THE LOGIC OF QUANTUM PHYSICS *

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The logical structure of quantum mechanics is so different from that of earlier physical theories that there are to this day physicists who believe the present day quantum mechanics cannot be correct, on the grounds that it creates an artifical division between the microcosm, where new forms of laws are required, and the macrocosm, where classical logical forms are known to be satisfactory. It is bad enough to have to speak of electron diffraction, to be compelled to speak of the diffraction of elephants seems unbearable; yet quantum mechanics in its present form leads to the use of wave-functions even for elephants, and it is easy to devise a simple experiment that splits the wave-function of an elephant into two packets separated by any number of miles. (One lures the elephant into a freight train controlled by a device which responds to the result of a one-electron diffraction experiment. The exact arrangement of photomultipliers, peanuts, *etc.* is left to the reader.) Yet we only believe in that packet which moves with the elephant, whichever indeterminate way the creature is finally transported. Since Schroedinger's equation (or its like) will never cause the ghostly second elephant-packet to disappear, there must be something besides Schroedinger's equation active in nature, the argument goes; for example, nonlinear corrections that are important for elephants but not for electrons. This is called the problem of the collapsing wave-function.

Since I think the difficulties of this problem have already been well worked out, this paper is expository – putting the main points of the argument out for all to see again – and formal – setting up a useful and suggestive mathematical scheme for the quantum logic.

The Class Calculus

It seems that we understand the actual by contrast with the

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virtual or possible. By the time we come to asking the properties of an actual, individual, system, we have set up a glossary of the properties that it *might* have, on the basis of our total experience with the world. Thus in classical mechanics we do not consider one system by itself, we imbed it as a point in a *phase space* of "all possible" states of the system, even though perhaps only one of these points is ultimately to be regarded as actual. It is an expression of the Laplacian world picture that all questions about properties of the world – whether a rose is red, or a violet is blue, sugar is sweet or so are you – correspond to *subsets* or *classes* A, B, C, D,... in such a space. We suppose that each class we consider corresponds to a possible experimental test, which would pass only specimens belonging to that class. This test can also be used to prepare members of the class. Diagrammatically we can represent the test associated with class A by a flow diagram:

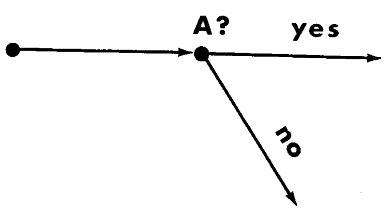


FIGURE 1.

showing an input stream of subjects, some failing and some passing. This is our provisional substitute for Venn diagrams, which depict a class as a region of a plane:

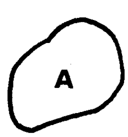


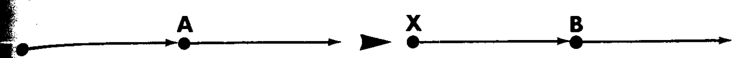
FIGURE 2.

Venn diagrams take too much for granted, flow diagrams too little. A better scheme will be used later. We are now going to consider relations among such classes, setting up a *calculus of classes.*

We are in a delicate position, using logic to study the need for changing logic. We suppose at first that for macroscopic systems the classical laws are valid, and work as much as possible with such systems.

Thus, we take as our basic relation among classes, inclusion,

(A ⊂ B), a relation which is understood to be testable by the following macroscopic scheme:



A⊂B

FIGURE 3.

Here X is any source of subjects, all of whose output is verified to pass A; then A is replaced by B. If for all such X, B also passes the entire output, we say A ⊂ B.

We suppose a kind of physical induction to be valid: once it has been found that a large sample of the output of the source X passes the test A, we expect yet untested members of the output to pass A. We have

$$A \subset A. \tag{1}$$

We verify that

$$\text{if } A \subset B \text{ and } B \subset A \text{ then } A = B \tag{2}$$

is a consistent identification, and that

$$\text{if } A \subset B \text{ and } B \subset C \text{ then } A \subset C. \tag{3}$$

We designate the trivial test by I:

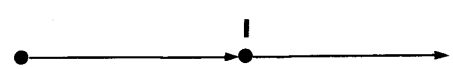


FIGURE 4.

and the impossible test by O:

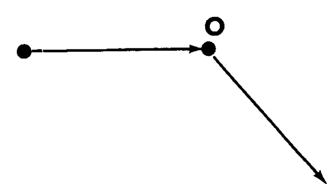


FIGURE 5.

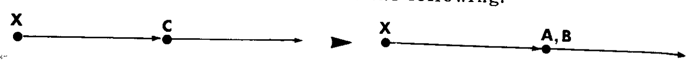
That is, we suppose the existence of classes O, I such that for all classes A, we have

$$O \subset A \subset I. \qquad (4)$$

We can now use the inclusion *relation* ⊂ to set up two binary *operations* on classes, also having observational meaning. We define first the *meet* of A and B:

A ∩ B = C means a. C ⊂ A, C ⊂ B
b. if D ⊂ A and D ⊂ B then D ⊂ C.

Thus, part a. of the definition of the relation A ∩ B = C is checked experimentally by two schemes like the following:



a. C ⊂ A, C ⊂ B

FIGURE 6.

We leave the construction for part b. as an exercise. Notice that we do not give an experimental test for A ∩ B itself. We do not, for example, show how to test A ∩ B when test A and B are known. This is because in some cases the two obvious schemes

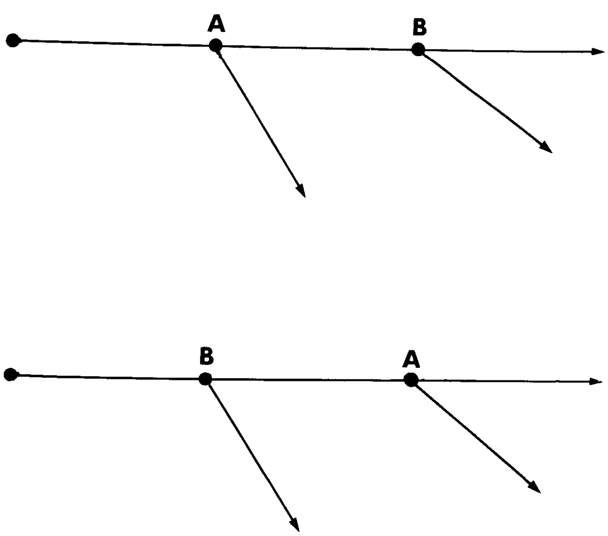


FIGURE 7.

are not tests at all (do not have the property of reproducibility) and moreover are different in their outputs. We do not guarantee the

existence of a C = A ∩ B yet. But we have given an experimental meaning to the relation A ∩ B = C. Moreover if C exists it is evidently unique.

The same ideas are involved in the *join* A ∪ B, for which we define:

A ∪ B means a. A ⊂ C and B ⊂ C
b. A ⊂ D and B ⊂ D then C ⊂ D.

Briefly A ∩ B is the g.l.b., A ∪ B the l.u.b. of A and B.

We now can state another law of nature accepted in classical and quantum physics alike:

$$A \cup B,\ A \cap B \text{ always exist.} \qquad (5)$$

It is evident that $A \cup A = A \cap A = A$, and that $A \cap B \subset A \subset A \cup B$.

For technical reasons we will limit ourselves to classes of a finite nature expressed by the assumption that:

There exists an integer $|A|$ associated with each class A, called the measure of A, such that if $A \subset B$ and $A \neq B$ then $|A| < |B|$, and always

$$|A| + |B| = |A \cup B| + |A \cap B|. \qquad (6)$$

Without loss of generality, we may take 0 and 1 as the least and next-least values of this measure. In classical physics $|A|$ would designate the number of elements of the class A.

Classical and quantum physics agree also in supporting the existence of an operation of taking the *complement* $\bar{A}$ of a class A, with the following properties:

$$\text{If } \bar{A} = B, \text{ then } \bar{B} = A,\ A \cap B = O, \text{ and } A \cup B = I.$$
$$\text{If } A \subset B \text{ then } \bar{B} \subset \bar{A}. \qquad (7)$$

The complement must be regarded as an additional element of structure, for the complement operation is not uniquely defined by the above properties. It corresponds to the following diagram:

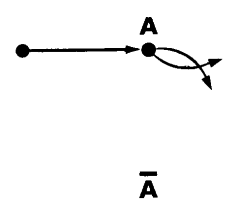


FIGURE 8.

Now let us indicate some of the possibilities admitted by principles (1-7). In the following diagrams, each point represents a class. The relation $A \subset B$ exists when there is an ascending path from A to B. The complement operation in each case is effected by reflecting the diagram in the central point X. Under each diagram we give $|I|$ and on the vertical scale the value of the measure $|A|$ for each class A.

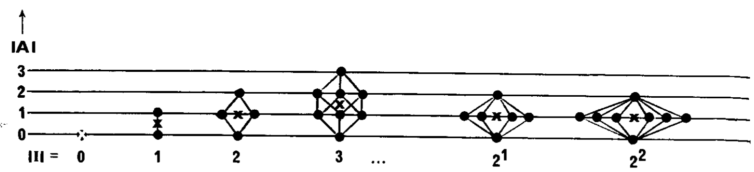


FIGURE 9.

The diagrams labelled $|I| = 0, 1, 2, 3$ are respectively just the 2^n subclasses of a universe (or phase space) of $n = 0, 1, 2, 3$ points respectively. (Their diagrams are just perspectives of the n-dimensional cell, we notice: the line, square, and cube are represented.) These correspond to "Boolean Algebra," the classical calculus of classes. What do we make of 2^1? The difference between 2 and 2^1 is the difference between the classical and quantum calculus of classes. In the classical cases the following distributive law is obeyed:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \quad (8)$$

but in the case of the lattice 2^1 there is no such law. Instead in 2^1 any pair A, B of the four classes of measure unity are *complementary,* meaning that

$$A \neq (A \cap B) \cup (A \cap \bar{B}).$$

This is sometimes confused with a breakdown of the *tertium non datur,* $A \cup \bar{A} = I$, of (7).

The concept of complementarity is so crucial that we shall go into it at greater length before proceeding further.

The Meaning of the Wave Function

In each of the diagrams in the preceeding figure is embodied a calculus of classes. It is now useful to consider relations between these diagrams, setting up, at least in rudimentary form, a calculus of calculi – or diagram calculus. Inspection shows that each of the diagrams (0, 1, 2, 3, 4,...) is a rendering on the plane of a cell of n-dimensions: point, line, square, cube, tesseract, *etc.* Each cell in this sequence can be built up systematically from the preceeding one by duplicating and connecting; this is an example of the basic operation of our diagram calculus, which would be written

$$C^{n+1} = C^1 + C^n.$$

In general one defines the sum of two of our diagrams L_1, L_2 as a new diagram

$$L = L_1 + L_2$$

whose classes A, are all the ordered pairs (A_1, A_3), of classes A_1 from L_1, A_2 from L_2; whose inclusion

$$A \subset B \text{ means } A_1 \subset A_2 \text{ and } B_1 \subset B_2,$$

so that $\cap$ and $\cup$, 0 and 1 are defined for L; and whose complement is

$$\bar{A} = (\overline{A_1, A_2}) = (\bar{A}_1, I_2) \cap (I_1, \bar{A}_2).$$

The sum $L = L_1 + L_2$ is the class calculus of an individual with two exclusive alternatives: to belong to the universe of discourse of L_1 *or* to belong to the universe of discourse of L_2. This diagram addition is a commutative process that can be used to make complicated class calculuses out of simpler ones.

It is evidently a great simplification to consider only *coherent* diagrams, those which are *not* expressible as sums except by $L = L + O$. For example this cuts off the infinite sequence of Boolean diagrams 0, 1, 2, *etc.,* at the second term. But the diagram marked 2^1 in our diagram is coherent, and survives. A simple way to check coherence is as follows: for every class A and B of measure 1 there exists a class C of measure 1 such that

$$A \cup B = B \cup C = C \cup A.$$

This is a necessary and sufficient condition for coherence, and inspection shows it to be satisfied for the diagrams labeled 0, 1, 2^1. But in every case but 0, 1 this condition is in strong conflict with classical intuition. If $A \neq B$ then $A \cup B$ has measure 2 – it should contain only two individuals. Whence C? C is of the kind known in quantum mechanics as a (coherent) superposition of A and B: present in $A \cup B$ yet distinct from A and from B.

This relation among A, B, C provides the simplest instance of complementary classes. Let B be chosen to belong to $\bar{A}$ (excluding thereby the too little diagrams 0, 1). Then we have

$$C \cap A = O,$$

and $$C \cap \bar{A} = O,$$

but $$C \cap (A \cup \bar{A}) = C \neq (C \cap A) \cup (C \cap \bar{A}),$$

verifying the condition we have called complementarity between A and C.

The specimen 2^1 is too puny for quantum mechanics, even as a term in a large sum. The physical world never has the alternative, it seems reasonable to suppose, of existing in a universe of discourse of measure 2. A remarkable thing happens when we move to coherent diagrams with $|I| > 2$, however. The *classes* in such a class calculus are realizable exactly as the totality of *subspaces* of a vector space uniquely determined by the class calculus or diagram. The vector space is constructed by choosing a number system (numbers $\alpha, \beta, \ldots$ with a sum $\alpha + \beta$, a product $\alpha\beta$, an inverse α^{-1} and a conjugation $\bar{\alpha}$), which we will take to be the complex numbers, and by taking sequences $a = (a_1, \ldots, a_n)$ to be *vectors* of the space. For discussing the process of complement $\bar{A}$ we need the "scalar product" $a^*b = \Sigma\, \bar{a}_i b_i$, for $\bar{A}$ is defined to be the totality of vectors b that are perpendicular to A in the sense that $a^*b = 0$ for all a in A. The dimension is $n = |I|$

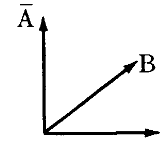
Thus, we are very close to quantum mechanics in the usual formulation. The *wave-functions* or *state vectors* of quantum mechanics correspond to subspaces of dimension equal to 1 in this vector space, and to *classes* of measure 1 in the calculus of classes represented by the vector space.

We can recognize complementary classes A, B as subspaces *A, B* that are not "square" to each other, in that the projection upon one subspace *B* of some vector *a* in the other subspace *A* is neither a nor 0.

The process of projecting a vector upon a subspace *A* is so useful in discussing the subspace that we designate it by the same letter: *Ax* represents the projection of the vector *x* on *A*. *Ax* defines a *linear operator A*, which can be used to represent the class A just as well as the subspace *A*, the correspondences between classes, subspaces, and projection operators being one-to-one.

Now we are deprived of the graphic representation of our calculus of classes that was used in FIGURE 9. There are nondenumerably infinite classes of measure 1 even in the example of a

universe of discourse measure of two, because there are so many different vectors (a_1, a_2) in two dimensions. But we have something better to take its place: the vector diagram. Thus a quantum system with just two mutually exclusive unit states A, $\bar{A}$ can be visualized in a vector diagram of the following kind:



$$B \subset A \cup \bar{A}$$

$$B \cap A = B \cap \bar{A} = O$$

FIGURE 10.

The trivial class I is the whole plane; the impossible class O is the origin; and most of the unit classes, like B, are complementary to A, being superpositions of A and $\bar{A}$. It would be more accurate if we drew two complex dimensions instead of two real ones; but the main features of quantum logic are already represented in the simpler diagram.

It is possible to say that this quantum class calculus is "merely" a calculus of *tests*, not really of classes; and that there exist classes not yet associated with tests but obeying familiar laws. This position is akin to the position that general relativity, for example, is "merely" a theory of lightrays, etc., not really of straight lines; and that there exist in space straight lines not yet accessible to measurement but obeying classical laws. Such positions are in principle not disprovable, it seems; which is a serious short-coming in a supposedly physical hypothesis. Moreover they appear quite arbitrary and prejudiced in their selection of the laws that the unobserved elements are to obey.

Probability

There has been no mention of probability. We concerned ourselves with yes-or-no judgments entirely. This seems necessary. A statement of probability of the usual kind for a single system

$$P = |\psi^*\psi|^2$$

presumably implies a statement of virtual certainty about the number of occurances in an ensemble of very many replicas of the individual, according to the frequency interpretation. But a statement of virtual certainty had better agree with one of the yes-or-no propositions we have formulated, according to a principle of continuity.

In other words, the "expectation value formula" is very nearly a consequence of the "eigenvalue principle", rather than an independent assumption as often presented. Leaving the logical sequence of development for a moment, we can express this as an elementary theorem.

For an individual system S, let x be a self-adjoint observable, ψ any nomalized state-vector in its domain. Then consider an ensemble of replicas $S_1,\ldots, S_N$ all in the corresponding states $\psi_1,\ldots, \psi_N$, so that

$$\Psi = \psi_1 \times \ldots \times \psi_N$$

is the state of the ensemble. Let

$$X = (x_1 + \ldots + x_N)/N$$

be the observable representing the mean value of x. Then there exists a unique number ξ such that

$$\lim_{N \to \infty} || X\Psi - \xi\Psi || \equiv \lim \Delta = 0$$

namely

$$\xi = \psi^* x \psi.$$

That is, even if the individual ψ is far from an eigenstate of the individual x, the ensemble Ψ is nearly an eigenstate of the mean X, as measured by the error Δ.

The proof is a straighforward calculation:

$$\Delta^2 = || X\Psi - \xi\Psi ||^2 = \frac{N-1}{N}[\psi^* x\psi - \xi]^2 + \frac{1}{N}[\psi^* x^2\psi - (\psi^* x\psi)^2].$$

Since the first bracket must vanish if $\Delta \to 0$, ξ must have the asserted value; and then the second bracket gives

$$\Delta \sim \Delta x/\sqrt{N},$$

which approaches 0.)

Moreover, the inclusion of a formula for probability in the foundations of quantum mechanics makes it seem as though quantum mechanics is incorrect for systems that have no known replicas but are unique, e.g. cosmological theories. In the absence of ensembles it is difficult to understand what is meant by probabilities other than 0 or 1. Therefore it is pleasing that it is possible to found quantum mechanics exclusively with yes-or-no, 0 or 1 probability, principles, and derive the intermediate probabilities at least as well as in classical statistical mechanics.

But of course we cannot maintain the appearance of empiricism – as when we formulated our class calculus of tests – in speaking of intrinsically unique systems, e.g. the whole universe. The relations between classes were *empirically* defined in terms of *real* ensembles provided by various kinds of sources, generators, populations. In going to unique systems we give this up and deal with *virtual* ensembles, theoretical constructs. But this is evidently a common feature of the logic of classical and quantum physics and will not be discussed further here.

The Propositional Calculus

Now we wish to set up a model quantum mechanical *propositional* calculus of the most elementary kind: no quantifiers, just a sufficiently long but finite list of individuals a,b,c, etc. These individuals are the stuff on which the "tests" of Part I were carried out – they are instances of observations, of either one system or an actual ensemble. The calculus must cope with propositions that express a result of observation on the individuals, such as $a \epsilon A$, and propositions that express a relation between the classes, A,B,C,... such as $A \subset B$; and we must make from these new propositions that can be confronted with experience. Since the propositional calculus is to formalize the actual way that quantum physicists proceed, and quantum physicists believe they use the classical laws of thought (when they consider the matter at all) it seems that the propositional calculus will not have the paradoxical structure that the class calculus has. On the other hand, quantum physicists have learned that there are some seemingly well-posed questions that theory cannot answer, but only observation. The simplest case is one in which we observe $a \epsilon A$, know that $A \cap B = O$, and inquire whether $a \epsilon B$. Classically, $A \cap B = O$ would imply that $A \subset \bar{B}$, and we would infer $\sim(a \epsilon B)$. Quantum class calculus permits $A \cap B = O$ without $A \subset \bar{B}$ when A and B are complementary, and the quantum physicist in that case does not seek a yes-or-no theoretical prediction about $a \epsilon B$. So we could expect the non-distributivity of the calculus of classes to become an incompleteness of the calculus of propositions.

If P,Q,R,... are propositions, we write, as usual,

$$P \to Q, \; P \wedge Q, \; P \vee Q$$

for compound propositions "P implies Q", "P and Q", "P or Q";

and

$$\sim P$$

for the negation of P.

We suppose given the "table" of the relations $A \subset B$ and the operation $\sim A$, and a list of the individuals a,b, etc. Every true relation $A \subset B$ among classes leads to a variety of true propositions, namely we suppose

$$A \subset B \longleftrightarrow [(a\epsilon A \rightarrow a\epsilon B) \wedge$$
$$(b\epsilon A \rightarrow b\epsilon B) \wedge \ldots].$$

Similarly

$$A = \bar{B} \longleftrightarrow [\ (a\epsilon A\ \longleftrightarrow [\ (a\epsilon A \longleftrightarrow \sim a\epsilon B) \wedge \text{etc.}]$$

where $\longleftrightarrow$ stands for mutual implication: $P \longleftrightarrow Q$ means $P \rightarrow Q \ \Delta\ Q \rightarrow P$. We suppose

$$(a\epsilon A \wedge A \subset B) \rightarrow (a\epsilon B)$$

The main question is about propositions like the following:

$$a\epsilon A \wedge a\epsilon B \longleftrightarrow a\epsilon\ A \cap B?$$
$$a\epsilon A \vee a\epsilon B \longleftrightarrow a\epsilon\ A \cup B?$$

Before deciding, let us consider a simple example, a system of spin 1/2. Here, we briefly take for granted the way to describe composite systems by multiplying wave vectors and the theory of the evolution of systems in time expressed by a unitary transformation obeying Schroedinger's equation. Let the classes (subspaces, projections) belonging to $\sigma_z = \pm 1$ be designated by $|\uparrow|$ and $|\downarrow|$ respectively; similarly for $\sigma_x = \pm 1$ and $|\rightarrow|$, $|\leftarrow|$. We can introduce corresponding state vectors $|\uparrow\rangle$, etc. so that

$$|\uparrow| = |\uparrow\rangle\langle\uparrow|,\ |\downarrow| = \ldots$$

and also, because the spin is 1/2,

$$\sqrt{2}\ |\rightarrow\rangle = |\uparrow\rangle + |\downarrow\rangle.$$

These vectors are shown in this figure:

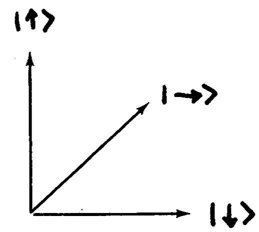


FIGURE 11.

First let σ_z be measured (Stern-Gerlach) and found to be + 1/2:

$$a\epsilon\ |\uparrow|.$$

Now let a measuring apparatus, initially in the class $|A_0| = |A_0\rangle\langle A_0|$, be coupled to the spin so as to measure σ_x. This requires a Hamiltonian that would cause the evolution of state vectors

$$|A_0\rangle\ |\rightarrow\rangle \rightarrow |A_+\rangle\ |\rightarrow\rangle$$
$$|A_0\rangle\ |\leftarrow\rangle \rightarrow |A_-\rangle\ |\leftarrow\rangle\ ,$$

where $|A_\pm\rangle$ is the measuring apparatus recording the result $\sigma_x = \pm 1$, to take place during the measurement. Therefore the second measurement produces the evolution

$$|A_0\rangle\ |\uparrow\rangle = |A_0\rangle\ (|\rightarrow\rangle + |\leftarrow\rangle)/\sqrt{2}$$
$$\rightarrow (|A_+\rangle\ |\rightarrow\rangle + |A\rangle\ |\leftarrow\rangle)/\sqrt{2} \equiv |1\rangle.$$

Suppose the *actual* result of the second measurement to be that associated with $|A_+\rangle\ |\rightarrow\rangle \equiv |2\rangle$. Then we have simultaneously

$$a\epsilon\ |1|$$

as a result of the first measurement and

$$a\epsilon\ |2|$$

from the second measurement. But it is well known that in this case

$$|1| \cap |2| = 0,$$

expressing only the fact that the vectors $|1\rangle$ and $|2\rangle$ cannot be exactly parallel. If we keep the above questioned rules then we would infer $\sim(x\epsilon|1| \wedge x\epsilon|2|)$. If we keep the classical rules of the propositional calculus this would in turn imply $x\epsilon|1| \rightarrow \sim x\epsilon|2|$, which seems to flatly contradict experience. There are several choices. We shall *not* suppose that the outcome $|2|$ of the measurement is indeed compatable with $\sim x\epsilon 2$ – this would be to destroy the bridge between the theory and reality. I think the usual practice amounts to keeping the classical propositional calculus, in which $\sim(P \wedge Q) \rightarrow (P \rightarrow \sim Q)$, and dropping the questioned correspondence between the class calculus and the propositional calculus. One declares that the disjunction

$$x\epsilon A \vee x\epsilon \bar{A}$$

(e.g. the election spin is up or it is down) is actually meaningless unless an observation is made on the election spin; then it is true. I call this the minimal form of the quantum propositional calculus: It dispenses with the above form of the law of the excluded middle, and thereby avoids violations of the distributive law, whose various terms are often regarded as meaningless instead. With the remaining rules of the propositional calculus it does not seem that contradictions with practice or nature will arise. But one is deprived of the means to deduce many reasonable and harmless assertions about the system being considered, such as

$$a\epsilon A \vee a\epsilon \bar{A};\ \sim (a\epsilon A \wedge a\epsilon \bar{A}).$$

Neither these nor their negations seem accessible with the enumerated rules. This seems to be one way in which those anomalous situations in the quantum class calculus where $A \cap B = 0$ without $A \subset \sim B$, influence the calculus of propositions.

Instead we shall accept the questioned rules, so that the propositional calculus faithfully follows the class calculus. But we insist that the *interpretation* – the comparison with reality – be made exclusively through the scheme

$$\begin{array}{c} P \\ P \rightarrow Q \\ \hline Q \end{array},$$

where the first two lines represent data, either from an observation or from theory – based on general experience – and Q represents a prediction of the theory. From this interpretive scheme we can indeed develop others such as

$$\frac{P}{P \vee Q} \qquad \frac{P \wedge Q}{P}$$

but *not* the forbidden ones

$$\begin{array}{c} P \\ \sim(P \wedge Q) \\ \hline \sim Q \end{array} \qquad \begin{array}{c} P \\ \sim P \vee Q \\ \hline Q \end{array} \qquad \begin{array}{c} P \\ Q \\ \hline P \wedge Q \end{array}$$

in cases where the nondistributivity is important. Thus, our form of quantum logic, which we call maximal in contrast, consists of the quantum class calculus outlined already; a quantum propositional calculus that parallels it faithfully, with the usual strict correspondence between the two; additional laws to express the validity of the classical distributive propositional calculus for macroscopic propositions like $A \subset B$, as opposed to microscopic ones like $a\epsilon A$; and the familiar rule of deduction: *Modus ponens*, the scheme

$$\vdash P \ \&\ \vdash P \rightarrow Q \Rightarrow \ \vdash Q$$

(if the propositions P and $P \rightarrow Q$ are deduced, then deduce Q). Many other deduction schemes, such as

$$\vdash P \ \&\ \vdash Q \Rightarrow \ \vdash P \wedge Q$$

are legitimizable from these. But the distinction between these rules for *deducing* general propositions, and the preceding schemes for *interpreting* them, must be retained: As observed instance of $x\epsilon A$ and also of $x\epsilon B$ is simply not an instance of $x\epsilon A \wedge B$, in general. This is already implicit in the earlier flow diagram discussion.

Non-Disturbing Observations

Know thyself, the maxim says; and I think some physical systems obey the maxim. This creates some embarrassment for the minimal form of the quantum propositional calculus. The notion that there are two distinct modes of evolution of a physical system – one dynamical given by Schroedinger's equation, the other statistical given by the outcomes of measurements – is tolerable when used for open systems. For such systems the act of measurement is an external disturbance, a breaking of the internal law. But when we come to self-knowing systems – I am one, I think you are another – this is not so. When I take note of my surroundings I only obey the dynamical laws appropriate to the admittedly complicated system of surroundings and myself. The attempt to excuse a supposed breakdown of the dynamical laws in the case of such self-knowing systems leads some to assume a rather unphysical infinite regression of observers within observers: I watch my surroundings, my nervous system watches my retina, my brain watches my optic nerve,.... It is supposed that this sequence can be indefinitely extended, to smaller and smaller volumes of the brain, but never stopped. "That this boundary can be pushed arbitrarily deeply into the interior of the body of the actual observer is the content of the principle of the psycho-physical parallelism – but this does not change the fact that in each method of description the boundary must be put somewhere, if the method is not to proceed vacuously, i.e., if a comparison with experiment is to be possible. Indeed experience only makes statements of this type: an observer has made a certain (subjective) observation; and never any like this: A physical quantity has a certain value." (von Neumann, p. 420) This seems at least unnecessary. It is much more natural to me to suppose that my knowledge is not something gleaned from observations of (= interactions with) my brain but is actually identified with a state of my brain as a physical system. For example in an electron diffraction experiment that "splits my wave-function into two packets", I am not violating but obeying the dynamical laws that ordinarily govern me, my apparatus, and the electron; and yet our wave-function is "reduced", i.e. I can use the observed outcome of the experiment to eleminate one of our two wave-packets in computing our future. Therefore, it seems preferable not to regard the state-vector or wave-function as a physical attribute of an individual system. Indeed in the interpretation put forward here, it is as unphysical to suppose an individual system "has" a wave-function, that undergoes various evolutions and catastrophes, as to suppose in classical mechanics that an individual system is accompanied in phase space by a class (set) that mysteriously waxes and waves without information about the system. The glossary of admissible physical questions about an individual system may permit one to ask for the energy, momentum, and perhaps even for the position; never for the wave function, which merely expresses the outcomes of other, admissible, questions.

Thus, it is preferable, though I cannot assert that it is necessary, to suppose that the truth of such propositions as the law of the excluded middle, in the form $x \epsilon A \vee x \epsilon \bar{A}$ here, does not wait on an observation of the system but is independent of circumstances. This however is only possible with a nonclassical quantum logic, for example the one we have described here.

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