7. Exercises

Exercise 7.1.

(i) Solve the system of equations (2) for mutualism model,

$$\dot{x} = ay, \quad \dot{y} = bx, \quad a, b > 0,$$

and show that the general solution is given by the equations

$$\begin{aligned} x &= C_1 \mathrm{e}^{\alpha t} + C_2 \mathrm{e}^{-\alpha t}, & \alpha &= \sqrt{ab}, \\ y &= \beta[C_1 \mathrm{e}^{\alpha t} - C_2 \mathrm{e}^{-\alpha t}], & \beta &= \sqrt{b/a}. \end{aligned} \tag{1}$$

(ii) Verify by differentiation that functions (1) solve Eqs. (2).
(iii) Single out from (1) the solution satisfying the initial conditions $x(t_0) = x_0,\ y(t_0) = y_0$.

Exercise 7.2.

Integrate equation

$$m\ddot{x} + mg = 0, \tag{2}$$

for free fall of a body.

Exercise 7.3. (i) Verify that the function

$$x(t) = C_2 - \frac{g}{\gamma}\, t - C_1\, \mathrm{e}^{-\gamma t}$$

given in Part 1

$$x(t) = C_2 - \frac{g}{\gamma}\, t - C_1\, \mathrm{e}^{-\gamma t}, \quad v(t) = -\frac{g}{\gamma} + \gamma\, C_1\, \mathrm{e}^{-\gamma t}, \tag{3}$$

solves the equation

$$m\ddot{x} + \gamma m\dot{x} + mg = 0, \quad \gamma = \mathrm{const.} > 0. \tag{4}$$

for fall of a body in Earth's atmosphere.

(ii) Work out equations

$$E_\gamma = (g + \gamma v)\mathrm{e}^{-(\gamma/g)\,v-(\gamma^2/g)\,x}\,. \tag{5}$$

for the solution satisfying the initial conditions $x(t_0) = x_0, \quad \dot{x}(t_0) = v_0$.

Exercise 7.4. (See Sect. 1.7.2). Show that equations

$$C_1 + C_2 + C_3 = 0, \quad C_1 + C_2 - C_3 = 0$$

yield $C_1 + C_2 = 0,\ C_3 = 0$.

Exercise 7.5.

(See Sect. 1.7.3). Show that if $C_1(\mathrm{e}^{\alpha l} - \mathrm{e}^{-\alpha l}) = 0$ and $\alpha \neq 0, \; l \neq 0$ then $C_1 = 0$.

Exercise 7.6. Verify that the quantity E_γ given by Eq. (6),

$$E_\gamma = (g + \gamma v)\mathrm{e}^{-(\gamma/g)\, v - (\gamma^2/g)\, x}\,, \tag{6}$$

satisfies the conservation law for Eq. (4), $m\ddot{x} + \gamma m\dot{x} + mg = 0$, i.e.

$$\left.\frac{dE_\gamma}{dt}\right|_{(4)} = 0. \tag{7}$$

Exercise 7.7.

(See Sect. 3.4). Verify that the Laplace vector (8),

$$\mathbf{A} = [\mathbf{v} \times \mathbf{M}] + (\alpha/r)\,\mathbf{x}\,. \tag{8}$$

satisfies the conservation equation

$$\left.\frac{d\mathbf{A}}{dt}\right|_{(9)} = 0$$

for mathematical model for motion of planets

$$m\,\frac{d^2\mathbf{x}}{dt^2} = \frac{\alpha}{r^3}\,\mathbf{x}\,, \quad \alpha = \text{const.} \tag{9}$$

$m\,\ddot{\mathbf{x}} = (\alpha/r^3)\,\mathbf{x}\,.$

Exercise 7.8.

Verify that the substitution

$$y = x^{\lambda}, \quad \lambda = \text{const.} \tag{10}$$

reduces Euler's equation

$$x^2 y'' + (A+1)xy' + By = 0, \quad A, B = \text{const.} \tag{11}$$

into the quadratic equation

$$\lambda^2 + A\lambda + B = 0. \tag{12}$$

Exercise 7.9. Verify that the change of independent variable

$$t = \ln x \tag{13}$$

maps Euler's equation (11) to constants coefficient equation

$$y'' + A\, y' + B\, y = 0. \tag{14}$$

Exercise 7.10.

(See Sect. 4.7). Verify that the substitution (15),

$$y = \sqrt{1+x^2}\, \mathrm{e}^{\lambda \arctan x} \tag{15}$$

converts Eq. (16),

$$\left(1+x^2\right)^2 y'' + \left(1+x^2\right) A\, y' + (B - Ax - 1) y = 0 \tag{16}$$

to the quadratic equation (12) $\lambda^2 + A\lambda + B = 0$.

Exercise 7.11. Verify that the function

$$y = x\left[C_1 \cos\left(\frac{\omega}{x}\right) + C_2 \sin\left(\frac{\omega}{x}\right)\right]. \tag{17}$$

solves the equation

$$y'' + \frac{\omega^2}{x^4}\, y = 0. \tag{18}$$

Exercise 7.12. Show that the dilation generator

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

is written in the variables $t = \ln|x|,\ u = y/x$ in the form

$$X = \frac{\partial}{\partial t}\cdot$$

Exercise 7.13.

Reduce the equarion

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \tag{19}$$

to the integrable form

$$\frac{du}{dt} = f(u) - u$$

by introducing the canonical variables $t = \ln|x|,\ u = y/x$.

Exercise 7.14. Show that the dilation $\overline{x} = x\,\mathrm{e}^{a}, \quad \overline{y} = y\,\mathrm{e}^{-a}$ is admitted by the equation

$$y' + y^2 - \frac{2}{x^2} = 0. \tag{20}$$

Exercise 7.15. Show that the dilation

$$\overline{x} = x\,\mathrm{e}^{a}, \quad \overline{y} = y\,\mathrm{e}^{-a} \tag{21}$$

satisfies the group properties $T_0: \quad \varphi(x, y, 0) = x, \quad \psi(x, y, 0) = y,$

$$T_b T_a = T_{a+b}. \tag{22}$$

Exercise 7.16. Derive the generator

$$X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\,\cdot \tag{23}$$

for the dilation group

$$\overline{x} = x\,\mathrm{e}^{a}, \quad \overline{y} = y\,\mathrm{e}^{-a}. \tag{24}$$

Exercise 7.17. Derive canonical variables

$$t = \ln|x|, \quad u = xy \tag{25}$$

for generator

$$X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\,\cdot \tag{26}$$

Thank you!