

## Inflationary Spacetimes Are Incomplete in Past Directions

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Many inflating spacetimes are likely to violate the weak energy condition, a key assumption of singularity theorems. Here we offer a simple kinematical argument, requiring no energy condition, that a cosmological model which is inflating—or just expanding sufficiently fast—must be incomplete in null and timelike past directions. Specifically, we obtain a bound on the integral of the Hubble parameter over a past-directed timelike or null geodesic. Thus inflationary models require physics other than inflation to describe the past boundary of the inflating region of spacetime.

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*I. Introduction.*—Inflationary cosmological models [1–3] are generically eternal to the future [4,5]. In these models, the Universe consists of postinflationary, thermalized regions coexisting with still-inflating ones. In comoving coordinates the thermalized regions grow in time and are joined by new thermalized regions, so the *comoving volume* of the inflating regions vanishes as  $t \rightarrow \infty$ . Nonetheless, the inflating regions expand so fast that their *physical volume* grows exponentially with time. As a result, there is never a time when the Universe is completely thermalized. In such spacetimes, it is natural to ask if the Universe could also be past-eternal. If it could, eternal inflation would provide a viable model of the Universe with no initial singularity. The Universe would never come into existence. It would simply exist.

This possibility was discussed in the early days of inflation, but it was soon realized [6,7] that the idea could not be implemented in the simplest model in which the inflating Universe is described by an exact de Sitter space. More general theorems showing that inflationary spacetimes are geodesically incomplete to the past were then proved [8]. One of the key assumptions made in these theorems is that the energy-momentum tensor obeys the weak energy condition. Although this condition is satisfied by all known forms of classical matter, subsequent work has shown that it is likely to be violated by quantum effects in inflationary models [9,10]. Such violations must occur whenever quantum fluctuations result in an increase of the Hubble parameter  $H$ —i.e., when  $dH/dt > 0$ —provided that the spacetime and the fluctuation can be approximated as locally flat. Such upward fluctuations in  $H$  are essential for the future-eternal nature of chaotic inflation. Thus, the weak energy condition is generally violated in an eternally inflating universe. These violations appear to open the door again to the possibility that inflation, by itself, can eliminate the need for an initial singularity. Here we argue that this is not the case. In fact, we show that the general situation is very similar to that in de Sitter space.

The intuitive reason why de Sitter inflation cannot be past-eternal is that, in the full de Sitter space, exponential expansion is preceded by exponential contraction. Such a contracting phase is not part of standard inflationary models and does not appear to be consistent with the physics of inflation. If thermalized regions were able to form all the way to past infinity in the contracting spacetime, the whole Universe would have been thermalized before inflationary expansion could begin. In our analysis we exclude the possibility of such a contracting phase by considering spacetimes for which the past region obeys an *averaged expansion condition*, by which we mean that the average expansion rate in the past is greater than zero:

$$H_{\text{av}} > 0. \quad (1)$$

With a suitable definition of  $H$  and the region over which the average is to be taken, we show that the averaged expansion condition implies past-incompleteness.

It is important to realize that the terms expansion and contraction refer to the behavior of congruences of timelike geodesics (the potential trajectories of test particles). It is meaningless to say that a spacetime is expanding at a single point, since in the vicinity of any point one can always construct congruences that expand or contract at any desired rate. We will see, however, that nontrivial consequences can result if we assume the existence of a single congruence with a positive average expansion rate throughout some specified region.

While the past of an inflationary model is a matter of speculation, the attractor nature of the inflationary equations implies that many properties of the future can be deduced unambiguously. According to the standard picture of inflation, all physical quantities are slowly varying on the scale of  $H^{-1}$ . In the vicinity of any point  $P$  in the inflating region, we can choose an approximately homogeneous, isotropic, and flat spacelike surface which can serve as the starting point for the standard analysis of stochastic evolution [11]. A simple pattern of expansion is

established, in which the comoving geodesics  $\mathbf{x} = \text{const}$  in the synchronous gauge form a congruence with  $H \gtrsim \sqrt{(8\pi/3)G\rho_0}$ , where  $\rho_0$  is the minimum energy density of the inflationary part of the potential energy function. This congruence covers the future light cone of  $P$ . While large fluctuations can drive  $H$  to negative values, such fluctuations are extremely rare. Once inflation ends in any given region, however, many of the geodesics are likely to develop caustics as the matter clumps to form galaxies and black holes. If we try to describe inflation that is eternal into the past, it would seem reasonable to assume that the past of  $P$  is like the inflating region to the future, which would mean that a congruence that is expanding everywhere, except for rare fluctuations, can be defined throughout that past.

For the proof of our theorem, however, we find that it is sufficient to adopt a much weaker assumption, requiring only that a congruence with  $H_{\text{av}} > 0$  can be continuously defined along some past-directed timelike or null geodesic.

In Sec. II, we illustrate our result by showing how it arises in the case of a homogeneous, isotropic, and spatially flat universe. In the course of the argument we shed some light on the meaning of an incomplete null geodesic by relating the affine parametrization to the cosmological redshift. In Sec. III we present our main, model-independent argument. In Sec. IV we offer some remarks on the interpretation and possible extensions of our result.

*II. A simple model.*—Consider a model in which the metric takes the form

$$ds^2 = dt^2 - a^2(t)d\vec{x}^2. \quad (2)$$

We first examine the behavior of null geodesics and then timelike ones.

From the geodesic equation one finds that a null geodesic in the metric (2), with affine parameter  $\lambda$ , obeys the relation

$$d\lambda \propto a(t) dt. \quad (3)$$

Alternatively, we can understand this equation by considering a physical wave propagating along the null geodesic. In the short wavelength limit the wave vector  $k^\mu$  is tangential to the geodesic and is related to the affine parametrization of the geodesic by  $k^\mu \propto dx^\mu/d\lambda$ . This allows us to write  $d\lambda \propto dt/\omega$ , where  $\omega \equiv k^0$  is the physical frequency as measured by a comoving observer. In an expanding model the frequency is redshifted as  $\omega \propto 1/a(t)$ , so we recover the result of Eq. (3).

From Eq. (3), one sees that if  $a(t)$  decreases sufficiently quickly in the past direction, then  $\int a(t)dt$  can be bounded and the maximum affine length must be finite. To relate this possibility to the behavior of the Hubble parameter  $H$ , we first normalize the affine parameter by choosing  $d\lambda = [a(t)/a(t_f)]dt$ , so  $d\lambda/dt \equiv 1$  when  $t = t_f$ ,

where  $t_f$  is some chosen reference time. Using  $H \equiv \dot{a}/a$ , where a dot denotes a derivative with respect to  $t$ , we can multiply Eq. (3) by  $H(\lambda)$  and then integrate from some initial time  $t_i$  to the reference time  $t_f$ :

$$\int_{\lambda(t_i)}^{\lambda(t_f)} H(\lambda)d\lambda = \int_{a(t_i)}^{a(t_f)} \frac{da}{a(t_f)} \leq 1, \quad (4)$$

where equality holds if  $a(t_i) = 0$ . Defining  $H_{\text{av}}$  to be an average over the affine parameter,

$$H_{\text{av}} \equiv \frac{1}{\lambda(t_f) - \lambda(t_i)} \int_{\lambda(t_i)}^{\lambda(t_f)} H(\lambda)d\lambda \leq \frac{1}{\lambda(t_f) - \lambda(t_i)}, \quad (5)$$

we see that any backward-going null geodesic with  $H_{\text{av}} > 0$  must have a finite affine length, i.e., is past-incomplete.

A similar argument can be made for timelike geodesics, parametrized by the proper time  $\tau$ . For a particle of mass  $m$ , the four-momentum  $P^\mu \equiv m dx^\mu/d\tau$ , so we can write  $d\tau = (m/E)dt$ , where  $E \equiv P^0$  is the energy of the particle as measured by a comoving observer. If we define the magnitude of the three-momentum  $p$  by  $p^2 \equiv -g_{ij}P^iP^j$ , where  $i$  and  $j$  are summed 1 to 3, then  $E = \sqrt{p^2 + m^2}$ . For a comoving trajectory we have  $P^i = 0$ , and therefore  $d\tau = dt$ . For all others,  $p \propto 1/a(t)$  [12], so we can write  $p(t) = [a(t_f)/a(t)]p_f$ , where  $p_f$  denotes the value of the three-momentum  $p$  at the reference time  $t_f$ . Combining all this, we find

$$\begin{aligned} \int_{t_i}^{t_f} H(\tau)d\tau &= \int_{a(t_i)}^{a(t_f)} \frac{mda}{\sqrt{m^2a^2 + p_f^2a^2(t_f)}} \\ &\leq \ln\left(\frac{E_f + m}{p_f}\right) = \frac{1}{2}\ln\left(\frac{\gamma + 1}{\gamma - 1}\right), \end{aligned} \quad (6)$$

where the inequality becomes an equality when  $a(t_i) = 0$ . Here  $E_f \equiv \sqrt{p_f^2 + m^2}$  and  $\gamma \equiv 1/\sqrt{1 - v_{\text{rel}}^2}$ , where  $v_{\text{rel}} \equiv p_f/E_f$  is the speed of the geodesic relative to the comoving observers at time  $t_f$ . Since the integral is bounded, the argument used for null geodesics can be repeated, with the average taken over proper time.

*III. The main argument.*—In this section we show that the inequalities of Eqs. (4) and (6) can be established in arbitrary cosmological models, making no assumptions about homogeneity, isotropy, or energy conditions. To achieve this generality, we need a definition of the Hubble parameter  $H$  that applies to arbitrary models and which reduces to the standard one ( $H = \dot{a}/a$ ) in simple models.

Consider a timelike or null geodesic  $\mathcal{O}$  (“the observer”). We assume that a congruence of timelike geodesics (“comoving test particles”) has been defined along  $\mathcal{O}$  [13], and we will construct a definition for  $H$  that depends only on the relative motion of the observer and test particles.

In order to motivate what we do, we first consider the case of nonrelativistic velocities in Minkowski space. Suppose that the observer measures the velocities of the test particles as a function of the time  $\tau$  on his own clock. At time  $\tau_1$  particle 1 passes with velocity  $\vec{u}(\tau_1)$ , and at time  $\tau_2 = \tau_1 + \Delta\tau$  particle 2 passes with velocity  $\vec{u}(\tau_2)$ . What expansion rate could he infer from these measurements? The separation vector between the positions  $\vec{r}_1$  and  $\vec{r}_2$  of the two particles at  $\tau_2$  is  $\Delta\vec{r} = \vec{r}_1 - \vec{r}_2 = \vec{u}\Delta\tau$ , and its magnitude is  $\Delta r \equiv |\Delta\vec{r}|$ . Their relative velocity is  $\Delta\vec{u} = \vec{u}_1 - \vec{u}_2 = -(d\vec{u}/d\tau)\Delta\tau$ . The Hubble expansion rate is defined in terms of the rate of separation of these particles, which in turn depends on the radial component of their relative velocity,  $\Delta u_r = \Delta\vec{u} \cdot \Delta\vec{r}/\Delta r$ . The inferred Hubble parameter  $H$  is then

$$H \equiv \frac{\Delta u_r}{\Delta r} = \frac{\Delta\vec{u} \cdot \Delta\vec{r}}{|\Delta\vec{r}|^2} = -\frac{\vec{u} \cdot (d\vec{u}/d\tau)}{|\vec{u}|^2}, \quad (7)$$

which will equal the standardly defined Hubble parameter for the case of a homogeneous, isotropic universe. The expression for  $H$  may be simplified to  $H = -d \ln v_{\text{rel}}/d\tau$ , where  $v_{\text{rel}} = |\vec{u}|$ . The fact that  $H$  is the total derivative of a function of  $v_{\text{rel}}$  implies that the variation of  $v_{\text{rel}}$  is determined completely by the local value of  $H$ , even if the Universe is inhomogeneous and anisotropic.

We can now generalize this idea to the case of arbitrary velocities in curved spacetime. Let  $v^\mu = dx^\mu/d\tau$  be the four velocity of the geodesic  $\mathcal{O}$ , where we take  $\tau$  to be the proper time in the case of a timelike observer, or an affine parameter in the case of a null observer. Let  $u^\mu(\tau)$  denote the four velocity of the comoving test particle that passes the observer at time  $\tau$ . We define  $\gamma \equiv u_\nu v^\nu$ , so in the timelike case  $\gamma$  may be viewed as the relative Lorentz factor ( $1/\sqrt{1 - v_{\text{rel}}^2}$ ) between  $u^\mu$  and  $v^\mu$ . In the null case,  $\gamma = dt/d\tau$ , where  $t$  is the time as measured by comoving observers, and  $\tau$  is the affine parameter of  $\mathcal{O}$ .

Consider observations made by  $\mathcal{O}$  at times  $\tau_1$  and  $\tau_2 = \tau_1 + \Delta\tau$ , as shown in Fig. 1, where  $\Delta\tau$  is infinitesimal. Let  $\Delta r^\mu$  be a vector that joins the world lines of the two

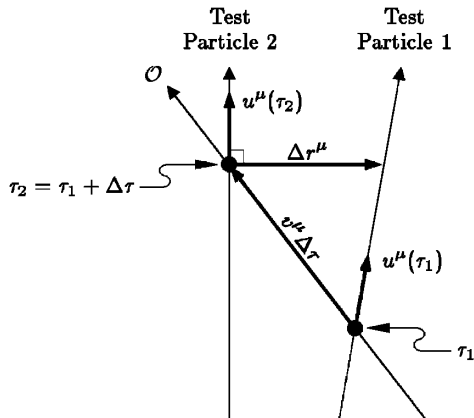


FIG. 1. The observer's world line  $\mathcal{O}$  and two test particles.

test particles at equal times in their own rest frame. Such a vector is perpendicular to the world lines and can be constructed by projecting the vector  $-v^\mu \Delta\tau$  to be perpendicular to  $u^\mu$ :  $\Delta r^\mu = -v^\mu \Delta\tau + \gamma u^\mu \Delta\tau$ . In the rest frame of the observer this answer reduces for small velocities to its nonrelativistic counterpart,  $\Delta r^\mu = (0, u^i \Delta\tau)$ . This is a spacelike vector of length  $\Delta r \equiv |\Delta r^\mu| = \sqrt{\gamma^2 - \kappa} \Delta\tau$ , where  $\kappa \equiv v^\mu v_\mu$  is equal to 1 for the timelike case and 0 for the null case. The separation velocity will be  $\Delta u^\mu = -(Du^\mu/d\tau)\Delta\tau$ , where  $D/d\tau$  is the covariant derivative along  $\mathcal{O}$ . The covariant derivative allows us to compare via parallel transport vectors defined at two different points along  $\mathcal{O}$  and can be justified by considering the problem in the free-falling frame, for which the affine connection vanishes on  $\mathcal{O}$  at  $\tau = \tau_1$ . The radial component of this velocity will be  $\Delta u_r = -(\Delta u^\mu \Delta r_\mu)/\Delta r$ , where the sign arises from the Lorentz metric. We define the Hubble parameter as [14]

$$H \equiv \frac{\Delta u_r}{\Delta r} = \frac{-v_\mu (Du^\mu/d\tau)}{\gamma^2 - \kappa}. \quad (8)$$

Since  $\mathcal{O}$  is a geodesic, we have  $(Dv^\mu/d\tau) = 0$ , and therefore

$$H = \frac{-d\gamma/d\tau}{\gamma^2 - \kappa} = \frac{d\tau}{d\tau} F(\gamma(\tau)), \quad (9)$$

where

$$F(\gamma) = \begin{cases} \gamma^{-1} & \text{null observer } (\kappa = 0), \\ \frac{1}{2} \ln \frac{\gamma+1}{\gamma-1} & \text{timelike observer } (\kappa = 1). \end{cases} \quad (10)$$

As in Sec. II, we now integrate  $H$  along  $\mathcal{O}$  from some initial  $\tau_i$  to some chosen  $\tau_f$ :

$$\int_{\tau_i}^{\tau_f} H d\tau = F(\gamma_f) - F(\gamma_i) \leq F(\gamma_f). \quad (11)$$

In the null case  $F(\gamma_f) = \gamma_f^{-1}$ , which is equal to the value of  $d\tau/dt$  at  $t_f$ , normalized in Sec. II to unity.

Equation (11) therefore reproduces exactly the results of Eqs. (4) and (6), but in a much more general context. Again we see that if  $H_{\text{av}} > 0$  along any null or noncomoving timelike geodesic, then the geodesic is necessarily past-incomplete.

*IV. Discussion.*—Our argument shows that null and timelike geodesics are, in general, past-incomplete in inflationary models, whether or not energy conditions hold, provided only that the averaged expansion condition  $H_{\text{av}} > 0$  holds along these past-directed geodesics. This is a stronger conclusion than the one arrived at in previous work [8] in that we have shown under reasonable assumptions that almost all causal geodesics, when extended to the past of an arbitrary point, reach the boundary of the inflating region of spacetime in a finite proper time (finite affine length, in the null case).

What can lie beyond this boundary? Several possibilities have been discussed, one being that the boundary of

the inflating region corresponds to the beginning of the Universe in a quantum nucleation event [15]. The boundary is then a closed spacelike hypersurface which can be determined from the appropriate instanton.

Whatever the possibilities for the boundary, it is clear that unless the averaged expansion condition can somehow be avoided for all past-directed geodesics, inflation alone is not sufficient to provide a complete description of the Universe, and some new physics is necessary in order to determine the correct conditions at the boundary [16]. This is the chief result of our Letter. The result depends on just one assumption: the Hubble parameter  $H$  has a positive value when averaged over the affine parameter of a past-directed null or noncomoving timelike geodesic.

The class of cosmologies satisfying this assumption is not limited to inflating universes. Of particular interest is the recycling scenario [18], in which each comoving region goes through a succession of inflationary and thermalized epochs. Since this scenario requires a positive true vacuum energy  $\rho_v$ , the expansion rate will be bounded by  $H_{\min} = \sqrt{8\pi G\rho_v/3}$  for locally flat or open equal-time slicings, and the conditions of our theorem may be satisfied. One must look carefully, however, at the possibility of discontinuities where the inflationary and thermalized regions meet. This issue requires further analysis.

Our argument can be straightforwardly extended to cosmology in higher dimensions. For example, in the model of Ref. [19] brane worlds are created in collisions of bubbles nucleating in an inflating higher-dimensional bulk spacetime. Our analysis implies that the inflating bulk cannot be past-complete.

We finally comment on the cyclic Universe model [20] in which a bulk of four spatial dimensions is sandwiched between two three-dimensional branes. The effective  $(3+1)$ -dimensional geometry describes a periodically expanding and recollapsing universe, with curvature singularities separating each cycle. The internal brane spacetimes, however, are nonsingular, and this is the basis for the claim [20] that the cyclic scenario does not require any initial conditions. We disagree with this claim.

In some versions of the cyclic model the brane spacetimes are everywhere expanding, so our theorem immediately implies the existence of a past boundary at which boundary conditions must be imposed. In other versions, there are brief periods of contraction, but the net result of each cycle is an expansion. For null geodesics each cycle is identical to the others, except for the overall normalization of the affine parameter. Thus, as long as  $H_{\text{av}} > 0$  for a null geodesic when averaged over one cycle, then  $H_{\text{av}} > 0$  for any number of cycles, and our theorem would imply that the geodesic is incomplete.

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  - [13] We do not require that this congruence be defined throughout spacetime, just along  $\mathcal{O}$ . Away from  $\mathcal{O}$  the members of this congruence may cross or focus, but such behavior does not affect our argument.
  - [14] Our definition of  $H$  in Eq. (8) can also be expressed as  $H = -u_{;\nu}^\mu n_\mu n^\nu$ , where  $n^\mu$  is a unit vector in the direction of  $\Delta r^\mu$ . This can be shown by substituting  $v^\mu = \gamma u^\mu - \sqrt{\gamma^2 - \kappa} n^\mu$  into (8) and using the relations  $u_{\mu;\nu} u^\mu = u_{\mu;\nu} u^\nu = 0$ . This expression shows that  $H$  depends on the direction but not the speed of  $\mathcal{O}$ , as seen in the rest frame of the comoving test particles. Furthermore, the frequently used definition  $\tilde{H} = (1/3)u_{;\mu}^\mu$  is obtained from ours by averaging over all directions.
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