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O Botafumeiro: Parametric pumping in the Middle Ages

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$$\tau(\epsilon) = \frac{2\pi}{\omega_1(\epsilon)} = 2\pi \frac{l_0}{c_0} \frac{1}{\alpha_1(\epsilon)} = \frac{\pi}{2} \frac{\tau_0}{\alpha_1(\epsilon)}. \quad (3.16)$$

Since $\alpha_1(\epsilon) \to \pi/2$ as $\epsilon \to \infty$, we can set

$$\alpha_1(\epsilon) = \frac{\pi}{2}[1 - \delta(\epsilon)]$$

in Eq. (3.15) to obtain

$$\alpha_1(\epsilon) \underset{\epsilon \to \infty}{\to} \frac{\pi}{2}\left(1 - \frac{1}{\epsilon}\right). \quad (3.17)$$

When this expression for $\alpha_1(\epsilon)$ is put into Eq. (3.16), the result is

$$\tau(\epsilon) \underset{\epsilon \to \infty}{\to} \tau_0\left(1 + \frac{1}{\epsilon}\right). \quad (3.18)$$

Of course, Eq. (3.14) estimates the time at which the end of the spring will return to the neighborhood of its initial position and be at rest while Eq. (3.18) estimates the period of the lowest normal mode. They are not, and need not be, the same, but they are close to each other for large ϵ.

[1]J. T. Cushing, Am. J. Phys. **52**, 925 (1984).

[2]J. M. Bowen, Am. J. Phys. **50**, 1145 (1982).

[3]Reference 1, Eqs. (2.4)–(2.10).

[4]If we introduce the variables $r = \xi + c_0 t$, $s = \xi - c_0 t$ and transform the wave equation to $\partial^2 w/\partial r \partial s = 0$, then we obtain Eq. (2.1) by direct integration.

[5]An integration constant, say A, from integrating Eq. (2.3) can be added onto Eq. (2.4), but then it must be subtracted from Eq. (2.5) so that $w_1(\xi,t)$ is unaffected. One easily verifies that all subsequent expressions for the $w_j(\xi,t)$ remain the same no matter what the value chosen for A. Therefore, we take $A = 0$.

[6]Since we have just established the existence of a $w(\xi,t)$ by explicitly exhibiting it, we can repeat the usual argument for a $\overline{w}(\xi,t)$, which represents the difference of two assumed distinct solutions. However, for that case $\overline{f}(\xi) \equiv 0 \equiv \overline{g}(\xi)$ so that our construction yields $\overline{w}(\xi,t) \equiv 0$. That is, the solution exists and is unique.

[7]Reference 1, Eqs. (2.11) and (2.12).

[8]In fact, *for the* $M = 0$ *case*, if $f(\xi)$ is a polynomial of degree n and $g(\xi)$ a polynomial of degree m, *all* of the $F_j(\eta)$ and $G_j(\eta)$ are polynomials of *at worst* degree $\max(n, m+1)$ in η. This is evident from Eqs. (2.4), (2.5), (2.9), and the condition (2.17) [which is just Eq. (2.12) for $\kappa = 0$]. This means that for Eqs. (2.21) and (2.22) the $w_j(\xi,t)$ can be just first-order polynomials in ξ and t and can be written down at once by making the $w_j(\xi,t)$ continuous across the region boundaries of Fig. 1. This leads directly to the results stated in Eq. (2.23) and in Fig. 2. In the $M = 0$ case the derivatives of $w_j(\xi,t)$ will *not* be continuous across the characteristics (or boundaries) since an impulse travels along the spring. (See the discussion at the end of Sec. III of Ref. 1.) For $M \neq 0$ the first derivatives will be continuous.

[9]As a minor point on Bowen's very nice paper, it should be noted that the initial condition which he applies can be somewhat artificial since, taken literally, it corresponds to an unstretched spring in free fall (in a uniform gravitational field) but instantaneously at rest with the upper end held fixed as the rest of the spring falls to set the spring into oscillation. [Bowen's $U(x,t)$ is the same as our $w(\xi,t)$ and, in our notation, his initial condition is $w(\xi,t=0) = y_0(\xi;M=0)$, where $y_0(\xi;M=0)$ is given in Ref. 16 of Ref. 1 with $M = 0$.] That is, a loosely coiled spring will *not* remain undisturbed when it is held at rest in a vertical position in a uniform gravitational field. While the spring is being held at rest so that its *overall* length remains l_0, the appropriate initial configuration with $\overline{y}_0(0) = 0 = \overline{y}_0(\xi = l_0)$ is $y_0(\xi) = \frac{1}{2}(g/c_0^2)\xi(l_0 - \xi)$. If the lower end is released from rest at $t = 0$, the appropriate initial condition on $w(\xi,t)$ is

$$w(\xi,t=0) = \overline{y}_0(\xi) - y_0(\xi;M=0) = -(gl_0/2c_0^2)\xi.$$

This is of the same form as our initial condition (2.21) with a replaced by $-gl_0^2/2c_0^2$. The source of the downward impulse at $t = 0$ is the unbalanced force $\frac{1}{2}mg$ which the spring exerts on its free end once the external force supporting the end has been removed.

O Botafumeiro: Parametric pumping in the Middle Ages

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The pendular motion of a giant censer (O Botafumeiro) that hangs in the transept of the cathedral of Santiago de Compostela, and is cyclically pumped by men who pull at the supporting rope, is analyzed. Maximum angular amplitude attainable, and number of cycles and time needed to attain it, are calculated; the results agree with observed values ($\sim 82°$, ~ 17 cycles, ~ 80 seconds) to the few percent accuracy of both the analysis and the observations and parameter measurements. The energy gain in a pumping cycle is obtained for an arbitrary pumping procedure to two orders in the small fractional change of pendular length; the relevance of the ratio (characteristic radial acceleration during pumping)/g to the gain is discussed. Effects due to rope mass, air drag on both Censer and rope, and the fact that the Censer is not a point mass, are considered. If the pumping cycle is inverted once the maximum amplitude has been attained, the Censer could be swiftly brought to rest, avoiding the usual violent stop. Historically recorded accidents, rope shape, and the influence of relevant parameters on the motion are discussed.

I. INTRODUCTION

Santiago de Compostela, a town in Galicia, the northwest region of Spain, was a pilgrims' shrine famous throughout Christendom during the Middle Ages.[1] The Cathedral of Santiago, where the remains of St. James the Major are supposed to lie, presents a singular feature dating from those times: In some liturgical functions a giant

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censer, O Botafumeiro, is hung by a rope from a structure high over the crossing and then set into motion along the transept.[2–4] As it sways like a pendulum after being moved slightly off the vertical, a team of men who pull at cords tied at the other end of the rope cyclically decrease and increase its length at the lowest and highest points of the oscillation, respectively. In this way they finally get the Censer to the vaults. This is clearly an example of parametric amplification by pumping,[5] like the familiar swing where a child squats and stands up periodically.

The rite of pumping O Botafumeiro appears to be about 700 years old. The actual Cathedral, lying on the site of an older one razed by muslim leader Almanzor in 997 A.D., was built between the years 1075 and 1122 (though towers and facades were later substantially modified).[6,7] A Codex, Liber Sancti Jacobi,[8] was donated to the Cathedral within the following quarter century; Chapter 3 of its Book III describes the ceremony of a procession inside the Cathedral, now still part of the liturgy of some feasts: no mention of the Censer is made. But a note in Latin, added at the quarter of the 14th century on the margin of folio CLXII of the Codex, just after that description, depicts the swinging of O Botafumeiro and gives notice that "it is now" part of the ceremony.[9] Hence, there can be no doubt that the rite started between the years 1150 and 1325, closer to the last date probably. The next record on the Censer appears in a 1426 Inventory of the Cathedral[10] though the rope was at times said[11] to have been used in the murder of Archbishop Suero Gómez de Toledo, a deed which occurred on 29 June 1366.[12]

The procedure to follow in pumping is far from trivial for laymen who, if asked to explain it, usually do handwaving. The discovery of the procedure, four centuries before the pendulum was studied, was a really nice feat. There is no similarity with a swing in this respect. Swings are probably as old as children, but then every child would learn to pump by himself, much the same way he now learns to ride a bicycle, unconscious perception of his own motion being part of the process; a rule for pumping swings was never needed, and was probably not discussed until recent times.[13–15] Pumping O Botafumeiro, on the other hand, is a team effort, its head, the chief verger of the Cathedral, calling orders where required; body self-perception is not involved. Corrections to the orders were no doubt introduced in the early times since pulling at the cords can be performed in a number of inappropriate ways (shortening and lengthening the rope continuously along the entire oscillation, or at the highest and lowest points, respectively, or only at either one at alternate times) and there was no 13th century dynamics to ask help from. At some point the procedure became formal and explicit knowhow, in the sense that it belonged to a circle of experts and could be transmitted.[16] In fact, transmission does appear to have occurred: Annals of the Chapter of the Cathedral of Orense (100 km to the southeast of Santiago), dating from A.D. 1503, prove that a giant censer used to be swung in its transept at that time.[17] Tradition holds the same for Tuy, 100 km to the south of Santiago.[3] As a contrast, notice that a big, gold censer was hung at old St. Peter's in Rome, in the 7th century, by orders of Pope Sergius I, and no record exists of it having ever been set into motion.[18]

O Botafumeiro used at the present time is made of silvered brass, and dates from 1852 when it replaced an iron one.[3] The Censer described in the Codex was made of silver. A new silver Censer was later donated to the Cathedral by French King Louis XI, when Dauphin; a Bull threatening excommunication to anyone who would rob it, was pronounced by Pope Nicholas V on 27 September 1447.[2] Either Censer broke to pieces in A.D. 1499 in a violent fall (that we shall discuss in Sec. VI). Records on a Censer appear on and off for the following three and a half centuries and notice is given of it being made of silver as late as 1615.[3] There is no record however of its replacement by the iron one. The silver Censer was said[19] to have been taken by French troops in a requisition during their 1809 campaign in Galicia,[6] but it does not appear in the Catalogs, held in the Archives of the Cathedral, of objects of precious stones or metals requisitioned.[3]

The actual structure supporting the Censer, an iron frame resting through four iron corbels on the great piers of the crossing tower, was built in 1602.[20] At that time the primitive support, a system of planks set across the tower at the height of nave and transept vaults, was thought to obstruct excessively the light coming from the windows above. A narrow gallery, circling the inside of the tower one meter above the frame, was built at the same time. (The original crossing tower had been fortified around 1310 and then reformed into its extant form during a period of 50 years about one century later.[7]) The frame supports two coaxial chestnut rollers on which the rope is wound. When an old rope must be replaced, last time for the 1975 Jubilee, a ladder is let to rest on the frame, from the gallery[21]; access to it is gained from outside, through the windows, after walking the stone roof of the nave from an exit just behind a tower in the west facade.

O Botafumeiro had to be an impressive sight in the Middle Ages. A German, J. Munzer, recorded it in a Latin chronicle of his travel through Spain in 1494 A.D., at the dawn of America.[22] The sight remains impressive at present: a mass of 57 kg, getting to a vault 21 m high through an arch 65 m long, the rope almost horizontal, then falling down to pass half a meter above the ground, at 68 km/h. Contrary to the case of swings, O Botafumeiro appears amenable to a quantitatively detailed and precise analysis.

We measured parameters relevant to the problem and made observations of the motion of the Censer; these experimental data are given in Sec. II. In Sec. III we obtain the lowest-order approximation to the energy gain in a cycle, considering both pumping and air drag as weak. In Sec. IV we obtain next-order corrections to the energy gain; we take into account rope mass and Censer size effects, and determine the shape of the rope. This analysis is used in Sec. V to make predictions on the motion of the Censer, comparing them to the observations of Sec. II. Historically recorded accidents are discussed in Sec. VI. In Sec. VII we summarize the results and comment on how they are affected by pertinent changes in some parameters.

II. RELEVANT MEASUREMENTS AND OBSERVATIONS

O Botafumeiro, drawn in Fig. 1, weighs[23] 53 kg and its center of mass lies 50 cm above its base, when vertical. The rope presently used has a diameter $h = 4.5$ cm and a mass per unit length $\lambda = 0.81$ kg/m. Almost 3 m of rope are tied to the ring up in the Censer in a gross, heavy knot reaching 25 cm above its top.

Figure 2 shows the structure supporting O Botafumeiro: the four corbels starting in the great crossing piers, the iron frame in which they end, and the two rollers, of diameters

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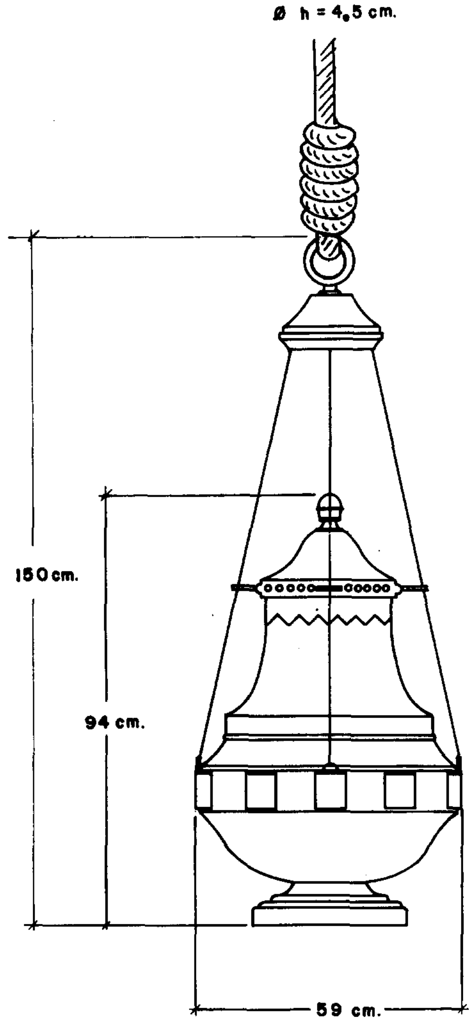

Fig. 1. O Botafumeiro.

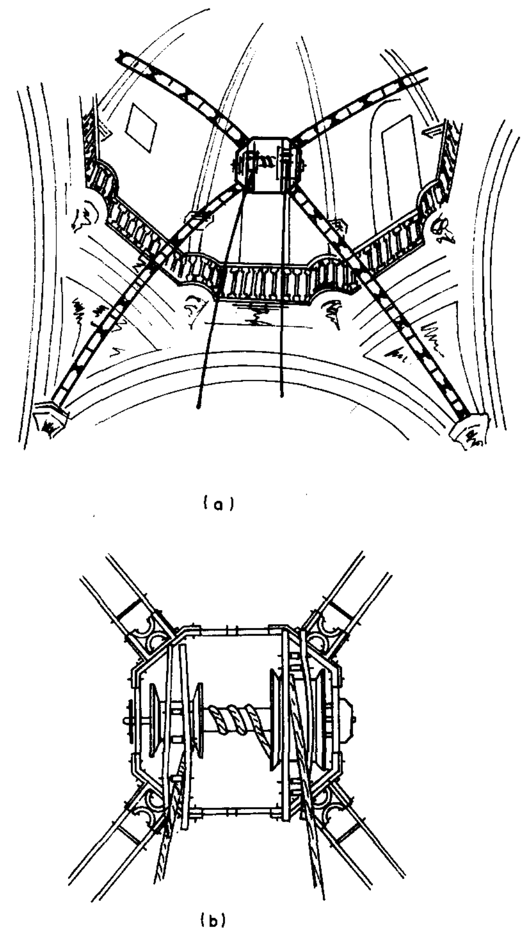

Fig. 2. (a) General view of the structure supporting the Censer; (b) detailed view of frame, rollers, and guides for the rope.

60 and 30 cm, respectively. Wooden guides, hanging loosely from the frame to avoid friction, help maintain the plane of the oscillation. The axis of the rollers stands 21.8 m above the ground.

Once the bowl holding live coals, and then incense, is introduced in the Censer, and considering the knot as part of it, a nearly rigid body of mass $M = 56.5$ kg is obtained. Its center of mass C lies 55 cm above the base and passes 1.2 m above the ground in the lowest point of the oscillation. Thus L, as given in Fig. 3, measures 20.6 m. The length of rope hanging down to the knot, $L - l = 19.4$ m, weighs 15.7 kg.

When the men pull down on their side of the rope, they uncoil about 1.45 m of rope from the thin roller. The axis, common to both rollers, makes close to two turns, winding up a length $\Delta L = 2.9$ m of rope on the thick one. The amplitude of the oscillation when pumping begins, once the Censer is moved off the vertical and given a push, is 13°. After 80 s and 17 pumping cycles (half-periods of the Censer as a pendulum), a maximum amplitude of about 82° is attained, C reaching within 1 m of the vault.

III. LOWEST-ORDER ANALYSIS

Here we neglect both rope mass and Censer size ($\lambda L / M \to 0, l/L \to 0$). Ignoring pumping and air drag the Censer is a pendulum of constant energy

$$E = T + U \equiv \tfrac{1}{2} ML^2 \dot{\theta}^2 + MgL(1 - \cos\theta), \qquad (1)$$

where θ is the angle between the rope and the vertical and U is the potential energy measured from the lowest position of C. Introducing the oscillation amplitude θ_m, where $\dot{\theta} = 0$, we have

$$E = MgL(1 - \cos\theta_m), \qquad (2)$$

$$\dot{\theta}^2 = 2\omega_0^2(\cos\theta - \cos\theta_m), \quad \omega_0^2 \equiv g/L. \qquad (3)$$

The period is $\tau = 4\omega_0^{-1} K(\sin\theta_m/2)$, where K is the elliptic integral of the first kind; $\tau \to 2\pi/\omega_0 \simeq 9.1$ s for $\theta_m \to 0$, $\tau \simeq 10.8$ s for $\theta_m \simeq \pi/2$.

To analyze pumping consider the equations for a pendulum of variable length r,

$$M(2\dot{r}\dot{\theta} + r\ddot{\theta}) = -Mg\sin\theta, \qquad (4)$$

$$M(\ddot{r} - r\dot{\theta}^2 - g\cos\theta) = -F, \qquad (5)$$

where F is the rope tension. The energy equation, $dE = -F\,dr$, follows from adding (4)$\times r\,d\theta$ and (5)$\times dr$; Eq. (4)$\times r$ gives the time derivative of angular momentum. Assume now that r goes through a pumping cycle (Fig. 4). Using just (5) we get the energy gain per cycle,

$$\Delta E_p = -\int_1^5 F\,dr = -\int_1^5 M(r\dot{\theta}^2 + g\cos\theta)dr \qquad (6)$$

($\dot{r}$ vanishes at both 1 and 5). To first order in $\Delta L/L$ we use

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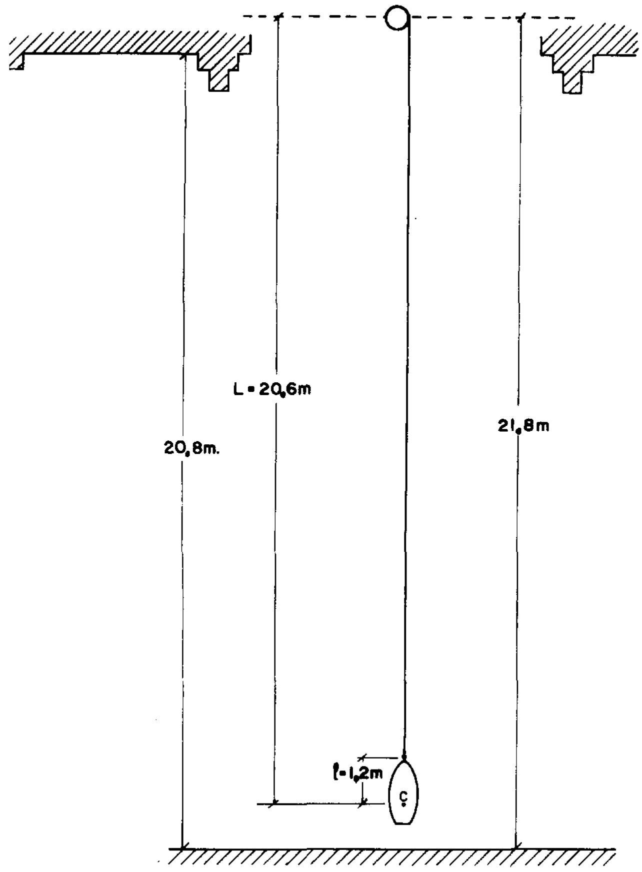

Fig. 3. Schematics showing the Censer hanging in the transept; C is the center of mass.

$r \simeq L$ and Eq. (3) in (6) to obtain

$$\Delta E_p = -3Mg \int_1^5 \cos\theta \, dr . \tag{7}$$

If pumping occurs in steps around $\dot\theta = 0$ and $\theta = 0$, and if the duration δt of each step is short compared with τ, (7) becomes

$$\Delta E_p = 3Mg\Delta L (1 - \cos\theta_m) . \tag{8}$$

Pendulum, or swing, pumping is often said to be based on conservation of angular momentum, $Mr^2\dot\theta$, around $\theta = 0$[24]; from the corresponding change in T, $\Delta\theta_m$ for the cycle can be determined.[13] To find ΔE, changes in U at each step must be taken into account [Haag[14] ignored them and wrongly got $\Delta E_p = 2Mg\Delta L (1 - \cos\theta_m)$]. Note that $Mr^2\dot\theta =$ const follows from (4) *only* if $|\dot r\dot\theta| \gg g|\theta|$, that is, if $\Delta L/g\delta t^2 \gg 1$. Curry[15] derived (8) in the opposite limit: He assumed that for $\Delta L/L$ small F would be the same as in absence of pumping; Eq. (5) shows that this requires $|\ddot r| \sim \Delta L/\delta t^2 \ll g$. Our analysis proves that (8) is valid for $\Delta L/L$ and $\delta t/\tau$ small, and $\Delta L/g\delta t^2$ *arbitrary*. Equation (7) generalizes (8) to $\delta t/\tau$ arbitrary, and clearly shows that ΔE_p is maximum if pumping is instantaneous.

To analyze air drag, we let $\Delta L/L \to 0$. The drag on the Censer is written as[25]

$$\bar F_D = -\tfrac{1}{2}\rho_a v_c^2 S C_D (\bar v_c / v_c) , \tag{9}$$

where S is a convenient area and C_D the drag coefficient; we took $S \equiv \pi(l/3)^2$. Compressibility effects are negligible ($v_c < 20$ m/s). Thus C_D only depends on a Reynolds number, say $R_e \equiv 2lv_c/3\nu$, ν being the air kinematic viscosity

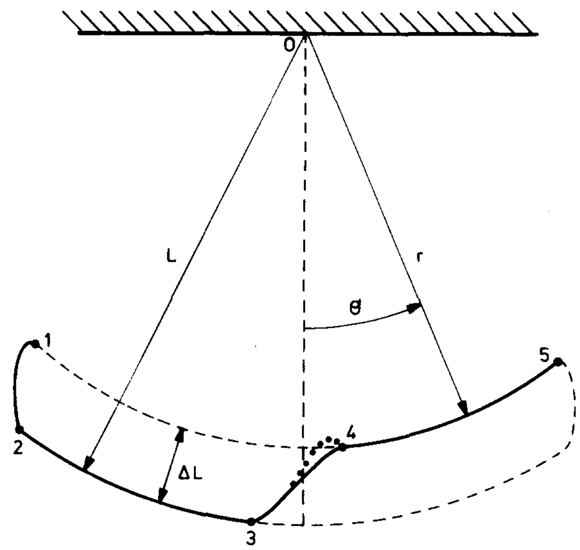

Fig. 4. Schematics of a pumping cycle (1→5) for a simple pendulum. The dotted line represents overshooting.

($\simeq 0.15$ cm^2/s). The range of interest is $10^5 < R_e < 10^6$. Table I shows $C_D(R_e)$, measured in a wind tunnel using a commercial model of the Censer (scale 1/4.5 within a few percent); the lift (toward the base) is also shown.[26] At the highest R_e the turbulent boundary layer is already detached. In using (9) we assumed that the Censer moves in steady translation normal to its axis: The speeds of characteristic points of the Censer lie within the narrow range $v_c(1 \pm l/3L) \simeq v_c(1 \pm 0.02)$; again, the characteristic time for air to flow around it, $l/3v_c$, is much less (by a factor $l/3L$) than the acceleration time $v_c/v_c^2 L^{-1}$.

We write the drag on the element δs of the rope at distance s from its top as

$$\delta \bar f_D(s) = -\tfrac{1}{2}\rho_a v_s^2 h \delta s c_D(s)(\bar v_s / v_s) , \quad \bar v_s = \bar v_c s/L \tag{10}$$

and use the coefficient c_D of an infinite cylinder in steady translation of velocity $\bar v_s$ normal to its axis. This approximation just requires h/s to be small. The Reynolds number is $R_{es} = hv_s/\nu \simeq 0.06 R_e s/L$.

The energy loss is then

$$\frac{dE}{dt} = -\frac{1}{2}\rho_a v_c^3 \left(SC_D + h \int_0^L ds \frac{s^3}{L^3} c_D \right) . \tag{11}$$

Using (3), $v_c^2 = L^2\dot\theta^2$, and $dt = d\theta/\dot\theta$, and integrating from $-\theta_m$ to θ_m we get the loss per cycle

$$\Delta E_D = -2MgL\epsilon P(\theta_m)(1 - \cos\theta_m) , \tag{12}$$

Table I. Drag and lift coefficients of Censer versus Reynolds number. R_e and C_D defined in the text, $C_L \equiv C_D \times$ lift/drag.

R_e	C_D	C_L
1.65×10^5	0.51	...
2.3×10^5	0.61	...
3.3×10^5	0.54	0.05
4.8×10^5	0.52	0.12
6.0×10^5	0.60	0.10
6.9×10^5	0.62	0.10

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where

$$P(\theta_m) \equiv (\sin\theta_m - \theta_m\cos\theta_m)/(1-\cos\theta_m)\,, \qquad (13)$$

$$\epsilon \equiv \rho_a L(S\overline{C}_D + hL\bar{c}_D/4)/M\,; \qquad (14)$$

$\overline{C}_D = 0.59$ and $\bar{c}_D = 1.15$[27] are average drag coefficients, accurate within a few percent. From Eqs. (2), (8), and (12), we finally obtain $\Delta E = \Delta E_p + \Delta E_D$, or

$$\frac{\Delta(1-\cos\theta_m)}{1-\cos\theta_m} = \frac{3\Delta L}{L} - 2\epsilon P(\theta_m)\,. \qquad (15)$$

IV. SECOND-ORDER RESULTS

A. Drag and pumping corrections

The first-order energy changes found in Sec. III for the cycle 1→5 (Fig. 4) can be easily separated into four stages,

$$E\,|_1^2 = -Mg\Delta L\cos\theta_m\,, \quad E\,|_3^4 = Mg\Delta L(3-2\cos\theta_m)\,, \qquad (16)$$

$$E\,|_2^3 = E\,|_4^5 = -MgL\epsilon P(\theta_m)(1-\cos\theta_m)\,, \qquad (17)$$

from which (8) and (12) are recovered. The oscillation amplitudes at points 2, 3, and 4 can be obtained from (16) and (17) and from the obvious expressions $E(1) = MgL - Mg(L-\Delta L)\cos\theta_{m1}$, $E(2) = MgL - MgL\cos\theta_{m2}$, etc.:

$$\cos\theta_{m2} = \cos\theta_{m1} \equiv \cos\theta_m\,,$$

$$\cos\theta_{m3} = \cos\theta_m + \epsilon P(\theta_m)(1-\cos\theta_m)\,,$$

$$\cos\theta_{m4} = \cos\theta_{m3} - (3\Delta L/L)(1-\cos\theta_m)\,.$$

Since $\delta t/\tau$ is small the energy changes take place successively. Hence, to get drag-pumping cross terms we just replace θ_m in (16) and (17) by the modified amplitude at the start of each stage; for stage 4→5 we also use $L-\Delta L$ instead of L. The second-order cross terms to be added to (8) and (12) are then

$$Mg\Delta L\,[\epsilon_r P(\theta_m) - 3\epsilon\theta_m](1-\cos\theta_m)\,,$$
$$\epsilon_r \equiv \rho_a L^2 h\bar{c}_D/4M\,. \qquad (18)$$

To get second-order drag terms we ignore pumping and exactly solve Eq. (11), which can be rewritten as

$$\frac{dE}{d\theta} = -\frac{1}{2}\rho_a L^3\Big(S\overline{C}_D + \frac{hL}{4}\bar{c}_D\Big)\dot\theta^2 \equiv G(E,\theta)\,;$$

$\dot\theta^2$ is given by (1) so that G is linear in E. We integrate from $\theta = -\theta_m$, $E = MgL - Mg(L-\Delta L)\cos\theta_m$, to the angle where $\dot\theta$ vanishes again. Expanding the result in powers of ϵ, we obtain the desired correction to (12),

$$2MgL\epsilon^2\theta_m P(\theta_m)(1-\cos\theta_m)\,. \qquad (19)$$

Finally, we ignore drag and integrate $(L-\Delta L)d\theta\times$Eq. (4) between points 1 and 5 to get

$$[\tfrac{1}{2}M(L-\Delta L)r\dot\theta^2 - Mg(L-\Delta L)\cos\theta]\,|_1^5$$
$$= -\frac{3}{2}\int_1^5 M(L-\Delta L)\dot\theta^2\,dr\,. \qquad (20)$$

The left-hand side is ΔE. Subtracting $2\times$Eq. (20) from $3\times$Eq. (6) and using (3) in the integral $\int_1^5[r-(L-\Delta L)]\dot\theta^2\,dr$ we obtain

$$\Delta E_p = -3Mg\int_1^5\Big(1 + 2\frac{r-(L-\Delta L)}{L}\Big)\cos\theta\,dr\,; \qquad (21)$$

Eq. (21) carries the general result (7) to second order in $\Delta L/L$. Taking $(\cos\theta - \cos\theta_m)$ small in step 1→2, and $(1-\cos\theta) \simeq \theta^2/2$ small in step 3→4, Eq. (21) yields (8) + corrections terms,

$$3Mg\frac{\Delta L^2}{L}(1-\cos\theta_m) + 3Mg\int_1^2(\cos\theta_m - \cos\theta)dr$$
$$+\frac{3}{2}Mg\int_3^4\theta^2\,dr\,. \qquad (22)$$

The integrals do depend on the pumping process, and in particular, on the ratio $\Delta L/g\delta t^2$; they vanish for instantaneous pumping ($\Delta L/g\delta t^2 \to \infty$). Tea and Falk[15] took that limit [$\Delta L/L = O(1)$, $\delta t/\tau$ small]. Unfortunately, such an approximation is invalid for the Censer, and for swings that hang by ropes: $\ddot r$ cannot exceed $3g$ during the positive $\ddot r$ motions at $\theta = 0$ and $\theta = 0$. Usually, $\Delta L/g\delta t^2 = O(1)$ in swings with rigid bars too (F/Mg is of order unity).

Step 1→2 seems to be close to a free fall. Making this approximation, (22) transforms into

$$-Mg\Delta L\left[\left(\frac{8\Delta L}{L}\right)^{1/2}\phi(y_m)(1-\cos\theta_m) + 3I\frac{\Delta L}{L}\right]$$
$$\times(1-\cos\theta_m)\,, \qquad (22')$$

where

$$\phi \equiv (y_m^2+1)^{3/2} - y_m(y_m^2 + \tfrac{3}{2})\,,$$
$$y_m \equiv \cos\theta_m/(2\Delta L/L)^{1/2}\,. \qquad (23)$$

Note that (22') is of order $(\Delta L/L)^{3/2}$ for $\cos\theta_m$ small enough. The integral from 3 to 4 was rearranged by setting $t = 0$ at $\theta = 0$, using $\theta^2 = \dot\theta^2 t^2$ and (3), and defining $I = -\int_3^4 gt^2\,dr/\Delta L^2$. To estimate I we solve (5) with the model

$$\ddot r = -bg\,, \qquad t_3 < t < t_*\,,$$
$$\ddot r = (3-2\cos\theta_m)g \text{ (free fall)}\,, \qquad t_* < t < t_4\,.$$

Initial conditions are $r_3 = L$, $\dot r_3 = 0$; r nd $\dot r$ are continuous at t_*. We determine t_* and t_4 by setting $r = L - \gamma\Delta L$ at $\dot r = 0 (\gamma > 1$, Fig. 4) and $r_4 = L - \Delta L$, respectively. We can thus calculate $I(\theta_m, b, \gamma, gt_3^2/\Delta L)$, and θ_3 and θ_4 as well. Bounds on $b, \gamma, gt_3^2/\Delta L$ can be inferred from observations and reasonable estimates of maximum force exerted by the men at the rope; typically $-0.2 < I < 0.2$ (the overshooting can make I negative). In the following we set $I = 0$.

B. $\lambda L/M$ and l/L corrections

Figure 5 shows the Censer as a compound, double pendulum hanging by a heavy rope. First take $l/L = 0$, and let $\bar r(s,t)$ be the vector from point 0 to the element s of the rope. The equation for $\bar r$ is

$$\lambda\frac{\partial^2\bar r}{\partial t^2} = \lambda\bar g + \frac{\partial}{\partial s}\Big(F_s\frac{\partial\bar r}{\partial s}\Big) - \frac{1}{2}\rho_a h\bar c_D\Big(\frac{\partial\bar r}{\partial t}\Big)^2\frac{\partial\bar r/\partial t}{|\partial\bar r/\partial t|}\,. \qquad (24)$$

To lowest order both $\partial\bar r/\partial s$ and the tension F_s are independent of s. Consider orthonormal vectors $\bar u$ and $\bar w$; $\bar u$ points from 0 to A ($\equiv C$ for $l/L = 0$) to all orders. Writing $\bar r = s\bar u(t) + \bar r_1(s,t) + \ldots$, we use $\bar r(s=0) = 0$, $\bar r(s=L)\cdot\bar w = 0$, and $|\partial\bar r/\partial s| = 1$ to get $\bar r_1 = r_1\bar w$ and $r_1(s=L) = 0$. Thus $\bar r_C = L\bar u$. To obtain the total energy we add the rope energy (where $\bar r \simeq s\bar u$) to (1):

$$E = \tfrac{1}{2}\lambda gL^2 + MgL(1 + \lambda L/2M)(1-\cos\theta_m)\,, \qquad (25)$$

$$\dot\theta^2 = 2\omega_0^2(1 + \lambda L/6M)(\cos\theta - \cos\theta_m)\,. \qquad (26)$$

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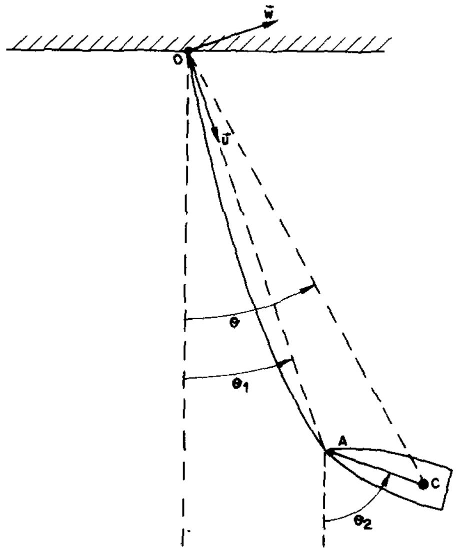

Fig. 5. The Censer as a compound, double pendulum, hanging by a heavy rope.

For θ_m small (26) gives the well-known frequency $\omega_0(1+\lambda L/12M)$.[28,29]

To find corrections to Eqs. (8) and (12), we solve (24) to first order; $F\equiv F_s(s=L)$ is obtained from (5), using $r=L$, $\ddot{r}=0$, and (26). Then

$$\frac{r_1}{L}=-\frac{(1-s/L)s/3L}{3\cos\theta-2\cos\theta_m}\left[\frac{\lambda L}{M}\left(1-\frac{s}{2L}\right)\sin\theta + \epsilon_r\left(1+\frac{s}{L}+\frac{s^2}{L^2}\right)(\cos\theta-\cos\theta_m)\right], \quad (27)$$

$$F_s=Mg(3\cos\theta-2\cos\theta_m)+\lambda Lg\left[\left(\frac{4}{3}-\frac{s^2}{L^2}\right)\times(\cos\theta-\cos\theta_m)+\left(1-\frac{s}{L}\right)\cos\theta\right]. \quad (28)$$

For $\Delta L/g\delta t^2\to 0$ we have $\Delta E_p=\Delta L\,[F_s(s=0,\theta=0)-F_s(s=0,\theta=\theta_m)]$; the correction to (8) is then

$$7\lambda Lg\Delta L\,(1-\cos\theta_m)/3\,. \quad (29)$$

For $\Delta L/g\delta t^2\to\infty$, conservation of total angular momentum $(ML^2\dot\theta+\lambda L^3\dot\theta/3)$ at $\theta=0$ leads to a kinetic energy change; adding the potential energy changes at $\theta=-\theta_m$ and $\theta=0$ for both Censer and rope, (29) is again recovered. We shall use (29) for $\Delta L/g\delta t^2$ arbitrary. Corrections to (12) arise from using (26) instead of (3) in $v_c^2=L^2\dot\theta^2$, and using $\bar{v}_s=\dot\theta(s+\partial r_1/\partial\theta)\overline{w}-\dot\theta r_1\bar{u}$ instead of $\bar{v}_s=s\bar{v}_c/L=\dot\theta s\overline{w}$; the $\lambda L/M$ correction to (12) is then

$$(\lambda gL^2/3)(\tfrac{4}{5}Q\epsilon_r-\epsilon)P(\theta_m)(1-\cos\theta_m)\,. \quad (30)$$

Q is a function of θ_m satisfying $0.92<Q<1.10$ for $13°<\theta_m<82°$; we shall use $Q=1$ in the following. There is no drag–drag correction due to rope curvature because the ϵ_r term in (27) is an even function of θ.

Now take $\lambda L/M=0$, $l/L\neq 0$. We find a correction to (12),

$$8Mgl\epsilon_r P(\theta_m)(1-\cos\theta_m)\,; \quad (31)$$

it arises from using a rope length $L-l$ instead of L in the rope drag. We find (1) valid to within 1%. {Compound pendulum effects are negligible simply because the moment of inertia with respect to an axis through C is $I\sim 10^{-3}ML^2$. Double pendulum effects would be much larger if the amplitude of the relative oscillation, $\psi\equiv\theta_2-\theta_1$, were comparable to θ_m [its frequency is of the order of $\omega_0(L/l)^{1/2}\gg\omega_0$]. Fortunately, the ratio ψ_m/θ_m is observed to be small. To see why, consider the equation for ψ, $\ddot\psi=-lF\psi/I$; in the absence of pumping the frequency, and thus the energy, of ψ just oscillates with the long period τ.[30] Pumping is accompanied by sudden changes in F, and thus in the frequency of ψ; the sudden change in oscillation energy will depend on the value of the phase at the time. Hence, we would expect ψ_m to exhibit fluctuations instead of monotonous growth as the motion proceeds.}

V. UP TO THE VAULT AND DOWN

Let $\theta_n\equiv\theta_m(n)$ be the amplitude at the end of the nth pumping cycle. Then Eq. (15) takes the form

$$\cos\theta_{n+1}-\cos\theta_n = -[3\Delta L/L-2\epsilon P(\theta_n)](1-\cos\theta_n)\,. \quad (32)$$

With the corrections (18), (19), (22′), (29)–(31), and (25), and setting $I=0$, $Q=1$, we have instead

$$\left(1+\frac{\lambda L}{2M}\right)(\cos\theta_{n+1}-\cos\theta_n) = -\left\{\frac{3\Delta L}{L}\left[1+\frac{7\lambda L}{9M}-\left(\frac{8\Delta L}{9L}\right)^{1/2}\phi(y_n)(1-\cos\theta_n)\right] - 2\epsilon P(\theta_n)\left[1+\frac{\lambda L}{6M}-\frac{\epsilon_r}{\epsilon}\left(\frac{\Delta L}{2L}+\frac{2\lambda L}{15M}+\frac{4l}{L}\right) + \frac{3\theta_n\Delta L}{2LP(\theta_n)}-\epsilon\theta_n\right]\right\}(1-\cos\theta_n)\,; \quad (33)$$

$\phi(y)$ is given by (23). Figure 6 shows the solution to Eq. (33) with initial condition $\theta_0=13°$, using $\rho_a=1.25\ \mathrm{kg/m^3}$ and the data of Secs. II and III; when $n\to\infty$, θ_n tends to a limit θ_∞ for which the bracket vanishes. Also shown is the time t_n at the end of each cycle; t_n is obtained from the equation $\Delta t=\int_{\theta_1}^{\theta_5}d\theta/\dot\theta$ (Fig. 4), which may be manipulated into the

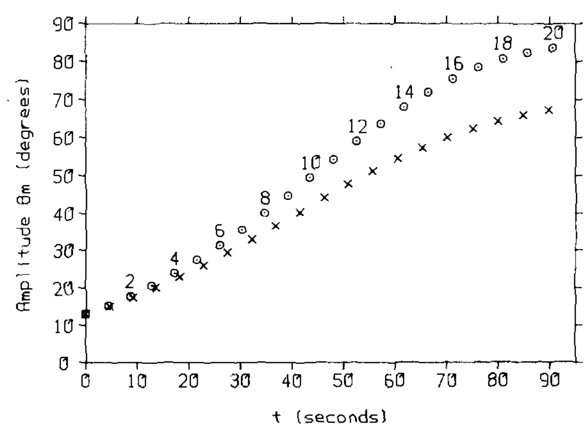

Fig. 6. Oscillation amplitude and time elapsed at end of each pumping cycle (numbers displayed are cycle numbers). (⊙) Obtained from Eqs. (33) and (34); (×) obtained from a gross approximation (Sec. V), and shown for comparison.

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form

$$t_{n+1} = t_n + \frac{2}{\omega_0} K\left(\sin\frac{\theta_n}{2}\right)\left[1 - \frac{\lambda L}{12M} - \frac{\Delta L}{4L} + \left(\frac{\Delta L}{2L} - \frac{\epsilon}{3} P(\theta_n)\right) Q_1(\theta_n)\right]. \quad (34)$$

Equation (34) exhibits first-order corrections to the half-period of a simple pendulum. The last term, where Q_1 involves complete elliptic integrals, can be neglected for all θ_n here. For comparison Fig. 6 also shows θ_n and t_n as given by the gross approximations (32) and $t_{n+1} = t_n + 2\omega_0^{-1} K(\sin\theta_n/2)$.

The fact that the amplitude cannot exceed a maximum value θ_∞ is confirmed by experience (the men who pump are really unable to hit the vault with the Censer).[21] The observed value of the maximum ($82 \pm 2°$) is in good agreement with our prediction ($\theta_\infty \simeq 85°$), which is also accurate to within 2 or 3 degrees: The bracket of (33), an expansion to two orders in the small parameters $\Delta L/L$, $\lambda L/M$, etc., has an accuracy of a few percent. The uncertainty in ΔL (2.9 ± 0.3 m) is large, but we checked that a 10% error in ΔL leads to a 3% error in θ_∞. Few percent uncertainties in other quantities (ρ_a, M, $\overline{C}_D$, and so on) are found to hardly affect the value of θ_∞.

The observed number of pumping cycles (17 ± 1) is in good agreement with the value n_∞ (~ 18) required in Fig. 6 to get within a few degrees from θ_∞. Again the major uncertainty is ΔL; it may lead to an error of ± 1 in n_∞. Figure 6 predicts a time t_{17} [weakly dependent on ΔL according to Eq. (34)] that agrees within a few percent with the measured duration of 17 cycles (~ 80 s).

Once pumping ceases, the oscillation amplitude decreases due to air drag. The decrease is naturally very slow at low amplitudes as shown in Fig. 7, obtained by taking $\theta_0 = 82°$ and setting $\Delta L = 0$; we used Eq. (32) for simplicity. In actual fact the head of the team finally halts the Censer by sheer force. Note however that the Censer could be efficiently braked by inverting the pumping cycle of Fig. 4 (rope lengthening at $\theta = 0$, shortening at $\dot{\theta} = 0$). This would amount to changing the sign of ΔL in (32); the relation $\theta_m(n)$ that ensues is shown in Fig. 7 for comparison. Since (32) was used, Fig. 7 is only approximate.

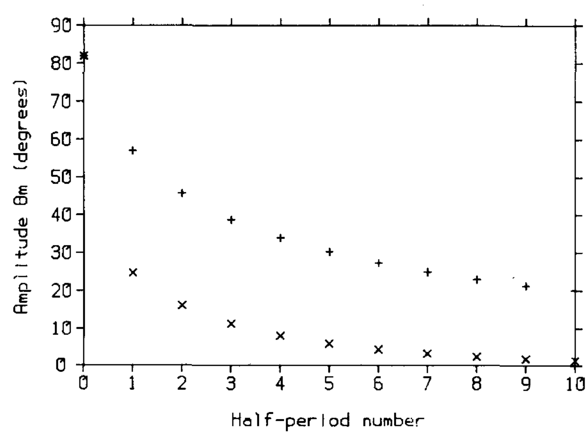

Fig. 7. Amplitude decay from its maximum value, versus half-period number, using Eq. (32) for simplicity: (+) no pumping, $\Delta L \to 0$; ($\times$) inverted pumping cycle, $\Delta L \to -\Delta L$.

VI. ACCIDENTS

If pumping is ignored the tension F in the rope is largest when $\theta = 0$, at high θ_m [Eq. (28)]. Pumping changes this: accidents in which the Censer would break loose are in fact more probable when $\theta = \theta_m$, at low θ_m. This is because a large tension must develop to stop the fall $1 \to 2$ (Fig. 4) in each pumping cycle; the work by F during the stop, $Mg\Delta L \cos\theta_m$, is smaller at higher amplitudes. Such an accident did actually occur in the past. On 23 May 1622, the rope broke and the Censer fell vertically (implying $\theta \simeq \theta_m$) near the men who pull at the rope (implying low θ_m: the men stand almost below the supporting frame).[3]

Pumping at $\theta = 0$ is a second occasion for accidents. At high θ_m the tension during the positive $\ddot{r}$ motion (Sec. IV A) could easily be twice as large as the maximum tension ($\sim 3Mg$) in the absence of pumping. History has recorded just one more accident, and it seems to fit those conditions. The Censer hangs by four chains from the top ring (Fig. 1). On 25 July 1499, the chains broke and the Censer flew to one side of the transept, crushing at the door.[2,31] Note that if one of the chains broke during pumping around $\theta = 0$, and if θ_m was high, the angle θ_b at which the Censer got loose needed not be small. We now show that the door, distant 32.5 m from the center, could indeed be reached only if θ_m was high and θ_b moderate. To this end consider the free motion after breaking, neglect air drag, and set $\cos\theta_m = 0$. When the Censer gets level with the lowest point of oscillation, its distance D to the center is given by

$$D = (L - \Delta L)[\sin\theta_b(1 + 2\cos^2\theta_b) + 2\cos^{3/2}\theta_b(1 - \cos^3\theta_b)^{1/2}].$$

D presents a maximum, ~ 42 m, at $\theta_b = 41°$. The maximum would be reduced by air drag. Clearly D could be less than 32.5 m if $\cos\theta_m$ was not small or if θ_b departed sensibly from its optimum value.

Figure 8 shows quantitatively why accidents involving the rope and the first arch in the transept do not occur. Equation (27) was used to draw the shape of the rope at maximum amplitude. Due to its own weight the rope bends just enough to miss the arch.

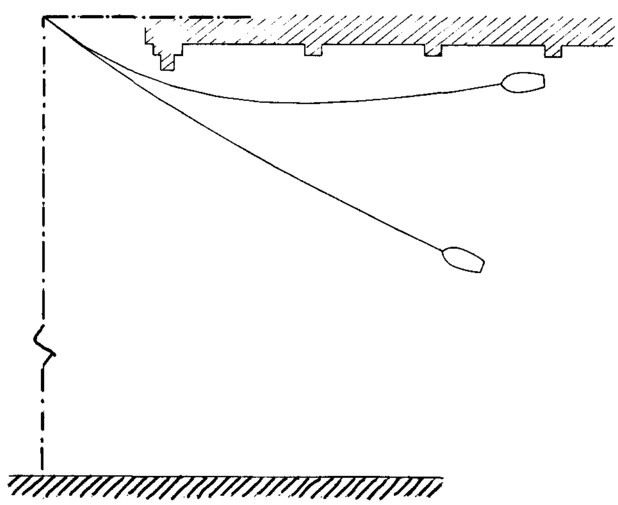

Fig. 8. Shape of the rope at $\theta = \theta_m$, for $\theta_m = 60°$ and $\theta_m = 82°$ (maximum amplitude).

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VII. SUMMARY AND COMMENTS

Our analysis of O Botafumeiro as a pumped pendulum yielded the angular amplitude, and the time elapsed, at the end of each pumping cycle. We found that as the number of cycles increases the amplitude goes to a limit value θ_∞, just below the vault. We obtained the time and the number of cycles required to get within a few degrees from θ_∞. To the accuracy achieved (a few percent) all results are in agreement with observations.

We carried out a quite general analysis of the pumping process. The energy gain per cycle for an arbitrary manner of pumping was obtained to second order in the ratio $\Delta L / L$, L and $L - \Delta L$ being the maximum and minimum pendular lengths. The ratio $\Delta L / g\delta t^2$, where δt is the duration of each pumping step, is the parameter affecting the result (which is optimum for $\Delta L / g\delta t^2 \to \infty$). We took into account air drag, both on the Censer and the rope, and the rope mass effects, as well as the fact that the Censer is not a point mass. Historically recorded falls and rope bending were also considered. We finally suggested that once the Censer reaches the vault, it could be efficiently braked so as to avoid the violent final stop, by inverting the pumping cycle.

Some comments on the values of relevant parameters are appropriate. One may presume that when the pumping procedure was discovered seven centuries ago a single roller was used to coil on and support the rope. The value of ΔL had to be then the length of rope that the men pull down (~ 1.5 m, the largest to allow confortable pumping). For $\Delta L \sim 1.5$ m, Eq. (33) yields $\theta_\infty \sim 55°$, well below the vault. Probably the need to increase ΔL was felt soon and led to the actual use of two rollers with diameter ratio of 2: according to the oldest record (mentioned in Sec. I) the Censer used to reach the vault at the time.

The present rope, of diameter $h = 4.5$ cm, replaced one with $h = 4$ cm in 1975. A photograph taken in the first third of this century shows a rope with $h = 3.2$ cm.[4] Records from the 17th century[3] suggest an even thinner diameter at that time. Aside from a major probability of breaking a thin rope produces weak accidents. For $h = 3.2$ cm, say, rope deflection is about half as large as at present (Fig. 8); at maximum amplitude the rope would hinge on the first arch of the transept. Further, both energy pumped in *and* rope drag decrease with h; it appears that lowering h sufficiently would increase θ_∞ by a few degrees: the Censer would touch the vault, a fact now impossible that did happen in the past.[3]

On occasions the men at the rope have tried and failed to pump a light silver ornament that usually hangs in the crossing in the place of the Censer.[21] This can be easily explained. At high amplitudes the energy pumped in per cycle is of the order of $Mg\Delta L$, where M is the mass of the object being pumped; the energy loss due to rope drag is of the order of $\rho_a g L^3 h$, the remaining drag loss is of the order of $\rho_a g L^2 S$ ($S \equiv$object front area). For M small enough θ_∞ will clearly be small. A censer too light cannot be pumped.

Pumping a giant censer in a Gothic cathedral would have been an almost impossible feat. Gothic vaults are typically twice as high (~ 40 m) as the Romanesque vault of Santiago. Doubling L would increase the energy loss well over one order of magnitude; M, and thus both S and h, would have to be larger. A team of more than 50 men would be needed and the mass of the Censer would exceed 500 kg, a definite hardship on the supporting structure. Note that enlarging ΔL (instead of M) by a factor of 10 to about $3L/4$, would make the pumping cycle unfeasible.

ACKNOWLEDGMENTS

The author is grateful to the chief verger of the Cathedral, A. Raposo Guldrin, who provided information and gave support for the measurements; to P. Paseiro Losada who helped in carrying out the measurements; and to A. Feo Palacios who measured the drag coefficient of the Censer.

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Reversibility and step processes: An experiment for the undergraduate laboratory

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An experiment with a spring is described to illustrate the fact that the irreversibility involved in a physical process in transforming the system from an initial state to a terminal state bears an inverse relationship to the number of discrete steps in which it is carried out, leading to the conclusion that the process becomes reversible as the number of steps tends to infinity. A similar relationship is shown to hold for processes like charging of a capacitor and compression of a perfect gas.

I. INTRODUCTION

The concept of reversibility in thermodynamics is an intricate one for students at the undergraduate level. A good number of introductory texts[1] state that in the transformation of a system from an initial state to a terminal state by indefinitely increasing the number of changes and correspondingly decreasing the size of each change,[2] we arrive at an ideal process in which the system passes through a continuous succession of equilibrium states and that a transformation of this kind is reversible. This statement is only intuitive and no experimental effort to establish it seems to have been made to our knowledge.

Recently Calkin and Kiang[3] have shown that a transformation from $\Delta S > 0$ to $\Delta S = 0$, in the process of raising the temperature of a given amount of water from T_a to T_b, may be carried out as the number of steps are increased to infinity.

In this paper we report an experiment which gives a concrete visualization of this fact. We believe that this can be a useful experiment for an undergraduate laboratory in teaching this concept. The experiment consists of stretching a spring from an initial state to a final stretched state in a varied number of steps. It is shown that the increase in entropy in the process is an inverse function of the number of steps in which the process is performed. From this it is inferred that the process becomes reversible when the number of steps tends to infinity.

In Sec. II, the theory of the experiment is discussed. Section III describes the experimental setup and finally in Sec. IV, experimental procedure and the results are discussed. Appendices are added to establish a similar correlation for the processes of charging of a capacitor and compression of a perfect gas.

II. THEORY

Let a linear[4] spring of force constant K be loaded with a total mass M but in N equal steps each time by a mass $m = M/N$. When the spring is loaded by a mass, it not only undergoes extension but also performs damped oscillations about the new position of equilibrium, gradually dissipating its energy into heat. Let h and H be the static extensions of the spring due to masses m and M, respectively. Then $H = Nh$.

The loss in gravitational energy of the loaded mass is given by

$$mgh + 2mgh + \cdots + Nmgh = \tfrac{1}{2} MgH(1 + 1/N). \qquad (1)$$

The elastic energy stored in the spring due to loading is equal to

$$\tfrac{1}{2} KH^2 = \tfrac{1}{2} MgH. \qquad (2)$$

Therefore the energy dissipated in the form of heat is given by

$$\tfrac{1}{2} MgH(1 + 1/N) - \tfrac{1}{2} MgH = \frac{1}{2}\frac{MgH}{N}. \qquad (3)$$

Assuming that the temperature T of the surroundings does not change,[5] the increase in entropy in this process is given by

$$\Delta S = \tfrac{1}{2} MgH/NT. \qquad (4)$$

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