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ALBERT V. CAROZZI

HAMILTON, WILLIAM ROWAN (*b.* Dublin, Ireland, 4 August 1805; *d.* Dunsink Observatory [near Dublin], 2 September 1865), *mathematics, optics, mechanics.*

His father was Archibald Rowan Hamilton, a Dublin solicitor, whose most important client was the famous Irish patriot Archibald Hamilton Rowan. Both Hamilton and his father took one of their Christian names from Rowan, and the matter is further complicated by the fact that Rowan's real name was Hamilton as well. But there is no evidence of kinship.

The fourth of nine children, Hamilton was raised and educated from the age of three by his uncle James Hamilton, curate of Trim, who quickly recognized his fabulous precocity. By his fifth year he was proficient in Latin, Greek, and Hebrew; and during his ninth year his father boasted of his more recent mastery of Persian, Arabic, Sanskrit, Chaldee, Syriac, Hindustani, Malay, Marathi, Bengali, "and others."

Mathematics also interested Hamilton from an early age, but it was the more dramatic skill of rapid calculation that first attracted attention. In 1818 he competed unsuccessfully against Zerah Colburn, the American "calculating boy"; he met him again in 1820. At about this time he also began to read Newton's *Principia* and developed a strong interest in astronomy, spending much time observing through his own telescope. In 1822 he noticed an error in Laplace's *Mécanique céleste.* His criticism was shown by a friend to the astronomer royal of Ireland, the Reverend John Brinkley, who took an interest in Hamilton's progress and was later instrumental in getting Hamilton appointed as his successor at Dunsink Observatory.

Hamilton's enthusiasm for mathematics caught fire in 1822, and he began studying furiously. The result was a series of researches on properties of curves and surfaces that he sent to Brinkley. Among them was "Systems of Right Lines in a Plane," which contained the earliest hints of ideas that later were developed into his famous "Theory of Systems of Rays." On 31 May 1823 Hamilton announced to his cousin Arthur Hamilton that he had made a "very curious discovery" in optics,[1] and on 13 December 1824 he presented a paper on caustics at a meeting of the Royal Irish Academy with Brinkley presiding. The paper was referred to a committee which reported six months later that it was "of a nature so very abstract, and the formulae so general, as to require that the reasoning by which some of the conclusions have been obtained should be more fully developed. . . ."[2] Anyone who has struggled with Hamilton's papers can sympathize with the committee, but to Hamilton it was a discouraging outcome. He returned to his labors and expanded his paper on caustics into the "Theory of Systems of Rays," which he presented to the Academy on 23 April 1827, while still an undergraduate at Trinity College. Hamilton considered his "Systems of Rays" to be merely an expansion of his paper on caustics. Actually the papers were quite different. The characteristic function appeared only in the "Theory of Systems of Rays," while "On Caustics" investigated the properties of a general rectilinear congruence.

Hamilton had taken the entrance examination for Trinity on 7 July 1823 and, to no one's surprise, came out first in a field of 100 candidates. He continued this auspicious beginning by consistently winning extraordinary honors in classics and science throughout his college career. Trinity College, Dublin, offered an excellent curriculum in mathematics during Hamilton's student years, owing in large part to the work of Bartholomew Lloyd, who became professor at the college in 1812 and instituted a revolution in the teaching of mathematics. He introduced French textbooks and caused others to be written in order to bring the students up to date on Continental methods. These reforms were essentially completed when Hamilton arrived at Trinity.

On 10 June 1827 Hamilton was appointed astronomer royal at Dunsink Observatory and Andrews professor of astronomy at Trinity College. He still had not taken a degree, but he was chosen over several well-qualified competitors, including George Biddell Airy.

As a practical astronomer Hamilton was a failure. He and his assistant Thompson maintained the instruments and kept the observations with the somewhat reluctant help of three of Hamilton's sisters who lived at the observatory. After his first few years Hamilton did little observing and devoted himself entirely to theoretical studies. On one occasion in 1843 he was called to task for not having maintained a satisfactory program of observations, but this protest did not seriously disturb his more congenial mathematical researches.

Life at the observatory gave Hamilton time for his mathematical and literary pursuits, but it kept him somewhat isolated. His reputation in the nineteenth century was enormous; yet no school of mathematicians grew up around him, as might have been expected if he had resided at Trinity College. In the scientific academies Hamilton was more active. He joined the Royal Irish Academy in 1832 and served as its president from 1837 to 1845. A prominent early member of the British Association for the Advancement of Science, he was responsible for bringing the annual meeting of the association to Dublin in 1835. On that occasion he was knighted by the lord lieutenant. In 1836 the Royal Society awarded him the Royal Medal for his work in optics at the same time that Faraday received the medal for chemistry. A more signal honor was conferred in 1863, when he was placed at the head of fourteen foreign associates of the new American National Academy of Sciences.

In 1825 Hamilton fell in love with Catherine Disney, the sister of one of his college friends. When she refused him, he became ill and despondent and was close to suicide. The pain of this disappointment stayed with Hamilton throughout his life and was almost obsessive. In 1831 he was rejected by Ellen De Vere, sister of his good friend the poet Aubrey De Vere, and in 1833 he married Helen Bayly. It was an unfortunate choice. Helen suffered from continual ill health and an almost morbid timidity. She was unable to run the household, which eventually consisted of two sons and a daughter, and absented herself for long periods of time.

When Catherine Disney was dying in 1853, Hamilton visited her twice. He was desperate to get mementos from her brother—locks of hair, poetry, a miniature that he secretly had copied in Dublin—and relieved his distress by writing his confessions to close friends, often daily, sometimes twice a day. Harassed by guilt over his improper feelings and the fear that his secret would become known, Hamilton sought further release in alcohol, and for the rest of his life he struggled against alcoholism.

It would be a mistake to picture Hamilton's life as constant tragedy, however. He was robust and energetic, with a good sense of humor. He possessed considerable eloquence; but his poetry, which he greatly prized, was surprisingly bad. Hamilton had many acquaintances in the Anglo-Irish literary community. He was a frequent visitor at the home of the novelist Maria Edgeworth, and in 1831 he began his long correspondence with De Vere, with whom he shared his metaphysical and poetical ideas and impressions. But his most important literary connection was with William Wordsworth, who took to Hamilton and seemed to feel an obligation to turn him from his poetic ambitions to his natural calling as a mathematician. Hamilton insisted, however, that the "spirit of poetry" would always be essential to his intellectual perfection.

A more important philosophical influence was Samuel Taylor Coleridge. Hamilton was greatly impressed by *The Friend* and *Aids to Reflection,* and it was through Coleridge that he became interested in the philosophy of Immanuel Kant. Hamilton's first serious venture into idealism came in 1830, when he began a careful reading of the collected works of George Berkeley, borrowed from Hamilton's friend and pupil Lord Adare. A letter written in July of the same year mentions Berkeley together with Rudjer Bošković. By 1831 he was struggling with Coleridge's distinction between Reason and Understanding as it appeared in the *Aids to Reflection.* A draft of a letter to Coleridge in 1832, which remained unsent, proclaims his adherence to Bošković's theory of point atoms in space. Coleridge had been very critical of atomism in the *Aids to Reflection,* and Hamilton inquired whether the mathematical atomism, which he believed was required for the undulatory theory of light, was acceptable to Coleridge. He had obtained a copy of Kant's *Critique of Pure Reason* in October 1831 and set about reading it with enthusiasm.

By 1834 Hamilton's idealism was complete. In the introduction to his famous paper "On a General Method in Dynamics" (1834), he declared his support for Bošković and argued for a more abstract and general understanding of "force or of power acting by law in space and time" than that provided by the atomic theory. In a letter also of the same year he wrote: "Power, acting by law in Space and Time, is the ideal base of an ideal world, into which it is the problem of physical science to refine the phenomenal world, that so we may behold as one, and under the forms of our own understanding, what had seemed to be manifold and foreign."[3]

While Hamilton was strongly attracted to the ideas of Kant and Bošković, it is difficult to see how they had any direct effect on his system of dynamics. The "General Method in Dynamics" of 1834 was based directly on the characteristic function in optics, which he had worked out well before he studied Kant or Bošković. The most significant contribution of his philosophical studies was to confirm him in his search for the most general application of mathematics to the physical world. It was this high degree of generality and abstraction that permitted him to include wave optics, particle optics, and dynamics in the same mathematical theory. In reading Kant, Hamilton claimed that his greatest pleasure was in finding his own opinions confirmed in Kant's works. It was more "recognizing" than "discovering."[4] He had the same reaction in talking to Faraday. Faraday, the eminently experimental chemist, had arrived at a view as antimaterialistic as his own, although his own view came completely from theoretical studies.[5]

Scientific Work. Hamilton's major contributions were in the algebra of quaternions, optics, and dynamics. He spent more time on quaternions than on any other subject. Next in importance was optics. His dynamics, for which he is best known, was a distant third. His manuscript notebooks and papers contain many optical studies and drafts of published papers, while there is relatively little on his dynamical theories. One is forced to conclude that the papers on dynamics were merely extensions of fundamental ideas developed in his optics.

The published papers are very difficult to read. Hamilton gave few examples to illustrate his methods, and his exposition is completely analytical with no diagrams. His unpublished papers are quite different. In working out his ideas he tested them on practical problems, often working through lengthy computations. A good example is his application of the theory of systems of rays to the symmetrical optical system—a very valuable investigation that remained in manuscript until his optical papers were collected and published in 1931.

In the "Theory of Systems of Rays" (1827) Hamilton continued the work of his paper "On Caustics" (1824), but he applied the analysis explicitly to geometrical optics and introduced the characteristic function. Only the first of three parts planned for the essay was actually published, but Hamilton continued his analysis in three published supplements between 1830 and 1832. In the "Theory of Systems of Rays," Hamilton considered the rays of light emanating from a point source and being reflected by a curved mirror. In this first study the medium is homogeneous and isotropic, that is, the velocity of light is the same at every point and for every direction in the medium. Under these conditions the rays filling space are such that they can be cut orthogonally by a family of surfaces, and Hamilton proved that this condition continues to hold after any number of reflections and refractions.

Malus had proved the case only for a single reflection or refraction. There had been more general proofs given subsequently by Dupin, Quetelet, and Gergonne, but Hamilton was apparently unaware of their work.[6] He proved the theorem by a modification of the principle of least action which he later called the principle of varying action. The principle of least or stationary action (which was identical to Fermat's principle in the optical case) determined the path of the ray between any fixed end points. By varying the initial point on a surface perpendicular to the ray, Hamilton was able to demonstrate that the final points of the original and varied rays fall on a surface perpendicular to both of them. Therefore, at any time after several reflections the end points of the rays determine a surface perpendicular to the rays.

Hamilton called these surfaces "surfaces of constant action," a term that made sense if light was considered as particles, since all the particles emanating together from the source reached the surface at the same time. But Hamilton continually insisted that this "remarkable analogy" between the principle of least action and geometrical optics did not require the assumption of any hypothesis about the nature of light, because the stationary integral to be found by the calculus of variations was of the same form whether light was considered as particles (in which case the integral is the action $\int v\,ds$) or as waves (in which case the integral is the optical length $\int \mu\,ds$).

From the property that all rays are cut perpendicularly by a family of surfaces, Hamilton showed that the differential form

$$\alpha\, dx + \beta\, dy + \gamma\, dz$$

has to be derived, where α, β, γ are direction cosines of the ray and are taken as functions of the coordinates (x,y,z). The direction cosines must then equal the partial differential coefficients of a function V of x, y, z, so that

$$\alpha = \frac{\partial V}{\partial x},\ \beta = \frac{\partial V}{\partial y},\ \gamma = \frac{\partial V}{\partial z}.$$

From the relation between the direction cosines $\alpha^2 + \beta^2 + \gamma^2 = 1$ Hamilton obtained the expression

$$\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 = 1$$

by substitution. He noticed that a solution of this equation is obtained by making V the length of the ray. It is the function V that Hamilton called the "characteristic function," and he declared that it was "the most complete and simple definition that could be given of the application of analysis to optics." The characteristic function contains "the whole of mathematical optics." [7]

The long third supplement of 1832 was Hamilton's most general treatment of the characteristic function in optics and was essentially a separate treatise. Where the initial point was previously fixed so that V was a function only of the coordinates of the final point, the initial coordinates were now added as variables so that V became a function of both initial and final coordinates. The characteristic function now completely described the optical system, since it held for any set of incident rays rather than only for those from a given initial point. Hamilton further generalized his investigation by allowing for heterogeneous and anisotropic media, and this greater generality allowed him to introduce an auxiliary function T, which, in the case of homogeneous initial and final media, is a function of the directions of the initial and final rays.

The importance of the characteristic function came from the fact that it described the system as a function of variables describing the initial and final rays. The principle of least action determined the optical path between fixed points. The characteristic function made the optical length a function of variable initial and final points.

At the end of the third supplement, Hamilton applied his characteristic function to the study of Fresnel's wave surface and discovered that for the case of biaxial crystals there exist four conoidal cusps on the wave surface. From this discovery he predicted that a single ray incident in the correct direction on a biaxial crystal should be refracted into a cone in the crystal and emerge as a hollow cylinder. He also predicted that if light were focused into a cone incident on the crystal, it would pass through the crystal as a single ray and emerge as a hollow cone. Hamilton described his discovery to the Royal Irish Academy on 22 October 1832 and asked Humphrey Lloyd, professor of natural philosophy at Trinity College, to attempt an experimental verification. Lloyd had some difficulty obtaining a satisfactory crystal, but two months later he wrote to Hamilton that he had found the cone.

Hamilton's theoretical prediction of conical refraction and Lloyd's verification caused a sensation. It was one of those rare events where theory predicted a completely unexpected physical phenomenon. Unfortunately it also involved Hamilton in an unpleasant controversy over priority with his colleague James MacCullagh, who had come very close to the discovery in 1830. MacCullagh was persuaded not to push his claim, but after this incident Hamilton was very sensitive about questions of priority.

Hamilton's theory of the characteristic function had little impact on the matter of greatest moment in optics at the time—the controversy between the wave and particle theories of light. Since his theory applied equally well to both explanations of light, his work in a sense stood above the controversy. He chose, however, to support the wave theory; and his prediction of conical refraction from Fresnel's wave surface was taken as another bit of evidence for waves. He entered into the debates at the meetings of the British Association and took part in an especially sharp exchange at the Manchester meeting of 1842, where he defended the wave theory against attacks by Sir David Brewster.

Shortly after completion of his third supplement to the "Theory of Systems of Rays," Hamilton undertook to apply his characteristic function to mechanics as well as to light. The analogy was obvious from his first use of the principle of least action. As astronomer royal of Ireland he appropriately applied his theory first to celestial mechanics in a paper entitled "On a General Method of Expressing the Paths of Light and of the Planets by the Coefficients of a Characteristic Function" (1833). He subsequently bolstered this rather general account with a more detailed study of the problem of three bodies using the characteristic function. The latter treatise was not published, however, and Hamilton's first general statement of the characteristic function applied to dynamics was his famous paper "On a General Method in Dynamics" (1834), which was followed the next year by a second essay on the same subject.

These papers are difficult to read. Hamilton presented his arguments with great economy, as usual, and his approach was entirely different from that now commonly presented in textbooks describing the method. In the two essays on dynamics Hamilton first applied the characteristic function V to dynamics just as he had in optics, the characteristic function being the action of the system in moving from its initial to its final point in configuration space. By his law of varying action he made the initial and final coordinates the independent variables of the characteristic function. For conservative systems, the total energy H was constant along any real path but varied if the initial and final points were varied, and so the characteristic function in dynamics became a function of the $6n$ coordinates of initial and final position (for n particles) and the Hamiltonian H.

The function V could be found only by integrating the equations of motion—a formidable task, as Hamilton realized. His great achievement, as he saw it, was to have reduced the problem of solving $3n$ ordinary differential equations of the second order (as given by Lagrange) to that of solving two partial differential equations of the first order and second degree. It was not clear that the problem was made any easier, but as Hamilton said: "Even if it should be thought that no practical facility is gained, yet an intellectual pleasure may result from the reduction of the most complex . . . of all researches respecting the forces and motions of body, to the study of one characteristic function, the unfolding of one central relation."[8]

The major part of the first essay was devoted to methods of approximating the characteristic function in order to apply it to the perturbations of planets and comets. It was only in the last section that he introduced a new auxiliary function called the principal function (S) by the transformation $V = tH + S$, thereby adding the time t as a variable in place of the Hamiltonian H.

The principal function could be found in a way analogous to the characteristic function; that is, it had to satisfy the following two partial differential equations of the first order and second degree:

$$\frac{\partial S}{\partial t} + \sum \frac{1}{2m}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right] = U$$

$$\frac{\partial S}{\partial t} + \sum \frac{1}{2m}\left[\left(\frac{\partial S}{\partial a}\right)^2 + \left(\frac{\partial S}{\partial b}\right)^2 + \left(\frac{\partial S}{\partial c}\right)^2\right] = U_0.$$

The variables x, y, z of the first equation represent the position of the particles at some time t; and the variables a, b, c of the second equation are the initial coordinates. U is the negative of the potential energy.

In the second essay, Hamilton deduced from the principal function the now familiar canonical equations of motion and immediately below showed that the same function S was equal to the time integral of the Lagrangian between fixed points

$$S = \int_0^t (T + U)\,dt = \int_0^t L\,dt.$$

The statement that the variation of this integral must be equal to zero is now referred to as Hamilton's principle.

A solution to Hamilton's principal function was very difficult to obtain in most actual cases, and it was K. G. J. Jacobi who found a much more useful form of the same equation.[9] In Jacobi's theory the S function is a generating function which completely characterizes a canonical transformation even when the Hamiltonian depends explicitly on the time. Since the canonical transformation depends on a single function, Jacobi was able to drop the second of Hamilton's two equations and the problem was reduced to the solution of the single partial differential equation

$$\frac{\partial S}{\partial t} + H\left(q_1, \cdots, q_n; \frac{\partial S}{\partial q_1}, \cdots, \frac{\partial S}{\partial q_n}; t\right) = 0,$$

which is usually referred to as the Hamilton-Jacobi equation. Hamilton had shown that the principal function S, defined as the time integral of the Lagrangian L, was a special solution of a partial differential equation; but it was Jacobi who demonstrated the converse, that by the theory of canonical transformations any complete solution of the Hamilton-Jacobi equation could be used to describe the motion of the mechanical system.[10]

The difficulty of solving the Hamilton-Jacobi equation gave little advantage to the Hamiltonian method over that of Lagrange in the nineteenth century. The method had admirable elegance but little practical advantage. With the rise of quantum mechanics, however, Hamilton's method suddenly regained importance because it was the one form of classical mechanics that carried over directly into the quantum interpretation. A great advantage of the Hamiltonian method was the close analogy between mechanics and optics that it contained; and this analogy was exploited by Louis de Broglie and Schrödinger in their formulations of wave mechanics.

Hamilton's tendency to pursue his studies in their greatest generality led to other important contributions. He extended his general method in dynamics to create a "calculus of principal relations," which permitted the solution of certain total differential equations by the calculus of variations. Another important contribution was the hodograph, the curve defined by the velocity vectors of a point in orbital motion taken as drawn from the origin rather than from the moving point.

Hamilton also attempted to apply his dynamics to the propagation of light in a crystalline medium. Previous authors, said Hamilton, had written much on the preservation of light vibrations in different media, but no one had attempted to investigate the propagation of a wave front into an undisturbed medium; or as he explained it to John Herschel, "Much had been done, perhaps, in the dynamics of *light;* little, I thought, in the dynamics of *darkness.*"[11] This new science of the dynamics of darkness he named "skotodynamics." He was actually hampered by his enthusiasm for Bošković's theory because it led him to study the medium as a series of attracting points rather than as a continuum; but his research, as usual, led to important new ideas. One was the distinction between group velocity and phase velocity. Another was his valuable study of "fluctuating functions," an extension of Fourier's theorem, which in turn led him to give the first complete asymptotic expansion for Bessel functions.

All of Hamilton's work in optics and dynamics depended on a single central idea, that of the characteristic function. It was the first of his two great "discoveries." The second was the quaternions, which he discovered on 16 October 1843 and to which he devoted most of his efforts during the remaining twenty-two years of his life. In October 1828 Hamilton complained to his friend John T. Graves about the shaky foundations of algebra. Such notions as negative and imaginary numbers, which appeared to be essential for algebra, had no real meaning for him; and he argued that a radical rewriting of the logical foundations of algebra was badly needed.[12] In the same year John Warren published *A Treatise on the Geometrical Representation of the Square Roots of Negative Quantities.* Hamilton read it in 1829 at Graves's instigation. Warren's book described the so-called Argand diagram by which the complex number is represented as a point on a plane with one rectangular axis representing the real part of the number and the other axis representing the imaginary part. This geometrical representation of complex numbers raised two new questions in Hamilton's mind: (1) Is there any other algebraic representation of complex numbers that will reveal all valid operations on them? (2) Is it possible to find a hypercomplex number that is related to three-dimensional space just as a regular complex number is related to two-dimensional space? If such a hypercomplex number could be found, it would be a "natural" algebraic representation of space, as opposed to the artificial and somewhat arbitrary representation by coordinates.

On 4 November 1833 Hamilton read a paper on algebraic couples to the Royal Irish Academy in which he presented his answer to the first question. His algebraic couples consisted of all ordered pairs of real numbers, for which Hamilton defined rules of addition and multiplication. He then demonstrated that these couples constituted a commutative associative division algebra, and that they satisfied the rules for operations with complex numbers. For some mathematicians the theory of number couples was a more significant contribution to mathematics than the discovery of quaternions.[13] On 1 June 1835 Hamilton presented a second paper on number couples entitled *Preliminary and Elementary Essay on Algebra as the Science of Pure Time,* in which he identified the number couples with steps in time. He combined this paper with his earlier paper of 1833, added some *General Introductory Remarks,* and published them in the *Proceedings of the Royal Irish Academy* of 1837.

This was a time of intense intellectual activity for Hamilton. He was deeply involved in the study of dynamics as well as the algebra of number couples. This was also the time when he was most involved in the study of Kant and was forming his own idealistic philosophy. Manuscript notes from 1830 and 1831 (before he read Kant) already contained Hamilton's conviction that the foundation for algebra was to be found in the ordinal character of numbers, and that this ordering had an intuitive basis in time. Kant's philosophy must certainly have strengthened this conviction.[14] It was through the concept of time that Hamilton hoped to correct the weaknesses in the logical foundations of algebra. He recognized three different schools of algebra. The first was the practical school, which considered algebra as an instrument for the solution of problems; therefore it sought rules of application. The second, or philological school, considered algebra as a language consisting of formulas composed of symbols which could be arranged only in certain specified ways. The third was the theoretical school, which considered algebra as a group of theorems upon which one might meditate. Hamilton identified himself with the last school and insisted that in algebra it was necessary to go beyond the signs of the formalist to the things signified. Only by relating numbers to some real intuition could algebra be truly called a "science."

In the *Critique of Pure Reason,* Kant argued that the ordering of phenomena in space was an operation of the mind and that this ordering had to be part of the mode of perceiving things. The science that

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studied this aspect of perception in its purest form was geometry; therefore geometry could well be called the "science of pure space." According to Hamilton the intuition of order in time was even more deep-seated in the human mind than the intuition of order in space. We have an intuitive concept of pure or mathematical time more fundamental than all actual chronology or ordering of particular events. This intuition of mathematical time is the real referent of algebraic symbolism. It is "co-extensive and identical with Algebra, so far as Algebra itself is a Science."[15] Hamilton presented this idea to his fellow mathematicians Graves and De Morgan with some hesitation and received an unenthusiastic response, which he probably anticipated. Although the idea had been with him for some time Hamilton mentioned it casually to Graves for the first time in a letter of 11 July 1835 where he referred to it as this "crochet of mine."[16] But in spite of the adverse reaction Hamilton never wavered in his conviction that the intuition of time was the foundation of algebra.

Hamilton had less success in answering the second question posed above, whether it would be possible to write three-dimensional complex numbers or, as he called them, "triplets." Addition of triplets was obvious, but he could find no operation that would follow the rules of multiplication. Thirteen years after Hamilton's death G. Frobenius proved that there is no such algebra and that the only possible associative division algebras over the real numbers are the real numbers themselves, complex numbers, and real quaternions. In searching for the elusive triplets, Hamilton sought some way of making his triplets satisfy the law of the moduli, since any algebra obeying this law is a division algebra. The modulus of a complex number is that number multiplied by its complex conjugate, and the law of the moduli states that the product of the moduli of two complex numbers equals the modulus of the product. By analogy to complex numbers, Hamilton wrote the triplet as $x + iy + jz$ with $i^2 = j^2 = -1$ and took as its modulus $x^2 + y^2 + z^2$. The product of two such moduli can be expressed as the sum of squares; but it is the sum of four squares not the sum of three squares, as would be the case if it were the modulus of a triplet.

The fact that he obtained the sum of four squares for the modulus of the product must have indicated to Hamilton that possibly ordered sets of four numbers, or "quaternions," might work where the triplets failed. Thus he tested hypercomplex numbers of the form $(a + ib + jc + kd)$ to see if they satisfied the law of the moduli. They worked, but only by sacrificing the commutative law. Hamilton had to make the product $ij = -ji$.[17] Hamilton's great insight came in realizing that he could sacrifice commutativity and still have a meaningful and consistent algebra. The laws for multiplication of quaternions then followed immediately:

$$\begin{aligned} ij &= k = -ji, \\ jk &= i = -kj, \\ ki &= j = -ik, \\ i^2 &= j^2 = k^2 = ijk = -1. \end{aligned}$$

The quaternions came to Hamilton in one of those flashes of understanding that occasionally occur after long deliberation on a problem. He was walking into Dublin on 16 October 1843 along the Royal Canal to preside at a meeting of the Royal Irish Academy, when the discovery came to him. As he described it, "An electric circuit seemed to close."[18] He immediately scratched the formula for quaternion multiplication on the stone of a bridge over the canal. His reaction must have been in part a desire to commemorate a discovery of capital importance, but it was also a reflection of his working habits. Hamilton was an inveterate scribbler. His manuscripts are full of jottings made on walks and in carriages. He carried books, pencils, and paper everywhere he went. According to his son he would scribble on his fingernails and even on his hard-boiled egg at breakfast if there was no paper handy.

Hamilton was convinced that in the quaternions he had found a natural algebra of three-dimensional space. The quaternion seemed to him to be more fundamental than any coordinate representation of space, because operations with quaternions were independent of any given coordinate system. The scalar part of the quaternion caused difficulty in any geometrical representation and Hamilton tried without notable success to interpret it as an extraspatial unit. The geometrical significance of the quaternion became clearer when Hamilton and A. Cayley independently showed that the quaternion operator rotated a vector about a given axis.[19]

The quaternions did not turn out to be the magic key that Hamilton hoped they would be, but they were significant in the later development of vector analysis. Hamilton himself divided the quaternion into a real part and a complex part which he called a vector. The multiplication of two such vectors according to the rules for quaternions gave a product consisting again of a scalar part and a complex part.

$$\alpha = xi + yj + zk$$
$$\alpha' = x'i + y'j + z'k$$

$$\alpha\alpha' = -(xx' + yy' + zz') + i(yz' - zy') + j(zx' - xz') + k(xy' - yx')$$

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The scalar part, which he wrote as $S.\ \alpha\alpha'$, is recognizable as the negative of the scalar or dot product of vector analysis, and the vector part, which he wrote as $V.\ \alpha\alpha'$, is recognizable as the vector or cross product. Hamilton frequently used these symbols as well as a new operator which he introduced,

$$\lhd = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}$$

and

$$-\lhd^2 = \left(\frac{d}{dx}\right)^2 + \left(\frac{d}{dy}\right)^2 + \left(\frac{d}{dz}\right)^2,$$

and called attention to the fact that the applications of this new operator in physics "must be extensive to a high degree." [20] Gibbs suggested the name "del" for the same operator in vector analysis and this is the term now generally used.

Hamilton was not the only person working on vectorial systems in the mid-nineteenth century.[21] Hermann Günther Grassmann working independently of Hamilton published his *Ausdehnungslehre* in 1844 in which he treated n-dimensional geometry and hypercomplex systems in a much more general way than Hamilton; but Grassmann's book was extremely difficult and radical in its conception and so had very few readers. Hamilton's books on quaternions were also too long and too difficult to attract much of an audience. His *Lectures on Quaternions* (1853) ran to 736 pages with a sixty-four-page preface. Any reader can sympathize with John Herschel's request that Hamilton make his principles "clear and familiar down to the level of ordinary unmetaphysical apprehension" and to "introduce the new phrases as strong meat gradually given to babes." [22] His advice was ignored and the *Lectures* bristles with complicated new terms such as *vector, vehend, vection, vectum, revector, revehend, revection, revectum, provector, transvector,* etc.[23] Herschel replied with a cry of distress, but it did no good and the *Elements of Quaternions,* which began as a simple manual, was published only after Hamilton's death and was even longer than the *Lectures.*

The first readable book on quaternions was P. G. Tait's *Elementary Treatise on Quaternions* (1867). Tait and Hamilton had been in correspondence since 1858, and Tait had held up the publication of his book at Hamilton's request until after the *Elements* appeared. Tait was Hamilton's most prominent disciple, and during the 1890's entered into a heated controversy with Gibbs and Heaviside over the relative advantages of quaternions and vectors. One can sympathize with Tait's commitment to quaternions and his dissatisfaction with vector analysis. It was difficult enough to give up the commutative property in quaternion multiplication, but vector analysis required much greater sacrifices. It accepted *two* kinds of multiplication, the dot product and the cross product. The dot product was not a real product at all, since it did not preserve closure; that is, the product was not of the same nature as the multiplier and the multiplicand. Both products failed to satisfy the law of the moduli, and both failed to give an unambiguous method of division. Moreover the cross product (in which closure was preserved) was neither associative nor commutative.[24] No wonder a devout quaternionist like Tait looked upon vector analysis as a "hermaphrodite monster." [25] Nevertheless vector analysis proved to be the more useful tool, especially in applied mathematics. The controversy did not entirely die, however, and as late as 1940 E. T. Whittaker argued that quaternions "may even yet prove to be the most natural expression of the new physics [quantum mechanics]." [26]

The quaternions were not the only contribution that Hamilton made to mathematics. In 1837 he corrected Abel's proof of the impossibility of solving the general quintic equation and defended the proof against G. B. Jerrard, who claimed to have found such a solution. He also became interested in the study of polyhedra and developed in 1856 what he called the "Icosian Calculus," a study of the properties of the icosahedron and the dodecahedron. This study resulted in an "Icosian Game" to be played on the plane projection of a dodecahedron. He sold the copyright to a Mr. Jacques of Piccadilly for twenty-five pounds. The game fascinated a mathematician like Hamilton, but it is unlikely that Mr. Jacques ever recovered his investment.

In spite of Hamilton's great fame in the nineteenth century one is left with the impression that his discoveries had none of the revolutionary impact on science that he had hoped for. His characteristic function in optics did not hit at the controversy then current over the physical nature of light, and it became important for geometrical optics only sixty years later when Bruns rediscovered the characteristic function and called it the method of the eikonal.[27] His dynamics was saved from oblivion by the important additions of Jacobi, but even then the Hamiltonian method gained a real advantage over other methods only with the advent of quantum mechanics. The quaternions, too, which were supposed to open the doors to so many new fields of science turned out to be a disappointment. Yet quaternions were the seed from which other noncommutative algebras grew. Matrices and even vector analysis have a parent in quaternions. Over the long run the success of Hamilton's work has justified his efforts. The high degree of abstraction and generality that made his papers so difficult to read has also made them stand the test of time, while more specialized researches with greater immediate utility have been superseded.

NOTES

1. Graves, *Life of Sir William Rowan Hamilton,* I, 141.
2. *Ibid.,* 186.
3. Hamilton to H. F. C. Logan, 27 June 1834, Graves, II, 87-88.
4. Hamilton to Lord Adare, 19 July 1834, Graves, II, 96; and to Wordsworth, 20 July 1834, Graves, II, 98.
5. Graves, II, 95-96.
6. *Mathematical Papers,* I, 463, editor's note.
7. *Ibid.,* 17, 168.
8. *Ibid.,* II, 105.
9. *Crelle's Journal,* **17** (1837), 97-162.
10. The differences between Hamilton's and Jacobi's formulations are described in detail in the *Mathematical Papers,* II, 613-621, editor's app. 2; and in Lanczos, *Variational Principles,* 229-230, 254-262.
11. *Mathematical Papers,* II, 599.
12. Graves, I, 303-304.
13. C. C. MacDuffee, "Algebra's Debt to Hamilton," in *Scripta mathematica,* **10** (1944), 25.
14. MS notebook no. 25, fol. 1, and notebook no. 24.5, fol. 49. See also Graves, I, 229, where Hamilton in 1827 referred to "the sciences of Space and Time (to adopt here a view of Algebra which I have elsewhere ventured to propose)."
15. *Mathematical Papers,* III, 5.
16. Graves, II, 143.
17. E. T. Whittaker, "The Sequence of Ideas in the Discovery of Quaternions," in *Proceedings of the Royal Irish Academy,* **50A** (1945), 93-98.
18. Graves, II, 435.
19. *Mathematical Papers,* III, 361-362.
20. *Ibid.,* 262-263.
21. Crowe, *History of Vector Analysis,* pp. 47-101.
22. Graves, II, 633.
23. Crowe, p. 36.
24. *Ibid.,* pp. 28-29.
25. *Ibid.,* p. 185.
26. E. T. Whittaker, "The Hamiltonian Revival," in *Mathematical Gazette,* **24** (1940), 158.
27. J. L. Synge, "Hamilton's Method in Geometrical Optics," in *Journal of the Optical Society of America,* **27** (1937), 75-82.

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I. ORIGINAL WORKS. Hamilton's mathematical papers have been collected in three volumes, *The Mathematical Papers of Sir William Rowan Hamilton* (Cambridge, 1931-1967). These volumes are carefully edited with short introductions and very valuable explanatory appendices and notes. The collection is not complete, but the editors have selected the most important papers, including many that were previously unpublished. A complete bibliography of Hamilton's published works appears at the end of vol. III of Robert P. Graves, *Life of Sir William Rowan Hamilton,* 3 vols. (Dublin, 1882-1889). Graves collected Hamilton's papers and letters for his biography shortly after Hamilton's death. The bulk of these manuscripts is now at the library of Trinity College, Dublin, with a smaller collection at the National Library of Ireland, Dublin. The manuscript collection at Trinity is very large, containing approximately 250 notebooks and a large number of letters and loose papers.

II. SECONDARY LITERATURE. R. P. Graves's biography is composed largely of letters which have been edited to remove much of the mathematical content. Graves also suppressed some correspondence that he considered too personal.

Most of the secondary literature on Hamilton has been written by mathematicians interested in the technical aspects of his work. An exception is Robert Kargon, "William Rowan Hamilton and Boscovichean Atomism," in *Journal of the History of Ideas,* **26** (1965), 137-140. The best introduction to Hamilton's optics is John L. Synge, *Geometrical Optics; an Introduction to Hamilton's Method* (Cambridge, 1937). Also valuable are his "Hamilton's Method in Geometrical Optics," in *Journal of the Optical Society of America,* **27** (1937), 75-82; G. C. Steward, "On the Optical Writings of Sir William Rowan Hamilton," in *Mathematical Gazette,* **16** (1932), 179-191; and George Sarton, "Discovery of Conical Refraction by Sir William Rowan Hamilton and Humphrey Lloyd (1833)," in *Isis,* **17** (1932), 154-170.

Hamilton's work in dynamics is described in René Dugas, "Sur la pensée dynamique d'Hamilton: origines optiques et prolongements modernes," in *Revue scientifique,* **79** (1941), 15-23; and A. Cayley, "Report on the Recent Progress of Theoretical Dynamics," in *British Association Reports* (1857), pp. 1-42. Another valuable exposition of the method is in Cornelius Lanczos, *The Variational Principles of Mechanics,* 3rd ed. (Toronto, 1966).

The centenary of Hamilton's discovery of quaternions was the occasion for two very important collections of articles, in *Proceedings of the Royal Irish Academy,* **50A,** no. 6 (Feb. 1945), 69-121; and in *Scripta mathematica,* **10** (1944), 9-63. These collections cover not only the quaternions, but also contain biographical notices, an article on the mathematical school at Trinity College, Dublin, and articles on Hamilton's dynamics, his optics, and his other contributions to algebra. The relationship between quaternions and vector analysis is described in great detail in Michael Crowe, *A History of Vector Analysis; the Evolution of the Idea of a Vectorial System* (Notre Dame, Ind., 1967); and in Reginald J. Stephenson, "Development of Vector Analysis From Quaternions," in *American Journal of Physics,* **34** (1966), 194-201; and Alfred M. Bork, "'Vectors Versus Quaternions'—the Letters in Nature," in *American Journal of Physics,* **34** (1966), 202-211.

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Hampson was educated at Manchester Grammar School and Trinity College, Oxford, graduating M.A. in 1881. He went to the Inner Temple, evidently with the intention of becoming a barrister; but he does