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Spherically Symmetrical Models in General Relativity†

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The field equations of general relativity are applied to pressure-free spherically symmetrical systems of particles. The equations of motion are determined without the use of approximations and are compared with the Newtonian equations. The total energy is found to be an important parameter, determining the geometry of 3-space and the ratio of effective gravitating to invariant mass. The Doppler shift is discussed and is found to contain both the velocity shift and the Einstein shift combined in a rather complex expression.

1. INTRODUCTION

The field equations of the general theory of relativity are very complex. The only non-static solutions which have so far been obtained are either approximations or are of the cosmological type. Since approximate solutions apply only in the cases where the field is almost Newtonian, their use in pointing out intrinsic consequences of the theory is somewhat restricted. Similarly cosmological solutions suffer from the disadvantage that the spatial part of space-time is supposed to be homogeneous and isotropic. Therefore it is often difficult, owing to the lack of independent variables, to disentangle the causes of various effects.

The main purpose of the present paper is to derive the equation of motion and to describe various properties of pressure-free spherically symmetrical systems. A rigorous solution of the field equations has been ob-

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tained and it is hoped that the model presented can be of use in illustrating and clarifying various points of interest in the theory.

The work may be considered to be an extension of the work of McCrea, McVittie and Lemaître[2] on the problem of condensation. The applicability of the work of these authors is somewhat restricted by the fact that they consider only small deviations from an Einstein universe. The work in the present paper is more general in that the system need approximate neither to a Newtonian system nor to an Einstein universe, but is more restricted in its assumptions of zero pressure and of spherical symmetry. Since the general trend in recent work in this field seems to be towards pressure-free systems the former assumption is not as restrictive as might appear at first sight.

Tolman[3] considers a system identical with ours and derives equivalent equations of motion. His discussion of it however is concerned with properties of the system very different from those to which detailed consideration is given in this paper.

The assumption of spherical symmetry supplies us with a model which lies between the completely homogeneous models of cosmology and the actual universe with its irregularities. In this sense an advance has been made which, though small, suffices to show up a number of significant features such as the shift of the spectral lines discussed in Section 7 and the connection between the geometry of 3-space and the energy of the spherical shells of matter (Section 5).

The equation of motion obtained is very simple, and is, but for a different interpretation of the constants, identical with the Newtonian equation of energy (Section 4). The extreme simplicity of this result is a very attractive feature of the theory.

In connection with the problem of the equations of motion in general relativity, it is interesting to observe that the postulate of the motion of matter along geodesics does not lead to any contradictions with the field equations in our system, which seems to be more complicated than any other system yet examined without the use of approximations.

The question of boundary conditions at infinity does not arise in our model; the condition that the centre of the system is an ordinary point (we exclude point masses) is found to be sufficient to determine the solutions

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[2] W. H. McCrea and G. C. McVittie, *M. N.* **91**, 128, 1930; **92**, 7, 1931; G. C. McVittie *M. N.* **91**, 274, 1931; **92**, 500, 1932; **93**, 325, 1933; G. Lemaître, *M. N.* **91**, 483,490, 1931. In the last named paper Lemaître studies a problem very closely related to ours and many equations given in the appendix can be found in the paper.

[3] R. C. Tolman, *Proc. Nat. Acad. Sci.* **20**, 169, 1934.

of the equations uniquely.

Gravitational units are used throughout the paper, i.e. the velocity of light and the constant of gravitation are put equal to unity.

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2. THE METRIC

We now proceed to give a list of the assumptions made in order to specify the system.

(i) The system is and remains spherically symmetrical, i.e. the mass density and particle velocity are functions of a radial coordinate r and a time coordinate t only, and the motion of each particle is purely radial.

(ii) Each particle moves under the influence of gravity only. This implies that there are no electromagnetic forces acting on the particles and that there are no pressures.

(iii) The orbits of particles do not intersect. This means that they do not overtake each other. If an imaginary observer moves outwards from the centre he will therefore always pass the particles in the same order irrespective of his starting-time and speed, provided he is faster than any outward-moving particle. The exclusion of intersecting orbits does not lack physical meaning, since if they did intersect pressures would certainly arise.

(iv) The mass density is everywhere finite.

In addition to these assumptions we will also usually put the cosmological constant $\lambda = 0$. This simplifies the mathematics and leaves most of the essential features of the theory unimpaired. All the important formulae will however also be stated in the form they take when λ does not vanish.

In order to define our system of coordinates, suppose that we have a permanent source of light at the centre O of our spherically symmetrical system and surround this source by a small sphere. By assumption (iv) we can use a Galilean system of coordinates in the neighbourhood O and we can therefore introduce spherical polars on this sphere.

We now define the coordinates θ, ϕ of any event in the following way: consider the ray of light which went from O to the event. The θ, ϕ coordinates of the event are the θ, ϕ coordinates of the point of intersection of this ray and the small sphere.

In order to define the coordinates t, r, we observe that by assumption (ii) and by the fundamental postulates of general relativity our particles move along geodesics. If there are any points of space unoccupied by particles we will imagine these regions to be filled by very fine dust of negligible mass moving so as to satisfy (i), (ii) and (iii). We will assume that this is possible. On this understanding we have a family of non-intersecting geodesics, such that there is one and only one member of the family passing through each point of space-time. Accordingly we can draw a family of hypersurfaces, orthogonal to this family of geodesics, one such hypersurface passing through each point of space-time. We now define these hypersurfaces to be the hypersurfaces of constant t. As is well known in cosmology, the fact that our orthogonal trajectories are geodesics allows us to choose t so that it measures proper time along each member of our family of geodesics.

With assumption (i) and our choice of θ and ϕ, we already have the θ and ϕ coordinates constant along each geodesic. We now choose our r coordinate so that it too is constant along each geodesic and so as to make the surfaces of constant r orthogonal to the other coordinate surfaces. This is evidently possible. We also specify that r is positive and that roughly speaking it increases with distance from the origin. More precisely a point $(t_1, r_1, \theta_1, \phi_1)$ is assumed to be between the origin O and a point $(t_1, r_2, \theta_1, \phi_1)$ if, and only if, $r_1 < r_2$. Otherwise r is arbitrary.

In concluding this definition of our system of coordinates it must be mentioned that if assumption (iii) (non-intersection) does not hold throughout all space-time, but only throughout a finite or infinite region of space-time including O for some period, then we can still introduce our coordinates in at least part of that region and the theory will be valid there.

It follows from our assumption and our definition of the coordinates that our metric is

$$ds^2 = dt^2 - X^2(r,t)dr^2 - Y^2(r,t)(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{1}$$

where $X(r,t)$ and $Y(r,t)$ are functions of r and t only.

Accordingly the metric tensor $g_{\mu\nu}$ is[4]

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -X^2 & 0 & 0 \\ 0 & 0 & -Y^2 & 0 \\ 0 & 0 & 0 & -Y^2\sin^2\theta \end{bmatrix}. \tag{2}$$

The field equations of general relativity establish a connection between this tensor and the energy tensor $T^{\mu\nu}$. Since our system is without pressure

$$T^{\mu\nu} = \rho \frac{dx^\mu}{ds}\frac{dx^\nu}{ds}, \tag{3}$$

[4] We use the suffixes (0, 1, 2, 3) for (t, r, θ, ϕ) in that order.

where ρ is the invariant density and ds^{μ}/ds is the velocity of the matter. In our system of coordinates each particle moves in a way which keeps its r, θ, ϕ coordinates constant and makes the time component of its velocity equal to unity. Accordingly

$$T^{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4}$$

ρ is of course a function of r and t.

We will find it advantageous to use in our work not ρ but M, the sum of the invariant masses of all the particles with radial coordinate less than r. By our non-intersection assumption this mass depends on r only, so that $M = M(r)$.

Clearly

$$M(r) = \int_0^{\tau} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi\, T\sqrt{-g} = 4\pi \int_0^{r} dr\, \rho X Y^2, \tag{5}$$

so that

$$\frac{dM}{dr} = 4\pi\rho X Y^2. \tag{6}$$

Then the field equations of general relativity take the form[5]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi T_{\mu\nu} = \begin{cases} -\frac{2M'(r)}{XY^2} & (\mu = \nu = 0), \\ 0 & (\text{otherwise}), \end{cases} \tag{7}$$

where $R_{\mu\nu}$ is the Einstein tensor.

3. THE DEFINITION OF DISTANCE

Before we discuss the solution of the field equations, it will be desirable to investiage the main features of the propagation of light in our system, so that we can define distance.

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[5] In the appendix a more general problem is also discussed, viz. what system of pressures is compatible with the motion of particles along our geodesics. While such a system of pressures is in general more or less arbitrary, it is interesting to note that the assumption of isotropic pressure (i.e. $T_1^1 = T_2^2 = T_3^3$) is easily seen to imply that the pressure is a function of our time coordinate t only.

Astronomically, the most important definition of distance is probably luminosity distance.[6] The apparent luminosity of a source of known absolute luminosity is measured and is corrected for Doppler shift. The distance of the source is then defined as being proportional to the square root of the ratio of absolute and corrected apparent luminosity. As a result of the researches of Tolman, v. Laue and Robertson,[7] it is known that in a homogeneous universe the square of the Doppler shift has to be taken as the correcting factor. We shall now show that in our spherically symmetrical universe with the standard source at its centre, it is still correct to use the square of the Doppler shift, and that the luminosity distance of this standard source for an oberver at (t, r, θ, ϕ) is $Y(r, t)$.

By the definition of our system of coordiantes a ray of light travelling outwards from the centre satisfies the equations

$$\frac{dt}{dr} = X(r, t), \tag{8}$$

$$\theta = \text{const.}, \qquad \phi = \text{const.} \tag{9}$$

Consider now two rays with the same θ and ϕ values, and let the equation of the first ray be

$$t = T(r), \tag{10}$$

while the equation of the second ray is

$$t = T(r) + \tau(r). \tag{11}$$

We shall also assume that $\tau(r)$ is small. Then by (8)

$$\frac{dT(r)}{dr} = X\{r, T(r)\}, \qquad \frac{d\tau(r)}{dr} = \tau(r)\left(\frac{\partial X}{\partial t}\right)_{r,T(r)}. \tag{12}$$

This result gives the equation of a ray and the rate of variation of $\tau(r)$ along the ray. If we take $\tau(0)$ to be the period of oscillation of some spectral line at the origin, the Doppler shift in the Hubble-Tolman notation will be

$$\frac{\tau(r)}{\tau(0)} = \frac{\nu_0}{\nu} = 1 + z. \tag{13}$$

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[6] W. H. McCrea, *Z. f. Ap.* **9**, 290, 1935.

[7] R. C. Tolman *Proc. Nat. Acad. Sci.* **16**, 511 (para. 6), 1930; M. v. Laue, *Z. f. Ap.*, **12**, 208,1936; H. P. Robertson, *Z. f. Ap.*, **15**, 69, 1937.

Considering now z not as a function of r along a ray but as a function of t and r throughout space-time, we see that it satisfies the partial differential equation

$$\frac{\partial z}{\partial r} + X\frac{\partial z}{\partial t} = (1+z)\frac{\partial X}{\partial t}; \qquad z = 0 \quad \text{at} \quad r = 0. \tag{14}$$

A more detailed analysis of equations (12) will be given in Section 7; for our present purposes equation (14) is sufficient.

In order to investigate the variation of intensity with distance, we adopt Robertson's procedure.[8]

Consider an observer at $t_1, r_1, \theta_1, \phi_1$. For measurements in his neighbourhood he will use a local Galilean system $\bar{t}, \bar{x}, \bar{y}, \bar{z}$ with

$$\left.\begin{aligned} \bar{t} &= t - t_1, \quad \bar{x} = X(r_1, t_1)(r - r_1), \quad \bar{y} = Y(r_1, t_1)(\theta - \theta_1), \\ \bar{z} &= Y(r_1, t_1)(\phi - \phi_1)\sin\theta_1, \\ ds^2 &= d\bar{t}^2 - d\bar{x}^2 - d\bar{y}^2 - d\bar{z}^2. \end{aligned}\right\} \tag{15}$$

Assuming the wave-length of light to be minute compared with the dimensions of our system, the wave coming from the origin will appear to him to be plane, and hence his measurement of the electromagnetic energy tensor will give

$$\overline{E^{00}} = \overline{E^{01}} = \overline{E^{11}} = U \quad \text{(say)}, \tag{16}$$

while all other components vanish, U will be his measurement of the apparent luminosity of the source at O.

In our usual system of coordinates

$$E^{00} = U, \qquad E^{01} = \frac{1}{X}U, \qquad E^{11} = \frac{1}{X^2}U. \tag{17}$$

while all other $E^{\mu\nu}$ vanish. Applying the relativistic conservation law $(E^{\mu\nu})_\nu = 0$, we obtain one equation for U, viz.,[9]

$$\frac{\partial U}{\partial r} + X\frac{\partial U}{\partial t} + 2\left(\frac{\partial X}{\partial t} + \frac{X}{Y}\frac{\partial Y}{\partial t} + \frac{1}{Y}\frac{\partial Y}{\partial r}\right) = 0. \tag{18}$$

Putting

$$U = \frac{C}{Y^2(1+z)^2}, \tag{19}$$

[8] H. P. Robertson, *Z. f. Ap.*, **15**, 69, 1937.
[9] The three index symbols are listed in the appendix (equation (3)).

and making use of (14), we find

$$\frac{\partial C}{\partial r} + X\frac{\partial C}{\partial t} = 0. \tag{20}$$

This means that C is constant along each ray. If the source at the centre does not vary, then C will be constant throughout all space-time. Hence, by (19), Y can be found by observing the apparent luminosity of a source of known absolute luminosity, and applying the square of the Doppler shift as correcting factor.

The significance of this result is twofold: First we note that the important concept of luminosity distance is equivalent to our Y, which, as we shall see, is mathematically the most convenient dynamical variable in our system.

Secondly it is interesting to find that in our extension of the previously known use of the correcting factor, we can still use the whole of the Doppler shift, the effects of the Einstein and velocity shifts being indistinuishable.

Two other definitions of distance may be mentioned. If an observer at O measures the distance of an object of known size at (t, r, θ, ϕ) by measuring its apparent size, he will evidently obtain Y. If an observer at (t, r, θ, ϕ) measures his distance from O by measuring the parallax of O, it can be shown that this result will be $Y(X/Y')^{1/2}$. As we shall see in the next section, X/Y', the square of the ratio of this "distance" to Y, is, as a consequence of the field equations, a function of r only, i.e. it is a constant for every observer moving with the particles.

Finally, consider a test particle following an arbitrary geodesic. It can easily be shown that we can turn the system of coordinates so that the geodesic lies entirely in the surface $\theta = \pi/2$, and then there is an integral of angular momentum which is

$$Y^2\frac{d\phi}{ds} = \text{const.}$$

Again Y takes the part of the classical radius.

We see therefore that Y is a variable of considerable significance occupying in many ways a position corresponding to the classical concept of distance. Accordingly, we shall refer to $Y(r, t)$ as the distance of the particle (r, θ, ϕ) from the origin at time t and, since t measures the particle's proper time, we shall refer to $\partial Y/\partial t$ as its velocity.

4. THE EQUATION OF MOTION

The equations (7) are discussed in the appendix, where it is shown that they are equivalent to

$$X = \frac{1}{W(r)} \frac{\partial Y}{\partial r}, \tag{21}$$

$$\left(\frac{\partial Y}{\partial t} \right)^2 = W^2(r) - 1 + \frac{2}{Y} \int_0^r M'(r) W(r) dr, \tag{22}$$

where $W(r)$ is an arbitrary function of r. (21) merely expresses X in terms of Y, while (22) supplies us with the equation of motion.[10]

In order to compare (22) with the Newtonian approximation, we have to consider the case of small velocities and small masses. Accordingly W will be near unity and we put

$$W^2(r) = 1 + 2E(r), \tag{23}$$

where E is small. Then, negleciting the product of M' and E, we have

$$\frac{1}{2}\left(\frac{\partial Y}{\partial t} \right)^2 - \frac{M(r)}{Y} = E(r), \tag{24}$$

which is identical with the Newtonian equation of energy, E representing the total energy per unit mass. The exact equation (22) may be re-written

$$\frac{1}{2}\left(\frac{\partial Y}{\partial t} \right)^2 - \frac{1}{Y} \int_0^r dr M'(r) \{1 + 2E(r)\}^{1/2} = E(r), \tag{25}$$

and we see, then, that with our definition of distance and velocity the only difference between our equation and the Newtonian equation is that the effective gravitating mass is not the invariant mass. The most interesting point of this result is that the ratio of effective gravitating mass and invariant mass depends not on the kinetic energy but on the total[11] energy $E(r)$. This suggests that the total energy and hence the potential energy have a rather greater significance in general relativity than hitherto supposed, a point which will be more fully discussed in the next section.

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[10] Equation (22) can be integrated again. The integration is carried out in the appendix, but the result is of little importance for us.

[11] E. T. Whittaker, *Proc. Roy. Soc.*, A. **149**, 384, 1935, obtains similar results.

The acceleration $\partial^2 Y / \partial t^2$ can be determined by differentiating (25). The result is

$$\frac{\partial^2 Y}{\partial t^2} = -\frac{1}{Y^2}\int_0^r M'(r)dr\{1 + 2E(r)\}^{1/2}. \tag{26}$$

Hence we still have an inverse square law. However, it is now impossible to derive (25) from (26) without ambiguity, since $E(r)$, the constant of integration is already contained in (26).

An important similarity between our equations and those of the Newtonian theory is that in both theories the spherical shells of matter further away from O than a particle P do not effect the motion of P at all.

We are now in a position to discuss how $W(r)$ (and hence $E(r)$) are determined. As is usual in dynamics, our system will be fully defined only if, at some instant, the positions and velocities of all particles are given. It will be seen that in our case it is unnecessary that the velocities and positions of different particles are given at instants related in any particular way.

Let $t = t_0(r)$ prescribe the value of t at which the position and velocity of the particles at r are given as well as the density in their neighbourhood. We must of course assume that $t_0(r)$ is a single valued function. Let

$$Y\{r, t_0(r)\} = R(r), \tag{27}$$

$$\left(\frac{\partial Y}{\partial t}\right)_{r, t_0(r)} = V(r), \tag{28}$$

and let us assume that the mass distribution is given by giving $M(r)$. (27) is really only an equation defining r.

Then consider the equation of motion (22) at each point at the moment $t = t_0(r)$. We have

$$V^2 = W^2 - 1 + \frac{2}{R}\int_0^r W(r)M'(r)dr. \tag{29}$$

Multiplying by R and differentiating (as we are allowed to do, since each function in (29) is a function of r only),

$$RW\frac{dW}{dr} + \frac{1}{2}W^2\frac{dR}{dr} + WM' = \frac{1}{2}\frac{d}{dr}\{R(V^2 + 1)\}. \tag{30}$$

This first-order equation combined with the boundary condition $W = 1$ at $r = 0$, determines W. ($W = 1$ at $r = 0$ because, by our assumptions

$V(0) = 0$ and $M'(r) = 0(r^2)$ near $r = 0$). It is easily proved that the equation and the boundary condition determine W uniquely in spite of the singularity at $r = 0$.

If we are not given $M(r)$ but $\rho(r, t_0)$, then (29) takes the form

$$V^2 = W^2 - 1 + \frac{8\pi}{R} \int_0^r \rho\{r, t_0(r)\} dr R^2 \left\{ \frac{dR}{dr} - V \frac{dt_0}{dr} \right\}, \tag{31}$$

and this immediately determines W. The simplicity of this equation is a direct consequence of (6) and (21).

Some remarks must be made about the sign of W. At $r = 0$ $W = 1$. If (as we will assume) V is continuous, then W must be continuous and hence cannot change sign without vanishing at some r. By the definition of our metric $W = 0$ constitutes an impenetrable barrier, since $ds^2 = -\infty$ for any dr. It might be argued that this could be avoided if $Y' = 0$ at the same point. It can however be easily proved from the equation of motion that if $W(r)$ has a nth order zero at some r, then Y' may have a permanent $(n-1)$th order zero there but not a permanent nth order zero.

Hence in all the parts of our system which are connected with the origin, we must have $W > 0$.

The theory can easily be extended so as to include the cosmological constant λ. As is well known, the field equations with λ are

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = -8\pi T_{\mu\nu}. \tag{32}$$

With these field equations we still have

$$X = \frac{1}{W} \frac{\partial Y}{\partial r}, \tag{33}$$

but the equation of motion is now

$$\left(\frac{\partial Y}{\partial t} \right)^2 = \frac{1}{3} \lambda Y^2 + W^2 - 1 + \frac{2}{Y} \int_0^r W M' dr. \tag{34}$$

This equation brings out very clearly the non-classical character of the λ expansion of large regions. No term corresponding to the λ term can be found in the Newtonian approximation.

5. POTENTIAL ENERGY

One of the most remarkable feats of the general theory of relativity is that the laws of conservation of mass, energy and momentum are combined in the law of conservation of the tensor $T^{\mu\nu}$ and that hence it is the purely kinetic energy which is conserved. The potential energy of Newtonian theory is relegated to the position of a pseudo-tensor which can be made to vanish by a suitable choice of the system of coordinates. It is also well known that in many systems which are approximately Newtonian, potential energy is directly connected with g_{00} if the most obvious system of coordinates is used.

As will be shown in this section, total (and hence potential) energy occupies a significant position in our theory too, although our system need not be approximately Newtonian.

In Section 2 space-time was divided into space and time in a very significant way. This division (by choosing the surfaces t = const. to be orthogonal to the world lines of the particles) is possible whenever the particles follow non-intersecting geodesics and is independent of the assumption of spherical symmetry. This division is of physical significance, since it is determined by the orbits of the particles. Accordingly some significance can be attached to the 3-space so defined. In our model this 3-space has the metric

$$d\sigma^2 = X^2 dr^2 + Y^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{35}$$

where

$$X = \frac{1}{W(r)} \frac{\partial Y}{\partial r}.$$

Since we are dealing with a fixed time section, Y may be regarded as a function of r only. Moreover it is a monotonic function of r and hence may be introduced as coordinate.

Then

$$d\sigma^2 = (dY)^2/H^2 + Y^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{36}$$

where

$$H(Y) = W(r).$$

The Riemann-Christoffel tensor is easily seen to be given by

$$\begin{aligned} R_{121}{}^2 &= R_{131}{}^3 = -\frac{1}{HY}\frac{dH}{dY}, \qquad R_{232}{}^3 = 1 - H^2, \\ R_{122}{}^3 &= R_{121}{}^3 = R_{132}{}^3 = 0. \end{aligned} \tag{37}$$

Hence the Einstein tensor is

$$\begin{aligned} R_1^1 &= 2\frac{H}{Y}\frac{dH}{dY}, \qquad R_2^2 = R_3^3 = \frac{H^2 - 1}{Y^2} + \frac{H}{Y}\frac{dH}{dY}, \\ R &= \frac{2}{Y^2}\frac{d}{dY}\{Y(H^2 - 1)\}. \end{aligned} \tag{38}$$

The most striking consequence of these equations is that the 3-space is flat if, and only if, $H = 1$, i.e. $W = 1$. If $W > 1$ and does not vary too rapidly (or more exactly if $Y(H^2 - 1)$ is an increasing function of Y), then the curvature of the 3-space is positive, while in the opposite case it is negative.

It should be clearly understood that these statements need not be applied to the whole of space but may be applied to any group of particles occupying a finite interval of r. If e.g. all the particles in the range $r_1 \leq r \leq r_2$ have zero total energy, then they are embedded in a flat section of 3-space extending (at least) from r_1 to r_2.

Hence the curvature of the 3-space is entirely determined by the total energy of the particles. In our model, total (and hence potential) energy has a direct geometrical significance. We must remember that our division of space and time, while by no means invariant, is a physically significant division.

In addition to this geometrical interpretation of W (i.e. of total energy), W is also (as mentioned in Section 3) the ratio of effective gravitating to invariant mass.

We see, therefore, that we can, in our model, assign a definite place to total (and hence potential) energy even in cases far removed from nearly Newtonian ones.

6. COSMOLOGICAL MODELS

Our theory can easily be linked with certain cosmological models.[12] In these models the 3-space is supposed to be homogeneous and isotropic and this is clearly a special case of spherical symmetry. On the other hand, our theory deals only with pressure-free systems, so that we see that the pressure-free models of relativistic cosmology must be special cases of our models and we proceed to derive them.

Evidently ρ is a function of the time only and hence by (6) and (21)

$$\int_0^r W M' dr = \frac{4\pi}{3} \rho Y^3. \qquad (39)$$

It also follows from our expression for the curvature of the 3-space (which must be independent of position) that

$$W^2(r) = 1 + Y^2 f(t).$$

[12] Cf. H. P. Robertson's article "Relativistic Cosmology", *Rev. Mod. Phys.*, **5**, No. 1, 1933.

Therefore Y must be a product of a function of t only and a function of r only, so that we may put

$$Y = rg(t) \tag{40}$$

and

$$W^2 = 1 - kr^2, \tag{41}$$

and hence

$$\dot{g}^2 = -k + \tfrac{1}{3}g^2\{\lambda + 8\pi\rho(t)\}. \tag{42}$$

Since $\int_0^r WM'dr$ is a function of r only, it follows from (39) and (40) that ρg^3 is a constant. Therefore (42) takes the form

$$3g(\dot{g}^2 + k) - \lambda g^3 = 8\pi g^3\rho = \text{constant}.$$

The various cases of relativistic cosmology arise for different values of the constants. The metric is given by

$$ds^2 = dt^2 - [g(t)]^2\left\{\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right\}. \tag{43}$$

The substitution $r = R/(1 + \tfrac{1}{4}kR^2)$ turns (43) into the more familiar form

$$ds^2 = dt^2 - \left\{\frac{g(t)}{1 + \tfrac{1}{4}kR^2}\right\}^2\{dR^2 + R^2 d\theta^2 + R^2\sin^2\theta\, d\phi^2\}.$$

7. THE DOPPLER SHIFT

In Section 3 we found that $Y(r, t)$ had direct physical significance as the value an observer at (r, t) would assign to his distance from O as a result of his measurement of apparent luminosity of a source at O corrected for Doppler shift. $\partial Y/\partial t$ was then called his velocity, being the rate of change of distance from O with proper time. Now evidently his measurement of the Doppler shift supplies him with another definition of velocity and the relation between the Doppler shift and his velocity $\partial Y/\partial t$ will be discussed in this section.

We saw in Section 3 that the equation of a ray of light travelling outwards from O was

$$\theta = \text{const.}, \qquad \phi = \text{const.}, \qquad t = T(r),$$

where

$$\frac{dT(r)}{dr} = X\{r, T(r)\}, \tag{44}$$

and that the Doppler shift $1 + z$ satisfied

$$\frac{dz}{dr} = (1 + z)\left(\frac{\partial X}{\partial t}\right)_{r,T(r)} \qquad z = 0 \quad \text{at} \quad r = 0.$$

Hence the Doppler shift $1 + z_1$ at $r = r_1$, $t = t_1$ is given by

$$\log(1 + z_1) = \int_0^{r_1} dr\left(\frac{\partial X}{\partial t}\right)_{r,T(r)} = \int_0^{r_1} dr\frac{dr}{W(r)}\left(\frac{\partial^2 Y}{\partial r \partial t}\right)_{r,T(r)}, \tag{45}$$

where $T(r)$ is the solution of (44) passing through $r = r_1$, $t = t_1$.

We can find a more significant expression for (45) if $W + \dot{Y} > 0$ (as will usually be the case).[13] For then $Y\{r, T(r)\}$ is an increasing function of r (its differential quotient is $Y' + T'\dot{Y} = Y'(W + \dot{Y})/W$) and hence we are allowed to re-label our r coordinate in such as way that $Y\{r, T(r)\} = r$.

Now let

$$\left(\frac{\partial Y}{\partial t}\right)_{r,T(r)} = v(r), \qquad \left(\frac{\partial^2 Y}{\partial t^2}\right)_{r,T(r)} = f(r). \tag{46}$$

Then expanding Y in a Taylor series in $t - T$ we have

$$Y(r, t) = r + v(r)\{t - T(r)\} + \tfrac{1}{2}f(r)\{t - T(r)\}^2 + \ldots. \tag{47}$$

Equation (44) for T becomes

$$WT' = 1 - vT' \quad \text{or} \quad T' = \frac{1}{W + v}, \tag{48}$$

while, from the equation of motion,

$$v^2 = W^2 - 1 + \frac{2}{r}\int_0^r WM'dr, \qquad f = -\frac{1}{r^2}\int_0^r WM'dr.$$

Substituting into (45) we have

$$\log(1 + z_1) = \int_0^{r_1} dr\frac{v' - fT'}{W} = \int_0^{r_1} dr\frac{Wv' + vv' - f}{W(W + v)}. \tag{49}$$

[13] The meaning of this restriction is discussed at the end of this section.

Substituting for $vv' - f$ from the equation of motion we have

$$\log(1 + z_1) = \int_0^{r_1} dr \frac{v' + W' + \frac{M'}{r}}{v + W}$$
$$= \log(v_1 + W_1) + \int_0^{r_1} dr \frac{M'}{r(W + v)}, \qquad (50)$$

since $v_0 = 0$, $W_0 = 1$.

A little care is required in interpreting this expression. The shift of spectral lines is due to two causes, viz. the velocity shift due to the relative motion of source and observer, and the Einstein shift due to the difference between the potential energy per unit mass at the source and at the observer. The velocity shift is, in our units, just equal to v_1 and is easily identified in (50).

The Einstein shift is of a more complicated type. We have so far identified

$$-\int_0^{r_1} M'W\, dr/r_1,$$

with the potential energy per unit mass at r_1 but this is only true in a very restricted sense. For in bringing a particle from infinity to r_1 we have tacitly assumed (by virture of the non-intersection hypothesis) that all the spherical shells of matter outside r_1 were moved to their positions from infinity in such a way that they were always beyond our particle.

We obtained the correct equation of motion for our particles with this definition of potential energy only because the particle orbits do not intersect. For a ray of light the situation is radically different, since it passes matter on its way. Accordingly we now require a new definition of potential energy, which we will first obtain in the Newtonian analogue.

There the force per unit mass is $-M/r^2$ and accordingly the difference in potential energy between the origin and r_1 is

$$\int_0^{r_1} \frac{M(r)}{r^2} dr = -\frac{M(r_1)}{r_1} + \int_0^{r_1} \frac{M'dr}{r}, \qquad (51)$$

by integration by parts.

But for small masses and velocities (50) becomes

$$v_1 + W_1 + \int_0^{r_1} \frac{M'}{r} dr - 1 = v_1 + \frac{1}{2} v_1^2 - \frac{M(r_1)}{r_1} + \int_0^{r_1} \frac{M'dr}{r}, \qquad (52)$$

since $W - 1$ is the energy per unit mass.

The first term on the right-hand side is the ordinary velocity shift, the second term is small compared with it, while the last two terms describe the Einstein shift in terms of the Newtonian potential (51).

We see then that the exact expression (50) gives us an expression for the combination of velocity shift and Einstein shift in our model.

An interesting, but as we have seen not a radically new, point in (50) is the fact that the spectral shift does not only depend on conditions at the source and at the observer but also on the distribution of matter in the intervening space. Note that any empty part of space does not contribute to the integral in (50).

The sign of the velocity shift depends on the sign of v_1, but the Einstein shift is easily seen to be towards the red, at least for reasonably small masses and velocities.

For light proceeding in the opposite direction the velocity shift has the same sign as before, but the Einstein shift changes sign. An analysis similar to the one above gives for light travelling from r_1 to the origin

$$\log \frac{\tau_0}{\tau_1} = -\log(W_1 - v_1) - \int_0^{r_1} \frac{M' dr}{r(W - v)} .$$

Finally it might be mentioned that if we re-introduce the cosmological constant λ, no change is made in (50) or any of the subsequent arguments. Hence there is no shift of the spectral lines explicitly due to λ.

In this discussion of Doppler shift we have so far restricted ourselves to the case $W + \dot{Y} > 0$. What is the significance of this conditon?

As we move along the ray, Y changes and

$$\frac{dY}{dt} = \frac{\partial Y}{\partial t} + \frac{\partial Y}{\partial r}\frac{dr}{dt} = W + \frac{\partial Y}{\partial t} . \tag{53}$$

Hence Y increases along the ray only as long as $W + Y > 0$. This is due to the curious way light is dragged along by matter (which is also exemplified by equation (48)). If matter is falling into the origin at very high speed it may stop the ray and even reverse its direction. Of course this applies only to the Y picture; the ray continues to reach matter with higher and higher r values but only when this new matter has moved sufficiently near to the origin. In the Y picture there is a barrier to outward-moving rays of light and therefore our definition of Y does not apply beyond this region. This barrier is not nearly as impenetrable as the $W = 0$ barrier mentioned in Section 3, since e.g. inward-moving rays of light can easily cross it.

It might be mentioned that this reversal of the direction of light can only occur if we have very high densities throughout very large regions. For if $-\dot{Y} > W > 0$ then by (22)

$$W^2 - 1 + \frac{2}{Y}\int_0^r M'W\,dr > W^2,$$

and hence we must have

$$\int_0^r M'(r)W(r)dr = 4\pi\int_0^r \rho Y^2 \frac{\partial Y}{\partial r}\,dr > \frac{1}{2}Y. \tag{54}$$

It is a remarkable fact that while inward-moving matter can in such extreme circumstances reverse the direction of an outward-travelling ray, such a ray will always catch up with outward-moving matter (no matter how large its $\dot{Y}$ may be) provided it has not been held up previously by the above-mentioned barrier.

8. LIMITATIONS OF THE THEORY

With the exception of the non-intersection hypothesis our assumptions, if correct initially, will necessarily remain correct as the motion progresses. However, a serious limitation of the validity of the theory arises because the system may develop in such a way that the non-intersection hypothesis, although initially true, is later violated.

We have to consider separately the case where the violation of the hypothesis occurs at the origin and the case in which it occurs elsewhere.

If the orbits of particles intersect at any point other than O, this fact would show itself in our notation by Y' vanishing at some point. For by our definition of our metric this would imply that the distance between the particles of two different shells has become zero. Accordingly our equations apply only up to the minimum value of t for which $Y'(r, t)$, considered now as a function of r, has a zero. If however we assume that this intersection does not upset the spherical symmetry, it seems that our theory will remain valid even for larger t for all $r < r_1(t)$, where $r_1(t)$ is the least r at which $Y'(t, r) = 0$. Our theory remains valid because the gravitational field at a point is independent of the spherical shells of matter beyond it.

A more srious violation of the non-intersection assumption may occur at O. For if matter near O moves into O it will either pass through O and re-emerge on the other side, its orbits intersecting the orbits of the incoming matter, or a point mass will be formed at O. In both cases the theory breaks down completely, in the first case because of the intersections in the neighbourhood of O, in the second case because of the singularity at O. Although this singularity might formally be included in the equations the breakdown of the physical interpretation of Y would rob the theory of much of its significance.

No such difficulty at O will arise if originally there is a small empty region round O and if the matter nearest to O does not move inwards at first (for then it will never move inwards).

APPENDIX

Our first step must be to find the energy tensor corresponding to the metric

$$ds^2 = dt^2 - X^2(r,t)dr^2 - Y^2(r,t)d\theta^2 - Y^2(r,t)\sin^2\theta\, d\phi^2. \tag{1}$$

In this work we will use the field equations including λ.

We put $(t, r, \theta, \phi) = (x_0, x_1, x_2, x_3)$. A dot will denote differentiation with respect to t. Then the Christoffel symbols

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\alpha}\left(\frac{\partial g_{\nu\alpha}}{\partial x_\mu} + \frac{\partial g_{\mu\alpha}}{\partial x_\nu} - \frac{\partial g_{\mu\nu}}{\partial x_\alpha}\right) \tag{2}$$

are

$$\left.\begin{aligned}
&\Gamma^1_{11} = \frac{X'}{X},\quad \Gamma^0_{11} = X\dot{X},\quad \Gamma^1_{01} = \frac{\dot{X}}{X},\quad \Gamma^2_{02} = \Gamma^3_{03} = \frac{\dot{Y}}{Y},\quad \Gamma^0_{22} = Y\dot{Y},\\
&\Gamma^2_{12} = \Gamma^3_{13} = \frac{Y'}{Y},\quad \Gamma^1_{22} = -\frac{YY'}{X^2},\quad \Gamma^0_{33} = Y\dot{Y}\sin^2\theta,\\
&\Gamma^1_{33} = -\frac{YY'}{X^2}\sin^2\theta,\quad \Gamma^2_{33} = -\sin\theta\cos\theta,\quad \Gamma^3_{23} = \cot\theta.
\end{aligned}\right\} \tag{3}$$

All other $\Gamma^\sigma_{\mu\nu}$ vanish.

The mixed Einstein tensor has components

$$\left.\begin{aligned}
&R^0_0 = \frac{\ddot{X}}{X} + \frac{2\ddot{Y}}{Y},\quad R^1_1 = \frac{\ddot{X}}{X} + 2\frac{\dot{X}\dot{Y}}{XY} + 2\frac{X'Y'}{X^3Y} - 2\frac{Y''}{X^2Y},\\
&R^2_2 = R^3_3 = \frac{\ddot{Y}}{Y} + \frac{1+\dot{Y}^2}{Y^2} + \frac{\dot{X}\dot{Y}}{XY} - \frac{1}{X^2}\left(\frac{Y''}{Y} + \frac{Y'^2}{Y^2} - \frac{X'Y'}{XY}\right),\\
&R^0_1 = -X^2R^1_0 = 2\left(\frac{\dot{Y}'}{Y} - \frac{\dot{X}Y'}{XY}\right),\\
&R = 2\frac{\ddot{X}}{X} + 4\frac{\ddot{Y}}{Y} + 4\frac{\dot{X}\dot{Y}}{XY} + 2\frac{1+\dot{Y}^2}{Y^2} - \frac{2}{X^2}\left(\frac{2Y''}{Y} + \frac{Y'^2}{Y^2} - 2\frac{X'Y'}{XY}\right).
\end{aligned}\right\} \tag{4}$$

All other components vanish.

Hence the components of the energy tensor are

$$\left.\begin{aligned}
\lambda + 8\pi T_0^0 &= 2\frac{\dot{X}\dot{Y}}{XY} + \frac{1+\dot{Y}^2}{Y^2} - \frac{1}{X^2}\left(\frac{2Y''}{Y} + \frac{Y'^2}{Y^2} - 2\frac{X'Y'}{XY}\right),\\
\lambda + 8\pi T_1^1 &= 2\frac{\ddot{Y}}{Y} + \frac{1+\dot{Y}^2}{Y^2} - \frac{Y'^2}{X^2Y^2},\\
\lambda + 8\pi T_2^2 = \lambda + 8\pi T_3^3 &= \frac{\ddot{X}}{X} + \frac{\ddot{Y}}{Y} + \frac{\dot{X}\dot{Y}}{XY} - \frac{1}{X^2}\left(\frac{Y''}{Y} - \frac{X'Y'}{XY}\right),\\
8\pi T_1^0 = -8\pi X^2 T_0^1 &= -2\left(\frac{\dot{Y}'}{Y} - \frac{\dot{X}Y'}{XY}\right).
\end{aligned}\right\} \tag{5}$$

Owing to the assumption that matter moves without altering its (r, θ, ϕ) coordinates we must have, even in the presence of pressures, $T_1^0 = 0$.

Hence

$$X(r,t) = \frac{1}{W(r)} \frac{\partial Y(r,t)}{\partial r}, \tag{6}$$

where $W(r)$ is an arbitrary function of r.

Then

$$\left.\begin{aligned}
\lambda + 8\pi T_0^0 &= \frac{U'}{Y^2Y'},\\
\lambda + 8\pi T_1^1 &= \frac{\dot{U}}{Y^2\dot{Y}},\\
\lambda + 8\pi T_2^2 = \lambda + 8\pi T_3^3 &= \frac{1}{2YY'}\frac{\partial}{\partial r}\left(\frac{\dot{U}}{\dot{Y}}\right),\\
4\lambda + 8\pi T &= \frac{1}{Y^2Y'}\frac{\partial}{\partial r}\left(Y\frac{\dot{U}}{\dot{Y}} + U\right),\\
\text{where} \quad U &= Y(1 + \dot{Y}^2 - W^2).
\end{aligned}\right\} \tag{7}$$

As explained in Section 2 this set of equations defines the most general set of pressures which will move the particles along our specified geodesics.

If we assume the pressure to be isotropic then

$$T_1^1 = T_2^2 = T_3^3, \tag{8}$$

and therefore, by (7),

$$\frac{\dot{U}}{Y^2\dot{Y}} = \frac{1}{2YY'}\frac{\partial}{\partial r}\left(\frac{\dot{U}}{\dot{Y}}\right).$$

Hence

$$\frac{\partial}{\partial r}\left(\frac{\dot{U}}{Y^2\dot{Y}}\right) = 0, \tag{9}$$

and so

$$T_1^1 = T_2^2 = T_3^3 = F(t). \tag{10}$$

This means that the pressure is uniform.

We are mainly interested in the case of zero pressure, i.e. $T_1^1 = T_2^2 = T_3^3 = 0$. Then

$$\dot{U} = \lambda Y^2 \dot{Y},$$

so that

$$U = \tfrac{1}{3}\lambda Y^3 + S(r). \tag{11}$$

Hence

$$8\pi T_0^0 = 8\pi T = S'(r)/\, Y^2 Y' = S'(r)/X\ Y^2 W(r). \tag{12}$$

By (6) of Section 2 this implies

$$S(r) = 2\int_0^r M'(r) W(r) dr, \tag{13}$$

since we have $U = 0$ at $r = 0$.

Hence

$$\dot{Y}^2 = W^2 - 1 + \frac{1}{3}\lambda Y^2 + \frac{2}{Y}\int_0^r M'(r) W(r) dr. \tag{14}$$

In the case $\lambda = 0$ this reduces to

$$\dot{Y}^2 = W^2 - 1 + \frac{2}{Y}\int_0^r M'(r) W(r) dr. \tag{15}$$

These equations may be integrated again, viz.

$$t = \int^Y \left\{\frac{1}{3}\lambda u^2 + (W^2 - 1) + \frac{2G}{u}\right\}^{-1/2} du, \tag{16}$$

where

$$G(r) = \int_0^r M'W\, dr.$$

The lower limit of integration is an arbitrary function of r. It is frequently convenient to have $Y\{r, t_0(r)\} = r$, where $t_0(r)$ is an arbitrary function of r.

Then

$$t = t_0(r) + \int^{Y} \left\{ \frac{1}{3}\lambda u^2 + (W^2 - 1) + \frac{2G}{u} \right\}^{-1/2} du. \tag{17}$$

The integral is elliptic unless $\lambda = 0$ when it is elementary. On carrying out the integration for $\lambda = 0$ we find that, when $W > 1$,

$$t = t_0(r) + \frac{2G(r)}{(W^2 - 1)^{3/2}} \left\{ F\left(\frac{2G}{Y(W^2 - 1)} \right) - F\left(\frac{2G}{r(W^2 - 1)} \right) \right\}, \tag{18}$$

where

$$F(x) = \frac{\sqrt{1 + x}}{x} + \frac{1}{2} \log \frac{\sqrt{1 + x} - 1}{\sqrt{1 + x} + 1},$$

while when $W < 1$,

$$t = t_0(r) + \frac{2G}{(1 - W^2)^{3/2}} \left\{ H\left(\frac{2G}{Y(W^2 - 1)} \right) - H\left(\frac{2G}{r(W^2 - 1)} \right) \right\}, \tag{19}$$

where

$$H(x) = \frac{\sqrt{x - 1}}{x} + \tan^{-1} \sqrt{x - 1}.$$

(In this case we must always have $Y \leq \frac{2G}{1 - W^2}$).

Finally, if $W = 1$,

$$t = t_0(r) + \frac{Y^{3/2} - r^{3/2}}{\sqrt{\frac{9}{2} G(r)}}. \tag{20}$$

We may conclude the appendix by giving the components of the Riemann-Christoffel tensor.

We find that

$$\left.\begin{aligned}
R_{1212} = \frac{R_{1313}}{\sin^2\theta} &= -\frac{Y'^2}{W^2}\left(\frac{\lambda}{3}Y^2 + \frac{G'}{Y'} - \frac{G}{Y}\right),\\
R_{2323} &= -Y^2 \sin^2\theta \left(\frac{\lambda}{3}Y^2 + \frac{2G}{Y}\right),\\
R_{0202} = \frac{R_{0303}}{\sin^2\theta} &= \frac{\lambda}{3}Y^2 - \frac{G}{Y},\\
R_{0101} &= \frac{Y'^2}{W^2 Y^2}\left(\frac{\lambda}{3}Y^2 - \frac{G'}{Y'} + 2\frac{G}{Y}\right).
\end{aligned}\right\} \tag{21}$$

All other independent components vanish.

The invariant $B = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is given by

$$B = \frac{8}{3}\lambda^2 + \frac{8}{3}\lambda\frac{G'}{Y^2Y'} + 12\frac{G'^2}{Y^4Y'^2} - 32\frac{GG'}{Y^5Y'} + 48\frac{G^2}{Y^6}. \qquad (22)$$

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