6. Solved problems

Problem 6.1.

Integrate the following equation from Part 1:

$$\dot{P} = \alpha P, \quad \alpha = \text{const.} > 0. \tag{1}$$

Obtain the solution

$$P(t) = P_0 \mathrm{e}^{\alpha(t-t_0)}, \tag{2}$$

using initial condition $P(t_0) = P_0$. Verify by differentiation that function (2) solves Eq. (1).

Solution. We write Eq. (1) in the form

$$\frac{dP}{dt} = \alpha P,$$

separate the variables:

$$\frac{dP}{P} = \alpha dt,$$

integrate and obtain $\ln P = \alpha\, t + \ln C$, where constant of integration is written as $\ln C$ in order to write general solution in compact form

Hence $C = P_0 \mathrm{e}^{-\alpha t_0}$. Substituting it in general solution we obtain (2):

$$P(t) = P_0 \mathrm{e}^{\alpha(t - t_0)}.$$

Verification of the solution. Differentiation of the above function gives

$$\frac{dP(t)}{dt} = \alpha\, P_0 \mathrm{e}^{\alpha(t - t_0)} = \alpha P(t).$$

Hence $P(t)$ satisfies Eq. (1). Initial condition is also satisfied:

$$P(t_0) = P_0 \mathrm{e}^{\alpha(t_0 - t_0)} = P_0 \mathrm{e}^0 = P_0.$$

Problem 6.2.

(i). Integrate the system of equations (3) for model of predator and prey:

$$\dot{x} = ay, \quad \dot{y} = -bx, \qquad a, b > 0. \tag{3}$$

(ii). Single out the solution satisfying initial conditions

$$x(t_0) = x_0, \; y(t_0) = y_0.$$

Solution. (i). Easy way to integrate our system is to reduce **two** first-order linear equations (3) to **one** second-order linear equation. We differentiate first equation of the system, $\ddot{x} = a\dot{y}$, substitute $\dot{y}$ from second equation and obtain $\ddot{x} = -abx$, i.e. equation for *harmonic oscillations:*

$$\ddot{x} + \alpha^2 x = 0, \quad \alpha = \sqrt{ab}. \tag{4}$$

We find particular solutions of Eq. (4) by *Euler's substitution* (5):

$$x = \mathrm{e}^{\lambda t}. \tag{5}$$

Substitution (5) reduces *differential equation* (4) to *quadratic equation*

$$\lambda^2 + \alpha^2 = 0. \tag{6}$$

This equation has two complex roots, $\lambda_1 = i\alpha$ and $\lambda_2 = -i\alpha$. Hence Eq. (5) provides two complex solutions to Eq. (4), $\tilde{x}_1 = \mathrm{e}^{i\alpha t},\ \tilde{x}_2 = \mathrm{e}^{-i\alpha t}$. Using Euler's formula (7) with $r = 0$,

$$\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta \tag{7}$$

we rewrite the solutions $\tilde{x}_1 = \mathrm{e}^{i\alpha t},\ \tilde{x}_2 = \mathrm{e}^{-i\alpha t}$ in the form

$$\tilde{x}_1 = \cos(\alpha t) + i\sin(\alpha t), \quad \tilde{x}_2 = \cos(\alpha t) - i\sin(\alpha t),$$

take their linear combinations $x_1 = (\tilde{x}_1 + \tilde{x}_2)/2$ and $x_2 = (\tilde{x}_1 - \tilde{x}_2)/(2i)$ and obtain real solutions $x_1 = \cos(\alpha t),\ x_2 = \sin(\alpha t)$. Their linear combination provides the general solution to Eq. (4):

$$x = C_1\cos(\alpha t) + C_2\sin(\alpha t). \tag{8}$$

We differentiate x given by Eq. (8), use first equation of system (3), $\dot{x} = ay$, and obtain

$$y = \frac{1}{a}\,\dot{x} = \frac{\alpha}{a}\left[C_2 \cos(\alpha t) - C_1 \sin(\alpha t)\right].$$

Since $\alpha = \sqrt{ab}$, the expression for y is written

$$y = \sqrt{b/a}\left[C_2 \cos(\alpha t) - C_1 \sin(\alpha t)\right]. \tag{9}$$

Eqs. (8) - (9) give the general solution (10) to Eqs. (3):

$$\begin{aligned} x &= C_1 \cos(\alpha t) + C_2 \sin(\alpha t), & \alpha &= \sqrt{ab}, \\ y &= \beta\left[C_2 \cos(\alpha\, t) - C_1 \sin(\alpha\, t)\right], & \beta &= \sqrt{b/a}. \end{aligned} \tag{10}$$

(ii). Let us single out the solution satisfying initial conditions $x(t_0) = x_0, \; y(t_0) = y_0$. Taking $t = t_0$ in Eqs. (10) and solving resulting system of two linear equations for C_1, C_2, one obtains

$$C_1 = x_0 \cos(\alpha t_0) - \frac{y_0}{\beta} \sin(\alpha t_0),$$

$$C_2 = x_0 \sin(\alpha t_0) + \frac{y_0}{\beta} \cos(\alpha t_0).$$

Inserting these C_1, C_2 in Eqs. (10), one obtains the solution:

$$\begin{aligned} x(t) &= x_0 \cos[\alpha(t - t_0)] + \frac{y_0}{\beta} \sin[\alpha(t - t_0)], \\ y(t) &= y_0 \cos[\alpha(t - t_0)] - \beta x_0 \sin[\alpha(t - t_0)]. \end{aligned} \tag{11}$$

Problem 6.3.

Integrate the equation

$$m\ddot{x} + \gamma m\dot{x} + mg = 0, \quad \gamma = \text{const.} > 0, \tag{12}$$

modeling fall of a body in Earth'a atmosphere.

Solution. Since Eq. (12) contains the term with $\dot{x}$ and does not have the term with x, it has a solution linear in t, i.e. $x_* = k\,t$. Eq. (12) yields $k = -g/\gamma$. Hence $x_* = -(g/\gamma)\,t$. Let us solve homogeneous equation $m\ddot{x} + \gamma m\dot{x} = 0$. Euler's substitution (5) $x = \mathrm{e}^{\lambda t}$ reduces it to $\lambda(\lambda + \gamma) = 0$ whence $\lambda_1 = -\gamma, \lambda_2 = 0$. Whence two solutions

$$x_1 = \mathrm{e}^{-\gamma t}, \quad x_2 = 1$$

of homogeneous equation. Taking their linear combination in the form $C_2 x_2 - C_1 x_1$ and adding particular solution x_* of the nonhomogeneous equation we obtain general solution of Eq. (12) (see Eqs. (13)):

$$x = C_2 - \frac{g}{\gamma}\,t - C_1\,\mathrm{e}^{-\gamma t}. \tag{13}$$

Problem 6.4.

Verify that energy E given by the equation

$$E = \frac{1}{2} m v^2 + mgx \tag{14}$$

satisfies conservation equation

$$\left.\frac{dE}{dt}\right|_{(16)} = 0 \tag{15}$$

for the equation

$$m\ddot{x} + mg = 0 \tag{16}$$

of the free fall.

Solution. Differentiating E and invoking that $v = \dot{x}$, we obtain

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + mg\dot{x},$$

or

$$\frac{dE}{dt} = \dot{x}[m\ddot{x} + mg].$$

Problem 6.5.

Verify that the vector of angular momentum

$$\mathbf{M} = m(\mathbf{x} \times \mathbf{v}), \tag{17}$$

satisfies conservation equation

$$\left.\frac{d\mathbf{M}}{dt}\right|_{(18)} = 0$$

for mathematical model for motion of planets (18),

$$m\,\ddot{\mathbf{x}} = (\alpha/\mathbf{r}^3)\,\mathbf{x}\,.$$

Solution. Differentiation of $\mathbf{M} = m(\mathbf{x} \times \mathbf{v})$ gives

$$\dot{\mathbf{M}} = m(\dot{\mathbf{x}} \times \mathbf{v}) + m(\mathbf{x} \times \dot{\mathbf{v}}).$$

Since $\dot{\mathbf{x}} = \mathbf{v}$ and $\mathbf{v} \times \mathbf{v} = 0$ the above equation becomes

$$\dot{\mathbf{M}} = m(\mathbf{v} \times \dot{\mathbf{v}}).$$

Now we write $\dot{\mathbf{v}} = \ddot{\mathbf{x}}$ and substitute $m\ddot{\mathbf{x}} = \alpha \mathbf{x}/r^3$ due to the equation (Part 3)

$$m \frac{d^2\mathbf{x}}{dt^2} = \frac{\alpha}{r^3}\,\mathbf{x}\,, \quad \alpha = \mathrm{const.} \tag{18}$$

Hence

$$\dot{\mathbf{M}} = \frac{\alpha}{r^3}(\mathbf{x} \times \mathbf{x}).$$

Since $\mathbf{x} \times \mathbf{x} = 0$ we arrive at conservation equation

$$\dot{\mathbf{M}}\big|_{(18)} = 0.$$

Problem 6.6.

Show that Eq. (**??**),

$$y' = \frac{y}{x} + \frac{y^3}{x^3}\,.$$

admits the dilation group (**??**), $\overline{x} = x\mathrm{e}^{a}, \quad \overline{y} = y\mathrm{e}^{a}$.

Solution. Since in (**??**) the variables $\overline{x}$ and $\overline{y}$ have the same constant factor e^{a}, we have:

$$\frac{d\overline{y}}{d\overline{x}} = \frac{dy}{dx}\,, \quad \frac{\overline{y}}{\overline{x}} = \frac{y}{x}\,, \quad \frac{\overline{y}^3}{\overline{x}^3} = \frac{y^3}{x^3}\,.$$

Hence, Eq. (**??**) implies that variables $\overline{y}, \overline{x}$ satisfy the same equation (**??**):

$$\frac{d\overline{y}}{d\overline{x}} = \frac{\overline{y}}{\overline{x}} + \frac{\overline{y}^3}{\overline{x}^3}\,.$$

Thus, Eq. (**??**) admits the dilation group (**??**).

Thank you!