Measures on the Closed Subspaces of a Hilbert Space

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1. Introduction. In his investigations of the mathematical foundations of quantum mechanics, MACKEY[2] has proposed the following problem: Determine all measures on the closed subspaces of a Hilbert space. A measure on the closed subspaces means a function μ which assigns to every closed subspace a non-negative real number such that if $\{A_i\}$ is a countable collection of mutually orthogonal subspaces having closed linear span B, then

$$\mu(B) = \sum \mu(A_i).$$

It is easy to see that such a measure can be obtained by selecting a vector v and, for each closed subspace A, taking $\mu(A)$ as the square of the norm of the projection of v on A. Positive linear combinations of such measures lead to more examples and, passing to the limit, one finds that, for every positive semi-definite self-adjoint operator T of the trace class,

$$\mu(A) = \text{trace}\,(TP_A),$$

where P_A denotes the orthogonal projection on A, defines a measure on the closed subspaces. It is the purpose of this paper to show that, in a separable Hilbert space of dimension at least three, whether real or complex, every measure on the closed subspaces is derived in this fashion.

If we regard the measure as being defined, not on the closed subspaces, but on the orthogonal projections corresponding, then the problem can be significantly generalized as follows: Determine all measures on the projections in a factor. We solve this problem for factors of type I, but our methods are not applicable to factors of types II and III.

For factors of type I the problem is simplified by the existence of minimal subspaces. In view of the complete additivity demanded in the definition, it is quite obvious that a measure on the closed subspaces of a separable Hilbert

[1]The author has been partially supported by the Office of Ordnance Research, Contract No. DA 19-020-ORD-3778. He is indebted also to R. V. KADISON for helpful remarks.

[2]See his forthcoming article, Quantum mechanics and Hilbert space, *Amer. Math. Monthly.*

Journal of Mathematics and Mechanics, Vol. 6, No. 6 (1957).

space is determined by its values on the one-dimensional subspaces. This leads us to the study of what we shall call frame functions.

Definition. *A frame function of weight W for a separable Hilbert space $\mathcal{H}$ is a real-valued function f defined on the (surface of the) unit sphere of $\mathcal{H}$ such that if $\{x_i\}$ is an orthonormal basis of $\mathcal{H}$ then*

$$\sum f(x_i) = W.$$

While we are ultimately interested in non-negative frame functions, it is convenient to consider those with negative values, particularly in the finite-dimensional case. Here no convergence questions can arise since the sum in the definition is finite.

If S is a closed subspace of $\mathcal{H}$, then any frame function for $\mathcal{H}$ becomes one for S by restriction, the weight being probably changed. A one-dimensional S leads us immediately to the following observation: If f is any frame function and $|\lambda| = 1$, then $f(\lambda x) = f(x)$.

Definition. *A frame function f is regular if and only if there exists a self-adjoint operator T defined on $\mathcal{H}$ such that*

$$f(x) = (Tx, x) \qquad \textit{for all unit vectors } x.$$

Our objective is to prove that all frame functions are regular, at least with suitable additional hypotheses. In dimension one it is obvious that every frame function is regular. In dimension two a frame function can be defined arbitrarily on a closed quadrant of the unit circle in the real case, and similarly in the complex case. In higher dimensions the orthonormal sets are intertwined and there is more to be said. However, if f is a finite-dimensional frame function and g is a discontinuous endomorphism of the real numbers as an additive group, $g \circ f$ is also a frame function. This construction produces a great class of wildly discontinuous frame functions, all of which are unbounded. This suggests the additional hypothesis bounded. Slightly stronger, and closer to our goals, is the hypothesis non-negative. In the finite-dimensional case these are essentially equivalent since a constant function is a frame function and the frame functions form a linear set. We shall show that every non-negative frame function in three or more dimensions is regular.

2. Frame functions in three-dimensional real Hilbert spaces.

2.1. Lemma. *In a finite-dimensional real Hilbert space a frame function is regular if and only if it is the restriction to the unit sphere of a quadratic form.*

Proof: Obvious.

2.2. Lemma. *Consider the functions on the unit circle in R^2 given in the usual angular coordinate by* $\cos n\theta$, *n an integer. Such a function is a frame function if and only if $n = 0$ or $n \equiv 2 \pmod 4$.*

Proof: If the weight is W we must have

$$\cos n\theta + \cos n\left(\theta + \frac{\pi}{2}\right) = \left(1 + \cos n\frac{\pi}{2}\right)\cos n\theta - \sin n\frac{\pi}{2}\sin n\theta = W$$

for all θ. This is true if and only if $n = 0$ or $1 + \cos \frac{1}{2}n\pi = 0$.

2.3. Theorem. *Every continuous frame function on the unit sphere in R^3 is regular.*

Proof: Let $\mathcal{C}$ denote the space of continuous functions on the unit sphere S of R^3 endowed with the usual norm. The rotation group G of R^3 is represented as a group of linear operators on $\mathcal{C}$ if we define

$$U_\sigma h = h \circ \sigma^{-1} \quad \text{where} \quad \sigma \in G,\ h \in \mathcal{C}.$$

Let Q_n denote the space of surface harmonics of degree n; these functions may be characterized as the restrictions to S of the homogeneous polynomial functions of degree n which satisfy Laplace's equation in R^3. These spaces are irreducible G-invariant subspaces of $\mathcal{C}$ and furthermore they are the only irreducible G-invariant subspaces.

Let F be the space of continuous frame functions on S. It is readily seen that F is a closed G-invariant linear subspace of $\mathcal{C}$. From the general theory of representations of compact groups (or the older theory of continuous functions on S) it follows that F is the closed linear span of certain of the Q_n. Now Q_0 consists of constant functions, so $Q_0 \subset F$. Since Q_1 is made up of linear functions on R^3 restricted to S, $Q_1 \not\subset F$ (these functions change sign on passing to antipodes, frame functions do not). The space Q_2 contains the restrictions to S of quadratic forms of trace 0; these are all frame functions of weight 0, so $Q_2 \subset F$. Suppose $n > 2$; then, using the characterization given in the previous paragraph, we may check that Q_n contains the restrictions to S of the explicit functions on R^3 given in *cylindrical* coordinates by

$$\rho^n \cos n\theta \quad \text{and} \quad [\rho^2 - 2(n-1)z^2]\rho^{n-2}\cos(n-2)\theta.$$

If $Q_n \subset F$, then both of these functions would restrict to be frame functions not only on S but on the unit circle in the x-y plane. This contradicts lemma 2.2, so $Q_n \not\subset F$. This proves that F is the closed linear span of Q_0 and Q_2. Since these spaces have finite dimension $F = Q_0 + Q_2$. The functions of Q_0, although constant, are nonetheless restrictions of quadratic forms because $x^2 + y^2 + z^2 = 1$ on S. This proves that every continuous frame function is the restriction of a quadratic form, which is a paraphrase of the theorem.

2.4. It is convenient to describe constructions on the sphere in terms of the ordinary latitude-longitude coordinates. When such a system has been selected we designate by N the closed northern hemisphere. Through every point q of N other than the north pole there is a unique great circle tangent to the circle of latitude through q; we shall call it the EW great circle through q.

If f is a real-valued function defined on the set X we write osc (f, X) for sup $\{f(x) \mid x \in X\}$ − inf $\{f(x) \mid x \in X\}$.

2.5. Lemma. *Suppose z is a point of N other than the north pole. Consider the set X of all points $x \in N$ such that for some y*

(a) *y is on the EW great circle through x,*

(b) *z is on the EW great circle through y.*

Then X has a non-empty interior.

Proof: This is very easily seen geometrically. Analytically we may take the point z to have orthonormal coordinates $\langle \cos\theta, 0, \sin\theta \rangle$ where $0 < \theta < \frac{1}{2}\pi$, and then we verify that the locus L of points $y = \langle \xi, \eta, \zeta \rangle$ satisfying (b) is given by

$$\psi = (\xi^2 + \eta^2)\sin\theta - \xi\zeta\cos\theta = 0.$$

Now if x is any point at which the form ψ is negative, then the EW circle through x must cross the locus L, since on the equator $\psi > 0$. Thus X contains at least the open set of points at which ψ is negative, which is not empty because $\langle \cos\phi, 0, \sin\phi \rangle$ is in it if $\theta < \phi < \frac{1}{2}\pi$.

2.6. Lemma. *Suppose that f is a frame function on the unit sphere S in R^3 and that, for a certain neighborhood U of p,* osc $(f, U) = \alpha$. *Then every point of the great circle with pole p has a neighborhood V for which* osc $(f, V) \leqq 2\alpha$.

Proof: We take latitude-longitude coordinates with p as north pole. Suppose U contains all points in latitudes above $\frac{1}{2}\pi - \theta$. Let q_0 be any point on the equator and let r be the point in latitude $-\frac{1}{2}\theta$ due south of q_0. Let C_0 be the great circle connecting r and q_0 and let r_0' and q_0' be orthogonal to r and q_0 respectively in $C_0 \cap N$. Both of these points fall in U; furthermore, the same will be true if q_0 is replaced by any point q in a certain neighborhood V of q_0, keeping r fixed.

If now q_1 and q_2 are in V, let C_i be the great circle connecting r and q_i and take r_i' and q_i' on $C_i \cap N$ so that $r_i' \perp r$, $q_i' \perp q_i$ $(i = 1, 2)$. Then we will have

$$f(r) + f(r_i') = f(q_i) + f(q_i'), \qquad i = 1, 2.$$

Subtracting these equations

$$|f(q_1) - f(q_2)| = |f(r_1') - f(r_2') + f(q_2') - f(q_1')| \leqq 2\alpha$$

since $r_1', r_2', q_1', q_2' \in U$. This shows that osc $(f, V) \leqq 2\alpha$.

2.7. Lemma. *Suppose that f is a frame function on the unit sphere S in R^3 and that, for a certain non-empty open set U,* osc $(f, U) = \alpha$. *Then every point of S has a neighborhood W such that* osc $(f, W) \leqq 4\alpha$.

Proof: From any point p of U we can reach any point of S in two steps of arc length $\frac{1}{2}\pi$; hence this lemma follows from the preceding.

2.8. Theorem. *Every non-negative frame function on the unit sphere S in R^3 is regular.*

Proof: Let f be a non-negative frame function of weight W on S. We may subtract a constant from f and it will remain a frame function; hence it is no loss of generality to suppose that $\inf f(x) = 0$. The proof would be considerably shortened if we knew that f achieved the value 0, but we consider the general case.

Let ϵ be a positive number and put $\eta = \epsilon/88$. We can find a point p such that $f(p) \leqq \eta$. Take latitude-longitude coordinates with p at the north pole. Let σ be the polar rotation through angle $\frac{1}{2}\pi$, and set

$$g(x) = f(x) + f(\sigma x).$$

Evidently g is a non-negative frame function of weight $2W$. For any point q on the equator, p, q, and σq form an orthonormal set so $g(q) = f(q) + f(\sigma q) = W - f(p)$; thus g is constant on the equator.

Consider any point $r \in N - \{p\}$. Let C be the EW great circle through r; it meets the equator at a point q orthogonal to r; therefore $2W \geqq g(r) + g(q) = g(r) + W - f(p)$, whence

$$g(x) \leqq W + f(p) \leqq W + \eta \quad \text{for all} \quad x \in N - \{p\}. \tag{1}$$

Continuing, consider any point $s \in C \cap N$ and an orthogonal point $t \in C \cap N$. We have $g(r) + W - f(p) = g(s) + g(t) \leqq g(s) + W + \eta$ giving

$$g(r) \leqq g(s) + 2\eta \tag{2}$$

for any point $r \in N - \{p\}$ and any point s on the EW circle through r.

Let $\beta = \inf \{g(x) \mid x \in N - \{p\}\}$ and take a point $z \in N - \{p\}$ for which $g(z) \leqq \beta + \eta$. If $x \in N - \{p\}$ is a point such that for some y

(a) y is on the EW great circle through x,

(b) z is on the EW great circle through y,

then

$$g(x) \leqq g(y) + 2\eta,$$
$$g(y) \leqq g(z) + 2\eta$$

and therefore

$$\beta \leqq g(x) \leqq g(z) + 4\eta \leqq \beta + 5\eta.$$

The set of points x satisfying the condition has a non-void interior U by lemma 2.5 and the last display shows $\operatorname{osc}(g, U) \leqq 5\eta$. By lemma 2.7 there is a neighborhood V of p such that $\operatorname{osc}(g, V) \leqq 20\eta$. Since $g(p) = 2f(p) \leqq 2\eta$, $\sup \{g(x) \mid x \in V\} \leqq 22\eta$. Since f is non-negative and $f \leqq g$ pointwise, $\operatorname{osc}(f, V) \leqq 22\eta$. Applying 2.7 once again, any point $u \in S$ has a neighborhood W such that $\operatorname{osc}(f, W) \leqq 88\eta = \epsilon$. Since ϵ can be arbitrarily small this proves that f is continuous and the theorem now follows from theorem 2.3.

3. Higher dimensions and complex Hilbert spaces.

3.1. *We shall say that a real-linear subspace $\mathcal{K}$ of a Hilbert space $\mathcal{H}$ is completely real if the inner product takes only real values on $\mathcal{K} \times \mathcal{K}$.*

A closed completely real subspace is itself a real Hilbert space with respect to the restriction of the inner product of $\mathcal{H}$. It is clear that if every pair of vectors in a set X has real inner product, then the real-linear subspace spanned by X is completely real and so is its closure. In particular, an orthonormal set of vectors spans a completely real subspace. Conversely, an orthonormal subset of a completely real subspace is an orthonormal subset of $\mathcal{H}$. It follows from these remarks that a frame function for $\mathcal{H}$ becomes a frame function when restricted to a completely real subspace.

3.2. Lemma.[3] *If f is a non-negative regular frame function of weight W on a real Hilbert space, then for any unit vectors x and y*

$$|f(x) - f(y)| \leqq 2W\,||x - y||.$$

Proof: Since f is regular there is a symmetric operator T such that $f(x) = (Tx, x)$. Because f is non-negative we have $0 \leqq (Tx, x) \leqq W$ for all unit vectors x and therefore $||T|| \leqq W$.

Now, for any unit vectors x and y, $(Tx, y) = (Ty, x)$, so $f(x) - f(y) = (T(x + y), x - y)$ and therefore

$$|f(x) - f(y)| \leqq ||T||\,||x + y||\,||x - y|| \leqq 2W\,||x - y||.$$

3.3. Lemma. *Suppose that f is a non-negative frame function on a two-dimensional complex Hilbert space which is regular on every completely real subspace. Then f is regular.*

Proof: Suppose W is the weight of f and M is its least upper bound. We can choose unit vectors x_n so that $f(x_n) \to M$ and we can arrange that $x_n \to y$, because the unit sphere is compact. Let $\lambda_n = (y, x_n)/|(y, x_n)|$; we have $\lambda_n \to 1$ and $\lambda_n x_n \to y$. Since $|\lambda_n| = 1$, $f(\lambda_n x_n) = f(x_n)$. Moreover, $(\lambda_n x_n, y)$ is real, so $\lambda_n x_n$ and y are in a completely real subspace. By lemma 3.2 we have

$$\begin{aligned}|f(y) - M| \leqq |f(y) - f(\lambda_n x_n)| + |f(x_n) - M|& \\ \leqq 2W\,||y - \lambda_n x_n|| + |f(x_n) - M|&,\end{aligned}$$

from which we see that $f(y) = M$.

Define F on H by

$$F(v) = ||v||^2 f\left(\frac{v}{||v||}\right) \quad \text{if} \quad v \neq 0,$$

$$F(0) = 0.$$

The hypotheses concerning f imply that F becomes a quadratic form when restricted to any completely real subspace. Furthermore, since $f(\lambda v) = f(v)$ whenever $|\lambda| = 1$, $F(\lambda v) = |\lambda|^2 F(v)$ for all scalars λ and vectors v.

Let z be any unit vector orthogonal to y. Then $F(y) = f(y) = M$ and $F(z) =$

[3]The present version of this lemma and its proof are due to R. S. PALAIS, who was kind enough to read the first draft of this paper.

$f(z) = W - f(y) = W - M$. On the completely real subspace determined by y and z, F is a quadratic form whose maximum value on the unit circle is attained at y; therefore the matrix for F relative to the basis y, z is diagonal. Hence

$$F(\alpha y + \beta z) = \alpha^2 F(y) + \beta^2 F(z) = M\alpha^2 + (W - M)\beta^2$$

if α and β are real.

If λ and μ are non-zero complex numbers and $z' = (\mu/|\mu|)(|\lambda|/\lambda)z$, then z' is also a unit vector orthogonal to y; therefore

$$F(\lambda y + \mu z) = F((|\lambda|/\lambda)(\lambda y + \mu z))$$
$$= F(|\lambda|\, y + |\mu|\, z') = M\, |\lambda|^2 + (W - M)\, |\mu|^2.$$

The exceptional cases, λ or μ zero, present no difficulty, so we see that

$$F(x) = (Tx, x)$$

for any vector x, where T is the self-adjoint operator whose matrix, relative to y, z, is

$$\begin{Vmatrix} M & 0 \\ 0 & W - M \end{Vmatrix}.$$

This shows that f is regular.

3.4. Lemma.[4] *Suppose that f is a non-negative frame function for a Hilbert space $\mathcal{H}$ (either real or complex) and that f is regular when restricted to any two-dimensional subspace of $\mathcal{H}$. Then f is regular.*

Proof: We give the proof in a form which covers both the real and complex cases simultaneously. Define F as in lemma 3.3. On each two-dimensional subspace S of $\mathcal{H}$, there is a form A_s (either bilinear or Hermitian) such that $F(x) = A_s(x, x)$ for $x \in S$. We define A on all of $\mathcal{H} \times \mathcal{H}$ by

$$A(x, y) = A_s(x, y) \quad \text{if} \quad x \in S,\ y \in S.$$

(Usually there will only be one two-dimensional subspace S containing both x and y, but if say $x = \lambda y$, then $A_s(x, y) = \lambda A_s(y, y) = \lambda F(y)$ which is independent of the choice of S.) Because only two-dimensional subspaces of $\mathcal{H}$ are involved we derive the following relations from the forms A_s :

(1) $A(\alpha x, y) = \alpha A(x, y)$

(2) $A(x, y) = \overline{A(y, x)}$

(3) $4 \operatorname{Re} A(x, y) = F(x + y) - F(x - y)$

(4) $2F(x) + 2F(y) = F(x + y) + F(x - y)$

for all vectors x, y and scalars α.

[4] This lemma is due to JORDAN & VON NEUMANN, On inner products in linear metric space, *Annals of Math.* **36** (1935), pp. 719–723.

Now

$$
\begin{aligned}
8\,\mathrm{Re}\, A(x, z) + 8\,\mathrm{Re}\, A(y, z) & \\
&= 2F(x + z) - 2F(x - z) + 2F(y + z) - 2F(y - z) \\
&= F(x + y + 2z) + F(x - y) - F(x + y - 2z) - F(x - y) \\
&= 4\,\mathrm{Re}\, A(x + y, 2z) \\
&= 8\,\mathrm{Re}\, A(x + y, z).
\end{aligned}
$$

Replacing x and y by ix and iy and using (1) we find also

$$\mathrm{Im}\, A(x, z) + \mathrm{Im}\, A(y, z) = \mathrm{Im}\, A(x + y, z)$$

giving

$$A(x, z) + A(y, z) = A(x + y, z) \tag{5}$$

which, together with (1) and (2), shows that A is bilinear or Hermitian on all of $\mathcal{H} \times \mathcal{H}$.

Take vectors x and y with $||x|| \leqq 1$, $||y|| \leqq 1$; with a proper choice of ω, where $|\omega| = 1$, we have

$$
\begin{aligned}
4\,|A(x, y)| = 4A(\omega x, y) = 4\,\mathrm{Re}\, A(\omega x, y) &= F(\omega x + y) - F(\omega x - y) \\
&\leqq M(||\omega x + y||^2 + ||\omega x - y||^2) = 2M(||\omega x||^2 + ||y||^2) \leqq 4M
\end{aligned}
$$

where $M = \sup \{|f(u)| \mid ||u|| = 1\}$. Thus A is bounded and there exists a bounded self-adjoint operator T such that

$$A(x, y) = (Tx, y) \quad \text{for all} \quad x, y \in \mathcal{H}.$$

Finally, $f(x) = F(x) = A(x, x) = (Tx, x)$ for all unit vectors x which concludes the proof.

3.5. Theorem. *Every non-negative frame function on either a real or complex Hilbert space of dimension at least three is regular.*

Proof: A frame function for $\mathcal{H}$ becomes a frame function for any completely real subspace of $\mathcal{H}$ by restriction. Every completely real two-dimensional subspace of $\mathcal{H}$ can be embedded in a completely real three-dimensional subspace, since $\dim \mathcal{H} \geqq 3$. Therefore theorem 2.8 shows that any non-negative frame function f is regular on every completely real two-dimensional subspace of $\mathcal{H}$. Lemma 3.3 shows that f is regular on every two-dimensional subspace; hence f is regular by the last lemma.

4. The main result.

4.1. Theorem. *Let μ be a measure on the closed subspaces of a separable (real or complex) Hilbert space $\mathcal{H}$ of dimension at least three. There exists a positive* *semi-definite self-adjoint operator T of the trace class such that for all closed subspaces A of $\mathfrak{H}$*

$$\mu(A) = \text{trace}\,(TP_A)$$

where P_A is the orthogonal projection of $\mathfrak{H}$ onto A.

Proof: If B_x is the one-dimensional subspace spanned by the unit vector x, then $f(x) = \mu(B_x)$ defines a non-negative frame function f. There is a self-adjoint operator T such that $\mu(B_x) = (Tx\,,\,x)$, for all unit vectors x. Since $(Tx\,,\,x) \geqq 0$ for all unit vectors x, T is positive semi-definite. If $\{x_i\}$ is an orthonormal basis for $\mathfrak{H}$,

$$\mu(\mathfrak{H}) = \sum \mu(B_{x_i}) = \sum (Tx_i\,,\,x_i).$$

Since the latter sum converges, T is in the trace class, indeed trace $T = \mu(\mathfrak{H})$.

If A is an arbitrary closed subspace, we can choose an orthonormal basis $\{y_i\}$ for A and adjoin further vectors $\{z_j\}$ so that $\{y_i\,,\,z_j\}$ is an orthonormal basis for $\mathfrak{H}$. Then $P_A y_i = y_i$ for all i and $P_A z_j = 0$ for all j so

$$\begin{aligned}\mu(A) = \sum \mu(B_{y_i}) &= \sum_i (Ty_i\,,\,y_i) \\ &= \sum_i (TP_A y_i\,,\,y_i) + \sum_j (TP_A z_j\,,\,z_j) = \text{trace}\,(TP_A).\end{aligned}$$

The theorem is proved.

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