The Mathematical Power of Epicyclical Astronomy

By *Norwood Russell Hanson**

T HIS PAPER has two objectives. The first is to describe more graphically than has yet been done the elegance of the ancient technique of epicycle-on-deferent. The second is to expose as erroneous an implicit contention of several historians — namely, that the inadequacy of Ptolemaic astronomy is somehow connected with its formally weak computational equipment.[1] When compared with the powerful calculational devices of our twentieth century, epicycle-on-deferent astronomy apparently comes out as a laughably primitive attempt to predict planetary perturbations. But to reason thus is fallacious.

Actually, these objectives will be achieved simultaneously — if they are achieved at all. To see the comprehensive theoretical power of this ancient geometrical device just *is* to see its elegance.

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First then, a graphic account of the flexible beauty of epicycle-on-deferent. Let Ptolemy himself introduce the subject:

> . . . the epicycle is made to move in longitude in the order of the signs in a circle concentric with the Zodiac [i.e., from west to east], while the planet moves on the epicycle at a velocity which is the same as that of the anomaly; on that part of the epicycle farthest from the earth the motion is direct.[2]

Ptolemy here describes that example of epicyclical motion which is now the staple illustration in history of science textbooks (Figure 1).

It is said of Herakleides that he envisaged such a motion for Mercury and Venus. Thus, Chalcidius:

[1] So prevalent is this view that as late as 1880 De Morgan had to write: "On this theory of epicycles . . . the common notion is that it was a cumbrous and useless apparatus, thrown away by the moderns" (*Dictionary of Greek and Roman Biography*, compiled by Sir William Smith [London, 1880], s.v. Claudius Ptolemaeus, p. 576). Dampier does not help to correct "the common notion" forcibly or clearly enough. He writes: "Its [the theory of Hipparchus and Ptolemy] one fault *from* the geometric point of view was the complications of cycles and epicycles it involved" (*A History of Science*, 4th ed. [Cambridge], p. 109, my italics). Even an authority like Abetti is misleading in this connection: "It [the Copernican system] has the weakness of the epicycles, which could not explain the variable direction of the planet, due to its elliptic motion around the sun . . ." (*The History of Astronomy* [New York, 1952], p. 80). The present paper purports to demonstrate the untenability of this position.

[2] Claudius Ptolemy, *Syntaxis mathematica,* ed. Heiberg (Leipzig, 1898-1903), III, 3.

Herakleides of Pontus, in describing the path of Venus and the sun, and assigning one midpoint for both, showed how Venus is sometimes above and sometimes below the sun.[3]

Vitruvius amplifies this:

Mercury and Venus make their retrogradations and retardations around the rays of the sun, making a crown, as it were, by their courses about the sun as center.[4]

Martianus Capella is even more explicit:

Venus and Mercury . . . place the center of their orbits in the sun; so that they sometimes move above it and sometimes below it, i.e., nearer the earth . . . the circles of this star [Mercury] and Venus are epicycles. That is to say, they do not include the round earth within their own orbit, but are carried around it latterly, as it were.[5]
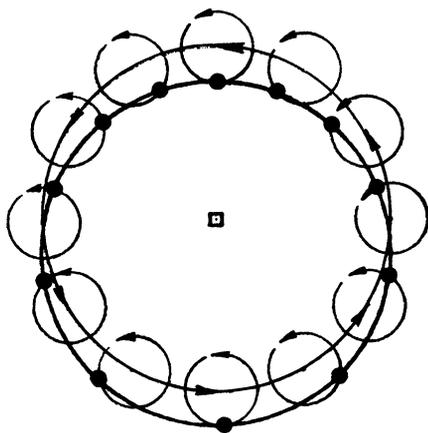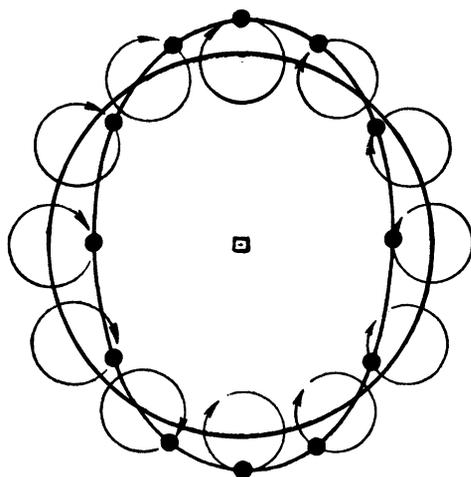

FIGURE 1

FIGURE 2

As textbooks readily show, by reversing the direction in which the planet turns on the epicycle, an elliptical orbit results, an effect of which Copernicus was well aware (Figure 2).

It is demonstrable, of course, that this same effect could have been obtained without the epicycle — simply by letting the deferential circle's center itself move in a circle just the size of the epicycle above (Figure 3).

As Ptolemy puts it:

the center of the excentric revolves . . . whilst the planet moves on the excentric in the opposite direction . . .[6]

Indeed, Ptolemy actually superimposes both demonstrations to reveal their perfect equivalence:

[3] *Commentary on Plato's Timaeus,* 109.
[4] *On Architecture,* IX, 1.6.

[5] *On the Marriage of Philology and Mercury,* VIII, 859, 879.
[6] *Syntaxis mathematica,* III, 3.

... according to either hypothesis it will appear possible for the planets seemingly to pass, in equal periods of time, through unequal arcs of the ecliptic circle which is concentric with the cosmos. ... It must be understood that all the appearances can be cared for interchangeably according to either hypothesis, when the same ratios are involved in each. In short, the hypotheses are interchangeable.[7]

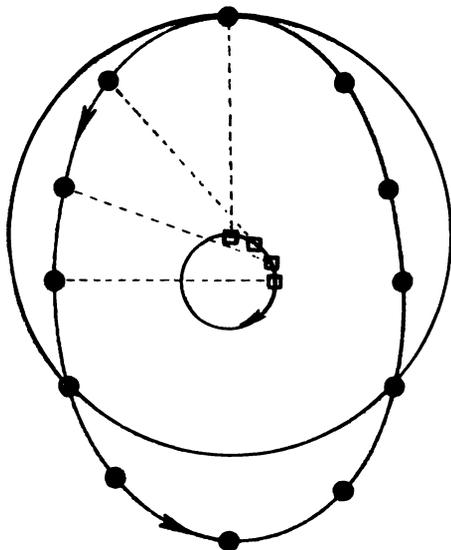Textbook expositions usually reach their zenith with Figure 4.
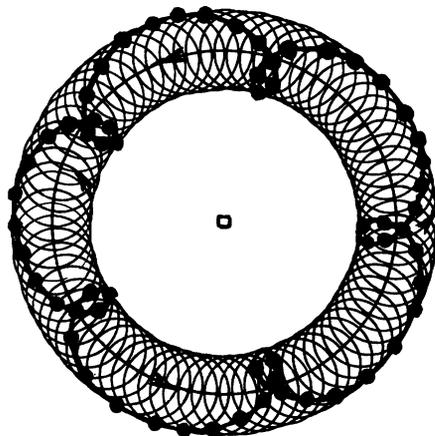

FIGURE 3   FIGURE 4

With only this much of the story of epicycle-on-deferent told, it would indeed appear that the ancient's astronomical tasks were Sisyphean. The complex trajectories of comets, and the erratic wobblings of double stars, not to mention the retrogradations-in-latitude of Mars and Venus, all seem to be motions which would *in principle* elude such a primitive technique. But this is wrong. And it is wrong ever to let it even sound as if there were dynamically significant motions which could not be captured by Ptolemy's methods.

For example, merely by letting the length of the epicycle's radius approximate to that of the deferent one can attain perfect rectilinear motion along a diameter of the deferent[8] (Figure 5).

Historically, this motion is significant. Kepler, in *De motibus stellae Martis* makes Mars librate rectilinearly across its epicycle.[9] Incidentally, it was at this point in his deliberations that Kepler realized that such a librational

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[7] *Ibid.*, and compare book IX.

[8] Compare Copernicus, *De revolutionibus orbium coelestium* (Thoruni, 1873), III, 4: "someone will ask how the regularity of these librations is to be understood, since it was said in the beginning that the celestial movement was regular, or composed of regular and circular movements ..."; "movement along a straight line is compounded of two circular

movements which compete with one another . . . a reciprocal and irregular movement is composed of regular movements ..." (pp. 165-166). Cf. also, C. B. Boyer, "Note on Epicycles and the Ellipse from Copernicus to Lahire," *Isis*, 1947, *38:* 54-56.

[9] Cf. *De Motibus Stellae Martis*, IV, ch. 58, in *Gesammelte Werke* (München, 1937).

hypothesis could be made equivalent to the elliptical orbit hypothesis, independently discovered useful for calculating Mars' longitudes.[10]

By altering the speed of the planet's east-to-west motion on an epicycle travelling west-to-east, one can approximate to a triangular figure (Figure 6).
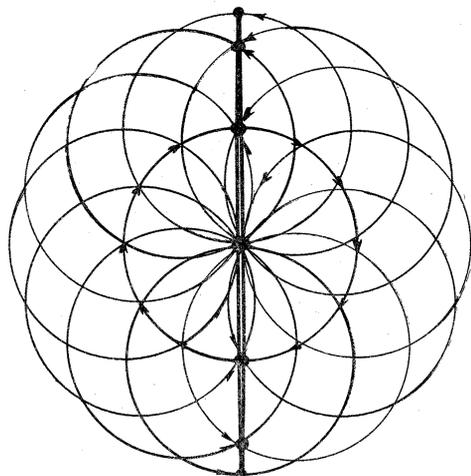

FIGURE 5
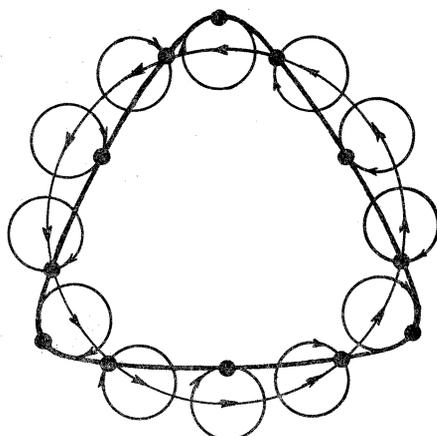
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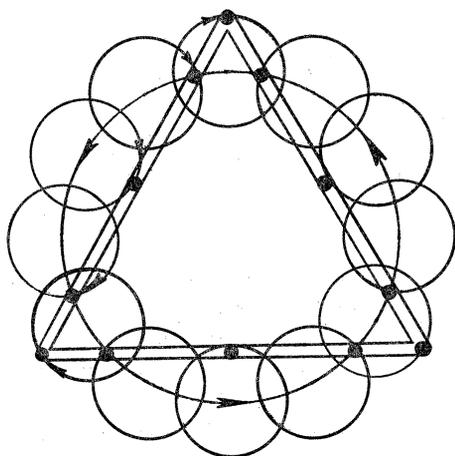
FIGURE 7
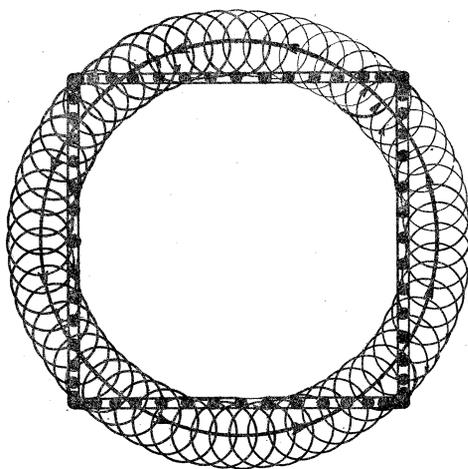
FIGURE 8

Indeed, one can take this to its rectilinear limit (Figure 7).

When confronted with this triangular "orbit," and the square one which follows (Figure 8), several historians and philosophers of my acquaintance

[10] *Ibid.*, ch. 59. Kepler does not actually undertake the detailed construction we have just set out above; but inasmuch as the librations in question are always "librati in dia-metro epicycli," it is clear that Kepler's "justification" of this rectilinear Martian motion would have to be precisely what has been drawn above.

have registered startled, incredulous reactions. Before having seen these figures, they were inclined to regard such resultant "orbits" as non-constructible by the use of epicycle and deferent alone. Observers had to be assured that no trick, or juggling of the epicycle speeds has been responsible for these sharp-cornered figures.

It will be noted, however, that both "orbits" immediately above are described with a rather broad line. This is theoretically significant. But since the second objective of this paper is embodied in the explanation of why this line is broadened, let us return later to a detailed account of the matter.

Consider, first, a few more orbital possibilities contained within the ancient technique. By letting a second epicycle ride along on the first, a complex variety of ellipses are constructible (Figure 9).
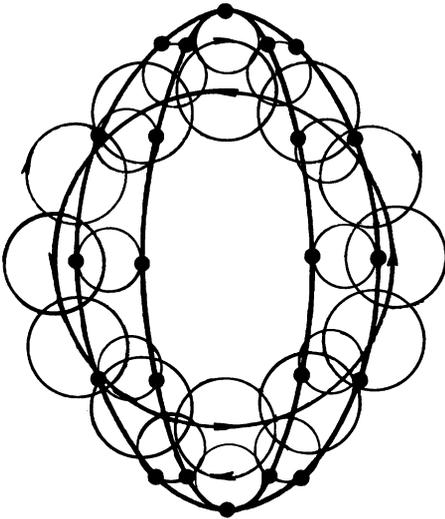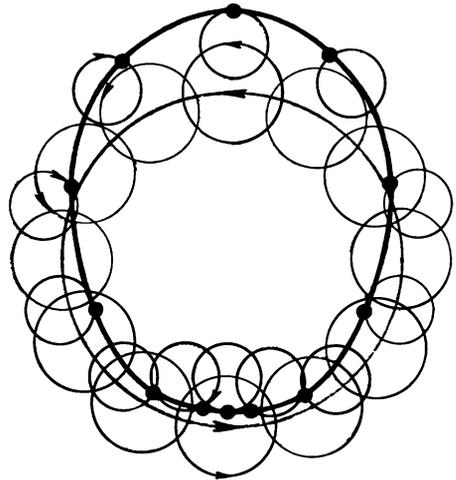

FIGURE 9   FIGURE 10

Indeed, by varying the revolutional speeds of the secondary epicycles, a virtual infinitude of bilaterally-symmetrical curves can be produced (Figure 10). Even Kepler's intractable oviform curve, to determine the equations of which he implored the help of the world's geometers,[11] can be approximated quite closely with a third epicycle (Figure 11).

Some extraordinarily complex, periodically-repetitive configurations are also constructible (Figure 12).

In fact, as will be proved presently, an infinitude of non-periodic "orbits" can also be generated. Simply increase the size of the bundle of epicycles riding on a given deferent, and vary at will the speeds and directions of revolution of the component epicycles. Clearly, the range of complexity for the orbits resulting from the epicycle device is unlimited. *There is no bilaterally-symmetrical, nor excentrically-periodic curve used in any branch of astro-*

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[11] Cf. *De motibus stellae Martis*, IV, ch. 47,   eorumque opem imploro."
p. 297 (ed. cit.): "... appello Geometras

physics or observational astronomy today which could not be smoothly plotted
as the resultant motion of a point turning within a constellation of epicycles,
finite in number, revolving upon a fixed deferent.
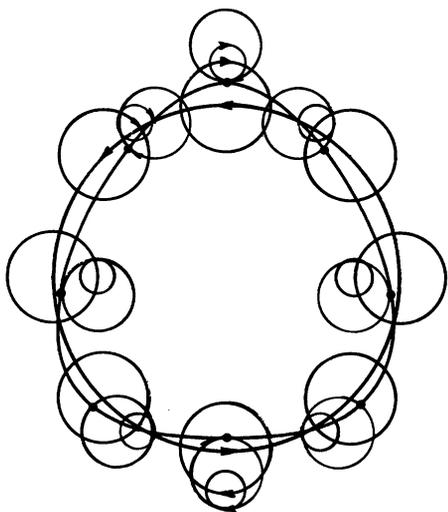

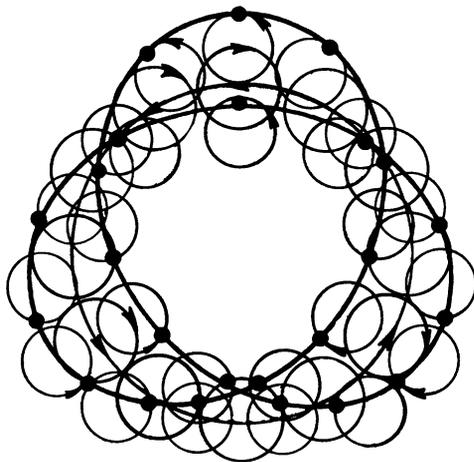FIGURE 11   FIGURE 12

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Bilaterally-symmetrical, and excentrically periodic orbits that are curvilinear — these are one thing, but *rectilinear* polygons are quite another.

Let us return now to the square and the triangular "orbits" depicted above. No one will doubt the formal power of an astronomical-geometrical technique which could even permit a planet to move in a square if need be. We have never put so heavy a demand on our own contemporary techniques — though we could do so if the heavens required it. The point is that Apollonius, Hipparchus, and Ptolemy could, in principle, have done the same thing as ourselves, had the heavens required it of them.

The algebraic demonstration to follow will show that, for all practical astronomical purposes (in the twentieth century as well as in the second), one could, with a finite number of epicycles, generate a square orbit.

More precisely, if the square in question is a *visible* square — i.e., if the lines of which it is constructed have some detectable breadth as well as length — then that square is the possible resultant orbit of some finite combination of epicycles. Choose some arbitrarily small number ϵ, as small as you please but greater than zero; it is possible to get the sinusoidal "wiggle" (typical of epicyclical constructions) along the lines constituting the square so small that the amplitude of the wiggle, i.e., the distance between its crests and troughs, will be smaller than ϵ. And all this with a finite number of epicycles. Theoretically, of course, there must always be some slight cusp at the vertices of the square so long as the epicycles remain finite in number. The cusps will disappear, theoretically, only when the number of constituent epicycles has gone to infinity. But in this case the number ϵ will have gone to the limit zero; the

square in question would then be composed of *Euclidean* straight lines (i.e., one-dimensional curves), in which case the square would no longer be visible anyhow. Thus, no matter how finely the square is drawn, if it can be *seen,* there is some finite number of epicycles of whose resultant motion the square is the construct. The formal proof of this is as follows.

The Representation of an Arbitrary Periodic Motion in the Plane Considered as a Superposition of Circular Motions

A. Let us represent points in the plane by complex numbers $z = x + iy = re^{i\theta}$; the corresponding Cartesian coordinates will be (x,y), and the corresponding polar coordinates will be (r,θ).

Remember that the addition of two complex numbers

$$(z,z') \rightarrow z + z'$$

corresponds to vector addition of the position vectors of the corresponding points (Figure 13).
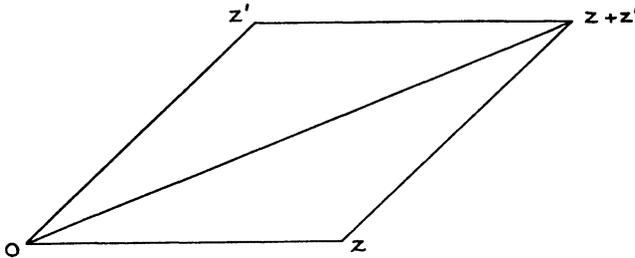

FIGURE 13

B. A uniform circular motion with center c, radius ρ, and period T, may be represented by:

$$z = c + \rho e^{(2\pi it/T) + i\alpha},$$

where t denotes time, and α represents the initial phase of the point.

C. Suppose now that a point A is moving in the way described by the equation:

$$z = f(t)$$

Suppose also that B is moving relative to A in a circle of radius ρ, with period T and initial phase α; then the motion of B is given by the equation:

$$z = f(t) + \rho e^{(2\pi it/T) + i\alpha}$$

We can then think of B as moving on an epicycle carried by A.

D. We see immediately that the superposition of a new epicycle (one now carried by B) is equivalent to the addition of a new term:

$$\rho e^{(2\pi it/T) + i\alpha}$$

This is added to the expression for z. This term can also be written:

$$\rho e^{i\alpha} e^{2\pi it/T}$$

or, more briefly,

$$a e^{ikt}$$

where a is a non-zero complex number, and k is real.

E. Note that any form of retrograde motion corresponds simply to taking T (or k) negative.

F. A motion given by the superposition of n epicycles would be given by an equation of this form:

$$z = a_1 e^{ik_1 t} + a_2 e^{ik_2 t} + \cdots + a_n e^{ik_n t}$$

G. Now suppose that we are just *given* a periodic motion in the plane by the equation:

$$z = f(t)$$

We may assume now that the period of motion is 2π (even if it becomes necessary to change the time scale somewhat).

Suppose that $f(t)$ is a sufficiently well-behaved function [it will be enough that $f(t)$ is continuous and of bounded variation—this is a natural enough condition to impose; that $f(t)$ is taken as continuous and of bounded variation means only that the resultant orbit is a continuous curve of finite length.] Then it is well known that we can write:

$$f(t) = c_0 + (c_1 e^{it} + c_{-1} e^{-it}) + (c_2 e^{2it} + c_{-2} e^{-2it}) = \sum_{n=-\infty}^{\infty} c_n e^{int},$$

this series being *uniformly* convergent.

Let us now write

$$S_N(t) = \sum_{n=-N}^{N} c_n e^{int}$$

Choose now some small number $\epsilon > 0$. Since the series is uniformly convergent, we can now find N_0 such that

$$|f(t) - S_N(t)| < \epsilon$$

for all $N \geqq N_0$ and all t. In other words, if we consider the two orbits:

$$z = f(t), \qquad z = S_N(t),$$

the distance between corresponding points of these two orbits remains *less than* ϵ for all time.

Thus the original orbit, $z = f(t)$, can be replaced, with as small a loss in accuracy as we please, by an orbit $z = S_N(t)$; that is, some finite superposition of epicycles.

H. The function defining a square or a triangular "orbit" satisfies these conditions completely; the above considerations therefore apply without qualification to these special cases illustrated earlier in this paper.

I. The superpositions we have used so far are special ones; for the periods of the epicycles are:

$$\pm 2\pi, \ \pm \pi, \ \pm 2/3\pi, \ \pm 1/2\pi, \ \pm 2/5\pi,\ldots$$

In particular, they are commensurable with one another.

J. However, it must be stressed here that even non-periodic orbits can be represented by such a superposition of epicycles, provided only that we allow incommensurable periods. The basic theorem would be:

Let $z = f(t)$ be a complex-valued function of the real variable t, and suppose

that $f(t)$ is a uniformly almost-periodic function of t (see H. Bohr, *Fastperiodische Funktionen* (1932) for all the necessary definitions); then we can approximate to this motion as closely as we please for *all time* by an expression of the form:

$$z = a_1 e^{i\lambda_1 t} + a_2 e^{i\lambda_2 t} + \cdots + a_n e^{i\lambda_n t},$$

i.e., a superposition of epicycles of radii $|a_1|$, ..., $|a_n|$, and periods $\dfrac{2\pi}{\lambda_1}, \cdots, \dfrac{2\pi}{\lambda_n}$.

To conclude this examination of the elegance, flexibility and power of epicycle-on-deferent astronomy, three further observations are in order.

1. All of our diagrams have been presented in the plane. But this is only a technical limitation, and should not obscure the fact that geocentric astronomy was designed as much to cope with a planet's aberrations in latitude as with those in longitude. Merely by inclining the epicycle's axis of rotation, further variations normal to the plane are easily achieved. By a simple extension of the proofs set out above, it would be possible to move a point (within a finite framework of epicycles) over the surface of a cube, a pyramid, an ovoid—or indeed, over any solid figure of which the foregoing diagrams may be construed as sections. This possibility is built into the ancient technique.

2. Formally speaking, epicycle theory is far from being "a closed book." The famous "brachistochrone" problem introduced by Johann Bernoulli (*Acta eruditorum*, 1696) has affinities with epicyclical motion considered as under the influence of gravity alone. This problem, along with the so-called "isoperimetric" problem, led historically to the development of the calculus of variations by Euler and Lagrange, and ultimately (through Maupertuis, Fermat and Hamilton) into the general theory of variational principles which has become such a powerful formal tool in mechanics, optics, electrodynamics, and in relativity and quantum theory. In more than one place the modern physicist must face dynamical problems which are closely analogous to the very difficulties which the intrepid Ptolemy faced. Indeed, some of his problems require, even today, the most careful analysis for their full solution.

3. It is hardly necessary to point out that our own lunar motions, as well as those of the satellites of Mars, Jupiter and the other planets, are almost purely epicyclical. Our moon faithfully describes what is basically an epicycle moving on a deferent (the earth's orbit) excentric to the sun. Moreover, subtle variations in the ellipsoidal character of the earth's orbit, as well as in the moon's own librations and perturbations, stretch epicycle theory to its utmost as a descriptive astronomical tool. Had he possessed all the data we have, Ptolemy would have been quite at home with our problems of lunar dynamics.

These points are mentioned only to stress further what is behind this article, viz., that Ptolemy's mathematics was, in principle, as powerful, at least for the special problems before him then, as is our own in dealing with these same problems. The ancient technique of epicycle-on-deferent, far from being but a relic of the past, is not dispensable even today.