$$\text{where} \quad g(x,y) = \frac{-D_f\begin{pmatrix} x & \xi_1, & \xi_2, & \cdots, & \xi_n \\ y & \eta_1, & \eta_2, & \cdots, & \eta_n \end{pmatrix}}{D_f\begin{pmatrix} \xi_1, & \xi_2, & \cdots, & \xi_n \\ \eta_1, & \eta_2, & \cdots, & \eta_n \end{pmatrix}}$$

and $\{a_i : i = 1,2, \cdots, n\}$ is a set of arbitrary constants.

Thus, Fredholm proved that the analogy between the matrix equation $(I + F)U = V$ and equation (1) was complete and even included an alternative theorem for the integral equation. Yet he showed more. His result meant that the solution $\phi(x)$ for equation (1) could be developed in a power series in the complex variable λ

$$\phi(x) = \psi(x) + \sum_{p=1}^{\infty} \psi_p(x)\lambda^p$$

which is a meromorphic function of λ for every λ satisfying $D_{\lambda f} \neq 0$. (To see this, replace $f(x,y)$ with $\lambda f(x,y)$ in equation [1].) This result was so important that, unable to prove it, Henri Poincaré was forced to assume it in 1895–1896 in connection with his studies of the partial differential equation $\Delta u + \lambda u = h(x,y)$.

Fredholm's work did not represent a dead end. His colleague Erik Holmgren carried Fredholm's discovery to Göttingen in 1901. There David Hilbert was inspired to take up the study; he extended Fredholm's results to include a complete eigenvalue theory for equation (1). In the process he used techniques that led to the discovery of Hilbert spaces.

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M. BERNKOPF

FREGE, FRIEDRICH LUDWIG GOTTLOB (*b.* Wismar, Germany, 8 November 1848; *d.* Bad Kleinen, Germany, 26 July 1925), *logic, foundations of mathematics.*

Gottlob Frege was a son of Alexander Frege, principal of a girl's high school, and of Auguste Biallo-blotzky. He attended the Gymnasium in Wismar, and from 1869 to 1871 he was a student at Jena. He then went to Göttingen and took courses in mathematics, physics, chemistry, and philosophy for five semesters. In 1873 Frege received his doctorate in philosophy at Göttingen with the thesis, *Ueber eine geometrische Darstellung der imäginaren Gebilde in der Ebene.* The following year at Jena he obtained the *venia docendi* in the Faculty of Philosophy with a dissertation entitled "Rechungsmethoden, die sich auf eine Erweitung des Grössenbegriffes gründen," which concerns one-parameter groups of functions and was motivated by his intention to give such a definition of quantity as gives maximal extension to the applicability of the arithmetic based upon it. The idea presented in the dissertation of viewing the system of an operation *f* and its iterates as a system of quantities, which in the introduction to his *Grundlagen der Arithmetik* (1884) Frege essentially ascribes to Herbart, hints at the notion of *f*-sequence expounded in his *Begriffsschrift* (1879).

After the publication of the *Begriffsschrift,* Frege was appointed extraordinary professor at Jena in 1879 and honorary professor in 1896. His stubborn work toward his goal—the logical foundation of arithmetic—resulted in his two-volume *Grundgesetze der Arithmetik* (1893–1903). Shortly before publication of the second volume Bertrand Russell pointed out in 1902, in a letter to Frege, that his system involved a contradiction. This observation by Russell destroyed Frege's theory of arithmetic, and he saw no way out. Frege's scientific activity in the period after 1903 cannot be compared with that before 1903 and was mainly in reaction to the new developments in mathematics and its foundations, especially to Hilbert's axiomatics. In 1917 he retired. His *Logische Untersuchungen,* written in the period 1918–1923, is an extension of his earlier work.

In his attempt to give a satisfactory definition of number and a rigorous foundation to arithmetic, Frege found ordinary language insufficient. To overcome the difficulties involved, he devised his *Begriffsschrift* as a tool for analyzing and representing mathematical proofs completely and adequately. This tool has gradually developed into modern mathematical logic, of which Frege may justly be considered the creator.

The *Begriffsschrift* was intended to be a formula language for pure thought, written with specific symbols and modeled upon that of arithmetic (i.e., it develops according to definite rules). This is an essential difference between Frege's calculus and, for example, Boole's or Peano's, which do not formalize mathematical proofs but are more flexible in expressing the logical structure of concepts.

One of Frege's special symbols is the assertion sign $\vdash$ (properly only the vertical stroke), which is interpreted if followed by a symbol with judgeable content. The interpretation of $\vdash\!\!-\!\!-\, A$ is "A is a fact." Another symbol is the conditional $\begin{array}{l}\top\!\!- \\ \llcorner\!\!- \end{array}$, and $\begin{array}{l}\top\!\!- A \\ \llcorner\!\!- B\end{array}$ is to be read as "B implies A." The assertion $\vdash\!\!\begin{array}{l}\top\!\!- A \\ \llcorner\!\!- B\end{array}$ is justified in the following cases: (1) A and B are true; (2) A is true and B is false; (3) A and B are false.

Frege uses only one deduction rule, which consists in passing from $\vdash\!\!-\!\!-\, B$ and $\vdash\!\!\begin{array}{l}\top\!\!- A \\ \llcorner\!\!- B\end{array}$ to $\vdash\!\!-\!\!-\, A$. The assertion that A is not a fact is expressed by $\vdash\!\!\!\top\!\!\!-\, A$, i.e., the small vertical stroke is used for negation. Frege showed that the other propositional connectives, "and" and "or," are expressible by means of negation and implication, and in fact developed propositional logic on the basis of a few axioms, some of which have been preserved in modern presentations of logic. Yet he did not stop at propositional logic but also developed quantification theory, which was possible because of his general notion of function. If in an expression a symbol was considered to be replaceable, in all or in some of its occurrences, then Frege calls the invariant part of the expression a function and the replaceable part its argument. He chose the expression $\Phi(A)$ for a function and, for functions of more than one argument, $\Psi(A,B)$. Since the Φ in $\Phi(A)$ also may be considered to be the replaceable part, $\Phi(A)$ may be viewed as a function of the argument Φ. This stipulation proved to be the weak point in Frege's system, as Russell showed in 1902.

Generality was expressed by

$$\vdash\!\!\smile^{\mathfrak{a}}\!\!-\, \Phi(\mathfrak{a})$$

which means that $\Phi(a)$ is a fact, whatever may be chosen for the argument. Frege explains the notion of the scope of a quantifier and notes the allowable transition from $\vdash\!\!-\!\!-\, X(a)$ to $\vdash\!\!\smile^{\mathfrak{a}}\!\!-\, X(\mathfrak{a})$, where a occurs only as argument of $X(a)$, and from $\vdash\!\!\begin{array}{l}\top\!\!- \Phi(a) \\ \llcorner\!\!- A\end{array}$ to $\vdash\!\!\begin{array}{l}\top\!\!\smile^{\mathfrak{a}}\!\!- \Phi(\mathfrak{a}) \\ \llcorner\!\!-\!\!-\!\!-\!\!- A\end{array}$, where a does not occur in A and in $\Phi(a)$ occurs only in the argument places.

Existence was expressed by $\vdash\!\!\!\top\!\!\smile^{\mathfrak{a}}\!\!\top\!\!-\, \Lambda(\mathfrak{a})$. There was no explicitly stated rule of substitution.

It should be observed that Frege did not construct his system for expressing pure thought as a formal system and therefore did not raise questions of completeness or consistency. Frege applied his *Begriffsschrift* to a general theory of sequences, and in part III he defines the ancestral relation on which he founded mathematical induction. This relation was afterward introduced informally by Dedekind and formally by Whitehead and Russell in *Principia mathematica*.

The *Begriffsschrift* essentially underlies Frege's definition of number in *Grundlagen der Arithmetik* (1884), although it was not used explicitly. The greater part of this work is devoted to a severe and effective criticism of existing theories of number. Frege argues that number is something connected with an assertion concerning a concept; and essential for the notion of number is that of equality of number (i.e., he has to explain the sentence "The number which belongs to the concept F is the same as that which belongs to G."). He settled on the definition "The number which belongs to the concept F is the extension of the concept of being equal to the concept F," where equality of concepts is understood as the existence of a one-to-one correspondence between their extensions. The number zero is that belonging to a concept with void extension, and the number one is that which belongs to the concept equal to zero. Using the notion of f-sequence, natural numbers are defined, with ∞ the number belonging to the notion of being a natural number.[1]

Frege's theories, as well as his criticisms in the *Begriffsschrift* and the *Grundlagen*, were extended and refined in his *Grundgesetze*, in which he incorporated the essential improvements on his *Begriffsschrift* that had been expounded in the three important papers "Funktion und Begriff" (1891), "Über Sinn und Bedeutung" (1892), and "Über Begriff und Gegenstand" (1892). In particular, "Über Sinn und Bedeutung" is an essential complement to his *Begriffsschrift*. In addition, it has had a great influence on philosophical discussion, specifically on the development of Wittgenstein's philosophy. Nevertheless, the philosophical implications of the acceptance of Frege's doctrine have proved troublesome.

An analysis of the identity relation led Frege to the distinction between the sense of an expression and its denotation. If a and b are different names of the same object (refer to or denote the same object), we can legitimately express this by $a = b$, but $=$ cannot be considered to be a relation between the objects themselves.

Frege therefore distinguishes two aspects of an expression: its denotation, which is the object to which it refers, and its sense, which is roughly the thought expressed by it. Every expression expresses its sense. An unsaturated expression (a function) has no denotation.

These considerations led Frege to the conviction that a sentence denotes its truth-value; all true sentences denote the True and all false sentences denote the False—in other words, are names of the True and the False, respectively. The True and the False are to be treated as objects. The consequences of this distinction are further investigated in "Über Begriff und Gegenstand." There Frege admits that he has not given a definition of concept and doubts whether this can be done, but he emphasizes that concept has to be kept carefully apart from object. More interesting developments are contained in his "Funktion und Begriff." First, there is the general notion of function already briefly mentioned in the *Grundlagen,* and second, with every function there is associated an object, the so-called *Wertverlauf,* which he used essentially in his *Grundgesetze der Arithmetik.*

Since a function is expressed by an unsaturated expression $f(x)$, which denotes an object if x in it is replaced by an object, there arises the possibility of extending the notion of function because sentences denote objects (the True $[T]$ and the False $[F]$), and one arrives at the conclusion that, e.g., $(x^2 = 4) = (x > 1)$ is a function. If one replaces x by 1, then, because $1^2 = 4$ denotes F, as does $1 > 1$, it follows that $(1^2 = 4) = (1 > 1)$ denotes T.

Frege distinguishes between first-level functions, with objects as argument, and second-level functions, with first-level functions as arguments, and notes that there are more possibilities. For Frege an object is anything which is not a function, but he admits that the notion of object cannot be logically defined. It is characteristic of Frege that he could not take the step of simply postulating a class of objects without entering into the question of their nature. This would have taken him in the direction of a formalistic attitude, to which he was fiercely opposed. In fact, at that time formalism was in a bad state and rather incoherently maintained. Besides, Frege was not creating objects but was concerned mainly with logical characterizations. This in a certain sense also holds true for Frege's introduction of the *Wertverlauf,* which he believed to be something already there and which had to be characterized logically.

In considering two functions, e.g., $x^2 - 4x$ and $x(x - 4)$, one may observe that they have the same value for the same argument. Therefore their graphs are the same. This situation is expressed by Frege true for Frege's introduction of the *Wertverlauf,* as $x^2 - 4x$." Without any further ado he goes on to speak of the *Wertverlauf* of a function as being something already there, and introduced a name for it. The *Wertverläufe* of the above mentioned functions $x(x - 4)$ and $x^2 - 4x$ are denoted by $\grave{\alpha}(\alpha - 4)$ and $\grave{\epsilon}(\epsilon^2 - 4\epsilon)$ respectively, and in general $\grave{\epsilon}f(\epsilon)$ is used to denote the *Wertverlauf* of function $f(\xi)$. This *Wertverlauf* is taken to be an object, and Frege assumes the basic logical law characterizing equality of *Wertverläufe:*

$$(\grave{\epsilon}f(\epsilon) = \grave{\alpha}\, g(\alpha)) = (\neg\!\smile^{\mathfrak{a}}\!\!-\, f(\mathfrak{a}) = g(\mathfrak{a})).$$

Frege extends this to logical functions (i.e., concepts), which are conceived of as functions whose values are truth-values, and thus extension of a concept may be identified with the *Wertverlauf* of a function assuming only truth-values. Therefore, e.g., $\grave{\epsilon}(\epsilon^2 = 1) = \grave{\alpha}(\alpha + 1)^2 = 2(\alpha + 1)$.

In the appendix to volume II of his *Grundgesetze,* Frege derives Russell's paradox in his system with the help of the above basic logical law. Russell later succeeded in eliminating his paradox by assuming the theory of types.

It is curious that the man who laid the most suitable foundation for formal logic was so strongly opposed to formalism. In volume II of the *Grundgesetze,* where he discusses formal arithmetic at length, Frege proves to have a far better insight than its exponents and justly emphasizes the necessity of a consistency proof to justify creative definitions. He is aware that because of the introduction of the *Wertverläufe* he may be accused of doing what he is criticizing. Nevertheless, he argues that he is not, because of his logical law concerning *Wertverläufe* (which proved untenable).

When Hilbert took the axiomatic method a decisive step further, Frege failed to grasp his point and attacked him for his imprecise terminology. Frege insisted on definitions in the classic sense and rejected Hilbert's "definition" of a betweenness relation and his use of the term "point." For Frege geometry was still the theory of space. But even before 1814 Bolzano had already reached the conclusion that for an abstract theory of space, one may be obliged to assume the term point as a primitive notion capable of various interpretations. Hilbert's answer to Frege's objections was quite satisfactory, although it did not convince Frege.

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B. VAN ROOTSELAAR

FREIESLEBEN, JOHANN KARL (*b.* Freiberg, Saxony, Germany, 14 June 1774; *d.* Nieder-Auerbach, Saxony, Germany, 20 March 1846), *geology, mineralogy, mining.*

Freiesleben came from an old Freiberg mining family. This circumstance, coupled with the active, centuries-old mining industry in the vicinity of Freiberg, determined his choice of mining science as his profession. While still a secondary school student he worked as a miner in pits and galleries. Freiesleben attended the Mining Academy in Freiberg from 1790 to 1792, and there he found a patron in Abraham Gottlob Werner, professor of geology and mineralogy. When Leopold von Buch and Alexander von Humboldt came to Freiberg in order to study under Werner, Freiesleben became friendly with both of them. He made his first scientific journey through Saxony and Thuringia with Buch, and with his friend E. F. von Schlotheim he explored the Thuringian Forest. With Humboldt he journeyed to Bohemia, and in 1795 they traveled in the Swiss Jura, the Alps, and Savoy. A lasting friendship developed between the two men. From 1792 to 1795 Freiesleben studied jurisprudence in Leipzig and often visited the Harz Mountains.

Upon returning from the journey to Switzerland, Freiesleben obtained a position as a mining official in Marienberg and Johanngeorgenstadt. In the latter city he married Marianne Caroline Beyer, the daughter of a clergyman, in 1800. In the same year he became director of the copper and silver mines in Eisleben. In this capacity he did much work in the technical aspects of mining and in science. He returned to Freiberg in 1808. There he joined the Bureau of Mines and was entrusted with the management of various governmental and corporate mines and metallurgical works. In 1838 he was placed in charge of all Saxon mining operations. He was pensioned in 1842. He died in 1846 after a short illness, while on an official tour.

Freiesleben was awarded a doctorate by the University of Marburg in 1817, and in 1828 he became a member of the Prussian Academy of Sciences in Berlin. He was a loyal friend and an adviser and benefactor to the lonely and needy; he was, moreover, closely bound to his family.

A product of Freiesleben's trips to the Harz Mountains was one of his first major works: *Bemerkungen über den Harz* (1795), in which mineralogical and technical mining observations and descriptions stand out. Freiesleben completed his most important work, *Beitrag zur Kenntniss des Kupferschiefergebirges* (1807–1815), during his stay in Eisleben. In this he presents a painstaking and detailed description of the Permian Kupferschiefer and of the accompanying formations of the Zechstein and the Lower Permian Rotliegend, as well as of the Triassic. For decades this work remained indispensable to science and technology. Only when he attempted to trace individual formations into neighboring regions did he make errors. Thus he equated the limestone in the Swiss Jura and the Swabian Alb (now known as Malm) with the similar-appearing dolomite of the Zechstein (Upper Permian) in Thuringia and Saxony because of its many caverns. He also compared the Cretaceous Alpine limestone with the Upper Permian limestone, and the Cretaceous Quadersandstein of the northern edge of the Harz with the Triassic Bunter sandstone of Saxony.