useful information is also found in H. Mayhew, *The Story of the Peasant-Boy Philosopher* (London, 1857).

LAURENS LAUDAN

FERMAT, PIERRE DE (*b.* Beaumont-de-Lomagne, France, 20 August 1601; *d.* Castres, France, 12 January 1665), *mathematics.*

Factual details concerning Fermat's private life are quite sparse.[1] He apparently spent his childhood and early school years in his birthplace, where his father, Dominique Fermat, had a prosperous leather business and served as second consul of the town. His uncle and godfather, Pierre Fermat, was also a merchant. To the family's firm financial position Fermat's mother, Claire de Long, brought the social status of the parliamentary *noblesse de robe.* Hence, his choice of law as his profession followed naturally from the social milieu into which he was born. Having received a solid classical secondary education locally, Fermat may have attended the University of Toulouse, although one can say with certainty only that he spent some time in Bordeaux toward the end of the 1620's before finally receiving the degree of Bachelor of Civil Laws from the University of Orleans on 1 May 1631.

Returning to Toulouse, where some months earlier he had purchased the offices of *conseiller* and *commissaire aux requêtes* in the local *parlement,* Fermat married his mother's cousin, Louise de Long, on 1 June 1631. Like his in-laws, Fermat enjoyed as *parlementaire* the rank and privileges of the *noblesse de robe;* in particular he was entitled to add the "de" to his name, which he occasionally did. Fermat's marriage contract, the price he paid for his offices, and several other documents attest to the financial security he enjoyed throughout his life.

Five children issued from Fermat's marriage. The oldest, Clément-Samuel, apparently was closest to his father. As a lawyer he inherited his father's offices in 1665 and later undertook the publication of his father's mathematical papers.[2] Fermat's other son, Jean, served as archdeacon of Fimarens. The oldest daughter, Claire, married; her two younger sisters, Catherine and Louise, took holy orders. These outward details of Fermat's family life suggest that it followed the standard pattern for men of his social status. The direct male line ended with the death of Clément-Samuel's son, Jean-François, from whom Claire's grandson inherited the offices originally bought by Fermat.

As a lawyer and *parlementaire* in Toulouse, Fermat seems to have benefited more from the high rate of mortality among his colleagues than from any outstanding talents of his own. On 16 January 1638 he rose to the position of *conseiller aux enquêtes* and in 1642 entered the highest councils of the *parlement:* the criminal court and then the Grand Chamber. In 1648 he acted as chief spokesman for the *parlement* in negotiations with the chancellor of France, Pierre Séguier. However, Fermat's letters to Séguier and to his physician and confidant, Marin Cureau de La Chambre,[3] suggest that Fermat's performance in office was often less than satisfactory; and a confidential report by the *intendant* of Languedoc to Colbert in 1664 refers to Fermat in quite deprecatory terms. A staunch Catholic, Fermat served also—again probably by reason of seniority—as member and then president of the Chambre de l'Édit, which had jurisdiction over suits between Huguenots and Catholics and which convened in the Huguenot stronghold of Castres.

In addition to his fame as a mathematician, Fermat enjoyed a modest reputation as a classical scholar. Apparently equally fluent in French, Italian, Spanish, Latin, and Greek, he dabbled in philological problems and the composition of Latin poetry (see appendixes to his *Oeuvres,* I).

Except for an almost fatal attack of the plague in 1652, Fermat seems to have enjoyed good health until the years immediately preceding his death. He died in Castres, two days after having signed his last *arrêt* for the Chambre de l'Édit there. At first buried in Castres, his remains were brought back to the family vault in the Church of the Augustines in Toulouse in 1675.

The Development of Fermat's Mathematics. Fermat's letters and papers, most of them written after 1636 for friends in Paris, provide the few available hints regarding his development as a mathematician. From them one can infer that his stay in Bordeaux in the late 1620's most decisively shaped his approach to mathematics; almost all of his later achievements derived from research begun there. It was apparently in Bordeaux that Fermat studied in depth the works of François Viète. From Viète he took the new symbolic algebra and theory of equations that served as his basic research tools. More important, however, Viète's concept of algebra as the "analytic art" and the program of research implicit in that concept largely guided Fermat's choice of problems and the manner in which he treated them. Fermat himself viewed his work as a continuation of the Vietan tradition.

From Viète, Fermat inherited the idea of symbolic algebra as a formal language or tool uniting the realms of geometry and arithmetic (number theory). An algebraic equation had meaning in both realms, depending only on whether the unknowns denoted line segments or numbers. Moreover, Viète's theory of equations had shifted attention away from solutions of specific equations to questions of the relationships between solutions and the structures of their parent equations or between the solutions of one equation and those of another. In his own study of the application of determinate equations to geometric constructions, Viète laid the groundwork for the algebraic study of solvability and constructibility. Fermat sought to build further on this foundation. An overall characteristic of his mathematics is the use of algebraic analysis to explore the relationships between problems and their solutions. Most of Fermat's research strove toward a "reduction analysis" by which a given problem could be reduced to another or identified with a class of problems for which the general solution was known. This "reduction analysis," constituted from the theory of equations, could be reversed in most cases to operate as a generator of families of solutions to problems.

At first Fermat, like Viète, looked to the Greek mathematicians for hints concerning the nature of mathematical analysis. Believing that the so-called "analytical" works cited by Pappus in book VII of the *Mathematical Collection,* most of which were no longer extant,[4] contained the desired clues, Fermat followed Viète and others in seeking to restore those lost texts, such as Apollonius' *Plane Loci* (*Oeuvres,* I, 3–51) and Euclid's *Porisms* (*Oeuvres,* I, 76–84). Another supposed source of insight was Diophantus' *Arithmetica,* to which Fermat devoted a lifetime of study. These ancient sources, together with the works of Archimedes, formed the initial elements in a clear pattern of development that Fermat's research followed. Taking his original problem from the classical sources, Fermat attacked it with the new algebraic techniques at his disposal. His solution, however, usually proved more general than the problem that had inspired it. By skillful application of the theory of equations in the form of a "reduction analysis," Fermat would reformulate the problem in its most general terms, often defining thereby a class of problems; in many cases the new problem structure lost all contact with its Greek forebear.

In Fermat's papers algebra as the "analytic art" achieved equal status with the traditional geometrical mode of ancient mathematics. With few exceptions he presented only the algebraic derivation of his results, dispensing with their classical synthetic proofs. Convinced that the latter could always be provided, Fermat seldom attempted to carry them out, with the result in several cases that he failed to see how the use of algebra had led to the introduction of concepts quite foreign to the classical tradition.

In large part Fermat's style of exposition characterized the unfinished nature of his papers, most of them brief essays or letters to friends. He never wrote for publication. Indeed, adamantly refusing to edit his work or to publish it under his own name, Fermat thwarted several efforts by others to make his results available in print. Showing little interest in completed work, he freely sent papers to friends without keeping copies for himself. Many results he merely entered in the margins of his books; e.g., his "Observations on Diophantus," a major part of his work on number theory, was published by his son on the basis of the marginalia in Fermat's copy of the Bachet edition of the *Arithmetica.* Some other work slipped into print during Fermat's lifetime, although only by virtue of honoring his demand for anonymity. This demand allows no clear or obvious explanation. Fermat knew of his reputation and he valued it. He seemed to enjoy the intellectual combat of the several controversies to which he was a party. Whatever the reason, anonymity and refusal to publish robbed him of recognition for many striking achievements and toward the end of his life led to a growing isolation from the main currents of research.

Fermat's name slipped into relative obscurity during the eighteenth century. In the mid-nineteenth century, however, renewed interest in number theory recalled him and his work to the attention of mathematicians and historians of mathematics. Various projects to publish his extant papers culminated in the four-volume edition by Charles Henry and Paul Tannery, from which the extent and importance of Fermat's achievements in fields other than number theory became clear.

Analytic Geometry. By the time Fermat began corresponding with Mersenne and Roberval in the spring of 1636, he had already composed his "Ad locos planos et solidos isagoge" (*Ouevres,* I, 91–103), in which he set forth a system of analytic geometry almost identical with that developed by Descartes in the *Géométrie* of 1637. Despite their simultaneous appearance (Descartes's in print, Fermat's in circulated manuscript), the two systems stemmed from entirely independent research and the question of priority is both complex and unenlightening. Fermat received the first impetus toward his system from an attempt to reconstruct Apollonius' lost treatise *Plane Loci* (loci that are either straight lines or circles). His completed restoration, although composed in the traditional style of Greek geometry, nevertheless gives clear evidence that Fermat employed algebraic analysis in seeking demonstrations of the theorems listed by Pappus. This application of algebra, combined with the peculiar nature of a geometrical locus and the slightly different proof procedures required by locus demonstrations, appears to have revealed to Fermat that all of the loci discussed by Apollonius could be expressed in the form of indeterminate algebraic equations in two unknowns, and that the analysis of these equations by means of Viète's theory of equations led to crucial insights into the nature and construction of the loci. With this inspiration from the *Plane Loci,* Fermat then found in Apollonius' *Conics* that the *symptomata,* or defining properties, of the conic sections likewise could be expressed as indeterminate equations in two unknowns. Moreover, the standard form in which Apollonius referred the *symptomata* to the cone on which the conic sections were generated suggested to Fermat a standard geometrical framework in which to establish the correspondence between an equation and a curve. Taking a fixed line as axis and a fixed point on that line as origin, he measured the variable length of the first unknown, *A*, from the origin along the axis. The corresponding value of the second unknown, *E*, he constructed as a line length measured from the end point of the first unknown and erected at a fixed angle to the axis. The end points of the various lengths of the second unknown then generated a curve in the *A,E* plane.

Like Descartes, then, Fermat did not employ a coordinate system but, rather, a single axis with a moving ordinate; curves were not plotted, they were generated. Within the standard framework

> Whenever two unknown quantities are found in final equality, there results a locus [fixed] in place, and the end point of one of these unknown quantities describes a straight line or a curve ["Isagoge," *Oeuvres,* I, 91].

The crucial phrase in this keystone of analytic geometry is "fixed in place";[5] it sets the task of the remainder of Fermat's treatise. Dividing the general second-degree equation $Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$ into seven canonical (irreducible) forms according to the possible values of the coefficients, Fermat shows how each canonical equation defines a curve: $Dx = Ey$ (straight line), $Cxy = F$ (equilateral hyperbola), $Ax^2 \pm Cxy = By^2$ (straight lines), $Ax^2 = Ey$ (parabola), $F - Ax^2 = Ay^2$ (circle), $F - Ax^2 = By^2$ (ellipse), and $F + Ax^2 = By^2$ (axial hyperbola). In each case he demonstrates that the constants of the equation uniquely fix the curve defined by it, i.e., that they contain all the data necessary to construct the curve. The proof relies on the construction theorems set forth in Euclid's *Data* (for the straight line and circle, or "plane loci") or Apollonius' *Conics* (for the conic sections, or "solid loci"). In a corollary to each case Fermat employs Viète's theory of equations to establish the family of equations reducible to the canonical form and then shows how the reduction itself corresponds to a translation (or expansion) of the axis or the origin or to a change of angle between axis and ordinate. In the last theorem of the "Isagoge," for example, he reduces the equation $b^2 - 2x^2 = 2xy + y^2$ to the canonical form $2b^2 - u^2 = 2v^2$, where $u = \sqrt{2}x$ and $v = x + y$. Geometrically, the reduction shifts the orthogonal x,y system to a skew u,v system in which the u-axis forms a 45° angle with the x-axis and the v-ordinate is erected at a 45° angle on the u-axis. The curve, as Fermat shows, is a uniquely defined ellipse.

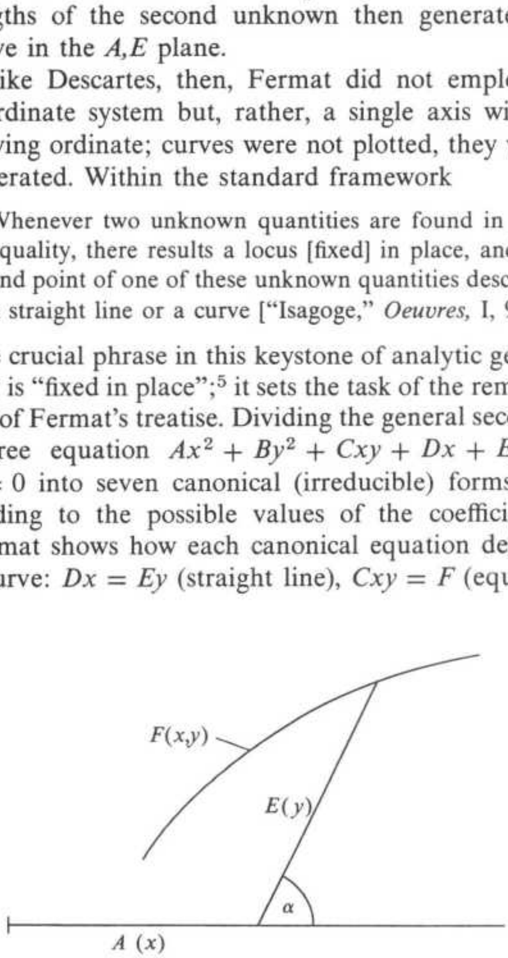

FIGURE 1

Although the analytic geometries of Descartes and Fermat are essentially the same, their presentations differed significantly. Fermat concentrated on the geometrical construction of the curves on the basis of their equations, relying heavily on the reader's knowledge of Viète's algebra to supply the necessary theory of equations. By contrast, Descartes slighted the matter of construction and devoted a major portion of his *Géométrie* to a new and more advanced theory of equations.

In the years following 1636, Fermat made some effort to pursue the implications of his system. In an appendix to the "Isagoge," he applied the system to the graphic solution of determinate algebraic equations, showing, for example, that any cubic or quartic equation could be solved graphically by means of a parabola and a circle. In his "De solutione problematum geometricorum per curvas simplicissimas et unicuique problematum generi proprie convenientes dissertatio tripartita" (*Oeuvres,* I, 118–131), he took issue with Descartes's classification of curves in the *Géométrie* and undertook to show that any determinate algebraic equation of degree $2n$ or $2n - 1$ could be solved graphically by means of curves determined by indeterminate equations of degree n.

In 1643, in a memoir entitled "Isagoge ad locos ad superficiem" (*Oeuvres,* I, 111–117), Fermat attempted to extend his plane analytic geometry to solids of revolution in space and perhaps thereby to restore the content of Euclid's *Surface Loci,* another text cited by Pappus. The effort did not meet with success because he tried to reduce the three-dimensional problem to two dimensions by determining all possible traces resulting from the intersection of a given solid by an arbitrary plane. The system required, first of all, an elaborate catalog of the possible traces for various solids. Second, the manipulation of the equation of any trace for the purpose of deriving the parameters that uniquely determine the solid requires methods that lay beyond Fermat's reach; his technique could at best define the solid qualitatively. Third, the basic system of the 1636 "Isagoge," lacking the concept of coordinates referred to two fixed orthogonal axes, presented substantial hurdles to visualizing a three-dimensional correlate.

Although Fermat never found the geometrical framework for a solid analytic geometry, he nonetheless correctly established the algebraic foundation of such a system. In 1650, in his "Novus secundarum et ulterioris ordinis radicum in analyticis usus" (*Oeuvres,* I, 181–188), he noted that equations in one unknown determine point constructions; equations in two unknowns, locus constructions of plane curves; and equations in three unknowns, locus constructions of surfaces in space. The change in the criterion of the dimension of an equation—from its degree, where the Greeks had placed it, to the number of unknowns in it—was one of the most important conceptual developments of seventeenth-century mathematics.

The Method of Maxima and Minima. The method of maxima and minima, in which Fermat first established what later became the algorithm for obtaining the first derivative of an algebraic polynomial, also stemmed from the application of Viète's algebra to a problem in Pappus' *Mathematical Collection.* In a lemma to Apollonius' *Determinate Section,* Pappus sought to divide a given line in such a way that certain rectangles constructed on the segments bore a minimum ratio to one another,[6] noting that the ratio would be "singular." In carrying out the algebraic analysis of the problem, Fermat recognized that the division of the line for rectangles in a ratio greater than the minimum corresponded to a quadratic equation that would normally yield two equally satisfactory section points. A "singular" section point for the minimum ratio, he argued, must mean that the particular values of the constant quantities of the equation allow only a single repeated root as a solution.

Turning to a simpler example, Fermat considered the problem of dividing a given line in such a way that the product of the segments was maximized. The algebraic form of the problem is $bx - x^2 = c$, where b is the length of the given line and c is the product of the segments. If c is the maximum value of all possible products, then the equation can have only one (repeated) root. Fermat then sought the value for c in terms of b for which the equation yielded that (repeated) root. To this end he applied a method of Viète's theory of equations called "syncrisis," a method originally devised to determine the relationships between the roots of equations and their constant parameters. On the assumption that his equation had two distinct roots, x and y, Fermat set $bx - x^2 = c$ and $by - y^2 = c$, whence he obtained $b = x + y$ and $c = xy$. Taking these relationships to hold generally for any quadratic equation of the above form, he next considered what happened in the case of a repeated root, i.e., when $x = y$. Then, he found, $x = b/2$ and $c = b^2/4$. Hence, the maximum rectangle results from dividing the given line in half, and that maximum rectangle has an area equal to one-quarter of the square erected on the given line b.

Amending his method in the famous "Methodus ad disquirendam maximam et minimam" (*Oeuvres,* I, 133–136), written sometime before 1636, Fermat expressed the supposedly distinct roots as A and $A + E$ (that is, x and $x + y$), where E now represented the difference between the roots. In seeking, for example, the maximum value of the expression $bx^2 - x^3$, he proceeded as follows:

$$bx^2 - x^3 = M^3$$
$$b(x+y)^2 - (x+y)^3 = M^3,$$
whence $$2bxy + by^2 - 3x^2y - 3xy^2 - y^3 = 0.$$

Division by y yields the equation

$$2bx + by - 3x^2 - 3xy - y^2 = 0,$$

which relates the parameter b to two roots of the equation via one of the roots and their difference. The relation holds for any equation of the form $bx^2 - x^3 = M^3$, but when M^3 is a maximum the equation has a repeated root, i.e., $x = x + y$, or $y = 0$. Hence, for that maximum, $2bx - 3x^2 = 0$, or $x = 2b/3$ and $M^3 = 4b^3/27$.

Fermat's method of maxima and minima, which is clearly applicable to any polynomial $P(x)$, originally rested on purely finitistic algebraic foundations.[7] It assumed, counterfactually, the inequality of two equal roots in order to determine, by Viète's theory of equations, a relation between those roots and one of the coefficients of the polynomial, a relation that was fully general. This relation then led to an extreme-value solution when Fermat removed his counterfactual assumption and set the roots equal. Borrowing a term from Diophantus, Fermat called this counterfactual equality "adequality."

Although Pappus' remark concerning the "singularity" of extreme values provided the original inspiration for Fermat's method, it may also have prevented him from seeing all its implications. Oriented toward unique extreme values and dealing with specific problems that, taken from geometrical sources and never exceeding cubic expressions, failed to yield more than one geometrically meaningful solution, Fermat never recognized the distinction between global and local extreme values or the possibility of more than one such value. This block to an overall view of the problem of maxima and minima vitiates an otherwise brilliant demonstration of Fermat's method, which he wrote for Pierre Brûlard de St.-Martin in 1643 (*Oeuvres,* supp., 120–125) and which employs the sophisticated theory of equations of Descartes's *Géométrie.* There Fermat established what today is termed the "second derivative criterion" for the nature of an extreme value ($f''(x) < 0$ for a maximum, $f''(x) > 0$ for a minimum), although his lack of a general overview forestalled investigation of points of inflection ($f''(x) = 0$).

The original method of maxima and minima had two important corollaries. The first was the method of tangents[8] by which, given the equation of a curve, Fermat could construct the tangent at any given point on that curve by determining the length of the subtangent. Given some curve $y = f(x)$ and a point (a,b) on it, Fermat assumed the tangent to be drawn and

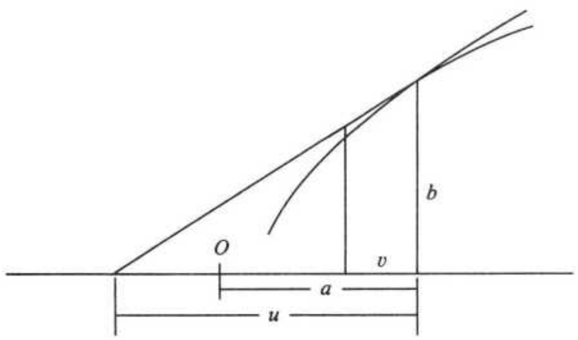

FIGURE 2

to cut off a subtangent of length u on the x-axis. Taking an arbitrary point on the tangent and denoting the difference between the abscissa of that point and the abscissa a by v, he counterfactually assumed that the ordinate to the point on the tangent was equal to the ordinate $f(a - v)$ to the curve, i.e., that the two ordinates were "adequal." It followed, then, from similar triangles that

$$\frac{b}{u} \approx \frac{f(a-v)}{u-v}.$$

Fermat removed the adequality, here denoted by $\approx$, by treating the difference v in the same manner as in the method of maxima and minima, i.e., by considering it as ultimately equal to zero. His method yields, in modern symbols, the correct result, $u = f(a)/f'(a)$, and, like the parent method of maxima and minima, it can be applied generally.

From the method of maxima and minima Fermat drew as a second corollary a method for determining centers of gravity of geometrical figures (*Oeuvres,* I, 136–139). His single example—although again the method itself is fully general—concerns the center of gravity of a paraboloidal segment. Let CAV be the generating parabola with axis AI and base CV. By symmetry the center of gravity O of the paraboloidal segment lies on axis $AI = b$ at some distance $AO = x$

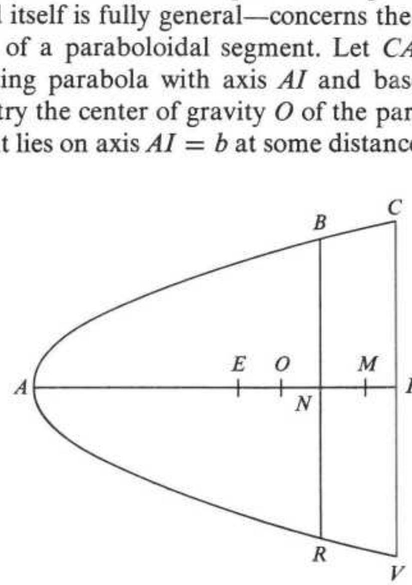

FIGURE 3

from the vertex A. Let the segment be cut by a plane parallel to the base and intersecting the axis at an arbitrary distance y from point I. Let E and M denote the centers of gravity of the two resulting subsegments. Since similar figures have similarly placed centers of gravity (Archimedes), $b/x = (b - y)/AE$, whence $EO = x - AE = xy/b$. By the definition of the center of gravity and by the law of the lever, segment $CBRV$ is to segment BAR as EO is to OM. But, by Archimedes' *Conoids and Spheroids,* proposition 26, paraboloid CAV is to paraboloid BAR as AI^2 is to AN^2, or as b^2 is to $(b - y)^2$, whence

$$\frac{EO}{OM} = \frac{CBRV}{BAR} = \frac{b^2 - (b-y)^2}{(b-y)^2}.$$

Here Fermat again employed the notion of adequality to set OM counterfactually equal to OI, whence

$$OI = b - x \approx OM = \left(\frac{xy}{b}\right)\left(\frac{b^2 - 2by + y^2}{2by - y^2}\right).$$

He removed the adequality by an application of the method of maxima and minima, i.e., by dividing through by y and then setting $y\ (= OI - OM)$ equal to zero, and obtained the result $x = 2b/3$. In applying his method to figures generated by curves of the forms $y^q = kx^p$ and $x^p y^q = k$ (p,q positive integers), Fermat employed the additional lemma that the similar segments of the figures "have the same proportion to corresponding triangles of the same base and height, even if we do not know what that proportion is,"[9] and argued from that lemma that his method of centers of gravity eliminated the problem of quadrature as a prerequisite to the determination of centers of gravity. Such an elimination was, of course, illusory, but the method did not depend on the lemma. It can be applied to any figure for which the general quadrature is known.

Fermat's method of maxima and minima and its corollary method of tangents formed the central issue in an acrid debate between Fermat and Descartes in the spring of 1638. Viewing Fermat's methods as rivals to his own in the *Géométrie,* Descartes tried to show that the former were at once paralogistic in their reasoning and limited in their application. It quickly became clear, however, that, as in the case of their analytic geometries, Fermat's and Descartes's methods rested on the same foundations. The only substantial issue was Descartes's disapproval of mathematical reasoning based on counterfactual assumptions, i.e., the notion of adequality. Although the two men made formal peace in the summer of 1638, when Descartes admitted his error in criticizing Fermat's methods, the bitterness of the dispute, exacerbated by the deep personal hatred Descartes felt for Fermat's friend and spokesman, Roberval, poisoned any chance for cooperation between the two greatest mathematicians of the time. Descartes's sharp tongue cast a pall over Fermat's reputation as a mathematician, a situation which Fermat's refusal to publish only made worse.[10] Through the efforts of Mersenne and Pierre Hérigone, Fermat's methods did appear in print in 1642, but only as bare algorithms that, by setting the difference y of the roots equal to zero from the start, belied the careful thinking that originally underlay them. Moreover, other mathematicians soon were publishing their own, more general algorithms; by 1659, Huygens felt it necessary to defend Fermat's priority against the claims of Johann Hudde. In time, Fermat's work on maxima and minima was all but forgotten, having been replaced by the differential calculus of Newton and Leibniz.

Methods of Quadrature. Fermat's research into the quadrature of curves and the cubature of solids also had its beginnings in the research that preceded his introduction to the outside mathematical world in 1636. By that time, he had taken the model of Archimedes' quadrature of the spiral[11] and successfully extended its application to all spirals of the forms $\rho = (a\vartheta)^m$ and $R/R - \rho = (\alpha/\vartheta)^m$. Moreover, he had translated Archimedes' method of circumscription and inscription of sectors around and within the spiral into a rectangular framework. Dividing a given ordinate y_0 (or the corresponding abscissa x_0) of a curve $y = f(x)$ into N equal intervals and drawing lines parallel to the axis, Fermat determined that Area I

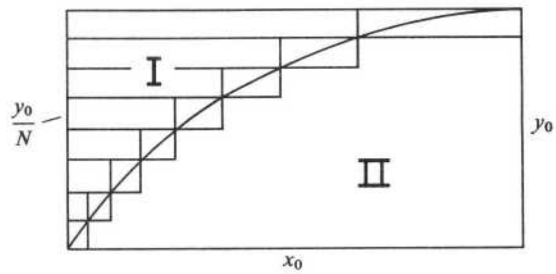

FIGURE 4

in Figure 4 lay between limits $\frac{y_0}{N}\sum_{i=1}^{N} x_i$ and $\frac{y_0}{N}\sum_{i=1}^{N-1} x_i$, where x_i is the abscissa that corresponds to ordinate $(i/N)y_0$. Since he possessed a recursive formula for determining $\sum_{=1}^{N} i^m$ for any positive integer m, Fermat could prove that

$$\frac{1}{N^{m+1}}\sum_{i=1}^{N} i^m > \frac{1}{m+1} > \frac{1}{N^{m+1}}\sum_{i=1}^{N-1} i^m$$

for all values of N. In each case the difference between the bounds is $1/N$, which can be made as small as one wishes. Hence, for any curve of the form $y^m = kx$, Fermat could show that the curvilinear Area I $= [1/(m+1)]x_0y_0$ and the curvilinear Area II $= [m/(m+1)]x_0y_0$. As an immediate corollary, he found that he could apply the same technique to determine the volume of the solid generated by the rotation of the curve about the ordinate or axis, with the restriction in this case that m be an even integer.

Sometime before 1646 Fermat devised a substantially new method of quadrature, which permitted the treatment of all curves of the forms $y^q = kx^p$ and $x^p y^q = k$ (p,q positive integers; in the second equation $p + q > 2$). The most striking departure from the earlier method is the introduction of the concept of adequality, now used in the sense of "approximate equality" or "equality in the limiting case." In the first example given in his major treatise on quadrature[12] Fermat derives the shaded area under the curve $x^2y = k$ in Figure 5 as follows (we use modern

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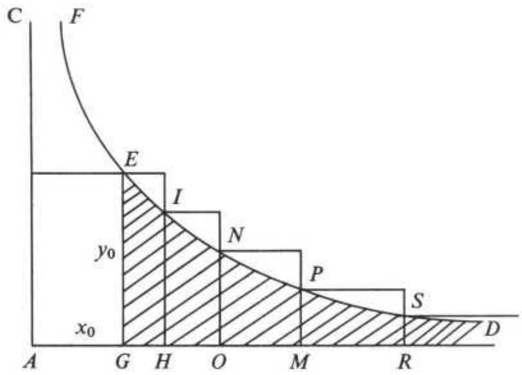

FIGURE 5

notation to abbreviate Fermat's lengthy verbal description while preserving its sense): the infinite x-axis is divided into intervals by the end points of a divergent geometric sequence of lengths $AG, AH, AO, \cdots$, or $x_0, (m/n)x_0, (m/n)^2x_0, \cdots$, where $m > n$ are arbitrary integers. Since $(m/n)^i - (m/n)^{i-1} = (m/n)^{i-1}(m/n - 1)$, each interval can, by suitable choice of m and n, be made as closely equal to another as desired and at the same time can be made as small as desired. Fermat has, then, $GH \approx HO \approx OM \approx \cdots$ and $GH \rightarrow 0$. From the curve and the construction of the intervals, it follows directly that the approximating rectangles erected on the intervals form the convergent geometric series

$$(m/n - 1)x_0y_0, (n/m)(m/n - 1)x_0y_0, (n/m)^2(m/n - 1)x_0y_0, \cdots .$$

Its sum is $(m/n - 1)x_0y_0 + x_0y_0$, which is "adequal" to the shaded area. It approaches the curved area ever more closely as the size of the intervals approaches zero, i.e., as $m/n \rightarrow 1$. In the limiting case, the sum will be x_0y_0, which in turn will be the exact area of the shaded segment. Generalizing the procedure for any curve $x^py^q = k$ and a given ordinate y_0, Fermat determined that the area under the curve from y_0 on out is $[q/(p - q)]x_0y_0$. Adapting the procedure to curves $y^q = kx^p$ (by dividing the finite axis from 0 to x_0 by a convergent geometric sequence of intervals), he was also able to show that the area under the curve is $[q/(p + q)]x_0y_0$.

In the remainder of his treatise on quadrature, Fermat shifted from the geometrical style of exposition to the algebraic and, on the model of Viète's theory of equations, set up a "reduction analysis" by which a given quadrature either generates an infinite class of quadratures or can be shown to be dependent on the quadrature of the circle. To carry out the project he introduced a new concept of "application of all y^n to a given segment," by which he meant the limit-sum of the products $y^n\triangle x$ over a given segment b of the x-axis as $\triangle x \rightarrow 0$ (in the absence of any notation by Fermat, we shall borrow from Leibniz and write $Omn_b y^n$ to symbolize Fermat's concept). Fermat then showed by several concrete examples that for any curve of the form $y^n = \Sigma a_i x^i/b_j x^j$ the determination of $Omn_b y^n$ follows directly from setting $y^n = \Sigma u_i$, where $u_i = a_i x^i/b_j x^j$. For each i the resulting expression $u_i = f(x)$ will denote a curve of the form $u_i^q = kx^p$ or the form $x^pu_i^q = k$. For each curve, the determination of $Omn_b u_i$ corresponds to the direct quadrature set forth in the first part of the treatise, and hence it is determinable. Therefore $Omn_b y^n = \Sigma Omn_b u_i$ is directly determinable.

Fermat next introduced the main lemma of his treatise, an entirely novel result for which he characteristically offered no proof. For any curve $y = f(x)$ decreasing monotonically over the interval $0,b$, where $f(0) = d$ and $f(b) = 0$, $Omn_b y^n = Omn_d nxy^{n-1}$. This result is equivalent to the modern statement

$$\int_0^b y^n\,dx = n\int_0^d xy^{n-1}\,dy.$$

One example from Fermat's treatise on quadrature suffices to display the subtlety and power of his reduction analysis. Can the area beneath the curve $b^3 = x^2y + b^2y$ (i.e., the "witch of Agnesi") be squared algebraically? Two transformations of variable and an application of the main lemma supply the answer. From $by = u^2$ and $bv = xu$, it follows first that $Omn_x y = 1/b\ Omn_x u^2 = 2/b\ Omn_u xu = 2\ Omn_u v$. Hence, the quadrature of the original area depends on that of the transformed curve $F(u,v)$. But substitution of variables yields $b^2 = u^2 + v^2$, the equation of a circle. Therefore, the quadrature of the original area depends on the quadrature of the circle and cannot be carried out algebraically.

Fermat's treatise first circulated when it was printed in his *Varia opera* of 1679. By then much of its contents had become obsolescent in terms of the work of Newton and Leibniz. Even so, it is doubtful what effect the treatise could have had earlier. As sympathetic a reader as Huygens could make little sense of it.[13] In addition, Fermat's method of quadrature, like his method of tangents, lacks even the germ of several concepts crucial to the development of the calculus. Not only did Fermat not recognize the inverse relationship between the two methods, but both methods, conceptually and to some extent operationally, steered away from rather than toward the notion of the tangent or the area as a function of the curve.

Fermat's one work published in his lifetime, a treatise on rectification appended to a work on the cycloid by Antoine de La Loubère,[14] was a direct corollary of the method of quadrature. Cast, however, in the strictly geometrical style of classical Greek mathematics, it hid all traces of the underlying algebraic analysis. In the treatise Fermat treated the length of a curve as the limit-sum of tangential segments $\triangle S$ cut off by abscissas drawn through the end points of intervals $\triangle y$ on a given y ordinate. In essence, he showed that for any curve $y = y(x)$,

$$\frac{\triangle S^2}{\triangle y^2} = [x'(y)]^2 + 1.$$

Taking $u^2 = [x'(y)]^2 + 1$ as an auxiliary curve, Fermat used the relation $S = Omn_y u$ to reduce the problem of rectification to one of quadrature. He used the same basic procedure to determine the area of the surface generated by the rotation of the curve about an axis or an ordinate, as the results in a 1660 letter to Huygens indicate.

Number Theory. As a result of limited circulation in unpublished manuscripts, Fermat's work on analytic geometry, maxima and minima and tangents, and quadrature had only moderate influence on contemporary developments in mathematics. His work in the realm of number theory had almost none at all. It was neither understood nor appreciated until Euler revived it and initiated the line of continuous research that culminated in the work of Gauss and Kummer in the early nineteenth century. Indeed, many of Fermat's results are basic elements of number theory today. Although the results retain fundamental importance, his methods remain largely a secret known only to him. Theorems, conjectures, and specific examples abound in his letters and marginalia. But, except for a vague outline of a method he called "infinite descent," Fermat left no obvious trace of the means he had employed to find them. He repeatedly claimed to work from a method, and the systematic nature of much of his work would seem to support his claim.

In an important sense Fermat invented number theory as an independent branch of mathematics. He was the first to restrict his study in principle to the domain of integers. His refusal to accept fractional solutions to problems he set in 1657 as challenges to the European mathematics community (*Oeuvres,* II, 332–335) initiated his dispute with Wallis, Frénicle, and others,[15] for it represented a break with the classical tradition of Diophantus' *Arithmetica,* which served as his opponents' model. The restriction to integers explains one dominant theme of Fermat's work in number theory, his concern with prime numbers and divisibility. A second guiding theme of his research, the determination of patterns for generating families of solutions from a single basic solution, carried over from his work in analysis.

Fermat's earliest research, begun in Bordeaux, displays both characteristics. Investigating the sums of the aliquot parts (proper divisors) of numbers, Fermat worked from Euclid's solution to the problem of "perfect numbers"—$\sigma(a) = 2a$, where $\sigma(a)$ denotes the sum of all divisors of the integer a, including 1 and a—to derive a complete solution to the problem of "friendly" numbers—$\sigma(a) = \sigma(b) = a + b$—and to the problem $\sigma(a) = 3a$. Later research in this area aimed at the general problem $\sigma(a) = (p/q)a$, as well as $\sigma(x^3) = y^2$ and $\sigma(x^2) = y^3$ (the "First Challenge" of 1657). Although Fermat offered specific solutions to the problem $\sigma(a) = na$ for $n = 3, 4, 5, 6$, he recorded the algorithm only for $n = 3$. The central role of primeness and divisibility in such research led to several corollaries, among them the theorem (announced in 1640) that $2^k - 1$ is always a composite number if k is composite and may be composite for prime k; in the latter case, all divisors are of the form $2mk + 1$.

Fermat's interest in primeness and divisibility culminated in a theorem now basic to the theory of congruences; as set down by Fermat it read: If p is prime and a^t is the smallest number such that $a^t = kp + 1$ for some k, then t divides $p - 1$. In the modern version, if p is prime and p does not divide a, then $a^{p-1} \equiv 1 \pmod{p}$. As a corollary to this theorem, Fermat investigated in depth the divisibility of $a^k \pm 1$ and made his famous conjecture that all numbers of the form $2^{2^n} + 1$ are prime (disproved for $n = 5$ by Euler). In carrying out his research, Fermat apparently relied on an extensive factual command of the powers of prime numbers and on the traditional "sieve of Eratosthenes" as a test of primeness. He several times expressed his dissatisfaction with the latter but seems to have been unable to find a more efficient test, even though in retrospect his work contained all the necessary elements for one.

A large group of results of fundamental importance to later number theory (quadratic residues, quadratic forms) apparently stemmed from Fermat's study of the indeterminate equation $x^2 - q = my^2$ for nonsquare m. In his "Second Challenge" of 1657, Fermat claimed to have the complete solution for the case $q = 1$. Operating on the principle that any divisor of a number of the form $a^2 + mb^2$ (m not a square) must itself be of that form, Fermat established that all primes of the form $4k + 1$ (but not those of the form $4k + 3$) can be expressed as the sum of two squares, all primes of the form $8k + 1$ or $8k + 3$ as the sum of a square and the double of a square, all primes of the form $3k + 1$ as $a^2 + 3b^2$, and that the product of any two primes of the form $20k + 3$ or $20k + 7$ is expressible in the form $a^2 + 5b^2$.

Another by-product of this research was Fermat's claim to be able to prove Diophantus' conjecture that any number can be expressed as the sum of at most four squares. Extending his research on the decomposition of numbers to higher powers, Fermat further claimed proofs of the theorems that no cube could be expressed as the sum of two cubes, no quartic as the sum of two quartics, and indeed no number a^n as the sum of two powers b^n and c^n (the famous "last theorem," mentioned only once in the margin of his copy of Diophantus' *Arithmetica*). In addition, he claimed the complete solution of the so-called "four-cube problem" (to express the sum of two given cubes as the sum of two other cubes), allowing here, of course, fractional solutions of the problem.

To prove his decomposition theorems and to solve the equation $x^2 - 1 = my^2$, Fermat employed a method he had devised and called "infinite descent." The method, an inverse form of the modern method of induction, rests on the principle (peculiar to the domain of integers) that there cannot exist an infinitely decreasing sequence of integers. Fermat set down two rather vague outlines of his method, one in his "Observations sur Diophante" (*Oeuvres,* I, 340–341) and one in a letter to Carcavi (*Oeuvres,* II, 431–433). In the latter Fermat argued that no right triangle of numbers (triple of numbers a, b, c such that $a^2 + b^2 = c^2$) can have an area equal to a square ($ab/2 = m^2$ for some m), since

> If there were some right triangle of integers that had an area equal to a square, there would be another triangle less than it which had the same property. If there were a second, less than the first, which had the same property, there would be by similar reasoning a third less than the second which had the same property, and then a fourth, a fifth, etc., ad infinitum in decreasing order. But, given a number, there cannot be infinitely many others in decreasing order less than it (I mean to speak always of integers). From which one concludes that it is therefore impossible that any right triangle of numbers have an area that is a square [letter to Carcavi, *Oeuvres,* II, 431–432].

Fermat's method of infinite descent did not apply only to negative propositions. He discovered that he could also show that every prime of the form $4k + 1$ could be expressed as the sum of two squares by denying the proposition for some such prime, deriving another such prime less than the first, for which the proposition would again not hold, and so on. Ultimately, he argued, this decreasing sequence of primes would arrive at the least prime of the form $4k + 1$—namely, 5—for which, by assumption, the proposition would not hold. But $5 = 2^2 + 1^2$, which contradicts the initial assumption. Hence, the proposition must hold. Although infinite descent is unassailable in its overall reasoning, its use requires the genius of a Fermat, since nothing in that reasoning dictates how one derives the next member of the decreasing sequence for a given problem.

Fermat's letters to Jacques de Billy, published by the latter as *Doctrinae analyticae inventum novum,*[16] form the only other source of direct information about Fermat's methods in number theory. In these letters Fermat undertook a complete treatment of the so-called double equations first studied by Diophantus. In their simplest form they required the complete solution of the system $ax + b = \square$, $cx + d = \square$. By skillful use of factorization to determine the base solution and the theorem that, if a is a solution, then successive substitution of $x + a$ for x generates an infinite family of solutions, Fermat not only solved all the problems posed by Diophantus but also extended them as far as polynomials of the fourth degree.

The importance of Fermat's work in the theory of numbers lay less in any contribution to contemporary developments in mathematics than in their stimulative influence on later generations. Much of the number theory of the nineteenth century took its impetus from Fermat's results and, forced to devise its own methods, contributed to the formulation of concepts basic to modern algebra.

Other Work. *Probability.* Fermat shares credit with Blaise Pascal for laying the first foundations of the theory of probability. In a brief exchange of correspondence during the summer of 1654, the two men discussed their different approaches to the same solution of a problem originally posed to Pascal by a gambler: How should the stakes in a game of chance be divided among the players if the game is prematurely ended? In arriving at specific, detailed solutions for several simple games, Fermat and Pascal operated from the basic principle of evaluating the expectation of each player as the ratio of outcomes favorable to him to the total number of possible outcomes. Fermat relied on direct computations rather than general mathematical formulas in his solutions, and his results and methods quickly became obsolete with the appearance in 1657 of Christiaan Huygens' mathematically more sophisticated *De ludo aleae.*

Optics (Fermat's Principle). In 1637, when Fermat was engaged with traditional and rather pedestrian problems in geostatics, he read Descartes's *Dioptrique.* In a letter to Mersenne, which opened the controversy between Descartes and Fermat mentioned above, Fermat severely criticized the work. Methodologically, he could not accept Descartes's use of mathematics to make a priori deductions about the physical world. Philosophically, he could not agree with Descartes that "tendency to motion" (Descartes's basic definition of light) could be understood and analyzed in terms of actual motion. Physically, he doubted both the assertion that light traveled more quickly in a denser medium (he especially questioned the meaning of such a statement together with the assertion of the instantaneous transmission of light) and Descartes's law of refraction itself. Mathematically, he tried to show that Descartes's demonstrations of the laws of reflection and refraction proved nothing that Descartes had not already assumed in his analysis, i.e., that Descartes had begged the question. The ensuing debate in the fall of 1637 soon moved to mathematics as Descartes launched a counterattack aimed at Fermat's method of tangents, and Fermat returned to the original subject of optics only in the late 1650's, when Claude Clerselier reopened the old argument while preparing his edition of Descartes's *Lettres.*

Fermat, who in his earlier years had fervently insisted that experiment alone held the key to knowledge of the physical world, nonetheless in 1662 undertook a mathematical derivation of the law of refraction on the basis of two postulates: first, that the finite speed of light varied as the rarity of the medium through which it passed and, second, that "nature operates by the simplest and most expeditious ways and means." In his "Analysis ad refractiones" (*Oeuvres,* I, 170–172), Fermat applied the second postulate (Fermat's principle) in the following manner: In Figure 6 let the upper half of the circle represent the rarer of two media and let the lower half represent the denser; further, let *CD* represent a given incident ray. If the "ratio of the resistance of the denser medium to the resistance of the rarer medium" is expressed as the ratio of the given line *DF* to some line *M*, then "the motions which occur along lines *CD* and *DI* [the refracted ray to be determined] can be

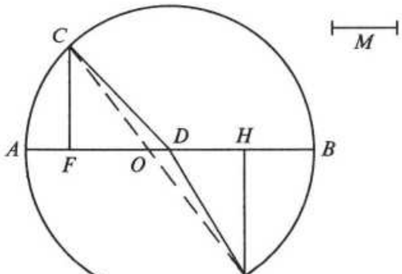

FIGURE 6

measured with the aid of the lines *DF* and *M;* that is, the motion that occurs along the two lines is represented comparatively by the sum of two rectangles, of which one is the product of *CD* and *M* and the other the product of *DI* and *DF*" ("Analysis ad refractiones," pp. 170–171). Fermat thus reduces the problem to one of determining point *H* such that that sum is minimized. Taking length *DH* as the unknown *x*, he applies his method of maxima and minima and, somewhat to his surprise (expressed in a letter to Clerselier), arrives at Descartes's law of refraction.

Although Fermat took the trouble to confirm his derived result by a formal, synthetic proof, his interest in the problem itself ended with his derivation. Physical problems had never really engaged him, and he had returned to the matter only to settle an issue that gave rise to continued ill feeling between him and the followers of Descartes.

In fact, by 1662 Fermat had effectively ended his career as a mathematician. His almost exclusive interest in number theory during the last fifteen years of his life found no echo among his junior contemporaries, among them Huygens, who were engaged in the application of analysis to physics. As a result Fermat increasingly returned to the isolation from which he had so suddenly emerged in 1636, and his death in 1665 was viewed more as the passing of a grand old man than as a loss to the active scientific community.

NOTES

1. All published modern accounts of Fermat's life ultimately derive from Paul Tannery's article in the *Grande encyclopédie,* repr. in *Oeuvres,* IV, 237–240. Some important new details emerged from the research of H. Blanquière and M. Caillet in connection with an exhibition at the Lycée Pierre de Fermat in Toulouse in 1957: *Un mathématicien de génie, Pierre de Fermat 1601–1665* (Toulouse, 1957).
2. *Diophanti Alexandrini Arithmeticorum libri sex et de numeris multangulis liber unus. Cum commentariis C. G. Bacheti V. C. et observationibus D. P. de Fermat Senatoris Tolosani* (Toulouse, 1670); *Varia opera mathematica D. Petri de Fermat Senatoris Tolosani* (Toulouse, 1679; repr. Berlin, 1861; Brussels, 1969).
3. Cureau shared Fermat's scientific interests and hence provided a special link to the chancellor. There is much to suggest that the *parlement* of Toulouse took advantage of Fermat's ties to Cureau.
4. Regarding book VII and its importance for Greek geometrical analysis, see M. S. Mahoney, "Another Look at Greek Geometrical Analysis," in *Archive for History of Exact Sciences,* **5** (1968), 318–348. On its influence in the early seventeenth century, see Mahoney, "The Royal Road" (diss., Princeton, 1967), ch. 3.
5. Fermat's original Latin reads: *fit locus loco.* The last word is not redundant, as several authors have thought; rather, the phrase is elliptic, lacking the word *datus.* Fermat's terminology here comes directly from Euclid's *Data* (*linea positione data:* a line given, or fixed, in position).

Regarding the algebraic symbolism that follows here and throughout the article, note that throughout his life Fermat employed the notation of Viète, which used the capital vowels for unknowns and the capital consonants for knowns or parameters. To avoid the confusion of an unfamiliar notation, this article employs Cartesian notation, translating Fermat's *A* uniformly as *x*, *E* as *y*, etc.

6. Pappus, *Mathematical Collection* VII, prop. 61. The geometrical formulation is too complex to state here without a figure and in addition requires some interpretation. In Fermat's algebraic formulation, the problem calls for the determination of the minimum value of the expression

$$\frac{bc - bx + cx - x^2}{ax - x^2},$$

where *a*, *b*, *c* are given line segments.
7. The modern foundation of Fermat's method is the theorem that if $P(x)$ has a local extreme value at $x = a$, then $P(x) = (x - a)^2R(x)$, where $R(a) \neq 0$.
8. Fermat's original version of the method is contained in the "Methodus ad disquirendam maximam et minimam" (*Oeuvres*, I, 133-136); in its most finished form it is described in a memoir sent to Descartes in June 1638 (*Oeuvres*, II, 154-162).
9. Fermat to Mersenne, 15 June 1638 (*Oeuvres*, supp., pp. 84-86).
10. Descartes's most famous remark, made to Frans van Schooten, who related it to Huygens (*Oeuvres*, IV, 122), was the following: "Monsieur Fermat est Gascon, moi non. Il est vrai, qu'il a inventé plusieurs belles choses particulières, et qu'il est homme de grand esprit. Mais quant à moi j'ai toujours étudié à considerer les choses fort généralement, afin d'en pouvoir concluire des règles, qui aient aussi ailleurs de l'usage." The connotation of "troublemaker" implicit in the term "Gascon" is secondary to Descartes's charge, believed by some of his followers, that Fermat owed his reputation to a few unsystematic lucky guesses.
11. In his treatise *On Spirals*.
12. "De aequationum localium transmutatione et emendatione ad multimodam curvilineorum inter se vel cum rectilineis comparationem, cui annectitur proportionis geometricae in quadrandis infinitis parabolis et hyperbolis usus" (*Oeuvres*, I, 255-285). The treatise was written sometime between 1657 and 1659, but at least part of it dates back to the early 1640's.
13. Huygens to Leibniz, 1 September 1691 (*Oeuvres*, IV, 137).
14. "De linearum curvarum cum lineis rectis comparatione dissertatio geometrica. Autore M.P.E.A.S." The treatise was published with La Loubère's *Veterum geometria promota in septem de cycloide libris, et in duabus adjectis appendicibus* (Toulouse, 1660).
15. The dispute is recorded in Wallis' *Commercium epistolicum de quaestionibus quibusdam mathematicis nuper habitum* (Oxford, 1658). The participants were William Brouncker, Kenelm Digby, Fermat, Bernard Frénicle, Wallis, and Frans van Schooten.
16. Published as part of Samuel Fermat's edition of Diophantus in 1670 (see note 2).

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Michael S. Mahoney

FERMI, ENRICO (*b.* Rome, Italy, 29 September 1901; *d.* Chicago, Illinois, 28 November 1954), *physics.*

His father, Alberto Fermi, was an administrative employee of the Italian railroads; his mother, Ida de Gattis, was a schoolteacher. Fermi received a traditional education in the public schools of Rome, but his scientific formation was due more to the books he read than to personal contacts. It is possible to gather exact information on his readings from an extant notebook, and later in life he mentioned having studied such works as Poisson's *Traité de mécanique*, Richardson's *Electron Theory of Matter*, Planck's *Vorlesungen über Thermodynamik*, and several by Poincaré.

Fermi was fundamentally an agnostic, although he had been baptized a Catholic. In 1928 he married Laura Capon, the daughter of an admiral in the Italian navy. His wife's family was Jewish and was severely persecuted during the Nazi-Fascist period. Fermi, who enjoyed excellent health until his fatal illness, led a very simple, frugal life with outdoor activities as his main recreations. His unusual physical strength and endurance enabled him to hike, play tennis, ski, and swim; although in none of these sports was he outstanding.

Fermi was a member of a great many academies and scientific societies, including the Accademia dei Lincei, the U.S. National Academy of Sciences, and the Royal Society of London. He received the Nobel Prize in 1938 and the Fermi Prize, named for him,