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# CATEGORICAL FOUNDATION OF QUANTUM MECHANICS AND STRING THEORY

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The unification of quantum mechanics and general relativity remains the primary goal of theoretical physics, with string theory appearing as the only plausible unifying scheme. In the present work, in a search of the conceptual foundations of string theory, we analyze the relational logic developed by C. S. Peirce in the late 19th century. The Peircean logic has the mathematical structure of a category with the relation $R_{ij}$ among two individual terms $S_i$ and $S_j$, serving as an arrow (or morphism). We introduce a realization of the corresponding categorical algebra of compositions, which naturally gives rise to the fundamental quantum laws, thus indicating category theory as the foundation of quantum mechanics. The same relational algebra generates a number of group structures, among them $W_\infty$. The group $W_\infty$ is embodied and realized by the matrix models, themselves closely linked with string theory. It is suggested that relational logic and in general category theory may provide a new paradigm, within which to develop modern physical theories.

*Keywords*: Relational logic; category theory; foundations of quantum mechanics; string theory.

The raison d' être of physics is to understand the wonderful variety of nature in a unified way. A glance at the history of physics is revealing: the unification of terrestrial and celestial mechanics by Newton in the 17th century; of optics with the theories of electricity and magnetism by Maxwell in the 19th century; of space–time geometry and the theory of gravitation by Einstein in the years 1905 to 1916; and of thermodynamics and atomic physics through the advent of quantum mechanics in the 1920's.[1] The next leap in this on-going process is the unification of the two pillars of modern physics: quantum mechanics and general relativity. String theory, in this respect, appears as the most promising example of a candidate unified theory.[2]

Strings emerged in the study of strong interactions, modeling the flux tubes between quark–antiquark pairs in hadronic collisions, in the Regge limit, nicely described by the Veneziano amplitude,[3] which can be reproduced from a relativistic string theory.[4] In a similar vein, the hadronic structure functions in the small

$x$ Bjorken limit are most conveniently described via colored dipoles.[5] A precise and profound analysis of a string dual of QCD has been provided by 't Hooft.[6] 't Hooft considered a generalization of QCD by replacing the gauge group SU(3) by SU($N$). The limit $N \to \infty$ with $\lambda \equiv g^2_{\text{YM}} N$ kept fixed, leads to a topological expansion. The leading order (in $1/N$) Feynmann diagrams can be drawn on a planar surface and higher-order diagrams on surfaces of higher genus. In a most interesting development an holographic analogy[7,8] has been established between matter or open strings on a D-brane and gravity or closed strings in the bulk.[9] We realize that string theory is a tantalizing rich theory, since on one hand is connected to the dynamics of the space–time continuum, and on the other hand the discrete modes of string vibrations represent the totality of elementary particles.

Every single physical theory is corroborated or disproved by experiment. The early hope of making direct contact between experiment and string theory has long since dissipated, and there is as yet no experimental program for finding even indirect manifestations of underlying string degrees of freedom in nature.[10] Particle/string theorists under these conditions focused their attention in searching for the internal coherence and the physical principles governing string theory. This search is of paramount importance. While in developing general relativity Einstein was guided by the principle of equivalence, we are lacking a foundational principle for either string theory or quantum mechanics.[1,11] In the present work we suggest that a form of logic, relational logic developed by C. S. Peirce in the second half of the 19th century, may serve as the conceptual foundation of quantum mechanics and string theory.

Peirce, a most original mind, made important contributions in science, philosophy, semiotics and notably in logic, where he invented and elaborated novel system of logical syntax and fundamental logical concepts. The starting point is the binary relation $S_i R S_j$ between the two "individual terms" (subjects) $S_j$ and $S_i$. In a short hand notation we represent this relation by $R_{ij}$. Relations may be composed: whenever we have relations of the form $R_{ij}$, $R_{jl}$, a third transitive relation $R_{il}$ emerges following the rule:[12,13]

$$R_{ij} R_{kl} = \delta_{jk} R_{il} \,. \tag{1}$$

In ordinary logic the individual subject is the starting point and it is defined as a member of a set. Peirce, in an original move, considered the individual as the aggregate of all its relations

$$S_i = \sum_j R_{ij} \,. \tag{2}$$

It is easy to verify that the individual $S_i$ thus defined is an eigenstate of the $R_{ii}$ relation

$$R_{ii} S_i = S_i \,. \tag{3}$$

The relations $R_{ii}$ are idempotent

$$R_{ii}^2 = R_{ii} \tag{4}$$

and they span the identity

$$\sum_i R_{ii} = \mathbf{1}\,. \tag{5}$$

The Peircean logical structure bears great resemblance to category theory, a remarkably rich branch of mathematics developed by Eilenberg and Maclane in 1945.[14] In categories the concept of transformation (transition, map, morphism or arrow) enjoys an autonomous, primary and irreducible role. A category[15] consists of objects $A$, $B$, $C, \ldots$ and arrows (morphisms) $f$, $g$, $h, \ldots$. Each arrow $f$ is assigned an object $A$ as domain and an object $B$ as codomain, indicated by writing $f : A \to B$. If $g$ is an arrow $g : B \to C$ with domain $B$, the codomain of $f$, then $f$ and $g$ can be "composed" to give an arrow $gof : A \to C$. The composition obeys the associative law $ho(gof) = (hog)of$. For each object $A$ there is an arrow $1_A : A \to A$ called the identity arrow of $A$. The analogy with the relational logic of Peirce is evident, $R_{ij}$ stands as an arrow, the composition rule is manifested in Eq. (1) and the identity arrow for $A \equiv S_i$ is $R_{ii}$. There is an important literature on possible ways the category notions can be applied to physics; specifically to quantizing space–time,[16] attaching a formal language to a physical system,[17] studying topological quantum field theories.[18,19]

$R_{ij}$ may receive multiple interpretations: as a transition from the $j$ state to the $i$ state, as a measurement process that rejects all impinging systems except those in the state $j$ and permits only systems in the state $i$ to emerge from the apparatus, as a transformation replacing the $j$ state by the $i$ state. We proceed to a representation of $R_{ij}$

$$R_{ij} = |r_i\rangle\langle r_j|\,, \tag{6}$$

where state $\langle r_i|$ is the dual of the state $|r_i\rangle$ and they obey the orthonormal condition

$$\langle r_i|r_j\rangle = \delta_{ij}\,. \tag{7}$$

It is immediately seen that our representation satisfies the composition rule Eq. (1). The completeness, Eq. (5), takes the form

$$\sum_n |r_i\rangle\langle r_i| = \mathbf{1}\,. \tag{8}$$

All relations remain satisfied if we replace the state $|r_i\rangle$ by $|\varrho_i\rangle$, where

$$|\varrho_i\rangle = \frac{1}{\sqrt{N}} \sum_n |r_i\rangle\langle r_n| \tag{9}$$

with $N$ the number of states. Thus we verify Peirce's suggestion, Eq. (2), and the state $|r_i\rangle$ is derived as the sum of all its interactions with the other states. $R_{ij}$ acts as a projection, transferring from one $r$ state to another $r$ state

$$R_{ij}|r_k\rangle = \delta_{jk}|r_i\rangle\,. \tag{10}$$

We may think also of another property characterizing our states and define a corresponding operator

$$Q_{ij} = |q_i\rangle\langle q_j| \tag{11}$$

with

$$Q_{ij}|q_k\rangle = \delta_{jk}|q_i\rangle \tag{12}$$

and

$$\sum_n |q_i\rangle\langle q_i| = \mathbf{1}\,. \tag{13}$$

Successive measurements of the $q$-ness and $r$-ness of the states is provided by the operator

$$R_{ij}Q_{kl} = |r_i\rangle\langle r_j|q_k\rangle\langle q_l| = \langle r_j|q_k\rangle S_{il} \tag{14}$$

with

$$S_{il} = |r_i\rangle\langle q_l|\,. \tag{15}$$

Considering the matrix elements of an operator $A$ as $A_{nm} = \langle r_n|A|r_m\rangle$, we find for the trace

$$\mathrm{Tr}(S_{il}) = \sum_n \langle r_n|S_{il}|r_n\rangle = \langle q_l|r_i\rangle\,. \tag{16}$$

From the above relation we deduce

$$\mathrm{Tr}(R_{ij}) = \delta_{ij}\,. \tag{17}$$

Any operator can be expressed as a linear superposition of the $R_{ij}$

$$A = \sum_{i,j} A_{ij}R_{ij} \tag{18}$$

with

$$A_{ij} = \mathrm{Tr}(AR_{ji})\,. \tag{19}$$

The individual states can be redefined

$$|r_i\rangle \to e^{i\varphi_i}|r_i\rangle\,, \tag{20}$$

$$|q_i\rangle \to e^{i\theta_i}|q_i\rangle \tag{21}$$

without affecting the corresponding composition laws. However, the overlap number $\langle r_i|q_j\rangle$ changes and therefore we need an invariant formulation for the transition $|r_i\rangle \to |q_j\rangle$. This is provided by the trace of the closed operation $R_{ii}Q_{jj}R_{ii}$

$$\mathrm{Tr}(R_{ii}Q_{jj}R_{ii}) \equiv p(q_j, r_i) = |\langle r_i|q_j\rangle|^2\,. \tag{22}$$

The completeness relation, Eq. (13), guarantees that $p(q_j, r_i)$ may assume the role of a probability since

$$\sum_j p(q_j, r_i) = 1\,. \tag{23}$$

We discover that starting from the relational logic of Peirce we obtain all the essential laws of quantum mechanics. Our derivation underlines the outmost relational nature of quantum mechanics and goes in parallel with the analysis of the quantum algebra of microscopic measurement presented by Schwinger.[20]

Further insights are obtained if we consider the simplified case of only two states ($i = 1, 2$). We define

$$R_z = \frac{1}{2}(R_{11} - R_{22}) \tag{24}$$

and

$$R_+ = R_{12}\,, \qquad R_- = R_{21}\,. \tag{25}$$

These operators satisfy the SU(2) commutation relations

$$[R_z, R_\pm] = \pm R_\pm\,, \qquad [R_+, R_-] = 2R_z \tag{26}$$

and the quadratic Casimir operator

$$R^2 = R_z^2 + \frac{1}{2}(R_+R_- + R_-R_+) \tag{27}$$

can be written as

$$R^2 = \frac{1}{2}\left(\frac{1}{2} + 1\right)\mathbf{1}\,. \tag{28}$$

The underlying dynamics is analogous to an "angular momentum 1/2 particle" and the SU(2) algebra is realized in a way reminiscent of the Schwinger scheme.[21,22] A matrix representation of $R_{ij}$, for the two-states case, is provided by

$$R_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad R_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{29}$$

$$R_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad R_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{30}$$

The matrices

$$\exp(sR_{12}) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \tag{31}$$

$$\exp(tR_{21}) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \tag{32}$$

perform shear transformations in a two-dimensional space,[23] while the matrix

$$\exp[\eta(R_{11} - R_{22})] = \begin{pmatrix} e^\eta & 0 \\ 0 & e^{-\eta} \end{pmatrix} \tag{33}$$

generates squeeze transformations.

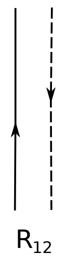


Fig. 1. The relation $R_{12}$. Solid (dashed) line stands for the state 1 (2). A downward (upward) arrow is attached to an impinging (emerging) state.

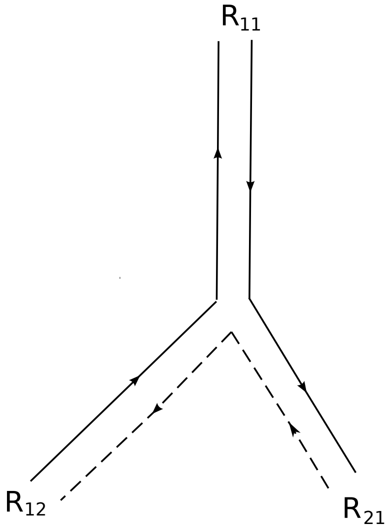


Fig. 2. Pictorial representation of the composition rule $R_{12}R_{21} = R_{11}$.

For the general case of $N$ available states the $R_{ij}$ satisfy the $W_\infty$ algebra

$$[R_{ij}, R_{kl}] = \delta_{jk}R_{il} - \delta_{li}R_{kj} \, . \tag{34}$$

The $W_\infty$ algebras are bosonic extensions of the Virasoro algebra, containing generating currents of higher conformal-spin, in addition to the spin-2 stress tensor of Virasoro (for a review see Ref. 24). They are linked to the area-preserving diffeomorphisms of two-dimensional surfaces.[25,26] $W_\infty$ symmetries are exhibited by a number of systems, among them, $QCD_2$,[27,28] gravity in two-dimensions,[29] bosonic string in four-dimensional Minkowski space.[30] We may proceed to a pictorial representation of the operation $R_{ij}$. Each distinct state $i$ is represented by a specific line (solid, dashed, ...), with a downward (upward) arrow attached to the annihilated (created) state. In this sense we picture $R_{12}$ by a double line, Fig. 1, while the composition rule, for example $R_{12}R_{21} = R_{11}$, is represented by the diagram of Fig. 2. The similarity with string theory, string joining and string splitting, is

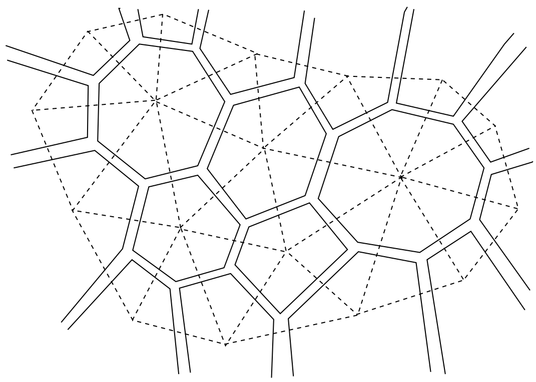

Fig. 3. Random partition of a surface. Each triangle (dashed lines) is dual to a cubic vertex.

obvious. The "cubic-string" interaction may be repeated an indefinite number of times, with vertices connected together and giving rise to different forms of polygons (see Fig. 3). These types of structures can be generated by a random matrix model[10]

$$Z = \int [dM] \exp\left\{-N \operatorname{tr}\left(\frac{1}{2}M^2 + gM^3\right)\right\}, \tag{35}$$

where $M$ are $N \times N$ random matrices. A perturbative expansion of this integral leads to 't Hooft-type Feynman diagrams with cubic vertices. Each such diagram specifies a unique surface topology, with faces arbitrary $n$-gons. The corresponding dual lattice has $n$ lines meeting at a point but the faces are triangles. The result is a triangulated Riemann surface (Fig. 3). An expansion of $Z$ in inverse powers of $N$ is equivalent to a topological expansion, selecting diagrams of specific genus $h$

$$Z = \sum_{h=0}^{\infty} Z_h(g) N^{2-2h}\,. \tag{36}$$

As $g$ is increased successive contributions $Z_h$ diverge at the same critical value $g = g_c$. The partition function can be reorganized into

$$Z = \sum_h F_h g_s^{2h-2}\,, \tag{37}$$

where the "renormalized" string coupling $g_s$ is given by

$$g_s = \frac{1}{N(g-g_s)^{\frac{2-\gamma}{2}}} \tag{38}$$

with $\gamma$ the critical exponent. The continuum two-dimensional string theory is obtained in the double scaling limit $N \to \infty$, $g \to g_c$ with $g_s$ kept fixed.[31–33]

Modern physics is marked by two impressive theoretical constructions: quantum mechanics and string theory. Each of them is an elaborate and detailed theory providing understanding or insights to a host of different problems. Yet, we are lacking a conceptual foundation for these theories. In the present work we have indicated that a form of logic, relational logic developed by C. S. Peirce, may serve as the foundation of both quantum mechanics and string theory. The starting point is that the concept of relation is an irreducible basic datum. All other terms or objects are defined in terms of relations, transformations, morphisms, arrows, structures. Usually, we adhere to mathematical considerations derived within set theory. A set is deprived of any structure, being a plurality of structureless individuals, qualified only by membership (or nonmembership). Accordingly a set-theoretic enterprise is analytic, atomistic, arithmetic. On the other hand, a relational or categorical formulation is bound to be synthetic, holistic, geometric. It appears that quantum theory, string theory and eventually the physical theories to come, are better conceived, analyzed and comprehended within a new paradigm inspired by relational and categorical principles.

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