1. Some mathematical models of real world phenomena given by ordinary differential equations

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Mathematical models give approximate description of real processes. Improvements of approximations or their extension to more general situations may increase complexity of mathematical models significantly.

Used Literature: [1] Ibragimov, N.H. A practical course in differential equations and mathematical modeling. Higher Education Press (China), 2009. Translated into Chinese in 2010.

Remark on terminology. The word **integration**, when it is applied to ordinary differential equations, will mean **computation of the general solution** of the equations in question.

1.1. Malthusian principle of population growth

Malthus, Thomas Robert (1766–1834), the pioneer in mathematical treatment of demographic problems. T.R. Malthus formulates his principle of population in Chapter 1 of his "Essay" as follows:

"Assuming then my postulata granted, I say, that the power of population is indefinitely greater than the power in the earth to produce subsistence for man. Population, when unchecked, increases in a geometrical ratio. Subsistence increases only in an arithmetical ratio. A slight acquaintance with numbers will show the immensity of the first power in comparison of the second."

Title page of revised edition of *"An essay on the principle of population as it affects the future improvement of society, with remarks on the speculations of Mr. Godwin, M. Condorcet, and other writers"* by Thomas Malthus. First published in 1798. Second revised ed., 1803.

AN ESSAY

PRINCIPLE OF POPULATION;

A VIEW OF ITS PAST AND PRESENT EFFECTS

HUMAN HAPPINESS;

THE REV. T. R. MALTHUS, A.M. F.R.S.

SIXTH EDITION

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Malthusian principle of population growth is formulated in modern textbooks as an assumption that the rate $\dot{P} = dP/dt$ of growth of population P is proportional to the population:

$$\dot{P} = \alpha P, \quad \alpha = \text{const.} > 0. \tag{1}$$

Integration of the model equation (1) gives Malthusian principle:

$$P(t) = P_0 \mathrm{e}^{\alpha(t - t_0)}, \tag{2}$$

where P_0 and $P(t)$ denote the population in millions at the initial time $t = t_0$ and at an arbitrary time t, respectively. See Presentation 6, Problem 6.1.

1.2. Predator and prey

The simplest model is given by the system of first-order equations

$$\dot{x} = ay, \quad \dot{y} = -bx, \quad a, b > 0. \tag{3}$$

Here x and y denote population densities of predator and prey species, respectively. General solution of system (3) is

$$\begin{aligned} x &= C_1 \cos(\alpha t) + C_2 \sin(\alpha t), \qquad & \alpha = \sqrt{ab}, \\ y &= \beta\,[C_2 \cos(\alpha\, t) - C_1 \sin(\alpha\, t)], \quad & \beta = \sqrt{b/a}, \end{aligned} \tag{4}$$

where C_1 and C_2 are arbitrary constants. The solution satisfying initial conditions $x(t_0) = x_0,\ y(t_0) = y_0$ has the form (see Presentation 6, Problem 6.2(ii))

$$\begin{aligned} x(t) &= x_0 \cos[\alpha(t - t_0)] + \frac{y_0}{\beta} \sin[\alpha(t - t_0)], \\ y(t) &= y_0 \cos[\alpha(t - t_0)] - \beta x_0 \sin[\alpha(t - t_0)]. \end{aligned} \tag{5}$$

1.3. Lotka-Volterra model

The model (3) is formulated by linear equations. Its value as a general "law" of population growth is extremely limited as far as complicated societies (like human population) is concerned. A more realistic model of predator and prey, known as the Lotka -Volterra model (A.J. Lotka, 1925, V. Volterra, 1926), is formulated by the following system of nonlinear ordinary differential equations of the first order:

$$\dot{x} = (a - by)x, \quad \dot{y} = (Ax - B)y, \tag{6}$$

where a, b, A, and B are positive constants. Qualitative analysis of solutions of the system (6) shows, e.g., that any biological system described by these equations ultimately approaches either a constant or periodic population (See Eqs. (4)).

1.4. Mutualism model, for example ants and aphids

An aphid is known also as a plant louse.

The model is obtained from (3) by changing sign in second equation:

$$\dot{x} = ay, \quad \dot{y} = bx, \quad a, b > 0. \tag{7}$$

The solution of the system (7) is given by

$$x = C_1\,\mathrm{e}^{\alpha t} + C_2\,\mathrm{e}^{-\alpha t}, \qquad \alpha = \sqrt{ab},$$
$$y = \beta[C_1\,\mathrm{e}^{\alpha t} - C_2\,\mathrm{e}^{-\alpha t}], \qquad \beta = \sqrt{b/a}. \tag{8}$$

1.5. Free fall of a body

Free fall means motion of a body in absence of forces other than gravity. Free fall of a body with constant mass m under the Earth's gravitation is governed by equation

$$m\ddot{x} + mg = 0, \tag{9}$$

where $x = x(t)$ is trajectory of the body, $\ddot{x}$ is the second derivative of $x(t)$ and g is the acceleration of gravity.
Integrating Eq. (9) twice, we obtain the trajectory and velocity:

$$x(t) = -\frac{g}{2}t^2 + C_1 t + C_2, \quad v(t) = -gt + C_1. \tag{10}$$

Letting $t = 0$ in Eqs. (10) one obtains that C_2 is the initial position $x_0 = x(0)$ of the body and C_1 is its initial velocity $v_0 = v(0)$.

Hence trajectory $x = x(t)$ and velocity $v = dx(t)/dt$ of the body with initial position x_0 and initial velocity v_0 are given by

$$x = x_0 + tv_0 - \frac{g}{2}t^2, \quad v = v_0 - gt. \tag{11}$$

Eliminating time t from Eqs. (10) one obtains energy

$$E = \frac{1}{2}\,mv^2 + mgx \tag{12}$$

which is constant on the trajectory. The latter statement means that the derivative of quantity (12) vanishes on solutions of Eq. (9). Hence the following **conservation law for Eq. (9)** is satisfied:

$$\left.\frac{dE}{dt}\right|_{(9)} = 0. \tag{13}$$

1.6. Fall of a body in the Earth's atmosphere

In practice one should take into account that the Earth's atmosphere is stratified. Assumption on free fall is acceptable when falling body is far from the Earth where the Earth's atmosphere is rarified and can be neglected. When a body enters into dense layers of atmosphere, the body warms up due to friction in atmosphere.

A reasonable modification of the equation (9) for free fall that allows to take into account friction in atmosphere with a good approximation, is given by the following equation with empirical constant γ :

$$m\ddot{x} + \gamma m\dot{x} + mg = 0, \quad \gamma = \text{const.} > 0. \tag{14}$$

Integration of Eq. (14) shows that Eqs. (10) are replaced by

$$x(t) = C_2 - \frac{g}{\gamma}\, t - C_1\, \mathrm{e}^{-\gamma t}, \quad v(t) = -\frac{g}{\gamma} + \gamma\, C_1\, \mathrm{e}^{-\gamma t}, \tag{15}$$

or in terms of initial position $x_0 = x(0)$ and initial velocity $v_0 = v(0)$:

$$x = x_0 + \frac{v_0}{\gamma} + \frac{g}{\gamma^2} - \frac{g}{\gamma} t - \left(\frac{v_0}{\gamma} + \frac{g}{\gamma^2}\right)\mathrm{e}^{-\gamma t}, \quad v = -\frac{g}{\gamma} + \left(v_0 + \frac{g}{\gamma}\right)\mathrm{e}^{-\gamma t}.$$

Energy (12) for free fall is replaced with **conserved** (See Presentation 7, Exercise 7.6) quantity

$$E_\gamma = (g + \gamma v)\mathrm{e}^{-(\gamma/g)\, v - (\gamma^2/g)\, x}. \tag{16}$$

Using the Taylor series expansion

$$\mathrm{e}^{-\gamma t} = 1 - \gamma t + \frac{1}{2}\,\gamma^2 t^2 - \frac{1}{6}\,\gamma^3 t^3 + \cdots$$

we obtain approximate expressions for trajectory and velocity:

$$\begin{aligned} x &\approx x_0 + t v_0 - \frac{1}{2}\,g t^2 + \frac{\gamma}{6}\left(g t^3 - 3 v_0 t^2\right) + o(\gamma)\,,\\ v &\approx v_0 - g t + \frac{\gamma}{2}\left(g t^2 - 2 v_0 t\right) + o(\gamma). \end{aligned} \tag{17}$$

In this approximation, conserved quantity (16) is written as follows:

$$E_\gamma \approx \frac{1}{2} m v^2 + m g x - \frac{\gamma m}{3g} v^3 + o(\gamma). \tag{18}$$

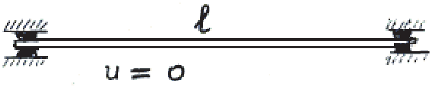
Comparison of Eq. (18) with Eq. (12) shows that loss of energy of a falling body due to friction in atmosphere is approximately equal to

$$\frac{\gamma m}{3g}\, v^3.$$

1.7. Collapse of driving shafts in motor ships: description of the problem

At beginning of constructing motor ships, constructors came across troublesome phenomenon of a seemingly accidental "beating" of shafts in power transmission systems that might lead to collapse of the shaft. The strange phenomenon was explained by considering mathematical model in terms of differential equation. Integration of the modeling equation gave a clear instruction on how to elude the "beating".

1.7.1. Formulation of modeling differential equation



Shaft of length l revolves at two bearings located at $x = 0$ and $x = l$. Denote by $u = u(x)$ shaft's displacement at point x from equilibrium position. Then $u|_{x=0} = u|_{x=l} = 0$. It is clear from physics that bearings are points of inflection of the function $u(x)$. Therefore second derivative of $u(x)$ vanishes at $x = 0$ and $x = l$. Hence

$$u\Big|_{x=0} = 0, \quad \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\Big|_{x=0} = 0, \quad u\Big|_{x=l} = 0, \quad \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\Big|_{x=l} = 0. \tag{19}$$

Mathematical model of transversal vibrations of slender rods shows that positions of equilibrium of rotating cylindrical shaft are described by forth-order ordinary differential equation (Book [1], Sec. 2.3.3)

$$\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} = \alpha^4 u \tag{20}$$

with positive constant coefficient

$$\alpha^4 = \frac{p\omega^2}{g\mu}, \tag{21}$$

where p is the weight of the shaft per unit length, g is the acceleration of gravity, the positive constant μ is connected with the material of the shaft and ω is a constant angular velocity of the rotating shaft.

The phenomenon of "beating" occurs when the problem defined by Eq. (20) and boundary conditions (19) has "nontrivial solution", i.e. solution $u = u(x)$ not vanishing identically in interval $0 \le x \le l$.

A beating shaft

1.7.2. Solution of modeling equation

It was shown by Leonard Euler in 1743 that particular solutions of equations with constant coefficients can be found by substitution

$$u = \mathrm{e}^{\lambda x}, \quad \lambda = \text{const.} \tag{22}$$

Euler's substitution (22) reduces differential equation (20) to the algebraic equation (called the *characteristic equation*)

$$\lambda^4 - \alpha^4 = 0. \tag{23}$$

Eq. (23) has two real roots $\lambda_1 = \alpha, \ \lambda_2 = -\alpha$ and two complex roots, $\lambda_3 = i\alpha, \ \lambda_4 = -i\alpha$. Hence Euler's substitution (22) provides two real solutions

$$u_1 = \mathrm{e}^{\alpha x}, \quad u_2 = \mathrm{e}^{-\alpha x}$$

and two complex solutions

$$\tilde{u}_3 = \mathrm{e}^{i\alpha x}, \quad \tilde{u}_4 = \mathrm{e}^{-i\alpha x}.$$

Using Euler's formula for the trigonometric representation of the complex exponent:

$$\mathrm{e}^{r+i\theta} = \mathrm{e}^{r}(\cos\theta + i\sin\theta) \tag{24}$$

we rewrite the solutions $\tilde{u}_3,\ \tilde{u}_4$ in the form

$$\tilde{u}_3 = \cos(\alpha x) + i\sin(\alpha x), \quad \tilde{u}_4 = \cos(\alpha x) - i\sin(\alpha x),$$

then take their linear combinations

$$u_3 = \frac{\tilde{u}_3 + \tilde{u}_4}{2}, \quad u_4 = \frac{\tilde{u}_3 - \tilde{u}_4}{2i}$$

and obtain real solutions

$$u_3 = \cos(\alpha x), \quad u_4 = \sin(\alpha x).$$

Altogether, we have now four linearly independent real solutions

$$u_1 = \mathrm{e}^{\alpha x}, \quad u_2 = \mathrm{e}^{-\alpha x}, \quad u_3 = \cos(\alpha x), \quad u_4 = \sin(\alpha x).$$

Their linear combination provides the general solution to Eq. (20):

$$u = C_1 \mathrm{e}^{\alpha x} + C_2 \mathrm{e}^{-\alpha x} + C_3 \cos(\alpha x) + C_4 \sin(\alpha x), \quad C_i = \text{const.} \tag{25}$$

Now we use boundary conditions (19) at $x = 0$.
Substitution of (25) in the first two equations from (19) gives

$$C_1 + C_2 + C_3 = 0, \quad C_1 + C_2 - C_3 = 0$$

whence $C_1 + C_2 = 0$, $C_3 = 0$. See Presentation 7, Exercise 7.4.
Now boundary conditions at $x = l$ give

$$C_1(\mathrm{e}^{\alpha l} - \mathrm{e}^{-\alpha l}) + C_4 \sin(\alpha l) = 0, \; C_1(\mathrm{e}^{\alpha l} - \mathrm{e}^{-\alpha l}) - C_4 \sin(\alpha l) = 0,$$

whence

$$C_1(\mathrm{e}^{\alpha l} - \mathrm{e}^{-\alpha l}) = 0, \quad C_4 \sin(\alpha l) = 0. \tag{26}$$

1.7.3. Solution of the problem of driving shafts

First equation in (26) gives $C_1 = 0$ (See Presentation 7, Exercise 7.5). Then $C_2 = 0$ due to equation $C_1 + C_2 = 0$. So we have obtained

$$C_1 = C_2 = C_3 = 0.$$

If $C_4 = 0$ then Eq. (25) gives $u = 0$, i.e. shaft is stable. Hence $C_4 \neq 0$ and second equation in (26) gives

$$\sin(\alpha l) = 0,$$

whence $\alpha l = n\pi$ with positive integers n. Thus

$$\alpha = \frac{n\pi}{l}, \quad n = 1, 2, 3, \dots . \tag{27}$$

Using definition (21) of α,

$$\alpha^4 = \frac{p\omega^2}{g\mu},$$

we write

$$\omega = \alpha^2 \sqrt{\frac{g\mu}{p}},$$

substitute here α given by Eq (27) for each $n = 1, 2, 3, \ldots$ and obtain

$$\omega_n = \frac{n^2\pi^2}{l^2}\sqrt{\frac{g\mu}{p}}, \quad n = 1, 2, 3, \ldots . \tag{28}$$

Conclusion: Collapse of the shaft is possible whenever its angular velocity approaches any one of the critical values (28).