

On the Existence of Undistorted Progressive Waves (UPWs) of Arbitrary Speeds $0 \leq v < \infty$ in Nature

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We present the theory, the experimental evidence and fundamental physical consequences concerning the existence of families of undistorted progressive waves (UPWs) of arbitrary speeds $0 \leq v < \infty$, which are solutions of the homogeneous wave equation, the Maxwell equations, and Dirac, Weyl, and Klein-Gordon equations.

1. INTRODUCTION

In this paper we present the theory, the experimental evidence, and the fundamental physical consequences concerning the existence of families of undistorted progressive waves (UPWs)³ moving with arbitrary speeds⁴ $0 \leq v < \infty$. We show that the main equations of theoretical physics, namely the scalar *homogeneous wave equation* (HWE), the *Klein-Gordon equation* (KGE), the *Maxwell equations*, and the *Dirac and Weyl equations*, have UPWs solutions in a homogeneous medium, including the vacuum. By UPW, following Courant and Hilbert⁽¹⁾ we mean that the UPW waves are distortion-free, i.e., they are translationally invariant and thus do not spread, or they reconstruct their original form after a certain period of time. Explicit examples of how to construct the UPWs solutions for the

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³ UPW is used for the singular, i.e., for undistorted progressive wave.

⁴ We use units where $c = 1$, c being the so-called velocity of light in vacuum.

HWE are found in Appendix A. The UPWs solutions to any field equations have infinite energy. However, using the finite aperture approximation (FAA) for diffraction (Appendix A), we can project quasi-undistorted progressive waves (QUPWs) for any field equation which have finite energy and can then in principle be launched in physical space.

In Sec. 2 we show results of a recent experiment proposed and realized by us where the measurement of the speeds of a FAA to a *subluminal*⁵ Bessel pulse [Eq. (2.1)] and of the FAA to a *superluminal X-wave* [Eq. (2.5)] are done. The results are in excellent agreement with the theory.

In Sec. 3 we discuss some examples of UPW solutions of Maxwell equations; (i) subluminal solutions which are interesting concerning some recent attempts appearing in the literature⁽²⁻⁴⁾ of construction of *purely electromagnetic particles* (PEP) and (ii) a superluminal UPW solution of Maxwell equations called the superluminal electromagnetic *X-wave*⁽⁵⁾ (SEXW). We briefly discuss how to launch a FAA to SEXW. In view of the experimental results presented in Sec. 2 we are confident that such electromagnetic waves will be produced in the next few years. In Sec. 4 we discuss the important question concerning the speed of propagation of the energy carried by superluminal UPW solutions of Maxwell equations, clearing some misconceptions found in the literature. In Sec. 5 we show that the experimental production of a superluminal electromagnetic wave implies a breakdown of the principle of relativity. In Sec. 6 we present our conclusions.

Appendix B presents a *unified theory* of how to construct UPWs of arbitrary speeds $0 \leq v < \infty$ which are solutions of Maxwell, Dirac, and Weyl equations. Our unified theory is based on the Clifford bundle formalism⁽⁶⁻¹⁰⁾ where all fields quoted above are represented by objects of the same mathematical nature. We take the care of translating all results in the standard mathematical formalisms used by physicists in order for our work to be useful for a larger audience.

Before starting the technical discussions, it is worthwhile to briefly recall the history of the UPWs of arbitrary speeds $0 \leq v < \infty$, which are solutions of the main equations of theoretical physics.

To the best of our knowledge, H. Bateman⁽¹¹⁾ in 1913 was the first person to present a subluminal UPW solution of the HWE. This solution corresponds to what we called the subluminal *spherical Bessel beam* in Appendix A [see Eq. (A.31)]. Apparently this solution has been rediscovered and used in diverse contexts many times in the literature. It appears, e.g., in the papers of Mackinnon⁽¹²⁾ of 1978 and of Gueret and Vigier,⁽¹³⁾ and

⁵ In this experiment the waves are sound waves in water and, of course, the meaning of the words subluminal, luminal, and superluminal in this case is that the waves travel with speed less, equal, or greater than c_s , the so-called velocity of sound in water.

more recently in the papers of Barut and collaborators.^(14, 15) In particular, Barut⁽¹⁴⁾ also showed that the HWE has superluminal solutions. In 1987 Durnin and collaborators rediscovered a subluminal UPW solution of the HWE in cylindrical coordinates.⁽¹⁶⁻¹⁸⁾ These are the Bessel beams of Sec. A4 [see Eq. (A.41)]. We said rediscovered because these solutions are known at least since 1941, as they are explicitly written down in Stratton's book.⁽¹⁹⁾ The important point here is that Durnin⁽¹⁶⁾ and collaborators constructed an optical subluminal Bessel beam. At that time they did not have the idea of measuring the speed of the beams, since they were interested in the fact that the FAA to these beams were quasi-UPWs and could be very useful for optical devices. Indeed they used the term "diffraction-free beams," which has been adopted by some other authors later. Other authors still use for UPWs the term *nondispersive beams*. We quote also that Hsu and collaborators⁽²⁰⁾ realized a FAA to the J_0 Bessel beam [Eq. (A.41)] with a narrow-band PZT ultrasonic transducer of non-uniform polishing. Lu and Greenleaf⁽²¹⁾ produced the first J_0 nondiffracting annular array transducers with a PZT ceramic/polymer composite and applied it to medical acoustic imaging and tissue characterization.^(22, 23) Also Campbell *et al.*⁽²⁴⁾ used an annular array to realize a FAA to a J_0 Bessel beam and compared the J_0 beam to the so-called *axicon beam*.⁽²⁵⁾ For more on this topic, see Ref. 26.

Luminal solutions of a new kind for the HWE and Maxwell equations, also known as focus wave mode [FWM] (see Appendix A), have been discovered by Brittingham⁽²⁷⁾ (1983), and his work inspired many interesting and important studies as, e.g., Refs. 29 to 40.

To our knowledge, the first person to write about the possibility of a *superluminal* UPW solution of HWE and, more important, of Maxwell equations was Band.^(41, 42) He constructed a superluminal electromagnetic UPW from the modified Bessel beam [Eq. (A.42)], which was used to generate in an appropriate way an electromagnetic potential in the Lorentz gauge. He suggested that his solution could be used to eventually launch a superluminal wave in the exterior of a conductor with cylindrical symmetry with appropriate charge density. We discuss further some of Band's statements in Sec. 4.

In 1992, Lu and Greenleaf⁽⁴³⁾ presented the first superluminal UPW solution of the HWE for acoustic waves which could be launched by a physical device.⁽⁴⁴⁾ They discovered the so-called *X-waves*, a name due to their shape (see Fig. 3). In the same year Donnelly and Ziolkowski⁽⁴⁵⁾ presented a thoughtful method for generating UPW solutions of homogeneous partial equations. In particular, they studied also UPW solutions for the wave equation in a lossy infinite medium and to the KGE. They clearly stated also how to use these solutions to obtain through the

Hertz potential method (see Appendix B3) UPW solutions of Maxwell equations.

In 1993 Donnelly and Ziolkowski⁽⁴⁶⁾ reinterpreted their study of Ref. 45 and obtained subluminal, luminal, and superluminal UPW solutions of the HWE and of the KGE. In Appendix A we make use of the methods of this important paper in order to obtain some UPW solutions. Also in 1992 Barut and Chandola⁽⁴⁷⁾ found superluminal UPW solutions of the HWE. In 1995 Rodrigues and Vaz⁽⁴⁸⁾ discovered in quite an independent way⁶ subluminal and superluminal UPW solutions of Maxwell equations and the Weyl equation. At that time Lu and Greenleaf⁽⁵⁾ proposed also to launch a superluminal electromagnetic X -wave.⁷

In September 1995 Professor Ziolkowski took knowledge of⁽⁴⁸⁾ and informed one of the authors [WAR] of his publications and also of Lu's contributions. Soon a collaboration with Lu started which produced this paper. To end this introduction, we must call to the reader's attention that in the last few years several important experiments concerning the superluminal tunneling of electromagnetic waves appeared in the literature.^(51, 52) Particularly interesting is Nimtz's paper⁽⁵³⁾ announcing that he transmitted Mozart's Symphony #40 at 4.7c through a rectangular waveguide. The solutions of Maxwell equations in a waveguide led to solutions of Maxwell equations that propagate with subluminal or superluminal speeds. These solutions can be obtained with the methods discussed in this paper and will be discussed in another publication.

2. EXPERIMENTAL DETERMINATION OF THE SPEEDS OF ACOUSTIC FINITE-APERTURE BESSEL PULSES AND X -WAVES

In Appendix A we show the existence of several UPW solutions to the HWE, in particular the subluminal UPWs Bessel beams [Eq. (A.36)] and the superluminal UPWs X -waves [Eq. (A.52)]. Theoretically the UPW X -waves, both the broad-band and band-limited [see Eq. (2.4)], travel with speed $v = c_s / \cos \eta > 1$. Since only FAA to these X -waves can be launched with appropriate devices, the question arises if these FAA X -waves travel also with speed greater than c_s , which can be answered only by experiment.

⁶ Rodrigues and Vaz are interested in obtaining solutions of Maxwell equations characterized by non-null field invariants, since solutions of this kind are^(49, 50) necessary in proving a surprising relationship between Maxwell and Dirac equations.

⁷ A version of Ref. 5 was submitted to *IEEE Trans. Antennas Propag.* in 1991. See Ref. 40 of Ref. 43.

Here we present the results of measurements of the speeds of a FAA to a broad-band Bessel beam, called a Bessel pulse (see below), and of a FAA to a band-limited X -wave, both moving in water. We write the formulas for these beams, inserting into the HWE the parameter c_s known as the speed of sound in water. In this way the dispersion relation [Eq. (A.37)] must read

$$\frac{\omega^2}{c_s^2} - k^2 = \alpha^2 \tag{2.1}$$

Then we write for the Bessel beams

$$\Phi_{J_n}^<(t, \vec{x}) = J_n(\alpha\rho) e^{i(kz - \omega t + n\theta)}, \quad n = 0, 1, 2, \dots \tag{2.2}$$

Bessel pulses are obtained from Eq. (2.2) by weighting it with a transmitting transfer function $T(\omega)$ and then linearly superposing the result over angular frequency ω , i.e., we have

$$\Phi_{JBB_n}^<(t, \vec{x}) = 2\pi e^{in\theta} J_n(\alpha\rho) \mathcal{F}^{-1}[T(\omega) e^{ikz}] \tag{2.3}$$

where \mathcal{F}^{-1} is the inverse Fourier transform. The FAA to $\Phi_{JBB_n}^<$ will be denoted by $\text{FAA}\Phi_{JBB_n}^<$ (or $\Phi_{FAJ_n}^<$).

We recall that the X -waves are given by Eq. (A.52), i.e.,

$$\Phi_{X_n}^>(t, \vec{x}) = e^{in\theta} \int_0^\infty B(\bar{k}) J_n(\bar{k}\rho \sin \eta) e^{-\bar{k}[a_0 - i(z \cos \eta - c_s t)]} d\bar{k} \tag{2.4}$$

where $\bar{k} = k/\cos \eta$, $\bar{k} = \omega/c_s$. By choosing $B(\bar{k}) = a_0$ we have the infinite-aperture broad-bandwidth X -wave [Eq. (A.53)] given by

$$\Phi_{XBB_n}^>(t, \vec{x}) = \frac{a_0(\rho \sin \eta)^n e^{in\theta}}{\sqrt{M}(\tau + \sqrt{M})^n} \tag{2.5}$$

$$M = (\rho \sin \eta)^2 + \tau^2, \quad \tau = [a_0 - i(z \cos \eta - c_s t)]$$

A FAA to $\Phi_{XBB_n}^>$ will be denoted by $\text{FAA}\Phi_{XBB_n}^>$. When $B(\bar{k})$ in Eq. (2.4) is different from a constant, e.g., if $B(\bar{k})$ is the Blackman window function, we denote the X -wave by $\Phi_{XBL_n}^>$, where BL means band limited. A FAA to $\Phi_{XBL_n}^>$ will be denoted $\text{FAA}\Phi_{XBL_n}^>$. Also when $T(\omega)$ in Eq. (2.3) is the Blackman window function, we denote the respective wave by $\Phi_{JBL_n}^>$.

As discussed in Appendix A and detailed in Refs. 26 and 44 to produce a FAA to a given beam, the aperture of the transducer used must be finite. In this case the beams produced, in our case $\text{FAA}\Phi_{JBL_0}^>$ and

FAA Φ_{XBB_0} , have a finite depth of field⁽²⁶⁾ (DF)⁸ and can be approximately produced by truncating the infinite-aperture beams Φ_{JBL_0} and Φ_{XBB_0} (or Φ_{XBL_0}) at the transducer surface ($z=0$). Broad-band pulses for $z>0$ can be obtained by first calculating the fields at all frequencies with Eq. (A.28), i.e.,

$$\begin{aligned} \tilde{\Phi}_{FA}(\omega, \vec{x}) = & \frac{1}{i\lambda} \int_0^a \int_{-\pi}^{\pi} \rho' dp' d\theta' \tilde{\Phi}(\omega, \vec{x}') \frac{e^{i\vec{k}R}}{R^2} z \\ & + \frac{1}{2\pi} \int_0^a \int_{-\pi}^{\pi} \rho' dp' d\theta' \tilde{\Phi}(\omega, \vec{x}') \frac{e^{i\vec{k}R}}{R^3} z \end{aligned} \quad (2.6)$$

where the aperture weighting function $\tilde{\Phi}(\omega, \vec{x}')$ is obtained from the temporal Fourier transform of Eqs. (2.3) and (2.4). If the aperture is circular of radius a [as in Eq. (2.6)], the depth of field of the FAA Φ_{JBL_0} pulse, denoted BZ_{\max} , and the depth of field of the FAA Φ_{XBB_0} or FAA Φ_{XBL_0} denoted by XZ_{\max} are given by⁽²⁶⁾

$$BZ_{\max} = a \sqrt{(\omega_0/c_s \alpha)^2 - 1}, \quad XZ_{\max} = a \cot \eta \quad (2.7)$$

For the FAA Φ_{JBL_0} pulse we choose $T(\omega)$ as the Blackman window function⁽⁵⁴⁾ that is peaked at the central frequency $f_0=2.5$ MHz with a relative bandwidth of about 81% (-6 dB bandwidth divided by the central frequency). We have

$$B(\bar{k}) = \begin{cases} a_0 \left[0.42 - 0.5 \frac{\pi \bar{k}}{k_0} + 0.08 \cos \frac{2\pi \bar{k}}{k_0} \right], & 0 \leq \bar{k} \leq 2\bar{k}_0 \\ 0, & \text{otherwise} \end{cases} \quad (2.8)$$

The “scaling factor” in the experiment is $\alpha = 1202.45 \text{ m}^{-1}$ and the weighting function $\tilde{\Phi}_{JBB_0}(\omega, \vec{x})$ in Eq. (2.6) is approximated with stepwise functions. In practice this is done with the 10-element annular array transfer built by Lu and Greenleaf.^(26, 44) The diameter of the array is 50 mm. Figure 1 shows the block diagram for the production of FAA $\Phi_{XBL_0}^>$ and FAA $\Phi_{JBL_0}^<$.⁹ The measurement of the speed of the FAA Bessel pulse has been done by comparing the speed with which the peak of the FAA Bessel pulse travels with the speed of the peak of a pulse produced by a small circular element of the array (about 4 mm or 6.67λ in diameter, where λ is 0.6 mm in water). This pulse travels with speed $c_s = 1.5 \text{ mm}/\mu\text{s}$. The distance between the peaks and the surface of the transducer are

⁸ DF is the distance where the field maximum drops to half the value at the surface of the transducer.

⁹ Reprinted with permission from Fig. 2 of Ref. 44.

104.33(9) mm and 103.70(5) mm for the single-element wave and the Bessel pulse, respectively, at the same instant t of measurement. The results can be seen in the pictures taken from of the experiment in Fig. 2. As predicted by the theory developed in Appendix A, the speed of the Bessel pulse is 0.611(3)% slower than the speed c_s of the usual sound wave produced by the single element.

The measurement of the speed of the central peak of the $FAA\Phi_{XBL_0}^>$ wave obtained from Eq. (2.4) with a Blackman window function [Eq. (2.8)] has been done in the same way as for the Bessel pulse. The $FAA\Phi_{XBL_0}$ wave was produced by the 10-element array transducer of 50 mm diameter with the techniques developed by Lu and Greenleaf.^(26, 44) The distances traveled at the same instant t by the single-element wave and the X -wave are respectively 173.48(9) mm and 173.77(3) mm. Figure 3 shows the pictures taken from the experiment. In this experiment the axicon angle is $\eta = 4^\circ$. The theoretical speed of the infinite aperture X -wave is predicted to be 0.2242% greater than c_s . We found that the $FAA\Phi_{XBB_0}$ wave traveled with speed 0.267(6)% greater than c_s !

These results, which we believe are the first experimental determination of the speeds of subluminal and superluminal quasi-UPWs $FAA\Phi_{JBL_0}^>$ and $FAA\Phi_{JBB_0}^<$ solutions of the HWE, together with the fact that, as already quoted, Durnin⁽¹⁶⁾ produced subluminal *optical* Bessel beams, give us confidence that electromagnetic subluminal and superluminal waves

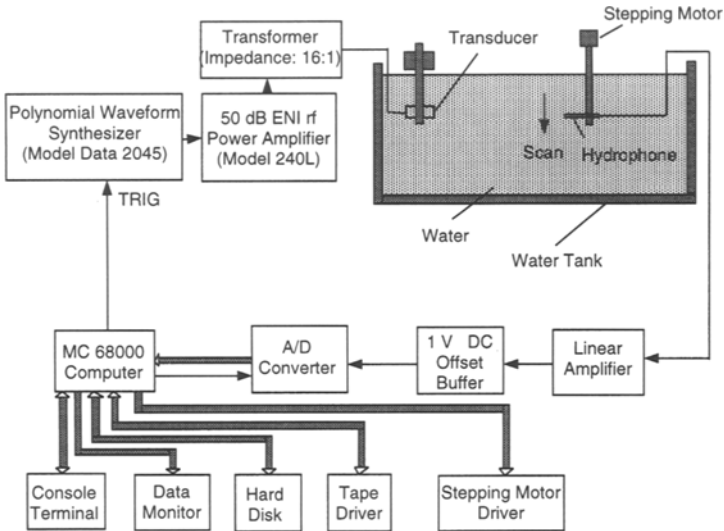


Fig. 1. Block diagram of acoustic production of the Bessel pulse and X -waves.

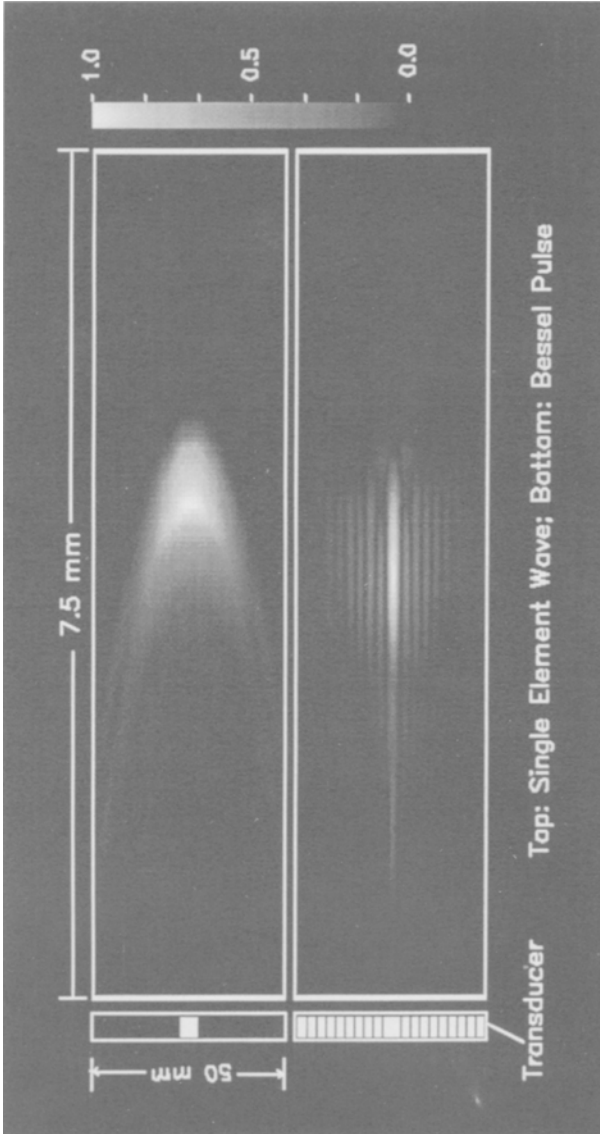


Fig. 2. Propagation speed of the peak of the Bessel pulse and its comparison with that of a pulse produced by a small circular element (about 4 mm or 6.67λ in diameter, where λ is 0.6 mm in water). The Bessel pulse was produced by a 50-mm-diameter transducer. The distances between the peaks and the surface of the transducer are 104.339 and 103.705 mm for the single-element wave and the Bessel pulse, respectively. The time used by these pulses is the same. Therefore, the speed of the peak of the Bessel pulse is 0.611(3)% slower than that of the single-element wave.

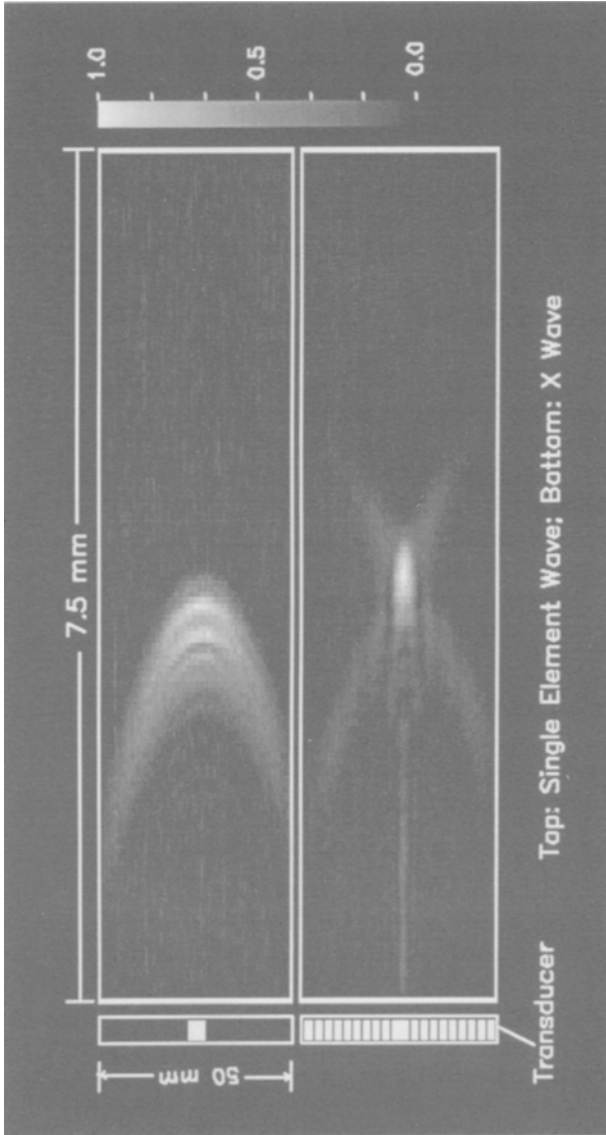


Fig. 3. Propagation speed of the peak of the X-wave and its comparison with that of a pulse produced by a small circular element (about 4 mm or 6.67λ , where λ is 0.6 mm in water). The X-wave was produced by a 50-mm -diameter transducer. The distance between the peaks and the surface of the transducer are 173.489 and 173.773 mm for the single-element wave and the X-wave, respectively. The time used by these pulses is the same. Therefore, the speed of the peak of the X-wave is $0.2441(8)\%$ faster than that of the single-element wave. The theoretical ratio for X-waves and the speed of sound is $(c_s/\cos \eta - c_s)/c_s = 0.2442\%$ for $\eta = 4^\circ$.

may be physically launched with appropriate devices. In the next section we study in particular the superluminal electromagnetic X -wave (SEXW).

It is important to observe here the following crucial points: (i) The $\text{FAA}\Phi_{XBB_n}$ is produced by the source (transducer) in a short period of time Δt . However, different parts of the transducer are activated at different times, from 0 to Δt , calculated from Eqs. (A.9) and (A.28). As a result the wave is born as an integral object for time Δt and propagates with the same speed as the peak. This is exactly what has been seen in the experiments and is corroborated by the computer simulations we did for the superluminal electromagnetic waves (see Sec. 3). (ii) One can find in almost all textbooks that the velocity of transport of energy for waves obeying the scalar wave equation

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi = 0 \quad (2.9)$$

is given by

$$\vec{v}_e = \frac{\vec{S}}{u} \quad (2.10)$$

where \vec{S} is the flux of momentum and u is the energy density, given by

$$\vec{S} = \nabla\phi \frac{\partial\phi}{\partial t}, \quad u = \frac{1}{2} \left[(\nabla\phi)^2 + \frac{1}{c^2} \left(\frac{\partial\phi}{\partial t}\right)^2 \right] \quad (2.11)$$

from which it follows that

$$v_e = \frac{|\vec{S}|}{u} \leq c_s \quad (2.12)$$

Our acoustic experiment shows that for the X -waves the speed of transport of energy is $c_s/\cos\eta$, since it is the energy of the wave that activates the detector (hydrophone). This shows explicitly that the definition of v_e is meaningless. This fundamental experimental result must be kept in mind when we discuss the meaning of the velocity of transport of electromagnetic waves in Sec. 4.

3. SUBLUMINAL AND SUPERLUMINAL UPW SOLUTIONS OF MAXWELL EQUATIONS (ME)

In this section we make full use of the Clifford bundle formalism (CBF) resumed in Appendix B, but we give translation of all the main

results in the standard vector formalism used by physicists. We start by reanalyzing in Sec. 3.1 the plane wave solutions (PWS) of ME with the CBF. We clarify some misconceptions and explain the fundamental role of the duality operator γ_5 and the meaning of $i = \sqrt{-1}$ in standard formulations of electromagnetic theory. Next, in Sec. 3.2 we discuss subluminal UPW solutions of ME and an unexpected relation between these solutions and the possible existence of purely electromagnetic particles (PEPs) envisaged by Einstein,⁽⁵⁵⁾ Poincaré,⁽⁵⁶⁾ and Ehrenfest⁽⁵⁷⁾ and recently discussed by Waite, Barut, and Zeni.^(2,3) In Sec. 3.3 we discuss in detail the theory of superluminal electromagnetic *X*-waves (SEXWs) and how to produce these waves by appropriate physical devices.

3.1. Plane Wave Solutions of Maxwell Equations

We recall that Maxwell equations in vacuum can be written as [Eq. (B.6)]

$$\partial F = 0 \tag{3.1}$$

where $F \in \sec \wedge^2(M) \subset \sec \mathcal{C}\ell(M)$. The well-known PWS of Eq. (3.1) are obtained as follows. We write in a given Lorentzian chart $\langle x^\mu \rangle$ of the maximal atlas of M (Sec. B2) a PWS moving in the z -direction

$$F = f e^{\gamma_5 k x} \tag{3.2}$$

$$k = k^\mu \gamma_\mu, \quad k^1 = k^2 = 0, \quad x = x^\mu \gamma_\mu \tag{3.3}$$

where $k, x \in \sec \wedge^1(M) \subset \sec \mathcal{C}\ell(M)$ and where f is a constant 2-form. From Eqs. (3.1) and (3.2) we obtain

$$k F = 0 \tag{3.4}$$

Multiplying Eq. (3.4) by k we get

$$k^2 F = 0 \tag{3.5}$$

and since $k \in \sec \wedge^1(M) \subset \sec \mathcal{C}\ell(M)$, then

$$k^2 = 0 \leftrightarrow k^0 = \pm |\vec{k}| = \pm k^3 \tag{3.6}$$

i.e., the propagation vector is lightlike. Also

$$F^2 = F \cdot F + F \wedge F = 0 \tag{3.7}$$

as can be easily seen by multiplying both members of Eq. (3.4) by F and taking into account that $k \neq 0$. Equation (3.7) says that the field invariants are null.

It is interesting to understand the fundamental role of the volume element γ_5 (duality operator) in electromagnetic theory. In particular, since $e^{\gamma_5 kx} = \cos kx + \gamma_5 \sin kx$, $\gamma_5 \equiv \mathbf{i}$, writing $F = \vec{E} + \mathbf{i}\vec{B}$ [see Eq. (B.17)], $f = \vec{e}_1 + \mathbf{i}\vec{e}_2$, we see that

$$\vec{E} + \mathbf{i}\vec{B} = \vec{e}_1 \cos kx - \vec{e}_2 \sin kx + \mathbf{i}(\vec{e}_1 \sin kx + \vec{e}_2 \cos kx) \quad (3.8)$$

From this equation, using $\partial F = 0$, it follows that $\vec{e}_1 \cdot \vec{e}_2 = 0$, $\vec{k} \cdot \vec{e}_1 = \vec{k} \cdot \vec{e}_2 = 0$ and then

$$\vec{E} \cdot \vec{B} = 0 \quad (3.9)$$

This equation is important because it shows that we must take care with the $i = \sqrt{-1}$ that appears in the usual formulations of Maxwell theory using complex electric and magnetic fields. The $i = \sqrt{-1}$ in many cases unfolds a secret that can only be known through Eq. (3.8). It also follows that $\vec{k} \cdot \vec{E} = \vec{k} \cdot \vec{B} = 0$, i.e., PWS of ME are *transverse* waves. We can rewrite Eq. (3.4) as

$$k\gamma_0\gamma_0 F\gamma_0 = 0 \quad (3.10)$$

and since $k\gamma_0 = k_0 + \vec{k}$, $\gamma_0 F\gamma_0 = -\vec{E} + \mathbf{i}\vec{B}$ we have

$$\vec{k}f = k_0 f \quad (3.11)$$

Now, we recall that in $\mathcal{C}\ell^+(M)$ (where, as we say in Appendix B, the typical fiber is isomorphic to the Pauli algebra $\mathcal{C}\ell_{3,0}$) we can introduce the operator of space conjugation denoted by $*$ such that writing $f = \vec{e} + \mathbf{i}\vec{b}$ we have

$$f^* = -\vec{e} + \mathbf{i}\vec{b}, \quad k_0^* = k_0, \quad \vec{k}^* = -\vec{k} \quad (3.12)$$

We can now interpret the two solutions of $k^2 = 0$, i.e. $k_0 = |\vec{k}|$ and $k_0 = -|\vec{k}|$, as corresponding to the solutions $k_0 f = \vec{k}f$ and $k_0 f^* = -\vec{k}f^*$; f and f^* correspond in quantum theory to “photons” of positive or negative helicities. We can interpret $k_0 = |\vec{k}|$ as a particle and $k_0 = -|\vec{k}|$ as an antiparticle.

Summarizing we have the following important facts concerning PWS of ME: (i) the propagation vector is lightlike, $k^2 = 0$; (ii) the field invariants are null, $F^2 = 0$; (iii) the PWS are transverse waves, i.e., $\vec{k} \cdot \vec{E} = \vec{k} \cdot \vec{B} = 0$.

3.2. Subluminal Solutions of Maxwell Equations and Purely Electromagnetic Particles

We take $\Phi \in \sec(\wedge^0(M) \oplus \wedge^4(M)) \subset \sec \mathcal{C}\ell(M)$ and consider the following Hertz potential $\pi \in \sec \wedge^2(M) \subset \sec \mathcal{C}\ell(M)$ [Eq. (B.25)]

$$\pi = \Phi \gamma^1 \gamma^2 \tag{3.13}$$

We now write

$$\Phi(t, \vec{x}) = \phi(\vec{x}) e^{\gamma_5 \Omega t} \tag{3.14}$$

Since π satisfies the wave equation, we have

$$\nabla^2 \phi(\vec{x}) + \Omega^2 \phi(\vec{x}) = 0. \tag{3.15}$$

Solutions of Eq. (3.15) (the Helmholtz equation) are well known. Here we consider the simplest solution in spherical coordinates,

$$\phi(\vec{x}) = C \frac{\sin \Omega r}{r}, \quad r = \sqrt{x^2 + y^2 + z^2} \tag{3.16}$$

where C is an arbitrary real constant. From the results of Appendix B we obtain the following stationary electromagnetic field, which is at rest in the reference frame Z where $\langle x^\mu \rangle$ are naturally adapted coordinates (Sec. B2):

$$\begin{aligned} F_0 = \frac{C}{r^3} [& \sin \Omega t (\alpha \Omega r \sin \theta \sin \varphi - \beta \sin \theta \cos \theta \cos \varphi) \gamma_0 \gamma_1 \\ & - \sin \Omega t (\alpha \Omega r \sin \theta \cos \varphi + \beta \sin \theta \cos \theta \sin \varphi) \gamma_0 \gamma_2 \\ & + \sin \Omega t (\beta \sin^2 \theta - 2\alpha) \gamma_0 \gamma_3 + \cos \Omega t (\beta \sin^2 \theta - 2\alpha) \gamma_1 \gamma_2 \\ & + \cos \Omega t (\beta \sin \theta \cos \theta \sin \varphi + \alpha \Omega r \sin \theta \cos \varphi) \gamma_1 \gamma_3 \\ & + \cos \Omega t (-\beta \sin \theta \cos \theta \cos \varphi + \alpha \Omega r \sin \theta \sin \varphi) \gamma_2 \gamma_3] \end{aligned} \tag{3.17}$$

with $\alpha = \Omega r \cos \Omega r - \sin \Omega r$ and $\beta = 3\alpha + \Omega^2 r^2 \sin \Omega r$. Observe that F_0 is regular at the origin and vanishes at infinity. Let us rewrite the solution using the Pauli algebra in $\mathcal{C}\ell^+(M)$. Writing ($\mathbf{i} \equiv \gamma_5$)

$$F_0 = \vec{E}_0 + \mathbf{i} \vec{B}_0 \tag{3.18}$$

we get

$$\vec{E}_0 = \vec{W} \sin \Omega t, \quad \vec{B}_0 = \vec{W} \cos \Omega t \tag{3.19}$$

with

$$\vec{W} = -C \left(\frac{\alpha \Omega y}{r^3} - \frac{\beta xz}{r^5}, -\frac{\alpha \Omega x}{r^3} - \frac{\beta yz}{r^5}, \frac{\beta(x^2 + y^2)}{r^5} - \frac{2\alpha}{r^3} \right) \quad (3.20)$$

We verify that $\text{div } \vec{W} = 0$, $\text{div } \vec{E}_0 = \text{div } \vec{B}_0 = 0$, $\text{rot } \vec{E}_0 + \partial \vec{B}_0 / \partial t = 0$, $\text{rot } \vec{B}_0 - \partial \vec{E}_0 / \partial t = 0$, and

$$\text{rot } \vec{W} = \Omega \vec{W} \quad (3.21)$$

Now, from Eq. (B.88) we know that $T_0 = \frac{1}{2} \vec{F} \gamma_0 F$ is the 1-form representing the energy density and the Poynting vector. It follows that $\vec{E}_0 \times \vec{B}_0 = 0$, i.e., the solution has zero angular momentum. The energy density $u = S^{00}$ is given by

$$u = \frac{1}{r^6} [\sin^2 \theta (\Omega^2 r^2 \alpha^2 + \beta^2 \cos^2 \theta) + (\beta \sin^2 \theta - 2\alpha)^2] \quad (3.22)$$

Then $\iiint_{\mathbb{R}^3} u \, d\mathbf{v} = \infty$. As explained in Sec. A6 a finite energy solution can be constructed by considering “wave packets” with a distribution of intrinsic frequencies $F(\Omega)$ satisfying appropriate conditions. Many possibilities exist, but they will not be discussed here. Instead, we prefer to direct our attention to Eq. (3.21). As is well known, this is a very important equation (called the force-free equation⁽²⁾) that appears, e.g., in hydrodynamics and in several different situations in plasma physics.⁽⁵⁸⁾ The following considerations are more important.

Einstein,⁽⁵⁵⁾ among others (see Ref. 3 for a review), studied the possibility of constructing purely electromagnetic particles (PEPs). He started from Maxwell equations for a PEP configuration described by an electromagnetic field F_p and a current density J_p , where

$$\partial F_p = J_p \quad (3.23)$$

and rightly concluded that the condition for existence of PEPs is

$$J_p \cdot F_p = 0 \quad (3.24)$$

This condition implies in vector notation

$$\rho_p \vec{E}_p = 0, \quad \vec{j}_p \cdot \vec{E}_p = 0, \quad \vec{j}_p \times \vec{B}_p = 0 \quad (3.25)$$

From Eq. (3.24) Einstein concluded that the only possible solution of Eq. (3.23) with the subsidiary condition given by Eq. (3.24) is $J_p = 0$.

However, this conclusion is correct, as pointed in Refs. 2 and 3, only if $J_p^2 > 0$, i.e., if J_p is a timelike current density. However, if we suppose that J_p can be spacelike, i.e., $J_p^2 < 0$, there exists a reference frame where $\rho_p = 0$, and a possible solution of Eq. (3.24) is

$$\rho_p = 0, \quad \vec{E}_p \cdot \vec{B}_p = 0, \quad \vec{j}_p = KC\vec{B}_p \tag{3.26}$$

where $K = \pm 1$ is called the chirality of the solution and C is a real constant. In Refs. 2 and 3 static solutions of Eqs. (3.22) and (3.23) are exhibited where $\vec{E}_p = 0$. In this case we can verify that \vec{B}_p satisfies

$$\nabla \times \vec{B}_p = KC\vec{B}_p \tag{3.27}$$

Now, if we choose $F_0 \in \sec \wedge^2(M) \subset \sec \mathcal{C}\ell(M)$ such that

$$\begin{aligned} F_0 &= \vec{E}_0 + \mathbf{i}\vec{B}_0 \\ \vec{E}_0 &= \vec{B}_p \cos \Omega t, \quad \vec{B}_0 = \vec{B}_p \sin \Omega t \end{aligned} \tag{3.28}$$

and $\Omega = KC > 0$, we immediately realize that

$$\partial F_0 = 0 \tag{3.29}$$

This is an amazing result, since it means that the free Maxwell equations may have stationary solutions that may be used to model PEPs. In such solutions the structure of the field F_0 is such that we can write

$$\begin{aligned} F_0 &= F'_p + \vec{F} = \mathbf{i}\vec{W} \cos \Omega t - \vec{W} \sin \Omega t \\ \partial F'_p &= -\partial \vec{F} = J'_p \end{aligned} \tag{3.30}$$

i.e., $\partial F_0 = 0$ is equivalent to a field plus a current. This fact opens several interesting possibilities for modeling PEPs (see also Ref. 4) and we discuss this issue further in another publication.

We observe that moving subluminal solutions of ME can be easily obtained choosing as Hertz potential, e.g.,

$$\pi_{<}(t, \vec{x}) = C \frac{\sin \Omega \xi_{<}}{\xi_{<}} \exp[\gamma_5(\omega_{<} t - k_{<} z)] \gamma_1 \gamma_2 \tag{3.31}$$

$$\begin{aligned} \omega_{<}^2 - k_{<}^2 &= \Omega^2 \\ \xi_{<} &= [x^2 + y^2 + \gamma_{<}^2(z - v_{<} t)^2] \end{aligned} \tag{3.32}$$

$$\gamma_{<} = \frac{1}{\sqrt{1 - v_{<}^2}}, \quad v_{<} = d\omega_{<} / dk_{<}$$

We are not going to write explicitly the expression for $F^<$ corresponding to $\pi^<$ because it is very long and will not be used in what follows.

We end this section with the following observations: (i) In general for subluminal solutions of ME (SSME) the propagation vector satisfies an equation like Eq. (3.30). (ii) As can be easily verified, for a SSME the field invariants are non-null. (iii) A SSME is not a transverse wave. This can be seen explicitly from Eq. (3.21). Conditions (i), (ii), and (iii) are in contrast with the case of the PWS of ME. In Refs. 49 and 50 Rodrigues and Vaz showed that for free electromagnetic fields ($\partial F = 0$) such that $F^2 \neq 0$, there exists a Dirac–Hestenes equation (see Sec. A8) for $\psi \in \sec(\wedge^0(M) + \wedge^2(M) + \wedge^4(M)) \subset \sec \mathcal{C}\ell(M)$ where $F = \psi \gamma_1 \gamma_2 \bar{\psi}$. This was the reason why Rodrigues and Vaz discovered subluminal and superluminal solutions of Maxwell equations (and also of Weyl equation)⁽⁴⁸⁾ which solve the Dirac–Hestenes equation [Eq. (B.40)].

3.3. The Superluminal Electromagnetic X-Wave (SEXW)

To simplify the matter, in what follows we now suppose that the functions Φ_{X_n} [Eq. (A.52)] and Φ_{XBB_n} [Eq. (A.53)], which are superluminal solutions of the scalar wave equation, are 0-form sections of the complexified Clifford bundle $\mathcal{C}\ell_c(M) = \mathbb{C} \otimes \mathcal{C}\ell(M)$ (see Sec. B4). We rewrite Eqs. (A.52) and (A.53) as¹⁰

$$\Phi_{X_n}(t, \vec{x}) = e^{in\theta} \int_0^\infty B(\vec{k}) J_n(\vec{k}\rho \sin \eta) e^{-\vec{k}[a_0 - i(z \cos \eta - t)]} d\vec{k} \quad (3.33)$$

and choosing $B(\vec{k}) = a_0$, we have

$$\Phi_{XBB_n}(t, \vec{x}) = \frac{a_0(\rho \sin \eta)^n e^{in\theta}}{\sqrt{M}(\tau + \sqrt{M})^n} \quad (3.34)$$

$$M = (\rho \sin \eta)^2 + \tau^2, \quad \tau = [a_0 - i(z \cos \eta - t)] \quad (3.35)$$

As in Sec. 2, when a finite broadband X-wave is obtained from Eq. (3.31) with $B(\vec{k})$ given by the Blackman spectral function [Eq. (2.8)] we denote the resulting X-wave by Φ_{XBL_n} (BL means band-limited wave). The finite aperture approximation (FAA) obtained with Eq. (A.28) to Φ_{XBL_n} will be denoted $\text{FAA}\Phi_{XBL_n}$ and the FAA to Φ_{XBB_n} will be denoted $\text{FAA}\Phi_{XBB_n}$. We use the same nomenclature for the electromagnetic fields derived from these functions. Further, we suppose now that the Hertz potential π , the vector

¹⁰ In what follows, $n = 0, 1, 2, \dots$

potential A , and the corresponding electromagnetic field F are appropriate sections of $\mathcal{C}\ell_C(M)$. We take

$$\pi = \Phi \gamma_1 \gamma_2 \in \sec \mathbb{C} \otimes \bigwedge^2 (M) \subset \sec \mathcal{C}\ell_C(M) \tag{3.36}$$

where Φ can be Φ_{X_n} , Φ_{XBB_n} , Φ_{XBL_n} , $FAA\Phi_{XBB_n}$, or $FAA\Phi_{XBL_n}$. Let us start by giving the explicit form of the F_{XBB_n} , i.e., the SEXWs. In this case Eq. (B.26) gives $\pi = i\vec{\pi}_m$ and

$$\vec{\pi}_m = \Phi_{XBB_n} \mathbf{z} \tag{3.37}$$

where \mathbf{z} is the versor of the z -axis. Also, let $\boldsymbol{\rho}$, $\boldsymbol{\theta}$ be respectively the versors of the ρ and θ directions, where (ρ, θ, z) are the usual cylindrical coordinates. Writing

$$F_{XBB_n} = \vec{E}_{XBB_n} + \gamma_5 \vec{B}_{XBB_n} \tag{3.38}$$

we obtain from Eqs. (A.53) and (B.25):

$$\vec{E}_{XBB_n} = -\frac{\boldsymbol{\rho}}{\rho} \frac{\partial^2}{\partial t \partial \theta} \Phi_{XBB_n} + \boldsymbol{\theta} \frac{\partial^2}{\partial t \partial \rho} \Phi_{XBB_n} \tag{3.39}$$

$$\vec{B}_{XBB_n} = \boldsymbol{\rho} \frac{\partial^2}{\partial \rho \partial z} \Phi_{XBB_n} + \boldsymbol{\theta} \frac{1}{\rho} \frac{\partial^2}{\partial \theta \partial z} \Phi_{XBB_n} + \mathbf{z} \left(\frac{\partial^2}{\partial z^2} \Phi_{XBB_n} - \frac{\partial^2}{\partial t^2} \Phi_{XBB_n} \right) \tag{3.40}$$

Explicitly we get for the components in cylindrical coordinates:

$$(\vec{E}_{XBB_n})_\rho = -\frac{1}{\rho} n \frac{M_3}{\sqrt{M}} \Phi_{XBB_n} \tag{3.41a}$$

$$(\vec{E}_{XBB_n})_\theta = \frac{1}{\rho} i \frac{M_6}{\sqrt{M} M_2} \Phi_{XBB_n} \tag{3.41b}$$

$$(\vec{B}_{XBB_n})_\rho = \cos \eta (\vec{E}_{XBB_n})_0 \tag{3.41c}$$

$$(\vec{B}_{XBB_n})_0 = -\cos \eta (\vec{E}_{XBB_n})_\rho \tag{3.41d}$$

$$(\vec{B}_{XBB_n})_z = -\sin^2 \eta \frac{M_7}{\sqrt{M}} \Phi_{XBB_n} \tag{3.41e}$$

The functions M_i , $i = 2, \dots, 7$, in (3.41) are

$$M_2 = \tau + \sqrt{M} \tag{3.42a}$$

$$M_3 = n + \frac{1}{\sqrt{M}} \tau \tag{3.42b}$$

$$M_4 = 2n + \frac{3}{\sqrt{M}} \tau \tag{3.42c}$$

$$M_5 = \tau + n \sqrt{M} \tag{3.42d}$$

$$M_6 = \left(\rho^2 \sin^2 \eta \frac{M_4}{M} - nM_3 \right) M_2 + n\rho^2 \frac{M_5}{M} \sin^2 \eta \tag{3.42e}$$

$$M_7 = (n^2 - 1) \frac{1}{\sqrt{M}} + 3n \frac{1}{M} \tau + 3 \frac{1}{\sqrt{M}^3} \tau^2 \tag{3.42f}$$

We immediately see from Eqs. (3.41) that the F_{XBB_n} are indeed superluminal UPW solutions of ME, propagating with speed $1/\cos \eta$ in the z -direction. That F_{XBB_n} are UPWs is trivial and that they propagate with speed $c_1 = 1/\cos \eta$ follows because F_{XBB_n} depends only on the combination of variables $(z - c_1 t)$ and any derivatives of Φ_{XBB_n} will keep the $(z - c_1 t)$ dependence structure.

Now, the Poynting vector \vec{P}_{XBB_n} and the energy density u_{XBB_n} for F_{XBB_n} are obtained by considering the real parts of \vec{E}_{XBB_n} and \vec{B}_{XBB_n} . We have

$$(\vec{P}_{XBB_n})_\rho = -\text{Re}\{(\vec{E}_{XBB_n})_\theta\} \text{Re}\{(\vec{B}_{XBB_n})_z\} \tag{3.43a}$$

$$(\vec{P}_{XBB_n})_\theta = \text{Re}\{(\vec{E}_{XBB_n})_\rho\} \text{Re}\{(\vec{B}_{XBB_n})_z\} \tag{3.43b}$$

$$(\vec{P}_{XBB_n})_z = \cos \eta [|\text{Re}\{(\vec{E}_{XBB_n})_\rho\}|^2 + |\text{Re}\{(\vec{E}_{XBB_n})_\theta\}|^2] \tag{3.43c}$$

$$u_{XBB_n} = (1 + \cos^2 \eta) [|\text{Re}\{(\vec{E}_{XBB_n})_\rho\}|^2 + |\text{Re}\{(\vec{E}_{XBB_n})_\theta\}|^2 + |\text{Re}\{(\vec{B}_{XBB_n})_z\}|^2] \tag{3.44}$$

The total energy of F_{XBB_n} is then

$$\varepsilon_{XBB_n} = \int_{-\pi}^{\pi} d\theta \int_{-\infty}^{+\infty} dz \int_0^\infty \rho d\rho u_{XBB_n} \tag{3.45}$$

Since as $z \rightarrow \infty$, \vec{E}_{XBB_n} decreases as $1/|z - t \cos \eta|^{1/2}$, what occurs for the X -branches of F_{XBB_n} , ε_{XBB_n} may not be finite. Nevertheless, as in the case

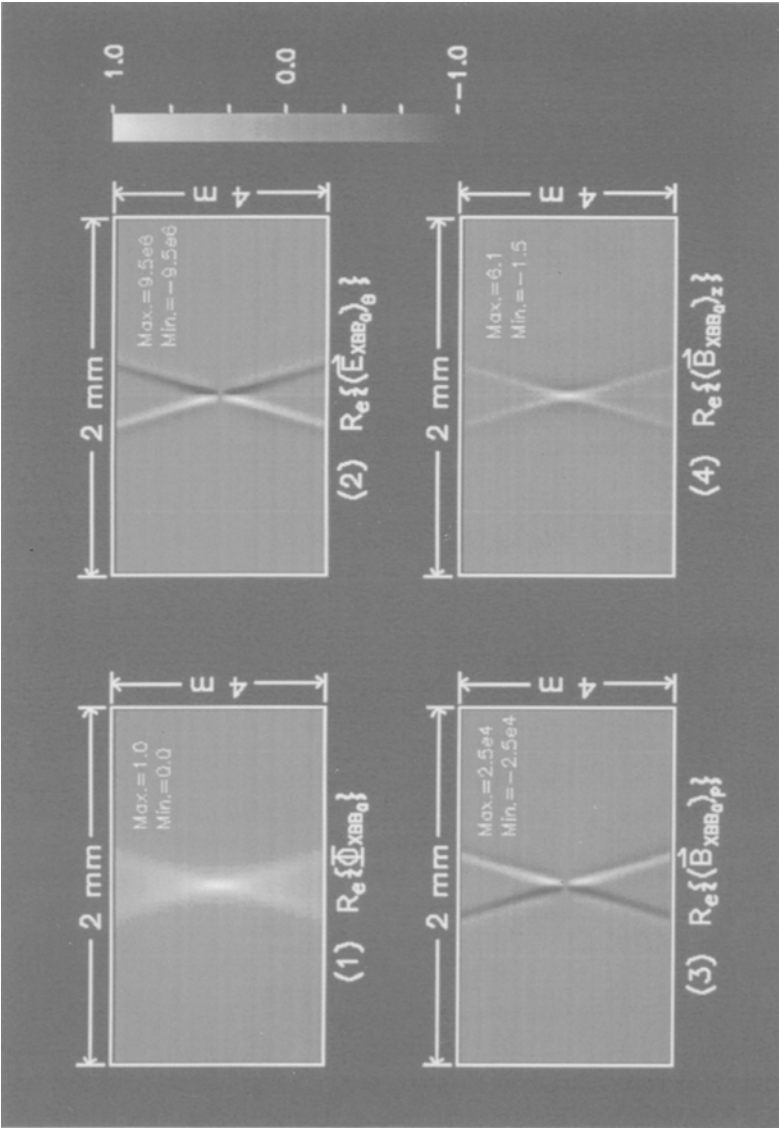


Fig. 4. Real part of the field components of the exact solution superluminal electromagnetic X-wave at distance $z = ct/\cos \eta$ ($\eta = 0.005^\circ$, $a_0 = 0.05$ mm, $n = 0$).

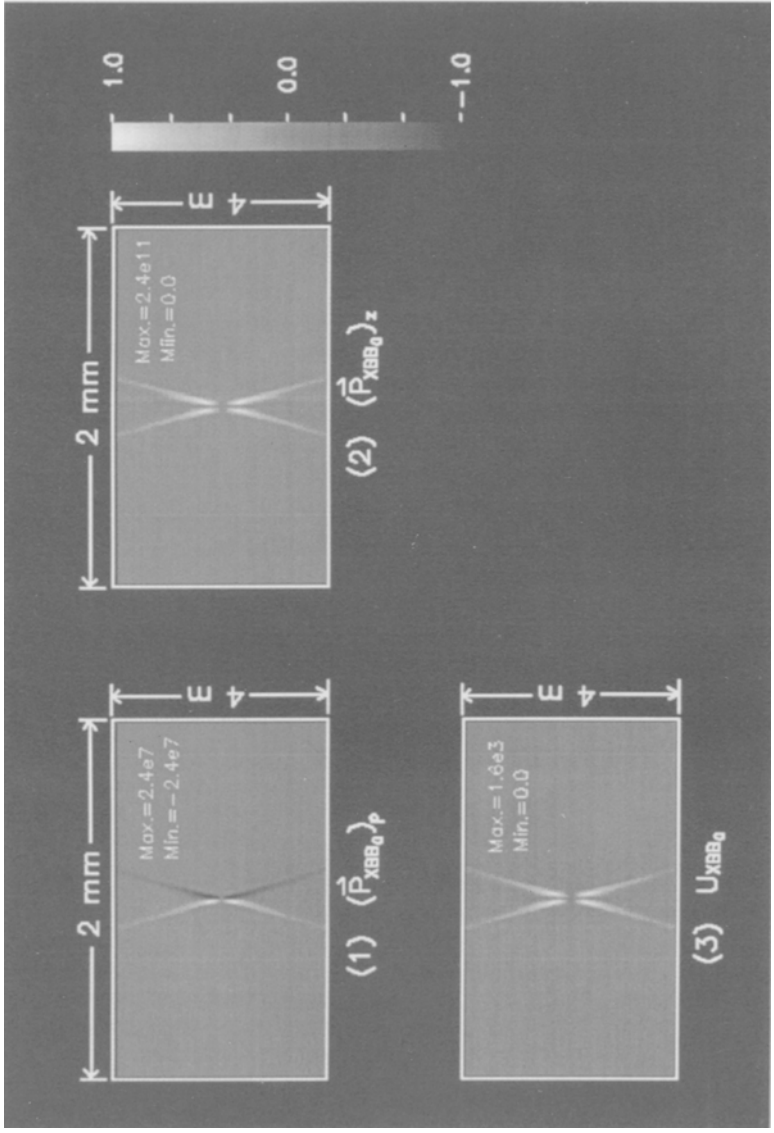


Fig. 5. Poynting flux and energy density of the exact solution superluminal electromagnetic X-wave at distance $z = ct/\cos \eta$, ($\eta = 0.005^\circ$, $u_0 = 0.05$ mm, $n = 0$).

Table I. Maxima and Minima of the Zeroth-Order Nondiffracting Electromagnetic X Waves (Units: MKSA)

	$\text{Re}\{\Phi_{XBB_0}\}$	$\text{Re}\{\vec{E}_{XBB_0}\}_\theta$	$\text{Re}\{(\vec{B}_{XBB_0})_\rho\}$	$\text{Re}\{(\vec{B}_{XBB_0})_z\}$	$(\vec{P}_{XBB_0})_\rho$	$(\vec{P}_{XBB_0})_z$	u_{XBB_0}
max	1.0	9.5×10^6	2.5×10^4	6.1	2.4×10^7	2.4×10^{11}	1.6×10^3
min	0.0	-9.5×10^6	-2.5×10^4	-1.5	-2.4×10^7	0.0	0.0

of the acoustic X -waves discussed in Sec. 2, we are quite sure that a $FAAF_{XBL_n}$ can be launched over a large distance. Obviously in this case the total energy of the $FAAF_{XBL_n}$ is finite.

We now restrict our attention to F_{XBB_0} . In this case from Eq. (3.40) and Eqs. (3.43) we see that $(\vec{E}_{XBB_0})_\rho = (\vec{B}_{XBB_0})_\theta = (\vec{P}_{XBB_0})_\theta = 0$. In Fig. 4¹¹ we see the amplitudes of $\text{Re}\{\Phi_{XBB_0}\}$ (1), $\text{Re}\{(\vec{E}_{XBB_0})_\theta\}$ (2), $\text{Re}\{(\vec{B}_{XBB_0})_\rho\}$ (3), and $\text{Re}\{(\vec{B}_{XBB_0})_z\}$ (4). Figure 5 shows respectively $(\vec{P}_{XBB_0})_\rho$ (1), $(\vec{P}_{XBB_0})_z$ (2), and u_{XBB_0} (3). The size of each panel in Figs. 4 and 5 is 4 m (ρ -direction) \times 2 mm (z -direction) and the maxima and minima of the images in Figs. 4 and 5 (before scaling) are shown in Table I, in MKSA units.¹²

Figure 6 shows the beam plots of F_{XBB_0} in Fig. 4 along one of the X -branches (from left to right). Figure 6(1) represents the beam plots of $\text{Re}\{\Phi_{XBB_0}\}$ (full line), $\text{Re}\{(\vec{E}_{XBB_0})_\theta\}$ (dotted line), $\text{Re}\{(\vec{B}_{XBB_0})_\rho\}$ (dashed line), and $\text{Re}\{(\vec{B}_{XBB_0})_z\}$ (long dashed line). Figure 6(2) represents the beam plots of $(\vec{P}_{XBB_0})_\rho$ (full line), $(\vec{P}_{XBB_0})_z$ (dotted line), and u_{XBB_0} (dashed line).

3.4. Finite Aperture Approximation to F_{XBB_0} and F_{XBL_0}

From Eqs. (3.39) (3.40), (3.43), and (3.44) we see that \vec{E}_{XBB_0} , \vec{B}_{XBB_0} , \vec{P}_{XBB_0} , and u_{XBB_0} are related to the scalar field Φ_{XBB_0} . It follows that the depth of the field⁽⁵⁾ (or nondiffracting distance—see Sec. 2) of the $FAAF_{XBB_0}$ and of the $FAAF_{XBL_0}$, which of course are to be produced by a finite-aperture radiator, are equal and given by

$$Z_{\max} = D/2 \cot \eta \tag{3.46}$$

where D is the diameter of the radiator and η is the axicon angle. It can be proved also⁽⁵⁾ that, for Φ_{XBL_0} (and more generally for Φ_{XBL_n}), Z_{\max} is independent of the central frequency of the spectrum $B(\vec{k})$ in Eq. (3.47).

¹¹ Figures 4-6 were reprinted with permission from Ref. 5.

¹² Reprinted with permission from Table I of Ref. 5.

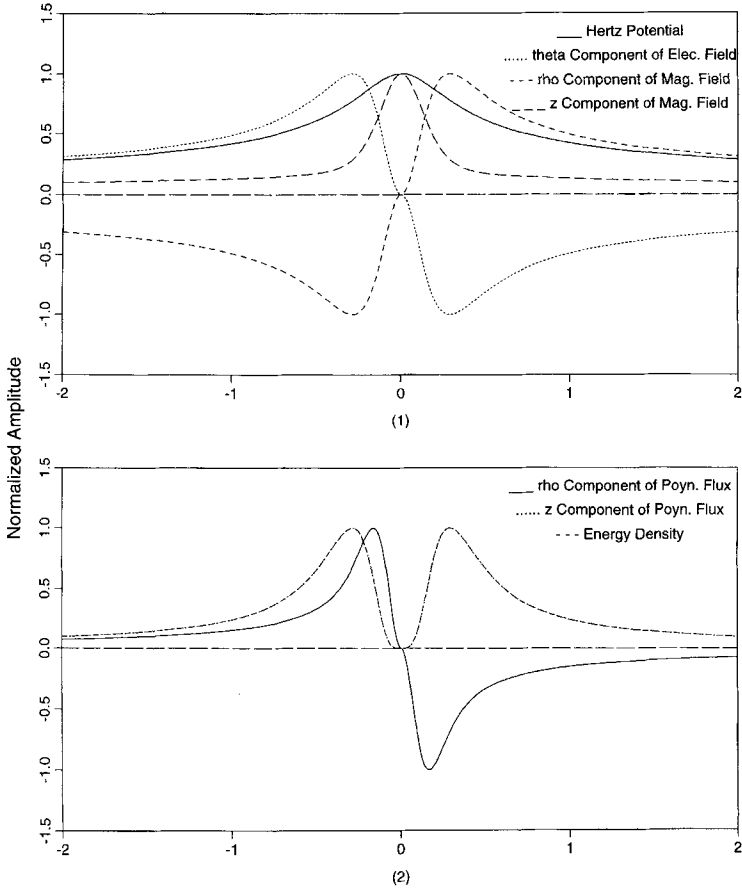


Fig. 6. (1) Beam plots along the X -branches of F_{XBB_0} for $\text{Re}\{\Phi_{XBB_0}\}$ or Hertz potential, $\text{Re}\{(\vec{E}_{XBB_0})_\rho\}$, $\text{Re}\{(\vec{B}_{XBB_0})_\rho\}$, and $\text{Re}\{(\vec{B}_{XBB_0})_z\}$. (2) Beam plots for $(\vec{P}_{XBB_0})_\rho$ (full line), $(\vec{P}_{XBB_0})_z$ (dotted line), and u_{XBB_0} (dashed line).

Then if we want, e.g., that F_{XBB_0} or F_{XBL_0} travel 115 km with a 20-m-diameter radiator, we need $\eta = 0.005^\circ$.

Figure 7 shows the envelope of $\text{Re}\{\text{FAA}\Phi_{XBB_0}\}$ obtained with the finite aperture approximation (FAA) given by Eq. (A.28), with $D = 20$ m, $a_0 = 0.05$ mm, and $\eta = 0.005^\circ$, for distances $z = 10$ km (1) and $z = 100$ km (2), respectively, from the radiator which is located at the plane $z = 0$. Figures 7(3) and 7(4) show the envelope of $\text{Re}\{\text{FAA}\Phi_{XBL_0}\}$ for the same distances and the same parameters (D , a_0 , and η) where $B(\vec{k})$ is the

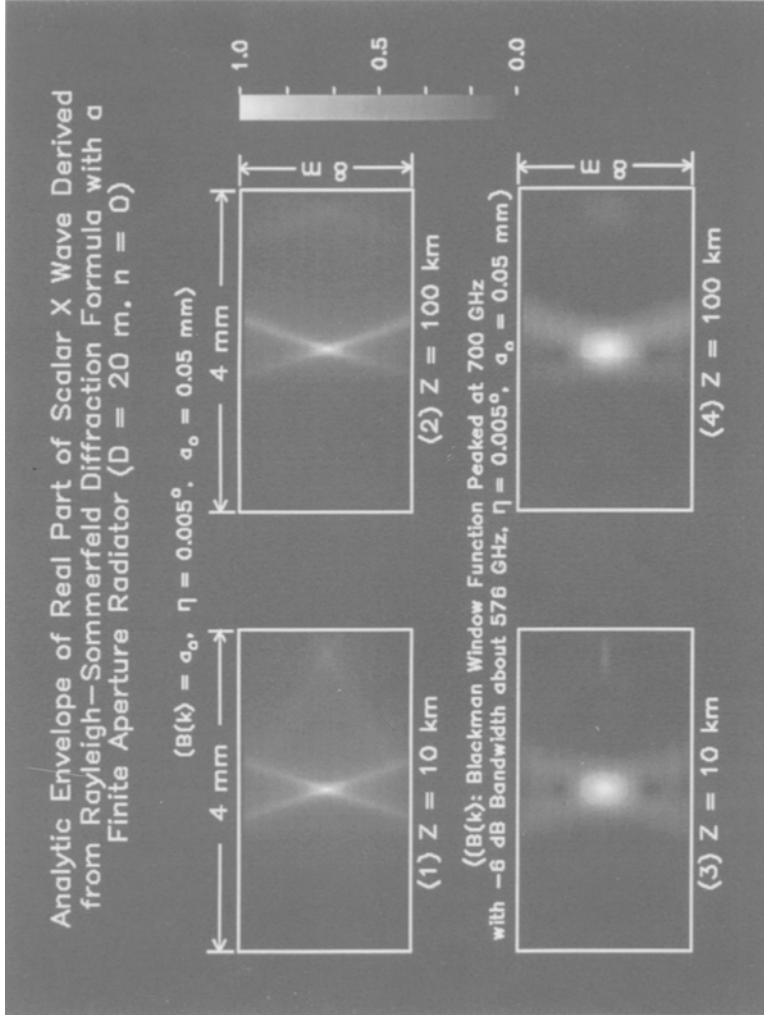


Fig. 7. (1, 2) The real part of $FAA\Phi_{\chi Bk_0}$ at distances $z = 10$ km and $z = 100$ km from the radiator located at the plane $z = 0$ with $D = 20$ m and $\eta = 0.005^\circ$. (3, 4) The real parts of $FAA\Phi_{\chi Bk_0}$ for the same distances.

following Blackman window function, peaked at the frequency $f_0 = 700$ GHz with a 6-dB bandwidth about 576 GHz:

$$B(\bar{k}) = \begin{cases} a_0 \left[0.42 - 0.5 \cos \frac{\pi \bar{k}}{\bar{k}_0} + 0.08 \cos \frac{2\pi \bar{k}}{\bar{k}_0} \right], & 0 \leq \bar{k} \leq 2\bar{k}_0 \\ 0, & \text{otherwise} \end{cases} \quad (3.47)$$

where $\bar{k}_0 = 2\pi f_0/c$ ($c = 300,000$ km/s). From Eq. (3.46) it follows that for the above choice of D , a_0 , and η

$$Z_{\max} = 115 \text{ km} \quad (3.48)$$

Figures 8(1) and 8(2) show the lateral beam plots and Figs. 8(3) and 8(4) show the axial beam plots respectively for $\text{Re}\{\text{FAA}\Phi_{XBB_0}\}$ and for $\text{Re}\{\text{FAA}\Phi_{XBL_0}\}$ used to calculate F_{XBB_0} and F_{XBL_0} . The full and dotted lines represent X -waves at distances $z = 10$ and 100 km, respectively. Figure 9 shows the peak values of $\text{Re}\{\text{FAA}\Phi_{XBB_0}\}$ (full line) and $\text{Re}\{\text{FAA}\Phi_{XBL_0}\}$ (dotted line) along the z -axis from $z = 3.45$ to 230 km. The dashed line represents the result of the exact Φ_{XBB_0} solution. The 6-dB lateral and axial beam widths of Φ_{XBB_0} , which can be measured in Figs. 7(1) and 7(2), are about 1.96 m and 0.17 mm respectively, and those of the $\text{FAA}\Phi_{XBL_0}$ are about 2.5 m and 0.48 mm as can be measured from Figs. 7(3) and 7(4). For Φ_{XBB_0} we can calculate^(43, 26) the theoretical values of the 6-dB lateral (BW_L) and axial (BW_A) beam widths, which are given by

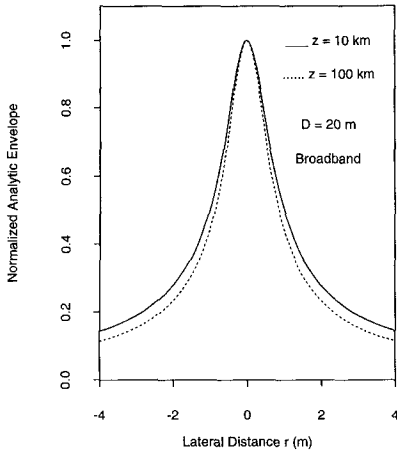
$$BW_L = \frac{2\sqrt{3} a_0}{|\sin \eta|}, \quad BW_A = \frac{2\sqrt{3} a_0}{|\cos \eta|} \quad (3.49)$$

With the values of D , a_0 , and η given above, we have $BW_L = 1.98$ m and $BW_A = 0.17$ mm. These are to be compared with the values of these quantities for the $\text{FAA}\Phi_{XBL_0}$.

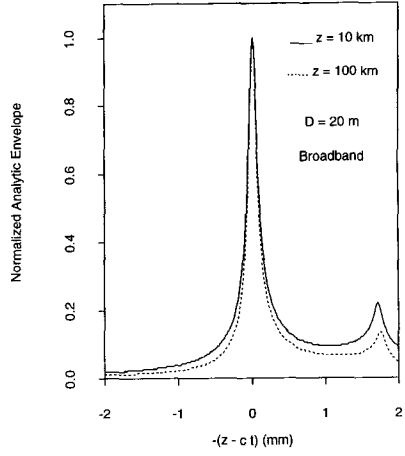
We remark also that Eq. (3.46) says that Z_{\max} does not depend on a_0 . Then we can choose an arbitrarily small a_0 to increase the localization (reduced BW_L and BW_A) of the X -wave without altering Z_{\max} . Smaller a_0 requires that the $\text{FAA}\Phi_{XBL_0}$ be transmitted with broader bandwidth. The depths of field of Φ_{XBB_0} and of Φ_{XBL_0} that we can measure in Fig. 9 are approximately 109 and 110 km, very close to the value given by Eq. (3.46), which is 115 km.

We conclude this section with the following observations.

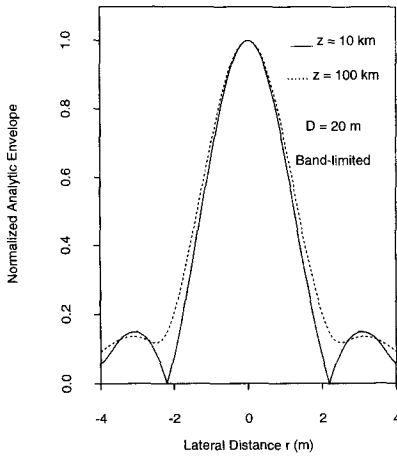
(i) In general both subluminal and superluminal UPW solutions of ME have non-null field invariants and are not transverse waves. In



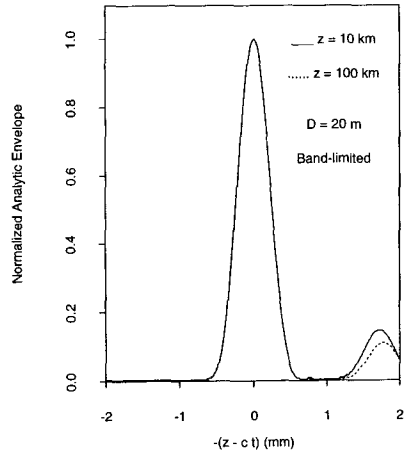
(1)



(3)



(2)



(4)

Fig. 8. Beam plots of scalar X -waves (finite aperture).

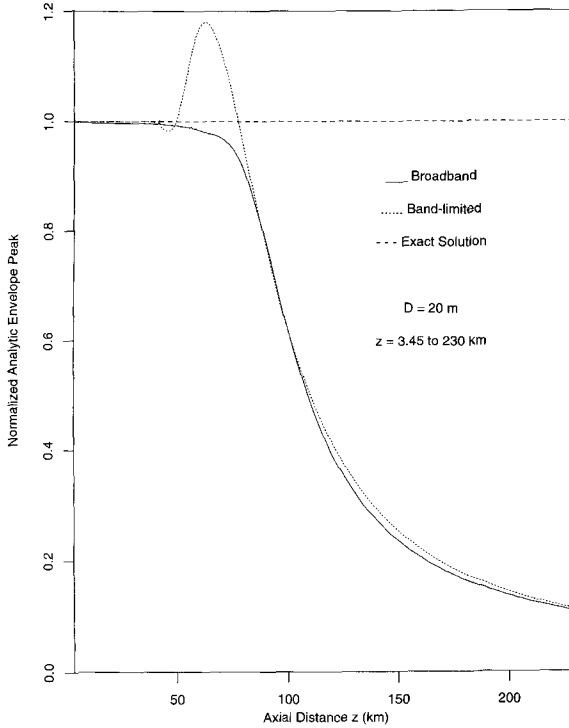


Fig. 9. Peak magnitude of X -waves along the z axis.

particular, our solutions have a longitudinal component along the z -axis. This result is important because it shows that, contrary to the speculations of Evans,⁽⁵⁹⁾ we do not need an electromagnetic theory with a nonzero photon-mass, i.e., with F satisfying the Proca equation in order to have an electromagnetic wave with a longitudinal component. Since Evans presents evidence⁽⁵⁹⁾ of the existence on longitudinal magnetic fields in many different physical situations, we conclude that the theoretical and experimental study of subluminal and superluminal UPW solutions of ME must be continued.

(ii) We recall that in microwave and optics, as is well known, the electromagnetic intensity is approximately represented by the magnitude of a scalar field solution of the HWE. We already quoted in the introduction that Durnin⁽¹⁶⁾ produced an optical J_0 -beam, which as seen from Eq. (3.1) is related to Φ_{XBB_0} (Φ_{XBL_0}). If we take into account this fact together with the results of the acoustic experiments described in Sec. 2, we arrive at

the conclusion that subluminal electromagnetic pulses J_0 and also superluminal X -waves can be launched with appropriate antennas using present technology.

(iii) If we take a look at the structure of, e.g., the $FAA\Phi_{XBB_0}$ [Eq. (3.40)] plus Eq. (A.28) we see that it is a “packet” of wavelets, each one traveling with speed c . Nevertheless, the electromagnetic X -wave that is an interference pattern is such that its peak travels with speed $c/\cos \eta > 1$. (This indeed happens in the acoustic experiment with $c \mapsto c_s$; see Sec. 2.) Since, as discussed above, we can project an experiment to launch the peak of the $FAA\Phi_{XBB_0}$ from a point z_1 to a point z_2 , the question arises: Is the existence of superluminal electromagnetic waves in conflict with Einstein’s special relativity? We give our answer to this fundamental issue in Sec. 5, but first we discuss in Sec. 4 the speed of propagation of the energy associated with a superluminal electromagnetic wave.

4. THE VELOCITY OF TRANSPORT OF ENERGY OF THE UPW SOLUTIONS OF MAXWELL EQUATIONS

Motivated by the fact that the acoustic experiment of Sec. 2 shows that the energy of the FAA X -wave travels with speed greater than c_s and since we found in this paper UPW solutions of Maxwell equations with speeds $0 \leq v < \infty$, the following question arises naturally: Which is the velocity of transport of the energy of a superluminal UPW (or quasi-UPW) solution of ME?

We can find in many physics textbooks (e.g., Ref. 10) and in scientific papers⁽⁴¹⁾ the following argument. Consider an arbitrary solution of ME in vacuum, $\partial F = 0$. Then if $F = \vec{E} + i\vec{B}$ [see Eq. (B.17)] it follows that the Poynting vector and the energy density of the field are

$$\vec{P} = \vec{E} \times \vec{B}, \quad u = \frac{1}{2}(\vec{E}^2 + \vec{B}^2) \tag{4.1}$$

It is obvious that the following inequality always holds:

$$v_e = \frac{|\vec{P}|}{u} \leq 1 \tag{4.2}$$

Now, the conservation of energy-momentum reads, in integral form over a finite volume V with boundary $S = \partial V$,

$$\frac{\partial}{\partial t} \left\{ \iiint_V d\mathbf{v} \frac{1}{2}(\vec{E}^2 + \vec{B}^2) \right\} = \oint_S d\vec{S} \cdot \vec{P} \tag{4.3}$$

Equation (4.3) is interpreted saying that $\oint_S d\vec{S} \cdot \vec{P}$ is the field energy flux across the surface $S = \partial V$, so that \vec{P} is the flux density—the amount of field energy passing through a unit area of the surface in unit time. For plane-wave solutions of Maxwell equations,

$$v_\varepsilon = 1 \quad (4.4)$$

and this result gives origin to the “dogma” that free electromagnetic fields transport energy at speed $v_\varepsilon = c = 1$.

However, $v_\varepsilon \leq 1$ is true even for subluminal and superluminal solutions of ME, as the ones discussed in Sec. 3. The same is true for the superluminal modified Bessel beam found by Band⁽⁴¹⁾ in 1987. There he claims that since $v_\varepsilon \leq 1$, there is no conflict between superluminal solutions of ME and relativity theory, since what relativity forbids is the propagation of energy with speed greater than c .

Here we challenge this conclusion. The fact is that, as is well known, \vec{P} is not uniquely defined. Equation (4.3) continues to hold true if we substitute $\vec{P} \mapsto \vec{P} + \vec{P}'$ with $\nabla \cdot \vec{P}' = 0$. But, of course, we can easily find for subluminal, luminal, or superluminal solutions of Maxwell equations a \vec{P}' such that

$$\frac{|\vec{P} + \vec{P}'|}{u} \geq 1 \quad (4.5)$$

We come to the conclusion that the question of the transport of energy in superluminal UPW solutions of ME is an experimental question. For the acoustic superluminal X -solution of the HWE (see Sec. 2) the energy around the peak area flows together with the wave, i.e., with speed $c = c_s/\cos \eta$ (although the “canonical” formula [Eq. (2.0)] predicts that the energy flows with $v_\varepsilon < c_s$). Since we can see no possibility for the field energy of the superluminal electromagnetic wave to travel outside the wave, we are confident to state that the velocity of energy transport of superluminal electromagnetic waves is superluminal.

Before ending, we give another example to illustrate that Eq. (4.2) [as is the case of Eq. (2.10)] is devoid of physical meaning. Consider a spherical conductor in electrostatic equilibrium with uniform superficial charge density (total charge Q) and with a dipole magnetic moment. Then, we have

$$\vec{E} = Q \frac{\mathbf{r}}{r^2}, \quad \vec{B} = \frac{C}{r^3} (2 \cos \theta \mathbf{r} + \sin \theta \boldsymbol{\theta}) \quad (4.6)$$

and

$$\vec{P} = \vec{E} \times \vec{B} = \frac{CQ}{r^5} \sin \theta \hat{\phi}, \quad u = \frac{1}{2} \left(\frac{Q^2}{r^4} + \frac{C^2}{r^6} (3 \cos^2 \theta + 1) \right) \quad (4.7)$$

Thus

$$\frac{|\vec{P}|}{u} = \frac{2CQr \sin \theta}{r^2 Q^2 + C^2(3 \cos^2 \theta + 1)} \neq 0 \quad \text{for } r \neq 0 \quad (4.8)$$

Since the fields are static, the conservation law [Eq. (4.3)] continues to hold true, as there is no motion of charges and for any closed surface containing the spherical conductor we have

$$\oint_S d\vec{S} \cdot \vec{P} = 0 \quad (4.9)$$

But *nothing* is in motion! In view of these results we must investigate whether the existence of superluminal UPW solutions of ME is compatible or not with the principle of relativity. We analyze this question in detail in the next section.

To end this section we recall that in Sec. 2.19 of his book Stratton⁽¹⁹⁾ presents a discussion of the Poynting vector and energy transfer which essentially agrees with the view presented above. Indeed he finished that section with the words: "By this standard there is every reason to retain the Poynting-Heaviside viewpoint until a clash with new experimental evidence shall call for its revision."¹³

5. SUPERLUMINAL SOLUTIONS OF MAXWELL EQUATIONS AND THE PRINCIPLE OF RELATIVITY

In Sec. 3 we showed that it seems possible with present technology to launch in free space superluminal electromagnetic waves (SEXWs). We show in the following that the physical existence of SEXWs implies a breakdown of the principle of relativity (PR). Since this is a fundamental issue, with implications for all branches of theoretical physics, we will examine the problem with great care. In Sec. 5.1 we give a rigorous mathematical definition of the PR, and in Sec. 5.2 we present the proof of the above statement.

¹³ Thanks are due to the referee for calling our attention to this point.

5.1. Mathematical Formulation of the Principle of Relativity and Its Physical Meaning

In Appendix B we define Minkowski spacetime as the triple $\langle M, g, D \rangle$, where $M \simeq \mathbb{R}^4$, g is a Lorentzian metric, and D is the Levi-Civita connection of g .

Consider now G_M , the group of all diffeomorphisms of M , called the manifold mapping group. Let \mathbf{T} be a geometrical object defined in $A \subseteq M$. The diffeomorphism $h \in G_M$ induces a deforming mapping $h_* : \mathbf{T} \mapsto h_* \mathbf{T} = \mathbf{T}$ such that:

- (i) If $f: M \supseteq A \rightarrow \mathbb{R}$, then $h_* f = f \circ h^{-1}: h(A) \rightarrow \mathbb{R}$
- (ii) If $\mathbf{T} \in \text{sec } T^{(r,s)}(A) \subset \text{sec } T(M)$, where $T^{(r,s)}(A)$ is the sub-bundle of tensors of type (r, s) of the tensor bundle $T(M)$, then

$$(h_* \mathbf{T})_{he} (h_* \omega_1, \dots, h_* \omega_r, h_* X_1, \dots, h_* X_s) = \mathbf{T}_e(\omega_1, \dots, \omega_r, X_1, \dots, X_s) \\ \forall X_i \in T_e A, \quad i = 1, \dots, s, \quad \forall \omega_j \in T_e^* A, \quad j = 1, \dots, r, \quad \forall e \in A$$

- (iii) If D is the Levi-Civita connection and $X, Y \in \text{sec } TM$, then

$$(h_* D_{h_* X} h_* Y)_{he} h_* f = (D_X Y)_e f, \quad \forall e \in M \quad (5.1)$$

If $\{f_\mu = \partial/\partial x^\mu\}$ is a coordinate basis for TA and $\{\theta^\mu = dx^\mu\}$ is the corresponding dual basis for T^*A and if

$$\mathbf{T} = T_{\nu_1 \dots \nu_s}^{\mu_1 \dots \mu_r} \theta^{\nu_1} \otimes \dots \otimes \theta^{\nu_s} \otimes f_{\mu_1} \otimes \dots \otimes f_{\mu_r} \quad (5.2)$$

then

$$h_* \mathbf{T} = [T_{\nu_1 \dots \nu_s}^{\mu_1 \dots \mu_r} \circ h^{-1}] h_* \theta^{\nu_1} \otimes \dots \otimes h_* \theta^{\nu_s} \otimes h_* f_{\mu_1} \otimes \dots \otimes h_* f_{\mu_r} \quad (5.3)$$

Suppose now that A and $h(A)$ can be covered by the local chart (U, η) of the maximal atlas of M , and $A \subseteq U$, $h(A) \subseteq U$. Let $\langle x^\mu \rangle$ be the coordinate functions associated with (U, η) . The mapping

$$x'^\mu = x^\mu \circ h^{-1}: h(U) \rightarrow \mathbb{R} \quad (5.4)$$

defines a coordinate transformation $\langle x^\mu \rangle \mapsto \langle x'^\mu \rangle$ if $h(U) \supseteq A \cup h(A)$. Indeed $\langle x'^\mu \rangle$ are the coordinate functions associated with the local chart (V, φ) where $h(U) \subseteq V$ and $U \cap V \neq \emptyset$. Now, since it is well known that under the above conditions $h_* \partial/\partial x^\mu \equiv \partial/\partial x'^\mu$ and $h_* dx^\mu \equiv dx'^\mu$, Eqs. (5.3) and (5.4) imply that

$$(h_* \mathbf{T})_{\langle x'^\mu \rangle} (he) = \mathbf{T}_{\langle x'^\mu \rangle} (e) \quad (5.5)$$

where $\mathbf{T}_{\langle x^\mu \rangle}(e)$ means the components of \mathbf{T} in the chart $\langle x^\mu \rangle$ at the event $e \in M$, i.e., $\mathbf{T}_{\langle x^\mu \rangle}(e) = T^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s}(x^\mu(e))$ and where $\bar{\mathbf{T}}^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s}(x'^\mu(he))$ are the components of $\bar{\mathbf{T}} = h_* \mathbf{T}$ in the basis $\{h_* \partial/\partial x^\mu = \partial/\partial x'^\mu\}$, $\{h_* dx^\mu = dx'^\mu\}$, at the point $h(e)$. Then Eq. (5.6) reads

$$\bar{T}^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s}(x'^\mu(he)) = T^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s}(x^\mu(e)) \tag{5.6}$$

or, using Eq. (5.5),

$$\bar{T}^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s}(x'^\mu(e)) = (A^{-1})^{\mu_1}_{\alpha_1} \dots A^{\beta_s}_{\nu_s} T^{\alpha_1 \dots \alpha_r}_{\beta_1 \dots \beta_s}(x^\mu(h^{-1}e)) \tag{5.7}$$

where $A^\mu_\alpha = \partial x^\mu / \partial x^\alpha$, etc.

In Appendix B we introduce the concept of inertial reference frames $I \in \text{sec } TU$, $U \subseteq M$ by

$$g(I, I) = 1 \quad \text{and} \quad DI = 0 \tag{5.8}$$

A general frame Z satisfies $g(Z, Z) = 1$, with $DZ \neq 0$. If $\alpha = g(Z,) \in \text{sec } T^*U$, we have

$$(D\alpha)_e = a_e \otimes \alpha_e + \sigma_e + \omega_e + \frac{1}{3}\theta_e h_e, \quad e \in U \subseteq M \tag{5.9}$$

where $a = g(A,)$, $A = D_Z Z$ is the acceleration and where ω_e is the rotation tensor, σ_e is the shear tensor, θ_e is the expansion, and $h_e = g|_{H_e}$ where

$$T_e M = [Z_e] \oplus [H_e] \tag{5.10}$$

H_e is the rest space of an *instantaneous observer* at e , i.e., the pair (e, Z_e) . Also $h_e(X, Y) = g_e(pX, pY)$, $\forall X, Y \in T_e M$, and $p: T_e M \rightarrow H_e$. (For the explicit form of ω , σ , θ , see Ref. 60). From Eqs. (5.9) and (5.10) we see that an inertial reference frame has no acceleration, no rotation, no shear, and no expansion.

We introduce also in Appendix B the concept of a (nacs/ I). A (nacs/ I) $\langle x^\mu \rangle$ is said to be in the Lorentz gauge if x^μ , $\mu = 0, 1, 2, 3$ are the usual Lorentz coordinates and $I = \partial/\partial x^0 \in \text{sec } TM$. We recall that it is a theorem that putting $I = e_0 = \partial/\partial x^0$, there exist three other fields $e_i \in \text{sec } TM$ such that $g(e_i, e_i) = -1$, $i = 1, 2, 3$, and $e_i = \partial/\partial x^i$.

Now, let $\langle x^\mu \rangle$ be Lorentz coordinate functions as above. We say that $l \in G_M$ is a *Lorentz mapping* if and only if

$$x'^\mu(e) = A^\mu_\nu x^\nu(e) \tag{5.11}$$

where $A^\mu_\nu \in \mathcal{L}^\uparrow_+$ is a Lorentz transformation. With abuse of notation we denote the subset $\{l\}$ of G_M such that Eq. (5.12) holds true also by $\mathcal{L}^\uparrow_+ \subset G_M$.

When $\langle x^\mu \rangle$ are Lorentz coordinate functions, $\langle x'^\mu \rangle$ are also Lorentz coordinate functions. In this case we denote

$$e_\mu = \partial/\partial x^\mu, \quad e'_\mu = \partial/\partial x'^\mu, \quad \gamma_\mu = dx^\mu, \quad \gamma'_\mu = dx'^\mu \quad (5.12)$$

When $l \in \mathcal{L}^\uparrow_+ \subset G_M$, we say that $l_*\mathbf{T}$ is the *Lorentz deformed version* of \mathbf{T} . Let $h \in G_M$. If for a geometrical object \mathbf{T} we have

$$h_*\mathbf{T} = \mathbf{T} \quad (5.13)$$

then h is said to be a symmetry of \mathbf{T} , and the set of all $\{h \in G_M\}$ such that Eq. (5.13) holds is said to be the symmetry group of \mathbf{T} . We can immediately verify that for $l \in \mathcal{L}^\uparrow_+ \subset G_M$

$$l_*g = g, \quad l_*D = D \quad (5.14)$$

i.e., the special restricted orthochronous Lorentz group \mathcal{L}^\uparrow_+ is a symmetry group of g and D .

In Ref. 62 we maintain that a physical theory τ is characterized by:

- (i) the theory of a certain “species of structure” in the sense of Boubarki⁽⁶³⁾;
- (ii) its physical interpretation;
- (iii) its present meaning and present applications.

We recall that in the mathematical exposition of a given physical theory τ , the postulates or basic axioms are presented as definitions. Such definitions mean that the physical phenomena described by τ behave in a certain way. Then, the definitions require more motivation than the pure mathematical definitions. We call coordinative definitions the physical definitions, a term introduced by Reichenbach.⁽⁶⁴⁾ It is necessary also to make clear that completely convincing and genuine motivations for the coordinative definitions cannot be given, since they refer to nature as a whole and to the physical theory as a whole.

The theoretical approach to physics behind (i), (ii), and (iii) above is then to admit the mathematical concepts of the “species of structure” defining τ as primitives, and define coordinatively the observation entities from them. Reichenbach assumes that “*physical knowledge* is characterized by the fact that concepts are not only defined by other concepts, but are also coordinated to real objects.” However, in our approach, each physical theory, when characterized as a species of structure, contains some implicit geometric objects, like some of the reference frame fields defined above, that cannot in general be coordinated to real objects. Indeed, it would be

absurd to suppose that all the infinity of IRF that exist in M must have a material support.

We define a *spacetime* theory as a theory of a species of structure such that, if $\text{Mod } \tau$ is the class of models of τ , then each $Y \in \text{Mod } \tau$ contains a substructure called spacetime (ST). More precisely, we have

$$Y = (\text{ST}, \mathbf{T}_1, \dots, \mathbf{T}_m) \tag{5.15}$$

where ST can be a very general structure.⁽⁶²⁾ In what follows we suppose that $\text{ST} = \mathcal{M} = (M, g, D)$, i.e., that ST is Minkowski spacetime. The \mathbf{T}_i , $i = 1, \dots, m$, are (explicit) geometrical objects defined in $U \subseteq M$ characterizing the physical fields and particle trajectories that cannot be geometrized in Y . Here, to be geometrizable means to be a metric field or a connection on M or objects derived from these concepts as, e.g., the Riemann tensor or the torsion tensor.

The reference frame fields will be called the *implicit* geometrical objects of τ , since they are mathematical objects that do not necessarily correspond to properties of a physical system described by τ .

Now, with the Clifford bundle formalism we can formulate in $\mathcal{C}\ell(M)$ all modern physical theories (see Appendix B) including Einstein's gravitational theory.⁽⁶⁾ We introduce now the Lorentz–Maxwell electrodynamics (LME) in $\mathcal{C}\ell(M)$ as a theory of a species of structure. We say that LME has as model

$$Y_{LME} = \langle M, g, D, F, J, \{\varphi_i, m_i, e_i\} \rangle \tag{5.16}$$

where (M, g, D) is Minkowski spacetime, $\{\varphi_i, m_i, e_i\}$, $i = 1, 2, \dots, N$, is the set of all charged particles, m_i and e_i being the masses and charges of the particles and $\varphi_i: \mathbb{R} \supset I \rightarrow M$ being the world lines of the particles characterized by the fact that if $\varphi_{i*} \in \text{sec } TM$ is the velocity vector, then $\check{\varphi}_i = g(\varphi_{i*}, \cdot) \in \text{sec } \Lambda^1(M) \subset \text{sec } \mathcal{C}\ell(M)$ and $\check{\varphi}_i \cdot \check{\varphi}_i = 1$. $F \in \text{sec } \Lambda^2(M) \subset \text{sec } \mathcal{C}\ell(M)$ is the electromagnetic field and $J \in \text{sec } \Lambda^1(M) \subset \text{sec } \mathcal{C}\ell(M)$ is the current density. The proper axioms of the theory are

$$\begin{aligned} \partial F &= J \\ m_i D_{\varphi_{i*}} \check{\varphi}_i &= e_i \check{\varphi}_i \cdot F \end{aligned} \tag{5.17}$$

From a mathematical point of view it is a trivial result that τ_{LME} has the following property: If $h \in G_M$ and if Eqs. (5.16) have a solution $\langle F, J, (\varphi_i, m_i, e_i) \rangle$ in $U \subseteq M$, then $\langle h_* F, h_* J, (h_* \varphi_i, m_i, e_i) \rangle$ is also a solution of Eqs. (5.16) in $h(U)$. Since the result is true for any $h \in G_M$, it is true for $l \in \mathcal{L}^1_+ \subset G_M$, i.e., for any Lorentz mapping.

We must now make it clear that $\langle F, J, \{\varphi_i, m_i, e_i\} \rangle$, which is a solution of Eq. (5.16) in U , can be obtained only by imposing *mathematical boundary conditions* which we denote by BU . The solution will be realizable in nature if and only if the mathematical boundary conditions are physically realizable. This is indeed a nontrivial point⁽⁶²⁾ for in particular it says to us that even if $\langle h_*F, h_*J, \{h_*\varphi_i, m_i, e_i\} \rangle$ can be a solution of Eqs. (5.16) with mathematical boundary conditions $Bh(U)$, it may happen that $Bh(U)$ cannot be physically realizable in nature. The following statement, denoted PR_1 , is usually presented⁽⁶²⁾ as the principle of (special) relativity in active form:

PR_1 : Let $l \in \mathcal{L}_+^\dagger \subset G_M$. If for a physical theory τ and $Y \in \text{Mod } \tau$, $Y = \langle M, g, D, \mathbf{T}_1, \dots, \mathbf{T}_m \rangle$ is a possible physical phenomenon, then $l_*Y = \langle M, g, D, l_*\mathbf{T}_1, \dots, l_*\mathbf{T}_m \rangle$ is also a possible physical phenomenon.

It is clear that hidden in PR_1 is the assumption that the boundary conditions that determine l_*Y are physically realizable. Before we continue we introduce the statement denoted PR_2 , known as the principle of (special) relativity in passive form⁽⁶²⁾:

PR_2 : “All inertial reference frames are physically equivalent or indistinguishable.”

We now give a precise mathematical meaning to the above statement.

Let τ be a spacetime theory and let $ST = \langle M, g, D \rangle$ be a substructure of $\text{Mod } \tau$ representing spacetime. Let $I \in \text{sec } TU$ and $I' \in \text{sec } TV$, $U, V \subseteq M$, be two inertial reference frames. Let (U, η) and (V, φ) be two Lorentz charts of the maximal atlas of M that are naturally adapted respectively to I and I' . If $\langle x^\mu \rangle$ and $\langle x'^\mu \rangle$ are the coordinate functions associated with (U, η) and (V, φ) , we have $I = \partial/\partial x^0$, $I' = \partial/\partial x'^0$.

Definition. Two inertial reference frames I and I' as above are said to be physically equivalent according to τ if and only if the following conditions are satisfied:

$$(i) \quad G_M \supset \mathcal{L}_+^\dagger \ni l: U \rightarrow l(U) \subseteq V, \quad x'^\mu = x^\mu \circ l^{-1} \Rightarrow I' = l_*I$$

When $Y \in \text{Mod } \tau$, $Y = \langle M, g, D, \mathbf{T}_1, \dots, \mathbf{T}_m \rangle$, is such that g and D are defined over all M and $\mathbf{T}_i \in \text{sec } \mathcal{L}(U) \subset \text{sec } \mathcal{L}(M)$, calling $o = \langle g, D, \mathbf{T}_1, \dots, \mathbf{T}_m \rangle$, o solves a set of differential equations in $\eta(U) \subset \mathbb{R}^4$ with a given set of boundary conditions denoted $b^{\circ\langle x^\mu \rangle}$, which we write as

$$D_{\langle x^\mu \rangle}^\alpha (o_{\langle x^\mu \rangle})_e = 0, \quad b^{\circ\langle x^\mu \rangle}, \quad e \in U \tag{5.18}$$

and we must have:

(ii) If $Y \in \text{Mod } \tau \Leftrightarrow l_* Y \in \text{Mod } \tau$, then necessarily

$$l_* Y = \langle M, g, D, l_* \mathbf{T}_1, \dots, l_* \mathbf{T}_m \rangle \tag{5.19}$$

is defined in $l(U) \subseteq V$ and calling $l_* o \equiv \{g, D, l_* \mathbf{T}_1, \dots, l_* \mathbf{T}_m\}$ we must have

$$D_{\langle x'^\mu \rangle}^\alpha (l_* o_{\langle x'^\mu \rangle})|_{le} = 0, \quad b^{l_* o \langle x'^\mu \rangle}, \quad le \in l(U) \subseteq V \tag{5.20}$$

In Eqs. (5.18) and (5.20) $D_{\langle x^\mu \rangle}^\alpha$ and $D_{\langle x'^\mu \rangle}^\alpha$ mean $\alpha = 1, 2, \dots, m$ sets of differential equations in \mathbb{R}^4 . The system of differential equations (5.19) must have the same functional form as the system of differential equations (5.17), and $b^{l_* o \langle x'^\mu \rangle}$ must be relative to $\langle x'^\mu \rangle$ in the same as manner $b^{o \langle x^\mu \rangle}$ is relative to $\langle x^\mu \rangle$, and if $b^{o \langle x^\mu \rangle}$ is physically realizable, then $b^{l_* o \langle x'^\mu \rangle}$ must also be physically realizable. We say under these conditions that $I \sim I'$ and that $l_* o$ is the Lorentz deformed version of the phenomena described by o .

Since in the above definition $l_* Y = \langle M, g, D, l_* \mathbf{T}_1, \dots, l_* \mathbf{T}_m \rangle$, it follows that when $I \sim I'$, then $l_* g = g, l_* D = D$ (as we already know), and this means that the spacetime structure does not give a preferred status to I or I' according to τ .

5.2. Proof that the Existence of SEXWs Implies a Breakdown of PR_1 and PR_2

We are now able to prove the statement presented at the beginning of this section, that the existence of SEXWs implies a breakdown of the principle of relativity in both its active (PR_1) and passive (PR_2) versions.

Let $l \in \mathcal{L}_+^\uparrow \subset G_M$ and let $F, \bar{F} \in \text{sec } A^2(M) \subset \text{sec } \mathcal{C}\ell(M)$, $\bar{F} = l_* F$. Let $\bar{F} = l_* F = R\check{F}R^{-1}$, where $\check{F}_e = (1/2) F_{\mu\nu}(x^\delta(l^{-1}e)) \gamma^\mu \gamma^\nu$ and where $R \in \text{sec } \text{Spin}_+(1, 3) \subset \text{sec } \mathcal{C}\ell(M)$ is a Lorentz mapping, such that $\gamma'^\mu = R\gamma^\mu R^{-1} = A_\alpha^\mu \gamma^\alpha$, $A_\alpha^\mu \in \mathcal{L}_+^\uparrow$, and let $\langle x^\mu \rangle$ and $\langle x'^\mu \rangle$ be Lorentz coordinate functions as before such that $\gamma^\mu = dx^\mu, \gamma'^\mu = dx'^\mu$ and $x'^\mu = x^\mu \circ l^{-1}$. We write

$$F_e = \frac{1}{2} F_{\mu\nu}(x^\delta(e)) \gamma^\mu \gamma^\nu \tag{5.21a}$$

$$F_e = \frac{1}{2} F'_{\mu\nu}(x'^\delta(e)) \gamma'^\mu \gamma'^\nu \tag{5.21b}$$

$$\bar{F}_e = \frac{1}{2} \bar{F}_{\mu\nu}(x^\delta(e)) \gamma^\mu \gamma^\nu \tag{5.22a}$$

$$\bar{F}_e = \frac{1}{2} \bar{F}'_{\mu\nu}(x'^\delta(e)) \gamma'^\mu \gamma'^\nu \tag{5.22b}$$

From (5.21a) and (5.21b) we get that

$$F'_{\alpha\beta}(x'^{\delta}(e)) = (A^{-1})^{\mu}_{\alpha} (A^{-1})^{\nu}_{\beta} F_{\mu\nu}(x^{\delta}(e)) \tag{5.23}$$

From (5.21a) and (5.22b) we also get

$$\bar{F}_{\alpha\beta}(x^{\delta}(e)) = A^{\mu}_{\alpha} A^{\nu}_{\beta} F_{\mu\nu}(x^{\delta}(l^{-1}e)) \tag{5.24}$$

Now, suppose that F is a superluminal solution of Maxwell equation, in particular a SEXW as discussed in Sec. 3. Suppose that F has been produced in the inertial frame I with $\langle x^{\mu} \rangle$ as (nacs/ I), with the physical device described in Sec. 3. F is generated in the plane $z = 0$ and is traveling with speed $c_1 = 1/\cos \eta$ in the negative z -direction. It will then travel to the future in spacetime, according to the observers in I . Now, there exists $l \in \mathcal{L}^{\uparrow}_+$ such that $l_* F = \bar{F} = RFR^{-1}$ will be a solution of Maxwell equations and such that if the velocity 1-form of F is $v_F = (c_1^2 - 1)^{-1/2} (1, 0, 0, -c_1)$, then the velocity 1-form of \bar{F} is $v_{\bar{F}} = (c_1'^2 - 1)^{-1/2} (-1, 0, 0, -c_1')$, with $c_1' > 1$, i.e., $v_{\bar{F}}$ is pointing to the past. As is well known, \bar{F} carries negative energy according to the observers in the I frame.

We then arrive at the conclusion that to assume the validity of PR_1 is to assume the physical possibility of sending to the past waves carrying negative energy. This seems to the authors an impossible task, and the reason is that there do not exist physically realizable boundary conditions that would allow the observers in I to launch \bar{F} in spacetime and such that it traveled to its own past.

We now show that there is also a breakdown of PR_2 , i.e., that it is not true that all inertial frames are physically equivalent. Suppose we have two inertial frames I and I' as above, i.e., $I = \partial/\partial x^0$, $I' = \partial/\partial x'^0$.

Suppose that F is a SEXW which can be launched in I with velocity 1-form as above and suppose \bar{F} is a SEXW built in I' at the plane $z' = 0$ and with velocity 1-form relative to $\langle x'^{\mu} \rangle$ given by $v_{\bar{F}} = v'^{\mu} \gamma'_{\mu}$ and

$$v_{\bar{F}} = \left(\frac{1}{\sqrt{c_1'^2 - 1}}, 0, 0, -\frac{c_1'}{\sqrt{c_1'^2 - 1}} \right) \tag{5.25}$$

If F and \bar{F} are related as above, we see (see Fig. 10) that \bar{F} , which has positive energy and is traveling to the future according to I' , can be sent to the past of the observers at rest in the I frame. Obviously this is impossible and we conclude that \bar{F} is not a physically realizable phenomenon in nature. It cannot be realized in I' but F can be realized in I . It follows that PR_2 does not hold.

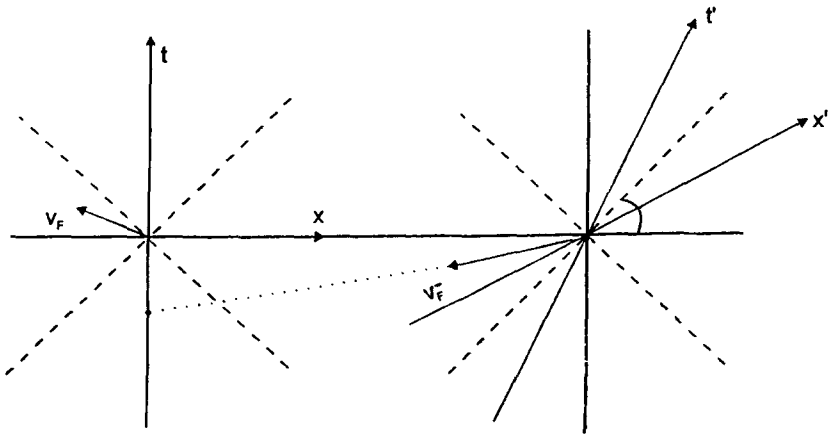


Fig. 10. \bar{F} cannot be launched by I' .

If the elements of the set of inertial reference frames are not equivalent, then there must exist a fundamental reference frame. Let $I \in \text{sec } TM$ be that fundamental frame. If I' is moving with speed V relative to I , i.e.,

$$I' = \frac{1}{\sqrt{1-V^2}} \frac{\partial}{\partial t} - \frac{V}{\sqrt{1-V^2}} \frac{\partial}{\partial z} \tag{5.26}$$

then, if observers in I' are equipped with a generator of SEXWs and if they prepare their apparatus in order to send SEXWs with different velocity 1-forms in all possible directions in spacetime, they will find a particular velocity 1-form in a given spacetime direction in which the device stops working. A simple calculation yields then, for the observes in I' , the value of V !

In Ref. 65 Recami argued that the principle of relativity continues to hold true even though superluminal phenomena exist in nature. In this theory of tachyons there exists, of course, a situation completely analogous to the one described above (called the Tolman–Regge paradox), and according to Recami’s view PR_2 is valid because I' must interpret \bar{F} as being an anti-SEXW carrying positive energy and going into the future according to him. In his theory of tachyons Recami was able to show that the dynamics of tachyons implies that no detector *at rest* in I can detect a tachyon (the same would be valid for a SEXW like \bar{F}) sent by I' with velocity 1-form given by Eq. (4.26). Thus he claimed that PR_2 is true. At first sight the argument seems good, but it is at least incomplete. Indeed, a detector in I does not need to be at rest in I . We can imagine a detector in periodic

motion in I which could absorb the \bar{F} wave generated by I' if this was indeed possible. It is enough for the detector to have relative to I the speed V of the I' frame in the appropriate direction at the moment of absorption. This simple argument shows that there is no salvation for PR_2 (and for PR_1) if superluminal phenomena exist in nature.

The attentive reader at this point probably has the following question in his/her mind: How could the authors start with Minkowski spacetime, with equations carrying the Lorentz symmetry, and yet arrive at the conclusion that PR_1 and PR_2 do not hold?

The reason is that the Lorentzian structure of $\langle M, g, D \rangle$ can be seen to exist directly from the Newtonian spacetime structure as proved in Ref. 66. In that paper Rodrigues and collaborators show that even if \mathcal{L}_+^1 is not a symmetry group of Newtonian dynamics it is a symmetry group of the only possible coherent formulation of Lorentz–Maxwell electrodynamic theory compatible with experimental results that can be formulated in the Newtonian spacetime.¹⁴

We finish by calling to the reader's attention that there are some experiments reported in the literature which suggest also a breakdown of PR_2 for the roto-translational motion of solid bodies. A discussion and references can be found in Ref. 67.

6. CONCLUSIONS

In this paper we presented a unified theory showing that the homogeneous wave equation, the Klein–Gordon equation, Maxwell equations, and the Dirac and Weyl equations have solutions with the form of undistorted progressive waves (UPWs) of arbitrary speeds $0 \leq v < \infty$.

We present also the results of an experiment which confirms that finite aperture approximations to a Bessel pulse and to an X -wave in water move as predicted by our theory, i.e., the Bessel pulse moves with speed less than c_s and the X -wave moves with speed greater than c_s , c_s being the sound velocity in water.

We exhibit also some subluminal and superluminal solutions of Maxwell equations. We showed that subluminal solutions can in principle be used to model purely electromagnetic particles. A detailed discussion is given about the superluminal electromagnetic X -wave solution of Maxwell equations and we showed that it can in principle be launched with available technology. Here a point must be clear: the X -waves, both

¹⁴ We recall that Maxwell equations have, as is well known, many symmetry groups besides \mathcal{L}_+^1 .

acoustic and electromagnetic, are signals in the sense defined by Nimtz.⁽⁷⁴⁾ It is a widespread misunderstanding that signals must have a front. A front can be defined only mathematically because it implies an infinite frequency spectrum. Every real signal does not have a well-defined front.

The existence of superluminal electromagnetic waves implies the breakdown of the principle of relativity.¹⁵ We observe that besides its fundamental theoretical implications, the practical implications of the existence of UPW solutions of the main field equations of theoretical physics (and their finite aperture realizations) are very important. This practical importance ranges from applications in ultrasound medical imaging to the project of electromagnetic bullets and new communication devices.⁽³³⁾ Also we would like to conjecture that the existence of subluminal and superluminal solutions of the Weyl equation may be important to solve some of the mysteries associated with neutrinos. Indeed, if neutrinos can be produced in subluminal or superluminal modes—see Refs. 75 and 76 for some experimental evidence concerning superluminal neutrinos—they can eventually escape detection on earth after leaving the sun. Moreover, for neutrinos in a subluminal or superluminal mode it would be possible to define a kind of “effective mass.” Recently some cosmological evidence that neutrinos have a nonvanishing mass has been discussed by, e.g., Primack *et al.*⁽⁷⁷⁾ One such “effective mass” could be responsible for the cosmological evidence, and in such a way that we can still have a left-handed neutrino since it would satisfy the Weyl equation. We discuss this issue further in another publication.

APPENDIX A. SOLUTIONS OF THE (SCALAR) HOMOGENEOUS WAVE EQUATION AND THEIR FINITE APERTURE REALIZATIONS

In this appendix we first recall briefly some well-known results concerning the fundamental (Green’s functions) and the general solutions of the (scalar) *homogeneous wave equation* (HWE) and the theory of their *finite aperture approximation* (FAA). FAA is based on the *Rayleigh–Sommerfeld formulation of diffraction* (RSFD) by a plane screen. We show that under certain conditions the RSFD is useful for designing physical devices to launch waves that travel with the characteristic velocity in a homogeneous

¹⁵ It is important to recall that there exists the possibility of propagation of superluminal signals inside hadronic matter. In this case the ingenious construction of Santilli’s isominkowskian spaces (see Refs. 68–73) is useful.

medium (i.e., the speed c that appears in the wave equation). More important, RSFD is also useful for projecting physical devices to launch some of the subluminal and superluminal solutions of the HWE (i.e., waves that propagate in a homogeneous medium with speeds respectively less and greater than c) that we present in this appendix. We use units such that $c=1$ and $\hbar=1$, where c is the so-called velocity of light in vacuum and \hbar is Planck's constant divided by 2π .

A1. Green's Functions and the General Solution of the (Scalar) HWE

Let Φ in what follows be a complex function in Minkowski spacetime M :

$$\Phi: M \ni x \mapsto \Phi(x) \in \mathbb{C} \quad (\text{A.1})$$

The inhomogeneous wave equation for Φ is

$$\square \Phi = \left(\frac{\partial^2}{\partial t^2} - \nabla^2 \right) \Phi = 4\pi\rho \quad (\text{A.2})$$

where ρ is a complex function in Minkowski spacetime. We define a two-point Green's function for the wave equation (A.2) as a solution of

$$\square G(x-x') = 4\pi\delta(x-x') \quad (\text{A.3})$$

As it is well known, the fundamental solutions of (A.3) are:

$$\textit{Retarded Green's function: } G_R(x-x') = 2H(x-x') \delta[(x-x')^2] \quad (\text{A.4a})$$

$$\textit{Advanced Green's function: } G_A(x-x') = 2H[-(x-x')] \delta[(x-x')^2] \quad (\text{A.4b})$$

where $(x-x')^2 \equiv (x^0-x'^0)^2 - (\vec{x}-\vec{x}')^2$, $H(x) = H(x^0)$ is the step function, and $x^0 = t$, $x'^0 = t'$.

We can rewrite Eqs. (A.4) as ($R = |\vec{x}-\vec{x}'|$):

$$G_R(x^0-x'^0; \vec{x}-\vec{x}') = \frac{1}{R} \delta(x^0-x'^0-R) \quad (\text{A.4c})$$

$$G_A(x^0-x'^0; \vec{x}-\vec{x}') = \frac{1}{R} \delta(x^0-x'^0+R) \quad (\text{A.4d})$$

We define the Schwinger function by

$$G_S = G_R - G_A = 2\varepsilon(x) \delta(x^2); \quad \varepsilon(x) = H(x) - H(-x) \quad (\text{A.5})$$

It has the properties

$$\square G_S = 0, \quad G_S(x) = -G_S(-x), \quad G_S(x) = 0, \quad \text{if } x^2 < 0 \quad (\text{A.6a})$$

$$G_S(0, \vec{x}) = 0, \quad \left. \frac{\partial G_S}{\partial x^i} \right|_{x^i=0} = 0, \quad \left. \frac{\partial G_S}{\partial x^0} \right|_{x^0=0} = \delta(\vec{x}) \quad (\text{A.6b})$$

For the reader who is familiar with the material presented in Appendix B, we observe that these equations can be rewritten in a very elegant way in $\mathcal{C}\ell_C(M)$. (If you have not read Appendix B, go to Eq. (A.8').) We have

$$\int_{\sigma} \star dG_S(x-y) = - \int_{\sigma} dG_S(x-y) \gamma^5 = 1, \quad \text{if } y \in \sigma \quad (\text{A.7})$$

where σ is any spacelike surface. Then if $f \in \text{sec } \mathbb{C} \otimes \wedge^0(M) \subset \text{sec } \mathcal{C}\ell_C(M)$ is any function defined on a spacelike surface σ , we can write

$$\int_{\sigma} [\star dG_S(x-y)] f(x) = - \int_{\sigma} dG_S(x-y) f(x) \gamma^5 = f(y) \quad (\text{A.8})$$

Equations (A.7) and (A.8) appear in textbooks on field theory as

$$\int_{\sigma} \partial_{\mu} G_S(x-y) d\sigma^{\mu}(x) = 1, \quad \int_{\sigma} f(x) \partial_{\mu} G_S(x-y) d\sigma^{\mu}(x) = f(y) \quad (\text{A.8}')$$

We now express the general solution of Eq. (A.2), including the initial conditions, in a bounded constant time spacelike hypersurface σ characterized by $\gamma^1 \wedge \gamma^2 \wedge \gamma^3$ in terms of G_R . We write the solution in the *standard vector notation*. Let the constant time hypersurface σ be the volume $V \subset \mathbb{R}^3$ and $\partial V = S$ its boundary. We have

$$\begin{aligned} \Phi(t, \vec{x}) = & \int_0^{t^+} dt' \iiint_V d\mathbf{v}' G_R(t-t', \vec{x}-\vec{x}') \rho(t', \vec{x}') \\ & + \frac{1}{4\pi} \iiint_V d\mathbf{v}' \left[G_R|_{t'=0} \frac{\partial \Phi}{\partial t'}(t', \vec{x}')|_{t'=0} \right. \\ & \left. - \Phi(t', \vec{x}')|_{t'=0} \frac{\partial}{\partial t'} G_R|_{t'=0} \right] \\ & + \frac{1}{4\pi} \int_0^{t^+} dt' \iint_S d\vec{S}^i \cdot (G_R \text{grad}' \Phi - \Phi \text{grad}' G_R) \quad (\text{A.9}) \end{aligned}$$

where grad' means that the gradient operator acts on \vec{x}' , and where t_+ means that the integral over t' must end on $t' = t + \varepsilon$ in order to avoid ending the integral exactly at the peak of the δ -function. The *first term* in Eq. (A.9) represents the effects of the sources, the *second term* represents the effects of the initial conditions (Cauchy problem), and the *third term* represents the effects of the boundary conditions on the space boundaries $\partial V = S$. This term is essential for the theory of diffraction and in particular for the RSFD.

Cauchy problem: Suppose that $\Phi(0, \vec{x})$ and $(\partial/\partial t)\Phi(t, \vec{x})|_{t=0}$ are known at every point in space, and assume that there are no sources present, i.e., $\rho = 0$. Then the solution of the HWE becomes

$$\Phi(t, \vec{x}) = \frac{1}{4\pi} \iiint d\mathbf{v}' \left[G_R|_{t'=0} \frac{\partial}{\partial t'} \Phi(t', \vec{x}')|_{t'=0} - \frac{\partial}{\partial t'} G_R|_{t'=0} \Phi(0, \vec{x}') \right] \quad (\text{A.10})$$

The integration extends over all space and we explicitly assume that the third term in Eq. (A.9) vanishes at infinity.

We can give an intrinsic formulation of Eq. (A.10). Let $x \in \sigma$, where σ is a spacelike surface *without* boundary. Then the solution of the HWE can be written

$$\begin{aligned} \Phi(x) &= \frac{1}{4\pi} \int_{\sigma} \{ G_S(x-x') [\star d\Phi(x')] - [\star dG_S(x-x')] \Phi(x') \} \\ &= \frac{1}{4\pi} \int_{\sigma} d\sigma^{\mu}(x) [G_S(x-x') \partial_{\mu} \Phi(x') - \partial_{\mu} G_S(x-x') \Phi(x')] \end{aligned} \quad (\text{A.11})$$

where G_S is the Schwinger function [see Eqs. (A.7, A.8)]. $\Phi(x)$ given by Eq. (A.11) corresponds to “causal propagation” in the usual Einstein sense, i.e., $\Phi(x)$ is influenced only by points of σ which lie in the backward (forward) light cone of x' , depending on whether x is “later” (“earlier”) than σ .

A2. Huygen’s Principle; the Kirchhoff and Rayleigh–Sommerfeld Formulations of Diffraction by a Plane Screen⁽⁷⁹⁾

Huygen’s principle is essential for understanding Kirchhoff’s formulation and the Rayleigh–Sommerfeld formulation (RSF) of diffraction by a plane screen. Consider again the general solution [Eq. (A.9)] of the HWE

which is non-null in the surface $S = \partial V$ and suppose also that $\Phi(0, \vec{x})$ and $(\partial/\partial t) \Phi(t, \vec{x})|_{t=0}$ are null for all $\vec{x} \in V$. Then Eq. (A.9) gives

$$\Phi(t, \vec{x}) = \frac{1}{4\pi} \iint_S d\vec{S}' \cdot \left[\frac{1}{R} \text{grad}' \Phi(t', \vec{x}') + \frac{\vec{R}}{R^3} \Phi(t, \vec{x}') - \frac{\vec{R}}{R^2} \frac{\partial}{\partial t'} \Phi(t', \vec{x}') \right]_{t' = t - R} \tag{A.12}$$

From Eq. (A.12) we see that if S is along a wavefront and the rest of it is at infinity or where Φ is zero, we can say that the field value Φ at (t, \vec{x}) is caused by the field Φ in the wave front at time $(t - R)$ earlier. This is Huygen's principle.

Kirchhoff's Theory. Now, consider a screen with a hole as in Fig. 11.

Suppose that we have an *exact solution* of the HWE that can be written as

$$\Phi(t, \vec{x}) = F(\vec{x}) e^{i\omega t} \tag{A.13}$$

where we define also

$$\omega = \bar{k} \tag{A.14}$$

and \bar{k} is not necessarily the propagation vector (see below). We want to find the field at $\vec{x} \in V$, with $\partial V = S_1 + S_2$ (Fig. 11), with $\rho = 0 \ \forall \vec{x} \in V$.

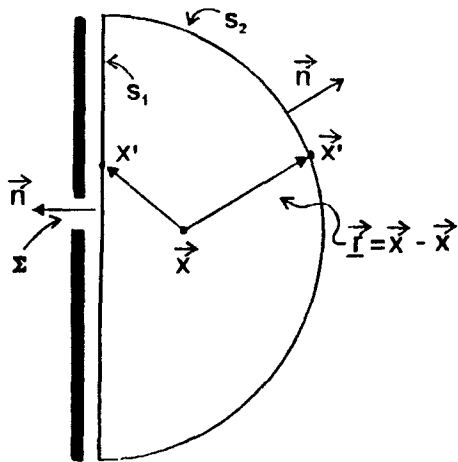


Fig. 11. Diffraction from a finite aperture.

Kirchhoff proposed to use Eq. (A.12) to give an approximate solution for the problem. Under the so-called Sommerfeld radiation condition,

$$\lim_{r \rightarrow \infty} r \left(\frac{\partial F}{\partial n} - ikF \right) = 0 \quad (\text{A.15})$$

where $r = |\vec{r}| = \vec{x} - \vec{x}'$, \vec{x}' being a point of S_2 , the integral in Eq. (A.12) is null over S_2 . Then, we get

$$F(\vec{x}) = \frac{1}{4\pi} \iint_{S_1} dS' \left(\frac{\partial F}{\partial n} G_K - F \frac{\partial G_K}{\partial n} \right) \quad (\text{A.16})$$

$$G_K = \frac{e^{-ikR}}{R}, \quad R = |\vec{x} - \vec{x}'|, \quad \vec{x}' \in S_1 \quad (\text{A.17})$$

Now, the “source” is opaque, except for the aperture which is denoted by Σ in Fig. 11. It is reasonable to suppose that the major contribution to the integral arises from points of S_1 in the aperture $\Sigma \subset S_1$. Kirchhoff then proposed the conditions:

(i) Across Σ , the fields F and $\partial F/\partial n$ are exactly the same as they would be in the absence of sources.

(ii) Over the portion of S_1 that lies in the geometrical shadow of the screen the field F and $\partial F/\partial n$ are null.

Conditions (i) plus (ii) are called *Kirchhoff boundary conditions*, and we end with

$$F_K(\vec{x}) = \iint_{\Sigma} dS' \left(\frac{\partial F}{\partial n} G_K - F \frac{\partial}{\partial n} G_K \right) \quad (\text{A.18})$$

where $F_K(\vec{x})$ is the Kirchhoff approximation to the problem. As is well known, F_K gives results that agree very well with experiments, if the dimensions of the aperture are large compared with the wavelength. Nevertheless, Kirchhoff's solution is inconsistent, since under the hypothesis given by Eq. (A.13), $F(\vec{x})$ becomes a solution of the Helmholtz equation

$$\nabla^2 F + \omega^2 F = 0 \quad (\text{A.19})$$

and as is well known it is illicit for this equation to impose simultaneously arbitrary boundary conditions for both F and $\partial F/\partial n$.

A further shortcoming of F_K is that it fails to reproduce the assumed boundary conditions when $\vec{x} \in \Sigma \subset S_1$. To avoid such inconsistencies Sommerfeld proposed to eliminate the necessity of imposing boundary

conditions on both F and $\partial F/\partial n$ simultaneously. This gives the so-called *Rayleigh–Sommerfeld formulation of diffraction by a plane screen* (RSFD). RSFD is obtained as follows. Consider again a solution of Eq. (A.18) under Sommerfeld radiation condition [Eq. (A.15)]

$$F(\vec{x}) = \frac{1}{4} \iint_{S_1} \left(\frac{\partial F}{\partial n} G_{RS} - F \frac{\partial G_{RS}}{\partial n} \right) dS' \tag{A.20}$$

where now G_{RS} is a Green function for Eq. (A.19) different from G_K . G_{RS} must provide an exact solution of Eq. (A.19), but we want in addition that G_{RS} or $\partial G_{RS}/\partial n$ vanish over the entire surface S_1 , since as we already said we cannot impose the values of F and $\partial F/\partial n$ simultaneously.

A solution for this problem is to take G_{RS} as a three-point function, i.e., as a solution of

$$(\nabla^2 + \omega^2) G_{RS}^-(\vec{x}, \vec{x}', \vec{x}'') = 4\pi\delta(\vec{x} - \vec{x}') - 4\pi\delta(\vec{x} - \vec{x}'') \tag{A.21}$$

We get

$$G_{RS}^-(\vec{x}, \vec{x}', \vec{x}'') = \frac{e^{i\vec{k}R}}{R} - \frac{e^{i\vec{k}R'}}{R'} \tag{A.22}$$

$$R = |\vec{x} - \vec{x}'|, \quad R' = |\vec{x} - \vec{x}''| \tag{A.23}$$

where $\vec{x} \in S_1$ and $\vec{x}'' = -\vec{x}'$ are mirror image points relative to S_1 . This solution gives $G_{RS}^-|_{S_1} = 0$ and $\partial G_{RS}^-/\partial n|_{S_1} \neq 0$.

Another solution for our problem such that $G_{RS}^+|_{S_1} \neq 0$ and $\partial G_{RS}^+/\partial n|_{S_1} = 0$ is realized for G_{RS}^+ satisfying

$$(\nabla^2 + \omega^2) G_{RS}^+(\vec{x}, \vec{x}', \vec{x}'') = 4\pi\delta(\vec{x} - \vec{x}') + 4\pi\delta(\vec{x} - \vec{x}'') \tag{A.24}$$

Then

$$G_{RS}^+(\vec{x}, \vec{x}', \vec{x}'') = \frac{e^{i\vec{k}R}}{R} + \frac{e^{i\vec{k}R'}}{R'} \tag{A.25}$$

with R and R' as in Eq. (A.23).

We now use G_{RS}^+ in Eq. (A.25) and take S_1 as being the $z=0$ plane. In this case $\vec{n} = -\hat{k}$, \hat{k} being the versor of the z direction, $\vec{R} = \vec{x} - \vec{x}'$, $\vec{R} \cdot \vec{n} = z' - z \cos(\vec{n}, \vec{R}) = (z' - z)/R$, and we get

$$F(\vec{x}) = -\frac{1}{2\pi} \iint_{S_1} dS' F(x', y', 0) \left[i\vec{k}z \frac{e^{i\vec{k}R}}{R^2} - \frac{e^{i\vec{k}R}}{R^3} z \right] \tag{A.26}$$

A3. Finite Aperture Approximation for Waves Satisfying

$$\Phi(t, \vec{x}) = F(\vec{x}) e^{-i\omega t}$$

The finite aperture approximation to Eq. (A.26) consists in integrating only over $\Sigma \subset S_1$, i.e., we suppose $F(\vec{x}) = 0$, $\forall \vec{x} \in (S_1 \setminus \Sigma)$. Taking into account that

$$\bar{k} = 2\pi/\lambda, \quad \omega = \bar{k} \quad (\text{A.27})$$

we get

$$F_{FAA} = \frac{1}{\lambda} \iint_{\Sigma} dS' F(x', y', 0) \frac{e^{i\bar{k}R}}{R^2} z + \frac{1}{2\pi} \iint_{\Sigma} dS' F(x', y', 0) \frac{e^{i\bar{k}R}}{R^3} z \quad (\text{A.28})$$

In Sec. A4 we show some subluminal and superluminal solutions of the HWE and then discuss for which solutions the FAA is valid. We show that there are indeed subluminal and superluminal solutions of the HWE for which (A.28) can be used. *Even more important*, we describe in Sec. 2 the results of recent experiments, conducted by us, that confirm the predictions of the theory for acoustic waves in water.

A4. Subluminal and Superluminal Solutions of the HWE

Consider the HWE ($c = 1$)

$$\frac{\partial^2}{\partial t^2} \Phi - \nabla^2 \Phi = 0 \quad (\text{A.2'})$$

We now present some subluminal and superluminal solutions of Eq. (A.2').⁽⁸⁰⁾

Subluminal and Superluminal Spherical Bessel Beams. To introduce these beams we define the variables

$$\xi_{<} = [x^2 + y^2 + \gamma_{<}^2 (z - v_{<} t)^2]^{1/2} \quad (\text{A.29a})$$

$$\gamma_{<} = \frac{1}{\sqrt{1 - v_{<}^2}}, \quad \omega_{<}^2 - k_{<}^2 = \Omega_{<}^2, \quad v_{<} = \frac{d\omega_{<}}{dk_{<}} \quad (\text{A.29b})$$

$$\xi_{>} = [-x^2 - y^2 + \gamma_{>}^2 (z - v_{>} t)^2]^{1/2} \quad (\text{A.29c})$$

$$\gamma_{>} = \frac{1}{\sqrt{v_{>}^2 - 1}}, \quad \omega_{>}^2 - k_{>}^2 = -\Omega_{>}^2, \quad v_{>} = d\omega_{>}/dk_{>} \quad (\text{A.29d})$$

We can now easily verify that the functions $\Phi_{<}^{lm}$ and $\Phi_{>}^{lm}$ below are respectively subluminal and superluminal solutions of the HWE (see example 3 below for how to obtain these solutions). We have

$$\Phi_p^{lm}(t, \vec{x}) = C_l j_l(\Omega_p \xi_p) P_m^l(\cos \theta) e^{im\theta} e^{i(\omega_p t - k_p z)} \tag{A.30}$$

where the index $p = <, >$, C_l are constants, j_l are the spherical Bessel functions, P_m^l are the Legendre functions, and (r, θ, φ) are the usual spherical coordinates. $\Phi_{<}^{lm}$ [$\Phi_{>}^{lm}$] has phase velocity $(w_{<}/k_{<}) < 1$ [$(w_{>}/k_{>}) > 1$] and the modulation function $j_l(\Omega_{<} \xi_{<})$ [$j_l(\Omega_{>} \xi_{>})$] moves with group velocity $v_{<}$ [$v_{>}$], where $0 \leq v_{<} < 1$ [$1 < v_{>} < \infty$]. Both $\Phi_{<}^{lm}$ and $\Phi_{>}^{lm}$ are *undistorted progressive waves* (UPWs). This term was introduced by Courant and Hilbert⁽¹⁾; however, they did not suspect that UPWs move with speeds greater than $c=1$. For use in the main text we write the explicit form of $\Phi_{<}^{00}$ and $\Phi_{>}^{00}$, which we denote simply by $\Phi_{<}$ and $\Phi_{>}$:

$$\Phi_p(t, \vec{x}) = C \frac{\sin(\Omega_p \xi_p)}{\xi_p} e^{i(\omega_p t - k_p z)}, \quad p = < \text{ or } > \tag{A.31}$$

When $v_{<} = 0$, we have $\Phi_{<} \rightarrow \Phi_0$,

$$\Phi_0(t, \vec{x}) = C \frac{\sin \Omega_{<} r}{r} e^{i\Omega_{<} t}, \quad r = (x^2 + y^2 + z^2)^{1/2} \tag{A.32}$$

When $v_{>} = \infty$, $\omega_{>} = 0$, and $\Phi_{>}^0 \rightarrow \Phi_\infty$,

$$\Phi_\infty(t, \vec{x}) = C_\infty \frac{\sinh \rho}{\rho} e^{i\Omega_{>} z}, \quad \rho = (x^2 + y^2)^{1/2} \tag{A.33}$$

We observe that if our interpretation of phase and group velocities is correct, then there must be a Lorentz frame where $\Phi_{<}$ is at rest. It is trivial to verify that in the coordinate chart $\langle x'^{\mu} \rangle$ which is a (nacs/ I'), where $I' = (1 - v_{<}^2)^{-1/2} \partial/\partial t + (v_{<}/\sqrt{1 - v_{<}^2}) \partial/\partial z$ is a Lorentz frame moving with speed $v_{<}$ in the z direction relative to $I = \partial/\partial t$, Φ_p goes in $\Phi_0(t', \vec{x}')$ given by Eq. (A.32) with $t \mapsto t'$, $\vec{x} \mapsto \vec{x}'$.

Subluminal and Superluminal Bessel Beams. The solutions of the HWE in cylindrical coordinates are well known.⁽¹⁹⁾ Here we recall how these solutions are obtained in order to present new subluminal and superluminal solutions of the HWE. In what follows the cylindrical coordinate functions are denoted by (ρ, θ, z) , $\rho = (x^2 + y^2)^{1/2}$, $x = \rho \cos \theta$, $y = \rho \sin \theta$. We write for Φ :

$$\Phi(t, \rho, \theta, z) = f_1(\rho) f_2(\theta) f_3(t, z) \tag{A.34}$$

Inserting (A.34) in (A.2') gives

$$\rho^2 \frac{d^2}{d\rho^2} f_1 + \rho \frac{d}{d\rho} f_1 + (B\rho^2 - \nu^2) f_1 = 0 \quad (\text{A.35a})$$

$$\left(\frac{d^2}{d\theta^2} + \nu^2 \right) f_2 = 0 \quad (\text{A.35b})$$

$$\left(\frac{d^2}{dt^2} - \frac{\partial^2}{\partial z^2} + B \right) f_3 = 0 \quad (\text{A.35c})$$

In these equations B and ν are separation constants. Since we want Φ to be periodic in θ , we choose $\nu = n$ an integer. For B we consider two cases:

(i) *Subluminal Bessel solution*, $B = \Omega_{<}^2 > 0$. In this case (A.35a) is a Bessel equation and we have

$$\Phi_{J_n}^{<}(t, \rho, \theta, z) = C_n J_n(\rho \Omega_{<}) e^{i(k_{<} z - \omega_{<} t + n\theta)}, \quad n = 0, 1, 2, \dots \quad (\text{A.36})$$

where C_n is a constant, J_n is the n th order Bessel function, and

$$\omega_{<}^2 - k_{<}^2 = \Omega_{<}^2 \quad (\text{A.37})$$

In Ref. 43 the $\Phi_{J_n}^{<}$ are called the n th-order nondiffracting Bessel beams.¹⁶

Bessel beams are examples of undistorted progressive waves (UPWs). They are “subluminal” waves. Indeed, the group velocity for each wave is

$$v_{<} = d\omega_{<}/dk_{<}, \quad 0 < v_{<} < 1 \quad (\text{A.38})$$

but the phase velocity of the wave is $(\omega_{<}/k_{<}) > 1$. That this interpretation is correct follows from the results of the acoustic experiment described in Sec. 2.

It is convenient for what follows to define the variable η , called the axicon angle,⁽²⁶⁾

$$k_{<} = \bar{k}_{<} \cos \eta, \quad \Omega_{<} = \bar{k}_{<} \sin \eta, \quad 0 < \eta < \pi/2 \quad (\text{A.39})$$

Then

$$\bar{k}_{<} = \omega_{<} > 0 \quad (\text{A.40})$$

¹⁶ The only difference is that $k_{<}$ is denoted by $\beta = \sqrt{\omega_{<}^2 - \Omega_{<}^2}$ and $\omega_{<}$ is denoted by $k' = \omega/c > 0$. (We use units where $c = 1$.)

and Eq. (A.36) can be rewritten as $\Phi_{A_n}^< \equiv \Phi_{J_n}^<$, with

$$\Phi_{A_n}^< = C_n J_n(\bar{k}_< \rho \sin \eta) e^{i(\bar{k}_< z \cos \eta - \omega_< t + n\theta)} \tag{A.41}$$

In this form the solution is called in Ref. 43 the n th order nondiffracting portion of the *axicon beam*. The phase velocity $v^{ph} = 1/\cos \eta$ is independent of $\bar{k}_<$, but, of course, it is dependent on $k_<$. We shall show below that waves constructed from the $\Phi_{J_n}^<$ beams can be *subluminal* or *superluminal*!

(ii) *Superluminal (Modified) Bessel Solution*, $B = -\Omega_>^2 < 0$. In this case (A.35a) is the modified Bessel equation and we denote the solutions by

$$\Phi_{K_n}^>(t, \rho, \theta, z) = C_n K_n(\Omega_> \rho) e^{i(k_> z - \omega_> t + n\theta)}, \quad n = 0, 1, \dots \tag{A.42}$$

where K_n are the modified Bessel functions, C_n are constants, and

$$\omega_>^2 - k_>^2 = -\Omega_>^2 \tag{A.43}$$

We see that $\Phi_{K_n}^>$ are also examples of UPWs, each of which has group velocity $v_> = d\omega_>/dk_>$ such that $1 < v_> < \infty$ and phase velocity $0 < (\omega_>/k_>) < 1$. As in the case of the spherical Bessel beam [Eq. (A.31)], we see again that our interpretation of phase and group velocities is correct. Indeed, for the superluminal (modified) Bessel beam there is no Lorentz frame where the wave is stationary.

The $\Phi_{K_0}^>$ beam was discussed by Band⁽⁴¹⁾ in 1988 as an example of superluminal motion. Band proposed to launch the $\Phi_{K_0}^>$ beam in the exterior of a cylinder of radius r_1 on which there is an appropriate superficial charge density. Since $K_0(\Omega_> r_1)$ is nonsingular, his solution works. In Sec. 3 we discuss some of Band's statements.

We are now prepared to present some other very interesting solutions of the HWE, in particular the so-called *X-waves*,⁽⁴³⁾ which are superluminal, as proved by the acoustic experiments described in Sec. 2.

Theorem [Lu and Greenleaf]⁽⁴³⁾. The three functions below are families of exact solutions of the HWE [Eq. (A.2')] in cylindrical coordinates:

$$\Phi_\eta(s) = \int_0^\infty T(\bar{k}_<) \left[\frac{1}{2\pi} \int_{-\pi}^\pi A(\phi) f(s) d\phi \right] d\bar{k}_< \tag{A.44}$$

$$\Phi_K(s) = \int_{-\pi}^\pi D(\eta) \left[\frac{1}{2\pi} \int_{-\pi}^\pi A(\phi) f(s) d\phi \right] d\eta \tag{A.45}$$

$$\Phi_L(\rho, \theta, z - t) = \Phi_1(\rho, \theta) \Phi_2(z - t) \tag{A.46}$$

where

$$s = \alpha_0(\bar{k}_<, \eta) \rho \cos(\theta - \phi) + b(\bar{k}_<, \eta)[z \pm c_1(\bar{k}_<, \eta) t] \quad (\text{A.47})$$

and

$$c_1(\bar{k}_<, \eta) = \sqrt{1 + [\alpha_0(\bar{k}_<, \eta)/b(\bar{k}_<, \eta)]^2} \quad (\text{A.48})$$

In these formulas $T(\bar{k}_<)$ is any complex function (well behaved) of $\bar{k}_<$ and could include the *temporal frequency transfer function* of a radiator system, $A(\phi)$ is any complex function (well behaved) of ϕ and represents a *weighting function* of the integration with respect to ϕ , $f(s)$ is any complex function (well behaved) of s [solution of Eq. (A.29)], $D(\eta)$ is any complex function (well behaved) of η and represents a weighting function of the integration with respect to η , called the axicon angle [see Eq. (A.39)], $\alpha_0(\bar{k}_<, \eta)$ is any complex function of $\bar{k}_<$, and η , $b(\bar{k}_<, \eta)$ is any complex function of $\bar{k}_<$ and η .

As in the previous solutions, we take $c = 1$. Note that $\bar{k}_<$, η , and the wave vector $k_<$ of the $f(s)$ solution of Eq. (A.29) are related by Eq. (A.39). Also $\Phi_2(z - t)$ is any complex function of $(z - t)$ and $\Phi_1(\rho, \theta)$ is any solution of the transverse Laplace equation, i.e.,

$$\left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \theta^2} \right] \Phi_1(\rho, \theta) = 0 \quad (\text{A.49})$$

The proof is obtained by direct substitution of Φ_η , Φ_K , and Φ_L in the HWE. Obviously, the exact solution Φ_L is an example of a luminal UPW, because if one “travels” with the speed $c = 1$, i.e., with $z - t = \text{const}$, both the lateral and axial components $\Phi_1(\rho, \theta)$ and $\Phi_2(z - t)$ will be the same for all time t and distance z . When $c_1(\bar{k}, \eta)$ in Eq. (A.47) is real, (\pm) represent respectively backward and forward propagating waves.

We recall that $\Phi_\eta(s)$ and $\Phi_K(s)$ represent families of UPWs if $c_1(\bar{k}_<, \eta)$ is independent of $\bar{k}_<$ and η respectively. These waves travel to infinity at speed c_1 . $\Phi_\eta(s)$ is a generalized function that contains some of the UPWs solutions of the HWE derived previously. In particular, if $T(\bar{k}_<) = \delta(\bar{k}_< - \bar{k}'_<)$, $\bar{k}'_< = \omega > 0$ is a constant and if $f(s) = e^s$, $\alpha_0(\bar{k}_<, \eta) = -i\Omega_<$, $b(\bar{k}_<, \eta) = i\beta = i\omega/c_1$, one obtains Durnin’s UPW beam⁽¹⁶⁾

$$\Phi_{\text{Durnin}}(s) = \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} A(\phi) e^{-i\Omega_< \rho \cos(\theta - \phi)} d\phi \right] e^{i(\beta z - \omega t)} \quad (\text{A.50})$$

If $A(\phi) = i^n e^{in\phi}$ we obtain the n th order UPW Bessel beam $\Phi_{J_n}^<$ given by Eq. (A.36). $\Phi_{A_n}^<(s)$ is obtained in the same way with the transformation $k_< = \bar{k}_< \cos \eta$; $\Omega_< = \bar{k}_< \sin \eta$.

The X-waves. We now present a superluminal UPW wave which, as discussed in Sec. 2, is physically realizable in an approximate way (FAA) in the acoustic case and can be used to generate Hertz potentials for the electromagnetic field (see Sec. 3). We take in Eq. (A.44):

$$T(\bar{k}_<) = B(\bar{k}_<) e^{-a_0 \bar{k}_<}, \quad A(\phi) = i^n e^{in\phi}, \quad \alpha_0(\bar{k}_<, \eta) = -i \bar{k}_< \sin \eta$$

$$b(\bar{k}_<, \eta) = ik \cos \eta, \quad f(s) = e^s \tag{A.51}$$

We then get

$$\Phi_{X_n}^> = e^{in\theta} \int_0^\infty B(\bar{k}_<) J_n(\bar{k}_< \rho \sin \eta) e^{-\bar{k}_< [a_0 - i(z \cos \eta - t)]} d\bar{k}_< \tag{A.52}$$

In Eq. (A.52) $B(\bar{k}_<)$ is any well-behaved complex function of $\bar{k}_<$ and represents a *transfer function of a practical radiator*, $\bar{k}_< = \omega$ and a_0 is a constant, and η is again called the axicon angle.⁽²⁶⁾ Equation (A.52) shows that $\Phi_{X_n}^>$ is represented by a Laplace transform of the function $B(\bar{k}_<) J_n(\bar{k}_< \rho \sin \eta)$ and an azimuthal phase term $e^{in\theta}$. The name *X-waves* for the $\Phi_{X_n}^>$ comes from the fact that these waves have an *X*-like shape in a plane containing the axis of symmetry of the waves (the *z*-axis; see Fig. 4(1) in Sec. 3).

The $\Phi_{XBB_n}^>$ Waves. This wave is obtained from Eq. (A.44) putting $B(\bar{k}_<) = a_0$. It is called the *X-wave* produced by an *infinite* aperture and *broad bandwidth*. We use in this case the notation $\Phi_{XBB_n}^>$. Under these conditions we get

$$\Phi_{XBB_n}^> = \frac{a_0 (\rho \sin \eta)^n e^{in\theta}}{\sqrt{M} (\tau + \sqrt{M})^n}, \quad n = 0, 1, 2, \dots \tag{A.53}$$

where the subscript denotes “broadband.” Also

$$M = (\rho \sin \eta)^2 + \tau^2 \tag{A.54}$$

$$\tau = [a_0 - i(z \cos \eta - t)] \tag{A.55}$$

For $n = 0$ we get $\Phi_{XBB_0}^>$:

$$\Phi_{XBB_0}^> = \frac{a_0}{\sqrt{(\rho \sin \eta)^2 + [a_0 - i(z \cos \eta - t)]^2}} \tag{A.56}$$

It is clear that all $\Phi_{XBB_n}^>$ are UPWs which propagate with speed $c_1 = 1/\cos \eta > 1$ in the z -direction. Our statement is justified for as can be easily seen (as in the modified superluminal Bessel beam) there is no Lorentz frame where $\Phi_{XBB_n}^>$ is at rest. Observe that this is the real speed of the wave; phase and group velocity concepts are not applicable here. Equation (A.56) does not give any dispersion relation.

The $\Phi_{XBB_n}^>$ waves cannot be produced in practice as they have infinite energy (see Sec. A7), but as we shall show a good approximation for them can be realized with *finite-aperture* radiators.

A5. Construction of $\Phi_{J_n}^<$ and X-Waves with Finite-Aperture Radiators

In Sec. A3 we study the condition under which the Rayleigh–Sommerfeld solution to HWE [Eq. (A.24)] can be derived. The condition is just that the wave Φ must be written as $\Phi(t, \vec{x}) = F(\vec{x}) e^{-i\omega t}$, which is true for the Bessel beams $\Phi_{J_n}^<$. In Sec. 2 we show that a finite-aperture approximation (FAA) to a broad-band Bessel beam or Bessel pulse denoted $\text{FAA}\Phi_{BBJ_n}$ or Φ_{FAJ_n} [see Eq. (2.3)] can be physically realized and moves as predicted by the theory.

At first sight it is not obvious that for the Φ_{X_n} waves we can use Eq. (A.26), but actually we can. This happens because we can write

$$\Phi_{X_n}^>(t, \vec{x}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{\Phi}_{X_n}^>(\omega, \vec{x}) e^{-i\omega t} d\omega \tag{A.57}$$

$$\tilde{\Phi}_{X_n}^>(\omega, \vec{x}) = 2\pi e^{in\theta} B(\omega) J_n(\omega\rho \sin \eta) H(\omega) e^{-\omega(a_0 - iz \cos \eta)}, \quad n = 0, 1, 2, \dots \tag{A.58}$$

where $H(\omega)$ is the step function and each $\tilde{\Phi}_{X_n}^>(\omega, \vec{x})$ is a solution of the transverse Helmholtz equation. Then the Rayleigh–Sommerfeld approximation can be written and the FAA can be used. Denoting the FAA to $\Phi_{X_n}^>$ by $\Phi_{FAX_n}^>$ and using Eq. (A.28) we get

$$\begin{aligned} \Phi_{FAX_n}^>(\vec{k}_<, \vec{x}) &= \frac{1}{i\lambda} \int_0^{2\pi} d\theta' \int_0^{D/2} \rho' d\rho' \tilde{\Phi}_{X_n}^>(\vec{k}_<, \rho', \phi') \frac{e^{i\vec{k}_< \cdot \vec{R}}}{R^2} z \\ &\quad + \frac{1}{2\pi} \int_0^{2\pi} d\theta' \int_0^{D/2} \rho' d\rho' \tilde{\Phi}_{X_n}^>(\vec{k}_<, \rho', \theta') \frac{e^{i\vec{k}_< \cdot \vec{R}}}{R^3} z \end{aligned} \tag{A.59}$$

$$\Phi_{FAX_n}^>(t, \vec{x}) = \mathcal{F}^{-1}[\Phi_{FAX_n}^>(\omega, \vec{x})], \quad n = 0, 1, 2, \dots \tag{A.60}$$

where λ is the wavelength and $R = |\vec{x} - \vec{x}'|$. \mathcal{F}^{-1} represents the inverse Fourier transform. The first and second terms in Eq. (A.59) represent

respectively the contributions from high- and low-frequency components. We attached the symbol $>$ to $\tilde{\Phi}_{FA\mathcal{X}_t}$, meaning as before that the wave is superluminal. This is justified from the results of the experiment described in Sec. 2.

A6. The Donnelly–Ziolkowski Method (DZM) for Designing Subluminal, Luminal, and Superluminal UPWs Solutions of the HWE and the Klein–Gordon Equation (KGE)

Consider first the HWE for Φ [Eq. (A.2')] in a homogeneous medium. Let $\tilde{\Phi}(\omega, \vec{k})$ be the Fourier transform of $\Phi(t, \vec{x})$, i.e.,

$$\tilde{\Phi}(\omega, \vec{k}) = \int_{R^3} d^3x \int_{-\infty}^{+\infty} dt \Phi(t, \vec{x}) e^{-i(\vec{k}\vec{x} - \omega t)} \tag{A.61a}$$

$$\Phi(t, \vec{x}) = \frac{1}{(2\pi)^4} \int_{R^3} d^3\vec{k} \int_{-\infty}^{+\infty} d\omega \tilde{\Phi}(\omega, \vec{k}) e^{i(\vec{k}\vec{x} - \omega t)} \tag{A.61b}$$

Inserting (A.61a) in the HWE we get

$$(\omega^2 - \vec{k}^2) \tilde{\Phi}(\omega, \vec{k}) = 0 \tag{A.62}$$

and we are going to look for solutions of the HWE and Eq. (A.62) in the sense of distributions. We rewrite Eq. (A.62) as

$$(\omega^2 - k_z^2 - \Omega^2) \tilde{\Phi}(\omega, \vec{k}) = 0 \tag{A.63}$$

It is then obvious that any $\tilde{\Phi}(\omega, \vec{k})$ of the form

$$\tilde{\Phi}(\omega, \vec{k}) = \Xi(\Omega, \beta) \delta[\omega - (\beta + \Omega^2/4\beta)] \delta[k_z - (\beta - \Omega^2/4\beta)] \tag{A.64}$$

where $\Xi(\Omega, \beta)$ is an arbitrary weighting function, is a solution of Eq. (A.63) since the δ -functions imply that

$$\omega^2 - k_z^2 = \Omega^2 \tag{A.65}$$

In 1985 Ziolkowski⁽³⁰⁾ found a *luminal* solution of the HWE called the focus wave mode. To obtain this solution, we choose, e.g.,

$$\Xi_{FWM}(\Omega, \beta) = \frac{\pi^2}{i\beta} \exp(-\Omega^2 z_0/4\beta) \tag{A.66}$$

whence we get, assuming $\beta > 0$ and $z_0 > 0$,

$$\Phi_{FWM}(t, \vec{x}) = e^{i\beta(z+t)} \frac{\exp\{-\rho^2\beta/[z_0 + i(z-t)]\}}{4\pi i[z_0 + i(z-t)]} \tag{A.67}$$

Despite the velocities $v_1 = +1$ and $v_2 = -1$ appearing in the phase, the modulation function of Φ_{FWM} has very interesting properties, as discussed in detail in Ref. 46. It remains to observe that Eq. (A.67) is a special case of Brittingham’s formula.⁽²⁶⁾

Returning to Eq. (A.64) we see that the δ -functions make any function of the Fourier transform variables ω , k_z and Ω lie in a line on the surface $\omega^2 - k_z^2 - \Omega^2 = 0$ [Eq. (A.63)]. Then, the support of the δ -functions is the line

$$\omega = \beta + \Omega^2/4\beta, \quad k_z = \beta - \Omega^2/4\beta \tag{A.68}$$

The projection of this line in the (ω, k_z) plane is a straight line of slope $+1$ ending at the point (β, β) . When $\beta = 0$ we must have $\Omega = 0$, and in this case the line is $\omega = k_z$ and $\Phi(t, \vec{x})$ is simply a superposition of plane waves, each one having frequency ω and traveling with speed $c = 1$ in the positive z direction.

Luminal UPWs solutions can be easily constructed by the ZM⁽⁴⁶⁾ but will not be discussed here. Instead, we now show how to use ZM to construct subluminal and superluminal solutions of the HWE.

First Example. Reconstruction of the subluminal Bessel Beams $\Phi_{J_0}^<$ and the superluminal $\Phi_{XBB_0}^>$ (X -wave). Starting from the “dispersion relation” $\omega^2 - k_z^2 - \Omega^2 = 0$, we define

$$\tilde{\Phi}(\omega, \vec{k}) = \Xi(\vec{k}, \eta) \delta(k_z - \vec{k} \cos \eta) \delta(\omega - \vec{k}) \tag{A.69}$$

This implies that

$$k_z = \vec{k} \cos \eta, \quad \cos \eta = k_z/\omega, \quad \omega > 0, \quad -1 < \cos \eta < 1 \tag{A.70}$$

We take moreover

$$\Omega = \vec{k} \sin \eta, \quad \vec{k} > 0 \tag{A.71}$$

We recall that $\vec{\Omega} = (k_x, k_y)$, $\vec{\rho} = (x, y)$ and we choose $\vec{\Omega} \cdot \vec{\rho} = \Omega \rho \cos \theta$. Now, putting Eq. (A.69) in Eq. (A.61) we get

$$\Phi(t, \vec{x}) = \frac{1}{(2\pi)^4} \int_0^\infty d\vec{k} \vec{k} \sin^2 \eta \left[\int_0^{2\pi} d\theta \Xi(\vec{k}, \eta) e^{i\vec{k}\rho \sin \eta \cos \theta} \right] e^{i(\vec{k} \cos \eta z - \vec{k}t)} \tag{A.72}$$

Choosing

$$\Xi(\bar{k}, \eta) = (2\pi)^3 \frac{z_0 e^{-\bar{k}z_0 \sin \eta}}{\bar{k} \sin \eta} \tag{A.73}$$

where $z_0 > 0$ is a constant, we obtain

$$\Phi(t, \vec{x}) = z_0 \sin \eta \int_0^\infty d\bar{k} e^{-\bar{k}z_0 \sin \eta} \left[\frac{1}{2\pi} \int_0^{2\pi} d\theta e^{i\bar{k}\rho \sin \eta \cos \theta} \right] e^{i\bar{k}(\cos \eta z - t)}$$

Calling $z_0 \sin \eta = a_0 > 0$, the last equation becomes

$$\Phi_{X_0}^>(t, \vec{x}) = a_0 \int_0^\infty d\bar{k} e^{-\bar{k}a_0} J_0(\bar{k}\rho \sin \eta) e^{i\bar{k}(\cos \eta z - t)} \tag{A.74}$$

Writing $\bar{k} = \bar{k}_<$ and taking into account Eq. (A.41) we see that

$$J_0(\bar{k}_< \rho \sin \eta) e^{i\bar{k}_<(z \cos \eta - t)} \tag{A.75}$$

is a subluminal Bessel beam, a solution of the HWE moving in the positive z direction. Moreover, a comparison of Eq. (A.74) with Eq. (A.52) shows that (A.74) is a particular superluminal X -wave, with $B(\bar{k}_<) = e^{-a_0 \bar{k}_<}$. In fact, it is the $\Phi_{XBB_0}^>$ UPW given by Eq. (A.56).

Second Example. Choosing in (A.72)

$$\Xi(\bar{k}, \eta) = (2\pi)^3 e^{-z_0 |\cos \eta| \bar{k}} \cot \eta \tag{A.76}$$

gives

$$\Phi^>(t, \vec{x}) = \cos^2 \eta \int_0^\infty d\bar{k} \bar{k} e^{-z_0 |\cos \eta| \bar{k}} J_0(\bar{k}\rho \sin \eta) e^{-i\bar{k}(\cos \eta z - t)} \tag{A.77a}$$

$$= \frac{[z_0 - i \operatorname{sgn}(\cos \eta)(z - t/\cos \eta)]}{[\rho^2 \tan^2 \eta + [z_0 + i \operatorname{sgn}(\cos \eta)(z - t/\cos \eta)]^2]^{3/2}} \tag{A.77b}$$

Comparing Eq. (A.77a) with Eq. (A.52), we discover that the ZM produced in this example is a more general $\Phi_{X_0}^>$ wave where $B(\bar{k}_<) = e^{-z_0 |\cos \eta| \bar{k}_<}$. Obviously $\Phi^>(t, \vec{x})$ given by Eq. (A.77b) moves with superluminal speed $(1/\cos \eta)$ in the positive or negative z -direction depending on the sign of $\cos \eta$, denoted $\operatorname{sgn}(\cos \eta)$.

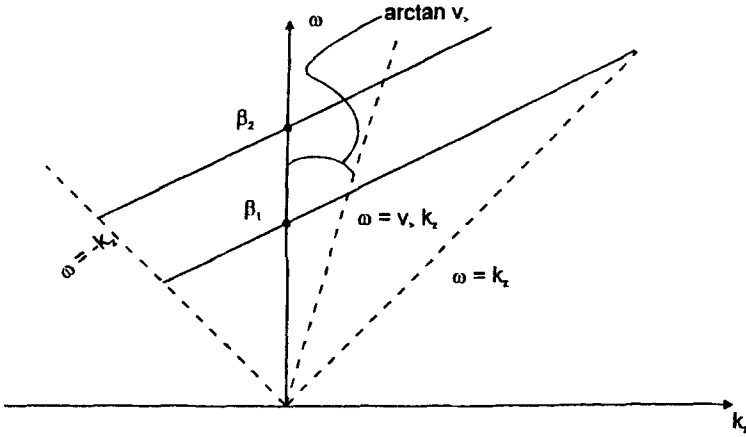


Fig. 12. Projection of the support lines of the transforms of two members of a family of subluminal solutions of the HWE.

In both examples studied above we see that the projection of the supporting line of Eq. (A.69) in the (ω, k_z) plane is the straight line $k_z/\omega = \cos \eta$, and $\cos \eta$ is its reciprocal slope. This line is inside the “light cone” in the (ω, k_z) plane.

Third Example. Consider two arbitrary lines with the same reciprocal slope that we denote by $v > 1$, both running between the lines $\omega = \pm k_z$ in the upper half-plane $\omega > 0$ and each one cutting the ω -axis at different values β_1 and β_2 (Fig. 12)). The two lines are projections of members of a family of HWE solution lines and each one can be represented as a portion of the straight lines (between the lines $\omega = \pm k_z$)

$$k_z = v(\omega - \beta_1), \quad k_z = v(\omega - \beta_2) \tag{A.78}$$

It is clear that on the solution line of the HWE, Ω takes values from zero up to a maximum value that depends on v and β and then back to zero.

We see also that the maximum value of Ω , given by $\beta v/\sqrt{v^2 - 1}$, on any HWE solution line occurs for those values of ω and k_z where the corresponding projection lines cut the line $\omega = vk_z$. It is clear that there are two points on any HWE solution line with the same value of Ω in the interval

$$0 < \Omega < \beta v/\sqrt{v^2 - 1} = \Omega_0 \tag{A.79}$$

It follows that in this case the HWE solution line breaks into two segments, as is the case of the projection lines. We can then associate two different weighting functions, one for each segment. We write

$$\begin{aligned}
 \tilde{\Phi}(\Omega, \omega, k_z) &= \Xi_1(\Omega, v, \beta) \delta \left[k_z - \frac{v[\beta + \sqrt{\beta^2 v^2 - \Omega^2(v^2 - 1)}]}{(v^2 - 1)} \right] \\
 &\times \delta \left[\omega - \frac{[\beta v^2 + \sqrt{v^2 \beta^2 - \Omega^2(v^2 - 1)}]}{(v^2 - 1)} \right] \\
 &+ \Xi_2(\Omega, v, \beta) \delta \left[k_z - \frac{v[\beta + \sqrt{\beta^2 v^2 - \Omega^2(v^2 - 1)}]}{(v^2 - 1)} \right] \\
 &\times \delta \left\{ \omega - \frac{[\beta v^2 + \sqrt{v^2 \beta^2 - \Omega^2(v^2 - 1)}]}{(v^2 - 1)} \right\} \tag{A.80}
 \end{aligned}$$

Now, choosing

$$\Xi_1(\Omega, v, \beta) = \Xi_2(\Omega, v, \beta) = (2\pi)^{3/2} \sqrt{\Omega_0^2 - \Omega^2}$$

we get

$$\begin{aligned}
 \Phi_{v, \beta}(t, \rho, z) &= \Omega_0 \exp \left(\frac{i\beta v(z - vt)}{\sqrt{v^2 - 1}} \right) \int_0^\infty d\chi \chi J_0(\Omega_0 \rho \chi) \\
 &\times \cos \left\{ \frac{\Omega_0 v}{\sqrt{v^2 - 1}} \frac{(z - t/v)}{\sqrt{1 - \chi^2}} \right\}
 \end{aligned}$$

Then

$$\Phi_{v, \beta}(t, \rho, z) = \exp \left[i\beta \frac{v(z - vt)}{\sqrt{v^2 - 1}} \right] \frac{\sin \{ \Omega_0 \sqrt{(v^2/(v^2 - 1))(z - t/v)^2 + \rho^2} \}}{\{ \Omega_0 \sqrt{(v^2/(v^2 - 1))(z - t/v)^2 + \rho^2} \}} \tag{A.81}$$

If we call $v_{<} = 1/v < 1$ and take into account the value of Ω_0 given by Eq. (A.79), we can write Eq. (A.81) as

$$\begin{aligned}
 \Phi_{v_{<}}(t, \rho, z) &= \frac{\sin(\Omega_0 \xi_{<})}{\xi_{<}} e^{i\Omega_0(z - vt)} \\
 \xi_{<} &= \left[x^2 + y^2 + \frac{1}{1 - v_{<}^2} (z - v_{<} t)^2 \right]^{1/2} \tag{A.82}
 \end{aligned}$$

which we recognize as the subluminal spherical Bessel beam of Sec. 4 [Eq. (A.31)].

Klein–Gordon Equation (KGE). We show here the existence of subluminal, luminal, and superluminal UPW solutions of the KGE. We want to solve

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right) \Phi^{KG}(t, \vec{x}) = 0, \quad m > 0 \quad (\text{A.83})$$

with the Fourier transform method. We obtain for $\tilde{\Phi}^{KG}(\omega, \vec{k})$ (a generalized function) the equation

$$\{\omega^2 - k_z^2 - (\Omega^2 + m^2)\} \tilde{\Phi}^{KG}(\omega, \vec{k}) = 0 \quad (\text{A.84})$$

As in the case of the HWE, any solution of the KGE will have a transform $\tilde{\Phi}(\omega, \vec{k})$ such that its support line lies on the surface

$$\omega^2 - k_z^2 - (\Omega^2 + m^2) = 0 \quad (\text{A.85})$$

From Eq. (A.85), calling $\Omega^2 + m^2 = K^2$, we see that we are in a situation identical to the HWE for which we showed the existence of subluminal, superluminal, and luminal solutions. We write down as examples one solution of each kind.

Subluminal UPW solution of the KGE. To obtain this solution it is enough to change in Eq. (A.81) $\Omega_0 = v\beta/\sqrt{v^2 - 1} \rightarrow \Omega_0^{KG} = [(v\beta/\sqrt{v^2 - 1})^2 - m^2]^{1/2}$. We have

$$\begin{aligned} \Phi_{<}^{KG}(t, \rho, z) &= \exp\left\{\frac{i\beta v(z - vt)}{\sqrt{v^2 - 1}}\right\} \frac{\sin(\Omega_0^{KG} \xi_{<})}{\xi_{<}} \\ \xi_{<} &= \left[x^2 + y^2 + \frac{1}{1 - v_{<}^2} (z - v_{<} t)^2\right]^{1/2}, \quad v_{<} = 1/v \end{aligned} \quad (\text{A.86})$$

Luminal UPW solution of the KGE. To obtain a solution of this type it is enough, as in Eq. (A.64), to write

$$\tilde{\Phi}^{KG} = \Xi(\Omega, \beta) \delta[k_z - (\Omega^2 + (m^2 - \beta^2)/2\beta)] \delta[\omega - (\Omega^2 + (m^2 + \beta^2)/2\beta)] \quad (\text{A.87})$$

Choosing

$$\Xi(\Omega, \beta) = \frac{(2\pi)^2}{\beta} \exp(-z_0 \Omega^2/2\beta), \quad z_0 > 0 \quad (\text{A.88})$$

gives

$$\begin{aligned} \Phi_{\beta}^{KG}(t, \vec{x}) &= \exp(iz(m^2 - \beta^2)/2\beta) \exp(-it(m^2 + \beta)/2\beta) \\ &\times \frac{\exp\{-\rho^2\beta/2[z_0 - i(z-t)]\}}{[z_0 - i(z-t)]} \end{aligned} \quad (\text{A.89})$$

Superluminal UPW solution of the KGE. To obtain a solution of this kind we introduce a parameter v such that $0 < v < 1$ and write for $\tilde{\Phi}^{KG}$ in (A.84)

$$\begin{aligned} \tilde{\Phi}_{v,\beta}^{KG}(\omega, \Omega, k_z) &= \Xi(\Omega, v, \beta) \delta \left[\omega - \frac{(-\beta v^2 + \sqrt{(\Omega^2 + m^2)(1 - v^2) + v^2\beta^2})}{1 - v^2} \right] \\ &\times \delta \left[k_z - \frac{v(-\beta + \sqrt{(\Omega^2 + m^2)(1 - v^2) + v^2\beta^2})}{1 - v^2} \right] \end{aligned} \quad (\text{A.90})$$

Next we choose

$$\Xi(\Omega, v, \beta) = \frac{(2\pi)^3 \exp(-z_0 \sqrt{\Omega_0^2 + \Omega^2})}{\sqrt{\Omega_0^2 + \Omega^2}} \quad (\text{A.91})$$

where $z_0 > 0$ is an arbitrary parameter, and where

$$\Omega_0^2 = \frac{\beta^2 v^2}{1 - v^2} + m^2 \quad (\text{A.92})$$

Then introducing $v_{>} = 1/v > 1$ and $\gamma_{>} = 1/\sqrt{v_{>}^2 - 1}$, we get

$$\begin{aligned} \Phi_{v,\beta}^{KG>}(t, \vec{x}) &= \exp\{i(\Omega_0^2 - m^2)(z - vt)\} \\ &\times \frac{\exp\{-\Omega_0 \sqrt{[z_0 - i\gamma_{>}(z - v_{>}t)]^2 + x^2 + y^2}\}}{\sqrt{[z_0 - i\gamma_{>}(z - v_{>}t)]^2 + x^2 + y^2}} \end{aligned} \quad (\text{A.93})$$

which is a superluminal UPW solution of the KGE moving with speed $v_{>}$ in the z direction. From Eq. (A.93) it is an easy task to reproduce the superluminal spherical Bessel beam which is a solution of the HWE [Eq. (A.30)].

A7. On the Energy of the UPW Solutions of the HWE

Let $\Phi_r(t, \vec{x})$ be a real solution of the HWE. Then, as is well known, the energy of the solution is given by

$$\varepsilon = \iiint_{\mathbb{R}^3} d\mathbf{v} \left[\left(\frac{\partial \Phi_r}{\partial t} \right)^2 - \Phi_r \nabla^2 \Phi_r \right] + \lim_{R \rightarrow \infty} \iint_{S(R)} dS \Phi_r \vec{n} \cdot \nabla \Phi_r \quad (\text{A.94})$$

where $S(R)$ is the 2-sphere of radius R .

We can easily verify that the real or imaginary parts of all UPW solutions of the HWE presented above have infinite energy. The question arises of how to project superluminal waves, solutions of the HWE, with finite energy. This can be done if we recall that all UPWs discussed above can be indexed by at least one parameter that here we call α . Then, calling $\Phi_\alpha(t, \vec{x})$ the real or imaginary parts of a given UPW solution, we may form “packets” of these solutions as

$$\Phi(t, \vec{x}) = \int d\alpha F(\alpha) \Phi_\alpha(t, \vec{x}) \quad (\text{A.95})$$

We now may test for a given solution Φ_α and for a weighting function $F(\alpha)$ if the integral in Eq. (A.94) is convergent. We can explicitly show for some (but not all) of the solutions shown above (subluminal, luminal, and superluminal) that for weighting functions satisfying certain integrability conditions the energy ε is finite. It is particularly important in this context to quote that the finite aperture approximations for all UPWs have, of course, finite energy. For the case in which Φ given by Eq. (A.95) is used to generate solutions for, e.g., Maxwell or Dirac fields (see Appendix B), the conditions for the energy of these fields to be finite will in general be different from the condition that gives for Φ a finite energy. This problem will be discussed in more detail in another paper.

APPENDIX B. A UNIFIED THEORY FOR CONSTRUCTION OF UPW SOLUTIONS OF MAXWELL, DIRAC, AND WEYL EQUATIONS

In this appendix we briefly recall the main results concerning the theory of Clifford algebras (and bundles) and their relationship with the Grassmann algebras (and bundles). Also the concept of Dirac–Hestenes spinors and their relationship with the usual Dirac spinors used by physicists is clarified. We introduce moreover the concepts of the Clifford and Spin–Clifford bundles of spacetime and the Clifford calculus. As we

shall see, this formalism provides a unified theory for the construction of UPW subluminal, luminal, and superluminal solutions of Maxwell, Dirac, and Weyl equations. More details on the topics of this appendix can be found in Refs. 6–9 and Ref. 81.

B1. Mathematical Preliminaries

Let $\mathcal{M} = (M, g, D)$ be Minkowski spacetime. (M, g) is a four-dimensional time-oriented and space-oriented Lorentzian manifold, with $M \simeq \mathbb{R}^4$ and $g \in \text{sec}(T^*M \times T^*M)$ being a Lorentzian metric of signature $(1, 3)$. T^*M [TM] is the cotangent [tangent] bundle. $T^*M = \bigcup_{x \in M} T_x^*M$ and $TM = \bigcup_{x \in M} T_xM$, and $T_xM \simeq P_x^*M \simeq \mathbb{R}^{1,3}$, where $\mathbb{R}^{1,3}$ is the Minkowski vector space.^(60–62) D is the Levi-Civita connection of g , i.e., $Dg = 0$, $\mathbf{T}(D) = 0$. Also $\mathbf{R}(D) = 0$, \mathbf{T} and \mathbf{R} being respectively the torsion and curvature tensors. Now, the Clifford bundle of differential forms $\mathcal{C}\ell(M)$ is the bundle of algebras $\mathcal{C}\ell(M) = \bigcup_{x \in M} \mathcal{C}\ell(T_x^*M)$, where $\forall x \in M$, $\mathcal{C}\ell(T_x^*M) = \mathcal{C}\ell_{1,3}$, the so-called spacetime algebra.^(9, 81–85) Locally as a linear space over the real field \mathbb{R} , $\mathcal{C}\ell(T_x^*(M))$ is isomorphic to the Cartan algebra $\wedge(T_x^*M)$ of the cotangent space and $\wedge(T_x^*M) = \sum_{k=0}^4 \wedge^k(T_x^*M)$, where $\wedge^k(T_x^*M)$ is the $\binom{4}{k}$ -dimensional space of k -forms. The Cartan bundle $\wedge(M) = \wedge(T_x^*M)$ can then be thought of as “embedded” in $\mathcal{C}\ell(M)$. In this way sections of $\mathcal{C}\ell(M)$ can be represented as a sum of inhomogeneous differential forms. Let $\{e_\mu = \partial/\partial x^\mu\} \in \text{sec } TM$, $(\mu = 0, 1, 2, 3)$ be an orthonormal basis $g(e_\mu, e_\nu) = \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and let $\{\gamma^\nu = dx^\nu\} \in \text{sec } \wedge^1(M) \subset \text{sec } \mathcal{C}\ell(M)$ be the dual basis. Then, the fundamental Clifford product (denoted in what follows by juxtaposition of symbols) is generated by $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}$ and if $\mathcal{C} \in \text{sec } \mathcal{C}\ell(M)$ we have

$$\mathcal{C} = s + v_\mu \gamma^\mu + \frac{1}{2!} b_{\mu\nu} \gamma^\mu \gamma^\nu + \frac{1}{3!} a_{\mu\nu\rho} \gamma^\mu \gamma^\nu \gamma^\rho + p \gamma^5 \tag{B.1}$$

where $\gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3 = dx^0 dx^1 dx^2 dx^3$ is the volume element and $s, v_\mu, b_{\mu\nu}, a_{\mu\nu\rho}, p \in \text{sec } \wedge^0(M) \subset \text{sec } \mathcal{C}\ell(M)$. For $A_r \in \text{sec } \wedge^r(M) \subset \text{sec } \mathcal{C}\ell(M)$, $B_s \in \text{sec } \wedge^s(M)$ we define^(9, 82) $A_r \cdot B_s = \langle A_r B_s \rangle_{|r-s|}$ and $A_r \wedge B_s = \langle A_r B_s \rangle_{r+s}$, where $\langle \rangle_k$ is the component in $\wedge^k(M)$ of the Clifford field.

Besides the vector bundle $\mathcal{C}\ell(M)$ we also need to introduce another vector bundle $\mathcal{C}\ell_{\text{Spin}_+(1,3)}(M)$ [$\text{Spin}_+(1,3) \simeq \text{SL}(2, \mathbb{C})$] called the Spin-Clifford bundle.^(8, 81, 84) We can show that $\mathcal{C}\ell_{\text{Spin}_+(1,3)}(M) \simeq \mathcal{C}\ell(M)/\mathcal{R}$, i.e., it is a quotient bundle. This means that sections of $\mathcal{C}\ell_{\text{Spin}_+(1,3)}(M)$ are equivalence classes of sections of the Clifford bundle, i.e., they are equivalence sections of nonhomogeneous differential forms (see Eqs. (B.2, B.3) below).

Now, as is well known, as electromagnetic field is represented by $F \in \sec \wedge^2(M) \subset \sec \mathcal{C}\ell(M)$. How to represent the Dirac spinor fields in this formalism? We can show that the even sections of $\mathcal{C}\ell_{\text{Spin}_+(1,3)}(M)$, called Dirac–Hestenes spinor fields, do the job. If we fix two orthonormal basis $\Sigma = \{\gamma^\mu\}$ as before, and $\tilde{\Sigma} = \{\tilde{\gamma}^\mu = R\gamma^\mu \tilde{R} = A_\nu^\mu \gamma^\nu\}$ with $A_\nu^\mu \in \text{SO}_+(1,3)$ and $R(x) \in \text{Spin}_+(1,3) \forall x \in M$, $R\tilde{R} = \tilde{R}R = 1$, and where $\tilde{}$ is the reversion operator in $\mathcal{C}\ell_{1,3}$, then^(8, 81) the representatives of an even section $\psi \in \sec \mathcal{C}\ell_{\text{Spin}_+(1,3)}(M)$ are the sections ψ_Σ and $\psi_{\tilde{\Sigma}}$ of $\mathcal{C}\ell(M)$ related by

$$\psi_\Sigma = \psi_{\tilde{\Sigma}} R \tag{B.2}$$

and

$$\psi_\Sigma = s + \frac{1}{2!} b_{\mu\nu} \gamma^\mu \gamma^\nu + p \gamma^5 \tag{B.3}$$

Note that ψ_Σ has the correct number of degrees of freedom in order to represent a Dirac spinor field, which is not the case with the so-called Dirac–Kähler spinor field (see Refs. 8 and 81).

Let \star be the Hodge star operator $\star: \wedge^k(M) \rightarrow \wedge^{4-k}(M)$. We can show that if $A_p \in \sec \wedge^p(M) \subset \sec \mathcal{C}\ell(M)$ we have $\star A = \tilde{A} \gamma^5$. Let d and δ be respectively the differential and Hodge codifferential operators acting on sections of $\wedge(M)$. If $\omega_p \in \sec \wedge^p(M) \subset \sec \mathcal{C}\ell(M)$, then $\delta \omega_p = (-1)^p \star^{-1} d \star \omega_p$, with $\star^{-1} \star = \text{identity}$.

The Dirac operator acting on sections of $\mathcal{C}\ell(M)$ is the invariant first-order differential operator

$$\partial = \gamma^\mu D_{e_\mu} \tag{B.4}$$

and we can show the very important result (see, e.g., Ref. 6):

$$\partial = \partial \wedge + \partial^\star = d - \delta \tag{B.5}$$

With these preliminaries we can write Maxwell and Dirac equations as follows^(82, 85):

$$\partial F = 0 \tag{B.6}$$

$$\partial \psi_\Sigma \gamma^1 \gamma^2 + m \psi_\Sigma \gamma^0 = 0 \tag{B.7}$$

We discuss further this last equation (Dirac–Hestenes equation) in Sec. B4. If $m = 0$ we have the massless Dirac equation

$$\partial \psi_\Sigma = 0 \tag{B.8}$$

which is Weyl's equation when ψ_Σ is reduced to a Weyl spinor field (see Eq. (B.12) below). Note that in this formalism Maxwell equations condensed in a single equation! Also, the specification of ψ_Σ depends on the frame Σ . When no confusion arises we represent ψ_Σ simply by ψ .

When $\psi_\Sigma \tilde{\psi}_\Sigma \neq 0$, where $\tilde{\cdot}$ is the reversion operator, we can show that ψ_Σ has the following canonical decomposition:

$$\psi_\Sigma = \sqrt{\rho} e^{\beta\gamma_5/2} R \tag{B.9}$$

where $\rho, \beta \in \sec \wedge^0(M) \subset \sec \mathcal{C}\ell(M)$ and $R \in \text{Spin}_+(1, 3) \subset \mathcal{C}\ell_{1,3}^+, \forall x \in M$. β is called the Takabayasi angle.⁽⁸⁾

If we want to work in terms of the usual spinor field formalism, we can translate our results by choosing, for example, the standard matrix representation of $\{\gamma^\mu\}$, and for ψ_Σ given by Eq. (B.3) we have the following (standard) matrix representation^(8, 49):

$$\Psi = \begin{pmatrix} \phi_1 & -\phi_2^* \\ \phi_2 & \phi_1^* \end{pmatrix} \tag{B.10}$$

where

$$\phi_1 = \begin{pmatrix} s - ib_{12} & b_{13} - ib_{23} \\ -b_{13} - ib_{23} & s + ib_{12} \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} -b_{03} + ip & -b_{01} + ib_{02} \\ -b_{01} - ib_{02} & b_{03} + ip \end{pmatrix}, \tag{B.11}$$

with s, b_{12}, \dots real functions; * denotes the complex conjugation. Right multiplication by

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

gives the usual Dirac spinor field.

We need also the concept of Weyl spinors. By definition, $\psi \in \sec \mathcal{C}\ell^+(M)$ is a Weyl spinor if⁽⁸³⁾

$$\gamma_5 \psi = \pm \psi \gamma_{21} \tag{B.12}$$

The positive [negative] "eigenstate" of γ_5 will be denoted ψ_+ [ψ_-]. For a general $\psi \in \sec \mathcal{C}\ell^+(M)$ we can verify that

$$\psi_\pm = \frac{1}{2} [\psi \mp \gamma_5 \psi \gamma_{21}] \tag{B.13}$$

are Weyl spinors with eigenvalues ± 1 of Eq. (B.12).

We recall that the even subbundle $\mathcal{C}\ell^+(M)$ of $\mathcal{C}\ell(M)$ is such that its typical fiber is the Pauli algebra $\mathcal{C}\ell_{3,0} \equiv \mathcal{C}\ell_{1,3}^+$ (which is isomorphic to $\mathbb{C}(2)$, the algebra of 2×2 complex matrices). The isomorphism $\mathcal{C}\ell_{3,0} \equiv \mathcal{C}\ell_{1,3}^+$ is exhibited by putting $\sigma_i = \gamma_i \gamma_0$, whence $\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}$. We recall also^(8, 81) that the Dirac algebra is $\mathcal{C}\ell_{4,1} \equiv \mathbb{C}(4)$ (see Sec. B4) and $\mathcal{C}\ell_{4,1} \equiv \mathbb{C} \otimes \mathcal{C}\ell_{1,3}$.⁽⁸⁶⁾

B2. Inertial Reference Frames (I), Observers, and Naturally Adapted Coordinate Systems

Let $\mathcal{M} = (M, g, D)$ be Minkowski spacetime. An *inertial reference frame* (irf) I is a timelike vector field $I \in \text{sec } TM$ pointing into the future such that $g(I, I) = 1$ and $DI = 0$. Each integral line of I is called an *inertial observer*. The coordinate functions $\langle x^\mu \rangle$, $\mu = 0, 1, 2, 3$, of the maximal atlas of M are said to be a naturally adapted coordinate system to I (nacs/ I) if $I = \partial/\partial x^0$.^(61, 62) Putting $I = e_0$ we can find $e_i = \partial/\partial x^i$, $i = 1, 2, 3$, such that $g(e_\mu, e_\nu) = \eta_{\mu\nu}$ and the coordinate functions x^μ are the usual Einstein–Lorentz ones and have a precise operational meaning: $x^0 = ct$,¹⁷ where t is measured by “ideal clocks” at rest on I and synchronized “à la Einstein,” and x^i , $i = 1, 2, 3$, are determined with ideal rules.^(61, 62) (We use units where $c = 1$.)

B3. Maxwell Theory in $\mathcal{C}\ell(M)$ and the Hertz Potential

Let $e_\mu \in \text{sec } TM$ be an orthonormal basis, $g(e_\mu, e_\nu) = \eta_{\mu\nu}$ and $e_\mu = \partial/\partial x^\mu$ ($\mu, \nu = 0, 1, 2, 3$), such that e_0 determines an IRF. Let $\gamma^\mu \in \text{sec } \wedge^2(M) \subset \text{sec } \mathcal{C}\ell(M)$ be the dual basis and let $\gamma_\mu = \eta_{\mu\nu} \gamma^\nu$ be the reciprocal basis to γ^μ , i.e., $\gamma^\mu \cdot \gamma_\nu = \delta^\mu_\nu$. We have $\gamma^\mu = dx^\mu$.

As is well known, the electromagnetic field is represented by a two-form $F \in \text{sec } \wedge^2(M) \subset \text{sec } \mathcal{C}\ell(M)$. We have

$$F = \frac{1}{2} F^{\mu\nu} \gamma_\mu \gamma_\nu, \quad F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} \quad (\text{B.14})$$

¹⁷ c is the constant called velocity of light in vacuum. In view of the superluminal and subluminal solutions of Maxwell equations found in this paper we do not think the terminology to be still satisfactory.

where (E^1, E^2, E^3) and (B^1, B^2, B^3) are respectively the Cartesian components of the electric and magnetic fields. Let $J \in \sec \wedge^1(M) \subset \sec \mathcal{C}\ell(M)$ be such that

$$J = J^\mu \gamma_\mu = \rho \gamma_0 + J^1 \gamma_1 + J^2 \gamma_2 + J^3 \gamma_3 \tag{B.15}$$

where ρ and (J^1, J^2, J^3) are respectively the Cartesian components of the charge and of the three-dimensional current densities.

We now write the Maxwell equation given by B6 in $\mathcal{C}\ell^+(M)$, the even subalgebra of $\mathcal{C}\ell(M)$. The typical fiber of $\mathcal{C}\ell^+(M)$, which is a vector bundle, is isomorphic to the Pauli algebra (see Sec. B1). We put

$$\vec{\sigma}_i = \gamma_i \gamma_0, \quad \mathbf{i} = \vec{\sigma}_1 \vec{\sigma}_2 \vec{\sigma}_3 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \gamma_5 \tag{B.16}$$

Recall that \mathbf{i} commutes with bivectors and since $\mathbf{i}^2 = -1$ it acts like the imaginary unit $i = \sqrt{-1}$ in $\mathcal{C}\ell^+(M)$. From Eq. (B.14), we get

$$F = \vec{E} + \mathbf{i} \vec{B} \tag{B.17}$$

with $\vec{E} = E^i \vec{\sigma}_i$, $\vec{B} = B^j \vec{\sigma}_j$, $i, j = 1, 2, 3$. Now, since $\partial = \gamma_\mu \partial^\mu$ we get $\partial \gamma_0 = \partial / \partial x^0 + \vec{\sigma}_i \partial^i = \partial / \partial x^0 - \nabla$. Multiplying Eq. (B.6) on the right by γ_0 , we find

$$\begin{aligned} \partial \gamma_0 \gamma_0 F \gamma_0 &= J \gamma_0 \\ (\partial / \partial x^0 - \nabla)(-\vec{E} + \mathbf{i} \vec{B}) &= \rho + \vec{J} \end{aligned} \tag{B.18}$$

where we used $\gamma^0 F \gamma_0 = -\vec{E} + \mathbf{i} \vec{B}$ and $\vec{J} = J^i \vec{\sigma}_i$. From Eq. (B.18) we get

$$-\partial_0 \vec{E} + \mathbf{i} \partial_0 \vec{B} + \nabla \vec{E} - \mathbf{i} \nabla \vec{B} = \rho + \vec{J} \tag{B.19}$$

$$-\partial_0 \vec{E} + \mathbf{i} \partial_0 \vec{B} + \nabla \cdot \vec{E} + \nabla \wedge \vec{E} - \mathbf{i} \nabla \cdot \vec{B} - \mathbf{i} \nabla \wedge \vec{B} = \rho + \vec{J} \tag{B.20}$$

We also have

$$-\mathbf{i} \nabla \wedge \vec{A} \equiv \nabla \times \vec{A} \tag{B.21}$$

since the usual vector product between two vectors $\vec{a} = a^i \vec{\sigma}_i$, $\vec{b} = b^j \vec{\sigma}_j$ can be identified with the dual of the bivector $\vec{a} \wedge \vec{b}$ through the formula $\vec{a} \times \vec{b} = -\mathbf{i}(\vec{a} \wedge \vec{b})$. Observe that in this formalism $\vec{a} \times \vec{b}$ is a true vector and not the meaningless pseudovector of the Gibbs vector calculus. Using Eq. (B.21) and equating the terms with the same grade we obtain

$$\begin{aligned} \nabla \cdot \vec{E} &= \rho, & \nabla \times \vec{B} - \partial_0 \vec{E} &= \vec{J} \\ \nabla \times \vec{E} + \partial_0 \vec{B} &= 0, & \nabla \cdot \vec{B} &= 0 \end{aligned} \tag{B.22}$$

which are Maxwell equations in the usual vector notation.

We now introduce the concept of Hertz potential⁽¹⁹⁾ which permits us to find nontrivial solutions of the free “vacuum” Maxwell equation

$$\partial F = 0 \tag{B.23}$$

once we know nontrivial solutions of the scalar wave equation,

$$\square \Phi = (\partial^2/\partial t^2 - \nabla^2) \Phi = 0, \quad \Phi \in \text{sec } \bigwedge^0(M) \subset \text{sec } \mathcal{C}\ell(M) \tag{B.24}$$

Let $A \in \text{sec } \bigwedge^1(M) \subset \text{sec } \mathcal{C}\ell(M)$ be the vector potential. We fix the Lorentz gauge, i.e., $\partial \cdot A = -\delta A = 0$ such that $F = \partial A = (d - \delta) A = dA$. We have the following important result:

Theorem. Let $\pi \in \text{sec } \bigwedge^2(M) \subset \text{sec } \mathcal{C}\ell(M)$ be the so-called Hertz potential. If π satisfies the wave equation, i.e., $\square \pi = \partial^2 \pi = (d - \delta)(d - \delta) \pi = -(d\delta + \delta d) \pi = 0$ and if we take $A = -\delta \pi$, then $F = \partial A$ satisfies the Maxwell equation $\partial F = 0$.

The proof is trivial. Indeed, $A = -\delta \pi$ implies $\delta A = -\delta^2 \pi = 0$ and $F = \partial A = dA$. Then $\partial F = (d - \delta)(d - \delta) A = d(d\delta \pi) = -\delta^2 d\pi = 0$, since $\delta d\pi = -d\delta \pi$ from $\partial^2 \pi = 0$.

From this result we see that if $\Phi \in \text{sec } \bigwedge^0(M) \subset \text{sec } \mathcal{C}\ell(M)$ satisfies $\partial^2 \Phi = 0$, then we can find a nontrivial solution of $\partial F = 0$, using a Hertz potential given, e.g., by

$$\pi = \Phi \gamma_1 \gamma_2 \tag{B.25}$$

In Sec. 3 this equation is used to generate the superluminal electromagnetic X -wave.

We now express the Hertz potential and its relation with the \vec{E} and \vec{B} fields, in order for our reader to see more familiar formulas. We write π as a sum of electric and magnetic parts, i.e.,

$$\begin{aligned} \pi &= \vec{\pi}_e + \mathbf{i} \vec{\pi}_m \\ \vec{\pi}_e &= -\pi^{0i} \vec{\sigma}_i, \quad \vec{\pi}_m = -\pi^{23} \vec{\sigma}_1 + \pi^{13} \vec{\sigma}_2 - \pi^{12} \vec{\sigma}_3 \end{aligned} \tag{B.26}$$

Then, since $A = \partial \pi$ we have

$$A = \frac{1}{2}(\partial \pi - \pi \vec{\partial}) \tag{B.27}$$

$$A \gamma_0 = -\partial_0 \vec{\pi}_e + \nabla \cdot \vec{\pi}_e - (\nabla \times \vec{\pi}_m) \tag{B.28}$$

and, since $A = A^\mu \gamma_\mu$, we also have

$$A^0 = \nabla \cdot \vec{\pi}_e, \quad \vec{A} = A^i \vec{\sigma}_i = -\frac{\partial}{\partial x^0} \vec{\pi}_e - \nabla \times \vec{\pi}_m$$

Since $\vec{E} = -\nabla A^0 - (\partial/\partial x^0) \vec{A}$ and $\vec{B} = \nabla \times \vec{A}$, we obtain

$$\vec{E} = -\partial_0(\nabla \times \vec{\pi}_m) + \nabla \times \nabla \times \vec{\pi}_e \tag{B.29}$$

$$\vec{B} = \nabla \times (-\partial_0 \vec{\pi}_e - \nabla \times \vec{\pi}_m) = -\partial_0(\nabla \times \vec{\pi}_e) - \nabla - \nabla \times \vec{\pi}_m \tag{B.30}$$

We define $\vec{E}_e, \vec{B}_e, \vec{E}_m, \vec{B}_m$ by

$$\begin{aligned} \vec{E}_e &= \nabla \times \nabla \times \vec{\pi}_e, & \vec{B}_e &= -\partial_0(\nabla \times \vec{\pi}_e) \\ \vec{E}_m &= -\partial_0(\nabla \times \vec{\pi}_m), & \vec{B}_m &= -\nabla \times \nabla \times \vec{\pi}_m \end{aligned} \tag{B.31}$$

We now introduce the 1-forms of stress-energy. Since $\partial F = 0$ we have $\tilde{F}\tilde{\partial} = 0$. Multiplying the first of these equation on the left by \tilde{F} and the second on the right by \tilde{F} and summing, we have

$$(1/2)(\tilde{F}\partial F + \tilde{F}\tilde{\partial}F) = \partial_\mu((1/2)\tilde{F}\gamma^\mu F) = \partial_\mu T^\mu = 0 \tag{B.32}$$

where $\tilde{F}\tilde{\partial} \equiv -(\partial_\mu \frac{1}{2} F_{\alpha\beta} \gamma^\alpha \gamma^\beta) \gamma^\mu$. Now,

$$-\frac{1}{2}(F\gamma^\mu F) \gamma^\nu = -\frac{1}{2}(F\gamma^\mu F\gamma^\nu) \tag{B.33}$$

Since $\gamma^\mu \cdot F = \frac{1}{2}(\gamma^\mu F - F\gamma^\mu) = F \cdot \gamma^\mu$, we have

$$\begin{aligned} T^{\mu\nu} &= -\langle (F \cdot \gamma^\mu) F\gamma^\nu \rangle_0 - \frac{1}{2}\langle \gamma^\mu F^2 \gamma^\nu \rangle_0 \\ &= -(F \cdot \gamma^\mu) \cdot (F \cdot \gamma^\nu) - \frac{1}{2}(F \cdot F) \gamma^\mu \cdot \gamma^\nu \\ &= F^{\mu\alpha} F^{\lambda\nu} \eta_{\alpha\lambda} + \frac{1}{4}\eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \end{aligned} \tag{B.34}$$

which we recognize as the stress-energy momentum tensor of the electromagnetic field, and $T^\mu = T^{\mu\nu} \gamma_\nu$.

By writing $F = \vec{E} + i\vec{B}$ as before, we can immediately verify that

$$\begin{aligned} T_0 &= -\frac{1}{2}F\gamma_0 F \\ &= [\frac{1}{2}(\vec{E}^2 + \vec{B}^2) + (\vec{E} \times \vec{B})] \gamma_0 \end{aligned} \tag{B.35}$$

We have already shown that $\partial_\mu T^\mu = 0$, and we can easily show that

$$\partial \cdot T^\mu = 0 \tag{B.36}$$

We now define the density of *angular momentum*. Choose as before a Lorentzian chart $\langle x^\mu \rangle$ of the maximal atlas of M and consider the 1-form $x = x^\mu \gamma_\mu = x_\mu \gamma^\mu$. Define

$$M_\mu = x \wedge T_\mu = \frac{1}{2}(x_\alpha T_{\mu\nu} - x_\nu T_{\alpha\mu}) \gamma^\alpha \wedge \gamma^\nu$$

It is trivial to verify that as $T_{\mu\nu} = T_{\nu\mu}$ and $\partial_\mu T^{\mu\nu} = 0$, we have

$$\partial^\mu M_\mu = 0 \quad (\text{B.37})$$

The *invariants* of the electromagnetic field F are $F \cdot F$ and $F \wedge F$ and

$$\begin{aligned} F^2 &= F \cdot F + F \wedge F \\ F \cdot F &= -\frac{1}{2} F^{\mu\nu} F_{\mu\nu}, \quad F \wedge F = -\gamma_5 F^{\mu\nu} F^{\alpha\beta} \epsilon_{\mu\nu\alpha\beta} \end{aligned} \quad (\text{B.38})$$

Writing as before $F = \vec{E} + \mathbf{i}\vec{B}$ we have

$$F^2 = (\vec{E}^2 - \vec{B}^2) + 2\mathbf{i}\vec{E} \cdot \vec{B} = F \cdot F + F \wedge F \quad (\text{B.39})$$

B4. Dirac Theory in $\mathcal{C}\ell(M)$

Let $\Sigma = \{\gamma^\mu\} \in \sec \wedge^1(M) \subset \sec \mathcal{C}\ell(M)$ be an orthonormal basis. Let $\psi_\Sigma \in \sec(\wedge^0(M) + \wedge^2(M) + \wedge^4(M)) \subset \sec \mathcal{C}\ell(M)$ be the representative of a Dirac–Hestenes spinor field in the basis Σ . Then, the representative of the Dirac equation in $\mathcal{C}\ell(M)$ is the following equation ($\hbar = c = 1$):

$$\partial \psi_\Sigma \gamma_2 \gamma_1 + m \psi_\Sigma \gamma_0 = 0 \quad (\text{B.40})$$

The proof is as follows:

Consider the *complexification* $\mathcal{C}\ell_C(M)$ of $\mathcal{C}\ell(M)$ called the *complex Clifford bundle*. Then $\mathcal{C}\ell_C(M) = \mathbb{C} \otimes \mathcal{C}\ell(M)$ and by the results of Sec. B1 it is trivial to see that the typical fiber of $\mathcal{C}\ell_C(M)$ is $\mathcal{C}\ell_{4,1} = \mathbb{C} \otimes \mathcal{C}\ell_{1,3}$, the Dirac algebra. Now let $\{\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4\} \subset \sec \wedge^1(M) \subset \sec \mathcal{C}\ell_C(M)$ be an orthonormal basis with

$$\begin{aligned} \Gamma_a \Gamma_b + \Gamma_b \Gamma_a &= 2g_{ab} \\ g_{ab} &= \text{diag}(+1, +1, +1, +1, -1) \end{aligned} \quad (\text{B.41})$$

Let us identify $\gamma_\mu = \Gamma_\mu \Gamma_4$ and call $I = \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4$. Since $I^2 = -1$ and I commutes with all elements of $\mathcal{C}\ell_{4,1}$ we identify I with $i = \sqrt{-1}$ and γ_μ with the fundamental set of $\mathcal{C}\ell(M)$. Then if $\mathcal{A} \in \sec \mathcal{C}\ell_C(M)$ we have

$$\mathcal{A} = \Phi_s + A_C^\mu \gamma_\mu + \frac{1}{2} B_C^{\mu\nu} \gamma_\mu \gamma_\nu + \frac{1}{3!} \tau_C^{\mu\nu\rho} \gamma_\mu \gamma_\nu \gamma_\rho + \Phi_p \gamma_5 \quad (\text{B.42})$$

where $\Phi_s, \Phi_p, A_C^\mu, B_C^{\mu\nu}, \tau_C^{\mu\nu\rho} \in \sec \mathbb{C} \otimes \wedge^0(M) \subset \sec \mathcal{C}l_C(M)$, i.e., $\forall x \in M, \Phi_s(x), \Phi_p(x), A_C^\mu(x), B_C^{\mu\nu}(x), \tau_C^{\mu\nu\rho}(x)$ are complex numbers.

Now,

$$f = \frac{1}{2}(1 + \gamma_0) \frac{1}{2}(1 + i\gamma_1\gamma_2), \quad f^2 = f$$

is a primitive idempotent field of $\mathcal{C}l_C(M)$. We can show that $if = \gamma_2\gamma_1f$. From (B.40) we can write the following equation in $\mathcal{C}l_C(M)$:

$$\partial\psi_\Sigma \gamma_2\gamma_1f + m\psi_\Sigma \gamma_0f = 0 \tag{B.43}$$

$$\Rightarrow \partial\psi_\Sigma if + m\psi_\Sigma f = 0 \tag{B.44}$$

and we have the following equation for $\Psi = \psi_\Sigma f$:

$$i\partial\Psi + m\Psi = 0 \tag{B.45}$$

Using for γ_μ the standard matrix representation (denoted here by $\underline{\gamma}_\mu$) we get that the matrix representation of Eq. (B.45) is

$$i\underline{\gamma}^\mu \partial_\mu |\Psi\rangle + m |\Psi\rangle = 0 \tag{B.46}$$

where now $|\Phi\rangle$ is a usual Dirac spinor field.

We now define a *potential* for the Dirac–Hestenes field ψ_Σ . Since $\psi_\Sigma \in \sec \mathcal{C}l^+(M)$ it is clear that there exist A and $B \in \sec \wedge^1(M) \subset \sec \mathcal{C}l(M)$ such that

$$\psi_\Sigma = \partial(A + \gamma_5 B) \tag{B.47}$$

since

$$\partial(A + \gamma_5 B) = \partial \cdot A + \partial \wedge A - \gamma_5 \partial \cdot B - \gamma_5 \partial \wedge B \tag{B.48}$$

$$= S + B + \gamma_5 P \tag{B.49}$$

$$S = \partial \cdot A, \quad B = \partial \wedge A - \gamma_5 \partial \wedge B, \quad P = -\partial \cdot B$$

We see that when $m = 0$, ψ_Σ satisfies the *Weyl equation*

$$\partial\psi_\Sigma = 0 \tag{B.50}$$

Using Eq. (B.50) we see that

$$\partial^2 A = \partial^2 B = 0 \tag{B.51}$$

This last equation allows us to find UPW solutions for the Weyl equation once we know UPW solutions of the scalar wave equation $\square\Phi=0$, $\Phi \in \sec \wedge^0(M) \subset \sec \mathcal{C}\ell(M)$. Indeed, it is enough to put $\mathcal{A} = (A + \gamma_5 B) = \Phi(1 + \gamma_5)v$, where v is a constant 1-form field. This result has been used in Ref. 48 to present subluminal and superluminal solutions of the Weyl equation.

We know (see Appendix A5) that the Klein–Gordon equation has superluminal solutions. Let $\Phi_{>}$ be a superluminal solution of $\square\Phi_{>} + m^2\Phi_{>} = 0$. Suppose $\Phi_{>}$ is a section of $\mathcal{C}\ell_C(M)$. Then in $\mathcal{C}\ell_C(M)$ we have the following factorization:

$$(\square + m^2)\Phi = (\partial + im)(\partial - im)\Phi = 0 \quad (\text{B.52})$$

Now

$$\Psi_{>} = (\partial - im)\Phi_{>} f \quad (\text{B.53})$$

is a Dirac spinor field in $\mathcal{C}\ell_C(M)$, since

$$(\partial + im)\Psi_{>} = 0 \quad (\text{B.54})$$

If we use for Φ in Eq. (B.52) a subluminal or a luminal UPW solution and then use Eq. (B.53), we see that the Dirac equation also has UPW solutions with arbitrary speed $0 \leq v < \infty$.

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