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DAVID FINKELSTEIN*

MATTER, SPACE AND LOGIC**

ARGUMENT

Physics has a warp and a woof, like a fabric stretched across many levels of abstraction and woven out two millenia long. Across the fabric is a pattern persistent over the entire length in which the levels tend to group themselves into three levels of increasing abstraction: theories of matter and mechanics, theories of space and geometry, and theories of logic. Running along the woof is a second pattern, a sequence of discovery pursued first at the most concrete level and then retraced at deeper levels. In this evolutionary process, the theory first passes from its earliest, most 'rigid', form into a different but still rigid form (fracture), and then into a non-rigid or 'flexible' theory with a continuum of freedom (flow). This process of fracture and flow of physical theories has attacked the deepest levels, those concerned with the logic of the physical world, only in this century and has yet to run its course there. Its working out at these levels is a principal motif of the present and of the immediate future of physics.

I. INTRODUCTORY REMARKS CONCERNING THE PHILOSOPHY, THE STRUCTURE AND THE EVOLUTION OF PHYSICS

For me the philosophy of science is a tool I use in determining what research to carry out. As I see it, we find ourselves standing on the relatively shifting ground of present knowledge faced by a wall which is visible only a short stretch above our heads; and the first step then is to take the biggest leap we can and grab hold and scramble like hell. We can take a bigger leap if we go back 2000 years and make a running start. Some people are able to dispense with this, but I find these questions very hard and have to go over them in great explicit detail.

In what I suppose is a very simplistic analysis of Physics, fundamental physical theory at any time breaks up into three levels of increasing

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theoretical abstraction, generality, and depth. (1) At the top, there are those statements which concern themselves directly with the distribution of matter as we see it around us: roughly speaking, the theory of Mechanics. I will have least to say about this level. (2) This theory employs the concepts of a deeper theory, Geometry, to tell us about what matter is distributed in: space. The fundamental concepts of world geometry are *cause* and *measure*. (Before relativity we would have included spatial concepts of length or distance, but now we understand that space-time is more like a time-axis then a Euclidean space[1], in that the partial ordering of events by the relation of causal antecedence defines 9 parts of the metrical structure of the world, and the concept of world measure or volume element defines the tenth part.) (3) Moreover, underlying Geometry is a still deeper physical theory, seldom formulated explicitly. Geometry itself is after all an exercise in Logic, classically. I count as part of the logical level the apparatus of set theory and arithmetic as well as the propositional calculus.

These are the three levels of physical theory of which I speak. It is now a commonplace that Mechanics and Geometry are empirical sciences insofar as they deal with reality. I shall emphasize the empirical aspects of Logic in this talk.

In the course of time interesting things have happened first at the top level, then at the next, and then at the bottom. In Figure 1, each row corresponds to one of these levels of theory and historical time moves across the columns from left to right. Let's begin with ancient Greece. I put a name in each level just to provide a mental tag. The earliest Mechanics I consider is the *Ptolemaic* picture of the astronomical world, which was formulated in the frame-work of *Euclidean* geometry, which in turn was an exercise in what we can call *Aristotelian* logic. (I will be more precise in a moment in what I mean by the logic underlying a physical theory.)

To begin with man has attempted to fill in, at each of these levels, a structure which is completely and categorically defined, absolute, necessary. He attempts to set up a doctrine or theory which is rigid, or technically, *categorical*. Examples will suggest the definition: Euclidean geometry is rigid; i.e. it has only one realization in the sense that any two realizations of Euclidean geometry are isomorphic. But there are many Riemannian geometries, so Riemannian geometry is not rigid but flexible. If one speaks of *the* theory it is generally a categorical one. We say *the* Euclidean geometry but *a* Riemannian geometry, *the* Aristotelian logic, but *a* non-Aristotelian logic. And if Ptolemy had succeeded, there would be *the* Ptolemaic mechanics, a particular configuration of heavenly crystalware governing the distribution of matter in a unique way. (Even the instantaneous epoch in Ptolemaic astronomy was intended to have absolute significance. One sees traces of this even in today's astrological remnants of early astronomy.)

	⋮	⋮	⋮	
. . .	Ptolemy	Copernicus	Newton	. . .
. . .	Euclid	Minkowski	Einstein	?
. . .	Aristotle	Bohr	?	?

Fig. 1. Fracture and flow in physical theories. The boxes in the top line stand for representative theories of the planetary orbits, as illustrative of stages in the evolution of Mechanics. The middle level represents theories of space-time structure, stages in the evolution of physical Geometry. Minkowski's name tags the flat space-time of special relativity, Einstein's the curved one of general relativity. On the bottom level, Aristotle stands for all of the classical calculus of propositions and sets from his syllogistic logic until the introduction of complementarity by Bohr, who correctly labels the next box; for when we extract the logic of Bohr's physics, we are concerned with what Bohr does, not what he says, and the fact that he disowned the general idea of a non-Aristotelian logic is quite irrelevant. Incidentally, I may be doing Aristotle a grave injustice. I am not actually aware that he ever expounded the distributive law or any equivalent, and if he did not then in all fairness the present logic of quanta too must be termed Aristotelian in a strict historical sense, and some later, lesser name, such as Boole, should be substituted for Aristotle in this figure. Notice also how much quicker Physics flows on its surface than in its depths, with this structuring. The flow on each level is complete before the lower levels stir at all.

With the passing of time one finds two successive kinds of change in the structure. The first step at each level is the recognition that there is a flaw in the structure, and the replacement of one rigid structure by another which is still rigid, but different. Call this process fracture. Fracture is explicitly seen at the level of geometry in the consideration of the first non-Euclidean geometries, the spherical and pseudospherical. These are still each categorical: e.g., there is only one spherical geometry.

Again, at the level of Mechanics, the early forms of the geocentric theory still had the categorical tendency of Ptolemy's. I do not think Copernicus had the concept that the laws of motion of the planets were what were to be determined. He too felt that there was something special about the circle as a motion. One way of recognizing the elevation of a concept to the level of a law, or an element of a rigid theory, is that it is invested with the symmetries that one thinks the universe should have. As long as the path itself is regarded as the law, then obviously the path must be circular, otherwise one direction would be singled out. Copernicus too revered circularity in the orbits, and I think he would have been in sympathy with the efforts of Kepler to find divine relationships between the radii of the heavenly spheres. They sought rigid theories.

Let me trace the development that I see at the highest level, before descending. I will now state my major premise: *After fracture comes flow.* Having recognized say that a postulate could be changed at all, someone is bound to take the next step, to suppose that perhaps there is not a fixed postulate governing that element of the theory once and for all, but that perhaps it is a *conditional* element. E.g., the structure in question may in part be self-determined and propagate itself in the passage of time, depending on the other elements in the theory. Thus in Newtonian mechanics the shape of the orbit itself becomes at least in part conditional, accidental, determined by initial conditions. It breaks up in fact into two parts, one of which, the law of force, is to be sought in principle through induction from the study of many cases, and once found is eternal; and another part of which, the initial data, is left for ever in the domain of the naturalist, subject to ever finer and finer observation, but in principle measured in just one case. This, the top line of Figure 1, is the two-step sequence I am speaking of: fracture followed by flow.

At the level of Geometry, this process occurred again. Once we recognized the possibility of curvature at all, it then became a natural step, historically, to consider that the curvature might vary from place to place, that the real laws of Geometry, in the sense of the eternal truths of Geometry, consisted simply in statements which told how matter effected the geometry or how the geometry effected matter, rather than the specification of what the geometry was, once and for all. The possibility of a flexible geometry we associate in mathematics above all with Riemann, but it was immediately put into a physical context by Riemann himself, and perhaps more dramatically by Clifford, who even suggested that perhaps all of physics was a process of geometry effecting itself and propagating itself into the future.[2] The transition from three-dimensional Euclidean geometry to four-dimensional Minkowskian geometry is a transition from one rigid to another rigid doctrine, a fracture this time in physics. And the subsequent transition to Einsteinian or general relativistic geometry corresponds once more to the second kind of step, flow. So much by way of introduction. I hope you will agree that the concepts of the fracture and flow of physical theories are useful ones.

II. THE NATURE AND STRUCTURE OF THE EMPIRICAL LOGIC, AND ITS FRACTURE

Now to the lowest level. I will argue that something like fracture has also occurred at the very deepest level of physical theory, a level which I have assigned to Logic (I will define what I mean); that the name to tag this event with, above all, is Bohr; and that the introduction of complementarity is a revolution at a deeper level than the step to special relativity or curved geometry. This is my minor premise: *there has been a fracture in physical logic.*

What do I mean by the Logic of a physical theory?

I think that one of the reasons that it is so difficult to understand quantum mechanics is that our teachers fail to tell us it is illogical, violates the canons of classical logic. It does so in the following sense.[3]

Suppose a theorizer is carefully and fully formulating a physical theory, including the relation to experience, the interpretation. In order to compare his theory to others, and in order to use it to understand nature, I would demand a list from him of all empirically ascertainable yes-or-no properties $P, Q, R, \ldots$ he ascribes to the physical system under consideration. Then I would like to ask how he orders these properties in the sense of empirical implication or inclusion $\subset$. This ordering constitutes the empirical content of the theory, upon which it must stand or fall. For which pairs of observations P, Q, does he claim that the presence of one entails the presence of the other, $P \subset Q$? Then in terms of this list and this partial ordering relation, I can proceed empirically to construct the algebra of sets and the propositional calculus of the physical theory.

In addition there's one other element of logic the theorizer has to specify in order for me to tell him the logic of his physical theory, and that is the operation of negation, or complement, $\sim P$. For each property there must be another one, the not-having of that property, the opposite or complemental property.

E.g., if I were discussing astronomy, the list of all possible properties of the bodies under consideration that are relevant to astronomy is really implicitly given in the description of the heavenly bodies by means of phase space. Implicit in the classical-mechanical world picture is the belief that each of the relevant physical properties of this mechanical system corresponds to some subset of an appropriate phase space. Every property corresponds to a statement that the positions and momenta lie in a certain set. Pushed to its extreme this world picture says, e.g., that every question about an object is a question about the location of its atoms in their phase space. In this case, the list of properties corresponds to the list of all subsets of the phase space. The relevant relationship of implication or inclusion is simply the relationship between two sets of the space, of one being included in the other.

Then in terms of this $\subset$ and $\sim$ we can define the logic, can define in an obvious way such concepts as $\cup$ (*or*) or $\cap$ (*and*). For example without any further ado I can define a $\cup$-adjunction of two properties, $A \cup B$, as a property C (which may or may not exist) obeying the following two conditions: First, $A \subset C$ and $B \subset C$. Second, C is minimal with respect to the first condition, i.e., is included in every other C' that obeys the first condition. Briefly, C minimally includes both A and B.

Whether such an adjunction exists is not an idle question. There is given the list of physical properties, there is a relation $\subset$ among them which has physical meaning, one inspects the list and sees whether the adjunction exists or not.

Similarly with a conjunction $A \cap B$, which is included in each of the terms A, B and is maximal in respect to this property.

I will use the terminology of set theory for the properties of the system, identifying each property with a set of virtual systems having that property. If P, Q are such sets, then $P \subset Q$ is not a set but a proposition, but $\sim P$, $P \cup Q$ and $P \cap Q$ are sets (the complement, union, and intersection, respectively). The quantities of a system can be expressed in terms of qualities (sets) of the system. A quantity Q is a labelled collection of sets

Q_λ that is exhaustive, $\cup Q_\lambda = \mathbf{1}$, and mutually exclusive, $Q_\lambda \subset \sim Q_{\lambda'}$ $(\lambda \neq \lambda')$. The set Q_λ is the set of those cases of the system in which the quantity Q takes on the value λ. The reasons that make one λ-labelling of a particular family of sets better than another, compelling us in one case to consider λ more fundamental than, say, λ^3, lie outside the domain of logic. The important thing about one of these quantities Q is that it always makes sense to ask 'What is the value of Q?' even though this is not a yes-or-no question. We have made sure that Q always has a value (is exhaustive) and only one value (is exclusive).

We can understand this algebra in many ways. Let us be operational for a moment. Take a population, more properly, a technic for obtaining samples of the population, say electrons in beams, or photons in light rays, or people in the U.S.A. Imagine a test-device which divides any sample of the population subjected to it into two sub-samples, one which passes and one which 'fails' the test. To minimize assumptions about the effect of observation on the thing observed, we will proceed as if all tests were totally destructive. We are then dealing with *sources* and *sinks* of samples, which are both represented by black boxes in Figure 2. There are all sorts of tests, easy and hard, fair and unfair. E.g., one can stand by the test-device flipping a coin and when it comes up heads, accept the next sample of the population that comes along, when it comes up tails, reject. This test is unfair, is based on something completely foreign to the members of the population being tested. In this way, one gets a very general concept of sample source and tests. A fair or objective test has the pleasant property that we know a way to make samples that pass with certainty. (How can we know this? By physical induction, about which I can say nothing new.) Then we believe that this test is really looking at an objective property of the individual. The coin-flipping experiment is not found to be an objective test, but the Stern-Gerlach experiment in which atoms are sorted according to their spin is such an experiment.

In this context, the relation of inclusion $A \subset B$ among tests A, B simply means that every source of members that all can pass the test A (as determined by sampling and physical induction) also provides a population which passes test B (as determined the same way) (Figure 2). This relation is an empirical thing.

The definitions of $\cap$ and $\cup$ do not tell you how to find that test C, e.g. which has the properties of the union. A priori, C may not exist.[4] They

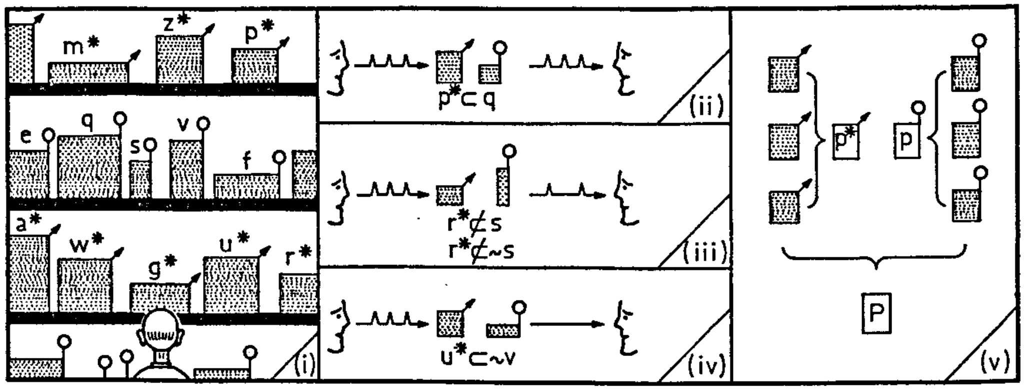

Fig. 2. Towards an operational logic. (i) The black boxes, male and female, stand for the experimenter's entire repertory of emitters (sources, starred) and absorbers (sinks, unstarred) of specimen systems. Three typical experiments are shown. In (ii), the experimenter signals source p^* three times and sink q signals back all three times. This is evidence for $p^* \subset q$, which means all systems emitted by p^* actuate q. Then $a^* \subset b^*$ is defined to mean that for all sinks x, if $b^* \subset x$ then $a^* \subset x$ (the definition of $a \subset b$ is dual to this). In (iii) conclusive experimental disproof of both $r^* \subset s$ and $r^* \subset \sim s$ is shown. Experimental evidence for $u^* \subset \sim v$ is shown in (iv). The white boxes of (v) are more abstract and stand for equivalence classes of black boxes, two black-box sources, a^* and b^*, being identified, $a^* \equiv b^*$, if for all x the empirical relations $a^* \subset x$ and $b^* \subset x$ are true or false together. (The identification $a \equiv b$ of sinks is the dual to this.) Evidently it is to be determined empirically that $\equiv$ is indeed an equivalence relation. A property P is obtained by further abstraction from the classes of (v), identifying a source-class p^* with a sink class p when for all $x, p \subset x$ if and only if $p^* \subset x$. The complement property $\sim P$ is defined by the conditions that (1) no system from a P source is accepted by a $\sim P$ sink, and no system from a $\sim P$ source is accepted by a P sink, and (2) $\sim P$ is maximal with respect to the property (i) in the partial ordering by $\subset$.

do provide you, however, with an empirical way of testing whether a given C is the union, or intersection. In that respect, these too are empirical concepts. I will not fully give the empirical meaning of the statement $P = \sim Q$, i.e. of negation. Classically, $\sim Q$ is completely defined by the conditions $Q \cap \sim Q = \mathbf{0}$ (= the absurd test that all fail) and $Q \cup \sim Q = \mathbf{1}$ (= the trivial test that all pass); but not quantum-mechanically, where a little more must be said. But $\sim$ is still empirical in content (see Figure 2).

Before we go on to the example of quantum mechanics, the important one, I want to say that if one carries out this process of constructing the logic behind what physicists do in the case of mechanics, the resulting logic is just the calculus of subsets of phase space, is just the Boolean algebra of the phase space.

Here is the way we would extract the logic of quantum mechanics. (This is all assuming that quantum mechanics is perfectly right as it stands and using the normal interpretation of quantum mechanics to answer the applicable physical questions. There is no change in quantum mechanics being suggested at this point.) The sets of the physical system according to quantum mechanics correspond to the yes-or-no, the one-or-zero, dynamical variables. They are the projection operators of subspaces of a Hilbert space of state-functions (or state-vectors) associated with the physical system. (What I am doing now is taking a well-known theory and distilling its logic as an example. In a moment I will discuss how to reverse the procedure and build up quantum mechanics from its primitive logical elements.) According to the usual rules, if every sample that belongs to the set P belongs to the set Q, then every eigenvector of P with eigenvalue 1 is an eigenvector of Q with eigenvalue 1, and the subspace of P is included in the subspace of Q. Thus it follows that the relation of inclusion for sets goes over to the relation of inclusion for subspaces. The intersection, given two sets as projection operators, is represented by the projection operator on the intersection of their two subspaces. Finally is there a projection operator meeting the requirements of the adjunction? A subspace, you recall, which includes either of these two subspaces and which is minimal with respect to this. Here the first obvious break in the parallel with the classical procedure appears. The union of two subspaces is not a subspace. If e.g. we believed that the wave-functions themselves were properties of the system, that we were measuring wave-functions, if the system 'had' a wave-function in the sense that it 'has' (say) position or energy, then the underlying empirical logic would be one in which the properties simply consisted of sets in the space of wave-functions, and the union of two sets in the space of wave-functions is a perfectly well-defined object. But according to the interpretation of quantum mechanics which is customary, the union is a set which has its own projection operator, and its subspace is not the union but the *span* of the two subspaces, the smallest subspace containing the two of them.

This is just to emphasize that you should not think of the space of wave-functions of a quantum-mechanical system as being too analogous to the phase space of a classical mechanical system. I have already emphasized that a Schroedinger particle 'has' a position, even if we do not know it, and shown how from the list of sets it is possible to construct all quantities that the system 'has'. Now I would emphasize that the system does not 'have' a wave-function, though for each particular wave-function it is meaningful to ask whether the system has it.

To some the fact that there are vectors neither in A nor in $\sim A$ suggests that there is not a two-valued logic in quantum mechanics, that the *tertium non datur* breaks down. But we have already seen that even in quantum mechanics $A \cup \sim A = \mathbf{1}$, $A \cap \sim A = \mathbf{0}$, expressing the fact that the *tertium non datur* was not the weak point of classical logic. Yet we also see that there must be a difference between the quantum-logical structure and that of classical mechanics, or else it too would be realizable as a calculus of all sets of some suitably chosen space. And in fact the fracture is in the distributive law. All the anomalies of quantum mechanics, all the things that make it so hard to understand, complementarity, interference, etc., are instances of non-distributivity. Let us see how the distributive law breaks down in quantum mechanics. The example I shall use is in the important paper of Birkhoff and Von Neumann. First let me write the distributive law: does

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)?$$

Let us imagine a particle in a box (Fig. 3) and take two physical quantities which are very rough estimates of position and momentum,

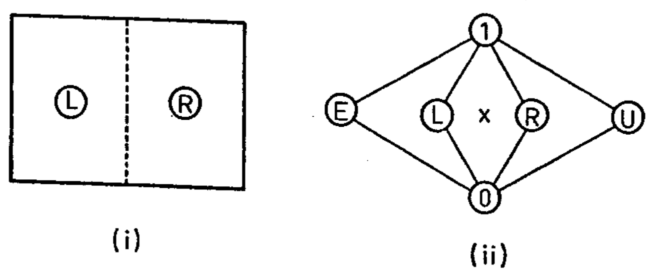

Fig. 3. A quantum in a box. (i) L is the property of being in the left-hand half of the box, R is the property of being in the right-hand half. Evenness E and oddness $U = \sim E$ cannot be shown in this diagram. (ii) This diagram represents part of the algebra of sets of a quantum in a box. Each circle stands for a set of virtual quanta, or a quantum property; the relation of inclusion, $\subset$, is indicated by an upward line, and the complement, $\sim$, is found by reflection about the center. $\mathbf{0}$ stands for the null set and $\mathbf{1}$ for the universal set. There are four unit sets: R, $L = \sim R$, a superposition E (even) and another $U = \sim E$ (odd). No present-day quantum theory is this simple; there are always an infinite number of unit sets, at least one for every complex number. However, all the axioms of quantum sets discussed are realized by this lattice, which is not distributive, except that $|\mathbf{1}| = 2$.[5]

which are two well-known complementary pairs. For position all we will take is the determination of whether the particle is on the left, L, or on the right, $R = \sim L$. For momentum all we will take is the determination of whether the particle is in an even state, E, or an odd state, $\sim E$, with respect to reflection around the middle, which is determined by the parity of a certain quantum number n proportional to the momentum. (According to whether n is even or odd, the wave-function is invariant or is changed in sign by reflection in the midplane.) Now let us work out both sides of the distributive non-identity, taking $A = E$, $B = L$, $C = \sim L$. It looks like this:

$$\underbrace{E \cap \underbrace{(L \cup R)}_{1}}_{E} \neq \underbrace{\underbrace{(E \cap L)}_{0} \cup \underbrace{(E \cap R)}_{0}}_{0}.$$

First, on the left, I have indicated the replacement $L \cup R = 1$, the identity. Either the particle is on the left, 'or' it is not; this is a meaningful, empirical statement, using 'or' in the way we defined it. Then the left-hand side of this distributive law is just E, the property of being in an even state. And now the right-hand side: the intersection of 'even' and 'left' appears. What is the subspace of wave-functions for this particle that are even and vanish on the right? They then vanish on the left as well, and so it is the zero subspace. With the definite meaning that intersection has been given here, one finds that evenness and leftness for a particle in the box are contradictory in the sense that their intersection vanishes, $E \cap L = 0$. Similarly, evenness and rightness are contradictory. Then the right-hand side of this identity is just 0, so the distributive law is false.

I will suppose we can still count. A set P is called a zero set if it is the null set, $P = 0$. A set P is called a unit set (or a pure case, elsewhere) if P includes 0 minimally, i.e. $0 \subset P$; and from

$$0 \subset X \subset P$$

it follows that either $X = P$ or X is a zero set. And so forth: a set P is called an n-set, and we write $|P| = n$, if P includes an $(n-1)$-set minimally. (Technically, I am here supposing 'modularity'.) For simplicity, I will confine my next remarks to theories in which every set is finite. For all we know, maybe every set *is* finite, after all.

From a very formal point of view, complementarity appeared when we threw out the set union of subspaces as a possible property of the physical system, and demanded the span be taken as the minimal subspace containing both of these. If we permitted merely the union, that would have corresponded exactly to making the wave-function itself the primitive dynamical element of the physical theory, like a point in phase space in classical mechanics. But if we look at what is experimentally possible, the repertory of observations we can make on the particle in the box is not as rich as the totality of subsets of the function space consisting of wave functions. The difference between the quantum mechanics of a particle and the classical mechanics of a Ψ field is not in the equations but in the logic.

Simply dropping a law is very unsatisfying, and generally non-categorical. In fact, as I indicated before, the transition to quantum mechanics is a transition from one categorical to another categorical collection of logical laws. What the quantum logic has that classical logic does not is called *coherence*[6]: An incoherent set is one which non-trivially 'splits' into disjoint sets, $P=P_1 \cup P_2$, $P_2 \subset \sim P_1$, in the sense that every subset of P is a union of a subset of P_1 and a subset of P_2. Otherwise P is called coherent. In classical physics every non-trivial universe of discourse is incoherent. In quantum physics the universe of discourse is coherent (slight reservations being made for 'superselection principles').

Now to the inverse question: reconstructing quantum mechanics from its logic. Suppose we stand back and look at the totality of those laws of classical sets which remain true, plus a law of coherence. Question: How much of quantum mechanics can we construct out of this? Answer: Every realization of the axioms I have sketched for you is equivalent to the collection of subspaces of some Hilbert space, in which the relation of negation is represented by the orthocomplement. What is not pinned down is simply the dimension of the Hilbert space and the nature of the underlying coefficients. These need not be the complex numbers, but could be an arbitrary conjugated number field.[6] The fact that it is a field, however, and in particular the law of multiplication and addition, reflect directly the logic from which one has started. Every conjugated number field can be used to build a Hilbert space, using that number field for '*c*-numbers', even though they may not commute; and then the subspaces of that Hilbert space realize the axioms I have indicated. Inversely, every realization of the axioms leads uniquely to a definite conjugated number field and Hilbert space.

This work is a modern outgrowth of the ancient 'coordinatization of synthetic projective geometry'. The relations I have discussed here as logical relations among sets are obviously very close to those relations treated as fundamental in the discussion of projective geometry. The new mathematical ingredient is the treatment of negation, the fact that negation leads not merely to a linear vector space but to a Hilbert space and to an underlying field which has a conjugation.[3]

Many things look simpler from this point of view. Take the problem of measurement: why are there two laws governing the evolution of the state-function, the continuous one of Schroedinger for dynamic evolution, and a discontinuous one of projection for measurement? Are not measurements dynamic processes, governed by the usual laws of dynamics? The answer is that any dynamical process, everything that happens to a system, is governed by a Schroedinger equation, that for the entire system. The second kind of change is part of the rules of interpretation, and tells us how our empirical knowledge about the system is to be expressed in the language of the theory. No matter how beautiful and orderly the equations of a theory may be, they are not physics and cannot be tested until their relation to the physical world is spelled out, and this spelling-out cannot be done by the equations themselves. Even in classical physics, therefore, there are two laws of development in the theory of the system, one for the system and one specifying the interpretation. Take a collection of planets. Specification of a property at time $t=0$ is made by giving a set in the phase space, or equivalently a distribution function $f(q, p, 0)$ obeying $f^2=f$. The laws of dynamical evolution make this property imply another property at a later time, $f(q, p, t)$, obeying a certain Liouville equation

$$\partial f/\partial t + [H, f] = 0.$$

The boundaries of the set move around in the course of time. If, however, we obtain any further information about the planets, we replace the set by another one, whether or not any interaction has taken place. I have never seen any attempt to calculate on a dynamical basis the rate at which the boundary of the set moves from its initial shape to its final shape during changes of this second kind....

When we work problems in quantum mechanics, we use classical logic in carrying out our computations, and this can obscure the fact that some of the expressions are themselves statements in a non-classical logic. This confusion is understandable, but avoidable. We must merely remember that a state-function is a statement about an electron, say, and is not an electron itself. The distinction is not a particularly subtle one: electrons are emitted by cathodes, state-functions are emitted by physicists. Therefore a property of a state-function is not the same thing as a property of an electron, and the two obey quite different logics. E.g., it is a true statement of quantum logic ($\hbar = 1$ being assumed) that if a quantum is in the state represented by the state function $\cos kx$, then its momentum is either k or $-k$, two quantum properties represented by state-functions $\exp ikx$ and $\exp -ikx$, respectively. But it is patently a false statement that if a state-function is $\cos kx$ then the state-function is either $\exp ikx$ or $\exp -ihx$. Rather, the relevant true statement is

$$\cos kx = (\exp ikx + \exp - ikx)/2,$$

and this statement is combined with other statements of a like kind according to purely classical logic. We are frequently cautioned that the word is not the thing. In the past, however, the two have at least been subjects of the same kind of logic. Now we move in a domain where the word is frequently much more substantial than the thing itself. The electron is microscopic. 'The electron' is macroscopic.

This section, I insist, is the non-speculative part of this paper. Certainly the interpretation I am proposing is far less revolutionary than the Bohr interpretation. I am extending our customary logic into the microworld at the expense of one law, the distributive. (All the discussions I have seen that assert the conventional nature of logic, dwell on how unthinkable it would be to give up the laws of *modus ponens* and the excluded middle. No one ever sticks up for the distributive law, so I suppose no one will miss it much.) The concepts of the logic are empirical, invariant under changes of names or observers: if P and Q as properties are identified with definite arrangements of hardware that verify them, then $P \cap Q$ and $P \cup Q$ are also definite arrangements of hardware. The Bohr interpretation gives up the possibility of a micro-logic altogether. Either P and Q, definite properties of a micro-system, are held to be meaningless at times, or else the combinations $P \cap Q$ and $P \cup Q$ are held to be meaningless at times. In the two-slit diffraction experiment, did an electron reaching the screen have to go either through one slit *or* the other? Common sense says yes; we say yes; Bohr says (I think) that the question is meaningless. It is quite as though, having discovered that the laws of Euclidean geometry are invalid in strong gravitational fields for any objectively definable concept of straight line, Einstein had entirely refused to extend the concepts of geometry into such regions, or abandoned objectivity (invariance), rather than introduce the concept of curvature. Conventionalists like Poincaré insisted that this is the proper attitude toward geometry, and I am sure there are some who have the same attitude toward logic. The former are helpless near the sun, the latter near the atom.

III. SPECULATIONS ON THE FUTURE FLOW OF PHYSICAL LOGIC TOWARD A FLEXIBLE THEORY

Let me summarize. Major premise: Fracture in physical theories is followed by flow. Minor premise: there has been a fracture in physical logic. Conclusion: there will be a flow in the physical logic. Of course, I advance this at present as an amusing theoretical speculation. It amounts to saying, let us tamper a little with the existing quantum mechanics.

I think the mathematics suggests trying other number fields for probability amplitudes. If one believes the quantum-logical laws will preserve their validity for at least a while, this is the obvious next step to try, and constitutes a further fracture. This has been tried without much success. Nature seems to like complex numbers.

A still bigger step that is suggested by this way of looking at things is a step from a rigid, albeit non-classical, logic to a logic which is itself a dynamic ingredient in the physical theory, an actor rather than part of the stage. (This is just another way of expressing the flow that we see happening at various levels of physics: what used to be stage props and setting suddenly gets up and takes part in the drama itself; turns out to be an actor.) What would it be like to begin to think about a logic which was capable of evolving, which was conditioned by the actual state of affairs in the universe?

I can only go by analogy. In the case of geometry, the possibility of an evolving and variable geometry exists because even though we do not change the geometry in the small, the geometry of the tangent space, when we go over to differential geometry we do change the way the tangent spaces are tied together to make the entire space. To describe this tying, we introduce the concept of *transport* of geometric quantities from one tangent space to another, relative to a path joining them. In the quantum theory of fields, we have at each point of space-time, at least in imagination, a separate physical system consisting of a field at that point. This physical system is ordinarily treated by the laws of quantum mechanics. We imagine a Hilbert space associated (at least in a formal way) with each point in space-time to be able to make statements about the values of the field and its derivatives at that point in a way consistent with complementarity. The Hilbert space for the entire theory is then obtained at present by binding all these together in a specially simple way. It is in the relation between these separate Hilbert spaces that a more variable logic can enter. Today we speak of a curved geometry; the best term I can think of for such a new logic is warped. New fundamental fields would enter to describe the way propositions at one point are combined logically with propositions at another point. If the Hilbert space at a point has dimension m, then the unitary group SU_m would figure as a new invariance group in physics, and basic m-component fields would enter. This kind of generalization has also been tried. Its simplest form may lead to a pleasing unification of electromagnetic and weak interactions, which are attributed to logical warps much as gravitational forces are attracted to geometrical curvature, but the higher symmetries do not work out right.

I personally do not feel the speculations I have entertained in this section are crazy enough. Too much of the usual conceptual structure is being kept intact at the higher level of geometry while we turn the foundations over at the deepest level. I am strongly tempted by the example of Clifford. If a flexible logic is possible at all, it may be rich enough to account for much more of the phenomena we see at the higher levels than we usually regard as logical in origin. I find a strong and encouraging resemblance between appropriately chosen basic concepts of world geometry (causal antecedence and measure) and certain basic logical notions (implication or inclusion, and number). Work in this direction is much too raw to justify extending this section any further.

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REFERENCES

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[1] This time-like rather than space-like tendency of the modern world geometry has been well noted by M. Capek, *Philosophical Impact of Contemporary Physics*, Van Nostrand Co., Princeton, N.J., 1961, especially Chapter 11.

[2] W. K. Clifford, *On the Space Theory of Matter*, paper presented before the Cambridge Philosophical Society, 1870. Collected in James R. Newman, *The World of Mathematics*, Simon & Schuster, New York, 1956.

[3] The mathematical development used extensively here is that of G. Birkhoff and J. von Neumann, 'The Logic of Quantum Mechanics', *Ann. Math.* **37** (1936) 823.

[4] G. W. Mackey, *Mathematical Foundations of Quantum Mechanics*, Benjamin, New York, 1963, regards the existence of $A \cup B$ and $A \cap B$ as an unnatural assumption of present-day quantum theory, and the postulate most likely to be abandoned next.

[5] This representation theorem assumes the system has more than two states, in the sense that $|\mathbf{1}| > 2$. If $|\mathbf{1}| = 2$, there are discrete special cases such as the algebra of sets of Figure 3ii, corresponding to 'finite projective geometries', in addition to the familiar algebras of sets represented in two-dimensional Hilbert space.

[6] Following J. M. Jauch.

[7] M. Tavel, Quaternion Theory of Weak and Electromagnetic Interactions, Ph.D. Thesis (Unpublished), Yeshiva University (1964).