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Peirce, Clifford, and Dirac

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There is a clear line of progression from the "logic of relations" of Charles Sanders Peirce through the algebras of William Kingdon Clifford. Further, it has been shown how one can obtain the nonrelativistic quantum theory of spin one-half particles from Peirce logic. Continuing the hypothetical history, it is demonstrated here that the relativistic Dirac theory can also be related to Peirce logic. The most natural way to accomplish this is to represent the Dirac wave functions themselves as Clifford numbers rather than as spinors. The wave functions can thus appear as 4×4 matrices. All quantities in this quantum theory can actually be expressed in terms of the Clifford basis, independent of a specific matrix representation.

KEY WORDS: Peirce logic; Clifford algebra; Dirac equation.

1. INTRODUCTION

In a recent paper (Beil and Ketner, 2003) the personal and professional association of the American physicist and philosopher Charles Sanders Peirce (1839–1914) and the British mathematician William Kingdon Clifford (1845–1874) was recounted. A relationship between the logical system which Peirce called his "logic of relations," on the one hand, and the well-known Clifford algebras, in particular C(3, 0), on the other, was also established. The Peirce logic was shown to translate into a linear associative algebra which is related to quaternions. The elements of the Peirce logic are Clifford numbers and are simple linear combinations of the quaternion basis.

A correspondence with nonrelativistic quantum mechanics was also demonstrated. This follows from the fact that the Pauli matrices are a representation of the basis of the C(3, 0) algebra. The quantum correspondence is made even closer by the representation of the wave functions themselves as matrices instead of spinors. That is, instead of wave functions as spinors which are represented as column or row matrices, the wave functions are taken to be, say, 2×2 matrices (in the Pauli

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basis) and just Clifford numbers themselves. Thus, all quantities in this quantum theory are Clifford elements and all quantum equations can be expressed in terms of the Clifford basis, independent of any specific matrix representation.

The use of a Clifford number interpretation of a relativistic wave function was actually discussed very early in the history of the Dirac theory (Eddington, 1928; Proca, 1930; Sauter, 1930; Sommerfeld, 1939). An outline of this development is given by Snygg (1997). As in the Pauli theory, the wave functions are represented by square matrices, however, in the Dirac case the matrices are 4×4. The Clifford algebra is C(1, 3) (Göckeler and Schücker, 1987).

This unified approach was recalled briefly by Ross (1986), but not explored in detail.

So the purpose here is to outline, in similar fashion as in Beil and Ketner (2003), how the Peirce logic relates to Dirac wave functions as elements of a Clifford algebra. This gives a quantum theory where all quantities are expressed as Clifford multivectors. Some alternative Dirac solutions appear which may lead to new physical interpretations.

2. PEIRCE AND CLIFFORD

Peirce's logic of relations might best be explained by an example he himself gave (Peirce, 1933): Take two mutually exclusive classes of individuals, say, teachers, u_1, and pupils, u_2. In general, there could be more than two classes, with individuals labeled u_i. These individuals are called "absolute terms" by Peirce, which we will shorten to "absolutes." Absolutes are one type of element in the logic.

The absolutes can be operated on linearly by a second type of element called "dual relatives" by Peirce. A better name for present purposes is "relative operators" or just "operators." In Peirce's scheme, for two classes of absolute, there are four operators, u_{11}, u_{12}, u_{21}, u_{22}, which act on the absolutes as follows:

$$u_{11}u_1 = u_1 \qquad u_{12}u_2 = u_1$$
$$u_{21}u_1 = u_2 \qquad u_{22}u_2 = u_2 \tag{1a}$$

A verbal statement of, say, $u_{12}u_2 = u_1$, would be "u_{12} acting on a pupil produces a teacher." Note that the first and last of the equations in (1a) have the form of eigenvalue equations.

In addition, there are four other operator results which are included:

$$u_{11}u_2 = 0 \qquad u_{12}u_1 = 0$$
$$u_{21}u_2 = 0 \qquad u_{22}u_1 = 0 \tag{1b}$$

A verbal statement of, say, $u_{12}u_1 = 0$ would be "u_{12} acting on a teacher produces nothing."

Table I. Multiplication Table for Relative Operators

	u_{11}	u_{12}	u_{21}	u_{22}
u_{11}	u_{11}	u_{12}	0	0
u_{12}	0	0	u_{11}	u_{12}
u_{21}	u_{21}	u_{22}	0	0
u_{22}	0	0	u_{21}	u_{22}

The general rule for the action of operators on absolutes is

$$u_{ij}u_k = \delta_{jk}u_i \tag{2}$$

The operators can also act on each other according to the multiplication rule

$$u_{ij}u_{kl} = \delta_{jk}u_{il} \tag{3}$$

For two classes of absolutes the multiplication table is given in Table I. The operator in the left column acts from the left on the operator in the top row and produces the result in the corresponding box. A simple matrix representation of these operators has been given by Lenzen (1975):

$$u_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad u_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad u_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad u_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{4}$$

The four operators can obviously be represented by a basis for 2×2 matrices. This means that, with complex coefficients, linear combinations of the four operators can be constructed which are the quaternion basis:

$$\mathbf{I} = u_{11} + u_{22}, \quad \mathbf{i} = -iu_{12} - iu_{21}, \quad \mathbf{j} = u_{21} - u_{12}, \quad \mathbf{k} = -iu_{11} + iu_{22} \tag{5}$$

So the u_{ij}'s are numbers in the quaternion system and are also Clifford numbers. It will be shown presently that this produces a direct relation between the operators of Peirce logic and the Pauli matrices of quantum theory. The absolutes u_i can also be placed in this algebraic scheme.

Peirce himself gave an expression for absolutes as a sum of operators:

$$u_i = \sum_j u_{ij} \tag{6}$$

For the case of two classes of individuals and the matrix representation (4),

$$\begin{aligned} u_1 &= u_{11} + u_{12} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \\ u_2 &= u_{21} + u_{22} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \end{aligned} \tag{7}$$

The prescription for constructing operators from absolutes uses the Hermitian conjugates:

$$u_1^* = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \qquad u_2^* = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \tag{8}$$

so that the operators are

$$\begin{aligned} u_{11} &= \frac{1}{2} u_1 u_1^* \qquad u_{12} = \frac{1}{2} u_1 u_2^* \\ u_{21} &= \frac{1}{2} u_2 u_1^* \qquad u_{22} = \frac{1}{2} u_2 u_2^* \end{aligned} \tag{9}$$

It was shown in Beil and Ketner (2003) how the algebra derived from this logic of relations corresponds to nonrelativistic quantum theory. The teacher–pupil classification becomes spinup–spindown. Note that all quantities in the theory, including the absolutes and operators of the logic, on the one hand, and the elements of the quantum theory (both wave functions and operators) on the other, can correspond to quantities in the Clifford algebra.

The specific linear transformation from elements of the Peirce logic to the usual representation of the Pauli matrices is

$$\begin{aligned} I = u_{11} + u_{22} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = u_{12} + u_{21} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma_2 = -iu_{12} + iu_{21} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = u_{11} - u_{22} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{10}$$

This shows explicitly how a quantum theory can be constructed from the classical Peirce logic. Peirce gave (in 1870) a version of the transformation (5) from the basis elements u_{ij} of this logic to a general quaternion basis (see Peirce, 1933, p. 81). This is a C(3, 0) Clifford algebra. Of course, this was over 50 years before the introduction of quantum theory. The Peirce/Clifford scaffolding stood neglected for that half century. When it was finally used almost no one remembered the names of the builders.

Peirce certainly recognized that his operators were related to noncommuting algebras such as quaternions. However, he never applied this back to his logic and, as far as the written record divulges, never actually developed a "quantum logic."

The discussion so far relates to nonrelativistic quantum theory. The same sort of development can give a relativistic theory.

In order to do this one can enlarge the logic to include another pair of classes. If this pair is positive energy–negative energy, this can lead to the Dirac particle theory, as will now be shown.

The logic in the case of positive energy–negative energy classes can be represented by a scheme which closely duplicates the above system with u_1 as spinup and u_2 as spindown. One needs only to take, say, t_1, as positive energy and t_2 as negative energy.

There are two independent operator matrices, u_{ij} and t_{ab}. A standard way to combine the t and u spaces is to use a tensor product $t \otimes u$. The basis of this space can be represented in 4×4 matrix form:

$$
\begin{aligned}
&t_{11} \otimes u_{11} = \begin{pmatrix} u_{11} & (0) \\ (0) & (0) \end{pmatrix} \; t_{11} \otimes u_{12} = \begin{pmatrix} u_{12} & (0) \\ (0) & (0) \end{pmatrix} \; t_{12} \otimes u_{11} = \begin{pmatrix} (0) & u_{11} \\ (0) & (0) \end{pmatrix} \; t_{12} \otimes u_{12} = \begin{pmatrix} (0) & u_{12} \\ (0) & (0) \end{pmatrix} \\
&t_{11} \otimes u_{21} = \begin{pmatrix} u_{21} & (0) \\ (0) & (0) \end{pmatrix} \; t_{11} \otimes u_{22} = \begin{pmatrix} u_{22} & (0) \\ (0) & (0) \end{pmatrix} \; t_{12} \otimes u_{21} = \begin{pmatrix} (0) & u_{21} \\ (0) & (0) \end{pmatrix} \; t_{12} \otimes u_{22} = \begin{pmatrix} (0) & u_{22} \\ (0) & (0) \end{pmatrix} \\
&t_{21} \otimes u_{11} = \begin{pmatrix} (0) & (0) \\ u_{11} & (0) \end{pmatrix} \; t_{21} \otimes u_{12} = \begin{pmatrix} (0) & (0) \\ u_{12} & (0) \end{pmatrix} \; t_{22} \otimes u_{11} = \begin{pmatrix} (0) & (0) \\ (0) & u_{11} \end{pmatrix} \; t_{22} \otimes u_{12} = \begin{pmatrix} (0) & (0) \\ (0) & u_{12} \end{pmatrix} \\
&t_{21} \otimes u_{21} = \begin{pmatrix} (0) & (0) \\ u_{21} & (0) \end{pmatrix} \; t_{21} \otimes u_{22} = \begin{pmatrix} (0) & (0) \\ u_{22} & (0) \end{pmatrix} \; t_{22} \otimes u_{21} = \begin{pmatrix} (0) & (0) \\ (0) & u_{21} \end{pmatrix} \; t_{22} \otimes u_{22} = \begin{pmatrix} (0) & (0) \\ (0) & u_{22} \end{pmatrix} \\
&(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned} \tag{11}
$$

The u matrices are taken from (4).

The index system can be changed to a 4×4 form by using g_{AB} $(A, B = 1, 2, 3, 4)$. The index A corresponds to the number of the row of the array in (11), the index B corresponds to the number of the column of the array. This gives a multiplication rule

$$g_{AB}g_{CD} = \delta_{BC}g_{AD} \tag{12a}$$

which is equivalent to the rule

$$(t_{ab} \otimes u_{ij})(t_{cd} \otimes u_{kl}) = (\delta_{bc}t_{ad} \otimes \delta_{jk}u_{il}) \tag{12b}$$

The rules (12a) and (12b) apply to the multiplication of 4×4 matrices.

The construction of the Peirce absolutes and the consequent derivation of the operators can now proceed along the lines of (7), (8), and (9). This will be postponed until after the definition of the Dirac states so the Dirac results will be available.

Just as linear combinations of the u matrices using complex coefficients gave the Pauli matrices (10), so also can the Dirac vectors be obtained in the C(1, 3) Clifford algebra from the 4×4 matrix basis above. The algebraic expressions for a common representation are

$$\gamma^0 = g_{11} + g_{22} - g_{33} - g_{44}$$
$$\gamma^1 = g_{14} + g_{23} - g_{32} - g_{41}$$

$$\gamma^2 = -i(g_{14} - g_{23} - g_{32} + g_{41})$$
$$\gamma^3 = g_{13} - g_{24} - g_{31} + g_{42} \tag{13}$$

and the matrix representations are the standard set which can be obtained immediately from (13).

These numbers satisfy the Clifford algebra rules

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \tag{14}$$

where η is diag$(1, -1, -1, -1)$, a Minkowski metric.

This clearly shows how the classical basis g_{AB} leads to the quantum basis γ^μ. In a sense, the classical matrices (and the Peirce algebra) are more fundamental since they are simpler. It must be pointed out, however, that the quantum operators cannot be considered to be a special case of the classical operators.

3. DIRAC

A vector with contravariant components $p^\mu = (p_0; p_1, p_2, p_3)$ and covariant components $p_\mu = (p_0; -p_1, -p_2, -p_3)$, such as the momentum vector, is expressed in the Dirac basis as

$$\mathbf{p} = p_0\gamma^0 - p_i\gamma^i = p_\mu\gamma^\mu \quad (i = 1, 2, 3;\ \mu = 0, 1, 2, 3) \tag{15}$$

The usual Einstein sums over up–down indices are implied. Note that these indices refer to the Dirac basis and not the matrix indices used above.

The Dirac equation is

$$i\hbar\partial\psi = mc\psi \tag{16}$$

where ∂ is the gradient

$$\partial = \frac{\partial}{\partial x^\mu}\gamma^\mu \tag{17}$$

Usually, ψ is called a spinor (or bispinor) and is represented by some 4-element column matrix. But these spinors, as mentioned in the Introduction, are not Clifford numbers. The choice is made instead to use solutions ψ which are Clifford numbers and have 4×4 matrix representation. Note that this approach diverges from that of Marchuk (1998) who considers states with various numbers of columns with no special consideration of solutions which are actually Clifford numbers. The states here are also different from Rarita–Schwinger states since there are no separate vector indices. The total algebra is only a Clifford algebra while Rarita–Schwinger involves the tensor product of a Clifford algebra and a 4-vector space with certain subsidiary conditions.

The assumption then, is that the form of the solution is

$$\psi_+ = \mathbf{u} \exp\left(-\frac{i}{\hbar} p_\mu x^\mu\right) \tag{18}$$

where **u** is some Clifford number with constant elements. It follows that

$$i\hbar\partial\psi_+ = \mathbf{p}\psi_+ = \mathbf{pu} \exp\left(-\frac{i}{\hbar} p_\mu x^\mu\right) \tag{19}$$

which leads to the condition

$$\mathbf{pu} = mc\mathbf{u} \tag{20}$$

This is an eigenvalue equation. Since

$$\mathbf{pp} = m^2c^2\mathbf{I} \tag{21a}$$

then a **u** which satisfies (20) is

$$\mathbf{u} = mc\mathbf{I} + \mathbf{p} \tag{21b}$$

This is a Clifford number, actually a scalar plus a vector. The complete set of a multivector basis is given in several texts, for example, Snygg (1997).

A matrix representation for **u** is

$$\mathbf{u} = \begin{pmatrix} mc + p_0 & 0 & -p_3 & -p_- \\ 0 & mc + p_0 & -p_+ & p_3 \\ p_3 & p_- & mc - p_0 & 0 \\ p_+ & -p_3 & 0 & mc - p_0 \end{pmatrix} \tag{22}$$

where $p_\pm = p_1 \pm p_2$. Other representations could be used.

If the Peirce projection operators $g_{11}, g_{22}, g_{33}, g_{44}$ are applied on **u** on the right, then four 4×4 matrices with only one nonzero column are obtained. These are just the four columns of (22). They are left ideals and correspond to the usual Dirac spinors, except that the spinors are 1×4 column matrices. The Peirce operators are identical with the standard projection operators

$$\begin{aligned} g_{11} &= \frac{1}{4}(\mathbf{I} + \gamma^0)(\mathbf{I} + \mathbf{s}) \quad g_{22} = \frac{1}{4}(I + \gamma^0)(\mathbf{I} - \mathbf{s}) \\ g_{33} &= \frac{1}{4}(\mathbf{I} - \gamma^0)(\mathbf{I} + \mathbf{s}) \quad g_{44} = \frac{1}{4}(\mathbf{I} - \gamma^0)(\mathbf{I} - \mathbf{s}) \\ \mathbf{s} &= i\gamma^1\gamma^2 \end{aligned} \tag{23}$$

The bivector **s** is also used as a spin operator and its eigenvalue determines whether the particle is in spinup ↑ or spindown ↓ state. However, the complete matrix solution (22), although it is an eigenmatrix for **p** (see (20)) is not an eigenstate for **s**. This can easily be verified by explicitly computing **su**. A remedy for the problem

(Grandy, 1991; Messiah, 1966) is to assume the particle momentum is only in the 3-direction. Thus, the solution matrix **u** becomes

$$\mathbf{u} = \begin{pmatrix} mc + p_0 & 0 & -p_3 & 0 \\ 0 & mc + p_0 & 0 & p_3 \\ p_3 & 0 & mc - p_0 & 0 \\ 0 & -p_3 & 0 & mc - p_0 \end{pmatrix} \tag{24}$$

The four matrix solutions

$$\mathbf{u}_1 = \mathbf{u}g_{11} \quad \mathbf{u}_2 = \mathbf{u}g_{22} \quad \mathbf{u}_3 = \mathbf{u}g_{33} \quad \mathbf{u}_4 = \mathbf{u}g_{44} \tag{25a}$$

are eigenstates for **s** and can be partially distinguished by their eigenvalues for spin.

$$\mathbf{su}_1 = +\mathbf{u}_1 \quad \mathbf{su}_2 = -\mathbf{u}_2 \quad \mathbf{su}_3 = +\mathbf{u}_3 \quad \mathbf{su}_4 = -\mathbf{u}_4 \tag{25b}$$

It is apparent that the state $\mathbf{u}_1$ has the same eigenvalues as the state $\mathbf{u}_3$. Similarly, $\mathbf{u}_2$ and $\mathbf{u}_4$ have the same eigenvalues. This has been noticed by several authors (Grandy, 1991; Messiah, 1966; Weinberg, 1995) but no satisfactory resolution has been suggested. Usually, the extra states are just ignored.

Another approach is to introduce a second set of column matrix solutions. These can be obtained by taking plane waves of the form

$$\psi_- = \mathbf{v} \exp\left(\frac{i}{\hbar} p_\mu x^\mu\right) \tag{26}$$

where **v** is a Clifford number with constant elements.

In the same way as for (19)

$$i\hbar\partial\psi_- = -\mathbf{p}\psi_- = -\mathbf{pv} \exp\left(\frac{i}{\hbar} p_\mu x^\mu\right) = mc\psi_- \tag{27}$$

Since

$$\mathbf{pv} = -mc\mathbf{v} \tag{28}$$

there are solutions with

$$\mathbf{v} = mc\mathbf{I} - \mathbf{p} \tag{29}$$

Like **u**, the **v** are the sum of a Clifford scalar and vector.

A matrix representation of **v** which corresponds to (24) is

$$\mathbf{v} = \begin{pmatrix} mc - p_0 & 0 & p_3 & 0 \\ 0 & mc - p_0 & 0 & -p_3 \\ -p_3 & 0 & mc + p_0 & 0 \\ 0 & p_3 & 0 & mc + p_0 \end{pmatrix} \tag{30}$$

Again, the momentum is assumed to be only in the 3-direction.

The same Peirce projection operators from (23) are applied on **v** on the right to produce similar 4×4 matrices with one nonzero column. One has, then, eight solutions represented by these Dirac spinors. They include (25a) and $\mathbf{v}_1 = \mathbf{v}g_{11}$, $\mathbf{v}_2 = \mathbf{v}g_{22}$, $\mathbf{v}_3 = \mathbf{v}g_{33}$, $\mathbf{v}_4 = \mathbf{v}g_{44}$.

A novel assumption is now introduced. This is to choose as Dirac wave functions the matrix states which consist of *two* nonzero columns. This is to be contrasted with the ordinary spinor solutions which are comparable to matrices which have one nonzero column or the full matrix solutions which have four nonzero columns.

The solutions are

$$\mathbf{u}_+ = \begin{pmatrix} mc + p_0 & 0 & -p_3 & 0 \\ 0 & 0 & 0 & 0 \\ p_3 & 0 & mc - p_0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{31a}$$

$$\mathbf{u}_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & mc + p_0 & 0 & p_3 \\ 0 & 0 & 0 & 0 \\ 0 & -p_3 & 0 & mc - p_0 \end{pmatrix} \tag{31b}$$

$$\mathbf{v}_+ = \begin{pmatrix} mc - p_0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & 0 \\ -p_3 & 0 & mc + p_0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{31c}$$

$$\mathbf{v}_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & mc - p_0 & 0 & -p_3 \\ 0 & 0 & 0 & 0 \\ 0 & p_3 & 0 & mc + p_0 \end{pmatrix} \tag{31d}$$

These four solutions are suggested by the Peirce logic, as will be seen. There are several favorable properties of these solutions.

First, they are easily expressed in terms of the spinor solutions:

$$\begin{aligned} \mathbf{u}_+ &= \mathbf{u}_1 + \mathbf{u}_3 \\ \mathbf{u}_- &= \mathbf{u}_2 + \mathbf{u}_4 \\ \mathbf{v}_+ &= \mathbf{v}_1 + \mathbf{v}_3 \\ \mathbf{v}_- &= \mathbf{v}_2 + \mathbf{v}_4 \end{aligned} \tag{32}$$

Second, they are simply related to the full matrix solutions:

$$\begin{aligned}\mathbf{u} &= \mathbf{u}_+ + \mathbf{u}_- \\ \mathbf{v} &= \mathbf{v}_+ + \mathbf{v}_-\end{aligned} \tag{33}$$

or also

$$\begin{aligned}2\mathbf{u}_+ &= \mathbf{u}(\mathbf{I} + \mathbf{s}) \\ 2\mathbf{u}_- &= \mathbf{u}(\mathbf{I} - \mathbf{s}) \\ 2\mathbf{v}_+ &= \mathbf{v}(\mathbf{I} + \mathbf{s}) \\ 2\mathbf{v}_- &= \mathbf{v}(\mathbf{I} - \mathbf{s})\end{aligned} \tag{34}$$

Third, they are a complete orthogonal set, as is easily verified. The normalization will be discussed later.

Fourth, they are eigenfunctions of both the momentum and spin operators

$$\begin{aligned}\mathbf{s}\mathbf{u}_+ &= +\mathbf{u}_+ \\ \mathbf{s}\mathbf{u}_- &= -\mathbf{u}_- \\ \mathbf{s}\mathbf{v}_+ &= +\mathbf{v}_+ \\ \mathbf{s}\mathbf{v}_- &= -\mathbf{v}_-\end{aligned} \tag{35}$$

$$\begin{aligned}\mathbf{p}\mathbf{u}_+ &= mc\mathbf{u}_+ \\ \mathbf{p}\mathbf{u}_- &= mc\mathbf{u}_- \\ \mathbf{p}\mathbf{v}_+ &= -mc\mathbf{v}_+ \\ \mathbf{p}\mathbf{v}_- &= -mc\mathbf{v}_-\end{aligned} \tag{36}$$

Also, **u** and **v** are, themselves, proportional to the projection operators for positive and negative energy solutions, as can be inferred by the orthogonality results.

Fifth, a complete set of commuting operators can be derived from the four two-column states. To demonstrate this one can begin with the assumption that if ψ is a Dirac solution then it can be represented as a matrix state which corresponds to a ket state $|\psi\rangle$. The Hermitian conjugate ψ^* of ψ is defined as a matrix with columns transposed into rows and the complex conjugate taken of each element. For example, for (22),

$$\mathbf{u}^* = \begin{pmatrix} mc + p_0 & 0 & p_3 & p_- \\ 0 & mc + p_0 & p_+ & -p_3 \\ -p_3 & -p_- & mc - p_0 & 0 \\ -p_+ & p_3 & 0 & mc - p_0 \end{pmatrix} \tag{37}$$

The actual bra state $\langle\psi|$ is the matrix $\psi^{\#} = \psi^{*}\gamma^{0}$. This is usually called the adjoint. The matrix representation is

$$\mathbf{u}^{\#} = \mathbf{u}^{*}\gamma^{0} = \begin{pmatrix} mc + p_0 & 0 & -p_3 & -p_- \\ 0 & mc + p_0 & -p_+ & p_3 \\ -p_3 & -p_- & p_0 - mc & 0 \\ -p_+ & p_3 & 0 & p_0 - mc \end{pmatrix} \tag{38}$$

An alternate way of computing the adjoint for the full solution matrices **u** or **v** is $\mathbf{u}^{\#} = \gamma^{0}\mathbf{u}$ or $\mathbf{v}^{\#} = \gamma^{0}\mathbf{v}$ as can easily be verified. The reader should be cautioned that $\psi^{\#} = \gamma^{0}\psi$ does not always hold (an example of this is $\mathbf{u}_+$ when p_+ or p_- is nonzero). However, the alternate adjoint computation will hold for all solutions subsequently used.

The general form of the matrix of product elements is $\langle\psi^{\#} \mid \psi\rangle$. This is, for example

$$\mathbf{u}^{\#}\mathbf{u} = \gamma^{0}\mathbf{u}\mathbf{u} = 2mc\gamma^{0}\mathbf{u} = 2mc\mathbf{u}^{\#} \tag{39}$$

This product of matrices corresponds to the matrix of products. Each element of the matrix is an expectation value in the traditional sense.

Also, operators in this matrix scheme take the general form $|\psi\rangle\langle\psi^{\#}|$. For example

$$\mathbf{u}\mathbf{u}^{\#} = \mathbf{u}\gamma^{0}\mathbf{u} = 2p_0\mathbf{u} \tag{40}$$

So one can write the operators also as matrices. These matrices are proportional to the solution matrices themselves. For the canonical solutions (31),

$$\begin{aligned} \mathbf{u}_+\mathbf{u}_+^{\#} &= 2p_0\mathbf{u}_+ \\ \mathbf{u}_-\mathbf{u}_-^{\#} &= 2p_0\mathbf{u}_- \\ \mathbf{v}_+\mathbf{v}_+^{\#} &= 2p_0\mathbf{v}_+ \\ \mathbf{v}_-\mathbf{v}_-^{\#} &= 2p_0\mathbf{v}_- \end{aligned} \tag{41}$$

So the sixth favorable property of the solutions (31) is that the four matrices form a set of orthogonal operators. From these operators one can construct a complete set of commuting operators (the fifth property already mentioned):

$$\begin{aligned} \mathbf{u}_+ + \mathbf{u}_- + \mathbf{v}_+ + \mathbf{v}_- &= 2mc\mathbf{I} \\ \mathbf{u}_+ + \mathbf{u}_- - \mathbf{v}_+ - \mathbf{v}_- &= 2\mathbf{p} \\ \mathbf{u}_+ - \mathbf{u}_- + \mathbf{v}_+ - \mathbf{v}_- &= 2mc\mathbf{s} \\ \mathbf{u}_+ - \mathbf{u}_- - \mathbf{v}_+ + \mathbf{v}_- &= 2\mathbf{s}\mathbf{p} = 2\mathbf{h} \end{aligned} \tag{42}$$

These are, in order, Clifford scalar, vector, bivector, and trivector. The trivector **h** is often labeled as the helicity . It has eigenvalues,

$$
\begin{aligned}
\mathbf{h}\mathbf{u}_+ &= mc\mathbf{u}_+ \\
\mathbf{h}\mathbf{u}_- &= -mc\mathbf{u}_- \\
\mathbf{h}\mathbf{v}_+ &= -mc\mathbf{v}_+ \\
\mathbf{h}\mathbf{v}_- &= mc\mathbf{v}_-
\end{aligned} \tag{43}
$$

This completes the list of eigenvalues from (35) and (36), except for a discussion of the scalar operator.

At this juncture one might recall that no mention has been made yet of the Hamiltonian. The usual identification is with the operator

$$\mathbf{H}_0 = c(\mathbf{u}\gamma^0 - p_0\mathbf{I}) \tag{44a}$$

This does satisfy $\mathbf{H}_0\mathbf{u} = p_0\mathbf{u}$ and $\mathbf{H}_0\mathbf{v} = -p_0\mathbf{v}$, however, it has several problems. It does not commute with **p**. It is a Clifford scalar plus vector plus bivector and thus does not have a simple Clifford identification. Also, there are lingering interpretation difficulties of the commutators of this $\mathbf{H}_0$ with physical quantities such as position and angular momentum. Finally, this Hamiltonian leads to the confusing topic of *Zitterbewegung* (Grandy, 1991).

All of this is solved by the *Ansatz*

$$\mathbf{H}_0 = i\hbar\frac{\partial}{\partial x^0} \tag{44b}$$

This satisfies

$$
\begin{aligned}
\mathbf{H}_0\psi_+ &= p_0\psi_+ \\
\mathbf{H}_0\psi_- &= -p_0\psi_-
\end{aligned} \tag{45}
$$

and this gives a clear distinction of positive and negative energy states. This Hamiltonian is a Clifford scalar and commutes with each of the operators in the complete set. Thus, it fits into the scheme of (42) as a physically significant scalar operator. It also acts appropriately on the position vector $\mathbf{r} = r_\mu\gamma^\mu$:

$$[\mathbf{H}_0, \mathbf{r}]\phi = [\mathbf{H}_0\mathbf{r} - \mathbf{r}\mathbf{H}_0]\phi = i\hbar\left[\frac{\partial}{\partial x^0}(\mathbf{r}\phi) - \mathbf{r}\frac{\partial}{\partial x^0}\phi\right] = i\hbar\frac{\partial\mathbf{r}}{\partial x^0}\phi \tag{46}$$

Since this holds for any general wave function ϕ,

$$[\mathbf{H}_0, \mathbf{r}] = i\hbar\frac{\partial\mathbf{r}}{\partial x^0} \tag{47}$$

which is a desirable result.

The argument can now be brought full circle. That is, the Dirac solutions (31) can be directly related to the Peirce logic of the two pairs of the absolutes as outlined by (11).

The four two-column solutions are transformed to the particle rest frame where $p_3 = 0$ and $p_0 = mc$. With suitable normalization this gives

$$\mathbf{u}_+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{u}_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{v}_+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{v}_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{48}$$

This corresponds to the pairs of matrices (u is spin, t is energy):

For $\mathbf{u}_+$ (spin up, positive energy),

$$u = u_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad t = t_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{49a}$$

For $\mathbf{u}_-$ (spin down, positive energy),

$$u = u_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad t = t_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{49b}$$

For $\mathbf{v}_+$ (spin up, negative energy),

$$u = u_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad t = t_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{49c}$$

For $\mathbf{v}_-$ (spin down, negative energy),

$$u = u_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad t = t_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{49d}$$

These are examples of elements of the tensor products (11). The γ matrices and consequently the entire Dirac theory can be expressed in terms of the $t \otimes u$ tensor products.

Finally, the Peirce absolutes for this scheme are easily identified, recalling (6). The absolute for spinup is $\mathbf{u}_+ + \mathbf{v}_+$. The absolute for spindown is $\mathbf{u}_- + \mathbf{v}_-$. The absolute for positive energy is just $\mathbf{u}$ (see (33)) and the absolute for negative energy is just $\mathbf{v}$.

Proceeding in the other direction, one could start from the rest frame solutions and arrive at the solutions (31) by means of a Lorentz transformation. Of course, Einstein was born in 1879 and this option was not suspected by either Peirce or Clifford.

4. DISCUSSION

In a previous paper it was indicated how Peirce's logic of relations could proceed to the C(3, 0) Clifford algebra and further to the nonrelativistic quantum theory represented by the Pauli matrices. The Peirce logic containing two classes of individuals corresponds to the case of two spin states of the particle.

Here, it has been shown how a Peirce logic with two sets of classes of two individuals can correspond to the C(1, 3) Clifford algebra and consequently to the relativistic quantum theory of Dirac. The two sets are spin, with classes up and down, and energy, with classes positive and negative. The combination of the spaces representing the two sets is accomplished by a tensor product and leads to a 4×4 representation. This representation is a linear transformation of the common Dirac formalism. The Dirac spinors relate directly to 4×4 matrices with one nonzero column. It was found that a representation with two nonzero columns has several advantages.

This comparison has historical significance in revealing that an idea of Peirce from the 1870s reappears in a quantum context with Dirac in the 1920s. The common language is the algebra of Clifford. There is, however, no known direct lineage. It is clear, however, that there is a classical correspondence to quantum theory. For example, operators in the Peirce logic are exactly the quantum projection operators. Also, the rest frame quantum solutions (48) are exactly the Peirce operators (23). Of course, Peirce himself never made the quantum leap and never used the noncommuting operators in his logic.

Since a systematic procedure has been developed to derive new Dirac solutions, it is reasonable to ask if there is any physical significance to be associated with these solutions. Some tentative ideas have emerged and will be the subject of future investigation.

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