos\varepsilon]\frac{x'-x}{r}ds,$$
$$Y\,ds' = JJ'\,ds'\int[f(r)\cos\theta\cos\theta' + g(r)\sin\theta\sin\theta'\cos\varepsilon]\frac{y'-y}{r}ds,$$
$$Z\,ds' = JJ'\,ds'\int[f(r)\cos\theta\cos\theta' + g(r)\sin\theta\sin\theta'\cos\varepsilon]\frac{z'-z}{r}ds,$$

$$\mathbf{M}ds' = JJ'\,ds'\int[f(r)\cos\theta\cos\theta' + g(r)\sin\theta\sin\theta'\cos\varepsilon]\frac{\mathbf{x'}-\mathbf{x}}{r}ds,$$

equations which can also be written

$$\begin{cases} X\,ds' = JJ'\,ds' \displaystyle\int \left[A(r)\frac{\partial r}{\partial s}\frac{\partial r}{\partial s'} + B(r)\frac{\partial^2 r}{\partial s\,\partial s'}\right]\frac{x'-x}{r}ds, \\ Y\,ds' = JJ'\,ds' \displaystyle\int \left[A(r)\frac{\partial r}{\partial s}\frac{\partial r}{\partial s'} + B(r)\frac{\partial^2 r}{\partial s\,\partial s'}\right]\frac{y'-y}{r}ds, \\ Z\,ds' = JJ'\,ds' \displaystyle\int \left[A(r)\frac{\partial r}{\partial s}\frac{\partial r}{\partial s'} + B(r)\frac{\partial^2 r}{\partial s\,\partial s'}\right]\frac{z'-z}{r}ds. \end{cases} \tag{4.5}$$

$$\left\{ \mathbf{M}\,ds' = JJ'\,ds' \int \left[A(r)\frac{\partial r}{\partial s}\frac{\partial r}{\partial s'} + B(r)\frac{\partial^2 r}{\partial s\,\partial s'}\right]\frac{\mathbf{x'}-\mathbf{x}}{r}ds \right.$$

Let (ξ,η,ζ) $[\equiv \boldsymbol{\xi}]$ be a fixed point taken on the circuit s; let ρ be the distance of this point to the point (x',y',z') $[\equiv \mathbf{x'}]$.

We suppose that the circuit s is the contour of a convex area all of whose dimensions are infinitely small in the first order. One easily sees that we can, by altering only X, Y, Z $[\equiv \mathbf{M}]$ with infinitely small quantities of the second order, replace equations (4.5) by the following:

$$\begin{aligned} X\,ds' &= JJ'\,ds'\left[A(\rho)\frac{\partial\rho}{\partial s'}\frac{x'-\xi}{\rho}\int\frac{\partial r}{\partial s}ds + B(\rho)\frac{x'-\xi}{\rho}\int\frac{\partial^2 r}{\partial s\,\partial s'}ds\right], \\ Y\,ds' &= JJ'\,ds'\left[A(\rho)\frac{\partial\rho}{\partial s'}\frac{y'-\eta}{\rho}\int\frac{\partial r}{\partial s}ds + B(\rho)\frac{y'-\eta}{\rho}\int\frac{\partial^2 r}{\partial s\,\partial s'}ds\right], \\ Z\,ds' &= JJ'\,ds'\left[A(\rho)\frac{\partial\rho}{\partial s'}\frac{z'-\zeta}{\rho}\int\frac{\partial r}{\partial s}ds + B(\rho)\frac{z'-\zeta}{\rho}\int\frac{\partial^2 r}{\partial s\,\partial s'}ds\right]. \end{aligned}$$

$$\mathbf{X}\,ds' = JJ'\,ds'\left[A(\rho)\frac{\partial\rho}{\partial s'}\frac{\mathbf{x'}-\boldsymbol{\xi}}{\rho}\int\frac{\partial r}{\partial s}ds + B(\rho)\frac{\mathbf{x'}-\boldsymbol{\xi}}{\rho}\int\frac{\partial^2 r}{\partial s\,\partial s'}ds\right]$$

The two quantities r and $\frac{\partial r}{\partial s'}$ varying continuously along the curve s, on has for the entire close closed curve

$$\int\frac{\partial r}{\partial s}ds = 0, \qquad \int\frac{\partial}{\partial s}\frac{\partial r}{\partial s'}ds = 0,$$

and the preceding equations become

$$X = 0, \qquad Y = 0, \qquad Z = 0.$$

$$\mathbf{X} = 0$$

It suffices to alter the three quantities X, Y, Z [$\equiv \mathbf{X}$] with infinitely small quantities of the same order as the area enclosed by the closed circuit to make them equal to 0. The quantities X, Y, Z [$\equiv \mathbf{X}$] are thus comprised of infinitely small ones of the same order as this area, which is the principle of sinuous currents.

Thus the first four hypotheses and the second experimental law that Ampère invokes lead to the principle of sinuous currents. This means that one can do without this latter to establish Ampère's law.

In effect, in the demonstration of Ampère's law, the principle of sinuous currents only serves to establish equation (3.3) and we have seen that this equation (3.3) can be established without invoking the principle of sinuous currents.

Chapter 5
On the real meaning that should be attributed to the principle of sinuous currents

To the demonstrations of the propositions that we have just established, J. Bertrand[1] attaches the following considerations:

> Allow me to add a remark related to the plausibility of the fundamental hypothesis, so natural in itself, accepted by Ampère: the action of two elements is directed along the line that joins them.
>
> Suppose that Ampère, who experimentally discovered the first and second law, and who, by reasoning alone, just as we did, deduced the first law; he could have said: if the action of two elements is, as it seems plausible, directed along the line that joins them, it is necessary that a sinuous conductor exerts the same action as a rectilinear conductor along the same direction. Would not the experiment, later coming to confirm this prediction, be reasonably regarded as a very strong proof in favor of the hypothesis that leads to it? Do the order in which truths have been discovered and the time when they have indicated their mutual dependence change anything about their plausibility?

In reality, to look into the matter closely, Ampère's classic experiment on the action of sinuous currents could not have the importance that J. Bertrand attributes to it in the passage we have just cited.

We keep Ampère's first hypothesis, set aside the second one, and modify the third one in the following way:

The magnitude and direction of the action exerted by the element ds on the element ds' depends uniquely on the intensities of the currents that traverse these two elements, on their lengths, and on their relative position.

We will see that these two hypotheses, the least questionable of all the principles on which Ampère's theory rests, lead to the law of sinuous currents, so that the experimental verification of this law satisfies only the two hypotheses in question.

Consider an element ds', in a given position with respect to the axes OX, OY, OZ. The components of the action of the element ds on the element ds' can be cast, in virtue of the two preceding hypotheses, in the following form:

[1] J. Bertrand, *Démonstration des théorèmes relatifs aux actions électrodynamiques* (*Journal de Physique*, 1st series, t. 3, p. 300; 1874).

$$
\begin{aligned}
X\,ds &= JJ'\Phi\,ds\,ds',\\
Y\,ds &= JJ'\Psi\,ds\,ds',\\
Z\,ds &= JJ'\mathrm{X}\,ds\,ds',
\end{aligned}
$$

the three quantities Φ, Ψ, X being, for a given direction of the element ds', functions of the elements that set the relative position of the two elements ds, ds'.

From these equations one immediately deduces the following result:

The three functions Φ, Ψ, X change sign, without changing their absolute value, when one reverses the direction of traversal of the element ds without changing the sens of traversal of the element ds'.

To this theorem, we add these two evident propositions:

1. *The action of any closed and uniform current on any element ds' is the product of ds' by a finite quantity, such that it must be the same for the three quantities*

$$\int \Phi\,ds, \qquad \int \Psi\,ds, \qquad \int \mathrm{X}\,ds,$$

where the integration is evaluated over a closed current.

2. *The integrals*

$$\int \Phi\,ds, \qquad \int \Psi\,ds, \qquad \int \mathrm{X}\,ds,$$

evaluated over an infinitely small closed contour, vary continuously when this contour is deformed and displaced continuously.

By reasoning similarly to what was presented on pages 98-99[2], we will arrive at the following conclusion:

The three integrals

$$\int \Phi\,ds, \qquad \int \Psi\,ds, \qquad \int \mathrm{X}\,ds,$$

are infinitely small in the second order when the integral

$$\int ds$$

is infinitely small in the first order.

This proposition, one can easily see, is none other than the principle of sinuous currents.

[2] [Duhem (1892, 98-99)]

Chapter 6
On the electrodynamic potential

Suppose that two closed conductors are present.

The mutual repulsive action of an element ds of the first conductor and of an element ds' of the second is, when denoting the intensities of the currents that traverse them by J, J',

$$F = \frac{\mathfrak{A}^2}{2} \frac{JJ'\, ds\, ds'}{r^2} (\cos\theta \cos\theta' - 2 \sin\theta \sin\theta' \cos\varepsilon). \tag{3.9}$$

But one has

$$\sin\theta \sin\theta' \cos\varepsilon = \cos\omega - \cos\theta \cos\theta'.$$

Thus one can write

$$F = \frac{\mathfrak{A}^2}{2} \frac{JJ'\, ds\, ds'}{r^2} \left(\frac{3}{2} \cos\theta \cos\theta' - \cos\omega \right).$$

If, in a modification, the distance r increases by δr, the mutual action performs a work

$$F\, \delta r,$$

and the mutual actions of the two conductors perform a work

$$d\mathfrak{T} = \mathfrak{A}^2 JJ' \iint \frac{1}{r^2} \left(\frac{3}{2} \cos\theta \cos\theta' - \cos\omega \right) \delta r\, ds\, ds'. \tag{6.1}$$

We have demonstrated in a general way in the Appendix to Book 13, equation (27),[1] that this equation can be written

$$d\mathfrak{T} = \frac{\mathfrak{A}^2}{2} JJ' \delta \iint \frac{1}{r} \cos\theta \cos\theta'\, ds\, ds', \tag{6.2}$$

which can be written, according to equation (1.10), as

[1] [Duhem (1892, 192)]

$$d\mathfrak{T} = \frac{\mathfrak{A}^2}{2} J J' \delta \iint \frac{1}{r} \cos\theta' \, ds \, ds'. \tag{6.2b}$$

According to equations (6.2) and (6.2b), *the mutual actions of two closed and uniform currents of invariable intensities admit a potential, which one can represent by either of the two expressions*

$$\Pi = -\frac{\mathfrak{A}^2}{2} J J' \iint \frac{\cos\theta \cos\theta'}{r} ds \, ds', \tag{6.3}$$

$$\Pi = -\frac{\mathfrak{A}^2}{2} J J' \iint \frac{\cos\omega}{r} ds \, ds', . \tag{6.3b}$$

This fundamental theorem was, for the first time, demonstrated by F.-E. Neumann[2]

This theorem, we have seen, includes the solution of all the problems that the experimental study of uniform currents can pose. One can thus ask if it is possible to obtain it directly, without invoking Ampère's law. One can, indeed, give the following demonstration, which rests on five hypotheses and an experimental law.

First hypothesis. — *The mutual actions of the two closed and uniform currents whose intensities are kept constant admit a potential.*

Second hypothesis. — *This potential is of the form*

$$\Pi = \sum \Psi_{12},$$

the quantity Ψ_{12} *depending on the intensities* J_1, J_2 *of the currents which traverse the elements* ds_1, ds_2, *the lengths of these elements, and the parameters that fix their relative position; the sign* $\sum$ *as assumed to extend over all the combinations obtained in taking one element of the first circuit and another of the second circuit.*

Third hypothesis. — *The quantity* Ψ_{12} *does not change if one replaces the element* ds_2 *by the element* ds_2', *symmetric to* ds_2 *with respect to a plane containing the element* ds_1 *and a point of the element* ds_2.

By reasons analogous to those which we presented at the beginning of (chapter 3), we will prove that Ψ_{12} is of the form

$$\Psi_{12} = \Phi_{12} J_1 J_2 \, ds_1 \, ds_2,$$

Φ_{12} depending only on the mutual position of the two elements ds_1, ds_2.

By considerations similar to those of the preceding paragraph, we will show that the quantity $\int \Phi_{12} ds_1$ is infinitely small in the second order, when $\int ds_1$ is infinitely small in the first order, and that the quantity $\int \Phi_{12} ds_2$ is infinitely small in the second order when $\int ds_2$ is infinitely small in the first order.

Then by reasoning as we did on pages 102 to 105,[3] we will see that

---

[2] F.-E. Neumann, *Ueber ein allgemeines Princip der mathematischen Theorie inducrter elektrisher Ströme*, read for the Academy of Berlin, 9 August 1847.

[3] [Duhem (1892, 102-105)]

$$\Phi_{12} = J_1 J_2\, ds_1\, ds_2 [F(r) \cos\theta_1 + G(r) \cos\omega]. \tag{6.4}$$

Fourth hypothesis. — *The two functions $F(r)$ and $G(r)$ are of the form*

$$F(r) = \frac{A}{r^n}, \qquad G(r) = \frac{B}{r^n},$$

n being a positive integer, and A and B, two constants.

These equations give equation (6.4) the form

$$\Phi_{12} = \frac{J_1 J_2\, ds_1\, ds_2}{r^n}(A \cos\theta_1 \cos\theta_2 + B \cos\omega). \tag{6.5}$$

Experimental law: *The third experimental law Ampère invokes.* — Consider two closed conductors C_1, C_2, traversed by uniform currents of intensities J_1, J_2. We give to the various points (x, y, z) [$\equiv \mathbf{x}$], … of the conductor C_2 a system of virtual displacements δx, δy, δz, … .

The actions of the conductor C_1 on the conductor C_2 perform a virtual work

$$d\mathfrak{T} = -J_1 J_2 \delta \iint \frac{A \cos\theta_1 \cos\theta_2 + B \cos\omega}{r^n} ds_1\, ds_2. \tag{6.6}$$

We next consider two conductors C_1', C_2', similar to the conductors C_1, C_2 and similarly positioned. Let K be the similarity ratio of the second system to the first. Give to the point (x', y', z') [$\equiv \mathbf{x'}$], homologous, on the conductor C_2', to the point (x, y, z) [$\equiv \mathbf{x}$] of the conductor C_2, a virtual displacement

$$\delta x' = K\, \delta x, \qquad \delta y' = K\, \delta y, \qquad \delta z' = K\, \delta z.$$

$$\delta \mathbf{x'} = K\, \delta \mathbf{x}$$

The virtual work, performed by the actions of the conductor C_1' on conductor C_2', will have the value

$$d\mathfrak{T}' = -J_1 J_2 \delta \iint \frac{A \cos\theta_1' \cos\theta_2' + B \cos\omega'}{r'^n} ds_1'\, ds_2'. \tag{6.7}$$

It is easy to see that one has

$$\begin{aligned}
\cos\theta_1' &= \cos\theta_1, & \cos\theta_2' &= \cos\theta_2, & \cos\omega' &= \cos\omega,\\
r' &= Kr, & ds_1' &= K\, ds_1, & ds_2' &= K\, ds_2,\\
\delta\cos\theta_1' &= \delta\cos\theta_1, & \delta\cos\theta_2' &= \delta\cos\theta_2, & \delta\cos\omega' &= \delta\cos\omega,\\
\delta r' &= K\,\delta r, & \delta ds_1' &= K\delta\, ds_1, & \delta ds_2' &= K\delta\, ds_2,
\end{aligned}$$

such that equation (6.7), compared to equation (6.6), gives

$$d\mathfrak{T}' = K^{(2-n)}\,d\mathfrak{T}.$$

But, the action suffered by an element of the conductors C_1', C_2' being assumed equal to the action suffered by the homologous element of the conductors C_1, C_2, one should evidently have

$$d\mathfrak{T}' = K\,d\mathfrak{T}.$$

One thus has

$$n = 1,$$

and the formula (6.5) becomes

$$\Phi_{12} = \frac{J_1 J_2\,ds_1\,ds_2}{r}(A\cos\theta_1\cos\theta_2 + B\cos\omega).$$

The mutual electrodynamic potential of the two closed and uniform currents has, consequently, the value

$$\Pi = JJ'\iint \frac{A\cos\theta' + B\cos\omega}{r}ds\,ds'.$$

If we note that we have equation (1.10),

$$\iint \frac{\cos\theta\cos\theta'}{r}dsds' = \iint \frac{\cos\omega}{r}dsds',$$

we see that we will be able to write either

$$\Pi = (A+B)JJ'\iint \frac{\cos\omega\cos\omega'}{r}ds\,ds', \tag{6.8}$$

or

$$\Pi = (A+B)JJ'\iint \frac{\cos\omega}{r}ds\,ds'. \tag{6.8b}$$

Fifth hypothesis. — *The constant* $(A+B)$ *is negative.*

If we then put

$$A + B = -\frac{\mathfrak{A}^2}{2},$$

equations (6.8) and (6.8b) give back equations (6.3) and (6.3b).

Chapter 7
On the determination of the function of distance in Ampère's formula

The formula of the electrodynamic actions being cast in the form

$$F = JJ' \, ds \, ds' \left\{ g(r) \sin\theta \sin\theta' \cos\varepsilon + \frac{1}{2} \frac{d}{dr}[rg(r)] \cos\theta \cos\theta' \right\}, \qquad (3.5)$$

Ampère hypothesized that the function $g(r)$ is of the form

$$g(r) = \frac{A}{r^n},$$

A being a constant, and n, a positive number. This hypothesis seems very arbitrary. One can replace it with an experimental law that is easy to verify, as J. Bertrand[1] has shown.

The equations, frequently invoked,

$$\cos\theta = -\frac{\partial r}{\partial s}, \qquad \cos\theta' = \frac{\partial r}{\partial s'}, \qquad \sin\theta \sin\theta' \cos\epsilon = -r \frac{\partial^2 r}{\partial s \, \partial s'},$$

transform formula (3.5) into

$$F = -JJ' \, ds \, ds' \left\{ rg(r) \frac{\partial^2 r}{\partial s \, \partial s'} + \frac{1}{2} \frac{d}{dr}[rg(r)] \frac{\partial r}{\partial s} \frac{\partial r}{\partial s'} \right\}. \qquad (7.1)$$

Consider a function $\psi(r)$ defined by the equation

$$\frac{d\psi(r)}{dr} = [rg(r)]^{\frac{1}{2}}. \qquad (7.2)$$

We will then have

[1] J. Bertrand, *Démonstration des théorèmes relatifs aux actions électrodynamiques* (*Journal de Physique*, 1st series, t. 3, p. 335; 1874).

$$rg(r) = \left(\frac{d\psi}{dr}\right)^2,$$

$$\frac{d}{dr}[rg(r)] = 2\frac{d\psi}{dr}\frac{d^2\psi}{dr^2},$$

and equation (7.1) will become

$$F = -JJ'\,ds\,ds'\frac{d\psi}{dr}\left(\frac{d\psi}{dr}\frac{\partial^2 r}{\partial s\,\partial s'} + \frac{d^2\psi}{dr^2}\frac{\partial r}{\partial s}\frac{\partial r}{\partial s'}\right)$$

or

$$F = -JJ'\,ds\,ds'\frac{d\psi}{dr}\frac{\partial^2\psi}{\partial s\,\partial s'}. \tag{7.3}$$

The work, performed by the mutual actions of the closed and uniform currents in any displacement of these currents, will have the value

$$d\mathfrak{T} = -JJ'\iint \frac{d\psi}{dr}\frac{\partial^2\psi}{\partial s\,\partial s'}\delta r\,ds\,ds'.$$

By reasoning on this double integral exactly as on page 191,[2] we reasoned on the integral

$$\iint \frac{dr^{\frac{1}{2}}}{dr}\frac{\partial^2 r^{\frac{1}{2}}}{\partial s\,\partial s'}\delta r\,ds\,ds',$$

which is a particular form of it obtained by putting $\psi(r) = r^{\frac{1}{2}}$, we will arrive at this result:

The elementary work between two closed and uniform currents has the value

$$d\mathfrak{T} = -\frac{1}{2}JJ'\delta\iint\left(\frac{d\psi}{dr}\right)^2 \cos\theta\cos\theta'\,ds\,ds'$$

or, in virtue of equation (7.2),

$$d\mathfrak{T} = -\frac{1}{2}JJ'\delta\iint rg(r)\cos\theta\cos\theta'\,ds\,ds'.$$

In other words, two closed, uniform, and constant currents exert on each other actions that admit for potential the quantity

$$\Pi = \frac{1}{2}JJ'\iint rg(r)\cos\theta\cos\theta'\,ds\,ds'. \tag{7.4}$$

We are thus brought to the following general question:

Knowing that the mutual actions of two closed, uniform, and constant currents admit a potential of the form

[2] [Duhem (1892, 191)]

$$\Pi = JJ' \iint [F(r)\cos\theta\cos\theta' + G(r)\cos\omega] ds\, ds', \tag{7.5}$$

determine the form of the functions $F(r)$ and $G(r)$.

The experimental law that J. Bertrand proposes to take as a proper principle to resolve this question is the following:

The action of a closed solenoid on any element of current is equal to 0.

This proposition can, as can easily be seen, be replaced by the following:

The mutual electrodynamic potential of a closed solenoid and a closed infinitely small current not enclosing the axis of the solenoid is equal to 0.

We adopt this proposition and see which conditions it imposes on the functions $F(r)$ and $G(r)$.

Consider two functions, $\varphi(r)$ and $\psi(r)$, defined by the equations

$$\frac{1}{r}\varphi r - \frac{d\varphi(r)}{dr} + F(r) = 0, \tag{7.6}$$

$$G(r) - \frac{1}{r}\varphi(r) = \psi(r). \tag{7.7}$$

Equation (7.5) will be able to be written

$$\Pi = JJ' \iint \left\{ \psi(r)\cos\omega + \frac{\varphi(r)}{r}\cos\omega + \left[\frac{d\varphi(r)}{dr} - \frac{\varphi r}{r}\right]\cos\theta\cos\theta' \right\} ds\, ds'.$$

If we put

$$\frac{d\Phi(r)}{dr} = -\varphi(r),$$

this equality will become

$$\Pi = JJ' \iint \left[\varphi(r)\cos\omega + \frac{\partial^2\Phi(r)}{\partial s\,\partial s'}\right] ds\, ds',$$

or simply

$$\Pi = JJ' \iint \psi(r)\cos\omega\, ds\, ds'. \tag{7.8}$$

Suppose that s and s' are two closed infinitely small currents; that Ω, Ω' are the areas of two surfaces corresponding to these currents; that N, N' are the normals to the positive faces of these areas.

We will have, by equation (3.5),

$$\begin{cases} \iint \psi(r)\cos\omega\, ds\, ds' = -\left(\dfrac{1}{r}\dfrac{d\psi}{dr} + \dfrac{d^2\psi}{dr^2}\right)\cos(\mathrm{N},\mathrm{N}')\Omega\Omega' \\ \qquad\qquad - \left(\dfrac{1}{r}\dfrac{d\psi}{dr} - \dfrac{d^2\psi}{dr^2}\right)\cos(\mathrm{N},r)\cos(\mathrm{N}',r)\Omega\Omega'. \end{cases} \tag{7.9}$$

Suppose that the area Ω is that of an infinitely small current part of a solenoid; let D be the distance of the two rings of the rings of the solenoid; let $\Phi = \frac{\Omega J}{D}$ be the power of the solenoid; let l be the directrix[3] of the solenoid. The electrodynamic potential of this solenoid on the small closed current s' will have, according to equations (7.8) and (7.9), the value

$$\begin{cases} \Pi = -\Phi J'\Omega' \displaystyle\int \Bigg[\left(\frac{1}{r}\frac{d\psi}{dr} + \frac{d^2\psi}{dr^2} \right) \cos(l, \mathrm{N}') \\ \qquad\qquad + \left(\frac{1}{r}\frac{d\psi}{dr} - \frac{d^2\psi}{dr^2} \right) \cos(l, r)\cos(\mathrm{N}', r) \Bigg] dl. \end{cases} \tag{7.10}$$

One has, moreover,

$$\cos(l, r) = -\frac{\partial r}{\partial l}, \qquad \cos(\mathrm{N}', r) = \frac{\partial r}{\partial \mathrm{N}'},$$
$$\cos(l, \mathrm{N}') = -\frac{\partial r}{\partial l}\frac{\partial r}{\partial \mathrm{N}'} - r\frac{\partial^2 r}{\partial l\,\partial \mathrm{N}'}.$$

The preceding equation thus becomes

$$\Pi = \Phi J'\Omega' \int \left[\frac{d}{dr}\left(r\frac{d\psi}{dr} \right) \frac{\partial^2 r}{\partial l\,\partial \mathrm{N}'} + \frac{2}{r}\frac{d\psi}{dr}\frac{\partial r}{\partial l}\frac{\partial r}{\partial \mathrm{N}'} \right] dl. \tag{7.11}$$

In virtue of the admitted experimental law, it is necessary and sufficient that the curve l is closed so that this quantity is equal to 0; in other words, the quantity under the $\int$ sign should be of the form

$$\frac{\partial}{\partial l}\Psi\left(r, \frac{\partial r}{\partial \mathrm{N}'} \right) dl,$$

Ψ being a uniform and continuous function of r and $\frac{\partial r}{\partial \mathrm{N}'}$.

This condition is equivalent to

$$\frac{\partial}{\partial r}\left[\frac{d}{dr}\left(r\frac{d\psi}{dr} \right) \right] = \frac{\partial}{\frac{\partial r}{\partial \mathrm{N}'}}\left(\frac{2}{r}\frac{d\psi}{dr}\frac{\partial r}{\partial \mathrm{N}'} \right).$$

Put

$$\Theta = r\frac{d\psi}{dr}, \tag{7.12}$$

and this equation will become the new equation

$$r^2\frac{d^2\Theta}{dr^2} = \frac{2\Theta}{r^2}.$$

By adding to both members the quantity $2r\frac{d\Theta}{dr}$, one can write this last equation

[3] ["*directrice*" (Hadamard, 1901, 114)]

$$\frac{d}{dr}\left(r^2\frac{d\Theta}{dr}\right) = 2\frac{d}{dr}(r\Theta).$$

In this form, it can be integrated once, giving the equation

$$r^2\frac{d\Theta}{dr} - 2r\Theta + C = 0,$$

C being a constant.

This equation, in turn, can be written

$$\frac{1}{r^2}\frac{d\Theta}{dr} - \frac{2}{r^3}\Theta + \frac{C}{r^4} = 0$$

or

$$\frac{d}{dr}\left(\frac{\Theta}{r^2}\right) + \frac{C}{r^4} = 0.$$

From which is deduced

$$\Theta = \frac{C}{4r} + C'r^2,$$

C being a new constant.

If one plugs this value of Θ into equation (7.12), we find

$$\frac{d\psi}{dr} = \frac{C}{2r^2} + C'r.$$

This equation shows us that ψ should be of the form

$$\psi = \frac{A}{r} + Br^2 + C, \tag{7.13}$$

A, B, C being three constants.

Thus, if the electrodynamic potential of two closed currents is to be of the form (7.5), the experimental law that we have just invoked requires, in virtue of equations (7.6), (7.7), and (7.13), that we have

$$G(r) = \frac{A}{r} + Br^2 + C + \frac{1}{r}\varphi(r), \tag{7.14}$$

$$F(r) = \frac{1}{r}\varphi(r) + \frac{d\varphi(r)}{dr}, \tag{7.15}$$

$\varphi(r)$ being an arbitrary function of r.

In other words, the most general form of the electrodynamic potential of two closed and uniform currents, which is compatible with the experimental law that we invoked, is the following:

$$\begin{cases} \Pi = JJ' \left\{ \iint \left(\frac{A}{r} + Br^2 + C \right) \cos\omega \, ds \, ds' \right. \\ \left. \quad + \iint \left[\frac{\varphi(r)}{r} (\cos\omega - \cos\theta\cos\theta') + \frac{d\varphi(r)}{dr} \cos\theta\cos\theta' \right] ds \, ds' \right\}. \end{cases} \tag{7.16}$$

Let us return to the question that served as the point of departure for these considers.

We suppose that one has demonstrated that the form of the mutual electrodynamic potential of two closed and uniform currents is

$$\Pi = \frac{1}{2} JJ' \iint rg(r) \cos\theta \cos\theta' \, ds \, ds'. \tag{7.4}$$

What form must be attributed to the function $g(r)$ so that the action of any closed solenoid on any element of current is equal to 0?

Formula (7.4) is deduced from formula (7.5) by making

$$G(r) = 0,$$
$$F(r) = \frac{1}{2} rg(r).$$

Therefore, equations (7.14) and (7.15) become

$$\varphi(r) = -(A + Cr + Br^3),$$
$$\frac{1}{2} rg(r) = \frac{A}{r} - 2Br^2.$$

Thus if the fifth hypothesis and the third experimental law that Ampère invoked are replaced by the experimental law that a closed solenoid does not act on any element of current, we arrive at this conclusion: *the function g(r) is of the form*

$$\frac{1}{2} g(r) = \frac{A}{r^2} - 2Br.$$

If we then invoke the following hypothesis:

The mutual action of any two elements of current tend toward 0 *when their distance grows beyond any limit,*one will be constrained to take for $g(r)$ the form

$$g(r) = \frac{2A}{r^2},$$

and one will regain Ampère's law.

Formula (7.16) allows us to modify the laws of electrodynamics expressed in the preceding paragraph analogously to how J. Bertrand modified Ampère's demonstration.

We do not invoke, as in the preceding paragraph, the hypothesis that the two functions $F(r)$ and $G(r)$ are of the form $\frac{A}{r^n}$, $\frac{B}{r^n}$; neither do we invoke the experimental law of the actions that are exerted between similar conductors. Instead of this law, we invoke: *The action of a closed solenoid on any current element is equal to* 0.

Formula (7.16), for the electrodynamic potential of any two closed and uniform currents, will result.

Now we have

$$\begin{aligned}&\frac{\varphi(r)}{r}(\cos\omega-\cos\theta\cos\theta')+\frac{d\varphi(r)}{dr}\cos\theta\cos\theta'\\&\qquad=-\left[\varphi(r)\frac{\partial^2 r}{\partial s\,\partial s'}+\frac{d\varphi(r)}{dr}\frac{\partial r}{\partial s}\frac{\partial r}{\partial s'}\right]=-\frac{\partial}{\partial s'}\left[\varphi(r)\frac{\partial r}{\partial s}\right],\end{aligned}$$

such that the quantity

$$\iint\left[\frac{\varphi(r)}{r}(\cos\omega-\cos\theta\cos\theta')+\frac{d\varphi(r)}{dr}\cos\theta\cos\theta'\right]ds\,ds'$$

is equal to 0, whatever the function $\varphi(r)$ is.

The choice of the function $\varphi(r)$ that appears in equation (7.16) being indifferent, one can take

$$\varphi(r)=-(A+Br^3+C),$$

and equation (7.16) then gives

$$\Pi=JJ'\iint\left(\frac{A}{r}-2Br^2\right)\cos\theta\cos\theta'\,ds\,ds'. \tag{7.17}$$

This formula, compared to formula (7.4), shows us that *the mutual actions of two closed and uniform currents are the same as if any two elements of uniform currents repel each other with a force directed along the line that joins them, and represented by the equation*

$$F=JJ'\,ds\,ds'\left\{g(r)\sin\theta\sin\theta'\cos\varepsilon+\frac{1}{2}\frac{d}{dr}[rg(r)]\cos\theta\cos\theta'\right\}, \tag{3.5}$$

where the function g(r) is of the form

$$g(r)=\frac{2A}{r^2}-4Br.$$

If, like J. Bertrand, one hypothesizes that this force should tend toward 0 when the two elements move away from each other beyond any limit, this force becomes identical to the elementary force that Ampère admitted.

We thus see that one can set aside the hypothesis that the actions of two closed and uniform currents are decomposed into elementary actions subject to the law of the equality of action and reaction; nor invoke the difficult-to-verify law that a closed and uniform current exerts on any current element an action normal to this element, and replace these hypotheses of Ampère with the much less questionable one that the mutual actions of two closed, uniform, and constant currents admit a potential. To determine the form of this potential, one will be able to follow the methods indicated by either Ampère or J. Bertrand to determine the form of the elementary action.

Appendix A
On Ampère's Law[1]

Gauss first stated the following proposition:

> There exist infinitely many laws for the action of an element of current on another element of current, such that the action of a closed current on an element of current is identical to the action determined by Ampère's law; but, among all these laws, one alone, Ampère's law, is such that the action of an element of current on another element of current is reduced to a unique force directed along the line that joins the two elements.

Since when Gauss discovered this proposition, it has been given several demonstrations. The following seems to us particularly simple.

The proposition in question immediately results from this one: *The action of an element of current on another element of current es completely determined when know knows the action of a closed and uniform current on an element of current and when one knows, moreover, that the elementary action is directed along the lines the joins the elements.*

Let ds, ds' be the elements of current; let i, i' be the intensities of the currents that traverse them. Suppose that one can admit for the action of the element ds on the element ds' two distinct expressions, both subject to the restrictions indicated in the previous statement. According to the first expression, the components of the action of ds on ds' will have the values

$$ii' X \, ds \, ds',$$
$$ii' Y \, ds \, ds',$$
$$ii' Z \, ds \, ds'.$$

$$ii'\mathbf{M} \, ds \, ds'$$

According to the second expression, these same components will have the value

[1] [Duhem (1886)]

$$ii'X'\,ds\,ds',$$
$$ii'Y'\,ds\,ds',$$
$$ii'Z'\,ds\,ds'.$$

$$ii'\mathbf{M_1}\,ds\,ds'$$

Suppose that the element ds is part of any closed and uniform current. The action of this current on the element ds' should be the same, regardless which of the two elementary laws one accepts. One should thus have

$$\int X\,ds = \int X_1\,ds,$$
$$\int Y\,ds = \int Y_1\,ds,$$
$$\int Z\,ds = \int Z_1\,ds,$$

$$\int \mathbf{M}\cdot d\mathbf{s} = \int \mathbf{M_1}\cdot d\mathbf{s}$$

the integrals being curvilinear integrals evaluated over any close contour.

According to the known properties of curvilinear integrals, these equations can be replaced by the following:

$$\begin{cases} (X - X_1)ds = d\mathfrak{F}(x, y, z), \\ (Y - Y_1)ds = d\mathfrak{G}(x, y, z), \\ (Z - Z_1)ds = d\mathfrak{H}(x, y, z), \end{cases} \tag{A.1}$$

$$(\mathbf{X} - \mathbf{X_1})ds = d\boldsymbol{\mathfrak{F}}(\mathbf{M})$$

$\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$ [$\equiv \boldsymbol{\mathfrak{F}}$] being any functions of coordinates x, y, z [$\equiv \mathbf{M}$] of a point of the element ds.

If the two considered actions are directed along the line that joins the two elements ds, ds', one must have, in denoting the coordinates of a point of the element ds' by x', y', z' [$\equiv \mathbf{M'}$],

$$
\begin{aligned}
(y'-y)Z-(z'-z)Y&=0,\\
(z'-z)X-(x'-x)Z&=0,\\
(x'-x)Y-(y'-y)X&=0;
\end{aligned}
$$

$$(\mathbf{M'}-\mathbf{M})\times\mathbf{X}=0$$

and also

$$
\begin{aligned}
(y'-y)Z_1-(z'-z)Y_1&=0,\\
(z'-z)X_1-(x'-x)Z_1&=0,\\
(x'-x)Y_1-(y'-y)X_1&=0.
\end{aligned}
$$

$$(\mathbf{M'}-\mathbf{M})\times\mathbf{X_1}=0$$

We then deduce from equations (A.1) the following relations:

$$
\begin{cases}
(y'-y)\mathfrak{H}(x,y,z)-(z'-z)\mathfrak{G}(x,y,z)=0,\\
(z'-z)\mathfrak{F}(x,y,z)-(x'-x)\mathfrak{H}(x,y,z)=0,\\
(x'-x)\mathfrak{G}(x,y,z)-(y'-y)\mathfrak{F}(x,y,z)=0.
\end{cases}
\tag{A.2}
$$

$$(\mathbf{M'}-\mathbf{M})\times\boldsymbol{\mathfrak{F}}(\mathbf{M})=0$$

Put

$$
\begin{aligned}
F(x,y,z)&=(y'-y)\mathfrak{H}(x,y,z)-(z'-z)\mathfrak{G}(x,y,z),\\
G(x,y,z)&=(z'-z)\mathfrak{F}(x,y,z)-(x'-x)\mathfrak{H}(x,y,z),\\
H(x,y,z)&=(x'-x)\mathfrak{G}(x,y,z)-(y'-y)\mathfrak{F}(x,y,z).
\end{aligned}
$$

$$\mathbf{F}(\mathbf{M})=(\mathbf{M'}-\mathbf{M})\times\boldsymbol{\mathfrak{F}}(\mathbf{M})$$

Equations (A.2) allow us to write:

$$
\begin{cases}
\mathfrak{H}(x,y,z)dy-\mathfrak{G}(x,y,z)dz=-dF(x,y,z),\\
\mathfrak{F}(x,y,z)dz-\mathfrak{H}(x,y,z)dx=-dG(x,y,z),\\
\mathfrak{G}(x,y,z)dx-\mathfrak{F}(x,y,z)dy=-dH(x,y,z).
\end{cases}
\tag{A.3}
$$

$$\mathfrak{F} \times d\mathbf{M} = -d\mathbf{F}(\mathbf{M})$$

Examine the first equality: the total differential of $F(x, y, z)$ does not contain a term in dx; the function $F(x, y, z)$ is independent of x. It is the same for the partial derivatives $\mathfrak{G}(x, y, z)$ and $\mathfrak{H}(x, y, z)$. By reasoning similarly for the other two equations, one arrive at the following conclusion:

$\mathfrak{F}$ is a function only of the variable x,

$\mathfrak{G}$ is a function only of the variable y,

$\mathfrak{H}$ is a function only of the variable z.

But equations (A.3) also give us, in expressing that the first members are total differentials,

$$\begin{cases} \dfrac{\partial \mathfrak{H}}{\partial z} + \dfrac{\partial \mathfrak{G}}{\partial y} = 0 \\ \dfrac{\partial \mathfrak{F}}{\partial x} + \dfrac{\partial \mathfrak{H}}{\partial z} = 0 \\ \dfrac{\partial \mathfrak{G}}{\partial y} + \dfrac{\partial \mathfrak{F}}{\partial x} = 0 \end{cases} \tag{A.4}$$

From which we deduce

$$\frac{\partial \mathfrak{F}}{\partial x} = 0,$$
$$\frac{\partial \mathfrak{G}}{\partial y} = 0,$$
$$\frac{\partial \mathfrak{H}}{\partial z} = 0.$$

If one joins these results to those that have just been obtained, one sees that the quantities $\mathfrak{F}$, $\mathfrak{G}$, $\mathfrak{H}$ $[\equiv \boldsymbol{\mathfrak{F}}]$ are of simple constants. By referring to equations (A.1), we find

$$X = X_1,$$
$$Y = Y_1,$$
$$Z = Z_1,$$

$$\mathbf{X} = \mathbf{X_1}$$

which demonstrates the stated proposition.

References

Translator's Bibliography

Ampère AM (2015) Ampère's electrodynamics: analysis of the meaning and evolution of Ampère's force between current elements, together with a complete translation of his masterpiece: Theory of electrodynamic phenomena, uniquely deduced from experience. Apeiron, Montreal, URL http://www.ifi.unicamp.br/~assis/Amperes-Electrodynamics.pdf

Duhem PMM (1886) Sur la loi d'Ampère. J Phys Theor Appl 5(1):26–29, DOI 10.1051/jphystap:01886005002601, URL https://hal.archives-ouvertes.fr/jpa-00238641

Duhem PMM (1892) Leçons sur l'électricité et le magnétisme: Les courants linéaires, vol 3. Gauthier-Villars, Paris, URL https://archive.org/details/leonssurllec03duheuoft

Hadamard J (1901) Leçons de géométrie élementaire: Géométrie dans l'espace, vol 2. Armand Colin, Paris, URL https://books.google.com/books?id=M8nA-HjyTZQC, OCLC: 609259792

Hadamard J (2008) Lessons in Geometry: Plane Geometry, vol 1. American Mathematical Society Education Development Center, Providence, R.I.; Newton, MA, URL http://dx.doi.org/10.1090/mbk/057, OCLC: 946298478

Katz VJ (1979) The History of Stokes' Theorem. Mathematics Magazine 52(3):146, DOI 10.2307/2690275, URL http://www.jstor.org/stable/10.2307/2690275?origin=crossref

Neumann C (1898) Die elektrischen Kräfte: Darlegung und Erweiterung der von A. Ampére, F. Neumann, W. Weber, G. Kirchhoff entwickelten mathematischen Theorieen. B.G. Teubner, Leipzig, URL https://archive.org/details/DieElektrischenKrafte, OCLC: 9977133