THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

VOLUME XXV | MAY, 1918 | NUMBER 5

ORIGIN OF THE NAME "MATHEMATICAL INDUCTION."

By FLORIAN CAJORI, Colorado College.

The process of reasoning called "mathematical induction" has had several independent origins. It has been traced back to the Swiss Jakob (James) Bernoulli,[1] the Frenchmen B. Pascal[2] and P. Fermat,[3] and the Italian F. Maurolycus.[4] The process of Fermat differs somewhat from the ordinary mathematical induction; in it there is a *descending* order of progression, leaping irregularly over perhaps several integers from n to $n - n_1$, $n - n_1 - n_2$, etc. Such a process was used still earlier by J. Campanus in his proof of the irrationality of the golden section, which he published in his edition of *Euclid*, 1260. By reading a little bit between the lines one can find traces of mathematical induction still earlier, in the writings of the Hindus and the Greeks, as, for instance, in the "cyclic method" of Bhaskara, and in Euclid's proof[5] that the number of primes is infinite.

[1] Jacobi Bernoulli, Basileensis, *Opera*, Tomus I, Genevæ, MDCCXLIV, p. 282, reprinted from *Acta eruditorum*, Lips., 1686, p. 360. See also Jakob Bernoulli's *Ars conjectandi*, 1713, p. 95; or a German translation by Haussner in *Ostwald's Klassiker*, No. 107, p. 96.

[2] *Œuvres complètes de Blaise Pascal*, Vol. 3, Paris, 1866, p. 248. See also M. Cantor, *Vorlesungen über Geschichte der Mathematik*, 2 Aufl., 1900, pp. 749, 756, 757; H. G. Zeuthen, *Geschichte der Mathematik im XVI. und XVII. Jahrhundert*, deutsch von R. Meyer, Leipzig, 1903, pp. 168, 172; W. H. Bussey, in AM. MATH. MONTHLY, Vol. 24, 1917, pp. 203–215.

[3] Charles S. Peirce in the *Century Dictionary*, Art. "Induction," and in the *Monist*, Vol. 2, 1892, pp. 539, 545; see also F. Cajori in *Bulletin Am. Math. Soc.*, Vol. 15, 1909, pp. 407, 408. Peirce called mathematical induction the "Fermatian inference."

[4] G. Vacca, *Bulletin Am. Math. Soc.*, Vol. 16, 1909, pp. 70–73; M. Cantor in *Zeitschr. f. Math. u. Naturwiss. Unterricht*, Vol. 33, 1902, p. 536; W. H. Bussey, AM. MATH. MONTHLY, Vol. 24, pp. 200–203.

[5] *Elements*, Bk. 9, Prop. 20.

But no one, to my knowledge, has before this time traced historically the origin of the *name* "mathematical induction." Why should a process of argumentation in its essence so foreign to the "induction" known to natural science be called "mathematical induction"? Maurolycus, Pascal and Fermat gave no special names to their logical processes. It is our purpose in this article to show that one of the several origins of the *process* lay in the process of "induction" from isolated facts, and that historically the *name* "mathematical induction" took its origin in the *name* "induction" of natural science.

The earliest mathematicians who appear in the history of the name of the processes in question were John Wallis and Jakob Bernoulli. Wallis, in his *Arithmetica infinitorum* (Oxford, 1656), page 15, Proposition XIX, proceeds to find "per modum inductionis" the ratio of the sum of the squares 0, 1, 4, 9, $\cdots$ n^2 to the product $n^2(n+1)$. He writes down by inspection six relations, of which the first three are as follows:

$$\frac{0+1=1}{1+1=2}=\frac{3}{6}=\frac{1}{3}+\frac{1}{6},$$

$$\frac{0+1+4=5}{4+4+4=12}=\frac{1}{3}+\frac{1}{12},$$

$$\frac{0+1+4+9=14}{9+9+9+9=36}=\frac{7}{18}=\frac{1}{3}+\frac{1}{18}.$$

He observes that in all six cases the ratio is greater than $\frac{1}{3}$, but the excess decreases steadily as the respective number of squares increases; for the six cases which he writes down the excesses are respectively $\frac{1}{6}, \frac{1}{12}, \frac{1}{18}, \frac{1}{24}, \frac{1}{30}, \frac{1}{36}$. He takes the law of decrease, that is shown in the first six cases to hold when n exceeds six and increases without limit; the excess is seen to approach the limit zero. This process, and phrases like "fiat investigatio per modum inductionis," are repeated in the study of ratios like the above involving the sums of the first, second and third powers of numbers. The statement is added that the method applies equally well to higher powers. He speaks, p. 33, of "rationes inductione repertas" and freely relies upon incomplete "induction" in the manner followed in natural science.

Wallis's incomplete "induction" brought both praise and blame. Twenty-nine years later he says in his *Treatise of Algebra*, London, 1685, Chapter 78 (page 298): "Those Propositions in my Arithmetick of Infinites, are (some of them) demonstrated by way of Induction: which is plain, obvious, and easy; and where things proceed in a clear regular order (as here they do), very satisfactory." On page 310 Wallis comments on Bullialdus, the author of *Ad arithmeticam infinitorum:* "He must either waive all such Progressions (as not of use, or at least, as not demonstrated); or else must be content to close his Induction (as I do) . . . he doth allow the Doctrine, as sound and good (and much applauds it) and the Demonstration (by Induction) as sufficient, (by admitting it himself). Only he thinks I have not done my invention so much honour as it doth deserve."

Wallis states (page 306) that Fermat "blames my demonstration by Induction, and pretends to amend it. . . . I look upon Induction as a very good method of Investigation; as that which doth very often lead us to the easy discovery of a General Rule."

Another criticism came from Jakob Bernoulli,[1] who takes one of Wallis's problems and proceeds to show how the procedure can be improved by introducing the argument from n to $n + 1$. Thus, in Bernoulli's mind, incomplete induction, because of its incompleteness, gave birth to the mathematical induction.

Jakob Bernoulli ranks as one of the inventors of this argument, but he gave it no special name. In his posthumous *Ars conjectandi*, 1713, he enters upon a more detailed criticism of Wallis and applies the argument from n to $n + 1$ to the proof of the binomial theorem.

For about 140 years after Jakob Bernoulli, the term "induction" was used by mathematicians in a double sense: (1) "Induction" used in mathematics in the manner in which Wallis used it; (2) "Induction" used to designate the argument from n to $n + 1$. Neither usage was widespread. The former use of "induction" is encountered, for instance, in the Italian translation (1800) of Bossut and Lalande's dictionary,[1] article "Induction (term in mathematics)." The binomial formula is taken as an example; its treatment merely by verification, for the exponents $m = 1$, $m = 2$, $m = 3$, etc., is said to be by "Induction." We read that "it is not desirable to use this method, except for want of a better method." H. Wronski in a similar manner classed "méthodes inductionnelles" among the presumptive methods ("méthodes présomptives") which lack absolute rigor.[2] P. Barlow[3] makes a statement, which is copied verbatim by J. Mitchell,[4] to this effect: "Induction is a term used by mathematicians to denote those cases in which the generality of any law, or form, is inferred from observing it to have obtained in several cases. Such inductions, however, are very deceptive, and ought to be admitted with the greatest caution." The argument from n to $n + 1$ is not described by Barlow and Mitchell.

The second use of the word "induction" (to indicate proofs from n to $n + 1$) was less frequent than the first. More often the process of mathematical induction was used without the assignment of a name. In Germany A. G. Kästner[5] uses this new "genus inductionis" in proving Newton's formulas on the sums of

---

[1] *Acta eruditorum*, Leipzig, 1686, p. 360.

[1] *Dizionario enciclopedico delle matematiche delli signori AB. Bossut, La Lande ec.* Traduzione dal Francese, arricchita di annotazioni, ed aggiunte del traduttore italiano. Padova, MDCCC, Art. "Induzione (termine di Matematica)." Of the binomial formula, as an example, it says: "Quello che senza conoscere la maniera esatta, e generale di dimostrare questa formula la concludesse per averla verificata nel caso di $m = 1$; $m = 2$; $m = 3$, ec. giudicherebbe per *induzione*. Non conviene dunque servirsi di questo metodo che in mancanza di un più esatto."

[2] Art. "Méthode" in A. S. de Montferrier, *Dictionnaire des sciences mathématiques pures et appliquées*, Vol. 2, Paris, 1836.

[3] P. Barlow, *Dictionary of Pure and Mixed Mathematics*, London, 1814, Art. "Induction."

[4] J. Mitchell, *Dictionary of the Mathematical and Physical Sciences*, London, 1823, Art. "Induction."

[5] A. G. Kästner, *Dissertationes Mathematicæ et Physicæ*, Altenburgi, (1771), p. 3.

the powers of roots. G. S. Klügel[1] in his dictionary refers to the demonstrative weakness of Wallis's Induction, then explains Jakob Bernoulli's proof from n to $n + 1$, but gives it no name. In England, Thomas Simpson[2] uses the n to $n + 1$ proof without designating it by a name, as does much later also George Boole.[3]

A special name was first given by English writers in the early part of the nineteenth century. George Peacock, in his *Treatise on Algebra*, Cambridge, 1830, under permutations and combinations, speaks (page 201) of a "law of formation extended by induction to any number," using "induction," as yet in the sense of "divination." Later he explains the argument from n to $n + 1$ and calls it "demonstrative induction" (page 203). We suspect that Peacock prepared this chapter when he was under the influence of Jakob Bernoulli; he himself acknowledges indebtedness to German writers.

The next publication is one of vital importance in the fixing of names; it is Augustus De Morgan's article "Induction (Mathematics)" in the *Penny Cyclopedia*, London, 1838. He suggests a new name, namely "successive induction," but at the end of the article he uses incidentally the term "mathematical induction." This is the earliest use of this name that we have seen. Peacock's and De Morgan's designations were adopted and popularized by Isaac Todhunter,[4] who, in introducing the reader to this method of proof, used both names, "mathematical induction" and "demonstrative induction," but in the chapter-heading used only the former. Both names are used by Jevons[5] and Joseph Ficklin[6] while other popular textbook writers, for instance H. S. Hall and S. R. Knight,[7] W. S. Aldis,[8] G. Chrystal,[9] use only "mathematical induction." The term "demonstrative induction" has become obsolete.

In the United States the proof from n to $n + 1$ was at first used without assigning it a name; for instance, by Benjamin Greenleaf,[10] Charles Davies,[11] and G. A. Wentworth.[12] As we have seen, Ficklin of the University of Missouri used the designation "mathematical induction" in 1874; he expresses his indebtedness to Todhunter in the exposition of the method. The term is used also by G. A. Wentworth in his *Elements of Algebra* (complete edition), Boston, 1884 (Preface, 1881), p. 343. But not until the twentieth century did American textbooks in algebra regularly introduce the name.

On the European continent the name "mathematical induction" is used, but

[1] *Mathematisches Wörterbuch*, Erste Abtheilung, Vol. I, Leipzig, 1803, p. 303.

[2] T. Simpson, *Treatise of Algebra*, London, 1755, p. 205.

[3] *Calculus of Finite Differences*, ed. J. F. Moulton, London, 1880, p. 12.

[4] I. Todhunter, *Algebra*, 4th ed., Cambridge, 1866, p. 280.

[5] W. S. Jevons, *Elementary Lessons in Logic*, London & New York, 1882, p. 220.

[6] J. Ficklin, *Complete Algebra*, 1874, p. 299.

[7] Hall & Knight, *Algebra*, ed. by F. L. Sevenoak, New York, 1898, p. 343.

[8] W. S. Aldis, *Textbook of Algebra*, Oxford, 1887, p. 253.

[9] G. Chrystal, *Algebra*, Part II, p. 2.

[10] B. Greenleaf, *New Higher Algebra*, Boston, 1866, p. 282.

[11] J. E. Worcester, *Dictionary of the English Language*, Boston, 1860, under "Induction." See also Davies's Bourdon's *Algebra*, New York and Chicago, 1871, p. 53.

[12] G. A. Wentworth, *College Algebra*, 1902, p. 222.

is uncommon. The usual German expression is "vollständige Induktion." In criticism of this term Federigo Enriques[1] says: "We should not confound *mathematical induction,* namely the argument from n to $n + 1 \cdots$ with the *complete induction* of Aristotle." In this Aristotelian sense the term "vollständige Induktion" is used in 1840 in the article "Induction" in Ersch and Gruber's *Encyklopädie,* where we find the example: If two sides are found to be greater than the third side in plane triangles with a right angle, and with an obtuse angle, and also with only acute angles, and this inequality is shown to be true likewise of spherical triangles, then the inequality can be asserted to be true of *all* triangles. Here a "vollständige Induktion" is quite different from the argument from n to $n + 1$. The use of the same name for two different types of induction is objectionable. The name "vollständige Induktion" was used by R. Dedekind in his *Was sind und was sollen die Zahlen,* 1887, §§ 59 and 80. Through him the method received great emphasis in Germany.[2] The English equivalent of "vollständige Induktion," namely, "complete induction," is seldom used.[3] According to A. Haas[4] the designation "höhere Induktion" is also employed. Poincaré, in his *Science et Hypothèse,* does not restrict himself to any one name, but is partial to the phrases "démonstration par récurrence," "raisonnement par récurrence."

MISCELLANEA.

By AUBREY J. KEMPNER, University of Illinois.

§ I. Concerning Greatest and Least Absolute Values of Analytic Functions.

The following theorems are known:

Ia. If $f(z)$ is an analytic (single-valued) function of the complex variable $z = x + iy$, regular in the interior and on the boundary of a circle C about a point a of the complex plane as center, then $|f(a)| \leqq M$, where M is the largest value which $|f(z)|$ assumes on the boundary of C.

Ib. Under the same conditions for $f(z)$, $|f(c)| \leqq M$, where c is any interior point of the circle C.[5]

Ic. In Ib the circular region may be replaced by a region S, not necessarily simply connected, but the boundary of which we shall assume, in this as in the

[1] F. Enriques, *Principles of Science,* transl. by K. Royce, Chicago and London, 1914, p. 133.

[2] An excellent article in the English language on "Mathematical Induction" is that by C. J. Keyser in the *Americana.*

[3] See, however, W. H. Bussey, "The Origin of Mathematical Induction," Am. Math. Monthly, Vol. 24, 1917, p. 199.

[4] A. Haas, *Lehrbuch über den binomischen und polynomischen Lehrsatz,* Bremerhaven, 1906, p. 6.

[5] For Ia and Ib compare, among others: Fouët, *Leçons sur la théorie des fonctions analytiques,* 1910, Vol. II, p. 79; Osgood, *Lehrbuch der Funktionentheorie,* 1912, Vol. I, p. 300; Vivanti-Gutzmer, *Theorie der eindeutigen analytischen Funktionen,* 1906, p. 81.