P H Y S I C A L R E V I E W L E T T E R S

Time Drift of Cosmological Redshifts as a Test of the Copernican Principle

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We present the time drift of the cosmological redshift in a general spherically symmetric spacetime. We demonstrate that its observation would allow us to test the Copernican principle and so determine if our Universe is radially inhomogeneous, an important issue in our understanding of dark energy. In particular, when combined with distance data, this extra observable allows one to fully reconstruct the geometry of a spacetime describing a spherically symmetric underdense region around us, purely from background observations.

I. INTRODUCTION

Cosmological data is usually interpreted under the assumption that the Universe is spatially homogeneous and isotropic. This is justified by the Copernican principle, stating that we are not located at a favored position in space. Combined with the observed isotropy, this leads to a Robertson-Walker (RW) geometry [1], at least on the scale of the observable Universe.

This implies that the spacetime metric reduces to a single function of the cosmic time, the scale factor $a(t)$. This function can be Taylor expanded as $a(t) = a_0 + H_0(t - t_0) - \frac{1}{2} q_0 H_0^2 (t - t_0)^2 + \cdots$, where H_0 is the Hubble parameter and q_0 the deceleration parameter. Low redshift observations [2] combined with the assumption of almost flatness of the spatial sections, justified mainly by the cosmic microwave background data [3,4], lead to the conclusion that $q_0 < 0$: the expansion is accelerating. This conclusion involves no hypothesis about the theory of gravity or the matter content of the Universe [5], as long as the Copernican principle holds. This has stimulated a growing interest in possible explanations [5,6], ranging from new matter fields dominating the dynamics at late times to modifications of general relativity.

While many tests of general relativity on astrophysical scales have been designed [7], the verification of the Copernican principle has attracted little attention, despite the fact that relaxing this assumption may be the most conservative way, from a theoretical perspective, of explaining the recent dynamics of the Universe without introducing new physical degrees of freedom [8].

This possibility that we may be living close to the center (because isotropy around us seems well established observationally) of a large underdense region has attracted considerable interest. In particular, the low redshift (background) observations such as the magnitude-redshift relation can be matched [9] by a nonhomogeneous spacetime

of the Lemaître-Tolman-Bondi (LTB) family (that is, spherically symmetric solution of Einstein equations sourced by pressureless matter and no cosmological constant). Unfortunately, this simple extension of the RW universes depends on two free functions (see below for details) so that the reconstruction is underdetermined and one must fix one function by hand. Thus, one needs at least one extra independent observation to reconstruct the geometry of an LTB universe. A limitation to this reconstruction arises because most data lie on our past light cone. This takes us back to the observational cosmology program [10] and the question [11] of how to extract as much information as possible about our spacetime from cosmological data alone. Among many results, it was demonstrated [12] that the two free functions of a LTB spacetime can be reconstructed from the angular distance and number counts, even though evolution effects make it impossible to be conclusive [13].

Recently, two new ideas were proposed. First, it was realized [14] that the distortion of the Planck spectrum of the cosmic microwave background allows one to test the Copernican principle. Second, a consistency relation between distances on the null cone and Hubble rate measurements in RW universes was derived [15], based on the fact that the curvature is constant; this also serves as an observational test of the Copernican principle.

In this Letter, we reconsider the time drift of cosmological redshift in spacetimes with less symmetries than the RW universe and we demonstrate how, when combined with distance data, it can be used to test the Copernican principle, mainly because observing the thickening of our past light cone brings new information. As pointed out by Sandage and then McVittie [16], one should expect to observe such a time drift in any expanding spacetime. This may lead to a better understanding of the physical origin of the recent acceleration [17,18], or to tests of the

variation of fundamental constants [19]. This measurement, while challenging, may be achieved with extremely large telescopes (ELT) and, in particular, it is one of the main science drivers in design of the Cosmic Dynamics Experiment (CODEX) spectrograph [20]. Our result may strengthen the scientific case for this project.

We start by introducing observational coordinates, which allow us to derive the general expression for the time drift in a spherically symmetric, but not necessarily spatially homogeneous, universe [see Eq. (6)]. We show that observation of both the luminosity distance and the redshift drift allows one to probe the Copernican principle at low redshifts, when "dark energy" dominates [see the consistency relation (8)]. We demonstrate that this expression may be used with distance data to fully reconstruct the geometry of a LTB spacetime [see Eq. (9)].

II. $\dot{z}$ IN A SPHERICALLY SYMMETRIC UNIVERSE

Observational coordinates.—We consider a spherically symmetric spacetime in observational coordinates $\{w, y, \vartheta, \varphi\}$, where w labels the past light cones of events along the wordline $\mathcal{C}$ of the observer, assumed to lie at the center so that w is constant on each past light cone, with $u^a \partial_a w > 0$, u^a being the four-velocity of the cosmic fluid, $u_a u^a = -1$. y is a comoving radial distance coordinate specified down the past light cone of an event $\mathcal{O}$ on $\mathcal{C}$. Many choices are possible, such as the affine parameter down the null geodesics from $\mathcal{O}$, the area distance, or the redshift; whatever choice we assume on the past light cone of $\mathcal{O}$, it is specified on other past light cones through being comoving with the cosmic fluid, i.e., $y_{,a} u^a = 0$. (ϑ, φ) are angular coordinates based at $\mathcal{C}$ and propagated parallelly along the past light cone. The metric in these observational coordinates is

$$ds^2 = -A^2(w, y)dw^2 + 2A(w, y)B(w, y)dydw$$
$$+ C^2(w, y)d\Omega^2, \tag{1}$$

which is clearly spherically symmetric around the worldline $\mathcal{C}$ defined by $y = 0$. The requirement that the 2-spheres $\{w, y\} = $ const behave regularly around $\mathcal{C}$ when $y \rightarrow 0$ implies [10] that $A(w, y) \rightarrow A(w, 0) \neq 0$, $B(w, y) \rightarrow B(w, 0) \neq 0$, and $C(w, y) = B(w, 0)y + \mathcal{O}(y^2)$.

There remain two coordinate freedoms in possible rescalings of w and y. However, once specified on $\mathcal{C}$, w is determined on the other worldlines by the condition that $\{w = \text{const}\}$ are past light cones of events on $\mathcal{C}$. This allows us to arbitrarily choose $A(w, 0)$. Also, once specified on one past light cone, y is determined on all the others because it is a coordinate comoving with the fluid. This allows us to choose $B(w_0, y)$ for a given value of $w = w_0$.

On each past light cone, the cross-sectional area of a source is related to the solid angle $d\Omega^2$ under which it is observed by an observer on $\mathcal{C}$ at $w = w_0$ by $C^2(w_0, y)d\Omega^2$. This implies that C is the angular distance, D_A, i.e., $D_A(y) = C(w_0, y)$. The distance duality relation [21] then

implies that the luminosity distance is given by $D_L(y) = (1 + z)^2 D_A$. The redshift is given by

$$1 + z = \frac{(u_a k^a)_{\text{emission}}}{(u_a k^a)_{\text{observer}}} = \frac{A(w_0, 0)}{A(w_0, y)}, \tag{2}$$

where the matter velocity and photon wave vector are given by $u^a = A^{-1}\delta^a_w$ and $k^a = (AB)^{-1}\delta^a_y$, respectively. We deduce that the isotropic expansion rate, defined by $3H = \nabla_a u^a$, is given by

$$H(w, y) = \frac{1}{3A}\left[\frac{\partial_w B(w, y)}{B(w, y)} + 2\frac{\partial_w C(w, y)}{C(w, y)}\right]. \tag{3}$$

For the central observer, who sees the Universe isotropic, H is simply the Hubble expansion rate. At small redshifts, $H(w, y) = \frac{\partial_w B(w,0)}{B(w,0)A(w,0)} + \mathcal{O}(y)$, so the Hubble constant is $H_0 = \partial_w B(w_0, 0)/B(w_0, 0)A(w_0, 0)$.

In the particular case of a dust dominated universe, the acceleration and vorticity vanish and the fluid four-velocity can be expressed as the gradient of the proper time along the matter worldlines: $u_a = -\partial_a t$. Since we also have $u_a = -A\partial_a w + B\partial_a y$ we deduce that $dt = Adw - Bdy$ so that $A = \partial_w t$ and $B = \partial_y t$. The surfaces of simultaneity are thus given by $Adw = Bdy$ and we have the integrability condition $\partial_y A + \partial_w B = 0$.

The covariant derivative of u_a is therefore of the form $\nabla_a u_b = H(g_{ab} + u_a u_b) + \sigma_{ab}$, where the shear σ_{ab} is symmetric, traceless, and satisfies $u^a \sigma_{ab} = 0$. The scalar shear $\sigma^2 = \sigma^{ab}\sigma_{ab}/2$ is consequently the only nonvanishing kinematical variable and is given by

$$\sigma(w, y) = \frac{2}{\sqrt{3}}\frac{1}{A}\left(\frac{\partial_w B}{B} - \frac{\partial_w C}{C}\right), \tag{4}$$

where an arbitrary sign has been chosen. The regularity conditions imply $\sigma(w, 0) = 0$, which is expected since the expansion is observed to be isotropic about the central worldline.

Expression of the redshift drift.—From the expression (2), it is straightforward to deduce that $\dot{z} \equiv \frac{\delta z}{\delta w}(w_0, y)$ is given by

$$\dot{z}(w_0, y) = (1 + z)\left[\frac{\partial_w A(w_0, 0)}{A(w_0, 0)} - \frac{\partial_w A(w_0, y)}{A(w_0, y)}\right]. \tag{5}$$

Now, we can choose w such that $A(w_0, 0) = 1$. Then, on our past light cone, we can choose y such that $\partial_w \ln B(w_0, y) = \partial_w \ln A(w_0, y)$. Note however that such a choice is not possible for all w. It follows that

$$\dot{z}(w_0, y) = (1 + z)H_0 - H(w_0, y) - \frac{1}{\sqrt{3}}\sigma(w_0, y). \tag{6}$$

This is the general expression for the time drift of the redshift as it would be measured by an observer at the center of a spherically symmetric universe.

Robertson-Walker case.—Since $\sigma = 0$ for a RW spacetime, Eq. (6) reduces to the standard Sandage-McVittie formula. Let us consider the RW metric in conformal coordinates

$$ds^2 = a^2(\eta)[-d\eta^2 + d\chi^2 + f_K^2(\chi)d\Omega^2], \quad (7)$$

where χ is the radial comoving coordinate and $f_K = (\sin\chi, \chi, \sinh\chi)$ according to the curvature of the spatial sections. Now, with $w = \eta + \chi$ and $y = \chi$, this leads to the form (1) for the metric (7) with $A = B = a(w - y)$ and $C = a(w - y)f_K(y)$. H is thus constant on each constant time hypersurface and $\sigma = 0$ everywhere. Since $\partial_w A|_{y=\text{const}} = \partial_\eta a$, Eq. (5) gives $\dot{z} = (1 + z)H_0 - H(z)$ where we have shifted to cosmic time ($dt = ad\eta$), using $1 + z = a_0/a$.

We understand from this exercise why the observational coordinates are adapted to the computation of $\dot{z}$ in an arbitrary spacetime. They are just a generalization to less symmetric spacetimes of the conformal coordinates, where the expression is easily obtained in the RW case.

We also conclude that since the three functions (A, B, C) are expressed in terms of a unique function, all background observations depend on some function of H, $\dot{z}$ being no exception.

Consistency relation.—It follows that, in a RW universe, we can determine a consistency relation between several observables. From the metric (7) and the relation for $\chi(z)$ that follows, one deduces that $H^{-1}(z) = D'(z)[1 + \Omega_{K0}H_0^2 D^2(z)]^{-1/2}$, where a prime stands for ∂_z and $D(z) = D_L(z)/(1 + z)$. This relation is the basis of the test of the RW structure, as recently proposed in Ref. [15], arguing there that knowledge of $H(z)$ at different redshifts, from, e.g., baryon acoustic oscillations or differential age estimates of passively evolving galaxies, could then be used to check this yields the same value of Ω_{K0}. Here, we argue that it can be implemented using $\dot{z}(z)$ as an observational input. Defining

$$\text{Cop}[D_L(z), \dot{z}(z), z] \equiv 1 + \Omega_{K0}H_0^2 D(z)^2$$
$$- [H_0(1 + z) - \dot{z}(z)]^2[D'(z)]^2, \quad (8)$$

we must have $\text{Cop}[D_L(z), \dot{z}(z), z] = 0$ whatever the matter content of the Universe and the field equations, since it derives from a purely kinematical relation that does not rely on the dynamics (i.e., the Friedmann equations). This is a *consistency relation* between independent observables that holds in *any* Robertson-Walker spacetime.

Spherically symmetric spacetimes.—Writing the LTB metric in observational coordinates requires the solution of the null geodesic equation, which is in general possible only numerically. Consider an LTB spacetime with metric

$$ds^2 = -dt^2 + S^2(r, t)dr^2 + R^2(r, t)d\Omega^2,$$

where $S(r, t) = R'/\sqrt{1 + 2E(r)}$ and $\dot{R}^2 = 2M(r)/R(r, t) + 2E(r)$, using a dot and prime to refer to derivatives with respect to t and r. The Einstein equations can be solved parametrically as $\{R(r, \eta), t(\eta, r)\} = \{\frac{M(r)}{\mathcal{E}(r)}\Phi'(\eta), T_0(r) + \frac{M(r)}{[\mathcal{E}(r)]^{3/2}}\Phi(\eta)\}$, where Φ is defined by $\Phi(\eta) = (\sinh\eta - \eta, \eta^3/6, \eta - \sin\eta)$, and $\mathcal{E}(r) = (2E, 2, -2E)$ according to whether E is positive, null, or negative.

This solution depends on three arbitrary functions of r only, $E(r)$, $M(r)$, and $T_0(r)$. Their choice determines the model completely. For instance, $(E, M, T_0) = (-K_0r^2, M_0r^3, 0)$ corresponds to a RW universe. One can further use the freedom in the radial coordinate to fix one of the three functions at will so that one effectively has only two arbitrary independent functions. Assume we fix $M(r)$. We want to determine $\{E(r), T_0(r)\}$ to reproduce some observables on our past light cone. This can be represented parametrically as $\{r(z), E(z), T_0(z)\}$.

Let us sketch the reconstruction and use r as the integration coordinate, instead of z. Our past light cone is defined as $t = \hat{t}(r)$ and we set $\mathcal{R}(r) = R[\hat{t}(r), r]$. The time derivative of R is given by $\dot{R}[\hat{t}(r), r] \equiv \mathcal{R}_1 = \sqrt{2M_0r^3/\mathcal{R}(r) + 2E(r)}$. Then we get $R'[\hat{t}(r), r] \equiv \mathcal{R}_2(r) = -\{\mathcal{R}(r) - 3[\hat{t}(r) - T_0(r)]\mathcal{R}_1(r)/2\}E'/E - \mathcal{R}_1(r)T_0'(r) + \mathcal{R}(r)/r$. Finally, more algebra leads to $\dot{R}'[\hat{t}(r), r] \equiv \mathcal{R}_3(r) = \{\mathcal{R}_1(r) - 3M_0r^3[\hat{t}(r) - T_0(r)]/\mathcal{R}^2(r)\}E'(r)/2E(r) + M_0r^3T_0'(r)/\mathcal{R}^2 + \mathcal{R}_1(r)/r$. Thus, $\dot{R}$, R', and $\dot{R}'$ evaluated on the light cone are just functions of $\mathcal{R}(r)$, $E(r)$, $T_0(r)$, and their first derivatives. Now, the null geodesic equation gives that

$$\frac{d\hat{t}}{dr} = -\frac{\mathcal{R}_2(r)}{\sqrt{1 + 2E(r)}}, \qquad \frac{dz}{dr} = \frac{1 + z}{\sqrt{1 + 2E(r)}}\mathcal{R}_3(r),$$

and

$$\frac{d\mathcal{R}}{dr} = \left[1 - \frac{\mathcal{R}_1(r)}{\sqrt{1 + 2E(r)}}\right]\mathcal{R}_2(r).$$

These are three first-order differential equations relating five functions $\mathcal{R}(r)$, $\hat{t}(r)$, $z(r)$ $E(r)$, and $T_0(r)$. To reconstruct the free functions we thus need two observational relations. $\mathcal{R}(z) = D_A(z)$ is the obvious choice. Then, from $\dot{z}(z)$, we have the new relation

$$\frac{1}{9}\left(\frac{\mathcal{R}_3}{\mathcal{R}_2} - \frac{\mathcal{R}_1}{\mathcal{R}}\right)^2 = \left[\dot{z} - (1 + z)H_0 + \frac{1}{3}\left(\frac{\mathcal{R}_3}{\mathcal{R}_2} + 2\frac{\mathcal{R}_1}{\mathcal{R}}\right)\right]^2. \quad (9)$$

It was shown [9] that the observed $D_L(z)$ can be reproduced from the function $T_0(r)$ assuming $E = 0$, or from the function $E(r)$ assuming $T_0 = 0$. Here, we have shown that $\{E(r), T_0(r)\}$ can be completely reconstructed from the data without assumptions.

III. DISCUSSION

In this Letter, we have shown that observation of both the luminosity distance and time drift of the redshift as a function of z allows one to construct a test of the Copernican principle. We have derived the general expression for $\dot{z}(z)$ in a spherically symmetric spacetime [see Eqs. (5) and (6)]. This extends the standard computation which was restricted to RW spacetimes and was extended to almost RW spacetimes only recently [22].

As a by-product, this also allowed us to derive the consistency relation (8) between the observed $D_L(z)$ and $\dot{z}(z)$, thereby extending the result of Ref. [15] to an alternate observable. That is, while $H(z)$ characterizes the local isotropic expansion rate, $\dot{z}(z)$ gives access to the expansion between us and a source. In RW models these are trivially related but in general the shear enters these observables differently, thereby presenting a test of the Copernican principle using only background observations. We have shown that we can extract the shear as a function of z and demonstrated that it allows one to close the reconstruction problem for a LTB spacetime.

$D_L(z)$ can be measured from the observation of type Ia supernovae, particularly with actual projects such as JDEM, up to redshifts of order 1.5. $\dot{z}(z)$ has a typical amplitude of order $\delta z \sim -5 \times 10^{-10}$ on a time scale of $\delta t = 10$ yr, for a source at redshift $z = 4$. This measurement is challenging, and impossible with present-day facilities. However, it was recently revisited [23] in the context of ELT, arguing they could measure velocity shifts of order $\delta v \sim 1$–10 cm/s over a 10 yr period from the observation of the Lyman-α forest. It is one of the science drivers in design of the CODEX spectrograph [20] for the future European ELT. The study of the precision to which we can check the Copernican principle with these two data sets is beyond the scope of this Letter. Indeed, many effects, such as proper motion of the sources, local gravitational potential, or acceleration of the Sun may contribute to the time drift of the redshift. It was shown [22], however, that these contributions can be brought to a 0.1% level so that the cosmological redshift is actually measured.

Future high precision data may thus allow a test of the Copernican principle, even though observations are localized on our past light cone. While important in its own right for understanding the foundations of our cosmological model, it is also critical for our understanding of the acceleration of the Universe—it will permit us to be confident that any such acceleration is not simply a misinterpretation of the data because of incorrectly assuming the geometry of our Universe at low redshift.

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