# The Born rule and its interpretation

The ***Born rule*** provides a link between the mathematical formalism of quantum theory and experiment, and as such is almost single-handedly responsible for practically all predictions of quantum physics. In the history of science, on a par with the Heisenberg uncertainty relations (→ indeterminacy relations) the Born rule is often seen as a turning point where indeterminism entered fundamental physics. For these two reasons, its importance for the practice and philosophy of science cannot be overestimated.

The Born rule was first stated by Max Born (1882-1970) in the context of scattering theory [1], following a slightly earlier paper in which he famously omitted the absolute value squared signs (though he corrected this is a footnote added in proof). The application to the position operator (cf. (5) below) is due to Pauli, who mentioned it to Heisenberg and Jordan, the latter publishing Pauli's suggestion with acknowledgment [6] even before Pauli himself spent a footnote on it [8]. The general formulation (6) below is due to von Neumann (see §III.1 of [7]), following earlier contributions by Dirac [2] and Jordan [5,6].

Both Born and Heisenberg acknowledge the profound influence of Einstein on the probabilistic formulation of quantum mechanics. However, Born and Heisenberg as well as Bohr, Dirac, Jordan, Pauli and von Neumann differed with Einstein about the (allegedly) fundamental nature of the Born probabilities and hence on the issue of → determinism. Indeed, whereas Born and the others just listed after him believed the outcome of any individual quantum measurement to be unpredictable in principle, Einstein felt this unpredictability was just caused by the incompleteness of quantum mechanics (as he saw it). See, for example, the invaluable source [3]. Mehra & Rechenberg [20] provide a very detailed reconstruction of the historical origin of the Born rule within the context of quantum mechanics, whereas von Plato [22] embeds a briefer historical treatment of it into the more general setting of the emergence of modern probability theory and probabilistic thinking.

Let $a$ be a quantum-mechanical → observable, mathematically represented by a self-adjoint operator on a Hilbert space $H$ with inner product denoted by $(\ ,\ )$. For the simplest formulation of the Born rule, assume that $a$ has non-degenerate discrete spectrum: this means that $a$ has an orthonormal basis of eigenvectors $(e_i)$ with corresponding eigenvalues $\lambda_i$, i.e. $ae_i = \lambda_i e_i$. A fundamental assumption underlying the Born rule is that a → measurement of the observable $a$ will produce one of its eigenvalues $\lambda_i$ as a result. In what follows, $\Psi \in H$ is a unit vector and hence a (pure) state in the usual sense. Then the Born rule states:

> If the system is in a sate $\Psi$, then the probability $P(a = \lambda_i \mid \Psi)$ that the eigenvalue $\lambda_i$ of $a$ is found when $a$ is measured is
>
> $$P(a = \lambda_i \mid \Psi) = |(e_i, \Psi)|^2. \tag{1}$$

In other words, if $\Psi = \sum_i c_i e_i$ (with $\sum_i |c_i|^2 = 1$), then $P(a = \lambda_i \mid \Psi) = |c_i|^2$.

The general formulation of the Born rule (which is necessary, for example, to discuss observables with continuous spectrum such as the position operator $x$ on $H = L^2(\mathbb{R})$ for a particle moving in one dimension) relies on the spectral theorem for self-adjoint operators on Hilbert space (see, e.g., [21]). According to this theorem, a self-adjoint operator $a$ defines a so-called spectral measure (alternatively called a projection-valued measure or PVM) $B \mapsto p^{(a)}(B)$ on $\mathbb{R}$. Here $B$ is a (Borel) subset of $\mathbb{R}$ and $p^{(a)}(B)$ is a projection on $H$. (Recall that a projection on a Hilbert space $H$ is a bounded operator $p : H \to H$ satisfying $p^2 = p^* = p$; such operators correspond bijectively to their images $pH$, which are closed subspaces of $H$.) The spectral measure $p^{(a)}$ turns out to be concentrated on the spectrum $\sigma(a) \subset \mathbb{R}$ of $a$ in the sense that if $B \cap \sigma(a) = \emptyset$, then $p^{(a)}(B) = 0$ (hence $p^{(a)}$ is often defined on $\sigma(a)$ instead of $\mathbb{R}$). The map $B \mapsto p^{(a)}(B)$ satisfies properties such as $p^{(a)}(A \cup B) = p^{(a)}(A) + p^{(a)}(B)$ when $A \cap B = \emptyset$ (and a similar property for a countable family of disjoint sets) and $p^{(a)}(\mathbb{R}) = 1$ (i.e. the unit operator on $H$). Consequently, a self-adjoint operator $a$ and a unit vector $\Psi \in H$ jointly define a probability measure $P_\Psi^{(a)}$ on $\mathbb{R}$ by

$$P_\Psi^{(a)}(B) := (\Psi, p^{(a)}(B)\Psi) = \|p^{(a)}(B)\Psi\|^2, \tag{2}$$

where $\|\cdot\|$ is the norm derived from the inner product on $H$. The properties of $p^{(a)}$ just mentioned then guarantee that $P_\Psi^{(a)}$ indeed has the properties of a probability measure, such as $P_\Psi^{(a)}(A \cup B) = P_\Psi^{(a)}(A) + P_\Psi^{(a)}(B)$ when $A \cap B = \emptyset$ (and a similar property for a countable family of disjoint sets) and $P_\Psi^{(a)}(\mathbb{R}) = 1$. Again, the probability measure $P_\Psi^{(a)}$ is concentrated on $\sigma(a)$.

For example, if $a$ has discrete spectrum, then $\sigma(a) = \{\lambda_1, \lambda_2, \ldots\}$ and $p^{(a)}(B)$ projects onto the space spanned by all eigenvectors whose eigenvalues lie in $B$. In particular, if $\Psi = \sum_i c_i e_i$ as above, then $P_\Psi^{(a)}(\{\lambda_i\}) = |c_i|^2$. In the case of the position operator $x$ as above, $\sigma(x) = \mathbb{R}$ and $p^{(x)}(B)$ equals the characteristic function $\chi_B$, seen as a multiplication operator on $L^2(\mathbb{R})$. The image of $p^{(x)}(B)$ consists of functions vanishing (almost everywhere) outside $B$, and the measure $P_\Psi^{(x)}$ is given by

$$P_\Psi^{(x)}(B) = \int_{\mathbb{R}} dx\, \chi_B(x)|\Psi(x)|^2 = \int_B dx\, |\Psi(x)|^2. \tag{3}$$

The general statement of the Born rule, then, is as follows:

> If the system is in a state $\Psi \in H$, then the probability $P(a \in B \mid \Psi)$ that a result in $B \subset \mathbb{R}$ is found when $a$ is measured equals
>
> $$P(a \in B \mid \Psi) = P_\Psi^{(a)}(B). \tag{4}$$

For discrete non-degenerate spectrum this reduces to (1). For the position operator in one dimension, (4) yields

$$P(x \in B \mid \Psi) = \int_B dx\, |\Psi(x)|^2 \tag{5}$$

for the probability that the particle is found in the region $B$.

Note that it follows from the general Born rule (4) that with probability one a measurement of $a$ will lead to a result contained in its spectrum, since $P_\Psi^{(a)}(B) = 0$ whenever $B \cap \sigma(a) = \emptyset$. Curiously, however, the probability $P(a = \lambda \mid \Psi)$ of finding any specific number $\lambda$ in the continuous spectrum of $a$ is zero! As a case in point, the probability $P(x = x_0 \mid \Psi)$ of finding the particle at any given point $x_0$ vanishes. Of course, this phenomenon also occurs in classical probability theory (e.g., the probability of any given infinite sequence of results of a coin flip is zero).

The rule (4) is easily extended to $n$ commuting self-adjoint operators $a_1, \ldots, a_n$ [7]:

> The probability that the observables $a_1, \ldots, a_n$ simultaneously take some value in a subset $B_1 \times \cdots \times B_n \subset \mathbb{R}^n$ upon measurement in a state $\Psi$ is
>
> $$P_\Psi(a_1 \in B_1, \ldots, a_n \in B_n) = \|p^{(a_1)}(B_1) \cdots p^{(a_n)}(B_n)\Psi\|^2. \tag{6}$$

This version of the Born rule is needed, for example, in order to generalize (5) to three dimensions. Indeed, the ensuing formula is practically the same, this time with $B \subset \mathbb{R}^3$ and $x$ replaced by $(x, y, z)$.

The statement that the expectation value of an observable $a$ in a state $\Psi$ equals $(\Psi, a\Psi)$ is equivalent to the Born rule. To see this, we identify projections with yes-no questions [7], identifying the answer 'yes' with eigenvalue 1 and 'no' with eigenvalue 0. The expectation value $(\Psi, p\Psi) = \|p\Psi\|^2$ of a projection then simply becomes the probability of the answer 'yes'. Taking $p = p^{(a)}(B)$ then reproduces (4), since the probability of 'yes' to the question $p^{(a)}(B)$ is nothing but $P(a \in B \mid \Psi)$. In this fashion, the Born rule may be generalized from pure states to mixed ones (i.e. → density matrices in the standard formalism we are considering here), by stipulating that the expectation value of $a$ in a state $\rho$ (i.e. a positive trace-class operator with trace one) is $\mathrm{Tr}(\rho a)$. For a further generalization in this direction see → Algebraic quantum mechanics.

Finally, another formulation of the Born rule is as follows:

> The transition probability $P(\Psi, \Phi)$ from a state $\Psi$ to a state $\Phi$, or, in other words, the probability of a 'quantum jump' from $\Psi$ to $\Phi$, is
>
> $$P(\Psi, \Phi) = |(\Psi, \Phi)|^2. \tag{7}$$

This related to the first formulation above, in that in standard measurement theory one assumes a 'collapse of the wave-function' in the sense that $\Psi$ changes to $e_i$ after a measurement of $a$ yielding $\lambda_i$. The transition probability $P(\Psi, e_i)$ is then precisely equal to $P(a = \lambda_i \mid \Psi)$ as stated above.

The ***Born interpretation*** of quantum mechanics is usually taken to be the statement that the empirical content of the theory (and particularly of the quantum state) is given by the Born rule. However, this is not really an interpretation at all until it is is specified what the notions of measurement and probability mean. The pragmatic attitude taken by most physicists is that measurements are what experimentalists perform in the laboratory and that probability is given the frequency interpretation [15, 17] (which is neutral with respect to the issue whether the probabilities are fundamental or due to ignorance). Given that firstly the notion of a quantum measurement is quite subtle and hard to define, and that secondly the frequency interpretation is held in rather low regard in the philosophy of probability [17, 18], it is amazing how successful this attitude has been! Going beyond pragmatism requires a mature → interpretation of quantum mechanics, however. Each such interpretation hinges on some interpretation of probability and will contain its own perspective on the Born rule.

The nature of the Born rule comes out particularly well in the → Copenhagen interpretation, especially if this approach is combined with → Algebraic quantum mechanics. In the algebraic approach, a quantum system is modeled by a non-commutative $C^*$-algebra of observables. The simplest illustration of this is the algebra $M_n$ of all complex $n \times n$ matrices. This contains the commutative $C^*$-algebra $D_n$ of all diagonal matrices as a subalgebra. A unit vector $\Psi \in \mathbb{C}^n$ determines a pure state $\psi$ on $M_n$ in the algebraic sense by $\psi(a) = (\Psi, a\Psi)$. The latter may be restricted to a state $\psi_{|D_n}$ on $D_n$, which turns out to be mixed: if $\Psi = \sum_{i=1}^n c_i e_i$ and $d_\lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is the diagonal matrix with entries $(\lambda_1, \ldots, \lambda_n)$, then

$$\psi_{|D_n}(d_\lambda) = \sum_{i=1}^n |c_i|^2 \lambda_i \tag{8}$$

yields the expectation value of $d_\lambda$ in the state $\psi$. In particular, if $p_i \in D_n$ is the projection $p_i = \mathrm{diag}(0, \ldots, 1, \ldots, 0)$ having 1 on the $i$'th diagonal entry and zeros elsewhere, then $\psi_{|D_n}(p_i) = |c_i|^2$ yields the Born probability of obtaining $\lambda_i$ upon measuring $D_\lambda$.

Similarly, one may regard a wave function $\Psi \in L^2(\mathbb{R})$ as an algebraic state $\psi$ on the $C^*$-algebra $B(L^2(\mathbb{R}))$ of all bounded operators on the Hilbert space $L^2(\mathbb{R})$. This $C^*$-algebra contains the commutative subalgebra $C_0(\mathbb{R})$ given by all multiplication operators on $L^2(\mathbb{R})$ defined by continuous functions of $x \in \mathbb{R}$ that vanish at infinity (roughly speaking, this is the $C^*$-algebra generated by the position operator). The restriction $\psi_{|C_0(\mathbb{R})}$ of $\psi$ to $C_0(\mathbb{R})$ is given by

$$\psi_{|C_0(\mathbb{R})}(f) = \int_{\mathbb{R}} dx\, |\Psi(x)|^2 f(x). \tag{9}$$

The probability measure $P_{\psi_{|C_0(\mathbb{R})}}$ on $\mathbb{R}$ associated to the functional $\psi_{|C_0(\mathbb{R})}$ by the Riesz representation theorem [21] is just $P_{\psi_{|C_0(\mathbb{R})}} = P_\Psi^{(x)}$, cf. (3). Hence the restricted state $\psi_{|C_0(\mathbb{R})}$precisely yields the Born–Pauli probability (5).

Finally, to recover (4) (assuming for simplicity that the operator $a : H \to H$ is bounded), one considers the commutative $C^*$-algebra $C^*(a)$ of $B(H)$ generated by $a$ and the unit operator. It can be shown [21] that $C^*(a) \cong C(\sigma(a))$. Hence a unit vector $\Psi \in H$ defines a state $\psi$ on $B(H)$, whose restriction $\psi_{|C^*(a)}$ to $C^*(a)$ yields a probability measure $P_{\psi_{|C^*(a)}}$ on the spectrum $\sigma(a)$ of $a$. It easily follows that

$$P_{\psi_{|C^*(a)}} = P_\Psi^{(a)}, \tag{10}$$

which reproduces (2).

The physical relevance of these constructions derives from Bohr's doctrine of classical concepts, which is an essential ingredient of the Copenhagen interpretation [24]. In particular, if it is to serve its function, a measurement apparatus has to be described as if it were classical. This implies that if it is used as a measuring device, the apparatus (which a priori is quantum mechanical) has to be described by a commutative subalgebra $D$ of its full non-commutative algebra $A$ of quantum-mechanical observables.

Upon the identifications explained above, the Born probability measure then comes out to be just the restriction of the total state on $A$ to the 'classical' subalgebra $D$ thereof that Bohr calls for.

This account does not provide a derivation of the Born rule from first principles, but it does clarify its mathematical and physical origin. In particular, in the Copenhagen interpretation probabilities arise because we look at the quantum world through classical glasses:

> "One may call these uncertainties [i.e. the Born probabilities] objective, in that they are simply a consequence of the fact that we describe the experiment in terms of classical physics; they do not depend in detail on the observer. One may call them subjective, in that they reflect our incomplete knowledge of the world." (Heisenberg [4], pp. 53–54)

In other words, one cannot say that the Born probabilities are either subjective (i.e. Bayesian, or due to ignorance) or objective (i.e. fundamentally ingrained in nature and independent of the observer). Instead, the situation is more subtle and has no counterpart in classical physics or probability theory: the choice of a particular classical description is subjective, but once it has been made the ensuing probabilities are objective and the particular outcome of an experiment compatible with the chosen classical context is unpredictable. Or so Bohr and Heisenberg say...

In most interpretations of quantum mechanics, some version of the Born rule is simply *postulated*. This is the case, for example, in the → Consistent histories interpretation, the → Modal interpretation and the → Orthodox interpretation. Attempts to *derive* the Born rule from more basic postulates of quantum theory go back to Finkelstein [16] and Hartle [19], whose work was corrected and extended in [14]. These authors study infinite sequences of measurements and prove that the ensuing relative frequencies automatically satisfy the Born rule. It is controversial, however, to what extent this argument really derives the Born rule or is eventually circular [11, 12]. In the version of the → Many worlds interpretation developed by Deutsch [13] and his followers [23, 26], the authors claim to derive the Born rule using arguments from decision theory, but once again the charge of circularity has been raised [9, 10]. See also [27, 25] for a similar debate in the context of → Decoherence. The conclusion seems to be that no generally accepted derivation of the Born rule has been given to date, but this does not imply that such a derivation is impossible in principle.

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