should be an almost periodic function in the sense of Bohr, it is necessary and sufficient that

$$f(x) \equiv ic + g(x),$$

where c is a real constant and $\int^x g(x)\,dx$ *is bounded.*

5. We can complete our result by a statement about the moduli of the a.p. functions $f(x)$, $y(x)$. The modulus of $G(x)$ is equal to the modulus of $f(x)$, and therefore, by the second part of our lemma, the modulus of $y(x)$ is contained in any modulus containing both the number c and the modulus of $f(x)$. On the other hand, it was shown by Bohr (*loc. cit.*) that, for any a.p. functions $y(x)$, $G(x)$ which are in the relation (2), the number c and the modulus of $g(x)$ are both contained in the modulus of $y(x)$. Thus we have the

COMPLETION OF THE THEOREM. *The modulus of* $y(x)$ *is the smallest modulus containing the number c and the modulus of* $f(x)$.

THE CONSTRUCTION OF DECIMALS NORMAL IN THE SCALE OF TEN

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A decimal $\cdot S$ is said to be normal in the scale of ten if, when γ_ρ is an arbitrary sequence of an arbitrary number ρ of digits, and $G(x)$ denotes the number of times that γ_ρ occurs as ρ consecutive digits in the first x digits of S,

$$G(x) = 10^{-\rho}x + o(x)$$

as $x \to \infty$. Rules have been given for the construction of such decimals, but these have always been somewhat involved.

Actually, a very simple construction is adequate; we shall, in fact, show in the course of this paper that the decimal $\cdot 123456789101112\ldots$, composed of the natural sequence of numbers counting from 1 upwards, is itself normal in the scale of ten.

First, we shall prove

THEOREM I. *If* s_r *denotes the sequence*

$$\cdot 00..0, 00..1, 00..2, \ldots\ldots, 99..9,$$

* Received 19 April, 1933; read 27 April, 1933.

consisting of the 10^r *possible arrangements of r digits, ranked in ascending order of magnitude (so that, for example,* s_1 *denotes* 0,1,2,3,4,5,6,7,8,9, *and* s_2 *denotes* 00,01,02,...,98,99), *then the decimal*

$$\cdot S = \cdot s_1 s_2 s_3 \ldots s_r \ldots$$

is normal in the scale of ten.

We shall then deduce

THEOREM II. *If* s_r *is defined as above, and if* ${}_\mu s_r$ *denotes the sequence formed by repeating* s_r μ *times,* μ *being any fixed positive integer, then the decimal*

$$\cdot{}_\mu S = \cdot{}_\mu s_1 {}_\mu s_2 \ldots {}_\mu s_r \ldots$$

is normal in the scale of ten.

This will enable us to obtain the result quoted above, namely

THEOREM III. *The decimal* ·12345678910111213... *is normal in the scale of ten.*

In addition, we shall show how to prove from first principles the easier but less interesting

THEOREM IV. *If* ${}_\mu s_r$ *is defined as above, then the decimal*

$$\cdot{}_r S = \cdot{}_1 s_1 {}_2 s_2 \ldots {}_r s_r \ldots$$

is normal in the scale of ten.

Finally, we shall enunciate certain further theorems of this type, but since they represent somewhat trivial deductions from Theorem III, and need for their establishment tedious lemmas and an involved notation, no attempt at a proof will be advanced.

In the course of the work we shall use the following notation:

S_r and S will denote respectively the sequences $s_1 s_2 \ldots s_r$ and $s_1 s_2 \ldots s_r s_{r+1} \ldots$;

x_r and X_r will denote the number of digits in s_r and S_r;

g_r and G_r will denote the number of times that γ_ρ occurs in s_r and S_r respectively;

$g_r(x)$ and $G(x)$ will denote the number of times that γ_ρ occurs in the first x digits of s_r and of S respectively.

We shall prove Theorem I in the following manner.

(i) We shall estimate g_r and G_r and obtain

$$g_r = 10^{-\rho} x_r + o(x_r), \quad G_r = 10^{-\rho} X_r + o(X_r) \quad (\text{as } r \to \infty).$$

(ii) We shall estimate $g_r(x)$ and obtain

$$g_r(x) = 10^{-\rho} x + o(x_r)$$

as $r \to \infty$ and x varies in any manner consistent with the existence of $g_r(x)$.

(iii) We shall deduce an estimate for $G(x)$ to show that the decimal $\cdot S$ is normal in the scale of ten.

We have defined s_r to consist of 10^r consecutive members of r digits each; it will be convenient to place commas between consecutive members of $\cdot S$. Thus

$$s_1 = 0,1,2,3,4,5,6,7,8,9; \quad \cdot S = \cdot 0,1, \ldots,9,00,01, \ldots,$$

so that $\cdot S$ contains an endless succession of commas.

Consider an individual occurrence of γ_ρ in the sequence. If γ_ρ occurs with a comma between two of its digits, we shall say that γ_ρ occurs *divided*; if it occurs with no comma between any two of its digits, we shall say that γ_ρ occurs *undivided*. Thus 37 occurs in s_2, undivided at ...,36,37,38, ... and divided at ...,72,73,74, ...

Now (A) *if $r < \rho$, γ_ρ cannot occur undivided in s_r; but, if $r \geqslant \rho$, γ_ρ occurs undivided in s_r exactly $(r-\rho+1)\,10^{r-\rho}$ times.*

The assertion for $r < \rho$ is obvious. If $r \geqslant \rho$, there are $r-\rho+1$ positions which γ_ρ may occupy undivided within a member of s_r, since the first digit of γ_ρ may occur as any of the first $r-\rho+1$ digits of the member of s_r (thus 37.. .37. ..37). Having determined the position, we may choose the remaining $r-\rho$ digits of the member in $10^{r-\rho}$ different ways. Thus γ_ρ will occur in a given position in $10^{r-\rho}$ distinct members of s_r. Hence γ_ρ will occur undivided in s_r exactly $(r-\rho+1)\,10^{r-\rho}$ times.

A further numerical example may make this clearer: 37 occurs undivided in s_3 twenty times, viz. ten times in the first position ...,370,371, ...,378,379, ... and ten times in the second position

$$\ldots,037, \ldots,137, \ldots\ldots,837, \ldots,937, \ldots.$$

Further, s_r does not contain more than 10^r commas and γ_ρ cannot occur divided by any one given comma as many as ρ times. Hence (B) γ_ρ *cannot occur divided in s_r more than $\rho 10^r$ times.*

By (A) and (B),

$$g_r = (r-\rho+1)\,10^{r-\rho} + O(10^r) \quad (\text{as } r \to \infty).$$

But $x_r = r10^r$; hence

$$g_r = 10^{-\rho} x_r + o(x_r). \tag{i}$$

Now
$$G_r = \sum_{s=1}^{r} g_s + O(r), \quad X_r = \sum_{s=1}^{r} x_s,$$

so that
$$G_r = 10^{-\rho} X_r + o(X_r). \tag{i}$$

In order to estimate $g_r(x)$ we consider how often in each position γ_ρ may occur undivided in the first x digits of s_r.

We may suppose that the x-th digit of s_r occurs within the member $p_{r-1} p_{r-2} \dots p_1 p_0$ of s_r, so that we have

$$x = r \sum_{t=0}^{r-1} p_t 10^t + \theta r \quad (0 < \theta \leqslant 1). \tag{C}$$

Now let $g_{rk}(x)$ denote the number of times that γ_ρ occurs undivided in the first x digits of s_r, with the first digit of γ_ρ as the k-th digit of a member of s_r. Then, if $k > r - \rho + 1$,

$$g_{rk}(x) = 0, \tag{D}$$

and, if $k \leqslant r - \rho + 1$,

$$g_{rk}(x) = 10^{r-\rho-k+1} \left\{ \sum_{t=r-k+1}^{r-1} p_t 10^{t+k-r-1} + \theta' \right\} \quad (0 \leqslant \theta' \leqslant 1). \tag{D}$$

For, with γ_ρ fixed in position in the member, we may choose the last $r - \rho - k + 1$ digits of the member in $10^{r-\rho-k+1}$ ways. Having chosen these, in order to ensure that the member lies as required in the sequence $00 \dots 0,\ 00 \dots 1,\ \dots,\ p_{r-1} \dots p_\lambda$ consisting of the first x digits of s_r, we shall be able to choose the first $k-1$ digits of the member, either in

$$\sum_{t=r-k+1}^{r-1} p_t 10^{t+k-r-1}$$

ways, or in
$$\sum_{t=r-k+1}^{r-1} p_t 10^{t+k-r-1} + 1$$

ways, so that the total number of times that γ_ρ may occur in the k-th position undivided, in the first x digits of s_r, is correctly estimated by the formulae (D).

A numerical example may be welcome: 37 will occur in the second position ($k = 2$) in the first 7987 digits of s_4 (*i.e.* in 0000, 0001, ..., 1995, 199.) exactly twenty times ($= 10[1+1]$ times), for, when we have chosen the last digit (8 say), we are left with two choices 1 or 0 for the first digit; but, if we were considering the first 5000 digits only of s_4 (*i.e.* 0000, ..., 1249.), we could no longer choose 1 as our first digit, and 37 would occur only ten times in the second position (viz. 0370, 0371, ..., 0379) in the first 5000 digits of s_4.

Referring to the formulae (D) we have

$$g_{rk}(x) = 10^{-\rho}\left\{\sum_{t=r-k+1}^{r-1} p_t 10^t + \theta' 10^{r-k+1}\right\},$$

so that

$$\sum_{k=1}^{r-\rho+1} g_{rk}(x) = 10^{-\rho}\left\{\sum_{k=1}^{r-\rho+1}\sum_{t=r-k+1}^{r-1} p_t 10^t\right\} + O(10^r)$$

$$= 10^{-\rho}\sum_{t=\rho}^{r-1}\{(t+1-\rho)p_t 10^t\} + O(10^r). \tag{E}$$

Now γ_ρ cannot occur divided in s_r more than $\rho 10^r$ times; hence, by (C) and (E),

$$g_r(x) = 10^{-\rho}x + O(10^r) = 10^{-\rho}x + o(x_r) \tag{ii)*}$$

as $r \to \infty$ and x varies in any manner consistent with the existence of $g_r(x)$.

To obtain $G(x)$ from the results (i), (ii), we suppose the x-th digit of S to occur as the y-th digit of s_r. Then

$$x = X_{r-1} + y; \quad G(x) = G_{r-1} + g_r(y) + O(1) \quad (\text{as } x \to \infty).$$

Hence, by (i) and (ii),

$$G(x) = 10^{-\rho}X_{r-1} + 10^{-\rho}y + O(10^r),$$

so that

$$G(x) = 10^{-\rho}x + o(x), \tag{iii}$$

and $\cdot S$ is normal in the scale of ten. This proves Theorem I.

In order to prove Theorem II we extend our notation.

${}_\mu s_r$ will denote the sequence s_r repeated μ times;

${}_\mu S_r$ will denote the sequence ${}_\mu s_1\, {}_\mu s_2\, {}_\mu s_3 \ldots {}_\mu s_r$;

${}_\mu S$ will denote the sequence ${}_\mu s_1\, {}_\mu s_2\, {}_\mu s_3 \ldots {}_\mu s_r\, {}_\mu s_{r+1} \ldots$;

${}_\mu x_r$ and ${}_\mu X_r$ will denote the number of digits in ${}_\mu s_r$ and ${}_\mu S_r$;

${}_\mu g_r$ and ${}_\mu G_r$ will denote the number of occurrences of γ_ρ in the sequences ${}_\mu s_r$ and ${}_\mu S_r$ respectively;

${}_\mu g_r(x)$ and ${}_\mu G(x)$ will denote the number of occurrences of γ_ρ in the first x digits of ${}_\mu s_r$ and ${}_\mu S$ respectively.

Then to estimate ${}_\mu G(x)$ we suppose, on the same lines as in the proof of Theorem I, that the x-th digit of ${}_\mu S$ occurs as the y-th digit of ${}_\mu s_r$; we

* The formulae (i) can easily be deduced from the result (ii), but an independent proof of the simpler formulae is given since they are used again in the proof of Theorem IV, in which no appeal to the result (ii) is necessary.

further suppose that the y-th digit of ${}_\mu s_r$ occurs as the z-th digit of some s_r of ${}_\mu s_r$, so that

$$x = {}_\mu X_{r-1} + y = {}_\mu X_{r-1} + qx_r + z \quad (0 \leqslant q < \mu,\ 0 < z \leqslant x_r). \tag{F}$$

Also, as $x \to \infty$,

$$ {}_\mu G(x) = {}_\mu G_{r-1} + qg_r + g_r(z) + O(r). \tag{G}$$

Hence, in virtue of the relations (i) and (ii),

$$ {}_\mu G(x) = 10^{-\rho}({}_\mu X_{r-1} + qx_r + z) + o(x);$$

whence, by (F),

$$ {}_\mu G(x) = 10^{-\rho} x + o(x). \tag{iv}$$

This proves Theorem II, that the decimal $\cdot {}_\mu S$ is normal in the scale of ten.

To prove Theorem III, we use the particular case of Theorem II, that the decimal $\cdot {}_9 S$ is normal in the scale of ten.

We show that, if we insert one extra digit after each comma in $\cdot {}_9 S$, the new decimal $\cdot {}_9 S'$ so obtained will also be normal in the scale of ten. Thus, let r denote the number of digits in the member of ${}_9 S$ in which the x-th digit of ${}_9 S$ occurs and let $C(x)$ denote the number of commas among the first x digits of ${}_9 S$. Then, since $r10^r = O(x)$ as $x \to \infty$,

$$C(x) = O(10^r) = o(x). \tag{H}$$

Let the x-th digit of ${}_9 S$ become the x'-th digit of ${}_9 S'$. Then x is defined as a function of the positive integer x' except in the cases where the x'-th digit of ${}_9 S'$ is one of the new digits. In this case, we define the corresponding value of x to be the same as that corresponding to $x'-1$. Then

$$x' = x + C(x) + O(1) = x + o(x). \tag{I}$$

Again, the insertion of one new digit in ${}_9 S$ cannot alter ${}_9 G(x)$ by more than ρ. Hence, if ${}_9 G'(x')$ denotes the number of occurrences of γ_ρ in the first x' digits of ${}_9 S'$,

$$ {}_9 G'(x') = {}_9 G(x) + O\{C(x)\} = 10^{-\rho} x + o(x) = 10^{-\rho} x' + o(x'), \tag{v}$$

and $\cdot {}_9 S'$ is normal in the scale of ten.

Now by suitable choice of the new digits we can arrange that $\cdot {}_9 S'$ is the decimal

$$\cdot 10,11, \ldots, 19,20,21, \ldots, 29,30, \ldots\ldots, 99,100,101, \ldots\ldots, 999,1000, \ldots .$$

Hence this decimal is normal in the scale of ten, and Theorem III, that the decimal

$$\cdot 1234567891011121314 \ldots .$$

is normal in the scale of ten, follows directly.

Theorem IV can be proved by means of the equations (i), without appeal to the equation (ii). For let

${}_r s_r$ denote the sequence s_r repeated r times;

${}_r S_r$ and ${}_r S$ denote the sequence ${}_1 s_1 \, {}_2 s_2 \dots {}_r s_r$ and ${}_1 s_1 \, {}_2 s_2 \dots {}_r s_r \dots$;

${}_r x_r$ and ${}_r X_r$ denote the number of digits in ${}_r s_r$ and ${}_r S_r$ respectively;

${}_r g_r$ and ${}_r G_r$ denote the number of occurrences of γ_ρ in ${}_r s_r$ and ${}_r S_r$;

${}_r g_r(x)$ and ${}_r G(x)$ denote the number of occurrences of γ_ρ in the first x digits of ${}_r s_r$ and of ${}_r S$ respectively.

Then we can express any positive integer x in the form

$$x = {}_{r-1}X_{r-1} + qx_r + y = (r-1)\, X_{r-1} + qx_r + y, \tag{J}$$

where $0 \leqslant q < r$, $0 < y \leqslant x_r = o(x)$ as $x \to \infty$. Also

$$_rG(x) = {}_{r-1}G_{r-1} + qg_r + Y' = (r-1)\, G_{r-1} + qg_r + Y, \tag{K}$$

where $Y = O(x_r) = o(x)$ as $x \to \infty$. Hence, by equations (i),

$$_rG(x) = 10^{-\rho}\{(r-1)\, X_{r-1} + qx_r\} + o(x),$$

so that, by (J),

$$_rG(x) = 10^{-\rho}\, x + o(x), \tag{vi}$$

and the decimal $\cdot {}_r S$ is normal in the scale of ten. This proves Theorem IV.

By an extension of similar methods it is possible to prove that various other types of decimal are normal in the scale of ten. Thus it is possible to prove

THEOREM V. *The decimal* ·46891012141516182021... *formed of the sequence of composite numbers is normal in the scale of ten**.

THEOREM VI. *If a is any positive number and a_r denotes the integral part of ar, then the decimal $\cdot a_1 a_2 \dots a_r \dots$ is normal in the scale of ten.*

THEOREM VII. *If L_r denotes the integral part of $r \log r$, then the decimal $\cdot L_1 L_2 \dots L_r \dots$ is normal in the scale of ten.*

It would be reasonable to suppose that the decimal formed by the sequence of prime numbers is also normal in the scale of ten, but of this I have no proof.

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* In order to prove Theorem V, we use the theorem that $\pi(x) = o(x)$ as $x \to \infty$, where $\pi(x)$ denotes the number of primes in the first x integers.