Wien, Phil.-hist. Kl., **149**, no. 4 (1904), and **151**, no. 1 (1906); K. Sudhoff, "Die kurze 'Vita' und das Verzeichnis der Arbeiten Gerhards v. Cremona, von seinem Schülern und Studien genossen kurz nach dem Tode des Meisters (1187) zu Toledo verabfasst," in *Archiv für Geschichte der Medizin*, **8** (1915), 73–92; H. Suter, "Die Mathematiker und Astronomen der Araber und ihre Werke," *Abhandlungen zur Geschichte der mathematischen Wissenschaften*, **10** (1900); "Nachträge und Berichtigungen . . . ," *ibid.*, **14** (1902), 155–185; H. Suter, "Über einige noch nicht sicher gestellte Autorennamen in den Übersetzungen des Gerhard von Cremona," in *BM*, 3rd ser., **4** (1903), 19–27; L. Thorndike, *A History of Magic and Experimental Science*, II (New York, 1923), see index; "The Latin Translations of Astrological Works by Messahalla," in *Osiris*, **12** (1956), 49–72; and "John of Seville," in *Speculum*, **34** (1959), 20–38; Manfred Ullmann, *Die Medizin im Islam* (Leiden–Cologne, 1970); and *Die Natur- und Geheimwissenschaften im Islam* (Leiden, 1972); and F. Wüstenfeld, "Die Übersetzungen arabischer Werke in das lateinische seit dem XI. Jahrhundert," in *Abhandlungen der Gesellschaft der Wissenschaften zu Göttingen*, **22** (1877), 55–81.

RICHARD LEMAY

GRASSMANN, HERMANN GÜNTHER (*b.* Stettin, Pomerania [now Szczecin, Poland], 15 April 1809; *d.* Stettin, 26 September 1877), *mathematics.*

Life and Works. Grassmann came from a family of scholars. His father, Justus Günther Grassmann, studied theology, mathematics, and physics. He was a minister for a short time, then became a teacher of mathematics and physics at the Gymnasium in Stettin, where he did much to raise the level of education. He also wrote elementary mathematics textbooks and did research on problems in physics and crystallography. Grassmann's mother was Johanne Medenwald, a minister's daughter, from Klein-Schönfeld.

The third of twelve children, Grassmann received his earliest instruction from his mother and at a private school before attending the Stettin Gymnasium. He also learned to play the piano. At the age of eighteen he passed the final secondary school examination, ranking second. With his eldest brother, Gustav, Hermann studied theology for six semesters at the University of Berlin, where his teachers included August Neander and Friedrich Schleiermacher. At the same time he studied classical languages and literature and attended the lectures of August Böckh.

Grassmann returned to Stettin in the fall of 1830 and began intensive independent study of mathematics and physics. In December 1831, at Berlin, he took the examination for teaching at the Gymnasium level; the examiners stated, however, that Grassmann had to display greater knowledge of his subjects before he could be considered qualified to teach the higher grades. At Easter 1832 he obtained a post as assistant teacher at the Stettin Gymnasium, and two years later he passed the first-level theology examination given by the Lutheran church council of Stettin.

Grassmann did not become a minister, however. In the autumn of 1834 he was hired as senior master at the Gewerbeschule in Berlin, succeeding Jakob Steiner, who had been called to the University of Berlin. A year later Grassmann was appointed to the faculty of the newly founded Otto Schule in Stettin, where he taught mathematics, physics, German, Latin, and religion. Meanwhile he pursued his studies in theology, mathematics, and natural science. In 1839 he passed the second-level theology examination in Stettin and, the following year at Berlin, an examination in mathematics, physics, chemistry, and mineralogy that fully qualified him to teach all grades of secondary school.

The latter examination included a written portion to be done at home, the subject of which was the theory of the tides. This assignment proved decisive for Grassmann's career. In 1832 he had begun to work on a new geometric calculus. With its aid he was now able to give a simplified exposition of the mathematical developments in Lagrange's *Mécanique analytique* and to derive, in an original manner different from that of Laplace, the portions of the latter's *Mécanique céleste* relevant to the theory of the tides. Although Grassmann used the new methods only to the extent necessary to solve the problem at hand, he was undoubtedly aware of the far-reaching significance of his creation; and by 1840 he had decided to concentrate entirely on mathematical research.

At first, however, Grassmann devoted considerable effort to teaching. He wrote several brief textbooks for use in secondary school, some of which were frequently reprinted. They included *Grundriss der deutschen Sprachlehre* (1842) and *Leitfaden für den ersten Unterricht in der lateinischen Sprache* (1842). In collaboration with W. Langbein he published *Deutsches Lesebuch für Schüler von acht bis zwölf Jahren* (1846). Reorganizations of the Stettin school system led to his being transferred on several occasions (the Otto Schule, the Gymnasium, the Friedrich Wilhelm Schule; at the last he received the title *Oberlehrer* in May 1847). In 1852 Grassmann succeeded his father (who had died in March of that year) as fourth-ranking teacher at the Stettin Gymnasium, a post that brought with it the title of professor.

Meanwhile, in the fall of 1843, Grassmann had completed the manuscript of the first volume of his chief work, *Die lineale Ausdehnungslehre*, which appeared the following year as *Die Wissenschaft der extensiven Grösse oder die Ausdehnungslehre*. Its fundamental significance was not grasped by contemporaries; and even a mathematician of the caliber of A. F. Möbius—whose own geometric research was to some extent related to Grassmann's—did not fully understand the author's intentions. Disapproving of the many new concepts and certain philosophical formulations, he declined to write a review of the book, and thus it was totally disregarded by the experts.

As an application of the *Ausdehnungslehre*, which is based on the general concept of connectivity, Grassmann published *Neue Theorie der Elektrodynamik* (1845), in which he replaced Ampère's fundamental law for the reciprocal effect of two infinitely small current elements with a law requiring less arbitrary assumptions. Thirty years later Clausius independently rediscovered Grassmann's law but acknowledged his priority as soon as he was apprised of it. In a series of articles published between 1846 and 1856 Grassmann applied his theory to the generation of algebraic curves and surfaces, in the hope that these papers, which were much less abstract than his book, would inspire mathematicians to read the *Ausdehnungslehre*. His hopes were not fulfilled.

On the other hand, Grassmann received rapid public recognition for a work submitted in 1846, at Möbius' suggestion, to the Fürstlich Jablonowsky'sche Gesellschaft der Wissenschaften in Leipzig. In it he solved the problem posed by the society of establishing the geometric characteristic, first outlined by Leibniz, for designating topological relations, without recourse to metric properties. His entry, *Geometrische Analyse*, was awarded the full prize and was published by the society in 1847. Möbius, who was one of the judges, justly criticized both the abstract manner in which Grassmann introduced his new concepts and his neglect of intuitive aids, defects that made the last part of the text particularly difficult to read. Accordingly, Möbius incorporated into the book his own essay, "Die Grassmann'sche Lehre von den Punktgrössen und den davon abhängigen Grössenformen," in which he explained Grassmann's *Scheingrössen* as abbreviated expressions of intuitively interpretable quantities.

In May 1847 Grassmann wrote to the Prussian ministry of education, requesting that he be considered for a post as university professor, in the event that one became available. The ministry thereupon requested an opinion of Grassmann's prize essay from E. E. Kummer, a mathematician at Breslau. Kummer's severe judgment that the work contained "commendably good material expressed in a deficient form" led to the rejection of Grassmann's application.

A man of broad interests and a strong sense of political responsibility, Grassmann participated in the political events leading to the Revolution of 1848. With his brother Robert, his scientific collaborator for many years, he founded the *Deutsche Wochenschrift für Staat, Kirche und Volksleben*, which was soon replaced by the daily *Norddeutsche Zeitung*. Advocates of a Germany united under Prussian leadership, the brothers hoped for the establishment of a constitutional monarchy, ruled by the king in cooperation with the Reichstag. Revolution and civil war, they contended, were not the proper means of winning greater freedom. In articles published in 1848 and 1849 Grassmann considered chiefly problems of constitutional law; but with the restoration he became increasingly dissatisfied and withdrew from the paper.

On 12 April 1849 Grassmann married Marie Therese Knappe, the daughter of a Pomeranian landowner. Of their eleven children, two died in early childhood and two others somewhat later. His sons Justus and Max became mathematics teachers at the Stettin Gymnasium; Ludolf, a physician; Hermann, professor of mathematics at the University of Giessen; and Richard, professor of mechanical engineering at the Technische Hochschule in Karlsruhe.

As a student Grassmann had been admitted to the Freemason lodge in Stettin, and from 1856 he held the post of treasurer. In 1857 he became a member of the board of directors of the Pommersche Hauptverein für die Evangelisierung Chinas. Founded in 1850, this society published a journal and occasionally sent missionaries to China. Under Grassmann's chairmanship it was unified with the Rheinische Missions-Gesellschaft in Barmen in 1873.

Soon after the political unrest of 1848–1849 had subsided, Grassmann began to study Sanskrit and then Gothic, Lithuanian, Old Prussian, Old Persian, Russian, and Church Slavonic—investigations that laid the foundations for his studies in comparative linguistics. Profiting from his acute sense of hearing and his capacity for making careful observations, Grassmann developed a theory of the physical nature of speech sounds (1854). He had recognized that each vowel sound arises through definite overtones that are specifically characteristic of it and planned to substantiate his theory experimentally, using a tone generator instead of the human voice. The device he intended to use, however, did not furnish sufficiently pure vibrations—that is, it was not sufficiently free of overtones.

In "Zur Theorie der Farbenmischung" (1853) Grassmann opposed certain conclusions that Helmholtz had drawn from experiments on the mixing of colors. According to Grassmann's theory, each color can be represented by a weighted point plotted on a circular surface; the position of the point indicates its hue and saturation, and the weight, its intensity. If the colors to be mixed are thus represented, then the center of gravity, in which the entire mass is seen as being concentrated, gives the intensity of the visual impression of the mixture. Helmholtz later acknowledged the correctness of the center-of-gravity construction but retained reservations concerning the circular form of the overall field.

Around the middle of 1854 Grassmann resumed work on the *Ausdehnungslehre*. He may have been stimulated in this effort by a remark made by Möbius, who informed him of two papers in which equations with several unknowns were solved by means of an approach incorporating *clefs algébriques* that corresponded to the one that Grassmann had developed in sections 45, 46, and 93 of his book. Indeed, Möbius insisted that Grassmann assert his priority against the authors of these papers, A. B. de Saint-Venant (in *Comptes rendus . . . de l'Académie des sciences*, **21** [1845], 620–625) and Cauchy (*ibid.*, **36** [1853], 70–76, 129–136). Grassmann subsequently addressed himself to the Paris Academy as well as to both authors, without charging plagiarism. Whether Cauchy, in particular, knew of the *Ausdehnungslehre* when he wrote his paper cannot be established.

Rather than writing a second volume of the *Ausdehnungslehre*, as he had originally intended, Grassmann decided to rework the text; and *Die Ausdehnungslehre. Vollständig und in strenger Form bearbeitet* was published at Berlin in 1862. The new version, unfortunately, fared no better than the first; Grassmann, in presenting a systematic foundation of his theory, had failed to see that a different approach was necessary for mathematicians to acquaint themselves with the new concepts. Given the failure of the first edition, it is all the more astonishing that Grassmann did not attempt to emphasize the advantages of his ideas by demonstrating them through specific examples. Again, no serious attempts were made by mathematicians to work through the peculiar terminology of this strange theory, and the revised edition was long ignored. Although the Leopoldina elected Grassmann to membership in 1864, it was in recognition of his achievements in physics—not in mathematics.

Disappointed by his continued lack of success, Grassmann gradually turned away from mathematical research. Besides a few minor articles, in 1865 he published a supplement to his *Lehrbuch der Arithmetik* (1860) entitled *Lehrbuch der Trigonometrie für höhere Lehranstalten* (a planned third part on geometry never appeared) and "Grundriss der Mechanik," in the *Stettiner Gymnasialprogramm* of 1867.

Grassmann also concentrated increasingly on linguistic research. Works on phonetics that were based on the historical study of language (1860, 1862) were followed by the important "Über die Aspiranten und ihr gleichzeitiges Vorhandensein im An- und Auslaute der Wurzeln" (1863), in which he formulated the law of aspirates that is named for him. A crucial contribution to the study of the Germanic sound shift, this law has become part of the science of comparative linguistics.

Linguistic research of a different kind forms the basis of *Deutsche Pflanzennamen* (1870), which Grassmann wrote with his brother Robert and his brother-in-law Christian Hess. The goal of the work was to introduce German names for all plants grown in the German-language area, terms as precise as the Latin and Greek forms that would be derived etymologically from the various Germanic languages. Grassmann hoped that the effort of collecting and explicating involved in the project would prove useful to both botanists (especially biology teachers) and linguists.

An achievement of a much higher order is represented by Grassmann's work on the Sanskrit language. Realizing that research in comparative linguistics must take the oldest Indic languages as a starting point, Grassmann began an intensive study of the hymns of the *Rig-Veda* about 1860. Unlike his mathematical work, which was greatly ahead of its time, these studies benefited from opportune timing. Theodor Aufrecht's authoritative text of the *Rig-Veda* and the St. Petersburg Sanskrit dictionary compiled by Otto von Böthlingk and W. R.

von Roth were also published at this time. Grassmann's complete glossary of the *Rig-Veda* was modeled on the Biblical concordances, and the entry for each word indicates the grammatical form in which it appears. Although criticized on points of detail, the six-part *Wörterbuch zum Rigveda* (1873–1875) was generally praised by specialists and remained the standard work for many years.

Although Grassmann had originally planned to follow the glossary with a grammar, he came to consider it more important to publish a translation of the hymns, believing it essential to their interpretation; it appeared in two parts as *Rig-Veda. Übersetzt und mit kritischen Anmerkungen versehen* (1876–1877). Alfred Ludwig's simultaneously published complete translation of the texts (1876), although far superior philologically to Grassmann's work, was presented in a German that was difficult to understand. Grassmann, in contrast, seeking to reproduce not only the meanings of the words but also the overall feeling of the original, retained a metrical form—at the sacrifice of a faithful rendering in certain passages.

Unlike his mathematical works, Grassmann's linguistic research was immediately well received by scholars. A year before his death he became a member of the American Oriental Society, and the Faculty of Philosophy of the University of Tübingen awarded him an honorary doctorate.

In the meantime, a few mathematicians had become aware of Grassmann's work. His election on 2 December 1871 as a corresponding member of the Göttingen Academy of Sciences encouraged him to publish some short mathematical papers during the last years of his life. In 1877 he prepared another edition of the *Ausdehnungslehre* of 1844 for publication; it appeared posthumously in 1878. Despite steadily increasing infirmity, he continued to work until succumbing to cardiac insufficiency.

Influence. Toward the end of his life, and even more after his death, a growing recognition of Grassmann's accomplishments was observed among specialists. Alfred Clebsch, the leading German mathematician of the time, for example, made a special effort to call attention to Grassmann during a memorial address that he delivered in honor of Julius Plücker in 1871. H. Hankel was the first to discuss the *Ausdehnungslehre* in a book: *Theorie der komplexen Zahlensysteme* (1867). The obituary of Grassmann by Sturm, Schröder, and Sohncke displays considerable appreciation for his work. At about the same time there appeared a short biography of him by V. Schlegel, who had previously used Grassmann's ideas in his *System der Raumlehre* (1872–1875).

The geometry that Grassmann established on the basis of the *Ausdehnungslehre* was later elaborated as *Punktrechnung*, first in Peano's *Calcolo geometrico* (1888). Grassmann's most gifted son, Hermann, wrote three textbooks on analytic geometry based on the approach of the *Ausdehnungslehre*; and in 1909 he applied the latter to the theory of gyroscopic motion. The other most important German advocates of the book were R. Mehmke and his student A. Lotze. French mathematicians were introduced to Grassmann's concepts by F. Caspary, who used them in his research on the generation of algebraic curves. The spread of Grassmann's ideas was further aided by the publication of a collected edition of his works and of papers written on the centenary of his birth. An early advocate of Grassmann's *Ausdehnungslehre* was the American physicist J. W. Gibbs (1839–1903). Well-known, among many other achievements, for his popularization of a vector algebra as it may be derived from Grassmann's and W. R. Hamilton's ideas, he emphatically preferred—in contrast to Tait's rival claims for Hamilton's quaternions—Grassmann's less limited concepts. Gibbs created what he called "dyadics," an operational approach that found favor with theoretical physicists until it was replaced by modern vector and tensor analysis. In this dyadics, a matrix in its operational application is understood as a sum of more simple operators, the so-called simple dyads. As Gibbs himself explained, the germ of these ideas must be seen in certain indeterminate products (*Lückenausdrücke*), which Grassmann introduced in a note at the end of the first edition of his *Ausdehnungslehre* of 1844. Thus the birth of linear matrix algebra, often associated with the publication of Cayley's classic "Memoir on the Theory of Matrices" in 1858, may be said to have occurred already in 1844. A century later H. G. Forder's *Calculus of Extension* (1941) testified to the continuing appeal of the *Ausdehnungslehre* among mathematicians in the English-speaking world.

The Ausdehnungslehre. The *Ausdehnungslehre* concerns geometric analysis, a border region between analytic geometry, which uses only the algebra of coordinates and equations, and synthetic geometry, which dispenses with all algebraic aids. The first to conceive of this kind of geometric analysis was Leibniz. In a letter of 1679 to Huygens (see Grassmann, *Werke*, I, 417) he wrote that in order to develop such an analysis, one should work directly with the symbols of geometric concepts—

such as points, straight lines, and planes—that is, without the intermediary of coordinates; further, loci and other properties should always be expressed through algebraic expressions written in the symbols of these basic concepts. Leibniz did not fully elaborate his idea, and systems of geometric analysis did not appear until the nineteenth century, after analytic and synthetic geometry had undergone considerable development. On this question one may consult three articles by H. Rothe, A. Lotze, and C. Betsch in the *Encyklopädie der mathematischen Wissenschaften* (1916–1923); the article by Lotze deals especially with the *Ausdehnungslehre*.

All systems of geometric analysis share the characteristic that one of their fundamental elements is the geometric addition of directed line segments, an operation borrowed from mechanics. Accordingly, one may place among these systems the description of Euclidean plane geometry by means of complex numbers that was formulated around 1800. Although known as the theory of the Gaussian plane, it was also elaborated independently of Gauss by Wessel and Argand. A purely geometric computation of line segments in the plane, in which even the complex numbers do not appear explicitly, had to await the appearance of the method of equipollences elaborated by Bellavitis in the 1830's. Another example of geometric analysis is Möbius' barycentric calculus (1827). The barycentric coordinates of Euclidean space are special systems of projective coordinates obtained with the aid of concepts drawn from point mechanics. More important in the present context, however, is the fact that in 1844, in a work devoted to establishing the barycentric calculus on a firmer basis, Möbius conceived of the line segment as the difference between two points, a notion that played an important role in the calculus of points based on the *Ausdehnungslehre*.

The theory of quaternions developed by W. R. Hamilton between 1843 and 1853 originated in an attempt to generalize the complex numbers in a manner that would preserve, if possible, all the laws of arithmetic. This generalization could be effected, however, only by giving up the commutative law of multiplication. Doing so gave rise to the system of quaternions named for Hamilton. In current terminology, this system is a skew field of a fourth-rank algebra over the field R of the real numbers.

During this same period, Grassmann developed his *Ausdehnungslehre*. Its algebraic entities are the extensive quantities. These consist, in the first instance, of quantities of the first rank of the base domain S_n^1; this, in modern terms, is an n-dimensional vector space over R, which is expressed here for the first time in such generality. Its base vectors are $e_1, \cdots, e_n$. In his attempt to find a suitable multiplication of two quantities of S_n^1, Grassmann proceeded differently from Hamilton. He did not seek to make S_n^1 into a ring but, instead, added to S_n^1 a domain $S^2{}_{\binom{n}{2}}$ of quantities of the second rank —that is, a vector space of dimension $\binom{n}{2}$ with base quantities e_{ij} $(1 \leq i < j \leq n)$. The product of two quantities of S_n^1—which he called the outer product—is so constructed that it lies in $S^2{}_{\binom{n}{2}}$. This outer multiplication is to be distributive, so it need be defined only for the e_i. If the multiplication is designated by brackets, then

$$[e_ie_j] = -[e_je_i] = e_{ij} \qquad (1 \leq i < j \leq n)$$

and

$$[e_ie_i] = 0 \qquad (1 \leq i \leq n).$$

Then, for arbitrary r where $1 \leq r \leq n$, Grassmann established the domain $S^r{}_{\binom{n}{r}}$ of the quantities of the r-th rank with base $e_{i_1 \ldots i_r}$ $(1 \leq i_1 < \cdots < i_r \leq n)$. By using the formulation $[e_{j_1} \cdots e_{j_r}] = +e_{i_1 \ldots i_r}$, *or* $= -e_{i_1 \ldots i_r}$, or $= 0$—according as $(j_1 \cdots j_r)$ is an even permutation of $(i_1 \cdots i_r)$ or an odd permutation of it, or whether the j_ν are not all different—the outer product of r base quantities of S_n^1 can be expressed as a quantity of $S^r{}_{\binom{n}{r}}$. At this stage one can immediately calculate the outer product of r arbitrary quantities of S_n^1. Grassmann called "simple" the special quantities of $S^r{}_{\binom{n}{r}}$ that arise in this manner. Since he set $e_{1 \ldots n} = 1$, he was again able to conceive S_n^1 as the scalar domain R.

Through reduction to the basic unities and use of the associative law, it can be shown that for arbitrary unities of S^r and S^s $[e_{i_1 \ldots i_r} e_{j_1 \ldots j_s}] = 0$, if i and j are not all different; that this expression equals $+1$ or -1 in the cases, respectively, that $(i_1 \cdots i_r\, j_1 \cdots j_s)$ is an even or an odd permutation of a combination $(n_1 \cdots n_r \cdots n_{r+s})$, where $1 \leq n_1 < \cdots < n_{r+s} \leq n$. From this result one may easily obtain, through distributive multiplication, the progressive product of the quantities $A^r \in S^r$ and $B^s \in S^s$. If one takes $e_{1 \ldots n}$ as an independent unity and adds quantities of arbitrary rank, which Grassmann avoided doing, one obtains, in modern terms, the Grassmann algebra of rank 2^n over S_n^1. In this algebra $[A^r, B^s]$ vanishes for $r + s > n$. Grassmann therefore constructed a "regressive" product of A^r and B^s. To this end he associated to each unity $e_{i_1 \ldots i_r}$ its supplement $| e_{i_1 \ldots i_r} = \pm e_{i_{r+1} \ldots i_n}$.

Here $(i_1 \cdots i_n)$ is a permutation of $(1 \cdots n)$, and equals +1 or −1 according as the latter is even or odd. From this it can be shown that the extension $|A^r$ for every $A^r \in S^r$ is a quantity of S^{n-r}.

The inner product of two quantities A^r, B^r of S^r is given by $P = [A \cdot |B]$, which Grassmann considered a scalar, since it is a quantity in S^n. At this point he could explain the notation of orthogonality and absolute value in S^r. He interpreted the regressive product of the quantities $A^r \in S^r$ and $B^s \in S^s$ (where $r + s > n$), for the unities ϵ^r, η^s, as that unity of rank $2n - (r + s)$ the supplement of which is the progressive product of the supplement of ϵ^r and η^s. This makes it easy to define the regressive product of arbitrary $A^r \in S^r$ and $B^s \in S^s$, where $r + s > n$.

On the basis of the preceding steps, Grassmann could obtain the outer products of arbitrarily many extensive quantities. These are "pure" if the multiplications to be performed are either all progressive or all regressive, and "mixed" if this is not the case. In general, a mixed product is neither commutative nor associative, although it does fulfill certain computational rules that Grassmann established. In addition to the outer products, Grassmann developed a multiplication that he termed "algebraic," which obeys the law $e_ie_j = e_je_i$ for $i = 1 \cdots n$ and leads to what is today known as the polynomial ring.

Grassmann derived the ideas of his *Ausdehnungslehre,* as well as his new way of forming products, essentially from geometry, particularly from the geometry of n dimensions, which was then still in its infancy. In his calculus of points, the points of R_n, which are provided with weights (that is, numbers different from zero), are conceived as the fundamental domain S^1_{n+1} of a totality of extensive quantities. The points that are assigned weight 1 are also called "simple." $\lambda a + \mu b$ then yields all points of the straight line A spanned by the simple points a, b, those points having the weight $\lambda + \mu$. This is identical to the scheme developed by Möbius, where (λ, μ) are the barycentric coordinates of the relevant point of A. The only difference is that no weighted point of A corresponds to a-b but, rather, to the vector leading from b to a. In Grassmann's calculus, therefore, the vector space V_n over R appears as a subset of the domain S^1_{n+1} of the point range. If $a_0 \cdots a_k$ and $b_0 \cdots b_k$ are each linearly independent simple points that span the same $R_k \subset R_n$, then the outer products $[a_0 \cdots a_k]$ and $[b_0 \cdots b_k]$ do not vanish; and they differ by, at most, a scalar factor $\lambda \neq 0$. They are therefore associated with this R_k. Their components, related to a base $e_{i_0 \cdots i_k}$, are now called the Grassmann coordinates of R_k. They fulfill a system of quadratic relations that Grassmann gave for $k = 1$; and if they are interpreted as points of a projective space, they describe the manifold $G_{n,k}$ that is named for Grassmann.

If the simple quantities $[a_0 \cdots a_r]$ and $[b_0 \cdots b_s]$ are associated with the spaces R^a_r and R^b_s of R_n, and if these have no finite or infinite point in common, then the progressive outer product $[a_0 \cdots a_r b_0 \cdots b_s]$ corresponds to the connection space R_{r+s+1} of R^a_r and R^b_s. If R^a_r and R^b_s span the entire R_n, then the regressive product of $[a_0 \cdots a_r]$ and $[b_0 \cdots b_s]$ is associated with the intersection space $R^a_r \cap R^b_s$. In works directly inspired by Grassmann the perceptual spaces R_2, R_3 are treated in detail with these methods, as are the projective spaces P_2, P_3 over R. For example, in them the line vector (also known as the rod), which is bound to the connecting line A of a,b, is described by means of the outer product $[a,b]$ of the points a,b. Then $[abc]$ defines an oriented surface element bound to the plane containing a,b,c; its content can be either positive or negative, or zero when a, b, and c are collinear. The outer product of two free vectors yields a bivector.

Apart from details, the *Ausdehnungslehre* appears in retrospect as a very general and comprehensive treatise, the implications of which reached far beyond the "state of the art" in 1844 or even in 1862. In geometry, mathematicians were still thinking in terms of a "real" three-dimensional space and saw no need to occupy themselves with such a theory of "extended magnitudes" in a fictional n-dimensional space. Hamilton's goal had been to find a consistent algebra of rotations and vectors in three-dimensional space; when he reached it in his system of quaternions he was forced to sacrifice the traditional principle of commutativity for multiplication. Grassmann, on the other hand, had not only immediately considered manifolds of an arbitrary number of dimensions, but also had introduced new, seemingly artificial kinds of multiplication for its various types of elements. Nobody in his day could foresee that, in its general algebraic aspects, the *Ausdehnungslehre* did much more than merely accomplish for any finite number of dimensions what Hamilton's quaternions were designed to do for Euclidean space of three dimensions. Beyond that it anticipated (or even included) wide areas of modern linear algebra and of matrix, vector, and tensor analysis. Thus, three lines of later development may be distinguished in connection with Grassmann's principal work: first, the generalization of the geometrical concept of space (also anticipated at much the same time by Cayley and by Riemann); second, the influence on Gibbs and thus on the creation of vector analysis; and third, the important anticipation of fundamental parts of modern algebra, though this was not immediately noticed by the mathematical public.

Other Mathematical Works. Despite the long neglect of his ideas, Grassmann was always convinced of their importance. In several works he attempted to show how the theory of quaternions and invariant theory (then called modern algebra) can be understood on the basis of the *Ausdehnungslehre*. Still more important, however, are his writings on the "lineal" generation of algebraic entities, in which he also draws on his theory. This group of publications deals, for example, with the theory of constructing points of algebraic curves and surfaces by simply drawing straight lines and planes through given points, as well as with the determination of intersection points of known straight lines.

As early as 1721 Maclaurin had demonstrated that given the three points a, b, c and the two straight lines A, B of general position in the plane, the locus of the third vertices of all triangles the first two vertices of which lie, respectively, on A and B and the sides of which pass, correspondingly, through a, b, c is a conic section. In terms of the calculus of points, this statement means that the mixed outer product $(a \times AbB \times c)$ vanishes. Grassmann made the important discovery that in this way every plane algebraic curve C can be generated lineally. As a result, if C is of order n, it can be described by setting equal to zero an outer product in which, in addition to symbols for certain fixed points and straight lines, the expression for the variable point x of C appears n times. A cubic can be expressed, accordingly, as $(xaA) \cdot (xbB) \cdot (xcC) = 0$. This signifies that the locus of the point x of the plane is a cubic, if the line connecting x with three fixed points a, b, c cuts the three fixed straight lines A, B, C in three collinear points. Moreover, one can obtain every plane cubic in this manner. Grassmann was thus able to refute Plücker's assertion that curves higher than the second degree could be conceived only in terms of coordinate geometry. In writings collected in volume II, part 1, of the *Werke*, Grassmann considered, in particular, the lineal generations of plane cubics and quartics, as well as of third-degree spatial surfaces. (One of these generations bears his name.) He demonstrated that by setting equal to zero the products he designated as planimetric or stereometric, all these generations could be obtained from the *Ausdehnungslehre*.

A large portion of the *Ausdehnungslehre* is devoted to analysis. Grassmann treats functions of n real variables as functions of extensive quantities of a base domain S_n^1. Since he introduced a metric into S_n^1 in the form of the inner product, he was able to derive Taylor expansions, remainder formulas, and other items. His most important studies in analysis concern Pfaff's problem—that is, the theory of the integration of a Pfaffian equation

$$\omega = A_1(x_1 \cdots x_n)\,dx_1 + \cdots + A_n(x_1 \cdots x_n)\,dx_n = 0.$$

This question had interested leading nineteenth-century mathematicians both before and after Grassmann, especially Pfaff and Jacobi. Grassmann contributed the following important theorem: If one calls k the class of ω—that is, the minimum number of variables into which ω can be transformed—then, when $k = 2h$, ω can be transformed into the normal form

$$z_{h+1}\,dz_1 + \cdots + z_{2h}\,dz_h^1$$

and, when $k = 2h - 1$, into

$$p \cdot (dz_h + z_{h+1}\,dz_1 + \cdots + z_{2h-1}\,dz_{h-1}),$$

where p is a function of $z_1 \cdots z_{2h-1}$. Even these results, however, which appeared in the 1862 edition of the *Ausdehnungslehre* and surpassed Jacobi's achievements, obviously did not attract much attention. Recognition had to await their translation into the more customary language of analysis by F. Engel in his commentary on Grassmann's works.

The calculus of differential forms, which is based on Grassmann's outer multiplication, occupies a firm position in modern analysis. This calculus has enabled mathematicians to develop differential geometry in an elegant manner, as is particularly evident in the work of E. Cartan.

BIBLIOGRAPHY

Grassmann's writings were collected as *Mathematische und physikalische Werke*, F. Engel, ed., 3 vols. in 6 pts. (Leipzig, 1894–1911).

Biographical and historical works are E. T. Bell, *The Development of Mathematics* (New York–London, 1945), 198–206; M. J. Crowe, *A History of Vector Analysis* (Notre Dame, Ind.–London, 1967), ch. 3; A. E. Heath, "Hermann Grassmann. The Neglect of

His Work. The Geometric Analysis and Its Connection with Leibniz' Characteristic," in *Monist*, **27** (1917), 1-56; F. Engel, "H. Grassmann," in *Jahresberichte der Deutschen Mathematikervereinigung*, **18** (1909), 344-356; and "Grassmanns Leben," which is Grassmann's *Werke*, III, pt. 2 (1911); G. Sarton, "Grassmann—1844," in *Isis*, **35** (1944), 326-330; V. Schlegel, *Hermann Grassmann, sein Leben und seine Werke* (Leipzig, 1878); and *Die Grassmannsche Ausdehnungslehre. Ein Beitrag zur Geschichte der Mathematik in den letzten 50 Jahren* (Leipzig, 1896); and R. Sturm, E. Schröder, and L. Sohncke, "H. Grassmann. Sein Leben und seine mathematisch-physikalischen Arbeiten," in *Mathematische Annalen*, **14** (1879), 1-45, with bibliography of his works.

Works on geometrical analysis in general include H. Hankel, *Theorie der komplexen Zahlensysteme* (Leipzig, 1867); A. F. Möbius, *Der baryzentrische Kalkül* (Leipzig, 1827), also in his *Gesammelte Werke*, I (1885), 1-388; and "Über die Zusammensetzung gerader Linien und eine daraus entspringende neue Begründung des baryzentrischen Calculs," in *Journal für die reine und angewandte Mathematik*, **28** (1844), 1-9; and three articles collectively entitled "Systeme geometrischer Analyse," in *Encyklopädie der mathematischen Wissenschaften*, III, pt. 1 (1916-1923): by H. Rothe, 1277-1423; A. Lotze, 1425-1550; and C. Betsch, 1550-1595.

Works based on Grassmann's *Ausdehnungslehre* include F. Caspary, "Über die Erzeugung algebraischer Raumkurven durch veränderliche Figuren," in *Journal für die reine und angewandte Mathematik*, **100** (1887), 405-412; and "Sur une méthode générale de la géométrie, qui forme le lieu entre la géométrie synthétique et la géométrie analytique," in *Bulletin des sciences mathématiques et astronomiques*, 2nd ser., **13** (1889), 202-240; H. G. Forder, *Calculus of Extension* (Cambridge, 1941); H. Grassmann, Jr., "Über die Verwertung der Streckenrechnung in der Kreiseltheorie," in *Sitzungsberichte der Berliner mathematischen Gesellschaft*, **8** (1909), 100-114; and *Projektive Geometrie der Ebene, unter Verwendung der Punktrechnung dargestellt*, 2 vols. in 3 pts. (Leipzig, 1909-1923); F. Kraft, *Abriss des geometrischen Calculs nach H. G. Grassmann* (Leipzig, 1893); A. Lotze, *Die Grundgleichungen der Mechanik, neu entwickelt mit Grassmanns Punktrechnung* (Leipzig, 1922); and *Punkt- und Vektorenrechnung* (Berlin-Leipzig, 1929); R. Mehmke, *Vorlesungen über Punkt- und Vektorenrechnung*, I (Leipzig-Berlin, 1913); G. Peano, *Calcolo geometrico* (Turin, 1888); and V. Schlegel, *System der Raumlehre nach den Prinzipien der Grassmann'schen Ausdehnungslehre*, 2 vols. (Leipzig, 1872-1875). A bibliography was compiled by G. Peano: "Elenco bibliografico sull' 'Ausdehnungslehre' di H. Grassmann," in *Rivista di matematica*, **5** (1895), 179-182.

W. Burau
C. J. Scriba

GREEN, GEORGE (*b.* Nottingham, England, July 1793 [baptized 14 July]; *d.* Sneinton, near Nottingham, 31 May 1841), *mathematics, natural philosophy.*

Although Green left school at an early age to work in his father's bakery, he had probably already developed an interest in mathematics that was fostered by Robert Goodacre, the leading private schoolmaster of Nottingham and author of a popular arithmetic textbook. Virtually self-taught, Green acquired his knowledge of mathematics through extensive reading. Many of the works he studied were available in Nottingham at the Bromley House Subscription Library, which he joined in 1823. By that time the family had moved to Sneinton, a suburb, where his father had established a successful milling business; Green used the top story of the mill as a study.

Green's most important work, *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*, was published by subscription in March 1828. Apparently, almost all of the fifty-two subscribers were patrons and friends of Green's; a local baronet, Edward ffrench Bromhead of Thurlby, assisted Green later but was not an early promoter. Until other evidence is available, one can only conjecture that Green's supporters included some of the leading members of the Bromley House Library; the list of subscribers suggests only limited circulation outside Nottingham.

In the preface Green indicated that his "limited sources of information" preventing his giving a proper historical sketch of the mathematical theory of electricity, and indeed, he cites few sources. Among them are Cavendish's single-fluid theoretical study of electricity of 1771, two memoirs by Poisson of 1812 on surface electricity and three on magnetism (1821-1823), and contributions by Arago, Laplace, Fourier, Cauchy, and T. Young. The preface concludes with a request that the work be read with indulgence, in view of the limitations of the author's education.

The *Essay* begins with introductory observations emphasizing the central role of the potential function. Green coined the term "potential" to denote the results obtained by adding the masses of all the particles of a system, each divided by its distance from a given point. The general properties of the potential function are subsequently developed and applied to electricity and magnetism. The formula connecting surface and volume integrals, now known as Green's theorem, was introduced in the work, as was "Green's function," the