a fundamental study that lists sources for and titles of thirteen of Archigenes' lost writings, and "Archigenes," in Pauly-Wissowa, *Real-Encyclopädie der classischen Altertumswissenschaft*, II (1896), cols. 484–486.

JERRY STANNARD

ARCHIMEDES (*b.* Syracuse, *ca.* 287 B.C.; *d.* Syracuse, 212 B.C.), *mathematics, mechanics.*

Few details remain of the life of antiquity's most celebrated mathematician. A biography by his friend Heracleides has not survived. That his father was the astronomer Phidias we know from Archimedes himself in his *The Sandreckoner* (Sect. I. 9). Archimedes was perhaps a kinsman of the ruler of Syracuse, King Hieron II (as Plutarch and Polybius suggest). At least he was on intimate terms with Hieron, to whose son Gelon he dedicated *The Sandreckoner.* Archimedes almost certainly visited Alexandria, where no doubt he studied with the successors of Euclid and played an important role in the further development of Euclidian mathematics. This visit is rendered almost certain by his custom of addressing his mathematical discoveries to mathematicians who are known to have lived in Alexandria, such as Conon, Dositheus, and Eratosthenes. At any rate Archimedes returned to Syracuse, composed most of his works there, and died there during its capture by the Romans in 212 B.C. Archimedes' approximate birth date of 287 B.C. is conjectured on the basis of a remark by the Byzantine poet and historian of the twelfth century, John Tzetzes, who declared (*Chiliad* 2, hist. 35) that Archimedes "worked at geometry until old age, surviving seventy-five years." There are picturesque accounts of Archimedes' death by Livy, Plutarch, Valerius Maximus, and Tzetzes, which vary in detail but agree that he was killed by a Roman soldier. In most accounts he is pictured as being engaged in mathematics at the time of his death. Plutarch tells us (*Marcellus,* Ch. XVII) that Archimedes "is said to have asked his friends and kinsmen to place on his grave after his death a cylinder circumscribing a sphere, with an inscription giving the ratio by which the including solid exceeds the included." And indeed Cicero (see *Tusculan Disputations,* V, xxiii, 64–66), when he was Quaestor in Sicily in 75 B.C.,

> . . . tracked out his grave. . . . and found it enclosed all around and covered with brambles and thickets; for I remembered certain doggerel lines inscribed, as I had heard, upon his tomb, which stated that a sphere along with a cylinder had been put upon the top of his grave. Accordingly, after taking a good look all around (for there are a great quantity of graves at the Agrigentine Gate), I noticed a small column arising a little above the bushes, on which there was the figure of a sphere and a cylinder. . . . Slaves were sent in with sickles. . . and when a passage to the place was opened we approached the pedestal in front of us; the epigram was traceable with about half of the lines legible, as the latter portion was worn away.

No surviving bust can be certainly identified as being of Archimedes, although a portrait on a Sicilian coin (whatever its date) is definitely his. A well-known mosaic showing Archimedes before a calculating board with a Roman soldier standing over him was once thought to be a genuine survival from Herculaneum but is now considered to be of Renaissance origin.

Mechanical Inventions. While Archimedes' place in the history of science rests on a remarkable collection of mathematical works, his reputation in antiquity was also founded upon a series of mechanical contrivances which he is supposed to have invented and which the researches of A. G. Drachmann tend in part to confirm as Archimedean inventions. One of these is the water snail, a screwlike device to raise water for the purpose of irrigation, which, Diodorus Siculus tells us (*Bibl. hist.,* V, Ch. 37), Archimedes invented in Egypt. We are further told by Atheneus that an endless screw invented by Archimedes was used to launch a ship. He is also credited with the invention of the compound pulley. Some such device is the object of the story told by Plutarch in his life of *Marcellus* (Ch. XIV). When asked by Hieron to show him how a great weight could be moved by a small force, Archimedes "fixed upon a three-masted merchantman of the royal fleet, which had been dragged ashore by the great labors of many men, and after putting on board many passengers and the customary freight, he seated himself at a distance from her, and without any great effort, but quietly setting in motion a system of compound pulleys, drew her towards him smoothly and evenly, as though she were gliding through the water." It is in connection with this story that Plutarch tells us of the supposed remark of Archimedes to the effect that "if there were another world, and he could go to it, he could move this one," a remark known in more familiar form from Pappus of Alexandria (*Collectio,* Bk. VIII, Prop. 11): "Give me a place to stand on, and I will move the earth." Of doubtful authenticity is the oft-quoted story told by Vitruvius (*De architectura,* Bk. IX, Ch. 3) that Hieron wished Archimedes to check whether a certain crown or wreath was of pure gold, or whether the goldsmith had fraudulently alloyed it with some silver.

> While Archimedes was turning the problem over, he chanced to come to the place of bathing, and there,

> as he was sitting down in the tub, he noticed that the amount of water which flowed over by the tub was equal to the amount by which his body was immersed. This indicated to him a method of solving the problem, and he did not delay, but in his joy leapt out of the tub, and, rushing naked towards his home, he cried out in a loud voice that he had found what he sought, for as he ran he repeatedly shouted in Greek, *heurēka, heurēka.*

Much more generally credited is the assertion of Pappus that Archimedes wrote a book *On Sphere-making,* a work which presumably told how to construct a model planetarium representing the apparent motions of the sun, moon, and planets, and perhaps also a closed star globe representing the constellations. At least, we are told by Cicero (*De re publica,* I, XIV, 21–22) that Marcellus took as booty from the sack of Syracuse both types of instruments constructed by Archimedes:

> For Gallus told us that the other kind of celestial globe [that Marcellus brought back and placed in the Temple of Virtue], which was solid and contained no hollow space, was a very early invention, the first one of that kind having been constructed by Thales of Miletus, and later marked by Eudoxus of Cnidus . . . with the constellations and stars which are fixed in the sky. . . . But this newer kind of globe, he said, on which were delineated the motions of the sun and moon and of those five stars which are called the wanderers . . . contained more than could be shown on a solid globe, and the invention of Archimedes deserved special admiration because he had thought out a way to represent accurately by a single device for turning the globe those various and divergent courses with their different rates of speed.

Finally, there are references by Polybius, Livy, Plutarch, and others to fabulous ballistic instruments constructed by Archimedes to help repel Marcellus. One other defensive device often mentioned but of exceedingly doubtful existence was a burning mirror or combination of mirrors.

We have no way to know for sure of Archimedes' attitude toward his inventions. One supposes that Plutarch's famous eulogy of Archimedes' disdain for the practical was an invention of Plutarch and simply reflected the awe in which Archimedes' theoretical discoveries were held. Plutarch (*Marcellus,* Ch. XVII) exclaims:

> And yet Archimedes possessed such a lofty spirit, so profound a soul, and such a wealth of scientific theory, that although his inventions had won for him a name and fame for superhuman sagacity, he would not consent to leave behind him any treatise on this subject, but regarding the work of an engineer and every art that ministers to the needs of life as ignoble and vulgar, he devoted his earnest efforts only to those studies the subtlety and charm of which are not affected by the claims of necessity. These studies, he thought, are not to be compared with any others; in them, the subject matter vies with the demonstration, the former supplying grandeur and beauty, the latter precision and surpassing power. For it is not possible to find in geometry more profound and difficult questions treated in simpler and purer terms. Some attribute this success to his natural endowments; others think it due to excessive labor that everything he did seemed to have been performed without labor and with ease. For no one could by his own efforts discover the proof, and yet as soon as he learns it from him, he thinks he might have discovered it himself, so smooth and rapid is the path by which he leads one to the desired conclusion.

Mathematical Works. The mathematical works of Archimedes that have come down to us can be loosely classified in three groups (Arabic numbers have been added to indicate, where possible, their chronological order). The first group consists of those that have as their major objective the proof of theorems relative to the areas and volumes of figures bounded by curved lines and surfaces. In this group we can place *On the Sphere and the Cylinder* (5); *On the Measurement of the Circle* (9); *On Conoids and Spheroids* (7); *On Spirals* (6); and *On the Quadrature of the Parabola* (2), which, in respect to its Propositions 1–17, belongs also to the second category of works. The second group comprises works that lead to a geometrical analysis of statical and hydrostatical problems and the use of statics in geometry: *On the Equilibrium of Planes,* Book I (1), Book II (3); *On Floating Bodies* (8); *On the Method of Mechanical Theorems* (4); and the aforementioned propositions from *On the Quadrature of the Parabola* (2). Miscellaneous mathematical works constitute the third group: *The Sandreckoner* (10); *The Cattle-Problem*; and the fragmentary *Stomachion.* Several other works not now extant are alluded to by Greek authors (see Heiberg, ed., *Archimedis opera,* II, 536–554). For example, there appear to have been various works on mechanics that have some unknown relationship to *On the Equilibrium of Planes.* Among these are a possible work on *Elements of Mechanics* (perhaps containing an earlier section on centers of gravity, which, however, may have been merely a separate work written before *Equilibrium of Planes,* Book I), a tract *On Balances,* and possibly one *On Uprights.* Archimedes also seems to have written a tract *On Polyhedra,* perhaps one *On Blocks and Cylinders,* certainly one on *Archai* or *The Naming of Numbers* (a work preliminary to *The Sandreckoner*), and a work on *Optics* or *Catoptrics.* Other works are attributed to Archimedes by Arabic authors, and, for the most part, are extant in Arabic manuscripts (the titles for which manuscripts are known are indicated by an asterisk; see Bibliography): *The Lemmata**, or *Liber assumptorum* (in its present form certainly not by Archimedes since his name is cited in the proofs), *On Water Clocks**, *On Touching Circles**, *On Parallel Lines*, *On Triangles**, *On the Properties of the Right Triangle**, *On Data*, and *On the Division of the Circle into Seven Equal Parts**.

But even the genuine extant works are by no means in their original form. For example, *On the Equilibrium of Planes*, Book I, is possibly an excerpt from the presumably longer *Elements of Mechanics* mentioned above and is clearly distinct from Book II, which was apparently written later. A solution promised by Archimedes in *On the Sphere and the Cylinder* (Bk. II, Prop. 4) was already missing by the second century A.D. *On the Measurement of the Circle* was certainly in a much different form originally, with Proposition II probably not a part of it (and even if it were, it would have to follow the present Proposition III, since it depends on it). The word *parabolēs* in the extant title of *On the Quadrature of the Parabola* could hardly have been in the original title, since that word was not yet used in Archimedes' work in the sense of a conic section. Finally, the tracts *On the Sphere and the Cylinder* and *On the Measurement of the Circle* have been almost completely purged of their original Sicilian-Doric dialect, while the rest of his works have suffered in varying degrees this same kind of linguistic transformation.

In proving theorems relative to the area or volume of figures bounded by curved lines or surfaces, Archimedes employs the so-called Lemma of Archimedes or some similar lemma, together with a technique of proof that is generally called the "method of exhaustion," and other special Greek devices such as *neuseis*, and principles taken over from statics. These various mathematical techniques are coupled with an extensive knowledge of the mathematical works of his predecessors, including those of Eudoxus, Euclid, Aristeus, and others. The Lemma of Archimedes (*On the Sphere and Cylinder*, Assumption 5; cf. the Preface to *On the Quadrature of the Parabola* and the Preface to *On Spirals*) assumes "that of two unequal lines, unequal surfaces, and unequal solids the greater exceeds the lesser by an amount such that, when added to itself, it may exceed any assigned magnitude of the type of magnitudes compared with one another." This has on occasion been loosely identified with Definition 4 of Book V of the *Elements* of Euclid (often called the axiom of Eudoxus): "Magnitudes are said to have a ratio to one another which are capable, when multiplied, of exceeding one another."

But the intent of Archimedes' assumption appears to be that if there are two unequal magnitudes capable of having a ratio in the Euclidian sense, then their difference will have a ratio (in the Euclidian sense) with any magnitude of the same kind as the two initial magnitudes. This lemma has been interpreted as excluding actual infinitesimals, so that the difference of two lines will always be a line and never a point, the difference between surfaces always a surface and never a line, and the difference between solids always a solid and never a surface. The exhaustion procedure often uses a somewhat different lemma represented by Proposition X.1 of the *Elements* of Euclid: "Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out." This obviously reflects the further idea of the continuous divisibility of a continuum. One could say that the Lemma of Archimedes justifies this further lemma in the sense that no matter how far the procedure of subtracting more than half of the larger of the magnitudes set out is taken (or also no matter how far the procedure of subtracting one-half the larger magnitude, described in the corollary to Proposition X.1, is taken), the magnitude resulting from the successive division (which magnitude being conceived as the difference of two magnitudes) will always be capable of having a ratio in the Euclidian sense with the smaller of the magnitudes set out. Hence one such remainder will some time be in a relationship of "less than" to the lesser of the magnitudes set out.

The method of exhaustion, widely used by Archimedes, was perhaps invented by Eudoxus. It was used on occasion by Euclid in his *Elements* (for example, in Proposition XII.2). Proof by exhaustion (the name is often criticized since the purpose of the technique is to avoid assuming the complete exhaustion of an area or a volume; Dijksterhuis prefers the somewhat anachronistic expression "indirect passage to the limit") is an indirect proof by reduction to absurdity. That is to say, if the theorem is of the form $A = B$, it is held to be true by showing that to assume its opposite, namely that A is not equal to B, is impossible since it leads to contradictions. The method has several forms. Following Dijksterhuis, we can label the two main types: the compression method and the approximation method. The former is the most widely used and exists in two forms, one that depends upon taking decreasing differences and one that depends on taking decreasing ratios. The fundamental procedure of both the "difference" and the "ratio" forms starts with the successive inscription and circumscription of regular figures within or without the figure for which the area or volume is sought. Then in the "difference" method the area or volume of the inscribed or circumscribed figure is regularly increased or decreased until the difference between the desired area or volume and the inscribed or circumscribed figure is less than any preassigned magnitude. Or to put it more specifically, if the theorem is of the form $A = B$, A being the curvilinear figure sought and B a regular rectilinear figure the formula for the magnitude of which is known, and we assume that A is greater than B, then by the exhaustion procedure and its basic lemma we can construct some regular rectilinear inscribed figure P such that P is greater than B; but it is obvious that P, an included figure, is in fact always less than B. Since P cannot be both greater and less than B, the assumption from which the contradiction evolved (namely, that A is greater than B) must be false. Similarly, if A is assumed to be less than B, we can by the exhaustion technique and the basic lemma find a circumscribed figure P that is less than B, which P (as an including figure) must always be greater than B. Thus the assumption of A less than B must also be false. Hence, it is now evident that, since A is neither greater nor less than B, it must be equal to B. An example of the exhaustion procedure in its "difference" form is to be found in *On the Measurement of the Circle:* [1]

Proposition 1

The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle.

Let $ABCD$ be the given circle, K the triangle described.

Then, if the circle is not equal to K, it must be either greater or less.

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I. If possible, let the circle be greater than K.

Inscribe a square $ABCD$, bisect the arcs AB, BC, CD, DA, then bisect (if necessary) the halves, and so on, until the sides of the inscribed polygon whose angular points are the points of division subtend segments whose sum is less than the excess of the area of the circle over K.

Thus the area of the polygon is greater than K.

Let AE be any side of it, and ON the perpendicular on AE from the centre O.

Then ON is less than the radius of the circle and therefore less than one of the sides about the right angle in K. Also the perimeter of the polygon is less than the circumference of the circle, i.e. less than the other side about the right angle in K.

Therefore the area of the polygon is less than K; which is inconsistent with the hypothesis.

Thus the area of the circle is not greater than K.

II. If possible, let the circle be less than K.

Circumscribe a square, and let two adjacent sides, touching the circle in E, H, meet in T. Bisect the arcs between adjacent points of contact and draw the tangents at the points of bisection. Let A be the middle point of the arc EH, and FAG the tangent at A.

Then the angle TAG is a right angle.

Therefore $$TG > GA$$ $$> GH.$$

It follows that the triangle FTG is greater than half the area $TEAH$.

Similarly, if the arc AH be bisected and the tangent at the point of bisection be drawn, it will cut off from the area GAH more than one-half.

Thus, by continuing the process, we shall ultimately arrive at a circumscribed polygon such that the spaces intercepted between it and the circle are together less than the excess of K over the area of the circle.

Thus the area of the polygon will be less than K.

Now, since the perpendicular from O on any side of the polygon is equal to the radius of the circle, while the perimeter of the polygon is greater than the circum-

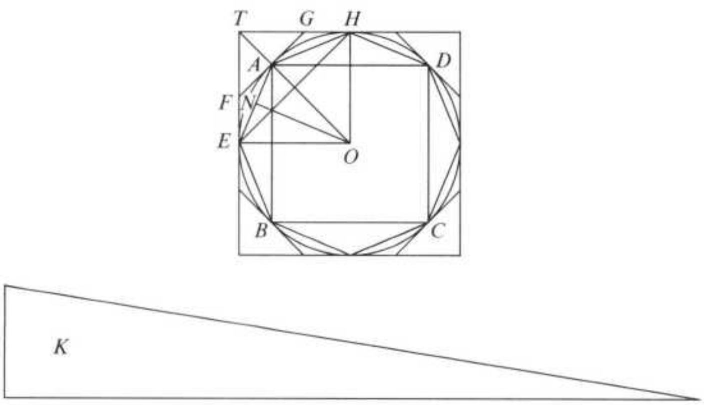

FIGURE 1

ference of the circle, it follows that the area of the polygon is greater than the triangle K; which is impossible.

Therefore the area of the circle is not less than K.

Since then the area of the circle is neither greater nor less than K, it is equal to it.

Other examples of the "difference" form of the exhaustion method are found in *On Conoids and Spheroids* (Props. 22, 26, 28, 30), *On Spiral Lines* (Props. 24, 25), and *On the Quadrature of the Parabola* (Prop. 16).

The "ratio" form of the exhaustion method is quite similar to the "difference" form except that in the first part of the proof, where the known figure is said to be less than the figure sought, the ratio of circumscribed polygon to inscribed polygon is decreased until it is less than the ratio of the figure sought to the known figure, and in the second part the ratio of circumscribed polygon to inscribed polygon is decreased until it is less than the ratio of the known figure to the figure sought. In each part a contradiction is shown to follow the assumption. And thus the assumption of each part must be false, namely, that the known figure is either greater or less than the figure sought. Consequently, the known figure must be equal to the figure sought. An example of the "ratio" form appears in *On the Sphere and the Cylinder* (Bk. I):[2]

Proposition 14

The surface of any isosceles cone excluding the base is equal to a circle whose radius is a mean proportional between the side of the cone [a generator] and the radius of the circle which is the base of the cone.

Let the circle A be the base of the cone; draw C equal to the radius of the circle, and D equal to the side of the cone, and let E be a mean proportional between C, D.

Draw a circle B with radius equal to E.

Then shall B be equal to the surface of the cone (excluding the base), which we will call S.

If not, B must be either greater or less than S.

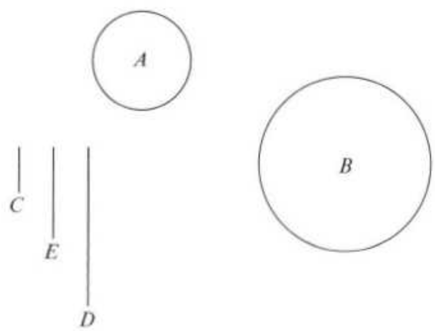

FIGURE 2

I. Suppose $B < S$.

Let a regular polygon be described about B and a similar one inscribed in it such that the former has to the latter a ratio less than the ratio $S:B$.

Describe about A another similar polygon, and on it set up a pyramid with apex the same as that of the cone.

Then (polygon about A):(polygon about B)
$= C^2:E^2$
$= C:D$
= (polygon about A):(surface of pyramid excluding base). Therefore

$$(\text{surface of pyramid}) = (\text{polygon about } B).$$

Now (polygon about B):(polygon in B) $< S:B$.

Therefore

$$(\text{surface of pyramid}):(\text{polygon in } B) < S:B,$$

which is impossible (because the surface of the pyramid is greater than S, while the polygon in B is less than B).

Hence $\qquad B \not< S.$

II. Suppose $B > S$.

Take regular polygons circumscribed and inscribed to B such that the ratio of the former to the latter is less than the ratio $B:S$.

Inscribe in A a similar polygon to that inscribed in B, and erect a pyramid on the polygon inscribed in A with apex the same as that of the cone.

In this case

$$(\text{polygon in } A):(\text{polygon in } B) = C^2:E^2$$
$$= C:D$$

$>$ (polygon in A):(surface of pyramid excluding base).

This is clear because the ratio of C to D is greater than the ratio of the perpendicular from the center of A on a side of the polygon to the perpendicular from the apex of the cone on the same side.

Therefore

$$(\text{surface of pyramid}) > (\text{polygon in } B).$$

But $\qquad$ (polygon about B):(polygon in B) $< B:S$.

Therefore, *a fortiori*,

(polygon about B):(surface of pyramid) $< B:S$; which is impossible.

Since therefore B is neither greater nor less than S,

$$B = S.$$

Other examples of the "ratio" form of the exhaustion method are found in *On the Sphere and the Cylinder*, (Bk. I, Props. 13, 33, 34, 42, 44.)

As indicated earlier, in addition to the two forms of the compression method of exhaustion, Archimedes used a further technique which we may call the approximation method. This is used on only one occasion, namely, in *On the Quadrature of the Parabola* (Props. 18–24). It consists in approximating from below the area of a parabolic segment. That is to say, Archimedes continually "exhausts" the parabola by drawing first a triangle in the segment with the same base and vertex as the segment. On each side of the triangle we again construct triangles. This process is continued as far as we like. Thus if A_1 is the area of the original triangle, we have a series of inscribed triangles whose sum converges toward the area of parabolic segment: A_1, $1/4\,A_1$, $(1/4)^2A_1, \cdots$ (in the accompanying figure A_1 is $\triangle PQq$ and $1/4\,A_1$ or A_2 is the sum of triangles Prq and PRQ and A_3 is the sum of the next set of inscribed triangles—not shown on the diagram but equal to $[1/4]^2A_1$). In order to prove that K, the area of the parabolic segment, is equal to $4/3\,A_1$, Archimedes first proves in Proposition 22 that the sum of any finite number of terms of this series is less than the area of the parabolic segment. He then proves in Proposition 23 that if we have a series of terms $A_1, A_2, A_3, \cdots$ such as those given above, that is, with $A_1 = 4A_2, A_2 = 4A_3, \cdots$, then

$$A_1 + A_2 + A_3 + \cdots + A_n + \frac{1}{3}\cdot A_n = \frac{4}{3}\cdot A_1,$$

or

$$A_1\left[1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots + \left(\frac{1}{4}\right)^{n-1} + \frac{1}{3}\cdot\left(\frac{1}{4}\right)^{n-1}\right] = \frac{4}{3}\cdot A_1$$

With modern techniques of series summation we would simply say that as n increases indefinitely $(1/4)^{n-1}$ becomes infinitely small and the series in brackets tends toward 4/3 as a limit and thus the parabolic segment equals $4/3 \cdot A_1$. But Archimedes followed the Greek *reductio* procedure. Hence he showed that if we assume $K > 4/3 \cdot A_1$ on the basis of a corollary to Proposition 20, namely, that by the successive inscription of triangles "it is possible to inscribe in the parabolic segment a polygon such that the segments left over are together less than any assigned area" (which is itself based on Euclid, *Elements* X.1), a contradiction will ensue. Similarly, a contradiction results from the assumption of $K < 4/3 \cdot A_1$. Here in brief is the final step of the proof (the reader is reminded that the terms $A_1, A_2, A_3, \cdots, A_n$, which were used above, are actually rendered by $A, B, C, \cdots, X$):[3]

Proposition 24

Every segment bounded by a parabola and a chord Qq is equal to four-thirds of the triangle which has the same base as the segment and equal height.

Suppose $\quad K = \frac{4}{3}\triangle PQq,$

where P is the vertex of the segment; and we have then to prove that the area of the segment is equal to K.

For, if the segment be not equal to K, it must either be greater or less.

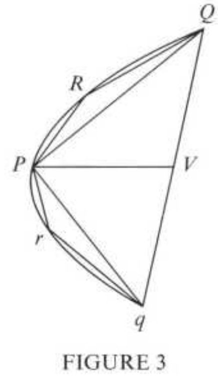

FIGURE 3

I. Suppose the area of the segment greater than K.

If then we inscribe in the segments cut off by PQ, Pq triangles which have the same base and equal height, i.e. triangles with the same vertices R, r as those of the segments, and if in the remaining segments we inscribe triangles in the same manner, and so on, we shall finally have segments remaining whose sum is less than the area by which the segment PQq exceeds K [Prop. 20 Cor.].

Therefore the polygon so formed must be greater than the area K; which is impossible, since [Prop. 23]

$$A + B + C + \cdots + Z < \frac{4}{3}A,$$

where $\quad A = \triangle PQq.$

Thus the area of the segment cannot be greater than K.

II. Suppose, if possible, that the area of the segment is less than K.

If then $\triangle PQq = A$, $B = 1/4\,A$, $C = 1/4\,B$, and so on, until we arrive at an area X such that X is less than the difference between K and the segment, we have

$$A + B + C + \cdots + X + \frac{1}{3}X = \frac{4}{3}A \qquad \text{[Prop. 23]}$$
$$= K.$$

Now, since K exceeds $A + B + C + \cdots + X$ by an area less than X, and the area of the segment by an area greater than X, it follows that

$$A + B + C + \cdots + X > \text{(the segment)};$$

which is impossible, by Prop. 22. . . .

Hence the segment is not less than K.

Thus, since the segment is neither greater nor less than K,

$$\text{(area of segment } PQq) = K = \frac{4}{3}\triangle PQq.$$

In the initial remarks on the basic methods of Archimedes, it was noted that Archimedes sometimes used the technique of a *neusis* ("verging") construc-

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tion. Pappus defined a *neusis* construction as "Two lines being given in position, to place between them a straight line given in length and verging towards a given point." He also noted that "a line is said to verge towards a point, if being produced, it reaches the point." No doubt "insertion" describes the mathematical meaning better than "verging" or "inclination," but "insertion" fails to render the additional condition of inclining or verging toward a point just as the name *neusis* in expressing the "verging" condition fails to render the crucial condition of insertion. At any rate, the *neusis* construction can be thought of as being accomplished mechanically by marking the termini of the linear insertion on a ruler and shifting that ruler until the termini of the insertion lie on the given curve or curves while the ruler passes through the verging point. In terms of mathematical theory most of the Greek *neuseis* require a solution by means of conics or other higher curves. *Neusis* constructions are indicated by Archimedes in *On Spirals* (Props. 5–9). They are assumed as possible without any explanation. The simplest case may be illustrated as follows:[4]

Proposition 5

Given a circle with center O, and the tangent to it at a point A, it is possible to draw from O a straight line OPF, meeting the circle in P and the tangent in F, such that, if c be the circumference of any given circle whatever,

$$FP:OP < (\text{arc } AP):c.$$

Take a straight line, as D, greater than the circumference c. [Prop. 3]

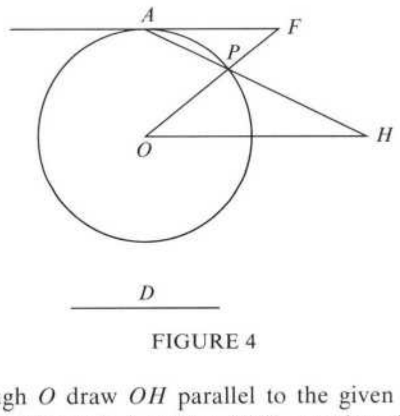

FIGURE 4

Through O draw OH parallel to the given tangent, and draw through A a line APH, meeting the circle in P and OH in H, such that the portion PH intercepted between the circle and the line OH may be equal to D [literally: "let PH be placed equal to D, verging toward A"]. Join OP and produce it to meet the tangent in F.

Then $FP:OP = AP:PH$, by parallels,
$= AP:D$
$< (\text{arc } AP):c$.

With the various methods that have been described and others, Archimedes was able to demonstrate a whole host of theorems that became a basic part of geometry. Examples beyond those already quoted follow: "The surface of any sphere is equal to four times the greatest circle in it" (*On the Sphere and the Cylinder,* Bk. I, Prop. 23); this is equivalent to the modern formulation $S = 4\pi r^2$. "Any sphere is equal to four times the cone which has its base equal to the greatest circle in the sphere and its height equal to the radius of the sphere" (*ibid.,* Prop. 34); its corollary that "every cylinder whose base is the greatest circle in a sphere and whose height is equal to the diameter of the sphere is 3/2 of the sphere and its surface together with its base is 3/2 of the surface of the sphere" is the proposition illustrated on the tombstone of Archimedes, as was noted above. The modern equivalent of Proposition 34 is $V = 4/3\,\pi r^3$. "Any right or oblique segment of a paraboloid of revolution is half again as large as the cone or segment of a cone which has the same base and the same axis" (*On Conoids and Spheroids,* Props. 21–22). He was also able by his investigation of what are now known as Archimedean spirals not only to accomplish their quadrature (*On Spirals,* Props. 24–28), but, in preparation therefore, to perform the crucial rectification of the circumference of a circle. This, then, would allow for the construction of the right triangle equal to a circle that is the object of *On the Measurement of a Circle* (Prop. I), above. This rectification is accomplished in *On Spirals* (Prop. 18): "If a straight line is tangent to the extremity of a spiral described in the first revolution, and if from the point of origin of the spiral one erects a perpendicular on the initial line of revolution, the perpendicular will meet the tangent so that the line intercepted between the tangent and the origin of the spiral will be equal to the circumference of the first circle" (see Fig. 5).

It has also been remarked earlier that Archimedes employed statical procedures in the solution of geometrical problems and the demonstration of theorems. These procedures are evident in *On the Quadrature of the Parabola* (Props. 6–16) and also in *On the Method.* We have already seen that in the latter part of *On the Quadrature of the Parabola* Archimedes demonstrated the quadrature of the parabola by purely geometric methods. In the first part of the tract he demonstrated the same thing by means of a balancing method. By the use of the law of the lever and a knowledge of the centers of gravity of triangles and trapezia, coupled with a *reductio* procedure, the quadrature is demonstrated. In *On the Method* the same statical procedures are used; but, in addition, an entirely new assumption is joined with them, namely, that a plane figure can be considered as the summation of its line elements (presumably infinite in number) and that a volumetric figure can be

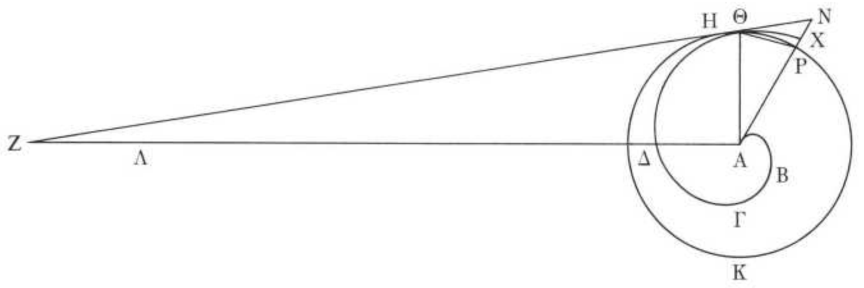

FIGURE 5

considered as the summation of its plane elements. The important point regarding this work is that it gives us a rare insight into Archimedes' procedures for discovering the theorems to be proved. The formal, indirect procedures that appear in demonstrations in the great body of Archimedes' works tell us little as to how the theorems to be proved were discovered. To be sure, sometimes he no doubt proved theorems that he had inherited with inadequate proof from his predecessors (such was perhaps the case of the theorem on the area of the circle, which he proved simply and elegantly in *On the Measurement of the Circle* [Prop. 1], as has been seen). But often we are told by him what his own discoveries were, and their relation to the discoveries of his predecessors, as, for example, those of Eudoxus. In the Preface of Book I of *On the Sphere and the Cylinder,* he characterizes his discoveries by comparing them with some established theorems of Eudoxus:[5]

> Now these properties were all along naturally inherent in the figures referred to . . ., but remained unknown to those who were before my time engaged in the study of geometry. Having, however, now discovered that the properties are true of these figures, I cannot feel any hesitation in setting them side by side both with my former investigations and with those of the theorems of Eudoxus on solids which are held to be most irrefragably established, namely, that any pyramid is one third part of the prism which has the same base with the pyramid and equal height, and that any cone is one third part of the cylinder which has the same base with the cone and equal height. For, though these properties also were naturally inherent in the figures all along, yet they were in fact unknown to all the many able geometers who lived before Eudoxus, and had not been observed by anyone. Now, however, it will be open to those who possess the requisite ability to examine these discoveries of mine.

Some of the mystery surrounding Archimedes' methods of discovery was, then, dissipated by the discovery and publication of *On the Method of Mechanical Theorems.* For example, we can see in Proposition 2 how it was that Archimedes discovered by the "method" the theorems relative to the area and volume of a sphere that he was later to prove by strict geometrical methods in *On the Sphere and the Cylinder:*[6]

Proposition 2

We can investigate by the same method the propositions that

(*1*) *Any sphere is* (*in respect of solid content*) *four times the cone with base equal to a great circle of the sphere and height equal to its radius; and*

(*2*) *the cylinder with base equal to a great circle of the sphere and height equal to the diameter is 1-1/2 times the sphere.*

(1) Let $ABCD$ be a great circle of a sphere, and AC, BD diameters at right angles to one another.

Let a circle be drawn about BD as diameter and in a plane perpendicular to AC, and on this circle as base let a cone be described with A as vertex. Let the surface of this cone be produced and then cut by a plane through C parallel to its base; the section will be a circle on EF as diameter. On this circle as base let a cylinder be erected with height and axis AC, and produce CA to H, making AH equal to CA.

Let CH be regarded as the bar of a balance, A being its middle point.

Draw any straight line MN in the plane of the circle $ABCD$ and parallel to BD. Let MN meet the circle in O, P, the diameter AC in S, and the straight lines AE, AF in Q, R respectively. Join AO.

Through MN draw a plane at right angles to AC; this plane will cut the cylinder in a circle with diameter MN, the sphere in a circle with diameter OP, and the cone in a circle with diameter QR.

Now, since $MS = AC$, and $QS = AS$,

$$\begin{aligned} MS \cdot SQ &= CA \cdot AS \\ &= AO^2 \\ &= OS^2 + SQ^2. \end{aligned}$$

And, since $HA = AC$,

$$\begin{aligned} HA:AS &= CA:AS \\ &= MS:SQ \\ &= MS^2:MS \cdot SQ \\ &= MS^2:(OS^2 + SQ^2), \text{ from above,} \\ &= MN^2:(OP^2 + QR^2) \\ &= (\text{circle, diam. } MN):(\text{circle, diam. } OP \\ &\quad + \text{circle, diam. } QR). \end{aligned}$$

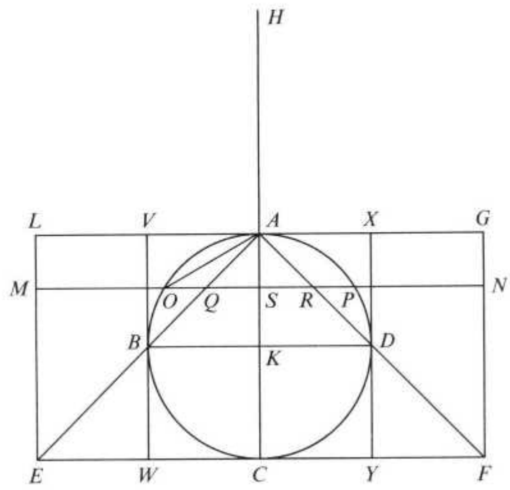

FIGURE 6

That is,
$HA:AS$ = (circle in cylinder):(circle in sphere + circle in cone).

Therefore the circle in the cylinder, placed where it is, is in equilibrium, about A, with the circle in the sphere together with the circle in the cone, if both the latter circles are placed with their centers of gravity at H.

Similarly for the three corresponding sections made by a plane perpendicular to AC and passing through any other straight line in the parallelogram LF parallel to EF.

If we deal in the same way with all the sets of three circles in which planes perpendicular to AC cut the cylinder, the sphere and the cone, and which make up those solids respectively, it follows that the cylinder, in the place where it is, will be in equilibrium about A with the sphere and the cone together, when both are placed with their centers of gravity at H.

Therefore, since K is the center of gravity of the cylinder,

$$HA:AK = (\text{cylinder}):(\text{sphere} + \text{cone } AEF).$$

But $HA = 2AK$;
therefore cylinder = 2 (sphere + cone AEF).

Now cylinder = 3 (cone AEF); [Eucl. XII. 10]

therefore cone AEF = 2 (sphere).

But, since $EF = 2BD$,

cone AEF = 8 (cone ABD);
therefore sphere = 4 (cone ABD).

(2) Through B, D draw VBW, XDY parallel to AC; and imagine a cylinder which has AC for axis and the circles on VX, WY as diameters for bases.

Then

cylinder VY = 2 (cylinder VD)
= 6 (cone ABD) [Eucl. XII. 10]
= $\frac{3}{2}$ (sphere), from above.

Q.E.D.

> From this theorem, to the effect that a sphere is four times as great as the cone with a great circle of the sphere as base and with height equal to the radius of the sphere, I conceived the notion that the surface of any sphere is four times as great as a great circle in it; for, judging from the fact that any circle is equal to a triangle with base equal to the circumference and height equal to the radius of the circle, I apprehended that, in like manner, any sphere is equal to a cone with base equal to the surface of the sphere and height equal to the radius.

It should be observed in regard to this quotation that the basic volumetric theorem was discovered prior to the surface theorem, although in their later formal presentation in *On the Sphere and the Cylinder,* the theorem for the surface of a sphere is proved first. By using the "method" Archimedes also gave another "proof" of the quadrature of the parabola—already twice proved in *On the Quadrature of the Parabola*—and he remarks in his preface (see the quotation below) that he originally discovered this theorem by the method. Finally, in connection with *On the Method,* it is necessary to remark that Archimedes considered the method inadequate for formal demonstration, even if it did provide him with the theorems to be proved more rigorously. One supposes that it was the additional assumption considering the figures as the summation of their infinitesimal elements that provoked Archimedes' cautionary attitude, which he presents so lucidly in his introductory remarks to Eratosthenes:[7]

> Seeing moreover in you, as I say, an earnest student, a man of considerable eminence in philosophy, and an admirer [of mathematical inquiry], I thought fit to write out for you and explain in detail in the same book the peculiarity of a certain method, by which it will be possible for you to get a start to enable you to investigate some of the problems in mathematics by means of mechanics. This procedure is, I am persuaded, no less useful even for the proof of the theorems themselves; for certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said method did not furnish an actual demonstration. But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge. This is a reason why, in the case of the theorems the proof of which Eudoxus was the first to discover, namely that the cone is a third part of the cylinder, and the pyramid of the prism, having the same base and equal height, we should give no small share of the credit to Democritus who was the first to make the assertion with regard to the said figure though he did not prove it. I am myself in the position of having first made the discovery of the theorem now to be

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> published [by the method indicated], and I deem it necessary to expound the method partly because I have already spoken of it and I do not want to be thought to have uttered vain words, but equally because I am persuaded that it will be of no little service to mathematics; for I apprehend that some, either of my contemporaries or of my successors, will, by means of the method when once established, be able to discover other theorems in addition, which have not yet occurred to me.

While Archimedes' investigations were primarily in geometry and mechanics reduced to geometry, he made some important excursions into numerical calculation, although the methods he used are by no means clear. In *On the Measurement of the Circle* (Prop. 3), he calculated the ratio of circumference to diameter (not called π until early modern times) as being less than 3-1/7 and greater than 3-10/71. In the course of this proof Archimedes showed that he had an accurate method of approximating the roots of large numbers. It is also of interest that he there gave an approximation for $\sqrt{3}$, namely, $1351/780 > \sqrt{3} > 265/153$. How he computed this has been much disputed. In the tract known as *The Sandreckoner,* Archimedes presented a system to represent large numbers, a system that allows him to express a number P^{10^8}, where P itself is $(10^8)^{10^8}$. He invented this system to express numbers of the sort that, in his words, "exceed not only the number of the mass of sand equal in magnitude to the earth . . ., but also that of a mass equal in magnitude to the universe." Actually, the number he finds that would approximate the number of grains of sand to fill the universe is a mere 10^{63}, and thus does not require the higher orders described in his system. Incidentally, it is in this work that we have one of the few antique references to Aristarchus' heliocentric system.

In the development of physical science, Archimedes is celebrated as the first to apply geometry successfully to statics and hydrostatics. In his *On the Equilibrium of Planes* (Bk. I, Props. 6–7), he proved the law of the lever in a purely geometrical manner. His weights had become geometrical magnitudes possessing weight and acting perpendicularly to the balance beam, itself conceived of as a weightless geometrical line. His crucial assumption was the special case of the equilibrium of the balance of equal arm length supporting equal weights. This postulate, although it may ultimately rest on experience, in the context of a mathematical proof appears to be a basic appeal to geometrical symmetry. In demonstrating Proposition 6, "Commensurable magnitudes are in equilibrium at distances reciprocally proportional to their weights," his major objective was to reduce the general case of unequal weights at inversely proportional distances to the special case of equal weights at equal distances. This was done by (1) converting the weightless beam of unequal arm lengths into a beam of equal arm lengths, and then (2) distributing the unequal weights, analyzed into rational component parts over the extended beam uniformly so that we have a case of equal weights at equal distances. Finally (3) the proof utilized propositions concerning centers of gravity (which in part appear to have been proved elsewhere by Archimedes) to show that the case of the uniformly distributed parts of the unequal weights over the extended beam is in fact identical with the case of the composite weights concentrated on the arms at unequal lengths. Further, it is shown in Proposition 7 that if the theorem is true for rational magnitudes, it is true for irrational magnitudes as well (although the incompleteness of this latter proof has been much discussed). The severest criticism of the proof of Proposition 6 is, of course, the classic discussion by Ernst Mach in his *Science of Mechanics,* which stresses two general points: (1) experience must have played a predominant role in the proof and its postulates in spite of its mathematical-deductive form; and (2) any attempt to go from the special case of the lever to the general case by replacing expanded weights on a lever arm with a weight concentrated at their center of gravity must assume that which has to be proved, namely, the principle of static moment. This criticism has given rise to an extensive literature and stimulated some successful defenses of Archimedes, and this body of literature has been keenly analyzed by E. J. Dijksterhuis (*Archimedes,* pp. 289–304). It has been pointed out further, and with some justification, that Proposition 6 with its proof, even if sound, only establishes that the inverse proportionality of weights and arm lengths is a sufficient condition for the equilibrium of a lever supported in its center of gravity under the influence of two weights on either side of the fulcrum. It is evident that he should also have shown that the condition is a necessary one, since he repeatedly applies the inverse proportionality as a necessary condition of equilibrium. But this is easily done and so may have appeared trivial to Archimedes. The succeeding propositions in Book I of *On the Equilibrium of Planes* show that Archimedes conceived of this part of the work as preparatory to his use of statics in his investigation of geometry of the sort that we have described in *On the Quadrature of the Parabola* and *On the Method.*

In his *On Floating Bodies,* the emphasis is once more largely on geometrical analysis. In Book I, a somewhat obscure concept of hydrostatic pressure is presented as his basic postulate:[8]

> Let it be granted that the fluid is of such a nature that of the parts of it which are at the same level and adjacent to one another that which is pressed the less is pushed away by that which is pressed the more, and that each of its parts is pressed by the fluid which is vertically above it, if the fluid is not shut up in anything and is not compressed by anything else.

As his propositions are analyzed, we see that Archimedes essentially maintained an Aristotelian concept of weight directed downward toward the center of the earth conceived of as the center of the world. In fact, he goes further by imagining the earth removed and so fluids are presented as part of a fluid sphere all of whose parts weigh downward convergently toward the center of the sphere. The surface of the sphere is then imagined as being divided into an equal number of parts which are the bases of conical sectors having the center of the sphere as their vertex. Thus the water in each sector weighs downward toward the center. Then if a solid is added to a sector, increasing the pressure on it, the pressure is transmitted down through the center of the sphere and back upward on an adjacent sector and the fluid in that adjacent sector is forced upward to equalize the level of adjacent sectors. The influence on other than adjacent sectors is ignored. It is probable that Archimedes did not have the concept of hydrostatic paradox formulated by Stevin, which held that at any given point of the fluid the pressure is a constant magnitude that acts perpendicularly on any plane through that point. But, by his procedures, Archimedes was able to formulate propositions concerning the relative immersion in a fluid of solids less dense than, as dense as, and more dense than the fluid in which they are placed. Proposition 7 relating to solids denser than the fluid expresses the so-called "principle of Archimedes" in this fashion: "Solids heavier than the fluid, when thrown into the fluid, will be driven downward as far as they can sink, and they will be lighter [when weighed] in the fluid [than their weight in air] by the weight of the portion of fluid having the same volume as the solid." This is usually more succinctly expressed by saying that such solids will be lighter in the fluid by the weight of the fluid displaced. Book II, which investigates the different positions in which a right segment of a paraboloid can float in a fluid, is a brilliant geometrical tour de force. In it Archimedes returns to the basic assumption found in *On the Equilibrium of Planes, On the Quadrature of the Parabola,* and *On the Method,* namely, that weight verticals are to be conceived of as parallel rather than as convergent at the center of a fluid sphere.

Influence. Unlike the *Elements* of Euclid, the works of Archimedes were not widely known in antiquity. Our present knowledge of his works depends largely on the interest taken in them at Constantinople from the sixth through the tenth centuries. It is true that before that time individual works of Archimedes were obviously studied at Alexandria, since Archimedes was often quoted by three eminent mathematicians of Alexandria: Hero, Pappus, and Theon. But it is with the activity of Eutocius of Ascalon, who was born toward the end of the fifth century and studied at Alexandria, that the textual history of a collected edition of Archimedes properly begins. Eutocius composed commentaries on three of Archimedes' works: *On the Sphere and the Cylinder, On the Measurement of the Circle,* and *On the Equilibrium of Planes.* These were no doubt the most popular of Archimedes' works at that time. The *Commentary on the Sphere and the Cylinder* is a rich work for historical references to Greek geometry. For example, in an extended comment to Book II, Proposition 1, Eutocius presents manifold solutions of earlier geometers to the problem of finding two mean proportionals between two given lines. *The Commentary on the Measurement of the Circle* is of interest in its detailed expansion of Archimedes' calculation of π. The works of Archimedes and the commentaries of Eutocius were studied and taught by Isidore of Miletus and Anthemius of Tralles, Justinian's architects of Hagia Sophia in Constantinople. It was apparently Isidore who was responsible for the first collected edition of at least the three works commented on by Eutocius as well as the commentaries. Later Byzantine authors seem gradually to have added other works to this first collected edition until the ninth century when the educational reformer Leon of Thessalonica produced the compilation represented by Greek manuscript A (adopting the designation used by the editor, J. L. Heiberg). Manuscript A contained all of the Greek works now known excepting *On Floating Bodies, On the Method, Stomachion,* and *The Cattle Problem.* This was one of the two manuscripts available to William of Moerbeke when he made his Latin translations in 1269. It was the source, directly or indirectly, of all of the Renaissance copies of Archimedes. A second Byzantine manuscript, designated as B, included only the mechanical works: *On the Equilibrium of Planes, On the Quadrature of the Parabola,* and *On Floating Bodies* (and possibly *On Spirals*). It too was available to Moerbeke. But it disappears after an early fourteenth-century reference. Finally, we can mention a third Byzantine manuscript, C, a palimpsest whose Archimedean parts are in a hand of the tenth century. It was not available to the Latin West in the Middle Ages, or indeed in modern times until its identification by Heiberg in 1906 at Constantinople (where it had been brought from Jerusalem). It contains large parts of *On the Sphere and the Cylinder,* almost all of *On Spirals,* some parts of *On the Measurement of the Circle* and *On the Equilibrium of Planes,* and a part of the *Stomachion.* More important, it contains most of the Greek text of *On Floating Bodies* (a text unavailable in Greek since the disappearance of manuscript B) and a great part of *On the Method of Mechanical Theorems,* hitherto known only by hearsay. (Hero mentions it in his *Metrica,* and the Byzantine lexicographer Suidas declares that Theodosius wrote a commentary on it.)

At about the same time that Archimedes was being studied in ninth-century Byzantium, he was also finding a place among the Arabs. The Arabic Archimedes has been studied in only a preliminary fashion, but it seems unlikely that the Arabs possessed any manuscript of his works as complete as manuscript A. Still, they often brilliantly exploited the methods of Archimedes and brought to bear their fine knowledge of conic sections on Archimedean problems. The Arabic Archimedes consisted of the following works: (1) *On the Sphere and the Cylinder* and at least a part of Eutocius' commentary on it. This work seems to have existed in a poor, early ninth-century translation, revised in the late ninth century, first by Isḥāq ibn Ḥunayn and then by Thābit ibn Qurra. It was reedited by Nasīr ad-Dīn al-Ṭūsī in the thirteenth century and was on occasion paraphrased and commented on by other Arabic authors (see Archimedes in Index of Suter's "Die Mathematiker und Astronomen"). (2) *On the Measurement of the Circle,* translated by Thābit ibn Qurra and reedited by al-Ṭūsī. Perhaps the commentary on it by Eutocius was also translated, for the extended calculation of π found in the geometrical tract of the ninth-century Arabic mathematicians the Banū Mūsā bears some resemblance to that present in the commentary of Eutocius. (3) A fragment of *On Floating Bodies,* consisting of a definition of specific gravity not present in the Greek text, a better version of the basic postulate (described above) than exists in the Greek text, and the enunciations without proofs of seven of the nine propositions of Book I and the first proposition of Book II. (4) Perhaps *On the Quadrature of the Parabola*—at least this problem received the attention of Thābit ibn Qurra. (5) Some indirect material from *On the Equilibrium of Planes* found in other mechanical works translated into Arabic (such as Hero's *Mechanics,* the so-called Euclid tract *On the Balance,* the *Liber karastonis,* etc.). (6) In addition, various other works attributed to Archimedes by the Arabs and for which there is no extant Greek text (see list above in "Mathematical Works"). Of the additional works, we can single out the *Lemmata* (*Liber assumptorum*), for, although it cannot have come directly from Archimedes in its present form, in the opinion of experts several of its propositions are Archimedean in character. One such proposition was Proposition 8, which employed a *neusis* construction like those used by Archimedes:[9]

Proposition 8

If we let line AB be led everywhere in the circle and extended rectilinearly [see Fig. 7], and if BC is posited as equal to the radius of the circle, and C is connected to the center of the circle D, and the line (CD) is produced to E, arc AE will be triple arc BF. Therefore, let us draw EG parallel to AB and join DB and DG. And because the two angles DEG, DGE are equal, $\angle GDC = 2\angle DEG$. And because $\angle BDC = \angle \mathrm{BCD}$ and $\angle CEG = \angle \mathrm{ACE}$, $\angle GDC = 2\angle CDB$ and $\angle BDG = 3\angle BDC$, and arc BG = arc AE, and arc $AE = 3$ arc BF; and this is what we wished.

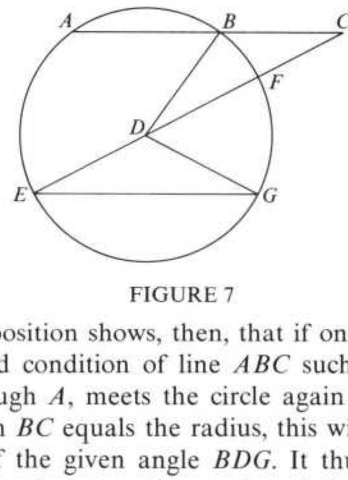

FIGURE 7

This proposition shows, then, that if one finds the position and condition of line ABC such that it is drawn through A, meets the circle again in B, and its extension BC equals the radius, this will give the trisection of the given angle BDG. It thus demonstrates the equivalence of a *neusis* and the trisection problem—but without solving the *neusis* (which could be solved by the construction of a conchoid to a circular base).

Special mention should also be made of the *Book on the Division of the Circle into Seven Equal Parts,* attributed to Archimedes by the Arabs, for its remarkable construction of a regular heptagon. This work stimulated a whole series of Arabic studies of this problem, including one by the famous Ibn al-Haytham (Alhazen). Propositions 16 and 17, leading to that construction, are given here in toto:[10]

Proposition 16

Let us construct square $ABCD$ [Fig. 8] and extend side AB directly toward H. Then we draw the diagonal

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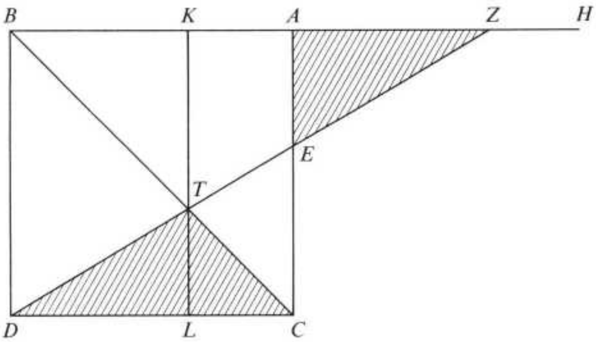

FIGURE 8

BC. We lay one end of a rule on point D. Its other end we make meet extension AH at a point Z such that $\triangle AZE = \triangle CTD$. Further, we draw the straight line KTL through T and parallel to AC. And now I say that $AB \cdot KB = AZ^2$ and $ZK \cdot AK = KB^2$ and, in addition, each of the two lines AZ and $KB > AK$.

Proof:

(1) $CD \cdot TL = AZ \cdot AE$ [given] Hence

(2) $\frac{CD(=AB)}{AZ} = \frac{AE}{TL}$

Since $\triangle ZAE \sim \triangle ZKT \sim \triangle TLD$, hence

(3) $\frac{AE}{TL} = \frac{AZ}{LD(=KB)}$, $\frac{AB}{AZ} = \frac{AZ}{KB}$, and

$\frac{TL(=AK)}{KT(=KB)} = \frac{LD(=KB)}{ZK}$. Therefore

(4) $AB \cdot KB = AZ^2$ and
$ZK \cdot AK = KB^2$

and each of the lines AZ and $KB > AK$. Q.E.D.

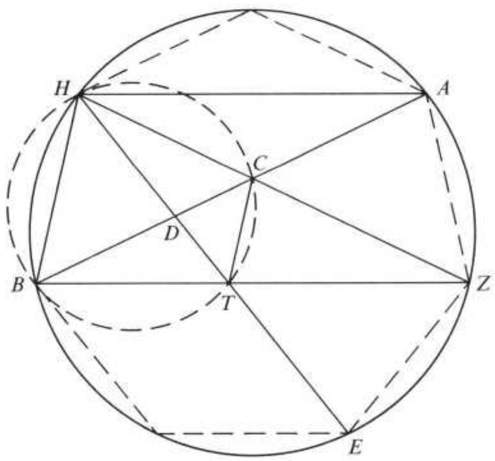

FIGURE 9

Proposition 17

We now wish to divide the circle into seven equal parts (Fig. 9). We draw the line segment AB, which we set out as known. We mark on it two points C and D, such that $AD \cdot CD = DB^2$ and $CB \cdot BD = AC^2$ and in addition each of the two segments AC and $DB > CD$, following the preceding proposition [i.e., Prop. 16]. Out of lines AC, CD and BD we construct $\triangle CHD$. Accordingly $CH = AC$, $DH = DB$ and $CD = CD$. Then we circumscribe about $\triangle AHB$ the circle $AHBEZ$ and we extend lines HC and HD directly up to the circumference of the circle. On their intersection with the circumference lie the points Z and E. We join B with Z. Lines BZ and HE intersect in T. We also draw CT. Since $AC = CH$, hence $\angle HAC = \angle AHC$, and arc AZ = arc HB. And, indeed, $AD \cdot CD = DB^2 = DH^2$ and [by Euclid, VI.8] $\triangle AHD \sim \triangle CHD$; consequently $\angle DAH = \angle CHD$, or arc ZE = arc BH. Hence BH, AZ and ZE are three equal arcs. Further, ZB is parallel to AH, $\angle CAH = \angle CHD = \angle TBD$; $HD = DB$, $CD = DT$, $CH = BT$. Hence, [since the products of the parts of these diagonals are equal], the 4 points B, H, C and T lie in the circumference of one and the same circle. From the similarity of triangles HBC and HBT, it follows that $CB \cdot DB = HC^2 = AC^2$ [or $HT/HC = HC/HD$] and from the similarity of $\triangle THC$ and $\triangle CHD$, it follows that $TH \cdot HD = HC^2$. And further $CB = TH$ [these being equal diagonals in the quadrilateral] and $\angle DCH = \angle HTC = 2\angle CAH$. [The equality of the first two angles arises from the similarity of triangles THC and CHD. Their equality with $2\angle CAH$ arises as follows: (1) $AHD = 2\angle CAH$, for $\angle CAH = \angle CHD = \angle CHA$ and $\angle AHD = \angle CHA + \angle CHD$; (2) $\angle AHD = \angle BTH$, for parallel lines cut by a third line produce equal alternate angles; (3) $\angle BTH = \angle DCH$, from similar triangles; (4) hence $\angle DCH = 2\angle CAH$.] [And since $\angle HBA = \angle DCH$, hence $\angle HBA = 2\angle CAH$.] Consequently, arc AH = 2 arc BH. Since $\angle DHB = \angle DBH$, consequently arc EB = 2 arc HB. Hence, each of arcs AH and EB equals 2 arc HB, and accordingly the circle $AHBEZ$ is divided into seven equal parts. Q.E.D. And praise be to the one God, etc.

The key to the whole procedure is, of course, the *neusis* presented in Proposition 16 (see Fig. 8) that would allow us in a similar fashion to find the points C and D in Proposition 17 (see Fig. 9). In Proposition 16 the *neusis* consisted in drawing a line from D to intersect the extension of AB in point Z such that $\triangle AZE = \triangle CTD$. The way in which the *neusis* was solved by Archimedes (or whoever was the author of this tract) is not known. Ibn al-Haytham, in his later treatment of the heptagon, mentions the Archimedean *neusis* but then goes on to show that one does not need the Archimedean square of Proposition 16. Rather he shows that points C and D in Proposition 17 can be found by the intersection of a parabola and a hyperbola.[11] It should be observed that all but two of Propositions 1–13 in this tract concern right triangles, and those two are necessary for propositions concerning right triangles. It seems probable, therefore, that Propositions 1–13 comprise the so-called *On* *the Properties of the Right Triangle* attributed in the *Fihrist* to Archimedes (although at least some of these propositions are Arabic interpolations). Incidentally, Propositions 7–10 have as their objective the formulation $K = (s - a) \cdot (s - c)$, where K is the area and a and c are the sides including the right angle and s is the semiperimeter, and Proposition 13 has as its objective $K = s(s - b)$, where b is the hypotenuse. Hence, if we multiply the two formulations, we have

$$K^2 = s(s - a) \cdot (s - b) \cdot (s - c)$$

or

$$K = \sqrt{s(s - a) \cdot (s - b) \cdot (s - c)},$$

Hero's formula for the area of a triangle in terms of its sides—at least in the case of a right triangle. Interestingly, the Arab scholar al-Bīrūnī attributed the general Heronian formula to Archimedes. Propositions 14 and 15 of the tract make no reference to Propositions 1–13 and concern chords. Each leads to a formulation in terms of chords equivalent to $\sin A/2 = \sqrt{(1 - \cos A)/2}$. Thus Propositions 14–15 seem to be from some other work (and at least Proposition 15 is an Arabic interpolation). If Proposition 14 was in the Greek text translated by Thābit ibn Qurra and does go back to Archimedes, then we would have to conclude that this formula was his discovery rather than Ptolemy's, as it is usually assumed to be.

The Latin West received its knowledge of Archimedes from both the sources just described: Byzantium and Islam. There is no trace of the earlier translations imputed by Cassiodorus to Boethius. Such knowledge that was had in the West before the twelfth century consisted of some rather general hydrostatic information that may have indirectly had its source in Archimedes. It was in the twelfth century that the translation of Archimedean texts from the Arabic first began. The small tract *On the Measurement of the Circle* was twice translated from the Arabic. The first translation was a rather defective one and was possibly executed by Plato of Tivoli. There are many numerical errors in the extant copies of it and the second half of Proposition 3 is missing. The second translation was almost certainly done by the twelfth century's foremost translator, Gerard of Cremona. The Arabic text from which he worked (without doubt the text of Thābit ibn Qurra) included a corollary on the area of a sector of a circle attributed by Hero to Archimedes but missing from our extant Greek text.

Not only was Gerard's translation widely quoted by medieval geometers such as Gerard of Brussels, Roger Bacon, and Thomas Bradwardine, it also served as the point of departure for a whole series of emended versions and paraphrases of the tract in the course of the thirteenth and fourteenth centuries. Among these are the so-called Naples, Cambridge, Florence, and Gordanus versions of the thirteenth century; and the Corpus Christi, Munich, and Albert of Saxony versions of the fourteenth. These versions were expanded by including pertinent references to Euclid and the spelling-out of the geometrical steps only implied in the Archimedean text. In addition, we see attempts to specify the postulates that underlie the proof of Proposition I. For example, in the Cambridge version three postulates (*petitiones*) introduce the text:[12] "[1] There is some curved line equal to any straight line and some straight line to any curved line. [2] Any chord is less than its arc. [3] The perimeter of any including figure is greater than the perimeter of the included figure." Furthermore, self-conscious attention was given in some versions to the logical nature of the proof of Proposition I. Thus, the Naples version immediately announced that the proof was to be *per impossibile,* i.e., by reduction to absurdity. In the Gordanus, Corpus Christi, and Munich versions we see a tendency to elaborate the proofs in the manner of scholastic tracts. The culmination of this kind of elaboration appeared in the *Questio de quadratura circuli* of Albert of Saxony, composed some time in the third quarter of the fourteenth century. The Hellenistic mathematical form of the original text was submerged in an intricate scholastic structure that included multiple terminological distinctions and the argument and counterargument technique represented by initial arguments ("principal reasons") and their final refutations.

Another trend in the later versions was the introduction of rather foolish physical justifications for postulates. In the Corpus Christi version, the second postulate to the effect that a straight line may be equal to a curved line is supported by the statement that "if a hair or silk thread is bent around circumference-wise in a plane surface and then afterwards is extended in a straight line, who will doubt—unless he is hare-brained—that the hair or thread is the same whether it is bent circumference-wise or extended in a straight line and is just as long the one time as the other." Similarly, Albert of Saxony, in his *Questio,* declared that a sphere can be "cubed" since the contents of a spherical vase can be poured into a cubical vase. Incidentally, Albert based his proof of the quadrature of the circle not directly on Proposition X.1 of the *Elements,* as was the case in the other medieval versions of *On the Measurement of the Circle,* but rather on a "betweenness" postulate: "I suppose that with two continuous [and comparable] quantities proposed, a magnitude greater than the 'lesser' can be cut from the 'greater.'" A similar postulate was employed in still another fourteenth-century version of the *De mensura circuli* called the Pseudo-Bradwardine version. Finally, in regard to the manifold medieval versions of *On the Measurement of the Circle,* it can be noted that the Florence version of Proposition 3 contained a detailed elaboration of the calculation of π. One might have supposed that the author had consulted Eutocius' commentary, except that his arithmetical procedures differed widely from those used by Eutocius. Furthermore, no translation of Eutocius' commentary appears to have been made before 1450, and the Florence version certainly must be dated before 1400.

In addition to his translation of *On the Measurement of the Circle,* Gerard of Cremona also translated the geometrical *Discourse of the Sons of Moses* (*Verba filiorum*) composed by the Banū Mūsā. This Latin translation was of particular importance for the introduction of Archimedes into the West. We can single out these contributions of the treatise: (1) A proof of Proposition I of *On the Measurement of the Circle* somewhat different from that of Archimedes but still fundamentally based on the exhaustion method. (2) A determination of the value of π drawn from Proposition 3 of the same treatise but with further calculations similar to those found in the commentary of Eutocius. (3) Hero's theorem for the area of a triangle in terms of its sides (noted above), with the first demonstration of that theorem in Latin (the enunciation of this theorem had already appeared in the writings of the *agrimensores* and in Plato of Tivoli's translation of the *Liber embadorum* of Savasorda). (4) Theorems for the volume and surface area of a cone, again with demonstrations. (5) Theorems for the volume and surface area of a sphere with demonstrations of an Archimedean character. (6) A use of the formula for the area of a circle equivalent to $A = \pi r^2$ in addition to the more common Archimedean form, $A = 1/2\,cr$. Instead of the modern symbol π the authors used the expression "the quantity which when multiplied by the diameter produces the circumference." (7) The introduction into the West of the problem of finding two mean proportionals between two given lines. In this treatise we find two solutions: (*a*) one attributed by the Banū Mūsā to Menelaus and by Eutocius to Archytas, (*b*) the other presented by the Banū Mūsā as their own but similar to the solution attributed by Eutocius to Plato. (8) The first solution in Latin of the problem of the trisection of an angle. (9) A method of approximating cube roots to any desired limit.

The *Verba filiorum* was, then, rich fare for the geometers of the twelfth century. The tract was quite widely cited in the thirteenth and fourteenth centuries. In the thirteenth, the eminent mathematicians Jordanus de Nemore and Leonardo Fibonacci made use of it. For example, the latter, in his *Practica geometrie,* excerpted both of the solutions of the mean proportionals problem given by the Banū Mūsā, while the former (or perhaps a continuator) in his *De triangulis* presented one of them together with an entirely different solution, namely, that one assigned by Eutocius to Philo of Byzantium. Similarly, Jordanus (or possibly the same continuator) extracted the solution of the trisection of an angle from the *Verba filiorum,* but in addition made the remarkably perspicacious suggestion that the *neusis* can be solved by the use of a proposition from Ibn al-Haytham's *Optics,* which solves a similar *neusis* by conic sections.

Some of the results and techniques of *On the Sphere and the Cylinder* also became known through a treatise entitled *De curvis superficiebus Archimenidis* and said to be by Johannes de Tinemue. This seems to have been translated from the Greek in the early thirteenth century or at least composed on the basis of a Greek tract. The *De curvis superficiebus* contained ten propositions with several corollaries and was concerned for the most part with the surfaces and volumes of cones, cylinders, and spheres. This was a very popular work and was often cited by later authors. Like Gerard of Cremona's translation of *On the Measurement of the Circle,* the *De curvis superficiebus* was emended by Latin authors, two original propositions being added to one version (represented by manuscript D of the *De curvis superficiebus*)[13] and three quite different propositions being added to another (represented by manuscript M of the *De curvis*).[14] In the first of the additions to the latter version, the Latin author applied the exhaustion method to a problem involving the surface of a segment of a sphere, showing that at least this author had made the method his own. And indeed the geometer Gerard of Brussels in his *De motu* of about the same time also used the Archimedean *reductio* procedure in a highly original manner.

In 1269, some decades after the appearance of the *De curvis superficiebus,* the next important step was taken in the passage of Archimedes to the West when much of the Byzantine corpus was translated from the Greek by the Flemish Dominican, William of Moerbeke. In this translation Moerbeke employed Greek manuscripts A and B which had passed to the pope's library in 1266 from the collection of the Norman kings of the Two Sicilies. Except for *The Sandreckoner* and Eutocius' *Commentary on the Measurement of the Circle,* all the works included in manuscripts A and B were rendered into Latin by William. Needless to say, *On the Method, The Cattle*

Problem, and the *Stomachion*, all absent from manuscripts A and B, were not among William's translations. Although William's translations are not without error (and indeed some of the errors are serious), the translations, on the whole, present the Archimedean works in an understandable way. We possess the original holograph of Moerbeke's translations (MS Vat. Ottob. lat. 1850). This manuscript was not widely copied. The translation of *On Spirals* was copied from it in the fourteenth century (MS Vat. Reg. lat. 1253, 14r–33r), and several works were copied from it in the fifteenth century in an Italian manuscript now at Madrid (Bibl. Nac. 9119), and one work (*On Floating Bodies*) was copied from it in the sixteenth century (MS Vat. Barb. lat. 304, 124r–141v, 160v–161v). But, in fact, the Moerbeke translations were utilized more than one would expect from the paucity of manuscripts. They were used by several Schoolmen at the University of Paris toward the middle of the fourteenth century. Chief among them was the astronomer and mathematician John of Meurs, who appears to have been the compositor of a hybrid tract in 1340 entitled *Circuli quadratura*. This tract consisted of fourteen propositions. The first thirteen were drawn from Moerbeke's translation of *On Spirals* and were just those propositions necessary to the proof of Proposition 18 of *On Spirals*, whose enunciation we have quoted above. The fourteenth proposition of the hybrid tract was Proposition 1 from Moerbeke's translation of *On the Measurement of the Circle*. Thus this author realized that by the use of Proposition 18 from *On Spirals*, he had achieved the necessary rectification of the circumference of a circle preparatory to the final quadrature of the circle accomplished in *On the Measurement of the Circle*, Proposition 1. Incidentally, the hybrid tract did not merely use the Moerbeke translations verbatim but also included considerable commentary. In fact, this medieval Latin tract was the first known commentary on Archimedes' *On Spirals*. That the commentary was at times quite perceptive is indicated by the fact that the author suggested that the *neusis* introduced by Archimedes in Proposition 7 of *On Spirals* could be solved by means of an *instrumentum conchoydeale*. The only place in which a medieval Latin commentator could have learned of such an instrument would have been in that section of the *Commentary on the Sphere and the Cylinder* where Eutocius describes Nicomedes' solution of the problem of finding two mean proportionals (Bk. II, Prop. 1). We have further evidence that John of Meurs knew of Eutocius' *Commentary* in the Moerbeke translation when he used sections from this commentary in his *De arte mensurandi* (Ch. VIII, Prop. 16), where three of the solutions of the mean proportionals problem given by Eutocius are presented. Not only did John incorporate the whole hybrid tract *Circuli quadratura* into Chapter VIII of his *De arte mensurandi* (composed, it seems, shortly after 1343) but in Chapter X of the *De arte* he quoted verbatim many propositions from Moerbeke's translations of *On the Sphere and the Cylinder* and *On Conoids and Spheroids* (which latter he misapplied to problems concerning solids generated by the rotation of circular segments). Within the next decade or so after John of Meurs, Nicole Oresme, his colleague at the University of Paris, in his *De configurationibus qualitatum et motuum* (Part I, Ch. 21) revealed knowledge of *On Spirals*, at least in the form of the hybrid *Circuli quadratura*. Further, Oresme in his *Questiones super de celo et mundo*, quoted at length from Moerbeke's translation of *On Floating Bodies*, while Henry of Hesse, Oresme's junior contemporary at Paris, quoted briefly therefrom. (Before this time, the only knowledge of *On Floating Bodies* had come in a thirteenth-century treatise entitled *De ponderibus Archimenidis sive de incidentibus in humidum*, a Pseudo-Archimedean treatise prepared largely from Arabic sources, whose first proposition expressed the basic conclusion of the "principle of Archimedes": "The weight of any body in air exceeds its weight in water by the weight of a volume of water equal to its volume.") Incontrovertible evidence, then, shows that at the University of Paris in the mid-fourteenth century six of the nine Archimedean translations of William of Moerbeke were known and used: *On Spirals, On the Measurement of the Circle, On the Sphere and the Cylinder, On Conoids and Spheroids, On Floating Bodies*, and Eutocius' *Commentary on the Sphere and the Cylinder*. While no direct evidence exists of the use of the remaining three translations, there has been recently discovered in a manuscript written at Paris in the fourteenth century (BN lat. 7377B, 93v–94r) an Archimedean-type proof of the law of the lever that might have been inspired by Archimedes' *On the Equilibrium of Planes*. But other than this, the influence of Archimedes on medieval statics was entirely indirect. The anonymous *De canonio*, translated from the Greek in the early thirteenth century, and Thābit ibn Qurra's *Liber karastonis*, translated from the Arabic by Gerard of Cremona, passed on this indirect influence of Archimedes in three respects: (1) Both tracts illustrated the Archimedean type of geometrical demonstrations of statical theorems and the geometrical form implied in weightless beams and weights that were really only geometrical magnitudes. (2) They gave specific reference in geometrical language to the law of the lever (and in the *De canonio* the law of the lever is connected directly to Archimedes). (3) They indirectly reflected the centers-of-gravity doctrine so important to Archimedes, in that both treatises employed the practice of substituting for a material beam segment a weight equal in weight to the material segment but hung from the middle point of the weightless segment used to replace the material segment. Needless to say, these two tracts played an important role in stimulating the rather impressive statics associated with the name of Jordanus de Nemore.

In the fifteenth century, knowledge of Archimedes in Europe began to expand. A new Latin translation was made by James of Cremona in about 1450 by order of Pope Nicholas V. Since this translation was made exclusively from manuscript A, the translation failed to include *On Floating Bodies,* but it did include the two treatises in A omitted by Moerbeke, namely, *The Sandreckoner* and Eutocius' *Commentary on the Measurement of the Circle.* It appears that this new translation was made with an eye on Moerbeke's translations. Not long after its completion, a copy of the new translation was sent by the pope to Nicholas of Cusa, who made some use of it in his *De mathematicis complementis,* composed in 1453–1454. There are at least nine extant manuscripts of this translation, one of which was corrected by Regiomontanus and brought to Germany about 1468 (the Latin translation published with the *editio princeps* of the Greek text in 1544 was taken from this copy). Greek manuscript A itself was copied a number of times. Cardinal Bessarion had one copy prepared between 1449 and 1468 (MS E). Another (MS D) was made from A when it was in the possession of the well-known humanist George Valla. The fate of A and its various copies has been traced skillfully by J. L. Heiberg in his edition of Archimedes' *Opera.* The last known use of manuscript A occurred in 1544, after which time it seems to have disappeared. The first printed Archimedean materials were in fact merely Latin excerpts that appeared in George Valla's *De expetendis et fugiendis rebus opus* (Venice, 1501) and were based on his reading of manuscript A. But the earliest actual printed texts of Archimedes were the Moerbeke translations of *On the Measurement of the Circle* and *On the Quadrature of the Parabola* (*Tetragonismus, id est circuli quadratura etc.*), published from the Madrid manuscript by L. Gaurico (Venice, 1503). In 1543, also at Venice, N. Tartaglia republished the same two translations directly from Gaurico's work, and, in addition, from the same Madrid manuscript, the Moerbeke translations of *On the Equilibrium of Planes* and Book I of *On Floating Bodies* (leaving the erroneous impression that he had made these translations from a Greek manuscript, which he had not since he merely repeated the texts of the Madrid manuscript with virtually all their errors). Incidentally, Curtius Trioianus published from the legacy of Tartaglia both books of *On Floating Bodies* in Moerbeke's translation (Venice, 1565). The key event, however, in the further spread of Archimedes was the aforementioned *editio princeps* of the Greek text with the accompanying Latin translation of James of Cremona at Basel in 1544. Since the Greek text rested ultimately on manuscript A, *On Floating Bodies* was not included. A further Latin translation of the Archimedean texts was published by the perceptive mathematician Federigo Commandino in Bologna in 1558, which the translator supplemented with a skillful mathematical emendation of Moerbeke's translation of *On Floating Bodies* (Bologna, 1565) but without any knowledge of the long lost Greek text. Already in the period 1534–1549, a paraphrase of Archimedean texts had been made by Francesco Maurolico. This was published in Palermo in 1685. One other Latin translation of the sixteenth century by Antonius de Albertis remains in manuscript only and appears to have exerted no influence on mathematics and science. After 1544 the publications on Archimedes and the use of his works began to multiply markedly. His works presented quadrature problems and propositions that mathematicians sought to solve and demonstrate not only with his methods, but also with a developing geometry of infinitesimals that was to anticipate in some respect the infinitesimal calculus of Newton and Leibniz. His hydrostatic conceptions were used to modify Aristotelian mechanics. Archimedes' influence on mechanics and mathematics can be seen in the works of such authors as Commandino, Guido Ubaldi del Monte, Benedetti, Simon Stevin, Luca Valerio, Kepler, Galileo, Cavalieri, Torricelli, and numerous others. For example, Galileo mentions Archimedes more than a hundred times, and the limited inertial doctrine used in his analysis of the parabolic path of a projectile is presented as an Archimedean-type abstraction. Archimedes began to appear in the vernacular languages. Tartaglia had already rendered into Italian Book I of *On Floating Bodies,* Book I of *On the Sphere and the Cylinder,* and the section on proportional means from Eutocius' *Commentary on the Sphere and the Cylinder.* Book I of *On the Equilibrium of Planes* was translated into French in 1565 by Pierre Forcadel. It was, however, not until 1670 that a more or less complete translation was made into German by J. C. Sturm on the basis of the influential Greek and Latin edition of David Rivault (Paris, 1615). Also notable for its influence was the new Latin edition of Isaac Barrow (London, 1675). Of the many editions prior to the modern edition of Heiberg, the most important was that of Joseph Torelli (Oxford, 1792). By this time, of course, Archimedes' works had been almost completely absorbed into European mathematics and had exerted their substantial and enduring influence on early modern science.

NOTES

1. Heath, *The Works of Archimedes*, pp. 91–93. Heath's close paraphrase has been used here and below because of its economy of expression. While he uses modern symbols and has reduced the general enunciations to statements concerning specific figures in some of the propositions quoted below, he nevertheless achieves a faithful representation of the spirit of the original text.
2. *Ibid.*, pp. 19–20.
3. *Ibid.*, pp. 251–252.
4. *Ibid.*, pp. 156–57.
5. *Ibid.*, pp. 1–2.
6. *Ibid.*, Suppl., pp. 18–22.
7. *Ibid.*, pp. 13–14.
8. Dijksterhuis, *Archimedes*, p. 373.
9. Clagett, *Archimedes in the Middle Ages*, pp. 667–668.
10. Schoy, *Die trigonometrischen Lehren*, pp. 82–83.
11. *Ibid.*, pp. 85–91.
12. Clagett, *op. cit.*, p. 27. The succeeding quotations from the various versions of *On the Measurement of the Circle* are also from this volume.
13. *Ibid.*, p. 520.
14. *Ibid.*, p. 530.

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2. *The Arabic Archimedes* (the manuscripts cited are largely from Suter, "Die Mathematiker und Astronomen" [see Secondary Literature], and C. Brockelmann, *Geschichte der arabischen Literatur*, 5 vols., Vols. I–II [adapted to Suppl. vols., Leiden, 1943–1949], Suppl. Vols. I–III [Leiden, 1937–1942]). *On the Sphere and the Cylinder* and *On the Measurement of the Circle;* both appear in Nāṣir al-Dīn al-Ṭūsī, *Majmūʿ al-Rasā'il*, Vol. II (Hyderabad, 1940). Cf. MSS Berlin 5934; Florence Palat. 271 and 286; Paris 2467; Oxford, Bodl. Arabic 875, 879; India Office 743; and M. Clagett, *Archimedes in the Middle Ages*, I, 17, n. 8. The al-Ṭūsī edition also contains some commentary on Bk. II of *On the Sphere and the Cylinder. Book of the Elements of Geometry* (probably the same as *On Triangles*, mentioned in the *Fihrist*) and *On Touching Circles;* both appear in *Rasā'il Ibn Qurra* (Hyderabad, 1947, given as 1948 on transliterated title page). *On the Division of the Circle into Seven Equal Parts* (only Props. 16–17 concern heptagon construction; Props. 1–13 appear to be the tract called *On the Properties of the Right Triangle;* Props. 14–15 are unrelated to either of other parts). MS Cairo A.-N.8 H.-N. 7805, item no. 15. German translation by C. Schoy, *Die trigonometrischen Lehren des persischen Astronomen Abu 'l-Raihân Muh. ibn Ahmad al-Bîrûnî* (Hannover, 1927), pp. 74–84. The text has been analyzed in modern fashion by J. Tropfke, "Die Siebeneckabhandlung des Archimedes," in *Osiris*, **1** (1936), 636–651. *On Heaviness and Lightness* (a fragment of *On Floating Bodies*); Arabic text by H. Zotenberg in *Journal asiatique;* Ser. 7, **13** (1879), 509–515, from MS Paris, BN Fonds suppl. arabe 952 bis. A German translation was made by E. Wiedemann in the *Sitzungsberichte der Physikalisch-medizinischen Sozietät in Erlangen*, **38** (1906), 152–162. For an English translation and critique, see M. Clagett, *The Science of Mechanics in the Middle Ages* (Madison, Wis., 1959, 2nd pr., 1961), pp. 52–55. *Lemmata* (*Liber assumptorum*), see the edition in al-Ṭūsī, *Majmūʿ al-Rasā'il*, Vol. II (Hyderabad, 1940). MSS Oxford, Bodl. Arabic 879, 895, 939, 960; Leiden 982; Florence, Palat. 271 and 286; Cairo A.-N. 8 H.-N 7805. This work was first edited by S. Foster, *Miscellanea* (London, 1659), from a Latin translation of I. Gravius; Abraham Ecchellensis then retranslated it, the new translation being published in I. A. Borelli's edition of *Apollonii Pergaei Concicorum libri V, VI, VII* (Florence, 1661). Ecchellensis' translation was republished by Heiberg, *Opera*, II, 510–525. See also E. S. Stamates' effort to reconstruct the original Greek text in *Bulletin de la Societé Mathématique de Grèce*, new series, **6 II**, Fasc. 2 (1965), 265–297. *Stomachion*, a fragmentary part in Arabic with German translation in H. Suter, "Der Loculus Archimedius oder das Syntemachion des Archimedes," in *Abhandlungen zur Geschichte der Mathematik*, **9** (1899), 491–499. This is one of two fragments. The other is in Greek and is given by Heiberg, *Opera*, II, 416. Eutocius, *Commentary on the Sphere and the Cylinder*, a section of Bk. II. MSS Paris, BN arabe 2457, 44°; Bibl. Escor. 960; Istanbul, Fatīh Mosque Library Ar. 3414, 60v–66v; Oxford, Bodl. Arabic 875 and 895. Various tracts and commentaries *On the Sphere and the Cylinder*, Bk. II, in part paraphrased and translated by F. Woepcke, *L'Algebra d'Omar Alkhayâmmî* (Paris, 1851), pp. 91–116.

3. *The Medieval Latin Archimedes.* A complete edition and translation of the various Archimedean tracts arising

from the Arabic tradition have been given by M. Clagett, *Archimedes in the Middle Ages*, Vol. I (Madison, Wis., 1964). Vol. II will contain the complete text of Moerbeke's translations and other Archimedean materials from the late Middle Ages. Moerbeke's translation of *On Spirals* and brief parts of other of his translations have been published by Heiberg, "Neue Studien" (see below). See also M. Clagett, "A Medieval Archimedean-Type Proof of the Law of the Lever," in *Miscellanea André Combes*, II (Rome, 1967), 409–421. For the Pseudo-Archimedes, *De ponderibus* (*De incidentibus in humidum*), see E. A. Moody and M. Clagett, *The Medieval Science of Weights* (Madison, 1952; 2nd printing, 1960), pp. 35–53, 352–359.

II. SECONDARY LITERATURE. The best over-all analysis is in E. J. Dijksterhuis, *Archimedes* (Copenhagen, 1956), which also refers to the principal literature. The translations of Heath and Ver Eecke given above contain valuable evaluative and biographical materials. In addition, consult C. Boyer, *The Concepts of the Calculus* (New York, 1939; 2nd printing, 1949; Dover ed. 1959), particularly ch. 4 for the reaction of the mathematicians of the sixteenth and seventeenth centuries to Archimedes. M. Clagett, "Archimedes and Scholastic Geometry," in *Mélanges Alexandre Koyré*, Vol. I: *L'Aventure de la science* (Paris, 1964), 40–60; "The Use of the Moerbeke Translations of Archimedes in the Works of Johannes de Muris," in *Isis*, **43** (1952), 236–242 (the conclusions of this article will be significantly updated in M. Clagett, *Archimedes in the Middle Ages*, Vol. II); and "Johannes de Muris and the Problem of the Mean Proportionals," in *Medicine, Science and Culture, Historical Essays in Honor of Owsei Temkin*, L. G. Stevenson and R. P. Multhauf, eds. (Baltimore, 1968), 35–49. A. G. Drachmann, "Fragments from Archimedes in Heron's Mechanics," in *Centaurus*, **8** (1963), 91–145; "The Screw of Archimedes," in *Actes du VIII^e Congrès international d'Histoire des Sciences Florence-Milan 1956*, **3** (Vinci-Paris, 1958), 940–943; and "How Archimedes Expected to Move the Earth," in *Centaurus*, **5** (1958), 278–282. J. L. Heiberg, "Neue Studien zu Archimedes," in *Abhandlungen zur Geschichte der Mathematik*, **5** (1890), 1–84; and *Quaestiones Archimedeae* (Copenhagen, 1879). Most of the biographical references are given here by Heiberg. S. Heller, "Ein Fehler in einer Archimedes-Ausgabe, seine Entstehung und seine Folgen," in *Abhandlungen der Bayerischen Akademie der Wissenschaften. Mathematisch-naturwissenschaftliche Klasse*, new series, **63** (1954), 1–38. E. Rufini, *Il "Metodo" di Archimede e le origini dell'analisi infinitesimale nell'antichita* (Rome, 1926; new ed., Bologna, 1961). H. Suter, "Die Mathematiker und Astronomen der Araber und ihre Werke," in *Abhandlungen zur Geschichte der mathematischen Wissenschaften*, **10** (1892), *in toto;* "Das Mathematiker- Verzeichniss im Fihrist des Ibn Abī Ja'kûb an-Nadîm," *ibid.*, **6** (1892), 1–87. B. L. Van der Waerden, *Erwachende Wissenschaft*, 2nd German ed. (Basel, 1966), pp. 344–381. See also the English translation, *Science Awakening*, 2nd ed. (Groningen, 1961), pp. 204–206, 208–228. E. Wiedemann, "Beiträge zur Geschichte der Naturwissenschaften III," in *Sitzungsberichte der Physikalisch-medizinischen Sozietät in Erlangen*, **37** (1905), 247–250, 257. A. P. Youschkevitch, "Remarques sur la méthode antique d'exhaustion," in *Mélanges Alexandre Koyré*, I: *L'Aventure de la science* (1964), 635–653.

MARSHALL CLAGETT

ARCHYTAS OF TARENTUM (*fl.* Tarentum [now Taranto], Italy, *ca.* 375 B.C.), *philosophy, mathematics, physics.*

After the Pythagoreans had been driven out of most of the cities of southern Italy by the Syracusan tyrant Dionysius the Elder at the beginning of the fourth century B.C., Tarentum remained their only important political center. Here Archytas played a leading role in the attempt to unite the Greek city-states against the non-Greek tribes and powers. After the death of Dionysius the Elder, he concluded, through the agency of Plato, an alliance with his son and successor, Dionysius the Younger.

Archytas made very important contributions to the theory of numbers, geometry, and the theory of music. Although extant ancient tradition credits him mainly with individual discoveries, it is clear that all of them were connected and that Archytas was deeply concerned with the foundations of the sciences and with their interconnection. Thus he affirmed that the art of calculation (λογιστική) is the most fundamental science and makes its results even clearer than those of geometry. He also discussed mathematics as the foundation of astronomy.

A central point in Archytas' manifold endeavors was the theory of means (μεσότητες) and proportions. He distinguished three basic means: the arithmetic mean of the form $a - b = b - c$ or $a + c = 2b$; the geometric mean of the form $a:b = b:c$ or $ac = b^2$; and the harmonic mean of the form $(a - b):(b - c) = a:c$. Archytas and later mathematicians subsequently added seven other means.

A proposition and proof that are important both for Archytas' theory of means and for his theory of music have been preserved in Latin translation in Boethius' *De musica.* The proposition states that there is no geometric mean between two numbers that are in "superparticular" (ἐπιμόριος) ratio, i.e., in the ratio $(n + 1):n$. The proof given by Boethius is essentially identical with that given for the same proposition by Euclid in his *Sectio canonis* (Prop. 3). It presupposes several propositions of Euclid that appear in *Elements* VII as well as VIII, Prop. 8. Through a careful analysis of Books VII and VIII and their relation to the above proof, A. B. L. Van der Waerden has succeeded in making it appear very likely that many of the theorems in Euclid's *Elements* VII and their proofs