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OLEXA MYRON BILANIUK

LIE, MARIUS SOPHUS (*b.* Nordfjordeide, Norway, 17 December 1842; *d.* Christiania [now Oslo], Norway, 18 February 1899), *mathematics.*

Sophus Lie, as he is known, was the sixth and youngest child of a Lutheran pastor, Johann Herman Lie. He first attended school in Moss (Kristianiafjord), then, from 1857 to 1859, Nissen's Private Latin School in Christiania. He studied at Christiania University from 1859 to 1865, mainly mathematics and sciences. Although mathematics was taught by such people as Bjerknes and Sylow, Lie was not much impressed. After his examination in 1865, he gave private lessons, became slightly interested in astronomy, and tried to learn mechanics; but he could not decide what to do. The situation changed when, in 1868, he hit upon Poncelet's and Plücker's writings. Later, he called himself a student of Plücker's, although he had never met him. Plücker's momentous idea to create new geometries by choosing figures other than points—in fact straight lines—as elements of space pervaded all of Lie's work.

Lie's first published paper brought him a scholarship for study abroad. He spent the winter of 1869–1870 in Berlin, where he met Felix Klein, whose interest in geometry also had been influenced by Plücker's work. This acquaintance developed into a friendship that, although seriously troubled in later years, proved crucial for the scientific progress of both men. Lie and Klein had quite different characters as humans and mathematicians: the algebraist Klein was fascinated by the peculiarities of charming problems; the analyst Lie, parting from special cases, sought to understand a problem in its appropriate generalization.

Lie and Klein spent the summer of 1870 in Paris, where they became acquainted with Darboux and Camille Jordan. Here Lie, influenced by the ideas of the French "anallagmatic" school, discovered his famous contact transformation, which maps straight lines into spheres and principal tangent curves into curvature lines. He also became familiar with Monge's theory of differential equations. At the outbreak of the Franco-Prussian war in July, Klein left Paris; Lie, as a Norwegian, stayed. In August he decided to hike to Italy but was arrested near Fontainebleau as a spy. After a month in prison, he was freed through Darboux's intervention. Just before the Germans blockaded Paris, he escaped to Italy. From there he returned to Germany, where he again met Klein.

In 1871 Lie was awarded a scholarship to Christiania University. He also taught at Nissen's Private Latin School. In July 1872 he received his Ph.D. During this period he developed the integration theory of partial differential equations now found in many textbooks, although rarely under his name.

Lie's results were found at the same time by Adolph Mayer, with whom he conducted a lively correspondence. Lie's letters are a valuable source of knowledge about his development.

In 1872 a chair in mathematics was created for him at Christiania University. In 1873 Lie turned from the invariants of contact transformations to the principles of the theory of transformation groups. Together with Sylow he assumed the editorship of Niels Abel's works. In 1874 Lie married Anna Birch, who bore him two sons and a daughter.

His main interest turned to transformation groups, his most celebrated creation, although in 1876 he returned to differential geometry. In the same year he joined G. O. Sars and Worm Müller in founding the *Archiv för mathematik og naturvidenskab*. In 1882 the work of Halphen and Laguerre on differential invariants led Lie to resume his investigations on transformation groups.

Lie was quite isolated in Christiania. He had no students interested in his research. Abroad, except for Klein, Mayer, and somewhat later Picard, nobody paid attention to his work. In 1884 Klein and Mayer induced F. Engel, who had just received his Ph.D., to visit Lie in order to learn about transformation groups and to help him write a comprehensive book on the subject. Engel stayed nine months with Lie. Thanks to his activity the work was accomplished, its three parts being published between 1888 and 1893, whereas Lie's other great projects were never completed. F. Hausdorff, whom Lie had chosen to assist him in preparing a work on contact transformations and partial differential equations, got interested in quite different subjects.

This happened after 1886 when Lie had succeeded Klein at Leipzig, where, indeed, he found students, among whom was G. Scheffers. With him Lie published textbooks on transformation groups and on differential equations, and a fragmentary geometry of contact transformations. In the last years of his life Lie turned to foundations of geometry, which at that time meant the Helmholtz space problem.

In 1889 Lie, who was described as an open-hearted man of gigantic stature and excellent physical health, was struck by what was then called neurasthenia. Treatment in a mental hospital led to his recovery, and in 1890 he could resume his work. His character, however, had changed greatly. He became increasingly sensitive, irascible, suspicious, and misanthropic, despite the many tokens of recognition that were heaped upon him.

Meanwhile, his Norwegian friends sought to lure him back to Norway. Another special chair in mathematics was created for him at Christiania University, and in September 1898 he moved there. He died of pernicious anemia the following February. His papers have been edited, with excellent annotations, by F. Engel and P. Heegaard.

Lie's first papers dealt with very special subjects in geometry, more precisely, in differential geometry. In comparison with his later performances, they seem like classroom exercises; but they are actually the seeds from which his great theories grew. Change of the space element and related mappings, the lines of a complex considered as solutions of a differential equation, special contact transformations, and trajectories of special groups prepared his theory of partial differential equations, contact transformations, and transformation groups. He often returned to this less sophisticated differential geometry. His best-known discoveries of this kind during his later years concern minimal surfaces.

The crucial idea that emerged from his preliminary investigations was a new choice of space element, the contact element: an incidence pair of point and line or, in n dimensions, of point and hyperplane. The manifold of these elements was now studied, not algebraically, as Klein would have done—and actually did—but analytically or, rather, from the standpoint of differential geometry. The procedure of describing a line complex by a partial differential equation was inverted: solving the first-order partial differential equation

$$F\left(x, x_1, \cdots, x_{n-1}, \frac{\partial x}{\partial x_1}, \cdots, \frac{\partial x}{\partial x_{n-1}}\right) = 0$$

means fibering the manifold $F(x, x_1, \cdots, x_{n-1}, p_1, \cdots, p_{n-1}) = 0$ of $(2n-1)$-space by n-submanifolds on which the Pfaffian equation $dx = p_1 dx + \cdots + p_{n-1}\, dx_{n-1}$ prevails. This Pfaffian equation was interpreted geometrically: it means the incidence of the contact elements $\ulcorner x, x_1, \cdots, x_{n-1}, p_1, \cdots, p_{n-1}\urcorner$ and $\ulcorner x + dx, x_1 + dx_1, \cdots, x_{n-1} + dx_{n-1}, p_1 + dp_1, \cdots, p_{n-1} + dp_{n-1}\urcorner$. This incidence notion was so strongly suggested by the geometry of complexes (or, as one would say today, by symplectic geometry) that Lie never bothered to state it explicitly. Indeed, if it is viewed in the related $2n$-vector space instead of $(2n+1)$-projective space, incidence means what is called conjugateness with respect to a skew form. It was one of Lie's idiosyncrasies that he never made this skew form explicit, even after Frobenius had introduced it in 1877; obviously Lie did not like it because he had missed it. It is another drawback that Lie adhered mainly to projective formulations in $(2n-1)$-space, which led to clumsy formulas as soon as things had to be presented analytically; homogeneous formulations in $2n$-space are more elegant and make the ideas much clearer, so they will be used in the sequel such that the partial differential equation is written as $F(x_1, \cdots, x_n, p_1, \cdots p_n) = 0$, with $p_1dx_1 + \cdots + p_ndx_n$ as the total differential of the nonexplicit unknown variable. Then the skew form (the Frobenius covariant) has the shape $\sum(\delta p_idx_i - dp_i\delta x_i)$.

A manifold $z = f(x_1, \cdots, x_n)$ in $(n + 1)$-space, if viewed in the $2n$-space of contact elements, makes $\sum p_idx_i$ a complete differential, or, in geometrical terms, neighboring contact elements in this manifold are incident. But there are more such n-dimensional *Elementvereine:* a k-dimensional manifold in $(n + 1)$-space with all its n-dimensional tangent spaces shares this property. It was an important step to deal with all these *Elementvereine* on the same footing, for it led to an illuminating extension of the differential equation problem and to contact transformations. Finding a complete solution of the differential equation now amounted to fibering the manifold $F = 0$ by n-dimensional *Elementvereine.* In geometrical terms the Lagrange-Monge-Pfaff-Cauchy theory (which is often falsely ascribed to Hamilton and Jacobi) was refashioned: to every point of $F = 0$ the skew form assigns one tangential direction that is conjugate to the whole $(2n - 1)$-dimensional tangential plane. Integrating this field of directions, or otherwise solving the system of ordinary differential equations

$$\frac{dx_i}{dt} = \frac{\partial F}{\partial p_i}, \frac{dp_i}{dt} = -\frac{\partial F}{\partial x_i},$$

one obtains a fibering of $F = 0$ into curves, the "characteristic strips," closely connected to the Monge curves (touching the Monge cones). Thus it became geometrically clear why every complete solution also had to be fibered by characteristic strips.

Here the notion of contact transformation came in. First suggested by special instances, it was conceived of as a mapping that conserves the incidence of neighboring contact elements. Analytically, this meant invariance of $\sum p_idx_i$ up to a total differential. The characteristic strips appeared as the trajectories of such a contact transformation:

$$(F,\cdot) = \sum\left(\frac{\partial F}{\partial p_i}\frac{\partial}{\partial x_i} - \frac{\partial F}{\partial x_i}\frac{\partial}{\partial p_i}\right).$$

Thus characteristic strips must be incident everywhere as soon as they are so in one point. This led to a geometric reinterpretation of Cauchy's construction of one solution of the partial differential equation. From one $(n - 1)$-dimensional *Elementverein* on $F = 0$, which is easily found, one had to issue all characteristic strips. But even a complete solution was obtained in this way: by cross-secting the system of characteristics, the figure was lowered by two dimensions in order to apply induction. Solving the partial differential equation was now brought back to integrating systems of ordinary equations of, subsequently, 2, 4, $\cdots$, $2n$ variables. In comparison with older methods, this was an enormous reduction of the integration job, which at the same time was performed analytically by Adolph Mayer.

With the Poisson brackets $(F,\cdot)$ viewed as contact transformations, Jacobi's integration theory of systems

$$F_j(x_1, \cdots, x_n, p_1, \cdots p_n) = 0$$

was reinterpreted and simplified. Indeed, $(F,\cdot F_j)$ is nothing but the commutator of the related contact transformations. The notion of transformation group, although not yet explicitly formulated, was already active in Lie's unconscious. The integrability condition $(F_i, F_j) = \sum\rho_{ij}^k F_k$ (where the ρ_{ij}^k are functions) was indeed closely connected to group theory ideas, and it is not surprising that Lie called such a system a group. The theory of these "function groups," which was thoroughly developed for use in partial differential equations and contact transformations, was the last stepping-stone to the theory of transformation groups, which was later applied in differential equations.

Lie's integration theory was the result of marvelous geometric intuitions. The preceding short account is the most direct way to present it. The usual way is a rigmarole of formulas, even in the comparatively excellent book of Engel and Faber. Whereas transformation groups have become famous as Lie groups, his integration theory is not as well known as it deserves to be. To a certain extent this is Lie's own fault. The nineteenth-century mathematical public often could not understand lucid abstract ideas if they were not expressed in the analytic language of that time, even if this language would not help to make things clearer. So Lie, a poor analyst in comparison with his ablest contemporaries, had to adapt and express in a host of formulas, ideas which would have been said better without them. It was Lie's misfortune that by yielding to this urge, he rendered his theories obscure to the geometricians and failed to convince the analysts.

About 1870 group theory became fashionable. In 1870 C. Jordan published his *Traité des substitutions,* and two years later Klein presented his *Erlanger Programm.* Obviously Klein and Lie must have discussed group theory early. Nevertheless, to name a certain set of (smooth) mappings of (part of) n space, depending on r parameters, a group was still a new way of speaking. Klein, with his background in the theory of invariants, of course thought of very special groups, as his *Erlanger Programm* and later works prove.

Lie, however, soon turned to transformation groups in general—finite continuous groups, as he christened them ("finite" because of the finite number of parameters, and "continuous" because at that time this included differentiability of any order wanted). Today they are called Lie groups. In the mid-1870's this theory was completed, although its publication would take many years.

Taking derivatives (velocity fields) at identity in all directions creates the infinitesimal transformations of the group, which together form the infinitesimal group. The first fundamental theorem, providing a necessary and sufficient condition, tells how the derivatives at any parameter point $a_1, \cdots, a_r$ are linearly combined from those at identity. The second fundamental theorem says that the infinitesimal transformations will and should form what is today called a Lie algebra,

$$[X_i, X_j] = X_i X_j - X_j X_i = \sum_k c_{ij}^k X_k,$$

with some structure constants c_{ij}^k. Antisymmetry and Jacobi associativity yield the relations

$$c_{ij}^k + c_{ji}^k = 0,$$

$$\sum_k (c_{ij}^k c_{kl}^m + c_{jl}^k c_{ki}^m + c_{li}^k c_{kj}^m) = 0$$

between the structure constants. It cost Lie some trouble to prove that these relations were also sufficient.

From these fundamental theorems the theory was developed extensively. The underlying abstract group, called the parameter group, showed up. Differential invariants were investigated, and automorphism groups of differential equations were used as tools of solution. Groups in a plane and in 3-space were classified. "Infinite continuous" groups were also considered, with no remarkable success, then and afterward. Lie dreamed of a Galois theory of differential equations but did not really succeed, since he could not explain what kind of *ausführbare* operations should correspond to the rational ones of Galois theory and what solving meant in the case of a differential equation with no nontrivial automorphisms. Nevertheless, it was an inexhaustible and promising subject.

Gradually, quite a few mathematicians became interested in the subject. First, of course, was Lie's student Engel. F. Schur then gave another proof of the third fundamental theorem (1889–1890), which led to interesting new views; L. Maurer refashioned the proofs of all fundamental theorems (1888–1891); and Picard and Vessiot developed Galois theories of differential equations (1883, 1891). The most astonishing fact about Lie groups, that their abstract structure was determined by the purely algebraic phenomenon of their structure constants, led to the most important investigations. First were those of Wilhelm Killing, who tried to classify the simple Lie groups. This was a tedious job, and he erred more than once. This made Lie furious, and according to oral tradition he is said to have warned one of his students who was leaving: "Farewell, and if ever you meet that s.o.b., kill him." Although belittled by Lie and some of his followers, Killing's work was excellent. It was revised by Cartan, who after staying with Lie wrote his famous thesis (1894). For many years Cartan—gifted with Lie's geometric intuition and, although trained in the French tradition, as incapable as Lie of explaining things clearly—was the greatest, if not the only, really important mathematician who continued Lie's tradition in all his fields. But Cartan was isolated. Weyl's papers of 1922–1923 marked the revival of Lie groups. In the 1930's Lie's local approach gave way to a global one. The elimination of differentiability conditions in Lie groups took place between the 1920's and 1950's. Chevalley's development of algebraic groups was a momentous generalization of Lie groups in the 1950's. Lie algebras, replacing ordinary associativity by Jacobi associativity, became popular among algebraists from the 1940's. Lie groups now play an increasingly important part in quantum physics. The joining of topology to algebra on the most primitive level, as Lie did, has shown its creative power in this century.

In 1868 Hermann von Helmholtz formulated his space problem, an attempt to replace Euclid's foundations of geometry with group-theoretic ones, although in fact groups were never explicitly mentioned in that paper. In 1890 Lie showed that Helmholtz's formulations were unsatisfactory and that his solution was defective. His work on this subject, now called the Helmholtz-Lie space problem, is one of the most beautiful applications of Lie groups. In the 1950's and 1960's it was reconsidered in a topological setting.

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HANS FREUDENTHAL

LIEBER, THOMAS. See **Erastus, Thomas.**

LIEBERKÜHN, JOHANNES NATHANAEL (*b.* Berlin, Germany, 5 September 1711; *d.* Berlin, 7 December 1756), *anatomy*.

One of the most skillful German anatomists of the early eighteenth century, Lieberkühn was the son of Johannes Christianus Lieberkühn, a goldsmith. His father insisted that Johannes and his brother plan for careers in theology; thus prior to attending the academy at Jena, the boy was sent to the Halle Magdeburg Gymnasium.

At Jena, Lieberkühn studied mathematics, mechanics, and natural philosophy, coming under the influence of the physician-iatromathematician G. E. Hamberger (1697–1755). His interest in medicine was sharpened by this experience and he went on to study chemistry, anatomy, and physiology with Hermann Friedrich Teichmeyer (1685–1744) and Johann Adolph Wedel (1675–1747). Lieberkühn left Jena in 1733 and joined his brother at Rostock as a candidate to become a preacher. Here he delivered only a few sermons, choosing instead to continue studies of what most interested him. Johann Gustav Reinbech (1683–1741), the noted Protestant theologian, recognized Lieberkühn's aptitude for scientific studies and introduced him to the Prussian king, Frederick William I. After interviewing Lieberkühn, the king released him from the career set by his father, who had died in the meantime, so that he could devote full time to science and medicine.

Even before Lieberkühn returned to Jena in 1735, the Berlin Academy of Sciences enrolled him as a fellow as a result of his earlier work at Jena. Subsequent to his second period at Jena, Lieberkühn traveled and studied in other centers, including the Imperial Natural Sciences Academy in Erfurt, where its president, A. E. Buchner, made him a fellow.

He pursued further medical study, especially of anatomy and chemistry, at Leiden under Boerhaave, B. S. Albinus, J. D. Gaub, and Swieten. Lieberkühn's Leiden tutelage culminated in the award of a medical degree in 1739. His dissertation, *De valvula coli et usu processus vermicularis*, was commended by Boerhaave and Swieten. Another dissertation written at this time, "De plumbi indole," was not published and apparently is no longer extant.

Lieberkühn's fascination with anatomical structures and their mechanisms expressed itself in his *De fabrica et actione villorum intestinorum tenuium hominis* (1745). Here, for the first time, were described, in greatest detail, the structure and function of the numerous glands attached to the villi, appropriately called Lieberkühnian glands, as well as the structure and function of the villi found in the intestines.

All of these were made comprehensible by the meticulous and skillful injections of a mixture of wax, turpentine, and colophony or dark resin.

To explain the flow of fluids into these intestinal components Lieberkühn constructed a model. By means of an open curved brass tube, cone-shaped at both ends, with two outlet tubes placed toward the narrow center, each of which drained into a separate vessel, he demonstrated the flow of chyle from the arterioles to the villi and the ascent of the lymph from the villi into the small veins. He used tinted water to show the flow of lymph from the villus into the veins and plain water to emulate the flow of chyle to the villus. As a result of his excellent demonstrations of the intestinal contents of an experimental animal before the members of the Royal Society of London, Lieberkühn was made a fellow in 1740.

In the tradition of Hamberger, his first medical teacher at Jena, Lieberkühn explored the circulatory vessels, devising special microscopes to view in greater detail the intricacies of fluid motion within the living animal. One of these was the anatomical microscope used for viewing the circulation in frogs. The specimen was attached to the body of the microscope, which consisted of two thin silver plates between which was placed a small lens and around which were arranged hooks to hold and manipulate the animal. The part of the animal to be observed was fixed over the lens.

En route to Leiden, Lieberkühn had visited Amsterdam, where he saw a solar microscope similar to the one Fahrenheit made in 1736. Another microscope, for the invention of which Lieberkühn has been given credit, to be used in illuminating opaque objects, was based on the principle of Fahrenheit's solar microscope. It consisted of a small, concave, highly polished silver speculum, later termed a Lieberkühn, that provided intense reflection of the sun's rays directly upon the object. Although Descartes had shown a solar microscope in his *Dioptrique* as early as 1637, it