RELATIONAL QUANTUM MECHANICS*

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Abstract

We suggest that the inner syntax of Quantum Mechanics is relational logic, a form of logic developed by C. S. Peirce during the years 1870 – 1880. The Peircean logic has the structure of category theory, with relation serving as an arrow (or morphism). At the core of the relational logical system is the law of composition of relations. This law leads to the fundamental quantum rule of probability as the square of an amplitude. Our study of a simple discrete model, extended to the continuum, indicates that a finite number of degrees of freedom can live in phase space. This "granularity" of phase space is determined by Planck's constant h. We indicate also the broader philosophical ramifications of a relational quantum mechanics.

Introduction

Quantum mechanics (QM) stands out as the theory of the 20th century, shaping the most diverse phenomena, from subatomic physics to cosmology. All quantum predictions have been crowned with full success and utmost accuracy. Yet, the admiration

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we feel towards QM is mixed with surprise and uneasiness. QM defies common sense and common logic. Various paradoxes, including Schrodinger's cat and EPR paradox, exemplify the lurking conflict. The reality of the problem is confirmed by the Bell's inequalities and the GHZ equalities. We are thus led to revisit a number of old interlocked oppositions: operator – operand, discrete – continuous, finite –infinite, hardware – software, local – global, particular – universal, syntax – semantics, ontological – epistemological.

The logic of a physical theory reflects the structure of the propositions describing the physical system under study. The propositional logic of classical mechanics is Boolean logic, which is based on set theory. A set theory is deprived of any structure, being a plurality of structure-less individuals, qualified only by membership (or non-membership). Accordingly a set-theoretic enterprise is analytic, atomistic, arithmetic. It was noticed as early as 1936 by Neumann and Birkhoff that the quantum real needs a non-Boolean logical structure. On numerous cases the need for a novel system of logical syntax is evident. Quantum measurement bypasses the old disjunctions subject-object, observer-observed. The observer affects the system under observation and the borderline between ontological and epistemological is blurred. Correlations are not anymore local and a quantum system embodies multiple entanglements. The particular-universal dichotomy is also under revision. While a single quantum event is particular, a plethora of quantum events leads to universal patterns. Viewing the quantum system as a system encoding information, we understand that the usual distinction between hardware and software is not relevant. Most importantly, if we consider the opposing terms being-becoming, we realize that the emphasis is sifted to the becoming, the movement, the process. The underlying dynamics is governed by relational principles and we have suggested [1] that the relational logic of C. S. Peirce may serve as the conceptual foundation of QM.

Peirce, the founder of American pragmatism, made important contributions in science, philosophy, semeiotics and notably in logic. Many scholars (Clifford, Schröder, Whitehead, Lukasiewicz) rank Peirce with Leibniz and Aristotle in the history of thought. Logic, in its most general sense, is the formal science of representation, co-extensive with semeiotics. Algebraic logic attempts to express the laws of thought in the form of mathematical equations, and Peirce incorporated a theory of relations into algebraic logic [2, 3]. Relation is the primary irreducible datum and everything is expressed in terms of relations. A relational formulation is bound to be synthetic, holistic, geometric. Peirce invented also a notation for quantifiers and developed quantification theory, thus he is regarded as one of the principal founders of modern logic.

In the next section we present the structures of the relational logic and a representation of relation which will lead us to the probability rule of QM. In the third section we analyze a discrete system and demonstrate the non-commutation of conjugate operators. In the last section we present the conclusions and indicate directions for future work.

The logic of relations and the quantum rules

The starting point is the binary relation S_iRS_j between the two 'individual terms' (subjects) S_j and S_i. In a short hand notation we represent this relation by R_{ij}. Relations may be composed: whenever we have relations of the form R_{ij}, R_{jl}, a third transitive relation R_{il} emerges following the rule [2, 3]

$$R_{ij}R_{kl} = \delta_{jk}R_{il} \tag{1}$$

In ordinary logic the individual subject is the starting point and it is defined as a member of a set. Peirce, in an original move, considered the individual as the aggregate of all its relations

$$S_i = \sum_j R_{ij}. \tag{2}$$

It is easy to verify that the individual S_i thus defined is an eigenstate of the R_{ii} relation

$$R_{ii}S_i = S_i. \tag{3}$$

The relations R_{ii} are idempotent

$$R_{ii}^2 = R_{ii} \tag{4}$$

and they span the identity

$$\sum_i R_{ii} = \mathbf{1} \tag{5}$$

The Peircean logical structure bears great resemblance to category theory, a remarkably rich branch of mathematics developed by Eilenberg and Maclane in 1945 [4]. In categories the concept of transformation (transition, map, morphism or arrow) enjoys an autonomous, primary and irreducible role. A category [5] consists of objects A, B, C,... and arrows (morphisms) f, g, h,... . Each arrow f is assigned an object A as domain and an object B as codomain, indicated by writing $f : A \to B$. If g is an arrow $g : B \to C$ with domain B, the codomain of f, then f and g can be "composed" to give an arrow $gof : A \to C$. The composition obeys the associative law $ho(gof) = (hog)of$. For each object A there is an arrow $1_A : A \to A$ called the identity arrow of A. The analogy with the relational logic of Peirce is evident, R_{ij} stands as an arrow, the composition rule is manifested in eq. (1) and the identity arrow for $A \equiv S_i$ is R_{ii}. There is an important literature on possible ways the category notions can be applied to physics; specifically to quantising space-time [6], attaching a formal language to a physical system [7], studying topological quantum field theories [8, 9], exploring quantum issues and quantum information theory [10].

A relation R_{ij} may receive multiple interpretations: as the proof of the logical proposition i starting from the logical premise j, as a transition from the j state to the i state, as a measurement process that rejects all impinging systems except those in the state j and permits only systems in the state i to emerge from the apparatus. We proceed to a representation of R_{ij}

$$R_{ij} = |r_i\rangle \langle r_j| \tag{6}$$

where state $\langle r_i|$ is the dual of the state$|r_i\rangle$ and they obey the orthonormal condition

$$\langle r_i|\, r_j\rangle = \delta_{ij} \tag{7}$$

It is immediately seen that our representation satisfies the composition rule eq. (1). The completeness, eq.(5), takes the form

$$\sum_i |r_i\rangle \langle r_i| = \mathbf{1} \tag{8}$$

All relations remain satisfied if we replace the state $|r_i\rangle$ by $|\varrho_i\rangle$, where

$$|\varrho_i\rangle = \frac{1}{\sqrt{N}} \sum_n |r_i\rangle \langle r_n| \tag{9}$$

with N the number of states. Thus we verify Peirce's suggestion, eq. (2), and the state $|r_i\rangle$ is derived as the sum of all its interactions with the other states. R_{ij} acts as a projection, transferring from one r state to another r state

$$R_{ij}\, |r_k\rangle = \delta_{jk}\, |r_i\rangle\,. \tag{10}$$

We may think also of another property characterizing our states and define a corresponding operator

$$Q_{ij} = |q_i\rangle \langle q_j| \tag{11}$$

with

$$Q_{ij}\, |q_k\rangle = \delta_{jk}\, |q_i\rangle \tag{12}$$

and

$$\sum_i |q_i\rangle \langle q_i| = \mathbf{1}. \tag{13}$$

Successive measurements of the q-ness and r-ness of the states is provided by the operator

$$R_{ij}Q_{kl} = |r_i\rangle \langle r_j|\, q_k\rangle \langle q_l| = \langle r_j|\, q_k\rangle\, S_{il} \tag{14}$$

with

$$S_{il} = |r_i\rangle \langle q_l|\,. \tag{15}$$

Considering the matrix elements of an operator A as $A_{nm} = \langle r_n\, |A|\, r_m\rangle$ we find for the trace

$$Tr\left(S_{il}\right) = \sum_n \langle r_n\, |S_{il}|\, r_n\rangle = \langle q_l|\, r_i\rangle\,. \tag{16}$$

>From the above relation we deduce

$$Tr\left(R_{ij}\right) = \delta_{ij}. \tag{17}$$

Any operator can be expressed as a linear superposition of the R_{ij}

$$A = \sum_{i,j} A_{ij} R_{ij} \tag{18}$$

with

$$A_{ij} = Tr\left(A R_{ij}\right). \tag{19}$$

The individual states can be redefined

$$|r_i\rangle \;\rightarrow\; e^{i\varphi_i}\, |r_i\rangle \tag{20}$$

$$|q_i\rangle \;\rightarrow\; e^{i\theta_i}\, |q_i\rangle \tag{21}$$

without affecting the corresponding composition laws. However the overlap number $\langle r_i|\, q_j\rangle$ changes and therefore we need an invariant formulation for the transition $|r_i\rangle \rightarrow |q_j\rangle$. This is provided by the trace of the closed operation $R_{ii}Q_{jj}R_{ii}$

$$Tr\left(R_{ii}Q_{jj}R_{ii}\right) \equiv p\left(q_j, r_i\right) = \left|\langle r_i|\, q_j\rangle\right|^2. \tag{22}$$

The completeness relation, eq. (13), guarantees that $p\left(q_j, r_i\right)$ may assume the role of a probability since

$$\sum_j p\left(q_j, r_i\right) = 1. \tag{23}$$

We discover that starting from the relational logic of Peirce we obtain the essential law of Quantum Mechanics. Our derivation underlines the outmost relational nature of Quantum Mechanics and goes in parallel with the analysis of the quantum algebra of microscopic measurement presented by Schwinger [11].

The emergence of Planck's constant

Consider a chain of N discrete states $|a_k\rangle$, with $k = 1, 2, \ldots, N$. A relation R acts like a shift operator

$$R\,|a_k\rangle = |a_{k+1}\rangle \tag{24}$$
$$R\,|a_N\rangle = |a_1\rangle \tag{25}$$

N is the period of R

$$R^N = \mathbf{1} \tag{26}$$

The numbers which satisfy $a^N = 1$ are given by

$$a_k = \exp\left(2\pi i \frac{k}{N}\right) \qquad k = 1, 2, \ldots, N \tag{27}$$

Then we have

$$R^N - 1 = \left(\frac{R}{a_k}\right)^N - 1 = \left[\left(\frac{R}{a_k}\right) - 1\right] \sum_{j=0}^{N-1} \left(\frac{R}{a_k}\right)^j = 0 \tag{28}$$

R has a set of eigenfunctions

$$R\,|b_i\rangle = b_i\,|b_i\rangle \tag{29}$$

with b_i the N-th root of unity ($b_i = a_i$). It is decomposed like

$$R = \sum_j b_j\,|b_j\rangle\,\langle b_j| \tag{30}$$

Notice that we may write

$$|b_j\rangle\,\langle b_j| = \frac{1}{N} \sum_{k=1}^{N} \left(\frac{R}{b_j}\right)^k \tag{31}$$

The above projection operator acting upon $|a_N\rangle$ will give

$$|b_j\rangle\,\langle b_j|\,a_N\rangle = \frac{1}{N} \sum_{k=1}^{N} \left(\frac{1}{b_j}\right)^k |a_k\rangle \tag{32}$$

Matching from the right with $\langle a_N|$ we obtain

$$\langle a_N|\,b_j\rangle\,\langle b_j|\,a_N\rangle = \frac{1}{N} \tag{33}$$

We adopt the positive root

$$\langle b_j | \, a_N \rangle = \frac{1}{\sqrt{N}} \tag{34}$$

and equ. (32) becomes

$$|b_j\rangle = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \exp\left[-2\pi i \, \frac{jk}{N}\right] |a_k\rangle \tag{35}$$

Inversely we have the decomposition

$$|a_m\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \exp\left[2\pi i \, \frac{mn}{N}\right] |a_n\rangle \, . \tag{36}$$

We introduce another relation Q acting like shift operator

$$\langle b_k | \, Q = \langle b_{k+1} | \tag{37}$$

$$\langle b_N | \, Q = \langle b_1 | \tag{38}$$

The relation Q receives the decomposition

$$Q = \sum_j a_j \, |a_j\rangle \, \langle a_j | \tag{39}$$

Consider now

$$\langle b_k | \, QR = \langle b_{k+1} | \, R = \exp\left[2\pi i \frac{(k+1)}{N}\right] \langle b_{k+1} | \tag{40}$$

$$\langle b_k | \, RQ = \exp\left[2\pi i \frac{k}{N}\right] \langle b_k | \, Q = \exp\left[2\pi i \frac{k}{N}\right] \langle b_{k+1} | \tag{41}$$

We conclude that the conjugate operators R and Q do not commute

$$QR = \exp\left[2\pi i \frac{1}{N}\right] RQ \tag{42}$$

Similarly

$$Q^n R^m = \exp\left[2\pi i \frac{nm}{N}\right] R^m Q^n \tag{43}$$

In our discrete model the non-commutativity is determined by N. As $N \to \infty$ the relation-operators Q and R commute. However it would be hasty to conclude that as

$N \to \infty$ we reach the continuum. The transition from the discrete to the continuum is a subtle affair and many options are available. Let us define

$$L = Na \qquad p = \frac{2\pi}{L} \tag{44}$$

Then

$$\exp\left[2\pi i \frac{1}{N}\right] = \exp\left[ipa\right]. \tag{45}$$

What counts is the size of the available phase space and we may use Planck's constant h as a unit measuring the number of phase space cells. Using rather $\exp\left[\frac{i}{\hbar}pa\right]$, equ.(42) becomes

$$QR = \exp\left[\frac{i}{\hbar}pa\right] RQ \tag{46}$$

Approaching the continuum we may replace the discrete operators by exponential forms

$$R = \exp\left[\frac{i}{\hbar}pX\right] \tag{47}$$

$$Q = \exp\left[\frac{i}{\hbar}aP\right]. \tag{48}$$

With R and Q unitary operators, X and P are hermitian operators. From equs. (46), (47), (48), we deduce

$$[X, P] = i\hbar. \tag{49}$$

The foundational non-commutative law of Quantum Mechanics testifies that there is a limit size $\hbar \sim pa$ in dividing the phase space. With $p \sim mv \simeq mc$ we understand that a represents the Compton wavelength.

Conclusions

We are used first to wonder about particles or states and then about their interactions. First to ask about "what is it" and afterwards "how is it". On the other hand, quantum mechanics displays a highly relational nature. We are led to reorient our thinking and consider that things have no meaning in themselves, and that only the correlations between them are "real" [12]. We adopted the Peircean relational logic as a consistent framework to prime correlations and gain new insights into these theories. The logic of relations leads us naturally to the fundamental quantum rule, the probability as the square of an amplitude. The study of a simple discrete model, once extended to the continuum, reveals that only finite degrees of freedom can live in a given phase space. The "granularity" of phase space (how many cells reside within a given phase space) is determined by Planck's constant h.

Discerning the foundations of a theory is not simply a curiosity. It is a quest for the internal architecture of the theory, offering a better comprehension of the entire theoretical construction and favoring the study of more complex issues. We have indicated elsewhere [13] that a relation may be represented by a spinor. The Cartan – Penrose argument [14, 15], connecting spinor to geometry, allowed us to study geometries using spinors. Furthermore we have shown that space-time may emerge as the outcome of quantum entanglement [16].

It isn't inappropriate to connect category theory and relational logic, the conceptual foundations of quantum mechanics, to broader philosophical interrogations. Relational and categorical principles have been presented by Aristotle, Leibniz, Kant, Peirce, among others. Relational ontology is one of the cornerstones of Christian theology, advocated consistently by the Fathers (notably by Saint Gregory Palamas). We should view then science as a "laboratory philosophy" and always link the meaning of concepts to their operational or practical consequences.

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