Astron. Astrophys. 231, 301–308 (1990)

ASTRONOMY
AND
ASTROPHYSICS

Jeans mass of a cosmological coherent scalar field

M. Bianchi[1,2], D. Grasso[2,3], and R. Ruffini[2,3]

[1] Dipartimento di Fisica, II University of Rome "Tor Vergata", Via Orazio Raimondo, I-00173 Rome, Italy
[2] ICRA-International Center for Relativistic Astrophysics, University of Rome "La Sapienza", Piazzale Aldo Moro 2, I-00185 Rome, Italy
[3] Department of Physics, University of Rome "La Sapienza", Piazzale Aldo Moro 2, I-00185 Rome, Italy

Received April 20, 1988; accepted September 21, 1989

Abstract. Density fluctuations of a cosmological quantum real scalar field in a coherent state are interpreted as Bogoliubov compressional excitations of a condensed Bose-Einstein gas. The Jeans instability mechanism is generalized in this fully quantum context. The evolution of the Jeans mass in a FRW Universe that emerges from an inflationary stage is then studied in the non-relativistic and ultrarelativistic phases. The behavior of the Jeans mass is found to be qualitatively similar, although quantitatively different, to that of a fermion matter field. The introduction of the gauge-invariant formalism is necessary to find the time evolution of the perturbations in the radiation dominated stage.

Key words: cosmology – elementary particles

1. Introduction

The potential role that scalar fields might play in cosmology has recently attracted a great deal of interest from a purely theoretical point of view and may have some consequences for observations (see e.g. Linde, 1982; Albrecht and Steinhardt, 1982; Albrecht et al., 1982). Recent speculations in particle physics and cosmology have proposed various kinds of exotic scalar fields as candidates for a solution of the problem of dark matter in the Universe and particularly in galactic halos (e.g. Preskill et al., 1983; Turner, 1986). Quite apart from these cosmological considerations, attention has also been given to self-gravitating systems of bosons (Ruffini and Bonazzola, 1969). Further results in this field have followed from the introduction of nonlinear analysis (see e.g. Colpi et al., 1986 and Lee, 1987). The aim of the present paper is to analyze the possibility advanced by Baldeschi et al. (1983) that self-gravitating systems of bosons might play a fundamental role in astrophysics. Here we study the possibility of using an instability Jeans mechanism to form self-gravitating configurations from a scalar field in a coherent state. Classical real scalar field coherently oscillating have been studied in the literature in several contexts (see e.g. Turner, 1983; Piran and Williams, 1985). It is known that in the latest phases of the evolution of the Universe, this field behaves as conventional dust with vanishing temperature and pressure so that in many works (see e.g. Ipser and Sikivie, 1983) its Jeans mass is assumed to be zero. In our opinion, a complete physical understanding of this kind of system can be reached only in a fully quantum description.

In this view, we study a coherent state, as defined by Glauber (1963), of a quantum real scalar field gravitationally self-interacting at temperature $T=0$. A macroscopical set of scalar particles in this state coincide with a weakly nonideal condensed Bose-Einstein gas (see e.g. Umezawa et al., 1982). We interpret density perturbations of the quantum scalar field as excitations of this Bose-Einstein condensate. The hydrodynamical behavior of a quantum Bose-Einstein condensate leads us to interpret these excitations as small amplitude density waves (see E.P. Gross, 1966).

In Sect. 2, we show that the dispersion relation of the perturbations of a classical real scalar field obtained by Khlopov et al. (1985) coincides with the Bogoliubov's (1974) energy spectrum of the excitations of boson ground-state. In Sect. 3, we calculate the expectation values of the energy-tensor components on the coherent state of a set of scalar particles delimited in half wavelength of a density fluctuation in a FRW Universe. In this sense we generalize the Jeans theory to a scalar field. The physical mechanism that contributes to a novanishing Jeans mass has the same nature as that which accounts for the equilibrium of the boson stars as studied by Ruffini and Bonazzola (1969). In Sect. 4, we study the behavior of the Jeans mass in an ultrarelativistic limit using a gauge invariant perturbation model. Finally, in Sect. 5, we compare our results to the case of equilibrium of self-gravitating configurations. Section 6 contains our conclusions.

2. Density fluctuations as Bogoliubov excitations

The concept of Bose-Einstein condensation in quantum field theory is strictly connected with that of coherence. The concept of coherence was first introduced in quantum mechanics by E. Schrödinger (1962) to describe the particular state of an oscillator which minimizes the quantum indetermination. It was further developed by R. Glauber (1963) for electromagnetic radiation. The definition of coherent state, given by Glauber, as the state which yields factorization of the correlation functions, coincides with the one given by C.N. Yang (1962), for condensation of an ideal boson gas. The extension to nonideal boson gas (superfluid) was obtained by W. Cumming and J.K. Johnston (1966). In this article we analyze a linear density perturbation of a cosmological quantum coherent scalar field in its ground-state corresponding to a compressional excitation of a weakly nonideal

---

Send offprint requests to: R. Ruffini[3]

Bose-Einstein condensate. We note that the ground-state that we consider is completely equivalent to an axion field ground-state, as studied by Turner et al. (1983). The coherent oscillation of this field at temperature $T=0$ correspond in fact to a large assembly of particles in the same single-particle state with momentum $\boldsymbol{k}=0$, i.e. to a Bose-Einstein condensate.

The energy spectrum of the excitations of a weakly nonideal Bose-Einstein condensate at zero temperature, was first obtained by N.N. Bogoliubov (1974) in Hartee's perturbative scheme. The excitations frequency Ω_k is

$$\Omega_k=\left(\frac{k^4}{4m^2}+\frac{k^2 n_o V_k}{m}\right)^{1/2} \tag{1}$$

where n_o is the number density of particles in the ground-state, m the mass of the bosons, $\mathbf{k}$ the wave number of the excitation and V_k the Fourier transform of the pair potential energy (in this paper we set $\hbar=c=1$). It is easy to verify that in the long-wavelength limit, this expression leads to a linear (phonon-like) dispersion relation (see e.g. Fetter and Walecka, 1971). The Fourier transform of the Newtonian gravitational potential has, in a distributional sense the expression

$$V_k=-4\pi Gm^2\int_0^\infty \frac{\sin kr'}{k}dr'=-\frac{4\pi Gm^2}{k^2} \tag{2}$$

where G is the gravitational constant. The term associated with $\mathbf{k}=0$ has to be subtracted in Hartee's procedure to calculate the energy of the excitations because it corresponds to the interaction of a uniform distribution of mass. Since V_k is negative there is a direct consequence on the stability of the system: from Eq. (1) it follows that Ω_k can acquire imaginary values for sufficiently small values of $\mathbf{k}$. Consequently, the group velocity of sound, defined by $v_s=\partial\Omega_k/\partial k$, becomes imaginary for values of k approaching zero. It is sometimes claimed that this fact leads to a breakdown of the theory, which is considered applicable only for repulsive interactions (see e.g. Fetter and Walecka, 1971). In our opinion the appearance of imaginary values of Ω_k corresponds to a well defined physical instability of the system and the perturbation treatment introduced fails to describe only the development of such instability. Something similar happens for the ground-state of electrons in semiconductors: the introduction of the attractive electron-phonon interaction make the ground-state unstable and lead to the formation of bound states i.e. Copper pairs (see De Gennes, 1966). It is conceivable that a complete description of the development of instability in self-gravitating structures is possible only by taking into account nonlinear effects.

We show how this physical instability is of the same nature as the one studied by M. Yu Khlopov et al. (1985) for a classical real scalar field coherently oscillating in a static flat Universe. Although this field has not a quantum nature the comparison of the results obtained by Khlopov et al. with those that we obtain for a fully quantum coherent scalar field is meaningful in view of the classical behavior of a coherent field. The expected value of physical quantities on a coherent state fulfills, in fact, classical equations (see e.g. Schrödinger, 1962 and Glauber, 1963). In the limit of a weak gravitational field, expressed by the condition $G\rho\ll m^2$, they found for a real field

$$\Omega_k^2=-2m\sqrt{m^2+k^2+4\pi G\rho}+(k^2+2m^2) \tag{3}$$

where ρ is the energy density of the field. It is easy to verify that such a condition coincides with the applicability condition of the perturbative method used by Bogoliubov. Expanding Eq. (3) for $G\rho/m^2\ll 1$ we obtain

$$\Omega_k^2\simeq\frac{k^4}{4m^2}+2\pi G\rho\frac{k^2}{m^2}-4\pi G\rho \tag{4}$$

and using Eq. (2) we can write

$$\Omega_k^2=\frac{k^4}{2m^2}+\frac{V_k\rho k^2}{m^2}\left(1-\frac{k^2}{2m^2}\right) \tag{5}$$

Since we are interested in macroscopic fluctuations, we require that the wavelength of the density fluctuations be larger than the Compton wavelength of the scalar field, namely $k^2\ll m^2$. In this limit the last term on the right of Eq. (5) can be neglected and the Khlopov dispersion relation (given by Eq. 3) becomes equivalent to the excitation spectrum of Bogoliubov (given by Eq. 1) (note that $n_o m=\rho$). We conclude that the Khlopov and Bogoliubov excitations have the same nature. The critical value of the wave number found by Khlopov et al.

$$k_J=2(m\sqrt{\pi G\rho})^{1/2} \tag{6}$$

coincides with the one obtained for propagation of compressional excitations.

To clarify the physical meaning of how the Jeans instability occurs in a coherent scalar field we use a simple quantum mechanics analogue. We consider a single particle in a potential well created by a density fluctuation. The width of the well is $\frac{\lambda}{2}$, where $\lambda=2\pi/k$. The depth, that we approximate to be constant, is

$$\varphi\sim-\frac{GNm^2}{\lambda} \tag{7}$$

From elementary principles of quantum mechanics we deduce that in order to have negative energy levels, i.e. bound states, the depth of the potential well needs to be larger than

$$\varphi_{\rm crit}=\frac{\pi^2}{2m\lambda_{\rm min}^2} \tag{8}$$

This corresponds to the zero-point energy of the particle in the hole. If we identify $\varphi=\varphi_{\rm crit}$ we obtain, for m and N fixed, the minimal wavelength above which the gravitational field of the fluctuation is able to create bound states, that is

$$\lambda_J=\lambda_{\rm crit}\sim\frac{1}{Gm^3N} \tag{9}$$

and clearly coincides with Eq. (6). In fact, if we assume $V_J\sim\lambda_J^3$ we then have

$$\lambda_J\sim\frac{1}{(G\rho m^2)^{1/4}} \tag{10}$$

so that

$$k_J=\frac{2\pi}{\lambda_J}\sim(m\sqrt{G\rho})^{1/2} \tag{11}$$

Thus we can think that zero-point pressure plays the same role which, in the traditional Jeans treatment, is played by the pressure of a thermal gas in equilibrium against gravitational attraction.

In the next section we generalize these considerations to the case of a coherent scalar field in a Friedmann Universe, using the standard theory of linearized perturbations in general relativity.

3. Jeans mass for $H \ll m$

The Lagrangian density of a gravitationally interacting scalar field of mass m is

$$\mathscr{L} = (-g)^{1/2}\frac{1}{2}(g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} - m^2\phi^2) \tag{12}$$

The components of the energy-momentum tensor given by

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}L \tag{13}$$

where $L = (-g)^{-1/2}\mathscr{L}$, can be written as

$$T^{\mu\nu} = g^{\alpha\mu}g^{\beta\nu}\phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} + \frac{1}{2}g^{\mu\nu}m^2\phi^2 \tag{14}$$

We use a Friedmann metric corresponding to zero spatial curvature.

$$ds^2 = dt^2 - R^2(t)\delta_{ij}dx^i dx^j. \tag{15}$$

With this choice Eq. (14) can be written as

$$T^o_o = \frac{1}{2}(\dot\phi^2 - g^{ij}\phi_{,i}\phi_{,j} + m^2\phi^2) \tag{16a}$$

$$T^i_j = -\frac{1}{2}\delta^i_j(\dot\phi^2 + g^{kl}\phi_{,k}\phi_{,l} - m^2\phi^2) + g^{ki}\phi_{,k}\phi_{,j} \tag{16b}$$

$$T^i_o = g^{ii}\phi_{,i}\dot\phi \tag{16c}$$

The equation of motion of the field is

$$\ddot\phi + 3\frac{\dot R}{R}\dot\phi - \Delta\phi + m^2\phi = 0 \tag{17}$$

The field is quantized as

$$\hat\phi(x) = \int d^3k(\hat a_k\psi_k\psi_k(t) + \hat a^\dagger_k\psi^*_k(x)\psi^*_k(t)) \tag{18}$$

In order to have (18) as a solution of (17) we need that

$$\psi(t) = \frac{1}{(2R^3\omega)^{1/2}}e^{-i\int^t\omega dt} \tag{19a}$$

$$\psi_k(x) = e^{ik\cdot x} \tag{19b}$$

where $\omega^2 = (m^2 + k^2/R^2)$, discarding terms of the order of H/m, and k^2/R^2 is the kinetic term.

Considering a fluctuation of wave number k_1 of the background, i.e. on the homogeneous distribution of scalar particles in the same coherent state with momentum $k = 0$, we study the equilibrium of the region delimited by half wavelength of the fluctuation in analogy to the classical Jeans theory. If the perturbation is adiabatic and temperature is vanishing then all the N_1 particles contained in this region are in the same state, defined by

$$|\Psi_{k_1}\rangle = \frac{1}{4\pi\sqrt{N_1!}}\int d\Omega_{k_1}(\hat a^\dagger_{k_1})^{N_1}|0\rangle \tag{20}$$

where $|0\rangle$ is the vacuum state. The integration is performed for all the directions of $\mathbf{k}_1$. Note that $|\Psi_{k_0}\rangle$ must be not confused with the background ground-state that has $k = 0$. However a linear adiabatic fluctuation cannot destroy the coherence so $|\Psi_{k_1}\rangle$ is also a coherent state as we shall demonstrate in Sect. 5.

We calculate the expected values of the components of the energy-momentum tensor on $|\Psi_{k_1}\rangle$. The application of the classical Jeans theory is then conditioned by the vanishing of the expected values of the nondiagonal components of the energy-momentum tensor. From Eq. (16) we observe that the computation of the expectation value of the components of the stress tensor reduce to that of mixed products of the spatial and temporal derivates of ϕ. We find

$$\langle\phi^2\rangle = \frac{1}{(4\pi^2)^2}\int\int d\Omega_{k_o}\Omega_{k_o}\sum_{kk'}[\langle a_k a_{k'}\rangle\psi_k\psi_{k'}\psi^2_k(t) + \langle a_k a^\dagger_{k'}\rangle\psi^*_k\psi_{k'}|\psi_k(t)|^2 + \langle a^\dagger_k a_{k'}\rangle\psi_k\psi^*_{k'}|\psi_k(t)|^2 + \langle a^\dagger_k a^\dagger_{k'}\rangle\psi^*_k\psi^*_{k'}\psi^*_k(t)^2] \tag{21}$$

where $\langle\ldots\rangle \equiv \langle\Psi_{k_1}|\ldots|\Psi_{k_1}\rangle$. The second term in the integrand of (21) leads to a divergence. The nature of such divergence is well known: it is associated with the presence of zero-point energy terms in any mode of the field. It is also known that infinities are removed, introducing a normal ordering of the $\hat a$ and $\hat a^\dagger$ operators (Birrel and Davies, 1982). We note that our result should be substantially modified in a real expanding Universe by effects of creation of particles (see e.g. Parker, 1977). However these effects can be discarded if the expansion rate is considerably smaller than the oscillation frequency of the field i.e. $H \ll m$. In this way the second term becomes the same as the third term in the integral in (21).

Using the commutation rules for the $\hat a$ and $\hat a^\dagger$ operators

$$[a_k, a^\dagger_k] = \delta_{k,k} \quad [a_k, a_{k'}] = 0 \quad [a^\dagger_k, a^\dagger_k] = 0 \tag{22}$$

and the Bogoliubov's prescription, that, in virtue of the macroscopical occupation of $|\Psi_{k_1}\rangle$, transform a_{k_1} and $a^\dagger_{k_1}$ into two c-numbers equal to $\sqrt{N_1}$ we obtain

$$\langle\hat\phi^2\rangle = \frac{N_1}{R^3\omega} \tag{23a}$$

In the same way we obtain

$$\langle\hat\phi^2_o\rangle = \frac{N_1\omega}{R^3} \tag{23b}$$

$$\langle\hat\phi_{,i}\hat\phi_{,j}\rangle = \frac{1}{4\pi^2}\int d\Omega_{k_1}N_1(k_1)_i(k_1)_j|\psi_k(t)|^2 = \frac{1}{3}\delta_{ij}\frac{N_1\boldsymbol{k}^2_1}{R^3\omega} \tag{23c}$$

$$\langle\hat\phi_{,o}\hat\phi_{,i}\rangle = \frac{N_1}{4\pi^2}\dot\psi^*_{k_1}(t)\psi_{k_1}(t)\int d\Omega_{k_1}(k_1)_i + \text{c.c.} = 0 \tag{23d}$$

where c.c. mean the complex conjugate of the preceding term. The last two equations express the fact that, as follows from the isotropy of the system, only the quadratic terms in the ground-state momentum give nonvanishing contributions. Using (16) and (23) we finally obtain

$$\langle T^o_o\rangle = \frac{N_1}{2R^3}\left(\omega + \frac{k^2_1}{\omega R^2} + \frac{m^2}{\omega}\right) \tag{24a}$$

$$\langle T^i_i\rangle = -\frac{3N_1}{2R^3}\left(\omega + \frac{k^2_1}{3\omega R^2} - \frac{m^2}{\omega}\right) \tag{24b}$$

$$\langle T^i_j\rangle = 0 \quad \text{if } i \neq j \tag{24, c}$$

$$\langle T^i_o\rangle = 0 \tag{24d}$$

We observe that the vanishing of the nondiagonal terms of the expectation values of the components of $T_{\mu\nu}$ allows us to treat the scalar field in complete analogy to a perfect fluid. In fact, Eqs. (24)

fulfill the known relation

$$\langle T^{\mu\nu}\rangle = (\rho_1 + p_1)u^\mu u^\nu - p_1 g^{\mu\nu} \quad (25)$$

where $u^\mu = (1, 0, 0, 0)$ and, in a nonrelativistic regime with $k_{ph} \ll m$,

$$\rho_1 = \langle T^o_o\rangle = \frac{N_1 m}{R^3} \quad (26a)$$

$$p_1 = -1/3\langle T^i_i\rangle = \frac{1}{3}\frac{N_1}{R^3}\frac{k_1^2}{mR^2} \quad (26b)$$

are respectively the energy density and the pressure associated to the perturbation. For the ground-state (i.e. $\boldsymbol{k}=0$) it is easy to verify that

$$\rho = \frac{N_o m}{R^3} \quad (27a)$$

$$p = 0 \quad (27b)$$

where N_o is the total number of scalar particle of the background.

The straightforward application of the linearized theory of perturbations in general relativity (see Lifshits and Khalatnikov, 1963 and Weinberg, 1972) allows to develop the background metric as

$$g^{\mu\nu} = g^{(o)\mu\nu} + h^{\mu\nu} \quad (28)$$

A convenient gauge choice is given by the synchronous gauge $h_{oo}=1$, $h_{oi}=0$. Following the notation of Weinberg (1972) we obtain the coupled system of equations

$$\ddot{h} - 2H\dot{h} + 2\left(H^2 - \frac{\ddot{R}}{R}\right)h = -8\pi G R^2(p_1 + 3\rho_1) \quad (29a)$$

$$\dot{\rho}_1 + 3H(\rho_1 + p_1) = -(\rho + p)\left[\frac{\partial}{\partial t}\left(\frac{h}{2R^2}\right) + i\nabla\mathbf{u}\right] \quad (29b)$$

$$\frac{\partial}{\partial t}\left((\rho + p)R^5\nabla\mathbf{u}_1\right) = -R^3\nabla^2 p_1 \quad (29c)$$

where $\mathbf{u}$ is the spatial component of the 4-velocity ($u^o = 1$, $u^i = 0$), $h = \mathrm{tr}h_{ij}$ and p_1, ρ_1 and $\mathbf{u}_1$ are the first order fluctuation amplitudes of the corresponding quantities.

Since we are studying adiabatic perturbation the definition of the sound velocity is

$$v_s^2 \simeq p_1/\rho_1 = \frac{1}{3}\frac{k_{ph}^2}{m^2} \quad (30)$$

where $k_{ph} = k_1/R$. Assuming plane wave behavior, we get in the nonrelativistic regime ($k \ll m$)

$$\ddot{\delta} + 2H\dot{\delta} + \left(\frac{k_{ph}^4}{m^2} - 4\pi G\rho\right)\delta = 0 \quad (31)$$

where $\delta \equiv \delta\rho/\rho$.

We compare this equation with the one obtained by J. Ipser and P. Sikivie (1983) for a pressureless axion field, in which the sound velocity vanishes

$$\ddot{\delta}_a + 2H\dot{\delta}_a \simeq 4\pi G\rho_a\delta_a \quad (32)$$

As underlined by Stecker and Shafi (1983) in this case the Jeans mass of the field results to be zero so that a perturbation is free to grow as soon as it come inside the Hubble horizon.

In analogy with Weinberg (1972), Eq. (31) can be written

$$t^2\ddot{\delta} + \frac{4}{3}t\dot{\delta} + \left(\frac{\Lambda^2}{t^{2\gamma-8/3}} - \frac{2}{3}\right)\delta = 0 \quad (33)$$

where $\Lambda^2 \equiv t^{2\gamma-2/3}v_s^2 k_{ph}^2$ and $v_s^2 = k^2/3R^2m^2 \sim t^{-2/3} = t^{1-\gamma}$ with $\gamma = 5/3$. This equation has a solution of the form (see e.g. Smirnov Vol. II, 1977)

$$\delta = C_1 t^{-1/6} J_{5/2}(3\Lambda^{-1/3}) + C_2 t^{-1/6} J_{5/2}(3\Lambda t^{-1/3}) \quad (34)$$

where $J_{\pm 5/2}$ are first order Bessel functions. The Bessel functions, $J_n(x)$, oscillate for $x \gg 1$ and decrease with a power-law for $x \ll 1$. So we have a critical condition for $t^{1/3} \approx \Lambda$, i.e.

$$t^{-2} \sim 6\pi G\rho \sim v_s^2 k_{ph}^2 \quad (35)$$

which corresponds to the classical Jeans criterion. Inserting in Eq. (35) the sound velocity given by Eq. (30) we obtain

$$k_J^2 \sim m\sqrt{\pi G\rho} \quad (36)$$

in agreement with the result of Khlopov et al. (see Eq. (6)).

The Jeans mass of the perturbation is then given by

$$M_J = \frac{4}{3}\pi\rho\left(\frac{\lambda_J}{2}\right)^3 \approx 10^2\rho^{1/4}\left(\frac{m_{Pl}}{m}\right)^{3/2} \approx 10\left(\frac{m_{Pl}^2}{m}\right)\left(\frac{H}{m}\right)^{1/2} \quad (37)$$

and is, clearly, different from zero.

4. Jeans mass in a radiation dominated phase

The interest in scalar fields in modern cosmology is often connected to the inflationary models of evolution of the Universe. From Eq. (16), written for a Friedmann Universe, without fluctuations,

$$\rho = \langle T^o_o\rangle = \frac{1}{2}(|\dot{\phi}|^2 + m^2|\phi|^2) \quad (38a)$$

$$p = -\frac{1}{3}\langle T^i_i\rangle = \frac{1}{2}(|\dot{\phi}|^2 - m^2|\phi|^2) \quad (38b)$$

we see that a massive scalar field with a potential $V(\phi) = m^2\phi^2$ behaves like a pressureless nonrelativistic matter (dust) as long as it is in its coherently oscillating regime ($\phi \sim e^{imt}$) and $H \ll m$. In fact in this limit $|\dot{\phi}|^2 = m^2|\phi|^2$ and it is evident that $p=0$.

In another limit, when $|\dot{\phi}|^2 \ll m^2|\phi|^2$, the pressure is negative (see Belinsky et al., 1984; Piran and Williams, 1985) and a de Sitter phase of exponential expansion occurs spontaneously if the scalar field dominates the Universe's dynamics. The case with $|\dot{\phi}|^2 \gg m^2|\phi|^2$ corresponds to the equation of state $p = \rho$ and has been studied by Zeldovich (1962) and Belinsky and Khalatnikov (1973). Belinsky et al. (1984) have also shown that the probability of obtaining an era with such an equation of state is practically zero in a flat Universe dominated by a massive scalar field. Two radiation-like matter dominated phases are expected to preceed and to follow this era (see e.g. Branderberger, 1985).

In this section we study the development of gravitational instability in a phase of this kind. During it, we know that the dynamics of the Universe is described by a equation of state $p = \rho/3$ and the sound velocity is

$$v_s^2 = \frac{\partial p}{\partial \rho} = \frac{1}{3} \quad (39)$$

We note that now the scalar field must be considered as highly uncoherent. In fact, there is not a macroscopical occupation of a single state (temperature is now very high) and the field behaves as a classical ultra-relativistic matter (see e.g. Turner, 1983). It is impossible to apply Jean's calculations in their original form. It is known in fact, (see e.g. Branderberger, 1985) that in an inflationary Universe a density perturbation can occur outside the Hubble radius, also called the "effective particle horizon" $r_H \sim H^{-1}$. This radius defines the maximal distance over which the microphysics can act coherently; a synchronization operation is impossible for quantities extending to larger dimensions. For this reason, we need to introduce a "gauge invariant formalism" to give an unambiguous description of the evolution of such perturbations.

We use the approach pioneered by Bardeen (1980) and in particular we re-examine the works of M. Sasaki (1983) concerning scalar gauge-invariant perturbations of classical matter plus a classical real scalar field. Sasaki starts from Bardeen's gauge invariant expression of the energy density in a longitudinal gauge and, discarding the contribution of the classical matter, he obtains a second order relation describing the time evolution of the scalar density fluctuations. Our calculation although similar to that of Sasaki, leads to a different result. We found the following evolution relation

$$\ddot{f}+(2+3v_s^2)\left(\frac{\dot{R}}{R}\right)\dot{f}+\left(k_{\rm ph}^2 v_s^2-\frac{1}{2}\chi(p+\rho)\right)f=0 \tag{40}$$

where $\chi = 8\pi G$ and f is the gauge invariant form of the density contrast associated with the perturbation. It coincide with the well known time evolution equation of density perturbation in a FRW Universe dominated by classical matter (see e.g. Weinberg, 1972, Eq. (15.9.21)). It disagrees with Sasaki's result in the coefficient which multiplies $k_{\rm ph}^2 f$, equal to one in his case, that we found to be v_s^2. Using Eq. (38) in a radiation dominated phase, Eq. (40) can be written as

$$t^2\ddot{f}+\frac{3}{2}t\dot{f}+\left(\frac{k_{\rm ph}^2}{3}t^2-\frac{1}{2}\right)f=0 \tag{41}$$

This equation can be lead back to a Bessel equation and in analogy with the previous section, its solution has the form

$$f=C_1 t^{-1/4} J_{\sqrt{\frac{33}{4}}}\left(\sqrt{\frac{4}{3}}k_{\rm ph}t\right)+C_2 t^{-1/4} J_{-\sqrt{\frac{33}{4}}}\left(\sqrt{\frac{4}{3}}k_{\rm ph}t\right) \tag{42}$$

Again we have a critical condition for the growth of fluctuations that is

$$k_J \approx t^{-1} \approx H \tag{43}$$

so that

$$M_J=\frac{4}{3}\pi\rho\left(\frac{2\pi}{k_J}\right)^3=\frac{1}{6}m_{\rm Pl}^2 H^2\left(\frac{2\pi}{H^3}\right)\approx 10 m_{\rm Pl}^2 H^{-1} \tag{44}$$

As expected, the Jeans length is independent of the particle mass that is negligible in an ultra-relativistic phase. Further, it is easy to verify that (44) coincides with (37) when $H \sim m$. We can see that M_J behaves as $\rho^{-1/2}$ in this phase, similar to that of a fermion field. This fact is not surprising: in a ultrarelativistic regime no difference is expected between the equation of state of fermions and bosons. Thus, according to (44) and (37) (see Fig. 1) we conclude that the evolution of the Jeans mass of a cosmological scalar field that comes out of an inflationary stage, is qualitatively similar to that of a fermion field. The Jeans length increases like the Hubble scale and the density fluctuations are frozen. This occurs as long as the scalar field behaves like radiation and its wavelength is larger than horizon's radius. Later, when H becomes smaller than m and the field behaves like dust, the Jeans mass starts to go down and the fluctuations wavelength enters the cosmological horizon so that bound states can begin to form.

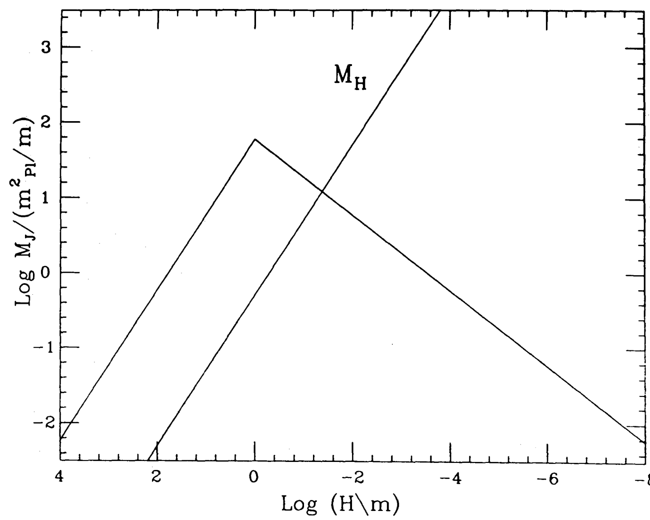

Fig. 1. The logarithm of the Jeans mass of a coherent scalar in $m_{\rm Pl}^2/m$ units is given as a function of the logarithm of the ratio of the Hubble parameter with m. M_J take the maximum value when $H=m$ at the end of an hypothetical ultrarelativistic phase. M_H is the mass of the scalar field contained in the Hubble horizon

5. Comparison with the mass of equilibrium configurations

The behavior of the Jeans mass of a coherent scalar field during the expansion of the Universe results qualitatively similar to the one of a fermion field. Previous works (see e.g. Bond et al., 1980; Ruffini et al., 1986) showed in fact that the Jeans mass of a neutrino field in a nonrelativistic phase is

$$M_J \simeq 6.5\cdot 10^{10} M_\odot \left(\frac{10\,{\rm eV}}{m_\nu}\right)^{7/2}(1+z)^{3/2}$$
$$=6.5\cdot 10^{10}\left(\frac{10\,{\rm eV}}{m_\nu}\right)^{7/2}\frac{H(t)}{H_o} \tag{45}$$

where m_ν is the neutrino mass, z the cosmological redshift $1+z(t)=R_o/R(t)$ and H_o today's Hubble constant. Thus, it results that M_J is proportional to $\rho_\nu^{1/2}$, where ρ_ν is the mass density of the neutrinos. When the field is relativistic we know that (see e.g. Weinberg, 1972)

$$M_J=1.75\cdot 10^{-2}\pi^{5/2}m_{\rm Pl}^3\rho^{-1/2}=8.86\cdot 10^{-1}\,m_{\rm Pl}^2 H^{-1} \tag{46}$$

The similarity of this equation with (44) is not surprising. In a nonrelativistic limit there is no difference between Bose and Fermi equation of state, and in the behavior of the associated Jeans masses. The same does not occur in a nonrelativistic phase. Thus, we obtain a $\rho^{1/4}$ and a $\rho^{1/2}$ behavior of M_J, respectively, for the boson and the fermion fields.

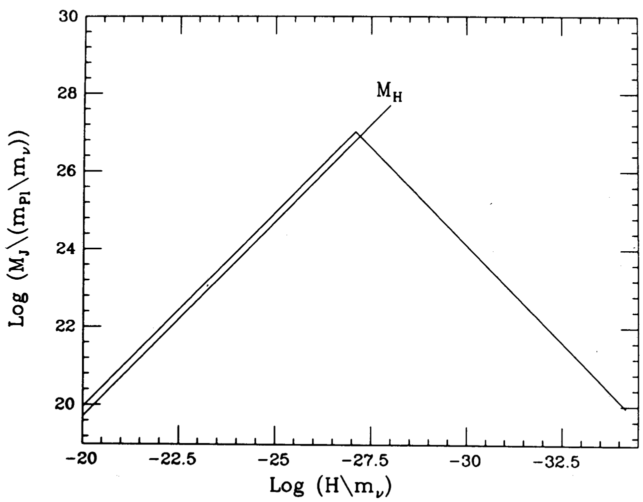

Fig. 2. This is the analogous of Fig. 1 for a neutrino field with mass of 30 GeV

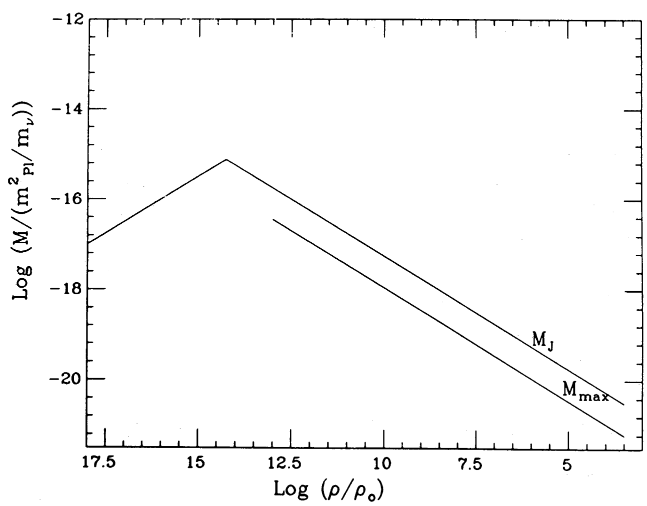

Fig. 3. The Jeans mass of a neutrino field with mass of 30 GeV, given as function of the energy-density of the field, is compared with the mass which maximize the capture rate of a self-gravitating neutrino configuration $M_{\max}$

Fabbri et al. (1982) have shown that the neutrino Jeans mass (in a neutrino dominated Universe) in the nonrelativistic limit, has the same behavior (see Fig. 3) as that of the mass of the configuration which maximizes the particle capture rate in a asymptotically flat space. This critical mass is equal to

$$M_{\max} = 1.3 \cdot 10^{10} M_\odot \left(\frac{10\mathrm{eV}}{m_\nu}\right)^{7/2} (1+z)^{3/2}$$
$$= 1.3 \cdot 10^{10} M_\odot \left(\frac{10\mathrm{eV}}{m_\nu}\right)^{7/2} \left(\frac{\rho}{\rho_o}\right)^{1/2} \quad (47)$$

The same behavior has been found by Ruffini and Bonazzola (1969) for the equilibrium mass of an ideal system of self-gravitating fermions in a Newtonian regime as a function of the central density ρ_c. This coincidence is found although the Jeans mass derives from a linearized treatment while the critical mass found by Fabbri et al. and the equilibrium mass found by Ruffini and Bonazzola arise from a fully nonlinear treatment in general relativity. This fact suggests that we search for a similar coincidence between the boson Jeans mass and the equilibrium mass of a fully condensed self-gravitating boson configuration in the limit $T=0$. This kind of configuration has been studied by Ruffini and Bonazzola (1969).

We note that the ground state of the condensate bosons configurations used by Ruffini and Bonazzola is a coherent state. This state is

$$|\Psi_o\rangle = \frac{1}{\sqrt{N!}} (\hat{a}^\dagger_{10})^N |0\rangle \quad (48)$$

where $\hat{a}^\dagger_{10}$ is the creation operator on the ground state and where $n=1$, $l=0$ are respectively the radial and angular quantum numbers. The coherence of the field is an essential condition for the control of the collapse. A similar fact was noted by Parker and Fulling (1973) with respect to the avoidance of singularities of a quantized scalar field in the first stages of the FWR Universe. Sasaki (1984) noted that $|\Psi_o\rangle$ is not a coherent state. A coherent state, in fact, must be an eigenstate of $\hat{a}_{10}$, in agreement with the definition of coherent state given by Glauber (1963). But, if we take into account the macroscopical occupation of the boson ground state, we recover coherence. Indeed, as noted Bogoliubov (1974), $\hat{a}_{10}$ becomes a c-number equal to $n^{1/2}$:

$$\hat{a}_{10}|\Psi_o\rangle = n^{1/2}|\Psi_o\rangle \quad (49)$$

and so it is evident that $|\Psi_o\rangle$ is a coherent state. In agreement with the result of Ruffini and Bonazzola, the limit central density of a self-gravitating system of bosons, reached in a fully condensate configuration, is (see Ingrosso and Ruffini, 1986)

$$\rho_{c\,\mathrm{lim}} = 10^{-3} m^4 N_o^4 G^3 \quad (50)$$

so we obtain

$$M_{\mathrm{lim}} \approx 10^3 \rho_{c\,\mathrm{lim}}^{1/4} \left(\frac{m_{\mathrm{Pl}}}{m}\right)^{3/2} \quad (51)$$

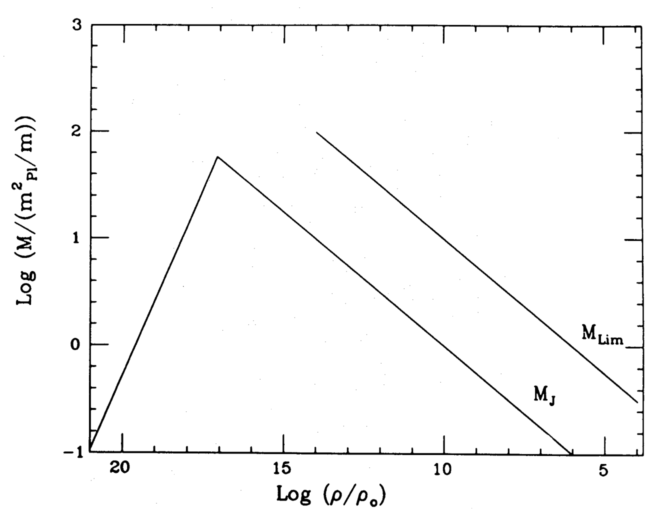

Fig. 4. The Jeans mass of a coherent scalar field with mass of 10^{-57} g, given as a function of the field energy-density, is compared with the mass of the fully condensate self-gravitating configurations in a Newtonian approximation

which is very close to (37) providing the central density ρ_c is replaced by the cosmological energy density of the scalar field (see Fig. 4).

As noted by Ruffini and Bonazzola this result, found in a Newtonian nonrelativistic treatment, is correct until the total mass of a configuration is smaller than a critical value equal to $M_{\rm crit} = m_{\rm Pl}^2/m$ after that, the limit radius of the configuration, equal to

$$R_{\rm lim} \sim \frac{1}{Gm^3 N} \tag{52}$$

becomes smaller than the Schwarzschild radius $R_S = 2GM$. Thus, it is impossible to leave out the effects of general relativity. It is simple to verify that this happens when

$$R_{\rm horizon} = R_{\rm lim} \sim R_{\rm Compton} = m^{-1} \tag{53}$$

It is interesting to note that an analogous behavior occurs in the cosmological case: the scalar field comes out of the relativistic phase just when the Hubble radius of the De-Sitter Universe is equal to the Compton length of the field.

6. Conclusions

We have investigated the stability against gravitational collapse of density fluctuations of a cosmological massive scalar field. The analogous nature of this field to a nonrelativistic coherent state of a superfluid has led us to treat these perturbations as Bogoliubov quantum excitations. In this spirit, the Jeans length found by Khlopov et al. is re-interpreted as the critical wavelength over which the Bogoliubov spectrum takes imaginary values.

The role of the coherent zero-point oscillations, which determines many of the peculiar properties of a superfluid, is also crucial in our case. We have shown in fact how the pressure associated with such oscillations plays the same role as the pressure of a thermal perfect fluid in the original Jeans theory.

The cosmological evolution of the Jeans mass has been analyzed. The vanishing of the expected values of the non-diagonal components of the energy-momentum tensor calculated on the boson ground state in the volume defined by the wavelength of the perturbation led us to write out the dynamic equations of the compressional fluctuations using the Lifshitz-Khalatnikov theory.

A consequence is the appearance of a nonvanishing Jeans mass for the coherent scalar field equal to

$$M_J = \frac{4}{3}\pi\rho\left(\frac{\lambda_J}{2}\right)^3 \approx 10^2 \rho^{1/4}\left(\frac{m_{\rm Pl}}{m}\right)^{3/2}$$

In the analysis of the field in the relativistic phase with $H \gg m$, we have introduced a gauge-invariant perturbations formalism. We have found a dispersion relation substantially different from the one of Sasaki. Consequently, the expression of the Jeans mass in this regime is given by

$$M_J \approx 10 m_{\rm Pl}^3 \rho^{-1/2}$$

The behavior of the Jeans mass of the scalar field which is qualitatively similar to the one of a fermion field has prompted us to search for analogies between the expression of the Jeans mass for a boson field and the mass of a self-gravitating boson configuration

$$M_{\rm lim} \approx 10^3 \rho_{c\,\rm lim}^{1/4}\left(\frac{m_{\rm Pl}}{m}\right)^{3/2}$$

The similarity found between these expressions opens new incentives to the research about nonlinear analysis of equilibrium configurations.

The positive results clearly point to the possibility of forming discrete self-gravitating systems of bosons in an expanding Universe.

Acknowledgements. The authors would like to thank Prof. C. Di Castro, Prof. M. Yu Khlopov, Prof. R.T. Jantzen, Prof. V.F. Mukhanov and Prof. H.G. Ohanian for their useful discussions.

References

Albrecht, A., Steinhardt, P.J.: 1982, *Phys. Rev. Lett.* **48**, 220
Albrecht, A., Steinhardt, P.J., Turner, M.S., Wilczek, F.: 1982, *Phys. Rev. Lett.* **48**, 1437
Baldeschi, M.R., Gelmini, G.B., Ruffini, R.: 1983, *Phys. Rev. Lett.* **112B**, 221
Bardeen, J.: 1980, *Phy. Rev.* **D22**, 1882
Belinsky, V.A., Grishchuk, L.P., Zeldovich, Ya.B., Khalatnikov, I.M.: 1984, in *Proceedings of the Third Seminar on Quantum Gravity*, eds. M.A. Markov, V.A. Berezin, V.P. Frolov, World Scientific
Belinsky, V.A., Khalatnikov, I.M.: 1973, *Sov. Phys. JEPT* **36**, 591
Birrel, N.D., Davies, P.C.W.: 1982, *Quantum Fields in Curved Space*, Cambridge University Press, Cambridge
Bogoliubov, N.N.: 1974, *J. Phys. U.S.S.R.* **11**, 23
Bond, J.R., Efstathion, G., Silk, J.: 1980, *Phys. Rev. Lett.* **45**
Brandenberger, R.H.: 1985, *Rev. of Mod. Phys.* **57**, 1
Colpi, M., Shapiro, S.L., Wasserman, I.: 1986, *Phys. Rev. Lett.* **57**, 2485
Cummings, F.W., Johnston, R.J.: 1966, *Phys. Rev.* **151**, 105
De Gennes, P.G.: 1966, *Superconductivity of Metals and Alloys*, cap. 4, Benjamin
Fabbri, R., Jantzen, R.T., Ruffini, R.: 1982, *Astron. Astrophys.* **114**, 219
Fetter, A.L., Walecka, J.D.: 1971, *Quantum Theory of many particles system*, McGraw-Hill, New York
Gibbons, G.W., Hawking, S.W.: 1977, *Phys. Rev.* **D15**, 2738
Glauber, R.: 1963, *Phys. Rev.* **130**, 2592
Glauber, R.: 1963, *Phys. Rev.* **131**, 2766
Gross, E.P.: 1966, in *Physics of many-particles systems*, ed. E. Meeron
Ipser, J., Sikivie, P.: 1983, *Phys. Rev. Lett.* **50**, 925
Ingrosso, G., Ruffini, R.: 1986, in *Proc. of fourth Marcel Grossmann Meeting*, ed. R. Ruffini, Elsevier, Amsterdam
Khlopov, M.Yu., Malomed, B.A., Zeldovich, Ya.B.: 1985, *Monthly Notices Roy. Astron. Soc.* **215**, 575
Lee, T.D.: 1987, *Phys. Rev.* **D35**, 3640
Lifshitz, E.M., Khalatnikov, I.M.: 1963, *Adv. Phys.* **12**, 185
Linde, A.D.: 1982, *Phys. Lett.* **108B**, 389
Parker, L., Fulling, S.A.: 1973, *Phys. Rev.* **D7**, 2357
Parker, L.: 1977, in *Asymptotic Structure of Space-Time*, eds. F.P. Esposito, L. Witten, Plenum Press, New York
Piran, T., Williams, R.: 1985, *Phys. Lett.* **163B**, 331
Preskill, J., Wise, M., Wilceck, F.: 1983, *Phys. Lett.* **120B**, 127

Ruffini, R., Bonazzola, S.: 1969, *Phys. Rev.* **187**, 1767
Ruffini, R., Song, D.J.: 1986, in *Gamow Cosmology*
Sasaki, M.: 1983, *Prog. Theor. Phys.* **70**, 394
Sasaki, M.: 1984, *Prog. Theor. Phys.* **72**, 1266
Schrödinger, E.: 1962, *Naturwissenschaften* **14**, 664
Smirnov, V.I.: 1964, *A Course of Higher Mathematics*, vol. II, Pergamon Press, New York
Stecker, F.W., Shafi, Q.: 1983, *Phys. Rev. Lett.* **50**, 928
Turner, M.S.: 1983, *Phys. Rev.* **D28**, 1243
Turner, M.S.: 1986, *Phys. Rev.* **D33**, 889
Turner, M.S., Wilczek, F., Zee, A.: 1983, *Phys. Lett.* **B125**, 35
Umezawa, H., Matsumoto, H., Tachiki, M.: 1982, *Thermofields Dynamics and condensate states*, North-Holland
Weinberg, S.: 1972, *Gravitation and cosmology*, Wiley, New York
Yang, C.N.: 1962, *Rev. Mod. Phys.* **34**, 694
Zeldovich, Ya.B.: 1962, *Sov. Phys. JEPT* **14**, 1143