3. Mathematical symmetry versus intuitive notion of symmetry, illustrated by motion of planets

3.1. Symmetry in planets motion: Intuitive perception

The apparent motions of the planets appear to be irregular and complicated. However, it was obvious in the remote past that the heavens ought to exemplify mathematical beauty. This would only be the case if the planets moved in circles. Indeed, in Greek science one can find a hypothesis that all the planets, including the Earth, go round the sun in circles. In the Middle Ages, a popular belief was that the Universe was made up of a number of spheres contained within one another, with the Earth at the center. The spheres were said to make musical sounds as they moved. These sounds were in harmony and were known as the *music of the spheres.*

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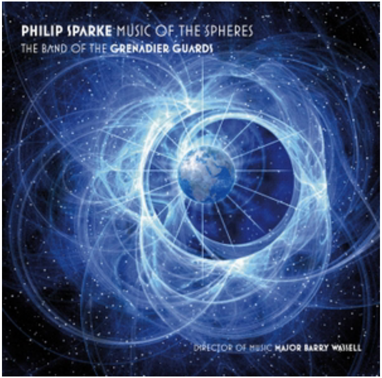

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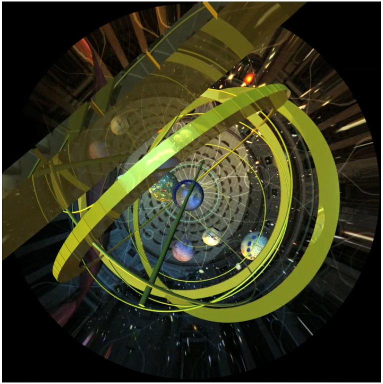

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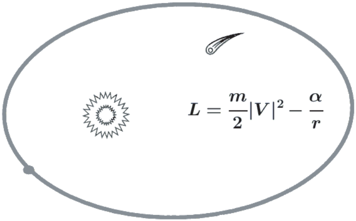
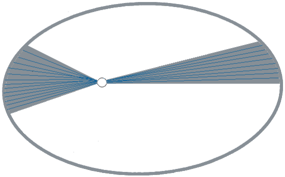
3.2. Discussion of Keplers laws

Johann Kepler (1571-1630) discovered, however, that **planets move in ellipses, not in circles, with the sun at a focus, not at the center.** He published his result in 1609. Since then this statement is known as Keplers first law.



Keplers second law states that

the areas swept out in equal times by the line joining the sun to a planet are equal.



Keplers third law, published in 1619, compares orbits of different planets. It asserts that the ratio T^2/R^3 of the square of the period T and the cube of the mean distance R from the sun is the same for all planets.

Keplers laws reduce the motion of planets to geometry and reveal, at a new level, a mathematical harmony in nature.

From practical point of view, it was important that Kepler gave an answer, based on empirical astronomy, to the question of **how the planets move.** The geometry of the heavens provided by Kepler's laws challenged scientists to answer the question of **why** the planets obey these laws. The question required an investigation of the dynamics of the Solar system. The necessary dynamics had been initiated by Galileo Galilei (1564-1642), astronomer and experimental philosopher. Galilei is universally recognized as the founder of modern science.
The mathematical model describing of the dynamics of the Solar system was developed by Isaac Newton (1642-1727) and published in his **Mathematical Principles of Natural Philosophy** in 1687.

3.3. Mathematical model of motion of planets

According to Newton's gravitation law, the force of attraction between the sun and a planet has the form

$$\mathbf{F} = \frac{\alpha}{r^3}\,\mathbf{x}, \quad \alpha = -gmM,$$

where g is the universal constant of gravitation, m and M are the masses of a planet and the sun, respectively, $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3)$ is the position vector of the planet considered as a particle, and

$$r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \tag{1}$$

is the distance of the planet from the sun. Hence, ignoring the motion of the sun under a planet's attraction, Newton's second law gives the following mathematical model for motion of planets:

$$m\,\frac{d^2\mathbf{x}}{dt^2} = \frac{\alpha}{r^3}\,\mathbf{x}\,, \quad \alpha = \text{const.} \tag{2}$$

Vector valued equation (2) is written in coordinate form as the following complicated **simultaneous** system of three ordinary differential equations:

$$
\begin{aligned}
m\,\ddot{x}^1 &= \frac{\alpha\,x^1}{[(x^1)^2+(x^2)^2+(x^3)^2]^{3/2}}\,,\\
m\,\ddot{x}^2 &= \frac{\alpha\,x^2}{[(x^1)^2+(x^2)^2+(x^3)^2]^{3/2}}\,,\\
m\,\ddot{x}^3 &= \frac{\alpha\,x^3}{[(x^1)^2+(x^2)^2+(x^3)^2]^{3/2}}\,,
\end{aligned}
\tag{3}
$$

where the dot denotes differentiation with respect to time t. Kepler's laws can be obtained by integrating the system of nonlinear equations (3). It can be shown however that the Kepler's laws are direct consequences of specific symmetries of Newton's gravitation force. This simple approach is provided by conservation laws.

3.4. Conservation laws responsible for 1st and 2nd Keplers laws

The second Kepler's law can be derived, without integrating the nonlinear equations (2), from conservation of angular momentum vector

$$\mathbf{M} = m(\mathbf{x} \times \mathbf{v}), \tag{4}$$

where the vector $\mathbf{v} = \dot{\mathbf{x}}$ is the velocity.
Furthermore, it was shown by P.S. Laplace in 1798 that the first Kepler's law, i.e. that the orbit of a planet is an ellipse, is a consequence of conservation of the vector

$$\mathbf{A} = [\mathbf{v} \times \mathbf{M}] + \frac{\alpha}{r}\,\mathbf{x}\,. \tag{5}$$

The vector (5) is known today as the **Laplace vector**.

3.5. Symmetries of Newton's gravitation law responsible for Keplers laws

All three Kepler's laws appear due to specific symmetries of Newton's gravitation law, namely invariance of the model equation (2) under certain transformation groups. Groups responsible for second and first Kepler's laws will be written as infinitesimal transformations $\bar{\mathbf{x}} = \mathbf{x} + \delta\mathbf{x}$ with small increment $\delta\mathbf{x}$.

Second Kepler's law: It was known since times of Euler that conservation of angular momentum $\mathbf{M}$ is due to the spherical symmetry of Newton's gravitation law, i.e. invariance of Eq. (2) under three infinitesimal rotations with increment

$$\delta\mathbf{x} = \mathbf{x} \times \mathbf{a},$$

where $\mathbf{a} = (a^1, a^2, a^3)$ is a small vector-parameter.

First Kepler's law: I have shown in 1983 that conservation of the Laplace vector $\mathbf{A}$ can be obtained as a consequence of a special symmetry of Newton's gravitation law under the transformations with increment

$$\delta \mathbf{x} = \mathbf{x} \times (\mathbf{v} \times \mathbf{a}) + (\mathbf{x} \times \mathbf{v}) \times \mathbf{a}.$$

Kepler's third law is valid due to invariance of Eq. (2) under scaling transformation

$$\bar{t} = a^3 t, \quad \bar{\mathbf{x}} = a^2 \mathbf{x}.$$

Compare these mathematical symmetries with intuitive perception of symmetry of planets motion discussed in Section 3.1.

Thank you!