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Cliffordons

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At higher energies the present complex quantum theory with its unitary group might expand into a real quantum theory with an orthogonal group, broken by an approximate *i* operator at lower energies. Implementing this possibility requires a real quantum double-valued statistics. A Clifford statistics, representing a swap (12) by a difference $\gamma_1 - \gamma_2$ of Clifford units, is uniquely appropriate. Unlike the Maxwell–Boltzmann, Fermi–Dirac, Bose–Einstein, and para-statistics, which are tensorial and single-valued, and unlike anyons, which are confined to two dimensions, Clifford statistics are multivalued and work for any dimensionality. Nayak and Wilczek such Clifford statistics for the fractional quantum Hall effect. We apply them to toy quanta here. A complex-Clifford example has the energy spectrum of a system of spin-1/2 particles in an external magnetic field. This supports the proposal that the double-valued rotations—spin—seen at current energies might arise from double-valued permutations—swap—to be seen at higher energies. Another toy with real Clifford statistics illustrates how an effective imaginary unit *i* can arise naturally within a real quantum theory. © *2001 American Institute of Physics.* [DOI: 10.1063/1.1379314]

I. INTRODUCTION: QUANTIFICATION PROCEDURES

Nayak and Wilczek[1] have proposed a startling new statistics for fractional quantum Hall effect carriers. It has great potential for even more fundamental applications to sub-particle structure.[2] To learn its properties we apply it here to some toy models.

The common statistics—Fermi–Dirac (FD), Bose-Einstein (BE), and Maxwell–Boltzmann (MB)—may be regarded as differing prescriptions for constructing the algebra of an ensemble of many individuals from the vector space of one individual. These procedures take qualitative yes-or-no questions about an individual into quantitative how-many questions about an ensemble of similar individuals. Such procedures were termed *quantification*. Now they are sometimes called ''second quantization,'' somewhat misleadingly.

We use a well-known operational formulation of quantum theory. The main point of quantum theory is that mathematical objects may be completely describable, since we make them up, but physical quanta are not. An electron, a physical entity, is not a spinor wave function, a linear operator, or any other mathematical object. But it seems that mathematical objects can usefully represent what we do to an electron. Kets represent input modes (preparation), bras represent outtake modes (registration), operators represent intermediate operations on quantum.[3]

Each of the usual statistics is defined by an associated linear mapping $Q^\dagger$ that maps any one-body initial mode ψ into a many-body creation operator:

$$Q^\dagger : V_I \to \mathcal{A}_S, \quad \psi \mapsto Q^\dagger \psi =: \hat{\psi}. \tag{1}$$

Here V_I is the initial-mode vector space of the individual I and $\mathcal{A}_S = \mathrm{End}\ V_S$ is the operator (or endomorphism) algebra of the quantified system S. The † in $Q^\dagger$ reminds us that $Q^\dagger$ is contragredient to the initial modes ψ. We write the mapping $Q^\dagger$ to the left of its argument ψ to respect the conventional Dirac order of cogredient and contragredient vectors in a contraction.

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Dually, the final modes $\psi^\dagger$ of the dual space $V_I^\dagger$ are mapped to annihilators in $\mathcal{A}_S$ by the linear operator Q

$$Q: V_I^\dagger \to \mathcal{A}_S, \quad \psi^\dagger \mapsto \psi^\dagger Q =: \hat{\psi}^\dagger. \tag{2}$$

We call the transformation Q the *quantifier* for the statistics. Q and $Q^\dagger$ are tensors of the type

$$Q = (Q^{aB}{}_C), \quad Q^\dagger = (Q^{\dagger C}{}_{aB}), \tag{3}$$

where a indexes a basis in the one-body space V_I and B,C index a basis in the many-body space V_S.

The basic creators and annihilators associated with an arbitrary basis $\{e_a | a=1,\ldots,N\} \subset V_I$ and its reciprocal basis $\{e^a | a=1,\ldots,N\} \subset V_I^\dagger$ are then

$$Q^\dagger e_a := \hat{e}_a =: Q^\dagger{}_a \tag{4}$$

and

$$e^a Q := \hat{e}^a =: Q^a. \tag{5}$$

The creator and annihilator for a general initial mode ψ are

$$\begin{aligned} Q^\dagger(e_a \psi^a) &= Q^\dagger{}_a \psi^a, \\ (\phi^\dagger{}_a e^a) Q &= \phi^\dagger{}_a Q^a, \end{aligned} \tag{6}$$

respectively.

We require that quantification respects the adjpoint †. This relates the two tensors Q and $Q^\dagger$

$$\psi^\dagger Q = (Q^\dagger \psi)^\dagger. \tag{7}$$

The rightmost † is the adjoint operation for the quantified system. Therefore,

$$\hat{e}_a^\dagger = M_{ab} \hat{e}^b, \tag{8}$$

with M_{ab} being the metric, the matrix of the adjoint operation, for the individual system.

We now generalize from the common statistics. A *linear statistics* shall be defined by a linear correspondence $Q^\dagger$ called the quantifier

$$Q^\dagger: V_I \to \mathcal{A}_S, \quad \psi \mapsto Q^\dagger \psi =: \hat{\psi}, \tag{9}$$

[compare (1)] from one-body modes to many-body operators, †-algebraically generating the algebra $\mathcal{A}_S := \mathrm{End}\ V_S$ of the many-body theory. We further require that the quantifier $Q^\dagger$ induce an isomorphism from the one-body unitary group U_I into the many-body unitary group U_S, as described in Sec. IV. This is the *representation principle* for quantifiers.

The representation principle implies bilinear algebraic commutation relations discussed below.

In general $Q^\dagger$ does not produce a creator and Q does not produce an annihilator, as they do in the common statistics.

We construct the quantified algebra $\mathcal{A}_S$ from the individual space V_I in three easy steps:

(1) We form the quantum algebra $\mathcal{A}(V_I)$, defined as the free † algebra generated by (the vectors of) V_I. Its elements are all possible iterated sums and products and †-adjoints of the vectors of V_I. We require that the operations $(+,\times,\dagger)$ of $\mathcal{A}(V_I)$ agree with those of V_I where both are meaningful;

(2) we construct the ideal $\mathcal{R} \subset \mathcal{A}$ of all elements of $\mathcal{A}(V_I)$ that vanish in virtue of the statistics. It is convenient and customary to define $\mathcal{R}$ by a set of expressions **R**, such that the commutation

relations between elements of $\mathcal{A}(V_I)$' have the form $r=0$ with $r\in\mathbf{R}$. Then $\mathcal{R}$ consists of all elements of $\mathcal{A}(V_I)$ that vanish in virtue of the commutation relations and the postulates of a †-algebra.

Let $\mathbf{R}$ be closed under †. Let $\mathcal{R}_0$ be the set of all evaluations of all the expressions in $\mathbf{R}$ when the variable vectors ψ in these expressions assume any values $\psi\in V_I$. Then $\mathcal{R}=\mathcal{A}(V_I)\mathcal{R}_0\,\mathcal{A}(V_I)$;

(3) we form the quotient algebra (actually, a residue algebra)

$$\mathcal{A}_S=\mathcal{A}(V_I)/\mathcal{R}, \tag{10}$$

by identifying elements of $\mathcal{A}(V_I)$ whose differences belong to $\mathcal{R}$.

Then $Q^\dagger$ maps each vector $\psi\in V_I$ into its residue class $\psi+\mathcal{R}$.

Historically, physicists carried out one special quantification first. Since classically one multiplies phase spaces when quantifying, they assumed that quantally one multiplies Hilbert spaces, forming the tensor product

$$V_S=\bigotimes_{p=0}^{N} V_I=V_I^N \tag{11}$$

of N individual spaces V_I. Then in order to improve agreement with experiment they removed degrees of freedom in the tensor product connected with permutations, reducing V_I^N to a subspace PV_I^N invariant under all permutations of individuals. Here P is a projection operator characterizing the statistics. The many-body algebra was then taken to be the algebra of linear operators on the reduced space: $\mathcal{A}_S=\mathrm{End}\;\;PV_I^N$.

We call a statistics built in that way on a subspace of the tensor algebra over the one-body initial mode space, a *tensorial* statistics. Tensorial statistics represents permutations in a single-valued way. The common statistics are tensorial.

Linear statistics is more general than tensorial statistics, in that the quotient algebra $\mathcal{A}_S=\mathcal{A}-\mathcal{R}$ defining a linear statistics need not be the operator algebra of any subspace of the tensor space $\mathrm{Ten}\,V_I$ and need not be single-valued. Commutation relations permit more general statistics than projection operators do. For example, anyon statistics is linear but not tensorial.

For another example, $\mathcal{A}_S$ may be the endomorphism algebra of a spinor space constructed from the quadratic space V_I. Such a statistics we call a *spinorial statistics*. Clifford statistics, the main topic of this paper, is a spinorial statistics. Linear statistics includes both spinorial and tensorial statistics.

The FD, BE, and MB statistics are readily presented as tensorial statistics. We give their quantifiers next.[3] We then generalize to spinorial, nontensorial, statistics.

II. STANDARD STATISTICS

A. Maxwell–Boltzmann statistics

Classical an MB aggregate is a sequence (up to isomorphism) and $Q=\mathrm{Seq}$, the *sequence*-forming quantifier. The quantum individual I has a Hilbert space $V=V_I$ over the field $\mathbb{C}$. The vector space for the q sequence is the (contravariant) tensor algebra $V_S=\mathrm{Ten}\,V_I$, whose product is the tensor product $\otimes$

$$V_S=\mathrm{Ten}\,V_I, \tag{12}$$

with the natural induced †. The kinematic algebra $\mathcal{A}_S$ of the sequence is the †-algebra of endomorphisms of $\mathrm{Ten}\,V_I$, and is generated by $\psi\in V_I$ subject to the generating relations

$$\hat{\psi}^\dagger\hat{\phi}=\psi^\dagger\phi. \tag{13}$$

The left-hand side is an operator product, and the right-hand side is the contraction of the dual vector $\psi^\dagger$ with the vector ϕ, with an implicit unit element $1\in\mathcal{A}_S$ as a factor.

B. Fermi–Dirac statistics

Here $Q=\mathrm{Set}$, the *set*-forming quantifier. The kinematic algebra for the quantum set has defining relations

$$\begin{aligned}\hat{\psi}\hat{\phi}+\hat{\phi}\hat{\psi}&=0,\\ \hat{\psi}^\dagger\hat{\phi}+\hat{\phi}\hat{\psi}^\dagger&=\psi^\dagger\phi.\end{aligned}\tag{14}$$

for all $\psi,\phi\in V_I$.

C. Bose–Einstein statistics

Here $Q=\mathrm{Sib}$, the *sib*-forming quantifier. The sib-generating relations are

$$\begin{aligned}\hat{\psi}\hat{\phi}-\hat{\phi}\hat{\psi}&=0,\\ \hat{\psi}^\dagger\hat{\phi}-\hat{\phi}\hat{\psi}^\dagger&=\psi^\dagger\phi,\end{aligned}\tag{15}$$

for all $\psi,\phi\in V_I$.

The individuals in each of the discussed quantifications, by construction, have the same (isomorphic) initial spaces. We call such individuals *isomorphic*.

III. RELATION TO THE PERMUTATION GROUP

A statistics is *abelian* if it represents the permutation group S_N on its N individuals by an abelian group of operators in the N-body mode space.

The FD or BE representations are not only abelian but scalar. They represent each permutation by a number, a projective representation of the identity operator. One calls entities with scalar statistics *indistinguishable*. Bosons and fermions are indistinguishable.

Non-abelian statistics describe distinguishable entities.

Nayak and Wilczek[1,4] give a spinorial statistics based on the work on nonabelions of Read and Moore.[5,6] Read and Moore use a subspace corresponding to the degenerate ground mode of some realistic Hamiltonian as the representation space for a nonabelian representation of the permutation group S_{2n} acting on the composite of $2n$ quasiholes in the fractional quantum Hall effect. This statistics, Wilczek showed, represents the permutation group on a spinor space, and permutations by noncommuting spin operators. The quasiholes of Read and Moore and of Wilczek and Nayak are distinguishable, but their permutations leave the ground subspace invariant.

Our own interest in the statistics of distinguishable entitities arises from a study of quantum space–time structure.[2] The dynamical process of any system is composite, it is generally believed, composed of isomorphic elementary actions going on all over, all the time. The first question that has to be answered in setting up an algebraic quantum theory of this composite process is: What statistics do the elementary actions have?

The elementary processes have ordinarily, though implicitly, been assumed to be distinguishable, being addressed by space–time coordinates, and to obey Maxwell–Boltzmann statistics. This repeats the history of particle statistics on the greater field of process statistics.

The Clifford statistics studied below is proposed primarily for the elementary processes of nature. We apply it here to toy models of particles in ordinary space–time to familiarize ourselves with its properties. In our construction, the representation space of the permutation group is the whole (spinor) space of the composite. The permutation group is not assumed to be a symmetry of the Hamiltonian or of its ground subspace any longer. It is used as a dynamical group, not a symmetry group.

IV. NO QUANTIFICATION WITHOUT REPRESENTATION

If we have defined how, for example, one translates individuals, this should define a way to translate the ensemble. We shall require of a quantification that any unitary transformation on an individual quantum entity induces a unitary transformation on the quantified system, defined by the quantifier.

This does not imply that, for example, the actual time-translation of an ensemble is carried out by translating the individuals. This would imply that the Hamiltonians combine additively, without interaction. There is still room for arbitrary interaction. The representation principle means only that there is a well-defined time-translation without interaction. This gives a physical meaning to interaction: it is the difference between the induced time translation generator and the actual one.

Thus we posit that an arbitrary (†-)unitary transformation $U: V_I \to V_I, \psi \mapsto U\psi$ of the individual ket-space V_I, also act naturally on the quantified mode space V_S through an operator $\hat{U}: V_S \to V_S$, defining a representation of the individual unitary group. This is the *representation principle*.

Then U also acts on the algebra $\mathcal{A}_S$ according to

$$\hat{U}: \mathcal{A}_S \to \mathcal{A}_S, \quad \hat{\psi} \mapsto \widehat{U\psi} = \hat{U}\hat{\psi}\hat{U}^{-1}. \tag{16}$$

Every unitary transformation $U: V_I \to V_I$ infinitesimally different from the identity is defined by a *generator* G

$$U = 1 + G\,\delta\theta, \tag{17}$$

where $G = -G^\dagger: V_I \to V_I$ is anti-Hermitian and $\delta\theta$ is an infinitesimal parameter. The infinitesimal anti-Hermitian generators G make up the Lie algebra dU_I of the unitary group U_I of the one-body theory.

By the representation principle, each individual generator G induces a *quantified generator* $\hat{G} \in \mathcal{A}_S$ of the quantified system, defined (up to an added constant) by its adjoint action on $\mathcal{A}_S$

$$\hat{G}: \hat{\psi} \mapsto \widehat{G\psi} = [\hat{G}, \hat{\psi}], \tag{18}$$

and (18) and (20) define a representation (Lie homomorphism) $R_Q: dU_I \to dU_S$ of the individual Lie algebra dU_I in the quantified Lie algebra dU_S.

Since

$$G = \sum_{a,b} e_a G^a{}_b e^b, \tag{19}$$

holds by the completeness of the basis e_a and the reciprocal basis e^a, we can express the quantified generator $\hat{G}$ by

$$\hat{G} := Q^\dagger G Q = \sum_{a,b} Q^\dagger{}_a G^a{}_b Q^b \equiv \sum_{a,b} \hat{e}_a G^a{}_b \hat{e}^b. \tag{20}$$

The representation principle holds for the usual statistics (MB, FD, BE) and for the Clifford statistics discussed below.

Proposition: If Q is a quantifier for a linear statistics then

$$[\hat{G}, Q^\dagger \psi] = G Q^\dagger \psi, \tag{21}$$

hold for all anti-Hermitian generators G.

Proof: We have

$$
\begin{aligned}
[\hat{G},Q^\dagger\psi] &= G^a{}_b\,(\hat{e}_a\,\hat{e}^b\,Q^\dagger\psi - Q^\dagger\psi\,\hat{e}_a\,\hat{e}^b)\\
&= G^a{}_b\,(\hat{e}_a\,(e^b\psi + (-1)^\kappa\,Q^\dagger\psi\,\hat{e}^b) - Q^\dagger\psi\,\hat{e}_a\,\hat{e}^b)\\
&= G^a{}_b\,\hat{e}_a\,e^b\psi = G^a{}_b\,\hat{e}_a\,\psi^b = G\,Q^\dagger\psi.
\end{aligned}
\tag{22}
$$

Here $\kappa=1$ for Fermi statistics and 0 for Bose.

If $\mathcal{A}$ is any algebra, by the *commutator algebra* $\Delta\mathcal{A}$ of $\mathcal{A}$ we mean the Lie algebra on the elements of $\mathcal{A}$ whose product is the commutator $[a,b]=ab-ba$ in $\mathcal{A}$. By the commutator algebra of a quantum system I we mean that of its operator algebra $\mathcal{A}_I$.

In the usual cases of Bose and Fermi statistics, and not in the cases of complex and real Clifford statistics discussed below, the quantification rule (20) defines a Lie isomorphism, $\Delta\mathcal{A}_I \to \Delta\mathcal{A}_S$, from the commutator algebra of the individual to that of the quantified system. Namely, if H and P are two (arbitrary) operators acting on the one-body ket-space, then

$$
\widehat{[H,P]} = [\hat{H},\hat{P}]. \tag{23}
$$

Explicitly

$$
\begin{aligned}
[\hat{H},\hat{P}] &= \hat{H}\hat{P} - \hat{P}\hat{H}\\
&= \hat{e}_r H^r{}_s \hat{e}^s \hat{e}_t P^t{}_u \hat{e}^u - \hat{e}_t P^t{}_u \hat{e}^u \hat{e}_r H^r{}_s \hat{e}^s\\
&= H^r{}_s P^t{}_u (\hat{e}_r \hat{e}^s \hat{e}_t \hat{e}^u - \hat{e}_t \hat{e}^u \hat{e}_r \hat{e}^s)\\
&= H^r{}_s P^t{}_u (\hat{e}_r(\delta^s_t \pm \hat{e}_t \hat{e}^s)\hat{e}^u - \hat{e}_t \hat{e}^u \hat{e}_r \hat{e}^s)\\
&= H^r{}_s P^t{}_u (\hat{e}_r \delta^s_t \hat{e}^u \pm \hat{e}_r \hat{e}_t \hat{e}^s \hat{e}^u - \hat{e}_t \hat{e}^u \hat{e}_r \hat{e}^s)\\
&= H^r{}_s P^t{}_u (\hat{e}_r \delta^s_t \hat{e}^u \pm \hat{e}_t \hat{e}_r \hat{e}^u \hat{e}^s - \hat{e}_t \hat{e}^u \hat{e}_r \hat{e}^s)\\
&= H^r{}_s P^t{}_u (\hat{e}_r \delta^s_t \hat{e}^u \pm \hat{e}_t(\mp \delta^u_r \pm \hat{e}^u \hat{e}_r)\hat{e}^s - \hat{e}_t \hat{e}^u \hat{e}_r \hat{e}^s)\\
&= H^r{}_s P^t{}_u (\hat{e}_r \delta^s_t \hat{e}^u - \hat{e}_t \delta^u_r \hat{e}^s)\\
&= \hat{e}_r (H^r{}_t P^t{}_u - P^r{}_t H^t{}_u)\hat{e}^u\\
&= \widehat{[H,P]}.
\end{aligned}
\tag{24}
$$

This implies that for BE and FD statistics, the quantification rule (20) can be extended from the unitary operators and their anti-Hermitian generators to the whole operator algebra of the quantified system.

V. CLIFFORD QUANTIFICATION

Now let the one-body mode space $V_I=\mathbb{R}^{N_+,N_-}=N_+\mathbb{R}\oplus N_-\mathbb{R}$ be a real quadratic space of dimension $N=N_++N_-$ and signature N_+-N_-. Denote the symmetric metric form of V_I by $g=(g_{ab}):=(e_a^\dagger e_b)$. We do not assume that g is positive-definite.

We define *Clifford quantification* (9) by:

(1) the Clifford-like generating relations

$$
\hat{\psi}\hat{\phi}+\hat{\phi}\hat{\psi}=\frac{\zeta}{2}\psi^\dagger\phi, \tag{25}
$$

for all $\phi,\psi\in V_I$, where ζ is a $\pm$ sign that can have either value;

(2) the Hermiticity condition (7)

$$
\hat{e}_a^\dagger = g_{ab}\hat{e}^b, \tag{26}
$$

(3) a rule for raising and lowering indices

$$
\hat{e}_a := \zeta'\,g_{ab}\hat{e}^b, \tag{27}
$$

where ζ' is another $\pm$ sign, and

(4) the definition (20) to quantify one-body generators.

Here $\zeta=\pm 1$ covers the two different conventions used in the literature. In Sec. VI we will see that $\zeta=\zeta'$, and that $\zeta=\zeta'=+1$ and $\zeta=\zeta'=-1$ are both allowed physically at the present theoretical stage of development. They lead to two different real quantifications, with either Hermitian or anti-Hermitian Clifford units.

For the quantified basis elements of V_I (25) leads to

$$\hat{e}_a\hat{e}_b+\hat{e}_b\hat{e}_a=\frac{\zeta}{2}g_{ab}\,. \tag{28}$$

The ψ's, which are assigned grade 1 and taken to be either Hermitian or anti-Hermitian, generate a graded †-algebra that we call the *free Clifford †-algebra* associated with $\mathbb{R}^{N_+,N_-}$ and write as $\mathrm{Cliff}(N_+,N_-)\equiv\mathrm{Cliff}(N_\pm)$. $\mathrm{Cliff}(N_\pm)$ contains a double-valued (or projective) representation of the permutation group S_N.

We call quanta obeying Clifford statistics *cliffordons*. Clifford statistics assembles cliffordons individually described by vectors into a composite described by spinors, which we call a *squadron*. We intend the -on suffix to remind us that unlike the common statistics the Clifford statistics has no classical correspondent.

A cliffordon is a hypothetical quantum-physical entity, like an electron, not to be confused with a mathematical object like a spinor or an operator. We cannot describe a cliffordon completely, but we represent our actions on a squadron of cliffordons adequately by operators in a Clifford algebra of operators. One encounters cliffordons only in permuting them, never in creating or annihilating them as individuals.

In assuming a real vector space of quantum modes instead of a complex one, we give up i-invariance but retain quantum superposition $a\psi+b\phi$ with real coefficients. Our theory is nonlinear from the complex point of view. Others considered nonlinear quantum theories, but gave up real superposition as well as i-invariance.[7,8] We are not *that* nonlinear.

VI. QUANTIFYING OBSERVABLES

In the usual statistics, the quantifier Q can be usefully extended from the Lie algebra of the individual to the commutator algebra of the individual; that is, from anti-Hermitian operators to all operators. This is not the case for Clifford quantification. There the quantification of any symmetric operator is a scalar, in virtue of Clifford's law, and so the commutator of any two operators is just the commutator of their antisymmetric parts. A straightforward calculation shows that

$$[\hat{H},\hat{P}]=\hat{H}\hat{P}-\hat{P}\hat{H}=\zeta\,\zeta'\big(\tfrac{1}{2}\widehat{[H,P]}+\tfrac{1}{4}(\widehat{[P,H^\dagger]}+\widehat{[P^\dagger,H]})\big). \tag{29}$$

The three simplest cases are:

(1) $H=H^\dagger,\ H'=H'^\dagger\Rightarrow[\hat{H},\hat{H}']=0$;
(2) $H=H^\dagger,\ G_1=-G_1^\dagger\Rightarrow[\hat{H},\hat{G}_1]=0$;
(3) $G=-G^\dagger,\ G'=-G'^\dagger\Rightarrow[\hat{G},\hat{G}']=\zeta\,\zeta'\,\widehat{[G,G']}$.

Thus Clifford quantification respects the commutation relations for anti-Hermitian generators if and only if $\zeta=\zeta'=+1$ or $\zeta=\zeta'=-1$; but not for Hermitian observables, contrary to the Bose and Fermi quantifications, which respect both.

VII. NAYAK–WILCZEK STATISTICS

The *complex* graded algebra generated by the ψ's with the relations (25) is called the *complex Clifford algebra* $\mathrm{Cliff}_{\mathbb{C}}(N)$ over $\mathbb{R}^{N_+,N_-}$. It is isomorphic to the full complex matrix algebra $\mathbb{C}(2^n)\otimes\mathbb{C}(2^n)$ for even $N=2n$, and to the direct sum $\mathbb{C}(2^n)\otimes\mathbb{C}(2^n)\oplus\mathbb{C}(2^n)\otimes\mathbb{C}(2^n)$ for odd $N=2n+1$. We regard $\mathrm{Cliff}_{\mathbb{C}}(N)$ as the kinematic algebra of the complex Clifford composite. As a vector space, it has dimension 2^N.

Schur[9] used complex spinors and complex Clifford algebra to represent permutations some years before Cartan used them to represent rotations. There is a fairly widespread view that spinors may be more fundamental than vectors, since vectors may be expressed as bilinear combinations of spinors. One of us took this direction in much of his work. Clifford statistics support the opposite view. There a vector describes an individual, a spinor an aggregate. Wilczek and Zee[10] seem to have been the first to recognize that spinors represent composites in a physical context, although this is implicit in the Chevalley construction of spinors within a Grassmann algebra.

For dimension $N=3$ spinors have as many parameters as vectors, but for higher N the number of components of the spinors associated with $\mathrm{Cliff}(N_{\pm})$ grows exponentially with N. The physical relevance of this irreducible double-valued (or projective) representation of the permutation group S_N was recognized by Nayak and Wilczek[1,4] in a theory of the fractional quantum Hall effect. We call the complex statistics based on $\mathrm{Cliff}_{\mathbb{C}}(N)$ the *Nayak–Wilczek* or NW statistics.

Clifford statistics, unlike the more familiar particle statistics,[11–13] provides no creators or annihilators. With each individual mode e_a of the quantified system they associate a Clifford unit $\gamma_a = 2Q^{\dagger}{}_a$.

We may represent any swap (transposition of two cliffordons, say 1 and 2) by the difference of the corresponding Clifford units

$$t_{12} := \frac{1}{\sqrt{2}}(\gamma_1 - \gamma_2), \tag{30}$$

and represent an arbitrary permutation, which is a product of elementary swaps, by the product of their representations. That is, as direct computation shows, this defines a projective homomorphism from S_N into the Clifford algebra generated by the γ_k.

By definition, the number N of cliffordons in a squadron is the dimensionality of the individual initial mode space V_I. N is conserved rather trivially, commuting with every Clifford element. We can change this number only by varying the dimensionality of the one-body space. In one use of the theory, we can do this, for example, by changing the space–time four-volume of the corresponding experimental region. Because our theory does not use creation and annihilation operators, an initial action on the squadron represented by a spinor ξ should be viewed as some kind of spontaneous transition condensation into a coherent mode, analogous to the transition from the superconducting to the many-vertex mode in a type-II superconductor. The initial mode of a set or sib of (FD or BE) quanta can be regarded as a result of possibly entangled creation operations. That of a squadron of cliffordons cannot.

As with (22), let us verify that definition (20) is consistent in the Clifford case:

$$[\hat{G}, Q^{\dagger}\psi] = G^a{}_b(\hat{e}_a \hat{e}^b Q^{\dagger}\psi - Q^{\dagger}\psi \hat{e}_a \hat{e}^b) = \tfrac{1}{2} G^a{}_b(\hat{e}_a \psi^b + \psi_a \hat{e}^b) = G\, Q^{\dagger}\psi. \tag{31}$$

This shows that $Q^{\dagger}\psi$ transforms correctly under the infinitesimal unitary transformation of $\mathbb{R}^{N_+, N_-}$ (cf. Ref. 14).

VIII. BREAKING *i* INVARIANCE

Thus we cannot construct useful Hermitian variables of a squadron by applying the quantifier to the Hermitian variables of the individual cliffordon.

This is closely related to fact that the real initial mode space $\mathbb{R}^{N_{\pm}}$ of a cliffordon has no special operator to replace the imaginary unit i of the standard complex quantum theory. The fundamental task of the imaginary element i in the algebra of complex quantum physics is precisely to relate conserved Hermitian observables H and anti-Hermitian generators G by

$$H = -i\hbar G. \tag{32}$$

To perform this function exactly, the operator i must commute exactly with all observables.

The central operators x and p of classical mechanics are contractions of noncentral operators $\breve{x}$ and $\breve{p}=-i\hbar\partial/\partial\breve{x}$.[2] In the limit of large numbers of individuals organized coherently into suitable condensate modes, the expanded operators of the quantum theory contract into the central operators of the classical theory. Condensations produce nearly commutative variables.

Likewise we expect the central operator i to be a contraction of a noncentral operator $\breve{i}$ similarly resulting from a condensation in a limit of large numbers. In the simpler expanded theory, $\breve{i}$, the correspondent of i, is not central.

One clue to the nature of $\breve{i}$ and the locus of its condensation is how the operator i behaves when we combine separate systems. Since infinitesimal generators $G,G',\ldots$ combine by addition, the imaginaries $i,i',\ldots$ of different individuals must combine by identification

$$i=i'=\cdots, \tag{33}$$

for (32) to hold exactly, and nearly so for (32) to hold nearly. The only other variables in present physics that combine by identification in this way are the time t of clasical mechanics and the space–time coordinates x^{μ} of field theories. All systems in an ensemble must have about the same i, just as all particles have about the same t in the usual instant-based formulation, and all fields have about the same space–time variables x^{μ} in field theory. We identify the variables t and x^{μ} for different systems because they are set by the experimenter, not the system. This suggests that the experimenter, or more generally the environment of the system, mainly defines the operator i. The central operators x,p characterize a small system that results from the condensation of many particles. The central operator i must result from a condensation in the environment; we take this to be the same condensation that forms the vacuum and the spatiotemporal structure represented by the variables x^{μ} of the standard model.

The existence of this contracted i ensures that at least approximately, every Lie commutation relation between dimensionless anti-Hermitian generators A,B,C of the standard complex quantum theory

$$[A,B]=C, \tag{34}$$

corresponds to a commutation relation between Hermitian variables $-i\hbar A$, $-i\hbar B$, $-i\hbar C$

$$[-i\hbar A,-i\hbar B]=-i\hbar(-i\hbar C). \tag{35}$$

It also tells us that this correspondence is not exact in nature.

Stückelberg[15] reformulated complex quantum mechanics in the real Hilbert space $\mathbb{R}^{2N}$ of twice as many dimensions by assuming a special real antisymmetric operator $J:\mathbb{R}^{2N}\to\mathbb{R}^{2N}$ commuting with all of the variables of the system.

A real † or Hilbert space has no such operator. For example, in $\mathbb{R}^2$ the operator

$$E:=\begin{bmatrix}\varepsilon_1 & 0\\ 0 & \varepsilon_2\end{bmatrix}, \tag{36}$$

is a symmetric operator with an obvious spectral decomposition representing, according to the usual interpretation, two selection operations performed on the system, and cannot be written in the form $G=-(J/\hbar)E$ relating it to some antisymmetric generator G for any real antisymmetric J commuting with E.

On the other hand, if we restrict ourselves to observable operators of the form

$$E':=\begin{bmatrix}\varepsilon & 0\\ 0 & \varepsilon\end{bmatrix}, \tag{37}$$

we can use the operator J

$$J := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{38}$$

to restore the usual connection between symmetry transformations and corresponding observables. This restriction can be generalized to any even number of dimensions.[15]

IX. BREAKDOWN OF THE EXPECTATION VALUE FORMULA

For a system described in terms of a general real Hilbert space there is no simple relation of the form $G=(i/\hbar)H$ between the symmetry generators and the observables: the usual notions of Hamiltonian and momentum are meaningless in that case. This amplifies our earlier observation that Clifford quantification $A \to \hat{A}$ respects the Lie commutation relations among anti-Hermitian generators, not Hermitian observables.

Operationally, this means that selective acts of individual and quantified cliffordons use essentially different sets of filters. This is not the case for complex quantum mechanics and the usual statistics. There some important filters for the composite are simply assemblies of filters for the individuals.

Again, in the complex case the expectation value formula for an assembly

$$Av\, X = \psi^\dagger X \psi / \psi^\dagger \psi, \tag{39}$$

is a consequence of the eigenvalue principle for individuals, rather than an independent assumption.[3,16] The argument presented in Refs. 3 and 16 assumes that the individuals over which the average is taken combine with Maxwell–Boltzmann statistics. For highly excited systems this is a good approximation even if the individuals have FD or BE or other tensorial statistics. It is not necessarily a good approximation for cliffordons, which have spinorial, not tensorial, statistics.

X. SPIN-1/2 COMPLEX CLIFFORD MODEL

In this section we present a simplest possible model of a complex Clifford composite. The resulting many-body energy spectrum is isomorphic to that of a sequence of spin-1/2 particles in an external magnetic field.

Recall that in the usual complex quantum theory the Hamiltonian is related to the infinitesimal time-translation generator $G=-G^\dagger$ by $G=iH$. Quantifying H gives the many-body Hamiltonian. In the framework of spinorial statistics, as discussed above, this does not work, and quantification in principle applies to the anti-Hermitian time-translation generator G, not to the Hermitian operator H. Our task now is to choose a particular generator and to study its quantified properties.

We assume an even-dimensional real initial-mode space $V_I=\mathbb{R}^{2n}$ for the quantum individual, and consider the dynamics with the simplest non-trivial time-translation generator

$$G := \varepsilon \begin{bmatrix} \mathbf{0}_n & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0}_n \end{bmatrix}, \tag{40}$$

where ε is a constant energy coefficient.

The quantified time-translation generator $\hat{G}$ then has the form

$$\begin{aligned} \hat{G} := \sum_{l,j}^{N} \hat{e}_l G^l{}_j \hat{e}^j &= -\varepsilon \sum_{k=1}^{n} (\hat{e}_{k+n}\hat{e}^k - \hat{e}_k \hat{e}^{k+n}) \\ &= +\varepsilon \sum_{k=1}^{n} (\hat{e}_{k+n}\hat{e}_k - \hat{e}_k \hat{e}_{k+n}) \\ &= 2\varepsilon \sum_{k=1}^{n} \hat{e}_{k+n}\hat{e}_k \\ &\equiv \frac{1}{2}\varepsilon \sum_{k=1}^{n} \gamma_{k+n}\gamma_k . \end{aligned} \tag{41}$$

By Stone's theorem, the generator $\hat{G}$ of time translation in the spinor space of the complex Clifford composite of $N=2n$ individuals can be factored into a Hermitian $H^{(N)}$ and an imaginary unit i that commutes strongly with $H^{(N)}$

$$\hat{G}=iH^{(N)}. \tag{42}$$

We suppose that $H^{(N)}$ corresponds to the Hamiltonian and seek its spectrum.

We note that by (41), $\hat{G}$ is a sum of n commuting anti-Hermitian algebraically independent operators $\gamma_{k+n}\gamma_k, k=1,2,\ldots,n$, $(\gamma_{k+n}\gamma_k)^\dagger=-\gamma_{k+n}\gamma_k$, $(\gamma_{k+n}\gamma_k)^2=-1^{(N)}$.

We use the well-known $2^n\times 2^n$ complex matrix representation of the γ-matrices of the complex universal Clifford algebra associated with the real quadratic space $\mathbb{R}^{2n}$ (Brauer and Weyl[17]):

$$\begin{gathered}
\gamma_{2j-1}=\boldsymbol{\sigma}_3\otimes\cdots\otimes\boldsymbol{\sigma}_3\otimes\boldsymbol{\sigma}_1\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1},\\
\gamma_{2j}=\boldsymbol{\sigma}_3\otimes\cdots\otimes\boldsymbol{\sigma}_3\otimes\boldsymbol{\sigma}_2\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1},\\
j=1,2,3,\ldots,n,
\end{gathered} \tag{43}$$

where σ_1, σ_2 occur in the jth position, the product involves n factors, and σ_1, σ_2, σ_3 are the Pauli matrices. The representation of the corresponding permutation group S_{2n} is reducible. We can simultaneously diagonalize the $2^n\times 2^n$ matrices representing the commuting operators $\gamma_{k+n}\gamma_k$, and use their eigenvalues, $\pm i$, to find the spectrum λ of $\hat{G}$, and consequently of $H^{(N)}$.

A simple calculation shows that the spectrum of $\hat{G}$ consists of the eigenvalues

$$\lambda_k=\tfrac{1}{2}\varepsilon(n-2k)i, \quad k=0,1,2,\ldots,n, \tag{44}$$

with multiplicity

$$\mu_k=C^n_k:=\frac{n!}{k!(n-k)!}. \tag{45}$$

The spectrum of eigenvalues of the Hamiltonian $H^{(N)}$ then consists of $n+1$ energy levels

$$E_k=\tfrac{1}{2}(n-2k)\varepsilon, \tag{46}$$

with degeneracy μ_k. Thus E_k ranges over the interval

$$-\tfrac{1}{4}N\varepsilon<E<\tfrac{1}{4}N\varepsilon, \tag{47}$$

in steps of ε, with the given degeneracies.

Thus the spectrum of the structureless N-body complex Clifford composite is the same as that of a system of N spin-1/2 Maxwell–Boltzmann particles of magnetic moment μ in a magnetic field $\mathbf{H}$, with the identification

$$\tfrac{1}{4}\varepsilon=\mu H. \tag{48}$$

Even though we started with such a simple one-body time-translation generator as (40), the spectrum of the resulting many-body Hamiltonian possesses some complexity, reflecting the fact that the units in the composite are distinguishable, and their swaps generate the dynamical variables of the system.

This spin-1/2 model does not tell us how to swap two Clifford units experimentally. Like the phonon model of the harmonic oscillator, the statistics of the individual quanta enters the picture only through the commutation relations among the fundamental operators of the theory.

XI. REAL CLIFFORD STATISTICS

Real Clifford quantification establishes a morphism (20) from the Lie algebra of the individual into that of the composite. The proof for real Clifford statistics parallels that for the complex Clifford case closely.

According to the Periodic Table of the Spinors,[18,19–21] the free (or universal) Clifford algebra $\mathrm{Cliff}_{\mathbb{R}}(N_+, N_-)$ is algebra-isomorphic to the endomorphism algebra of a module $\Sigma(N_+, N_-)$ over a ring $\mathcal{R}(N_+, N_-)$. We give the table to simplify reference to it (here $\zeta = -1$):

N_+ \ N_-	0	1	2	3	4	5	6	7	...
0	$\mathbb{R}$	$\mathbb{R}_2$	$2\mathbb{R}$	$2\mathbb{C}$	$2\mathbb{H}$	$2\mathbb{H}_2$	$4\mathbb{H}$	$8\mathbb{C}$	...
1	$\mathbb{C}$	$2\mathbb{R}$	$2\mathbb{R}_2$	$4\mathbb{R}$	$4\mathbb{C}$	$4\mathbb{H}$	$4\mathbb{H}_2$	$8\mathbb{H}$	...
2	$\mathbb{H}$	$\mathbb{C}_2$	$4\mathbb{R}$	$4\mathbb{R}_2$	$8\mathbb{R}$	$8\mathbb{C}$	$8\mathbb{H}$	$8\mathbb{H}_2$	...
3	$\mathbb{H}_2$	$2\mathbb{H}$	$4\mathbb{C}$	$8\mathbb{R}$	$8\mathbb{R}_2$	$16\mathbb{R}$	$16\mathbb{C}$	$16\mathbb{H}$	...
4	$2\mathbb{H}$	$2\mathbb{H}_2$	$4\mathbb{H}$	$8\mathbb{C}$	$16\mathbb{R}$	$16\mathbb{R}_2$	$32\mathbb{R}$	$32\mathbb{C}$	...
5	$4\mathbb{C}$	$4\mathbb{H}$	$4\mathbb{H}_2$	$8\mathbb{H}$	$16\mathbb{C}$	$32\mathbb{R}$	$32\mathbb{R}_2$	$64\mathbb{R}$	...
6	$8\mathbb{R}$	$8\mathbb{C}$	$8\mathbb{H}$	$8\mathbb{H}_2$	$16\mathbb{H}$	$32\mathbb{C}$	$64\mathbb{R}$	$64\mathbb{R}_2$	...
7	$8\mathbb{R}_2$	$16\mathbb{R}$	$16\mathbb{C}$	$16\mathbb{H}$	$16\mathbb{H}_2$	$32\mathbb{H}$	$64\mathbb{C}$	$128\mathbb{R}$	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

(49)

It shows that the ring of coefficients $\mathcal{R}(N_+, N_-)$ varies periodically with period 8 in each of the dimensionalities N_+ and N_- of V_I, and is a function of signature $N_+ - N_-$ alone. In the first cycle, $N_+ - N_- = 0,1,\ldots,7$, and $\mathcal{R} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{H}\oplus\mathbb{H}, \mathbb{H}, \mathbb{C}, \mathbb{R}, \mathbb{R}\oplus\mathbb{R}$, respectively. Then the cycle repeats ad infinitum.

In our application the module $\Sigma(N_+, N_-)$, the spinor space supporting $\mathrm{Cliff}_{\mathbb{R}}(N_+, N_-)$, serves as the initial mode space of a squadron of N real cliffordons. $\mathcal{R}(N_\pm)$ we call the *spinor coefficient ring* for $\mathrm{Cliff}_{\mathbb{R}}(N_+, N_-)$.

XII. PERMUTATIONS

In the standard statistics there is a natural way to represent permutations of individuals in the N-body composite. Each N-body ket is constructed by successive action of N creation operators on the special vacuum mode. Any permutation of individuals can be achieved by permuting these creation operators in the product. The identity and alternative representations of the permutation group S_N in the BE and FD cases then follow from the defining relations of Sec. II.

In the case of Clifford statistics, some things are different. There is still an operator associated with each cliffordon; now it is a Clifford unit. Permutations of cliffordons are still represented by operators on a many-body † space. But the mode space on which these operators act is now a spinor space, and its basis vectors are not constructed by creation operators acting on a special "vacuum" ket.

The Clifford representation of the permutation group that we have employed is reducible into two irreducible Schur representations. It is a bit easier to write than Schur's because our individual operators γ_i anticommute exactly, corresponding to exactly orthogonal directions in the one-body mode space, like the generators of Dirac's Clifford algebra. In Schur's irreducible representation (slightly simplified) these operators are replaced by their projections normal to the principle diagonal direction $n := \Sigma \gamma_i / \sqrt{N}$, which is invariant under all permutations. The corresponding angles are those subtended by the edges of a regular simplex of N vertices in $N-1$ dimensions as seen from the center. These angles are all determined by

$$\cos^2 \theta = \frac{1}{N-1}. \tag{50}$$

They differ from $\pi/2$ by an angle that vanishes for large N like $1/N$.

XIII. EMERGENCE OF A QUANTUM *i*

The Periodic Table of the Spinors (Sec. XI) suggests another origin for the complex i of quantum theory, and one that is not approximately central but exactly central. Some Clifford algebras $\mathrm{Cliff}_{\mathbb{R}}(N_+, N_-)$ have the spinor coefficient ring $\mathbb{C}$, containing an element i. Multiplication by this i then represents an operator in the center of the Clifford algebra, which we designate also by i. We may use i-multiplication to represent the top element $\gamma^{\uparrow}$ whenever $\gamma^{\uparrow}$ is central and has square -1. This $i \in \mathrm{Cliff}_{\mathbb{R}}(N_{\pm})$ corresponds to the i of complex quantum theory.

$\mathrm{Cliff}_{\mathbb{R}}(1,0)$ contains such an i but is commutative. According to the Periodic Table (with the choice of $\zeta = -1$), the smallest noncommmutative Clifford algebras of Euclidean signature with complex spinor coefficients are $\mathrm{Cliff}_{\mathbb{R}}(0,3)$ with negative Euclidean signature, and $\mathrm{Cliff}_{\mathbb{R}}(5,0)$ with positive Euclidean signature. Triads or pentads of such cliffordons could underlie the physical ''elementary'' particles, giving rise to complex quantum mechanics within the real. We consider these two cases in turn.

$\mathrm{Cliff}_{\mathbb{R}}(0,3) = \mathbb{C}(2)$ has the familiar Pauli representation $\gamma_1 := i\,\sigma_1$, $\gamma_2 := i\,\sigma_2$, $\gamma_3 := i\,\sigma_3$ with $\zeta = -1$. We choose a particular one-cliffordon dynamics of the form

$$G := \begin{bmatrix} 0 & V & 0 \\ -V & 0 & \varepsilon \\ 0 & -\varepsilon & 0 \end{bmatrix}. \tag{51}$$

Quantification (20) of G gives

$$\hat{G} = iH^{(-3)} \tag{52}$$

with the Hamiltonian

$$H^{(-3)} = \frac{1}{2}\begin{bmatrix} V & \varepsilon \\ \varepsilon & -V \end{bmatrix}. \tag{53}$$

This is also the Hamiltonian for a generic two-level quantum-mechanical system (with the energy separation ε) in an external potential field V, like the ammonia molecule in a static electric field discussed in Ref. 22.

$\mathrm{Cliff}_{\mathbb{R}}(5,0) = \mathbb{C}(4)$ has the matrix representation $\gamma_1 := i\,\sigma_1 \otimes \mathbf{1}$, $\gamma_2 := i\,\sigma_2 \otimes \mathbf{1}$, $\gamma_3 := i\,\sigma_3 \otimes \sigma_1$, $\gamma_4 := i\,\sigma_3 \otimes \sigma_2$, $\gamma_5 := i\,\sigma_3 \otimes \sigma_3$, again with $\zeta = -1$. Its top Clifford unit is $\gamma^{\uparrow} := \Pi_k \gamma_k = \gamma^{\uparrow\dagger} = \gamma^{\uparrow -1}$ with eigenvalues ± 1. We choose a specimen dynamics (for the individual cliffordon) in the form

$$G := \begin{bmatrix} 0 & V & 0 & 0 & 0 \\ -V & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{54}$$

Quantification (20) of G gives

$$\hat{G} = iH^{(5)}, \tag{55}$$

with the Hamiltonian

$$H^{(5)}=\frac{1}{2}V\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}. \tag{56}$$

The two examples considered above show how a squadron of several real cliffordons can obey a truly complex quantum theory.

XIV. FERMI AND CLIFFORD STATISTICS

The FD algebra of creators and annihilators is a special case of a Clifford algebra over a quadratic space with neutral quadratic form, called the quantum algebra by Saller[23] and the mother algebra by Doran *et al.*[24] Is FD statistics ever a special case of Clifford statistics? Specifically, are their †-algebras ever isomorphic?

From the N annihilators a_k of the complex FD statistics we can form a sequence of anticommuting hermitian square roots of unity

$$i_k=a_k+a_k^\dagger,\quad i_{k+N}=\frac{a_k-a_k^\dagger}{i}. \tag{57}$$

Moreover, the complex †-algebra that these generate is a Clifford †-algebra Cliff$(2N,0)$. The transformation from the FD generators to the Clifford is invertible. Therefore complex FD statistics and complex Clifford statistics have isomorphic †-algebras.

The graded †-algebras are obviously not isomorphic. The two grade operators do not even commute.

The question is more complicated for the real Fermi and Clifford quantifications. We follow Doran *et al.*,[24] among others.

In the real FD formulation we begin with a real one-fermion n-dimensional space $F\cong n\mathbb{R}$ with no metric or adjoint. The FD quantified algebra $\mathcal{A}$ has the bilinear associative product defined by the FD relations

$$\begin{aligned}f_if_j+f_jf_i&=0,\\ f_if^j+f^jf_i&=\delta_i^j,\end{aligned} \tag{58}$$

and the adjoint defined by

$$f_i^\dagger:=f^i. \tag{59}$$

The f_i are creation and f^j are annihilation operators.

To present $\mathcal{A}$ as a Clifford algebra we form the direct sum

$$W=F\oplus F^\dagger. \tag{60}$$

In a basis $\{f_i,f^i\}_{i=1}^n$ adapted to this direct sum, we define the following GL(V)-invariant metric for W

$$g\sim\begin{bmatrix}0&1/2\\1/2&0\end{bmatrix}, \tag{61}$$

corresponding to

$$f_i \cdot f_j = 0, \quad f^i \cdot f^j = 0, \quad f^i \cdot f_j = \tfrac{1}{2}\delta^i_j. \tag{62}$$

Since F supports a quantum theory it too has a quadratic form $*: F \otimes F \to \mathbb{R}$, which we assume to be Euclidean. We did not use $*$ in the construction of $\mathcal{A}$ and g.

We quantify this fermion by a mapping $Q^\dagger : W \to \mathcal{A}$ into the †-algebra of the composite. For brevity we write f_i for $Q^\dagger f_i$ as is also customary.

The quantification Q has the representation property. In the FD case this means that Q represents the orthogonal group $\mathrm{SO}(F,*)$ in $\mathcal{A}$; in fact it represents the larger group $\mathrm{GL}(F)$, for $*$ has not entered into the definition of Q.

The basis $\{\gamma_i, \tilde{\gamma}_i\}_{i=1}^n$ defined by

$$\gamma_i := f_i + f^i, \quad \tilde{\gamma}_i := f_i - f^i, \tag{63}$$

gives the metric g of W the diagonal form

$$g \sim \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{64}$$

corresponding to

$$\gamma_i \cdot \gamma_j = 1, \quad \tilde{\gamma}_i \cdot \tilde{\gamma}_j = -1, \quad \tilde{\gamma}_i \cdot \gamma_j = 0. \tag{65}$$

That is, $W = E \oplus \tilde{E}$ is a neutral quadratic space, with Eucidean subspace E and anti-Euclidean subspace $\tilde{E}$.

The γ's obey

$$\begin{aligned} \gamma_i \gamma_j + \gamma_j \gamma_i &= +2\delta_{ij}, \\ \tilde{\gamma}_i \tilde{\gamma}_j + \tilde{\gamma}_j \tilde{\gamma}_i &= -2\delta_{ij}, \\ \tilde{\gamma}_i \gamma_j + \gamma_j \tilde{\gamma}_i &= 0. \end{aligned} \tag{66}$$

Therefore the FD algebra (58) is isomorphic to a real Clifford algebra $\mathrm{Cliff}(W,\dagger) = \mathrm{Cliff}(E \oplus \tilde{E})$.

Are the Clifford and FD †-algebras also isomorphic?

With respect to the Fermi adjoint †, half of the Clifford generators (the γ_i) are Hermitian and the other half (the $\tilde{\gamma}_i$) are anti-Hermitian. In a Clifford † algebra, however, all the generators are anti-Hermitian or Hermitian together. Therefore the Clifford-algebra generators $\{\gamma_i, \tilde{\gamma}_i\}_{i=1}^n$ are not Clifford †-algebra generators.

In some cases we construct suitable generators using the top element $\tilde{\gamma}^\uparrow$ of $\tilde{E}$:

If the dimension n of E (and F) is a multiple of 4, then $\overline{\gamma_i} := \tilde{\gamma}^\uparrow \tilde{\gamma}_i$ anticommutes with the γ_j, and is Hermitian like the γ_j. Then the elements $\{\gamma_i, \overline{\gamma_i}\}_{i=1}^n$ generate a Clifford †-algebra with [cf. (66)]

$$\begin{aligned} \gamma_i \gamma_j + \gamma_j \gamma_i &= +2\delta_{ij}, \\ \overline{\gamma_i}\,\overline{\gamma_j} + \overline{\gamma_j}\,\overline{\gamma_i} &= +2\delta_{ij}, \\ \overline{\gamma_i} \gamma_j + \gamma_j \overline{\gamma_i} &= 0, \end{aligned} \tag{67}$$

which is isomorphic to the FD algebra of F. Then the Clifford-quantified †-algebra (the case $\zeta = \zeta' = +1$) is isomorphic to a Fermi-quantified one when $n = 4m$, and the adjoint of the one-cliffordon space is positive definite. The two quantified theories then predict the same transition amplitudes and spectra.

Analogously, when $\zeta=\zeta'=-1$ and all the Clifford generators are anti-Hermitian, and $n=4m$, the FD and Clifford statistics again give isomrphic † algebras.

They still differ in their grades. The FD quantified system has a grade G_F with spectrum $-N,\ldots,0,\ldots,N$, corresponding to the creation and annihilation fermions. The Clifford quantified system has a positive grade operator G_C with spectrum $0,1,\ldots 2N$. The operators G_C and G_F do not even commute. The FD and Clifford graded-algebras are not isomorphic.

This is merely a difference in language. The operators that are said to create and annihilate things in FD statistics are said to permute things in Clifford statistics. In Clifford statistics nothing is created or destroyed.

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[1] C. Nayak and F. Wilczek, Nucl. Phys. B **479**, 529 (1996).
[2] J. Baugh, D. Finkelstein, A. Galiautdinov, and H. Saller, J. Math. Phys. **42**, 1489 (2001).
[3] D. Finkelstein, *Quantum Relativity* (Springer-Verlag, New York, 1996).
[4] F. Wilczek, e-print hep-th/9806228 [LANL] (submitted for publication).
[5] G. Moore and N. Read, Nucl. Phys. B **360**, 362 (1990).
[6] N. Read and G. Moore, Prog. Theor. Phys. (Kyoto) Suppl. **107**, 157 (1992).
[7] S. Weinberg, Phys. Rev. Lett. **62**, 485 (1989).
[8] D. C. Brody and L. P. Hughston, J. Math. Phys. **39**, 6502 (1998); **40**, 12 (1999); **41**, 2586 (2000).
[9] I. Schur, Journal für die reine und angewandte Mathematik **139**, 155 (1911).
[10] F. Wilczek and A. Zee, Phys. Rev. D **25**, 553 (1982).
[11] F. A. Berezin, *The Method of Second Quantization* (Academic, New York, 1966)
[12] R. P. Feynman, *Statistical Mechanics: A Set of Lectures* (W. A. Benjamin, Inc., Massachusetts, 1972).
[13] J. W. Negele, H. Orland, *Quantum Many-Particle Systems* (Addison-Wesley, Reading, Mass., 1988).
[14] P. A. M. Dirac, *Spinors in Hilbert Space* (Plenum, New York, 1974).
[15] E. C. G. Stückelberg, Helv. Phys. Acta **33**, 727 (1960).
[16] D. Finkelstein, Trans. NY Acad. Sci. **25**, 621 (1963).
[17] R. Brauer and H. Weyl, Am. J. Math. **LVII(2)**, 425 (1935).
[18] P. Budinich, and A. Trautman, *The Spinorial Chessboard* (Springer, Berlin, 1988).
[19] P. Lounesto, *Clifford Algebras and Spinors* (Cambrige University Press, Cambridge, 1997).
[20] I. R. Porteous, *Clifford Algebras and the Classical Groups* (Cambrige University Press, Cambridge, 1995).
[21] J. Snygg, *Clifford Algebra: A Computational Tool for Physisicts* (Oxford University Press, Cambridge, 1997).
[22] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics. Quantum Mechanics* (Addison-Wesley, Reading, Mass., 1965).
[23] H. Saller, Nuovo Cimento **B108**, 603 (1993); **B109**, 255 (1994).
[24] C. Doran, D. Hestenes, F. Sommen, and N. Van Acker, J. Math. Phys. **34**, 3642 (1993).