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JOHN C. BURNHAM

CARTAN, ÉLIE (*b.* Dolomieu, France, 9 April 1869; *d.* Paris, France, 6 May 1951), *mathematics.*

Cartan was one of the most profound mathematicians of the last hundred years, and his influence is still one of the most decisive in the development of modern mathematics. He was born in a village in the French Alps. His father was a blacksmith, and at that time children of poor families had almost no opportunity to reach the university. Fortunately, while he was still in elementary school, his intelligence impressed the young politician Antonin Dubost, who was then an inspector of primary schools (and was later president of the French Senate); Dubost secured for Cartan a state stipend enabling him to attend the *lycée* in Lyons and later to enter the École Normale Supérieure in Paris. After graduation he started his research with his now famous thesis on Lie groups, a topic then still in its very early stages. He held teaching positions at the universities of Montpellier, Lyons, Nancy, and finally Paris, where he became a professor in 1912 and taught until his retirement in 1940. In 1931 he was elected a member of the French Academy of Sciences, and in his later years he received many honorary degrees and was elected a foreign member of several scientific societies.

Cartan's mathematical work can be described as the development of analysis on differentiable manifolds, which many now consider the central and most vital part of modern mathematics and which he was foremost in shaping and advancing. This field centers on Lie groups, partial differential systems, and differential geometry; these, chiefly through Cartan's contributions, are now closely interwoven and constitute a unified and powerful tool.

Cartan was practically alone in the field of Lie groups for the thirty years after his dissertation. Lie had considered these groups chiefly as systems of analytic transformations of an analytic manifold, depending analytically on a finite number of parameters. A very fruitful approach to the study of these groups was opened in 1888 when Wilhelm Killing systematically started to study the group in itself, independent of its possible actions on other manifolds. At that time (and until 1920) only local properties were considered, so the main object of study for Killing was the Lie algebra of the group, which exactly reflects the local properties in purely algebraic terms. Killing's great achievement was the determination of all simple complex Lie algebras; his proofs, however, were often defective, and Cartan's thesis was devoted mainly to giving a rigorous foundation to the "local" theory and to proving the existence of the "exceptional" Lie algebras belonging to each of the types of simple complex Lie algebras Killing had shown to be possible. Later Cartan completed the "local" theory by explicitly solving two fundamental problems, for which he had to develop entirely new methods: the classification of simple real Lie algebras and the determination of all irreducible linear representations of simple Lie algebras, by means of the notion of weight of a representation, which he introduced for that purpose. It was in the process of determining the linear representations of the orthogonal groups that Cartan discovered in 1913 the spinors, which later played such an important role in quantum mechanics.

After 1925 Cartan grew more and more interested in topological questions. Spurred by Weyl's brilliant results on compact groups, he developed new methods for the study of global properties of Lie groups; in particular he showed that topologically a connected Lie group is a product of a Euclidean space and a compact group, and for compact Lie groups he discovered that the possible fundamental groups of the underlying manifold can be read from the structure of the Lie algebra of the group. Finally, he outlined a method of determining the Betti numbers of compact Lie groups, again reducing the problem to an algebraic question on their Lie algebras, which has since been completely solved.

Cartan's methods in the theory of differential systems are perhaps his most profound achievement. Breaking with tradition, he sought from the start to formulate and solve the problems in a completely invariant fashion, independent of any particular choice of variables and unknown functions. He thus was able for the first time to give a precise definition of what is a "general" solution of an arbitrary differential system. His next step was to try to determine all "singular" solutions as well, by a method of "prolongation" that consists in adjoining new unknowns and new equations to the given system in such a way that any singular solution of the original system becomes a general solution of the new system. Although Cartan showed that in every example which he treated his method led to the complete determination of all singular solutions, he did not succeed in proving in general that this would always be the case for an arbitrary system; such a proof was obtained in 1955 by Kuranishi.

Cartan's chief tool was the calculus of exterior differential forms, which he helped to create and develop in the ten years following his thesis, and then proceeded to apply with extraordinary virtuosity to the most varied problems in differential geometry, Lie groups, analytical dynamics, and general relativity. He discussed a large number of examples, treating them in an extremely elliptic style that was made possible only by his uncanny algebraic and geometric insight and that has baffled two generations of mathematicians. Even now, some twenty years after his death, students of his results find that a sizable number of them are still in need of clarification; chief among these are his theory of "equivalence" of differential systems and his results on "infinite Lie groups" (which are not groups in the usual sense of the word).

Cartan's contributions to differential geometry are no less impressive, and it may be said that he revitalized the whole subject, for the initial work of Riemann and Darboux was being lost in dreary computations and minor results, much as had happened to elementary geometry and invariant theory a generation earlier. His guiding principle was a considerable extension of the method of "moving frames" of Darboux and Ribaucour, to which he gave a tremendous flexibility and power, far beyond anything that had been done in classical differential geometry. In modern terms, the method consists in associating to a fiber bundle E the principal fiber bundle having the same base and having at each point of the base a fiber equal to the group that acts on the fiber of E at the same point. If E is the tangent bundle over the base (which since Lie was essentially known as the manifold of "contact elements"), the corresponding group is the general linear group (or the orthogonal group in classical Euclidean or Riemannian geometry). Cartan's ability to handle many other types of fibers and groups allows one to credit him with the first general idea of a fiber bundle, although he never defined it explicitly. This concept has become one of the most important in all fields of modern mathematics, chiefly in global differential geometry and in algebraic and differential topology. Cartan used it to formulate his definition of a connection, which is now used universally and has superseded previous attempts by several geometers, made after 1917, to find a type of "geometry" more general than the Riemannian model and perhaps better adapted to a description of the universe along the lines of general relativity.

Cartan showed how to use his concept of connection to obtain a much more elegant and simple presentation of Riemannian geometry. His chief contribution to the latter, however, was the discovery and study of the symmetric Riemann spaces, one of the few instances in which the initiator of a mathematical theory was also the one who brought it to its completion. Symmetric Riemann spaces may be defined in various ways, the simplest of which postulates the existence around each point of the space of a "symmetry" that is involutive, leaves the point fixed, and preserves distances. The unexpected fact discovered by Cartan is that it is possible to give a complete description of these spaces by means of the classification of the simple Lie groups; it should therefore not be surprising that in various areas of mathematics, such as automorphic functions and analytic number theory (apparently far removed from differential geometry), these spaces are playing a part that is becoming increasingly important.

Cartan's recognition as a first-rate mathematician came to him only in his old age; before 1930 Poincaré and Weyl were probably the only prominent mathematicians who correctly assessed his uncommon powers and depth. This was due partly to his extreme modesty and partly to the fact that in France the main trend of mathematical research after 1900 was in the field of function theory, but chiefly to his extraordinary originality. It was only after 1930 that a younger generation started to explore the rich treasure of ideas and results that lay buried in his papers. Since then his influence has been steadily increasing, and with the exception of Poincaré and Hilbert, probably no one else has done so much to give the mathematics of our day its present shape and viewpoints.

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JEAN DIEUDONNÉ

CARTESIUS, RENATUS. See **Descartes, René.**

CARUS, PAUL (*b.* Ilsenburg, Germany, 18 July 1852; *d.* La Salle, Illinois, 11 February 1919), *philosophy.*