ALGEBRAIC QUANTUM FIELD THEORY

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INTRODUCTION

From the title of this Chapter, one might suspect that the subject is some *idiosyncratic* approach to quantum field theory (QFT). The approach is indeed idiosyncratic in the sense of demographics: only a small proportion of those who work on QFT work on *algebraic* QFT (AQFT). However, there are particular reasons why philosophers, and others interested in foundational issues, will want to study the "algebraic" approach.

In philosophy of science in the analytic tradition, studying the foundations of a theory T has been thought to presuppose some minimal level of clarity about the referent of T. (Moreover, to distinguish philosophy from sociology and history, T is not taken to refer to the activities of some group of people.) In the early twentieth century, it was thought that the referent of T must be a set of axioms of some formal, preferably first-order, language. It was quickly realized that not many interesting physical theories can be formalized in this way. But in any case, we are no longer in the grip of axiomania, as Feyerabend called it. So, the standards were loosened somewhat — but only to the extent that the standards were simultaneously loosened within the community of professional mathematicians. There remains an implicit working assumption among many philosophers that studying the foundations of a theory requires that the theory has a mathematical description. (The philosopher's working assumption is certainly satisfied in the case of statistical mechanics, special and general relativity, and nonrelativistic quantum mechanics.) In any case, whether or not having a mathematical description is mandatory, having such a description greatly facilitates our ability to draw inferences securely and efficiently.

So, philosophers of physics have taken their object of study to be theories, where theories correspond to mathematical objects (perhaps sets of models). But it is not so clear where "quantum field theory" can be located in the mathematical universe. In the absence of some sort of mathematically intelligible description of QFT, the philosopher of physics has two options: either find a new way to understand the task of interpretation, or remain silent about the interpretation of quantum field theory.[1]

It is for this reason that AQFT is of particular interest for the foundations of quantum field theory. In short, AQFT is our best story about where QFT lives

[1] For the first option, see [Wallace, forthcoming].

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in the mathematical universe, and so is a natural starting point for foundational inquiries.

1 ALGEBRAIC PROLEGOMENA

This first section provides a minimal overview of the mathematical prerequisites of the remainder of the Chapter.

1.1 von Neumann algebras

The standard definition of a von Neumann algebra involves reference to a topology, and it is then shown (by von Neumann's double commutant theorem) that this topological condition coincides with an algebraic condition (condition 2 in the Definition 2). But for present purposes, it will suffice to take the algebraic condition as basic.

DEFINITION 1. Let $\mathcal{H}$ be a Hilbert space. Let $\mathfrak{B}(\mathcal{H})$ be the set of bounded linear operators on $\mathcal{H}$ in the sense that for each $A \in \mathfrak{B}(\mathcal{H})$ there is a smallest nonnegative number $\|A\|$ such that $\langle Ax, Ax\rangle^{1/2} \leq \|A\|$ for all unit vectors $x \in \mathcal{H}$. [Subsequently we use $\|\cdot\|$ ambiguously for the norm on $\mathcal{H}$ and the norm on $\mathfrak{B}(\mathcal{H})$.] We use juxtaposition AB to denote the composition of two elements A, B of $\mathfrak{B}(\mathcal{H})$. For each $A \in \mathfrak{B}(\mathcal{H})$ we let A^* denote the unique element of $\mathfrak{B}(\mathcal{H})$ such that $\langle A^*x, y\rangle = \langle x, Ay\rangle$, for all $x, y \in \mathfrak{R}$.

DEFINITION 2. Let $\mathfrak{R}$ be a $*$-subalgebra of $\mathfrak{B}(\mathcal{H})$, the bounded operators on the Hilbert space $\mathcal{H}$. Then $\mathfrak{R}$ is a von Neumann algebra if

1. $I \in \mathfrak{R}$,
2. $(\mathfrak{R}')' = \mathfrak{R}$,

where $\mathfrak{R}' = \{B \in \mathfrak{B}(\mathcal{H}) : [B, A] = 0, \forall A \in \mathfrak{R}\}$.

DEFINITION 3. We will need four standard topologies on the set $\mathfrak{B}(\mathcal{H})$ of bounded linear operators on $\mathcal{H}$. Each of these topologies is defined in terms of a family of seminorms — see [Kadison and Ringrose, 1997, Chaps. 1,5] for more details.

- The uniform topology on $\mathfrak{B}(\mathcal{H})$ is defined in terms of a single norm:

  $$\|A\| = \sup\{\|Av\| : v \in \mathcal{H}, \|v\| \leq 1\},$$

  where the norm on the right is the given vector norm on $\mathcal{H}$. Hence, an operator A is a limit point of the sequence $(A_i)_{i\in\mathbb{N}}$ iff $(\|A_i - A\|)_{i\in\mathbb{N}}$ converges to 0.

- The weak topology on $\mathfrak{B}(\mathcal{H})$ is defined in terms of the family $\{p_{u,v} : u, v \in \mathcal{H}\}$ of seminorms where

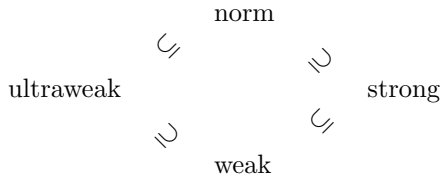
  $$p_{u,v}(A) = \langle u, Av\rangle.$$

The resulting topology is not generally first countable, and so the closure of a subset S of $\mathfrak{B}(\mathcal{H})$ is generally larger than the set of all limit points of sequences in S. Rather, the closure of S is the set of limit points of generalized sequences (nets) in S — see [Kadison and Ringrose, 1997, Chap. 1] for more details. A net $(A_i)_{i\in\mathcal{I}}$ in $\mathfrak{B}(\mathcal{H})$ converges weakly to A just in case $(p_{u,v}(A_i))_{i\in\mathcal{I}}$ converges to $p_{u,v}(A)$ for all $u, v \in \mathcal{H}$.

- The strong topology on $\mathfrak{B}(\mathcal{H})$ is defined in terms of the family $\{p_v : v \in \mathcal{H}\}$ of seminorms where
$$p_v(A) = \|Av\|.$$
Thus, a net $(A_i)_{i\in\mathcal{I}}$ converges strongly to A iff $(p_v(A_i))_{i\in\mathcal{I}}$ converges to $p_v(A)$, for all $v \in \mathcal{H}$.

- The ultraweak topology on $\mathfrak{B}(\mathcal{H})$ is defined in terms of the family $\{p_\rho : \rho \in \mathcal{T}(\mathcal{H})\}$ where $\mathcal{T}(\mathcal{H})$ is the set of positive, trace 1 operators ("density operators") on $\mathcal{H}$ and
$$p_\rho(A) = \mathrm{Tr}(\rho A).$$
Thus a net $(A_i)_{i\in\mathcal{I}}$ converges ultraweakly to A just in case $(\mathrm{Tr}(\rho A_i))_{i\in\mathcal{I}}$ converges to $\mathrm{Tr}(\rho A)$, for all $\rho \in \mathcal{T}(\mathcal{H})$.

FACT 4. The topologies are ordered as follows:



Since closed sets are just the complements of open sets, this means that a weakly closed set is ultraweakly closed, and an ultraweakly closed subset is norm closed. Furthermore, the four topologies on $\mathfrak{B}(\mathcal{H})$ coincide iff $\mathcal{H}$ is finite dimensional.

FACT 5. If S is a bounded, convex subset of $\mathfrak{B}(\mathcal{H})$, then the weak, ultraweak, and strong closures of S are the same.

FACT 6. For a $*$-algebra $\mathfrak{R}$ on $\mathcal{H}$ that contains I, the following are equivalent: (i) $\mathfrak{R}$ is weakly closed; (ii) $\mathfrak{R}'' = \mathfrak{R}$. This is von Neumann's double commutant theorem.

DEFINITION 7. Let $\mathfrak{R}$ be a subset of $\mathfrak{B}(\mathcal{H})$. A vector $x \in \mathcal{H}$ is said to be *cyclic* for $\mathfrak{R}$ just in case $[\mathfrak{R}x] = \mathcal{H}$, where $\mathfrak{R}x = \{Ax : A \in \mathfrak{R}\}$, and $[\mathfrak{R}x]$ is the closed linear span of $\mathfrak{R}x$. A vector $x \in \mathcal{H}$ is said to be *separating* for $\mathfrak{R}$ just in case $Ax = 0$ and $A \in \mathfrak{R}$ entails $A = 0$.

FACT 8. Let $\mathfrak{R}$ be a von Neumann algebra on $\mathcal{H}$, and let $x \in \mathcal{H}$. Then x is cyclic for $\mathfrak{R}$ iff x is separating for $\mathfrak{R}'$.

DEFINITION 9. A *normal* state of a von Neumann algebra $\mathfrak{R}$ is an ultraweakly continuous state. We let $\mathfrak{R}_*$ denote the normal state space of $\mathfrak{R}$.

1.2 C^*-algebras and their representations

DEFINITION 10. A C^*-algebra is a pair consisting of a $*$-algebra $\mathfrak{A}$ and a norm $\|\cdot\| : \mathfrak{A} \to \mathbb{R}$ such that

$$\|AB\| \leq \|A\| \cdot \|B\|, \qquad \|A^*A\| = \|A\|^2,$$

for all $A, B \in \mathfrak{A}$. We usually use $\mathfrak{A}$ to denote the algebra and its norm.

In this Chapter, we will only use C^*-algebras that contain a multiplicative identity I.

DEFINITION 11. A state ω on $\mathfrak{A}$ is a linear functional such that $\omega(A^*A) \geq 0$ for all $A \in \mathfrak{A}$, and $\omega(I) = 1$.

DEFINITION 12. A state ω of $\mathfrak{A}$ is said to be *mixed* if $\omega = \frac{1}{2}(\omega_1 + \omega_2)$ with $\omega_1 \neq \omega_2$. Otherwise ω is said to be *pure*.

DEFINITION 13. Let $\mathfrak{A}$ be a C^*-algebra. A *representation* of $\mathfrak{A}$ is a pair $(\mathcal{H}, \pi)$, where $\mathcal{H}$ is a Hilbert space and π is a $*$-homomorphism of $\mathfrak{A}$ into $\mathfrak{B}(\mathcal{H})$. A representation $(\mathcal{H}, \pi)$ is said to be *irreducible* if $\pi(\mathfrak{A})$ is weakly dense in $\mathfrak{B}(\mathcal{H})$. A representation $(\mathcal{H}, \pi)$ is said to be *faithful* if π is an isomorphism.

DEFINITION 14. Let $(\mathcal{H}, \pi)$ and $(\mathcal{K}, \phi)$ be representations of a C^*-algebra $\mathfrak{A}$. Then $(\mathcal{H}, \pi)$ and $(\mathcal{K}, \phi)$ are said to be:

1. *unitarily equivalent* if there is a unitary $U : \mathcal{H} \to \mathcal{K}$ such that $U\pi(A) = \phi(A)U$ for all $A \in \mathfrak{A}$.
2. *quasiequivalent* if the von Neumann algebras $\pi(\mathfrak{A})''$ and $\phi(\mathfrak{A})''$ are $*$-isomorphic.
3. *disjoint* if they are not quasiequivalent.

DEFINITION 15. A representation $(\mathcal{K}, \phi)$ is said to be a *subrepresentation* of $(\mathcal{H}, \pi)$ just in case there is an isometry $V : \mathcal{K} \to \mathcal{H}$ such that $\pi(A)V = V\phi(A)$ for all $A \in \mathfrak{A}$.

FACT 16. Two representations are quasiequivalent iff they have unitarily equivalent subrepresentations.

The famous Gelfand-Naimark-Segal (GNS) theorem shows that every C^*-algebraic state can be represented by a vector in a Hilbert space.

THEOREM 17 (GNS). *Let ω be a state of $\mathfrak{A}$. Then there is a representation $(\mathcal{H}, \pi)$ of $\mathfrak{A}$, and a unit vector $\Omega \in \mathcal{H}$ such that:*

1. *$\omega(A) = \langle \Omega, \pi(A)\Omega \rangle$, for all $A \in \mathfrak{A}$;*
2. *$\pi(\mathfrak{A})\Omega$ is dense in $\mathcal{H}$.*

Furthermore, the representation $(\mathcal{H}, \pi)$ is the unique one (up to unitarily equivalence) satisfying the two conditions.

Since we will later need to invoke the details of the GNS construction, we sketch the outlines of its proof here.

Sketch of proof. We construct the Hilbert space $\mathcal{H}$ from equivalence classes of elements in $\mathfrak{A}$, and the representation π is given by the action of left multiplication. In particular, define a bounded sesquilinear form on $\mathfrak{A}$ by setting

$$\langle A, B\rangle_\omega = \omega(A^*B), \qquad A, B \in \mathfrak{A}.$$

Let $\mathcal{H}_0$ be the quotient of $\mathfrak{A}$ induced by the norm $\|A\|_\omega = \langle A, A\rangle_\omega^{1/2}$. Let $\mathcal{H}$ be the unique completion of the pre-Hilbert space $\mathcal{H}_0$. Thus there is an inclusion mapping $j : \mathfrak{A} \to \mathcal{H}$ with $j(\mathfrak{A})$ dense in $\mathcal{H}$. Define the operator $\pi(A)$ on $\mathcal{H}$ by setting

$$\pi(A)j(B) = j(AB), \qquad B \in \mathfrak{A}.$$

One must verify that $\pi(A)$ is well-defined, and extends uniquely to a bounded linear operator on $\mathcal{H}$. One must also then verify that π is a $*$-homomorphism. Finally, if we let $\Omega = j(I)$, then Ω is obviously cyclic for $\pi(\mathfrak{A})$. ■

PROPOSITION 18. *Let ω be a state of $\mathfrak{A}$. The GNS representation $(\mathcal{H}, \pi)$ of $\mathfrak{A}$ induced by ω is irreducible iff ω is pure.*

Notes: Standard references on C^*-algebras include [Kadison and Ringrose, 1997] and [Takesaki, 2002].

1.3 Type classification of von Neumann algebras

DEFINITION 19. Two projections E, F in a von Neumann algebra $\mathfrak{R}$ are said to be *equivalent*, written $E \sim F$ just in case there is a $V \in \mathfrak{R}$ such that $V^*V = E$ and $VV^* = F$.

NOTE 20. If we were being really careful, we would replace "equivalent" in the previous definition with "equivalence modulo $\mathfrak{R}$", and similarly "$\sim$" with "$\sim_{\mathfrak{R}}$." But we will not run into trouble by omitting the reference to $\mathfrak{R}$. The operator V in the previous definition is called a *partial isometry* with *initial projection* E and *final projection* F.

DEFINITION 21. For von Neumann algebras $\mathfrak{R}_1$ and $\mathfrak{R}_2$, we let $\mathfrak{R}_1 \wedge \mathfrak{R}_2 = \mathfrak{R}_1 \cap \mathfrak{R}_2$. We let $\mathfrak{R}_1 \vee \mathfrak{R}_2$ denote the von Neumann algebra generated by $\mathfrak{R}_1$ and $\mathfrak{R}_2$, i.e. the intersection of all von Neumann algebras containing $\mathfrak{R}_1$ and $\mathfrak{R}_2$.

DEFINITION 22. $Z(\mathfrak{R}) = \mathfrak{R} \wedge \mathfrak{R}'$ is called the *center* of the von Neumann algebra $\mathfrak{R}$. A von Neumann algebra $\mathfrak{R}$ is called a *factor* just in case $Z(\mathfrak{R}) = \mathbb{C}I$, equivalently, $\mathfrak{R} \vee \mathfrak{R}' = \mathfrak{B}(\mathcal{H})$. A projection $E \in Z(\mathfrak{R})$ is called a *central projection* in $\mathfrak{R}$.

DEFINITION 23. Let $E \in \mathfrak{R}$ be a projection, and let $E\mathfrak{R}E = \{EAE : A \in \mathfrak{R}\}$. Then clearly, $E\mathfrak{R}E$ is a linear subspace of $\mathfrak{R}$. Furthermore, since for $A, B \in \mathfrak{R}$, $AEB \in \mathfrak{R}$ and $A^* \in \mathfrak{R}$, it follows that $E\mathfrak{R}E$ is closed under products, as well as under $*$. It is also not difficult to see that $E\mathfrak{R}E$ is weakly closed, and hence is a von Neumann algebra on $E\mathcal{H}$.

DEFINITION 24. Let $\mathfrak{R}$ be a von Neumann algebra. A projection $E \in \mathfrak{R}$ is said to be:

1. *minimal* just in case $\mathfrak{R}$ contains no proper subprojection of E.
2. *abelian* just in case the algebra $E\mathfrak{R}E$ is abelian.
3. *infinite* just in case there is a projection $E_0 \in \mathfrak{R}$ such that $E_0 < E$ and $E \sim E_0$.
4. *finite* just in case it is not infinite.
5. *properly infinite* just in case E is infinite and for each central projection P of $\mathfrak{R}$, either $PE = 0$ or PE is infinite.

FACT 25. We have the following relations for projections:

$$\begin{array}{c} \text{minimal} \implies \text{abelian} \implies \text{finite} \\ \text{properly infinite} \implies \text{infinite} \iff \neg\,\text{finite} \end{array}$$

For factors, the first arrows on both lines can be reversed.

We now give the Murray-von Neumann type classification of factors (for more on this, see [Kadison and Ringrose, 1997, Chap. 7] or [Sunder, 1987, Chap. 1]).

DEFINITION 26. A von Neumann factor $\mathfrak{R}$ is said to be:

1. type I if it contains an abelian projection;
2. type II if it contains a finite projection, but no abelian projection;
3. type III if it is neither type I nor type II.

The type I factors were already completely classified by Murray and von Neumann: for each cardinal number κ there is a unique (up to isomorphism) type I_κ factor, namely $\mathfrak{B}(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space of dimension κ. The type II factors can be further subdivided according to whether or not the identity projection I is finite (type II_1) or infinite (type II_∞). The type III factors can be subdivided into types III_λ with $\lambda \in [0, 1]$, although the basis for this subclassification depends on Tomita-Takesaki modular theory (see Section 1.4).

For general von Neumann algebras, the type classification must be just a bit more sophisticated: a type I algebra is defined as an algebra that has an abelian projection E such that no nontrivial projection in $Z(\mathfrak{R})$ majorizes E. Similarly, a

type II algebra is defined as an algebra having a finite projection E such that no nontrivial projection in $Z(\mathfrak{R})$ majorizes E. Thus we have:

PROPOSITION 27. *Let $\mathfrak{R}$ be a von Neumann algebra. Then $\mathfrak{R} = \mathfrak{R}_{\mathrm{I}} \oplus \mathfrak{R}_{\mathrm{II}} \oplus \mathfrak{R}_{\mathrm{III}}$, where $\mathfrak{R}_{\mathrm{X}}$ is type X for X=I,II,III.*

Proof. See [Kadison and Ringrose, 1997, Thm. 6.5.2]. ■

We will soon see that the local algebras in QFT are "typically" type III, and this has many interesting implications. The fact that type III algebras do not have abelian projections is connected to questions of locality in Section 3.3. The fact that the state space of type III_1 factors is homogeneous is also connected to questions of locality in Section 3.3. The fact that type III algebras do not contain representatives of their states (i.e. density operators) is connected to the modal interpretation of QFT in Section 5.

The following classification of von Neumann algebras is also natural, but it cuts across the Murray-von Neumann classification.

DEFINITION 28. A von Neumann algebra $\mathfrak{R}$ is said to be:

- of *infinite type* if I is infinite in $\mathfrak{R}$;
- *properly infinite* if I is properly infinite in $\mathfrak{R}$.
- *semi-finite* if the central projection E_{III} in $\mathfrak{R}$ (defined in Prop. 27) is zero.

The finite factors include the type I_n and type II_1 factors. The infinite factors include the type I_∞ factors as well as the type II_∞ and III factors. The distinction between finite and infinite factors coincides with the existence of a tracial state.

DEFINITION 29. A faithful normalized trace on a von Neumann algebra $\mathfrak{R}$ is a state ρ on $\mathfrak{R}$ such that:

1. ρ is tracial; i.e. $\rho(AB) = \rho(BA)$, for all $A, B \in \mathfrak{R}$,
2. ρ is faithful; i.e. $\rho(A^*A) = 0$ only if $A = 0$.

FACT 30. A von Neumann factor $\mathfrak{R}$ is finite iff there is a faithful normal tracial state ρ on $\mathfrak{R}$. A von Neumann factor $\mathfrak{R}$ is semifinite iff there is a "faithful normal semifinite trace" on $\mathfrak{R}$; but we do not pause here to define this notion.

1.4 Modular theory

We state here without proof some of the basic facts about Tomita-Takesaki modular theory. These facts are necessary in order to understand the classification of type III von Neumann algebras, which in turn is essential to understanding the mathematical structure of AQFT.

DEFINITION 31. Let $\mathfrak{R}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, and suppose that $\Omega \in \mathcal{H}$ is cyclic and separating for $\mathfrak{R}$. In such a case, we say that $(\mathfrak{R}, \Omega)$ is *in standard form*. Define an operator S_0 on $\mathcal{H}$ by setting

$$S_0 A\Omega = A^*\Omega, \qquad A \in \mathfrak{R}.$$

Then S_0 extends to a closed anti-linear operator S on $\mathcal{H}$. Let $S = J\Delta^{1/2}$ be the polar decomposition of S, so that Δ is positive (but generally unbounded), and J is anti-unitary. (Recall that a positive operator has spectrum in $\mathbb{R}^+$.) We call Δ the *modular operator* and J the *modular conjugation* associated with the pair $(\mathfrak{R}, \Omega)$.

THEOREM 32. *Let $\mathfrak{R}$ be a von Neumann algebra with cyclic and separating vector Ω. Then $J\Omega = \Omega = \Delta\Omega$ and*

$$\begin{aligned} \Delta^{it}\mathfrak{R}\Delta^{-it} &= \mathfrak{R}, \qquad \forall t \in \mathbb{R}, \\ J\mathfrak{R}J &= \mathfrak{R}'. \end{aligned}$$

Proof. See [Kadison and Ringrose, 1997, Thm. 9.2.9], or [Sunder, 1987, Thm. 2.3.3]. ■

DEFINITION 33. Let $(\mathfrak{R}, \Omega)$ be in standard form, and let ω be the state of $\mathfrak{R}$ induced by Ω. For each $t \in \mathbb{R}$, define the *modular automorphism* σ_t^ω of $\mathfrak{R}$ by

$$\sigma_t^\omega(A) = \Delta^{it} A \Delta^{-it}, \qquad A \in \mathfrak{R},$$

for all $A \in \mathfrak{R}$. Define a $*$ anti-isomorphism $\gamma : \mathfrak{R} \to \mathfrak{R}'$ by setting $\gamma(A) = JA^*J$, for all $A \in \mathfrak{R}$.

DEFINITION 34. If $\mathfrak{A}$ is a C^*-algebra, we let $\mathrm{Inn}\mathfrak{A}$ denote the group of inner automorphisms of $\mathfrak{A}$; i.e. $\alpha \in \mathrm{Inn}\mathfrak{A}$ just in case there is a unitary $U \in \mathfrak{A}$ such that $\alpha(A) = UAU^*$ for all $A \in \mathfrak{A}$.

The spectrum of the modular operator Δ gives a rough measure of the periodicity of the modular automorphism group $(\sigma_t^\omega)_{t\in\mathbb{R}}$; i.e. the smaller the spectrum of Δ, the closer the automorphism σ_t^ω is to the identity $\iota : \mathfrak{R} \to \mathfrak{R}$. In the extreme case, if $\mathrm{sp}\Delta = \{1\}$, then $\sigma_t^\omega = \iota$ for all $t \in \mathbb{R}$. Conversely, as Δ goes up to $\mathbb{R}^+$, the group $(\sigma_t^\omega)_{t\in\mathbb{R}}$ tends toward being ergodic (i.e. having no fixed points).

DEFINITION 35. Define the *modular spectrum* $S(\mathfrak{R})$ of $\mathfrak{R}$ by

$$S(\mathfrak{R}) = \bigcap_\omega \mathrm{sp}(\Delta_\omega),$$

where ω runs over the family of faithful normal states of $\mathfrak{R}$, and Δ_ω are the corresponding modular operators.

PROPOSITION 36. *Let $\mathfrak{R}$ be a von Neumann factor with cyclic and separating vector Ω. Then the following are equivalent:*

1. *$\mathfrak{R}$ is semifinite.*
2. *For all $t \in \mathbb{R}$, the modular automorphism σ_t^ω is inner; i.e. there is a unitary $U \in \mathfrak{R}$ such that $\sigma_t^\omega(A) = UAU^*$ for all $A \in \mathfrak{R}$.*
3. *$S(\mathfrak{R}) = \{1\}$.*

Proof. See [Takesaki, 2003, p. 122] and [Sunder, 1987, p. 111]. ■

We now proceed to Connes' subclassification of the type III factors. This subclassification uses the notion of the "period of the flow of weights" (where a weight is a generalization of the notion of a state). However, in order to bypass some background material, we use the following (provably equivalent) definition.

DEFINITION 37. A factor $\mathfrak{R}$ of type III is said to be:

1. Type III_0 if $S(\mathfrak{R}) = \{0, 1\}$.
2. Type III_λ, $\lambda \in (0, 1)$, if $S(\mathfrak{R}) = \{\lambda^n : n \in \mathbb{Z}\} \cup \{0\}$.
3. Type III_1 if $S(\mathfrak{R}) = \mathbb{R}^+$.

The conditions in Defn. 37 do not bear their physical interpretation on their sleeve. That is, it is not immediately clear how the physics of type III_λ algebras differs (if at all) from that of type III_μ algebras, for $\lambda \neq \mu$. However, a result of Connes and Størmer [1978] cashes out some of the significance of the distinctions between different types of factors.

DEFINITION 38. Let $\mathfrak{R}$ be a von Neumann algebra, and let $\mathfrak{R}_*$ be its normal state space. We define the *diameter of the state orbit space* $d(\mathfrak{R})$ by

$$d(\mathfrak{R}) = \sup\Big\{\inf\big\{\|(\omega_1 \circ \alpha) - \omega_2\| : \alpha \in \mathrm{Inn}\mathfrak{R}\big\} : \omega_1, \omega_2 \in \mathfrak{R}_*\Big\}.$$

Alternatively, let $[\omega]$ denote the norm closure of $\{\omega \circ \alpha : \alpha \in \mathrm{Inn}\mathfrak{R}\}$ (the orbit of the state under inner automorphisms), and let K denote the quotient of the normal state space $\mathfrak{R}_*$. Then $d(\mathfrak{R})$ is the diameter of K relative to the induced metric

$$\overline{d}([\omega_1], [\omega_2]) = \inf\{\|\omega_1' - \omega_2'\| : \omega_i' \in [\omega_i]\}.$$

Clearly $d(\mathfrak{R}) \in [0, 2]$, with $d(\mathfrak{R}) = 0$ iff the orbit of every state is dense in the normal state space. If $\mathfrak{R}$ is not a factor, then there are states ω_1, ω_2 such that $\|\omega_1 \circ \alpha - \omega_2\| = 2$ for all $\alpha \in \mathrm{Inn}\mathfrak{R}$, and so $d(\mathfrak{R}) = 2$. For type I_n factors, the distance between normal states is the same as the trace norm distance of the corresponding density operators. In this case, we have

$$d(\mathfrak{R}) = 2\Big(1 - \frac{1}{n}\Big) = \|\tau - \omega\|,$$

where τ is the trace and ω is any pure state. We also have $d(\mathfrak{R}) = 2$ for factors of type I_∞ and of type II [Takesaki, 2003, p. 430].

If $d(\mathfrak{R})$ gives some sort of measure of "how noncommutative" the algebra $\mathfrak{R}$ is, then type III_1 factors are the most noncommutative.

DEFINITION 39. A von Neumann algebra $\mathfrak{R}$ is said to be *countably decomposable* just in case any family of mutually orthogonal projection operators in $\mathfrak{R}$ is countable.

PROPOSITION 40. *If $\mathfrak{R}$ is a countably decomposable factor of type III_λ, then*

$$d(\mathfrak{R}) = 2\frac{1-\lambda^{1/2}}{1+\lambda^{1/2}}.$$

Proof. See [Connes and Størmer, 1978] and [Takesaki, 2003, p. 427]. ■

The function $f(\lambda) = 2(1-\lambda^{1/2})/(1+\lambda^{1/2})$ is monotonically decreasing on $[0, 1]$. In particular, $f(1) = 0$ so that, for type III_1 factors, the orbit of any normal state ω is norm dense in the state space. According to Connes [1994, p. 473] this means that "one cannot distinguish between two states of a factor of type III_1 by means of a property that is closed and invariant under inner automorphisms." In particular, since two unitarily equivalent states must be considered to be "equally mixed," there are no distinctions to be drawn in terms of the mixedness of states of a type III_1 factor.

Notes: For an overview of modular theory, see [Summers, ND] or [Connes, 1994]. For a full treatment, see [Takesaki, 2003]. For a detailed exposition of applications of modular theory in QFT, see [Borchers, 2000].

2 STRUCTURE OF THE NET OF OBSERVABLE ALGEBRAS

2.1 Nets of algebras, basic properties

AQFT proceeds by isolating some structural assumptions that hold in most known QFT models. It formalizes these structural assumptions, and then uses "abstract but efficient nonsense" to derive consequences of these assumptions.

The basic formalism of AQFT is a "net of local observable algebras" over spacetime. Although this formalism can be applied to a very wide class of spacetimes, we restrict attention in this Chapter mostly to Minkowski spacetime.

An open *double cone* in Minkowski spacetime is the intersection of the causal future of a point x with the causal past of a point y to the future of x. Let $\mathcal{K}$ be the set of open double cones in Minkowski spacetime, and let $O \mapsto \mathfrak{A}(O)$ be a mapping from $\mathcal{K}$ to C^*-algebras. We assume that all our C^*-algebras are unital, i.e. have a multiplicative identity. We assume that the set $\{\mathfrak{A}(O) : O \in \mathcal{K}\}$ of C^*-algebras (called a *net of observable algebras over Minkowski spacetime*) is an inductive system in the sense that:

If $O_1 \subseteq O_2$, then there is an embedding (i.e. an isometric $*$-homomorphism) $\alpha_{12} : \mathfrak{A}(O_1) \to \mathfrak{A}(O_2)$.

ASSUMPTION 41 (Isotony). The mapping $O \mapsto \mathfrak{A}(O)$ is an inductive system.

The isotony assumption is sometimes motivated by the idea that an observable measurable in a region O_1 is *a fortiori* measurable in any region O_2 containing O_1. But the isotony axiom is also justified by its utility: for, if $\{\mathfrak{A}(O) : O \in \mathcal{K}\}$ is an inductive system, then there is an inductive limit C^*-algebra $\mathfrak{A}$ generated by all the local algebras. We call $\mathfrak{A}$ the *quasilocal* algebra, because it contains observables that can be uniformly approximated by local observables.

NOTE 42. In some spacetimes, the set of double cones is not directed. In many such cases, it is still possible to define the quasilocal algebra by means of more sophisticated techniques [Fredenhagen, 1993].

Now we turn to the main *relativistic* assumption of AQFT.

ASSUMPTION 43 (Microcausality). A net $\mathfrak{A}$ of C^*-algebras is said to satisfy *microcausality* just in case if O_1, O_2 are spacelike separated double cones, then $[\mathfrak{A}(O_1), \mathfrak{A}(O_2)] = \{0\}$.

This assumption is thought to reflect the constraints on spacetime structure imposed by the theory of relativity.

NOTE 44. It is *not* a tenet of AQFT that quantities that are associated to spacelike separated regions must be represented by commuting operators. In fact, Fermi field operators assigned to spacelike separated regions will anticommute. So, AQFT has need of a distinction between observable (represented by elements of $\mathfrak{A}(O)$) and unobservable quantities (represented by "field operators"). For more on this distinction, see Sections 7.2 and following on DHR superselection theory.

In this Chapter, we will not attempt to justify or to dispute the microcausality assumption. However, we will briefly discuss its connection to issues of locality in Section 3.

2.2 Existence/uniqueness of vacuum states/representations

The existence of translation-invariant states

In this section, we inquire concerning the existence and uniqueness of vacuum states and representation. For this, recall that an *affine space* (e.g. Minkowski spacetime) is a triple consisting of a set S, a vector space V, and a map $+ : S \times V \to S$ satisfying certain properties. In this case, V is called the *translation group*.

ASSUMPTION 45 (Translation Covariance). If $\mathfrak{A}$ is a net of operator algebras on an affine space, then we assume that there is a faithful, continuous representation $x \mapsto \alpha_x$ of the translation group in the group Aut$\mathfrak{A}$ of automorphisms of $\mathfrak{A}$, and

$$\alpha_x(\mathfrak{A}(O)) = \mathfrak{A}(O + x),$$

for any double cone O, and translation x.

NOTE 46. For the case of Minkowski spacetime, the translation group is a subgroup of the Poincaré group. In many cases of physical interest, $x \to \alpha_x$ extends to a representation of the full Poincaré group in the group Aut$\mathfrak{A}$ of automorphisms of $\mathfrak{A}$. But we will only need that fact for one result (Prop. 132).

Translation invariance has traditionally been thought to be a necessary condition on a vacuum state.

FACT 47. If there is an action α of the translation group on $\mathfrak{A}$, then translation-invariant states of $\mathfrak{A}$ exist. Indeed, since the translation group is abelian, it has an invariant mean μ — i.e. a translation invariant, positive linear functional on the algebra $L^\infty(G)$ of essentially bounded measurable (with respect to the Haar measure) functions on the group G. Given a state ω of $\mathfrak{A}$, we can then define an averaged state ρ by

$$\rho(A) := \int \omega(\alpha_x A) d\mu(x).$$

The state ρ is translation invariant. (See Emch, this volume, Section 3.5.)

NOTE 48. The preceding argument cannot be used to show the existence of Lorentz invariant states. The Lorentz group is not *amenable*, and so does not admit an invariant mean. Hence, we cannot use these general methods to prove the existence of Lorentz invariant states. Of course, in concrete models (e.g. free Bose and Fermi fields) there are other way to establish the existence of such states.

Let G be a group acting by automorphisms on $\mathfrak{A}$. A generalization of the GNS theorem shows that a G-invariant state ω of $\mathfrak{A}$ gives rise to a GNS Hilbert space $\mathcal{H}$ that carries a unitary representation U of G, and the GNS vector Ω is invariant under the G-action on $\mathcal{H}$.

FACT 49. Let α be a strongly continuous action of the group G by automorphisms of $\mathfrak{A}$. If ω is a G-invariant state of $\mathfrak{A}$, then the GNS representation $(\mathcal{H}, \pi)$ of $\mathfrak{A}$ induced by ω is G-covariant in the sense that there is a strongly continuous representation U of G in the unitary group of $\mathfrak{B}(\mathcal{H})$ such that

1. $U(g)\pi(A)U(g)^* = \pi(\alpha_g(A))$, for all $A \in \mathfrak{A}$,
2. $U(g)\Omega = \Omega$ for all $g \in G$.

Only one vacuum per Hilbert space

NOTE 50. When considering the group Aut$\mathfrak{A}$ of automorphisms of a C^*-algebra, we take as our standard topology the strong topology on the set $L(\mathfrak{A})$ of bounded linear mappings on $\mathfrak{A}$ (considered as a Banach space). That is, α_i converges to α just in case for each $A \in \mathfrak{A}$, $\alpha_i(A)$ converges to $\alpha(A)$ in the norm on $\mathfrak{A}$.

DEFINITION 51. We use the GNS representation theorem (Thm. 17) to transfer terminology about representations (Defn. 14) to terminology about states. So, e.g., we say that two states are *disjoint* if their GNS representations are disjoint.

A vacuum state should be at least translation invariant. Furthermore, the microcausality assumption on the net $\mathfrak{A}$ entails that any two observables commute "in the limit" where one is translated out to spacelike infinity. That is, for any $A, B \in \mathfrak{A}$, and for any spacelike vector x,

$$\lim_{t\to\infty} \|[\alpha_{tx}(A), B]\| = 0.$$

This in turn entails that G acts on $\mathfrak{A}$ as a *large group of automorphisms* in the following sense:

> If ω is a G-invariant state and $(\mathcal{H}, \pi)$ is the GNS representation of $\mathfrak{A}$ induced by ω, then for any $A \in \mathfrak{A}$,
>
> $$\overline{\text{conv}}\{\pi(\alpha_g(A)) : g \in G\},$$
>
> has nonempty intersection with $\pi(\mathfrak{A})'$.

Here we use $\overline{\text{conv}}S$ to denote the weakly closed convex hull of S. (See [Størmer, 1970] for the relevant proofs.) Note however that we would also expect the same to be true in a non-relativistic setting, because we would expect observables associated with disjoint regions of space to commute. (We have not invoked the fact that any vector in Minkowski spacetime is the sum of two spacelike vectors.)

Thanks to extensive research on "C^*-dynamical systems," much is known about G-invariant states when G acts as a large group of automorphisms of $\mathfrak{A}$. In particular, the set of G-invariant states is convex and closed (in the weak* topology), hence the set has extreme points, called *extremal invariant states*. (Obviously if a pure state of $\mathfrak{A}$ is G-invariant, then it is extremal invariant.) Furthermore, we also have the following result concerning the disjointness of G-invariant states.

PROPOSITION 52. *Let ω be a G-invariant state of $\mathfrak{A}$, let $\mathcal{H}$ be its GNS Hilbert space, and let Ω be the GNS vector. Then the following are equivalent:*

1. *ω is clustering in the sense that*

$$\lim_{t\to\infty} \omega(\alpha_{tx}(A)B) = \omega(A)\omega(B).$$

2. *ω is extremal invariant.*
3. *If a G-invariant state ρ is quasiequivalent to ω, then $\rho = \omega$. In other words, no other G-invariant state is quasiequivalent to ω.*
4. *The ray spanned by Ω is the unique (up to scalar multiples) G-invariant subspace of $\mathcal{H}$.*

Proof. See [Størmer, 1970]. For related details, see also [Emch, 1972, pp. 183, 287] and Emch, this volume, Section 3. ■

So, if a (vacuum) state is clustering, then no other translation invariant state is in its folium (i.e. the set of states that are quasiequivalent to that state). Similarly, if a state is *extremal* invariant (*a fortiori* if it is pure) then it is the unique translation invariant state in its folium.

NOTE 53. The existence of disjoint vacua is related to spontaneous symmetry breaking. See Section 10.7.

NOTE 54. Prop. 52 plays a central role in the proof of "Haag's theorem" given in [Emch, 1972, p. 248]. In particular, the uniqueness of extremal G-invariant states is equated with the nonexistence of "vacuum polarization."

2.3 The Reeh-Schlieder Theorem

We have assumed that a vacuum state is translation invariant. But we expect a vacuum state to obey a stronger constraint that reflects the relativistic nature of the theory. In particular, the unitary representation defined in Fact 49 is generated infinitesimally by the four momentum operator $\mathbf{P}$. (The idea of a four momentum operator can be made precise in the "SNAG [Stone-Naimark-Ambrose-Gelfand] Theorem," which generalizes Stone's theorem on the existence of self-adjoint operators generating one-parameter unitary groups.) We require that the energy is positive in every Lorentz frame, equivalently, that the spectrum of $\mathbf{P}$ lies in the forward light cone.

We now generalize this requirement by abstracting away from the details of the forward lightcone. The forward lightcone G_+ has the following property: $G_+ \cap (-G^+) = \{0\}$ where $-G_+ = \{-g : g \in G_+\}$. So, the spectrum condition only requires that the unitary representation of the translation group has spectrum in a set that is asymmetric under taking additive inverses.

ASSUMPTION 55 (Spectrum Condition). Let G be the translation group, and let ω be a G-invariant state of $\mathfrak{A}$. We say that the pair $(\mathfrak{A}, \omega)$ satisfies the *spectrum condition* just in case: there is a subset G_+ of G such that $G_+ \cap (-G_+) = \{0\}$, and in the GNS representation $(\mathcal{H}, \pi)$ of $\mathfrak{A}$ induced by ω, the spectrum $\mathrm{sp}(U)$ of the induced unitary representation of G, is contained in G_+.

The Reeh-Schlieder Theorem shows that the spectrum condition entails that the vacuum vector Ω is cyclic for *every* local algebra. For this theorem, we suppose that a translation invariant vacuum state ω on $\mathfrak{A}$ has been chosen, and that $(\mathcal{H}, \pi)$ is the GNS representation of $\mathfrak{A}$ induced by ω. We then define a corresponding net $\mathfrak{R}$ of von Neumann algebras on $\mathcal{H}$ by

$$O \mapsto \mathfrak{R}(O) \equiv \pi(\mathfrak{A}(O))''.$$

If the net $\mathfrak{A}$ satisfies microcausality, then so will $\mathfrak{R}$. Since Ω is cyclic for $\pi(\mathfrak{A})$, the set $\{\mathfrak{R}(O)\Omega : O \in \mathcal{K}\}$ is dense in $\mathcal{H}$.

To prove the theorem, we need one additional assumption.

ASSUMPTION 56. The net $O \mapsto \mathfrak{R}(O)$ is said to satisfy *additivity* just in case for any double cone O, the set $\{\mathfrak{R}(O + x) : x \in G\}$ generates $\mathfrak{R}$ as a C^*-algebra. (Here again, G denotes the translation group.)

The additivity assumption is sometimes justified on the grounds that there should be no smallest length scale in the theory — i.e. any observable is generated by taking products, sums, etc. of observables from arbitrarily small regions.

THEOREM 57 (Reeh-Schlieder). *Suppose that the net $O \mapsto \mathfrak{R}(O)$ satisfies the spectrum condition and additivity. Then for all double cones O, Ω is cyclic for $\mathfrak{R}(O)$. If the net $\mathfrak{R}$ also satisfies microcausality, then Ω is separating for every local algebra.*

The Reeh-Schlieder (RS) Theorem has been one of the more intensely studied issues in the foundations of relativistic QFT. In a pair of articles [Redhead, 1995a; Redhead, 1995b], Redhead shows that the RS Theorem entails that the vacuum state displays nonlocal correlations. (See also [Halvorson and Clifton, 2000]). Redhead also points out since the vacuum is separating for each local algebra, every local event has a nonzero probability of occurring in the vacuum state; in particular, there can be no local number operators (since they would have the vacuum state as an eigenvector). Finally, [Fleming, 2000] argues that RS Theorem entails a pernicious sort of nonlocality, worse than the nonlocality in non-relativistic QM, and so indicates a need to revise the standard formulation of AQFT. (For one possible reply, see [Halvorson, 2001].)

Due to the use of the spectrum condition, it would seem that RS Theorem is a "purely relativistic result," without analogue in non-relativistic QM or QFT (see [Saunders, 1992]). Furthermore, we might expect that many other results of relativistic QFT that are derived from RS Theorem would fail for non-relativistic theories. Indeed, non-relativistic QFT *does* admit local number operators. However, a version of the spectrum condition, and consequently a version of RS Theorem has been shown to hold for non-relativistic theories [Requardt, 1986].

Notes: The original Reeh-Schlieder Theorem was formulated in the axiomatic approach to QFT, and can be found in [Reeh and Schlieder, 1961]. More up-to-date presentations of the theorem can be found in [Horuzhy, 1990; D'Antoni, 1990; Baumgärtel and Wollenberg, 1992], and [Araki, 1999].

2.4 The funnel property

DEFINITION 58. Let $\mathfrak{R}_1, \mathfrak{R}_2$ be von Neumann algebras on $\mathcal{H}$ such that $\mathfrak{R}_1 \subseteq \mathfrak{R}_2$. If there is a vector $\Omega \in \mathcal{H}$ that is cyclic and separating for $\mathfrak{R}_1, \mathfrak{R}_2$, and $\mathfrak{R}_1' \cap \mathfrak{R}_2$, then the pair $(\mathfrak{R}_1, \mathfrak{R}_2)$ is said to be a *standard inclusion* of von Neumann algebras.

NOTE 59. Let $O \mapsto \mathfrak{R}(O)$ be a net of von Neumann algebras on Minkowski spacetime. Suppose that the Reeh-Schlieder property holds for Ω, i.e. for each

double cone O, Ω is cyclic and separating for $\mathfrak{R}(O)$. Then if O_1, O_2 are double cones such that the closure $\overline{O}_1$ of O_1 is contained in O_2, then the pair $(\mathfrak{R}(O_1), \mathfrak{R}(O_2))$ is a standard inclusion of von Neumann algebras.

DEFINITION 60. Let $\mathfrak{R}_1, \mathfrak{R}_2$ be von Neumann algebras on $\mathcal{H}$ such that $\mathfrak{R}_1 \subseteq \mathfrak{R}_2$. The pair $(\mathfrak{R}_1, \mathfrak{R}_2)$ is said to be a *split inclusion* if there is a type I factor $\mathfrak{N}$ such that $\mathfrak{R}_1 \subseteq \mathfrak{N} \subseteq \mathfrak{R}_2$.

ASSUMPTION 61 (Funnel Property). The net $O \mapsto \mathfrak{R}(O)$ of von Neumann algebras is said to satisfy the *funnel property* if for any double cones O_1, O_2 with $\overline{O}_1$ contained in O_2, the pair $(\mathfrak{R}(O_1), \mathfrak{R}(O_2))$ is a split inclusion.

NOTE 62. A type I factor $\mathfrak{N}$ is countably decomposable iff $\mathfrak{N}$ is isomorphic to $\mathfrak{B}(\mathcal{H})$ with $\mathcal{H}$ separable iff $\mathfrak{N}$ is separable in the ultraweak topology (see [Kadison and Ringrose, 1997, Exercise 5.7.7]).

In our discussion of superselection theory (Sections 7.2–11), at one crucial juncture (Prop. 243, p. 813) we will have to invoke the assumption that the vacuum Hilbert space is separable. This will be the only place in the Chapter where we need to assume that a Hilbert space is separable. In particular, the separability assumption is needed to establish the correspondence between two notions of superselection sectors, one of which is physically motivated, and one of which is mathematically useful. The following result is the only attempt we will make to connect the separability assumption to something with (perhaps) more clear physical significance. (In general, we are highly suspicious of the physical warrant for the separability assumption; compare with Section 6, and with [Halvorson, 2004].)

PROPOSITION 63. *Let $\mathfrak{R}$ be a net of von Neumann algebras on $\mathcal{H}$, and suppose that $\Omega \in \mathcal{H}$ is cyclic and separating for all local algebras. If the net satisfies the funnel property, then $\mathcal{H}$ is separable.*

Proof. (Compare with Prop. 1.6 of [Doplicher and Longo, 1984].) Let O_1, O_2 be double cones with $\overline{O}_1 \subseteq O_2$. Let $\mathfrak{N}$ be a type I factor such that $\mathfrak{R}(O_1) \subseteq \mathfrak{N} \subseteq \mathfrak{R}(O_2)$, and let ω be the state of $\mathfrak{N}$ induced by Ω. Recall that $\mathfrak{N}$ is isomorphic to $\mathfrak{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$. Since $\mathfrak{N} \subseteq \mathfrak{R}(O_2)$ and Ω is separating for $\mathfrak{R}(O_2)$, ω is faithful and normal. Hence $\mathcal{K}$ is separable, and there is a countable set $\mathfrak{N}_0$ that is ultraweakly dense in $\mathfrak{N}$. Since $\mathfrak{R}(O_1) \subseteq \mathfrak{N}$, and Ω is cyclic for $\mathfrak{R}(O_1)$ it follows that $[\mathfrak{N}_0\Omega] = [\mathfrak{N}\Omega] = \mathcal{H}$. Hence $\mathcal{H}$ is separable. ■

If one wanted to justify an assumption that the vacuum Hilbert space is separable, Prop. 63 shows that it is enough to justify the funnel property. There are concrete models where the funnel property demonstrably does *not* hold [Horuzhy, 1990, p. 23]. But the physical significance of these models is not clear, and there are a couple of other considerations that might favor the funnel property: (i): In Section 3.3, we show that connection of the funnel property with issues about nonlocality. (ii): Buchholz and Wichmann [1986] argue that the funnel property is a *sufficient* condition for a particle interpretation of QFT. Of course, the interpreter of QFT will want to critically examine Buchholz and Wichmann's notion

of a "particle interpretation." (Compare with Section 4.5, where particle interpretations are discussed further. Compare also with Section 6.2, which hints at connections between nonseparable Hilbert space and field interpretations of QFT.)

NOTE 64. The funnel property for free fields is shown in [Buchholz, 1974].

2.5 Type of local algebras

We now collect the currently known information on the type of local algebras in physically relevant representations of the net of local observable algebras.

DEFINITION 65. Let $\mathfrak{R}_1$ and $\mathfrak{R}_2$ be nets of von Neumann algebras on a Hilbert space $\mathcal{H}$. We say that $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are *locally quasiequivalent* just in case for each double cone O there is an isomorphism $\varphi_O : \mathfrak{R}_1(O) \to \mathfrak{R}_2(O)$.

NOTE 66. Although it is not an "axiom" of AQFT, there are good reasons to believe that representations of physical interest (in particular for elementary particle physics) are locally quasiequivalent to some vacuum representation, where a vacuum representation is the GNS representation of some privileged (e.g. perhaps translation invariant) state. For example local quasi-equivalence holds between any two physical representations according to the selection criterion of Doplicher-Haag-Roberts (see Section 7.2 and following), and according to the more liberal selection criterion of [Buchholz and Fredenhagen, 1982]. Thus, any conclusion we draw concerning the structure of *local algebras* in a vacuum representation can be inferred to hold as well for these other representations.

Local algebras are properly infinite

Some relatively simple results narrow down the possible options for the type of local algebras. For this, we define the important "property B," because it is a consequence of plausible assumptions (viz. additivity and the spectrum condition), because it also makes sense in situations where there is no translation group (unlike the spectrum condition), and because it is all we need to infer various results, in particular that local algebras are properly infinite.

DEFINITION 67. Let $O \to \mathfrak{R}(O)$ be a net of von Neumann algebras on some Hilbert space $\mathcal{H}$. We say that the net $\mathfrak{R}$ satisfies *property B* just in case for any two double cones O_1 and O_2 such that $\overline{O}_1 \subseteq O_2$, if $E \in \mathfrak{R}(O_1)$ is a nonzero projection, then E is equivalent in $\mathfrak{R}(O_2)$ to the identity projection I; i.e. there is an isometry $V \in \mathfrak{R}(O_2)$ such that $VV^* = E$.

NOTE 68. If for each O, the algebra $\mathfrak{R}(O)$ is type III, then the net $\mathfrak{R}$ satisfies property B.

We expect property B to hold for a net of observable algebras because it follows from the physically motivated postulates of weak additivity and the spectrum condition.

PROPOSITION 69. *Let $O \mapsto \mathfrak{R}(O)$ be a net of von Neumann algebras satisfying microcausality, the spectrum condition, and weak additivity. Then the net $O \mapsto \mathfrak{R}(O)$ satisfies property B.*

Proof. For the original proof, see [Borchers, 1967]. For a recent exposition, see [D'Antoni, 1990]. ■

ASSUMPTION 70 (Nontriviality). A net $O \mapsto \mathfrak{A}(O)$ of C^*-algebras is said to satisfy *non-triviality* just in case for each double cone O, $\mathfrak{A}(O) \neq \mathbb{C}I$.

PROPOSITION 71. *Let $O \to \mathfrak{R}(O)$ be a net of von Neumann algebras that satisfies microcausality, property B, and non-triviality. Then for every double cone O, the von Neumann algebras $\mathfrak{R}(O)$ and $\mathfrak{R}(O')'$ are properly infinite.*

Proof. We first show that $\mathfrak{R}(O)$ is properly infinite; that is, that every central projection in $\mathfrak{R}(O)$ is infinite. Let C be a central projection in $\mathfrak{R}(O)$. Choose a nontrivial double cone O_1 whose closure is contained in O. Then by property B, for each nonzero projection $E \in \mathfrak{R}(O_1)$, E is equivalent to I modulo $\mathfrak{R}(O)$. Since $\mathfrak{R}(O_1) \neq \mathbb{C}I$, there is a projection $E \in \mathfrak{R}(O_1)$ such that $E \sim (I - E) \sim I$ modulo $\mathfrak{R}(O)$. It then follows that $EC \sim (I - E)C \sim C$ modulo $\mathfrak{R}(O)$. It is clear that $EC = CEC \leq C$. If $EC = C$ then $(I - E)C = 0$, a contradiction. Therefore $EC < C$ and $EC \sim C$ modulo $\mathfrak{R}(O)$. That is, C is an infinite projection in $\mathfrak{R}(O)$, and $\mathfrak{R}(O)$ is properly infinite. By microcausality, $\mathfrak{R}(O_1) \subseteq \mathfrak{R}(O')'$; thus the preceding argument also shows that $\mathfrak{R}(O')'$ is properly infinite. ■

In particular, the preceding proposition rules out the cases of type I_n and type II_1 von Neumann algebras. Already this result has implications for questions about nonlocality; see Prop. 102 in Section 3.3. However, the previous proposition leaves open the possibility that local algebras might be type I_∞ factors, and it also leaves open the case that local algebras might be direct sums of heterogeneous types of von Neumann algebras.

Local algebras are hyperfinite

We will shortly see that the best results we have point toward the fact that local algebras are type III, which were originally thought to be unruly anomalies with no relevance for physics. However, we first show that under some physically plausible conditions, local algebras are approximated by finite-dimensional algebras (i.e. they are "hyperfinite"), which shows that after all they are not so unruly.

DEFINITION 72. Let $\mathfrak{R}$ be a von Neumann algebra. Then $\mathfrak{R}$ is said to be *hyperfinite* just in case there is a family $(\mathfrak{R}_a)_{a\in\mathbb{A}}$ of finite dimensional von Neumann algebras in $\mathfrak{R}$ such that $\mathfrak{R} = (\cup_{a\in\mathbb{A}}\mathfrak{R}_a)''$.

Hyperfiniteness turns out to be an extremely useful condition for mathematical purposes. Indeed, hyperfiniteness is intimately linked to the existence of normal conditional expectations (see [Kadison and Ringrose, 1997, Chap. 8]), and there

is a unique type II_1 hyperfinite factor, and a unique type III_1 hyperfinite factor. From a physical/foundational point of view, one might also think that a failure of hyperfiniteness for $\mathfrak{R}$ might make it difficult to find a correspondence between elements of the algebra $\mathfrak{R}$ and real-life laboratory procedures which can only involve a finite number of tasks.

FACT 73. Every type I von Neumann algebra is hyperfinite. See [Kadison and Ringrose, 1997, Exercise 8.7.26].

ASSUMPTION 74 (Inner/Outer Continuity). A net $O \mapsto \mathfrak{R}(O)$ of von Neumann algebras is said to be *inner continuous* if for any monotonically increasing net $(O_a)_{a\in\mathbb{A}}$ with least upper bound O, we have

$$\bigvee_{a\in\mathbb{A}} \mathfrak{R}(O_a) = \mathfrak{R}(O),$$

where $\mathfrak{R}_1 \vee \mathfrak{R}_2$ denotes the von Neumann algebra generated by $\mathfrak{R}_1$ and $\mathfrak{R}_2$. Outer continuity is defined by taking a decreasing net of regions, and the intersection of the corresponding von Neumann algebras.

NOTE 75. The condition that the net $\mathfrak{R}$ be continuous from the inside is satisfied whenever $\mathfrak{R}$ is the "minimal" net constructed in the standard way from underlying Wightman fields. See [Buchholz *et al.*, 1987]. Similarly, the maximal net satisfies outer continuity.

PROPOSITION 76. *Suppose that the net $O \mapsto \mathfrak{R}(O)$ satisfies the funnel property and either inner or outer continuity. Then for each double cone O, $\mathfrak{R}(O)$ is hyperfinite.*

Sketch of proof. (Compare [Buchholz *et al.*, 1987, p. 134].) We just look at the case where the net is inner continuous. By the funnel property there is a type I factor $\mathfrak{N}_i$ interpolating between $\mathfrak{R}(O_i)$ and $\mathfrak{R}(O)$. It then follows that the union of the ascending sequence $\mathfrak{N}_i$ of hyperfinite factors is dense in $\mathfrak{R}(O)$, hence $\mathfrak{R}(O)$ is hyperfinite. ■

Local algebras are type III_1 factors

A series of results, accumulated over a period of more than thirty years, indicates that the local algebras of relativistic QFT are type III von Neumann algebras, and more specifically, hyperfinite type III_1 factors. We gather some of these results in this section. The first result, due to Longo [1979], improved on some earlier results by Driessler.

PROPOSITION 77. *Let $\mathfrak{R}$ be a von Neumann algebra acting on $\mathcal{H}$, $\Omega \in \mathcal{H}$ a separating unit vector for $\mathfrak{R}$, G a locally compact abelian group with dual Γ, and U a continuous unitary representation of G on $\mathcal{H}$ such that $U\Omega = \Omega$ and the ray $\mathbb{C}\,\Omega$ is the unique $U(G)$ invariant subspace of $\mathcal{H}$. Suppose that there exist subsets $G_+ \subseteq G$ and $\Gamma_+ \subseteq \Gamma$ such that*

1. $G_+ \cup (-G_+) = G$ *and* $U(g)\mathfrak{R}U(g)^* \subseteq \mathfrak{R}$, *for all* $g \in G_+$.
2. $\Gamma_+ \cap (-\Gamma_+) = \{0\}$ *and* $\mathrm{sp}(U) \subseteq \Gamma_+$.

Then either $\mathfrak{R} = \mathbb{C}I$ *or* $\mathfrak{R}$ *is a type* III_1 *factor.*

Sketch of proof. (See [Longo, 1979, p. 203] for details.) Let ω be the state of $\mathfrak{R}$ given by $\omega(A) = \langle \Omega, A\Omega \rangle$. The proof of this result proceeds by showing that $\mathfrak{R}_\omega = \mathbb{C}I$, where $\mathfrak{R}_\omega$ is the centralizer of the state ω. In particular, let E be a projection in $\mathfrak{R}_\omega$, and define the function $f : G \to \mathbb{C}$ by

$$f(g) = \langle \Omega, EU(g)E\Omega \rangle = \langle \Omega, EU(g)EU(-g)\Omega \rangle.$$

Using the constraint on $\mathrm{sp}(U)$, it can be shown that f is constant, and hence $U(g)E\Omega = E\Omega$ for all $g \in G$. Since $\mathbb{C}\,\Omega$ is the unique invariant subspace under $U(G)$, it follows that $E\Omega = \Omega$, and since Ω is separating for $\mathfrak{R}$, $E = 0$ or $E = I$. ■

The preceding proposition applies to algebras of the form $\pi(\mathfrak{A}(W))''$, where W is a wedge region, and π is a vacuum representation of the quasilocal algebra $\mathfrak{A}$. Indeed, we can take G_+ to be a one-parameter semi-group of lightlike translations with origin at the apex of W, in which case $\mathbb{R} = G_+ \cup (-G_+)$. Let ω be a translation invariant state on $\mathfrak{A}$ such that $(\mathfrak{A}, \omega)$ satisfies that spectrum condition (Assumption 55). We then have that the dual group Γ of G in $\mathbb{R}^4$ is also a lightlike line, and hence the spectrum condition entails that there is a subset Γ_+ of Γ, namely those vectors that point toward the future, such that $\Gamma_+ \cap (-\Gamma_+) = \{0\}$. Finally, we saw in Section 2.2 that when ω is extremal invariant, the ray $\mathbb{C}\,\Omega$ is the unique $U(G)$ invariant subspace of $\mathcal{H}$.

For results relevant to *local* algebras, we must impose one further condition on the net $\mathfrak{R}$. The first result ([Buchholz *et al.*, 1987]) requires reference to axiomatic QFT with unbounded operators smeared by test-functions (see [Streater and Wightman, 1964]). That is, we must assume that the net $\mathfrak{R}$ arises from an underlying Wightman field theory that satisfies a certain condition — asymptotic scale invariance.

Recall that in the axiomatic approach, fields are essentially self-adjoint operators of the form $\Phi(f)$, where f is a test-function on spacetime. The presence of these test-functions allows the definition of a notion of asymptotic scale invariance.

DEFINITION 78. Let $N : \mathbb{R}^+ \to \mathbb{R}^+$ be a monotone function. Then a scaling transformation of the test-functions is given by $f \mapsto f_\lambda$, where $f_\lambda(x) = N(\lambda)f(\lambda^{-1}x)$. Let Φ_α be a set of Wightman fields generating the net $O \mapsto \mathfrak{R}(O)$. We say that the fields satisfy *asymptotic scale invariance* just in case there is some field Φ with vanishing vacuum expectation values:

$$\langle \Omega, \Phi(f)\Omega \rangle = 0,$$

and for a suitable choice of $N(\lambda)$, the scaled field operators $\Phi(f_\lambda)$ have the following properties:

1. The expectation values $\langle \Omega, \Phi(f_\lambda)^* \Phi(f_\lambda)\Omega \rangle$ converge for all test-functions in the limit $\lambda \to 0$, and are nonzero for some f;

2. The norms $\|\Phi(f_\lambda)^* \Phi(f_\lambda)\Omega\|$ and $\|\Phi(f_\lambda)\Phi(f_\lambda)^*\Omega\|$ stay bounded in this limit.

When a net of von Neumann algebras arises from a Wightman theory with asymptotic scale invariance, it follows that local algebras are hyperfinite type III_1 factors.

PROPOSITION 79 ([Buchholz *et al.*, 1987]). *Let $\mathfrak{R}$ be a net of von Neumann algebras that satisfies microcausality, the spectrum condition, and the funnel property. Suppose also that $\mathfrak{R}$ can be constructed from an underlying Wightman theory that satisfies asymptotic scale invariance. Then for each double cone O, $\mathfrak{R}(O) = \mathfrak{M} \overline{\otimes} \mathfrak{Z}$, where $\mathfrak{M}$ is the unique type III_1 hyperfinite factor and $\mathfrak{Z}$ is the center of $\mathfrak{R}(O)$.*

NOTE 80. In [Buchholz *et al.*, 1987], the funnel property is derived from a more basic postulate called "nuclearity," which imposes bounds on the number of local degrees of freedom.

Of course, one wishes for a result that is more intrinsic to AQFT. Such a result is provided in [Buchholz and Verch, 1995], using the method of scaling algebras that allows the computation of the short distance (scaling) limit of a net $\mathfrak{A}$ of local observables. (For a short exposition of scaling algebras, we refer the reader to [Buchholz, 1998].) In summary, besides the basic assumptions on the net, the only additional assumption needed to derive the type III_1 property is that the net has a nontrivial scaling limit.

NOTE 81. In some concrete models, it can be shown directly that local algebras are the unique type III_1 hyperfinite factor. For example, for the free Bose field of mass $m = 0$ (in the Minkowski vacuum representation), local algebras are isomorphic to algebras for wedge regions. Thus Prop. 77 shows that local algebras are type III_1 factors. Furthermore, the free Bose field of mass $m > 0$ is locally quasiequivalent to the case of $m = 0$, and so its local algebras are also type III_1 hyperfinite factors. See [Horuzhy, 1990, p. 254].

The derivation of the type III_1 property is one of the most surprising and interesting results of contemporary mathematical physics. But what is the foundational significance of the result? How would the world be different if local algebras were, say, type $\mathrm{III}_{1/2}$, or even more radically different, if they were type II_∞? For one, there is a crucial difference between the structure of states on familiar type I algebras, and the structure of states on type III algebras: since type III algebras have no atomic projections, and the support projection of a pure normal state is atomic, it follows that type III algebras have no pure normal states. (But of course the same is true for type II algebras.) As pointed out in [Clifton and Halvorson, 2001b] and [Ruetsche, 2004], this absence of pure states is a further obstacle to an ignorance interpretation of quantum probabilities. (See also Section 3.4.)

Yngvason [2005] makes several interesting claims about the conceptual importance of type III algebras, especially in relation to questions of nonlocality. First, according to Yngvason, "type I intuitions" can lead to paradoxes, such as that encountered in Fermi's famous two-atom system. However, claims Yngvason, these paradoxes disappear if we model these situation appropriately with type III algebras. Second, Yngvason claims that the homogeneity of the state space of a type III_1 factor $\mathfrak{R}$ can be interpreted as saying that for any two states ω_1, ω_2 on $\mathfrak{R}$, ω_2 can be prepared from ω_1 (within arbitrarily good accuracy) via a unitary operation. Such an operation is, of course, nonselective, and so does not change the statistics of measurements of observables in $\mathfrak{R}'$. So, in one sense, an observer with a type III algebra has more control over his state space than an observer with a type I algebra.

3 NONLOCALITY AND OPEN SYSTEMS IN AQFT

NOTE 82. For this section, we use the following notational conventions: uppercase roman letters for algebras, lowercase roman letters for operators, and $\mathbb{1}$ for the multiplicative identity in an algebra.

It is a basic assumption of AQFT that the observable algebras $A(O_1)$ and $A(O_2)$ are mutually commuting when O_1 and O_2 are spacelike separated. This requirement — which we have called "microcausality" — is sometimes also called "Einstein causality," because of a suggested connection between the commutativity of the algebras $A(O_1), A(O_2)$ and the relativistic prohibition on "superluminal signaling." Implicit in this connection is a claim that if $[a, b] \neq 0$ for $a \in A(O_1)$ and $b \in A(O_2)$, then a measurement of a could change the statistics of a measurement of b.

Despite the fact that nonrelativistic QM makes no reference to spacetime, it has a footprint of the relativistic prohibition of superluminal signalling. In particular, the state space of two distinct objects is a tensor product $H_1 \otimes H_2$, and their joint algebra of observables is $B(H_1) \otimes B(H_2)$. In this tensor product construction we represent observables for system A as simple tensors $a \otimes \mathbb{1}$ and observables of system B as $\mathbb{1} \otimes b$. Thus, we have a version of microcausality. But we also have stronger independence properties. For example, for every state φ_1 of system A and state φ_2 of system B, there is a state φ of $A \otimes B$ such that $\varphi|_A = \varphi_1$ and $\varphi|_B = \varphi_2$.

In this section, we investigate the extent to which two local algebras $A(O_1), A(O_2)$ can be thought to represent distinct, independent parts of reality. In Sections 3.1 and 3.2, we discuss the relations between microcausality and other independence assumptions for the algebras $A(O_1), A(O_2)$. In Section 3.3, we summarize some results concerning violation of Bell's inequality in AQFT. Finally, in Section 3.4 we ask whether a local algebra $A(O)$ can be isolated from the influences of its environment.

3.1 Independence of C^* and von Neumann algebras

We first consider notions of independence between a general pair of von Neumann or C^*-algebras.

DEFINITION 83. If e, f are projection operators on a Hilbert space $\mathcal{H}$, then we let $e \wedge f$ denote the projection onto the closed subspace $e(\mathcal{H}) \cap f(\mathcal{H})$.

FACT 84. Let R be a von Neumann algebra acting on $\mathcal{H}$. If $e, f \in R$ then $e \wedge f \in R$.

DEFINITION 85 (Schlieder Property). Let R_1, R_2 be von Neumann algebras acting on the Hilbert space $\mathcal{H}$. We say that the pair (R_1, R_2) satisfies the *Schlieder property* just in case if $e \in R_1$ and $f \in R_2$ are nonzero projections, then $e \wedge f \neq 0$.

The Schlieder property entails that for $e \in R_1, f \in R_2$, if $e, f \neq 0$ and $e, f \neq \mathbb{1}$ then:

$$e \wedge f \neq 0, \quad \neg e \wedge \neg f \neq 0, \quad e \wedge \neg f \neq 0, \quad \neg e \wedge f \neq 0,$$

where $\neg x = \mathbb{1} - x$ is the projection onto the orthogonal complement of $x(\mathcal{H})$. Hence if "$\wedge$" is the analogue of conjunction in classical logic, then the Schlieder property is the analogue of logical independence.

DEFINITION 86. If A, B are C^*-subalgebras of some C^*-algebra C, we let $A \vee B$ denote the C^*-algebra generated by $A \cup B$.

DEFINITION 87 (C^*-Independence). Let A, B be C^*-algebras. We say that the pair (A, B) is C^*-*independent* just in case for any state ω_1 of A and any state ω_2 of B, there is a state ω of $A \vee B$ such that $\omega|_A = \omega_1$ and $\omega|_B = \omega_2$. In other words, each state of A is compatible with each state of B.

The C^*-independence assumption has an obvious operationalist motivation: if Alice is an observer at O_1 and Bob is an observer at O_2, then C^*-independence amounts to the claim that Alice's choice to prepare a state cannot in any way obstruct Bob's ability to prepare a state. Indeed, [Summers and Buchholz, 2005] claim that a failure of C^*-independence could be detected by local observers. On the other hand, C^*-independence could also be regarded as an explication of the notion of the independence of objects:

> *Two objects A, B are truly independent just in case any state of A is compatible with any state of B; i.e. there are no logical relations between predications of states to A and B.*

Unfortunately, C^*-independence does not imply microcausality.

EXAMPLE 88. We show that C^*-independence does not entail microcausality. (Compare with [Napiórkowski, 1972].) Consider the finite dimensional $*$-algebra $C(\mathbb{Z}_4) \oplus M_2$, where $C(\mathbb{Z}_4)$ is the abelian $*$-algebra of dimension 4, and M_2 is the 2×2 matrices over $\mathbb{C}$. The projection lattice of $C(\mathbb{Z}_4)$ is the Boolean algebra with two atoms; hence it contains logically independent elements e_1, e_2. Now choose two projections $f_1, f_2 \in M_2$ such that $[f_1, f_2] \neq 0$, and let R_i be the abelian $*$-subalgebra of $C(\mathbb{Z}_4) \oplus M_2$ generated by the projection $e_i \oplus f_i$.

To see that (R_1, R_2) is C^*-independent, let ω_i be states on the R_i, and let $\lambda_i = \omega_i(e_i \oplus f_i)$. By the logical independence of e_1, e_2, there is a state ρ of $C(\mathbb{Z}_4)$ such that $\rho(e_i) = \lambda_i$. Then the state $\rho \oplus 0$ on $C(\mathbb{Z}_4) \oplus M_2$ is a common extension of the ω_i since

$$(\rho \oplus 0)(e_i + f_i) = \rho(e_i) = \lambda_i,$$

and a state's value on $e_i \oplus f_i$ determines its value on R_i. Therefore, (R_1, R_2) is C^*-independent. On the other hand, $[e_1 + f_1, e_2 + f_2] = [f_1, f_2] \neq 0$, whence (R_1, R_2) does not satisfy microcausality.

In the previous example, the algebras R_1 and R_2 share a common superselection sector: each commutes with the projection $p = \mathbb{1} \oplus 0$. However, the reduced algebras pR_ip are not C^*-independent. In fact, the diagnosis of this example can be generalized into the following result.

PROPOSITION 89. *Let R_1 and R_2 be von Neumann algebras acting on a Hilbert space $\mathcal{H}$. If for every projection $e \in Z(R_1 \vee R_2)$, the pair (eR_1e, eR_2e) is C^*-independent, then $[R_1, R_2] = \{0\}$.*

Proof. See [Summers and Buchholz, 2005]. ■

DEFINITION 90 (Split Property). Let R_1 and R_2 be von Neumann algebras on $\mathcal{H}$ such that $R_1 \subseteq R_2'$. Then the pair (R_1, R_2) is said to satisfy the *split property* just in case there is a type I factor M such that $R_1 \subseteq M \subseteq R_2'$.

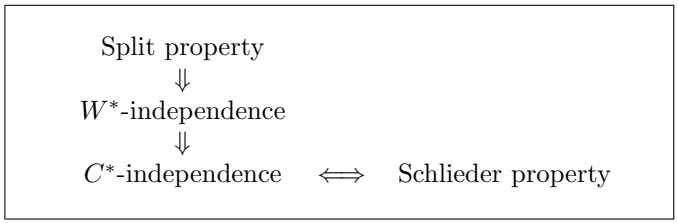
REMARK 91. (i): It is clear that the previous definition is equivalent to saying that (R_1, R_2') is a 'split inclusion' as per Definition 60.

(ii): If (R_1, R_2) satisfies the split property, then under some fairly standard conditions (e.g. R_1 or R_2 is type III), there is a natural $*$-isomorphism α between $\overline{R_1 \vee R_2}$ and the von Neumann algebra tensor product $R_1 \overline{\otimes} R_2$; by saying that α is 'natural', we mean that it extends the map $AB \mapsto A \otimes B$. Furthermore, the $*$-isomorphism α is spatial, i.e. there is a unitary operator u such that $\alpha(x) = uxu^*$. See [Summers, 1990, p. 212].

(iii): On the other hand, suppose that R is a factor, so that $R \cup R'$ generates $B(\mathcal{H})$ as a von Neumann algebra, i.e. $\overline{R \vee R'} = B(\mathcal{H})$. Then R' is of the same type (I, II, or III) as R [Kadison and Ringrose, 1997, Thm. 9.1.3], and so the von Neumann algebra tensor product $R \overline{\otimes} R'$ is of the same type as R [Kadison and Ringrose, 1997, p. 830]. So if R is type II or III, then $\overline{R \vee R'}$ is strictly larger than, and not isomorphic to $R \overline{\otimes} R'$.

DEFINITION 92 (W^*-Independence). Let R_1 and R_2 be von Neumann algebras acting on $\mathcal{H}$. The pair (R_1, R_2) is said to be *W^*-independent* just in case for every normal state φ_1 of R_1 and for every normal state φ_2 of R_2, there is a normal state φ of $R_1 \vee R_2$ such that $\varphi|_{R_i} = \varphi_i$.

With the assumption of the mutual commutativity of R_1 and R_2 (i.e. microcausality), we have the following implications (see [Summers, 1990, p. 222]):



3.2 Independence of local algebras

We now consider which independence properties hold between pairs of algebras associated with spacelike separated regions. In general, not much can be said about the independence of such algebras. In order to get such results off the ground, we need a stronger notion of spacelike separation.

DEFINITION 93. Two double cones O_1, O_2 are said to be *strictly spacelike separated* just in case there is a neighborhood N of zero such that $O_1 + x$ is spacelike separated from O_2 for all $x \in N$.

PROPOSITION 94. *Suppose that the net $O \mapsto R(O)$ satisfies microcausality, weak additivity, and the spectrum condition. If O_1 and O_2 are strictly spacelike separated, then $(R(O_1), R(O_2))$ satisfies the Schlieder property.*

Proof. See [Schlieder, 1969]. ■

In terms of logical strength, the following concept lies between spacelike separation and strict spacelike separation; furthermore, this concept makes sense for spacetimes without a translation group.

DEFINITION 95. Two double cones O_1 and O_2 are said to be *strongly spacelike separated* just in case there are double cones $\tilde{O}_i$ such that $\overline{O}_i \subseteq \tilde{O}_i$, and $\tilde{O}_1, \tilde{O}_2$ are spacelike.

FACT 96. If O_1 and O_2 are strictly spacelike separated, then they are strongly spacelike separated.

Of course, the assumptions of Proposition 94 (microcausality, additivity, spectrum) are precisely what is used to derive property B for the net (Proposition 69). So, it is perhaps illustrative to give a simple derivation of the Schlieder property from property B. (Such a result also applies in contexts — e.g. QFT on curved spacetime — where the spectrum condition does not make sense.)

PROPOSITION 97. *Suppose that the net $O \mapsto R(O)$ of von Neumann algebras satisfies microcausality and property B. If O_1 and O_2 are strongly spacelike separated, then $(R(O_1), R(O_2))$ satisfies the Schlieder property.*

Proof. Let O_1 and O_2 be strongly spacelike separated, and let $e_i \in \mathfrak{R}(O_i)$ be projections. Then there are regions $\tilde{O}_i$ such that $\overline{O}_i \subseteq \tilde{O}_i$, and $\tilde{O}_1$ is spacelike to $\tilde{O}_2$. By property B, there are isometries $v_i \in R(\tilde{O}_i)$ such that $v_i v_i^* = e_i$.

Furthermore, $[v_1, v_2] = 0$ and hence $e_1 e_2 = v_1 v_2 (v_1 v_2)^*$. But $v_1 v_2$ is an isometry, and so $v_1 v_2 (v_1 v_2)^* \neq 0$. ■

The split property clearly does not hold for $(R(W), R(W'))$ where W is a wedge region and W' is its causal complement. Indeed, since $R(W)$ and $R(W')$ are type III_1 factors, there can be no $*$-isomorphism between $R(W) \overline{\otimes} R(W')$ and $\overline{R(W) \vee R(W)'} = B(H)$. However, if the funnel property holds for the net $O \mapsto R(O)$, then $(R(O_1), R(O_2))$ satisfies the split property when O_1 and O_2 are strictly spacelike separated double cones.

3.3 Bell correlation between von Neumann algebras

We first define a generalized notion of Bell type measurements for a pair of von Neumann algebras.

DEFINITION 98. Let A and B be mutually commuting C^*-subalgebras of some C^*-algebra C. Then we set

$$\mathbb{B}(A,B) \equiv \Big\{(1/2)[a_1(b_1+b_2)+a_2(b_1-b_2)] : a_i = a_i^* \in A, b_i = b_i^* \in B, \\ -\mathbb{1} \leq a_i, b_i \leq \mathbb{1}\Big\}.$$

Elements of $\mathbb{B}(A,B)$ are called *Bell operators* for (A,B).

Let r be a Bell operator for (A,B). It can be shown that $|\varphi(r)| \leq \sqrt{2}$ for each state φ on C [Summers and Werner, 1987]. It is also straightforward to check that if φ is a separable state (i.e. a mixture of product states) then $|\varphi(r)| \leq 1$. Indeed, the Bell measurement correlations in the state φ can be reproduced by a local hidden variable model iff $|\varphi(r)| \leq 1$ [Summers and Werner, 1987; Baez, 1987].

DEFINITION 99. Define the Bell correlation coefficient of a state φ of $A \vee B$ by

$$\beta(\varphi, A, B) = \sup\{\, |\varphi(r)| : r \in \mathbb{B}(A,B) \,\}.$$

If $|\beta(\varphi, A, B)| > 1$, then φ is said to *violate a Bell inequality*, or to be *Bell correlated*.

It is a straightforward exercise to show that if R_1 is an abelian von Neumann algebra and $R_1 \subseteq R_2'$, then for any state φ, $\beta(\varphi, R_1, R_2) \leq 1$. For a sort of converse, Landau [1987] shows that if R_1 and R_2 are nonabelian von Neumann algebras such that $R_1 \subseteq R_2'$, and if (R_1, R_2) satisfies the Schlieder property, then there is *some* state φ that violates Bell's inequality maximally relative to (R_1, R_2). Similarly, Bacciagaluppi [1994] shows that if A and B are C^*-algebras, then some state violates a Bell inequality for $A \otimes B$ iff both A and B are nonabelian.

When A and B have further properties, we can derive even stronger results. For present purposes, we will simply apply a couple of the known results to the case of AQFT. (See [Summers, 1990] for many more details.)

PROPOSITION 100. *Let R be a type III_1 factor acting on a separable Hilbert space $\mathcal{H}$. Then every normal state φ of $B(\mathcal{H})$ is maximally Bell correlated across (R, R'), that is $\beta(\varphi, R, R') = \sqrt{2}$.*

Proof. See [Summers and Werner, 1988; Summers and Werner, 1995]. ■

NOTE 101. Prop. 77 tells us that under quite generic conditions, the wedge algebra $R(W)$ is a type III_1 factor. In this case, Prop. 100 tells us that the vacuum is maximally Bell correlated across $(R(W), R(W)')$.

PROPOSITION 102. *Suppose that R_1 and R_2 are von Neumann algebras on $\mathcal{H}$ such that $R_1 \subseteq R_2'$, and (R_1, R_2) satisfies the Schlieder property. If R_1 and R_2 are properly infinite, then there is a dense set of vectors in $\mathcal{H}$ that induce Bell correlated states across (R_1, R_2).*

Proof. See [Halvorson and Clifton, 2000]. ■

NOTE 103. If a net $O \mapsto R(O)$ of von Neumann algebras on $\mathcal{H}$ satisfies property B and nontriviality, then the hypotheses of Prop. 102 apply to algebras $R(O_1)$ and $R(O_2)$ when O_1 and O_2 are strongly spacelike separated.

Notes: For a comprehensive review of pre-1990 results on independence of local algebras in AQFT, see [Summers, 1990]. For some more recent results, see [Summers, 1997; Florig and Summers, 1997; Rédei, 1998; Halvorson and Clifton, 2000; Summers and Buchholz, 2005].

3.4 Intrinsically entangled states

According to Clifton and Halvorson [2001b], the type III property of local algebras in AQFT shows that it is impossible to disentangle local systems from their environment. To see the argument, recall that it is a standard (perhaps somewhat justified) assumption that the general form of a dynamical evolution T of observables, represented by self-adjoint elements of a C^*-algebra A is given by a completely positive (CP) linear mapping T of A such that $T(\mathbb{1}) = \mathbb{1}$. (Such an assumption is certainly commonplace in, say, quantum information theory.) Here we recall the pertinent definition.

DEFINITION 104. Let A be a C^*-algebra. A linear map T of A is said to be *positive* if $T(a^*a) \geq 0$ for each $a \in A$. T is said to be *completely positive* if for each $n \in \mathbb{N}$, the map $T \otimes \mathrm{id}_n : A \otimes M_n \to A \otimes M_n$ defined on elementary tensors by

$$(T \otimes \mathrm{id}_n)(a \otimes b) = T(a) \otimes b,$$

is positive. Here M_n is the C^*-algebra of $n \times n$ matrices over $\mathbb{C}$.

NOTE 105. If $T : \mathfrak{A} \to \mathfrak{A}$ is positive and $T(\mathbb{1}) = \mathbb{1}$, then for each state ω of $\mathfrak{A}$, we define $T^*(\omega)$ by $T^*(\omega)(a) = \omega(T(a))$. It follows that T^* is an affine mapping of the state space into itself.

For type I factors, Kraus' theorem [Kraus, 1983] shows that CP maps are "inner."

THEOREM 106 (Kraus Representation). *If R is a type I_n factor then the following are equivalent for a linear map $T : R \to R$.*

1. *T is completely positive and $T(\mathbb{1}) = \mathbb{1}$.*
2. *T is the restriction of an automorphism $x \mapsto uxu^*$ on an algebra of the form $R \otimes B(H)$.*
3. *There are positive operators $a_1, \ldots, a_n \in R$ such that $\sum_{i=1}^n a_i = \mathbb{1}$ and*

$$T(x) = \sum_{i=1} a_i^{1/2} x a_i^{1/2}. \quad (1)$$

One special case of Eqn. (1) is the Lüders rule with projection operators e and $\mathbb{1} - e$:

$$T_e(x) = exe + (\mathbb{1} - e)x(\mathbb{1} - e).$$

Furthermore, if the algebra R is type I, we can choose $e \in R$ to be an abelian projection. We have the following result:

> If the local algebra R is a type I factor, then there is a universal disentangling operation T_e. That is, no matter what the initial state, the outcome of applying T_e is that the final state is separable.

However, suppose that R has no abelian projections (e.g. R is type III). Then for each nonzero projection $e \in R$, the algebras eRe and $eR'e$ are nonabelian, and hence there is some entangled state φ for the pair $(eRe, eR'e)$. This entangled state is the image under the operation $(T_e)^*$ of some state on $\overline{R \vee R'}$. Hence, the operation T_e does not disentangle all states.

This heuristic argument can be tightened up into a "proof" that no operation on R can disentangle the states of $\overline{R \vee R'}$. See [Clifton and Halvorson, 2001b] for details.

NOTE 107. (i): The Kraus representation theorem is not valid as it stands for type III algebras. Indeed, the Kraus representation theorem is a special case of the Stinespring decomposition theorem [Stinespring, 1955].

(ii): A CP operation on a von Neumann algebra is typically also assumed to be ultraweakly continuous. The continuity of T might be justified on the grounds that it is necessary if T^* is to map normal states to normal states. For objections to the continuity requirement, see [Srinivas, 1980].

4 PROSPECTS FOR PARTICLES

The main application of relativistic QFT is to fundamental *particle* physics. But it is not completely clear that fundamental particle physics is really about particles.

Indeed, despite initial signs that QFT permits a particle interpretation (via Fock space), there are many negative signs concerning the possibility of particle ontology of relativistic QFT. This section is devoted to assessing the status of particles from the point of view of AQFT.

4.1 Particles from Fock space

We begin our investigation of particles with the "story from mother's knee" about how to give QFT a particle interpretation. (See [Teller, 1995] for one philosopher's interpretation of this story.) The story begins with a special Hilbert space, called *Fock space*. Now Fock space is just another separable infinite dimensional Hilbert space (and so isomorphic to all its separable infinite dimensional brothers). But the key is writing it down in a fashion that *suggests* a particle interpretation. In particular, suppose that H is the one-particle Hilbert space, i.e. the state space for a single particle. Now depending on whether our particle is a Boson or a Fermion, the state space of a pair of these particles is either $E_s(H \otimes H)$ or $E_a(H \otimes H)$, where E_s is the projection onto the vectors invariant under the permutation $\Sigma_{H,H}$ on $H \otimes H$, and E_a is the projection onto vectors that change signs under $\Sigma_{H,H}$. For present purposes, we ignore these differences, and simply use $H \otimes H$ to denote one possibility or the other. Now, proceeding down the line, for n particles, we have the Hilbert space $H^n \equiv H \otimes \cdots \otimes H$, etc..

A state in H^n is definitely a state of n particles. To get disjunctive states, we make use of the direct sum operation "$\oplus$" on Hilbert spaces. So we define the Fock space $\mathcal{F}(H)$ over H as the infinite direct sum:

$$\mathcal{F}(H) = \mathbb{C} \oplus H \oplus (H \otimes H) \oplus (H \otimes H \otimes H) \oplus \cdots .$$

So, the state vectors in Fock space include a state where the are no particles (the vector lies in the first summand), a state where there is one particle, a state where there are two particles, etc.. Furthermore, there are states that are superpositions of different numbers of particles.

One can spend time worrying about what it means to say that particle numbers can be superposed. But that is the "half empty cup" point of view. From the "half full cup" point of view, it makes sense to count particles. Indeed, the positive (unbounded) operator

$$N = 0 \oplus 1 \oplus 2 \oplus 3 \oplus 4 \oplus \cdots ,$$

is the formal element of our model that permits us to talk about the number of particles.

NOTE 108. In the category of Hilbert spaces, all separable Hilbert spaces are isomorphic — there is no difference between Fock space and the single particle space. If we are not careful, we could become confused about the bearer of the name "Fock space."

The confusion goes away when we move to the appropriate category. According to Wigner's analysis [Wigner, 1939], a particle corresponds to an irreducible unitary representation of the identity component $\mathcal{P}$ of the Poincaré group. Then the single particle space and Fock space are distinct objects in the category of representations of $\mathcal{P}$. The underlying Hilbert spaces of the two representations are both separable (and hence isomorphic as Hilbert spaces); but the two representations are most certainly not equivalent (one is irreducible, the other reducible).

4.2 Fock space from the algebra of observables

The Fock space story is not completely abandoned within the algebraic approach to QFT. In fact, when conditions are good, Fock space emerges as the GNS Hilbert space for some privileged vacuum state of the algebra of observables. We briefly describe how this emergence occurs before proceeding to raise some problems for the naive Fock space story. (We look here only at the symmetric — Bosonic — case. A similar treatment applies to the antisymmetric — Fermionic — case.)

The algebraic reconstruction of Fock space arises from the algebraic version of canonical quantization. Suppose that S is a real vector space (equipped with some suitable topology), and that σ is a symplectic form on S. So, S represents a classical phase space (see Butterfield, this volume). The *Weyl algebra* $\mathfrak{A}[S,\sigma]$ is a specific C^*-algebra generated by elements of the form $W(f)$, with $f \in S$ and satisfying the canonical commutation relations in the Weyl-Segal form:

$$W(f)W(g) = e^{-i\sigma(f,g)/2}W(f+g).$$

Suppose that there is also some notion of spacetime localization for elements of S, i.e. a mapping $O \mapsto S(O)$ from double cones in Minkowski spacetime to subspaces of S. Then, if certain constraints are satisfied, the pair of mappings

$$O \mapsto S(O) \mapsto \mathfrak{A}(O) \equiv C^*\{W(f) : f \in S(O)\},$$

can be composed to give a net of C^*-algebras over Minkowski spacetime. (Here C^*X is the C^*-algebra generated by the set X.)

Now if we are given some dynamics on S, then we can — again, if certain criteria are satisfied — define a corresponding dynamical automorphism group α_t on $\mathfrak{A}[S,\sigma]$. There is then a unique dynamically stable pure state ω_0 of $\mathfrak{A}[S,\sigma]$, and we consider the GNS representation $(\mathcal{H},\pi)$ of $\mathfrak{A}[S,\sigma]$ induced by ω_0. To our delight, we find that the infinitesimal generators $\Phi(f)$ of the one-parameter groups $\{\pi(W(f))\}_{t\in\mathbb{R}}$ behave just like the field operators in the old-fashioned Fock space approach. Furthermore (now speaking non-rigorously), if we define operators

$$\begin{aligned} a(f) &= 2^{-1/2}\left(\Phi(f) + i\Phi(Jf)\right), \\ a^*(f) &= 2^{-1/2}\left(\Phi(f) - i\Phi(Jf)\right), \end{aligned}$$

we find that they behave like creation and annihilation operators of particles. (Here J is the unique "complex structure" on S that is compatible with the dynamics.)

In particular, by applying them to the vacuum state Ω, we get the entire GNS Hilbert space $\mathcal{H}$. Finally, if we take an orthonormal basis $\{f_i\}$ of S, then the sum

$$\sum_{i=1}^{\infty} a^*(f_i)a(f_i),$$

is the number operator N. Thus, the traditional Fock space formalism emerges as one special case of the GNS representation of a state of the Weyl algebra.

NOTE 109. The Minkowski vacuum representation $(\mathcal{H}_0, \pi_0)$ of $\mathfrak{A}$ is Poincaré covariant, i.e. the action $\alpha_{(a,\Lambda)}$ of the Poincaré group by automorphisms on $\mathfrak{A}$ is implemented by unitary operators $U(a, \Lambda)$ on $\mathcal{H}$. When we say that $\mathcal{H}$ is isomorphic to Fock space $\mathcal{F}(H)$, we do not mean the trivial fact that $\mathcal{H}$ and $\mathcal{F}(H)$ have the same dimension. Rather, we mean that the unitary representation $(\mathcal{H}, U)$ of the Poincaré group is a Fock representation.

Notes: See [Bratteli and Robinson, 1997, Section 5.2] for a detailed account of the reconstruction of Fock space from the Weyl algebra. See also [Clifton and Halvorson, 2001a] and [Halvorson, 2001] for shorter expositions.

4.3 Nonuniqueness of particle interpretations

If we have a representation $(\mathcal{H}, \pi)$ of the quasilocal algebra $\mathfrak{A}$ such that $\mathcal{H}$ is isomorphic to Fock space, then we can make sense of talk about particles. Furthermore, such representations exist, e.g., the GNS representation of the Minkowski vacuum state ω_0 of the free Bose field. So, in the most simple cases (e.g. free fields on flat spacetime), there is no problem concerning the existence of particle interpretations of the theory.

But there is a problem about *uniqueness*: there are unitarily inequivalent representations of $\mathfrak{A}$, each of which is isomorphic to Fock space. Furthermore, a result from [Chaiken, 1967; Chaiken, 1968] shows that two inequivalent Fock representations correspond to two number operators that cannot be thought of as notational variants of the same description of reality. Indeed, there are no states of $\mathfrak{A}$ that assign sharp values to both number operators. Hence, the particle interpretations provided by the two Fock representations are mutually exclusive.

The issue of inequivalent Fock representations is treated in depth in [Clifton and Halvorson, 2001a]. For present purposes, we simply note that this worry about nonuniqueness is tied in to a more general worry about inequivalent representations of the quasilocal C^*-algebra $\mathfrak{A}$. But this more general issue cannot be resolved without reference to recent developments in the theory of superselection sectors (see Sections 7.2 and following). We return to this question in Section 7.

4.4 Problems for localized particles

Suppose that we have settled the uniqueness problem that is raised in the previous subsection — e.g. we have found a good reason for preferring a particular Fock

representation $(\mathcal{H}, \pi)$ of $\mathfrak{A}$, and so we have a preferred global number operator N on $\mathcal{H}$. The next question is whether relativistic QFT is consistent with an ontology of *localized* particles — that is, whether it makes sense to talk about the number of particles in a bounded region O of space.

As pointed out in Section 2.3, the Reeh-Schlieder (RS) Theorem entails that the local algebras of AQFT do not contain operators that annihilate the vacuum. Hence if a number operator has the vacuum as an eigenstate, then there are no local number operators. That is perhaps enough to convince most readers that localized particles are not possible in relativistic QFT. Nonetheless, there have been attempts to bypass the RS Theorem, most notably the proposal of Newton and Wigner (recently resurrected in [Fleming, 2000]). It has been argued that such attempts are not promising [Halvorson, 2001]. Furthermore, it can be shown independently of the full framework of AQFT, and without the RS Theorem, that a positive energy condition combined with microcausality rules out local number operators [Halvorson and Clifton, 2002].

Despite the various No Go results for localized particles in relativistic QFT, the interpretation of experiments in high energy physics seems to require a notion of something causing clicks in detectors, and that a "detector" is fairly well localized in some bounded region of spacetime. A detector corresponds to a positive operator C in $\mathfrak{A}$, and is "completely reliable" only if it registers 0 identically in the vacuum state, i.e. $C\Omega = 0$. Hence the Reeh-Schlieder Theorem entails that C is not contained in any local algebra. Nonetheless, a notion of approximate localization of C can be salvaged: choose some $A \in \mathfrak{A}(O)$ with $0 \leq A \leq I$, and set

$$C = \int f(x)\alpha_x(A)dx,$$

where f is a smooth function whose Fourier transform has support in the complement of the forward light cone. (The function f automatically has unbounded support.) Then $C\Omega = 0$, and the function f can also be chosen so that C is "close" in the norm topology to an operator in $\mathfrak{A}(O)$.

The notion of approximately localized detectors is employed extensively in Haag-Ruelle scattering theory and recent developments thereof, to which we now turn.

4.5 Particle interpretations generalized: Scattering theory and beyond

It is not true that a representation $(\mathcal{K}, \pi)$ of $\mathfrak{A}$ must be a Fock representation in order for states in the Hilbert space $\mathcal{K}$ to have an interpretation as particle states. Indeed, one of the central tasks of "scattering theory," is to provide criteria — in the absence of full Fock space structure — for defining particle states. These criteria are needed in order to describe scattering experiments which cannot be described in a Fock representation, but which need particle states to describe the input and output states.

Haag and Swieca [1965] propose to pick out the n-particle states by means of localized detectors; we call this the *detector criterion*:

> A state with at least n-particles is a state that would trigger n detectors that are far separated in space.

Philosophers might worry that the detector criterion is too operationalist. Indeed, some might claim that detectors themselves are made out of particles, and so *defining* a particle in terms of a detector would be viciously circular.

If we were trying to give an analysis of the concept of a particle, then we would need to address such worries. However, scattering theory does not end with the detector criterion. Indeed, the goal is to tie the detector criterion back to some other more intrinsic definition of particle states. The traditional intrinsic definition of particle states is in terms of Wigner's symmetry criterion:

> A state of n particles (of spins s_i and masses m_i) is a state in the tensor product of the corresponding representations of the Poincaré group.

Thus, scattering theory — as originally conceived — needs to show that the states satisfying the detector criterion correspond to an appropriate representation of the Poincaré group. In particular, the goal is to show that there are isometries $\Omega^{\text{in}}, \Omega^{\text{out}}$ that embed Fock space $\mathcal{F}(H)$ into $\mathcal{K}$, and that intertwine the given representations of the Poincaré group on $\mathcal{F}(H)$ and $\mathcal{K}$.

Based on these ideas, detailed models have been worked out for the case where there is a mass gap. Unfortunately, as of yet, there is no model in which $\mathcal{H}^{\text{in}} = \mathcal{H}^{\text{out}}$, which is a necessary condition for the theory to have an S-matrix, and to define transition probabilities between incoming and outgoing states. (Here $\mathcal{H}^{\text{in}}$ is the image of Fock space in $\mathcal{K}$ under the isometry Ω^{in}, and similarly for $\mathcal{H}^{\text{out}}$.)

Recently, Buchholz and collaborators have claimed that Wigner's symmetry criterion is too stringent — i.e. there is a more general definition of particle states. They claim that it is only by means of this more general criterion that we can solve the "infraparticles" problem, where massive particles carry a cloud of photons (see [Buchholz *et al.*, 1991]).

Note: For a review of progress in scattering theory in AQFT, see [Haag, 1996, Chapter 6] and [Buchholz and Summers, ND].

5 THE PROBLEM OF VALUE-DEFINITENESS IN AQFT

The "measurement problem" of nonrelativistic QM shows that the standard approach to the theory is impaled on the horns of a dilemma: either (i) one must make ad hoc adjustments to the dynamics ("collapse") when needed to explain the results of measurements, or (ii) measurements do not, contrary to appearances, have outcomes (see Dickson, this volume, Section 5).

There are two main responses to the dilemma: On the one hand, some suggest that we abandon the unitary dynamics of QM in favor of stochastic dynamics that

accurately predicts our experience of measurement outcomes. On the other hand, some suggest that we maintain the unitary dynamics of the quantum state, but that certain quantities (e.g. position of particles) have values even though these values are not specified by the quantum state. (See Dickson, this volume, Section 5.5 for a more nuanced discussion of the possible responses.)

Both approaches — the approach that alters the dynamics, and the approach with additional values — are completely successful as responses to the measurement problem in nonrelativistic QM. But both approaches run into obstacles when it comes to synthesizing quantum mechanics with relativity. In particular, the additional values approach (e.g. the de Broglie–Bohm pilot-wave theory) appears to require a preferred frame of reference to define the dynamics of the additional values (see [Cushing, 1994, pp. 188–191, 196–198], [Holland, 1995], and [Bohm and Hiley, 1995, Chaps. 11 & 12]), and in this case it would fail the test of Lorentz invariance.

The "modal" interpretation of quantum mechanics is similar in spirit to the de Broglie–Bohm theory, but begins from a more abstract perspective on the question of assigning definite values to some observables. (Following [Bell, 1987], we might call these the "beables" of the theory.) Rather than making an intuitively physically motivated choice of the determinate values (e.g. particle positions), the modal interpretation makes the mathematically motivated choice of the spectral decomposition of the quantum state (i.e. the density operator) as determinate. (See [Dieks and Vermaas, 1998; Vermaas, 1999] for reviews of the modal interpretation; see [Clifton, 1995] for motivation.)

Unlike the de Broglie–Bohm theory, it is not obvious that the modal interpretation must violate the spirit or letter of relativistic constraints, e.g. Lorentz invariance [Dickson and Clifton, 1998, p. 9]. So, it seems that there should be some hope of developing a modal interpretation within the framework of AQFT. This is the starting point for Dieks' [2000] proposal for a modal interpretation of AQFT. Rather than expound Dieks' original proposal, we move directly to the criticism in [Clifton, 2000], to which we also refer the reader for further elaboration.

5.1 Clifton-Kitajima classification of modal algebras

Clifton's critique of the modal interpretation of AQFT is based on a remarkable theorem which classifies all possible "modal subalgebras" of a local von Neumann algebra $\mathfrak{R}(O)$ relative to a state ρ. According to Clifton — and the modal interpreters seem to agree on this point — the algebra $\mathfrak{D}$, $\mathfrak{D} \subseteq \mathfrak{R}(O)$ of definite local observables should satisfy the following constraints relative to a given state ρ of $\mathfrak{R}(O)$:

DEFINITION 110. Let $\mathfrak{R}$ be a von Neumann algebra, and let ρ be a state of $\mathfrak{R}$. Then a von Neumann subalgebra $\mathfrak{D}$ of $\mathfrak{R}$ is said to be a *modal algebra* for $(\mathfrak{R}, \rho)$ just in case:

1. (*Value definiteness*) The restricted state $\rho|_{\mathfrak{D}}$ is a mixture of dispersion-free

states. (Definition: A state is *dispersion free* iff it assigns each projection operator either 0 or 1.)

2. (*Definability*) $\mathfrak{D}$ is left invariant under all symmetries of $\mathfrak{R}$ that leave the state ρ invariant.

3. (*Maximality*) $\mathfrak{D}$ is maximal, subject to the first two conditions.

The last requirement is imposed simply to rule out trivial counterexamples to uniqueness — e.g. one could always pick the algebra $\mathbb{C}I$ of scalar multiples of the identity. The second requirement is supposed to explicate the idea that $\mathfrak{D}$ is "picked out by" (i.e. is definable in terms of) the state ρ. We have left the notion of a "symmetry" vague (and we will return to this question in the next subsection), but Clifton takes the symmetries to coincide with the $*$-automorphisms of $\mathfrak{R}$, and this is needed for the main result (Theorem 114).

To state this result, we need to define the notion of the centralizer of a state. The following proposition establishes the equivalence of two possible definitions of the centralizer.

PROPOSITION 111. *Let $\mathfrak{R}$ be a von Neumann algebra, let ω be a faithful normal state of $\mathfrak{R}$, and let σ_t^ω be the modular automorphism group of $\mathfrak{R}$. Then the following two sets are coextensive:*

1. $\{A \in \mathfrak{R} : \sigma_t^\omega(A) = A, \forall t \in \mathbb{R}\}$

2. $\{A \in \mathfrak{R} : \omega(AB) = \omega(BA), \forall B \in \mathfrak{R}\}$

The proof of Prop. 111 depends on the full apparatus of modular theory. We refer the reader to [Takesaki, 2003, Chap. 8] for details.

DEFINITION 112. It is clear that the set defined in the previous proposition is in fact a von Neumann subalgebra of $\mathfrak{R}$. We call this subalgebra the *centralizer* of ω in $\mathfrak{R}$, and we denote it by $\mathfrak{R}_\omega$.

EXAMPLE 113. Let $\mathfrak{R} = \mathfrak{B}(\mathcal{H})$, and let ω be a faithful normal state of $\mathfrak{R}$. Then ω has the form

$$\omega(A) = \mathrm{Tr}(DA), \qquad A \in \mathfrak{R},$$

for some density operator $D \in \mathfrak{R}$. Then $\mathfrak{R}_\omega = \{D\}'$, and $Z(\mathfrak{R}_\omega)$ is the abelian von Neumann algebra $\{D\}''$. In particular, if ω is the maximally mixed state of a type I_n factor, then $\mathfrak{R}_\omega = \mathfrak{B}(\mathcal{H})$, and $Z(\mathfrak{R}_\omega) = \mathbb{C}I$.

The Clifton-Kitajima Theorem shows that there is a unique modal algebra for $(\mathfrak{R}, \omega)$, and in the case that the state ω is faithful, it is $Z(\mathfrak{R}_\omega)$, the center of the centralizer of ω.

THEOREM 114 (Clifton-Kitajima). *Let $\mathfrak{R}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, and let ω be a normal state of $\mathfrak{R}$.*

1. *If ω is faithful then $Z(\mathfrak{R}_\omega)$ is the unique modal algebra for $(\mathfrak{R}, \omega)$.*

2. *Generally, the unique modal algebra for* $(\mathfrak{R}, \omega)$ *is* $\mathfrak{N} \oplus Z(\mathfrak{R}_\omega)E$*, where* E *is the smallest projection in* $\mathfrak{R}$ *such that* $\omega(E) = 1$*, and* $\mathfrak{N}$ *is the algebra of all bounded operators on* $(I - E)(\mathcal{H})$*.*

The result is proven for the case where ω is faithful in [Clifton, 2000], and for the general case in [Kitajima, 2004].

As pointed out by Clifton [2000], Thm. 114 spells trouble for a modal interpretation of AQFT, because there are many cases where the algebra $Z(\mathfrak{R}_\omega)$ is trivial. (See [Ruetsche and Earman, 2005] for further development of this point.)

1. Let W be a wedge region in Minkowski spacetime, and let Ω be the vacuum state. Then there are no fixed points in $\mathfrak{R}(W)$ of the modular automorphism group σ_t^ω (see the proof of Proposition 77, and also [Driessler, 1975]). Hence, $\mathfrak{R}_\omega = \mathbb{C}I$, and $Z(\mathfrak{R}_\omega) = \mathbb{C}I$.

2. In relativistic QFT, local algebras are the type III_1 hyperfinite factor $\mathfrak{R}$ (see Section 2.5). But $\mathfrak{R}$ has a dense set of ergodic states — states with trivial centralizer. For all these states, $Z(\mathfrak{R}_\omega) = \mathbb{C}I$.

Thus, it makes an enormous difference — at least for the feasibility of the modal interpretation — that local algebras are type III_1. For if local algebras were either type I_∞ or III_0, then there would be good news for the modal interpretation.

PROPOSITION 115. *Let* $\mathfrak{R}$ *be a type* I_∞ *factor. Then for every normal state* ω *of* $\mathfrak{R}$*, the unique modal algebra* $\mathfrak{D}_\omega$ *is nontrivial.*

Proof. We have $\mathcal{D}_\omega = Z(\mathfrak{R}_\omega) = \{D\}''$, where D is the density operator, i.e. the positive operator in $\mathfrak{R}$ that implements the state ω via the trace formula. Furthermore, when $\mathfrak{R}$ is type I_∞, D cannot be a multiple of the identity. ■

PROPOSITION 116. *Let* $\mathfrak{R}$ *be a type* III_0 *factor. Then for every faithful normal state* ω *of* $\mathfrak{R}$*, the unique modal algebra* $\mathfrak{D}_\omega$ *is nontrivial.*

Proof. Prop. 3.15 in [Takesaki, 2003, p. 402] entails that $\mathfrak{D}_\omega$ has no atomic projections, and hence is infinite dimensional. ■

5.2 What is a symmetry in AQFT?

We note here just one problem with application of the Clifton-Kitajima theorem to AQFT: the notion of symmetry invoked might be too liberal for the setting where we have a *net of algebras over spacetime*, as opposed to a single von Neumann algebra. Clifton's application of the theorem assumes that any automorphism of $\mathfrak{R}$ is a symmetry. However, if $\mathfrak{R} = \mathfrak{R}(O)$ is just one algebra of an entire net $O \mapsto \mathfrak{R}(O)$, then it is not clear that every automorphism of $\mathfrak{R}$ is a symmetry of the relevant system. What we need is a notion of a symmetry of the net $O \mapsto \mathfrak{R}(O)$.

NOTE 117. A partially ordered set $\mathcal{K}$ can be regarded as a category where for $x, y \in \mathcal{K}$, $\mathrm{Hom}(x, y) = \{(x, y)\}$ if $x \leq y$, and otherwise $\mathrm{Hom}(x, y) = \emptyset$. Let $\mathbf{C}^*$

be the category with C^*-algebras as objects and $*$-homomorphisms as arrows. On this conception, a net of C^*-algebras on Minkowski spacetime is a functor $\mathfrak{A} : \mathcal{K} \to \mathbf{C}^*$ where $\mathcal{K}$ is the category of double cones in Minkowski spacetime, ordered by inclusion, and such that $\mathfrak{A}(\mathrm{Hom}(O_1, O_2))$ is an isometry when $\mathrm{Hom}(O_1, O_2)$ is not empty. (For definitions of functors and natural transformations, see p. 866.)

DEFINITION 118. Let $\mathcal{K}$ be a partially ordered set (e.g. regions in some manifold ordered by inclusion). Let $O \mapsto \mathfrak{A}(O)$ and $O \mapsto \mathfrak{B}(O)$ be nets of C^*-algebras over $\mathcal{K}$. A *net morphism* $\alpha : \mathfrak{A} \to \mathfrak{B}$ is a natural transformation between the functors. That is, α consists of a collection of morphisms

$$\{\alpha_O : \mathfrak{A}(O) \to \mathfrak{B}(O) : O \in \mathcal{K}\},$$

that is natural in O. In other words, for each $f \in \mathrm{Hom}(O_1, O_2)$, $\alpha_{O_2} \circ \mathfrak{A}(f) = \mathfrak{B}(f) \circ \alpha_{O_1}$, which just means that the following diagram commutes

$$\begin{array}{ccc} \mathfrak{A}(O_1) & \xrightarrow{\alpha_{O_1}} & \mathfrak{B}(O_1) \\ \Big\downarrow{\scriptstyle \mathfrak{A}(f)} & & \Big\downarrow{\scriptstyle \mathfrak{B}(f)} \\ \mathfrak{A}(O_2) & \xrightarrow[\alpha_{O_2}]{} & \mathfrak{B}(O_2) \end{array}$$

FACT 119. Net automorphisms correspond to automorphisms of the quasilocal algebra that leave each local subalgebra globally invariant. To state this precisely, let $\mathfrak{A}$ denote the functor from $\mathcal{K}$ into $\mathbf{C}^*$, and let $\mathfrak{B}$ denote the inductive limit of $\mathfrak{A}$. We identify $\mathfrak{A}(O)$ with its image in $\mathfrak{B}$. Then α is a net automorphism of $\mathfrak{A}$ iff there is an automorphism β of $\mathfrak{B}$ such that

$$\beta|_{\mathfrak{A}(O)} = \alpha_O.$$

Now, given a net $\mathfrak{A}$ with inductive limit $\mathfrak{B}$, what should we consider as a symmetry of $\mathfrak{B}$?

PROPOSAL 120. A symmetry of the net $\mathfrak{A}$ corresponds to a net automorphism α; i.e. a natural transformation of $\mathfrak{A}$. That is, a symmetry of $\mathfrak{A}$ corresponds to an automorphism of the quasilocal algebra that leaves each local subalgebra globally invariant.

This first proposal is surely too strict, because it excludes the case of symmetries induced by underlying symmetries of the spacetime. But if $\mathcal{K}$ consists of an appropriate set of regions of spacetime M (i.e. a set that is closed under symmetries of the spacetime), then a symmetry of M will induce an order-preserving bijection F on $\mathcal{K}$. Note that since F is a functor, $\mathfrak{A} \circ F$ is also a functor. Thus, we consider the following liberalized definition.

PROPOSAL 121. A symmetry of the net $\mathfrak{A}$ consists of a pair (F, α) where F is an order-preserving bijection of $\mathcal{K}$, and α is a net morphism (natural transformation) from $\mathfrak{A}$ to $\mathfrak{A} \circ F$.

If we accept this proposal, then we must replace Clifton's definability condition with the following modified condition:

> *Definability-2:* Given $O \in \mathcal{K}$, let $\mathcal{K}_0$ be the full subcategory of $\mathcal{K}$ with objects $\{O_0 : O_0 \leq O\}$, and let $\mathfrak{R}_O$ denote the restriction of the von Neumann algebra valued functor $\mathfrak{R}$ to $\mathcal{K}_0$. Then the algebra $\mathfrak{D}$ must be left invariant by all symmetries of $\mathfrak{R}_O$ that preserve the state ρ on $\mathfrak{R}(O)$.

Since not all automorphisms of $\mathfrak{R}(O)$ are symmetries of the net $\mathfrak{R}_O$, the new definability condition is weaker than the old one: there will typically be more candidates for the role of $\mathfrak{D}$.

The Clifton-Kitajima Theorem does not apply under the revised definition of symmetries of $\mathfrak{R}(O)$. On the other hand, we are not aware of a positive result showing the existence and uniqueness of subalgebras of $\mathfrak{R}(O)$ that are definite in the state ω and invariant under all net automorphisms that preserve ω. There are suggestive hints such as the result in [Doplicher and Longo, 1984]:

PROPOSITION 122. *Let* $(\mathfrak{R}_1 \subseteq \mathfrak{R}_2, \omega)$ *be a standard split inclusion of von Neumann algebras. Then there is a* unique *type I factor* $\mathfrak{N}$ *such that: (i)* $\mathfrak{R}_1 \subseteq \mathfrak{N} \subseteq \mathfrak{R}_2$*, and (ii)* $\mathfrak{N}$ *is invariant under all automorphisms of* $\mathfrak{R}_2$ *that preserve both* $\mathfrak{R}_1$ *and the state* ω*.*

Of course, the algebra $\mathfrak{N}$ itself does not have dispersion-free states, and so cannot be the algebra of definite observables. However, the state $\omega|_{\mathfrak{N}}$ is normal, and since $\mathfrak{N}$ is a type I factor, there is a density operator $D \in \mathfrak{N}$ that induces the state in the sense that $\omega(A) = \mathrm{Tr}(DA)$, for all $A \in \mathfrak{N}$. Then assuming that $\mathfrak{R}_1$ must for some reason be left invariant under symmetries of $\mathfrak{R}_2$, the algebra $\mathfrak{D} = \{D\}''$ looks like a good candidate for the modal interpreter's set of definite-value observables in $\mathfrak{R}_2$ in the state ω.

To apply Prop. 122 to AQFT with $\mathfrak{R}_i = \mathfrak{R}(O_i)$, and $O_1 \subseteq O_2$, we would have to assume that the split property holds. Although the split property does not hold in every model, failure of the split property implies a sort of pathology, and it might not be too surprising if there were certain physically pathological cases where the modal interpretation yields a trivial set of definite quantities.

Notes: For recent discussions of adapting the modal interpretation to a relativistic setting, see [Myrvold, 2002; Ruetsche and Earman, 2005].

6 QUANTUM FIELDS AND SPACETIME POINTS

In standard/heuristic presentations of QFT, the fundamental physical quantities (observables, or more generally quantum fields) are operators indexed by space-

time points: $\Phi(x)$ (see t'Hooft, this volume). Based on this fact, at least one philosopher ([Teller, 1995]) describes the ontology of QFT in terms of the idea a *field of operators and their expectation values*. On the other hand, the mathematical approach to QFT (e.g. the Wightman approach) eschews the use of operators at points in favor of operators smeared over space(time) by test-functions: $\Phi(f)$. According to Arnteznius [2003], this fact supports the view that spacetime has no pointlike events, and *a fortiori* that there are no field values at spacetime points.

As QFT became more mathematically rigorous, an intuition developed that it is not only difficult to define the value of a field at a point, but that it is impossible to do so — such quantities simply do not exist. (Compare von Neumann's critique of Dirac's delta functions and the notion of pointlike localized particles.) This intuition has sometimes been buttressed by heuristic and operationalist arguments — e.g. Bohr and Petersen's [1950] argument that it is impossible to measure field strengths at a point. For example, Haag [1996, p. 58] claims that, "a quantum field $\Phi(x)$ at a point cannot be a proper observable." Even philosophers can be found claiming that, "field operators need to be 'smeared' in space" [Huggett, 2000, p. 631, fn. 8].

But the arguments against field operators at a point often confuse questions of measurability with questions of existence, and rarely rise to a level of rigor that is acceptable for drawing metaphysical conclusions. In this section, we review some of the rigorous arguments that exist for and against field quantities at points. We will see that these results do not decisively rule out field quantities at points, but they clarify the interpretive tradeoffs that must be made.

6.1 No Go theorems

In the following three subsections, we review No Go theorems for field operators at spacetime points.

Translation covariance rules out operators at a point

The first no go theorem shows that if there is a continuous unitary representation of the translation group, then for any fixed time t, the field configuration operators $\phi(x,t)$ commute with the field momentum operators $\pi(x',t)$, *even when these operators are associated with the same point*. This result is a serious problem, because $\phi(x,t)$ and $\pi(x,t)$ are supposed to be canonically conjugate (see [Ryder, 1996, p. 131], [Huggett, 1999]):

$$(2) \quad [\phi(x,t), \pi(x',t)] = i\delta(x - x').$$

Moreover, this bad outcome cannot be blamed on any sort of "conflict" between quantum mechanics and relativity, because the bad outcome also holds for non-relativistic theories.

THEOREM 123. *Let $\phi(x,t)$ and $\pi(y,t)$ be fields of operators, either bounded or unbounded and self-adjoint, such that*

$$[\phi(x,t), \pi(y,t)] = 0,$$

when $x \neq y$. (In the unbounded case, we mean that $\phi(x,t)$ and $\pi(y,t)$ are defined on a common dense set $\mathcal{D}$, and they commute on this set.) If $y \mapsto U(y)$ is a continuous representation of the translation group such that $U(y)\pi(x,t)U(y)^* = \pi(x+y,t)$, for all $x, y \in \mathbb{R}^3$, then

$$[\phi(x,t), \pi(x,t)] = 0,$$

for all $x \in \mathbb{R}^3$.

Proof. Since this proof only uses field operators on the same time slice, we will suppress reference to t. Suppose first that $\phi(x)$ and $\pi(y)$ are bounded operators. In this case, the mapping:

$$f(y) := [\phi(x), \pi(x+y)] = [\phi(x), U(y)\pi(x)U(y)^*], \tag{3}$$

is a weak-operator continuous function from $\mathbb{R}^3$ into the bounded operators on $\mathcal{H}$. Choose a sequence $(y_n)_{n\in\mathbb{N}}$ of nonzero vectors that converges to 0. Since f is continuous, and $f(y_n) = 0$ for all $n \in \mathbb{N}$,

$$[\phi(x), \pi(x)] = f(0) = \lim_{n\to\infty} f(y_n) = 0. \tag{4}$$

Now suppose that $\phi(x)$ and $\pi(y)$ are unbounded but self-adjoint. Then replace $\pi(x)$ with one of its spectral projections $E_S(x)$, where S is a Borel subset of $\mathbb{R}$, and replace $\pi(x)$ with one of its spectral projections $F_{S'}(y)$, where S' is a Borel subset of $\mathbb{R}$. By the preceding argument, $E_S(x)$ and $F_{S'}(y)$ commute. Since this is true for all such pairs of spectral projections, it follows that the spectral projections of $\phi(x)$ commute pairwise with the spectral projections of $\pi(x)$. Hence $\phi(x)$ and $\pi(x)$ are defined on a common dense set $\mathcal{D}$ in $\mathcal{H}$, and they commute on this dense set. ■

Poincaré covariance rules out operators at a point

For our next two no go theorems, we will need to gather a couple of classic results.

DEFINITION 124. A function $f : \mathbb{R}^n \to \mathbb{C}$ is said to be *of positive type* just in case for each $c_1, \ldots, c_n \in \mathbb{C}$, and each $x_1, \ldots, x_n \in \mathbb{R}^n$, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \overline{c}_j c_i f(x_i - x_j) \geq 0.$$

THEOREM 125 (Bochner). *Let $f : \mathbb{R}^n \to \mathbb{C}$ be a continuous function of positive type. Then the Fourier transform of f is a bounded measure on $\mathbb{R}^n$.*

Proof. For a proof of Bochner's theorem, see [Rudin, 1991, p. 303] and [Folland, 1995, p. 95]. ■

NOTE 126. Only the group structure of $\mathbb{R}^n$ is really needed for Bochner's theorem. So, we are not making any mistake by thinking of Minkowski spacetime as $\mathbb{R}^4$ in this context.

We will need the following key lemma in all of our subsequent results.

LEMMA 127. *Let f be a continuous positive definite function on $\mathbb{R}^n$. Then f is the constant 1 function iff the Fourier transform of f is the probability measure with support $\{0\}$.*

The proof of the above lemma is trivial: the Fourier transform of the measure μ with support $\{0\}$ is the function f defined by

$$f(x) = \int_{\mathbb{R}^n} e^{i(x\cdot p)} d\mu(p) = e^{i(x\cdot 0)} = 1.$$

But the Fourier transformation is a bijection between complex Radon measures on $\mathbb{R}^n$ and bounded continuous functions on $\mathbb{R}^n$.

DEFINITION 128. We say that a measure μ on Minkowski spacetime is *Lorentz invariant* just in case $\mu(\Lambda(S)) = \mu(S)$ for each Borel subset S of M, and each homogeneous Lorentz transformation Λ, where $\Lambda(S) = \{\Lambda(x) : x \in S\}$.

Clearly, the only Lorentz invariant probability measure on Minkowski spacetime is the measure supported on $\{0\}$ (the unique fixed point of the homogeneous Lorentz group). The following result is the "Fourier transformed" version of that fact.

LEMMA 129. *Let M be Minkowski spacetime. If $f : M \to \mathbb{C}$ is a continuous function of positive type such that $f(\Lambda x) = f(x)$ for each Lorentz transformation Λ, then f is constant.*

Sketch of proof. By Bochner's theorem, if $f : M \to \mathbb{C}$ is a continuous function of positive type, then f is the Fourier transform of a bounded measure μ on M. It is straightforward to verify that if f is Lorentz invariant then so is μ. But a bounded, Lorentz invariant measure is supported on $\{0\}$. By Lemma 127, the Fourier transform of μ is a constant function. Therefore, $f = 1$ is constant. ■

FACT 130. Let U be a unitary representation of the translation group on a Hilbert space $\mathcal{H}$. Then the following are equivalent:

1. The spectrum of the representation U is Δ;
2. For every $u, v \in \mathcal{H}$, the function $f : \mathbb{R}^n \to \mathbb{C}$ given by

$$f(x) = \langle u, U(x)v\rangle, \qquad x \in \mathbb{R}^n,$$

has Fourier transform with support in Δ.

Finally, the following is our core lemma for the next two results.

LEMMA 131. *Let $A : M \to \mathfrak{B}(\mathcal{H})$ be an operator valued function, and let U be a unitary representation of the translation group on $\mathcal{H}$ such that $U(x)A(0)U(x)^* = A(-x)$ for all $x \in M$. Define a function $f : M \to \mathbb{C}$ by*

$$f(x) = \langle \Omega, A(x)A(0)\Omega \rangle = \langle \Omega, U(x)^* A(0)U(x)A(0)\Omega \rangle.$$

If f is constant, then there is a $c \in \mathbb{C}$ such that $A(x)\Omega = c\,\Omega$ for each $x \in M$.

Proof. Let $\psi = A(0)\Omega$. Then $f(x) = f(0)$ is expressed as

$$\langle \psi, U(x)\psi \rangle = \langle \psi, \psi \rangle = \|\psi\|^2.$$

But we also have $\|\psi\| = \|U(x)\psi\|$ since $U(x)$ is unitary. Hence

$$\langle \psi, U(x)\psi \rangle = \|\psi\| \cdot \|U(x)\psi\|,$$

and the Cauchy-Schwartz inequality entails that $U(x)\psi = \psi$ for all x. That is, $U(x)A(0)\Omega = A(0)\Omega$. Note in addition that $U(x)A(y)\Omega = U(x+y)A(0)\Omega = A(0)\Omega$. Hence all vectors $A(x)\Omega$ are invariant under the translation group. ■

Now, the second no go theorem (due to [Wizimirski, 1966]) shows that there is no nontrivial Poincaré covariant field of bounded operators on Minkowski spacetime.

THEOREM 132. *Suppose that $A : M \to \mathfrak{B}(\mathcal{H})$ is an operator-valued function, and U is a continuous unitary representation of the Poincaré group on $\mathcal{H}$ such that:*

1. *$U(y,\Lambda)A(x)U(y,\Lambda)^* = A((\Lambda x) - y)$, for all $(y,\Lambda) \in \mathcal{P}$ and $x \in M$;*
2. *There is a unique (up to scalar multiples) translation-invariant vector $\Omega \in \mathcal{H}$.*

Then there is a $c \in \mathbb{C}$ such that $A(x)\Omega = c\,\Omega$ for all $x \in M$.

NOTE 133. (i): The assumption of the uniqueness of Ω might seem unwarranted. But under some fairly standard conditions, this assumption can be derived. See Section 2.2. (ii): This theorem makes no assumption about commutation relations between operators $A(x)$ and $A(y)$.

Proof [of Theorem 132.] Define a function $f : M \to \mathbb{C}$ by

$$f(x) = \big\langle \Omega, A(x)^* A(0)\Omega \big\rangle, \qquad x \in M.$$

By condition 2 we have $U(x)\Omega = \Omega$. Hence by condition 1 we have $A(x)^* = U(x)A(0)^*U(x)^*$, and hence

$$f(x) = \big\langle A(0)\Omega, U(x)^* A(0)\Omega \big\rangle,$$

which is obviously positive definite. Furthermore, since $x \mapsto U(x)^*$ is weakly continuous, f is continuous.

Now we establish that $f(\Lambda(x)) = f(x)$ for all $x \in M$ and all Lorentz transformations Λ. We have

$$\begin{aligned} f(\Lambda x) &= \langle \Omega, A(\Lambda x)^* A(0)\Omega \rangle \\ &= \langle \Omega, U(0,\Lambda) A^*(x) U(0,\Lambda)^{-1} A(0)\Omega \rangle \\ &= \langle U(0,\Lambda)^{-1}\Omega, A(x)^* U(0,\Lambda)^{-1} A(0) U(0,\Lambda)\Omega \rangle \\ &= \langle \Omega, A(x)^* A(\Lambda(0))\Omega \rangle \\ &= \langle \Omega, A(x)^* A(0)\Omega \rangle \\ &= f(x). \end{aligned}$$

Thus, Lemma 129 entails that f is constant, and Lemma 131 entails that there is a $c \in \mathbb{C}$ such that $A(x)\Omega = c\,\Omega$ for all $x \in M$. ■

Microcausality and Spectrum Condition rule out operators at a point

The final no go theorem, originally by Wightman [1964] invokes both microcausality and the spectrum condition. (See [Horuzhy, 1990, p. 46] and [Baumgärtel and Wollenberg, 1992, p. 115] for alternative proofs.)

THEOREM 134. *Suppose that $A : M \to \mathfrak{B}(\mathcal{H})$ is an operator valued function, and U is a continuous unitary representation of the translation group on $\mathcal{H}$ such that:*

1. *$[A(x), A(y)] = 0$ when x and y are spacelike separated;*
2. *$U(x)A(y)U(x)^* = A(y - x)$, for all $x, y \in M$;*
3. *U satisfies the spectrum condition.*
4. *There is a unique translation invariant vector $\Omega \in \mathcal{H}$.*

Then there is a $c \in \mathbb{C}$ such that $A(x)\Omega = c\,\Omega$ for all $x \in M$.

Proof. As above, define $f : M \to \mathbb{C}$ by

$$f(x) = \langle \Omega, A(x)A(0)\Omega \rangle, \qquad x \in M.$$

Fix a nonzero spacelike vector x. Then by condition 1,

$$U(x)^* A(0) U(x) A(0) = A(x)A(0) = A(0)A(x) = A(0)U(x)^* A(0)U(x).$$

Therefore,

$$\begin{aligned} f(x) &= \langle \Omega, U(x)^* A(0) U(x) A(0)\Omega \rangle = \langle \Omega, A(0)U(x)^* A(0)\Omega \rangle \\ &= \langle \Omega, A(0)U(-x)A(0)\Omega \rangle = f(-x). \end{aligned}$$

Now consider the function $F : \mathbb{R} \to \mathbb{C}$ given by $F(t) = f(tx)$, so that $F(t) = F(-t)$. By condition 3, the Fourier transform of f is supported in the forward

light cone. Hence, the Fourier transform of F is supported in $[0, +\infty)$. But since $F(t) = F(-t)$, the Fourier transform of F is also supported in $(-\infty, 0]$. Therefore, the Fourier transform of F is the point mass at $\{0\}$. By Lemma 129, F is constant. Finally, since any two points in M can be connected by two spacelike vectors, we can apply the previous procedure twice to show that f is constant. Therefore, by Lemma 131, there is a $c \in \mathbb{C}$ such that $A(x)\Omega = c\,\Omega$ for all $x \in M$. ■

COROLLARY 135. *Let $O \mapsto \mathfrak{R}(O)$ be a net of von Neumann algebras acting irreducibly on a Hilbert space $\mathcal{H}$, and let U be a strongly continuous unitary representation that implements the action of the translation group on the net $\mathfrak{R}$. Suppose that the net satisfies microcausality (assumption 43). Suppose that U satisfies the spectrum condition, and that there is a translation invariant vector $\Omega \in \mathcal{H}$. Then for each point $x \in M$,*

$$\bigcap_{\{O \in \mathcal{K} \,:\, x \in O\}} \mathfrak{R}(O) = \mathbb{C}I.$$

Proof. Fix $x \in M$, and fix a double cone $x \in O$. Choose an arbitrary operator, denoted by $A(x)$, in

$$\bigcap_{\{O \in \mathcal{K} \,:\, x \in O\}} \mathfrak{R}(O).$$

Now for general $y \in M$, define

$$A(y) = U(x - y)A(x)U(x - y)^*,$$

so that the mapping $A : M \to \mathfrak{B}(\mathcal{H})$ automatically satisfies condition 2 of Theorem 134. Furthermore, since the net $\mathfrak{R}$ satisfies microcausality, and the unitary group U implements the translations on $\mathfrak{R}$, the mapping A satisfies condition 1 of Theorem 134. It then follows that there is a $c \in \mathbb{C}$ such that $A(x) = cI$. Since x was an arbitrary element of M, the result is proven. ■

6.2 Go theorems

Why should we care if $\Phi(x)$ cannot be taken to denote any non-trivial operator on Hilbert space? Does this have any implications for the interpretation of QFT? After all, for any neighborhood O of x, we can find a test-function f that is supported in O, and we can replace the non-denoting term "$\Phi(x)$" with the denoting term "$\Phi(f)$". In fact, couldn't we think of "$\Phi(x)$" as a name for the sequence $\{\Phi(f_n)\}_{n=1}^{\infty}$, where $\{f_n\}_{n=1}^{\infty}$ is a sequence of test-functions that converges to the delta-function at x? More precisely, it seems that we could even attempt to define an expectation value for the pseudo-operator $\Phi(x)$ as follows: If ρ is a state of the quantum field, define:

$$\rho(\Phi(x)) := \lim_{n \to \infty} \rho(\Phi(f_n)). \tag{5}$$

In this section, we make this idea precise in two Go Theorems for field quantities at points. The first result we report from the work of Rehberg and Wollenberg [Rehberg and Wollenberg, 1986; Wollenberg, 1986] (see also [Fredenhagen and Hertel, 1981], [Bostelmann, 2000; Bostelmann, 2004]). This result shows that within the Wightman framework, a quantum field at a point can be represented by a sesquilinear form. The second result shows that if we drop the requirement of continuity on our representation of the translation group, then quantum fields at points can be represented by self-adjoint operators.

Quantum fields as sesquilinear forms

DEFINITION 136. Let $\mathcal{H}$ be a Hilbert space. A *sesquilinear form* on $\mathcal{H}$ is a linear subspace $D(t)$ of $\mathcal{H}$ and a mapping $t : D(t) \times D(t) \to \mathbb{C}$ that is antilinear in the first argument, and linear in the second argument. The form t is said to be *densely defined* just in case $D(t)$ is dense in $\mathcal{H}$. The form t said to be *symmetric* just in case $t(\varphi, \psi) = \overline{t(\psi, \varphi)}$ for all $\varphi, \psi \in D(t)$. The form t is said to be *positive* just in case $t(\psi, \psi) \geq 0$ for all $\psi \in D(t)$.

DEFINITION 137. If t is a sesquilinear form on $\mathcal{H}$ then we define the associated *quadratic form* by $t(\psi) = t(\psi, \psi)$ for all $\psi \in D(t)$. A positive quadratic form t is said to be *closed* just in case for any sequences $(\psi_n)_{n\in\mathbb{N}}$ in $D(t)$ if $\psi_n \to \psi$ and $t(\psi_n - \psi_m) \to 0$, then $\psi \in D(t)$ and $t(\psi_n - \psi) \to 0$.

NOTE 138. A densely defined, symmetric sesquilinear form is a *prima facie* candidate to represent a physical quantity or an observable. Since t is symmetric, the corresponding quadratic form is real-valued. Hence, for each unit vector $\psi \in D(t)$, we might say that the "expectation value" of t in state ψ is $t(\psi)$. Indeed, at first glance, the expectation value mapping $t \mapsto t(\psi)$ seems to have all the same properties as the corresponding expectation mapping for operators.

THEOREM 139. *Let $\Phi(\cdot)$ be a Wightman field on the Hilbert space $\mathcal{H}$. That is, Φ maps elements of a test-function space $\mathcal{S}(\mathbb{R}^4)$ to unbounded operators on $\mathcal{H}$ with some common dense domain $\mathcal{D}$. Let $(\delta_n)_{n\in\mathbb{N}}$ be a sequence of test-functions whose support shrinks to the point x. Then for each $u, v \in \mathcal{D}$, the sequence*

$$\langle u, \Phi(\delta_1)v\rangle, \langle u, \Phi(\delta_2)v\rangle, \langle u, \Phi(\delta_3)v\rangle, \ldots,$$

converges to a finite complex number, which we denote by $\langle u, \Phi(x)v\rangle$. The map $u, v \mapsto \langle u, \Phi(x)v\rangle$ is a sesquilinear form with domain $\mathcal{D}$, which we denote by $\Phi(x)$.

Proof. See [Baumgärtel and Wollenberg, 1992, p. 332] and [Rehberg and Wollenberg, 1986; Wollenberg, 1986]. ■

NOTE 140. One naturally wishes to have a version of this theorem in a more purely algebraic setting. Such a result might be available in the context of the scaling algebras of [Buchholz and Verch, 1995; Buchholz, 1998].

The result is surprising for a couple of reasons. We might have thought that the reason $\Phi(x)$ is not an operator is because the expectation values $\langle u, \Phi(\delta_n)u\rangle$ grow without bound as the test-functions δ_n shrink to a point — i.e. there is some sort of divergence. But Theorem 139 shows that conjecture to be false. The obstruction to $\Phi(x)$'s becoming an operator must lie somewhere else.

So, we have several No Go Theorems against quantum fields as operators (even unbounded operators), and one Go Theorem for quantum fields as sesquilinear forms. What should we conclude from these apparently conflicting results? Should we say that there is a field quantity at the point x, or not?

To answer this question we need to think harder about the relation between operators on a Hilbert space and physical quantities. Why was it that we thought that physical quantities correspond to operators? If we suppose that an unbounded operator can represent a quantity, then must that operator be self-adjoint (i.e. must A coincide with A^* on a common dense domain), or does it suffice to satisfy some weaker condition? Does any symmetric sesquilinear form have all the features necessary to represent a physical quantity? In order to get clear on these questions, it might help to get clear on the mathematical details of the relationship between sesquilinear forms and operators. Fortunately, there are quite a few results in this direction.

Clearly, every linear (possibly unbounded) operator T on $\mathcal{H}$ defines a sesquilinear form with domain $D(T)$ via the equation

$$t(\psi, \varphi) = \langle \psi, T\varphi\rangle. \tag{6}$$

On the other hand, it is less clear when an arbitrary form t corresponds an operator via Eqn. (6).

DEFINITION 141. A sesquilinear form t on $\mathcal{H}$ is said to be *bounded* just in case there is a $n \in \mathbb{N}$ such that $|t(\varphi, \psi)| \leq n$ whenever $\varphi, \psi \in D(t)$ with $\|\varphi\|, \|\psi\| \leq 1$.

PROPOSITION 142. *There is a one-to-one correspondence between densely defined, bounded sesquilinear forms on $\mathcal{H}$ and elements of $\mathfrak{B}(\mathcal{H})$. In particular, if t is bounded sesquilinear form on $\mathcal{H}$ then there is a unique operator $T \in \mathfrak{B}(\mathcal{H})$ such that $t(\varphi, \psi) = \langle \varphi, T\psi\rangle$ for all $\varphi, \psi \in \mathcal{H}$. Furthermore, t is symmetric iff T is self-adjoint.*

Proof. See [Kadison and Ringrose, 1997, Theorem 2.4.1]. ■

PROPOSITION 143. *If t is a densely defined, positive, closed, quadratic form, then there exists a unique positive operator T on $\mathcal{H}$ such that the domain of $T^{1/2}$ is $D(t)$ and*

$$t(\varphi, \psi) = \langle T^{1/2}\varphi, T^{1/2}\psi\rangle,$$

for all $\varphi, \psi \in D(t)$. In particular, $t(\varphi, \psi) = \langle \varphi, T\psi\rangle$ for all $\varphi, \psi \in D(t)$.

NOTE 144. The previous Proposition is useful in showing when a number operator N can be defined in a representation of the Weyl algebra $\mathfrak{A}[S, \sigma]$. For details, see [Clifton and Halvorson, 2001a] and [Bratteli and Robinson, 1997, p. 27].

The previous two propositions do not apply to the sesquilinear form $\Phi(x)$ because it is neither bounded nor positive. Furthermore, there is no known (to the author) characterization of when a symmetric sesquilinear form admits a representation as an operator — although there are some partial results in this direction (see [McIntosh, 1970]). It is clear that $\Phi(x)$ is not an operator; it is unclear what features operators have that $\Phi(x)$ lacks, and whether these features are necessary for a mathematical object to represent a quantity. Accordingly, it is unclear whether or not $\Phi(x)$ represents an element of reality.

Quantum fields as operators on non-separable Hilbert space

Our second Go result for quantum field operators at a point is really just a sketch of an example. We take a nonseparable Hilbert space H that can represent states of particles with point positions (compare with [Halvorson, 2004]). We then apply the standard second quantization procedure — which does not depend on the one-particle space being separable — to obtain a Fock space $\mathcal{F}(H)$, and self-adjoint field operators $\phi(x), \pi(x)$ indexed by points in $\mathbb{R}$.

Let $H = l_2(\mathbb{R})$ be the Hilbert space of square-summable sequences over $\mathbb{R}$; i.e. an element f of $l_2(\mathbb{R})$ is a mapping from $\mathbb{R}$ into $\mathbb{C}$ such that f vanishes at all but countably many points, and $\sum_{x\in\mathbb{R}} |f(x)|^2 < \infty$. The inner product on $l_2(\mathbb{R})$ is given by

$$\langle f, g\rangle = \sum_{x\in\mathbb{R}} \overline{f(x)} g(x). \tag{7}$$

Let $\mathcal{F}(H)$ be the Fock space over H. For each $x \in \mathbb{R}$, we let $\delta_x \in l_2(\mathbb{R})$ denote the characteristic function of $\{x\}$; the set $\{\delta_x : x \in \mathbb{R}\}$ is an (uncountably infinite) orthonormal basis for $l_2(\mathbb{R})$. For any $x \in \mathbb{R}$, we define the creation operator $a(x)$ by:

$$a(x)(f_1 \otimes \cdots \otimes f_n) := \delta_x \otimes f_1 \otimes \cdots \otimes f_n. \tag{8}$$

As in the standard case, we verify that $a^-(x) + ia^+(x)$ and $a^+(x) - ia^-(x)$ are preclosed, i.e. that the closure of the graphs of these operators are graphs of linear operators (see [Kadison and Ringrose, 1997, p. 155]), which we denote by

$$\phi(x) = \overline{a^-(x) + ia^+(x)}, \tag{9}$$
$$\pi(x) = \overline{a^+(x) - ia^-(x)}. \tag{10}$$

It then follows that $\phi(x)$ and $\pi(x)$ are self-adjoint, and on a dense domain $\mathcal{D}$ in $\mathcal{F}(H)$, we have

$$[\pi(x), \phi(x')] = i\,\langle \delta_x, \delta'_x\rangle = i\,\delta_0(x - x'), \tag{11}$$

where now δ_0 is a completely legitimate mathematical object — viz. the probability measure supported on $\{0\}$.

Consider the (discontinuous) representation $x \mapsto V(x)$ of the translation group on $l_2(\mathbb{R})$ defined on the basis elements $\{\delta_y : y \in \mathbb{R}\}$ by

$$V(x)\delta_y = \delta_{y-x}. \quad (12)$$

Let Γ be the 'Fock functor'; i.e. Γ maps a unitary operator V on the single particle space H to the corresponding operator

$$I \oplus V \oplus (V \otimes V) \oplus \cdots,$$

on $\mathcal{F}(H)$. Then $x \mapsto U(x) := \Gamma(V(x))$ is a *discontinuous* representation of the translation group on $\mathcal{F}(H)$, and

$$U(x)^* \phi(y) U(x) = \Phi(y - x). \quad (13)$$

Thus, $(\phi(\cdot), \pi(\cdot), \mathcal{F}(H), U)$ is a field system over $l_2(\mathcal{R})$, where $x \mapsto U(x)$ is a discontinuous unitary representation. We could then use the field system to define a net $O \mapsto \mathfrak{R}(O)$ of von Neumann algebras on $\mathcal{F}(H)$. But this net of course fails the spectrum condition, because the representation of the translation group is not continuous.

The model just described is probably too unwieldy to be of much use in describing real physical situations. Furthermore, there is no good reason to think that the procedure we followed generalizes to the case of interacting theories, where pointlike localized operators are needed to make sense of products of field operators. However, we hoped to show that it is conceivable that we can get by without some of the technical assumptions of the No Go Theorems. So, we should think very carefully before we try to use these theorems to draw conclusions about how QFT must be interpreted.

6.3 Field interpretations of QFT

In Section 4, we saw that there are severe obstacles to a particle interpretation of QFT. One might argue then, by a process of elimination, that we should adopt a "field interpretation" of QFT (see e.g. [Teller, 1995; Huggett, 2000]). But if we examine the field interpretation on its own merits, its not clear that it is better off than the particle interpretation.

In constructing canonical free theories (e.g. the free Bose and Fermi fields), one begins with a Hilbert space H which can be interpreted either as a "single particle space" (i.e. space of wavefunctions of a single quantum mechanical particle) or as a space of configurations of a classical field. Corresponding to these two interpretations, there are two ways to construct the Hilbert space of the quantum field theory:

1. Second quantization: the Hilbert space of the quantum field is the Fock space $\mathcal{F}(H)$ over H. (See Section 4.1).

2. Field quantization: the Hilbert space of the quantum field is is space $L_2(H, d)$ of 'square integrable' functions from H into $\mathbb{C}$ relative to the isonormal distribution d on H.

(In a rigorous treatment, elements of $L_2(H, d)$ are not really functions. See [Baez *et al.*, 1992, Section 1.3] for details.) The free field theories constructed by these two methods are known to be unitarily equivalent. However, the field quantization approach lends itself more naturally to a field interpretation. Indeed, in a recent survey of the foundations of QFT [Huggett, 2000], one finds the suggestive notation:

$$\Psi(\phi), \qquad \phi \in L_2(\mathbb{R}^{3n}),$$

for a function on the space $H := L_2(\mathbb{R}^{3n})$ of wavefunctions. Thus, it seems that a quantum field state can be interpreted as a superposition of classical field configurations in the same sense that a wavefunction of n particles can be interpreted as a superpositions of classical configurations of n particles.

However, there are difficulties with this approach. First, the field operators $\Phi(x)$ on $L_2(H, d)$ are the precise analogues of the position operators Q_i for a system of particles. That is, there is a natural interpretation of a function $\Psi \in L_2(H, d)$ as a probability distribution over the family $\{\Phi(x) : x \in M\}$ of mutually commuting operators. But the No Go theorems for the operators $\Phi(x)$ militate against interpreting Ψ as a probability distribution over classical field configurations. More directly, since d assigns zero measure to points in H (i.e. to individual field configurations), characteristic functions of singleton subsets of H — i.e. determinate field configurations — are identified with the zero vector in $L_2(H, d)$. That is, there is no state of the quantum field in which it is in a definite configuration.

It follows from the preceeding considerations that the No Go theorems for fields operators at a point undermine the field interpretation of QFT in the same way that No Go theorems for number operators undermine the particle interpretation. Thus, we should be wary of arguments for field interpretations based on problems with particle interpretations.

6.4 Points of time?

The preceding results were aimed at answering the question of whether there can be field operators at a spacetime point. Suppose that we concede that there cannot be, and we proceed with the standard mathematically rigorous approach to, say, the free Bose field, where field operators are smeared over *space* by test functions (see e.g. [Araki, 1963]). In this case, quantities are not tied to spacetime points, but they are tied to pointlike times. However, some claim that in general, the quantities will also have to be fuzzy in time. For example, according to Haag,

> Renormalization theory suggests that it is essential to smear out Φ both in space and time, in contrast to the case of free fields, where an averaging over 3-dimensional space at a fixed time is sufficient. Due to

> the stronger singularities, one cannot assume well-defined commutation relations of fields at equal time. [Haag, 1996, p. 59]

But such claims are speculative — we know of no theorems that prove that interacting fields must be smeared out in time. So, at the present time we have no particularly good reason to conclude that time is pointless.

7 THE PROBLEM OF INEQUIVALENT REPRESENTATIONS

The philosophy of local quantum physics (in Haag's terminology) is that the theoretical parts of QFT (e.g. unobservable fields, gauge group) should not count as part of the given data. Instead, the abstract net $\mathfrak{A}$ of observable algebras should be taken as the primitive. Following the terminology in [Ruetsche, 2002], we define 'Algebraic Imperialism' as the position that:

> The physical content of a quantum field theory is encoded in the net $O \mapsto \mathfrak{A}(O)$, the subgroup of $\mathrm{Aut}(\mathfrak{A})$ corresponding to physical symmetries (including dynamics), and the states on the quasilocal algebra $\mathfrak{A}$. A representation $(\mathcal{H}, \pi)$ of $\mathfrak{A}$ may be an aid to calculation, but has no ontological significance.

Such an attitude might seem incomprehensible to those steeped in the traditional Hilbert space formalism of QM. Indeed, where is the Hamiltonian, where are the transition probabilities, and how do we describe measurements? The very abstractness and generality of the algebraic formalism seems to empty it of a great deal of the content we expect in a physical theory.

However, some of these worries about lack of content of the abstract algebraic formalism are ill founded. Indeed, the GNS theorem (Thm. 17) shows that all the Hilbert spaces we will ever need are hidden inside the algebra itself. Furthermore, much of the vocabulary one learns to speak in elementary QM can be defined within this purely abstract setting. For example, for a definition of transition probabilities between states, see [Roberts and Roepstorff, 1968]; and for a definition of measurement probabilities, see [Wald, 1994].

But it is not true — at least on the face of it — that *all* pieces the traditional vocabulary of QFT can be reproduced in the algebraic setting. For example, the quasilocal algebra does not contain a number operator, and probability distributions over the spectrum of the number operator cannot be defined in terms of expectation values on $\mathfrak{A}$ (see [Clifton and Halvorson, 2001a]). What is perhaps even worse is that by beginning with a net $O \mapsto \mathfrak{A}(O)$ of *observable* algebras, we have effectively closed our eyes to the existence of unobservable fields, which do not generally commute with each other at spacelike separation. Thus, we seem to have no way to account for deep theoretical facts of QFT such as the connection between spin (commutation relations of field operators) and statistics.

Worries such as these might push us towards the second main position on the issue of representations, which Ruetsche [2002] calls Hilbert Space Conservatism:

> The theory is not the net $O \mapsto \mathfrak{A}(O)$, but the net plus one specific representation $(\mathcal{H}, \pi)$.

In fact, Hilbert Space Conservatism might be thought of as the default view of most workers in mainstream (Lagrangian) QFT, since the abstract algebra (and its representations) do not play a central role there.

But as with many realist views, the Conservative view faces epistemological difficulties: How do we decide which is the correct representation? In this case, the difficulty is particularly severe, because it can be proven mathematically that the predictions of states within any one representation can be reproduced to arbitrarily high accuracy by the states in *any other* representation.[2] (This is due to the fact that since $\mathfrak{A}$ is simple, Fell's theorem implies that the states in any folium are weak* dense in the state space.)

Nonetheless, it is tempting to think that the algebraic formalism is creating an interpretive problem. That is, it is tempting to think that if we stick to the old-fashioned way of doing QFT, the issue of inequivalent representations does not arise, and so neither does this interpretive dilemma. So, are inequivalent representations telling us something of foundational importance, or are they just mathematical playthings?

The motivating arguments for the algebraic approach have included the existence of inequivalent representations of the canonical commutation relations, as well as physical effects associated with Rindler-Fulling quanta. Nonetheless, these arguments have been resisted for various reasons, e.g. there is a suspicion that the Rindler vacuum representation is inferior, as a description of reality, to the Minkowski vacuum representation. So, in the following sections, we discuss another motivating argument for the algebraic approach — viz. superselection rules. It is in the analysis of superselection rules that the algebraic approach most clearly displays its beauty, utility, and foundational importance.

7.1 Superselection rules

In a now famous paper, Wick, Wightman, and Wigner [1952] argue that there is a physical system with state space H, and state vectors $\psi_1, \psi_2 \in H$ such that the linear combinations

$$2^{-1/2}(\psi_1 + e^{i\theta}\psi_2), \qquad \theta \in [0, 2\pi), \tag{14}$$

give rise to 'empirically indistinguishable' states. When this occurs, Wick *et al.* say that there is a 'superselection rule' between ψ_1 and ψ_2; alternatively, ψ_1 and ψ_2 lie in different 'superselection sectors.' We put 'empirically indistinguishable' in scare quotes, because the literature has been anything but clear about the nature

[2] This way of stating the problem is biased, and depends on taking "predictions of a representation" to mean expectation values of observables in the abstract algebra. If we also include expectation values of observables in the weak closure $\pi(\mathfrak{A})^-$, and expectation values of unbounded operators on $\mathcal{H}$, then the story becomes more complicated. Compare with [Clifton and Halvorson, 2001b].

of the relation between the states in Eqn. (14). Are the states in Eqn. (14) merely *empirically* indistinguishable, or is there a stronger sense in which these two states are equivalent? If the indistinguishability is empirical, how strong is the modal force? Do these states predict the same empirical phenomena in all physically possible worlds, or is their indistinguishability due to some special features (e.g. initial conditions) in our world? In this paper, we will not attempt to resolve these important questions about the nature of superselection rules.[3] Rather, we will content ourselves with explaining the proposal of Doplicher, Haag, and Roberts (DHR) for making Wick *et al.*'s notion precise within the context of AQFT.

The first approaches to superselection rules involved an *ad hoc* butchery of the state space and of the algebra of observables, with a resulting confusion about what should count as the states and observables (or quantities) of the resulting theory. One begins with a Hilbert space H with unit vectors giving pure states, and with self-adjoint elements of $B(H)$ as observables (or quantities). One then gives a heuristic argument for the claim that a superselection rule holds between the state vectors in some subspace H_1 and the state vectors in the complementary subspace $H_2 := (H_1)^{\perp}$. On the basis of this argument, the state space H is reduced to the union of H_1 and H_2; that is, a linear combination of a vector in H_1 and H_2 is no longer thought to be a possible (pure state) of the theory; the pure state vectors lie in *either* H_1 *or* H_2. Equivalently, the algebra of observables $B(H)$ is reduced to $B(H_1) \oplus B(H_2)$. The operators in $B(H_1 \oplus H_2)$ that could distinguish between the states in Eqn. (14) are demoted to the status of "unobservable." Thus, the algebra of observables is actually $B(H_1) \oplus B(H_2)$, and not $B(H_1 \oplus H_2)$.

Now, the algebraic approach provides two inversely related approaches to superselection rules.

1. First, we can follow the original "state space butchery" approach in a slightly more principled fashion: suppose that we are given some fields acting as operators on some big Hilbert space $\mathcal{H}$. Let $\mathfrak{F}$ denote the algebra of field operators. [Here $\mathfrak{F}$ is the analogue of the algebra $B(H_1 \oplus H_2)$, and $\mathcal{H}$ is the analogue of $H_1 \oplus H_2$. In this case, however, we are not given an *a priori* decomposition of $\mathcal{H}$ into a direct sum.] Suppose that we are also given a gauge group G that acts on the fields. We then *define* the observables as the gauge invariant fields. Let $\mathfrak{A}$ denote the algebra of observables. We also define the physical pure states as those vectors in $\mathcal{H}$ that transform irreducibly under the gauge group. A series of mathematical results (partially described in Section 9) shows then that $\mathcal{H}$ decomposes into a direct sum $\bigoplus \mathcal{H}_\xi$ of subspaces that transform irreducibly under the gauge group; and each subspace $\mathcal{H}_\xi$ corresponds to an irreducible representation of the algebra of observables $\mathfrak{A}$. We sketch this "top down" approach to superselection rules

[3]Superselection rules are also of foundational interest because they have been thought to help with the measurement problem — see e.g. [Beltrametti and Cassinelli, 1981, p. 74], [Landsman, 1995], [van Fraassen, 1991, pp. 264–272] — and more generally because of their connection with the emergence of a classical realm [Giulini, 2003]. However, we do not take up those specific issues in this Chapter.

in Section 9.

2. Instead of beginning with the field algebra $\mathfrak{F}$ and deriving the superselection structure (i.e. the set of physically interesting representations of the algebra $\mathfrak{A}$ of observables), we can begin with $\mathfrak{A}$ and consider its set of physical representations. What is a 'physical' representation of $\mathfrak{A}$? According to the criterion proposed by Doplicher, Haag, and Roberts (the DHR selection criterion), the physical representations are those that differ observably from the vacuum representation only in local regions. In this case, we still have the notion of superselection sectors, but we do not yet have a notion of fields or of a gauge group. It is not immediately clear that we have enough structure to explain the phenomena.

   However, it is at this point that the deep mathematical analysis begins. First, one proves that the category of DHR representations corresponds precisely to the set Δ of localized transportable endomorphisms of the observable algebra $\mathfrak{A}$ (see Section 8.2). Second, one proves that the set Δ naturally has the structure of a symmetric tensor $*$-category (see Section 8). Finally, the Doplicher-Roberts Reconstruction Theorem shows that the unobservable fields $\mathfrak{F}$ and gauge group G can be uniquely reconstructed from the category Δ.

The following sections outline some of the most important insights that have been gained in the study of superselection rules, and how this analysis bears on the foundational questions about the role of inequivalent representations. In short, our conclusion is that inequivalent representations are *not* irrelevant, and nor are they a problem. Rather, it is the *structure of the category of representations* that provides the really interesting theoretical content of QFT.

7.2 Minimal assumptions about the algebra of observables

For our discussion of superselection theory we need only a considerably pared down set of assumptions about the net of observable algebras. So, we now effectively cancel all assumptions we made about the net in Section 2. We begin with a *tabula rasa*, and add only those assumptions that we will need in the following sections.

By calling $\mathfrak{A}$ a "net," we are assuming that if $O_1 \subseteq O_2$ then $\mathfrak{A}(O_1) \subseteq \mathfrak{A}(O_2)$. But we do not promote this to the status of an Assumption.

ASSUMPTION 145 (Microcauality). If O_1 and O_2 are spacelike separated then $[\mathfrak{A}(O_1), \mathfrak{A}(O_2)] = \{0\}$.

ASSUMPTION 146 (Property B). The net $O \to \mathfrak{R}_0(O) \equiv \pi_0(\mathfrak{A}(O))''$ of von Neumann algebras satisfies property B, where $(\mathcal{H}_0, \pi_0)$ is the GNS representation of $\mathfrak{A}$ induced by ω_0.

ASSUMPTION 147 (Duality). The pair $(\mathfrak{A}, \omega_0)$ satisfies *Haag duality*, i.e.

$$\pi_0(\mathfrak{A}(O'))' = \pi_0(\mathfrak{A}(O))'',$$

for each double cone O, where $(\mathcal{H}_0, \pi_0)$ is the GNS representation of $\mathfrak{A}$ induced by ω_0.

ASSUMPTION 148 (Separability). The vacuum Hilbert space $\mathcal{H}_0$ is separable.

ASSUMPTION 149 (Nontriviality). For each double cone O, $\pi_0(\mathfrak{A}(O))$ contains an operator that is not a multiple of the identity; i.e. $\pi_0(\mathfrak{A}(O)) \neq \mathbb{C}I$.

A few remarks on these assumptions: (i) The first assumption is about the net $\mathfrak{A}$, but the remaining assumptions apply to a pair $(\mathfrak{A}, \omega_0)$, where $\mathfrak{A}$ is the quasilocal algebra and ω_0 is some fixed state. (ii) The duality assumption states that not only are the observables in $\mathfrak{R}_0(O')$ compatible with the observables in $\mathfrak{R}_0(O)$, but that $\mathfrak{R}_0(O')$ contains *all* observables that are compatible with the collection of observables in $\mathfrak{R}_0(O)$. We will assume in the following two sections (on DHR superselection theory) that the net $\mathfrak{A}$ satisfies duality relative to some privileged vacuum state ω_0. But, it does not follow from this that the net satisfies duality relative to every physical representation. In fact, a representation satisfies duality iff that sector has normal (Bose/Fermi) statistics; and every representation satisfies duality iff the gauge group is abelian. (iii) Duality in the vacuum sector is equivalent to the non-existence of spontaneously broken gauge symmetries. For the case of broken symmetries, we would impose a weaker requirement: essential duality. cf. Section 10.7. (iv) The separability assumption will only be invoked once — to show all superselection sectors induced by local fields are strongly locally equivalent (Prop. 243).

To be clear, note that we are now making no assumptions about the following: (i) No assumptions about the action of spacetime symmetries (e.g. translation symmetries, Lorentz symmetries) on the algebra $\mathfrak{A}$; (ii) No assumptions to the effect that the vacuum state ω_0 is translation invariant; (iii) No assumptions about the action of spacetime symmetries on the vacuum Hilbert space; (iv) No assumptions about the spectrum condition.

8 THE CATEGORY Δ OF LOCALIZED TRANSPORTABLE ENDOMORPHISMS

In this Section we study the category $\Delta(\mathfrak{A})$ of localized transportable endomorphisms of the observable algebra $\mathfrak{A}$. Since the physical motivation for this study might not be initially clear, we pause to note the relation between endomorphisms and representations.

Suppose that π_0 is a fixed representation of $\mathfrak{A}$ of some physical significance — e.g. the vacuum representation. Then for any endomorphism ρ of $\mathfrak{A}$, the composition $\pi_0 \circ \rho$ is also a representation of $\mathfrak{A}$. Thus, endomorphisms of $\mathfrak{A}$ correspond naturally to representations of $\mathfrak{A}$, and we can hope to gain insight into the structure of the representations of $\mathfrak{A}$ by studying the endomorphisms of $\mathfrak{A}$. However, the set End$\mathfrak{A}$ of endomorphisms of $\mathfrak{A}$ has more intrinsic structure than the set Rep$\mathfrak{A}$ of representations of $\mathfrak{A}$ — e.g., there is a product (viz. composition) operation on End$\mathfrak{A}$, and some endomorphisms have inverses. Thus, besides the traditional notions of equivalence and disjointness of representations, there are additional relations of physical importance on the set of representations of the form $\pi_0 \circ \rho$ with $\rho \in \text{End}\mathfrak{A}$.

If the problem of Lagrangian QFT is that there is only one Hilbert space, the problem of AQFT is that there are too many Hilbert spaces! Surely, not all of the representations of $\mathfrak{A}$ are physical. In Section 9, we look at the problem from a more traditional point of view. In particular, we begin with a field algebra $\mathfrak{F}$ of operators acting on a Hilbert space $\mathcal{H}$, and a gauge group G of unitary operators on $\mathcal{H}$. (We may suppose that G is the image of some representation of a fundamental symmetry group, e.g. $SU(2)$.) We also suppose that $\mathcal{H}$ contains a vacuum state Ω. We then *define* the observable algebra $\mathfrak{A}$ as the *gauge invariant fields*. But then we are again in the domain of AQFT: we have a reducible representation π of $\mathfrak{A}$ on $\mathcal{H}$, and the irreducible subrepresentations of π are the *superselection sectors that can be reached from the vacuum sector by the action of local (unobservable) fields*. Not all representations of $\mathfrak{A}$ appear in the decomposition of π — those that do not are surplus structure. However, all representations that appear in the decomposition of π are of the form $\pi_0 \circ \rho$, with ρ an endomorphism from the category $\Delta(\mathfrak{A})$! So, the motivation for studying these endomorphisms is that they correspond to representations that arise in this traditional, physically motivated way by acting on the vacuum representation with (unobservable) fields.[4]

There is yet another motivation for studying the DHR category: we want to understand the nature of gauge symmetries, and the DR Reconstruction Theorem provides crucial insight. In particular, the Theorem shows that DHR categories are in duality (in a mathematically precise sense) with compact groups. So, wherever there is a compact group, there is a DHR category, and vice versa. The study

[4]The DHR representations do *not* include those that can be reached from the vacuum by *nonlocal* fields, and so the domain of DHR superselection theory does not include theories with long range forces. But the case of local fields is already complicated enough, and is good training for the more general case.

of DHR categories and the study of compact gauge groups are one and the same; or, to take a more controversial stance, the structure of the category of physical representations of $\mathfrak{A}$ explains why there is a compact gauge group (see [Roberts, 1975]).

We now define the category $\Delta = \Delta(\mathfrak{A})$ and uncover some of its natural structures. As stated above, the objects of our category Δ will be a subset of the set End$\mathfrak{A}$ of $*$-endomorphisms of $\mathfrak{A}$.

DEFINITION 150. Let ρ be a $*$-endomorphism of $\mathfrak{A}$, i.e. $\rho : \mathfrak{A} \to \mathfrak{A}$ is a $*$-homomorphism (not necessarily surjective). Let O be a double cone in (Minkowski) spacetime. Then ρ is said to be *localized* in O just in case $\rho(A) = A$, for all $A \in \mathfrak{A}(O')$, where O' is the spacelike complement of O. We say that ρ is *localized* just in case there is a double cone O in which it is localized.

NOTE 151. By definition, a localized endomorphism satisfies $\rho(I) = I$, where I is the identity in $\mathfrak{A}$.

DEFINITION 152. If ρ is localized in O then ρ is said to be *transportable* just in case for any other double cone O_1, there is a morphism ρ_1 localized in O_1 and a unitary operator $U \in \mathfrak{A}$ such that $U\rho(A) = \rho_1(A)U$ for all $A \in \mathfrak{A}$.

DEFINITION 153. For each double cone $O \in \mathcal{K}$, we let $\Delta(O)$ denote the set of transportable morphisms that are localized in O, and we let $\Delta = \bigcup_{O\in\mathcal{K}} \Delta(O)$. Elements of Δ are the *objects* of the DHR category.

We must now define arrows between the objects.

DEFINITION 154. Let $\rho, \rho' \in \Delta$. We define the set $\mathrm{Hom}(\rho, \rho')$ of arrows between ρ and ρ' as follows:

$$\mathrm{Hom}(\rho, \rho') := \{T \in \mathfrak{A} : T\rho(A) = \rho'(A)T,\ \forall A \in \mathfrak{A}\}.$$

If $T \in \mathrm{Hom}(\rho, \rho')$ and $S \in \mathrm{Hom}(\rho', \sigma)$ then we define $S \circ T = ST$, which is obviously in $\mathrm{Hom}(\rho, \sigma)$.

Obviously, the identity $I \in \mathfrak{A}$ functions as the identity arrow for all objects; i.e. $I = \mathrm{id}_\rho \in \mathrm{End}(\rho)$ for all $\rho \in \mathrm{Obj}(\Delta)$. Occasionally, we will write I_ρ to indicate that we are considering I as the identity of $\mathrm{End}(\rho)$.

LEMMA 155. *Suppose that $\rho_i \in \Delta(O_i)$ for $i = 1, 2$, and that $T \in \mathrm{Hom}(\rho_1, \rho_2)$. Then for any double cone O containing $O_1 \cup O_2$, we have $T \in \mathfrak{A}(O)$.*

Proof. Let $B \in \mathfrak{A}(O')$. Then

$$TB = T\rho_1(B) = \rho_2(B)T = BT.$$

Hence $T \in \mathfrak{A}(O')'$. By duality in the vacuum sector, $T \in \mathfrak{A}(O)$. ■

PROPOSITION 156. *With the definition of hom-sets given above, Δ is a category.*

Proof. Completely straightforward. ■

So, we have shown that Δ is a category. In the remainder of this Section, we uncover more structure on Δ. We first show that Δ is a C^*-category; this involves showing that Δ has direct sums (an $\oplus$ operation), subobjects, and the hom-sets of Δ are vector spaces with a $*$-operation and norm $\|\cdot\|$ that obey the appropriate analogue of the C^*-algebra norm property. We then drop reference to the norms on the hom-sets, and show that there is a product operation $\otimes$ on Δ such that $(\Delta, \otimes, \iota)$ is a tensor $*$-category.

DEFINITION 157. A category $\mathcal{C}$ is said to be a *linear category* over the complex field $\mathbb{C}$, or a $\mathbb{C}$-*linear category*, just in case for all $X, Y \in \mathrm{Obj}(\mathcal{C})$, $\mathrm{Hom}(X, Y)$ is a complex vector space, and the composition $\circ$ of morphisms is bilinear. When speaking of $\mathbb{C}$-linear categories, we stipulate that all functors should be $\mathbb{C}$-linear.

DEFINITION 158. A $*$-*operation* on a $\mathbb{C}$-linear category $\mathcal{C}$ is a map which assigns to an arrow $s \in \mathrm{Hom}(X, Y)$ another arrow $s^* \in \mathrm{Hom}(Y, X)$. This map has to be antilinear, involutive ($s^{**} = s$), and contravariant ($(s \circ t)^* = t^* \circ s^*$). A $*$-operation is *positive* iff $s^* \circ s = 0$ implies $s = 0$. A $*$-*category* is a $\mathbb{C}$-linear category with a positive $*$-operation.

NOTE 159. If $\mathcal{C}$ is a $*$-category, then for each $X \in \mathrm{Obj}(\mathcal{C})$, $\mathrm{End}(X)$ is a $*$-algebra.

DEFINITION 160. A $*$-category is called a C^*-*category* if for all $X, Y \in \mathrm{Obj}(\mathcal{C})$, there is a norm $\|\cdot\|_{X,Y}$ on $\mathrm{Hom}(X, Y)$ such that $\langle \mathrm{Hom}(X, Y), \|\cdot\|_{X,Y} \rangle$ is a Banach space and

$$\begin{aligned} &\|s \circ t\|_{X,Z} \leq \|s\|_{Y,Z} \cdot \|t\|_{X,Y}, && \forall s \in \mathrm{Hom}(Y, Z), \forall t \in \mathrm{Hom}(X, Y) \\ &\|s^* \circ s\|_{X,X} = \|s\|^2_{X,Y}, && \forall s \in \mathrm{Hom}(X, Y). \end{aligned}$$

We borrow some definitions from the theory of $*$-algebras.

DEFINITION 161. Let $\mathcal{C}$ be a $*$-category. An arrow $f \in \mathrm{Hom}(X, Y)$ is said to be an *isometry* just in case $f^* \circ f = \mathrm{id}_X$. An arrow $f \in \mathrm{Hom}(X, Y)$ is said to be *unitary* just in case f and f^* are isometries. An arrow $p \in \mathrm{End}(Y) = \mathrm{Hom}(Y, Y)$ is said to be a *projection* if $p = p^*$ and $p \circ p = p$.

NOTE 162. If $s \in \mathrm{Hom}(Y, X)$ is an isometry then the arrow $p \equiv s \circ s^* \in \mathrm{End}(X)$ is a projection.

DEFINITION 163. Let $\mathcal{C}$ be a $*$-category. If $X, Y \in \mathrm{Obj}(\mathcal{C})$, then X is said to be a *subobject* of Y just in case there is an isometry $f \in \mathrm{Hom}(X, Y)$. (Roughly speaking, there is an isometric embedding of X into Y.) The $*$-category $\mathcal{C}$ is said to *have subobjects* just in case for each $Y \in \mathrm{Obj}(\mathcal{C})$ and projection $g \in \mathrm{End}(Y)$, there is an $X \in \mathrm{Obj}(\mathcal{C})$ and an isometry $f \in \mathrm{Hom}(X, Y)$ such that $f \circ f^* = g$. The $*$-category $\mathcal{C}$ is said to *have direct sums* just in case for any two objects X, Y in $\mathcal{C}$, there is an object Z in $\mathcal{C}$ and isometries $f \in \mathrm{Hom}(X, Z)$, $g \in \mathrm{Hom}(Y, Z)$ such that $f \circ f^* + g \circ g^* = \mathrm{id}_Z$.

We begin by verifying that the DHR category Δ is a $*$-category, i.e. the hom sets are vector spaces over $\mathbb{C}$, and there is a positive $*$-operation.

LEMMA 164. *The DHR category Δ is a $*$-category. That is, if $\rho, \sigma \in \mathrm{Obj}(\Delta)$, then $\mathrm{Hom}(\rho, \sigma)$ is a vector space over $\mathbb{C}$ with the operations inherited from $\mathfrak{A}$ (which is a vector space over $\mathbb{C}$), and the composition of arrows is bilinear. Furthermore, the $*$-operation inherited from $\mathfrak{A}$ is antilinear, involutive, contravariant, and positive.*

Proof. Completely straightforward. ■

PROPOSITION 165. *The DHR category Δ has direct sums.*

Proof. Let $\rho_1 \in \Delta(O_1)$, and let $\rho_2 \in \Delta(O_2)$. Choose a double cone O such that $(O_1 \cup O_2)^- \subseteq O$. Let E be a projection in $\mathfrak{A}(O_1)$. By property B, there are isometries $V_1, V_2 \in \mathfrak{A}(O)$ such that $V_1 V_1^* + V_2 V_2^* = I$. Define $\rho : \mathfrak{A} \to \mathfrak{A}$ by

$$\rho(A) = V_1 \rho_1(A) V_1^* + V_2 \rho_2(A) V_2^*, \qquad \forall A \in \mathfrak{A}.$$

Since $V_i V_j = \delta_{ij} I$, and $\sum_i V_i V_i^* = I$, it follows that ρ is a morphism. Since ρ_1, ρ_2 are localized in O, and $V_1, V_2 \in \mathfrak{A}(O)$, it follows that ρ is localized in O.

To see that ρ is transportable, let $\widetilde{O}$ be another double cone. Since the ρ_i are transportable, there are endomorphisms ρ_i' localized in $\widetilde{O}$, and unitary operators $U_i \in \mathrm{Hom}(\rho_i, \rho_i')$. As before, choose isometries V_1', V_2' in $\mathfrak{A}(\widetilde{O})$, and set $\rho' = V_1' \rho_1' V_1'^* + V_2' \rho_2' V_2'^*$. Then ρ' is localized in $\widetilde{O}$ and

$$V_1' U_1 V_1^* \in \mathrm{Hom}(\rho, \rho'), \qquad V_2' U_2 V_2^* \in \mathrm{Hom}(\rho, \rho').$$

If we set $W = V_1' U_1 V_1^* + V_2' U_2 V_2^*$, then $W \in \mathrm{Hom}(\rho, \rho')$ since it is a vector space. Furthermore,

$$\begin{aligned} W^* W &= [V_1' U_1 V_1^* + V_2' U_2 V_2^*]^* [V_1' U_1 V_1^* + V_2' U_2 V_2^*] \\ &= [V_1 U_1^* V_1'^* + V_2 U_2^* V_2'^*][V_1' U_1 V_1^* + V_2' U_2 V_2^*] \\ &= V_1 V_1^* + V_2 V_2^* = I, \end{aligned}$$

and similarly for WW^*. Therefore W is a unitary operator in $\mathrm{Hom}(\rho, \rho')$, showing that ρ is transportable. ■

DEFINITION 166. If $\rho_1, \rho_2 \in \Delta$, we denote their direct sum by $\rho_1 \oplus \rho_2$.

PROPOSITION 167. *The DHR category Δ has subobjects.*

Proof. Let $\rho \in \Delta(O)$, and let E be a projection in $\mathrm{End}(\rho)$; i.e. $E\rho(A) = \rho(A)E$, for all $A \in \mathfrak{A}$. Then for all $A \in \mathfrak{A}(O')$,

$$EA = E\rho(A) = \rho(A)E = AE.$$

Therefore, by duality in the vacuum sector $E \in \mathfrak{A}(O)$. Choose O_1 such that $O^- \subseteq O_1$. By property B, there is an isometry $V \in \mathfrak{A}(O_1)$ such that $VV^* = E$. Now define $\rho' : \mathfrak{A} \to \mathfrak{A}$ by

$$\rho'(A) = V^* \rho(A) V, \qquad \forall A \in \mathfrak{A}.$$

The isometry V embeds ρ' into ρ. Indeed,

$$\rho'(A)V^* = V^*\rho(A)VV^* = V^*\rho(A)E = V^*\rho(A).$$

and V is an isometry in $\mathrm{Hom}(\rho', \rho)$ such that $VV^* = E \in \mathrm{End}(\rho)$.

To see that ρ' is transportable, suppose that O_2 is an arbitrary double cone. Choose a double cone O_3 such that $O_3^- \subseteq O_2$. Since ρ is transportable, there is a morphism σ localized in O_3 and a unitary $U \in \mathrm{Hom}(\rho, \sigma)$. It then follows that $U\mathrm{End}(\rho)U^* = \mathrm{End}(\sigma)$, thus $E' = UEU^*$ is a projection in $\mathrm{End}(\sigma)$. Using property B, there is an isometry $V' \in \mathfrak{A}(O_1)$ such that $V'V'^* = E'$. Let $\sigma' = V'^*\sigma V'$. Clearly σ' is localized in O_1, and $W = V'^*UV \in \mathrm{Hom}(\rho', \sigma')$. Finally, W is unitary:

$$\begin{aligned} W^*W &= V^*U^*V'V'^*UV = V^*U^*E'UV \\ &= V^*EV = V^*VV^*V = I, \end{aligned}$$

and similarly for WW^*. Thus σ' is equivalent to ρ'. Since O_2 was an arbitrary double cone, $\rho' \in \Delta$. ■

DEFINITION 168. Suppose that $\mathcal{C}$ is a $\mathbb{C}$-linear category. An object X in $\mathcal{C}$ is said to be *irreducible* if it is nonzero and $\mathrm{End}(X) = \mathbb{C}\mathrm{id}_X$.

NOTE 169. Let ι be the identity endomorphism of $\mathfrak{A}$. Then $\iota \in \mathrm{Obj}(\Delta)$, and since the vacuum representation of $\mathfrak{A}$ is irreducible, ι is an irreducible object.

We now define a bifunctor $\otimes = (\otimes, \times)$ on the DHR category Δ, and verify that $(\Delta, \otimes, \iota)$ is a tensor $*$-category. But first we recall the pertinent definitions.

DEFINITION 170. A *bifunctor* on a category $\mathcal{C}$ consists of two mappings $F : \mathrm{Obj}(\mathcal{C}) \times \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{C})$ and $F : \mathrm{Hom}(\mathcal{C}) \times \mathrm{Hom}(\mathcal{C}) \to \mathrm{Hom}(\mathcal{C})$, such that for $s \in \mathrm{Hom}(X, Y)$ and $t \in \mathrm{Hom}(X', Y')$, $F(s, t) \in \mathrm{Hom}(F(X, X'), F(Y, Y'))$, and

$$\begin{aligned} F(s_1 \circ s_2, t) &= F(s_1, t) \circ F(s_2, t), \\ F(s, t_1 \circ t_2) &= F(s, t_1) \circ F(s, t_2), \\ F(\mathrm{id}_X, \mathrm{id}_{X'}) &= \mathrm{id}_{F(X,X')}. \end{aligned}$$

If $\mathcal{C}$ is a $*$-category, then a bifunctor F is also required to be bilinear and to commute with the $*$-operation. That is, for $s_i \in \mathrm{Hom}(X, X')$, $t_i \in \mathrm{Hom}(Y, Y')$ and $c \in \mathbb{C}$, we have

$$\begin{aligned} F(s_1 + s_2, t) &= F(s_1, t) + F(s_2, t), \\ F(s, t_1 + t_2) &= F(s, t_1) + F(s, t_2), \\ F(cs, t) &= cF(s, t) = F(s, ct), \end{aligned}$$

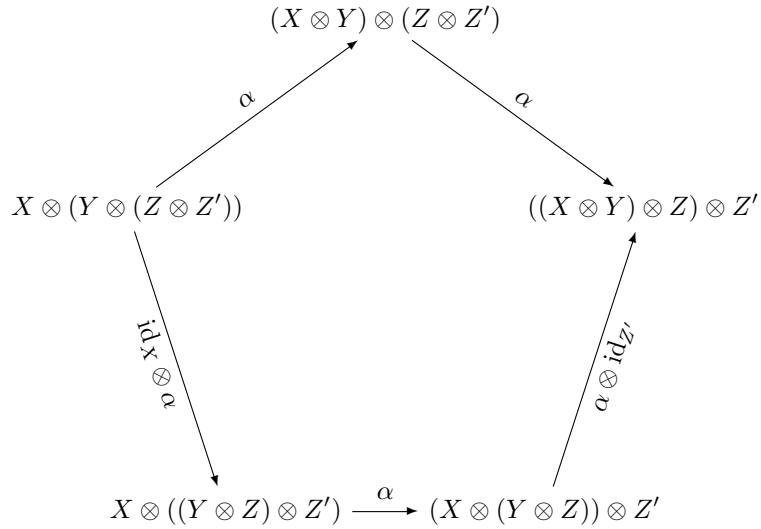
and

$$F(s, t)^* = F(s^*, t^*).$$

DEFINITION 171. Let $\otimes = (\otimes, \times)$ be a bifunctor on the category $\mathcal{C}$, and let $\mathbf{1} \in \mathrm{Obj}(\mathcal{C})$. Then $(\mathcal{C}, \otimes, \mathbf{1})$ is said to be a *tensor category* just in case $\otimes$ is associative up to a natural isomorphisms, and $\mathbf{1}$ is a two sided identity up to natural isomorphisms. The object $\mathbf{1}$ is called the *monoidal unit*. To be precise, to say that $\otimes$ is 'associative up to a natural isomorphisms' means that for each $X, Y, Z \in \mathrm{Obj}(\mathcal{C})$, there is an isomorphism $\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$ that is 'natural' for all X, Y, Z; i.e. if $s : X \to X'$ then

$$((s \otimes \mathrm{id}_Y) \otimes \mathrm{id}_Z) \circ \alpha_{X,Y,Z} = \alpha_{X',Y,Z} \circ (s \otimes (\mathrm{id}_Y \otimes \mathrm{id}_Z)), \tag{15}$$

and similarly for Y and Z. Furthermore, α is required to make the pentagonal diagram commute:

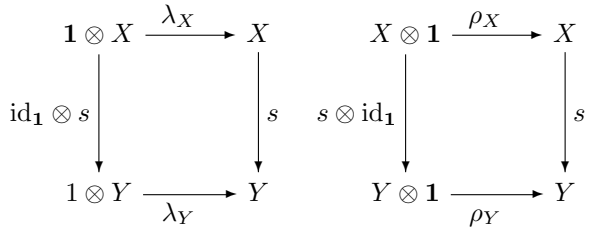


To say that $\mathbf{1} \in \mathcal{C}$ is a two sided identity up to natural isomorphisms means that for each object $X \in \mathrm{Obj}(\mathcal{C})$, there are isomorphisms $\lambda_X \in \mathrm{Hom}(\mathbf{1} \otimes X, X)$ and $\rho_X \in \mathrm{Hom}(X \otimes \mathbf{1}, X)$ such that:

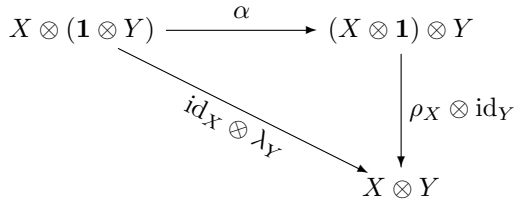
1. λ_X and ρ_X are natural in X; i.e. for any $s : X \to Y$,

$$s \circ \lambda_X = \lambda_Y \circ (\mathrm{id}_{\mathbf{1}} \otimes s), \tag{16}$$
$$s \circ \rho_X = \rho_Y \circ (s \otimes \mathrm{id}_{\mathbf{1}}). \tag{17}$$

In other words, the following two diagrams commute:



2. λ_X and ρ_X make the triangular diagram commute:



If $\mathcal{C}$ is also a $*$-category, there are two further requirements: (a.) the bifunctor $\otimes$ must be compatible with the operations of $+$ and $*$ (as required in the definition of bifunctor), and (b.) the monoidal unit $\mathbf{1}$ must be irreducible, i.e. $\mathrm{End}(\mathbf{1}) = \mathbb{C}\mathrm{id}_{\mathbf{1}}$. For a C^*-category $\mathcal{C}$ we require in addition that $\|s \times t\|_{X\otimes Y, X'\otimes Y'} \leq \|s\|_{X,X'} \cdot \|t\|_{Y,Y'}$.

Mac Lane's coherence theorem shows that we can without danger ignore the natural isomorphisms α, λ, and ρ. That is, we can treat $X\otimes(Y\otimes Z)$ and $(X\otimes Y)\otimes Z$ as the *same* object, and we can treat X, $\mathbf{1} \otimes X$, and $X \otimes \mathbf{1}$ as the same object. To be more precise, we define:

DEFINITION 172. A tensor category $\mathcal{C}$ is said to be *strict* if $\alpha_{X,Y,Z}, \lambda_X, \rho_X$ are identity morphisms for all $X, Y, Z \in \mathrm{Obj}(\mathcal{C})$.

For example, the tensor category $(\mathrm{Vect}, \otimes, \mathbb{C})$ of vector spaces is *not* strict, since e.g. $V \otimes \mathbb{C}$ is not literally the same vector space as V. On the other hand, a commutative monoid M can be thought of as a strict tensor category with one object and with arrows corresponding to elements of M. The coherence theorem can then be formulated as follows.

THEOREM 173 (Coherence Theorem). *Every tensor category is equivalent to a strict tensor category.*

Proof. See [Mac Lane, 1998]. ■

DEFINITION 174. If $\mathcal{C}$ is a tensor category, then we let $\mathcal{C}^{\mathrm{st}}$ denote its strictification.

With these definitions in hand, we proceed now to define a bifunctor on Δ, and to verify that it satisfies all of the relevant properties. Our product $\otimes$ of objects in Δ will be just the composition of endomorphisms.

PROPOSITION 175. *If* $\rho, \sigma \in \mathrm{Obj}(\Delta)$ *then* $\rho\sigma \in \mathrm{Obj}(\Delta)$.

Proof. It is clear that if ρ is localized in O_1 and σ is localized in O_2, then $\rho\sigma$ is localized in any double cone that contains $O_1 \cup O_2$.

To see that $\rho\sigma$ is transportable, let O_3 be an arbitrary double cone. Since ρ and σ are transportable, there are $\rho', \sigma' \in \Delta(O_3)$ and unitaries $U \in \mathrm{Hom}(\rho, \rho')$ and $V \in \mathrm{Hom}(\sigma, \sigma')$. Then $\rho'\sigma'$ is localized in O_3 and $U\rho(V)$ is a unitary in $\mathrm{Hom}(\rho\sigma, \rho'\sigma')$. Therefore, $\rho\sigma$ is transportable. ■

DEFINITION 176. Define $\otimes : \mathrm{Obj}(\Delta) \times \mathrm{Obj}(\Delta) \to \mathrm{Obj}(\Delta)$ by $\rho \otimes \sigma = \rho\sigma$.

The product $\times$ of arrows is slightly more complicated.

PROPOSITION 177. *If $S \in \mathrm{Hom}(\rho, \rho')$ and $T \in \mathrm{Hom}(\sigma, \sigma')$ then $S\rho(T) \in \mathrm{Hom}(\rho\otimes\sigma, \rho' \otimes \sigma')$.*

Proof. Since $S\rho(T) = \rho'(T)S$, it follows that for any $A \in \mathfrak{A}$,

$$\begin{aligned}(S\rho(T))\rho\sigma(A) &= S\rho(T\sigma(A)) = \rho'(T\sigma(A))S = \rho'(\sigma'(A)T)S \\ &= \rho'\sigma'(A)(\rho'(T)S) = \rho'\sigma'(A)(S\rho(T)).\end{aligned}$$

Therefore $S\rho(T) \in \mathrm{Hom}(\rho\sigma, \rho'\sigma')$. ■

DEFINITION 178. Define $\times : \mathrm{Hom}(\Delta) \times \mathrm{Hom}(\Delta) \to \mathrm{Hom}(\Delta)$ by: for $S \in \mathrm{Hom}(\rho, \rho')$ and $T \in \mathrm{Hom}(\sigma, \sigma')$, we set $S \times T = S\rho(T) \in \mathrm{Hom}(\rho \otimes \sigma, \rho' \otimes \sigma')$.

In the remainder of this section, we verify that $(\Delta, \otimes, \iota)$ is a tensor $*$-category.

$\otimes$ is a bifunctor on Δ

PROPOSITION 179. *For $S_1, S_2, T_1, T_2 \in \mathrm{Obj}(\Delta)$, if the source of T_i is the target of S_i (so that $T_i \circ S_i$ is defined) then*

$$(T_1 \times T_2) \circ (S_1 \times S_2) = (T_1 \circ S_1) \times (T_2 \circ S_2).$$

Proof. Straightforward calculation. ■

We must now check that $\times$ is compatible with $*$

PROPOSITION 180. *For all $S, T \in \mathrm{Hom}(\Delta)$,*

$$(S \times T)^* = S^* \times T^*.$$

Proof. Straightforward calculation. ■

ι is a monoidal unit

For each $\rho \in \mathrm{Obj}(\Delta)$, $\iota \otimes \rho$ must be naturally isomorphic to $\rho \otimes \iota$ and to ρ (as expressed in the monoidal unit diagrams). But in the present case, $\iota\otimes\rho = \rho\otimes\iota = \rho$, so this natural isomorphism holds trivially.

Natural associativity of $\otimes$

Next, the product operation $\otimes = (\otimes, \times)$ must be associative up to natural isomorphisms, as expressed by the pentagonal diagram. But this is trivial in the present case, because associativity holds strictly; that is:

PROPOSITION 181. *For all* $\rho_1, \rho_2, \rho_3 \in \mathrm{Obj}(\Delta)$,

$$\rho_1 \otimes (\rho_2 \otimes \rho_3) = (\rho_1 \otimes \rho_2) \otimes \rho_3,$$

and for all $T_1, T_2, T_3 \in \mathrm{Hom}(\Delta)$,

$$T_1 \times (T_2 \times T_3) = (T_1 \times T_2) \times T_3.$$

Proof. The first claim follows trivially from the fact that composition of endomorphisms is associative. The second claim can be verified by a straightforward calculation. ■

LEMMA 182. $(\Delta, \otimes, \iota)$ *is a* C^**-tensor category with the norms inherited from* $\mathfrak{A}$.

Proof. We must verify that $\mathrm{Hom}(\rho, \sigma)$ is closed in the norm on $\mathfrak{A}$. But this follows immediately from the fact that

$$\mathrm{Hom}(\rho, \sigma) = \{T \in \mathfrak{A} : T\rho(A) = \sigma(A)T,\ \forall A \in \mathfrak{A}\}.$$

It's clear that $\|s \circ t\| \leq \|s\|\|t\|$. Furthermore,

$$\|S \times T\| = \|S\rho(T)\| \leq \|S\| \cdot \|\rho(T)\| \leq \|S\| \cdot \|T\|.$$

■

To this point we have shown that (i): Δ is a C^*-category, and (ii): Δ is a tensor $*$-category. The following five Subsections are not linearly ordered. Subection 8.1 shows how to define the canonical braiding $\varepsilon_{\rho_1,\rho_2}$ on $(\Delta, \otimes, \iota)$ such that it is a 'braided' tensor $*$-category. Then in Subsection 8.2 we make good our claims about the motivation for studying the category Δ: we prove that there is a functoral correspondence between Δ and the category of representations that satisfy the DHR selection criterion. We then pick up some technical information about tensor $*$-categories that is essential for the physical interpretation of the corresponding representations. In Subsection 8.3 we see how to define a notion of the 'dimension' of an object in a tensor $*$-category, and we define the notion of 'conjugate' objects. In Subsection 8.4 we take a detour to talk about the relation of spacetime symmetries to the DHR representations. Finally, in Subsection 8.5 we give the intrinsic statistical classification of objects of Δ_f that corresponds to the intuitive distinction between Bosons and Fermions, or Bose fields and Fermi fields.

8.1 Δ is a braided tensor $*$-category

In this Subsection we define the canonical braiding on Δ; this gives us a grasp on what happens if we change the order in products, say $\rho \otimes \sigma$ versus $\sigma \otimes \rho$. We will also see that there is a remarkable connection between spacetime dimension and the properties of this braiding: if the spacetime has three or more dimensions, the braiding is a symmetry. We first recall the pertinent definitions.

DEFINITION 183. If $(\mathcal{C}, \otimes, \mathbf{1})$ is a tensor category then a *braiding* on $\mathcal{C}$ is a family of isomorphisms

$$\big\{c_{X,Y} \in \mathrm{Hom}(X \otimes Y, Y \otimes X) : X, Y \in \mathrm{Obj}(\mathcal{C})\big\},$$

satisfying the following two conditions:

1. $c_{X,Y}$ is natural in X and Y; i.e. for any $f \in \mathrm{Hom}(X, X')$ and $g \in \mathrm{Hom}(Y, Y')$,

   (18) $(g \times f) \circ c_{X,Y} = c_{X',Y'} \circ (f \times g)$.

2. $c_{X,Y}$ makes the following two hexagonal diagrams commute:

$$\begin{array}{ccc}
(X \otimes Y) \otimes Z & \xrightarrow{c_{X\otimes Y,Z}} & Z \otimes (X \otimes Y) \\
\Big\downarrow{\scriptstyle \alpha^{-1}} & & \Big\downarrow{\scriptstyle \alpha} \\
X \otimes (Y \otimes Z) & & (Z \otimes X) \otimes Y \\
\Big\downarrow{\scriptstyle \mathrm{id}_X \otimes c_{Y,Z}} & & \Big\downarrow{\scriptstyle c_{Z,X} \otimes \mathrm{id}_Y} \\
X \otimes (Z \otimes Y) & \xrightarrow[\alpha]{} & (X \otimes Z) \otimes Y
\end{array}$$

$$\begin{array}{ccc}
X \otimes (Y \otimes Z) & \xrightarrow{c_{X,Y\otimes Z}} & (Y \otimes Z) \otimes X \\
\Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \alpha^{-1}} \\
(X \otimes Y) \otimes Z & & Y \otimes (Z \otimes X) \\
\Big\downarrow{\scriptstyle c_{X,Y} \otimes \mathrm{id}_Z} & & \Big\downarrow{\scriptstyle \mathrm{id}_Y \otimes c_{Z,X}} \\
(Y \otimes X) \otimes Z & \xrightarrow[\alpha^{-1}]{} & Y \otimes (X \otimes Z)
\end{array}$$

That is, suppressing the associativity isomorphisms, $c_{X\otimes Y,Z}$ is expressed in terms of $c_{X,Y}$ and $c_{X,Z}$ as:

$$c_{X\otimes Y,Z} \;=\; (\mathrm{id}_Y \otimes c_{Z,X})^{-1} \circ (\mathrm{id}_X \otimes c_{Y,Z}),$$

and $c_{X,Y\otimes Z}$ is expressed in terms of $c_{X,Y}$ and $c_{Z,X}$ as:

$$c_{X,Y\otimes Z} \;=\; (\mathrm{id}_Y \otimes c_{Z,X})^{-1} \circ (c_{X,Y} \otimes \mathrm{id}_Z).$$

DEFINITION 184. A braiding $c_{X,Y}$ is called a *symmetry* if $(c_{X,Y})^{-1} = c_{Y,X}$ for all $X, Y \in \mathrm{Obj}(\mathcal{C})$.

DEFINITION 185. A tensor category with a privileged braiding (symmetry) is called a *braided (symmetric) tensor category.*

In order to find our braiding on Δ, we will need the following technical lemma.

LEMMA 186. *If $\rho \in \Delta(O_1)$ and $\sigma \in \Delta(O_2)$ where O_1 and O_2 are spacelike separated, then $\rho\sigma = \sigma\rho$.*

Proof. Since the union of $\{\mathfrak{A}(O) : O_1 \cup O_2 \subseteq O\}$ is dense in $\mathfrak{A}$, it suffices to show that $\rho\sigma(A) = \sigma\rho(A)$ whenever $A \in \mathfrak{A}(O)$ with $O_1 \cup O_2 \subseteq O$. Choose O_3, O_4 that are spacelike to O and such that $O_1 \cup O_3$ is spacelike to $O_2 \cup O_4$. (This may always be done, even in two dimensional spacetime.) Since ρ, σ are transportable, there are ρ', σ' localized in O_3 and O_4 respectively and unitary operators $U_1 \in \mathrm{Hom}(\rho, \rho')$ and $U_2 \in \mathrm{Hom}(\sigma, \sigma')$. Then

$$\sigma(A) = U_2\sigma'(A)U_2^* = U_2^* A U_2.$$

Furthermore, $U_2 \in \mathfrak{A}(O_1')$ by duality in the vacuum sector. Hence $\rho(U_2) = U_2$, and

$$\rho\sigma(A) = U_2 U_1 A U_1^* U_2^*.$$

Since $U_2U_1 = U_1U_2$, it follows that $\rho\sigma(A) = \sigma\rho(A)$. ■

We will not be able to define the braiding $\varepsilon_{\rho_1,\rho_2}$ in one shot. Rather, we first define arrows

$$\varepsilon_{\rho_1,\rho_2}(U_1, U_2) \in \mathrm{Hom}(\rho_1 \otimes \rho_2, \rho_2 \otimes \rho_1),$$

that depend on the choice of "spectator morphisms" $\tilde{\rho}_1, \tilde{\rho}_2$, and unitary intertwiners $U_i \in \mathrm{Hom}(\rho_i, \tilde{\rho}_i)$. We will then show that this definition is independent of the spectator morphisms and unitary intertwiners. (But, interestingly, when spacetime is two dimensional, the definition depends on the choice of a spatial orientation.)

DEFINITION 187. Suppose that $\rho_1 \in \Delta(O_1)$ and $\rho_2 \in \Delta(O_2)$. Let $\tilde{O}_1$ and $\tilde{O}_2$ be spacelike separated double cones. Since ρ_1 and ρ_2 are transportable, there are $\tilde{\rho}_i \in \Delta(\tilde{O}_i)$ and unitary operators $U_i \in \mathrm{Hom}(\rho_i, \tilde{\rho}_i)$. Thus $U_1 \times U_2 \in \mathrm{Hom}(\rho_1 \otimes \rho_2, \tilde{\rho}_1 \otimes \tilde{\rho}_2)$, and $U_2^* \times U_1^* \in \mathrm{Hom}(\tilde{\rho}_2 \otimes \tilde{\rho}_1, \rho_2 \otimes \rho_1)$. Since $\tilde{O}_1$ is spacelike to $\tilde{O}_2$,

Lemma 186 entails that $\tilde{\rho}_1 \otimes \tilde{\rho}_2 = \tilde{\rho}_2 \otimes \tilde{\rho}_1$. Thus, we may define $\varepsilon_{\rho_1,\rho_2}(U_1, U_2) \in \mathrm{Hom}(\rho_1 \otimes \rho_2, \rho_2 \otimes \rho_1)$ by

$$(19)\quad \varepsilon_{\rho_1,\rho_2}(U_1, U_2) := (U_2 \times U_1)^* \circ (U_1 \times U_2) = \rho_2(U_1^*)U_2^* U_1 \rho_1(U_2).$$

NOTE 188. Since endomorphisms preserve unitarity, $\varepsilon_{\rho_1,\rho_2}(U_1, U_2)$ is unitary.

To show that $\varepsilon_{\rho_1,\rho_2}(U_1, U_2)$ is independent of U_1 and U_2, we need the following Lemma, which shows that $\varepsilon_{\rho_1,\rho_2}(U_1, U_2)$ does not change under certain "perturbations" of U_1 and U_2.

LEMMA 189. *For $i = 1, 2$, let $\rho_i \in \Delta(O_i)$, let $\tilde{O}_1$ and $\tilde{O}_2$ be spacelike separated, let $\tilde{\rho}_i \in \Delta(\tilde{O}_i)$, and let $U_i \in \mathrm{Hom}(\rho_i, \tilde{\rho}_i)$. Then $\varepsilon_{\rho_1,\rho_2}(U_1, U_2)$ depends only on neighborhoods of U_1, U_2 in the following sense: if W_1, W_2 are unitaries such that $W_1 \in \mathfrak{A}(\tilde{O}_2')$, $W_2 \in \mathfrak{A}(\tilde{O}_1')$, and $W_1 W_2 = W_2 W_1$, then*

$$\varepsilon_{\rho_1,\rho_2}(W_1 U_1, W_2 U_2) = \varepsilon_{\rho_1,\rho_2}(U_1, U_2).$$

Proof. We must show that

$$(20)\quad (W_2 U_2 \times W_1 U_1)^* \circ (W_1 U_1 \times W_2 U_2) = (U_2^* \times U_1^*) \circ (U_1 \times U_2).$$

For any two unitary operators, $W_1, W_2 \in \mathfrak{A}$, we have

$$W_i U_i \times W_j U_j \quad = \quad W_i U_i \rho_i(W_j) \rho_i(U_j) = W_i \rho_i'(W_j)(U_i \times U_j).$$

Since $W_1 \in \mathfrak{A}(\tilde{O}_2')$ and $\tilde{\rho}_2$ is localized in $\tilde{O}_2$, $\tilde{\rho}_2(W_1) = W_1$; and similarly, $\tilde{\rho}_1(W_2) = W_2$. Hence, the left hand side of Eqn. 20 becomes

$$\begin{aligned}[(U_2 \times U_1)^* \tilde{\rho}_2(W_1^*) W_2^*]\,[W_1 \tilde{\rho}_1(W_2)(U_1 \times U_2)] &= (U_2 \times U_1)^* W_1^* W_2^* W_1 W_2 (U_1 \times U_2) \\ &= (U_2 \times U_1)^* (U_1 \times U_2),\end{aligned}$$

where we used $W_1 W_2 = W_2 W_1$ for the second equality. ■

LEMMA 190. *Let $\rho_i \in \Delta(O_i)$, $i = 1, 2$, and let $T \in \mathrm{Hom}(\rho_1, \rho_2)$. Then $T \in \mathfrak{A}(O)$ for any double cone containing $O_1 \cup O_2$.*

Proof. Let O be a double cone containing $O_1 \cup O_2$, and let $A \in \mathfrak{A}(O')$. Then $\rho_1(A) = \rho_2(A) = A$, and so

$$TA = T\rho_1(A) = \rho_2(A)T = AT.$$

Hence $T \in \mathfrak{A}(O')'$, and by duality in the vacuum sector, $T \in \mathfrak{A}(O)$. ■

Now we can show that $\varepsilon_{\rho_1,\rho_2}(U_1, U_2)$ depends only on the localization regions of the spectator morphisms.

PROPOSITION 191. *$\varepsilon_{\rho_1,\rho_2}(U_1, U_2)$ is definable in terms of ρ_1, ρ_2, and the regions $\tilde{O}_1, \tilde{O}_2$; and does not change if the latter are replaced by double cones $\tilde{\tilde{O}}_1, \tilde{\tilde{O}}_2$ such that $\tilde{O}_1 \subseteq \tilde{\tilde{O}}_1$ and $\tilde{O}_2 \subseteq \tilde{\tilde{O}}_2$.*

Proof. (1.) We show first that for a given pair $(\tilde{O}_1, \tilde{O}_2)$ of spacelike separated double cones, the definition $\varepsilon_{\rho_1,\rho_2}(U_1, U_2)$ is independent of spectator morphisms $(\tilde{\rho}_1, \tilde{\rho}_2)$, and unitary intertwiners (U_1, U_2). So, suppose that $\tilde{\tilde{\rho}}_i \in \Delta(\tilde{O}_i)$, and $U_i' \in \mathrm{Hom}(\rho_i, \tilde{\tilde{\rho}}_i)$. Let $W_i = U_i' U_i^* \in \mathrm{Hom}(\tilde{\rho}_i, \tilde{\tilde{\rho}}_i)$, so that $U_i' = W_i U_i$. Since W_i has left and right support in $\tilde{O}_i$, $W_i \in \mathfrak{A}(\tilde{O}_i) \subseteq \mathfrak{A}(\tilde{O}_j)'$. Thus, $W_1 W_2 = W_2 W_1$, and the hypotheses of Lemma 189 are satisfied. Therefore $\varepsilon_{\rho_1,\rho_2}(U_1, U_2) = \varepsilon_{\rho_1,\rho_2}(U_1', U_2')$.

(2.) Now let $\tilde{\tilde{O}}_1$ and $\tilde{\tilde{O}}_2$ be double cones such that $\tilde{\tilde{O}}_1 \perp \tilde{\tilde{O}}_2$, and $\tilde{O}_i \subseteq \tilde{\tilde{O}}_i$, for $i = 1, 2$. Choose $\tilde{\tilde{\rho}}_i \in \Delta(\tilde{\tilde{O}}_i)$, and unitaries $U_i' \in \mathrm{Hom}(\rho_i, \tilde{\tilde{\rho}}_i)$. But we also have $\tilde{\rho}_i \in \Delta(\tilde{O}_i) \subseteq \Delta(\tilde{\tilde{O}}_i)$. And the first part of the proof shows that for fixed support regions $(\tilde{\tilde{O}}_1, \tilde{\tilde{O}}_2)$, the definition of $\varepsilon_{\rho_1,\rho_2}$ is independent of the choice of spectator morphisms and unitary intertwiners. Therefore $\varepsilon_{\rho_1,\rho_2}(U_1, U_2) = \varepsilon_{\rho_1,\rho_2}(U_1', U_2')$. ■

NOTE 192. We can always choose the spectator morphisms to be localized in strictly spacelike separated regions. Indeed, given $\tilde{O}_1$ and $\tilde{O}_2$ that are spacelike separated, choose $\tilde{\tilde{O}}_1$ such that $(\tilde{\tilde{O}}_1)^- \subseteq \tilde{O}_1$. But then the Lemma implies (by switching $\tilde{\tilde{O}}_1$ with $\tilde{O}_1$, and setting $\tilde{\tilde{O}}_2 = \tilde{O}_2$) that we get the same definition from using $\tilde{\tilde{O}}_1$ or $\tilde{O}_1$. More generally, since ρ_1 is transportable, the regions $\tilde{O}_i$ can be chosen arbitrarily small.

NOTE 193. The previous note shows that a definition of $\varepsilon_{\rho_1,\rho_2}(U_1, U_2)$ is always equivalent to a definition using spectator morphisms localized in strictly spacelike separated regions. That is, there is a neighborhood N of zero such that $\tilde{O}_1 + x \subseteq \tilde{O}_2'$ for all $x \in N$. Again, since $\tilde{O}_1$ and $\tilde{O}_1 + x$ are contained in a double cone $\tilde{\tilde{O}}_1 \subseteq \tilde{O}_2'$, the previous Lemma (applied twice) entails that the pairs $(\tilde{O}_1, \tilde{O}_2)$ and $(\tilde{O}_1 + x, \tilde{O}_2)$ yield the same definition of $\varepsilon_{\rho_1,\rho_2}$.

By also shrinking $\tilde{O}_2$ if necessary, and repeating the above construction, we see that for any vector x, the pairs $(\tilde{O}_1 + x, \tilde{O}_2 + x)$ and $(\tilde{O}_1, \tilde{O}_2)$ yield the same definition of $\varepsilon_{\rho_1,\rho_2}$.

NOTE 194. In what follows, by "one dimensional spacetime", we mean one dimensional space with zero dimensional time. In this case, a double cone is just an open interval, and "spacelike separated" means disjoint.

PROPOSITION 195. *For spacetime of dimension two or less, $\varepsilon_{\rho_1,\rho_2}(U_1, U_2)$ is definable in terms of ρ_1, ρ_2 and the spatial orientation of $\tilde{O}_1$ with respect to $\tilde{O}_2$. That is, $\varepsilon_{\rho_1,\rho_2}(U_1, U_2)$ is independent of the choice of $(\tilde{O}_1, \tilde{O}_2)$, subject to the constraint of having the same spatial orientation.*

Proof. Let $\tilde{O}_i, \tilde{\tilde{O}}_i$ be given such that $\tilde{O}_1 \perp \tilde{O}_2$, $\tilde{\tilde{O}}_1 \perp \tilde{\tilde{O}}_2$, and $\tilde{O}_1$ is oriented with respect to $\tilde{O}_2$ as $\tilde{\tilde{O}}_1$ is with respect to $\tilde{\tilde{O}}_2$. Recall that translations of $(\tilde{O}_1, \tilde{O}_2)$ do not change $\varepsilon_{\rho_1,\rho_2}(U_1, U_2)$; nor does replacement of $\tilde{O}_i$ with a double cone either containing it or contained in it, and spacelike to $\tilde{O}_j$. But $(\tilde{O}_1, \tilde{O}_2)$ can be replaced by $(\tilde{\tilde{O}}_1, \tilde{\tilde{O}}_2)$ in a series of such moves. ■

DEFINITION 196. For spacetime of two dimensions or less, fix a spatial orientation, and use $O_1 < O_2$ to indicate that O_1 is to the left of O_2.

LEMMA 197. *For spacetimes of dimension two or less, if the spatial orientation of $\tilde{O}_1$ with respect to $\tilde{O}_2$ is the opposite of the spatial orientation of $\tilde{\tilde{O}}_1$ with respect to $\tilde{\tilde{O}}_2$, then*

$$\varepsilon_{\rho_1,\rho_2}(U_1,U_2) = [\varepsilon_{\rho_2,\rho_1}(U_2',U_1')]^*.$$

Proof. For defining $\varepsilon_{\rho_1,\rho_2}(U_1,U_2)$, we can choose $\tilde{O}_1 = O_1$, $\tilde{O}_2 < O_1$, $\tilde{\rho}_1 = \rho_1$, and $U_1 = I_{\rho_1} = I \in \mathrm{Hom}(\rho_1,\rho_1)$. In this case, the definition simplifies to

$$\varepsilon_{\rho_1,\rho_2}(I,U_2) = U_2^*\rho_1(U_2), \qquad (\tilde{O}_2 < \tilde{O}_1).$$

Using the same spectator morphisms, we have

$$\varepsilon_{\rho_2,\rho_1}(U_2,I) = \rho_1(U_2^*)U_2, \qquad (\tilde{O}_2 < \tilde{O}_1).$$

This latter expression uses the opposite spatial orientation. By the definability of $\varepsilon_{\rho_1,\rho_2}(U_1,U_2)$ in terms of spatial orientation (Prop. 195), we see that $\varepsilon_{\rho_1,\rho_2}(U_1, U_2) = [\varepsilon_{\rho_2,\rho_1}(U_2',U_1')]^*$ when the opposite spatial orientations are used for the two definitions. ■

DEFINITION 198 (The Canonical Braiding on Δ). For spacetimes of dimension two or less, we implement the convention that $\varepsilon_{\rho_1,\rho_2} = \varepsilon_{\rho_1,\rho_2}(U_1,U_2)$ with $\tilde{O}_2 < \tilde{O}_1$. The previous Lemma shows that if we define $\overline{\varepsilon}_{\rho_1,\rho_2}$ with the opposite convention, then $\overline{\varepsilon}_{\rho_1,\rho_2} = (\varepsilon_{\rho_2,\rho_1})^*$. For spacetimes of dimension three or more, we define $\varepsilon_{\rho_1,\rho_2} = \varepsilon_{\rho_1,\rho_2}(U_1,U_2)$ with $\tilde{O}_1$ and $\tilde{O}_2$ spacelike separated.

We now verify that $\varepsilon_{\rho_1,\rho_2}$ is a braiding on $(\Delta,\otimes,\iota)$.

PROPOSITION 199. *$\varepsilon_{\rho,\sigma}$ is a braiding on the DHR category $(\Delta,\otimes,\iota)$.*

Proof. (1) We first show that $\varepsilon_{\rho,\sigma}$ is natural in ρ and σ. For this it suffices to show that if $T \in \mathrm{Hom}(\rho,\rho')$ then

$$(21) \quad (I_\sigma \times T)\circ\varepsilon_{\rho,\sigma} = \varepsilon_{\rho',\sigma}\circ(T\times I_\sigma),$$
$$(22) \quad \varepsilon_{\sigma,\rho}\circ(I_\sigma\times T^*) = (T^*\times I_\sigma)\circ\varepsilon_{\sigma,\rho'}.$$

Let O_1,O_2,O_3 be double cones such that $\rho\in\Delta(O_1)$, $\sigma\in\Delta(O_2)$, and $\rho'\in\Delta(O_3)$. Choose a double cone O_4 that is spacelike to O_i, $i=1,2,3$; and if the spacetime dimension is less than three, choose O_4 to the left of all three. Choose $\sigma'\in\Delta(O_4)$, and $U\in\mathrm{Hom}(\sigma,\sigma')$. Then $\varepsilon(\rho,\sigma) = U^*\rho(U)$ and $\varepsilon(\rho',\sigma) = U^*\rho'(U)$. Since $T\in\mathrm{Hom}(\rho,\rho')\subseteq\mathfrak{A}(O_4')$, it follows that $\sigma'(T) = T$. Thus,

$$\sigma(T)U^*\rho(U) = U^*\sigma'(T)\rho(U) = U^*T\rho(U) = U^*\rho'(U)T.$$

This establishes Eqn. 21. The second equation can be established by a similar calculation.

(2) Now we show that $\varepsilon_{\rho,\sigma}$ makes the hexagonal diagrams commute. Since Δ is strict monoidal, we can omit the associativity isomorphisms. That is, it suffices to show that

$$(23)\quad \varepsilon_{\rho\otimes\sigma,\tau} = \big(\varepsilon_{\rho,\tau}\times I_\sigma\big)\circ\big(I_\rho\times\varepsilon_{\sigma,\tau}\big),$$
$$(24)\quad \varepsilon_{\rho,\sigma\otimes\tau} = \big(I_\sigma\times\varepsilon_{\rho,\tau}\big)\circ\big(\varepsilon_{\rho,\sigma}\times I_\tau\big).$$

Choose $\tau'\in\Delta$ such τ' is supported in a region that is spacelike to the support regions of ρ,σ,τ; for spacetimes of one or two dimensions, choose the support region of τ' to the left. Let $U\in\mathrm{Hom}(\tau,\tau')$. Then $\varepsilon_{\rho,\tau}=U^*\rho(U)$, $\varepsilon_{\sigma,\tau}=U^*\sigma(U)$, and $\varepsilon_{\rho\otimes\sigma,\tau}=U^*\rho\sigma(U)$. Furthermore,

$$U^*\rho(U)\rho[U^*\sigma(U)] = U^*\rho[UU^*\sigma(U)] = U^*\rho\sigma(U),$$

establishing Eqn. 23. The second equation is proven analogously. ■

PROPOSITION 200. *For spacetimes of dimension two or less, $\varepsilon_{\rho_1,\rho_2}$ is the unique braiding on $(\Delta,\otimes,\iota)$ such that $\varepsilon_{\rho_1,\rho_2}=I$ when $\rho_i\in\Delta(O_i)$ with $O_2<O_1$. For spacetimes of dimension three or more, $\varepsilon_{\rho_1,\rho_2}$ is the unique braiding on $(\Delta,\otimes,\iota)$ such that $\varepsilon_{\rho_1,\rho_2}=I$ when $\rho_i\in\Delta(O_i)$ with O_1 and O_2 spacelike separated.*

Proof. Choosing $\tilde{O}_2$ to the left of O_1 we can set $\varepsilon_{\rho_1,\rho_2}=\varepsilon_{\rho_1,\rho_2}(I,U_2)=U_2^*\rho_1(U_2)$ where $U_2\in\mathrm{Hom}(\rho_2,\rho_2')$. Now let c_{σ_1,σ_2} be another braiding on $(\Delta,\otimes,\iota)$ such that $c_{\sigma_1,\sigma_2}=I$ whenever σ_1 is localized in a region to the right of the localization region of σ_2. Then since c_{ρ_1,ρ_2} is natural in ρ_1 and ρ_2 and $c_{\rho_1,\rho_2'}=I$,

$$c_{\rho_1,\rho_2}=(U_2^*\times I_{\rho_1})\circ c_{\rho_1,\rho_2'}\circ(I_{\rho_1}\times U_2)=U_2^*\rho_1(U_2)=\varepsilon_{\rho_1,\rho_2}.$$

The proof for the higher dimensional case is structurally identical. ■

PROPOSITION 201. *For spacetime of dimension three or more, $\varepsilon_{\rho_1,\rho_2}=(\varepsilon_{\rho_2,\rho_1})^{-1}$, hence $\varepsilon_{\rho_1,\rho_2}$ is a symmetry on $(\Delta,\otimes,\iota)$.*

Proof. We first show that $\varepsilon_{\rho_1,\rho_2}(U_1,U_2)$ is independent of the choice $(\tilde{O}_1,\tilde{O}_2)$ of supports for the spectator morphisms. (Compare the proof of Prop. 195.) Since the spacetime has at least three dimensions, there is a sequence of double cones O_i, $i=1,\ldots,n$, such that: $O_1=\tilde{O}_2$, for each i, $O_i\cup O_{i+1}$ is contained in a double cone spacelike to $\tilde{O}_1$, and O_n has the opposite spatial orientation to $\tilde{O}_1$ as did $\tilde{O}_2$. Applying Prop. 191 repeatedly, we conclude that $\varepsilon_{\rho_1,\rho_2}(U_1,U_2)=\varepsilon_{\rho_1,\rho_2}(U_1,U_2')$, where $\tilde{\tilde{\rho}}_2\in\Delta(O_n)$ and $U_2'\in\mathrm{Hom}(\rho_2,\tilde{\tilde{\rho}}_2)$. Thus, $\varepsilon_{\rho_1,\rho_2}$ does not depend on the relative spatial orientation of $\tilde{O}_1$ and $\tilde{O}_2$. Prop. 195 shows that $\varepsilon_{\rho_1,\rho_2}(U_1,U_2)$ can depend on $(\tilde{O}_1,\tilde{O}_2)$ only through their relative spatial orientation. Therefore, $\varepsilon_{\rho_1,\rho_2}(U_1,U_2)$ is independent of $(\tilde{O}_1,\tilde{O}_2)$.

We can choose $\tilde{O}_1 = O_1$, $\tilde{O}_2 \perp O_1$, $\tilde{\rho}_1 = \rho_1$, and $U_1 = I_{\rho_1} = I \in \mathrm{Hom}(\rho_1, \rho_1)$ so that

$$\varepsilon_{\rho_1,\rho_2} = U^* \rho_1(U_2).$$

But given the independence of $\varepsilon_{\rho_1,\rho_2}$ from the orientation of $(\tilde{O}_1, \tilde{O}_2)$, we also have

$$\varepsilon_{\rho_2,\rho_1} = \rho_1(U_2)^* U_2 = (\varepsilon_{\rho_1,\rho_2})^*.$$

Since $\varepsilon_{\rho_1,\rho_2}$ is unitary, $\varepsilon_{\rho_1,\rho_2} = (\varepsilon_{\rho_2,\rho_1})^{-1}$. ■

NOTE 202. The preceding Proposition is the first place where we invoked the dimension of the underlying spacetime. We will be clear when subsequent results depend on assumptions about dimension.

DEFINITION 203. Let $\varepsilon_\rho := \varepsilon(\rho, \rho) \in \mathrm{End}(\rho \otimes \rho)$.

8.2 Relation between localized endomorphisms and representations

While the categories Δ and Δ_f defined in this section have very remarkable properties, their physical and philosophical relevance is certainly not obvious. We therefore relate the category Δ to a certain category of representations of the net $\mathfrak{A}$:

DEFINITION 204. Let $O \mapsto \mathfrak{A}(O)$ be a net of observables and $\pi_0 : \mathfrak{A} \to B(H_0)$ a vacuum representation. Then a DHR-representation (w.r.t. the vacuum representation π_0) is a $*$-representation $\pi : \mathfrak{A} \to B(H)$ such that $\pi|\mathfrak{A}(O') \cong \pi_0|\mathfrak{A}(O')$ for any double cone O. I.e., upon restriction to $\mathfrak{A}(O')$, the representations π and π_0 are unitarily equivalent. The category whose objects are DHR-representations of $\mathfrak{A}$ with bounded intertwining operators is denoted by $DHR(\mathfrak{A})$. It clearly is a C^*-category.

DEFINITION 205. Let $\mathfrak{A}$ be a net that is Poincaré covariant w.r.t. the positive energy representation $U_0 : \mathcal{P} \to \mathcal{U}(H_0)$. A representation (H, π) of $\mathfrak{A}$ is called covariant (with positive energy) if it is equipped with a strongly continuous unitary representation $U_\pi : \widehat{\mathcal{P}} \to \mathcal{U}(H)$ (with $\mathrm{spec} P^\mu \subset V_+$) of the universal covering of the Poincaré group such that $\mathrm{Ad}\, U_\pi(h) \circ \pi = \pi \circ \alpha_h$ for all $h \in \widehat{\mathcal{P}}$, where, omitting the covering map $\widehat{P} \to \mathcal{P}$ from the notation, $\alpha_h = \mathrm{Ad}\, U_0(h)$.

Note that the definition implies that the representation space H of a DHR representation must have the same dimension as the vacuum Hilbert space H_0.

PROPOSITION 206. *There is a functor $F : \Delta \to DHR(\mathfrak{A})$ of C^*-categories such that $F(\rho) = \pi_0 \circ \rho$ for the objects and $F(s) = \pi_0(s)$ for $s \in \mathrm{Hom}_\Delta(\rho, \sigma)$ for morphisms. This functor is an equivalence.*

Proof. We first note that these definitions make sense: $\rho \in \mathrm{Obj}(\Delta)$ maps $\mathfrak{A}$ into itself and can therefore be composed with the representation π_0, defining a new

representation. Furthermore, if S is an arrow in Δ, then Lemma 155 gives $S \in \mathfrak{A}$, thus $F(S) = \pi_0(S)$ makes sense. With $S \in \mathrm{Hom}_\Delta(\rho, \rho')$ we have

$$F(S)F(\rho)(A) = \pi_0(S)\pi_0(\rho(A)) = \pi_0(S\rho(A)) = \\ \pi_0(\rho'(A)S) = F(\rho')(A)F(S) \quad \forall A \in \mathfrak{A},$$

thus $F(S) \in \mathrm{Hom}(F(\rho), F(\rho'))$. Since id_ρ is the unit of $\mathfrak{A}$ we have $F(\mathrm{id}_\rho) = I_{H_0} = \mathrm{id}_{F(\rho)}$. The property $F(S \circ T) = F(S) \circ F(T)$ is obvious. Since π_0 is faithful, F is faithful. We must show that the representation $F(\rho) = \pi_0 \circ \rho$ satisfies the DHR criterion. Since $\rho \in \Delta$ is transportable, for every double cone O there exist $\rho_O \in \Delta$ localized in O and a unitary $U_O : \rho \to \rho_O$. Since ρ_O is localized in O, the representation $F(\rho_O) = \pi_0 \circ \rho_O$ coincides with π_0 on $\mathfrak{A}(O')$. Since $F(U_O) : F(\rho) \to F(\rho_O)$ is unitary, we have

$$F(\rho)|\mathfrak{A}(O') \cong F(\rho')|\mathfrak{A}(O') = \pi_0|\mathfrak{A}(O'),$$

implying $F(\rho) = \pi_0 \circ \rho \in DHR(\mathfrak{A})$. Now let $\rho, \rho' \in \mathrm{Obj}(\Delta)$ and $\tilde{S} \in \mathrm{Hom}(F(\rho), F(\rho'))$. If O is a double cone containing the localization regions of ρ, ρ',

$$\tilde{S}\pi_0(A) = \tilde{S}\pi_0(\rho(A)) = \tilde{S}F(\rho)(A) = F(\rho')(A)S = \pi_0(\rho'(A))\tilde{S} = \pi_0(A)\tilde{S}$$

for all $A \in \mathfrak{A}(O')$. Therefore, by Haag duality for π_0, $\tilde{S} \in \pi_0(\mathfrak{A}(O'))' = \pi_0(\mathfrak{A}(O))$. Thus there exists $S \in \mathrm{Hom}_\Delta(\rho, \rho')$ such that $\tilde{S} = F(S)$. This proves that the functor F is full. Finally, let $\pi \in DHR(\mathfrak{A})$ be a DHR representation on a Hilbert space H. Choose any double cone O. Then the DHR criterion implies the existence of a unitary $U : H \to H_0$ such that $U\pi(A) = \pi_0(A)U$ for all $A \in \mathfrak{A}(O')$. Define a new representation π' of $\mathfrak{A}$ on the vacuum Hilbert space H_0 by $\pi'(\cdot) = U\pi(\cdot)U^*$. By the very definition, we have $\pi'(A) = \pi_0(A)$ for all $A \in \mathfrak{A}(O')$. If now $\widehat{O}$ is any double cone containing O, and $A \in \mathfrak{A}(\widehat{O}')$ and $B \in \mathfrak{A}(\widehat{O})$ then

$$\pi'(B)\pi_0(A) = \pi'(BA) = \pi'(AB) = \pi'(A)\pi'(B) = \pi_0(A)\pi'(B),$$

implying $\pi'(\mathfrak{A}(\widehat{O})) \subset \pi_0(\mathfrak{A}(\widehat{O}'))' = \pi_0(\mathfrak{A}(O))$ by Haag duality for π_0. Thus π' maps the quasilocal algebra $\mathfrak{A}$ into $\pi_0(\mathfrak{A})$. Since π_0 is injective, we can define an endomorphism ρ of $\mathfrak{A}$ by $\rho = \pi_0^{-1} \circ \pi'$. By construction, ρ is localized in O, and we have $\pi' = \pi_0 \circ \rho = F(\rho)$. Repeating the argument with a different double cone $\tilde{O}$, we see that ρ is transportable, thus $\rho \in \Delta$. Since $\pi \cong \pi' = F(\rho)$, we have proven that every DHR representation is unitarily equivalent to one of the form $F(\rho)$ where $\rho \in \Delta$. Thus the functor F is essentially surjective, and therefore, cf. Appendix A an equivalence of categories. ■

NOTE 207. The significance of Proposition 206 is twofold. On the one hand, it provides an interpretation of the category Δ in terms of a class of representations of $\mathfrak{A}$. If one accepts for a moment that the category $DHR(\mathfrak{A})$ is worth studying, the above equivalence is a powerful tool. Namely, it permits us to pull the symmetric monoidal structure on Δ over to $DHR(\mathfrak{A})$ – which as defined initially is just a category – such as to make the functor $F : \Delta \to DHR(\mathfrak{A})$ an equivalence of

symmetric tensor C^*-categories. But once this is understood, it is much more convenient to work just with the category Δ rather than with $DHR(\mathfrak{A})$, since the tensor structure on $DHR(\mathfrak{A})$ will not be strict.

As to the physical motivation of the DHR condition, we give three arguments:

1. By an increasing sequence of double cones we mean a sequence $O_1 \subset O_2 \subset \cdots$ of double cones such that $\cup_i O_i = \mathbb{R}^d$ (typically $d = 4$). In the appendix of [Doplicher *et al.*, 1971], the following result (the converse of which is trivial) is proven:

   THEOREM 208. *Let ω be a state on $\mathfrak{A}$ such that*

   $$\lim_{n\to\infty} \|(\omega - \omega_0)|\mathfrak{A}(O_n')\| = 0$$

   and the GNS-representation π_ω associated with ω satisfies property B. Then there is a double cone O such that $\pi_\omega|\mathfrak{A}(O') \cong \pi_0|\mathfrak{A}(O')$.

2. In Section 9 we will show that the DHR criterion is satisfied by superselection sectors that are connected to the vacuum sector by a field net satisfying Bose-Fermi commutation relations. (See Section 9 for precise definitions and statements.) Conversely, in Section 10 we will prove that every DHR representation with finite dimension arises in this way. Together these results imply that DHR superselection sectors are precisely those induced by (graded) local fields. We refer to Sections 9-10 for further discussion.

3. Let (H, π) be a Poincaré covariant representation (in the sense of Definition 205) of $\mathfrak{A}$ such that H is separable and the spectrum $\mathrm{sp}\, P^\mu \subset \mathbb{R}^d$ of the momentum operator P^μ has an isolated mass shell $\{p \mid p^2 = m^2\}$ at its bottom, where $m > 0$. (Such a representation is called a massive one-particle representation.) Then, as proven in [Buchholz and Fredenhagen, 1982], for every 'spacelike cone' $\mathcal{C}$ one has a unitary equivalence $\pi|\mathfrak{A}(\mathcal{C}') \cong \pi_0|\mathfrak{A}(\mathcal{C}')$. (For the definition of spacelike cones cf. [Buchholz and Fredenhagen, 1982].) Despite the fact that this localization property is weaker than the one imposed by the DHR criterion, the category of representations that are localized in spacelike cones still can be equipped with a braided monoidal structure, cf. [Buchholz and Fredenhagen, 1982]. (The purely representation theoretic part of this theory was considerably simplified in [Doplicher and Roberts, 1990, Section 4].) In this theory, the dimension of spacetime must be $\geq 3 + 1$ in order for the braiding to be a symmetry! On the technical side the mathematical treatment is more complicated for the following reason: If π is a representation such that $\pi|\mathfrak{A}(\mathcal{C}') = \pi_0|\mathfrak{A}(\mathcal{C}')$, then Haag duality implies $\pi(\mathfrak{A}(\mathcal{C})) \subset \pi(\mathfrak{A}(\mathcal{C}))''$, but due to the weak closure the right hand side is not contained in the algebra $\mathfrak{A}$. The construction of a field net that we discuss in Section 10 can nevertheless be generalized to charges localized in spacelike cones, cf. [Doplicher and Roberts, 1990, Section 5]. On the grounds of the

cited results it seems evident that the cone-localized superselection sectors are physically better motivated than the more restrictive DHR sectors. The D(H)R theory expounded in Sections 7-10 remains useful as a technically easier 'mathematical laboratory'.

8.3 Dimension theory in tensor $*$-categories

For any tensor $*$-category, we can define a notion of "conjugates." The following is a simplified version of this definition for the case of a *strict* tensor $*$-category.

DEFINITION 209. Let $\mathcal{C}$ be a strict tensor $*$-category and $X \in \mathrm{Obj}(\mathcal{C})$. A *solution of the conjugate equations* is a triple $(\overline{X}, r, \overline{r})$, where $\overline{X} \in \mathrm{Obj}(\mathcal{C})$ and $r : \mathbf{1} \to \overline{X} \otimes X$, $\overline{r} : \mathbf{1} \to X \otimes \overline{X}$ satisfy

$$\begin{aligned} (\overline{r}^* \otimes \mathrm{id}_X) \circ (\mathrm{id}_X \otimes r) &= \mathrm{id}_X, \\ (r^* \otimes \mathrm{id}_{\overline{X}}) \circ (\mathrm{id}_{\overline{X}} \otimes \overline{r}) &= \mathrm{id}_{\overline{X}}. \end{aligned}$$

A strict tensor $*$-category $\mathcal{C}$ *has conjugates* if there is a solution of the conjugate equations for every $X \in \mathcal{C}$.

EXAMPLE 210. The definition of conjugates is exemplified in the (strictification of the) category $\mathrm{Rep}_f G$ of finite dimensional representations of a compact group. In particular, it is well known that for each representation (H, π) of G, there is a conjugate representation $(\overline{H}, \overline{\pi})$ of G. (There are several different constructions of this conjugate representation; see e.g. [Simon, 1996, p. 30].) In terms of universal properties, $(\overline{H}, \overline{\pi})$ is the unique irreducible representation of G such that $(H \otimes \overline{H}, \pi \otimes \overline{\pi})$ contains a copy of the trivial representation of G.

NOTE 211. Suppose that $(\overline{X}, r, \overline{r})$ is a conjugate for X, and that the tensor unit $\mathbf{1}$ is irreducible. Then $r^* \circ r \in \mathrm{End}(\mathbf{1}) = \mathbb{C}\mathrm{id}_{\mathbf{1}}$. Thus up to a scalar, r is an isometry, and therefore $\mathbf{1}$ is a direct summand of $\overline{X} \otimes X$. Furthermore, as can be shown using the conjugate equations, the map $\mathrm{End}(X) \to \mathrm{Hom}(\mathbf{1}, \overline{X} \otimes X)$, defined by $s \mapsto (\mathrm{id}_{\overline{X}} \otimes s) \circ r$, is an isomorphism of vector spaces. Therefore, if X is irreducible, the direct summand $\mathbf{1}$ appears with multiplicity 1 in $\overline{X} \otimes X$.

DEFINITION 212. Let $\mathcal{C}$ be a tensor $*$-category and $X \in \mathrm{Obj}(\mathcal{C})$. A solution $(\overline{X}, r, \overline{r})$ of the conjugate equations relative to X is called *normalized* if

$$r^* \circ r = \overline{r}^* \circ \overline{r},$$

and *standard* if

$$r^* \circ (\mathrm{id}_{\overline{X}} \otimes a) \circ r = \overline{r}^* \circ (a \otimes \mathrm{id}_{\overline{X}}) \circ \overline{r},$$

for all $a \in \mathrm{End}(X)$.

NOTE 213. If X, Y have (standard) conjugates, then $X \otimes Y$ and $X \oplus Y$ also have (standard) conjugates. If an object has a conjugate, then it has a standard conjugate. For more details, see the appendix.

DEFINITION 214. If an object $X \in \mathrm{Obj}(\mathcal{C})$ has a standard conjugate $(\overline{X}, r, \overline{r})$, we define its dimension $d(X)$ by

$$d(X)\mathrm{id}_{\mathbf{1}} = r^* \circ r.$$

If an object X does not have a conjugate, we formally say $d(X) = +\infty$.

NOTE 215. For all $X \in \mathrm{Obj}(\mathcal{C})$, $d(X) \geq 0$. Furthermore, if $X, Y \in \mathrm{Obj}(\mathcal{C})$ have conjugates then

$$d(\overline{X}) = d(X), \qquad d(X \otimes Y) = d(X) \cdot d(Y), \qquad d(X \oplus Y) = d(X) + d(Y),$$

and $d(\mathbf{1}) = 1$. (See the appendix for the discussion of these facts.)

DEFINITION 216. Let Δ be the DHR category. We define the full subcategory Δ_f of objects with finite dimension:

$$\mathrm{Obj}(\Delta_f) = \{\rho \in \mathrm{Obj}(\Delta) : d(\rho) < +\infty\}.$$

NOTE 217. By definition, Δ_f is a category with conjugates. It is closed under tensor products, direct sums and subobjects. In any C^*-tensor category with conjugates, the dimension of any object takes values in $[1, \infty)$, and in the interval $[1, 2]$ only values of the form $2\cos(\pi/n), n \geq 3$ can appear, cf. [Longo and Roberts, 1997]. In a symmetric C^*-tensor category, all dimensions are integers, as is proven in the Appendix.

PROPOSITION 218. *For each $X, Y \in \mathrm{Obj}(\Delta_f)$, $\mathrm{Hom}(X, Y)$ is a finite dimensional vector space. Every object $X \in \mathrm{Obj}(\Delta_f)$ is a finite direct sum of irreducible objects; i.e. the category Δ_f is semisimple.*

Proof. See the appendix. ■

NOTE 219. There is an important connection, discovered by Longo [1989] and explored further in [Longo and Roberts, 1997], between the dimension of a DHR sector $\rho \in \Delta$ and subfactor theory. Among many other things, the latter associates to any inclusion $N \subset M$ of factors an index $[M : N] \in [1, \infty]$. In order to apply this theory to AQFT we need to assume (or prove) that the local von Neumann algebras $\mathfrak{A}(O)$ are factors. (This is automatic, e.g., in conformally covariant theories.) If $\rho \in \Delta$ is localized in O, it restricts to a normal $*$-homomorphism of $\mathfrak{A}(O)$ into itself, giving rise to an inclusion $\rho(\mathfrak{A}(O)) \subset \mathfrak{A}(O)$. The index of this subfactor is related to the categorically defined dimension $d(\rho)$ by

$$[\mathfrak{A}(O) : \rho(\mathfrak{A}(O))] = d(\rho)^2. \tag{25}$$

Longo's result allows to give a very direct formula for the dimension of (the localized endomorphisms associated to) a DHR representation. Namely, all endomorphisms $\rho \in \Delta$ for which $\pi \cong \pi_0 \circ \rho$ have the same categorical dimension, justifying to write $d(\pi)$, and for any double cone O we have

$$d(\pi) = [\pi(\mathfrak{A}(O'))' : \pi(\mathfrak{A}(O))]^{1/2}.$$

This is seen as follows: π is unitarily equivalent to a representation $\pi' = \pi_0 \circ \rho$, where $\rho \in \Delta$ is localized in O. Then the inclusion $\pi(\mathfrak{A}(O)) \subset \pi(\mathfrak{A}(O'))'$ is unitarily equivalent to $\pi'(\mathfrak{A}(O)) \subset \pi(\mathfrak{A}(O'))'$, which equals $\pi_0(\rho(\mathfrak{A}(O))) \subset \pi_0(\mathfrak{A}(O))$. Now the claim follows by Eqn. (25) and the fact that the index is invariant under unitary transformations: $[UMU^* : UNU^*] = [M : N]$.

Another comment seems in order: The categorical definition of dimension of an object requires the existence of a conjugate object. On the other hand, assuming factoriality of the local algebras, the expressions $[\mathfrak{A}(O) : \rho(\mathfrak{A}(O))]$ (for an endomorphism localized in O) and $[\pi(\mathfrak{A}(O'))' : \pi(\mathfrak{A}(O))]$ (whose independence of O follows from mild additional axioms) do not presuppose the existence of a conjugate. In fact, one can show that finiteness of these subfactor indices implies the existence of a conjugate DHR representation, cf. [Guido and Longo, 1992].

8.4 Covariant representations

Since we decided to work with the category Δ of localized transportable endomorphisms rather than directly with DHR representations, we need the following

DEFINITION 220. Let $\mathfrak{A}$ be a Poincaré covariant net with covariant vacuum representation (H_0, π_0). An endomorphism $\rho \in \Delta(\mathfrak{A})$ is called covariant if there exists a strongly continuous positive energy representation $\pi_\rho : \widehat{\mathcal{P}} \to U(H_0)$ such that

$$\text{(26)} \quad \mathrm{Ad}\, U_\rho(h) \circ \pi_0 \circ \rho = \pi_0 \circ \rho \circ \beta_h \qquad \forall h \in \widehat{\mathcal{P}}.$$

The full subcategory of $\Delta(\mathfrak{A})$ consisting of the covariant morphisms is denoted by $\Delta_c(\mathfrak{A})$.

NOTE 221. For $\rho \in \Delta$, $h \in \mathcal{P}$ we define $\rho_h = \beta_h \circ \rho \circ \beta_h^{-1}$. If ρ is localized in the double cone O then ρ_h is localized in hO. If $\rho \in \Delta_c$ then Eqn. (26) can be restated as

$$\mathrm{Ad}(U(h)U_\rho(h)^*) \circ \pi_0 \circ \rho = \pi_0 \circ \beta_h \circ \rho \circ \beta_h^{-1} = \pi_0 \circ \rho_h \qquad \forall h \in \widehat{\mathcal{P}}.$$

Since ρ and ρ_h are both localized, it follows that $X_\rho(h) \equiv U(h)U_\rho(h)^* \in \mathrm{Hom}(\rho, \rho_h)$, thus $X_\rho(h) \in \mathfrak{A}$. This $\mathfrak{A}$-valued cocycle is very convenient since expressions like $\rho(U(h))$ don't make sense, whereas $\rho(X_\sigma(h))$ does. It satisfies the following cocycle equation:

$$\begin{aligned} X_\rho(gh) &= U(gh)U_\rho(gh)^* = U(g)U(h)U_\rho(h)^*U_\rho(g)^* \\ &= \beta_g(U(h)U_\rho(h)^*)U(g)U_\rho(g)^* = \beta_g(X_\rho(h))X_\rho(g). \end{aligned}$$

The same computation implies that, if $\rho \in \Delta$ and $h \mapsto X_\rho(h) \in \mathfrak{A}$ satisfies $X_\rho(gh) = \beta_g(X_\rho(h))X_\rho(g)$ for all $g, h \in \mathcal{P}$, then $U_\rho(h) := X_\rho(h)^*U(h)$ is a representation of $\mathcal{P}$ and Eqn. 26 holds, i.e. $\rho \in \Delta_c$.

PROPOSITION 222. *Δ_c is closed under tensor products, direct sums and subobjects.*

Proof. Let $\rho, \rho' \in \Delta_c$ with associated cocycles $X_\rho, X_{\rho'}$. Then

$$X_{\rho\rho'}(h) = X_\rho(h) \otimes X_{\rho'}(h) = X_\rho(h)\rho(X_{\rho'}(h)) \in \mathrm{Hom}(\rho \otimes \rho', \rho_h \otimes \rho'_h) \tag{27}$$

clearly satisfies the cocycle equation, thus $\rho\rho'$ is covariant. The proof for direct sums and subobjects is omitted, cf. [Roberts, 1990]. ■

If $T \in \mathrm{Hom}(\rho, \rho')$ then

$$\beta_h(T)\rho_h(A) = \beta_h(T\rho\beta_h^{-1}(A)) = \beta_h(\rho'\beta_h^{-1}(A)T) = \rho'_h(A)\beta_h(T),$$

thus $\beta_h(T) \in \mathrm{Hom}(\rho_h, \rho'_h)$.

Now we explore some consequences of finite dimensionality:

PROPOSITION 223. *Let $\rho, \rho' \in \Delta_{fc} := \Delta_f \cap \Delta_c$. Then*

(i) If $T \in \mathrm{Hom}(\rho, \rho')$ then $TU_\rho(h) = U_{\rho'}(h)T$ for all $h \in \widehat{\mathcal{P}}$.

(ii) Every $\rho \in \Delta_{fc}$ is covariant w.r.t. a unique representation U_ρ.

(iii) If $\rho, \rho' \in \Delta_{fc}$ and $T \in \mathrm{Hom}(\rho, \rho')$ then

$$\beta_h(T)X_\rho(h) = X_{\rho'}(h)T \qquad \forall h \in \widehat{\mathcal{P}}. \tag{28}$$

(iv) Δ_{fc} is closed under conjugates.

Proof. (i) For $h \in \widehat{\mathcal{P}}$ define $T_h = U_{\rho'}(h)TU_\rho(h)^*$. For any $A \in \mathfrak{A}$ we have

$$\begin{aligned} T_h\rho(A) &= U_{\rho'}(h)TU_\rho(h)^*\rho(A) = U_{\rho'}(h)T\rho(\beta_h^{-1}(A))U_\rho(h)^* \\ &= U_{\rho'}(h)\rho'(\beta_h^{-1}(A))TU_\rho(h)^* = \rho'(A)U_{\rho'}(h)TU_\rho(h)^* = T_h\rho'(A), \end{aligned}$$

thus $T_h \in \mathrm{Hom}(\rho, \rho')$. By assumption, ρ, ρ' have conjugates and thus $\mathrm{Hom}(\rho, \rho')$ is finite dimensional by Proposition 218. Thus $(h, T) \mapsto T_h$ is a finite dimensional representation of the Poincaré group $\mathcal{P}$. The claim $TU_\rho(h) = U_{\rho'}(h)T$ is equivalent to triviality of this representation. This triviality follows from the non-existence of finite dimensional unitary representations of $\widehat{\mathcal{P}}$ as soon as one produces a positive definite $\widehat{\mathcal{P}}$-invariant inner product on $\mathrm{Hom}(\rho, \rho')$. For this last step we refer, e.g., to [Roberts, 1990].

(ii) Apply (i) to $\rho' = \rho$, $U_{\rho'}(h) = \widetilde{U}_\rho(h)$, $T = \mathrm{id}_\rho = \mathbf{1}_{H_0}$ to conclude $U_\rho = \widetilde{U}_\rho$.

(iii) Using (i) we compute

$$\begin{aligned} \beta_h(T)X_\rho(h) &= (U(h)TU(h)^*)(U(h)U_\rho(h)^*) = U(h)TU_\rho(h^{-1}) \\ &= U(h)U_{\rho'}(h^{-1})T = X_{\rho'}(h)T, \end{aligned}$$

(iv) See [Roberts, 1990]. ■

NOTE 224. Under weak additional assumptions on the net $\mathfrak{A}$, it is shown in [Guido and Longo, 1992, Theorem 5.2] that every localized endomorphism of finite dimension is automatically covariant with positive energy! Equivalently, $\Delta_f \subset \Delta_c$, and therefore $\Delta_{fc} = \Delta_f$.

8.5 Statistics in braided tensor $*$-categories

DEFINITION 225. Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a tensor $*$-category with unitary braiding $c_{X,Y}$, and suppose that each $X \in \mathrm{Obj}(\mathcal{C})$ has a conjugate. For each $X \in \mathrm{Obj}(\mathcal{C})$, we define the *twist* of X, $\Theta_X \in \mathrm{End}(X)$, by

$$\Theta_X = (\overline{r}^* \otimes \mathrm{id}_X) \circ (\mathrm{id}_{\overline{X}} \otimes c_{X,X}) \circ (\overline{r} \otimes \mathrm{id}_X),$$

where $(\overline{X}, r, \overline{r})$ is a standard conjugate for X.

NOTE 226. For each $X \in \mathrm{Obj}(\mathcal{C})$, Θ_X is unitary. When X is irreducible, $\mathrm{End}(X) = \mathbb{C}\mathrm{id}_X$, and so $\Theta_X = \omega_X \mathrm{id}_X$, where ω_X is a complex number of unit modulus (called the *statistics phase*). In the case that $c_{X,Y}$ is a symmetry, then $(c_{X,X})^* = c_{X,X}$, and so $(\Theta_X)^* = \Theta_X$. Together with unitarity, this implies that $(\Theta_X)^2 = \mathrm{id}_X$.

DEFINITION 227. Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a tensor $*$-category with unitary symmetry $c_{X,Y}$. If $X \in \mathrm{Obj}(\mathcal{C})$ is irreducible, we say that X is a *Bosonic object* if $\omega_X = 1$, and we say that X is a *Fermionic object* if $\omega_X = -1$.

NOTE 228. We give a number of justifications for our focus on the category Δ_f of DHR representations with finite dimension.

(i): In the heuristic interpretation of the (unitary equivalence classes) of irreducible DHR representations as corresponding to the types of particles in a QFT, the conjugate $\overline{\rho}$ of a DHR representation ρ corresponds to the antiparticle. It may happen that a particle is its own antiparticle, i.e. $\rho \cong \overline{\rho}$; but the existence of antiparticles seems to be an integral part of relativistic quantum field theories.

(ii): The DHR sectors admitting a conjugate in the above sense are (rough) analogues in the operator algebraic approach to AQFT of Wightman fields with finitely many components. In the Wightman framework [Streater and Wightman, 1964] it is well known that infinite components behave 'pathologically' in that the PCT and spin-statistics theorems do not apply to them, and can in fact be violated. In algebraic QFT, these results are reflected in the fact that we cannot even define Bosonic and Fermionic objects that have dimension ∞, in the sense that they have no conjugates.

(iii): In [Fredenhagen, 1981] it was shown that every massive one-particle representation (cf. Note 207(iii)), which by the mentioned result of [Buchholz and Fredenhagen, 1982] is localizable in space-like cones, has a conjugate in the C^*-tensor category of cone-localizable representations. It therefore seems natural to require existence of conjugates also in the more restrictive setting of double cone localizable representations.

(iv): As pointed out in Note 224, DHR endomorphisms of finite dimension are automatically covariant, provided one accepts the additional conditions on the net $\mathfrak{A}$ needed for this result. Even if one doesn't wish to appeal to this result, finite dimensionality of the objects is needed (via finite dimensionality of the hom-sets) for the proof of Proposition 223. The latter will be crucial for lifting the Poincaré action from $\mathfrak{A}$ to the field theory $\mathfrak{F}$ in Section 10.

9 FROM FIELDS TO REPRESENTATIONS

In the current section we take the 'top down' approach to superselection rules. That is, we are given a field algebra $\mathfrak{F}$ and a gauge group G acting concretely on a Hilbert space $\mathcal{H}$. We then define the observables as the gauge invariant elements of $\mathfrak{F}$. The representation of $\mathfrak{F}$ on $\mathcal{H}$ then gives us a preferred set of representations of $\mathfrak{A}$; viz. those that can be 'created from the vacuum representation by the action of local fields.' Our main mathematical objective in the current section is to show that these representations satisfy the DHR selection criterion. Thus, all superselection sectors that arise in the traditional way — viz. by acting on the vacuum with fields — fall within the purview of DHR superselection theory. (But note: We are restricting attention to *local* fields.)

DEFINITION 229. Let ω_0 be a state on $\mathfrak{A}$, and let $(\mathcal{H}_0, \pi_0)$ be the corresponding GNS representation. A *field system with gauge symmetry* for $(\mathfrak{A}, \omega_0)$ is a quadruple $(\pi, \mathcal{H}, \mathfrak{F}, (G, k))$, where $(\mathcal{H}, \pi)$ is a representation of $\mathfrak{A}$, $O \mapsto \mathfrak{F}(O)$ is a net of von Neumann algebras acting irreducibly on $\mathcal{H}$, G is a strongly compact group of unitary operators on $\mathcal{H}$, k is a central element of G such that $k^2 = e$, and such that:

α) $(\mathcal{H}_0, \pi_0)$ is a subrepresentation of $(\mathcal{H}, \pi)$, i.e. there is an isometry $V : \mathcal{H}_0 \to \mathcal{H}$ such that $V\pi_0 = \pi V$;

β) V maps $\mathcal{H}_0$ into the subspace of G-invariant vectors of $\mathcal{H}$;

γ) the $U \in G$ induce automorphisms that leave each $\mathfrak{F}(O)$ globally fixed, and $\pi(\mathfrak{A}(O))'' \subseteq \mathfrak{F}(O)$ is the set of fixed points under the action of G on $\mathfrak{F}(O)$;

δ) for each $O \in \mathcal{K}$, $V(\mathcal{H}_0)$ is cyclic for $\mathfrak{F}(O)$;

ε) the fields are local relative to the observables, i.e. $\mathfrak{F}(O_1)$ and $\pi(\mathfrak{A}(O_2))$ commute elementwise whenever O_1 and O_2 are spacelike separated.

A few remarks on the definition of a field system: the fact that $\mathfrak{F}$ is generated by local algebras $\{\mathfrak{F}(O) : O \in \mathcal{K}\}$ means that elements of $\mathfrak{F}$ represent *local* fields — i.e., fields whose excitations can be localized within a bounded spacetime region. Furthermore:

δ) is the Reeh-Schlieder Condition: it states that each local region O carries a full set of fields in the sense that these local fields can reach each sector from the vacuum sector. [But note that Condition (δ) only guarantees that sectors in $\mathcal{H}$ can be reached from the vacuum sector. A stronger notion of

completeness would rely on some intrinsic criterion for physical sectors of $\mathfrak{A}$, and would require that all these sectors be contained in $\mathcal{H}$; see Definition 246.]

γ) can be interpreted as saying that the group G is an internal symmetry group of the field: it does not change the spacetime localization region of field operators.

ε) is the Relative Locality Condition. Since fields need not be observable, the field algebra is *not* required to satisfy microcausality. However, in the typical situation (i.e. normal commutation relations), field operators localized in one spacetime region either commute or anticommute with field operators localized in a spacelike separated region. Condition (ε) is a weakening of the requirement of normal (Bose/Fermi) commutation relations.

Since G is a compact group of unitary operators acting on $\mathcal{H}$, we can apply all of the apparatus of the theory of unitary representations of compact groups (see e.g. [Folland, 1995]). In particular, $\mathcal{H}$ decomposes into a direct sum of orthogonal subspaces $\mathcal{H}_\xi$ which reduce the action of G. Thus the reduced unitary representation U_ξ of G on $\mathcal{H}_\xi$ is factorial, i.e. the von Neumann algebra generated by the operators $\{g|_{\mathcal{H}_\xi} : g \in G\}$ is a factor. The representation U_ξ decomposes into a direct sum of *unitarily equivalent* irreducible representations of G. So, there is a privileged direct sum decomposition of $\mathcal{H}$:

$$\mathcal{H} = \bigoplus_\xi \mathcal{H}_\xi,$$

where the subspace $\mathcal{H}_\xi$ is generated by the vectors in $\mathcal{H}$ that transform according to the character ξ (unitary equivalence class of irreducible representations) of G.

In the present section our primary objectives are:

1. Show that the subspaces $\mathcal{H}_\xi$ reduce the action of the observable algebra $\mathfrak{A}$. So, the representation of $\mathfrak{A}$ on $\mathcal{H}$ decomposes into a direct sum $\bigoplus_\xi \pi_\xi$ of representations on the subspaces $\mathcal{H}_\xi$.

2. Show that each representation $(\mathcal{H}_\xi, \pi_\xi)$ of $\mathfrak{A}$ is factorial, so that the irreducible subrepresentations of $(\mathcal{H}_\xi, \pi_\xi)$ are mutually equivalent. (Hence each character ξ of G labels an equivalence class of irreducible representations of $\mathfrak{A}$.)

3. Show that $V(\mathcal{H}_0)$ is precisely the subspace of G-invariant vectors in $\mathcal{H}$. (Hence the character $\mathbf{1}$ of G labels the equivalence class of the vacuum representation of $\mathfrak{A}$.)

4. Show that each subrepresentation of $(\mathcal{H}, \pi)$ is a DHR representation. In slogan form, the sectors that can be reached from the vacuum by application of local fields correspond to DHR representations (i.e. representations that are equivalent, modulo any local region, to the vacuum representation).

Regarding objectives (1) and (2), it will suffice to show that $\pi(\mathfrak{A})'' = G'$, because then the von Neumann algebras $\pi(\mathfrak{A})''$ and G'' share the same central projections.

PROPOSITION 230. *If* $(\pi, \mathcal{H}, \mathfrak{F}, (G, k))$ *is a field system with gauge symmetry for* $(\mathfrak{A}, \omega_0)$ *then* $\pi(\mathfrak{A})' = G''$.

Our notation will henceforth be simplified if we use g and $U(g)$ ambiguously to denote elements of the unitary group G on $\mathcal{H}$. That is, we think of $g \to U(g)$ as the identity representation of G on $\mathcal{H}$.

Proof. Define $M : \mathfrak{B}(\mathcal{H}) \to G'$ by

$$M(A) = \int_G U(g) A U(g)^* \, d\mu(g),$$

where μ is the Haar measure on G. Then M is a faithful, normal projection of norm one from $\mathfrak{B}(\mathcal{H})$ onto G'. Since M is weakly continuous on the unit ball of $\mathfrak{B}(\mathcal{H})$, we have

$$G' = M(\mathfrak{B}(\mathcal{H})) = M(\overline{\mathfrak{F}}) = \overline{M(\mathfrak{F})} = \overline{\pi(\mathfrak{A})}. \tag{29}$$

Thus, $G'' = \pi(\mathfrak{A})'$. ■

It follows then that the factorial subrepresentations of the representation $(\mathcal{H}, \pi)$ of $\mathfrak{A}$ are in one to one correspondence with the factorial subrepresentations of the action of G on $\mathcal{H}$.

NOTE 231. Since G is compact each irreducible representation of G is finite dimensional. Let $\hat{G}$ be the set of characters (equivalence class of irreducible representations) of G, and for $\xi \in \hat{G}$, let $d(\xi)$ be the dimension of the underlying Hilbert space. Then the previous result gives a nice intuitive picture of the representation $(\mathcal{H}, \pi)$ of $\mathfrak{A}$. For each $\xi \in \hat{G}$, select an irreducible subrepresentation $(\mathcal{H}_\rho, \pi_\rho)$ of the factorial representation $(\mathcal{H}_\xi, \pi_\xi)$. Then we have

$$\pi(A) = \bigoplus_{\xi \in \hat{G}} d(\xi) \pi_\rho(A) = \bigoplus_{\xi \in \hat{G}} (\pi_\rho(A) \otimes I_\rho),$$

where $d(\xi)\pi_\rho(A) = \pi_\rho(A) \oplus \cdots \oplus \pi_\rho(A)$, $d(\xi)$ times, and I_ρ is the identity on an $d(\xi)$-dimensional Hilbert space.

LEMMA 232. *Let* $(\pi, \mathcal{H}, \mathfrak{F}, (G, k))$ *be a field system with gauge symmetry for* $(\mathfrak{A}, \omega_0)$. *Then* $\mathcal{H}_0$ *is separating for* $\mathfrak{F}$.

Proof. Let $F \in \mathfrak{F}$. If $F\mathcal{H}_0 = \{0\}$ then $\mathscr{E}(F^*F)\mathcal{H}_0 = \{0\}$. Since $\mathscr{E}(F^*F) \in \pi(\mathfrak{A})$ and π_0 is faithful, $\mathscr{E}(F^*F) = 0$. Since $\mathscr{E}$ is faithful, $F = 0$. Therefore, $\mathcal{H}_0$ is separating for $\mathfrak{F}$. ■

To obtain further information about the field system $(\pi, \mathcal{H}, \mathfrak{F}, (G, k))$, we identify "tensors under the action of G" in the field algebra $\mathfrak{F}$. To make sense of this idea, forget momentarily that $\mathfrak{F}$ has a product operation, and consider it merely

as a Banach space. The map $U \mapsto \mathrm{Ad}U$ is a (strongly) continuous representation of the compact group G in $\mathrm{Aut}\mathfrak{F}$, which is of course a subset of the invertible linear operators on $\mathfrak{F}$. As in the case of a representation of a compact group on a Hilbert space H, the representation of G on $\mathfrak{F}$ decomposes into a direct sum of disjoint representations. An operator $F \in \mathfrak{F}$ is said to *transform according to a representation ρ of G* just in case it is contained in a linear subspace H_ρ of $\mathfrak{F}$ carrying the corresponding representation of G. In fact, we will show that the irreducible subspaces in $\mathfrak{F}$ have a special algebraic property: they have support I.

LEMMA 233. *Let $(\pi, \mathcal{H}, \mathfrak{F}, (G, k))$ be a field system with gauge symmetry for $(\mathfrak{A}, \omega_0)$, and suppose that $\mathfrak{A}$ satisfies property B relative to ω_0. Then the net $O \mapsto \pi(\mathfrak{A}(O))''$ satisfies property B.*

Proof. We first establish that $\pi_0|_{\mathfrak{A}(O')}$ is quasiequivalent to $\pi|_{\mathfrak{A}(O')}$ for each double cone O.

By the Relative Locality Condition (ε), $\mathfrak{F}(O) \subseteq \pi(\mathfrak{A}(O'))'$. By the Reeh-Schlieder Condition (δ), $\mathcal{H}_0$ is a cyclic subspace for $\mathfrak{F}(O)$. Thus,

$$\mathcal{H} = [\mathfrak{F}(O)\mathcal{H}_0] \subseteq [\pi(\mathfrak{A}(O'))'\mathcal{H}_0].$$

Let E_0 be the orthogonal projection onto $\mathcal{H}_0$. The central support of E_0 in $\pi(\mathfrak{A}(O'))'$ is the projection onto $[\pi(\mathfrak{A}(O'))'E_0(\mathcal{H})]$ [Kadison and Ringrose, 1997, Prop. 5.5.2]. Thus E_0 has central support I in $\pi(\mathfrak{A}(O'))'$, and therefore $(\pi_0|_{\mathfrak{A}(O')}, \mathcal{H}_0)$ and $(\pi|_{\mathfrak{A}(O')}, \mathcal{H})$ are quasiequivalent [Kadison and Ringrose, 1997, Thm. 10.3.3].

Let O_1 be a double cone whose closure is contained in O, and let E be a nonzero projection in $\pi(\mathfrak{A}(O_1))''$. Choose a double cone O_2 that is spacelike separated from O. The preceding argument shows that there is a $*$-isomorphism φ from $\pi_0(\mathfrak{A}(O_2'))''$ to $\pi(\mathfrak{A}(O_2'))''$ such that $\varphi(\pi_0(A)) = \pi(A)$ for all $A \in \mathfrak{A}$. This isomorphism φ preserves the net structure: $\varphi[\pi_0(\mathfrak{A}(O_3))] = \pi(\mathfrak{A}(O_3))$ for any double cone O_3 contained in O_2'. Further, since φ is ultraweakly continuous [Kadison and Ringrose, 1997, Cor. 7.1.16], $\varphi[\pi_0(\mathfrak{A}(O_3))''] = \pi(\mathfrak{A}(O_3))''$. In particular, $\varphi(E)$ is a projection in $\pi_0(\mathfrak{A}(O_1))''$. By property B for π_0, there is an isometry $V \in \pi_0(\mathfrak{A}(O))''$ such that $VV^* = \varphi(E)$. Thus, $W := \varphi^{-1}(V) \in \pi(\mathfrak{A}(O))''$ is an isometry such that $WW^* = E$. Therefore the net $O \mapsto \pi(\mathfrak{A}(O))''$ satisfies property B. ■

DEFINITION 234. Consider the ordered n-tuple $(F_1, \ldots, F_n)$ of elements in $\mathfrak{F}$. We say that this n-tuple *transforms under the action of G according to character ξ* just in case:

1. $F_i^* F_j = 0$ if $i \neq j$; and
2. $\alpha_g(F_i) = \sum_{j=1}^n u_{ij}^\xi(g) F_j$, where u_{ij}^ξ is a set of matrix elements for ξ. That is, for some representation (H, ρ) of G of class ξ, and orthonormal basis $\{e_1, \ldots, e_n\}$ for H, $u_{ij}(g) := \langle e_i, \rho(g) e_j \rangle_H$.

NOTE 235. If $(F_1, \ldots, F_n)$ is a tensor in $\mathfrak{F}$ transforming according to ξ, then we can always replace the F_i's with partial isometries V_i with orthogonal ranges. Indeed, let $V_i|F_i|$ be the polar decomposition of F_i. When $i \neq j$, $F_i^* F_j = 0$, and so F_i^* and F_j^* have orthogonal ranges. Recall that if $F = V|F|$, then V annihilates the orthogonal complement of $\mathrm{r}(F^*) = \mathrm{r}(|F|)$ (see [Kadison and Ringrose, 1997, Thm. 6.1.2]). Thus $V_j|F_i| = \delta_{ij} F_j$, and

$$\Big(\sum_j u_{ij}^{\xi}(g) V_j\Big)|F_i| = \sum_j u_{ij}^{\xi}(g) V_j |F_j| = \sum_j u_{ij}^{\xi}(g) F_j = F_i.$$

By the uniqueness of the polar decomposition, $\sum_j u_{ij}^{\xi}(g) V_j = V_i$. Hence $(V_1, \ldots, V_n)$ is a tensor transforming according to ξ.

DEFINITION 236. Given $\varphi, \psi \in \mathcal{H}_\xi$, define a map $M_{\varphi,\psi}^{\xi} : \mathfrak{F} \to \mathfrak{F}$ by

$$M_{\varphi,\psi}^{\xi}(F) = \int_G \langle \varphi, U(g)\psi \rangle \alpha_g(F)\, d\mu(g),$$

where μ is the Haar measure on G.

FACT 237. Due to the invariance of μ we have $\alpha_g \circ M_{\varphi,\psi}^{\xi}(F) = M_{U(g)\varphi,\psi}^{\xi}(F)$.

LEMMA 238. *Let $(F_1, \ldots, F_n)$ be a tensor in $\mathfrak{F}(O)$ transforming as a unitary representation of class ξ. Then $F_i(\mathcal{H}_0) \subseteq \mathcal{H}_\xi$, where $\mathcal{H}_\xi$ is the subspace of vectors of $\mathcal{H}$ that transform according to ξ.*

Sketch of proof. Let $\varphi \in \mathcal{H}_0$, and let $g \in G$. Then

$$U(g)[M_{\varphi,\psi}^{\xi}(F)\varphi] = U(g) M_{\varphi,\psi}^{\xi}(F) U(g)^* \varphi = M_{U(g)\varphi,\psi}^{\xi}(F)\varphi.$$

Then a straightforward calculation using matrix elements for ξ establishes the result. ∎

LEMMA 239. *Let ξ be a character of G that occurs nontrivially in the decomposition of the action of G on $\mathcal{H}$. Then for each double cone O, there is a tensor $(F_1, \ldots, F_n)$ in $\mathfrak{F}(O)$ that transforms as a unitary representation of class ξ.*

Sketch of proof. Let $(\psi_1, \ldots, \psi_n)$ be an orthonormal basis from a G-irreducible subspace of $\mathcal{H}_\xi$. Let φ be a unit vector in this same subspace. Since $\mathcal{H}_0$ is cyclic for $\mathfrak{F}(O)$, and $\mathfrak{F}(O)$ is a von Neumann algebra, there is an $F \in \mathfrak{F}(O)$ and a vector $\varphi_0 \in \mathcal{H}_0$ such that $F\varphi_0 = \varphi$. Let $F_i = M_{\psi_i,\varphi}^{\xi}(F)$. One verifies then that $(F_1, \ldots, F_n)$ is the required tensor. ∎

LEMMA 240. *Let $F_1, \ldots, F_n \in \mathfrak{F}(O)$ such that $(F_1, \ldots, F_n)$ transforms according to the character ξ. Then if $\overline{O} \subseteq O_1$, there are $X_1, \ldots, X_n \in \mathfrak{A}(O_1)$ such that $(X_1, \ldots, X_n)$ transforms according to ξ and*

$$\sum_{i=1}^{n} X_i^* X_i = I.$$

Proof. First replace $F_1, \ldots, F_n$ with partial isometries $V_1, \ldots, V_n$, then let $V = \sum_{i=1}^n V_i$. Since the V_i have orthogonal ranges, V is a partial isometry, and $V^*V = \sum_{i=1}^n V_i^* V_i$. A straightforward calculation shows that $\alpha_g(V^*V) = V^*V$ for all $g \in G$. Thus, $E = V^*V$ is a projection in $\mathfrak{F}(O) \cap G' = \pi(\mathfrak{A}(O))''$. By Lemma 233, $O \mapsto \pi(\mathfrak{A}(O))''$ satisfies property B. Thus, there is an isometry $W \in \pi(\mathfrak{A}(O_1))''$ with $WW^* = E$. For $i = 1, \ldots, n$, let $X_i = V_i W$. Then the tensor $(X_1, \ldots, X_n)$ transforms according to ξ, and

$$\sum_{i=1}^n X_i^* X_i = W^* \Big(\sum_{i=1}^n V_i^* V_i \Big) W = I.$$

■

LEMMA 241. *Let $\mathcal{H}_\xi \subseteq \mathcal{H}$ be the range of a central projection in $\pi(\mathfrak{A})''$. Then for each double cone O, $\mathcal{H}_\xi$ is cyclic for $\mathfrak{F}(O)$.*

Proof. Let O_1 be a double cone such that $\overline{O}_1 \subseteq O$. By the Reeh-Schlieder Condition, $\mathcal{H}_0$ is cyclic for $\mathfrak{F}(O_1)$. By Lemma 239, there is a tensor $(F_1, \ldots F_n)$ in $\mathfrak{F}(O_1)$ that transforms according to the representation (H, ρ) of G. By Lemma 240, there is a tensor $(X_1, \ldots, X_n)$ in $\mathfrak{F}(O)$ that transforms that same way, and such that $\sum_{i=1}^n X_i^* X_i = I$. Then

$$\mathfrak{F}(O)\mathcal{H}_0 \;=\; \mathfrak{F}(O) \sum_{i=1}^n X_i^* X_i \mathcal{H}_0 \;\subseteq\; \mathfrak{F}(O)\mathcal{H}_\xi \,,$$

where the final inclusion follows by Lemma 238. Therefore $\mathcal{H}_\xi$ is cyclic for $\mathfrak{F}(O)$. ■

DEFINITION 242. Let $\mathrm{Rep}_{\mathfrak{F}}\mathfrak{A}$ be the category of subrepresentations of the representation $(\mathcal{H}, \pi)$ of $\mathfrak{A}$. We mean to take $\mathrm{Rep}_{\mathfrak{F}}\mathfrak{A}$ as a full subcategory of the category of all representations of $\mathfrak{A}$, i.e. the hom-sets between representations in $\mathrm{Rep}_{\mathfrak{F}}\mathfrak{A}$ are the same as the hom-sets in the larger category.

PROPOSITION 243. *Let $(\pi, \mathcal{H}, \mathfrak{F}, (G, k))$ be a field system with gauge symmetry for $(\mathfrak{A}, \omega_0)$. Then there is a faithful functor $F : \mathrm{Rep}_{\mathfrak{F}}\mathfrak{A} \to DHR(\mathfrak{A})$.*

Proof. Suppose that $(\mathcal{H}', \pi')$ is an object of $\mathrm{Rep}_{\mathfrak{F}}\mathfrak{A}$. That is, there is an isometry $V : \mathcal{H}' \to \mathcal{H}$ such that $V\pi' = \pi V$. We subsequently identify $\mathcal{H}'$ with its image in $\mathcal{H}$, and treat π' as mapping into $\mathfrak{B}(\mathcal{H})$. We must show that $(\mathcal{H}', \pi')$ is in $DHR(\mathfrak{A})$; that is, for any double cone O, $(\mathcal{H}', \pi'|_{\mathfrak{A}(O')})$ is unitarily equivalent to $(\mathcal{H}_0, \pi_0|_{\mathfrak{A}(O')})$.

Let $\overline{\pi} = \pi|_{\mathfrak{A}(O')}$. Since $E_\iota, E_\xi \in \pi(\mathfrak{A})' \subseteq \pi(\mathfrak{A}(O'))'$, E_ι and E_ξ reduce $\overline{\pi}$. We first establish that E_ι and E_ξ have the same central support in $\pi(\mathfrak{A}(O'))^-$, from which it follows that $E_\iota\overline{\pi}$ and $E_\xi\overline{\pi}$ are quasiequivalent [Kadison and Ringrose, 1997, Thm. 10.3.3].

By the Relative Locality Condition (ε), $\mathfrak{F}(O) \subseteq \pi(\mathfrak{A}(O'))'$. By the Reeh-Schlieder Condition (δ), $E_\iota\mathcal{H}$ is a cyclic subspace for $\mathfrak{F}(O)$. Thus,

$$\mathcal{H} = [\mathfrak{F}(O)E_\iota(\mathcal{H})] \subseteq [\pi(\mathfrak{A}(O'))'E_\iota(\mathcal{H})].$$

Similarly, Lemma 241 entails that $E_\xi\mathcal{H}$ is a cyclic subspace for $\mathfrak{F}(O)$, and so $[\pi(\mathfrak{A}(O'))'E_\xi(\mathcal{H})] = \mathcal{H}$. However, the central support of E_ι in $\pi(\mathfrak{A}(O'))'$ is the projection onto $[\pi(\mathfrak{A}(O'))'E_0(\mathcal{H})]$, and similarly for E_ξ [Kadison and Ringrose, 1997, Prop. 5.5.2]. Thus, E_ι and E_ξ have central support I in $\pi(\mathfrak{A}(O'))'$. Therefore, $(\pi_0|_{\mathfrak{A}(O')}, \mathcal{H}_0)$ and $(\pi_\xi|_{\mathfrak{A}(O')}, \mathcal{H}_\xi)$ are quasiequivalent, i.e. there is a $*$-isomorphism $\varphi : \pi_0(\mathfrak{A}(O')) \to \pi_\xi(\mathfrak{A}(O'))$ such that $\varphi(\pi_0(A)) = \pi_\xi(A)$ for all $A \in \mathfrak{A}(O')$.

The previous reasoning also shows (by replacing O with a spacelike separated double cone) that for each double cone O, $(\pi_0|_{\mathfrak{A}(O)}, \mathcal{H}_0)$ is quasiequivalent to $(\pi_\xi|_{\mathfrak{A}(O)}, \mathcal{H}_\xi)$. Thus, in particular, since the net $O \to \pi_0(\mathfrak{A}(O))''$ of von Neumann algebras satisfies property B (by assumption), so does the net $O \to \pi_\xi(\mathfrak{A}(O))''$.

To establish that $(\pi_0|_{\mathfrak{A}(O')}, \mathcal{H}_0)$ and $(\pi_\xi|_{\mathfrak{A}(O')}, \mathcal{H}_\xi)$ are unitarily equivalent, we will use the following result ([Kadison and Ringrose, 1997, Theorem 7.2.9]):

> *Let $\mathfrak{R}_j$, $j = 1, 2$, be von Neumann algebras acting on Hilbert spaces $\mathcal{H}_j$ respectively. Suppose that for $j = 1, 2$, there is a vector $x_j \in \mathcal{H}_j$ that is cyclic and separating for $\mathfrak{R}_j$. If $\alpha : \mathfrak{R}_1 \to \mathfrak{R}_2$ is a $*$ isomorphism then there is a unitary operator U from $\mathcal{H}_1$ to $\mathcal{H}_2$ that implements α.*

in conjunction with the fact ([Kadison and Ringrose, 1997, Exercise 9.6.32]):

> *If $\mathfrak{R}$ is a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$, and if $\mathfrak{R}'$ is properly infinite, then there is vector $x \in \mathcal{H}$ that is cyclic and separating for $\mathfrak{R}$.*

By Proposition 71, $\pi_0(\mathfrak{A}(O'))'$ and $\pi_\xi(\mathfrak{A}(O'))'$ are properly infinite. By assumption, $\mathcal{H}_0$ is separable. Thus, it will suffice to show that $\mathcal{H}_\xi$ is separable. Since π_ξ is irreducible, each non-zero vector $x \in \mathcal{H}_\xi$ is cyclic for $\pi_\xi(\mathfrak{A})$. Thus, $\mathcal{H}_\xi$ is the closure of the union of $\pi_\xi(\mathfrak{A}(O_n))x$ for an increasing sequence O_n of double cones. Hence it suffices to show that $\pi_\xi(\mathfrak{A}(O))x$ is separable for each $O \in \mathcal{K}$. Since $\mathcal{H}_0$ is separable, the unit ball of $\mathfrak{B}(\mathcal{H}_0)$ is compact metrizable [Kadison and Ringrose, 1997, Thm. 5.1.3; Exercise 5.7.7]. Since the unit ball of $\pi_0(\mathfrak{A}(O))''$ is a closed subset of the unit ball of $\mathfrak{B}(\mathcal{H}_0)$, it is also compact metrizable. But $\pi_\xi(\mathfrak{A}(O))''$ is $*$ isomorphic, hence ultraweakly homeomorphic, to $\pi_0(\mathfrak{A}(O))''$. Therefore, the unit ball of $\pi_\xi(\mathfrak{A}(O))''$ is compact metrizable, hence separable, in the weak operator topology. It follows that $\pi_\xi(\mathfrak{A}(O))''x$ is separable. ■

In Proposition 206 it was shown that there is a faithful, essentially surjective functor F' from the category $DHR(\mathfrak{A})$ of DHR representations to the category Δ of localized transportable morphisms of $\mathfrak{A}$. So, the previous Proposition entails that $F' \circ F$ is a faithful functor from $\mathrm{Rep}_{\mathfrak{F}}\mathfrak{A}$ into Δ. We subsequently replace $F' \circ F$ with just F.

Recall that Δ_f is the full subcategory of Δ of objects with conjugates. The final thing we need to show in this Section is that the image of each object in $\mathrm{Rep}_{\mathfrak{F}}\mathfrak{A}$ under F is isomorphic to an object in Δ_f. That is, we need to show that the image object has a conjugate.

Sketch of proof. One shows that the subrepresentations of G on $\mathcal{H}$ are closed under taking conjugates. This can be proven by noting a correspondence between the action of G on $\mathcal{H}$ and the action of G on $\mathfrak{F}$. Then use the fact that $\mathfrak{F}$ is a $*$-algebra. Thus, for each irreducible subrepresentation π_ρ of π, there is an irreducible subrepresentation $\pi_{\overline{\rho}}$ of π. Verify that $(F' \circ F)(\pi_{\overline{\rho}})$ is a conjugate for $(F' \circ F)(\pi_\rho)$. ■

Therefore, $F' \circ F$ is a faithful functor from $\mathrm{Rep}_{\mathfrak{F}}\mathfrak{A}$ into Δ_f. So we have shown:

> *Each representation of $\mathfrak{A}$ that arises from its being taken as the gauge invariant part of a field algebra is a representation of the form $\pi_0 \circ \rho$ with $\rho \in \mathrm{Obj}(\Delta_f)$.*

Thus, the study of Δ_f encompasses the study of representations that arise from the approach that begins with a field algebra.

We said above that in the "normal" situation, field operators in $\mathfrak{F}(O_1)$ with either commute or anticommute with field operators in $\mathfrak{F}(O_2)$ when O_1 and O_2 are spacelike separated. To be more precise, we would expect for a Bose field operator to commute both with other Bose field operators, as well as with Fermi field operators; and we would expect for a pair of Fermi field operators to anticommute. But what are Bose and Fermi field operators? The distinction between the two is defined in terms of the privileged element k of the gauge group G.

DEFINITION 244. If $\alpha_k(F) = F$ then F is said to be a *Bose field operator*; and if $\alpha_k(F) = -F$ then F is said to be a *Fermi field operator*.

We define a Bosonic sector in $\mathcal{H}$ to be a subspace $\mathcal{H}_\xi$ such that $U(k)|_{\mathcal{H}_\xi} = I$, and a Fermionic sector in $\mathcal{H}$ to be a subspace $\mathcal{H}_\xi$ such that $U(k)|_{\mathcal{H}_\xi} = -I$. It then follows that Bosonic field operators create Bosonic sectors from the vacuum, and Fermionic field operators create Fermionic sectors from the vacuum.

We can now make sense of the notion of normal commutation relations: Bose field operators should commute with each other and commute with Fermionic field operators. Fermionic field operators should anticommute with each other.

DEFINITION 245. A local operator algebra system of fields $(\pi, (G, k), \mathfrak{F})$ is said to *satisfy normal commutation relations* just in case the local field algebras satisfy *graded local commutativity:* If O_1 and O_2 are spacelike, and $F_\sigma \in \mathfrak{F}(O_1)$, $F'_\sigma \in \mathfrak{F}(O_2)$ are such that $\alpha_k(F_\sigma) = \sigma F_\sigma$ and $\alpha_k(F'_\sigma) = \sigma F_\sigma$, $(\sigma = \pm)$, then

$$F_+F'_+ = F'_+F_+, \qquad F_+F'_- = F'_-F_+, \qquad F_-F'_- = -F'_-F_-.$$

10 FROM REPRESENTATIONS TO FIELDS

The preceding section derives properties of representations of $\mathfrak{A}$, given that these representations are created from the vacuum representation by the action of local fields on the vacuum. But such an approach will seem at best heuristic to the Algebraic Imperialist. From the Imperialist's point of view, the entire content of the theory is contained in the abstract net $\mathfrak{A}$ of observable algebras.

On the one hand, the Imperialist might be an eliminativist about fields and gauge group. On the other hand, the Imperialist might claim that the fields and gauge group are physically significant, but only because they can be 'reconstructed' from the net of observable algebras. In order to justify this latter stance, the Imperialist would need to accomplish the following:

> **Task:** Try to reconstruct, in a mathematically rigorous fashion, the entire apparatus of QFT — fields, gauge groups, etc. — from the net of observable algebras.

A quixotic task indeed! For one, philosophers seemed to have settled that theory is always underdetermined by data; and so we should not expect to be able to find the full theoretical apparatus hidden within the net of observable algebras. But there is a surprise in store: the Task was undertaken, and was achieved. The DR Reconstruction Theorem shows in a fully rigorous and precise way that the DHR category encodes all the information needed to reconstruct, uniquely, the fields and the gauge group. This section provides the details of the reconstruction.

DEFINITION 246. A field system with gauge symmetry $(\pi, \mathcal{H}, \mathfrak{F}, (G, k))$ for $(\mathfrak{A}, \omega_0)$ is said to be *complete* if the representation π of $\mathfrak{A}$ contains copies of all representations in the DHR category $DHR(\mathfrak{A})$ of $\mathfrak{A}$.

DEFINITION 247. Two field systems with gauge symmetry $(\pi_1, \mathcal{H}_1, \mathfrak{F}_1, G_1)$ and $(\pi_2, \mathcal{H}_2, \mathfrak{F}_2, G_2)$ for $(\mathfrak{A}, \omega_0)$ are said to be *equivalent* if there exists a unitary operator $W : \mathcal{H}_1 \to \mathcal{H}_2$ such that:

$$\begin{aligned} W\pi_1(A) &= \pi_2(A)W, \quad \forall A \in \mathfrak{A}, \\ WU(G_1) &= U(G_2)W, \\ W\mathfrak{F}_1(O) &= \mathfrak{F}_2(O)W, \text{ for each double cone } O. \end{aligned}$$

Doplicher-Roberts Reconstruction Theorem. Let $(\mathfrak{A}, \omega_0)$ be a net of observable algebras satisfying duality and property B relative to a privileged 'vacuum' state ω_0. Then there exists a field system with gauge symmetry $(\pi, \mathcal{H}, \mathfrak{F}, (G, k))$ for $(\mathfrak{A}, \omega_0)$ that is complete, and that has normal commutation relations. Any complete, normal field system for $(\mathfrak{A}, \omega_0)$ is equivalent to $(\pi, \mathcal{H}, \mathfrak{F}, (G, k))$.

The proof of the reconstruction theorem is contained in [Doplicher and Roberts, 1989] and [Doplicher and Roberts, 1990]. In this article, we give an alternative proof, based on Deligne's embedding theorem [Deligne, 1990], and results obtained by Roberts [Roberts, ND] prior to obtaining the full proof of the reconstruction theorem.

In outline, the theorem shows first — as was essentially established in [Doplicher *et al.*, 1971] — that the DHR superselection sectors naturally have the structure of a braided tensor $*$-category with conjugates — and when the spacetime dimension is three or greater, we can replace "braided" with "symmetric." Now, until the late 1980's, this first result was merely suggestive: it is known that the category $\mathrm{Rep}_f G$ of representations of a compact group G on finite dimensional (super) Hilbert spaces is a symmetric tensor $*$-category with conjugates. Hence, the category of DHR superselection sectors seems to have all the structure of $\mathrm{Rep}_f G$ for some compact G. By the classical Tannaka-Krein duality theorem, it is possible to reconstruct G from $\mathrm{Rep}_f G$. Furthermore, Roberts [Roberts, ND] proved the conditional claim that *if* the category of superselection sectors was equivalent to the category $\mathrm{Rep}_f G$ for some compact G, then the field algebra $\mathfrak{F}$ could be reconstructed.

But there is a crucial difference between the category of superselection sectors and the category $\mathrm{Rep}_f G$. The category $\mathrm{Rep}_f G$ is "concrete" — it comes with an embedding into the category of Hilbert spaces, namely the forgetful functor, and hence its objects can be regarded as structured sets. It is also precisely the existence of such an embedding that is needed to construct a field algebra, because one needs the objects in the category to have "internal structure," as, for example, an object in the category $\mathcal{H}$ of Hilbert spaces is a structured set. Before we state the embedding theorem, whose proof is given in the Appendix, we need some preparatory definitions concerning 'supermathematics'.

10.1 Supermathematics and the embedding theorem

DEFINITION 248. A *super vector space*, alternatively a $\mathbb{Z}_2$*-graded vector space*, is a vector space V with a distinguished decomposition $V = V_+ \oplus V_-$. The subspace V_+ is called the *even* subspace, and V_- is called the *odd* subspace. Elements of $V_+ \cup V_- =: V_h$ are called *homogeneous*. Define the parity function ω on the

homogeneous elements by setting $\omega(v) = \pm 1$ if $v \in V_\pm$. A morphism between two super vector spaces is a linear mapping $T : V \to W$ such that $T(V_\pm) \subseteq W_\pm$. We let SVect denote the category of super vector spaces. A *super Hilbert space* is a super vector space V with a positive definite inner product such that $V_- \perp V_+$. We use $\mathcal{SH}$ to denote the category of super Hilbert spaces.

We now define operations that make SVect into a symmetric tensor category. It is straightforward to verify that the set $\mathrm{Mor}(V, W)$ of morphisms between two super vector spaces is a linear subspace of $B(V, W)$. Thus, SVect is a linear category.

If V and W are super vector spaces, then their direct sum is the vector space $V \oplus W$ with even subspace $V_+ \oplus W_+$ and odd subspace $V_- \oplus W_-$. We define the monoidal product in SVect as the vector space $V \otimes W$ whose even and odd subspaces are defined by

$$(V \otimes W)_\sigma = \bigoplus_{\sigma'\sigma''=\sigma} V_{\sigma'} \otimes W_{\sigma''},$$

where $\sigma = \pm$. Thus,

$$\begin{aligned} (V \otimes W)_+ &= (V_+ \otimes W_+) \oplus (V_- \otimes W_-), \\ (V \otimes W)_- &= (V_+ \otimes W_-) \oplus (V_- \otimes W_+). \end{aligned}$$

The monoidal unit is $\mathbb{C}$, with even subspace $\mathbb{C}$.

DEFINITION 249. For two super vector spaces V, W, we define the symmetry isomorphism

$$c_{V,W} : V \otimes W \to W \otimes V,$$

by setting

$$c_{V,W}(v \otimes w) = (-1)^{(1-\omega(v))(1-\omega(w))/4} w \otimes v, \qquad \forall v \in V_h, \forall w \in W_h.$$

on homogeneous simple tensors, and then by extending linearly.

PROPOSITION 250. *Both* $(\mathrm{SVect}, \otimes, \mathbb{C}, c_{V,W})$ *and* $(\mathcal{SH}, \otimes, \mathbb{C}, c_{V,W})$ *are symmetric tensor* $*$*-categories.*

NOTE 251. By the coherence theorem $\mathcal{SH}$ is equivalent to a strict symmetric tensor $*$-category, which we will also denote by $\mathcal{SH}$.

DEFINITION 252. A *supergroup* is a pair (G, k) where G is a group and k is a central element in G such that $k \cdot k = e$. A morphism between two supergroups (G_1, k_1) and (G_2, k_2) is a group homomorphism $\varphi : G_1 \to G_2$ such that $\varphi(k_1) = k_2$.

DEFINITION 253. A *(unitary) representation* π of a supergroup (G, k) is a super Hilbert space $V = V_+ \oplus V_-$ together with a (unitary) representation π of G on V such that $\pi(k)|_{V_\pm} = \pm \mathrm{id}_{V_\pm}$. The representations $\mathrm{Rep}(G, k)$ of (G, k) form a symmetric tensor $*$-category with tensor product and symmetry inherited from $\mathcal{SH}$, and monoidal unit the trivial representation of (G, k) on $\mathbb{C}$.

NOTE 254. Let $\mathcal{SH}_f$ be the full subcategory of finite dimensional super Hilbert spaces. For a supergroup (G,k) we denote by $\mathrm{Rep}_f(G,k)$ the full subcategory of finite dimensional representations of (G,k). The categories $\mathcal{SH}_f$ and $\mathrm{Rep}_f(G,k)$ are semisimple and have conjugates (see the Appendix for more on this terminology). Also, there is a canonical forgetful functor $K : \mathrm{Rep}_f(G,k) \to \mathcal{SH}_f$.

We now move on to the statement of the Embedding Theorem, which will be required for the reconstruction of the field algebra and gauge group. For more on supermathematics, we refer the reader to [Varadarajan, 2004; Deligne and Morgan, 1999]. (But note that DHR superselection theory is not concerned with supersymmetry in the sense of a symmetry transforming Bosonic and Fermionic fields into each other. Also, our definition of a supergroup is idiosyncratic.)

The Embedding Theorem. Let $\mathcal{SH}_f$ be the category of finite-dimensional super Hilbert spaces over $\mathbb{C}$. Let $(\mathcal{C}, \otimes, \mathbf{1}, c_{X,Y})$ be a tensor C^*-category with unitary symmetry $c_{X,Y}$, conjugates, direct sums, subobjects, and irreducible monoidal unit $\mathbf{1}$. (Such a category is called an STC^* in the Appendix.) Then

1. There is a faithful symmetric tensor $*$-functor $E : \mathcal{C} \to \mathcal{SH}_f$.
2. There is a compact supergroup (G,k), where G is the group of unitary natural monoidal transformations of E, and an equivalence $F : \mathcal{C} \to \mathrm{Rep}_f(G,k)$ of symmetric tensor $*$-categories such that $E = F \circ K$, where $K : \mathrm{Rep}_f(G,k) \to \mathcal{SH}_f$ is the forgetful functor.

NOTE 255. The embedding theorem is proven in Appendix B. In its proof we assume the tensor category $\mathcal{C}$ to be strict and we will work with the strictification $\mathcal{SH}$ of the category of super Hilbert spaces. In view of the coherence theorem for symmetric tensor categories the strictness assumptions do not limit the generality of the result. The tensor functor $F : \mathcal{C} \to \mathcal{SH}_f$ that we construct will, however, not be a strict tensor functor. In the construction of the field net below we do pretend for notational simplicity that F is strict. We will comment on this issue again at the end of this section.

10.2 Construction of the field net, algebraic aspects

We now apply the Embedding Theorem to the case of the DHR category Δ_f of localized transportable morphisms with finite dimension. In particular, we show that given an embedding $E : \Delta_f \to \mathcal{SH}_f$, it is possible to construct a local system of field algebras $(\pi, \mathcal{H}, \mathfrak{F}, (G,k))$. This strategy of reconstruction is based on the unpublished manuscript [Roberts, ND], which assumes the existence of an embedding (or fiber) functor. The actual existence theorem for the embedding

functor — which is based on the work of Tannaka and Deligne, but incorporates more recent simplifications — can be found in the Appendix.

DEFINITION 256. As a set, the field algebra $\mathfrak{F}_0$ consists of equivalence classes of triples (A, ρ, ψ), with $A \in \mathfrak{A}$, $\rho \in \mathrm{Obj}(\Delta_f)$, and $\psi \in E(\rho)$, modulo the equivalence relation

$$(AT, \rho, \psi) = (A, \rho', E(T)\psi),$$

for $T \in \mathrm{Hom}(\rho, \rho')$. Since $E(\lambda \mathrm{id}_\rho) = \lambda \mathrm{id}_{E(\rho)}$ we have $(\lambda A, \rho, \psi) = (A, \rho, \lambda\psi)$. Subsequently, we do not distinguish notationally between a triple (A, ρ, ψ) and its equivalence class.

PROPOSITION 257. *$\mathfrak{F}_0$ is a complex vector space under the operations:*

$$\lambda(A, \rho, \psi) := (\lambda A, \rho, \psi), \qquad \lambda \in \mathbb{C}, \tag{30}$$

and

$$(A_1, \rho_1, \psi_1) + (A_2, \rho_2, \psi_2) := (A_1 W_1^* + A_2 W_2^*, \rho, E(W_1)\psi_1 + E(W_2)\psi_2), \tag{31}$$

where $\psi_i \in E(\rho_i)$ and $W_i \in \mathrm{Hom}(\rho_i, \rho)$ are isometries with

$$W_1 W_1^* + W_2 W_2^* = \mathrm{id}_\rho. \tag{32}$$

In addition,

$$\begin{aligned} (A_1, \rho, \psi) + (A_2, \rho, \psi) &= (A_1 + A_2, \rho, \psi) \\ (A, \rho, \psi_1) + (A, \rho, \psi_2) &= (A, \rho, \psi_1 + \psi_2). \end{aligned}$$

Therefore, identifying $\mathfrak{A}$ with $\{(A, \iota, 1) : A \in \mathfrak{A}, 1 \in \mathbb{C} \equiv E(\iota)\}$, $\mathfrak{A}$ becomes a linear subspace of $\mathfrak{F}_0$; and identifying $E(\rho)$ with $\{(I, \rho, \psi) : \psi \in E(\rho)\}$, $E(\rho)$ becomes a linear subspace of $\mathfrak{F}_0$.

Proof. We first verify that the operations are well defined. Scalar multiplication is well defined since for any $T \in \mathrm{Hom}(\rho, \rho')$, $(\lambda A, \rho', E(T)\psi) = ((\lambda A)T, \rho, \psi) = (\lambda(AT), \rho, \psi)$. To show that addition is well defined, we first establish that Eqn. (31) is independent of the choice of W_1 and W_2. If $W_i' \in \mathrm{Hom}(\rho_i, \rho')$ is another such choice then

$$\begin{aligned} &\Big(A_1 W_1'^* + A_2 W_2'^*, \rho', E(W_1')\psi_1 + E(W_2')\psi_2\Big) \\ &\quad = \Big(A_1 W_1^* + A_2 W_2^* (W_1 W_1'^* + W_2 W_2'^*), \rho', E(W_1')\psi_1 + E(W_2')\psi_2\Big) \\ &\quad = \Big(A_1 W_1^* + A_2 W_2^*, \rho, E(W_1 W_1'^* + W_2 W_2'^*)E(W_1')\psi_1 + E(W_2')\psi_2\Big) \\ &\quad = \Big(A_1 W_1^* + A_2 W_2^*, \rho, E(W_1)\psi_1 + E(W_2)\psi_2\Big). \end{aligned}$$

To see that addition is independent of equivalence classes, let $T_i \in \mathrm{Hom}(\rho_i, \rho_i')$, W_i isometries in $\mathrm{Hom}(\rho_i, \rho)$, and W_i' isometries in $\mathrm{Hom}(\rho_i', \rho)$. Then,

$$\begin{aligned}
&(A_1T_1, \rho_1, \psi_1) + (A_2T_2, \rho_2, \psi_2) \\
&= \Big(A_1T_1W_1^* + A_2T_2W_2^*, \rho, E(W_1)\psi_1 + E(W_2)\psi_2\Big) \\
&= \Big((A_1W_1'^* + A_2W_2'^*)(W_1'T_1W_1^* + W_2'T_2W_2^*), \rho, E(W_1)\psi_1 + E(W_2)\psi_2\Big) \\
&= \Big(A_1W_1'^* + A_2W_2'^*, \rho, E(W_1')E(T_1)\psi_1 + E(W_2')E(T_2)\psi_2\Big) \\
&= (A_1, \rho_1', E(T_1)\psi_1) + (A_2, \rho_2', E(T_2)\psi_2).
\end{aligned}$$

To prove additivity in the first argument, choose $\sigma = \rho \oplus \rho$, and $W_i \in \mathrm{Hom}(\rho, \sigma)$ the corresponding isometries. Then

$$\begin{aligned}
(A_1, \rho, \psi) + (A_2, \rho, \psi) &= (A_1W_1^* + A_2W_2^*, \sigma, (E(W_1) + E(W_2))\psi) \\
&= (A_1W_1^* + A_2W_2^*, \sigma, E(W_1 + W_2)\psi) \\
&= ((A_1W_1^* + A_2W_2^*)(W_1 + W_2), \rho, \psi) \\
&= (A_1 + A_2, \rho, \psi).
\end{aligned}$$

Finally, to prove additivity in the second argument, choose $\sigma = \rho \oplus \rho$, and $W_i \in \mathrm{Hom}(\rho, \sigma)$ the corresponding isometries. Then

$$\begin{aligned}
(A, \rho, \psi_1) + (A, \rho, \psi_2) &= (AW_1^* + AW_2^*, \sigma, E(W_1)\psi_1 + E(W_2)\psi_2) \\
&= (A(W_1^* + W_2^*), \sigma, E(W_1)\psi_1 + E(W_2)\psi_2) \\
&= (A, \rho, E(W_1^* + W_2^*)(E(W_1)\psi_1 + E(W_2)\psi_2)) \\
&= (A, \rho, \psi_1 + \psi_2).
\end{aligned}$$

■

PROPOSITION 258. *The complex linear space $\mathfrak{F}_0$ becomes an algebra if we define*

$$(A_1, \rho_1, \psi_1)(A_2, \rho_2, \psi_2) := \Big(A_1\rho_1(A_2), \rho_1 \otimes \rho_2, \psi_1 \otimes \psi_2\Big), \tag{33}$$

where $\psi_i \in E(\rho_i)$, $i = 1, 2$. Furthermore, $\mathfrak{A}$ is a subalgebra of $\mathfrak{F}_0$, and the equivalence class of $(I, \iota, 1)$ is a multiplicative identity, where I is the multiplicative identity of $\mathfrak{A}$, and $1 \in E(\iota) = \mathbb{C}$.

Proof. We first verify that Eqn. (33) is well-defined on $\mathfrak{F}_0$. Let $T_i \in \mathrm{Hom}(\rho_i, \rho_i')$. Recalling that $T_1 \times T_2 = \rho_1'(T_2)T_1$, we have

$$\begin{aligned}
(A_1T_1, \rho_1, \psi_1)(A_2T_2, \rho_2, \psi_2) &= (A_1T_1\rho_1(A_2T_2), \rho_1 \otimes \rho_2, \psi_1 \otimes \psi_2) \\
&= (A_1\rho_1'(A_2T_2)T_1, \rho_1 \otimes \rho_2, \psi_1 \otimes \psi_2) \\
&= (A_1\rho_1'(A_2)\rho_1'(T_2)T_1, \rho_1 \otimes \rho_2, \psi_1 \otimes \psi_2) \\
&= (A_1\rho_1'(A_2)(T_1 \times T_2), \rho_1 \otimes \rho_2, \psi_1 \otimes \psi_2) \\
&= (A_1\rho_1'(A_2), \rho_1' \otimes \rho_2', E(T_1 \times T_2)(\psi_1 \otimes \psi_2)) \\
&= (A_1\rho_1'(A_2), \rho_1' \otimes \rho_2', E(T_1)\psi_1 \otimes E(T_2)\psi_2) \\
&= (A_1, \rho_1', E(T_1)\psi_1)(A_2, \rho_2', E(T_2)\psi_2).
\end{aligned}$$

A straightforward calculation shows that multiplication is associative. For distributivity, let $W_i \in \mathrm{Hom}(\rho_i, \rho)$. Then,

$$\begin{aligned}
&\Big[(A_1,\rho_1,\psi_1)+(A_2,\rho_2,\psi_2)\Big](A_3,\rho_3,\psi_3)\\
&= \big((A_1W_1^*+A_2W_2^*)\rho(A_3),\rho\otimes\rho_3,(E(W_1)\psi_1+E(W_2)\psi_2)\otimes\psi_3\big)\\
&= \big(A_1\rho_1(A_3)W_1^*+A_2\rho_2(A_3)W_2^*,\rho\otimes\rho_3,(E(W_1)\psi_1)\otimes\psi_3+(E(W_2)\psi_2)\otimes\psi_3\big)\\
&= \big(A_1\rho_1(A_3)(W_1^*\times I_{\rho_3})+A_2\rho_2(A_3)(W_2^*\times I_{\rho_3}),\rho\otimes\rho_3,(E(W_1)\psi_1)\otimes\psi_3+\\
&\quad +(E(W_2)\psi_2)\otimes\psi_3)\big)\\
&= (A_1,\rho_1,\psi_1)(A_3,\rho_3,\psi_3)+(A_2,\rho_2,\psi_2)(A_3,\rho_3,\psi_3).
\end{aligned}$$

■

We will need the following basic lemma from linear algebra.

DEFINITION 259. If H, H' are Hilbert spaces and $S \in \mathrm{Hom}(H\otimes H',\mathbb{C})$, then we define an antilinear mapping $\mathcal{J}S : H \to H'$ by setting

$$((\mathcal{J}S)x,x') = S(x\otimes x'), \qquad \forall x\in H, \forall x'\in H'.$$

LEMMA 260.

1. *$\mathcal{J}$ is antilinear: $\mathcal{J}(\lambda S)=\overline{\lambda}(\mathcal{J}S)$, and $\mathcal{J}(S_1+S_2)=\mathcal{J}S_1+\mathcal{J}S_2$.*
2. *If $T\in\mathrm{Hom}(H',H)$ then*

$$\begin{aligned}
T\circ(\mathcal{J}S) &= \mathcal{J}(S\circ(I_H\otimes T^*)),\\
(\mathcal{J}S)\circ T &= \mathcal{J}(S\circ(T\otimes I_{H'})).
\end{aligned}$$

3. *If $S'\in\mathrm{Hom}(H'\otimes H'',\mathbb{C})$ then $(\mathcal{J}S')\circ(\mathcal{J}S)=(S\otimes I_{H''})\circ(I_H\otimes S'^*)$.*
4. *Let $S_1\in\mathrm{End}(H_1\otimes H_1',\mathbb{C})$ and $S_2\in\mathrm{End}(H_2\otimes H_2',\mathbb{C})$. Then*

$$(\mathcal{J}S_2\otimes\mathcal{J}S_1)\circ\Sigma_{H_1,H_2} = \mathcal{J}[S_1\circ(I_{H_1}\otimes S_2\otimes I_{H_1'})].$$

Proof. Straightforward. A nice exercise in basic linear algebra. ■

NOTE 261. We will apply the previous Lemma to super Hilbert spaces. But we will take $\Sigma_{H,H'}$ to be the ordinary symmetry on the category $\mathcal{H}_f$ of finite dimensional Hilbert spaces.

LEMMA 262. *Let $T\in\mathrm{Hom}(\rho,\rho')$ and pick solutions $(\overline{\rho},R,\overline{R})$ and $(\overline{\rho}',R',\overline{R}')$ of the conjugate equations with respect to ρ and ρ'; that is, $R\in\mathrm{Hom}(\iota,\overline{\rho}\otimes\rho)$, $\overline{R}\in\mathrm{Hom}(\iota,\rho\otimes\overline{\rho})$ such that $(\overline{R}^*\times I_\rho)\circ(I_\rho\times R)=I_\rho$, $(R^*\times I_{\overline{\rho}})\circ(I_{\overline{\rho}}\times\overline{R})=I_{\overline{\rho}}$, and analogously for R' and $\overline{R}'$. Set*

$$\overline{T} := (I_{\overline{\rho}}\times\overline{R}'^*)\circ(I_{\overline{\rho}}\times T\times 1_{\overline{\rho}'})\circ(R\times I_{\overline{\rho}'}) = \overline{\rho}(\overline{R}'^*T)R.$$

Then $\overline{T} \in \mathrm{Hom}(\overline{\rho}', \overline{\rho})$ *and*

$$(34) \quad (I_{\overline{\rho}} \times T) \circ R = (\overline{T} \times I_{\rho'}) \circ R',$$

$$(35) \quad (I_{\rho} \times \overline{T}^*) \circ \overline{R} = (T^* \times I_{\overline{\rho}'}) \circ \overline{R}'.$$

Proof. For Eqn. (34), we have

$$\begin{aligned}(\overline{T} \times I_{\rho'}) \circ R' &= \overline{T}R' = \overline{\rho}(\overline{R}'^* T)RR' = \overline{\rho}(\overline{R}'^* T \rho(R'))R \\ &= \overline{\rho}(\overline{R}'^* \rho'(R')T)R,\end{aligned}$$

where we used the definition of $\times$ for the first equality, the definition of $\overline{T}$ for the second equality, $R \in \mathrm{Hom}(\iota, \overline{\rho} \otimes \rho)$ for the third equality, and $T \in \mathrm{Hom}(\rho, \rho')$ for the fourth equality. But by the conjugate equations, $\overline{R}'^* \rho'(R') = (\overline{R}'^* \times I_{\rho'}) \circ (I_{\rho'} \times R') = I_{\rho'} = I$, and hence $(\overline{T} \times I_{\rho'}) \circ R' = \overline{\rho}(T)R = (I_{\overline{\rho}} \times T) \circ R$. For Eqn. (35), we have

$$(36) \quad (I_{\rho} \times \overline{T}^*) \circ \overline{R} = \rho(\overline{T}^*)\overline{R} = \rho(R^*)\rho\overline{\rho}(T^*\overline{R}')\overline{R} = \rho(R^*)\overline{R}T^*\overline{R}',$$

where we used the definition of $\overline{T}$ for the second equality, and $\overline{R} \in \mathrm{Hom}(\iota, \rho\otimes\overline{\rho})$ for the third equality. But by the conjugate equations $\rho(R^*)\overline{R} = (I_\rho \times R^*) \circ (\overline{R} \times I_\rho) = I_\rho$, and hence $(I_\rho \times \overline{T}^*) \circ \overline{R} = T^*\overline{R}' = (T^* \times I_{\overline{\rho}'}) \circ \overline{R}'$. ∎

PROPOSITION 263. *The algebra* $\mathfrak{F}_0$ *becomes a* $*$*-algebra if we define*

$$(37) \quad (A, \rho, \psi)^* := (R^*\overline{\rho}(A)^*, \overline{\rho}, \mathcal{J}E(\overline{R}^*)\psi),$$

where $\psi \in E(\rho)$*, and* $(\overline{\rho}, R, \overline{R})$ *is a conjugate to* ρ*.*

Proof. We first show that the definition of $*$ is independent of the choice of conjugate to ρ. For this, let $(\overline{\rho}_1, R_1, \overline{R}_1)$ be any other choice. Define $W \in \mathrm{Hom}(\overline{\rho}, \overline{\rho}_1)$ by

$$(38) \quad W := (R^* \times I_{\overline{\rho}_1}) \circ (I_{\overline{\rho}} \times \overline{R}_1) = R^*\overline{\rho}(\overline{R}_1)$$

we have by the conjugate equations

$$W^{-1} := (R_1^* \times I_{\overline{\rho}}) \circ (I_{\overline{\rho}_1} \times \overline{R}) = R_1^*\overline{\rho}_1(\overline{R}).$$

Moreover,

$$\begin{aligned}(R_1^*\overline{\rho}_1(A)^*, \overline{\rho}_1, \mathcal{J}E(\overline{R}_1^*)\psi) &= (R^*W^{-1}\overline{\rho}_1(A)^*, \overline{\rho}_1, \mathcal{J}E(\overline{R}^*(I_\rho \times W^*))\psi) \\ &= (R^*\overline{\rho}(A)^*, \overline{\rho}, E(W^{-1})\mathcal{J}E(\overline{R}^*(I_\rho \times W^*))\psi) \\ &= (R^*\overline{\rho}(A)^*, \overline{\rho}, \mathcal{J}E(\overline{R}^*)\psi),\end{aligned}$$

where we used Lemma 260.3 for the final equality.

To see that the definition of $*$ is independent of equivalence classes, suppose that $T \in \mathrm{Hom}(\rho, \rho')$ and $\psi \in E(\rho)$. Then

$$\begin{aligned}
(AT, \rho, \psi)^* &= (R^*\overline{\rho}(T^*A^*), \overline{\rho}, \mathcal{J}E(\overline{R}^*)\psi) \\
&= (R'^*\overline{T}^*\overline{\rho}(A^*), \overline{\rho}, \mathcal{J}E(\overline{R}^*)\psi) \\
&= (R'^*\overline{\rho}'(A^*)\overline{T}^*, \overline{\rho}, \mathcal{J}E(\overline{R}^*)\psi) \\
&= (R'^*\overline{\rho}'(A^*), \overline{\rho}', E(\overline{T}^*)\mathcal{J}E(\overline{R}^*)\psi) \\
&= (R'^*\overline{\rho}'(A^*), \overline{\rho}', \mathcal{J}E(\overline{R}^* \circ (I_\rho \times \overline{T}))\psi) \\
&= (R'^*\overline{\rho}'(A^*), \overline{\rho}', \mathcal{J}E(\overline{R}'^* \circ (T \times I_{\overline{\rho}'}))\psi) \\
&= (R'^*\overline{\rho}'(A^*), \overline{\rho}', \mathcal{J}E(\overline{R}'^*)E(T)\psi) \\
&= (A, \rho', E(T)\psi)^*,
\end{aligned}$$

where we used Eqn. (34) for the second equality, the fact that $\overline{T}^* \in \mathrm{Hom}(\overline{\rho}, \overline{\rho}')$ for the third equality, Lemma 260.2 for the fifth equality, and Eqn. (35) for the sixth equality.

We verify that $*$ is involutive:

$$\begin{aligned}
(A, \rho, \psi)^{**} &= (R^*\overline{\rho}(A)^*, \overline{\rho}, \mathcal{J}E(\overline{R}^*)\psi)^* \\
&= (\overline{R}^*\rho(\overline{\rho}(A)R), \rho, \mathcal{J}E(R^*)\mathcal{J}E(\overline{R}^*)\psi) \\
&= (A\overline{R}^*\rho(R), \rho, \mathcal{J}E(R^*)\mathcal{J}E(\overline{R}^*)\psi) \\
&= (A, \rho, E((R^* \times I_{\overline{\rho}})(I_{\overline{\rho}} \times \overline{R}))\psi) \\
&= (A, \rho, \psi),
\end{aligned}$$

where we used Lemma 260.3 for the penultimate equality, and the conjugate equations for the final equality.

To verify that $*$ is antilinear, let $W_i \in \mathrm{Hom}(\rho_i, \rho)$. Then,

$$\begin{aligned}
&\big[(A_1, \rho_1, \psi_1) + (A_2, \rho_2, \psi_2)\big]^* = \big(A_1W_1^* + A_2W_2^*, \rho, E(W_1)\psi_1 + E(W_2)\psi_2\big)^* \\
&= (R^*\overline{\rho}(W_1A_1^* + W_2A_2^*), \overline{\rho}, \mathcal{J}E(\overline{R}^*)(E(W_1)\psi_1 + E(W_2)\psi_2)). \qquad (39)
\end{aligned}$$

But we may take $R = (\overline{W}_1 \times W_1) \circ R_1 + (\overline{W}_2 \times W_2) \circ R_2$, $\overline{R} = (W_1 \times \overline{W}_1) \circ \overline{R}_1 + (W_2 \times \overline{W}_2) \circ \overline{R}_2$, where $\overline{W}_i \in \mathrm{Hom}(\overline{\rho}_i, \overline{\rho})$ are isometries, $\overline{W}_1\overline{W}_1^* + \overline{W}_2\overline{W}_2^* = I_{\overline{\rho}}$. Then Eqn. (39) becomes

$$\begin{aligned}
&[(A_1, \rho_1, \psi_1) + (A_2, \rho_2, \psi_2)]^* \\
&= (R_1^*\overline{\rho}_1(A_1^*)\overline{W}_1^* + R_2^*\overline{\rho}_2(A_2^*)\overline{W}_2^*, \overline{\rho}, \mathcal{J}E(\overline{R}_1^*(I_{\rho_1} \times \overline{W}_1^*))\psi_1 + \\
&\quad \mathcal{J}E(\overline{R}_2^*(I_{\rho_2} \times \overline{W}_2^*))\psi_2) \\
&= (R_1^*\overline{\rho}_1(A_1^*)\overline{W}_1^* + R_2^*\overline{\rho}_2(A_2^*)\overline{W}_2^*, \overline{\rho}, E(\overline{W}_1)\mathcal{J}E(\overline{R}_1^*)\psi_1 + E(\overline{W}_2)\mathcal{J}E(\overline{R}_2^*)\psi_2) \\
&= (A_1, \rho_1, \psi_1)^* + (A_2, \rho_2, \psi_2)^*,
\end{aligned}$$

using Lemma 260 for the second equality.

Finally, we show that $[(A_1,\rho_1,\psi_1)(A_2,\rho_2,\psi_2)]^* = (A_2,\rho_2,\psi_2)^*(A_1,\rho_2,\psi_1)^*$. If $\rho = \rho_1\otimes\rho_2$ and $\rho' = \rho_1'\otimes\rho_2'$ then we may take $R = (I_{\overline{\rho}_2}\times R_1\times I_{\rho_2})\circ R_2$ and $\overline{R} = (I_{\rho_1}\times \overline{R}_2\times I_{\overline{\rho}_1})\circ \overline{R}_1$. Thus,

$$\begin{aligned}
[(A_1,\rho_1,\psi_1)(A_2,\rho_2,\psi_2)]^* &= (A_1\rho_1(A_2),\rho_1\otimes\rho_2,\psi_1\otimes\psi_2)^*\\
&= (R_2^*\overline{\rho}_2(R_1^*)\overline{\rho}_2\overline{\rho}_1(\rho_1(A_2^*)A_1^*),\overline{\rho}_2\otimes\overline{\rho}_1,\mathcal{J}E(\overline{R}_1^*\circ(I_{\rho_1}\times\overline{R}_2^*\times I_{\overline{\rho}_1}))\psi_1\otimes\psi_2)\\
&= (R_2^*\overline{\rho}_2(A_2^*)\overline{\rho}_2(R_1^*\overline{\rho}_1(A_1^*)),\overline{\rho}_2\otimes\overline{\rho}_1,\mathcal{J}E(\overline{R}_1^*\circ(I_{\rho_1}\times\overline{R}_2^*\times I_{\overline{\rho}_1}))\psi_1\otimes\psi_2)\\
&= (R_2^*\overline{\rho}_2(A_2^*)\overline{\rho}_2(R_1^*\overline{\rho}_1(A_1^*)),\overline{\rho}_2\otimes\overline{\rho}_1,\mathcal{J}E(\overline{R}_2^*)\psi_2\otimes\mathcal{J}E(\overline{R}_1^*)\psi_1)\\
&= (R_2^*\overline{\rho}_2(A_2^*),\overline{\rho}_2,\mathcal{J}E(\overline{R}_2^*)\psi_2)(R_1^*\overline{\rho}_1(A_1^*),\overline{\rho}_1,\mathcal{J}E(\overline{R}_1^*)\psi_1)\\
&= (A_2,\rho_2,\psi_2)^*(A_1,\rho_1,\psi_1)^*,
\end{aligned}$$

where the third equality follows from the fact that $R_1^*\in\mathrm{Hom}(\overline{\rho}_1\otimes\rho_1,\iota)$, and the fourth equality follows by Lemma 260.4. ■

PROPOSITION 264. *Let $E:\Delta_f\to\mathcal{SH}_f$ be the embedding functor from the DHR category Δ_f into the strictified category $\mathcal{SH}_f$ of finite dimensional super Hilbert spaces. Then the formula*

$$\alpha_g(A,\rho,\psi) = (A,\rho,g_\rho\psi),\qquad A\in\mathfrak{A},\psi\in E(\rho). \tag{40}$$

defines a group isomorphism $g\mapsto\alpha_g$ from the intrinsic group G of E into $\mathrm{Aut}_{\mathfrak{A}}\mathfrak{F}_0$, the group of $*$-automorphisms of $\mathfrak{F}_0$ leaving $\mathfrak{A}$ pointwise fixed.

Proof. Since g is a natural monoidal transformation, $g_\iota = \mathrm{id}_{E(\iota)} = \mathrm{id}_{\mathbb{C}}$. For any $g\in G$, α_g is well defined on $\mathfrak{F}_0$ since for $S\in\mathrm{Hom}(\rho,\rho')$,

$$\begin{aligned}
\alpha_g(AS,\rho,\psi) &= (AS,\rho,g_\rho\psi) = (A,\rho',E(S)g_\rho\psi)\\
&= (A,\rho',g(\rho')E(S)\psi) = \alpha_g(A,\rho',E(S)\psi).
\end{aligned}$$

Since $g_\iota=\mathrm{id}_{\mathbb{C}}$, α_g leaves $\mathfrak{A}\subset\mathfrak{F}_0$ pointwise fixed. Each g_ρ is linear so α_g is linear.

$$(A_1,\rho_1,g_{\rho_1}\psi_1)(A_2,\rho_2,g_{\rho_2}\psi_2) = \Big(A_1\rho_1(A_2),\rho_1\otimes\rho_2,(g_{\rho_1}\otimes g_{\rho_2})(\psi_1\otimes\psi_2)\Big),$$

but $g_{\rho_1\otimes\rho_2} = g_{\rho_1}\otimes g_{\rho_2}$ so

$$(A_1,\rho_1,g_{\rho_1}\psi_1)(A_2,\rho_2,g_{\rho_2}\psi_2) = \Big(A_1\rho_1(A_2),\rho_1\otimes\rho_2,g_{\rho_1\otimes\rho_2}(\psi_1\otimes\psi_2)\Big).$$

Thus,

$$\alpha_g(F_1)\alpha_g(F_2) = \alpha_g(F_1F_2). \tag{41}$$

To show that α_g is a $*$-homomorphism, recall that

$$(A,\rho,g_\rho\psi)^* = (R^*\overline{\rho}(A)^*,\overline{\rho},\mathcal{J}E(\overline{R}^*)g_\rho\psi). \tag{42}$$

If $\overline{\psi}\in E(\overline{\rho})$ then $E(\overline{R}^*)(g_\rho\psi\otimes g_{\overline{\rho}}\overline{\psi}) = E(\overline{R}^*)((g_\rho\otimes g_{\overline{\rho}})(\psi\otimes\overline{\psi}))$. Furthermore,

$$E(\overline{R}^*)(g_\rho\psi\otimes g_{\overline{\rho}}\overline{\psi}) = g_\iota E(\overline{R}^*)(\psi\otimes\overline{\psi}) = E(\overline{R}^*)(\psi\otimes\overline{\psi}).$$

Hence $g_{\overline{\rho}}^* \mathcal{J} E(\overline{R}^*) g_\rho = \mathcal{J} E(\overline{R}^*)$ and since $g_{\overline{\rho}}$ is unitary we get from (42),

$$(A, \rho, g_\rho \psi)^* = (R^* \overline{\rho}(A)^*, \overline{\rho}, \mathcal{J} E(\overline{R}^*) g_\rho \psi) = (R^* \overline{\rho}(A)^*, \overline{\rho}, g_{\overline{\rho}} \mathcal{J} E(\overline{R}^*) \psi),$$

so

$$\alpha_g(F^*) = \alpha_g(F)^*, \qquad F \in \mathfrak{F}. \tag{43}$$

Equations (41), (43) show that α_g is a $*$-homomorphism, its inverse is clearly $\alpha_{g^{-1}}$ so α_g defined by Eqn. (40) is an element of $\mathrm{Aut}_{\mathfrak{A}}\mathfrak{F}$. The mapping $g \mapsto \alpha_g$ is clearly a homomorphism.

Since G is a compact group, for every $g \neq e$, there exists a $(H, \pi) \in \mathrm{Rep}_f G$ such that $\pi(g) \neq \mathrm{id}_H$. Since the functor E is an equivalence, in particular essentially surjective, there exists a $\rho \in \mathrm{Obj}(\Delta_f)$ such that $E(\rho)$ is isomorphic to (H, π). Thus there exists $\psi \in E(\rho)$ such that

$$\pi(g)\psi = g_\rho \psi \neq \psi.$$

Defining $F = (I, \rho, \psi)$, we have $\alpha_g(F) \neq F$. This proves injectivity of $g \mapsto \alpha_g$.

It remains to show that $G \mapsto \mathrm{Aut}_{\mathfrak{A}}\mathfrak{F}_0$ is onto. Let $\alpha \in \mathrm{Aut}_{\mathfrak{A}}\mathfrak{F}_0$, $A \in \mathfrak{A}$ and $\psi \in E(\rho) \subset \mathfrak{F}_0$. Let $\Psi = (I, \rho, \psi)$. Then

$$(\alpha(\Psi))A = \alpha(\Psi A) = \alpha(\rho(A)\Psi) = \rho(A)\alpha(\Psi).$$

It is easily checked that this implies that $\alpha(\Psi)$ is of the form (I, ρ, ψ') with $\psi' \in E(\rho)$. Thus $\psi \mapsto \psi'$ is a linear map of $E(\rho)$ into $E(\rho)$ which we denote by g_ρ, and it remains to show that $g = (g_\rho)_{\rho \in \Delta_f}$ is monoidal natural transformation of E. For $S \in \mathrm{Hom}(\rho, \rho')$, we have

$$(S, \rho, g_\rho \psi) = \alpha(S, \rho, \psi) = \alpha(I, \rho', E(S)\psi) = (I, \rho', g_{\rho'} E(S)\psi).$$

Hence

$$E(S) g_\rho \psi = g_{\rho'} E(S)\psi, \qquad \psi \in E(\rho).$$

That is,

$$E(S) g_\rho = g_{\rho'} E(S),$$

and $g \in \mathrm{Nat}E$. To check monoidality, choose arbitrary $\psi_i \in E(\rho_i)$ and let $\Psi_i = (I, \rho_i, \psi_i)$. Then,

$$g_{\rho_1 \otimes \rho_2}(\psi_1 \otimes \psi_2) \;=\; \alpha(\Psi_1 \Psi_2) \;=\; \alpha(\Psi_1)\alpha(\Psi_2) \;=\; (g_{\rho_1} \otimes g_{\rho_2})(\psi_1 \psi_2).$$

Thus, $g \in \mathrm{Nat}_\otimes E$. It remains to show that g is unitary. For $\psi, \psi' \in E(\rho)$ and $\Psi = (I, \rho, \psi), \Psi' = (I, \rho, \psi')$ we have

$$\langle g_\rho \psi, g_\rho \psi' \rangle_{E(\rho)} I = \alpha(\Psi)^* \alpha(\Psi') = \alpha(\Psi^* \Psi') = \langle \psi, \psi' \rangle_{E(\rho)} I,$$

where the first and last equalities follow from Prop. 270. Hence g_ρ is unitary for each $\rho \in \mathrm{Obj}(\Delta_f)$. Therefore every $\alpha \in \mathrm{Aut}_{\mathfrak{A}}\mathfrak{F}_0$ is of the form α_g with $g \in G = \mathrm{Nat}_\otimes E$. ∎

DEFINITION 265. Given a double cone O, we define $\mathfrak{F}_0(O)$ to consist of those elements F in $\mathfrak{F}_0$ such that there exists $A \in \mathfrak{A}(O)$, $\rho \in \mathrm{Obj}(\Delta_f)$ localized in O, and $\psi \in E(\rho)$ with $F = (A, \rho, \psi)$.

PROPOSITION 266. *$\mathfrak{F}_0(O)$ is a $*$-subalgebra of $\mathfrak{F}_0$.*

Proof. Let $F_1 = (A_1, \rho_1, \psi_1)$ and $F_2 = (A_2, \rho_2, \psi_2)$ be in $\mathfrak{F}_0(O)$. Thus, the A_i can be chosen from $\mathfrak{A}(O)$ and the ρ_i can be chosen localized in O. Since $\rho_1(\mathfrak{A}(O)) \subseteq \mathfrak{A}(O)$, it follows that

$$F_1F_2 = (A_1\rho_1(A_2), \rho_1 \otimes \rho_2, \psi_1 \otimes \psi_2),$$

is also in $\mathfrak{F}_0(O)$. By transportability, $\overline{\rho}$ can be chosen localized in O, and in this case $\overline{\rho} \otimes \rho$ is localized in O. By Lemma 155, $R \in \mathfrak{A}(O)$. Hence,

$$F^* = (R^*\overline{\rho}(A)^*, \overline{\rho}, \mathcal{J}E(\overline{R}^*)\psi),$$

is in $\mathfrak{F}_0(O)$. Similarly, $\mathfrak{F}_0(O)$ is closed under the addition defined in Prop. 257 since ρ can also be chosen localized in O, and then the isometries W_1, W_2 are in $\mathfrak{A}(O)$ (by Lemma 155). ■

PROPOSITION 267. *The action of G on $\mathfrak{F}_0$ leaves $\mathfrak{F}_0(O)$ globally fixed.*

Proof. If $F \in \mathfrak{F}_0(O)$ then $F = (A, \rho, \psi)$ for some $A \in \mathfrak{A}(O)$ and ρ localized in O. Then clearly $\alpha_g(F) = (A, \rho, g_\rho\psi)$ is in $\mathfrak{F}_0(O)$. ■

NOTE 268. Having defined an action of the supergroup (G, k), the element $k \in G$ induces a $\mathbb{Z}_2$ grading on $\mathfrak{F}_0$ and on the local algebras $\mathfrak{F}_0(O)$.

PROPOSITION 269. *The field net $\mathfrak{F}_0$ satisfies normal commutation relations. That is, if O_1 and O_2 are spacelike, and $F_i \in \mathfrak{F}(O_i)$ are such that*

$$\alpha_k(F_i) = \sigma_i F_i,$$

then

$$F_1F_2 = (-1)^{(1-\sigma_1)(1-\sigma_2)/4}F_2F_1.$$

Proof. Choose $F_i = (A_i, \rho_i, \psi_i)$ with $A_i \in \mathfrak{A}(O_i)$ and ρ_i localized in O_i. Then $A_1A_2 = A_2A_1$, $\rho_1(A_2) = A_2$, $\rho_2(A_1) = A_1$, and $\varepsilon_{\rho_1,\rho_2} = \mathrm{id}_{\rho_1\otimes\rho_2}$. In view of the way G acts on $\mathfrak{F}_0$ we have

$$\sigma_i(A_i, \rho_i, \psi_i) = \alpha_k(A_i, \rho_i, \psi_i) = (A_i, \rho_i, k_{\rho_i}\psi_i),$$

and hence $k_{\rho_i}\psi_i = \sigma_i\psi_i$. That is, ψ_i is homogeneous and $\omega(\psi_i) = \sigma_i$. Furthermore, since E is a symmetric functor $E(\varepsilon_{\rho_1,\rho_2}) = \Sigma_{E(\rho_1),E(\rho_2)}$, where $\Sigma_{H,H'}$ is the symmetry on $\mathcal{SH}_f$ and therefore

$$\Sigma_{H,H'}(\psi_1 \otimes \psi_2) = (-1)^{(1-\sigma_1)(1-\sigma_2)/4}(\psi_2 \otimes \psi_1).$$

Hence

$$\begin{aligned} F_1F_2 &= (A_1\rho_1(A_2), \rho_1\otimes\rho_2, \psi_1\otimes\psi_2) \\ &= (A_1A_2\varepsilon_{\rho_2,\rho_1}, \rho_1\otimes\rho_2, \psi_1\otimes\psi_2) \\ &= (A_1A_2, \rho_2\otimes\rho_1, E(\varepsilon_{\rho_2,\rho_1})(\psi_1\otimes\psi_2)) \\ &= (A_2\rho_2(A_1), \rho_2\otimes\rho_1, E(\varepsilon_{\rho_2,\rho_1})(\psi_1\otimes\psi_2)) \\ &= (A_2\rho_2(A_1), \rho_2\otimes\rho_1, \Sigma_{E(\rho_2),E(\rho_1)}(\psi_1\otimes\psi_2)) \\ &= (-1)^{(1-\sigma_1)(1-\sigma_2)/4}(A_2\rho_2(A_1), \rho_2\otimes\rho_1, \psi_2\otimes\psi_1) \\ &= (-1)^{(1-\sigma_1)(1-\sigma_2)/4}F_2F_1. \end{aligned}$$

■

PROPOSITION 270. *For all* $\Psi = (I,\rho,\psi), \Psi' = (I,\rho,\psi')$ *with* $\psi,\psi' \in E(\rho)$ *we have*

$$\Psi A = \rho(A)\Psi, \tag{44}$$

$$\Psi^*\Psi' = \langle\psi,\psi'\rangle_{E(p)}. \tag{45}$$

For any orthonormal basis $\{\psi_i : i = 1,\dots,n\}$ *of* $E(\rho)$*, we have*

$$\sum_{i=1}^{n}\Psi_i\Psi_i^* = I. \tag{46}$$

Proof.

$$(I,\rho,\psi)(A,\iota,1) = (\rho(A),\rho,\psi) = (\rho(A),\iota,1)(I,\rho,\psi),$$

whence (44). For (45), we check:

$$\begin{aligned} (I,\rho,\psi)^*(I,\rho,\psi') &= (R^*,\overline{\rho}\otimes\rho,(\mathcal{J}E(\overline{R}^*)\psi)\otimes\psi') \\ &= (I,\iota,E(R^*)((\mathcal{J}E(\overline{R}^*)\psi)\otimes\psi')). \end{aligned}$$

Since $\mathcal{J}E(\overline{R}^*) : E(\rho)\to E(\overline{\rho})$ and $E(R^*) : E(\overline{\rho})\otimes E(\rho)\to\mathbb{C}$, it follows that $E(R^*)((\mathcal{J}E(\overline{R}^*)\psi)\otimes\psi')$ is a complex number. In fact, by the definition of $\mathcal{J}$ and Lemma 260.3,

$$\begin{aligned} E(R^*)((\mathcal{J}E(\overline{R}^*)\psi)\otimes\psi') &= \big\langle\mathcal{J}E(R^*)\circ\mathcal{J}E(\overline{R}^*)\psi,\psi'\big\rangle_{E(\rho)} \\ &= \big\langle\mathcal{J}E((R^*\times I_\rho)\circ(I_\rho\times\overline{R}^*))\psi,\psi'\big\rangle_{E(\rho)} \\ &= \big\langle\psi,\psi'\big\rangle_{E(\rho)}, \end{aligned}$$

where the final equality follows by the conjugate equations. So, combining the previous two equations we have

$$(I,\rho,\psi)^*(I,\rho,\psi') = \big(I,\iota,\langle\psi,\psi'\rangle_{E(\rho)}\big) = \langle\psi,\psi'\rangle_{E(\rho)}\,\big(I,\iota,1\big).$$

For Eqn. (46), we have

$$\sum_i (I,\rho,\psi_i)(I,\rho,\psi_i)^* = \Big(\rho(R)^*, \rho\otimes\overline{\rho}, \sum \psi_i \otimes \mathcal{J}E(\overline{R}^*)\psi_i\Big)$$
$$= (\rho(R)^*, \rho\otimes\overline{\rho}, E(\overline{R})1) = (\rho(R)^*\overline{R}, \iota, 1) \;=\; (I,\iota,1),$$

where the second equality follows from the definition of $\mathcal{J}$ and the final equality follows by the conjugate equations. ■

10.3 Completion of the field net

We now construct a representation $(\mathcal{H},\pi)$ of the $*$-algebra $\mathfrak{F}_0$, and show that $\pi|_{\mathfrak{A}}$ has a nontrivial subrepresentation equivalent to the GNS representation induced by the vacuum state ω_0. We do so by extending the state ω_0 from $\mathfrak{A}$ to $\mathfrak{F}_0$, and then by taking the GNS representation. In order to extend the state ω_0 from $\mathfrak{A}$ to $\mathfrak{F}_0$, it suffices to show that there is a positive linear map $m:\mathfrak{F}_0\to\mathfrak{A}$.

NOTE 271. Let $\rho\in \mathrm{Obj}(\Delta_f)$. Since Δ_f is semisimple (see Prop. 218), ρ is a finite direct sum $\rho=\rho_1\oplus\cdots\oplus\rho_n$ of irreducible objects in $\mathrm{Obj}(\Delta_f)$. Therefore, there is a projection $P_\iota^\rho\in\mathrm{End}(\rho)$ onto the direct sum of those irreducibles in this decomposition that are isomorphic to ι.

PROPOSITION 272. *Given $(A,\rho,\psi)\in\mathfrak{F}_0$, define*

(47) $\quad m(A,\rho,\psi) \;:=\; (AP_\iota^\rho,\rho,\psi).$

Then $m:\mathfrak{F}_0\to\mathfrak{A}$ is a faithful positive linear projection from $\mathfrak{F}_0$ onto $\mathfrak{A}$. Further,

(48) $\quad m(AF) = Am(F), \qquad A\in\mathfrak{A}, F\in\mathfrak{F}_0.$

Proof. We first show that m is well defined. If $T\in\mathrm{Hom}(\rho,\rho')$ then $TP_\iota^\rho = P_\iota^{\rho'}TP_\iota^\rho = P_\iota^{\rho'}T$, hence

$$m(AT,\rho,\psi) = (ATP_\iota^\rho,\rho,\psi) = (AP_\iota^{\rho'}T,\rho,\psi) = (AP_\iota^{\rho'},\rho',E(T)\psi) = m(A,\rho',E(T)\psi),$$

as required. m is clearly linear and satisfies Eqn. (48). We now show that m is positive. First, since ρ has finite dimension, ρ contains at most finitely many copies of the vacuum representation. Thus, $P_\iota^\rho=\sum_i S_iS_i^*$ where $S_i\in\mathrm{Hom}(\iota,\rho)$ and $S_i^*S_j=\delta_{ij}\mathrm{id}_\iota$. Thus,

$$m(A,\rho,\psi) = (AP_\iota^\rho,\rho,\psi) = \sum_i (AS_i,\iota,E(S_i^*)\psi).$$

However, $E(S_i^*)\psi=\lambda_i 1$ so that

$$m(A,\rho,\psi) = \sum_i \lambda_i (AS_i,\iota,1)\in\mathfrak{A}.$$

Since each $\rho \in \mathrm{Obj}(\Delta_f)$ is a finite direct sum of irreducible objects (Prop. 218), any $F \in \mathfrak{F}_0$ may be written as a finite sum $F = \sum_i F_i$, $F_i = (A_i, \rho_i, \psi_i)$, where $\psi_i \in E(\rho_i)$ with ρ_i irreducible and pairwise inequivalent. Thus,

$$m(F^*F) = \sum_{i,j} m(F_i^* F_j) = \sum_i m(F_i^* F_i).$$

Hence, to show that m is positive and faithful, it suffices to consider $m(F^*F)$ with $F = (A, \rho, \psi)$, $\psi \in E(\rho)$ and ρ irreducible. In this case,

$$(A, \rho, \psi)^*(A, \rho, \psi) = \big(R^*\overline{\rho}(A^*A), \overline{\rho} \otimes \rho, \mathcal{J}E(\overline{R}^*)(\psi \otimes \psi)\big).$$

Using $P_\iota^{\overline{\rho}\otimes\rho} = \|RR^*\|^{-1} RR^* = d(\rho)^{-1} RR^*$, we have

$$\begin{aligned} d(\rho)m(F^*F) &= (R^*\overline{\rho}(A^*A)RR^*, \overline{\rho} \otimes \rho, \mathcal{J}E(\overline{R}^*)\psi \otimes \psi) \\ &= (R^*\overline{\rho}(A^*A)R, \iota, E(R^*)\mathcal{J}E(\overline{R}^*)\psi \otimes \psi). \end{aligned}$$

Now,

$$E(R^*)\mathcal{J}E(\overline{R}^*)(\psi \otimes \psi) = \big\langle \mathcal{J}E(R^*)\mathcal{J}E(\overline{R}^*)\psi, \psi \big\rangle_{E(\rho)},$$

hence by Lemma 260,

$$\begin{aligned} d(\rho)m(F^*F) &= R^*\overline{\rho}(A^*A)R\,\big\langle E((\overline{R}^* \times I_\rho) \circ (I_\rho \times R))\psi, \psi \big\rangle_{E(\rho)} \\ &= R^*\overline{\rho}(A^*A)R\,\big\langle \psi, \psi \big\rangle_{E(\rho)}. \end{aligned}$$

Thus, $m(F^*F) \geq 0$ and $m(F^*F) = 0$ implies $\psi = 0$ or $\overline{\rho}(A)R = 0$. But $\overline{\rho}(A)R = 0$ only if

$$0 = \overline{R}^* \rho\overline{\rho}(A)\rho(R) = A\overline{R}^*\rho(R) = A.$$

Thus $m(F^*F) = 0$ implies $F = 0$, and m is a faithful positive linear projection from $\mathfrak{F}_0$ onto $\mathfrak{A}$. ■

LEMMA 273. *Let P_0^ρ be the projection in* $\mathrm{End}(E(\rho))$ *onto the subspace of G invariant vectors with respect to the action $\pi_\rho(g) = g_\rho$. Then $E(P_\iota^\rho) = P_0^\rho$. Furthermore, the conditional expectation m is G-invariant, i.e. $m(\alpha_g(F)) = m(F)$ for all $g \in G$ and $F \in \mathfrak{F}_0$.*

Proof. Recall that if (H, π) is an irreducible representation of a compact group G and π is not the trivial representation, then H contains no G-invariant vectors. If $\rho = \bigoplus \rho_i$ with ρ_i irreducible, then the previous observation implies that the G-invariant vectors in $E(\rho)$ are precisely those in the image of $E(P_\iota^\rho)$. Thus $E(P_\iota^\rho) = P_0^\rho$, implying $m(F) = \alpha_g(m(F))$. Furthermore,

$$\begin{aligned} m\alpha_g(A, \rho, \psi) &= m(A, \rho, g_\rho\psi) \;=\; (AP_\iota^\rho, \rho, g_\rho\psi) = (A, \rho, P_0^\rho g_\rho \psi) \\ &= (A, \rho, g_\rho P_0^\rho \psi) = (A, \rho, P_0^\rho \psi) \\ &= (AP_\iota^\rho, \rho, \psi) = m(A, \rho, \psi). \end{aligned}$$

■

In view of Prop. 272, $\omega_0 \circ m$ is a faithful state on the $*$-algebra $\mathfrak{F}_0$. Let $(\mathcal{H}, \pi)$ be the GNS representation of $\mathfrak{F}$ induced by $\omega_0 \circ m$, let $\mathfrak{F}$ be the norm closure of $\pi(\mathfrak{F}_0)$, and let $\mathfrak{F}(O)$ be the *weak* closure of $\pi(\mathfrak{F}_0(O))$. It is clear that $\mathfrak{F}$ is the C^*-inductive limit of the net $O \mapsto \mathfrak{F}(O)$. Since $\omega_0 \circ m$ is G-invariant by Lemma 273, there is a unitary representation U of G on $\mathcal{H}$ implementing the automorphisms α_g of $\mathfrak{F}_0$:

$$\pi(\alpha_g(F)) = U(g)\pi(F)U(g)^*, \qquad g \in G, F \in \mathfrak{F}_0,$$

and therefore it extends to $\mathfrak{F}$. Since $g \mapsto \alpha_g$ is injective, U is injective.

DEFINITION 274. Let $\sigma \in \hat{G}$ be an irreducible character of G. Define a map $\mathscr{E}_\sigma$ on $\mathfrak{B}(\mathcal{H})$ by

$$\mathscr{E}_\sigma(A) = \int_G \overline{\sigma(g)} U(g) A U(g)^* \, d\mu(g),$$

where μ is the Haar measure on $U(G)$.

NOTE 275. Let $F = (A, \rho, \psi) \in \mathfrak{F}_0$. Since the $U(g)$ implements α_g we have

$$\begin{aligned} \mathscr{E}_\sigma(\pi(F)) &= \int_G \overline{\sigma(g)} \Big(\pi(\alpha_g(A, \rho, \psi)) \Big) d\mu(g) = \int_G \overline{\sigma(g)} \pi(A, \rho, g_\rho \psi) d\mu(g) \\ &= \pi(A, \rho, P_\sigma^\rho \psi), \end{aligned} \tag{49}$$

where $P_\sigma^\rho \in \mathrm{End}(E(\rho))$ is the orthogonal projection onto the subspace transforming according to the irreducible representation σ. Since G is compact, $\mathscr{E}_\sigma$ is strongly continuous. Note that $\mathscr{E}_0(\pi(F)) = \pi[m(A, \rho, \psi)]$.

LEMMA 276. $\mathfrak{F}_0(O)\mathfrak{A} = \mathfrak{F}_0$.

Proof. Let $(A, \rho, \psi) \in \mathfrak{F}_0$. Since ρ is transportable, there is a unitary $T \in \mathrm{Hom}(\rho, \rho')$ with ρ' localized in O. Then

$$(A, \rho, \psi) = (AT^*, \rho', E(T)\psi) = (AT^*, \iota, 1)(I, \rho', E(T)\psi) = BF,$$

where $B \in \mathfrak{A}$ and $F \in \mathfrak{F}_0(O)$. Hence $\mathfrak{A}\mathfrak{F}_0(O) = \mathfrak{F}_0$. Since $\mathfrak{A}, \mathfrak{F}_0(O)$ and $\mathfrak{F}_0$ are $*$-algebras, $\mathfrak{F}_0(O)\mathfrak{A} = \mathfrak{F}_0$. ∎

THEOREM 277. *$(\pi, \mathcal{H}, \mathfrak{F}, (G, k))$ is a field system with gauge symmetry for $(\mathfrak{A}, \omega_0)$ with normal commutation relations (in the sense of Definitions 229 and 245).*

Proof. It is obvious that $\mathfrak{F}(O)$ is a G-stable von Neumann subalgebra of $\mathfrak{F}$. Also the net $O \mapsto \mathfrak{F}(O)$ satisfies normal commutation relations. We now run through the individual conditions in Definition 229.

(γ) We need to show that the fixed point algebra of $\mathfrak{F}(O)$ under the G action is $\pi(\mathfrak{A}(O))$. First note that $\mathscr{E}(\pi(\mathfrak{F}_0(O))) = \pi(m(\mathfrak{F}_0(O)))$. Thus,

$$\mathfrak{F}(O)^G = \mathscr{E}(\mathfrak{F}(O)) = \mathscr{E}\Big(\overline{\pi(\mathfrak{F}_0(O))}\Big) = \overline{\mathscr{E}(\pi(\mathfrak{F}_0(O)))} = \overline{\pi(m(\mathfrak{F}_0(O)))} = \overline{\pi(\mathfrak{A}(O))}.$$

The third equality follows by the normality of $\mathscr{E}$, and the last equality is due to the fact that m is a conditional expectation from $\mathfrak{F}_0$ to $\mathfrak{A}$.

(δ) Let $j : \mathfrak{F}_0 \to \mathcal{H}$ be the inclusion mapping derived from the GNS representation of $\omega_0 \circ m$. Since $\overline{j(\mathfrak{A})} = \mathcal{H}_0$ we have

$$\overline{\mathfrak{F}(O)\mathcal{H}_0} = \overline{\pi(\mathfrak{F}_0(O))\mathcal{H}_0} = \overline{\pi(\mathfrak{F}_0(O))j(\mathfrak{A})} = \overline{j(\mathfrak{F}_0(O)\mathfrak{A})} = \overline{j(\mathfrak{F}_0)} = \mathcal{H}.$$

(ε) Let O_1 and O_2 be spacelike separated. The subalgebra $\mathfrak{A}(O_1)$ of $\mathfrak{F}_0$ is pointwise invariant under the gauge transformations. In particular, $\alpha_k(A) = A$ for all $A \in \mathfrak{A}(O)$, i.e. elements of $\mathfrak{A}(O_1)$ are purely Bosonic. Therefore relative locality follows by normality of the commutation relations (Prop. 269).

Now we claim that $\mathrm{Aut}_{\mathfrak{A}}\mathfrak{F} = G$. By Eqn. (49), $\mathscr{E}_\sigma(\pi(\mathfrak{F}_0))$ is isomorphic as a Banach space to $\mathfrak{A} \otimes P^\rho_\sigma E(\rho)$, and so is a closed subspace of $\mathfrak{F}$, and so

$$\mathscr{E}_\sigma(\mathfrak{F}) = \mathscr{E}_\sigma(\overline{\pi(\mathfrak{F}_0)}) = \overline{\mathscr{E}_\sigma(\pi(\mathfrak{F}_0))} = \mathscr{E}_\sigma(\pi(\mathfrak{F}_0)).$$

Since for any $F \in \mathfrak{F}$ we have $F = \sum_{\sigma\in\hat{G}} \mathscr{E}_\sigma(F)$, and $\mathscr{E}_\sigma(F) \in \pi(\mathfrak{F}_0)$, it follows that an element $F \in \mathfrak{F}$ is in $\pi(\mathfrak{F}_0)$ if and only if $\mathscr{E}_\sigma(F) \neq 0$ for only finitely many $\sigma \in \hat{G}$. Together with linearity of α, this implies that $\alpha(\pi(\mathfrak{F}_0)) \subseteq \pi(\mathfrak{F}_0)$. Thus there exists a $g \in G$ such that $\alpha|_{\pi(\mathfrak{F}_0)} = \alpha_g$ (by Prop. 264). Since α_g is continuous and $\pi(\mathfrak{F}_0)$ is dense in $\mathfrak{F}$, α is the unique extension of α_g to $\mathfrak{F}$. ■

10.4 Poincaré covariance of the field net

Covariance considerations have played no prominent role in the DHR theory of Section 7 or in the above reconstruction of a field net $\mathfrak{F}$. We now show that the latter is Poincaré covariant if the underlying DHR sectors are. (Recall from Remark 224 that under favorable circumstances we have $\Delta_{fc} = \Delta_f$.)

THEOREM 278. *If in the construction of the field net $\mathfrak{F}$ we start from the category Δ_{fc} instead of Δ_f, the field net constructed above is covariant under an automorphic action of $\widehat{P}$. This action is implemented by a positive energy representation on the GNS representation space of $\mathfrak{F}$ corresponding to the state $\omega_0 \circ m$.*

Proof. Let $\beta_h = \mathrm{Ad}\, U(h)$ be the action of $\mathcal{P}$ on $\mathfrak{A}$. Recall from Note 221 that $\rho_h = \beta_h \circ \rho \circ \beta_h^{-1}$ and $X_\rho(h) \equiv U(h)U_\rho(h)^* \in \mathrm{Hom}(\rho, \rho_h)$ for all $h \in \widehat{\mathcal{P}}$. We define an action $\widehat{\beta}$ of $\widehat{\mathcal{P}}$ on $\mathfrak{F}_0$ by

$$\begin{aligned} \widehat{\beta}_h((A,\rho,\psi)) &\equiv (\beta_h(A), \rho_h, E(X_\rho(h))\psi) \\ &= (\beta_h(A)X_\rho(h), \rho, \psi) = (U(h)AU_\rho(h)^*, \rho, \psi). \end{aligned} \tag{50}$$

Let $\rho, \rho' \in \Delta_{fc}$ and $T \in \mathrm{Hom}(\rho, \rho')$. Then $\beta_h(T) \in \mathrm{Hom}(\rho_h, \rho'_h)$, and $TU_\rho(h) = U_{\rho'}(h)T$, cf. Section 8.4. Thus,

$$\begin{aligned} \beta_h(T)X_\rho(h) &= (U(h)TU(h)^*)(U(h)U_\rho(h)^*) = U(h)TU_\rho(h)^* \\ &= U(h)U_{\rho'}(h)T = X_{\rho'}(h)T, \end{aligned}$$

Using this equation, we compute

$$\begin{aligned}\widehat{\beta}_h((AT,\rho,\psi)) &= (\beta_h(AT),\rho_h,E(X_\rho(h))\psi) = (\beta_h(A),\rho'_h,E(\beta_h(T)X_\rho(h))\psi)\\ &= (\beta_h(A),\rho'_h,E(X_{\rho'}(h)T)\psi) = \widehat{\beta}_h((A,\rho',E(T)\psi)),\end{aligned}$$

thus $\widehat{\beta}_g$ is well defined. Let $i : A \mapsto (A,\iota,\mathbf{1})$ be the inclusion of $\mathfrak{A}$ in $\mathfrak{F}$. Then $\widehat{\beta}_h \circ i = i \circ \beta_g$, thus $\widehat{\beta}_g$ extends β_g. If $F \in \mathfrak{F}(O)$ then there exists a representation $F = (A,\rho,\psi)$ with $A \in \mathfrak{A}(O)$ and $\rho \in \Delta(O)$. Now it is evident from the definition that $\widehat{\beta}_h(F) \in \mathfrak{F}(hO)$. That $g \mapsto \widehat{\beta}_g$ is a group homomorphism is obvious from the r.h.s. of Eqn. (50). Now,

$$\begin{aligned}\widehat{\beta}_g((A_1,\rho_1,\psi_1)(A_2,\rho_2,\psi_2)) &= \widehat{\beta}_g((A_1\rho_1(A_2),\rho_1\rho_2,\psi_1\otimes\psi_2))\\ &= (U(h)A_1\rho_1(A_2)U_{\rho_1\rho_2}(h)^*,\rho_1\rho_2,\psi_1\otimes\psi_2)\\ &= (\beta_h(A_1)\rho_{1,h}(\beta_h(A_2))U(h)U_{\rho_1\rho_2}(h)^*,\rho_1\rho_2,\psi_1\otimes\psi_2)\\ &= (\beta_h(A_1)\rho_{1,h}(\beta_h(A_2))X_{\rho_1\rho_2}(h),\rho_1\rho_2,\psi_1\otimes\psi_2)\\ &= (\beta_h(A_1)\rho_{1,h}(\beta_h(A_2))X_{\rho_1}(h)\rho_1(X_{\rho_2}(h)),\rho_1\rho_2,\psi_1\otimes\psi_2)\\ &= (\beta_h(A_1)X_{\rho_1}(h)\rho_1(\beta_h(A_2)X_{\rho_2}(h)),\rho_1\rho_2,\psi_1\otimes\psi_2)\\ &= (U(h)A_1U_{\rho_1}(h)^*\rho_1(U(h)A_2U_{\rho_2}(h)^*),\rho_1\rho_2,\psi_1\otimes\psi_2)\\ &= (U(h)A_1U_{\rho_1}(h)^*,\rho_1,\psi_1)(U(h)A_2U_{\rho_2}(h)^*,\rho_2,\psi_2)\\ &= \widehat{\beta}_g((A_1,\rho_1,\psi_1))\widehat{\beta}_g((A_2,\rho_2,\psi_2)),\end{aligned}$$

where the fifth equality is due to Eqn. (27). Thus $\widehat{\beta}_g$ is an algebra homomorphism.

Let $\rho \in \Delta_{fc}$ and choose a conjugate $(\overline{\rho},R,\overline{R})$. Since the trivial morphism ι is covariant with $X_\iota = \mathrm{id}_\iota$, applying Eqn. (50) with $T = R^* \in \mathrm{Hom}(\overline{\rho}\rho,\iota)$ we get $R^* = \beta_h(R^*)X_{\overline{\rho}\rho}(h) = \beta_h(R^*)X_{\overline{\rho}}(h)\overline{\rho}(X_\rho(h))$, where we used Eqn. (27) again. This is equivalent to

$$R^*\overline{\rho}(X_\rho(h)^*) = \beta_h(R^*)X_{\overline{\rho}}(h), \tag{51}$$

which will be used below. Now we compute

$$\begin{aligned}(\widehat{\beta}_h(A,\rho,\psi))^* &= (U(h)AU_\rho(h)^*,\rho,\psi)^*\\ &= (R^*\overline{\rho}(U(h)AU_\rho(h)^*)^*,\overline{\rho},\mathcal{J}E(\overline{R}^*)\psi)\\ &= (R^*\overline{\rho}(U_\rho(h)A^*U(h)^*),\overline{\rho},\mathcal{J}E(\overline{R}^*)\psi)\\ &= (R^*\overline{\rho}(U_\rho(h)U(h)^*\beta_h(A^*)),\overline{\rho},\mathcal{J}E(\overline{R}^*)\psi)\\ &= (R^*\overline{\rho}(X_\rho(h)^*\beta_h(A^*)),\overline{\rho},\mathcal{J}E(\overline{R}^*)\psi)\\ &= (\beta_h(R^*)X_{\overline{\rho}}(h)\overline{\rho}(\beta_h(A))^*,\overline{\rho},\mathcal{J}E(\overline{R}^*)\psi)\\ &= (U(h)R^*U_{\overline{\rho}}(h)^*\overline{\rho}(\beta_h(A))^*,\overline{\rho},\mathcal{J}E(\overline{R}^*)\psi)\\ &= (U(h)R^*\overline{\rho}(A)^*U_{\overline{\rho}}(h)^*,\overline{\rho},\mathcal{J}E(\overline{R}^*)\psi)\\ &= \widehat{\beta}_h((R^*\overline{\rho}(A)^*,\overline{\rho},\mathcal{J}E(\overline{R}^*)\psi))\\ &= \widehat{\beta}_h((A,\rho,\psi)^*),\end{aligned}$$

thus $\widehat{\beta}_h$ is a $*$-homomorphism. (In the sixth equality we used Eqn. (51).)

In view of

$$\begin{aligned}\widehat{\beta}_h((A,\rho,\psi)) &= (U(h)AU_\rho(h)^*,\rho,\psi)\\ \alpha_g((A,\rho,\psi)) &= (A,\rho,\pi_{E(\rho)}(g)\psi)\end{aligned}$$

it is clear that $\widehat{\beta}_h\circ\alpha_g=\alpha_g\circ\widehat{\beta}_h$ for all $g\in G, h\in\widehat{\mathcal{P}}$. In view of $\pi\circ m=\mathcal{E}_0\circ\pi$, we have $\omega_0\circ\beta_h\circ m=\omega_0\circ m$. Thus the vacuum state of $\mathfrak{F}$ is $\widehat{\mathcal{P}}$-invariant, and $\widehat{\mathcal{P}}$ is unitarily implemented in the GNS representation. ■

10.5 Uniqueness of the field net

In the present section we have shown that, given a fiber functor $E:\Delta_f(\mathfrak{A})\to\mathcal{SH}$, there exists a field net with normal commutation relations that is complete, i.e. creates all representations in $\Delta_f(\mathfrak{A})$ from the vacuum. We call this the Roberts field net and denote it by $\mathfrak{F}^R_E$. We first consider the dependence of this construction on the functor E.

PROPOSITION 279. *Let $E_1,E_2:\Delta_f\to\mathcal{H}$ be two fiber functors. Then the Roberts field nets $\mathfrak{F}^R_{E_1},\mathfrak{F}^R_{E_2}$ constructed from them are unitarily equivalent.*

Proof. By Theorem 373 from the appendix, there exists a unitary monoidal natural isomorphism $\alpha:E_1\to E_2$. Based on this we define a map $\gamma:\mathfrak{F}^R_{0,1}\to\mathfrak{F}^R_{0,2}$ by $\gamma:(A,\rho,\psi)\mapsto(A,\rho,\alpha_\rho\psi)$. This makes sense since $\psi\in E_1(\rho)$ and $\alpha\in\mathrm{Hom}(E_1(\rho),E_2(\rho))$. γ is well defined since, for $T\in\mathrm{Hom}(\rho,\rho')$, we have

$$\begin{aligned}\gamma(AT,\rho,\psi) &= (AT,\rho,\alpha_\rho\psi)\\ &= (A,\rho',E_2(T)\circ\alpha_\rho\psi)\\ &= (A,\rho',\alpha_\rho\circ E_1(T)\psi))\\ &= \gamma(A,\rho',E_1(T)\psi).\end{aligned}$$

That γ is an algebra homomorphism follows from

$$\begin{aligned}\gamma((A_1,\rho_1,\psi_1))\gamma((A_2,\rho_2,\psi_2)) &= (A_1,\rho_1,\alpha_{\rho_1}\psi_1)(A_2,\rho_2,\alpha_{\rho_2}\psi_2)\\ &= (A_1\rho_1(A_2),\rho_1\otimes\rho_2,\alpha_{\rho_1}\psi_1\otimes\alpha_{\rho_2}\psi_2)\\ &= (A_1\rho_1(A_2),\rho_1\otimes\rho_2,\alpha_{\rho_1\otimes\rho_2}(\psi_1\otimes\psi_2))\\ &= \gamma((A_1\rho_1(A_2),\rho_1\otimes\rho_2,\psi_1\otimes\psi_2))\\ &= \gamma((A_1,\rho_1,\psi_1)(A_2,\rho_2,\psi_2)),\end{aligned}$$

where we have used monoidality $\alpha_{\rho_1\otimes\rho_2}=\alpha_{\rho_1}\otimes\alpha_{\rho_2}$ of α. Since an inverse can be obtained using the natural isomorphism α^*, γ is an isomorphism between the field algebras $\mathfrak{F}^R_{0,1}$ and $\mathfrak{F}^R_{0,2}$. It clearly respects the local structure, i.e. maps $\mathfrak{F}^R_{0,1}(O)$ to $\mathfrak{F}^R_{0,2}(O)$.

Next we claim that $m_2\circ\gamma=\gamma\circ m_1$, where m_1,m_2 are the projections defined earlier. Namely,

$$m_2\circ\gamma((A,\rho,\psi))=m_2((A,\rho,\alpha_\rho\psi))=(AP^\rho_\iota,\rho,\alpha_\rho\psi)=\gamma((AP^\rho_\iota,\rho,\psi))=\gamma\circ m_1((A,\rho,\psi)).$$

This implies that the states $\omega_0 \circ m_1$ and $\omega_0 \circ m_2 \circ \gamma$ on $\mathfrak{F}^R_{0,1}$ coincide, and therefore the isomorphism $\gamma : \mathfrak{F}^R_{0,1} \to \mathfrak{F}^R_{0,2}$ extends to a unitary equivalence of the norm completions in the GNS representations. ■

In order to study an arbitrary complete normal field net $\mathfrak{F}$, not a priori of the form $\overline{\mathfrak{F}^R}^{\|\cdot\|}$, we use the following

PROPOSITION 280. *Let $\mathfrak{F}$ be a complete normal field net for the observable net $\mathfrak{A}$. Then there exists a strict tensor functor $E_{\mathfrak{F}} : \Delta_f(\mathfrak{A}) \to \mathcal{SH}_f$ to the category of finite dimensional super Hilbert spaces. On the objects, $E_{\mathfrak{F}}$ is given by the vector space*

$$E_{\mathfrak{F}}(\rho) := \{F \in \mathfrak{F} \mid F\pi_0(A) = \pi_0(\rho(A))F \quad \forall A \in \mathfrak{A}\}.$$

The inner product is given by $\langle F, F'\rangle \mathbf{1} = F^*F'$ and the $\mathbb{Z}_2$-grading by the action of $k \in G$. For irreducible $\rho, \rho' \in \Delta_f$, we have $E(\varepsilon(\rho,\rho')) = \pm \sum_{i,j} \psi'_i \psi_j \psi_i'^* \psi_j^*$, where $\{\psi_i, i = 1, \ldots, d(\rho)\}$ and $\{\psi'_i, i = 1, \ldots, d(\rho')\}$ are orthonormal bases of $E(\rho)$ and $E(\rho')$, respectively, and the minus sign appears iff ρ and ρ are both fermionic.

Proof. (In this proof we write E instead of $E_{\mathfrak{F}}$.) For $s \in \mathrm{Hom}(\rho, \rho')$ we define $E(s) = \pi_0(s) \in \mathfrak{F}$. For $F \in E(\rho)$ we have $\pi_0(s)F\pi_0(A) = \pi_0(s)\pi_0(\rho(A))F = \pi_0(s\rho(A))F = \pi_0(\rho'(A)s)F = \pi_0(\rho'(A))\pi_0(s)F$ for all $A \in \mathfrak{A}$, thus $\pi_0(s)F \in E(\rho')$ and E is a functor. If $F, F' \in E(\rho)$ then $F^*F' \in \mathfrak{F} \cap \mathfrak{A}' = \mathbb{C}\mathbf{1}$, allowing us to define $\langle F, F'\rangle \mathbf{1} = F^*F'$. Let $s \in \mathrm{Hom}(\rho, \rho')$ and $F \in E(\rho), F' \in E(\rho')$. Then

$$\langle F', E(s)F\rangle = \langle F', \pi_0(s)F\rangle = F'^*\pi_0(s)F = (\pi_0(s)^*F')^*F = \langle \pi_0(s^*)F', F\rangle = \langle E(s^*)F', F\rangle,$$

where we have used that π_0 is $*$-preserving, shows that E is $*$-preserving. By Section 9 we have $E(\rho)E(\rho') = E(\rho \otimes \rho')$. If $S_i \in \mathrm{Hom}(\rho_i, \rho'_i)$, $F_i \in E(\rho_i)$ then

$$E(S_1 \times S_2)F_1F_2 = \pi_0(S_1\rho_1(S_2))F_1F_2 = \pi_0(S_1)F_1\pi_0(S_2)F_2 \in E(\rho'_1 \otimes \rho'_2),$$

thus $E(S_1 \times S_2) = E(S_1) \otimes E(S_2)$, thus E is a strict tensor functor. Completeness of the field net together with the discussion in Section 9 implies that E is faithful and satisfies $\dim E(\rho) = d(\rho)$. (The latter follows also by Proposition 344 of the appendix.) Finally, let $F \in E(\rho), F' \in E(\rho')$ be of norm one. Now let $\rho, \rho' \in \Delta_f$, and let $\psi_i, i = 1, \ldots, d(\rho)$ and $\psi'_i, i = 1, \ldots, d(\rho')$ be orthonormal bases of $E(\rho)$ and $E(\rho')$, respectively. Then

$$\tilde{c}(\rho, \rho') = \sum_{i,j} \psi'_i \psi_j \psi_i'^* \psi_j^*$$

is in $\mathfrak{F}^G$ and independent of the chosen bases. Furthermore, $\tilde{c}(\rho, \rho') \in \mathrm{Hom}(\rho \otimes \rho', \rho' \otimes \rho)$. The functoriality of E that was proven above implies that $\tilde{c}(\rho, \rho')$ is natural in both arguments. If now $\{\rho'' \in \Delta_f\}$ and $\psi''_k, k = 1, \ldots, d(\rho'')$ is an

orthonormal basis in $E(\rho'')$, then $\{\psi'_j\psi''_k\}$ is an orthonormal basis in $E(\rho'\otimes\rho'')$, thus

$$\begin{aligned}\tilde{c}(\rho,\rho'\otimes\rho'') &= \sum_{i,j,k}\psi'_j\psi''_k\psi_j\psi''^{*}_k\psi'^{*}_j\psi^{*}_i \\ &= \sum_{i,j,m,k,l}\psi'_m(\psi''_i\psi_j\psi''^{*}_i\psi^{*}_j)\psi'^{*}_m(\psi'_k\psi_l\psi'^{*}_k\psi^{*}_l) \\ &= \mathrm{id}_{\rho'}\otimes\tilde{c}(\rho,\rho'')\circ\tilde{c}(\rho,\rho')\otimes\mathrm{id}_{\rho''},\end{aligned}$$

which is one of the braid relations. One easily sees that $\tilde{c}(\rho,\rho')\tilde{c}(\rho',\rho)=\mathbf{1}$, thus $\tilde{c}(\cdot,\cdot)$ is a symmetry for the tensor category Δ_f. If ρ and ρ' are irreducible and localized spacelike to each other, the normal commutation relations of the corresponding fields imply that $\tilde{c}(\rho,\rho)=\pm1$, where the minus sign occurs iff ρ and ρ'' are fermionic. Now, for irreducible ρ,ρ' define $c(\rho,\rho')=\pm\tilde{c}(\rho,\rho')$, where we take the minus sign iff ρ and ρ' are fermionic, and extend c to reducible objects by naturality. Then $c(\rho,\rho')=1$ whenever ρ,ρ' are localized spacelike. Now it follows from the uniqueness result Proposition 200 that $E(\varepsilon(\rho,\rho'))=c(\rho,\rho')$. Thus $E_{\mathfrak{F}}$ is a symmetric tensor functor in the sense that it maps the symmetry ε of Δ_f to the symmetry c of the category $\mathcal{H}$ of Hilbert spaces. Equivalently, E is a symmetric tensor functor into the category of super Hilbert spaces equipped with the symmetry $\widetilde{c}$. ■

Thus every complete normal field net $\mathfrak{F}$ gives rise to a strict symmetric $*$-preserving fiber functor $E_{\mathfrak{F}}$. Denoting by $\mathfrak{F}^R_{E_{\mathfrak{F}}}$ the Roberts field net associated to the latter, our aim is to construct an isomorphism $\mathfrak{F}\cong\mathfrak{F}^R_{E_{\mathfrak{F}}}$.

THEOREM 281. *Let $\mathfrak{F}$ be a complete normal field net for $\mathfrak{A}$ and $E_{\mathfrak{F}}:\Delta_f\to\mathcal{SH}$ the fiber functor from Proposition 280. Then there is a unitary equivalence $\mathfrak{F}_{E_{\mathfrak{F}}}\to\mathfrak{F}$ of field nets.*

Proof. By Proposition 280, there is a symmetric $*$-preserving fiber functor $E_{\mathfrak{F}}:\Delta_f\to\mathcal{SH}$. By the concrete Tannaka theorem (Theorem 377 of the appendix), the compact group $G_{E_{\mathfrak{F}}}$ of unitary monoidal natural transformations of E is unitarily represented on the spaces $E_{\mathfrak{F}}(\rho)$. On the other hand, the compact group G coming with our field net $\mathfrak{F}$ also acts on these spaces, providing a homomorphism $G\to G_{E_{\mathfrak{F}}}$. This homomorphism is injective since G is concretely given as a group of unitaries on the Hilbert space H where $\mathfrak{F}$ lives. It is also surjective, since otherwise $\pi|\mathfrak{A}$ would contain representations that are not in Δ_f, contradicting the assumption that $\mathfrak{F}$ is a complete field net. Thus the given group G can be identified with the one reconstructed from the fiber functor $E_{\mathfrak{F}}$. For every $\sigma\in\widehat{F}$ we define a projection $\mathcal{E}_\sigma$ on $\mathfrak{F}$ as in Definition 274. We denote by $\mathfrak{F}_0$ the algebraic direct sum $\oplus_{\sigma\in\widehat{G}}\mathcal{E}_\sigma(\mathfrak{F})$, which is the same as $\{F\in\mathfrak{F}\mid\mathcal{E}_\sigma(F)=0$ for almost all $\sigma\in\widehat{G}\}$.

We now define a map $\gamma:\mathfrak{F}^R_{E_{\mathfrak{F}},0}\to\mathfrak{F}$ by $\gamma:(A,\rho,\psi)\mapsto\pi_0(A)\psi$. At first sight, this formula looks strange, but it makes perfect sense since $\psi\in E_{\mathfrak{F}}(\rho)$,

where $E_{\mathfrak{F}}(\rho)$ by definition is a subspace of $\mathfrak{F}$. As usual, γ is well defined since, for $T \in \mathrm{Hom}(\rho, \rho')$,

$$\gamma((AT, \rho, \psi)) = \pi_0(AT)\psi = \pi_0(A)E_{\mathfrak{F}}(T)\psi = \gamma((A, \rho', E_{\mathfrak{F}}(T)\psi)).$$

Furthermore,

$$\begin{aligned}\gamma((A_1, \rho_1, \psi_1)(A_2, \rho_2, \psi_2)) &= \gamma((A_1\rho_1(A_2), \rho_1\rho_2, \psi_1 \otimes \psi_2)) = \pi_0(A_1\rho_1(A_2))\psi_1\psi_2 \\ &= \pi_0(A_1)\psi_1\pi_0(A_2)\psi_2 = \gamma((A_1, \rho_1, \psi_1))\gamma((A_2, \rho_2, \psi_2)),\end{aligned}$$

where we have used $\psi_1 \in E_{\mathfrak{F}}(\rho_1) = \{F \in \mathfrak{F} \mid F\pi_0(A) = \pi_0(\rho(A))F\}$. Thus γ is an algebra homomorphism. This, together with $(A, \rho, \psi) = (A, \iota, 1)(\mathbf{1}, \rho, \psi)$ implies that γ is a $*$-homomorphism provided $\gamma(F^*) = \gamma(F)^*$ for $F = (\mathbf{1}, \rho, \psi)$. Now, using the $*$-operation on $\mathfrak{F}^R$ defined in Proposition 263, we have

$$\gamma((\mathbf{1}, \rho, \psi)^*) = \gamma((R^*, \overline{\rho}, (\mathcal{J}E(\overline{R}^*))\psi)) = \pi_0(R^*)(\mathcal{J}E(\overline{R}^*)\psi).$$

On the other hand, $\gamma((1, \rho, \psi))^* = \psi^*$, thus γ is a $*$-homomorphism provided $\psi^* = R^*(\mathcal{J}E(\overline{R}^*))\psi$ holds for all $\psi \in E(\rho)$.

Now, for any $\overline{\psi} \in E(\overline{\rho})$, we have $R^*\overline{\psi}\rho(A) = R^*\overline{\rho}\rho(A)\overline{\psi} = AR^*\overline{\psi}$, thus $(R^*\overline{\psi})^* \in E(\rho)$. Applying this to $\overline{\psi} = \mathcal{J}E(\overline{R}^*)\psi \in E(\overline{\rho})$, we see that $\psi^* = R^*(\mathcal{J}E(\overline{R}^*))\psi$ holds iff $\psi^*\psi' = R^*(\mathcal{J}E(\overline{R}^*)\psi)\psi'$ for all $\psi' \in E(\rho)$.

By Proposition 344 of the Appendix, $(E(\rho), E(R), E(\overline{R}))$ is a conjugate of $E(\rho)$ in the category of Hilbert spaces. (Or super Hilbert spaces. This doesn't matter since we don't use the symmetry.) Thus there are bases $\{e_i\}, \{f_i\}$ of $E(\rho)$ and $E(\overline{\rho})$, respectively, with dual bases $\{\widehat{e_i}\}, \{\widehat{f_i}\}$ in $\widehat{E(\rho)}, \widehat{E(\overline{\rho})}$ such that

$$E(R) = \sum_i f_i \otimes e_i, \quad E(\overline{R}) = \sum_i e_i \otimes f_i, \quad E(R)^* = \sum_i \widehat{f_i} \otimes \widehat{e_i}, \quad E(\overline{R})^* = \sum_i \widehat{e_i} \otimes \widehat{f_i}.$$

Thus, for $\psi \in E(\rho), \overline{\psi} \in E(\overline{\rho})$, we have

$$\langle \mathcal{J}E(\overline{R}^*)\psi, \overline{\psi} \rangle = \left(\sum_i \widehat{e_i} \otimes \widehat{f_i} \right)(\psi \otimes \overline{\psi}) = \sum_i \widehat{e_i}(\psi)\widehat{f_i}(\overline{\psi})$$

and therefore $\mathcal{J}E(\overline{R}^*)\psi = \sum_i \overline{\widehat{e_i}(\psi)} f_i$. Thus

$$E(R)^*((\mathcal{J}E(\overline{R}^*)\psi) \otimes \psi') = (\sum_i \widehat{f_i} \otimes \widehat{e_i})(\sum_j \overline{\widehat{e_j}(\psi)} f_j \otimes \psi') = \sum_i \overline{\widehat{e_i}(\psi)}\widehat{e_i}(\psi') = \langle \psi, \psi' \rangle.$$

Now, in $\mathfrak{F}$, the left hand side equals $R^*(\mathcal{J}E(\overline{R}^*)\psi)\psi'$ and the right hand side equals $\psi^*\psi'$, proving the desired identity $\psi^* = R^*(\mathcal{J}E(\overline{R}^*))\psi$.

Now, for $(A, \rho, \psi) \in \mathcal{F}^R_{E_{\mathfrak{F}},0}$ is is clear that $\gamma((A, \rho, \psi))$ is contained in a finite dimensional G-stable subspace of $\mathfrak{F}$ and thus in $\mathfrak{F}_0$. Every $F \in \mathfrak{F}_0$ is a sum of finitely many terms of the form $\mathcal{E}_\sigma(F)$ with $\sigma \in \widehat{G}$. Picking an irreducible

subspace H_σ of isometries in $\mathfrak{F}_0$ transforming according to the class σ, there is an endomorphism $\rho \in \Delta_f$ induced by the subspace H_σ. Since every $F \in \mathcal{E}_\sigma(\mathfrak{F})$ is a linear combination $\sum_i A_i\psi_i$ with $A_i \in \mathfrak{A}, \psi_i \in H_\sigma$, we have $F = \gamma(\sum_i(A, \rho, \psi_i)$, proving $\gamma(\mathfrak{F}^R_{E_\mathfrak{F},0}) = \mathfrak{F}_0$.

Let $(A, \rho, \psi) \in \mathfrak{F}^R_0$. By construction of $\mathfrak{F}_0$, we have a finite sum representation $(A, \rho, \psi) = \sum_i(A_i, \rho_i, \psi_i)$, where the ρ_i are irreducible and mutually non-isomorphic. Now $\gamma((A, \rho, \psi)) = \sum_i A_i\psi_i$, where the spaces $E(\rho_i) \subset \mathfrak{F}$ transform under mutually inequivalent irreducible representations of G. Thus $\gamma((A, \rho, \psi)) = 0$ iff $A_i\psi_i = 0$ for all i. isometries transforming according to a representation in the class σ. Since by harmonic analysis, every $F \in \mathfrak{F}$ has a unique representation of the form $F = \sum_\sigma A_{\sigma,i}\psi^\sigma_i$, this implies that for each i we have $(A_i, \rho_i, \psi_i) = 0$. Thus γ is injective.

We have thus proven that $\gamma : \mathfrak{F}^R_{E_\mathfrak{F},0} \to \mathfrak{F}_0$ is an isomorphism. Since the vacuum state $\omega^\mathfrak{F}_0 = (\Omega_0, \cdot\Omega)$ of $\mathfrak{F}$ is by assumption gauge invariant, the states $\omega^\mathfrak{F}_0 \circ \gamma$ and $\omega^\mathfrak{A}_0 \circ m$ on $\mathfrak{F}^R_{E_\mathfrak{F},0}$ coincide, implying that the completed nets are unitarily equivalent in their GNS representations. ■

COROLLARY 282. *Every complete normal field net $\mathfrak{F}$ is unitarily equivalent to a Roberts field net $\mathfrak{F}^R_E$, where it doesn't matter which fiber functor E we use.*

NOTE 283. As promised, we return to the issue of strictness of the functor $F : \mathcal{C} \to \mathcal{SH}_f$ that was assumed in the construction of the field net, but not proven in the appendix. In the latter, we constructed a non-strict fiber functor, i.e. a functor $E : \mathcal{C} \to \mathcal{SH}_f$ together with natural isomorphisms $d^E_{\rho,\rho'} : E(\rho) \otimes E(\rho') \to E(\rho \otimes \rho')$ and $e^E : \mathbf{1}_{\mathcal{SH}} \to E(\iota_\Delta)$ satisfying Eqns. (56), (57). The construction of the (algebraic) field algebra $\mathfrak{F}_0$ in Subsection 10.2 can easily be generalized to this situation: The product of fields is defined by

$$(A_1, \rho_1, \psi_1)(A_2, \rho_2, \psi_2) := \Big(A_1\rho_1(A_2), \rho_1 \otimes \rho_2, d^E_{\rho_1,\rho_2}(\psi_1 \otimes \psi_2)\Big)$$

and the unit is $(\mathbf{1}, \iota, e^E 1_\mathbb{C})$. Now associativity and the unit property are obvious consequences of Eqns. (56), (57). The rest of the constructions and proofs goes through as before, just carrying the unitaries d^E, e^E along. An interesting consequence of this and of Proposition 280 is that we can prove the existence of a strict fiber functor $E' : \Delta_f \to \mathcal{SH}'$, where $\mathcal{SH}'_f$ is a strictification of the category of finite dimensional super Hilbert spaces. This is consistent with strictification results in category theory. (Strictification of tensor categories is nicely treated in [Kassel, 1995, Chap. XI], but for strictification of tensor functors the best reference remains [Joyal and Street, 1993a, Sect. 1].)

10.6 Further relations between $\mathfrak{A}$ and $\mathfrak{F}$, and a Galois interpretation

In Section 9 we have discussed at length the structure of the superselection sectors of a net $\mathfrak{A}$ of observables in relation to the harmonic analysis of the action of a (global) gauge group on a field net $\mathfrak{F}$. Note that we did not claim that all

DHR representations of the fixed point net $\mathfrak{A} = \mathfrak{F}^G$ are connected to the vacuum representation by the fields in $\mathfrak{F}$. In order to see that this is in general false, consider a theory $\mathfrak{A}$ with non-trivial DHR-category and take $\mathfrak{F} := \mathfrak{A}$ as 'field net', acted upon by the trivial group $G = \{e\}$. Obviously, all DHR representations of $\mathfrak{A}$ are not created by the action of $\mathfrak{F}$ on H_0. In the special case where $\mathfrak{F}$ is Bosonic and itself satisfies all the requirements on an observable net, it may have non-trivial DHR sectors. Restricting a DHR representation π of $\mathfrak{F}$ with $d(\pi) < \infty$ to $\mathfrak{A}$, one obtains a DHR representation of $\mathfrak{A}$ of the same dimension, which therefore decomposes into a finite direct sum of irreducibles. If π is irreducible and inequivalent to the vacuum representation π_0 of $\mathfrak{F}$, then all the irreducible representations of $\mathfrak{A}$ obtained in this way are disjoint from those contained in $\pi_0|\mathfrak{A}$. We refrain from a further analysis of this issue. We do, however, wish to point out that one can specify conditions on a net $\mathfrak{F}$ implying that all DHR representations of $\mathfrak{A}$ are contained in $\pi_0|\mathfrak{A}$. This involves the net-cohomology or local 1-cohomology developed by J.E. Roberts and reviewed, e.g., in [Roberts, 1990, §3.4]. We refrain from even attempting to give precise statements and only say the following: If $\mathfrak{F}$ has 'quasi-trivial 1-cohomology' and is acted upon by a compact group G of global gauge symmetries, then the equivalent (by Proposition 206) categories $DHR_f(\mathfrak{A}) \simeq \Delta_f(\mathfrak{A})$ are equivalent, as symmetric tensor categories to $\mathrm{Rep}_f G$. In [Buchholz *et al.*, 1992] it is shown, e.g., that the theory of a free massive scalar field has quasi-trivial 1-cohomology. Thus, if one takes $\mathfrak{F}$ to be the direct product of N copies of such fields (of the same mass) then $SO(N)$ acts on $\mathfrak{F}$. Therefore, $\Delta_f(\mathfrak{A}) \simeq \mathrm{Rep}_f G$ whenever $G \subset SO(N)$ is a closed subgroup and $\mathfrak{A} = \mathfrak{F}^G$. In [Doplicher and Piacitelli, 2002] this observation is combined with a limit construction to prove that every (second countable) compact group arises as a DHR gauge group. In a similar fashion, one shows that if $\mathfrak{F}$ is the theory of a massive Fermion with its canonical $\mathbb{Z}/2$-symmetry, then $\Delta_f(\mathfrak{F}^{\mathbb{Z}/2}) \simeq \mathrm{Rep}_f \mathbb{Z}/2$.

There are results in the opposite direction, i.e. from the superselection structure of $\mathfrak{A}$ to that of $\mathfrak{F}$. By [Doplicher and Roberts, 1990, Theorem 3.6], which we have not covered entirely in Section 9, the field net reconstructed in [Doplicher and Roberts, 1990] and in Section 10 above satisfies 'twisted Haag duality'. In particular, if $\mathfrak{A}$ has no Fermionic representations then $\mathfrak{F}$ satisfies Haag duality. In this case, one can study the categories $DHR(\mathfrak{F})$ or $\Delta(\mathfrak{F})$. In [Conti *et al.*, 2001], the following has been proven:

THEOREM 284. *Let $\mathfrak{A}$ be a net of observables such that there are at most countably many DHR representations of finite dimension, all of which are Bosonic. Then the complete field net $\mathfrak{F}$ has no non-trivial DHR representations of finite dimension.*

Rather than trying to comment on the many other known results related to those treated in the preceding sections, we close this section by commenting on a very satisfactory *mathematical* interpretation of DHR/DR theory. We are referring to the evident analogy between this theory and the Galois theory of algebraic field extensions. (It should be clear that in the latter context, by 'field'

we mean the algebraic structure of which $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are examples, not the theory of classical or quantum fields.) A field $\mathbb{F}$ in the latter sense is called algebraically closed if every polynomial $P(x)$ with coefficients in $\mathbb{F}$ has a zero in $\mathbb{F}$. (P then is a product of linear factors $x - a$.) Every field $\mathbb{F}$ is a subfield of an essentially unique algebraically closed field $\overline{\mathbb{F}}$ that is an algebraic extension of $\mathbb{F}$. The latter means that $\overline{\mathbb{F}}$ is obtained by adjoining, typically transfinitely, solutions of polynomial equations to $\mathbb{F}$. The group $G_{\mathbb{F}} = \mathrm{Aut}_{\mathbb{F}}(\overline{\mathbb{F}})$ is compact, and one has a bijective correspondence between intermediate fields $\mathbb{F}' \subset \overline{\mathbb{F}}, \mathbb{F}' \supset \mathbb{F}$ and closed subgroups $H \subset G$. (The correspondence is given by $H \mapsto \overline{\mathbb{F}}^H$, $\mathbb{F}' \mapsto \mathrm{Aut}_{\mathbb{F}'}(\overline{\mathbb{F}})$.) A similar Galois correspondence holds in AQFT, cf. e.g. [Conti *et al.*, 2001; Carpi and Conti, 2001]. In view of Theorem 284, the construction of the complete DR field net is entirely analogous to that of the algebraic closure and can be considered as the passage to a simpler or better behaved theory. Conversely, just as taking the fixed field of an algebraically closed field $\mathbb{F}$ under the action of a closed subgroup $G \subset \mathrm{Aut}\mathbb{F}$ will result in an algebraically non-closed field $\mathbb{F}^G$, taking the G-fixed subnet of a net $\mathfrak{F}$ with trivial category $\Delta_f(\mathfrak{F})$ (more precisely, quasi-trivial 1-cohomology) will result in a net with non-trivial category $\Delta_f(\mathfrak{A})$. Thus the 'complication' manifested by a non-trivial DHR-category $\Delta_f(\mathfrak{A})$ indicates that the theory $\mathfrak{A}$ 'really' is just a subtheory of a simpler one.

Physically, however, it is not at all clear whether the 'observable' net $\mathfrak{A} = \mathfrak{F}^G$ with its non-trivial representation category $\Delta_f(\mathfrak{A})$ or the 'field net' $\mathfrak{F}$ with trivial $\Delta_f(\mathfrak{F})$ but non-trivial global symmetry group G is more fundamental – at least when $\mathfrak{F}$ is Bosonic. In [Haag, 1996] it is argued that the 'right' description of the physical situation should be in terms of a net without any global symmetries. (On the other hand, in [Haag, 1996, Section III.4.2] one finds a tentative postulate on a 'good' net $\mathfrak{A}$ of observables that implies triviality of $\Delta_f(\mathfrak{A})$. As the above discussion shows, it will be very hard to find a theory that has both a trivial DHR category Δ_f and trivial global symmetry group! The theory of a single free massive Bose field is one of the rare examples.) Whether or not one subscribes to these views, from a mathematical point of view, both nets $\mathfrak{A}$ and $\mathfrak{F}$ contain the same amount of information. This equivalence is in fact a useful tool, since it permits to view many problems from different angles. For example, while a spin statistics theorem can be proven in a 'field' framework, its physical interpretation may be clearer in the 'observable' setting.

10.7 Spontaneous symmetry breaking

So far, our entire analysis has presupposed the axiom of Haag duality for the theory $\mathfrak{A}$. Haag duality played an important rôle in our analysis of the category $\Delta(\mathfrak{A})$, but is needed also to establish the equivalence between the $\Delta(\mathfrak{A})$ and the representations satisfying the a priori physically motivated DHR criterion (Definition 204). Thus, while it seems that the study of DHR representations is physically motivated also for non-Haag dual nets, our mathematical analysis soon gets stuck. We will therefore briefly comment on an approach to resolve this issue, which turns

out to have a profound physical interpretation.

DEFINITION 285. Let $O \mapsto \mathfrak{R}(O)$ be a net of von Neumann algebras on a vacuum Hilbert space H_0. The *dual net* $\mathfrak{R}^d$ of $\mathfrak{R}$ is the assignment $O \mapsto \mathfrak{R}(O')'$.

If we have $O_1 \subset O_2$ then $O_2' \subset O_1'$, thus $\mathfrak{R}(O_2') \subset \mathfrak{R}(O_1')$, and therefore $\mathfrak{R}^d(O_1) \subset \mathfrak{R}^d(O_2)$. Thus the dual net really satisfies isotony. Microcausality of $\mathfrak{R}$ is equivalent to $\mathfrak{R}(O) \subset \mathfrak{R}(O')' = \mathfrak{R}^d(O)$, or briefly $\mathfrak{R} \subset \mathfrak{R}^d$, and Haag duality of $\mathfrak{R}$ is equivalent to $\mathfrak{R} = \mathfrak{R}^d$. If $\mathfrak{A}_1 \subset \mathfrak{A}_2$ (in the sense of an inclusion for every O) then $\mathfrak{A}_2^d \subset \mathfrak{A}_1^d$, thus $\mathfrak{A} \subset \mathfrak{A}^{dd}$, and a standard argument shows $\mathfrak{A}^d = \mathfrak{A}^{ddd}$. Note, however, that microcausality of $\mathfrak{R}$ does not imply microcausality of $\mathfrak{R}^d$! This motivates the following

DEFINITION 286. A net $O \mapsto \mathfrak{R}(O) \subset B(H_0)$ satisfies *essential duality* if both $\mathfrak{R}$ and the dual net $\mathfrak{R}^d$ (both indexed by double cones) satisfy microcausality.

LEMMA 287. *If $\mathfrak{R}$ satisfies essential duality then $\mathfrak{R}^d = \mathfrak{R}^{dd}$, i.e. $\mathfrak{R}^d$ satisfies Haag duality.*

NOTE 288. Essential duality can be difficult to verify. Fortunately, essential duality follows from *wedge duality*, to wit $\mathfrak{R}(W')' = \mathfrak{R}(W)$ for all wedge regions (the regions obtained from the standard wedge $W_0 = \{x \in \mathbb{R}^{1+s} \mid x_0 \geq |x_1|\}$ by Poincaré transformations). Besides being much easier to verify than essential duality, wedge duality is a very fundamental property that might well be required of any 'reasonable' QFT.

Assuming that $\mathfrak{R}$ satisfies essential duality, $\mathfrak{R}^d$ satisfies Haag duality and the D(H)R analysis applies to it. Thus we obtain a symmetric tensor $*$-category with conjugates $\Delta_f^d(\mathfrak{R}) := \Delta_f(\mathfrak{R}^d) \simeq DHR_f(\mathfrak{R}^d)$, and we can construct the complete DR field net $\mathfrak{F}$ associated with $(\mathfrak{R}^d, \Delta_f(\mathfrak{R}^d))$. One thus has an inclusion $\mathfrak{R} \subset \mathfrak{R}^d \subset \mathfrak{F}$ of nets. The DR gauge group acts on $\mathfrak{F}$ and we have $\mathfrak{F}^G = \mathfrak{R}^d$ and also $G = \mathrm{Aut}_{\mathfrak{R}^d}(\mathfrak{F})$. Since the group G is implemented by unitaries that leave the vacuum vector fixed, G consists of 'unbroken symmetries'. One can now define a larger group

$$\widehat{G} = \mathrm{Aut}_{\mathfrak{R}}(\mathfrak{F})$$

and topologize it suitably. Now $G \subset \widehat{G}$ consists precisely of the elements of $\widehat{G}$ that are unitarily implemented. The point is that the net $\mathfrak{R}$ acts irreducibly on H_0, thus a unitary whose adjoint action leaves all algebras $\mathfrak{R}(O)$ pointwise fixed must be a multiple of the identity also on $\mathfrak{R}^d$.

Concerning the categories associated with $\mathfrak{R}$, little can be said about the category $\Delta(\mathfrak{R})$, but Roberts proved the existence of an extension functor $K : DHR(\mathfrak{R}) \to DHR(\mathfrak{R}^d)$ such that $K(\pi)|\mathfrak{R} = \pi$ for every $\pi \in DHR(\mathfrak{R})$, cf. [Roberts, 1990, §3.4]. (Again, a crucial rôle is played by the theory of local 1-cohomology. Furthermore, this result breaks down in less than three spacetime dimensions due to the phenomenon of solitonic representations.) This functor actually is an equivalence, thus spontaneous symmetry breakdown doesn't manifest itself in the superselection structure.

For the detailed analysis we refer to [Roberts, 1974; Roberts, 1990] and the remarkable paper [Buchholz *et al.*, 1992], in which the Goldstone phenomenon is analyzed in the context of algebraic QFT.

Notes: DHR superselection theory originates in a four-paper series: [Doplicher *et al.*, 1969a] starts with a field algebra and gauge group and then derives properties of the superselection sectors. [Doplicher *et al.*, 1969b] reconstructs the field algebra and gauge group from the category of representations in the special case where the objects are all one dimensional (i.e. the equivalence classes of objects of Δ_f form an abelian group with the monoidal product and conjugates as inverses). [Doplicher *et al.*, 1971] defines the symmetry $\varepsilon_{\rho_1,\rho_2}$, and uses it to give the statistical classification of objects of Δ.

For surveys of DHR theory in general, see [Roberts, 1970; Roberts, 1990; Roberts, 2004; Fredenhagen, 1992; Fredenhagen, 1994], [Araki, 1999, Ch. 6], and [Haag, 1996, Ch. IV.2].

The full proof of the DR reconstruction theorem is distributed over [Doplicher *et al.*, 1974; Doplicher and Roberts, 1972; Doplicher and Roberts, 1989] and [Doplicher and Roberts, 1990]. The alternative approach to the reconstruction theorem that we use in this paper is based on [Roberts, ND] and [Deligne, 1990], incorporating simplifications of the latter due to [Bichon, 1998] and ourselves.

For informal expositions of the DR reconstruction theorem, see [Doplicher and Roberts, 1987; Doplicher, 1991; Doplicher, 1992; Doplicher, 1993; Doplicher, 1995]. For an interesting description of the goal of reconstructing fields and a gauge group, written before a solution was obtained, see [Roberts, 1975].

11 FOUNDATIONAL IMPLICATIONS OF THE RECONSTRUCTION THEOREM

We now return to the foundational questions (Section 7) that motivated our investigation. We also point out a few other cases where discussions in the philosophical literature (e.g. about permutation symmetry and the identity of particles) might benefit from the study of superselection theory.

11.1 Algebraic Imperialism and Hilbert Space Conservatism

DHR superselection theory sheds light on some questions that philosophers have asked about the role of inequivalent representations of the algebra of observables. But it will not answer all of our questions. We first bracket off those questions for which DHR theory provides little help, and then we proceed to develop a case for the relevance of DHR theory to foundational questions.

The DHR analysis requires that we fix a vacuum state ω_0, and hence a base representation $(\mathcal{H}_0, \pi_0)$. Inequivalent DHR representations do not correspond to different vacuua; rather, they correspond to different local excitations of one and the same vacuum state. So, DHR theory effectively ignores the question of how to choose a vacuum representation. (But note that the power of the DHR analysis

strongly suggests — against the Algebraic Imperialist — that representations *are* essential components of the physical content of the theory.)

Second, in some of the most familiar cases — e.g., the free Boson field — the DHR category is trivial. That is, $DHR(\mathfrak{A}) = \{\pi_0\}$, and so $\mathfrak{F} = \mathfrak{A}$. In this case, the vacuum representation is the *only* DHR representation (relative to itself). Thus, in such cases, the elaborate apparatus of DHR superselection theory seems to provide little insight into the physical importance of inequivalent representations. (However, if we are able to find a physical reason for choosing a preferred vacuum representation, then the DHR analysis suggests that no other representations are relevant for explaining the phenomena.)

Finally, even in cases where $DHR(\mathfrak{A})$ is nontrivial, the field algebra itself has inequivalent representations. (After all, it's just another large C^*-algebra.) And one might worry that the same Conservative versus Imperialist debate will arise with respect to the field algebra.

But DHR theory has something to say about inequivalent DHR representations, and about representations of the field algebra. First, the field algebra $\mathfrak{F}$ is constructed concretely as operators on a Hilbert space $\mathcal{H}$; i.e. $\mathfrak{F}$ comes with a preferred representation. (Recall that the preferred representation of $\mathfrak{F}$ is on a Hilbert space $\mathcal{H}$ that decomposes into a direct sum of superselection sectors for $\mathfrak{A}$.) Of course, we could consider other representations of $\mathfrak{F}$. But in other representations of $\mathfrak{F}$, we no longer have the intuitive interpretation of elements of $\mathfrak{F}$ as intertwiners between the DHR sectors of $\mathfrak{A}$. If the physically meaningful quantities have to have some connection to observable phenomena, then the interpretation of elements of $\mathfrak{F}$ in terms of $\mathfrak{A}$ might be thought to be a necessary condition for interpretation; and so the given representation of $\mathfrak{F}$ might be preferred for these reasons.

So, DHR theory suggests that the issue of inequivalent representations does not come up for the field algebra. Regarding the issue of inequivalent representation of the observable algebra, we can divide the problem into several questions:

1. Is there a physically privileged vacuum state/representation? What features pick it out?

2. Are all physical representations in the DHR category of some vacuum state? (We are ignoring for the time being theories with long range forces (see [Buchholz and Fredenhagen, 1982]). In more general settings, we expect the *form* of the question to remain the same: do we require physical states to be reachable from a fixed vacuum state by the action of an appropriate set of fields?)

3. If the answer to the second question is No, then how should we compare representations that are not connected to a vacuum representation by fields to representations that are?

Let's suppose that the first question has been answered in the affirmative, and that the vacuum representation $(\mathcal{H}_0, \pi_0)$ is fixed. Suppose also that $DHR(\mathfrak{A})$ is

nontrivial. Then how should we think about the inequivalent representations in $DHR(\mathfrak{A})$? A naive transcription of Hilbert Space Conservatism to the current context would tell us the following: the representations in $DHR(\mathfrak{A})$ are analogous to competing theories: one is correct, and the others are surplus structure. The naive transcription of Algebraic Imperialism to the current context would say: the representations in $DHR(\mathfrak{A})$ are surplus structure; the physical content of the theory is in $\mathfrak{A}$, the abstract algebra of observables.

Both Conservatism and Imperialism are based on an oversimplified view of the formalism: it is supposed that the elements of reality correspond to operators in the abstract algebra or in some Hilbert space representation thereof, and that the possible states are either all states on the abstract algebra or some particular folium of states. But the fundamental insight of DHR theory is that the set of representations itself has structure, and it is *this* structure that explains phenomena. So, a more adequate position would take all the representations more seriously. Hence, we propose that according to *Representation Realism*, the content of the theory is given by: (i) the net $O \mapsto \mathfrak{A}(O)$, (ii) the dynamics on the quasilocal algebra (i.e. the representation of the translation group in Aut$\mathfrak{A}$), and (iii) the symmetric tensor $*$-category $DHR(\mathfrak{A})$ of DHR representations.

Recall that the Conservative claims to have the advantage in terms of explanatory power: more structure (provided by choosing a representation) provides more elements of reality, and so more satisfying explanations. But DHR superselection theory shows that this claimed advantage is misleading: the focus on one representation ignores the most interesting structure, namely the relations between representations. Indeed, if we committed ourselves to one representation, and ignored others, we would have no field operators, no gauge group, no definition of Bose and Fermi fields, no definition of antiparticles, etc..

And yet there is a strong *prima facie* objection to Representation Realism: since the Hamiltonian is always an observable, no possible dynamical evolution can take us from a state in one representation to a state in an inequivalent representation. So, inequivalent representations are dynamically isolated from each other, and abstract relations between them cannot explain the features of states in the representation that best describes our universe.

The fact that the Hamiltonian is an observable — hence cannot map states from one sector to states in another — raises a further problem for our interpretation of field operators. Recall that we speak of "creating" states in a sector $\mathcal{H}_\rho$ by acting on the vacuum with elements from the field algebra. That is, we can choose $F \in H_\rho \subseteq \mathfrak{F}$ such that $F\Omega \in \mathcal{H}_\rho$, where $(\mathcal{H}_\rho, \pi_\rho)$ is disjoint from the vacuum representation $(\mathcal{H}_0, \pi_0)$. The talk of "creation" here suggests that we are talking about some sort of dynamical process. On the one hand, $F \in \mathfrak{F}$ can be chosen unitary, so structurally the map $\Omega \mapsto F\Omega$ looks like dynamics. But since the Hamiltonian is an observable, the transition $\Omega \mapsto F\Omega$ is not dynamically allowable. So, in what sense are states in $\mathcal{H}_\rho$ accessible from the vacuum? Is the key insight behind superselection rules that there are two notions of dynamic accessibility? If so, then how are we to understand the differences between these two notions?

11.2 Explanatory relations between representations

If we consider a C^*-algebra $\mathfrak{A}$ with no further structure, then the mathematically definable relations between representations (and hence, between states in the folia of these representations) are exhausted by the following table:

π_1 and π_2 are equivalent
π_1 and π_2 are quasiequivalent
π_1 and π_2 are disjoint
π_1 and π_2 are weakly equivalent

Table 1. Relations Between Representations of $\mathfrak{A}$

Outside of the fourth relation (which makes special reference to the topology of the state space), these relations are precisely those that can be defined in an arbitrary $*$-category $\mathcal{C}$ with subobjects. Two objects X, Y in $\mathcal{C}$ are equivalent if there is a unitary $u \in \mathrm{Hom}(X, Y)$; are quasiequivalent if there is an isometry $v \in \mathrm{Hom}(X, Y)$; and are disjoint just in case they are not quasiequivalent.

Consider now the normal state space K of a C^*-algebra $\mathfrak{A}$. The GNS theorem provides a map $\omega \mapsto (\mathcal{H}_\omega, \pi_\omega)$ from K into the representation category of $\mathfrak{A}$. We then use this map to induce relations corresponding to those in Table 1 on K: we talk about equivalent, quasiequivalent, and disjoint states. Furthermore, the individual folia (sets of states whose GNS representations are quasiequivalent) have a rich geometrical structure which corresponds exactly to the normal state space of $\mathfrak{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Thus, within a folium we have a notion of "transition probability" between pure states (corresponding to rays in $\mathcal{H}$), and a three place relation "ω is a superposition of ω_1 and ω_2." However, if two states lie in disjoint folia, then these relations trivialize. The transition probability between disjoint states is zero, and no state can be a superposition of states from a different folia. It seems that the only physically suggestive thing we can say about states from different folia is that they are "orthogonal."

It is precisely the preceding considerations that have lead philosophers to worry about inequivalent representations. The worry is based on the fact that disjoint representations seem to be *competitors*, and nothing more. In order to alleviate worries about the competition between representations, some philosophers [Clifton and Halvorson, 2001a; Halvorson, 2004] go so far as to claim that these representations are "complementary" descriptions of the phenomena (in the sense of Bohr). The word "complementarity" is of course meant to suggest that the representations are not *merely* competitors, and the choice of one does not need to be seen as completely ruling out the relevance of another.

We wish to replace suggestive — and possibly misleading — terminology with some real facts about the relationships between inequivalent representations. To illustrate what we mean by this, consider the case of *group* representations: let $\mathrm{Rep}_f G$ be the category of unitary representations of a compact group G. $\mathrm{Rep}_f G$ is

not only a $*$-category, but it has a monoidal product and conjugates. That is, for objects X, Y in $\mathrm{Rep}_f G$, there is a product object $X \otimes Y$, and a conjugate object $\overline{X}$. For our purposes, this is the crucial difference between group representations and the representations of an arbitrary C^*-algebra $\mathfrak{A}$. For an arbitrary C^*-algebra $\mathfrak{A}$, there is no product of representations, or conjugate of a representation.

In the case of compact group representations, typically $X \in \mathrm{Rep}_f G$ will be disjoint from both $X \otimes Y$ and $\overline{X}$. But in this case, we are not tempted to see X is *merely* as a competitor of $X \otimes Y$, or of $\overline{X}$; there are some interesting relations between these representations. Roughly speaking, information about X gives us information about $X \otimes Y$ and $\overline{X}$. Thus, although these representations are technically "disjoint," they are not completely unrelated to each other.[5]

One of the main accomplishments of the DHR analysis and DR reconstruction theorem is to show that the category Δ_f of physical representations is a tensor $*$-category with conjugates; indeed the Embedding Theorem (see the Appendix) shows that Δ_f is equivalent to the category $\mathrm{Rep}_f G$ for some compact group G. The obvious question then is whether these additional relations on the category of representations can help us get past the idea that disjoint representations are merely competing descriptions.

An analogy to states might be helpful. Consider a pair H_1, H_2 of Hilbert spaces, and let $\psi_i \in H_i$ be unit vectors. Now consider the following two "descriptions of reality":

1. The state is ψ_1.

2. The state is $\psi_1 \otimes \psi_2$.

What do we say here: are these competing descriptions? In one sense, (1) and (2) are competitors, because they cannot both be fully adequate descriptions of reality at the same time. However, (1) and (2) are not competitors in the same sense that, say, two orthogonal vectors in a single Hilbert space are competitors. The two state descriptions are not merely competitors, because there is an interesting sense in which ψ_1 is a "part" of $\psi_1 \otimes \psi_2$. Indeed, information about ψ_1 (e.g. expectation values it assigns to observables) does in fact give us information about $\psi_1 \otimes \psi_2$ because of the canonical mappings between H_1 and $H_1 \otimes H_2$.

Now let π_1, π_2 be objects in the DHR category Δ_f, and suppose (as will often be the case) that the representations π_1 and $\pi_1 \otimes \pi_2$ are disjoint. Are these competing descriptions? Again, π_1 and $\pi_1 \otimes \pi_2$ are competitors in the sense that if the state of an object (or of the universe?) is in $\mathcal{H}_{(\pi_1 \otimes \pi_2)}$ then it is not in $\mathcal{H}_{\pi_1}$. Nonetheless, π_1 and $\pi_1 \otimes \pi_2$ are not merely competitors, because in one sense π_1 is "part" of $\pi_1 \otimes \pi_2$.

[5]But note also: Philosophers of physics have so far not worried about inequivalent *group* representations as competing descriptions of reality. And for good reason, because group elements are not observables, and groups do not have states. Another insight of DHR theory is to show that physicist's intuitions about group representations are not totally baseless, because in fact the interesting (DHR) representations of the observable algebra correspond to representations of a compact group.

But two words of caution should be issued here. First, we must be cautious with the use of the "part" metaphor. For example, Δ_f can have a nontrivial representation π such that $\pi \otimes \pi$ is equivalent to the vacuum representation. Then it is not so clear that we should say that "π is part of $\pi \otimes \pi$." Second, there is one significant disanalogy between the case of states ψ_1 and $\psi_1 \otimes \psi_2$ and the case of representations π_1 and $\pi_1 \otimes \pi_2$: the two representations are GNS representations of states on a *single* C^*-algebra $\mathfrak{A}$. Hence we can directly compare the expectation values that these states assign to observables in $\mathfrak{A}$, and they will disagree significantly (indeed, for any $\varepsilon > 0$ there is an observable $A \in \mathfrak{A}$ such that $\|A\| \leq 1$ and $\|\omega_1(A) - \omega_2(A)\| > 2 - \varepsilon$). Thus, there is a clear, empirically verifiable sense in which states in π_1 are competitors with states in $\pi_1 \otimes \pi_2$.

Finally, there is an interesting physical relation between a DHR representation π and its conjugate $\overline{\pi}$, even though π and $\overline{\pi}$ are typically disjoint. In short, $\overline{\pi}$ is like an inverse of π: if π is irreducible, then $\overline{\pi}$ is the unique irreducible representation such that $\pi \otimes \overline{\pi}$ contains a copy of the vacuum representation. In fact, when $\pi = \pi_0 \circ \rho$ where ρ is a dimension 1 element of Δ_f, $d(\rho) = 1$, then this is the exact relation: ρ is an automorphism and $\overline{\rho} = \rho^{-1}$. In terms of field operators, if F creates the charge ξ, then $\overline{F}$ annihilates the charge ξ. Furthermore, when π admits a particle interpretation, then the states in the folium of $\overline{\pi}$ are the antiparticle states of the states in the folium of π [Doplicher *et al.*, 1969b].

11.3 Fields as theoretical entities, as surplus structure

From the standpoint of superselection theory, there is a sharp distinction between observable and unobservable fields, namely, a field operator is an observable iff it is invariant under all gauge transformations. To what extent does this distinction between fields and observables match up with the philosopher of science's distinction between theoretical and observational components of a theory? Even if the two notions are not exactly the same, the connection is suggestive. In particular, it seems interesting to ask about the extent to which the field plus gauge part of QFT is fixed by the observable algebra.

First, the notion of equivalent systems of field operators seems a fairly close analogue of the philosopher's notion of "theoretical equivalence."

DEFINITION 289. Let $\mathfrak{F}_1 = (\mathfrak{F}_1, \mathcal{H}_1, G_1)$ and $\mathfrak{F}_2 = (\mathfrak{F}_2, \mathcal{H}_2, G_2)$ be local field systems with gauge symmetry for $(\mathfrak{A}, \omega)$. (See Defn. 247 on p. 808.) Then $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are *theoretically equivalent* just in case they are unitarily equivalent as systems of local field operators. (See Defn. 10 on p. 816.)

REMARK 290. (i) This definition is not fully adequate, because it does not make reference to dynamics. For example, this definition entails that the free Bose field nets for different positive masses are theoretically equivalent. For a fully adequate definition, it would probably be necessary to require that the unitary mapping $W : \mathcal{H}_1 \to \mathcal{H}_2$ also intertwines the dynamical groups on the two Hilbert spaces. (ii) If $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are theoretically equivalent, then they are equivalent in *all* physically relevant senses (modulo dynamics): they have the same type of commutation

relations (either both have normal or abnormal commutation relations), they have isomorphic gauge groups, etc..

The working analogy also suggests that we define "observational equivalence" between two theories in terms of some equivalence between their nets of observable algebras. There are a myriad number of ways we could explicate the notion of observational equivalence in this setting; philosophers have their work cut out for them. The following two definitions give extremely weak notions of observational equivalence that do not take into account a representation of the algebra of observables.

DEFINITION 291. Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be two local field systems with gauge symmetry, and let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be the fixed point algebras; i.e.,

$$\mathfrak{A}_i = \{A \in \mathfrak{F}_i : \alpha_g(A) = A, \text{ for all } g \in G_i\}.$$

Then we say that $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are *weakly observationally equivalent* just in case there is a $*$-isomorphism α from the algebra $\mathfrak{A}_1$ onto the algebra $\mathfrak{A}_2$.

DEFINITION 292. Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be two local field systems with gauge symmetry, and let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be their fixed point nets; i.e. for each double cone O,

$$\mathfrak{A}_i(O) = \{A \in \mathfrak{F}_i(O) : \alpha_g(A) = A, \text{ for all } g \in G_i\}.$$

Then we say that $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are *observationally equivalent* just in case there is a net isomorphism $\alpha : \mathfrak{A}_1 \to \mathfrak{A}_2$ (see Defn. 118 on p. 767).

REMARK 293. The first definition is weaker because it does not require that the net structure be preserved by the $*$-isomorphism α.

Again, the definitions omit reference to dynamics, which would be an important component of a fully adequate treatment of observational equivalence. Nonetheless, even with these definitions, we can make some sense of remarks about underdetermination of fields by observables, or about the physical equivalence of different field theories.

1. (Construction of observationally equivalent theories) The DR reconstruction theorem provides a general, nontrivial recipe for constructing non-equivalent theories that are observationally equivalent: If $(\mathfrak{A}, \omega_0)$ has nontrivial DHR superselection sectors, then it can be embedded into two nonequivalent field algebras $\mathfrak{F}_1$ and $\mathfrak{F}_2$. Indeed, $\mathfrak{A}$ is always a field algebra over itself (but incomplete), but the field algebra $\mathfrak{F}$ from the DR reconstruction theorem is complete.

2. (Elimination of parafields) It has long been thought that parafields are a theoretical artifact. More precisely, it has been claimed that every parafield theory is "physically equivalent" to a theory with normal commutation relations (see [Araki, 1961]). The DR reconstruction theorem partially validates this claim by showing that every parafield theory is *observationally equivalent* to a theory with normal commutation relations. Indeed, suppose that

$\mathfrak{F}_1$ is a parafield theory. Then we can extract the observable algebra $\mathfrak{A}$ contained in $\mathfrak{F}_1$, and apply the DR reconstruction theorem to construct a field algebra $\mathfrak{F}_2$ with normal commutation relations. Since $\mathfrak{F}_1$ and $\mathfrak{F}_2$ have the same net of local observable algebras, they are observationally equivalent.

3. Some have claimed that the relation between quantum fields (the field algebra $\mathfrak{F}$) and observables (the observable algebra $\mathfrak{A}$) is analogous to relation between coordinates and a manifold in differential geometry. However, the DR reconstruction theorem shows that (subject to normal commutation relations), there is a *unique* field net $\mathfrak{F}$ and gauge group G compatible with the observable algebra $(\mathfrak{A}, \omega_0)$. Thus, there is a strong disanalogy between the two cases, since there seems to be no sense in which one coordinate system of a manifold is a better representation of reality than another coordinate system.

Finally, we are in a position to get a clear picture of the interrelations between questions about inequivalent representations and questions about gauge invariance.

According to a common interpretive stance, if two states of a physical system can be identified under a gauge symmetry, then those two states are different descriptions of the same underlying reality. So, for the purposes of counting states, we should look to the quotient of the state space under gauge orbits. Similarly, it is thought that the "real" quantities of a theory are the gauge invariant quantities (see [Earman, 2004]).

In the setting of DHR superselection theory, the algebra of observables $\mathfrak{A}$ is precisely the gauge invariant part of the field algebra $\mathfrak{F}$, that is,

$$\mathfrak{A} = \{A \in \mathfrak{F} : \alpha_g(A) = A, \text{ for all } g \in G\},$$

where G is the gauge group. This of course means that for any observable $A \in \mathfrak{A}$, there is no difference between a state ψ and the gauge transformed state $U(g)\psi$. (Of course, if ψ is a state vector in the vacuum representation, then $U(g)\psi = \psi$, since the representation of the gauge group there is trivial.) So, if the common interpretive stance is correct, the physical content of the theory is in the observable algebra $\mathfrak{A}$; the fields are "descriptive fluff."

So suppose that we ignore the field algebra $\mathfrak{F}$, and just look to the observable algebra $\mathfrak{A}$ to provide the physical content of the theory. But what should we say about the representations of $\mathfrak{A}$? Are they just descriptive fluff? If not, then is there one correct representation of $\mathfrak{A}$, or do we somehow need inequivalent representations in order to account for all of the physical facts?

The DR reconstruction theorem shows that the preceding two sets of questions — regarding the status of gauge variant quantities on the one hand, and representations on the other hand — are tightly intertwined. The full structure of the theory, field algebra $\mathfrak{F}$ and gauge group G, is uniquely recoverable (modulo completeness, and normal commutation relations) from the structure of the category $DHR(\mathfrak{A})$ of representations. The ontological significance of the gauge variant fields

is closely aligned with the ontological significance of inequivalent representations. (We will revisit this question in the next section when we discuss permutation symmetry.)

Of course, there is a crucial disanalogy between the global gauge symmetries in DHR superselection theory, and the local gauge symmetries of electromagnetism or general relativity. But it is not clear that this disanalogy makes the DR reconstruction theorem any less interesting for understanding the relation between gauge symmetry and superselection rules.

11.4 Statistics, permutation symmetry, and identical particles

Philosophers have taken an active interest in the differences between the Maxwell-Boltzmann statistics of classical physics, and the Bose-Fermi statistics of quantum physics. Indeed, it has been provocatively claimed that Bose-Fermi statistics is explained by permutation invariance — i.e. there are no physical differences between states with permuted particle labels — and that this entails that quantum particles are not "individuals" in the same sense that classical particles are. (See [French, 2000; French and Rickles, 2003] for an account of the argument.)

But such discussions can be hampered by an overly simplistic formalism. In particular, states are usually identified with unit vectors (or at best with rays) in a single Hilbert space, and no account is given of the status of non-permutation invariant operators. It might be helpful then to look at the issue from the perspective of a more flexible formalism that allows a distinction between fields and observables, and in which states can be represented by vectors in inequivalent representations of the algebra of observables.

There is another reason why the issue of permutation invariance should be revisited within the context of QFT. Some of the literature suggests that the metaphysical problem about the individuality of particles is *aufgehoben* in the transition to QFT, because: (i) QFT is about fields and not particles, and (ii) the Fock space formalism of QFT already identifies permuted states, and so rules out a notion of particles as individuals. We have already noted that it is not so clear how to make sense of the idea that QFT is about fields as opposed to particles. Furthermore, the DR reconstruction theorem shows precisely how to make sense of non-permutation invariant quantities and states in a theory that is manifestly permutation invariant.

It is not surprising that DHR theory is relevant for the issue of permutation invariance and statistics: one of the original goals of DHR was to clarify the role of statistics in QFT. Riding the crest of a wave of mathematical success, Roberts made the following bold claim about the DHR analysis of statistics:

> One of the insights provided by the study of superselection sectors concerns the origin of what is termed the 'statistics' of a particle. ... Now just as a theory should determine its particle states so should it determine the statistics of these particles. Ordinary quantum mechanics ignores this challenge saying in effect that the statistics of particles is

> one of the parameters determining the theory, the one telling you what symmetry the n-particle states have. QFT does a little better: it says that the statistics of the particles in the theory is determined by the commutation relations of fields at spacelike separations. ... In adopting the philosophy that the local observables determine the theory, we are forced to meet this challenge in full. [Roberts, 1976, p. 203]

In the remainder of the paper from which the quote is taken, Roberts shows how Bose-Fermi particle statistics emerges naturally from the DHR analysis of physical representations of the algebra of observables.

Roberts' claim is of crucial relevance to the philosophical debate about statistics and identical particles. The philosophers have asked, "what explains Bose-Fermi statistics?" Roberts' answer is that the explanation comes from the structure of the category of representations of the observable algebra.

Let us recall then how the Bose-Fermi distinction is supposed to emerge from the DHR analysis. In Section 8.3, an object ρ of the category Δ is shown to have an intrinsic dimension $d(\rho)$. The dimension is finite iff ρ has a conjugate; in this case we define a unitary operator $\Theta_\rho \in \mathrm{End}(\rho)$ called the *twist* of ρ. If ρ is irreducible, then $\Theta_\rho = \omega_\rho \mathrm{id}_\rho$ where $\omega_\rho = \pm 1$. We then stipulate that a "Bosonic" object is one with $\omega_\rho = 1$ and a "Fermionic" object is one with $\omega_\rho = -1$.

Of course, ρ is not the sort of thing that the philosophers have been calling Bosonic or Fermionic — it is not a wavefunction. To connect the two pieces of formalism, recall that an object of Δ_f (endomorphisms of the algebra of observables) corresponds to a representation $\pi_0 \circ \rho$ of the algebra of observables. So, we call the representation $\pi_0 \circ \rho$ Bosonic when ρ is Bosonic, and Fermionic when ρ is Fermionic. Finally, we then call a state ("wavefunction") Bosonic if it is in the folium of a Bosonic representation, and Fermionic if it is in the folium of a Fermionic representation. The claims to be assessed are: (i) does this stipulative definition adequately reproduce the distinction between Bosonic and Fermionic wavefunctions made in elementary nonrelativistic QM; and if so, (ii) what does this tell us about permutation invariance?

The Bose-Fermi distinction in nonrelativistic QM

In nonrelativistic QM, the state space of n identical particles is the tensor product $H \otimes \cdots \otimes H$ of n copies of a Hilbert space H. The Hilbert space $H \otimes \cdots \otimes H$ is spanned by *product states*, i.e. states of the form

$$\psi_1 \otimes \cdots \otimes \psi_n,$$

with $\psi_1, \ldots, \psi_n \in H$.

DEFINITION 294. We define the natural action of the permutation group S_n on $H \otimes \cdots \otimes H$ as follows. Let $\{\psi_1, \ldots, \psi_m\}$ be an orthonormal basis for H, and define for each permutation σ,

$$U(\sigma)(\psi_{i_1} \otimes \cdots \otimes \psi_{i_m}) = \psi_{\sigma(i_1)} \otimes \cdots \otimes \psi_{\sigma(i_m)},$$

and extend $U(\sigma)$ by linearity to all of H.

If $\dim H > 1$, then the representation U of S_n is reducible. It contains copies of the two one-dimensional representations of S_n, namely the identity representation $S_n \to \mathbf{1}$, and the alternating representation. The subspace of vectors of $H \otimes \cdots \otimes H$ transforming according to the identity representation is called the *symmetric* subspace, and the subspace of vectors transforming according to the alternating representation is called the *antisymmetric* subspace. Vectors in the symmetric subspace are called Bosonic, and vectors in the antisymmetric subspace are called Fermionic. These traditional definitions have served as the basis of discussions of permutation invariance in the philosophical literature.

In QM, states of n particles that differ only by permuting labels — for example, $\psi_1 \otimes \psi_2$ versus $\psi_2 \otimes \psi_1$ — should not be counted separately. For the purposes of statistical weighting, these two symbols represent one state. This has been stated as the principle of Permutation Invariance.

> **Permutation Invariance (PI):** Let $\mathfrak{A}$ be the observables for the n particle system. Then for each state ψ, and for each permutation $\sigma \in S_n$, we have
>
> $$\langle U(\sigma)\psi, AU(\sigma)\psi\rangle = \langle \psi, A\psi\rangle.$$

Permutation Invariance is sometimes also called the *Indistinguishability Postulate*: two states that differ by a permutation are indistinguishable (i.e. there is no measurement that can distinguish between the two). It has been claimed that PI entails that state has Bose or Fermi statistics, no states with "parastatistics" are allowed.

> **Dichotomy:** For each state vector ψ and permutation σ, we have $U(\sigma)\psi = \pm\psi$.

(See [van Fraassen, 1991, pp. 389ff] for an account of attempts to prove Dichotomy from PI. See [Butterfield, 1993] for further details.) In other words, the states that are not in the symmetric or antisymmetric subspaces are surplus structure.

This leaves us with a number of puzzles. First, what do we say about the vectors in $H \otimes \cdots \otimes H$ that are not in the symmetric or antisymmetric subspaces? Are they surplus structure? Are they possibilities that are contingently not realized in nature? More generally, not all vectors in $H \otimes \cdots \otimes H$ are of a definite symmetry type; and even among those that are of a definite symmetry type, not all are totally symmetric or totally antisymmetric. For any irreducible representation ξ of S_n we say that a wavefunction ψ in $H \otimes \cdots \otimes H$ is of symmetry type ξ just in case ψ is contained in the subspace corresponding to the representation ξ. Then $H \otimes \cdots \otimes H$ is the direct sum of subspaces of definite symmetry type vectors. But now the principle of plenitude suggests that there should be particles of every symmetry type. Why do we not see them?

An intrinsic characterization of symmetric and antisymmetric subspaces

We began with the full n-particle Hilbert space $H \otimes \cdots \otimes H$, and then we reduced to the symmetric and antisymmetric subspaces. We were then left wondering what to do with the remaining elements of $H \otimes \cdots \otimes H$.

The intrinsic description of the symmetric and antisymmetric subspaces is that they are representations of the group S_n. (In fact, they are quasiequivalent to the one dimensional irreducible representations of S_n.) So we can also work backwards. That is, if we are given a representation (H, π) of S_n, we can ask after the intrinsic characterization of this representation. Recall that the irreducible representations of S_n are in one-to-one correspondence with Young tableaux with n boxes (see [Simon, 1996]). There is a natural grouping of representations of S_n into para-Bose and para-Fermi categories: we specify the representation (H, π) by a pair of numbers $(d(\pi), \omega_\pi)$, with $d(\pi) \in \{1, 2, \ldots, n\}$ and $\omega_\pi = \pm 1$.

1. For $(d, +1)$, all Young tableaux whose columns have length less than or equal d. (In this case, we say that π has *Para-Bose statistics* of order d.)

2. For $(d, -1)$, all Young tableaux whose rows have length less than or equal d. (In this case, we say that π has *para-Fermi statistics* of order d.)

Clearly the division of representations into para-Bose and para-Fermi is mutually exclusive, but not exhaustive. (e.g., there are representations of S_n that contain copies of both the **1** representation and the alternating representation.)

Now suppose that we are in the following position (described vividly by Roberts in the opening quote): we are given a pure state ω of the algebra of observables $\mathfrak{A}$ and we asked whether its "intrinsic" statistics of its states is Bosonic or Fermionic. What can we do? First we construct the GNS representation $(\mathcal{H}, \pi)$ induced by ω. At least this makes things more concrete. But the Hilbert space $\mathcal{H}$ is not itself a tensor product, and so there is no natural representation of S_n on $\mathcal{H}$. Nor would it help to construct tensor products of elements of $\mathcal{H}$ in the normal way, because for $\psi \otimes \cdots \otimes \psi$ is trivially Bosonic. So, the obvious approach does not seem to tell us anything about the *intrinsic* symmetry type of elements of $\mathcal{H}$.

The key insight again comes from the DHR analysis: the representation $(\mathcal{H}, \pi)$ is naturally isomorphic to an object ρ of a symmetric tensor $*$-category, namely the category Δ_f of localized transportable endomorphisms. Since Δ_f has products we can construct $\rho \otimes \rho$, and the symmetry $\varepsilon_{\rho,\rho}$ gives us notion of permuting $\rho \otimes \rho$. [Recall that $\varepsilon_{\rho,\rho} \in \mathrm{Hom}(\rho \otimes \rho)$.] As we will see in the following section, this gives us a natural representation u of S_n in $\mathrm{End}(\rho \otimes \rho)$. Furthermore, the pair $(d(\rho), \omega_\rho)$, where $d(\rho)$ is the dimension of ρ, and ω_ρ is the statistical phase of ρ, coincide with the classification of u as a para-Bose or para-Fermi representation of S_n. We will also see that this natural representation u of S_n corresponds to a permutation of wavefunctions in the "larger" Hilbert space of the field algebra $\mathfrak{F}$.

Representations of S_n in a symmetric tensor $*$-category

Unitary representations of the permutation group S_n arise naturally in tensor $*$-categories with a unitary symmetry. Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a tensor $*$-category with unitary symmetry $c_{X,Y}$. Fix an object $X \in \mathrm{Obj}(\mathcal{C})$, and define a map $u : S_2 \to \mathrm{End}(X \otimes X)$ by setting

$$u((1)) = \mathrm{id}_{X\otimes X}, \qquad u((1,2)) = c_{X,X}.$$

Since $(c_{X,X})^2 = \mathrm{id}_{X\otimes X}$, u is a unitary representation of S_2 in $\mathrm{End}(X \otimes X)$. This construction can be iterated: define a map $u : S_n \to \mathrm{End}(X \otimes \cdots \otimes X)$ by setting

$$u((i, i+1)) = \mathrm{id}_X \otimes \cdots \otimes c_{X,X} \otimes \cdots \otimes \mathrm{id}_X.$$

It is easy to verify that u extends uniquely to a unitary representation of S_n in $\mathrm{End}(X \otimes \cdots \otimes X)$.

FACT 295. Let $\mathcal{C}$ be a tensor $*$-category with unitary symmetry and conjugates. Then for each irreducible object $X \in \mathrm{Obj}(\mathcal{C})$ the induced unitary representation u of S_n in $\mathrm{End}(X \otimes \cdots \otimes X)$ is para-Bose of order $d(X)$ if $\omega_X = +1$, and is para-Fermi of order $d(X)$ if $\omega_X = -1$. Furthermore, the statistical phase ω_X is the trace of $u((1,2)) = c_{X,X}$. (See Appendix B for more details.)

The physical interpretation becomes more clear in the presence of field operators. (Of course, the point of the Reconstruction Theorem is to show that such field operators are always available.) Let $(\mathcal{H}, \mathfrak{F}, (G, k))$ be a field system with gauge symmetry for the observable algebra $\mathfrak{A}$ and vacuum state ω. Let $O_1, \ldots, O_n$ be mutually spacelike separated regions. Let ρ be an irreducible object in Δ_f. Then using the transportability of ρ we can choose $F_i \in \mathfrak{F}(O_i)$ such that $F_i\Omega$ is in the sector $\mathcal{H}_{\hat{\rho}}$. (Recall that sectors are labeled by unitary equivalent classes $\hat{\rho}$ of objects in Δ_f.) In other words, F_i creates the charge $\hat{\rho}$ in the region O_i. Let σ be a permutation of $\{1, \ldots, n\}$ and consider the following two state vectors in $\mathcal{H}$:

$$\psi_1 \times \psi_2 \times \cdots \times \psi_n \equiv F_1F_2 \cdots F_n\Omega, \tag{52}$$

$$\psi_{\sigma(1)} \times \psi_{\sigma(2)} \times \cdots \times \psi_{\sigma(n)} \equiv F_{\sigma(1)}F_{\sigma(2)} \cdots F_{\sigma(n)}\Omega. \tag{53}$$

These two vectors are typically distinct. In fact, if the field net has normal commutation relations then we can compute the difference between the two vectors. Supposing that σ only permutes two numbers, the two vectors will be the same if ρ is Bosonic, and the two vectors will differ by a sign if ρ is Fermionic. However, the two vectors always induce the same state on the algebra of observables $\pi(\mathfrak{A})$. Indeed, if $\rho_i \in \Delta_f(O_i)$ are the corresponding morphisms, then the states induced by the two vectors, respectively, are

$$\omega \circ (\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_n) = \omega \circ (\rho_1\rho_2 \cdots \rho_n), \tag{54}$$

$$\omega \circ \left(\rho_{\sigma(1)} \otimes \rho_{\sigma(2)} \otimes \cdots \otimes \rho_{\sigma(n)}\right) = \omega \circ (\rho_{\sigma(1)}\rho_{\sigma(2)} \cdots \rho_{\sigma(n)}). \tag{55}$$

Since endomorphisms that are localized in spacelike separated regions commute, these two states are equal. Thus, permutation invariance holds for the observables, but not for the fields.

The interpretive upshot of the the DHR treatment of statistics is as follows: permutation invariance is a gauge symmetry in the sense that it leaves all observables' values invariant, but changes the values assigned to field operators. Are two states related by a permutation the same or different? Of course, the answer to the mathematical question is clear: the states of the observable algebra are the same, the states of the field algebra are different. So, whether or not we take permutations to correspond to "real" changes depends on what we think about the status of the field operators. So the issue of permutation invariance is just a special version of the issue of gauge invariance, and accordingly is tightly connected to the question of the status of inequivalent representations.

Parastatistics and nonabelian gauge groups

The abstract Tannaka Theorem (Appendix B) shows that each symmetric tensor $*$-category (STC*) $\mathcal{C}$ is equivalent to the representation category $\mathrm{Rep}_f(G,k)$ of a compact supergroup (G,k). Applied to our current topic, the theorem shows that the category Δ_f of localized transportable morphisms is equivalent to the representation category of the gauge group. Furthermore, Section B.9 shows that each object X of an STC* gives rise naturally to a unitary representation of the symmetric group S_n in $\mathrm{End}(X \otimes \cdots \otimes X)$, and this representation corresponds to the intrinsic statistical characterization of X. Now, we know that the categorical dimension of a representation (H,π) of (G,k) corresponds to the dimension of the underlying Hilbert space H. Hence:

LEMMA 296. *The category* $\mathrm{Rep}_f(G,k)$ *has irreducible objects of dimension greater than* 1 *iff* G *is nonabelian.*

Sketch of proof. The set of irreducible representations of G separates the elements of G. Hence for $gh \neq hg$, there is an irreducible representation (H,π) such that $\pi(g)\pi(h) \neq \pi(h)\pi(g)$. Therefore $\dim H \geq 2$. ■

It immediately follows from Fact 295, in conjunction with the fact that the embedding functor preserves dimension, that:

PROPOSITION 297. *There is an irreducible object* X *of* $\mathcal{C} \simeq \mathrm{Rep}_f(G,k)$ *with parastatistics iff the corresponding group* G *is nonabelian.*

Applied to our current case, this means that there are representations and states with parastatistics iff the gauge group G is nonabelian.[6] But we have good reasons to think that the case of nonabelian gauge groups is physically relevant. So, the DHR approach ignores worries about the supposed nonexistence of paraparticle states, and undermines claims that there is a proof of Dichotomy.

[6] But there is an ambiguity in "parastatistics." We mean para-Bose or para-Fermi statistics, not mixtures of Bose and Fermi statistics.

Braid group statistics

Recall from Section 8.1 that when spacetime is dimension 2, then $\varepsilon_{\rho_1,\rho_2}$ is not necessarily a symmetry on Δ_f, but only a braiding. In this case, objects in Δ_f are not classified according to representations of the symmetric group S_n; rather, objects in Δ_f are classified in terms of representations of the braid group B_n. In physical terms, states might not be permutation invariant, but satisfy the more general braid group statistics.

DEFINITION 298. The *braid group* B_n on n strands is the group generated by the set $\{\sigma_1, \ldots, \sigma_{n-1}\}$ satisfying the equations

$$(1) \quad \sigma_i\sigma_j = \sigma_j\sigma_i, \qquad |i-j| \geq 2,$$
$$(2) \quad \sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i.$$

The braid group on n strands can be given the following heuristic, geometric description: a braid consists of two sets $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ of base points, and n strands of yarn, where each yarn strand has one end attached to a_i, and the other end attached to b_j, and each base point touches only one strand of yarn. We identify two configurations if one can be transformed into the other without disconnecting the strands from the base points. In this picture, the identity braid has a_i connected to b_i, and no twists. The generating element σ_i can be thought of as the simple braid where: a_i is connected to b_{i+1}, a_{i+1} is connected to b_i, and these two strands are twisted only once. (Otherwise, the σ_i braid looks just like the identity braid.) Under this interpretation, the product gh of two braids is the braid formed by attaching the ending points of g to the beginning points of h.

PROPOSITION 299. *For each $n \in \mathbb{N}$, the mapping*

$$\varepsilon_\rho^{(n)}(\sigma_i) \quad = \quad \rho^{i-1}(\varepsilon_\rho) \quad = \quad I_\rho \times \cdots \times I_\rho \times \varepsilon_\rho \times I_\rho \times \cdots \times I_\rho,$$

defines a unitary representation of the braid group B_n in $\mathrm{End}\,(\rho \otimes \cdots \otimes \rho)$. *For each $i, j \in \mathbb{N}$ with $i \leq j$ we have*

$$\varepsilon_\rho^{(i)}(g) = \varepsilon_\rho^{(j)}(\varphi_{ij}(g)), \qquad \forall g \in S_i.$$

(A proof of this Proposition can be found in [Rehren *et al.*, 1990]. Alternatively, it is obvious given the considerations in Appendix B.) In other words, the product object $\rho \otimes \cdots \otimes \rho$ carries a unitary representation of the braid group, which is induced by the unitary operators of the form:

$$I_\rho \times \cdots \times \varepsilon_{\rho,\rho} \times \cdots \times I_\rho.$$

This represents an elementary permutation of the i-th and $(i+1)$-st copy of ρ.

There is a natural homomorphism of the braid group onto the symmetric group. This is obvious when we recall that the definition of the symmetric group is exactly

the same as the definition of the braid group with the additional condition that each generator is its own inverse. Hence, van Dyck's theorem [Hungerford, 1980, p. 78] entails that the obvious map f on generators extends uniquely to a group homomorphism $\overline{f} : B_n \to S_n$. So, each representation π of S_n yields a representation $\pi \circ \overline{f}$ of B_n. In slogan form: a system that obeys permutation statistics also obeys braid statistics.

Recall now the worrisome argument for the existence of paraparticles: There should be particles corresponding to all irreducible representations of S_n. For $n \geq 3$, there are non Bose or Fermi representations of S_n, so there should be paraparticles.

Now we can see that either something is wrong with this argument, or the problem is much more severe than we thought. Since any system that has S_n as a symmetry group also has B_n as a symmetry group, the argument commits us to predicting the existence of particles corresponding to all irreducible representations of B_n. But B_n has infinitely many irreducible representations. (Indeed, its representations have so far resisted classification.) Furthermore, we could now repeat the argument for *any* group K that can be mapped homomorphically onto B_n, and there is an infinite chain of larger groups with this property. Therefore, the principle of plenitude applied to group representations predicts more particles than we could ever possibly describe.

Notes: For discussions of statistics of DHR representations, see [Roberts, 1976; Doplicher, 1975].

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