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# Focus waves modes in homogeneous Maxwell's equations: Transverse electric mode

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This paper presents mathematical formulations for new, three-dimensional, packet-like solutions to the free-space homogeneous Maxwell's equations. These solutions are real, nonsingular, continuous functions which propagate in a straight line at light velocity. They remain focused for all time. The asymptotic behavior of the fields away from the moving pulse center has a magnitude which decreases as the inverse of the distance from the pulse centers.

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## I. INTRODUCTION

This paper presents the first mathematical representation of a family of three-dimensional, nondispersive, source-free, free-space, classical electromagnetic pulses which propagate in a straight line in free space at light velocity. The term "three-dimensional" means that the functional values fall off in all directions from the moving pulse centers; therefore, the pulse functions' magnitudes decrease in front of, behind, and transverse to the axis of propagation. The term "nondispresive" means that the pulse functions' envelope shapes remain fixed as they propagate; therefore, these functions do not spread as they propagate. In this case there is a ripple associated with the main pulse functions which causes the functional values to vary between consecutive points in the direction of propagation. However, the moving pulses have unique characteristics in that their functional values repeat identically at periodic points along the axis of propagation. Nondispersive pulse solutions have been observed both experimentally and mathematically in the soliton area.[1,2] These soliton solutions are plane waves in nature—not three-dimensional—because their functions are fixed-value transverse to the axis of propagation. It should be noted here that the soliton solutions satisfy nonlinear differential equations while the formulations presented in this paper are solutions to linear equations. The mathematical formulations presented below were found after a very extensive heuristical fit of various differential equation solutions. The author presents this paper hoping to initiate the mathematical study of three-dimensional, nondispersive pulses in known scientific areas which might extend present knowledge in these areas.

To convince the scientific community that these solutions are unique electromagnetic pulses, the mathematical formulations must (1) satisfy the homogeneous Maxwell's equations, (2) be continuous and nonsingular, (3) have a three-dimensional pulse structure, (4) be nondispersive for all time, (5) move at light velocity in straight lines, and (6) carry finite electromagnetic energy.

The first condition is due to the fact that source-free, classical electromagnetics are described by the homogeneous Maxwell's equations. Since the discontinuities and singularities in mathematical solutions represent charges in classical electromagnetics, the continuity and nonsingularity requirements are synonymous with source-free solutions. The three-dimensional and nondispersive conditions which were defined in the above paragraph make these solutions unique in the literature. An important physical quantity in all scientific areas is the energy associated with the phenomena. The amount of electromagnetic energy in the functions is an important classical electromagnetic quantity because it demonstrates the solution's worth when applied to the real world. The finite electromagnetic energy requirement suggests practical applicable wave functions. The paper begins by presenting pulse solutions which satisfy the above first five conditions and then modifies these solutions to satisfy all conditions.

The original mathematical formulations which are presented here depict pulses propagating along the $z$ axis. Since there are two electric field components which are transverse to the $z$ axis and three magnetic field components, they represent a transverse electric mode (TE). The pulse functions decrease in front of, behind, and transverse to the axis of propagation. Since they remain focused for all time as they propagate along the $z$ axis we term them Focus Wave Modes (FWM). The functions are composed of a three-term product; the first parts are three-dimensional pulses moving at light velocity, and the second are sinusoidal plane waves moving at a velocity less than light velocity along the same line of propagation. The third parts are sinusoidal functions in the angular variable around the axis of propagation. When viewed from a moving reference at light velocity, the first functions appear to be frozen in time while the second functions appear to be moving energy from the pulses' front to the rear. Because of the third function's angular nature around the axis of propagation, they do not effect the pulse shapes as the pulse moves through space. It is the three-dimensional, first pulse functions which dictate the asymptotic behavior of the total pulse functions which decrease as the inverse of the distance from the pulse centers. This far-field behavior is similar to that of a stationary dipole radiator in free space.[3] Like the stationary dipole radiator the original FWM mathematics has an infinite electromagnetic energy associated with the fields in the surrounding space. The term original notes in this paper the first solution which was found the by the author as compared to later modification of these solutions.

Since the original FWM formulations satisfy all the items listed above except the finite energy requirement, it is desirable to construct modified solutions which satisfy all six listed items; one modified solution is presented in this paper. This modified solution will be called the three-region extension to the original FWM formulations. The three-region extension consists of mathematically turning the wave off in front and behind by moving at light velocity two planes of discontinuities along and perpendicular to the $z$ axis. The fields are zero before the first plane, the FWM values between and zero beind the last plane. The three-region extensions are the combination of the original FWM pulses and the two planes of discontinuity all moving in unison. Will these composite pulses, once launched, change shapes as they propagate? The only way that these composite pulses could change shapes would be if the two planes of discontinuity distorted as they propagate. Therefore, the planes' stability needs to be demonstrated. The stability of the discontinuity planes is guaranteed by the work of Kline and Kay[4] who have studied the propagation of discontinuities in Maxwell's equations. These three-region extensions represent stable propagating pulses which satify all of the six items above.

This paper is presented in six sections with the first being this Introduction. The Maxwell's equations vector formulation, which must be satisfied, and the FWM mathematical formulations are presented in Sec. II. Section III discusses the mathematical behavior common to all the FWM formulations, such as lack of singularities and discontinuities, asymptotics, and pulse-like behavior. Section IV presents detailed discussions showing how the original FWM formulations satisfy Maxwell's equations in a cylindrical coordinate system along with the three-region extensions to the original FWM formulations. Section V presents the conclusions, while the last section contains the acknowledgments.

## II. MATHEMATICAL FORMULATIONS

In this section the homogeneous Maxwell's equations vector formulation is presented along with a few additional equations needed later. Then the original FWM mathematical expressions are presented in the cylindrical coordinate system.

### A. Defining equations

The homogeneous Maxwell's equations in vector form as outlined in Stratton[5] are shown below:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}, \tag{2}$$

$$\Delta \cdot \mathbf{D} = 0, \tag{3}$$

$$\Delta \cdot \mathbf{B} = 0, \tag{4}$$

where **E**, **D**, **H**, **B** and $t$ are the electric field, electric flux density, magnetic field, magnetic flux density, and time variable, respectively. In free space the electric quantities and magnetic quantities can be related by the equations[5]

$$\mathbf{D} = \epsilon_0 \mathbf{E}, \tag{5}$$

$$\mathbf{B} = \mu_0 \mathbf{H}, \tag{6}$$

where $\epsilon_0$ and $\mu_0$ are the free-space permittivity and permeability, respectively. The permittivity and permeability are related to the light velocity, $c$, by the equation[5]

$$\epsilon_0 \mu_0 = \frac{1}{c^2}. \tag{7}$$

Another important physical concept associated with classical electromagnetics is the pulse's total electrical energy. If the total electromagnetic energy in a single pulse is $\mathscr{E}$ then it can be expressed as[3]

$$\mathscr{E} = \frac{1}{2} \int_V (\epsilon_0 \mathbf{E} \cdot \mathbf{E} + \mu_0 \mathbf{H} \cdot \mathbf{H}) dv, \tag{8}$$

where $v$ is a volume integral over all space.

### B. Formulation

The mathematical expressions of the original FWM formulations will be presented in a cylindrical coordinate system, $[\rho, \phi, (z - ct)]$, which is shown in Fig. 1 along with a rectangular coordinate system $[x, y, (z - ct)]$. The $\rho$ and $\phi$ variables measure the radial distance in the $x$-$y$ plane from the origin to a point in this transverse plane and the angle from the $x$ axis to $\rho$ variable in this plane, respectively. The third space variable $z$, and the time variable, $t$, are placed together with light velocity, $c$, to write the retarded variable, $(z - ct)$. The retarded variable is used here instead of just the $z$ variable because it assists in demonstrating that the following functions remain unaltered as they move along the $z$ axis. Because these solutions represent packets in free space, the variables will have the following ranges: $0 \leqslant \rho \leqslant \infty$, $0 \leqslant \phi < 2\pi$, and $-\infty < (z - ct) < \infty$.

The various vector components of the FWM, TE-mode mathematical formulations can be written as

$$D_\rho(\rho, \phi, z, t) = \Psi_1 + \Psi_1^*, \tag{9}$$

$$D_\phi(\rho, \phi, z, t) = \Psi_2 + \Psi_2^*, \tag{10}$$

$$H_\rho(\rho, \phi, z, t) = \Psi_3 + \Psi_3^*, \tag{11}$$

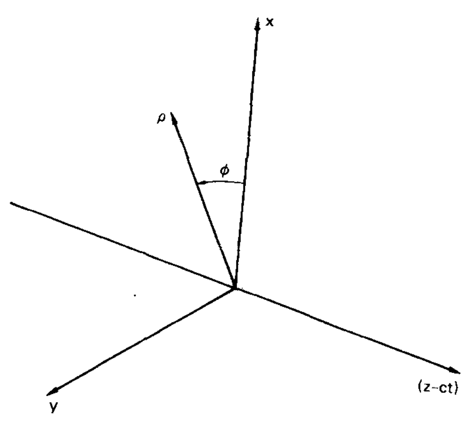


FIG. 1. Retarded coordinate system used to formulate three-dimensional packets.

$$H_\phi(\rho,\phi,z,t) = \Psi_4 + \Psi_4^*, \tag{12}$$

$$H_z(\rho,\phi,z,t) = \Psi_5 + \Psi_5^*, \tag{13}$$

where $\Psi_q$ are complex functions and $\Psi_q^*$ notes the complex conjugate. The functions are written as the sum of complex functions and their conjugates to guarantee that the fields are real. The subscripts on the field terms in Eqs. (9)–(13) denote the various vector components; note no $z$ component of the electric field appears because these are the TE-mode solutions. The parenthesis after the field terms define the various functional dependencies.

The $\Psi_q$ functions are written as

$$\Psi_q = A_q[\rho,(z-ct)]G_1[\rho,(z-ct)]G_2[(z-ct)] \times G_3[(z-c_1t)]\Phi'(\phi) \tag{14}$$

when $q = 1$ and 4, and

$$\Psi_q = A_q[\rho,(z-ct)]G_1[\rho,(z-ct)]G_2[(z-ct)] \times G_3[(z-c_1t)]\Phi(\phi), \tag{15}$$

when $q = 2,3,5$. The constant $c_1$ is a velocity which is not equal to light velocity. The brackets and parenthesis above note the various functional dependencies of each function defining the $\Psi_q$ functions. To avoid making the notation too clumsy, the remainder of this paper will neglect the functional dependencies unless needed to stress a point. The $A$ functions are

$$A_1 = \frac{D^{\mathrm{TE}}}{c^2}\left[\frac{(n+1)cg\rho^{n-1}}{F^{n+2}} - \frac{cg\rho^{n+1}}{4F^{n+3}} + \frac{(k_1c - k_2c_1)\rho^{n-1}}{F^{n+1}}\right], \tag{16}$$

$$A_2 = -\frac{D^{\mathrm{TE}}}{c^2}\left[\frac{n(n+1)cg\rho^{n-1}}{F^{n+2}} - \frac{(3n+4)cg\rho^{n+1}}{4F^{n+3}} + \frac{n(k_1c - k_2c_1)\rho^{n-1}}{F^{n+1}} + \frac{2cg\rho^{n+3}}{16F^{n+4}} - \frac{2(k_1c - k_2c_1)\rho^{n+1}}{4F^{n+2}}\right], \tag{17}$$

$$A_3 = -D^{\mathrm{TE}}\left[-\frac{n(n+1)g\rho^{n-1}}{F^{n+2}} + \frac{(3n+4)g\rho^{n+1}}{4F^{n+3}} + \frac{n(-k_1 + k_2)\rho^{n-1}}{F^{n+1}} - \frac{2g\rho^{n+3}}{16F^{n+4}} - \frac{2(-k_1 + k_2)\rho^{n+1}}{4F^{n+2}}\right], \tag{18}$$

$$A_4 = -D^{\mathrm{TE}}\left[-\frac{(n+1)g\rho^{n-1}}{F^{n+2}} + \frac{g\rho^{n+1}}{4F^{n+3}} + \frac{(-k_1 + k_2)\rho^{n-1}}{F^{n+1}}\right], \tag{19}$$

$$A_5 = -jD^{\mathrm{TE}}\left[\frac{\rho^{n+2}}{4F^{n+3}} - \frac{(n+1)\rho^n}{F^{n+2}}\right], \tag{20}$$

where $D^{\mathrm{TE}}$ is a real constant and

$$F = [jg(z-ct) + \xi]. \tag{21}$$

The term $j$, used above to define $A$ and $F$, is the square root of $-1$ while $g$ and $\xi$ are positive real constants.

The three $G$ functions used in $\Psi_q$ are

$$G_1 = \exp -\left(\frac{\rho^2}{4F}\right), \tag{22}$$

$$G_2 = \exp[-jk_1(z-ct)], \tag{23}$$

$$G_3 = \exp[jk_2(z-c_1t)], \tag{24}$$

where $k_1$ and $k_2$ are two different real propagation constants. The $\Phi$ functions in Eqs. (14) and (15) are

$$\Phi(\phi) = \begin{Bmatrix}\sin(n\phi)\\ \cos(n\phi)\end{Bmatrix}, \tag{25}$$

$$\Phi'(\phi) = \begin{Bmatrix}n\cos(n\phi)\\ -n\sin(n\phi)\end{Bmatrix}, \tag{26}$$

where the brace terms are here to depict two isolated cases in one single notation. When the first sinusoidal term in Eq. (25) is used with the first term in Eq. (26) a solution to the Maxwell's equations is found; the same is true for the second pair of terms. The $\Phi$ functions are the only case in this paper where the brace notation varies from the norm. The $n$ terms used in defining the $A_q$ functions come from the $\Phi$ functions. Since the $\Phi$ functions contain sinusoids in free space then $n$ must be a positive integer.

There is a set of two supplemental conditions which must be satisfied by the constants that help define the $\Psi_q$ functions. They are

$$2gk_2d_1 = 1, \tag{27}$$

$$k_2^2d_2 = \frac{k_1}{g}, \tag{28}$$

where

$$d_1 = \left(1 - \frac{c_1}{c}\right), \tag{29}$$

$$d_2 = \left(1 - \frac{c_1^2}{c^2}\right). \tag{30}$$

Note there are four constants $g$, $k_1$, $k_2$, and $c_1$ used in defining the $\Psi_q$ functions; these constants must satisfy the two supplemental conditions (27) and (28). By assuming that $g$, $k_1$ and $k_2$ are all positive, then from the two supplemental equations it is obvious that $0 < c_1 < c$. Therefore, $g$, $k_1$, $k_2$, and $c_1$ can be any positive real number as long as they satisfy Eqs. (27) and (28) and $c_1$ is less than $c$. The fact that there are four constants and only two supplemental equations allows a large range of constants which can be used. The multiplication constant, $D^{\mathrm{TE}}$, can by nonzero, real value. The $n$ values must be 0,1,2,...; these are dictated by the fact that in the physical world our functions must be single valued in the angle $\phi$. The $\xi$ will be any positive real constant.

## III. DISCUSSION OF THE ORIGINAL FWM SOLUTION STRUCTURE

Since the original FWM mathematical formulation presented in the last section might represent electromagnetic problems, certain characteristics common to each packet must be discussed. This section discusses the lack of singularities and discontinuities, asymptotic beahvior, and the packet-like shape of the solutions.

### A. Singularities and discontinuities

Since the $\Psi_q$ functions are to represent source-free solutions, they must be shown to contain no singularities and to be continuous. These properties of $\Psi$ will be discussed next. Looking back at Eqs. (9)–(26) it is observed that each $\Psi_q$ function is a sum of several terms each having the form

$$\frac{\rho^p}{F^{q'}} \exp - \left(\frac{\rho^2}{4F}\right) \exp[-jk_1(z-ct)] \exp[jk_2(z-c_1t)] \Phi_q(\phi), \tag{31}$$

where $p$ and $q'$ are positive integers and $\Phi_q$ are the appropriate $\Phi$ functions of Eqs. (25) and (26), which relates to Eqs. (14) and (15). It is this general function which must be demonstrated to contain no singularities and to be continuous. Since the complex exponential containing $k_1$ and $k_2$ give rise to sinusoidal terms which do not affect the function's singularities and continuities, the part of Eq. (31) that needs investigation is

$$\frac{\rho^p}{F^{q'}} \exp - \left(\frac{\rho^2}{4F}\right) \Phi_q(\phi). \tag{32}$$

The singular behavior of Eq. (32) will be studied first. The $1/F$ term is the one function in this expression which might cause singularities. If $F$ had a zero then $1/F$ would have a singularity; but since $g$, $\xi$, and $(z-ct)$ are real there is no singularity. Therefore, the $\Psi_q$ functions are nonsingular.

The complete continuity of Eq. (32) can be demonstrated by studying the continuity of each isolated component. The $\Phi_q$ functions which are sines and cosines of $\phi$ might give rise to discontinuities in the packet if the wrong function parameters are chosen. To demonstrate this, Fig. 2 shows a plane transverse to the axis of propagation. Note that

$$\sin(n\phi) = -\sin[n(\phi+\theta)], \tag{33}$$

if $n\theta = \pi$. From Eq. (33) and Fig. 2 it is obvious that for the same $\rho$ value there are more than one function values in Eq. (32). Therefore, if $\rho$ is allowed to approach zero from several different fixed $\phi$ values, the function in Eq. (32) could have different values at $\rho$ equal to zero. To avoid this problem the $\rho$-varying function in Eq. (32) must be zero at $\rho = 0$ which can be accomplished by making sure that $p > 0$. Since the lowest $p$ in Eq. (31) is $n-1$ it is necessary to make $n > 1$.

The functions shown in Eq. (32) where $n > 1$ have just been shown to be continuous near $\rho = 0$ because the functions are equal to zero here. There is another case, $n = 1$, which can be shown to be continuous at $\rho = 0$ by a different method. If one transforms the $\rho$- and $\phi$-vector components for the magnetic and electric fields into $x$- and $y$-vector components then one can show that when $n = 1$ these fields are continuous at the origin. The reason why this transform method works for $n = 1$ is because the field functions $\phi$ variation around the $z$ axis matches the transform $\phi$ variation. For the other cases, these two functional variations do not match; therefore, these fields functions would be discontinuous at $\rho = 0$ if not for the function's at this point. The unique nature of these fields functions near the origin make it necessary to use two separate proofs to demonstrate the continuity of all the useful cases here. The case where $n = 0$ will work because the coefficients multiplying the term $(n-1)$ are zero causing the $\rho$ function to have no singularities at $\rho = 0$. Therefore, the cases where the functions shown in Eq. (36) will be continuous at $\rho = 0$ are denoted by

$$n = 0,1,2,3\ldots. \tag{34}$$

This row leaves the following equation:

$$\frac{\rho^p}{F^{q'}} \exp - \left(\frac{\rho^2}{4F}\right) \tag{35}$$

from Eq. (32) that must be shown to be continuous in the $\rho-(z-ct)$ plane which is presented in Fig. 3. It is a trivial exercise to show that the $(1/F)^q$ function is continuous in $(z-ct)$; therefore, the last function which needs its continuity demonstrated is

$$\frac{\rho^p}{F^{q'}} \exp - \left\{\frac{\rho^2[\xi - jg(z-ct)]}{4[g^2(z-ct)^2+\xi^2]}\right\}. \tag{36}$$

The change in the exponential between Eqs. (35) and (36) is formed by multiplying and dividing it by $F^*$. The complex exponential above can be separated into two parts; when the real part of the exponential in Eq. (36) is retained, it gives

$$\frac{\rho^p}{F^{q'}} \exp - \left\{\frac{\rho^2 \xi}{4[g^2(z-ct)^2+\xi^2]}\right\} \times \cos\left\{\frac{g(z-ct)\rho^2}{4[g^2(z-ct)^2+\xi^2]}\right\}. \tag{37}$$

Since each function in Eq. (37) is continuous at each fixed

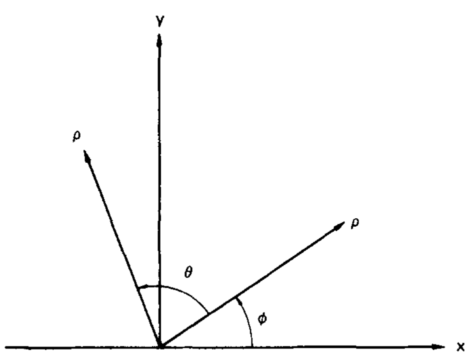

FIG. 2. $x$-$y$ plane demonstrating how a discontinuity might occur at $\rho = 0$.

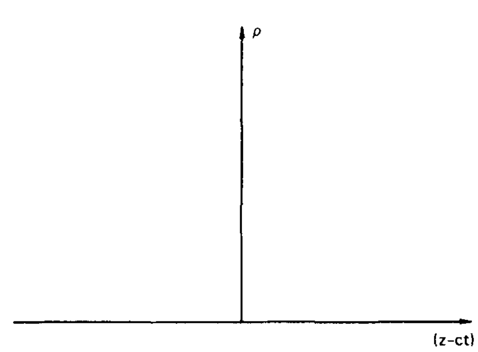

FIG. 3. $\rho-(z-ct)$ plane.

point in the $\rho - (z - ct)$ plane, then the product of these functions is continuous at each point. But, there are two reasons why the above equation is a well-behaved continous function in the entire $\rho - (z - ct)$ plane. The first is the exponential being negative for all values of $\rho$ and $(z - ct)$; the second is that $q'$ and $p$ are positive integers where $q'$ is greater than $p$. The first condition guarantees that the exponential makes the $\rho^p$ and cosine approach zero as $\rho$ approaches infinity for fixed $(z - ct)$. The second condition guarantees that the complex product approaches zero as both $\rho$ and $(z - ct)$ approach infinity and that the same function approaches zero as $\rho$ is fixed and $(z - ct)$ approaches infinity. The imaginary part of Eq. (36) is identical to Eq. (37) except it has a sine term instead of cosine function; therefore, its continuity can be demonstrated by a similar method. This proves that the $\Psi_q$ functions are continuous.

The $\Psi_q^*$ functions can be shown to be continuous and free of singularities by an identical process as used for $\Psi_q$.

## B. Asymptotics

Thus far, the FWM field functions presented above have been shown to have no singularities and to be continuous. To demonstrate that they are three-dimensional, packet-like solutions moving along the $z$ axis, the asymptotic nature of the functions must be investigated. Since the field functions in Eqs. (9)–(12) are composed of $\Psi_q$ and $\Psi_q^*$ these functions' asymptotic behavior needs to be discussed.

From Eqs. (9)–(26) it was observed that

$$\Psi_q = I_q[\rho,(z - ct)]\exp[jk_2(z - c_1 t))]\Phi_q, \qquad (38)$$

where

$$I_q = A_q[\rho,(z - ct)]G_1[\rho,(z - ct)]G_2[(z - ct)]. \qquad (39)$$

Looking at Eq. (38) it is obvious that $\Psi_q$ is the product of three functions. The first function is $I_q$ which varies with respect to $\rho$ and $(z - ct)$ and represents a fixed pulse shape that moves along the $z$ axis at light velocity. Since no other function in $\Psi_q$ contains the $\rho$ variation, it will be the $I_q$ function which contains the asymptotic behavior for large $\rho$. The second function in $\Psi_q$ is complex exponential that moves along the $z$ axis at a velocity less than light velocity. Since the complex exponential are actually sinusoids, these functions represent sine and cosine plane waves that will be superimposed on $I_q$ and moving at a velocity less than $I_q$'s velocity in the $z$ direction. The second function's sinusoidal nature causes this function to vary from $-1$ to 1; therefore, it does not affect the asymptotic behavior of $\Psi_q$. The third function in Eq. (38) is the $\Phi$ function which describes the polar angle variation in a plane transverse to the axis of motion; therefore, it also does not influence the asymptotic behavior of $\Psi_q$.

Since the $I_q$ functions govern the asymptotic behavior of the $\Psi_q$ functions, they will be studied here. From Eq. (39) it is observed that the last complex exponential function in the equations gives sinusoidal variations which move with the other functions at light velocity. Since these sinusoidal functions vary from $-1$ to $+1$, it will be the $A_q$ and $G_1$ functions which will control the asymptotic behavior of the $\Psi_q$ functions. Note from Eqs. (9)–(26) that the product of $A_q$ times $G_1$ gives multiple term sums. One of the most slowly varying asymptotic functions occurs when the power of $\rho$ is only one integer less than the power of $F$. An example of one such term from the $A_q$ times $G_1$ product is

$$\frac{a_1\rho^{n+1}}{F^{n+2}}\exp\left\{\frac{-\xi\rho^2}{4[g^2(z - ct)^2 + \xi^2]}\right\} \times \exp\left\{\frac{jg(z - ct)\rho^2}{4[g^2(z - ct)^2 + \xi^2]}\right\}, \qquad (40)$$

where $a_1$ is a real constant. In the above equation, the exponential of $G_1$ has been written as in Eq. (36). Since all the terms in the above real exponential are positive, its exponent will have a negative value for all $\rho$ and $(z - ct)$ values due to the negative sign. This negative character of the real exponential will guarantee that Eq. (40) decreases exponentially for fixed $(z - ct)$ and large $\rho$ values. For large $(z - ct)$ and fixed $\rho$ the $1/F^{n+2}$ functions will account for the decreasing behavior in front and behind the pulse. The slowest functional decrease in Eq. (40) occurs when both $\rho$ and $(z - ct)$ are allowed to simultaneously approach infinity. Along such a radial line from the pulses' center in the $\rho - (z - ct)$ plane it is observed that Eq. (40) decreases as

$$\frac{1}{R}, \qquad (41)$$

where $R = [\rho^2 + (z - ct)^2]^{1/2}$. The asymptotic behavior for the other terms in Eq. (39) can be found by an identical procedure as used above.

The $\Psi_q^*$ functions can be shown to have the same asymptotic behavior as $\Psi_q$ by using an identical procedure.

## C. Packet-like solutions

As has been stated, the original FWM mathematical formulations present the first three-dimensional, nondispersive pulse functions which satisfy Maxwell's homogeneous equations in free space. The general pulse shapes are discussed below. The pulse shapes can be best observed by looking at Eqs. (38) and (39). By following a similar argument given after these equations, one notes that the solutions are the product of three sets of functions; the first being three-dimensional functions, $I_q$, which move at light velocity along the $z$ axis. The second functions are sinusoidal plane waves moving at a velocity less than light velocity along the $z$ axis. The third functions are the angular variations around the axis of propagation. The first two functions truly govern the pulse functions shapes as they move in free space. The $I_q$ functions are the three-dimensional functions which describe the far-field behavior, while the second functions' sinusoidal plane wave terms propagate through the $I_q$ functions. Since the sinusoidal plane wave functions vary from $-1$ to 1, the $I_q$ function's asymptotic behavior is not affected by the plane waves propagating through them. If the fields are viewed in the stationary $(x,y,z)$ coordinates, then the $I_q$ functions are moving at light velocity along the $z$ axis while the other sinusoidal plane waves are moving energy from the pulses' front to the rear.

The original FWM mathematical formulations presented in this paper are three-dimensional, nondispersive packet-like solutions. The three-dimensional term means that as the solutions move through space at light velocity, the functions' values decrease in front, behind, and transversely to the pulse centers. Since the $I_q$ functions govern the pulses' asymptotic behavior then Eq. (41) guarantees that the pulses are three-dimensional. The nondispersive term means that the pulses do not spread as they propagate through free space. The $I_q$ functions are also the reason why the pulses do not spread. As has been discussed in the preceding paragraph, the $I_q$ functions represent fixed-shaped pulses moving along the $z$ axis at light velocity. Since the $I_q$ functions do not change with time then the pulse's functions will not spread. Since these are the first nondispersive, electromagnetic pulses appearing in the literature, it is natural to ask how much classical electromagnetic energy is associated with each pulse. This is the integral given in Eq. (8). Because of the asymptotic variation given in Eq. (41) the energy integral will be infinite. Even though these are the first pulse solutions to the homogeneous Maxwell's equations which have a peak value near the pulses' centers and decreases in all directions, the fields decrease too slowly to have a finite energy integral. A finite electromagnetic energy would require an asymptotic variation as $1/R^2$. The asymptotic values of the moving pulses, given in Eq. (41), have the same magnitude as the asymptotic values for a stationary dipole in free space.[3]

## IV. PROOFS

In this section, the original FWM formulations presented in Sec. II B will be shown to satisfy the homogeneous Maxwell's equations. Then the three-region extensions to the original FWM formulations will be presented and discussed.

### A. Maxwell's equations satisfied

By using the table in Plonsey and Collin[3] on the vector form of the homogeneous Maxwell's equations presented in Eqs. (1)–(4), it is observed that the TE modes in a cylindrical coordinate system must satisfy the following:

$$\frac{\partial E_\phi}{\partial z} = \frac{\partial B_\rho}{\partial t}, \tag{42}$$

$$\frac{\partial E_\rho}{\partial z} = -\frac{\partial B_\phi}{\partial t}, \tag{43}$$

$$\frac{1}{\rho}\frac{\partial(\rho E_\phi)}{\partial \rho} - \frac{1}{\rho}\frac{\partial E_\rho}{\partial \phi} = -\frac{\partial B_z}{\partial t}, \tag{44}$$

$$\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z} = \frac{\partial D_\rho}{\partial t}, \tag{45}$$

$$\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} = \frac{\partial D_\phi}{\partial t}, \tag{46}$$

$$\frac{1}{\rho}\frac{\partial(\rho H_\phi)}{\partial \rho} - \frac{1}{\rho}\frac{\partial H_\rho}{\partial \phi} = 0, \tag{47}$$

$$\frac{1}{\rho}\frac{\partial(\rho E_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial E_\phi}{\partial \phi} = 0, \tag{48}$$

$$\frac{1}{\rho}\frac{\partial(\rho H_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial H_\phi}{\partial \phi} + \frac{\partial H_z}{\partial z} = 0. \tag{49}$$

#### 1. Curl electric field

The curl of the electric fields equation requires that Eqs. (42)–(44) be satisfied. Before this is done, an additional notation needs presenting. Since each field is composed of the sum of a complex function and its conjugate, the superscripts 1 and 2 will now be used to note the complex function and its conjugate, respectively. When the appropriate $\Psi_q$ functions defined in Eqs. (9)–(26) are placed in Eqs. (42)–(44), the derivatives of the fields yield

$$\begin{aligned}\frac{\partial E_\phi^1}{\partial z} = -\frac{jD^{\mathrm{TE}}}{\epsilon_0 c^2}\Bigg[&-\frac{n(n^2+3n+2)cg^2\rho^{n-1}}{F^{n+3}} + \frac{(4n^2+14n+12)cg^2\rho^{n+1}}{4F^{n+4}} \\ &+ \frac{n(n+1)(-2k_1c+k_2c+k_2c_1)g\rho^{n-1}}{F^{n+2}} - \frac{(5n+12)cg^2\rho^{n+3}}{16F^{n+5}} \\ &+ \frac{(3n+4)(2k_1c-k_2c-k_2c_1)g\rho^{n+1}}{4F^{n+3}} + \frac{2cg^2\rho^{n+5}}{64F^{n+6}} \\ &+ \frac{2(-2k_1c+k_2c+k_2c_1)g\rho^{n+3}}{16F^{n+4}} + \frac{n(k_1c-k_2c_1)(-k_1+k_2)\rho^{n-1}}{F^{n+1}} \\ &- \frac{2(k_1c-k_2c_1)(-k_1+k_2)\rho^{n+1}}{4F^{n+2}}\Bigg] G_1G_2G_3\Phi,\end{aligned} \tag{50}$$

$$\begin{aligned}\frac{\partial B_\rho^1}{\partial t} = -j\mu_0 D^{\mathrm{TE}}\Bigg[&-\frac{n(n^2+3n+2)cg^2\rho^{n-1}}{F^{n+3}} + \frac{(4n^2+14n+12)cg^2\rho^{n+1}}{4F^{n+4}} \\ &+ \frac{n(n+1)(-2k_1c+k_2c+k_2c_1)g\rho^{n-1}}{F^{n+2}} - \frac{(5n+12)cg^2\rho^{n+3}}{16F^{n+5}} \\ &+ \frac{(3n+4)(2k_1c-k_2c-k_2c_1)g\rho^{n+1}}{4F^{n+3}} + \frac{2cg^2\rho^{n+5}}{64F^{n+6}} \\ &+ \frac{2(-2k_1c+k_2c+k_2c_1)g\rho^{n+3}}{16F^{n+4}} + \frac{n(k_1c-k_2c_1)(-k_1+k_2)\rho^{n-1}}{F^{n+1}} \\ &- \frac{2(k_1c-k_2c_1)(-k_1+k_2)\rho^{n+1}}{4F^{n+2}}\Bigg] G_1G_2G_3\Phi,\end{aligned} \tag{51}$$

$$\frac{\partial E_\rho^1}{\partial z} = \frac{jD^{\mathrm{TE}}}{\epsilon_0 c^2}\Big[ -\frac{(n^2+3n+2)cg^2\rho^{n-1}}{F^{n+3}} + \frac{(2n+4)cg^2\rho^{n+1}}{4F^{n+4}} - \frac{(n+1)(2k_1c - k_2c_1 - k_2c)g\rho^{n-1}}{F^{n+2}} - \frac{cg^2\rho^{n+3}}{16F^{n+5}} + \frac{(2k_1c - k_2c_1 - k_2c)g\rho^{n+1}}{4F^{n+3}} + \frac{(k_1c - k_2c_1)(-k_1+k_2)\rho^{n-1}}{F^{n+1}}\Big] \times G_1G_2G_3\Phi', \tag{52}$$

$$\frac{\partial B_\phi^1}{\partial t} = -j\mu_0 D^{\mathrm{TE}}\Big[ -\frac{(n^2+3n+2)cg^2\rho^{n-1}}{F^{n+3}} + \frac{(2n+4)cg^2\rho^{n+1}}{4F^{n+4}} - \frac{(n+1)(2k_1c - k_2c_1 - k_2c)g\rho^{n-1}}{F^{n+2}} - \frac{cg^2\rho^{n+3}}{16F^{n+5}} + \frac{(2k_1c - k_2c_1 - k_2c)g\rho^{n+1}}{4F^{n+3}} + \frac{(k_1c - k_2c_1)(-k_1+k_2)\rho^{n-1}}{F^{n+1}}\Big] \times G_1G_2G_3\Phi', \tag{53}$$

$$\frac{1}{\rho}\frac{\partial(\rho E_\phi^1)}{\partial\rho} - \frac{1}{\rho}\frac{\partial E_\rho^1}{\partial\phi} = -\frac{D^{\mathrm{TE}}}{\epsilon_0 c^2}\Big[ -\frac{(n^2+3n+2)cg\rho^n}{F^{n+3}} + \frac{(n+2)cg\rho^{n+2}}{2F^{n+4}} - \frac{(n+1)(k_1c - k_2c_1)\rho^n}{F^{n+2}} - \frac{cg\rho^{n+4}}{16F^{n+5}} + \frac{(k_1c - k_2c_1)\rho^{n+2}}{4F^{n+3}}\Big] G_1G_2G_3\Phi, \tag{54}$$

$$\frac{\partial B_z^1}{\partial t} = \mu_0 D^{\mathrm{TE}}\Big[ \frac{(2n+4)cg\rho^{n+2}}{4F^{n+4}} - \frac{(n^2+3n+2)cg\rho^n}{F^{n+3}} - \frac{cg\rho^{n+4}}{16F^{n+5}} + \frac{(k_1c - k_2c_1)\rho^{n+2}}{4F^{n+3}} - \frac{(n+1)(k_1c - k_2c_1)\rho^n}{F^{n+2}}\Big] G_1G_2G_3\Phi. \tag{55}$$

As noted above, the superscript 1 accounts for the $\Psi_q$ functions. By comparing alternate expressions in Eqs. (50)–(55) it can be shown that the first half of the field functions satisfy Eqs. (42)–(44). When the conjugation of the $\Psi_q$ functions defined in Eqs. (9)–(26) are placed in Eqs. (42)–(44), a set of equations similar to Eqs. (50)–(55) are obtained except it is the conjugate of these equations. When this conjugate set of equations are compared it is observed that the $\Psi_q^*$ functions also satisfy Eqs. (42)–(44). Since both $\Psi_q$ functions and $\Psi_q^*$ functions portions of the fields satisfy Eqs. (42)–(44), then their sum also satisfies the same equations because the Maxwell's equations are linear equations.

### 2. Curl magnetic field

Next the formulation presented in Sec. II B will be shown to satify the curl of magnetic fields equation presented in Eqs. (45)–(47). Before giving this proof it is convenient to present two modified forms of the two supplemental conditions given in Eqs. (27) and (28). The first equation is given below:

$$k_1 - k_2 + \frac{1}{2g} = k_1 - \frac{c_1}{c}k_2. \tag{56}$$

The above equation can be shown to be true by rearranging Eq. (27) then adding $k_1$ to both sides.

The second expression can be derived by a two-line proof. The first equation is

$$k_2^2\left(1 - \frac{c_1^2}{c^2}\right) = k_1\left[2k_2\left(1 - \frac{c_1}{c}\right)\right]; \tag{57}$$

it came from using Eq. (27) to eliminate the $g$ term from Eq. (28). Next, when Eq. (57) is rearranged, the following is obtained:

$$-k_1 + k_2 = k_1 - \left(\frac{c_1}{c}\right)k_2. \tag{58}$$

This now gives two altered identities of the two supplemental conditions which are needed in showing that the curl of the magnetic fields is satisfied.

When the appropriate derivatives of the fields shown in Eq. (45)–(47) are performed on the $\Psi_q$ functions defined in Eqs. (9)–(26), the following expressions are found:

$$\frac{1}{\rho}\frac{\partial H_z^1}{\partial\phi}-\frac{\partial H_\phi^1}{\partial z}=jD^{\rm TE}\Bigg[\frac{(n^2+3n+2)g^2\rho^{n-1}}{F^{n+3}}-\frac{(n+2)g^2\rho^{n+1}}{2F^{n+4}}+\frac{2(n+1)\left(k_1-k_2+\frac{1}{2g}\right)g\rho^{n-1}}{F^{n+2}}+\frac{g^2\rho^{n+3}}{16F^{n+5}}-\frac{2\left(k_1-k_2+\frac{1}{2g}\right)g\rho^{n+1}}{4F^{n+3}}+\frac{(-k_1+k_2)^2\rho^{n-1}}{F^{n+1}}\Bigg]\times G_1G_2G_3\Phi', \tag{59}$$

$$\frac{\partial D_\rho^1}{\partial t}=\frac{jD^{\rm TE}}{c^2}\Bigg[\frac{(n^2+3n+2)c^2g^2\rho^{n-1}}{F^{n+3}}-\frac{(n+2)c^2g^2\rho^{n+1}}{2F^{n+4}}+\frac{2(n+1)(k_1c-k_2c_1)cg\rho^{n-1}}{F^{n+2}}+\frac{c^2g^2\rho^{n+3}}{16F^{n+5}}-\frac{2(k_1c-k_2c_1)cg\rho^{n+1}}{4F^{n+3}}+\frac{(k_1c-k_2c_1)^2\rho^{n-1}}{F^{n+1}}\Bigg]G_1G_2G_3\Phi', \tag{60}$$

$$\frac{\partial H_\rho^1}{\partial z}-\frac{\partial H_z^1}{\partial\rho}=-jD^{\rm TE}\Bigg[\frac{n(n^2+3n+2)g^2\rho^{n-1}}{F^{n+3}}-\frac{(4n^2+14n+12)g^2\rho^{n+1}}{4F^{n+4}}+\frac{2n(n+1)\left(k_1-k_2+\frac{1}{2g}\right)g\rho^{n-1}}{F^{n+2}}+\frac{(5n+12)g^2\rho^{n+3}}{16F^{n+5}}-\frac{2(3n+4)\left(k_1-k_2+\frac{1}{2g}\right)g\rho^{n+1}}{4F^{n+3}}-\frac{2g^2\rho^{n+5}}{64F^{n+6}}+\frac{4\left(k_1-k_2+\frac{1}{2g}\right)g\rho^{n+3}}{16F^{n+4}}+\frac{n(-k_1+k_2)^2\rho^{n-1}}{F^{n+1}}-\frac{2(-k_1+k_2)^2\rho^{n+1}}{4F^{n+2}}\Bigg]G_1G_2G_3\Phi, \tag{61}$$

$$\frac{\partial D_\phi^1}{\partial t}=-\frac{jD^{\rm TE}}{c^2}\Bigg[\frac{n(n^2+3n+2)c^2g^2\rho^{n-1}}{F^{n+3}}-\frac{(4n^2+14n+12)c^2g^2\rho^{n+1}}{4F^{n+4}}+\frac{2n(n+1)(k_1c-k_2c_1)cg\rho^{n-1}}{F^{n+2}}+\frac{(5n+12)c^2g^2\rho^{n+3}}{16F^{n+5}}-\frac{2(3n+4)(k_1c-k_2c_1)cg\rho^{n+1}}{4F^{n+3}}-\frac{2c^2g^2\rho^{n+5}}{64F^{n+6}}+\frac{4(k_1c-k_2c_1)cg\rho^{n+3}}{16F^{n+4}}+\frac{n(k_1c-k_2c_1)^2\rho^{n-1}}{F^{n+1}}-\frac{2(k_1c-k_2c_1)^2\rho^{n+1}}{4F^{n+2}}\Bigg]G_1G_2G_3\Phi, \tag{62}$$

$$\frac{1}{\rho}\frac{\partial(\rho H_\phi^1)}{\partial\rho}=-D^{\rm TE}\Bigg[-\frac{n(n+1)g\rho^{n-2}}{F^{n+2}}+\frac{(3n+4)g\rho^n}{4F^{n+3}}+\frac{n(-k_1+k_2)\rho^{n-2}}{F^{n+1}}-\frac{2g\rho^{n+2}}{16F^{n+4}}-\frac{2(-k_1+k_2)\rho^n}{4F^{n+2}}\Bigg]\times G_1G_2G_3\Phi', \tag{63}$$

$$\frac{1}{\rho}\frac{\partial H_\rho^1}{\partial\phi}=-D^{\rm TE}\Bigg[-\frac{n(n+1)g\rho^{n-2}}{F^{n+2}}+\frac{(3n+4)g\rho^n}{4F^{n+3}}+\frac{n(-k_1+k_2)\rho^{n-2}}{F^{n+1}}-\frac{2g\rho^{n+2}}{16F^{n+4}}-\frac{2(-k_1+k_2)\rho^n}{4F^{n+2}}\Bigg]\times G_1G_2G_3\Phi'. \tag{64}$$

With the help of the identities given in Eqs. (56) and (58), by comparing alternate equations in the above set it can be shown that $\Psi_q$ portions of the fields satisfy Eqs. (45)–(47). When the $\Psi_q^*$ portion of the fields are placed into the curl of the magnetic fields equations, a conjugate set of Eqs. (59)–(64) is found, which in turn shows that the $\Psi_q^*$ portion of the fields satisfy Eqs. (45)–(47). Since both portions of the fields satisfy Eqs. (45)–(47), then their sum also satisfies the same equations because the Maxwell's equations are linear.

### 3. Divergence electric field

To show that the divergence of the electric fields is satisfied, Eq. (48) must be shown to be true. When the $\Psi_q$ functions of Eqs. (9)–(26) are placed into the derivatives of Eq. (48), it is found

$$\frac{1}{\rho}\frac{\partial(\rho E^1_\rho)}{\partial\rho} = \frac{D^{\rm TE}}{\epsilon_0 c^2}\Big[\frac{n(n+1)cg\rho^{n-2}}{F^{n+2}} - \frac{(3n+4)cg\rho^n}{4F^{n+3}} + \frac{n(k_1c-k_2c_1)\rho^{n-2}}{F^{n+1}} + \frac{2cg\rho^{n+2}}{16F^{n+4}} - \frac{2(k_1c-k_2c_1)\rho^n}{4F^{n+2}}\Big] G_1G_2G_3\Phi', \quad (65)$$

$$\frac{1}{\rho}\frac{\partial E^1_\phi}{\partial\phi} = -\frac{D^{\rm TE}}{\epsilon_0 c^2}\Big[\frac{n(n+1)cg\rho^{n-2}}{F^{n+2}} - \frac{(3n+4)cg\rho^n}{4F^{n+3}} + \frac{n(k_1c-k_2c_1)\rho^{n-2}}{F^{n+1}} + \frac{2cg\rho^{n+2}}{16F^{n+4}} - \frac{2(k_1c-k_2c_1)\rho^n}{4F^{n+2}}\Big] G_1G_2G_3\Phi'. \quad (66)$$

By comparing Eqs. (65) and (66) it can be shown that the $\Psi_q$ portions of the electric fields satisfy Eq. (48). The $\Psi_q^*$ portions of electric fields given a set of equations which are Eqs. (65) and (66) conjugated. From this set it can be shown that $\Psi_q^*$ portions of electric fields satisfy Eq. (48). Since the divergence is a linear operator, the sum of the $\Psi_q$ portions and $\Psi_q^*$ portions will also satisfy Eq. (48).

### 4. Divergence magnetic field

To show that the magnetic fields satisfy the divergence equation then Eq. (49) must be shown to be true. With the appropriate derivatives of the $\Psi_q$ functions in Eqs. (9)–(26), the following are found:

$$\frac{1}{\rho}\frac{\rho H^1_\rho}{\partial\rho} = -D^{\rm TE}\Big[-\frac{n^2(n+1)g\rho^{n-2}}{F^{n+2}} + \frac{(5n^2+12n+8)g\rho^n}{4F^{n+3}} + \frac{n^2(-k_1+k_2)\rho^{n-2}}{F^{n+1}} - \frac{(n+2)g\rho^{n+2}}{2F^{n+4}} - \frac{(n+1)(-k_1+k_2)\rho^n}{F^{n+2}} + \frac{4g\rho^{n+4}}{64F^{n+5}} + \frac{4(-k_1+k_2)\rho^{n+2}}{16F^{n+3}}\Big] G_1G_2G_3\Phi, \quad (67)$$

$$\frac{1}{\rho}\frac{\partial H^1_\phi}{\partial\phi} = n^2D^{\rm TE}\Big[-\frac{(n+1)g\rho^{n-2}}{F^{n+2}} + \frac{g\rho^n}{4F^{n+3}} + \frac{(-k_1+k_2)\rho^{n-2}}{F^{n+1}}\Big] G_1G_2G_3\Phi, \quad (68)$$

$$\frac{\partial H^1_z}{\partial z} = D^{\rm TE}\Big[-\frac{(2n+4)g\rho^{n+2}}{4F^{n+4}} + \frac{(n^2+3n+2)g\rho^n}{F^{n+3}} + \frac{g\rho^{n+4}}{16F^{n+5}} + \frac{(-k_1+k_2)\rho^{n+2}}{4F^{n+3}} - \frac{(n+1)(-k_1+k_2)\rho^n}{F^{n+2}}\Big] G_1G_2G_3\Phi. \quad (69)$$

When these above three equations are added together, then the $\Psi_q$ portions of the magnetic fields can be shown to satisfy Eq. (49). The $\Psi_q^*$ portions of the magnetic fields gives a conjugated set of Eqs. (67)–(69); therefore, this portion also satisfies Eq. (49). Since the divergence operator is a linear operator, the sum of the $\Psi_q$ portions and $\Psi_q^*$ portions will also satisfy Eq. (49).

## B. Three-region extensions to original FWM formulations

Since the original FWM energy integrals are infinite, a method is needed to formulate the solutions into fixed-shape pulses with finite energy integrals. One method to do this is by writing the original FWM formulations as a three-region problem. This is accomplished by allowing two infinitely extended surfaces of discontinuities to propagate at light velocity along the $z$ axis. The two surfaces of discontinuities are perpendicular to the $z$ axis. The three-region extensions' fields will be zero in front of the first moving surface, the original FWM formulations between the two moving surfaces, and zero behind the second moving surfaces. Since the surfaces of discontinuities are moving with the same velocity as the original FWM formulations, these three-region extensions will have fixed pulse shapes as they propagate. The three-region extensions represent three-dimensional, nondispersive, finite energy pulses which satisfy Maxwell's homogeneous equations. This section will discuss the three-region extensions' energy integrals along with the stability for the surfaces of discontinuities.

The energy integrals, Eq. (8), for the three-region extensions will now be shown to be finite. These integrations will be demonstrated on the $H_z$ fields; other terms require similar integrations. The integral which must be shown to be finite is

$$\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi}(H_z)^2\rho d\phi d\rho d(z-ct), \quad (70)$$

where the integration is over the space shown in Fig. 1. From Eqs. (13), (15), and (20) the $H_z$ functions can be written as

$$H_z = (K + K^*) \begin{Bmatrix} \sin(n\phi) \\ \cos(n\phi) \end{Bmatrix}, \tag{71}$$

where

$$K = \left( \frac{b_1 \rho^{n+2}}{F^{n+3}} + \frac{b_2 \rho^n}{F^{n+2}} \right) G_1 G_2 G_3, \tag{72}$$

where $b_1$ and $b_2$ are complex constants. The square of $H_z$ gives

$$H_z^2 = (K^2 + 2KK^* + K^{*2}) \begin{Bmatrix} \sin^2(n\phi) \\ \cos^2(n\phi) \end{Bmatrix}. \tag{73}$$

The various combinations of $K$ terms and $K^*$ terms needed in the above equations can be found by squaring Eq. (72) and its conjugate and multiplying it by its conjugate. Since each of the terms in Eq. (73) is shown to have finite energy by similar methods, then only one term needs to be present here. The second term in Eq. (73) is

$$KK^* = \left[ \frac{|b_1|^2 \rho^{2n+4}}{F^{n+3}(F^*)^{n+3}} + \frac{b_1 b_2^* \rho^{2n+2}}{F^{n+3}(F^*)^{n+2}} + \frac{b_1^* b_2 \rho^{2n+2}}{(F^*)^{n+3} F^{n+2}} + \frac{|b_2^2| \rho^{2n}}{F^{n+2}(F^*)^{n+2}} \right] G_1 G_1^*. \tag{74}$$

Looking at the above equation, note that the function is a summation of four terms each having a similar structure. Since the terms in the series have a similarity, only one term of the $KK^*$ term need to be integrated.

For the sake of demonstration, the first term in Eq. (74) will be used to show that it has finite energy. When the first term in Eq. (74) is substituted into the integral in Eq. (70) it gives

$$\int_{-b}^{b} \int_0^\infty \int_0^{2\pi} \frac{|b_1|^2 \rho^{2n+4}}{F^{n+3}(F^*)^{n+3}} \exp - \left[ \frac{\rho^2}{4} \left( \frac{F+F^*}{FF^*} \right) \right] \times \begin{Bmatrix} \sin^2(n\phi) \\ \cos^2(n\phi) \end{Bmatrix} \rho \, d\phi \, d\rho \, d(z-ct). \tag{75}$$

The term, $b$, used in the above limits of integration accounts for the fact that the present study is for the three-region extensions which are set to zero at some point in front and behind the pulse. From Eqs. (187) and (202) in Burington,[6] it is observed that the $\phi$ integrals can be integrated in closed form to give:

$$\pi |b_1|^2 \int_{-b}^{b} \int_0^\infty \frac{\rho^{(2n+4)+1}}{F^{n+3}(F^*)^{n+3}} \exp - \left[ \frac{\rho^2}{4} \left( \frac{F+F^*}{FF^*} \right) \right] \times d\rho \, d(z-ct). \tag{76}$$

In comparing Eqs. (76) to (75), note that brace term, which is part of the $\Phi$-function definition, is no longer needed here because it has been integrated. This is another way of saying that the equality between Eqs. (75) and (76) is the same for both sinusoids in $\Phi$ functions.

To perform the $\rho$ integration, Burington[6] must again be consulted. By using a change of variable on Eq. (380) in Burington[6] it is found

$$\int_0^\infty (\lambda)^{2p'+1} e^{-a\lambda^2} d\lambda = \frac{p'!}{a^{p'+1}}. \tag{77}$$

When Eq. (77) is used in Eq. (76) it is observed that Eq. (76) is equal to

$$\pi |b_1|^2 (4)^{n+3} (n+2)! \int_{-b}^{b} \left( \frac{1}{F+F^*} \right)^{n+3} d(z-ct). \tag{78}$$

When the definition of $F$ in Eq. (21) is used it is observed that $F$ plus its conjugate is a constant; therefore, the above integral is finite. By using a similar method each term in Eq. (73) (73) can be shown to have a finite integral for the three region extensions to the FWM. A similar method can also be used to show that the energy integrals for all fields are finite for the three-region extensions to the FWM formulations.

The surfaces of discontinuities are important to guarantee that the total pulse shapes in the three-region extensions propagates unaltered. The stability for the surfaces of discontinuities in Maxwell's equations used here is assured by the work of Kline and Kay.[4] Their work investigates how hypersurfaces of discontinuities propagate in Maxwell's equations. In this work they found that discontinuities across the hypersurface, defined by[4]

$$L(x,y,z,t) = 0, \tag{79}$$

must satisfy the four resulting differential equations:

$$\nabla L \times \mathbf{H}^d - \frac{L_t}{c} \mathbf{D}^d = 0, \tag{80}$$

$$\nabla L \times \mathbf{E}^d + \frac{L_t}{c} \mathbf{B}^d = 0, \tag{81}$$

$$\nabla L \cdot \left( \frac{1}{4\pi} \mathbf{D}_t^d \right) = 0, \tag{82}$$

$$\nabla L \cdot (\mathbf{B}^d) = 0, \tag{83}$$

on the hypersurface defined in Eq. (79). The above form equations are the source-free representations of Eq. (1.60) in this book.[4] The superscript, $d$, on the various fields components represents their functions' discontinuities across the hypersurface while the subscript, $t$, denotes the time derivative of those particular functions. In order to demonstrate the stability of the three-region extensions to the FWM, one needs a presentation of a moving function which satisfies Eq. (79) along two moving surfaces which are perpendicular to the $z$ axis. This function also must satisfy Eqs. (80)–(83) on the hypersurfaces. For our purpose, the function

$$L = \exp \left[ \frac{-1}{(z-ct)^2 - b^2} \right] \tag{84}$$

will work. Note that it is zero along the two planes $(z-ct) = b$ and $(z-ct) = -b$, and the gradient and time derivatives are zero along the same two planes. Therefore, the function satisfies Eqs. (79)–(83) along these two planes which move at light velocity along the $z$ axis. These unique features of Eq. (84) allow the two surfaces of discontinuities to move in stable unison with the original FWM formulations.

These three-region extensions to the FWM formulations gives three-dimensional, finite energy, nondispersive pulses which propagate at light velocity. The fields are zero in front of the first moving surface, and the original FWM formulations between the two surfaces, and zero behind the second moving surface. The fields are continuous everywhere except along the two moving surfaces of discontinuities. To get the energy integrals finite, these modifications had to accept two surfaces of discontinuities. The severity of these limitations will only be known from future applications of these functions.

## V. CONCLUSIONS

The original FWM formulations presented in Sec. II B have been shown to be real valued, three-dimensional, TE mode pulse solutions to the free-space, homogeneous Maxwell's equations when the parameters $g$, $k_1$, $k_2$, $\xi$, and $c_1$ are positive real constants, $0 < c_1 < c$, and satisfy two supplemental equations. These functions are continuous and free of singularities when $n = 0,1,2,3,4,\ldots$. They also move in a straight line at light velocity and do not disperse as they propagate. This means that the packet functions remain focused for all time. The pulses are composed of three parts: one part is a three-dimensional pulse moving at light velocity, the second part is a sinusoidal plane wave moving at a velocity less than light velocity, and the third is a sinusoidal function in the angular variable around the propagation axis. The original FWM far fields away from the moving pulse centers are similar in magnitude to the far fields of a stationary current dipole; therefore, it has infinite energy associated with the pulses. The original FWM formulations have been shown to satisfy the first five items listed in the introduction which describe unique electromagnetic pulses.

To obtain a formulation which satisfies all six items listed in the introduction, the original FWM formulations' leading and trailing edges were turned off by using two planes of discontinuity with the pulses. These are called three-region extensions to the original FWM formulations. They represent three-dimensional, nondispersive electromagnetic pulses with finite energy. Even though the planes of discontinuity are not presently propagatable from state-of-the-art antenna systems, they are unique, focused, classical electromagnetic pulses.

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