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Helmholtz's Polarization Theory of Ether and Matter

By 1890 most physicists on the Continent were well aware that something called "Maxwell's theory of electromagnetism" existed and that it had apparently received striking confirmation in 1888 with Heinrich Hertz's experiments on electric waves. Hertz's experiments precipitated a deep and widespread interest in Maxwell's theory, and by c. 1890 one finds, particularly in Germany, increasing numbers of articles in major journals which employ field equations of some kind. Some very few of these articles (e.g., those written by Hertz himself) approach the spirit of aspects of Maxwellian theory. However, by far the majority fundamentally mistook the core of British field physics while nevertheless appropriating from it those elements which could be assimilated to the prevailing Continental view (which had not changed in essence since the 1840s) that "electricity" is not an epiphenomenon but an entity in its own right. Indeed, to my knowledge there were almost no physicists in the 1890s who had not learned electromagnetic theory from Maxwell's *Treatise* and yet who had grasped the structure of Maxwellian theory (the sole exception, which fact I owe to Olé Knudsen, is Willard Gibbs). On the other hand, many Continental physicists of the period were intimately familiar with field equations.

The primary sources of Continental knowledge of field equations were almost certainly Poincaré's (1890) and Boltzmann's (1891) texts on the Maxwell theory, Föppl's (1894) account of Maxwell's theory, Helmholtz's (1897) lectures on the electromagnetic theory of light, Drude's (1894*b*) account of ether physics, and Volkmann's (1891) comprehensive optics text.[1] Some of these texts deal with electrostatics and electrodynamics as well as optics, while all discuss in detail the electromagnetic theory of light, which they refer to Maxwell. Indeed, both Poincaré and Boltzmann claim to explicate the structure of Maxwell's own work.

Yet one overriding attribute characterizes all but Föppl's text: each of them attempts to obtain the "Maxwell" field equations as a limiting case of Helmholtz's (1870; entirely non-Maxwellian) "polarization" theory of ether and matter. This theory (well known on the Continent by 1890 and also known in Britain) substantially determined the way in which Continental physicists understood Maxwell's theory. Even those few who no longer reached the Maxwell equations via Helmholtz's (like H. A. Lorentz after 1892 or Hertz after 1890) continued to bear unmistakeable marks of the Helmholtz polarization theory. Therefore, to understand the Continental reception of Maxwell's theory, we must begin with Helmholtz's (1870) equations.

1. Föppl's work (which was recommended to all by Heaviside [1897] in his review condemning Boltzmann [1891] as non-Maxwellian) is the only one of these six texts which does not rely on Helmholtzian theory but concentrates instead on fluxes and intensities. Nevertheless, it continues to regard electricity as substantial, in that "charge" is not referred directly to discontinuities in displacement. Of all the texts, though, Föppl's comes closest to proper Maxwillian concepts. This was one of the texts from which Einstein learned electromagnetism.

Helmholtz's theory has been discussed several times in recent years, and little to be said here differs substantially from those accounts insofar as the equations proper are concerned.[2] However, precisely because these equations have certain striking similarities to the Maxwell equations, modern accounts have followed the Continental argument that Maxwell's theory can be thought of as a limiting case of Helmholtz's, at least formally. This assertion is fundamentally mistaken, for there is an unbridgeable gulf between the two theories, a gulf which separates those who viewed electricity as a by-product of field processes from those who did not.

Helmholtz's theory consists of three interlocking components: first, expressions for electromagnetic potentials and the forces derived from them; second, a continuity equation linking electric "charge" and "current"; finally, a model for an electrically and magnetically polarizable medium.[3] We begin with Helmholtz's potentials.

There are two potential functions: one, a vector $\vec{U}$, depends solely on the current, $\vec{J}$, and distance; the other, a scalar, ϕ_f, depends on a net or (to use Helmholtz's word); "free" charge density ρ_f. To accommodate various possible forms for the potential $\vec{U}$ (all of which agree in yielding the correct force for closed conducting circuits), Helmholtz incorporated a constant k in $\vec{U}$ whose value must be fixed by experiments with open circuits:

(1) $$\vec{U}(\vec{x}) = \int[\vec{J}(\vec{x}')/|\vec{x} - \vec{x}'|]d^3x' + (1/2)(1 - k)\vec{\nabla}[\int J(\vec{x}') \cdot \vec{\nabla}_{x'}|\vec{x} - \vec{x}'|d^3x']$$

Helmholtz separately denoted the function whose gradient multiplies (1/2) (1 − k) as ψ:

(2) $$\psi(\vec{x}) = \int\vec{J}(\vec{x}') \cdot \vec{\nabla}_{x'} |\vec{x} - \vec{x}'|d^3x'$$

For ϕ_f Helmholtz wrote:

(3) $$\nabla^2\phi_f = -4\pi\rho_f \; . \; . \; . \; \phi_f(\vec{x}) = -(1/4\pi)\int\rho_f(\vec{x}')/|\vec{x} - \vec{x}'|d^3x'$$

Next Helmholtz presumed a continuity equation according to which all accumulations of "free" charge ρ_f are due to inhomogeneities in the current density $\vec{J}$:

(4) $$\partial\rho_f/\partial t = -\vec{\nabla} \cdot \vec{J}$$

Equations (3) and (4) establish a very important link between ϕ_f and $\vec{J}$:

(5) $$\partial/\partial t\nabla^2\phi_f = 4\pi\vec{\nabla} \cdot \vec{J}$$

By virtue of equation (5) it is possible to express the function ψ of equation (2)—a function whose gradient appears in $\vec{U}$, the electrodynamic potential—directly in terms of $\partial\phi_f/\partial t$ by partial integration of equation (2) under the assumptions that $\vec{J}$ and $\partial\phi_f/\partial t$ both vanish at infinity (see appendix 10 for derivation of equations [6]–[8]):

(6) $$\psi(\vec{x}) = (1/4\pi)\int[\partial/\partial t\phi_f(\vec{x}')]\nabla^2_{x'}|\vec{x} - \vec{x}'|d^3x'$$

2. See, e.g., Woodruff (1968), Hirosige (1969), and Rosenfeld (1956). I am indebted to Olé Knudsen for extensive discussions of Helmholtz's theory which we had in the fall of 1979. We employed an unpublished dissertation (Nielsen 1974) which succinctly covers important elements of the theory and which much aided our discussion.

3. Where the particular equations occur in what follows, I shall, for the most part, omit specific references to the loci in Helmholtz (1870) because they are readily identifiable.

In this way one can express $\nabla^2\vec{U}$ and $\vec{\nabla} \cdot \vec{U}$ as follows by virtue of the continuity equation (5) and the partial integration in the expression for $\vec{\nabla} \cdot \vec{U}$:

(7) $$\nabla^2\vec{U} = (1 - k)\vec{\nabla}\partial\phi_f/\partial t - 4\pi\vec{J}$$

(8) $$\vec{\nabla} \cdot \vec{U} = -k\partial\phi_f/\partial t$$

Equations (1) through (8) are the foundations of Helmholtz's electrodynamics. Their most significant property for our purposes is the appearance of terms containing not only the current $\vec{J}$ but also the quantity $\partial\phi_f/\partial t$ in the vector potential $\vec{U}$ and its derivatives. The time rate of change of ϕ_f affects $\vec{U}$, however, *solely* because of the continuity equation (5) which links ϕ_f with $\vec{J}$. That is, $\partial\phi_f/\partial t$ always derives from changing free charge densities ($\partial\rho_f/\partial t$) which, in turn, are always associated with inhomogeneous currents $\vec{J}$. This is a critical point, because the reason Helmholtz's theory cannot be reduced to Maxwell's is primarily due to the function $\partial\phi_f/\partial t$. In other words, the conflict between Helmholtz's and Maxwell's theories occurs where one would expect to find it—in the continuity equation.

We come now to Helmholtz's model for an electrically and magnetically polarizable medium. In essence, he assumed that inhomogeneities in the total electromotive force $\vec{E}_T$ in the medium determine a polarization charge density ρ_p and associated potential ϕ_p such that (with $\vec{P}$ the electric moment density):

(9) $$\rho_p = -\vec{\nabla} \cdot \vec{P} = -\vec{\nabla} \cdot (\chi\vec{E}_T) = (-1/4\pi)\nabla^2 \phi_P$$

Helmholtz distinguished in $\vec{E}_T$ the force $\vec{E}_E$ which engenders the polarization from the force $\vec{E}_P$, which arises as a result of the "distributed electricity"—the polarization charge ρ_p—produced by the action of $\vec{E}_E$. Introducing the potential ϕ_P of $\vec{E}_P$, he had:

(10) $$\vec{P} = \chi(\vec{E}_E - \vec{\nabla}\phi_P) = \chi\vec{E}_T$$

(Unfortunately, Helmholtz did not use different symbols to distinguish the ϕ_p of equation [10] from the ϕ_f of equation [3]. This leads to an apparent error in his final equation for the curl of the magnetic force, but this error was corrected without comment in Lorentz [1875], which was based on the Helmholtz theory.)

We have almost reached the final elements in Helmholtz's theory, but first we must introduce the magnetic force and polarization. Helmholtz assumed that induced electromotive forces may be produced by time changes in either the vector potential $\vec{U}$ due to currents or the potential $\vec{U}_M$ due to changing magnetization. The magnetization, $\vec{M}$, of the medium is taken as proportional to the total magnetic force $\vec{H}_T$:

(11) $$\vec{E}_{\text{ind}} = -A^2\partial\vec{U}/\partial t - A^2\partial\vec{U}_M/\partial t$$

(12) $$\vec{M} = \theta\vec{H}_T$$

The potential $\vec{U}_M$ is further defined as proportional to the curl of a vector $\vec{L}$ which is the potential of $\vec{M}$:

(13) $$A\vec{U}_M(\vec{x}) = \vec{\nabla} \times \int[\vec{M}(\vec{x}')/|\vec{x} - \vec{x}'|]d^3x' = \vec{\nabla} \times \vec{L}$$

(The distinction between $\vec{U}$ and $\vec{U}_M$ permits one to treat magnetization without recourse to a model linking it to current.)

The magnetization implicates a potential function ω_M, such that $\vec{\nabla} \cdot \vec{M}$ may be represented by $(1/4\pi)\nabla^2\omega_M$. The corresponding force is $-\vec{\nabla}\omega_M$, and the total force, $\vec{H}_T$, is then the sum of $\vec{\nabla}\omega_M$ and the force $\vec{B}_E$ due to currents, namely, $A\vec{\nabla} \times \vec{U}$:

$$\vec{M} = \theta(\vec{B}_E - \vec{\nabla}\omega_M) = \theta\vec{H}_T \tag{14}$$

$$\vec{B}_E = A\vec{\nabla} \times \vec{U} \tag{15}$$

We may now write the total electric force $\vec{E}_T$ by summing the various contributions and including a term $\vec{E}_{\text{ctm}}$ which represents the possible action of chemical, thermal, and other purely material processes. In doing so we shall not distinguish the fields attributed to polarization and true charges, as Helmholtz also did not, by representing both effects as the net potential ϕ_f due to Helmholtz's "free" charge; indeed, this free charge is just the sum of the modern conduction (ρ_c) and polarization charges. In modern terms:

$$\rho_f = -(1/4\pi)\nabla^2\phi_f = (1/4\pi)\vec{\nabla} \cdot \vec{E} = \rho_c + \rho_p = -(1/4\pi)\nabla^2(\phi_c - \phi_p)$$
$$= (1/4\pi)\vec{\nabla} \cdot \vec{D} - \vec{\nabla} \cdot \vec{P}$$

Whence, in terms of specific inductive capacity, we have:

$$\vec{E} = \vec{D} - 4\pi\vec{P} = \epsilon\vec{E} - 4\pi\chi\vec{E} \ldots \epsilon = 1 + 4\pi\chi$$

So in Helmholtz's theory:

$$\vec{P}/\chi = \vec{E}_T = -\vec{\nabla}\phi_f - A^2\partial\vec{U}/\partial t - A\partial/\partial t\vec{\nabla} \times \vec{L} + \vec{E}_{\text{ctm}} \tag{16}$$

To transform equation (16) into something like the Maxwellian form of the Faraday law, we must take its curl and express the second and third terms as functions of the magnetization $\vec{M}$. This is done simply with the help of equations (1), (14), and (15):

$$\vec{\nabla} \cdot \vec{L} = -\omega_M$$
$$\nabla^2\vec{L} = -4\pi\vec{M}$$
$$A\vec{\nabla} \times \partial\vec{U}/\partial t = \partial/\partial t(\vec{M}/\theta + \vec{\nabla}\omega_M)$$

Consequently, equation (16) yields the Helmholtzian version of the Faraday law as:

$$\vec{\nabla} \times \vec{E}_T = \vec{\nabla} \times \vec{P}/\chi = -A\partial/\partial t(1 + 4\pi\theta)\vec{H}_T + \vec{\nabla} \times \vec{E}_{\text{ctm}} \tag{17}$$

Evidently we may write:

$$\vec{H}_T = \vec{B} - 4\pi\vec{M} = \vec{M}/\theta \ldots \mu = 1 + 4\pi\theta$$

We turn next to the Ampère law, that is, the expression for $\vec{\nabla} \times \vec{H}_T$. Here we use equations (7), (8), (14), and (15):

$$\vec{\nabla} \times \vec{M}/\theta = \vec{\nabla} \times \vec{B}_E = A\vec{\nabla} \times (\vec{\nabla} \times \vec{U}) = A\vec{\nabla}(\vec{\nabla} \cdot \vec{U}) - A\nabla^2\vec{U}$$

Whence the Helmholtzian Ampère law reads:

$$\vec{\nabla} \times \vec{H}_T = \vec{\nabla} \times \vec{M}/\theta = -A\partial/\partial t\vec{\nabla}\phi_f + 4\pi A\vec{J} \tag{18}$$

Note that $\partial\vec{\nabla}\phi_f/\partial t$ appears in the Ampère law. Its presence derives not from $\vec{\nabla}\psi$ in the expression for $\vec{U}$ (since the curl of this vanishes) but from partial integration of the term in $\vec{J}$ through use of the continuity equation (5). (Helmholtz neglected here to distinguish between ϕ_f and ϕ_p, as mentioned above, though his "error" was, I believe, limited to equation [10], where Helmholtz used the same symbol for what is clearly ϕ_p that he had previously used for ϕ_f, namely ϕ. The corresponding equations [17] and [18] also use ϕ, and here ϕ must clearly denote ϕ_f since Helmholtz was considering an infinite conducting medium which can contain accumulations of conduction charge as well as polarization charge. In an infinite nonconducting medium it is possible to replace ϕ_f with ϕ_p).

Having established the Ampère and Faraday laws, we must now introduce an expression for the current $\vec{J}$ and an equation linking $\vec{J}$ with $\vec{E}_T$. For the conduction current part, $\vec{C}$, of $\vec{J}$, Helmholtz employed Ohm's law:

$$\kappa\vec{C} = \vec{E}_T = \vec{P}/\chi \tag{19}$$

Helmholtz next argued that the current consists of, in addition to $\vec{C}$, a polarization current $\partial\vec{P}/\partial t$ which does not obey Ohm's law—it is an unresisted electric motion:

$$\vec{J} = \vec{C} + \partial\vec{P}/\partial t \tag{20}$$

The continuity equation (5) holds for the total current $\vec{J}$ because accumulations of both, or either, conduction and polarization charge are possible. Combining equations (18), (19), and (20) we finally have:

$$\vec{\nabla} \times \vec{H}_T = \vec{\nabla} \times \vec{M}/\theta = -A\partial/\partial t\vec{\nabla}\phi_f + (4\pi A/\kappa)\vec{E}_T + (4\pi A\chi)\partial\vec{E}_T/\partial t \tag{21}$$

This, in essence, is the Helmholtz theory. If one uses $\vec{E}_T$, $\vec{H}_T$ instead of $\vec{P}$, M, as in equations (17) and (21), one can compare Helmholtz's equations with Maxwell's equations. We see at once that the two theories agree formally in the Faraday law but there is a glaring inconsistency in the Ampère law: instead of the term $(1 + 4\pi\chi)\partial\vec{E}_T/\partial t$, as in the Maxwell theory, Helmholtz has $4\pi\chi\partial E_T/\partial t$. It is precisely here that the elementary difference between the two theories appears, for this incompatibility reflects the fact that Helmholtz's theory does not, indeed cannot, incorporate even a formal equivalent of the Maxwellian displacement current. The ultimate reason for this is that in Helmholtz's theory all fields involve interactions between charge densities, and these interactions are *not* in fact propagated. Only the polarizations propagate; charge interactions are always instantaneous. We shall return to this point below. First, the formal incompatibility between the Maxwell and Helmholtz equations will be discussed in more detail.

Further examination of Helmholtz's Ampère law, equation (21), pinpoints its difference from the Maxwell Ampère law. Suppose we rewrite equation (21) as follows:

$$\vec{\nabla} \times \vec{H} = \vec{C} + \partial/\partial t(-\vec{\nabla}\phi_f + 4\pi\chi\vec{E}_T) \tag{21'}$$

Clearly if $\vec{E}_T$ were equal to $-\vec{\nabla}\phi_f$ we would have the Maxwell law since ϵ is just $1 + 4\pi\chi$. But $\vec{E}_T$ contains electrodynamic ($\vec{E}_{ind}$) as well as electrostatic forces, so that $-\vec{\nabla}\phi_f + 4\pi\chi\vec{E}_T$ is actually equal to the difference $\epsilon\vec{E}_T - \vec{E}_{ind}$. In other words, Helmholtz's law differs formally from Maxwell's by requiring the nonconducting part of the current to consist of the difference between the rate of change of displacement and $\vec{E}_{ind}$.

In Maxwellian terms this makes no sense at all, since it introduces an artificial distinction between conduction and displacement currents. But it does make sense in the Helmholtz theory because there one has no physical reason to choose $\partial\vec{D}/\partial t$ rather than $\partial(\vec{D} - \vec{E}_{ind})/\partial t$. Indeed, if equation (1) is chosen for $\vec{U}$, and equation (4) is the continuity equation, we necessarily obtain Helmholtz's expression. Most Continental physicists until c. 1900 understood Maxwell's theory in Helmholtzian terms, and they therefore sought formal conditions to transform the latter into the former. These are, superficially, readily found. What they amount to in the end is a conflation of displacement with polarization.

To find the conditions, we first obtain the wave equations for $\vec{P}$ by using equations (9), (16), (17), and (18), and by limiting ourselves to a nonconducting medium, wherein we may now legitimately replace ϕ_f with ϕ_p:

$$\vec{\nabla} \cdot \vec{P} = 0 \; . \; . \; . \; \nabla^2 \vec{P} = 4\pi\chi(1 + 4\pi\theta)A^2 \partial^2 \vec{P}/\partial t^2 \tag{22}$$

$$\vec{\nabla} \times \vec{P} = 0 \; . \; . \; . \; \nabla^2(\vec{\nabla} \cdot \vec{P}) = [4\pi\chi k A^2/(1 + 4\pi\chi)]\partial^2(\vec{\nabla} \cdot \vec{P})/\partial t^2 \tag{23}$$

Equations (22) and (23) respectively determine transverse and longitudinal waves of electric polarization. In Maxwell's theory longitudinal waves do not arise, so one condition to reach that theory is that the constant k be zero, which makes the longitudinal speed infinite. (Note that k cannot itself be infinite, though this also destroys the longitudinal wave, because this would make the vector potential negatively infinite.) However, this is not enough, even for optics, because the transverse wave speed implied by equation (22) is not the same as the Maxwellian speed.

This can most easily be seen by considering the refraction of a wave at the interface between two media which have constants χ_1, θ and χ_2, θ, respectively. Then the index of refraction will be:

$$n_{1,2} = \sqrt{[\chi_2/\chi_1]}$$

Now χ_2, χ_1 are not in fact directly measurable from electrostatic experiments because the polarization of the ether superposes on the polarization of the matter. However, by considering the force between measured charges one can show that the measurable $\bar{\chi}$ is related to the true constant χ as follows (χ_0 denotes the polarizability of the ether):

$$1 + 4\pi\bar{\chi} = (1 + 4\pi\chi)/(1 + 4\pi\chi_0)$$

Whence the index becomes:

$$n_{1,2} = \sqrt{[(1 + 4\pi\chi_0)\bar{\epsilon}_2 - 1]/[(1 + 4\pi\chi_0)\bar{\epsilon}_1 - 1]} \tag{24}$$

In equation (24) then, $\bar{\epsilon}_1$, $\bar{\epsilon}_2$ are the measured capacities. According to Maxwell's theory, however, $n_{1,2}$ should be just $\sqrt{[\bar{\epsilon}_2/\bar{\epsilon}_1]}$. The only way to reach this result from equation (24) is to assume that the polarizability χ_0 of the ether is effectively infinite, which means infinite χ also if $\bar{\chi}$ is to be finite.

In the eyes of those Continental physicists familiar with Helmholtz's work, these conditions (k zero and χ_0 infinite) were thought to lead directly to Maxwell's theory. Helmholtz (1870, 127) called these "Maxwell's limiting conditions," and Lorentz (1875, 275) actually wrote that Maxwell arrived "in this manner [viz., through these limiting conditions] at the result that transverse electric vibrations can propagate in air with speed equal to that of light." Moreover, the texts of the early 1890s mentioned above uniformly agree that one can arrive at Maxwell's theory in this way. But one cannot do so, and the reason is not hard to find. Simply put, taking the limit does not lead to Maxwell's theory; it only deprives Helmholtz's of physical significance.

We need not examine the questions raised by the vanishing of k because, as Poincaré later noted, the requirement that χ_0 be infinite alone suffices to grant the longitudinal wave infinite speed (Poincaré 1890, 2:112; cf. Woodruff 1968, 307). But the

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Yet until c. 1891, when Lorentz evidently first fully saw the profundity of this problem[4] (and long after that date for many Continental physicists), Helmholtz's theory was seen as the only route to Maxwell's. The reason for this is again not hard to find: no Continental physicist understood that in Maxwell's theory "charge" is an epiphenomenon due to discontinuities in induction, whereas everyone could grasp Helmholtz's theory because it was founded on a continuity equation that embodied the substantiality—though not the materiality—of charge. The complexity of Helmholtz's equations as compared with Maxwell's was more than compensated for by the familiar concepts they embodied. This essential difference was partly understood in Britain, where Maxwellian ideas had taken firm root, but it was never quite perceived on the Continent, despite the fact that those who read Maxwell's *Treatise*—and there were many after 1888—always experienced insuperable difficulties in grasping what he meant by the word "charge."

Let us now examine the underlying principles of Helmholtz's theory before we turn to several revealing Continental attempts in the 1890s to explicate Maxwell's theory. Begin with the fundamental equations (1)–(8). One can easily grasp the physical import of these equations by envisioning a current as a flow of charge, literally imagined. If the flow field is uniform then there will be no charge accumulation, and the electromagnetic potential will be due simply to solenoidal flow. If the current field is not solenoidal then charge density accumulates over time, and this changing density determines an additional current, $-(1/4\pi)(1 - k)\vec{\nabla}(\partial\phi/\partial t)$. There is nothing the least difficult about this on the model of a current as a substantial flow.

Now equations (1)–(8) by themselves do not imply waves because the charges act directly at a distance both statically and electrodynamically. However, when a polarizable medium is introduced, the state of polarization propagates at a finite rate because of electromagnetic induction. As polarization increases at one point, an elec-

4. Lorentz (1891). Hirosige (1969, 185) remarks:

> In the old [Helmholtzian] theory, too, one can assign to a medium an intervening role in electromagnetic phenomena, and thus arrive at an explanation of Hertz's experiment and the electromagnetic theory of light. But for this purpose the ratio between the quantity of electricity given to the condenser plate and the quantity transferred to the dielectric should not differ perceptibly from unity. It is difficult to make this requirement compatible with the one mentioned above that the quantity transferred to the dielectric should be smaller than the quantity supplied to the plate to give rise to the electric action of the parallel-plate condenser: [quoting Lorentz] "It is only through an artificial assumption that one could satisfy both requirements, and this is the second argument, to which I have already alluded, that seems to plead in favor of the new mode of conception."

The "artificial assumption" here is that the polarizability of all bodies, including the ether, must be effectively infinite and yet have finite ratios among one another.

tromotive force, it is true, acts at once throughout the medium, but it is of decreasing intensity with distance from the original locus. This temporally growing and spatially decreasing electromotive force in turn causes polarization buildup throughout the medium, and the electromotive force of each of these increasing polarizations necessarily acts at once upon all the others, including the first. Since these electromotive forces are decreasing functions of distance from their sources, the result is a pattern of electrodynamic interactions which, though the actions propagate infinitely rapidly, nevertheless imply that the polarization propagates as a wave. Nothing in this picture at all violates traditional concepts, including both substantiality of charge and action at a distance, anymore than the existence of waves in a point lattice whose elements interact with instantaneous central forces violates them. Here, however, there need be no analog of mass because finite transmission rates result directly from electromagnetic induction. Indeed, one could understand Helmholtz's theory simply in terms of a series of interacting circuits. It is hardly surprising that Continental physicists found the theory so convincing.

Moreover, it is also not surprising that they saw Maxwell's theory as a limit of Helmholtz's because, in their view, to quote Helmholtz:

> [Helmholtz's and Maxwell's] theories are opposed to each other in a certain sense, since according to the theory of magnetic induction originating with Poisson, which can be carried through in a fully corresponding way for the theory of dielectric polarisation of insulators, the action at a distance is diminished by the polarisation, while according to Maxwell's theory on the other hand the action at a distance is exactly replaced by the polarisation. . . . It follows . . . from these investigations that the remarkable analogy between the motion of electricity in a dielectric and that of the light ether does not depend on the particular form of Maxwell's hypotheses, but results also in a basically similar fashion if we maintain the older viewpoint about electrical actions at a distance. (Woodruff 1968, 307–308)

From this one sees that to Helmholtz—as to his Continental colleagues—the core of Maxwell's theory was the requirement that ether, indeed all dielectric media, be so highly polarizable that the immense charges produced at conductor-dielectric interfaces by polarization completely overwhelm and in fact cancel the conduction charges proper. In other words, it is not the case that polarization charge replaces conduction charge, not at all; for the electromotive force in a charged, isolated capacitor which engenders the polarization in the sandwiched dielectric is *due* to the conduction charge. However, the forces exerted directly at a distance by the conduction charges are, in the limit of the Helmholtz theory, canceled by the bounding polarization charges. In Maxwellian theory, by contrast, neither conduction nor polarization charge exerts forces because each is an epiphenomenon of, respectively, induction or intensity discontinuity. What Helmholtz and every other physicist on the Continent missed was this most elementary aspect of Maxwell's theory: its abolition of "charge" as a fundamental physical entity.

British Maxwellians seem to have grasped this difference between the two kinds of theories, though their outlook was sufficiently Maxwellian to preclude a complete understanding of the Helmholtz theory. Consider, for example, J. J. Thomson's

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(1885*a*) discussion of the relationships between the theories. Thomson read Helmholtz's χ as essentially the same in significance as Maxwell's ϵ, which it is not. He found that for consistency between the theories one must actually set $\bar{\epsilon}$ to χ/χ_0 and assume both χ and χ_0 to be infinite. Having said this much, Thomson at once turned to the locus of the basic difference between the theories, namely, the continuity equation, for here the incommensurability between the Maxwellian and Helmholtzian concepts of charge is strikingly evident.

Helmholtz, Thomson remarked, defined the total current $\vec{J}$ as $\vec{C} + \partial\vec{P}/\partial t$, and he wrote the continuity equation as:

$$\vec{\nabla} \cdot \vec{J} = -\partial\rho/\partial t \tag{24}$$

Here, Thomson remarked, "ρ is the volume density of the free electricity"—which latter Thomson thought should be $\vec{\nabla} \cdot \vec{D}$ since he assimilated ϵ to χ (Thomson 1885*a*, 134). But, he continued, in Maxwell's theory $\vec{\nabla} \cdot \vec{J}$ should be zero. Whence he concluded:

> . . . on Helmholtz's theory [conduction] currents behave like the flow of an incompressible fluid, while on Maxwell's theory it is the total current, which is the sum of the conduction currents and the dielectric currents, which behaves in this way. (Thomson 1885*a*, 134)

Thomson's conclusion is correct only if ρ is $\vec{\nabla} \cdot \vec{D}$, which, in Helmholtz's theory, it is not—it is $\vec{\nabla} \cdot \vec{E}$. We see that, though Thomson pinpoints at once the fundamental difference between the two theories—their treatments of charge and current as embodied in the continuity equation—he, as a Maxwellian, insists on treating "charge" as a discontinuity in displacement, not in the $\vec{E}$ field, for discontinuities in the latter ($\vec{\nabla} \cdot \vec{E}$) may occur without discontinuities in $\vec{D}$ at dielectric interfaces. Here we have a compelling instance of the unbridgeable gulf between the Maxwellian and Continental views: in Maxwellian theory "free" electricity necessarily denotes only one kind of charge, and it is an epiphenomenon of displacement discontinuity; in Helmholtzian theory a distinction is drawn between conduction and polarization charge, with "free" charge being their sum. This distinction is entirely foreign to Maxwellian theory, in which, at most, what Helmholtz meant by "free" charge would be termed "apparent" charge and have no basic physical significance since it is due only to discontinuities in intensity, not displacement. Whereas, for example, a Helmholtzian would analyze the force between the plates of a capacitor by summing the forces due to conduction and polarization charges, a Maxwellian would calculate the intensity engendered in the dielectric by the bounding displacement discontinuities and then compute the energy stored in the system as $(1/2)\int(\vec{E} \cdot \vec{D})d^3x$. J. J. Thomson had an inkling of this basic difference in outlook, for he concluded with the remark:

> We have seen that we can make certain equations which occur in Helmholtz's theory coincide with the corresponding ones in Maxwell's by giving particular values to certain constants. The difference in Helmholtz's and Maxwell's views as to the continuity of the currents is too serious to let us expect that we should ever get a complete agreement between their theories; and, in fact, make as many assumptions about the constants as we may, there are still differences between the theories. (Thomson 1885*a*, 138–39)

Just the year before R. T. Glazebrook (1884) had also compared the two theories (referring to Helmholtz's as that of "Helmholtz and Lorentz" because of Lorentz's use of the theory in 1875). His analysis is very like Thomson's, including its interpretation of "free" charge density as $\vec{\nabla} \cdot \vec{D}$. Glazebrook also points out that even if we replace Helmholtz's $\partial\vec{P}/\partial t$ in the Ampère law with Maxwell's $\partial\vec{D}/\partial t$, we still do not reach Maxwell's Ampère law because of the term $-A\partial/\partial t\vec{\nabla}\phi_f$ in the Helmholtz equation (21′). In an inhomogeneous body this term cannot vanish. However, in a homogeneous dielectric ϕ_f not only becomes ϕ_p (since there are no conduction currents) but also $\vec{\nabla}\phi_p$ will vanish since here $\vec{\nabla} \cdot \vec{P}$ is zero. So to reach the Maxwell theory from Helmholtz's, Glazebrook sees it necessary to replace Helmholtz's polarization current with Maxwell's displacement current and to assume homogeneity. This procedure violates the physical basis of the Helmholtz theory, in which there is no reason to consider $\partial\vec{E}/\partial t$ as well as $\partial\vec{P}/\partial t$ to be a part of the current because $\partial\vec{E}/\partial t$ derives in part from electrodynamic induction changes, and these changes alter forces but do not in themselves constitute currents.

In the end the Maxwell and Helmholtz theories are incommensurable; there is no way to pass between the two without altering the meaning of the word "charge"in addition to choosing limiting values of the Helmholtzian constants. Indeed, the very passage to the limit in the Helmholtz theory itself makes it extremely difficult to grant any meaning at all to the word "charge" because it obliterates the distinction between displacement and polarization. That is, whereas passage to the limit obliterates the dual aspect of charge insofar as the forces are concerned (since the bounding polarization charges now fully neutralize the conduction charges), it does not in itself provide a replacement for this duality because the concept of infinite polarizability lacks physical significance. This is the main problem which the Continentals encountered in trying to understand Maxwell's theory as a limit of Helmholtz's: in the limit the basic physical image of polarization as delimited charge shift, which underlies Helmholtz's theory, becomes deeply confused. This is the sort of thing one expects to happen when theories treat the same phenomena but are, in their deepest concepts, built upon radically different foundations. Equations which look similar in the two theories lead physicists to enforce comparisons in which the significance of the variables is lost, with the result that a sense of profound confusion necessarily occurs. The Maxwellians were somewhat better off here than their Continental colleagues because they at least were aware of the old Poisson-Mossotti charge theory of the dielectric upon which Helmholtz built, whereas the Continentals had only the Maxwellian articles and *Treatise* to read. But even the Maxwellians found it difficult, and perhaps impossible, to understand Helmholtz on his own terms because by the early 1880s the Maxwellian concepts of charge and current had, in Britain, thoroughly replaced the old ideas. Let us now turn to several significant Continental attempts, each based on a direct reading of Maxwell's *Treatise,* to explicate the Maxwellian concept of "charge."

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Appendix 10
Miscellaneous Derivations

The Helmholtz Polarization Theory

I shall derive equations (6)–(8) of part IV, chapter 21.

(21.6) $$\Psi(\vec{x}) = (1/4\pi)\int[\partial/\partial t\phi_f(\vec{x}')]\nabla^2_{x'}|\vec{x} - \vec{x}'|d^3x'$$

(21.7) $$\nabla^2\vec{U} = (1 - k)\vec{\nabla}\partial\phi_f/\partial t - 4\pi\vec{J}$$

(21.8) $$\vec{\nabla}\cdot\vec{U} = -k\partial\phi_f/\partial t$$

We are given:

(i) $$\Psi(\vec{x}) = \int\vec{J}(\vec{x}')\cdot\vec{\nabla}_{x'}|\vec{x} - \vec{x}'|d^3x'$$

(ii) $$\vec{U} = \int\vec{J}(\vec{x}')/|\vec{x} - \vec{x}'|d^3x' + (1/2)(1 - k)\vec{\nabla}\Psi$$

(iii) $$\partial/\partial t\nabla^2\phi_f = 4\pi\vec{\nabla}\cdot\vec{J}$$

For equation (21.6) we have from (i) and (iii) and two partial integrations in which we discard the surface integrals under the assumptions that both $\vec{J}$ and $\partial\phi_f/\partial t$ vanish at infinity (I thank Olé Knudsen for the succinct derivation which follows):

$$\begin{aligned}\Psi(\vec{x}) &= \int\vec{J}(\vec{x}')\cdot\vec{\nabla}_{x'}|\vec{x} - \vec{x}'|d^3x'\\ &= -\int\vec{\nabla}_{x'}\cdot\vec{J}(\vec{x}')|\vec{x} - \vec{x}'|d^3x'\\ &= (1/4\pi)\int[\partial/\partial t\nabla^2_{x'}\phi_f(\vec{x}')]|\vec{x} - \vec{x}'|d^3x'\\ &= -(1/4\pi)\int[\partial/\partial t\phi_f(\vec{x}')]\nabla^2_{x'}|\vec{x} - \vec{x}'|d^3x'\end{aligned}$$

We also obtain from (21.6):

(21.6′) $$\Psi(\vec{x}) = (1/2\pi)\int[\partial/\partial t\phi_f(\vec{x}')]/|\vec{x} - \vec{x}'|d^3x'$$

So we see that $(1/2\pi)\partial\phi_f/\partial t$ acts as a source density for the scalar Ψ, whence we have at once:

(21.6″) $$\nabla^2\Psi = 2\partial\phi_f/\partial t$$

Equation (21.7) for $\nabla^2\vec{U}$ follows immediately from (21.6″) and the fact that $\vec{J}(\vec{x}')$ acts in (ii) as a vector potential. We obtain equation (21.8) for $\vec{\nabla}\cdot\vec{U}$ as follows, performing only one partial integration under the assumption that $\vec{J}$ vanishes at infinity:

$$\begin{aligned}\vec{\nabla}\cdot\vec{U} &= \int\vec{\nabla}_x\cdot[\vec{J}(\vec{x}')/|\vec{x} - \vec{x}'|]d^3x' + (1/2)(1 - k)\nabla^2\Psi\\ &= \int[(\vec{J}(\vec{x}')\cdot\vec{\nabla}_x)(1/|\vec{x} - \vec{x}'|)]d^3x' + (1 - k)\partial\phi_f/\partial t\\ &= -\int[\vec{\nabla}_{x'}\cdot\vec{J}(\vec{x}')][1/|\vec{x} - \vec{x}'|]d^3x' + (1 - k)\,\partial\phi_f/\partial t\\ &= -(1/4\pi)\int[\partial/\partial t\nabla^2_{x'}\phi_f(\vec{x}')]/|\vec{x} - x'|]d^3x' + (1 - k)\partial\phi_f/\partial t\\ &= -k\partial\phi_f/\partial t\end{aligned}$$

Sissingh's Magneto-Optic Equations

Sissingh's goal was to use the observable angles to calculate the magneto-optic phase and amplitude. To see how this may be done we shall closely examine his deduction

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of the equations for observations in which the incident light is polarized in the plane of incidence (minimum) or is nearly so polarized (null). We shall assume the incident amplitude to be unity as a reference. The deduction yields equations which can be applied to all cases, so we shall use the symbol β for the angles and specialize later.

Begin by rotating the polarizer away from the plane of incidence through a small angle β_L^P, where β may be Γ (minimum) or γ (null), so that very nearly:

$$\sin\beta_L^P = \beta_L^P$$
$$\cos\beta_L^P = 1$$

Then the incident light consists of two components:
A_L^I = component amplitude parallel to plane of incidence = 1.
A_V^L = component amplitude normal to plane of incidence = β_L^P.
In reflection one has in general to consider four independent components if one takes it as given by previous experiments that the magneto-optic component is always perpendicular to the direction of incident polarization. They are: (1) the usual metallic reflection of A_L^I, (2) the usual metallic reflection of A_V^I, (3) the magneto-optic component in the plane of incidence and due to A_V^I, and (4) the magneto-optic component normal to the plane of incidence and due to A_L^I. Since, however, the magneto-optic amplitude depends on the incident amplitude, and since β_L^P is small, as is the effect itself, we may, with Sissingh, ignore the third component of the reflection. Consequently, if we take the phase of the usual metallic reflection component (1) as a reference, and denote its amplitude by A_L^R, the reflection consists of the following components:

In the plane of incidence.
amplitude A_L^R; reference phase.

Normal to the plane of incidence.
metallic: amplitude $A_V^R\beta_L^P$; phase ϕ_V^{met} relative to reference phase.
magneto-optic:
amplitude A_L^{mo}; phase ϕ_L^{mo} relative to reference phase.

(Here A_V^R is the amplitude of the usual metallic reflection normal to the plane of incidence.) All of A_V^R, A_L^R, ϕ_V^{met}, and the reference phase are determined by the usual metallic equations.

Suppose the analyzer is set at the small angle β_L^A with respect to the normal to the plane of incidence. Then, by projecting the three components of the reflection onto the plane of the analyzer, we see (fig. 39) that through it will pass the following three waves. (Here we define the signs of β_L^A and β_L^P as positive for clockwise rotations away from the plane of incidence; note that the subscript V on the analyzer angle and on the magneto-optic quantities always means that they derive from incident polarization parallel, or nearly so, to the plane of incidence.)

(i) $A_L^R\cos(\pi/2 + \phi_L^A) = -A_L^R\beta_L^A$; reference phase
(ii) $A_V^R\beta_L^P\cos\phi_L^A = A_V^R\beta_L^P$; relative phase ϕ_V^{met}
(iii) $A_L^{\text{mo}}\cos\beta_L^A = A_L^{\text{mo}}$; relative phase ϕ_L^{mo}

The sum of equations (i)–(iii), incorporating phases, is the optical vector R_L of the analyzed reflection for a wave with angular frequency ω:

$$R_L = -A_L^R\beta_L^A\cos(\omega t) + A_V^R\cos(\omega t - \phi_V^{\text{met}}) + A_L^{\text{mo}}\cos(\omega t - \phi_L^{\text{mo}})$$

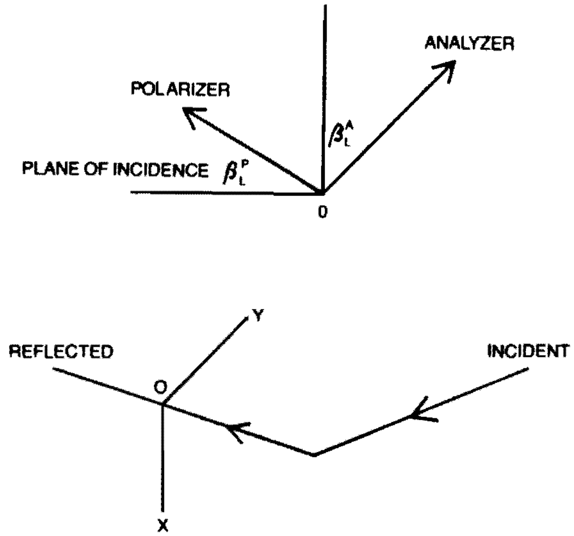

FIG. 39 Analyzer and polarizer positions in the Sissingh experiments

Whence, splitting R_L into two terms in $\cos(\omega t)$ and $\sin(\omega t)$, we find for the squared intensity, I_L^2, in the analyzer:

$$I_L^2 = a^2 + b^2 \text{ where}$$

$$a = -A_L^R\beta_L^A + A_V^R\beta_L^P\cos\phi_V^{met} + A_L^{mo}\cos\phi_L^{mo}$$

(1) $$b = A_V^R\beta_L^P\sin\phi_V^{met} + A_L^{mo}\sin\phi_L^{mo}$$

We can do precisely the same calculation for the squared intensity, I_V^2, for an incident wave in which the polarizer is set at an angle β_V^P near the normal to the plane of incidence:

$$I_V^2 = a'^2 + b'^2$$

$$a' = -A_L^R\beta_V^P + A_V^{mo}\cos\phi_V^{mo} + A_V^R\beta_V^A\cos\phi_V^{met}$$

(2) $$b' = A_V^{mo}\sin\phi_V^{mo} + A_V^R\beta_V^A\sin\phi_V^{met}$$

Here the magneto-optic component is parallel to the plane of incidence, and the analyzer is nearly parallel to it.

Sissingh could now find expressions for the phase and amplitude in both minimum and null experiments. In minimum experiments I_L or I_V are to be minimized as functions of either $\beta_{L,V}^P$ or $\beta_{L,V}^A$, depending on whether one first fixes, respectively, the analyzer or the polarizer and then rotates the other. Setting the variations $\delta I_{L,V}$ to zero, from equations (1) and (2) we have:

I_L^{min}

fixed polariser:

(3) $$-A_L^R\beta_L^A + A_V^R\beta_L^{FP}\cos\phi_V^{MET} + A_L^{MO}\cos\phi_L^{MO} = 0$$

fixed analyser:

(4) $$-A_L^R\beta_L^{FA}\cos\phi_V^{MET} + A_V^R\beta_L^P + A_L^{MO}\cos(\phi_V^{MET} - \phi_L^{MO}) = 0$$

I_V^{min}

fixed [illegible]

(5) $A_V^R\beta_V^A$ [illegible]

fixed [illegible]

(6) $-A_L\beta$ [illegible]

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I_V^{min}

fixed polariser:

(5) $A_V^R\beta_V^A - A_L^R\beta_V^{FP}\cos\phi_V^{MET} + A_V^{MO}\cos(\phi_V^{MET} - \phi_V^{MO}) = 0$

fixed analyser:

(6) $-A_L\beta_V^P + A_V^R\beta_V^{FA}\cos\phi_V^{MET} + A_V^{MO}\cos\phi_V^{MO} = 0$

Examining equations (3)–(6) we see that, if one assumes with Sissingh on purely empirical grounds that reversal of the magnetic field alters only the sign of the magneto-optic amplitude but has no phase effect, then, in general, the minimum positions of the polarizer and analyzer are not symmetrical about the principal planes (the plane of incidence and its normal plane). In order for symmetry to obtain, the fixed angles must accurately vanish. That is, the variable rotations can simply change sign on field reversal only if the fixed settings (denoted by superscript F in equations [3]–[6]) are accurately in or normal to the plane of incidence. This would make precise observation difficult because it is hard to set the positions exactly. However, we also see that, if field reversal merely alters the amplitude's sign, then, if we calculate the difference between, for example, $\beta_{L,V}^A$ for one field direction, and $\beta_{L,V}^A$ for the reversed direction, from equation (3), then the term in $\beta_{L,V}^{FP}$ is removed. This holds as well for equation (5), and, similarly, the term in $\beta_{L,V}^{FA}$ drops out of equations (4) and (6). Consequently, we do not need to set the fixed positions accurately because we can simply measure the angles between the variable positions which arise from field reversal. This is what Sissingh did; we shall, therefore, now redefine our previous angles Γ to denote the angular separation or double rotation obtained on field reversal. With this understanding, equations (3)–(6) may be solved for the amplitudes and phases; the results are equations (1)–(4) of part V, chapter 25.

The Goldhammer Theory

Recall that the final Goldhammer wave equation reads

(7) $$\partial^2\vec{H}/\partial t^2 = R^2e^{-2i\alpha}\nabla^2\vec{H} + (\vec{\lambda}' \times \partial\vec{H}/\partial t)$$

Here the vector $\vec{\lambda}'$ is intrinsically complex. But recall also that Goldhammer's theory in effect begins with equation (8):

(8) $$\partial^2\vec{A}'/\partial t^2 + \partial/\partial t\vec{\nabla}[\phi + \partial\omega/\partial t] = -\nabla^2[\vec{A}'/(\epsilon + ip/\omega) - \vec{\lambda}' \times \partial\vec{A}'/\partial t]$$

Now in order to reach equation (7) (which also holds for the vectors $\vec{A}$ and $\vec{A}'$ since $\vec{H}$ is $\vec{\nabla} \times \vec{A}$, which is equal to $\vec{\nabla} \times \vec{A}'$), Goldhammer had to remove the terms in ϕ and ω. To do so he first split ϕ into the sum $\phi' + \phi''$ and imposed three conditions:

(9) $$\nabla^2\phi' = 0$$

(10) $$\phi' + \partial\omega/\partial t = 0$$

(11) $$\phi'' = \vec{\lambda}' \cdot (\vec{\nabla} \times \vec{A}')$$

This involved Goldhammer's theory in great complications, including several mathematically doubtful steps. To see this, we shall take a simple example which derives from equation (15) of section 26.2 for the continuity of $\vec{A}$. Since the optical vector is $\vec{A}'$, or $\vec{A} - \vec{\nabla}\omega$, this continuity equation implicates ω, which satisfies the

Laplace equation. As a result Goldhammer had to introduce a complex exponential expression for ω as well as for $\vec{A}'$ in order to satisfy phase continuity.

Suppose, for example, that the plane of separation is $x = 0$. Let $z = 0$ be the plane of incidence. Then we know from equation (7) that there will be one reflected and two refracted waves:

$$\vec{A}'_I = (\vec{A}'^x_I, A'^y_I, A'^z_I)e^{i(xk^x_I + yk^y_I - at)}$$
$$\vec{A}'_R = (A'^x_R, A'^y_R, A'^z_R)e^{i(-xk^x_I + yk^y_I - at)}$$
$$\vec{A}'_R = \begin{cases} (A'^x_{R1}, A'^y_{R1}, A'^z_{R1})e^{i(xk^x_{R1} + yk^y_{R2} - at)} \\ + (A'^x_{R2}, , A'^y_{R2}, A'^z_{R2})e^{i(xk^x_{R2} + yk^y_{R2} - at)} \end{cases}$$

(12)

Moreover, as a result of absorption $\vec{k}_{R1}$ and $\vec{k}_{R2}$ are both complex. Introduce the real angle of incidence θ_I, equal to the angle of reflection, and the complex angles of refraction θ^1_R, θ^2_R, as well as the wavelengths λ_I, $\lambda^{1,2}_R$ in the media of incidence and refraction:

$$\vec{k}_I = (2\pi/\lambda_I)(\cos\theta_I, \sin\theta_I, 0)$$
$$\vec{k}^{1,2}_R = (2\pi/\lambda^{1,2}_R)(\cos\theta_R, \sin\theta_R, 0) \qquad (13)$$

This is, so far, standard in magneto-optics, but Goldhammer also had to introduce an expression for the scalar ω, which is not propagated since $\nabla^2\omega$ vanishes. To do so Goldhammer split ω into "reflected" (ω_R) and "refracted" ($\omega^{1,2}_R$) parts and assumed, despite the Laplace equation, that they can be expressed exponentially with corresponding "wavelengths" l_I, $l^{1,2}_I$:

$$\omega_R = iD_R e^{i\eta_R}$$
$$\omega^{1,2}_R = iD^{1,2}_R e^{i\eta^{1,2}_R} \qquad (14)$$

Here the D amplitudes are real and for the η we have:

$$\eta_R = 2\pi(-x\cos\Psi_I + y\sin\Psi_I)/l_I - at$$
$$\eta^{1,2}_R = 2\pi(x\cos\Psi^{1,2}_R + y\sin\Psi^{1,2}_R)/l^{1,2}_R - at \qquad (15)$$

However, since ω is not propagated, Goldhammer, in a rather doubtful step, required that the "wavelengths" l must be infinite:

(16) l_I and $l^{1,2}_R$ are both infinite

As a result of phase continuity at the boundary $x = 0$, Goldhammer obtained from equations (12)–(15):

$$\sin\theta^1_R/\lambda^1_R = \sin\theta^2_R/\lambda^2_R = \sin\theta_I/\lambda_I$$
$$= \sin\omega_I/l_I = \sin\Psi^1_R/l^1_R = \sin\Psi^2_R/l^2_R \qquad (17)$$

Equations (16) and (17) then provide relations between the θ and Ψ angles. (Goldhammer gave only the results—equations [22] and [23].) For example, since l_I is infinite we might argue as follows:

(18) $1/l_I\cos^2\Psi_I = 0$ (from [16])

(19) $(1/l^2_I)[1/\cos\Psi^2_I - 1] = -1/l^2_I$ (from [18])

(20) $(1/l_I)\sqrt{[1/\cos\Psi^2_I - 1]} = i/l_I$ (from [19])

(21) $\sin\Psi_I/l_I = i\cos\Psi_I/l_I$ (from [20])

Then equations (17)

(22)

Similarly we find:

(23)

Equations (22) and (
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(12)–(15) and (22)–(

(24) A'^x_I -

This was Goldhamm
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Then equations (17) and (21) together yield:

$$(22) \qquad \cos\Psi_I/l_I = -i\sin\theta_I/\lambda_I$$

Similarly we find:

$$(23) \qquad \cos\Psi_R^1/l_R^1 = \cos\Psi_R^2/l_R^2 = -i\sin\theta_I/\lambda_I$$

Equations (22) and (23) allowed Goldhammer to remove the Ψ, l variables from the boundary conditions. For example, continuity of A_x requires, by virtue of equations (12)–(15) and (22)–(23):

$$(24) \qquad A_I'^x + A_R'^x = A_{R1}'^x + A_{R2}'^x + 2\pi i\sin\theta_I(D_R + D_R^1 + D_R^2)/\lambda_I$$

This was Goldhammer's result.

We see that, although we can eliminate the peculiar angles and wavelengths Ψ, l from the boundary conditions, we still have the amplitudes D and the variables ϕ', ϕ''. To remove these Goldhammer turned to equations (9)–(11) and used the continuity of ϕ, assuming that $\vec{\lambda}'$ vanishes in the medium of incidence and that ϕ_I is zero. In this way he obtained:

$$(25) \qquad iD_R\partial/\partial te^{i\eta_R} = iD_R^1e^{i\eta_R^1} + iD_R^2e^{i\eta_R^2} - \vec{\lambda}'\cdot(\vec{\nabla}\times\vec{A}_{R1}') - \vec{\lambda}'\cdot(\vec{\nabla}\times\vec{A}_{R2}')$$

Since we can use equations (22) and (23) to eliminate the η phases, equation (25), together with the remaining boundary conditions, including continuity of $\vec{\nabla}A_{x,y,z}$, ultimately permits elimination also of the D amplitudes: in the end Goldhammer had fifteen amplitudes ($\vec{A}_I'$, $\vec{A}_R'$, $\vec{A}_{R1,2}'$, D_R, $D_R^{1,2}$) and twelve equations between them, which at once permit elimination of the D terms and expression of the remaining nine reflected and refracted components in terms of $\vec{A}_I'$, θ_I, $\vec{\lambda}'$, θ_R, and the complex Snel's law if approximations are introduced which permit the expression of $\vec{k}_{R1,2}$ in terms of the $\vec{k}_R$ which obtains when the magneto-optic vector vanishes and which corresponds to the usual complex angle of refraction θ_R. (See appendix 9 for an example of how the latter type of approximation works.)