Modeling by differential equations versus modeling by functions

Nail H. Ibragimov

Research Centre ALGA: Advances in Lie Group Analysis

Department of Mathematics and Natural Sciences
Blekinge Institute of Technology,
SE-371 79 Karlskrona, Sweden

www.bth.se/alga

March 11, 2016

2. Modeling by differential equations versus modeling by functions

2.1. Malthusian principle of population growth

Model by differential equation (1) and by function (2):

$$\frac{dP}{dt} = \alpha P, \tag{1}$$

$$P(t) = P_0 \mathrm{e}^{\alpha(t-t_0)}, \tag{2}$$

both are rather simple. Modeling by differential equation (1) is based on the natural assumption that **the rate of population growth is proportional to the population.** On the other hand, T.R. Malthus gave in his renowned book *An essay on the principle of population as it affects the future improvement of society, 1798*, an ingenious motivation of the unlimited growth of population expressed by the exponential function (2).

2.2. Predator and prey

Compare the mathematical model given by the system of first-order ordinary differential equations

$$\dot{x} = ay, \qquad \dot{y} = -bx, \qquad a, b > 0 \tag{3}$$

with modeling by functions:

$$\begin{aligned} x(t) &= x_0 \cos[\alpha(t - t_0)] + \frac{y_0}{\beta} \sin[\alpha(t - t_0)], \\ y(t) &= y_0 \cos[\alpha(t - t_0)] - \beta x_0 \sin[\alpha(t - t_0)]. \end{aligned} \tag{4}$$

Modeling by differential equations is given by differential equation

$$\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} = \alpha^4 u, \quad \alpha^4 = \frac{p\omega^2}{g\mu} \tag{5}$$

considered together with four boundary conditions:

$$u\big|_{x=0} = 0, \quad u\big|_{x=l} = 0, \quad \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\Big|_{x=0} = 0, \quad \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\Big|_{x=l} = 0. \tag{6}$$

Constant coefficient α^4 in Eq. (5) is expressed via angular velocity ω of rotating shaft, physical characteristics p and μ of the material of the shaft and acceleration gravity g (see Sect. 1.7.1).

Modeling by functions is given by

$$\omega_n = \frac{n^2\pi^2}{l^2}\sqrt{\frac{g\mu}{p}}\,, \quad n = 1, 2, 3, \ldots . \tag{7}$$

This form of modeling is simple for practical using. It shows that angular velocity of rotating shaft should not take values from the sequence

$$\omega_1 = \frac{\pi^2}{l^2}\sqrt{\frac{g\mu}{p}}\,, \quad \omega_2 = \frac{4\pi^2}{l^2}\sqrt{\frac{g\mu}{p}}\,, \quad \omega_3 = \frac{9\pi^2}{l^2}\sqrt{\frac{g\mu}{p}}\,, \ \ldots .$$

2.4. Falling body

Difference between two forms of modeling, (1) and (2), of Malthusian principle is not big. But situation is essentially different for falling body. In case of free fall (Part 1, Sect. 1.5), both ways of modeling are simple: given by function

$$x = x_0 + tv_0 - \frac{g}{2}t^2 \tag{8}$$

and by equation

$$m\ddot{x} + mg = 0. \tag{9}$$

In presence of Earth's atmosphere (Part 1, Sect. 1.6), modeling by function

$$x = x_0 + \frac{v_0}{\gamma} + \frac{g}{\gamma^2} - \frac{g}{\gamma}\,t - \left(\frac{v_0}{\gamma} + \frac{g}{\gamma^2}\right)\mathrm{e}^{-\gamma t}$$

is much more obscure than modeling by equation

$$m\ddot{x} + \gamma m\dot{x} + mg = 0. \tag{10}$$

Thank you!