XI. Diagrams and the Coding of System Structure

A. Signs

We are here concerned with a problem of communication -- specifically, the transmittal of the description of a reticulated system from one human mind to another. We seek a form of description which is complete yet sufficiently succinct, and of such a nature as to permit a verbal transmittal, over the telephone for example. Thus, an encoding of the schematic description is indicated.

To provide a background for this discussion we consider briefly the general theory of signs or semiotics. Charles Sanders PEIRCE states: "A Sign, or Representamen, is a First which stands in such a genuine triadic relation to a Second, called its Object, as to be capable of determining a Third, called its Interpretant, to assume the same triadic relation to its Object in which it stands itself to the same Object."

All sorts of human communication is accomplished by way of a sign-activity. That is, an individual A employs a sign S to communicate an idea of an object O to a second individual B in whose mind an interpretation I (also a sign) is evolved as a result of perceiving S. The situation is not uncommon in engineering analysis wherein the individuals A and B are the same person, and S is a sketch or diagram drawn as an aid in problem-solving -- a form of self-communication.

Peirce is to be credited with the trichotomy of signs into the classes: (i) Icons; (ii) Indices; (iii) Symbols. Quoting directly from Peirce:

> "A sign is either an icon, an index, or a symbol. An icon is a sign which would possess the character which renders it significant, even though its object had no existence; such as a lead-pencil streak as representing a geometrical line. An index is a sign which would, at once, lose the character which makes it a sign if its object were removed, but would not lose that character if there were no interpretant. Such, for instance, is a piece of mould with a bullet hole in it as a sign of a shot; for without the shot there would have been no hole; but there is a hole there, whether anybody has the sense to attribute it to a shot or not. A symbol is a sign which would lose the character which renders it a sign if there were no interpretant. Such is any utterance of speech which signifies what it does only by virtue of its being understood to have that signification."

Thus, an icon is a characterizing sign which exhibits in and by itself the properties which an object must possess to be denoted by it. Examples of icons are photographs, models, star charts, and chemical diagrams.

An index is a directing sign which refers to its object by a dynamical or spatial connection and otherwise bears no resemblance to the object. Sub- and superscripts, index marks, clocks and meters, and anything which focuses attention or startles may be considered an index.

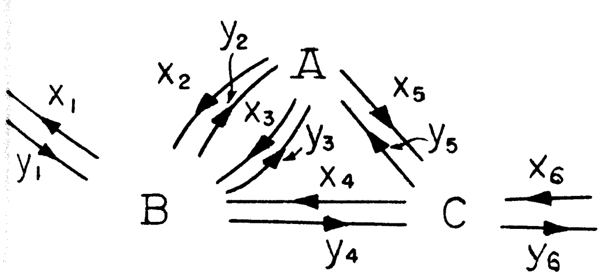
A symbol is a characterizing sign which always involves a rule or convention to establish the connection with the implied object. The utility relies utterly upon the mind of the interpreter to conjure up its meaning and significance. For example, names of people, things, stars, and elements, as well as code marks and mathematical notations, are all symbols.

A sign -- a schematic diagram, for example -- which refers to a physical system as its object, embodies all three classes of sign-action. The bare skeleton of the diagram is iconal, exhibiting directly certain properties of the system. This skeleton, however, is endowed with various labels, arrows, etc. which involve indicial and symbolic sign-action. For example, in a block diagram a component might be labeled "Ψ_1", which directs the reader's attention, or perhaps memory, to the previously made definition of this functional -- as distinguished from the definitions of Ψ_2, Ψ_3, etc. -- and thus involves both indicial and symbolic sign activity.

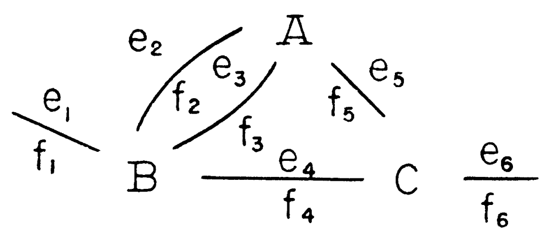
B. Communication of a Computing Structure

Schemata of various sorts -- block diagrams, signal flow graphs, etc. -- are invaluable aids to the description of systems and to the communication of their structure. We are specifically concerned with the problem of describing and communicating the nature of a computing structure, i.e., a network of computing functionals Γ_i. We desire a method which is sufficiently flexible to describe the most general types of nonlinear networks and which will lend itself to encoding for the purpose of verbal transmittal.

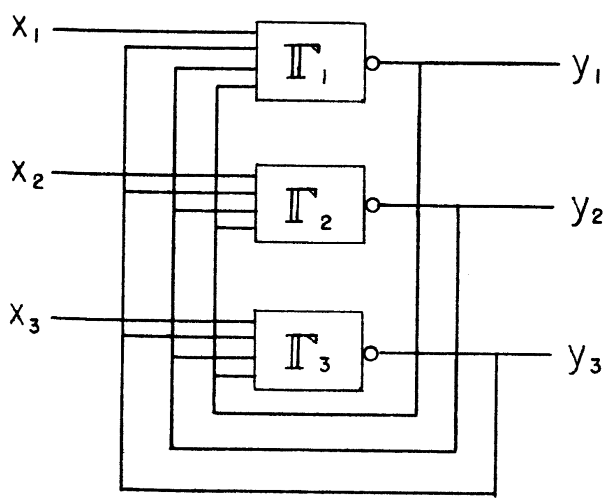
Two essential dichotomies may be discerned in the realm of schematic representations of system structure. The first is now familiar to us: the causal (bilateral signal flow) vs. the non-causal (energy bond) representations. The second dichotomy subdivides the large and variegated class of "branch-node" schemata into, on the one hand, those representations which identify the functional operators with the nodes and the signal variables with the branches (block diagrams); and, on the other hand, those representations which identify the variables with the nodes and the operators with the branches (Mason-Tustin signal flow graph).



Causal Bilateral Signal Flow Diagram



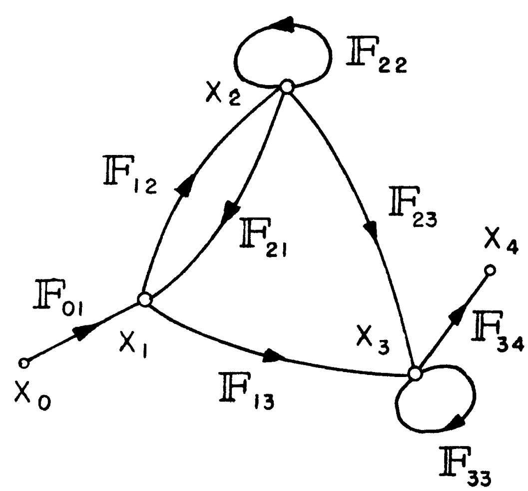
Non-Causal Energy Bond Diagram



Functional Block Diagram

Operators: Nodes

Variables: Branches



Signal Flow Graph

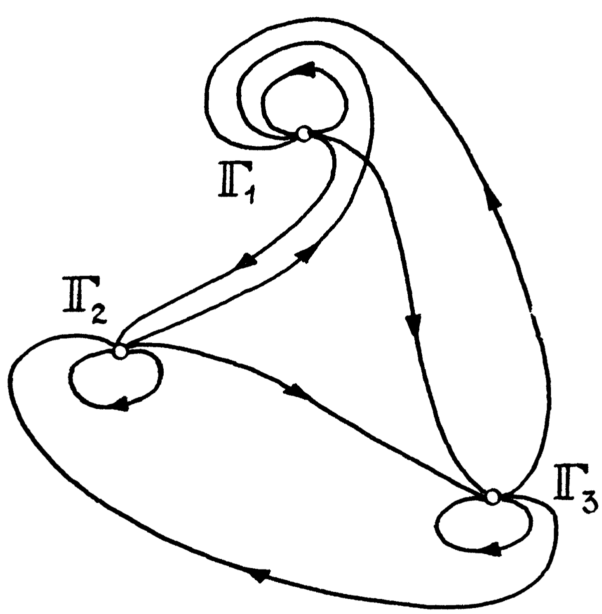
Variables: Nodes

Operators: Branches

The non-causal representation, a generalized circuit diagram, uncluttered and simple, enables the experienced analyst to visualize quickly the behavior of a system, while the causal description is essential for a detailed quantitative understanding of its performance. The block diagram is especially suited to determining the transfer characteristic of a structure of interconnected elements, provided the boundaries of the elements have been correctly chosen. In the case of a computing structure, which is our present concern, these boundaries are generally self evident. The block diagram has the distinct advantage of being applicable to the case of nonlinear as well as linear systems. The signal flow graph, on the other hand, may be used precisely only to describe linear networks since a summary action is implied at each of the nodes; that is, for example

$$x_1 = \mathbb{F}_{o1}\, x_o + \mathbb{F}_{21}\, x_2$$

For all these cases, however, we seek a representation which is capable of being encoded, and for this purpose the following branch-node structure suggests itself:



But this structure may be easily encoded by way of the following tabulation:

Y	Γ	X
1	Γ_1	1,2,3
2	Γ_2	2,3,1
3	Γ_3	3,1,2

Corresponding to each node there is a single output y, that results from the operation of the associated functional Γ upon the input signals, which in this case are simply the outputs of all three nodes. Thus, for example, the first row of the table might be read, "the signal y_1 results from the operation of Γ_1 upon y_1, y_2, and y_3". In actuality, of course, the entries in the Γ -column would indicate the nature of the functionals, say by way of a numerical coding: 1 for an upper selector, 2 for a lower selector, 3 for an integrator, etc. It is thus possible to communicate succinctly a complex structure in the form of a table or sequence of numbers only. The task of transforming this number sequence into a readable diagram and vice versa is almost trivial.

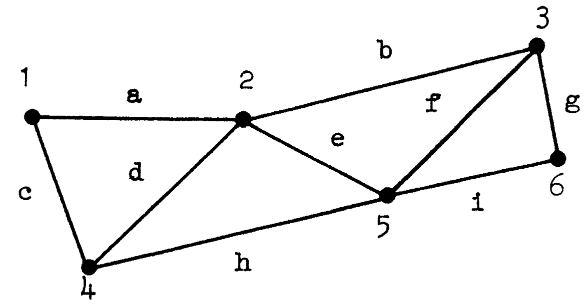
What we have done here is to treat a specific application of the broader theory of graphs, which in turn stems from the mathematical discipline of combinatorial topology. This general study deals with the ways in which the structural connexity of a space may be described and communicated; we recognize this as precisely the problem with which we have been concerned, wherein "the space" happens to include a computing system and the connectedness of interest to us embraces the functional relationships between the several computing components. In combinatorial topology connexity is communicated by way of incidence matrices, a condensed form of which are the coded tables here suggested for use in communicating system structure.

C. Combinatorial Topology - Incidence Matrix

A. W. TUCKER states: "Topology deals with the rudimentary geometrical properties which depend on continuity rather than on size and shape." The domain of discourse is a space in which the topologist attempts to establish theorems related to connexity and structure.

Henri POINCARÉ is generally cited as the originator of this branch of mathematics, which he named analysis situs.

Connexity is depicted by way of linear graphs or, alternatively, by incidence matrices. A linear graph is constituted from nodes and branches. A digraph (directed graph) is a linear graph in which the branches have been endowed with a directional sense. An example of an ordinary linear graph is given below:



In this graph there are nine branches and six nodes. The associated incidence matrix may be easily written:

	a	b	c	d	e	f	g	h	i
1	1	0	1	0	0	0	0	0	0
2	1	1	0	1	1	0	0	0	0
3	0	1	0	0	0	1	1	0	0
4	0	0	1	1	0	0	0	1	0
5	0	0	0	0	1	1	0	1	1
6	0	0	0	0	0	0	1	0	1

In this matrix an entry of "1" indicates a branch-node impingement, while an entry of "0" indicates no impingement. The elements are therefore labelled incidence numbers.

A topological space is a complex constituted from a number of simplexes or cells; these are labelled, according to convention, as follows:

0-cells	:	nodes
1-cells	:	branches
2-cells	:	loops

Hence, the incidence matrix discussed above, which depicted a node-branch structure, is called the "01" incidence matrix, or simply $\mathbb{I}_{01}$. Poincaré defined the numbers

$$a_k = \text{number of k-cells in a complex}$$

$$a_o = \text{number of 0-cells}$$
$$a_1 = \text{number of 1-cells}$$
$$a_2 = \text{number of 2-cells}$$

The rank of the incidence matrix $\mathbb{I}_{k,k+1}$ is denoted r_k. Since no significance has been attributed to $\mathbb{I}_{k,k+1}$ for $k=-1$ it is necessary to restrict this definition to hold only for $k = 0,1,2,\ldots$. Hence, we say that

$$r_k = \text{rank of } \mathbb{I}_{k-1,k} \ (\text{for } k = 0,1,2,\ldots) \ ; \ r_{-1} \equiv 0$$

We also define the k^{th} order Betti number

$$b_k = a_k - r_k - r_{k-1}$$

so that, in particular, the zeroeth and first Betti numbers are given by

$$b_o = a_o - r_o \quad ; \quad b_1 = a_1 - r_1 - r_o$$

which requires that some significance be attached to b_o. Accordingly, we define

$$b_o \equiv \text{number of separate connected parts in a complex.}$$

With this it is now convenient to define the rank R of the linear graph as

$$R \equiv r_o = a_o - b_o$$

which yields an alternative definition of the first Betti number for linear graphs, since $r_1 \equiv 0$, namely

$$b_1 = a_1 - a_o + b_o = a_1 - R$$

It is also propitious to observe that some authors refer to the first Betti number as the nullity, N.

The Euler characteristic is defined in terms of either the b_k or the a_k as follows:

$$K \equiv \sum_k b_k \quad (-1)^k \quad \equiv \sum_k \quad a_k \quad (-1)^k$$

The Euler characteristic for a connected linear graph of V nodes and B branches is simply

$$K = b_o - b_1 = a_o - a_1$$

or $$K = 1 - N = V - B$$

Since R = V - 1 we thus obtain the fundamental invariant relation for all linear graphs

$$\boxed{B = R + N}$$

which is identical to the previous result $b_1 = a_1 - R$.

By way of illustration of the significance of some of the above characteristic numbers three theorems are stated.

Theorem 1. If we start with the 0-cells of a linear graph and add the 1-cells one by one, the number of 1-cells added joining nodes not previously connected is r_o and the number of 1-cells added joining vertices already connected is b_1.

In connection with this theorem it is well to point out that a complex which contains no loops -- i.e., no closed paths within the structure but which would contain a loop with the addition of a single branch, is called a tree. A forest is a complex consisting of a number of disconnected trees.

Theorem 2. The first Betti number of a forest is zero.

Theorem 3. If the first Betti number of a graph is b_1, we can remove b_1 1-cells from it, but no fewer, which will reduce it to a forest.

These theorems are stated without proof for the purpose of illustration only. From them we observe the importance of the rank R and nullity N in the topological characterization of a space.

<u>Background Reading - Signs</u>

(1) PEIRCE, C. S. <u>Philosophical Writings</u>, (edited by J. Buchler), Logic as Semiotic: Theory of Signs.

Peirce presents his form of the theory of signs--the logic of semiotic. Much of the point of view adopted in this course originates with Peirce, although this subject has been taken up and colored by more recent thinkers in this field (and occasionally presented in more readable fashion).

(2) GALLIE, W. B. <u>Peirce and Pragmatism</u>

Gallie presents a compact summary of Peirce's semiosis and theory of signs.

(3) YOUNG, J. W. <u>Lectures on Fundamental Concepts of Algebra and Geometry</u>, pp. 226-239, (Growth of Algebraic Symbolism, by U. G. Mitchell)

The history of the use of symbols in algebra and arithmetic is traced.

(4) MORRIS, C. W. <u>Foundations of the Theory of Signs</u>

Morris presents (without adequate citation) much of Peirce's thought on this subject.

(5) CHERRY, C. <u>On Human Communication</u>, Chap. 3, pp. 219-226.

This is a modern text in which signs are discussed as a part of the broader subject of communication. Much of Peirce's thought is again represented.

(6) TRUXAL, J. G. <u>Automatic Feedback Control System Synthesis</u>, Chap. 2.

A discussion is given of the disadvantages of block diagrams and the Mason signal flow graph is presented as a useful tool in systems analysis.

Background Reading - Topology

(1) SYNGE, J. L. The Fundamental Theorem of Electrical Networks, Quarterly of Applied Math., July 1951, p. 113.

In his development of the theorems and concepts leading up to the "fundamental theorem" the author employs a very readable intuitive approach. Much of this development is purely a discussion of topology and digraphs which is direct support of the material on this subject presented herein.

(2) TUCKER, A. W. The Topological Concept of Space. (A lecture given at the Galois Institute of Mathematics).

Tucker discusses many of the essential concepts of topology without resorting to formal mathematical proofs. Thus, his approach lends itself to a deepening insight into this subject, beyond the superficial statements made in these notes.

(3) SINGER, James. One-Dimensional Analysis Situs. (A lecture given at the Galois Institute of Mathematics). (1935)

This reference contains much of the material used in these notes. The theorems merely stated herein are stated and proved by Singer, as are several additional theorems which concern the structure and connexity of linear graphs.

(4) SINGER, James. Two-Dimensional Analysis Situs. (A lecture given at the Galois Institute of Mathematics). (1936)

Many of the statements made by Tucker are discussed more thoroughly in this reference which extends, along intuitive lines, into the topology of two-dimensional spaces (surfaces).

D. Coded Representation of Graphs and Digraphs

The original branch-node incidence matrix of the previous section may be encoded in a simple array merely by condensing or collapsing either rows or columns in the following alternative fashions:

ROW CODE	
1	a c
2	a b d e
3	b f g
4	c d h
5	e f h i
6	g i

COLUMN CODE	
a	1 2
b	2 3
c	1 4
d	2 4
e	2 5
f	3 5
g	3 6
h	4 5
i	5 6

It is readily apparent that an encoding by rows gains rapidly in efficiency and simplicity as the connexity of the structure increases if the specific node and branch tags are both to be transmitted. Nevertheless we shall have frequent occasion to use both forms of coding as required.

Second (Branch-Loop) Incidence Matrix

In addition to the first (node-branch) incidence matrix, the matrix indicating the cyclic or closed-loop character of the system structure is also of fundamental topological interest. This circuital or branch-loop incidence may be determined for any reticulate system by indicating the incidence of all branches upon N + 1 independent loops where N is the nullity (i.e. the number of branches-out-of-tree) of the structure.

		LOOPS				
		I	II	III	IV	V
B	a	1	1	.	.	.
R	b	1	.	.	1	.
A	c	1	1	.	.	.
N	d	.	1	1	.	.
C	e	.	.	1	1	.
H	f	.	.	.	1	1
E	g	1	.	.	.	1
S	h	1	.	1	.	.
	i	1	.	.	.	1

Dual Graphs

For the graph previously depicted and discussed, the rank $R = 5$ and the nullity $N = 4$. Therefore, for the dual graph, the rank $R^* = 4$ and the nullity $N^* = 5$. This dual graph may be constructed directly from the transpose of the second incidence matrix, merely using the topological dual isomorphism:

$$(N + 1) \text{ Loops} \longleftrightarrow (R^* + 1) \text{ Dual Nodes}$$

$$(B) \text{ Branches} \longleftrightarrow (B^*) \text{ Dual Branches}$$

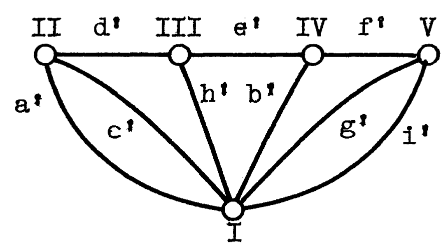
Thus the first incidence matrix of the dual graph is merely the transpose of the second incidence matrix of the original graph. This gives, in coded form:

I	a' b' c' g' h' i'
II	a' c' d'
III	d' e' h'
IV	b' e' f'
V	f' g' i'

corresponding to the graphical form:

DUAL:

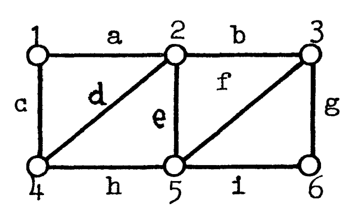
$$R^* = 4$$
$$N^* = 5$$
$$B^* = 9$$



by contrast to the original figure:

ORIGINAL:

$$R = 5$$
$$N = 4$$
$$B = 9$$



A well-known theorem of topology due to Hassler WHITNEY states that a dual graph can exist only for a planar graph (i.e. a linear graph which can be homeomorphically mapped onto a plane or a sphere).

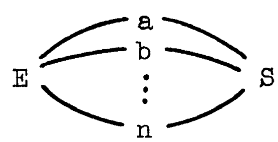
E. Coding the Energetic Structure of Multiport Systems

The previous incidence matrices and equivalent codes may be used for the topological structuring of multiport systems, provided that the system is closed, and the following correspondence is employed:

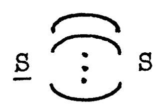
MULTIPORT ELEMENTS $\longleftrightarrow$ NODES

POWER BONDS $\longleftrightarrow$ BRANCHES

First, it is possible to close all otherwise open multiport systems by a simple artifice. Since an n-ported system S must necessarily be bonded to an n-ported environment E , we can always annex the environment to the system itself to form a necessarily closed system, in the fashion:



To emphasize the <u>complementary</u> aspect of the environment in this circumstance we may denote the environment of S by the underscored symbol, $\underline{S}$.
Thus the closed system becomes



The duality between S and $\underline{S}$ is complete since

$$\underline{\underline{S}} \equiv S$$

which means in words that the environment of the environment of a system is the system itself.

Thus if the system S is coded as:

S a b c ... n

then we may designate the environment $\underline{S}$ of S as

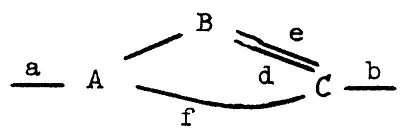
$\underline{S}$ a b c ... n

and the two subsystems as a single closed system becomes in coded form:

S a b c ... n
$\underline{S}$ a b c ... n

If the system is closed to begin with no complementary element, $\underline{S}$, is required for closure.

In the nontrivial case the internal energetic reticulation is also given. For example the structure:



might be encoded as

$\underline{S}$ a b

A a c f

B c d e

C b d e f

From the code itself we may infer the following facts, among others:

1) The overall system is a 2-port

2) It has been reticulated into 3 multiports, namely

   a) two 3-ports

   b) one 4-port

The assignment of CAUSALITY may be accomplished using the code alone as follows:

I ... EFFORT inputs are unmarked;

II ... FLOW inputs are underscored.

Thus a typical causality would be as follows:

$\underline{S}$ $\underline{a}$ b

A a $\underline{c}$ f

B c $\underline{d}$ $\underline{e}$

C $\underline{b}$ d e $\underline{f}$

This means that S itself is of the form

S a $\underline{b}$

since $\underline{S\ a\ \underline{b}} \equiv \underline{S}\ \underline{a}\ b$ is its complement.

The Mechanics of System Interconnection

Consider that we were assembling the original system from the subsystems A, B, C, whose ports have been assigned consistent causality. In this case we would have started with

$$A \quad a\ \underline{b}\ c \quad ; \quad B \quad a'\ \underline{b}'\ \underline{c}' \quad ; \quad C \quad \underline{a}''\ c''\ \underline{d}''$$

The primes would not generally appear in the separate listings and are here indicated only to prevent confusion.

The initial step requires the unique labelling or, better, numbering of all ordered ports, for example as follows:

$$A\ 1.\underline{2}.3. \qquad B\ 4.\underline{5}.\underline{6}. \qquad C\ \underline{7}.8.9.\underline{10}$$

This array corresponds to the (element-bond) incidence matrix

	1	2	3	4	5	6	7	8	9	10
A	1	$\underline{1}$	1	.	.	.	.	.	.	.
B	.	.	.	1	$\underline{1}$	$\underline{1}$	.	.	.	.
C	.	.	.	.	.	.	$\underline{1}$	1	1	$\underline{1}$

The particular interconnections given previously may now be expressed by

$$\begin{aligned} 2 &= 4 \\ 5 &= 8 \\ 6 &= 9 \\ 3 &= 10 \end{aligned}$$

These column identifications result in column additions in the system matrix and reduce the matrix to:

	1	4 + 2	10 + 3	8 + 5	9 + 6	7
A	1	$\underline{1}$	1	.	.	.
B	.	1	.	$\underline{1}$	$\underline{1}$	.
C	.	.	$\underline{1}$	1	1	$\underline{1}$
$\underline{S}$	$\underline{1}$	.	.	.	.	1
	a	b	c	d	e	f

where the environmental $\underline{S}$ row may be added such that the column sum vanishes identically for every column.

The corresponding coded system may now be written:

$$\begin{array}{l} \underline{S}\ \underline{a}\ f \\ A\ a\ \underline{b}\ c \\ B\ b\ \underline{d}\ \underline{e} \\ C\ \underline{c}\ d\ e\ \underline{f} \end{array}$$

To render this in a coded form identical to that of the original, only simple permutation of letters is required in the form:

$$\downarrow \ \frac{f\quad b\quad c}{b\quad c\quad f}$$

This yields the equivalent code

$$\begin{array}{l} \underline{S}\ \underline{a}\ b \\ A\ a\ \underline{c}\ f \\ B\ c\ \underline{d}\ \underline{e} \\ C\ \underline{f}\ d\ e\ \underline{b} \end{array}$$

which is merely a permutation of the first system and is therefore topologically or structurally identical.

Background Reading -- Graphs, Digraphs, and Networks

(1) CAYLEY, A. On the Analytical Forms called Trees, with Application to the Theory of Chemical Combinations, *Report of the British Association for the Advancement of Science*, pp. 257-305 (1875)

(2) KEMPE, A. B. A Memoir on the Theory of Mathematical Form, *Philosophical Transactions*, pp. 1-70 (1886).

A little known and truly remarkable anticipation of combinatorial topology whose origin is usually credited to the papers of POINCARE.

(3) KOENIG, D. *Theorie der Endlichen und Unendlichen Graphen*, Chelsea Publishing Co., New York (1950).

This relatively recent book has now become a classic in this field.

Background Reading -- Graphs, Digraphs, and Networks (continued)

(4) WHITNEY, H. Non-separable and Planar Graphs, Transactions of the American Mathematical Society, Vol. 34, pp. 339-362 (1932).

The author here proves for the first time that duals exist only for planar graphs and therefore for planar logical and electrical networks.

(5) HOHN, F. E., S. SESHU, and D. D. AUFENKAMP. The Theory of Nets, Transactions of the IRE, Vol. EC-6, No. 3, pp. 154-161 (September, 1957).

The authors generalize the concept of a digraph into a net to include certain higher order structural information. Many theorems and properties of universal value may then be adduced.

(6) SHIMBEL, A. Structure in Communication Nets, Proceedings of the Symposium on Information Networks, Polytechnic Institute of Brooklyn, Brooklyn, pp. 199-203 (1954).

This paper propounds concepts and methods which enable the determination of the minimum paths and resultant trees in any communication digraph.

(7) HARARY, F. Structural Duality, Behavioral Science, Vol. 2, No. 4, pp. 255-265 (October, 1957).

A very readable treatment of various duality transformations applied to graphs and digraphs.

(8) GRAYBEAL, T. D. Block Diagram Network Transformation, Electrical Engineering, pp. 985-990 (November, 1951).

(9) STOUT, T. M. A Block-Diagram Approach to Network Analysis, AIEE Transactions, pp. 255-260 (November, 1952).

(10) MASON, S. J., Feedback Theory--Some Properties of Signal Flow Graphs, Proc. Inst. Radio Engrs. 41, 1144-1156 (September, 1953).

(11) .... Feedback Theory--Further Properties of Signal Flow Graphs, Proc. Inst. Radio Engrs. 44, 920-926 (July, 1956).

The four papers above deal with transformations and equivalencies of flow graphs.

(12) SESHU, S. and REED, M. B. Linear Graphs and Electrical Networks .

This excellent text has a summary of much of the above and an excellent bibliography.