Poincaré's Rendiconti Paper on Relativity. Part I

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"Sur la dynamique de l'électron," published in 1906 in Rendiconti del Circolo Matematico di Palermo, is Poincaré's principal publication dealing with relativity. It is, among other things, of very considerable historical interest. To make it more readily accessible, the contents of this comprehensive paper will be presented, in a modernized version, in three parts that correspond to the three major topics treated in the paper. The first part, presented here, deals with the principle of relativity and the Lorentz group as well as with the action principle of Maxwell–Lorentz systems.

Henri Poincaré's comprehensive paper, "Sur la dynamique de l'électron" [Rendiconti del Circolo Matematico di Palermo **21,** 129 (1906); dated: Paris, July 1905] contains the illustrious mathematician's principal contributions to the theory of relativity. It is of considerable historic as well as intrinsic interest and deserves to be more fully and more widely known. It has not been easily accessible to physicists both because of the journal in which it is published and the style in which it is written. A presentation of the paper in the pages of this Journal, true to its content but modifying its form whenever indicated, is therefore in order.[1,2]

As its title indicates, the principal subject of the paper is the mechanics of the electron, the "elementary particle physics" topic around the turn of this century, which engaged the attention of leading theorists of that period such as J. J. Thomson, H. A. Lorentz, and M. Abraham. The underlying considerations relating to the principles of relativity and of the Lorentz group, as conceived by Poincaré, are presented in the Introduction and in introductory sections of the paper. In addition, the last section of the paper contains a discussion of a relativistically covariant theory of gravitation—the first such published theory.

The contributions of the paper fall thus into three groups: (I) foundations of special relativity, (II) theory of the electron, and (III) relativistic theory of gravitation. The present rendition of the Introduction and Pt. I is intended to be an accurate representation of both the text and the mathematical analysis. However, direct translations are given only of selected portions of the text. Nor is the mathematical part reproduced verbatim whenever changes in arrangement of the argument facilitate its understanding. To the same end the notation is modernized throughout, including the employment of the vector formalism.

These minor changes from the original, and also the different numbering of formulas which they have necessitated, need not introduce undue difficulties when comparison with the original is desired; because all the results in the original and the essential ideas in the derivations are left intact, and all comments that are not in the original are enclosed in braces or are given in footnotes. To facilitate further such comparisons, each formula that corresponds exactly to one in the original is numbered also as in the original, this numbering being enclosed in square brackets, with the first digit indicating the section (not shown in the original). In addition, a dictionary of symbols is presented in Table I. Last, for ease of future reference, some translated portions are listed by bracketed numbers.

ON THE DYNAMICS OF THE ELECTRON

Introduction

[1] "It seems at first sight that the aberration of light and the related optical and electrical phenomena would provide us with a means of determining the absolute motion of the earth, or rather its motion not with respect to the other stars, but with respect to the ether."

But all experiments including that of Michelson do not disclose such motion. Hence—

[2] "It seems that this impossibility to disclose experimentally the absolute motion of the earth is a general law of nature; we are led naturally to admit this law, which we shall call the *Postulate of Relativity*, and to admit it unrestrictedly. Although this postulate, which up till now agrees with experiment, must be confirmed or disproved by later more precise experiments, it is in any case of interest to see what consequences can flow from it."

[3] "An explanation has been proposed by Lorentz and by FitzGerald, who have introduced the hypothesis of a contraction experienced by all bodies in the direction of motion of the earth and proportional to the square of the aberration {i.e., $(v/c)^2$}; this contraction, which we shall call the *Lorentz contraction*, accounts for Michelson's experiment and for all other experiments performed to date."

Lorentz has sought to complete the contraction hypothesis so as to obtain full agreement with the postulate of relativity.

[4] "This is what he succeeded in accomplishing in his article entitled: 'Electromagnetic Phenomena in a System Moving with Any Velocity Smaller than that of Light' (Proceedings of the Amsterdam Academy, 27 May 1904)."

"The importance of the question impelled me to reconsider it; the results I have obtained agree with those of Mr. Lorentz in all the important points; I was led to modify and to complete them only in a few points of detail; the differences, which are of secondary importance, will be seen later on."

[5] "Lorentz's idea can be summarized thus: If it is possible to impress a common translation on an entire system without any of the sensible {"apparent" in the original} phenomena being altered, this means that the equations of an electrodynamic medium are unchanged by certain transformations, which we shall call *Lorentz transformations*; two systems, one stationary, the other in a state of translation, become thus the exact image of each other."

Lorentz's theory faces a serious obstacle in the problem of the "moving electron." In this respect Langevin's theory, involving an ellipsoidal electron of unchanging volume rather than unchanging axes normal to the electron's velocity, has the advantage of invoking only electromagnetic and binding forces. However, the theory disagrees with the relativity postulate, as was shown by Lorentz and as is shown here by a group-theoretical method. It is therefore necessary to complete Lorentz's theory of the electron—

[6] "It is necessary to assume a special force which explains simultaneously both the contraction and the constancy of two of the axes. I have sought to determine this force, and I have found that *it can be represented by a constant external pressure acting on the deformable and compressible electron, whose work is proportional to the change in volume of this electron.*"

If one adopts this postulate of constant pressure, and if one also assumes, as is suggested by Kaufman's experiments, that all of the electron's inertia is of purely electromagnetic origin, then the theory of the electron can be made to agree fully with the postulate of relativity. This is what is shown here using the principle of least action.

Lorentz has also deemed it necessary to assume that all forces, of whatever origin, transform under Lorentz transformations in the same way as the electromagnetic forces. This idea is applied here to the force of gravitation. It is shown that it follows that gravitational action propagates with the speed of light and that this conclusion can be reconciled with Laplace's proof of the impossibility of such propagation. The law of gravitation which is here developed satisfies Lorentz's condition and it reduces to the Newtonian law for sufficiently small velocities of the bodies. It also appears to agree with astronomical observations, "but the question can be decided only by a penetrating discussion."

[7] "But even admitting that this discussion would uphold the new hypothesis, what must we conclude? If gravitational propagation takes place with the speed of light, this cannot be fortuitous, it must be so because it is a function of the ether; and so, it becomes necessary to

penetrate the nature of this function, and to connect it with other functions of the fluid."

How are we to understand this conclusion from the relativity postulate that the speed of light enters both in the laws of electrodynamics and in the laws of gravitation or of any other type of force? Only two explanations are possible:

[8] "Either there is nothing in the world that is not of electromagnetic origin, or this part {i.e., the speed of light}, which is so to speak common to all physical phenomena, is only an appearance, something stemming from our methods of measurement. How do we make our measurements? By transporting to mutual juxtaposition objects considered as invariable solids, one would reply at first; but this is no longer true in the present theory if one admits Lorentzian contraction. In this theory two equal lengths are by definition two lengths which are traversed by light in equal times."

"Perhaps it would suffice to renounce this definition for the theory of Lorentz to be as completely overthrown as was the system of Ptolemy by the intervention of Copernicus. If this should happen one day, this would not prove the uselessness of Lorentz's effort; for Ptolemy, whatever one may think of him, was not useless to Copernicus."

TABLE I. Dictionary of notation employed in Poincaré's paper.

Quantity	Poincaré's notation	Present notation
Electric field[a]	(f, g, h)	$\mathbf{E}$
Magnetic field[a]	(α, β, γ)	$\mathbf{B}$
Scalar potential	ψ	ϕ
Vector potential	(F, G, H)	$\mathbf{A}$
Total electric current density[b]	(u, v, w)	$\mathbf{i}$
Velocity of electron[c]	(ξ, η, ζ)	$\mathbf{u}$
Displacement vector	(U, V, W)	$\boldsymbol{\xi}$
Force per unit volume	(X, Y, Z)	$\mathbf{f}$
Force per unit charge	(X_1, Y_1, Z_1)	$\mathbf{F}$
Relative velocity of two (inertial) reference frames[c]	ϵ	$-\beta$
$(1-\beta^2)^{-1/2}$	k	γ

[a] Electric *displacement* and magnetic *intensity* vectors in the original are here represented for the sake of agreement with present common practice, by current symbols for electric intensity and magnetic induction vectors, respectively.

[b] This includes Maxwell's displacement current density.

[c] In units of the speed of light *in vacuo* (i.e., in units such that $c=1$).

[9] "I have likewise not hesitated to publish these few partial results, even if at this very moment the entire theory may appear to be in danger by the discovery of cathode rays."[3]

1. Lorentz Transformation

The "fundamental formulas" {of the Maxwell–Lorentz theory}, using rationalized Gaussian units and choosing the units of length and of time so that the speed of light equals unity, are {see Table I}[4]:

$$\mathbf{i}=(\partial\mathbf{E}/\partial t)+\rho\mathbf{u}=\nabla\times\mathbf{B}, \tag{1}$$

$$\mathbf{B}=\nabla\times\mathbf{A},\qquad \mathbf{E}=-(\partial\mathbf{A}/\partial t)-\nabla\phi, \tag{2}$$

$$\partial\mathbf{B}/\partial t=-\nabla\times\mathbf{E}, \tag{3}$$

$$(\partial\rho/\partial t)+\nabla\cdot\rho\mathbf{u}=0, \qquad [1.1] \tag{4}$$

$$\nabla\cdot\mathbf{E}=\rho, \tag{5}$$

$$\partial\phi/\partial t+\nabla\cdot\mathbf{A}=0, \tag{6}$$

$$\Box\phi=-\rho,\qquad \Box\mathbf{A}=-\rho\mathbf{u},\quad [\Box\equiv\nabla^2-(\partial^2/\partial t^2)]; \tag{7}$$

$$\mathbf{f}=\rho[\mathbf{E}+(\mathbf{u}\times\mathbf{B})]. \qquad [1.2] \tag{8}$$

[10] "These equations admit of a remarkable transformation discovered by Lorentz, which is of interest because it explains why no experiment is capable of making known to us the absolute motion of the universe."

This transformation has the form {see Table I}

$$x'=\gamma l(x-\beta t),\qquad t'=\gamma l(t-\beta x),$$
$$y'=ly,\qquad z'=lz, \qquad [1.3] \tag{9}$$

where $l\equiv l(\beta)$ is, to begin with, an arbitrary constant.[5] The associated transformation of our Eqs. (1)–(8) can be deduced step by step, beginning with Eqs. (7).

We determine first the transformation of the charge density ρ. This is accomplished by combining the kinematic law of transformation of volumes with the following physical assumption[6]:

The charge of an electron is an invariant of our transformation. (10)

The law of transformation of volumes can be obtained as follows[7]:

"Let us consider a sphere carried along with the electron in its uniform translational motion, and let

$$(\mathbf{x}-\mathbf{u}t)^2=r^2$$

be the equation of this moving sphere, whose volume is $4\pi r^3/3$. The transformation will change it into an ellipsoid whose equation is easily found. In fact, one easily deduces from Eqs. (9)

$$\begin{aligned} x&=\gamma l^{-1}(x'+\beta t'),\\ t&=\gamma l^{-1}(t'+\beta x'),\\ y&=l^{-1}y',\\ z&=l^{-1}z'. \qquad [1.3\ bis] \quad (9') \end{aligned}$$

Thus the equation of the ellipsoid becomes

$$\gamma^2[x_1'+(\beta-u_1)t'-\beta u_1x']^2 + \sum_{j=2}^{3}[x_j'-\gamma u_j(t'+\beta x_1')]^2=l^2r^2.$$

This ellipsoid moves uniformly; for $t'=0$ it {i.e., its equation} reduces to

$$\gamma^2 x_1'^2(1-\beta u_1)^2+\sum_{j=2}^{3}(x_j'-\gamma\beta u_jx_1')^2=l^2r^2,$$

and it has the volume $(4\pi r^3/3)l^3/\gamma(1-\beta u_1)$."

From the ratio of volumes thus obtained and our assumption (10) it follows that the transformed density ρ' satisfies the equation

$$\rho'=\gamma l^{-3}(1-\beta u_1)\rho. \qquad [1.4] \quad (11)$$

Since by Eqs. (9) the transformation equations for the particle velocity vector $\mathbf{u}$ are given by the following "rule for addition of velocities,"

$$u_1'=(u_1-\beta)/(1-\beta u_1), \qquad u_j'=u_j/\gamma(1-\beta u_1) \quad (j=2,3), \quad (12)$$

we have also

$$\begin{aligned} \rho'u_1'&=\gamma l^{-3}(u_1-\beta)\rho,\\ \rho'u_j'&=l^{-3}\rho u_j \qquad (j=2,3). \quad [1.4\ bis] \quad (13) \end{aligned}$$

Comparing these results with those of Lorentz's paper [*loc. cit.*, Eqs. (7) and (8)], it is seen that there is agreement for the transformation of $\rho\mathbf{u}$, but not of ρ. In addition, the equation of conservation of charge, Eq. (4), cannot hold for the primed quantities of Lorentz since they do hold in the present case.

In fact, consider the Jacobian D of $(t+\lambda\rho, \mathbf{x}+\lambda\rho\mathbf{u})$ with respect to $(t, \mathbf{x})$. We find that

$$\partial D/\partial\lambda\,|_{\lambda=0}\equiv D_1=(\partial\rho/\partial t)+\nabla\cdot\rho\mathbf{u}. \qquad (14)$$

On the other hand, since $(t'+\lambda'\rho', \mathbf{x}'+\lambda'\rho'\mathbf{u}')$ is for $\lambda'=l^4\lambda$ connected with the corresponding unprimed quantities by the Lorentz transformation (9), it follows that $D'=D$.[8] Hence $D_1'=l^{-4}D_1$, and our result follows from Eq. (4) and the definition (14) of D_1.

Now, it is found from (9) that [notation in (7)]

$$\square'=l^{-2}\square. \qquad (15)$$

Hence it can be deduced from Eqs. (11) and (13) that the solution of Eqs. (7) in the primed quantities is[9]

$$\begin{aligned} \phi'&=\gamma l^{-1}(\phi-\beta A_1),\\ A_1'&=\gamma l^{-1}(A_1-\beta\phi),\\ A_j'&=l^{-1}A_j \qquad (j=2,3). \quad [1.7] \quad (16) \end{aligned}$$

[11] "These formulas differ considerably from those of Lorentz, but the difference bears in the last analysis only on the definitions."

Applying the relations (2) to the primed quantities, we obtain the transformation equations for the field vectors,

$$\begin{aligned} E_x'&=l^{-2}E_x, & B_x'&=l^{-2}B_x,\\ E_y'&=\gamma l^{-2}(E_y-\beta B_z), & B_y'&=\gamma l^{-2}(B_y+\beta E_z),\\ E_z'&=\gamma l^{-2}(E_z+\beta B_y), & B_z'&=\gamma l^{-2}(B_z-\beta E_y), \end{aligned} \qquad [1.9] \quad (17)$$

which are identical with those of Lorentz.

The proof of the validity of the remaining equations [1.1] in the primed quantities {i.e., the proof of their covariance under Lorentz transformations} now follows from the covariance results already obtained. In fact, Eq. (6) can be deduced from Eqs. (7) and (4); and Eqs. (1), (3), and (5) can be deduced from Eqs. (7), (2), and (6).

It remains to discover the law of transformation of the force density **f** of Eq. (8). This is accomplished by *assuming* the covariance of Eq. (8). Use of Eqs. (11), (13), (17), and (8) then yields the result

$$f_1' = \gamma l^{-5}(f_1 - \beta \mathbf{u}\cdot\mathbf{f}),$$
$$f_j' = l^{-5} f_j \qquad (j=2,3). \quad [1.11] \quad (18)$$

{For instance,

$$f_1' = l^{-5}\gamma\rho \times[(1-\beta u_1)E_1 + u_2(B_3 - \beta E_2) - u_3(B_2 - \beta E_3)]$$
$$= l^{-5}\gamma\rho(E_1 + u_2 B_3 - u_3 B_2 - \beta\mathbf{u}\cdot\mathbf{E}),$$

then using Eq. (8).}

By combining Eqs. (18) and (11) we obtain the transformation equations for the force **F** per unit charge:

$$F_1' = \gamma l^{-5}\rho\rho'^{-1}(F_1 - \beta\mathbf{u}\cdot\mathbf{F}),$$
$$F_j' = l^{-5}\rho\rho'^{-1}F_j \quad (j=2,3). \quad [1.11\ bis] \quad (19)$$

These equations differ significantly from those found by Lorentz [*loc. cit.*, Eq. (10)].

[12] "Before going further it is of importance to seek the cause of this significant divergence. It evidently derives from the fact that the formulas for u_i' are not the same, whereas the formulas for the electric and magnetic fields are the same."

Considering now the stability of the electron, it is seen that on the *assumption of an exclusively electromagnetic origin of its inertia*, the condition of its equilibrium requires the vanishing of **f** in its interior, and in view of Eqs. (18), this is indeed a covariant condition. Unfortunately, it is not an admissible condition; for instance when $\mathbf{u}=0$, we must have $\mathbf{E}=0$, and hence $\rho = \nabla\cdot\mathbf{E} = 0$. So we must look for additional *nonelectromagnetic forces* that insure the electron's equilibrium and find conditions that they must satisfy "in order that the equilibrium of electrons is not disturbed by the transformation."

2. Principle of Least Action

The principle of least action for Lorentz's equations is formulated here in a form that differs slightly from that presented by Lorentz.[10] We start with the action integral

$$J = \int dt \int d\tau [(\mathbf{E}^2 + \mathbf{B}^2)/2 - \mathbf{A}\cdot\mathbf{i}]$$
$$(d\tau \equiv d^3x), \quad [2.1] \quad (20)$$

where the quantities to be varied are taken to be connected by the relations

$$\nabla\cdot\mathbf{E} = \rho, \qquad \mathbf{B} = \nabla\times\mathbf{A}, \qquad \mathbf{i} = (\partial\mathbf{E}/\partial t) + \rho\mathbf{u}. \quad [2.2] \quad (21)$$

J is to be minimized subject to the following additional conditions:

The state of the system is fixed at the integration limits $t=t_0$ and $t=t_1$. (22)

"All our functions vanish at infinity."[11] (23)

Varying at first only **A**, we find after integrating by parts, and using the second relation in (21) and condition (23),

$$0 = \delta J = \int dt d\tau [\mathbf{B}\cdot(\nabla\times\delta\mathbf{A}) - \mathbf{i}\cdot\delta\mathbf{A}]$$
$$= -\int dt d\tau (\mathbf{i} - \nabla\times\mathbf{B})\cdot\delta\mathbf{A}, \quad (24)$$

which implies the field equation (1).

When the latter equation is substituted in Eq. (20), we obtain, again by an integration by parts and use of the second of Eqs. (21), the alternative expression

$$J = \int dt d\tau \tfrac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) \quad [2.4] \quad (25)$$

{which serves as the starting point for the relativistic discussion in the next section}.

Returning to the expression (20), we now vary the other quantities. We apply Eq. (1) [see Eq. (24)] and allow for the first of Eqs. (21) with the aid of the Lagrange multiplier ϕ. Our variation assumes then the form

$$\delta J = \int dt d\tau [(\mathbf{E}\cdot\delta\mathbf{E} - \mathbf{A}\cdot\delta\mathbf{i} - \phi(\nabla\cdot\delta\mathbf{E} - \delta\rho)].$$

Replacing here $\delta\mathbf{i}$ by the expression obtained from the last of Eqs. (21), and integrating by parts both spatially and with respect to t, we find upon applying conditions (22) and (23),

$$\delta J = \int dt d\tau \{[\mathbf{E} + (\partial\mathbf{A}/\partial t) + \nabla\phi]\cdot\delta\mathbf{E} + \phi\delta\rho - \mathbf{A}\cdot\delta(\rho\mathbf{u})\}. \quad [2.7] \quad (26)$$

When $\delta\rho=0$ and $\delta(\rho\mathbf{u})=0$, the vanishing of this variation yields the second of Eqs. (2).

If we now allow also for this equation, then Eq. (26) reduces to

$$\delta J=\int dt d\tau[\phi\delta\rho-\mathbf{A}\cdot\delta(\rho\mathbf{u})], \quad [2.9] \quad (27)$$

and this expression must lead to our remaining equation, Eq. (8). Indeed, if $\boldsymbol{\xi}\equiv\boldsymbol{\xi}(\mathbf{x}_0, t)$ denotes the displacement of the volume element $d\tau$ from a given initial position $\mathbf{x}_0$, and $\mathbf{f}$ is as defined in Table I, then we have by a generalization of D'Alembert's principle,

$$\delta J=-\int dt d\tau \mathbf{f}\cdot\delta\boldsymbol{\xi}, \quad [2.10] \quad (28)$$

where δJ is given in Eq. (27).

In order to transform δJ, Eq. (27), into a form involving only the variations $\delta\xi_i$, we apply the *principle of conservation of charge*, in the sense that the charge inside a given element of volume is unchanged during the variation in the position or shape of the volume, whether the variation be actual or virtual. It follows that in addition to the usual "equation of continuity" (i.e., equation of conservation of charge)

$$(\partial\rho/\partial t)+(\rho u_i)_{,i}=0$$

$[f_{,i}\equiv(\partial f/\partial x_i)$; repeated-index summation convention] (29)

we have the similar equation for the virtual motion of the charges (noting that $\mathbf{u}=d\boldsymbol{\xi}/dt$ since $\mathbf{x}=\mathbf{x}_0+\boldsymbol{\xi}$),

$$(\partial\rho/\partial\epsilon)+[\rho(d\xi_i/d\epsilon)]_{,i}=0, \quad (30)$$

where ϵ is the parameter for the virtual changes (as t is for the actual changes), and we distinguish by the respective symbols ∂ and d *local variation* (i.e., keeping x constant) and *convective variation* (i.e., keeping $\mathbf{x}_0$ constant). Hence

$$(\partial\rho/\partial\epsilon)\delta\epsilon\equiv\delta\rho=-(\rho\delta\xi_i)_{,i}, \quad (31)$$

since, of course, the $\delta\boldsymbol{\xi}$ in Eq. (28) are of convective character, so that,

$$\delta\boldsymbol{\xi}=(d\boldsymbol{\xi}/d\epsilon)\delta\epsilon. \quad (32)$$

Again, using the familiar connection between local and convective derivatives, and Eq. (32), we have

$$\begin{aligned}\delta u_i&=\delta\epsilon(\partial u_i/\partial\epsilon)\\&=\delta\epsilon[(du_i/d\epsilon)-u_{i,j}(d\xi_j/d\epsilon)]\\&=\delta\epsilon[(d^2\xi_i/d\epsilon dt)-\cdots]\\&=\delta\epsilon[(d^2\xi_i/dtd\epsilon)-\cdots]\\&=(d/dt)\delta\xi_i-u_{i,j}\delta\xi_j\\&=(\partial/\partial t)\delta\xi_i+(\delta\xi_i)_{,j}u_j-u_{i,j}\delta\xi_j.\end{aligned}$$

Hence, adding the zero represented by Eq. (30), and using Eq. (31), we find[12]

$$\begin{aligned}\delta(\rho u_i)&=u_i\delta\rho+\rho du_i\\&=-(\rho u_i\delta\xi_j)_{,j}+(\partial/\partial t)(\rho\delta\xi_i)\\&\quad+(\rho\delta\xi_i u_j)_{,j}. \quad [2.18] \quad (33)\end{aligned}$$

When we substitute the results (31) and (33) into Eq. (27) and perform the appropriate integrations by parts with due account of conditions (22) and (23), we obtain

$$\begin{aligned}\delta J&=\int dt d\tau\rho[\phi_{,j}-A_{i,j}u_i+(\partial A_j/\partial t)+A_{j,i}u_i]\delta\xi_j\\&=-\int dt d\tau\rho[-\phi_j-(\partial A_j/\partial t)+(A_{i,j}-A_{j,i})u_i]\delta\xi_j\\&=-\int dt d\tau\rho[\mathbf{E}+(\mathbf{u}\times\mathbf{B})]\cdot\delta\boldsymbol{\xi},\end{aligned}$$

Eq. (8) now follows at once upon substituting this result in Eq. (28).

3. Lorentz Transformation and the Principle of Least Action

[13] "Let us see if the principle of least action gives us the reason for the success of the Lorentz transformation."

We consider first the behavior of expression (25) under the transformations (9). We find that the Jacobian of x', y', z', t' with respect to x, y, z, t is l^4, so that

$$dt'd\tau'=l^4dtd\tau. \quad (34)$$

On the other hand, Eqs. (17) imply the identity

$$l^4(\mathbf{E}'^2-\mathbf{B}'^2)=\mathbf{E}^2-\mathbf{B}^2. \quad (35)$$

Consequently, if we take $t_0=-\infty$ and $t_1=\infty$, so that the limits of integration do not change under our transformations, then

$$J' = \iiiint_{-\infty}^{\infty} dt' d\tau' \tfrac{1}{2}(\mathbf{E}'^2 - \mathbf{B}'^2) = J. \qquad (35')$$

We investigate next the transformation properties of Eq. (28).

[14] "For this, it is at first necessary to compare $\delta\boldsymbol{\xi}'$ and $\delta\boldsymbol{\xi}$. Let us consider an electron whose initial coordinates are x_0, y_0, z_0; its coordinates at the instant t will be

$$x_i = x_{0i} + \xi_i \qquad (i = 1, 2, 3).$$

If one considers the corresponding electron after the Lorentz transformation, it will have the coordinates

$$x_1' = \gamma l(x_1 - \beta t), \qquad x_j' = l x_j \qquad (j = 1, 2),$$

where

$$x_i' = x_{0i} + \xi_i' \qquad (i = 1, 2, 3); \qquad (36)$$

but it will only attain these coordinates at the instant

$$t' = \gamma l(t - \beta x_1)."$$

It is therefore necessary to consider the variations δx_i when we also vary t; so that, using the same notation as before, we have

$$\delta \mathbf{x} = \delta\boldsymbol{\xi} + \mathbf{u}\delta t,$$

and correspondingly,

$$\delta \mathbf{x}' = \delta\boldsymbol{\xi}' + \mathbf{u}'\delta t'.$$

Combining the latter equation with the transformation equations (9) for $(\delta\mathbf{x}, \delta t)$, we obtain upon setting $\delta t = 0$, and using Eqs. (12) (recalling that $1 - \beta^2 = \gamma^{-2}$),

$$\delta\xi_1 = \gamma l^{-1}(1 - \beta u_1)\delta\xi_1', \qquad \delta\xi_j = l^{-1}(\delta\xi_j' - \beta\gamma u_j \delta\xi_1').$$

When these equations and Eq. (34) are applied to the integral in Eq. (28), the result is an integral of the same form in the primed quantities, provided f_i' are connected with f_i by Eqs. (18).

Thus "the principle of least action leads to the same result as the analysis in Sec. 1."

Referring to Eqs. (17),

[15] "we see that $\mathbf{E}^2 - \mathbf{B}^2$ is not changed by the Lorentz transformation, except for a constant factor; the same is not true of the expression $\mathbf{E}^2 + \mathbf{B}^2$ which appears in the energy."

4. The Lorentz Group

The Lorentz transformations form a group. In fact, if we combine the transformation (9) with the transformation

$$x_1'' = \gamma' l'(x_1' - \beta' t'), \qquad x_j'' = l' x_j,$$
$$t'' = \gamma' l'(t' - \beta' x'),$$

the resultant transformation is

$$x_1'' = \gamma'' l''(x_1 - \beta'' t), \qquad x_j'' = l'' x_j,$$
$$t'' = \gamma'' l''(t - \beta'' x),$$

where

$$\beta'' = (\beta + \beta')/(1 + \beta\beta'), \qquad l'' = ll',$$
$$\gamma'' = \gamma\gamma'(1 + \beta\beta') = (1 - \beta''^2)^{-1/2}.$$

If we set $l = 1$, and take $-\beta \equiv \epsilon$ infinitesimal, then by Eqs. (9) applied to $x_i' - x_i = \delta x_i$, $t' - t = \delta t$, we have

$$\delta x_1 \equiv \delta x = \epsilon t, \qquad \delta x_j = 0, \qquad \delta t' = \epsilon x$$

This is an "infinitesimal generating transformation" of the group, call it T_1. In Lie's notation we can write

$$T_1 = t(\partial/\partial x) + x(\partial/\partial t). \qquad (37)$$

If we take $\epsilon = 0$ and set $l = 1 + \delta l$, we find that $\delta x_i = x_i \delta l$, $\delta t = t\delta l$. We have thus another infinitesimal transformation of the group, T_0, where

$$T_0 = x_i(\partial/\partial x_i) + t(\partial/\partial t). \qquad (38)$$

Because the Maxwell–Lorentz equations are invariant under spatial rotations we also have the infinitesimal transformations

$$T_j = t(\partial/\partial x_j) + x_j(\partial/\partial t) \qquad (j = 2, 3). \qquad (39)$$

In addition, the Lie commutators, $[T_i, T_j] \equiv T_i T_j - T_j T_i$, reduce to

$$[T_i, T_j] = x_i(\partial/\partial x_j) - x_j(\partial/\partial x_i), \qquad (40)$$

which are easily verified to correspond to rotations in the x_i, x_j plane.

"We are thus led to envisage a continuous group, which we shall call the *Lorentz group.*" It is generated by the seven infinitesimal transformations represented in Eqs. (37)–(40).

Every transformation of this group can be decomposed into a transformation of the form

$$x_i'=lx_i, \qquad t'=lt$$

and a linear transformation that *leaves invariant the quadratic form* $\mathbf{x}^2-t^2$.

It can also be *represented by a transformation of the form* (9) *preceded and followed by a suitable rotation.*

[16] "But for our purpose we need only consider a portion of the transformations of this group; we have to suppose that l is a function of ϵ, and it is a question of choosing this function in such a way that this group portion, which I shall call P, forms also a group.

Let us turn the {coordinate} system by 180° around the y axis: we rediscover a transformation that must also belong to P. But this amounts to changing the sign of x, x', z, and z'; one finds thus {in our present notation, but writing here ϵ for $-\beta$}

$$x'=\gamma l(x-\epsilon t), \qquad y'=ly,$$
$$z'=lz, \qquad t'=\gamma l(t-\epsilon x) \quad [8.2] \quad (41)$$

Hence l does not change when ϵ is replaced by $-\epsilon$. On the other hand, if P is a group, then the transformation inverse to (9), which is

$$x'=\gamma l^{-1}(x-\epsilon t), \qquad y'=l^{-1}y,$$
$$z'=l^{-1}z, \qquad t'=\gamma l^{-1}(t-\epsilon x)$$

must also belong to P; it must therefore be identical to (2) {our (41)}; i.e., $l=l^{-1}$. One must thus have $l=1$."

[1] A mere outline of Poincaré's paper—as suggested some time ago [H. M. Schwartz, Amer. J. Phys. **33,** 170 (1965)]—would hardly serve the needs of historical research relating to special relativity. This is not contradicted, in my opinion, by a recent paper on the subject [C. Cuvaj, Amer. J. Phys. **36,** 1102 (1968)]. The extent of the richness of information provided by the Rendiconti paper on Poincaré's role in the history of relativity is made evident in a historic study now in preparation.

[2] After the manuscript of the present paper was completed my attention was called to a literal translation of Pts. I and III of Poincaré's paper in C. W. Kilmister, *Special Theory of Relativity* (Pergamon, New York, 1970). This now makes these two parts more readily physically accessible, but the utility of the present and projected précis remains; in particular, Pt. II (which forms 40% of the original paper), dealing with the frequently quoted "Poincaré stresses," is certainly of considerable interest.

[3] Poincaré uses here the curious term *rayons magnéto-cathodiques.*

[4] Equations (1)–(6) are of course not all independent, and their arrangement in [1.1] may appear strange at this point. Considerations developed in Secs. 2 and 3 provide at least partial clarification.

[5] The significance of the transformation parameter β is obvious from the context. Poincaré refers to both l and β as "two arbitrary constants," but the intension is obvious.

[6] Although this assumption is not set off in the original exposition, Poincaré's recognition of its primary significance is implicit in his discussion.

[7] Because of its historical interest, Poincaré's roundabout method of deriving this kinematic result is reproduced verbatim.

[8] The original text has $\lambda'=l^2\lambda$ (and later, $D_1'=l^{-2}D_1$) which is an obvious (and harmless) misprint.

That $D'=D$ can be seen, for instance, if we denote t by x_0, dx_α/dt by u_α ($\alpha=0, 1, 2, 3$), and write X_α for $x_\alpha+\lambda\rho u_\alpha$; we have then, letting L stand for the matrix of a Lorentz transformation (9), and using the repeated-index summation convention:

$$\partial X_\alpha'/\partial x_\beta'=L_{\alpha\gamma}(\partial X_\gamma/\partial x_\delta)L^{-1}{}_{\delta\beta},$$
$$\det \| \partial X_\alpha'/\partial x_\beta' \| \equiv D'=\det L\cdot\det L^{-1}\cdot D=D.$$

It is a little curious that Poincaré does not give here the much simpler and direct proof of the Lorentz covariance of Eq. (4), which in the above notation, and writing J_α for ρu_α, is as follows: $J_\alpha'=l^{-4}L_{\alpha\beta}J_\beta$,

$$l^4\partial J_\alpha'/\partial x_\alpha'=L_{\alpha\beta}\partial J_\beta/\partial x_\gamma L^{-1}{}_{\gamma\alpha}=\delta_{\gamma\beta}\partial J_\beta/\partial x_\gamma=\partial J_\beta/\partial x_\beta.$$

[9] In the notation of footnote 8, $l^{-2}\Box A_\alpha'=-l^{-4}L_{\alpha\beta}J_\beta=l^{-4}L_{\alpha\beta}\Box A_\beta$, i.e., $\Box(A_\alpha'-l^{-2}L_{\alpha\beta}A_\beta)=0$. An argument for discarding possible additive functions of the A_α that satisfy the wave equation is therefore required to complete the proof that $A_\alpha'=l^{-2}L_{\alpha\beta}A_\beta$. The same type of argument also enters in other similar deductions that follow.

[10] Poincaré has reference probably to Lorentz's variational treatment in his paper "Contributions to the Theory of Electrons," Proc. Acad. Amsterdam **5,** 608 (1903).

[11] Understood here of course is sufficient rapidity of tending to zero at spatial infinity, so that surface integrals obtained from integration by parts tend to zero.

[12] This derivation of Eq. (33) is a more direct version of that given in the original text.

[4] See, for example, Ref. 1 or W. H. Louisell, *Radiation and Noise in Quantum Electronics* (McGraw-Hill, New York, 1964), pp. 191 ff and 255 ff.

[5] The behavior of $b(t)$ or $\hat{b}(s)$, is of course much more complicated than this. That $\hat{b}(s)$ has, among other singularities, a simple pole at $s=-\gamma$ [see Eq. (3.13)] can be seen from the exponentially decaying behavior of $b(t)$ shown by Eqs. (4.1) and (5.1).

[6] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).

[7] This enhancement is not shown by the solutions of linearized equations of motion, which show the same behavior as the harmonic oscillator. See, for example, B. R. Mollow and M. M. Miller [Ann. Phys. **52**, 464 (1969)] who explicitly drop the higher order terms in the equations of motion, or H. Sauermann [Z. Physik **188**, 480 (1965)] who neglects the dependence of the Langevin forces on the state of the atomic system, and thus effectively linearizes the equations of motion.

Poincaré's Rendiconti Paper On Relativity. Part II

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This is a continuation of the modernized presentation of Poincaré's Rendiconti paper (begun in the November 1971 issue of this Journal). It covers Secs. 5–8 of that paper, dealing with its central theme, as indicated by its title, "On the dynamics of the electron," the subject being of interest to both the historian of the classical theory of the electron and the historian of relativity.

The purpose and scope of the present modernized rendition of Poincaré's Rendiconti paper is described in the introductory remarks to Pt. I.[1] A few additional words are needed to introduce Pt. II.

Although the contents of Pt. II (Sec 5-8 of the original work) deal with the structure and dynamics of "electrons," and are, therefore, in the first place, of interest to the historian of the development of the theory of charged particles at the turn of the century, there is also a great deal here which is relevant to the early history of relativity theory.

Because of the latter fact, the listing of some of the translated portions with the aid of bracketed numbers, begun in Pt. I, is continued. In general, all numberings in Pt. II (except for the references) are a continuation of those in Pt. I, so that in referring to Pt. I, no explicit mention of "Pt. I" is needed.

Although a number of new symbols arise in Pt. II, it seemed unnecessary to extend Table I of Pt. I. Wherever at all feasible, the original notation is retained, and the few nontrivial changes that are introduced are explained in the references. However, no reference is made (nor was it made in Pt. I) to the replacement of Poincaré's antiquated notation for different types of differentiation.

Because Pt. II contains material which is somewhat intricate and not now too familiar, and explanations and references in the original text are not always ample, the number of explanatory footnotes to Pt. II is rather large.

Helpful consultation on a number of points with my colleague, Lieber, is gratefully acknowledged.

ON THE DYNAMICS OF THE ELECTRON

5. Langevin's Waves

The inhomogeneous wave equations (7) are known to have the retarded-potential solutions

$$\phi = (4\pi)^{-1}\int(\rho_1/r)\,d\tau_1,$$

$$\mathbf{A} = (4\pi)^{-1}\int(\rho_1\mathbf{u}_1/r)\,d\tau_1 \qquad (d\tau_1 \equiv d^3x_1), \quad [5.2] \qquad (42)$$

where $\phi \equiv \phi(\mathbf{x}, t)$, $\rho_1 \equiv \rho(\mathbf{x}_1, t_1)$, etc., and

$$t_1 = t - r, \qquad r^2 = (\mathbf{x} - \mathbf{x}_1)^2. \qquad (43)$$

Langevin has shown that Eqs. (42) admit of a "particularly elegant form" when applied to the field "produced by a single electron."[2]

Let $\mathbf{x}_0 = \mathbf{x}_0(t)$ describe the motion of a given "molecule of the electron,"[3] and set $\mathbf{x}_1 \equiv \mathbf{x}_0(t_1) = \mathbf{x}_0 + \boldsymbol{\xi}$ for given fixed $\mathbf{x}$ and t. Then[4]

$$dx_{1i} = dx_{0i} + (\partial\xi_i/\partial x_{0j})\,dx_{0j} + (\partial\xi_i/\partial t_1)\,dt_1, \quad (44)$$

and by (43),

$$dt_1 = -dr = \mathbf{n}\cdot d\mathbf{x}_1, \qquad \mathbf{n} \equiv (\mathbf{x} - \mathbf{x}_1)/r. \quad (45)$$

Since $\partial\boldsymbol{\xi}/\partial t_1 = \mathbf{u}(t_1) \equiv \mathbf{u}_1$, we find from (44) that

$$(\delta_{1j} - u_{1i}n_j)\,dx_{1j} = (\delta_{ij} + \partial\xi_i/\partial x_{0j})\,dx_{0j}, \quad (46)$$

from which it follows that

$$J_1 d\tau_1 = J_0 d\tau_0, \qquad [5.3] \quad (47)$$

where J_1, J_0 are the determinants of the respective matrices in the left and right sides of (46). An immediate expansion gives[5]

$$J_1 = 1 + \omega, \qquad \omega \equiv -\mathbf{u}_1\cdot\mathbf{n}. \qquad (48)$$

On the other hand, if we denote by $\bar{\mathbf{x}}$ the position vectors of the "molecules of the electron" in the neighborhood of the chosen molecule, all taken at the time t_1 associated with this molecule, then the corresponding element of electric charge is[6]

$$dq_1 = \rho_1 d\bar{\tau} \qquad (d\bar{\tau} \equiv d^3\bar{x}). \qquad (49)$$

When we apply Eq. (44) to determine the $d\bar{x}_i$, the last term is absent since now $dt_1 = 0$, and instead of (47) we have the relation

$$d\bar{\tau} = \bar{J}_0 d\tau_0,$$

where $\bar{J}_0$ involves $\bar{\mathbf{x}} - \mathbf{x}_0$. In substituting this equation into Eq. (49) we may set $\bar{J}_0 = J_0$. Combining Eqs. (49), (47), (48), and (42) we obtain then the desired result:

$$\phi = (4\pi)^{-1}\int[dq_1/r(1+\omega)],$$

$$\mathbf{A} = (4\pi)^{-1}\int[\mathbf{u}_1 dq_1/r(1+\omega)]. \qquad (50)$$

"If we deal with a single electron, our integrals reduce to a single element, provided only sufficiently distant points $\mathbf{x}$ are considered for which r and ω have sensibly the same values at every point of the electron."

Referring to Eqs. (2), we see that the expressions for the electric and magnetic fields contain terms that involve the velocity but not the acceleration of the electron, as well as terms that involve the acceleration, and in fact, linearly. Langevin calls the former terms collectively "velocity wave," and the latter terms, "acceleration wave."

We consider first the velocity wave. By a Lorentz transformation we can bring the electron instantaneously to rest at the instant t_1. In this reference frame, S', Eqs. (50) reduce to[7]

$$\mathbf{A}' = 0, \qquad \phi' = e/4\pi r' \qquad (r' = |\mathbf{x}' - \mathbf{x}_1'|), \quad (50')$$

and therefore,

$$\mathbf{B}' = 0, \qquad \mathbf{E}' = (e/4\pi r'^3)(\mathbf{x}' - \mathbf{x}_1').$$

Transforming back to the original reference frame it is then found that[8]

$$\mathbf{B} = \boldsymbol{\beta} \times \mathbf{E},$$

$$\mathbf{E} = \gamma l^3(e/4\pi r'^3)[\mathbf{x} - \mathbf{x}_1 - \boldsymbol{\beta}(t - t_1)]. \qquad [5.4] \quad (51)$$

Turning to the acceleration wave, it will now be shown that the fields satisfy the following three conditions:

$$|\mathbf{E}| = |\mathbf{B}|, \qquad \mathbf{E}\cdot\mathbf{B}=0, \qquad \mathbf{E}\cdot\mathbf{n}=\mathbf{B}\cdot\mathbf{n}=0. \quad (52)$$

This result can likewise be derived by the application of a Lorentz transformation, in this instance, to an electron which oscillates in such a way "that the displacements and the velocities are infinitely small, but the accelerations are finite."[9] For such an electron the relations (52) follow from the corresponding results in "Hertz's celebrated paper *Die Kräfte elektrischer Schwingungen nach der Maxwell'schen Theorie*,[10] in the case of a very distant point."

In fact, the general validity (i.e., for finite electron velocity) of the first of Eqs. (52) is an immediate consequence of Eq. (35). Similarly, the second of Eqs. (52) follows from the relation $l^4\mathbf{E}'\cdot\mathbf{B}'=\mathbf{E}\cdot\mathbf{B}$, that is, deducible from Eqs. (17).

To prove the Lorentz covariance of the last of Eqs. (52), we note that the first two of these equations imply that

$$\mathbf{E}=\mathbf{B}\times\mathbf{n}, \qquad \mathbf{B}=\mathbf{n}\times\mathbf{E}, \; [5.6] \quad (53)$$

where $\mathbf{n}$ is defined in (45), and that by Eqs. (9) and (43),

$$r_x'\equiv x'-x_1'=\gamma l(r_x-\beta r),$$
$$r_y'=lr_y, \qquad r_z'=lr_z \quad [5.7] \quad (54)$$

so that[11]

$$\mathbf{E}\cdot\mathbf{r}'=\gamma l^{-1}\mathbf{E}\cdot\mathbf{r}, \qquad \mathbf{B}'\cdot\mathbf{r}'=\gamma l^{-1}\mathbf{B}\cdot\mathbf{r}. \quad (55)$$

Another derivation of the Lorentz covariance of the relations (52) is based on "simple considerations of homogeneity."[12]

> In fact, ϕ, A_i are functions of x_i-x_{1i} and $u_i=dx_{i1}/dt_1$, homogeneous of order -1 with respect to x_i, t, x_{1i}, t_1 and their differentials.
>
> Hence the derivatives of ϕ, A_i with respect to x_i, t (and consequently also the two fields $\mathbf{E}$ and $\mathbf{B}$) are homogeneous of degree -2 with respect to the same quantities, when we recall, moreover, that the expression $t-t_1=r=|\mathbf{x}-\mathbf{x}_1|$ is homogeneous {i.e., of degree 1} with respect to these quantities.
>
> Now these derivatives or these fields depend on the $\mathbf{x}-\mathbf{x}_1$, the velocities $d\mathbf{x}_1/dt_1$, and accelerations $d^2\mathbf{x}/dt_1^2$; they consist of a term independent of the accelerations (velocity wave) and a term linear with respect to the accelerations (acceleration wave). But $d\mathbf{x}_1/dt_1$ is homogeneous of degree 0 and $d^2\mathbf{x}_1/dt_1^2$ is homogeneous of degree -1; from which it follows that the velocity wave is homogeneous of degree -2 with respect to x_i-x_{1i} and the acceleration wave is homogeneous of degree -1. Therefore, at a very distant point the acceleration wave predominates and can therefore be identified with the total wave. Furthermore, the homogeneity law shows that the acceleration wave is similar to itself at a distant point and at an arbitrary point. It is therefore similar at an arbitrary point to the total wave at a distant point. But at a distant point the perturbation {i.e., the electromagnetic field} can propagate itself only in plane waves, so that the two fields must be equal, mutually perpendicular, and perpendicular to the direction of propagation.

6. Contraction of Electrons

> [17] Let us consider a uniformly and rectilinearly moving electron. We have seen that with the aid of a Lorentz transformation one can reduce the study of the field produced by such an electron to the case of a stationary electron; the Lorentz transformation replaces thus a moving *real* electron by a motionless *ideal* electron.[13]

We wish now to calculate the "electromagnetic masses of the electron." It is necessary, therefore, to determine the "total energy due to the motion of the electron" as well as the corresponding electromagnetic momentum. We cannot employ Eqs. (51) because they apply only to distant field points when it is permissible to treat the electron effectively as a point structure. But "the energy is located principally in the parts of the ether that are nearest to the electron."

> [18] Concerning this subject many hypotheses can be made. According to that of Abraham the electrons are spherical and nondeformable. Then upon applying the

Lorentz transformation since the real electron is spherical, the ideal electron will become an ellipsoid—

$$\gamma^{-2}x'^2+y'^2+z'^2=l^2r^2 \qquad (r \text{ is the radius of real electron}).^{14} \qquad (56)$$

Lorentz assumed, on the contrary, that it is the ideal electron which is spherical and that "electrons in motion become deformed." Thus it is the real electron which becomes an ellipsoid, whose semiaxes are $r/l\gamma$, r/l, r/l, where r is the radius of the ideal electron.

Now the field produced by the ideal electron, i.e., the field in the rest-frame S' of the electron, satisfies the equations[15]

$$\mathbf{B}'=0, \qquad \mathbf{E}'=-\nabla'\phi', \qquad (57)$$

and by (17), we have therefore for the "real field":

$$\left.\begin{array}{lll} B_1=0, & B_2=-\beta E_3, & B_3=\beta E_2 \\ E_1=l^2E_1', & E_j=\gamma l^2E_j' & (j=2,3) \end{array}\right\} [6.1] \qquad (58)$$

It follows that if we write[16]

$$A=\tfrac{1}{2}\int E_1^2 d\tau, \qquad B=\tfrac{1}{2}\int(E_2^2+E_3^2)d\tau,$$

$$C=\tfrac{1}{2}\int(B_2^2+B_3^2)d\tau, \qquad (59)$$

then, in the first place,

$$C'=0, \qquad C=\beta^2B, \qquad (60)$$

and since[17] $d\tau'=\gamma l^3 d\tau$,

$$A'=\gamma l^{-1}A, \qquad B'=\gamma^{-1}l^{-1}B. \qquad (61)$$

Moreover, by the spherical symmetry in the charge distribution of the Lorentz ideal electron,

$$B'=2A'. \qquad (62)$$

Thus, the "total energy," E, the "action per unit time,"[18] L, and the x component of the electromagnetic momentum, D, are given by the following equations [using (58) in the last equation]:

$$E=A+B+C=(3+\beta^2)l\gamma A',$$

$$L=A+B-C=3l\gamma^{-1}A',$$

$$D=\int(\mathbf{E}\times\mathbf{B})_x d\tau=2\beta B=4\beta\gamma lA'. \qquad (63)$$

"But one ought to have certain relations" between E, L, and D, namely,[19]

$$E=L-\beta(dL/d\beta), \qquad dD/d\beta=\beta^{-1}(dE/d\beta), \qquad (64)$$

"from which"[20]

$$D=-dL/d\beta, \qquad E=L+\beta D. \quad [6.2] \qquad (65)$$

[19] The second of equations (2) {Eqs. [6.2]} is always satisfied; but the first one is satisfied only if[21]

$$l=(1-\beta^2)^{1/6}=\gamma^{-1/3}, \qquad (66)$$

i.e., if the volume of the ideal electron is equal to that of the real electron, or also, if the electron's volume is a constant; this is Langevin's hypothesis.[22]

"This is in contradiction with the result of Sec. 4 and with the result obtained by Lorentz in a different way. It is this contradiction that needs to be explained."

To begin with, let us note that whatever the chosen hypothesis, we have [since by (60) and (61), $B-C=(1-\beta^2)B=l\gamma^{-1}B'$]

$$L=l\gamma^{-1}(A'+B'),$$

or since $C'=0$,

$$L=l\gamma^{-1}L'. \qquad [6.3] \qquad (67)$$

This result can be related to Eq. (35′) by noting that by (9), $t'=l\gamma^{-1}t-\beta x'$, and hence $dt'=l\gamma^{-1}dt$ $(dx'=0)$.

Let us consider now a general hypothesis, which embraces those of Abraham, Lorentz, and Langevin as special cases: the semiaxes of the *real* electron are

$$r, \theta r, \theta r,$$

and hence those of the *ideal* electron are

$$\gamma lr,\ l\theta r,\ l\theta r. \tag{68}$$

The three special models under consideration are then given by the following determinations:

$$\begin{array}{llll} & r=\text{const.}, & \theta=1 & \text{(Abraham)} \\ l=1, & \gamma r=\text{const.}, & \theta=\gamma & \text{(Lorentz)} \\ l=\gamma^{-1/3}, & \gamma lr=\text{const.}, & \theta=\gamma & \text{(Langevin)}. \end{array} \tag{69}$$

Whether the electron's charge is supposed to be distributed on its surface as on a conductor, or uniformly throughout its volume, we have[23]

$$E'=A'+B'=\varphi(\theta\gamma^{-1})/l\gamma r. \tag{70}$$

The function φ can be determined by combining Eqs. (70) and (67),

$$L=\frac{\varphi(\theta/\gamma)}{\gamma^2 r}, \tag{71}$$

and noting that for Abraham's model we have[23]

$$L=(a/r)[(\beta^2-1)/\beta]\ln[(1-\beta)/(1+\beta)]\ \text{(a const)}.$$

Taking into account the first set of Eqs. (69), it follows then that φ is defined by the relation

$$\varphi(\gamma^{-1})=(a/\beta)\ \ln[(1+\beta)/(1-\beta)]. \quad [6.5] \tag{72}$$

Now, all the electron-structure models (69) involve a constraint represented by a relation between r and θ of the form

$$r=b\theta^m \qquad (b\ \text{const}). \tag{73}$$

In fact, corresponding to the three cases in (69), we have, respectively,

$$m\ \text{arbitrary}, \qquad m=-1, \qquad m=-\tfrac{2}{3}. \tag{74}$$

By Eqs. (71) and (73),

$$L=\frac{\varphi(\theta/\gamma)}{b\gamma^2\theta^m}. \tag{75}$$

The condition of equilibrium for the electron "when one does not assume the intervention of any other forces except those of constraint" is given by the equation

$$\partial L/\partial\theta=0, \qquad [6.6] \tag{76}$$

which combined with Eq. (75), yields for the logarithmic derivative of φ the expression

$$\varphi'/\varphi=m\gamma/\theta. \tag{77}$$

On the other hand, by expansion in powers of β, we obtain from (72) that

$$(d/du)\ \ln\varphi(u)=-\tfrac{2}{3}, \qquad \text{when } u=1. \tag{77'}$$

Comparing with (77), we see [bearing in mind that the argument of φ in (77) in θ/γ] that if equilibrium is to hold for $\theta=\gamma$, then relation (73) must be combined with the last equation in (74), which represents Langevin's hypothesis.

Recalling the discussion concerning the relation (66), it is to be expected that the present result is related to the first of Eqs. (65), and we now show that there is in fact equivalence between the two results.

By the definition of D [see the last of Eqs. (63)], the equation of motion of the center of mass of the electron is[24]

$$dD/dt=\int X d\tau \qquad [X\equiv f_x\ \text{(see Pt. I, Table I]}, \tag{78}$$

and hence the principle of least action can be written (see Pt. I, Table I):

$$\delta J=\int X\delta\xi_x d\tau dt=\int X d\tau\delta\xi_x dt=\int(dD/dt)\delta\xi_x dt. \tag{79}$$

On the other hand,

$$J=\int L dt,$$

and by our assumption that $r=r(\theta)$, L can be considered as a function of θ and β. Hence,

$$\delta J=\int[(\delta L/\delta\beta)\delta\beta+(\delta L/\delta\theta)\delta\theta]dt. \tag{80}$$

Since $\delta\beta = d(\delta\xi_x)/dt$, we find by integration by parts,

$$-\int D\delta\beta dt = \int (dD/dt)\delta\xi_x dt.$$

Comparison of Eqs. (79) and (80) show that

$$-D = \partial L/\partial\beta, \qquad \partial L/\partial\theta = 0, \qquad (80')$$

and since

$$dL/d\beta = (\partial L/\partial\beta) + (\partial L/\partial\theta)(d\theta/d\beta),$$

we see that Eqs. (76) and the first of Eqs. (65) are indeed equivalent.

> [20] The conclusion is that if the electron is subject to a constraint between its three axes, *and if no other forces intervene except the forces of constraint*, then the shape assumed by this electron when it is in uniform motion, cannot be such that the corresponding ideal electron is spherical, except when the constraint is that the volume be constant, in agreement with Langevin's hypothesis.

What additional forces, then, are needed "to account for Lorentz's law, or more generally, for every law different from that of Langevin?"

> [21] The simplest hypothesis and the first that we should examine, is that these supplementary forces admit of a special potential depending on the three axes of the ellipsoid, and hence on θ and r.

Denoting this potential by $F(\theta, r)$, and retaining the symbol L for our previous Lagrangian function, the new expression for our action integral is

$$J = \int [L + F(\theta, r)] dt,$$

and the equilibrium conditions are represented by the equations

$$(\partial/\partial\theta)(L+F) = 0, \qquad (\partial/\partial r)(L+F) = 0. \qquad [6.8] \quad (81)$$

If we assume the connection (73), then only the first of Eqs. (81) needs to be retained, with L taking on the expression (75) so that $dL/d\theta = (-m\theta^{-1}\varphi + \gamma^{-1}\varphi')/b\gamma^2\theta^m$. By (72) and (77′),

$$\varphi(1) = a, \qquad \varphi'(1) = -(2/3)a. \qquad (82)$$

Therefore, for $\theta = \gamma$, $dF/d\theta = -dL/d\theta = (m + \frac{2}{3}) \times a/b\theta^{m+3}$, and hence (the integration constant being immaterial),

$$F = -(m + \tfrac{2}{3})a/(m+2)b\theta^{m+2}. \qquad (83)$$

For the Lorentz model [second of Eqs. (74)] this is

$$F = a/3b\theta. \qquad (84)$$

On the other hand,[25] if r, θ are treated as independent variables, then by Eqs. (71) and (81), we find for the case where $\theta = \gamma$ and relation (73) holds, that

$$\partial F/\partial r = a/b^2\theta^{2m+2}, \qquad \partial F/\partial\theta = \tfrac{2}{3}(a/b\theta^{m+3}). \qquad [6.9] \quad (85)$$

One can satisfy these equations by an expression of the form

$$F = Ar^\alpha\theta^\beta, \qquad [6.10] \quad (86)$$

where A, α, β are constants. For the case $\theta = \gamma$ and (73) one finds then in a few simple steps,

$$\alpha = 3s, \qquad \beta = 2s, \qquad s = -(m+2)/(3m+2),$$

$$A = a/\alpha b^{\alpha+1}. \qquad [6.11] \quad (87)$$

We now note, that since the volume of the ellipsoid is proportional to $r^3\theta^2$, it follows from Eqs. (86) and (87) that F is proportional to the volume raised to the power s. But for the Lorentz model, $s = 1$.

> [22] One recovers thus Lorentz's hypothesis provided one adds a supplementary potential proportional to the volume of the electron.

7. Quasistationary Motion

It remains to be seen if this hypothesis on the contraction of electrons accounts for the impossibility of manifesting absolute motion, and I shall begin with studying the quasistationary motion of an isolated electron, or one subjected only to the action of other distant electrons.

By *quasistationary motion* of an electron one understands "motion involving sufficiently small variations of the velocity so that the magnetic and electric energies due to the electron's motion differ little from what they would be in the case of uniform motion."

More precisely, and confining ourselves for the moment to the case of an isolated electron, we can define this motion in terms of our Lagrangian function L. When the electron motion is uniform, i.e., when the velocity, $\mathbf{u}$, of its center of mass is constant, then L depends only on the u_i and on the shape parameters r and θ. In the case of general nonuniform motion, L will depend not only on these five variables but also on their time derivatives of all orders. But for quasistationary motion "the partial derivatives of L with respect to the successive derivatives of u_i, r, θ are negligible compared with the partial derivatives of L with respect to the quantities u_i, r, θ themselves."

"The equations of such motion can be written":

$$(\partial/\partial\theta)(L+F)=(\partial/\partial r)(L+F)=0, \quad (88) \quad [7.1]$$

$$(d/dt)(\partial L/\partial u_i)=-\int f_i d\tau. \quad (89)$$

Here F has the same significance as in Sec. 6, and the force $\mathbf{f}$ is that produced by all the other electrons of our system.

Since L depends on the u_i only through $u\equiv|\mathbf{u}|$, therefore, retaining the symbol D for the magnitude of the electromagnetic momentum, we have,[26]

$$\partial L/\partial u_i=(\partial L/\partial u)(u_i/u)=-D(u_i/u), \quad (90)$$

and hence,

$$-\frac{d}{dt}\frac{\partial L}{\partial u_i}=\frac{D}{u}\dot{u}_i-D\frac{u_i}{u^2}\dot{u}+\frac{dD}{du}\frac{u_i}{u}\dot{u}, \quad [7.2] \quad (91)$$

where we use the dot to indicate differentiation with respect to t.

In particular, when $u_i=u\delta_{i1}$, then:

$$\begin{aligned}-(d/dt)(\partial L/\partial u_i)&=(dD/du)\dot{u} && (i=1)\\ &=(D/u)\dot{u}_i && (i=2,3),\end{aligned} \quad (92)$$

which, recalling (89), explains why Abraham calls dD/du and D/u the *longitudinal* and the *transverse* masses, respectively.

In the case of Lorentz's model we have[27]

$$D=-\partial L/\partial u=-dL/du. \quad (93)$$

On the other hand, upon substituting in L for r and θ their functions of u as deduced from Eqs. (88), we find that[28] $L=A(1-u^2)^{1/2}$, and by a proper choice of units,[29] we can arrange to have the constant $A=1$. Therefore, if we set

$$L=h\equiv(1-u^2)^{1/2} \quad (94)$$

and

$$M\equiv u(du/dt)\equiv\mathbf{u}\cdot\dot{\mathbf{u}},$$

we find by Eqs. (91) and (89) that our equation of quasistationary motion is of the form

$$h^{-1}\dot{\mathbf{u}}+h^{-3}M\mathbf{u}=\mathbf{F}, \quad [7.5] \quad (95)$$

where[30]

$$\mathbf{F}=\int\mathbf{f}d\tau'. \quad (95')$$

In proving Eq. (95) we make use of the following relations implied by Eqs. (93) and (94):

$$dD/du=h^{-3}, \qquad (1/u^2)(dD/du)-(D/u^3)=h^{-3}.$$

We now prove the theorem that Eq. (95) is covariant under Lorentz transformations.

First we note that Eqs. (12) can be written:

$$\mu u_1'=u_1-\beta, \qquad \mu u_j'=u_j/\gamma \qquad (j=2,3), \quad (96)$$

where

$$\mu\equiv 1-\beta u_1, \quad (96')$$

and that, therefore,

$$\mu h'=h/\gamma.$$

Using also the relation[31]

$$dt' = \gamma\mu dt,$$

we find that

$$du_1'/dt' = \dot{u}_1/\gamma^3\mu^3,$$

$$\frac{du_j'}{dt'} = \frac{[\dot{u}_j + (\beta/\mu)u_j\dot{u}_1]}{\gamma^2\mu^2} \qquad (j=2,3).$$

By somewhat lengthy but straightforward algebra, we then obtain the relations:

$$M' = -\beta h^2\dot{u}_1/\gamma^3\mu^4 + M/\gamma^3\mu^3,$$

$$h'^{-1}(du_1'/dt') + h'^{-3}u_1'M' = [\dot{u}_1 + h^{-2}(u_1-\beta)M]/\mu h \qquad [7.6]$$

$$(97)$$

$$h'^{-1}(du_j'/dt') + h'^{-3}u_j'M' = (\dot{u}_j + h^{-2}u_jM)/\mu h^2 \qquad [7.7]$$

Again, since Eq. (95) implies that $\mathbf{F}\cdot\mathbf{u} = M/h^3$, Eqs. (19) and (11) with $l=1$, and (96′) yield:

$$F_1' = (F_1 - \beta h^{-3}M)/\mu, \qquad F_j' = F_j/\gamma\mu \qquad (j=2,3). \qquad [7.9] \quad (98)$$

Combining Eqs. (97), (95), and (98), we see the truth of our theorem.

We now sharpen this theorem by showing that among the members of the class of models under consideration, that of Lorentz is the only one that satisfies the theorem.

First we observe that if one supposes that $L=h$ then one finds immediately that one must have $l=1$. In fact, with this assumption Eqs. (97), (98), and the primed version of (95) subsist with the right-hand sides multiplied, respectively, by l^{-1}, l^{-2}, and l^{-1}. If then Eq. (95) is to be identical with its primed version (i.e., if it is to be Lorentz covariant) then $l=1$ is obviously a necessary condition.

In general, we may of course in proving our uniqueness theorem restrict ourselves to the special case $u_j=0$ $(j=2,3)$ and hence deal with the equations of motion (92). Introducing the notation,

$$dD/du \equiv f(u), \qquad D/u \equiv g(u), \qquad (99)$$

we have then [recalling Eqs. (89) and (95′), and noting that $u_1=u$]:

$$f(u)\dot{u}_1 = F_1, \qquad g(u)\dot{u}_j = F_j \qquad (j=2,3), \qquad (100)$$

and for the transformed equations we have:

$$f(u')du'/dt' = F_1' = F_1(1-\beta u_1)/l^2\mu = F_1/l^2,$$

$$g(u')du_j'/dt' = F_j' = F_j/l^2\gamma\mu \qquad (j=2,3). \qquad (101)$$

Therefore,

$$f(u)\dot{u}_1 = l^2 f(u')du_1'/dt',$$

$$g(u)\dot{u}_j = l^2\gamma\mu g(u')du_j'/dt'. \qquad [7.11] \quad (102)$$

But, by Eqs. (96) and (96′),

$$du_1'/dt' = \dot{u}_1/\gamma^2\mu^3, \qquad du_j'/dt' = \dot{u}_j/\gamma^2\mu^2,$$

Hence we deduce from Eqs. (102) the identity

$$\gamma^2\mu^2 g(U)/g(u) \equiv f(U)/f(u), \qquad (103)$$

where

$$U \equiv (u-\beta)/(1-\beta u).$$

By letting

$$g(u)/f(u) \equiv \Omega(u),$$

the identity (103) in u and β, becomes

$$\Omega(U) \equiv \Omega(u)/\gamma^2\mu^2.$$

But for $u=0$ we have $U=-\beta$ and $\mu=1$. Hence,

$$\Omega(-\beta) = \Omega(0)/\gamma^2 = \Omega(0)(1-\beta^2). \qquad (104)$$

Since by Eq. (99),

$$\Omega(u) = \frac{D(u)}{udD/du},$$

we deduce from the differential equation (104), setting $-\beta = u$, and $\Omega(0) = m^{-1}$, that

$$D(u) = A[u(1-u^2)^{1/2}]^m, \qquad A \text{ const},$$

and hence from Eqs. (99) and (96) that[32]

$$g(u)=(A/u)[u/(1-u^2)^{1/2}]^m,$$

$$g(u')=\frac{A\mu}{u-\beta}\left(\frac{u-\beta}{(1-u^2)^{1/2}(1-\beta^2)^{1/2}}\right)^m.$$

But by Eqs. (101) and (100), $g(u')=g(u)\gamma\mu/l^2$. Hence,

$$(u-\beta)^{m-1}(1-\beta^2)^{-m/2}\equiv -u^{m-1}(1-\beta^2)^{-1/2}l^{-2}.$$

This identity clearly implies that $m=1$, $l=1$.

> [23] Thus, Lorentz's hypothesis is the only one which is compatible with the impossibility of manifesting absolute motion; if one admits this impossibility one must admit that electrons when in motion contract so as to become ellipsoids of revolution whose two axes remain constant; one must then admit, as we have shown in the preceding section, the existence of a supplementary potential proportional to the volume of the electron.

> [24] Lorentz's analysis is thus fully confirmed, but we can better account for the true reason of the matter under consideration; this reason must be sought among the considerations of Sec. 4. *The transformations which do not change the equations of motion must form a group*, and this cannot take place except when $l=1$.

Since we must not be able to recognize if an electron is at rest or in a state of absolute motion, it is necessary that when it is in motion, it undergoes a deformation which must be precisely that which is imposed on it by the corresponding transformation of the group.

8. Arbitrary Motion

The extension of the preceding results to the general case of arbitrary motion of an electron is easily accomplished with the aid of the action principle developed in Sec. 3. It suffices to add to the action J given by Eq. (25), "a term representing the supplementary potential F of Sec. 6," of the form

$$J_1=\int\sum F dt,$$

the $\sum$ extending over all the electrons of the system under consideration.

Since the covariance of J under Lorentz transformations has been proved in Sec. 3, that of the total action, $J+J_1$, will be established, when this covariance is shown ito hold for J_1. This is accomplished by noting first that inasmuch as for each electron,

$$F=\omega_0\tau,$$

"ω_0 being a coefficient special to the electron and τ its volume," we can write:

$$\sum F=\int\omega_0 d\tau,$$

where the integral is extended over all of space and the function ω_0 "vanishes outside of the electrons, and inside each electron it is equal to the coefficient appropriate to that electron."[33] Thus,

$$J_1=\int\omega_0 d\tau dt.$$

Now, by Eq. (34) for $l=1$, $d\tau' dt'=d\tau dt$. Hence, the equality of J_1 and the Loretnz-transformed action J_1' will be assured provided $\omega_0'=\omega_0$. But this is so—

> [25] because if a point belongs to an electron, the corresponding point after the Lorentz transformation still belongs to the same electron.[34]

This proof that $J'+J_1'=J+J_1$ provides also an answer to the question posed at the end of Sec. 1.

> [26] If the inertia of electrons is exclusively of electromagnetic origin, if they are subject only to forces of electromagnetic origin or to forces that generate the supplementary potential F, then no experiment can disclose absolute motion.

> What are these forces that generate the potential F? They can be evidently identified with a pressure acting in the interior of the electron; it is as if each electron were hollow and subject to a constant internal pressure (independent of volume); the work of such a pressure would evidently be proportional to the change in volume.

Now, by Eqs. (86) and (87) applied to Lorentz's model $(m=1)$,

$$F=Ar^3\theta^2,$$

where

$$A=a/3b^4, \qquad (105)$$

so that "our pressure is equal to A, up to a constant coefficient, which is moreover negative."

Let us now evaluate the "experimental mass of the electron" (i.e., its rest mass). By Eqs. (71), (82), and (73),

$$L=a(1-u^2)^{1/2}/b,$$

so that for $u\ll 1$,

$$L=(a/b)[1-(u^2/2)],$$

and conseqeuently "the mass whether longitudinal or transverse, will be a/b."

Since a is a "numerical constant," it follows from Eq. (105) that "the pressure which is generated by our supplementary potential is proportional to the fourth power of the experimental mass of the electron."

"Since the Newtonian attraction is proportional to this experimental mass, one is tempted to conclude that there exists some relationship between the causes which generate gravitation and this supplementary potential."

[1] H. M. Schwartz, Amer. J. Phys. **39**, 1287 (1971).

[2] P. Langevin, J. Phys. **4**, 165 (1905). Poincaré refers to this paper at the end of this section: "I shall content myself with referring for further details to the article of Mr. Langevin in the *Journal de Physique* (the year 1905)." He does not mention earlier derivations relating to Eqs. (50), but references to Lienard, Wiechert, and Schwartzschild are contained in Langevin's paper.

[3] A more literal translation of Poincaré's *molécule* in the present context would be "particle," but "particle of the electron" sounds no less strange to us. What is meant here, of course, is an infinitesimal element of charge at the point under consideration.

[4] We employ the repeated-indices summation convention. Free Roman indices range over 1, 2, 3.

[5] $\det\|\delta_{ij}+a_ib_j\|=\epsilon_{ijk}(\delta_{i1}+a_ib_1)(\delta_{j2}+a_jb_2)(\delta_{k3}+a_kb_3)$
$=1+a_ib_i$.

[6] Poincaré uses the symbol μ instead of q, and later [in Eq. (50′)] he introduces the symbol μ_1 for the electron charge, although the symbol e for that quantity was already then used by most investigators in electrodynamics (including Lorentz and Langevin). It should be noted that "electron" meant then an elementary charged particle of *either sign*.

[7] See Ref. 6.

[8] Equations [5.4] in the original text are written for the special case $\boldsymbol{\beta}=(\beta, 0, 0)$, when they follow from Eqs. (17) and (9): The first set, namely, $B_1=0$, $B_{2,3}=\mp\beta B_{3,2}$ is deducible directly from the transformation (17) and its inverse, while the second set is obtained by using these equations together with Eqs. (17) and (9). The general form given in (51) can be proved easily by using the general transformation formulas corresponding to Eqs. (17) and (9).

[9] The electron's motion may be thought of as represented by the equation $x=a\sin\omega t$ with $a\omega^2$ finite and a infinitesimal.

[10] H. Hertz, Ann. Phys. Chem. **36**, 1 (1888).

[11] By Eqs. (17) and (54),

$$\mathbf{E}'\cdot\mathbf{r}'=l^{-2}[E_x\gamma l(r_x-\beta r)+\gamma(E_y-\beta B_z)lr_y+\gamma(E_z+\beta B_y)lr_z]$$
$$=\gamma l^{-1}\{\mathbf{E}\cdot\mathbf{r}-\beta[rE_x-(\mathbf{B}\times\mathbf{r})_x]\},$$

and the first of Eqs. (55) follows from (53), remembering that $\mathbf{r}=r\mathbf{n}$. The second of Eqs. (55) is obtained in similar fashion.

[12] The contents of the present section deal not with new results, but with methods of proving old results. The preceding method serves to illustrate the application of Lorentz transformations. The following method shows the flavor of Poincaré's mathematical style. It is therefore presented *verbatim* (except for change in mathematical notation).

[13] The italics are not in the original. Note also that the statement [17] must have reference to the center of mass of the electron, unless the electron is assumed to be in some sense absolutely rigid.

[14] We arrive by means of our Lorentz transformation to the equation following Eq. (9′), which reduces to Eq. (56) when $u_1=\beta$, $u_2=u_3=0$.

[15] Cf. the last remark in Ref. 13.

[16] Our retention in (59) of Poincaré's symbols A, B need cause no confusion with the magnitudes of the vectors **A**, **B** (see Pt. I, Table I).

[17] The Jacobian $\partial(x')/\partial(x)$ for the transformation (9) when $t=$const is γl^3.

[18] That is, the field Lagrangian. Poincaré's symbol for this quantity, H, is even in conflict with his own symbol for A_z (see Pt. I, Table I).

[19] Unfortunately, Poincaré gives no indication as to how he arrived at these relations, nor any references. To present here any discussion of these developments would be inconsistent with the avowed purpose of the present work, as stated in the introduction to Part I. Perhaps the most useful single reference in this connection is M. Abraham, *Theorie der Elektrizität* (B. G. Teubner, 1905), Chap. 3, Vol. II, Eqs. (64) correspond to Eqs. (IIIa) and (IIIb) in the latter book. One must note that Abraham's and Poincaré's definitions of the Lagrangian function differ in sign, and bear in mind the differences in the underlying

basic assumptions. One must also remember that Poincaré's discussion in the present and in the following two sections presupposes (as does also Abraham's discussion) that the electron's mass is purely electromagnetic.

[20] The converse is also true, so that Eqs. (64) and (65) are actually equivalent. In fact, by the two equations in (64), $dE/d\beta = -\beta d^2L/d\beta^2 = \beta dD/d\beta$, i.e., $dD/d\beta = -d^2L/d\beta^2$ so that $D = -dL/d\beta + \text{const} = (E-L)/\beta + \text{const}$. But $(E-L)/\beta = 2C/\beta \to 0$ as $\beta \to 0$ [by (60) and (59) since E is finite]; therefore, since $D \to 0$ as $\beta \to 0$, the constant vanishes and we get the first of Eqs. (65). Then the second of Eqs. (65) is equivalent to the first of Eqs. (64). It follows then conversely that Eqs. (65) imply the first of Eqs. (64), and that the latter equation together with the first of Eqs. (65) imply the second of Eqs. (64).

[21] The second of Eqs. (65) is identically satisfied by virtue of the relations (63), but applying the latter to the first of Eqs. (65), we get the equation:

$$0 = D + dL/d\beta$$

$$A'[l(4\beta\gamma + 3d\gamma^{-1}/d\beta) + 3\gamma^{-1}dl/d\beta]$$

$$= A'(l\beta\gamma + 3\gamma^{-1}dl/d\beta),$$

or,

$$dl/l = -\beta d\beta/3(1-\beta^2).$$

Hence $l = (1-\beta^2)^{1/6}$, the multiplicative integration constant being unity, since obviously we must have $l=1$ when $\beta = 0$.

[22] See Ref. 17.

[23] M. Abraham, Göttinger Nachrichten **1**, 20 (1902), p. 37. See also Ref. 19.

[24] Bearing in mind also that a purely electromagnetic origin of the electron's inertia is being assumed.

[25] The purpose of this treatment is to obtain F explicitly in terms of r and θ so as to arrive at the result cited in [22]. The result (84) is in fact obtained from the general expression (86) [and (87)] when the conditions for the Lorentz model are introduced. Note also that the result (83) is consistent with Langevin's model involving no supplementary potential [see third equation in (74)].

[26] See Ref. 19. If we denote the electromagnetic momentum by $\mathbf{P}$, then $D = |\mathbf{P}|$, while in Sec. 6, $D = P_x$. The relation, $D = -\partial L/\partial u$, introduced in (90), is seen to be equivalent to the first equation in (80′), when we note that $\mathbf{u}$ can be identified there with $(\beta, 0, 0)$.

[27] See equation below (80′) and the second equation in (80′).

[28] By Eq. (73), since $\theta = \gamma$ and $m = -1$ for Lorentz's model.

[29] As is shown in Sec. 8, $A = a/b =$ rest mass of the electron.

[30] That we can use this notation and also apply (as we do presently) the transformation Eq. (19) without contradiction with the meaning of the symbol F given in Table I, follows from assumption (10).

[31] By (9) for $l=1$, $dt' = \gamma(dt - \beta dx) = \gamma dt(1-\beta u_1)$.

[32] Noting the identity:

$$(1-\beta u)^2 - (u-\beta)^2 \equiv (1-u^2)(1-\beta^2).$$

[33] The symbol ω_0 is thus now used as a δ-type function.

[34] This appears to involve the tacit assumption of the absolute invariance of the coefficient ω_0, similar to the invariance stated in (10) for electric charge.

Poincaré's Rendiconti Paper on Relativity. Part III

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This is the concluding part of a modernized rendition of Poincaré's Rendiconti paper on relativity, of which the first two parts appeared in the November 1971 and June 1972 issues of this Journal. It covers the last section of that paper, in which Poincaré develops in masterful, even if incomplete, fashion, a generalization of Newtonian gravitational theory, involving retarded action-at-a-distance interaction that is covariant under the Lorentz group. As the first such attempt it is of obvious historical significance. In addition, just as the first two parts, so this part, too, contains material of independent interest to the historian of the genesis of special relativity.

For the purpose and scope of the present modernized rendition of Poincaré's Rendiconti paper on relativity, and for the notation that is being employed, the reader is referred to the introductory remarks to Pt. 1 of this study [Amer. J. Phys. **39,** 1287 (1971)]. The additional remarks concerning notation made in the introduction to Pt. II [Amer. J. Phys. **40,** 862 (1972)] are also applicable here. Because this part of Poincaré's paper is of particular interest in connection with its methodological aspects, including an anticipation of the four-vector calculus, certain relevant portions of the original text are reproduced here more closely than would have been otherwise indicated.

As for the notes or comments—which, as in the earlier parts, are either given in footnotes or enclosed in braces in the text—these are intended in general, as previously, to serve only as explanation of the original text. In addition, there are included here a few footnotes that point out nontrivial misprints in existing French and English literal reproductions of Poincaré's paper and a footnote containing references to later work on the subject of Poincaré's pioneering investigation in relativistic gravitational theory.

9. HYPOTHESES CONCERNING GRAVITATION

[27] "Thus the impossibility of making evident the existence of absolute motion would be fully explained by Lorentz's theory, if all forces were of electromagnetic origin."

But there exist forces, such as gravitation, which are not of electromagnetic origin.

[28] "Lorentz was therefore obliged to complete his hypothesis by supposing that *forces of any origin, and in particular, gravitation, are affected by a translation* (or, if one prefers, by a Lorentz transformation) *in the same way as are the electromagnetic forces.*"

It follows from this assumption, as applied to gravitation, that we can no longer retain the Newtonian theory involving an attraction between two bodies that depends only on their relative position at each instant under consideration. The gravitational attraction must also depend on "the velocities of the two bodies." In addition, it is to be expected that "the force which acts on the attracted body at an instant t depends on the position and velocity of that body at the instant t; but also on the position and velocity of the *attracting* body, not at the instant t, but at an *earlier instant,* as if it took gravitation a certain time to propagate itself."

The equation for this propagation must therefore be of the form

$$\phi(t, \mathbf{x}, \mathbf{u}, \mathbf{u}_1) = 0, \quad [9.1] \quad (106)$$

where, $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_0$, $\mathbf{x}_0$ is the position vector of attracted body at time t_0, $\mathbf{x}_1$ is the position vector of attracting body at time $t_1 = t_0 + t$, and $\mathbf{u}$, $\mathbf{u}_1$ are the velocities of the attracted body at time t_0 and of the attracting body at time t_1.

Let now **F** represent the force exerted upon the attracted body at the time t_0.[1] It must be expressed in terms of

$$t,\ \mathbf{x},\ \mathbf{u},\ \mathbf{u}_1, \qquad [9.2] \quad (107)$$

and the following conditions must be satisfied:

(1) Equation (106) must be covariant under the Lorentz group.

(2) **F** must transform under the Lorentz transformations (9) in the same way as the electromagnetic force denoted in Sec. 1 by the same symbol, i.e., according to Eqs. (19).

(3) "When the two bodies are at rest one must regain the usual law of attraction." [Relation (106) becomes then, of course, irrelevant.]

These conditions obviously do not suffice. The following additional ones naturally come into consideration:

(4) "Since astronomical observations do not seem to disclose significant deviations from Newton's law, we shall choose the solution that deviates least from this law when the velocities of the two bodies are small."

(5) "We shall attempt to arrange for t to be always negative; for if, in fact, one conceives of the gravitational effect as requiring a certain time for its propagation, it is hard to understand how this effect could depend on the position which has *not yet been attained* by the attracting body."

[29] "There is one case when the indeterminacy of the problem disappears; this is the case of *relative* rest of the two bodies; i.e., when $\mathbf{u}=\mathbf{u}_1$; this then is the case which we shall examine first, on the assumption that these velocities are constant, so that the two bodies are involved in a common state of rectilinear, uniform, translational motion."

By choosing for the direction of our x_1 axis that of the common velocity of our bodies, so that $u_j=0$ ($j=2, 3$), then taking $\beta=u_1$ in Eqs. (9), the transformed reference frame S' becomes the rest frame of the bodies,[2] and consequently by condition (3), we have *to within a constant factor*,

$$\mathbf{F}'=-\mathbf{x}'/r'^3, \qquad r'^2=\mathbf{x}'^2. \quad [9.3] \quad (108)$$

In applying Eqs. (19) (where it is now being tacitly assumed that pertinent equations in Pt. I are taken with $l=1$), we note that Eq. (11) now gives

$$\rho'/\rho=\gamma(1-\beta u_1)=\gamma(1-\beta^2)=\gamma^{-1},$$

and that $\mathbf{F}\cdot\mathbf{u}=\beta F_1$, so that the transformation equations for **F** reduce to

$$F_1'=F_1, \qquad F_j'=\gamma F_j \qquad (j=2, 3).$$

Hence, using Eqs. (108) and (9), we find

$$F_1=-\gamma(x_1-u_1t)/r'^3, \qquad F_j=-x_j/\gamma r'^3 \qquad (j=2, 3) \quad [9.4] \quad (109)$$

or

$$\mathbf{F}=\nabla V, \qquad V=1/\gamma r', \quad [9.4\text{bis}] \quad (109')$$

where

$$r'^2=\gamma^2(x_1-u_1t)^2+x_2^2+x_3^2. \qquad (109'')$$

This result would appear to depend upon our choice of an hypothesis concerning t, "but it is easy to see that x_i-u_it, which alone appear in our formulas, do not depend on t."[3]

We also see that the force acting on the attracted body is normal to an ellipsoid whose center is at the position of the attracting body.

"In order to make further progress, it is necessary to look for the *invariants of the Lorentz group*."

"We know that the transformations of this group (taking $l=1$) are the linear transformations which do not change the quadratic form $\mathbf{x}^2-t^2$."[4] But this form can be written as $x_kx_k+(it)^2=x_\alpha x_\alpha$ ($\alpha=1, 2, 3, 4$), introducing the notation $x_4\equiv it$.[5] It can therefore be seen, since the quadruplets (x_α), (dx_α), $(dx_{1\alpha})$[6] transform in the same way under Lorentz transformations, that these quadruplets may be considered as "the coordinates of three points P, P', P'' in four-dimensional space," and that "the Lorentz transformation is but a rotation of this space about a fixed origin. It follows that the only independent invariants are "the six distances of the three points P, P', P'' from each other and from the origin"; in other words,[7] the six scalar products $x_\alpha x_\alpha$, $x_\alpha dx_\alpha$, etc., that can be formed from the four-vectors corresponding to P, P', and P''.

But what we actually need are not these invariants themselves but the invariant combinations which are homogeneous of degree zero with respect to the dx_α and the $dx_{1\alpha}$, since what we must find are suitable "invariant functions of the variables" (107). There are only four such combinations, namely,[8]

$$x_\alpha x_\alpha, \qquad (t-\mathbf{x}\cdot\mathbf{u})(1-\mathbf{u}^2)^{-1/2},$$
$$(t-\mathbf{x}\cdot\mathbf{u}_1)(1-\mathbf{u}_1^2)^{-1/2},$$
$$(1-\mathbf{u}\cdot\mathbf{u}_1)[(1-\mathbf{u}^2)(1-\mathbf{u}_1^2)]^{-1/2}. \quad [9.5] \quad (110)$$

Turning now our attention to the transformation properties of the force components, we are guided first by Eqs. (18), which show that if we write $\mathbf{f}\cdot\mathbf{u}\equiv f_0$, then

$$f_1'=\gamma(f_1-\beta f_0), \qquad f_j'=f_j \qquad (j=2, 3)$$
$$f_0'=\gamma(f_0-\beta f_1), \qquad [9.6] \quad (111)$$

so that f_ν ($\nu=0, 1, 2, 3$) are the components of a (real) four-vector. On the other hand,

$$F_\nu=f_\nu/\rho \qquad (\nu=0, 1, 2, 3; F_0\equiv\mathbf{F}\cdot\mathbf{u}), \quad (112)$$

and by Eq. (11) ($l=1$),

$$\rho/\rho'=1/\gamma(1-\beta u_1)=dt/dt'. \qquad (112')$$

Hence $(1-\mathbf{u}^2)^{-1/2}F_\nu$ are the components of a four-vector, and by reasoning similar to that used previously we find the additional four invariants[9]

$$(\mathbf{F}^2-F_0^2)(1-\mathbf{u}^2)^{-1}, \qquad (\mathbf{F}\cdot\mathbf{x}-F_0t)(1-\mathbf{u}^2)^{-1/2},$$
$$(\mathbf{F}\cdot\mathbf{u}-F_0)[(1-\mathbf{u}^2)(1-\mathbf{u}_1^2)]^{-1/2},$$
$$(\mathbf{F}\cdot\mathbf{u}-F_0)(1-\mathbf{u}^2)^{-1}, \quad [9.7] \quad (113)$$

of which the last vanishes identically by virtue of the definition of F_0 [in Eq. (112)].

We now have to satisfy the following conditions:

(a) The left-hand side of Eq. (106) must be a function of the four invariants (110).

(b) The invariants (113) must be functions of the invariants (110).

(c) "When the two bodies are in a state of absolute rest, $\mathbf{F}$ must have the value deduced from Newton's law, and when they are in a state of relative rest, it must have the value deduced from Eqs. (109)."

As to condition (a), "many hypotheses can obviously be made, of which we shall only examine two" [given by the vanishing of the first two invariants in Eq. (110)]:

$$\text{(A)} \quad \mathbf{x}^2\equiv r^2=t^2, \qquad \text{(B)} \quad \mathbf{x}\cdot\mathbf{u}=t.$$

At first sight it might appear that (A) has to be rejected on the basis of Laplace's proof that the propagation speed of gravitation, if not infinite, must exceed that of light—

[30] "But Laplace has examined the hypothesis of a finite propagation velocity *ceteris non mutatis*; here, on the contrary, this hypothesis is entangled with many others, and it can transpire that there exists between them a more or less perfect mutual compensation of the kind for which the applications of the Lorentz transformation have already provided us with so many examples."

At the same time, hypothesis (B) must be rejected because although it agrees with Laplace's result, it can in some instances conflict with condition (5).[10] Hypothesis (A), on the other hand, always agrees with that condition, upon our choice of the solution

$$t=-r. \qquad (114)$$

We therefore adopt hypothesis (A).

Combining now conditions (b) and (c) "for the case of absolute rest, the first two invariants (113) must reduce to $\mathbf{F}^2$ and $\mathbf{F}\cdot\mathbf{x}$, or, by Newton's law to

$$r^{-4}, \qquad -r^{-1}; \qquad (115)$$

on the other hand, by hypothesis (A) [i.e., by Eq. (114)] the second and third invariants (110) become

$$(-r-\mathbf{x}\cdot\mathbf{u})(1-\mathbf{u}^2)^{-1/2}, \quad (-r-\mathbf{x}\cdot\mathbf{u}_1)(1-\mathbf{u}_1^2)^{-1/2}, \qquad (116)$$

i.e., for absolute rest

$$-r, \qquad -r. \qquad (117)$$

We can therefore assume, *for example*, that the first two invariants (113) reduce to[11]

$$(1-\mathbf{u}_1^2)(r+\mathbf{x}\cdot\mathbf{u}_1)^{-4},$$
$$-(1-\mathbf{u}_1^2)^{1/2}(r+\mathbf{x}\cdot\mathbf{u}_1)^{-1};$$

but other combinations are possible."

"It is necessary to make a choice between these combinations, and we require also a third equation in order to determine **F**." We take now into account condition (4). First we note that if we neglect the squares of u_i and u_{1i}, and use Eq. (114), then the invariants (110) and (113) become, respectively,

$$0, \qquad -r-\mathbf{x}\cdot\mathbf{u}, \qquad -r-\mathbf{x}\cdot\mathbf{u}_1, \qquad 1, \quad (118)$$

and

$$\mathbf{F}^2, \qquad \mathbf{F}\cdot(\mathbf{x}+r\mathbf{u}), \qquad \mathbf{F}\cdot(\mathbf{u}_1-\mathbf{u}), \qquad 0. \quad (119)$$

But we must also bear in mind that in the *Newtonian* theory we have $t=0$, where t is defined in connection with Eq. (106), so that in the present approximation we can neglect higher powers of t than the first, and we may thus "proceed as if the motion were uniform." Consequently,[12]

$$\mathbf{x}=\mathbf{x}_1(0)+\mathbf{u}_1t, \qquad r(r-r_1)=\mathbf{x}\cdot\mathbf{u}_1t,$$

where $\mathbf{x}_1(0)$ is the position vector of the attracting body relative to the attracted body at the time t_0, $r_1=|\mathbf{x}_1(0)|$, and $r=|\mathbf{x}|$ [See (A)]; or by Eq. (114),

$$\mathbf{x}=\mathbf{x}_1(0)-\mathbf{u}_1r, \qquad r=r_1-\mathbf{x}\cdot\mathbf{u}_1. \quad (120)$$

Eqs. (118) and (119) thus become

$$0, \qquad -r_1+\mathbf{x}\cdot(\mathbf{u}_1-\mathbf{u}), \qquad -r_1, \qquad 1 \quad (121)$$

and [writing now $\mathbf{x}_1$ for $\mathbf{x}_1(0)$]

$$\mathbf{F}^2, \qquad \mathbf{F}\cdot[\mathbf{x}_1+(\mathbf{u}-\mathbf{u}_1)r_1], \qquad \mathbf{F}\cdot(\mathbf{u}_1-\mathbf{u}), \qquad 0, \quad (122)$$

where in the second of expressions (122) we have replaced r by r_1, since it is multiplied by $\mathbf{u}-\mathbf{u}_1$.

On the other hand, with **F** given by Newton's law, the expressions (119) take the form

$$r_1^{-4}, \quad -r_1^{-1}-\mathbf{x}_1\cdot(\mathbf{u}-\mathbf{u}_1)r_1^{-2}, \quad \mathbf{x}_1\cdot(\mathbf{u}-\mathbf{u}_1)r_1^{-3}, \quad 0.$$

"If then we denote the second and third invariants (110) by A and B, and the first three invariants (113) by M, N, P, we shall satisfy Newton's law to within terms of the second order in the velocities by putting"[13]

$$M=B^{-4}, \qquad N=AB^{-2}, \qquad P=(A-B)B^{-3}. \quad [9.8] \quad (123)$$

This solution is, however, not unique: since $(A-B)^2$ and $C-1$, where C is the fourth invariant (110), are of the second order in the velocities, "we may add to the right-hand sides of each of Eqs. (123) a term"

$$(C-1)f_1(A,B,C)+(A-B)^2f_2(A,B,C), \quad (124)$$

where f_1 and f_2 are arbitrary functions. On the other hand, the solution (123) as it stands is not acceptable, because it can lead in some cases to nonreal values of the F_i, since the quantities M, N, P are functions of the F_i as well as of $F_0=\mathbf{F}\cdot\mathbf{u}$.

"In order to avoid this inconvenience, we shall proceed in a different manner." We observe that the invariants (110) can be put [using Eq. (114)] in the form

$$0, \qquad A=-\gamma_0(r+\mathbf{x}\cdot\mathbf{u}), \qquad B=-\gamma_1(r+\mathbf{x}\cdot\mathbf{u}_1),$$
$$C=\gamma_0\gamma_1(1-\mathbf{u}\cdot\mathbf{u}_1),$$

where we have introduced the symbols

$$\gamma_0=(1-\mathbf{u}^2)^{-1/2}, \qquad \gamma_1=(1-\mathbf{u}_1^2)^{-1/2}, \quad (124')$$

"by analogy to the notation $\gamma=(1-\beta^2)^{-1/2}$ which appears in the Lorentz transformation," and that "the following systems of quantities

$$(\mathbf{x}, t=-r), \quad (\gamma_0\mathbf{F}, \gamma_0F_0), \quad (\gamma_0\mathbf{u}, \gamma_0), \quad (\gamma_1\mathbf{u}_1, \gamma_1)$$

undergo the *same* linear transformations when they are subjected to the transformations of the

Lorentz group. We are then led to put"

$$F_\nu = a\gamma_0^{-1}x_\nu + bu_\nu + c\gamma_0^{-1}\gamma_1 u_{1\nu}$$
$$(\nu = 0, 1, 2, 3; u_0 = u_{10} = 1), \quad [9.9] \quad (125)$$

which is obviously a four-vector provided a, b, c are four-scalars (i.e., Lorentz invariants).

"But for the compatibility of Eqs. (125) it is necessary {in order to agree with the *definition* of F_0 [see (112)]} that

$$\mathbf{F}\cdot\mathbf{u} - F_0 = 0,$$

which becomes upon replacing the F_ν by their values (125) and multiplying by γ_0^2"[14]:

$$aA + b + cC = 0. \quad [9.10] \quad (126)$$

"What we want is that if we neglect in comparison with the square of the velocity of light, the squares of the velocities u_i, as well as products of accelerations and distances, then the values of the F_ν remain in agreement with Newton's law."[15]

To this order of approximation, we have

$$\gamma_0 = \gamma_1 = 1, \qquad C = 1,$$
$$A = -r_1 + \mathbf{x}\cdot(\mathbf{u}_1 - \mathbf{u}), \qquad B = -r_1.$$

If we make then the simple choice [compatible with Eq. (126)]

$$b = 0, \qquad c = -aA/C,$$

we find, using Eqs. (120), that the three-vector part of Eq. (125) becomes

$$\mathbf{F} = a(\mathbf{x} - A\mathbf{u}_1) = a(\mathbf{x} + r\mathbf{u}_1) = a\mathbf{x}_1(0).$$

Since by Newton's law, $\mathbf{F} = -\mathbf{x}_1(0)/r_1^3$, "we must choose for the invariant a the quantity which reduces to $-r_1^{-3}$ within the adopted order of approximation, that is B^{-3}." Equations (125) assume then the form

$$F_\nu = \gamma_0^{-1}B^{-3}x_\nu - \gamma_0^{-1}\gamma_1 AB^{-3}C^{-1}u_{1\nu}. \quad [9.11] \quad (127)$$

[31] "We see at first that the corrected attraction consists of two components; one parallel to the vector joining the positions of the two bodies, the other parallel to the velocity of the attracting body."

"Let us recall that when we speak of the position or the velocity of the attracting body, we refer to its position or velocity at the instant the gravitational wave leaves it {i.e., at the *retarded* time, t_1}; whereas the position and velocity of the attracted body are referred to the instant when the gravitational wave reaches it, this wave being assumed to propagate with the velocity of light."

"I believe that it would be premature to wish to push the discussion of these formulas any further; I shall therefore confine myself to a few remarks."

1. "The solutions (127) are not unique" since we can add to the common factor B^{-3} the quantity (124); "or not take $b = 0$, but add arbitrary terms to a, b, c provided they satify condition (126) and are of second order in u_i as far as a is concerned, and of the first order as far as b and c are concerned."

2. The three-vector part of Eq. (127) can be written[16]

$$\mathbf{F} = \gamma_1 B^{-3}C^{-1}[(1 - \mathbf{u}\cdot\mathbf{u}_1)\mathbf{x} + (r + \mathbf{x}\cdot\mathbf{u})\mathbf{u}_1],$$
$$[9.11\text{bis}] \quad (128)$$

and the quantity in brackets can be written as

$$(\mathbf{x} + r\mathbf{u}_1) + [\mathbf{u} \times (\mathbf{u}_1 \times \mathbf{x})], \quad [9.12] \quad (129)$$

so that $\mathbf{F}$ appears to consist of two components, the first having "a vague analogy to the mechanical force due to the electric field," and the second to "the mechanical force due to the magnetic field." This analogy can be improved by getting rid of the factor C^{-1} in Eq. (128), the resulting expression depending then only linearly on $\mathbf{u}$. This can be done by applying remark 1 "to replace B^{-3} by CB^{-3} in Eqs. (127)."[17]

"Setting now

$$\gamma_1(\mathbf{x} + r\mathbf{u}_1) = \boldsymbol{\lambda}, \qquad \gamma_1(\mathbf{u}_1 \times \mathbf{x}) = \boldsymbol{\lambda}', \quad [9.13]$$

it follows, since C has disappeared from the denominator of (128), that

$$\mathbf{F} = B^{-3}\boldsymbol{\lambda} + B^{-3}(\mathbf{u} \times \boldsymbol{\lambda}'), \quad [9.14] \quad (130)$$

and one also has (as is easily checked)

$$B^2 = \lambda^2 - \lambda'^2. \quad [9.15]$$

Then λ or $B^{-3}\lambda$ is a kind of electric field, while λ' or, rather $B^{-3}\lambda'$, is a kind of magnetic field."

3. "The postulate of relativity would force us to adopt either the solution (127) or the solution (130), or any one of the solutions that can be deduced from them by using remark 1. But the primary question is whether they are consistent with astronomical observations. The deviation from Newton's law is of the order of u^2, that is, 10 000 times smaller than if it were of the order of u, that is, if the velocity of propagation were equal to that of light, *ceteris non mutatis*.[18] It is therefore permissible to hope that it will not be too great; however, only a more penetrating discussion could tell us that."[19]

[1] The original text contains here the phrase "at the instant t," a misprint which has not been corrected in either the French or English (partly edited) reproductions of the original paper, namely, those contained in H. Poincaré, *La Mécanique Nouvelle* [*conférence, mémoire et notes sur la théorie de relativité*. Introduction de m. Eduward Guillaume] (Gauthier–Villars, Paris, 1924), and in C. W. Kilmister, *Special Theory of Relativity* (Pergamon, New York, 1970). These will be referred to by the respective symbols (G) and (K).

[2] The original statement here reads: "the two bodies will be at rest after the transformation," i.e., the bodies will be in a state of *absolute* rest, this being clearly implicit in the wording of condition (3).

[3] Since assumption (3) is being used, we could simply set $t=0$ in Eqs. (109), or (109′) and (109″).

[4] See the discussion following the third paragraph after Eq. (40).

[5] No such symbol is introduced in the original text.

[6] The original symbols $\delta_1 x$, $\delta_1 y$, $\delta_1 z$ and $\delta_1 t$ are replaced here by $d\mathbf{x}_1$ and dt_1. The convenient symbol t_1 to represent t_0+t is not introduced in the original text, but from its context it is clear that $\delta_1 x \equiv \delta x_1$, etc., in Poincaré's notation for differentials. The reproduction of this notation in the present connection in (G) and (K) (see Ref. 1) is inconsistent with the editing of Poincaré's notation for derivatives found elsewhere in these references.

[7] We introduce at this point the convenient four-vector formalism. Had Poincaré adopted the ordinary vector calculus that was already in use by theoretical physicists—for example, Lorentz and Abraham—for some time, he would have in all likelihood introduced explicitly in the present connection the convenient four-dimensional vector calculus.

[8] The second expression in (110) can be written $\gamma_0(t-\mathbf{x}\cdot\mathbf{u})(\gamma_0^2-\gamma_0^2\mathbf{u}^2)^{-1/2}$, where γ_0 is defined later in (124′). Since $(\gamma_0, \gamma_0\mathbf{u})$ is a (real) four-vector (in fact, the velocity four-vector, in modern terminology), the invariance of this expression—and in a similar way, of the last two expressions—is apparent.

[9] Compare Ref. 8. The proof in the original text is based directly on Eq. (112′), which implies that F_ν and u_ν ($u_0=1$) transform in the same way under Lorentz transformations (as four-vectors except for the missing factor γ_0).

[10] There is a misprint here in the original text, which reads "but in certain cases, t could be negative" (instead of "positive"), which has been reproduced in (G) and (K).

[11] By choosing the second expression in (116) to compare with (117), and then taking account of (115). The misprints "invariants (4)" [for "invariant (7)"] is reproduced in (K) and changed to the wrong "invariants (5)" in (G).

[12] The second equation follows from the first, when higher powers than the first in the material velocities are neglected. We introduce here temporarily the symbol $x_1(0)$ to replace the symbol x_1 in the original text, because the latter symbol has been employed here previously in connection with Eq. (106).

[13] There is a misprint in (K) in the second equation of [9.8].

[14] With $a\cdot b \equiv a_0b_0 - a_ib_i$ $(\equiv a_\nu b^\nu)$, we have $\gamma_0^2F\cdot u = a\gamma_0 x\cdot u + b\gamma_0^2 u\cdot u + c\gamma_0\gamma_1 u_1\cdot u$, and by (110), $\gamma_0 x\cdot u = A$, $\gamma_0\gamma_1 u_1\cdot u = C$, while $\gamma_0^2 u\cdot u$ as the "square" of the "four-velocity" $\gamma_0 u$ is 1 (remembering that we are using units with $c=1$.)

[15] This is a more precise statement of condition (4) introduced in the beginning of this section.

[16] Recalling that $A=\gamma_0(t-\mathbf{x}\cdot\mathbf{u})$, $C=\gamma_0\gamma_1(1-\mathbf{u}\cdot\mathbf{u}_1)$, and using Eq. (114).

[17] By taking $f_1=B^{-3}$ and $f_2=0$ in expression (124).

[18] This sentence is rather obscure. Its meaning becomes clear when we read the corresponding part of the concluding paragraph in Poincaré's note on the subject of his Rendiconti article [Compt. Rend. **140**, 1504 (1905)]: "The deviation from the ordinary law of gravitation is, as I have said, of the order of ξ^2 {i.e., u^2}; if one only assumes, as was done by Laplace, that the velocity of propagation is that of light, this deviation would be of the order of ξ, that is, 10 000 times larger." (Cf. [30]).

[19] Such a discussion was presented a few years later by W. de Sitter [Monthly Notices Roy. Astron. Soc. **71**, 388 (1911)] and quite recently, as part of a general discussion of special relativistic theories of gravitation, by G. J. Whitrow and G. E. Morduch [Nature **118**, 790 (1960); also, "Relativistic Theories of Gravitation" in *Vistas in Astronomy*, edited by A. Beer, Editor, (Pergamon, New York, 1965), Vol. 6, pp. 1–68]. A brief summary of special relativistic theories of gravitation is contained in H. M. Schwartz, *Introduction to Special Relativity* (McGraw–Hill, New York, 1968), Appendix 7B (errata sheets can be obtained from the author).